ОБЪЕДИНЕННЫЙ
ИНСТИТУТ
ЯДЕРНЫХ
ИССЛЕДОВАНИЙ

Дубна

E2 - 4861

V.V. Babikov, R.M. Mir-Kasimov

# AN EQUATION FOR THE POTENTIAL SCATTERING AMPLITUDE

ЛАБОРАТОРИЯ ТЕОРЕТИЧЕСКОЙ ФИЗИКИ

1969

E2 - 4861



V.V. Babikov, R.M. Mir-Kasimov

# AN EQUATION FOR THE POTENTIAL SCATTERING AMPLITUDE

Submitted to "Physics Letters".

In many nuclear and atomic problems the so-called phase-function method[1-3] has been successfully exploited. The solutions of the equations of this method have a direct physical interpretation. They are the scattering phase shifts for parts of the given potential contained inside spheres of finite radii $r$. It seems to be of great interest to obtain an analogous equation for the total scattering amplitude without expanding it in partial waves. This would give a useful approach to many scattering problems, especially for potentials without spherical symmetry.

In this note an equation for the so-called scattering function $F(r, \vec{n}_1, k, \vec{n}_2)$ is obtained. This function is the scattering amplitude of a particle with energy $k^2$ ($\vec{n}_1, \vec{n}_2$ are the directions of the initial and final momenta) for a part of the potential $V(\vec{r}\,')\,\theta(r - r')$ contained in a sphere of radius $r$. The asymptotic value $F(\infty, \vec{n}_1, k, \vec{n}_2)$ is the scattering amplitude for the whole potential $V(\vec{r})$.

We start from the integral equation ($\hbar = 2m = 1$) for the wave function $\Psi(\vec{r}, k, \vec{n}_0)$

$$\Psi(\vec{r}, k, \vec{n}_0) = e^{ik\vec{r}\cdot\vec{n}_0} + \int d\vec{r}\,' G^{(+)}(\vec{r}, \vec{r}\,', k)\, V(\vec{r}\,')\, \Psi(\vec{r}\,', k, \vec{n}_0). \qquad (1)$$

The Green function $G^{(+)}(\vec{r},\vec{r}',k) = -\exp(ik|\vec{r}-\vec{r}'|)/4\pi|\vec{r}-\vec{r}'|$ can be represented in the form

$$G^{(+)}(\vec{r},\vec{r}',k) = -\frac{ik}{(4\pi)^2}\{\theta(r-r')\int d\vec{n}_3\, e^{-ikr'\vec{n}'\cdot\vec{n}_3} H^{(1)}(kr,\vec{n}\cdot\vec{n}_3) + \theta(r'-r)\int d\vec{n}_3\, e^{ikr\vec{n}\cdot\vec{n}_3} H^{(1)}(kr',-\vec{n}'\cdot\vec{n}_3)\}, \quad (2)$$

where the following quantity is used

$$H^{(1,2)}(kr,\vec{n}_1\cdot\vec{n}_2) = \frac{1}{kr}\sum_{\ell=0}^{\infty}(2\ell+1)i^{\ell}h_{\ell}^{(1,2)}(kr)P_{\ell}(\vec{n}_1\cdot\vec{n}_2) \quad (3)$$

being an analogue of the plane wave expansion[x]

$$e^{ikr\vec{n}_1\cdot\vec{n}_2} = \frac{1}{kr}\sum_{\ell=0}^{\infty}(2\ell+1)i^{\ell}j_{\ell}(kr)P_{\ell}(\vec{n}_1\cdot\vec{n}_2).$$

Let us note here that the functions $H^{(1,2)}(kr,\vec{n}_1\cdot\vec{n}_2)$ satisfy two important relations

$$\int H^{(1,2)}(kr,\vec{n}_1\cdot\vec{n}_2)\,d\vec{n}_2 = \pm 4\pi\frac{e^{\pm ikr}}{ikr},$$

$$H^{(1,2)}(kr,\vec{n}_1\cdot\vec{n}_2) \xrightarrow[kr\to\infty]{} \pm 4\delta(1\mp\vec{n}_1\cdot\vec{n}_2)\frac{e^{\pm ikr}}{ikr}. \quad (4)$$

---

[x] The Riccati-Bessel $j_{\ell}(kr)$ and Riccati-Hankel $h_{\ell}^{(1,2)}(kr)$ functions are defined here as in [1,2].

From (1)-(4) one finds that when $r \to \infty$ the wave function tends to a superposition of a plane wave propagating along the unit vector $\vec{n}_0$ and an outgoing spherical wave.

In the spirit of the phase-function approach we introduce the amplitude function $A(r, \vec{n}_1, k, \vec{n}_0)$ and the scattering function $F(r, \vec{n}_1, k, \vec{n}_2)$ considering the wave function as a linear superposition of two solutions of the potential free Schroedinger equation

$$\Psi(r, \vec{n}, k, \vec{n}_0) = \int d\vec{n}_1 A(r, \vec{n}_1, k, \vec{n}_0) \{ e^{ikr\vec{n}_1 \cdot \vec{n}} + \frac{ik}{4\pi} \int d\vec{n}_2 F(r, \vec{n}_1, k, \vec{n}_2) H^{(1)}(kr, \vec{n}\cdot\vec{n}_2) \} . \quad (5)$$

Comparing eqs. (1) and (5) we find

$$A(r,\vec{n}_1,k,\vec{n}_0) = \delta(\vec{n}_1 - \vec{n}_0) - \frac{ik}{(4\pi)^2} \int d\vec{r}\,' \theta(r' - r) H^{(1)}(kr', -\vec{n}\,'\cdot\vec{n}_1) V(\vec{r}\,') \Psi(\vec{r}\,', k, \vec{n}_0), \quad (6)$$

$$\int d\vec{n}_1 A(r,\vec{n}_1,k,\vec{n}_0) F(r,\vec{n}_1,k,\vec{n}_2) = -\frac{1}{4\pi} \int d\vec{r}\,' \theta(r - r') e^{-ikr'\vec{n}\,'\cdot\vec{n}_2} V(\vec{r}\,') \Psi(\vec{r}\,', k, \vec{n}_0). \quad (7)$$

Thus $A(\infty, \vec{n}_1, k, \vec{n}_0) = \delta(\vec{n}_1 - \vec{n}_0)$ and $F(\infty, \vec{n}_1, k, \vec{n}_2)$ is the total scattering amplitude for the potential $V(\vec{r})$.

Differentiating eqs. (6) and (7) with respect to $r$ and using eq. (5) we obtain the sought-for equation for $F$

$$\frac{\partial}{\partial r} F(r,\vec{n}_1,k,\vec{n}_2) = -\frac{r^2}{4\pi} \int d\vec{n} V(r,\vec{n}) \{ e^{ikr\vec{n}\cdot\vec{n}_1} + \frac{ik}{4\pi} \int d\vec{n}_3 F(r,\vec{n}_1,k,\vec{n}_3) H^{(1)}(kr, \vec{n}\cdot\vec{n}_3) \} \cdot$$
$$\cdot \{ e^{-ikr\vec{n}\cdot\vec{n}_2} + \frac{ik}{4\pi} \int d\vec{n}_4 F(r,\vec{n}_4,k,\vec{n}_2) H^{(1)}(kr, -\vec{n}\cdot\vec{n}_4) \} \quad (8)$$

with the boundary condition

$$F(0,\vec{n}_1,k,\vec{n}_2) = 0. \tag{9}$$

The integro-differential eq. (8) together with the boundary condition (9) is equivalent to an obvious integral equation.

From eq. (8) one can see that at any finite value $r=R$ the quantity $F(R,\vec{n}_1,k,\vec{n}_2)$ is equal to the scattering amplitude for a part of the potential $V(\vec{r})\,\theta(R-r)$ contained inside a sphere of radius $R$, because $F(R,\vec{n}_1,k,\vec{n}_2) = F(\infty,\vec{n}_1,k,\vec{n}_2)$ in that case.

It can be easily verified that the solution of eq. (8) satisfies the reciprocity relation

$$F(r,\vec{n}_1,k,\vec{n}_2) = F(r,-\vec{n}_2,k,-\vec{n}_1) \tag{10}$$

and for real potentials the unitarity condition

$$F(r,\vec{n}_1,k,\vec{n}_2) - F^*(r,\vec{n}_2,k,\vec{n}_1) = \frac{ik}{2\pi}\int d\vec{n}\,F(r,\vec{n}_1,k,\vec{n})\,F^*(r,\vec{n}_2,k,\vec{n}). \tag{11}$$

For a central potential $V(r)$ the scattering amplitude depends only on the scalar product $\vec{n}_1\cdot\vec{n}_2 = \cos\theta_{12}$ and eq. (8) reduces to

$$\frac{\partial}{\partial r}F(r,k,\cos\theta_{12}) = -\frac{r^2V(r)}{4\pi}\int d\vec{n}_0\Big\{e^{ikr\cos\theta_{01}} + \frac{ik}{4\pi}\int d\vec{n}_3\,F(r,k,\cos\theta_{13})H^{(1)}(kr,\cos\theta_{03})\Big\}\cdot \tag{12}$$

$$\cdot\Big\{e^{-ikr\cos\theta_{02}} + \frac{ik}{4\pi}\int d\vec{n}_4\,F(r,k,\cos\theta_{42})\,H^{(1)}(kr,-\cos\theta_{04})\Big\}.$$

Let us note that eq. (12) can be obtained also if one uses the partial wave expansion

$$F(r,k,\cos\theta)=\frac{1}{k}\sum_{\ell=0}^{\infty}(2\ell+1)\,f_\ell(r,k)\,P_\ell(\cos\theta) \tag{13}$$

and well-known /1-3/ equations for the partial scattering amplitudes

$$\frac{d}{dr}f_\ell(r,k)=-\frac{1}{k}V(r)[j_\ell(kr)+i\,f_\ell(r,k)\,h_\ell^{(1)}(kr)]^2,\; f_\ell(0,k)=0. \tag{14}$$

Eq. (8) can serve as the basis both for numerical computations and for various approximate treatments of scattering problems. In particular the Born approximation is obtained if one neglects in the right-hand side of eq. (8) the terms containing F.

A more complete discussion of eqs. (8), (12), different approaches to their solutions and also an investigation of the functions $H^{(1)}(kr,\cos\theta)$ and $H^{(2)}(kr,\cos\theta)$ will be given elsewhere.

## References


1. V.V.Babikov. Method fazovykh funktsi v kvantovoi mekhanike. Nauka, Moscow, 1968.
2. V.V.Babikov. UFN, 92, 3 (1967). (English transl. Soviet Physics Uspekhi, 10, 271 (1967).
3. F.Calogero. Variable Phase Approach to Potential Scattering. Acad. Press. N.Y., 1967.



Received by Publishing Department
on December, 17, 1969.