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ЛАБОРАТОРИЯ ТЕОРЕТИЧЕСКОЙ ФИЗИКИ

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It has been shown recently in ref./1/ that the equal-time commutation relations e.g. for $t = 0$, between the space integrals of the vector and axial current densities in the $U(3)$-scheme ($V_\mu^a(x)$ and $A_\mu^a(x)$, $\mu = 1,2,3,4$, $a = 0,1, \dots 8$) generate the algebra of $U(6) \times U(6)$. The integrals of the vector current time component and axial current space components generate the algebra of $U(6)$ and these are just the components that survive in the non-relativistic limit

$$I^a = \int V_4^a(x)\, d^3x \;, \qquad J_i^a = \int A_i^a(x)\, d^3x \qquad (1)$$

The integrals (1) do not contain the complete information on the properties of the system. We can obtain additional information, if we consider the current densities directly, instead of their integrals. Such an extension of the algebra was used in refs./2,6, /3/ to calculate the magnetic momenta. These papers were devoted to the algebra of the first momenta of the currents, i.e. the integrals of the vector products of the current densities and the coordinates.

We shall use a somewhat different approach and consider the algebra generated by the current density Fourier components $V_4^a(x)$ and $A_i^a(x)$ $i = 1,2,3$.x)

Note that the matrix elements of the current space integrals are non-zero, only equal total three-dimensional momenta in the initial and final states. Really, translational invariance implies that e.g.

$$\langle q \,|\, V_\mu(\vec{x}) \,|\, p \rangle = e^{i(\vec{p} - \vec{q})\vec{x}} \langle \vec{q} \,|\, V_\mu(0) \,|\, \vec{p} \rangle \;. \qquad (2)$$

and hence

$$\int d^3x \, \langle q \,|\, V_\mu(x) \,|\, p \rangle = (2\pi)^3 \delta(\vec{p} - \vec{q}) \langle q \,|\, V_\mu(0) \,|\, p \rangle \qquad (3)$$

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x) The other components cannot be considered in the approximation, introduced below.

Further, since I^a and J_i^a in (1) are generators of the group U(6), their matrix elements differ from zero only if the initial and final states belong to the same U(6) multiplet. Thus, when calculating e.g. the matrix element

$$\langle f | I^a, J_i^b | i \rangle = \sum_n \langle f | I^a | n \rangle\langle n | J_i^b | i \rangle \qquad (4)$$

where $\sum_n$ means a summation over all intermediate states, and $|i\rangle$ and $|f\rangle$ are states belonging to one and the same U(6) multiplet, only intermediate states $|n\rangle$ belonging to the same U(6) multiplet as the initial and final state, can contribute. Our further approximations will be based on this remark.

Let us consider the matrix elements of the current density Fourier transforms. From (2) we obtain e.g.

$$\int \langle q | V_\mu(x) | p \rangle e^{-i\vec{k}\vec{x}} d^3x = (2\pi)^3 \delta(\vec{p} - \vec{q} - \vec{k}) \langle q | V_\mu(0) | p \rangle \qquad (5)$$

We obtain the following commutation relation for the vector current density time components

$$[V_4^a(\vec{x}), V_4^b(\vec{y})] = \delta(\vec{x} - \vec{y}) f_c^{ab} V_4^c(\vec{x}) \qquad (6)$$

where f_c^{ab} are the structure constants. Multiplying the matrix elements of both sides of (6) between states with the total three-dimensional momenta $\vec{p}$ and $\vec{q}$ by $e^{i(\vec{\ell}\vec{y} - \vec{k}\vec{x})}$ using relation (4) for the matrix elements of operator products and integrating over $\vec{x}$ and $\vec{y}$ we obtain

$$\begin{aligned}\delta(\vec{p} + \vec{\ell} - \vec{q} - \vec{k}) [& \langle f, \vec{q} | V_4^a | n, \vec{q} - \vec{\ell} \rangle\langle n, \vec{p} - \vec{k} | V_4^b | i, \vec{p} \rangle - \\ & - \langle f, \vec{q} | V_4^b | n, \vec{q} + \vec{k} \rangle\langle n, \vec{\ell} + \vec{p} | V_4^a | i, \vec{p} \rangle] = \\ & = \delta(\vec{p} + \vec{\ell} - \vec{q} - \vec{k}) f_c^{ab} \langle \vec{q} | V_4^c | \vec{p} \rangle \end{aligned} \qquad (7)$$

Analogously we have

$$\begin{aligned}\delta(\vec{p} + \vec{\ell} - \vec{q} - \vec{k}) [& \langle f, \vec{q} | A_j^a | n, \vec{q} - \vec{\ell} \rangle\langle n, \vec{p} - \vec{k} | V_4^b | i, \vec{p} \rangle - \\ & - \langle f, \vec{q} | V_4^b | n, \vec{q} + \vec{k} \rangle\langle n, \vec{\ell} + \vec{p} | A_j^a | i, \vec{p} \rangle] = \\ & = \delta(\vec{p} + \vec{\ell} - \vec{q} - \vec{k}) f_c^{ab} \langle \vec{q} | V_j^c | \vec{p} \rangle \end{aligned} \qquad (8)$$

$$\begin{aligned}\delta(\vec{p} + \vec{\ell} - \vec{q} - \vec{k}) [& \langle f, \vec{q} | A_i^a | n, \vec{q} - \vec{\ell} \rangle\langle n, \vec{p} - \vec{k} | A_j^b | i, \vec{p} \rangle - \\ & - \langle f, \vec{q} | A_j^b | n, \vec{q} + \vec{k} \rangle\langle n, \vec{\ell} + \vec{p} | A_i^a | i, \vec{p} \rangle] = \\ & = \delta(\vec{p} + \vec{\ell} - \vec{q} - \vec{k}) [-\delta_{ij} f_c^{ab} \langle \vec{q} | V_4^c | \vec{p} \rangle - \\ & - \epsilon_{ijk} d_c^{ab} \langle \vec{q} | A_k^c | \vec{p} \rangle \end{aligned} \qquad (9)$$

Further we shall consider the commutation relations (7)-(9) for the initial and final state three-momenta equal to zero $\vec{p} = \vec{q} = 0$ and for $\vec{k} = \vec{\ell}$. It has already been mentioned, that for $\vec{k} = 0$ the only non-zero matrix elements in (7)-(9) are these between initial and final states, belonging to the same U(6) multiplet and that the only intermediate states contributing to the left sides, are those belonging to this multiplet.

Let us consider the matrix elements (7)-(9) for small $\vec{k}$. We shall assume that it is possible to neglect the contribution of intermediate states, belonging to other U(6) multiplets than the initial and final state, in the sums on the left-hand sides of (7)-(9) x/.

This will be a good approximation only for the commutation relations (7)-(9) between the generators of U(6), but will not be correct for the other commutation relations, even for $\vec{k} = 0$.

As a formal example let us consider the current matrix elements between states of the unitary triplet. We have:

$$\langle \vec{q} | V_\mu^a | \vec{p} \rangle = \bar{U}(q) [\gamma_\mu F_1[(p-q)^2] + \sigma_{\mu\nu} \frac{(p-q)_\nu}{2m} F_2[(p-q)^2]] \lambda^a U(p) \qquad (10)$$

$$\langle \vec{q} | A_\mu^a | \vec{p} \rangle = \bar{U}(q) [\gamma_\mu \gamma_5 G_1[(p-q)^2] + \frac{i(p-q)_\nu}{m} \gamma_5 G_2[(p-q)^2]] \lambda^a U(p) \qquad (11)$$

where λ^a is a unitary matrix, m - the baryon mass and $F_1(0)$, $G_1(0)$, are constants. Conservation of the vector current implies that $F_1(0) = 1$. Substituting (10) and (11) into (7)-(9) we obtain a system of equations:

$$\left(1 + \frac{k^2}{2m^2}\right)^{-1} \left(1 + \frac{k^2}{4m^2}\right)\left(F_1 - \frac{k^2}{4m^2} F_2\right)^2 = 1 \qquad (12)$$

$$\left(\frac{1 + k^2/4m^2}{1 + k^2/2m^2}\right)^{1/2} \left[G_1 \sigma_i - \frac{k_i}{m} \frac{\vec{\sigma}\vec{k}}{|\vec{k}|} G_2 \right] = G_1(0) \sigma_i \qquad (13)$$

$$\frac{1 + k^2/4m^2}{1 + k^2/2m^2} [G_1(k^2)]^2 = 1, \quad \text{where} \quad k = p - q \qquad (14)$$

which implies unambiguously that

$$F_1(k^2) - \frac{k^2}{4m^2} F_2(k^2) = \left(\frac{1 + \frac{k^2}{2m^2}}{1 + \frac{k^2}{4m^2}} \right)^{1/2} \qquad (15)$$

$$G_1(k^2) = \left(\frac{1 + k^2/2m^2}{1 + k^2/4m^2} \right)^{1/2} \qquad (16)$$

x/ An analogous approximation has been used also in refs /2,3/.

$$G_2(k^2) = 0 \qquad (17)$$

Differentiating (15) with respect to k^2 and setting $k^2 = 0$, we obtain

$$F_1'(0) - \frac{1}{4m^2} F_2(0) = \frac{1}{8m^2}$$

Thus the anomalous magnetic moment is connected with the electrical radius

$$\mu = -½ + 4m^2 F_1'(0) \qquad (18)$$

Similar results were obtained in refs. /2,3/ by introducing magnetic moment operators and their commutation relations. In contrast to Fourier component algebra, this algebra of the multipole moments is not closed.

The same method was used to investigate the form-factors of mesons belonging to the 36-dimensional representation of $U(6)$.

Let us consider the matrix elements

$$< V(q) | V_\mu | V(p) > = -i \bar{V}_\sigma \{ \frac{(p+q)_\mu}{2m} F_1^V(k^2)\, \delta_{\sigma\rho} +$$

$$+ \frac{1}{2m} [\delta_{\mu\sigma} k_\rho - \delta_{\mu\rho} k_\sigma] F_2^V(k^2) + \frac{(p+q)_\mu}{2m} \frac{k_\sigma}{m} \frac{k_\rho}{m} F_3^V(k^2) \} \lambda^a_F V_\rho \qquad (19)$$

$$< P(q) | V^a_\mu | P(p) > = -i(\bar{P} \lambda^a_F P)(p+q)_\mu F^{(p)} \qquad (20)$$

$$< P(q) | A^a_i | V(p) > = ((\bar{P} V_i)_F \lambda^a) H_1 + \frac{k_i k_\alpha}{m^2} ((\bar{P} V_\alpha)_F \lambda^a) H_2 + \frac{(p+q)_i k_\alpha}{m^2} (\bar{P} V_\alpha)_F \lambda^a)$$

Explicit calculations give the relations

$$F_1^V(k^2) = F_1^P(k^2) = \frac{(1 + k^2/2m^2)^{½}}{1 + k^2/4m^2} \qquad (22)$$

$$F_2^V - 2(1 + k^2/4m^2) F_3^V = (1 + k^2/2m^2)^{½} (1 + \frac{k^2}{4m^2})^{-1} \qquad (23)$$

$$H_2 = H_3 = -\frac{H_1(0)}{4} (1 + \frac{k^2}{2m^2})^{½} (1 + \frac{k^2}{4m^2})^{-1} \qquad (24)$$

$$H_1 = H_1(0)\, (1 + \frac{k^2}{2m^2})^{½} \qquad (25)$$

where m is the meson mass.

In particular, the anomalous magnetic moment μ of the vector meson is connected with their quadrupole moment Q by the relation

$$\mu - 2Q = 0 \qquad (26)$$

This relation (similarly as (18)) allows both μ and Q to be equal to zero and is compatible with the model considered in ref /4/.

An application of this theory to the 56-plet will be published separately.

References

1. R.P.Feynman, H.Gell-Mann, G.Zweig. Phys. Rev. Lett., 13 678 (1964).
2. R.Dashen and M.Gell-Mann. Preprint CALT 68-36.
   Y.Dothan, M.Gell-Mann and Y. Neeman. Preprint CALT 68-37.
3. B.W.Lee. Phys. Rev. Lett., 14, 676 (1965).
4. Н.Н.Боголюбов, Нгуен Ван Хьеу, Д.Стоянов, Б.В.Струминский, А.Н.Тавхелидзе, В.П.Шелест. Препринт ОИЯИ, Д-2075 (1965).

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Note added in proof.

For the baryon form-factors in the matrix elements

$$\langle N,\vec{q}|V^a_\mu|N,\vec{p}\rangle = \bar{u}(q)\{\gamma_\mu f^a_1[(p-q)^2] + \sigma_{\mu\nu}\frac{(p-q)_\nu}{2m} f^a_2[(p-q)^2]\}u(p) \quad (27)$$

$$\langle N,\vec{q}|A^a_\mu|N,\vec{p}\rangle = \bar{u}(q)\{\gamma_\mu\gamma_5 g^a_1[(p-q)^2] + \frac{i(p-q)_\mu}{m}\gamma_5 g^a_2[(p-q)^2]\}u(p) \quad (28)$$

$$\langle D,\vec{q}|V^a_\mu|N,\vec{p}\rangle = \bar{u}_\alpha(q)\{(\delta_{\mu\alpha}\gamma_5 - \frac{ip_\alpha}{2m}\gamma_\mu\gamma_5) f^a_3[(p-q)^2] + \frac{ip_\alpha}{m}\sigma_{\mu\lambda}\frac{k_\lambda}{m}\gamma_5 f^a_4[(p-q)^2] + [\frac{i(p-q)_\mu}{m}\frac{ip_\alpha}{m}\gamma_5 + \frac{(p-q)^2}{m^2}\gamma_5\delta_{\mu\alpha}] f^a_5[(p-q)^2]\}u(p) \quad (29)$$

$$\langle D,\vec{q}|A^a_\mu|N,\vec{p}\rangle = \bar{u}_\alpha(q)\{\delta_{\alpha\mu} g^a_3[(p-q)^2] + \frac{i(p_\mu+q_\mu)}{m}\cdot\frac{ip_\alpha}{m} g^a_4[(p-q)^2] + \frac{i(p-q)_\mu}{m}\cdot\frac{ip_\alpha}{m} g^a_5[(p-q)^2] + \frac{ip_\alpha}{m}\gamma_\mu g^a_6[(p-q)^2]\}u(p) \quad (30)$$

where u and u_μ are space wave functions of octet and decimet and

$$f^a_{1,2} = (\bar{N}\lambda^a N)_F F^F_{1,2} + (\bar{N}\lambda^a N)_D F^D_{1,2} + (\bar{N}\lambda^a N)_S F^S_{1,2} \quad (31)$$

$$g^a_{1,2} = (\bar{N}\lambda^a N)_F G^F_{1,2} + (\bar{N}\lambda^a N)_D G^D_{1,2} + (\bar{N}\lambda^a N)_S G^S_{1,2} \quad (32)$$

$$f^a_{3-5} = (\bar{D}\lambda^a N) F_{3-5} \quad (33)$$

$$g^a_{3-5} = (\bar{D}\lambda^a N) G_{3-5} \quad (34)$$

we have obtained the following relations:

$$F^F_1(k^2) - \frac{k^2}{4m^2} F^F_2(k^2) = \frac{(1+k^2/2m^2)^{1/2}}{(1+k^2/4m^2)^{1/2}} \quad (35)$$

$$F^D_1(k^2) - \frac{k^2}{4m^2} F^D_2(k^2) = 0 \quad (36)$$

$$\frac{k^2}{4m^2}[F_3(k^2) + \frac{k^2}{m^2} F_4(k^2) + 4(1+\frac{k^2}{4m^2}) F_5(k^2)] = 0 \quad (37)$$

$$G^D_1(k^2) = \frac{3}{2} G^F_1(k^2) = -3G^S_1(k^2) = \frac{(1+k^2/2m^2)^{1/2}}{(1+k^2/4m^2)^{1/2}} \quad (33)$$

$$G^F_2(k^2) = G^D_2(k^2) = G^S_2(k^2) = 0 \quad (39)$$

$$G_3(k^2) = \pm 2\frac{(1+k^2/2m^2)^{1/2}}{(1+k^2/4m^2)^{1/4}} \quad (40)$$

$$G_4(k^2) + G_5(k^2) = \mp\frac{(1+k^2/2m^2)^{1/2}}{(1+k^2/4m^2)^{3/2}} \quad (41)$$

$$G_6(k^2) = 0. \quad (42)$$

In particular the anomalous magnetic moments and the average quadratic radius are connected in the following manner

$$\mu^F = -\tfrac{1}{2} + \frac{2}{3}m^2\langle r^2\rangle^F_1 \quad (43)$$

$$\mu^D = \frac{2}{3}m^2\langle r^2\rangle^D_1 \quad . \quad (44)$$