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Noncommutative Stationary Processes

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Preface

Research on noncommutative stationary processes leads to an interesting interplay between operator algebraic and probabilistic topics. Thus it is always an invitation to an exchange of ideas between different fields. We explore some new paths into this territory in this book. The presentation proceeds rather systematically and elaborates many connections to already known results as well as some applications. It should be accessible to anyone who has mastered the basics of operator algebras and noncommutative probability but, concentrating on new material, it is no substitute for the study of the older sources (mentioned in the text at appropriate places). For a quick orientation see the Summary on the following page and the Introduction. There are also additional introductions in the beginning of each chapter.

The text is a revised version of a manuscript entitled 'Elements of a spatial theory for noncommutative stationary processes with discrete time index', which has been written by the author as a habilitation thesis (Greifswald, 2002). It is impossible to give a complete picture of all the mathematical influences on me which shaped this work. I want to thank all who have been engaged in discussions with me. Additionally I want to point out that B. Kümmerer and his students C. Hertfelder and T. Lang, sharing some of their conceptions with me in an early stage, influenced the conception of this work. Getting involved with the research of C. Köstler, B.V.R. Bhat, U. Franz and M. Schürmann broadened my thinking about noncommutative probability. Special thanks to M. Schürmann for always supporting me in my struggle to find enough time to write. Thanks also to B. Kümmerer and to the referees of the original manuscript for many useful remarks and suggestions leading to improvements in the final version. The financial support by the DFG is also gratefully acknowledged.

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Summary

In the first chapter we consider normal unital completely positive maps on von Neumann algebras respecting normal states and study the problem to find normal unital completely positive extensions acting on all bounded operators of the GNS-Hilbert spaces and respecting the corresponding cyclic vectors. We show that there exists a duality relating this problem to a dilation problem on the commutants. Some explicit examples are given.

In the second chapter we review different notions of noncommutative Markov processes, emphasizing the structure of a coupling representation. We derive related results on Cuntz algebra representations and on endomorphisms. In particular we prove a conjugacy result which turns out to be closely related to Kümmerer-Maassen-scattering theory. The extension theory of the first chapter applied to the transition operators of the Markov processes can be used in a new criterion for asymptotic completeness. We also give an interpretation in terms of entangled states.

In the third chapter we give an axiomatic approach to time evolutions of stationary processes which are non-Markovian in general but adapted to a given filtration. We call this an adapted endomorphism. In many cases it can be written as an infinite product of automorphisms which are localized with respect to the filtration. Again considering representations on GNS-Hilbert spaces we define adapted isometries and undertake a detailed study of them in the situation where the filtration can be factorized as a tensor product. Then it turns out that the same ergodic properties which have been used in the second chapter to determine asymptotic completeness now determine the asymptotics of nonlinear prediction errors for the implemented process and solve the problem of unitarity of an adapted isometry.

In the fourth chapter we give examples. In particular we show how commutative processes fit into the scheme and that by choosing suitable noncommutative filtrations and adapted endomorphisms our criteria give an answer to a question about subfactors in the theory of von Neumann algebras, namely when the range of the endomorphism is a proper subfactor.

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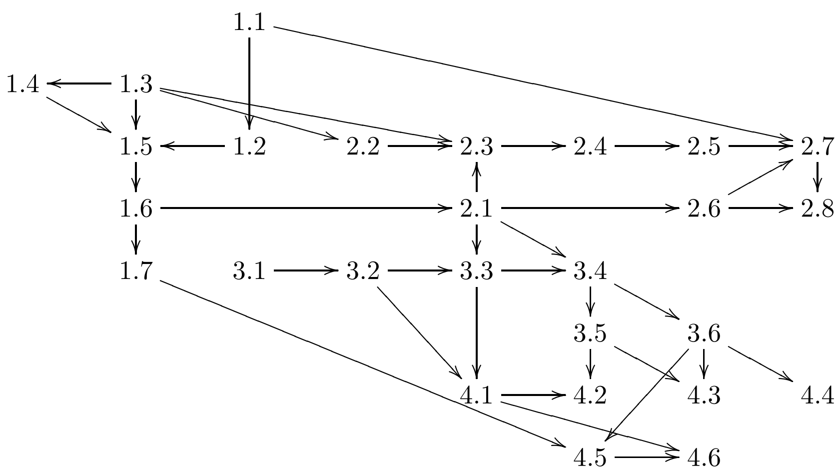
Flow Diagram for the Sections



Introduction

This work belongs to a field called quantum probability or noncommutative probability. The first name emphasizes the origins in quantum theory and the attempts to achieve a conceptual understanding of the new probabilistic features of this theory as well as the applications to physics which such a clarification can offer in return. The second name, which should be read as not necessarily commutative probability, puts the subject into the broader program of noncommutative mathematics and emphasizes the development of mathematical structures. The field has grown large and we do not intend to give a survey here but refer to the books [Da76, Me91, Pa92, Bi95, Ho01, QPC03] for different ways of approaching it. Probability theory in the usual sense appears as a part which is referred to as classical or commutative.

The core of classical probability consists of the theory of stochastic processes and in this respect noncommutative probability follows its predecessor. But the additional freedom to use noncommutative algebras offers vast new possibilities. From the beginning in quantum theory it has been realized that in particular operator algebras offer a rich source, i.e. algebras of operators on a Hilbert space. Especially since the eighties of the last century it has been shown that on a Hilbert space with a special structure, the Fock space, many aspects of classical probability and even rather advanced ones, can be reconstructed in the noncommutative framework in a revealing way. One of the highlights is a theory of noncommutative stochastic integration by R.L. Hudson and K.R. Parthasarathy which can be used as a tool to realize many noncommutative stochastic processes. Also the fundamental processes of classical probability, such as Brownian motion, appear again and they are now parts of noncommutative structures and processes in a very interesting way.

Other aspects come into play if one tries to use the theory of operator algebras more explicitly. This is also done in this work. An important starting point for us is the work done by B. Kümmerer since the eighties of the last century. Here the main idea has been to consider stationary Markov processes. In classical probability Markov processes received by far the most attention

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due to the richness of their algebraic and analytic properties. Stationarity, i.e. the dependence of probability distributions only on time differences, yields connections to further fields of mathematics such as dynamical systems and ergodic theory. The same is true in noncommutative probability. The structure theory of noncommutative stationary Markov processes generalizes many classical properties and exhibits new ones, giving also insights which relate probabilistic notions and models in quantum physics. Stationarity gives rise to time evolutions which are endomorphisms of operator algebras and thus provides a link between research in noncommutative probability and in operator algebras. In this theory the role of the Hilbert space becomes secondary and the abstract structure theory of operator algebras, especially von Neumann algebras, comes into view.

Here we have arrived at a very interesting feature of the theory of operator algebras. While they may be defined as algebras of operators on a Hilbert space, the most interesting of them, such as C^*-algebras or von Neumann algebras, also have intrinsic characterizations. Thus their theory can be developed intrinsically, what we have called abstract structure theory above, or one can study representation theory, also called spatial theory, which uses representations of the elements of the algebra as operators on a Hilbert space. Of course, many properties are best understood by cleverly combining both approaches.

Combining both approaches should also be useful in considering noncommutative stochastic processes. A general idea behind this work can be formulated as follows: For stationary Markov processes or stationary processes in general which can be defined in an abstract way, study some of their properties which become more accessible by including the spatial point of view.

Similar endeavours are of course implicit in many works on noncommutative probability, but starting from abstract stationary processes we can do it more explicitly. The text is based on the author's habilitation thesis with the more lengthy and more precise title 'Elements of a spatial theory for noncommutative stationary processes with discrete time index'. We have already explained what we mean by 'spatial'. The precise titel also makes clear that we do not intend to write a survey about all that is known about noncommutative processes. In particular the restriction to discrete time steps puts aside a lot of work done by quantum probabilists. While there are parts of this text where generalization to continuous time is rather obvious there are other parts where it is not, and it seems better to think about such things at a separate place.

On the other hand, by this restriction we open up the possibility to discard many technicalities, to concentrate on very basic problems and to discuss the issue how a systematic theory of noncommutative stationary processes may look like. Guided by the operator algebraic and in particular the corresponding spatial point of view we explore features which we think should be elements of a general theory. We will see analogies to the theory of commutative stationary processes and phenomena which only occur in the noncommutative setting.

It is encouraging that on our way we also achieve a better understanding of the already known approaches and that some applications to physics show up. It is clear however that many things remain to be done. The subject is still not mature enough for a definite top-down axiomatic treatment and there is much room for mental experimentation.

Now let us become more specific. Classical Markov processes are determined by their transition operators and are often identified with them, while for the noncommutative Markov processes mentioned above this is no longer the case. A very natural link between the classical and the noncommutative case occurs when they are both present together, related by extension respectively by restriction. Using spatial theory, more precisely the GNS-construction, we introduce the notion of an extended transition operator which acts on all bounded operators on the GNS-Hilbert space. This notion plays a central role in our theory and many sections study the delicate ways how extended transition encodes probabilistic information. While the original transition operator may act on a commutative or noncommutative algebra, the extended transition operator always acts on a noncommutative algebra and thus can only be considered as a probabilistic object if one includes noncommutative probability theory. In Chapter 1 we give the definitions and explore directly the relations between transition and extended transition. There exists a kind of duality with a dilation problem arising from the duality between algebras and commutants, and studying these problems together sheds some light on both. We introduce the concept of a weak tensor dilation in order to formulate a one-to-one correspondence between certain extensions and dilations. The study of this duality is the unifying theme of Chapter 1. We also give some examples where the extensions can be explicitly computed.

In Chapter 2 we study the significance of extended transition for Markov processes. In B. Kümmerer's theory of noncommutative stationary Markov processes their coupling structure is emphasized. Such a coupling representation may be seen as a mathematical structure theorem about noncommutative Markov processes or as a physical model describing the composition of a quantum system as a small open system acted upon by a large reservoir governed by noise. In this context we now recognize that the usefulness of extended transition lies mainly in the fact that it encodes information on the coupling which is not contained in the original transition operator of the Markov process. This encoding of the relevant information into a new kind of transition operator puts the line of thought nearer to what is usual in classical probability. This becomes even more transparent if one takes spatial theory one step further and extends the whole Markov process to an extended Markov process acting on all bounded operators on the corresponding GNS-Hilbert space. Here we notice a connection to the theory of weak Markov processes initiated by B.V.R. Bhat and K.R. Parthasarathy and elaborated by Bhat during the nineties of the last century. To connect Kümmerer's and Bhat's approaches by an extension procedure seems to be a natural idea which has not been studied up to now, and we describe how it can be done in our context.

For a future similar treatment of processes in continuous time this also indicates a link to the stochastic calculus on Fock space mentioned earlier. In fact, the invariant state of our extended process is a vector state, as is the Fock vacuum which is in most cases the state chosen to represent processes on Fock space. The possibility to get pure states by extension is one of the most interesting features of noncommutativity. Part of the interest in Fock space calculus always has been the embedding of various processes, such as Brownian motion, Poisson processes, Lévy processes, Markov processes etc., commutative as well as noncommutative, into the operators on Fock space. Certainly here are some natural possibilities for investigations in the future.

In Chapter 2 we also explore the features which the endomorphisms arising as time evolutions of the processes inherit from the coupling representation. This results in particular in what may be called coupling representations of Cuntz algebras. A common background is provided by the theory of dilations of completely positive maps by endomorphisms, and in this rerspect we see many discrete analogues of concepts arising in W. Arveson's theory of E_0-semigroups.

The study of cocycles and coboundaries connecting the full time evolution to the evolution of the reservoir leads to an application of our theory to Kümmerer-Maassen-scattering theory. In particular we show how this scattering theory for Markov processes can be seen in the light of a conjugacy problem on the extended level which seems to be somewhat simpler than the original one and which yields a new criterion for asymptotic completeness. An interpretation involving entanglement of states also becomes transparent by the extension picture. Quantum information theory has recently rediscovered the significance of the study of entanglement and of related quantities. Here we have a surprising connection with noncommutative probability theory. Some interesting possibilities for computations in concrete physical models also arise at this point.

Starting with Chapter 3 we propose a way to study stationary processes without a Markov property. We have already mentioned that stationarity yields a rich mathematical structure and deserves a study on its own. Further, an important connection to the theory of endomorphisms of operator algebras rests on stationarity and one can thus try to go beyond Markovianity in this respect. We avoid becoming too broad and unspecific by postulating adaptedness to a filtration generated by independent variables, and independence here means tensor-independence. This leads to the concept of an adapted endomorphism. There are various ways to motivate this concept. First, in the theory of positive definite sequences and their isometric dilations on a Hilbert space it has already been studied, in different terminology. Second, it is a natural generalization of the coupling representation for Markov processes mentioned above. Third, category theory encourages us to express all our notions by suitable morphisms and this should also be done for the notion of adaptedness. We study all these motivations in the beginning of Chapter 3 and then turn to applications for stationary processes.

It turns out that in many cases an adapted endomorphism can be written as an infinite product of automorphisms. The factors of this product give some information which is localized with respect to the filtration and can be thought of as building the endomorphism step by step. Such a successive adding of time steps of the process may be seen as a kind of 'horizontal' extension procedure, not to be confused with the 'vertical' extensions considered earlier which enlarge the algebras in order to encode better the information about a fixed time step. But both procedures can be combined. In fact, again it turns out that it is the spatial theory which makes some features more easily accessible.

The applications to stationary processes take, in a first run, the form of a structure theory for adapted isometries on tensor products of Hilbert spaces. Taking a hint from transition operators and extended transition operators of Markov processes we again define certain completely positive maps which encode properties in an efficient way. We even get certain dualities between Markov processes and non-Markovian processes with this point of view. These dualities rely on the fact that the same ergodic properties of completely positive maps which are essential for our treatment of asymptotic completeness in Kümmerer-Maassen scattering theory also determine the asymptotics of nonlinear prediction errors and answer the question whether an adapted endomorphism is an automorphism or not.

While such product representations for endomorphisms have occurred occasionally in the literature, even in the work of prominent operator algebraists such as A. Connes and V.F.R. Jones and in quantum field theory in the form developed by R. Longo, there exists, to the knowledge of the author, no attempt for a general theory of product representations as such. Certainly such a theory will be difficult, but in a way these difficulties cannot be avoided if one wants to go beyond Markovianity. The work done here can only be tentative in this respect, giving hints how our spatial concepts may be useful in such a program.

Probably one has to study special cases to find the most promising directions of future research. Chapter 4 provides a modest start and treats the rather abstract framework of Chapter 3 for concrete examples. This is more than an illustration of the previous results because in all cases there are specific questions natural for a certain class of examples, and comparing different such classes then leads to interesting new problems. First we cast commutative stationary adapted processes into the language of adapted endomorphisms, which is a rather uncommon point of view in classical probability. More elaboration of the spatial theory remains to be done here, but we show how the computation of nonlinear prediction errors works in this case. Noncommutative examples include Clifford algebras and their generalizations which have some features simplifying the computations. Perhaps the most interesting but also rather difficult case concerns filtrations given by tensor products of matrices. Our criteria can be used to determine whether the range of an adapted endomorphism is a proper subfactor of the hyperfinite II_1-factor, making contact to a field of research in operator algebras. However here we have included only the most immediate observations, and studying these connections is certainly a work on its own. We close this work with some surprising observations about extensions of adapted endomorphisms, exhibiting phenomena which cannot occur for Markov processes. Remarkable in this respect is the role of matrices which in quantum information theory represent certain control gates.

There is also an Appendix containing results about unital completely positive maps which occur in many places of the main text. These maps are the transition operators for noncommutative processes, and on the technical level it is the structure theory of these maps which underlies many of our results. It is therefore recommended to take an early look at the Appendix.

It should be clear by these comments that a lot of further work can be done on these topics, and it is the author's hope that the presentation in this book provides a helpful starting point for further attempts in such directions.

Preliminaries and notation

$\mathbb{N} := \{1, 2, 3, \ldots\}$ and $\mathbb{N}_0 := \{0, 1, 2, 3, \ldots\}$

Hilbert spaces are assumed to be complex and separable: $\mathcal{G},\ \mathcal{H},\ \mathcal{K},\ \mathcal{P}, \ldots$

The scalar product is antilinear in the first and linear in the second component.

Often $\xi \in \mathcal{G},\ \xi \in \mathcal{H},\ \eta \in \mathcal{K},\ \eta \in \mathcal{P}$.

Ω is a unit vector, often arising from a GNS-construction.

Isometries, unitaries: $v,\ u$

Projection on a Hilbert space always means orthogonal projection: $p,\ q$

p_ξ denotes the one-dimensional projection onto $\mathbb{C}\xi$. Sometimes we also use Dirac notation, for example $p_\xi = |\,\xi\rangle\langle\xi\,|$.

M_n denotes the $n \times n$-matrices with complex entries,

$\mathcal{B}(\mathcal{H})$ the bounded linear operators on $\mathcal{H}$.

'stop' means: strong operator topology

'wop' means: weak operator topology

$\mathcal{T}(\mathcal{H})$ trace class operators on $\mathcal{H}$

$\mathcal{T}^1_+(\mathcal{H})$ density matrices $= \{\rho \in \mathcal{T}(\mathcal{H}) : \rho \geq 0,\ Tr(\rho) = 1\}$

Tr is the non-normalized trace and tr is a tracial state.

Von Neumann algebras $\mathcal{A} \subset \mathcal{B}(\mathcal{G}),\ \mathcal{B} \subset \mathcal{B}(\mathcal{H}),\ \mathcal{C} \subset \mathcal{B}(\mathcal{K})$ with

normal states ϕ on $\mathcal{A}$ or $\mathcal{B}$, ψ on $\mathcal{C}$.

Note: Because $\mathcal{H}$ is separable, the predual $\mathcal{A}_*$ of $\mathcal{A} \subset \mathcal{B}(\mathcal{H})$ is separable and there exists a faithful normal state for $\mathcal{A}$, see [Sa71], 2.1.9 and 2.1.10.

By 'stochastic matrix' we mean a matrix with non-negative entries such that all the row sums equal one.

We use the term 'stochastic map' as abbreviation for 'normal unital completely positive map': $S,\ T$ (compare also A.1),

in particular $Z : \mathcal{B}(\mathcal{G}) \to \mathcal{B}(\mathcal{H})$.

$\mathcal{Z}$ denotes a certain set of stochastic maps, see 1.2.1.

$S : (\mathcal{A}, \phi_{\mathcal{A}}) \to (\mathcal{B}, \phi_{\mathcal{B}})$ means that the stochastic map S maps $\mathcal{A}$ into $\mathcal{B}$ and respects the states $\phi_{\mathcal{A}}$ and $\phi_{\mathcal{B}}$ in the sense that $\phi_{\mathcal{B}} \circ S = \phi_{\mathcal{A}}$.

Preadjoints of stochastic maps: $C,\ D, \ldots$

Homomorphism of a von Neumann algebra always means a (not necessarily unital) normal *−homomorphism: $j,\ J$

Unital endomorphisms: α

Conditional expectations: $P,\ Q$

If $w : \mathcal{G} \to \mathcal{H}$ is a linear operator, then we write $Ad\,w = w \cdot w^* : \mathcal{B}(\mathcal{G}) \to \mathcal{B}(\mathcal{H})$, even if w is not unitary.

General references for operator algebras are [Sa71, Ta79, KR83].

Probability spaces: (Ω, Σ, μ)

$\mathcal{M}(p, q)$ are the joint probability distributions for measures p, q and $\mathcal{S}(q, p)$ are the transition operators S with $p \circ S = q$, see Section 4.1.

Larger objects often get a tilde ˜ or hat ˆ, for example $\tilde{\mathcal{A}}$.

This should help to get a quick orientation but of course the conventions may be violated in specific situations and the reader has to look for the definition in the main text. We have made an attempt to invent a scheme of notation which provides a bridge between different chapters and sections and stick to it even if it is more clumsy than it would have been possible if the parts had been treated in isolation. We think that the advantages are more important. Besides the quick orientation already mentioned, the reader can grasp connections in this way even before they are explicitly formulated. Nevertheless, there is a moderate amount of repetition of definitions if the same occurs in different chapters to make independent reading easier.

Numbering of chapters, sections and subsections is done in the usual way. Theorems, propositions, lemmas etc. do not get their own numbers but are cross-referenced by the number of the subsection in which they are contained.

2

Markov Processes

We have already mentioned earlier that the stochastic maps considered in Chapter 1 can be interpreted as transition operators of noncommutative Markov processes. This will be explained in the beginning of Chapter 2. After some short remarks about the general idea of noncommutative stochastic processes we describe the approaches of B. Kümmerer [Kü85a, Kü88a, Kü03] and B.V.R. Bhat [Bh96, Bh01] to the noncommutative Markov property. This part is a kind of survey which we also use to prepare a connection between these approaches which we develop afterwards. Namely, Kümmerer's central idea of a coupling representation for the time evolution of a Markov process can also be used to analyze the structure of time evolutions for Bhat's weak Markov processes. This is not their original presentation, and thus we spend some time to work out the details. Because of the connections between Cuntz algebra representations and endomorphisms on $\mathcal{B}(\mathcal{H})$ [Cu77, BJP96], this also leads to a notion of coupling representation for Cuntz algebras. Besides many other ramifications mentioned in the text, it may be particularly interesting to consider these structures as discrete analogues to the theory of E_0-semigroups initiated by W. Arveson [Ar89, Ar03].

The point of view of coupling representations means to look at endomorphisms as perturbations of shifts. This is further worked out by a suitable notion of cocycles and coboundaries, and we succeed to characterize conjugacy between the shift and its perturbation by ergodicity of a stochastic map. Our motivation to look at this has been some work of B. Kümmerer and H. Maassen [KM00] on a scattering theory for Markov processes (in the sense of Kümmerer). We explain parts of this work and then show how it can be understood in the light of our work before. It is possible to construct weak Markov processes as extensions of these, essentially by GNS-construction, and then the conjugacy result mentioned above gives us an elegant new criterion for asymptotic completeness in the scattering theory. Moreover, here we have a link to Chapter 1. In fact, the stochastic map, which has to be examined for ergodicity, is an extension of the (dual of) the transition operator of the Markov process, exactly in the way analyzed in Chapter 1. In other words, the

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structure of the set of solutions for the extension problem is closely related to scattering theory and finds some nice applications there.

We have not included some already existing work about using Kümmerer-Maassen-scattering theory for systems in physics. See the remarks in 2.6.6. But in the last section of Chapter 2 we explain a way to look at coupling representations which emphasizes the physically important concept of entanglement for states. Asymptotic completeness of the scattering theory can be interpreted as a decay of entanglement in the long run.

2.1 Kümmerer's Approach

2.1.1 The Topic

B. Kümmerer's approach to noncommutative Markov processes (see [Kü85a, Kü85b, Kü88a, Kü88b, Kü03]) emphasizes so-called coupling representations which are considered to be the typical structure of such processes. Most of the work done concerns Markov processes which are also stationary. This is not so restrictive as it seems on first sight: see in particular [Kü03] for a discussion how a good understanding of the structure of such processes helps in the investigation of related questions. Compare also Section 3.3.

For a full picture of this theory the reader should consult the references above. Here we ignore many ramifications and concentrate on specifying a version of the theory which will be used by us later. Our version deals with discrete time steps and one-sided time evolutions.

2.1.2 Noncommutative Stochastic Processes

The classical probability space is replaced by a noncommutative counterpart, specified by a von Neumann algebra $\tilde{\mathcal{A}}$ with a faithful normal state $\tilde{\phi}$. We want to consider stochastic processes, i.e. families of random variables. Such a process can be specified by a von Neumann subalgebra $\mathcal{A} \subset \tilde{\mathcal{A}}$ and a family of unital injective $*-$homomorphisms $j_n : \mathcal{A} \to \tilde{\mathcal{A}}$ (with $n \in \mathbb{N}_0$), where j_0 is the identical embedding. The index n may be interpreted as time. The basic reference for this concept of a **noncommutative process** is [AFL82].

For $n \in \mathbb{N}$ the algebra $\mathcal{A}$ is translated inside of $\tilde{\mathcal{A}}$ by the j_n and we get subalgebras $\mathcal{A}_n := j_n(\mathcal{A}) \subset \tilde{\mathcal{A}}$, in particular $\mathcal{A}_0 = \mathcal{A}$. Thinking of selfadjoint elements as of real-valued variables (as discussed for example in [Me91]) we can in particular look at processes $(a_n := j_n(a))_{n\in\mathbb{N}_0}$ with $a \in \mathcal{A}$ selfadjoint. However it is useful to be flexible here and to include non-selfadjoint operators and also considerations on the algebras as a whole. The state $\tilde{\phi}$ specifies the probabilistic content: For any selfadjoint $\tilde{a} \in \tilde{\mathcal{A}}$ the value $\tilde{\phi}(\tilde{a})$ is interpreted as the expectation value of the random variable $\tilde{a}$.

2.1.3 Stationarity

A classical stochastic process is **stationary** if joint probabilities only depend on time differences. Instead of joint probabilities we can also consider (multi-) correlations between the random variables. Similarly for our noncommutative process we say that it is stationary if for elements $a_1, \ldots, a_k \in \mathcal{A}$ we always have

$$\tilde{\phi}(j_{n_1}(a_1) \ldots j_{n_k}(a_k)) = \tilde{\phi}(j_{n_1+n}(a_1) \ldots j_{n_k+n}(a_k))$$

for all $n_1, \ldots, n_k, n \in \mathbb{N}_0$. In particular $\tilde{\phi} \circ j_n = \phi$ for all n, where ϕ is the restriction of $\tilde{\phi}$ to $\mathcal{A}$. See [Kü03] for a detailed discussion. We also come back to the general theory of stationary processes in Section 3.3.

Here we only note the following important feature: Stationary processes have a time evolution. This means that on the von Neumann algebra $\mathcal{A}_{[0,\infty)}$ generated by all $\mathcal{A}_n$ with $n \in \mathbb{N}_0$ there is a unital $*-$endomorphism α with invariant state $\tilde{\phi}$ and such that $j_n(a) = \alpha^n(a)$ for all $a \in \mathcal{A}$ and $n \in \mathbb{N}_0$. If $\mathcal{A}_{[0,\infty)} = \tilde{\mathcal{A}}$ the process is called minimal. (This notion differs from the minimality in Section 2.2.) For a minimal stationary process it is possible to construct a two-sided extension to negative time in order to get an automorphic time evolution, but we shall concentrate our attention on the one-sided part.

2.1.4 Markov Property

To define the Markov property for noncommutative processes one assumes the existence of conditional expectations, for example $P = P_0 : \tilde{\mathcal{A}} \to \mathcal{A} = \mathcal{A}_0$ with $\phi \circ P = \tilde{\phi}$. This is an idempotent stochastic map which is a left inverse of the embedding j_0. Its existence is not automatic in the noncommutative setting. Compare 1.6.1 and [Sa71, Ta72, AC82]. If it exists, the conditional expectation (respecting the state) from $\tilde{\mathcal{A}}$ to $\mathcal{A}_{[m,n]}$, the von Neumann subalgebra of $\tilde{\mathcal{A}}$ generated by all $\mathcal{A}_k$ with $m \leq k \leq n$, is called $P_{[m,n]}$. Instead of $P_{[n,n]}$ we write P_n. Note that for a stationary process with (two-sided) automorphic time evolution it is enough to assume the existence of P_0 and the existence of all the other conditional expectations follows from that (see [Kü85a], 2.1.3).

Provided the conditional expectations exist we say, motivated by the classical notion, that the process is **Markovian** if $P_{[0,n]}(j_m(a)) = P_n(j_m(a))$ for all $a \in \mathcal{A}$ and all $m \geq n$ in $\mathbb{N}_0$. It suffices to check this for $m = n + 1$. Intuitively the Markov property means that the process has no memory. Some information about the transition from n to $n+1$ is contained in the transition operator $T_{n+1} : \mathcal{A} \to \mathcal{A}$ defined by $P_n(j_{n+1}(a)) = j_n(T_{n+1}(a))$. By iteration we find that $P j_n = T_1 \ldots T_n$. In particular, if the Markov process is **homogeneous**, i.e. if there is a stochastic map $T : \mathcal{A} \to \mathcal{A}$ such that $T_n = T$ for all n, then we see that $P j_n = T^n$ forms a semigroup. This corresponds to the Chapman-Kolmogorov equation in classical probability, and T is called the transition operator of the homogeneous Markov process.

2.1.5 Markov Dilation and Correlations

A stationary Markov process is always homogeneous and the state ϕ is invariant for T. Using the time evolution α we can also write $P\alpha^n|_{\mathcal{A}} = T^n$ for all $n \in \mathbb{N}_0$. For this reason a stationary Markov process with transition operator T is also called a **Markov dilation** of T (see [Kü85a, Kü88a]).

$$\begin{tikzcd}
(\mathcal{A}, \phi) \arrow[r, "T^n"] \arrow[d, "j_0"'] \arrow[dr, "j_n"] & (\mathcal{A}, \phi) \\
(\tilde{\mathcal{A}}, \tilde{\phi}) \arrow[r, "\alpha^n"] & (\tilde{\mathcal{A}}, \tilde{\phi}) \arrow[u, "P"']
\end{tikzcd}$$

Starting with T on $\mathcal{A}$, the larger algebra $\tilde{\mathcal{A}}$ where dilation takes place is not uniquely determined (even in the minimal case). In the quantum physics interpretation this non-uniqueness corresponds to different physical environments of the small system $\mathcal{A}$ which cannot be distinguished by observing the small system alone. Mathematically the non-uniqueness reflects the fact that the transition operator T only determines the so-called pyramidally time-ordered correlations, by the quantum regression formula

$$\tilde{\phi}(j_{n_1}(a_1^*) \dots j_{n_k}(a_k^*) j_{n_k}(b_k) \dots j_{n_1}(b_1))$$

$$= \phi(a_1^* \, T^{n_2 - n_1}(a_2^* \, T^{n_3 - n_2}(\dots T^{n_k - n_{k-1}}(a_k^* b_k) \dots) b_2) b_1)$$

if $a_1, \dots, a_k, b_1, \dots, b_k \in \mathcal{A}$ and $n_k \geq n_{k-1} \geq \dots \geq n_1$ in $\mathbb{N}_0$. But a complete reconstruction of the process from correlations requires the knowledge of correlations for arbitrary time orderings (see [AFL82]).

Not all stochastic maps $T : \mathcal{A} \to \mathcal{A}$ with invariant normal faithful state ϕ can be dilated in this way, the most immediate restriction being that T must commute with the modular automorphism group σ_t^ϕ of the state ϕ. More details and open problems on this kind of dilation theory can be found in [Kü88a].

2.1.6 Coupling Representations

Very often a Markov process exhibits a certain structure which is called a coupling representation. The terminology refers to the following procedure well-known in quantum physics: To investigate the behaviour of a small system, think of it as coupled to a larger system, a so-called reservoir or heat bath. The combined system is assumed to be closed and the usual laws of quantum physics apply (Schrödinger's equation etc.). Then using coarse graining arguments it is possible to derive results about the small system one is interested in.

We restrict ourselves to the case of tensor product couplings, although more general couplings are possible and important, also for the theory of noncommutative Markov processes. We say that a Markov process is given in a **coupling representation** (of tensor type) if the following additional ingredients are present:

There is another von Neumann algebra $\mathcal{C}$ with a faithful normal state ψ. We form the (minimal C^*-)tensor product $\bigotimes_{n=1}^{\infty} \mathcal{C}_n$, where each $\mathcal{C}_n$ is a copy of $\mathcal{C}$. We then define the von Neumann algebra $\mathcal{C}_{[1,\infty)}$ as the weak closure with respect to the product state $\psi_{[1,\infty)} := \bigotimes_{n=1}^{\infty} \psi_n$, where each ψ_n is a copy of ψ. The von Neumann algebra $\mathcal{C}_{[1,\infty)}$ represents the reservoir, and our assumption is that $(\tilde{\mathcal{A}}, \tilde{\phi})$ can be obtained in such a way that $\tilde{\mathcal{A}}$ is the weak closure of $\mathcal{A} \otimes \mathcal{C}_{[1,\infty)}$ with respect to the state $\tilde{\phi} = \phi \otimes \psi_{[1,\infty)}$. The algebras $\mathcal{A}$ and $\mathcal{C}_n$ are subalgebras of $\tilde{\mathcal{A}}$ in the obvious way.

Further it is assumed that there is a coupling, i.e. a unital injective $^*-$homomorphism $j_1 : \mathcal{A} \to \mathcal{A} \otimes \mathcal{C}_1$. Using the conditional expectation $P_{\psi_1} : \mathcal{A} \otimes \mathcal{C}_1 \to \mathcal{A}$, $a \otimes c \mapsto a\, \psi_1(c)$, we can define a stochastic map $T : \mathcal{A} \to \mathcal{A}$ given by $T := P_{\psi_1} j_1$. Then the coupling j_1 is a dilation (of first order) for T and the conditional expectation is of tensor type. In particular, it is a weak tensor dilation (of first order) in the sense introduced in Section 1.3. Additionally we have here a unital injective $^*-$homomorphism, a unital conditional expectation and the state used for conditioning is faithful.

Now let σ be the right tensor shift on $\bigotimes_{n=1}^{\infty} \mathcal{C}_n$ extended to the weak closure $\mathcal{C}_{[1,\infty)}$. Extend P_{ψ_1} in the obvious way to get a conditional expectation $P = P_0$ of tensor type from $\tilde{\mathcal{A}}$ to $\mathcal{A} = \mathcal{A}_0$. A time evolution α is defined by $\alpha := j_1 \sigma$. This notation means that for $a \in \mathcal{A}$ and $\tilde{c} \in \mathcal{C}_{[1,\infty)}$ we have $\alpha(a \otimes \tilde{c}) = j_1(a) \otimes \tilde{c} \in (\mathcal{A} \otimes \mathcal{C}_1) \otimes \mathcal{C}_{[2,\infty)}$. Thus α is actually a composition of suitable amplifications of j_1 and σ, which we have denoted with the same symbol.

$$\begin{tikzcd}
\mathcal{A} \arrow[dd, bend right, "j_1"'] & & & & & \\
\otimes & & {} \arrow[r, "\sigma"'] & {} & & \\
\mathcal{C} & \otimes & \mathcal{C} & \otimes & \mathcal{C} & \otimes \ \ldots
\end{tikzcd}$$

Define $j_n(a) := \alpha^n(a)$ for $a \in \mathcal{A}$ and $n \in \mathbb{N}_0$. If $j_1(a) = \sum_i a_i \otimes c_i \in \mathcal{A} \otimes \mathcal{C}_1$ then we get the recursive formula $j_n(a) = \sum_i j_{n-1}(a_i)\, \sigma^{n-1}(c_i)$. Denote by $Q_{[0,n]}$ the conditional expectation from $\tilde{\mathcal{A}}$ onto $\mathcal{A} \otimes \mathcal{C}_{[1,n]}$ (of tensor type). Then we conclude that

$$Q_{[0,n-1]}(j_n(a)) = j_{n-1}(T(a))$$

for all $a \in \mathcal{A}$ and $n \in \mathbb{N}$. This indicates that α may be considered as the time evolution of a homogeneous Markov process with transition operator T in a slightly generalized sense. In fact, this is enough to get $P j_n = T^n$ and if the conditional expectations $P_{[0,n]}$ exist, then we also have $P_{[0,n-1]}(j_n(a)) = j_{n-1}(T(a))$ for all a and n, which is the Markov property defined in 2.1.4.

If $(\phi \otimes \psi_1) \circ j_1 = \phi$, then $\tilde{\phi} \circ \alpha = \tilde{\phi}$ and the process is stationary. Recall from 1.6.3 that j_1 is an automorphic tensor dilation (of first order) if there is an automorphism α_1 of $\mathcal{A}\otimes\mathcal{C}_1$ such that $j_1(a) = \alpha_1(a\otimes\mathbb{1})$ for all $a \in \mathcal{A}$ and $\phi\otimes\psi_1$ is invariant for α_1. Thus from an automorphic tensor dilation (of first order) we can construct a stationary Markov process in a coupling representation. Note that in the automorphic case a two-sided automorphic extension of the time evolution of the Markov process to negative times can be written down immediately: just use the weak closure of $\bigotimes_{0\neq n\in\mathbb{Z}} \mathcal{C}_n$ (with respect to the product state) and a two-sided tensor shift σ (jumping directly from $n = -1$ to $n = 1$ in our notation, the index $n = 0$ is reserved for $\mathcal{A}$). In [Kü85a] this automorphic case is treated and simply called 'tensor dilation'. Up to these remarks our terminology is consistent with [Kü85a]. In the automorphic case the conditional expectations $P_{[0,n]}$ always exist: P_0 is of tensor type and the argument in ([Kü85a], 2.1.3) applies.

Summarizing, the main result is that it is possible to construct a stationary Markov process in a coupling representation from a stochastic map $T : \mathcal{A} \to \mathcal{A}$ with invariant state ϕ whenever one finds a tensor dilation of first order j_1 with $(\phi \otimes \psi_1) \circ j_1 = \phi$. Results in the converse direction, i.e. showing that a stationary Markov process exhibits a coupling structure, motivate and require the study of generalized Bernoulli shifts (replacing the tensor shift used here), see [Kü88a, Ru95].

2.2 Bhat's Approach

2.2.1 The Topic

In the following we review B.V.R. Bhat's notion of a weak Markov dilation (see [Bh96, Bh01]). As in Section 2.1 we send the reader to the references for the full picture and concentrate to single out a special version that will be used by us later. Again, as in Section 2.1, we consider discrete time steps and one-sided time evolutions.

2.2.2 Weak Markov Property

We want to dilate a stochastic map $Z : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$, where $\mathcal{H}$ is a Hilbert space. Suppose that $\mathcal{H} \subset \tilde{\mathcal{H}}$, where $\tilde{\mathcal{H}}$ is a larger Hilbert space with orthogonal projection $p_{\mathcal{H}}$ from $\tilde{\mathcal{H}}$ onto $\mathcal{H}$. A family $(J_n)_{n=0}^{\infty}$ of normal (and typically non-unital) $^*-$homomorphisms $J_n : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\tilde{\mathcal{H}})$ is called a **weak Markov dilation** of Z (or of the semigroup $(Z^n)_{n=0}^{\infty}$) if we have (with projections $p_{[0,n]} := J_n(\mathbb{1})$ onto $\hat{\mathcal{H}}_{[0,n]} \subset \tilde{\mathcal{H}}$):

(0) $J_0(x) = x\, p_{\mathcal{H}} \quad$ for all $x \in \mathcal{B}(\mathcal{H})$

(1) $p_{[0,n]}\, J_m(x)\, p_{[0,n]} = J_n(Z^{m-n}(x)) \quad$ for all $x \in \mathcal{B}(\mathcal{H}),\ m \geq n$ in $\mathbb{N}_0$.

We define $\hat{\mathcal{H}}$ to be the closure of $\bigcup_{n=0}^{\infty} \hat{\mathcal{H}}_{[0,n]}$. The dilation is called **primary** if $\hat{\mathcal{H}} = \tilde{\mathcal{H}}$.

We add some comments. Equation (0) means that J_0 acts identically on elements of $\mathcal{B}(\mathcal{H})$, embedding them into $\mathcal{B}(\tilde{\mathcal{H}})$ as vanishing on $\mathcal{H}^{\perp}$. Already here we see that the dilation procedure is non-unital, which is the main impact of the terminology 'weak'. Let us write p_0 instead of $p_{[0,0]}$ and $\mathcal{H}_0$ instead of $\hat{\mathcal{H}}_{[0,0]}$. Inserting $x = \mathbb{1}$ into equation (0) we find $p_0 = J_0(\mathbb{1}) = p_{\mathcal{H}}$ and $\mathcal{H}_0 = \mathcal{H}$. Inserting $x = \mathbb{1}$ into equation (1) we see that the sequence $(p_{[0,n]})_{n=0}^{\infty}$ is increasing, i.e. $p_{[0,m]} \geq p_{[0,n]}$ for $m \geq n$ in $\mathbb{N}_0$. Clearly (1) is a kind of Markov property similar to that in 2.1.4, also generalizing the Chapman-Kolmogorov equation of classical probability. But here the map which plays the role of the conditional expectation, namely $\mathcal{B}(\tilde{\mathcal{H}}) \ni \tilde{x} \mapsto p_{[0,n]}\, \tilde{x}\, p_{[0,n]}$, is not unital on $\mathcal{B}(\tilde{\mathcal{H}})$.

2.2.3 Correlations

It is peculiar to such weak dilations that the not time-ordered correlations can be reduced to the time-ordered ones and therefore by assuming minimality one gets a uniqueness result (contrary to the setting in Section 2.1). In detail, define for all $n \in \mathbb{N}_0$:

$$\begin{aligned}\mathcal{H}_{[0,n]} := \overline{span}\{ & J_{n_k}(x_k) J_{n_{k-1}}(x_{k-1}) \ldots J_{n_1}(x_1)\xi : \\ & n \geq n_k, \ldots, n_1 \in \mathbb{N}_0,\ x_k, \ldots, x_1 \in \mathcal{B}(\mathcal{H}),\ \xi \in \mathcal{H}\}\end{aligned}$$

and let $\hat{\mathcal{H}}^{min}$ be the closure of $\bigcup_{n=0}^{\infty} \mathcal{H}_{[0,n]}$. Then $\mathcal{H}_{[0,n]} \subset \hat{\mathcal{H}}_{[0,n]}$ for all n. If we have equality for all n and if further $\hat{\mathcal{H}}^{min} = \tilde{\mathcal{H}}$, then the dilation is called **minimal**. In [Bh01] it is shown that introducing a time ordering $n \geq n_k \geq n_{k-1} \ldots \geq n_1$ in the definition above does not change the space. Therefore a minimal weak Markov dilation of Z is unique up to unitary equivalence, by the quantum regression formula which here reads as follows:

$$\langle J_{n_k}(x_k) \ldots J_{n_1}(x_1)\xi,\ J_{n_k}(y_k) \ldots J_{n_1}(y_1)\eta \rangle$$

$$= \langle \xi, Z^{n_1}(x_1^*\, Z^{n_2-n_1}(x_2^* \ldots Z^{n_k-n_{k-1}}(x_k^* y_k) \ldots y_2) y_1)\eta \rangle$$

if $x_k, \ldots, x_1, y_k, \ldots, y_1 \in \mathcal{B}(\mathcal{H}),\ n \geq n_k \geq n_{k-1} \ldots \geq n_1$ in $\mathbb{N}_0,\ \xi, \eta \in \mathcal{H}$.

2.2.4 Recursive Construction of the Time Evolution

In [Bh96] a time evolution $\hat{J}^{min}$ is constructed which implements the minimal weak Markov dilation. In detail, there are Hilbert spaces $\mathcal{N}$, $\mathcal{P}$ with the following property: For all $n \in \mathbb{N}$ we have $\mathcal{H}_{[0,n]} = \mathcal{H}_{[0,n-1]} \oplus (\mathcal{N} \otimes \bigotimes_1^{n-1} \mathcal{P})$ (with $\mathcal{N} \otimes \bigotimes_1^{0} \mathcal{P} = \mathcal{N}$) and there are unitaries $w_n : \mathcal{H} \otimes \bigotimes_1^{n} \mathcal{P} \to \mathcal{H}_{[0,n]}$ such that

$$J_n(x) = w_n(x \otimes \mathbb{1}) w_n^*\, p_{[0,n]}.$$

The dimension of $\mathcal{P}$ is called the rank of Z. It is equal to the minimal number of terms in a Kraus decomposition of Z (see A.2.3). In fact, to construct $\hat{J}^{min}$ as in [Bh96] one starts with the minimal Stinespring representation for Z (see A.2.2), i.e. $Z(x) = (v_1')^*(x \otimes \mathbb{1})v_1'$, where $v_1' : \mathcal{H} \to \mathcal{H} \otimes \mathcal{P}$ is an isometry. (The notation v_1' is chosen in such a way that Z may be an extended transition operator as in 1.5.5. Indeed we want to exploit this point of view in Section 2.6. But for the moment Z is just an arbitrary stochastic map in $\mathcal{B}(\mathcal{H})$.) Then one takes $\mathcal{N}$ to be a Hilbert space with the same dimension as $(v_1'\mathcal{H})^\perp$ in $\mathcal{H} \otimes \mathcal{P}$ and defines $u_1^* : \mathcal{H} \oplus \mathcal{N} \to \mathcal{H} \otimes \mathcal{P}$ to be an arbitrary unitary extension of v_1'. (In [Bh96] $\mathcal{N}$ and u_1^* are constructed explicitly, but the description above also works.) We define w_n recursively by

$$w_1 := u_1, \quad w_n := (w_{n-1} \oplus (\mathbb{1} \otimes \mathbb{1}))\,(w_1 \otimes \mathbb{1}) \quad \text{for } n \geq 2$$

$\big($ with suitable identifications, in particular w_1 on the right side acts on the $n-$th copy of $\mathcal{P}$ and $(\mathcal{H} \oplus \mathcal{N}) \otimes \bigotimes_1^{n-1} \mathcal{P} = (\mathcal{H} \otimes \bigotimes_1^{n-1} \mathcal{P}) \oplus (\mathcal{N} \otimes \bigotimes_1^{n-1} \mathcal{P})$ $\big)$ and we can check that this yields a minimal dilation, see [Bh96].

$$\begin{tikzcd}
\mathcal{H} \arrow[r, "Z^n(x)"] \arrow[d] & \mathcal{H} \\
\mathcal{H}_{[0,n]} \arrow[r, "J_n(x)"'] & \mathcal{H}_{[0,n]} \arrow[u] \\
\mathcal{H} \otimes \bigotimes_1^n \mathcal{P} \arrow[r, "x \otimes 1"'] \arrow[u, "w_n"'] & \mathcal{H} \otimes \bigotimes_1^n \mathcal{P} \arrow[u, "w_n"']
\end{tikzcd}$$

Now consider inside of $\hat{\mathcal{H}}^{min} = \hat{\mathcal{H}}$ the subspaces

$$\begin{aligned}
\mathcal{H}^\perp \quad &= \mathcal{N} \qquad\qquad\quad \oplus (\mathcal{N} \otimes \mathcal{P}) \qquad\quad \oplus (\mathcal{N} \otimes \textstyle\bigotimes_1^2 \mathcal{P}) \quad \oplus \ldots \\
\mathcal{H}_{[0,n]}^\perp &= (\mathcal{N} \otimes \textstyle\bigotimes_1^n \mathcal{P}) \oplus (\mathcal{N} \otimes \bigotimes_1^{1+n} \mathcal{P}) \oplus (\mathcal{N} \otimes \bigotimes_1^{2+n} \mathcal{P}) \oplus \ldots
\end{aligned}$$

suggesting a canonical unitary from $\mathcal{H}^\perp \otimes \bigotimes_1^n \mathcal{P}$ onto $\mathcal{H}_{[0,n]}^\perp$. It can be used to extend w_n to a unitary $\hat{w}_n : \hat{\mathcal{H}} \otimes \bigotimes_1^n \mathcal{P} \to \hat{\mathcal{H}}$. Then there is an endomorphism Θ of $\mathcal{B}(\hat{\mathcal{H}})$ satisfying

$$\Theta^n(\hat{x}) = \hat{w}_n(\hat{x} \otimes \mathbb{1})\hat{w}_n^* \quad \text{for } \hat{x} \in \mathcal{B}(\hat{\mathcal{H}}),$$

and one finds that for $x \in \mathcal{B}(\mathcal{H}),\ n, m \in \mathbb{N}_0$:

$$\Theta^n(xp_{\mathcal{H}}) = \hat{w}_n(xp_{\mathcal{H}} \otimes \mathbb{1})\hat{w}_n^* = w_n(x \otimes \mathbb{1})w_n^*\, p_{[0,n]} = J_n(x),$$

$$\Theta^m(J_n(x)) = \Theta^m(\Theta^n(xp_{\mathcal{H}})) = \Theta^{m+n}(xp_{\mathcal{H}}) = J_{m+n}(x).$$

In other words, $\hat{J}^{min} := \Theta$ is a time evolution for the minimal weak Markov dilation. We have $Z^n(x) = p_{\mathcal{H}}\, \Theta^n(xp_{\mathcal{H}})\, p_{\mathcal{H}}$ for $x \in \mathcal{B}(\mathcal{H}),\ n \in \mathbb{N}_0$. One may also note that starting with a non-minimal Stinespring dilation and performing the construction above one would get a primary but non-minimal dilation (with different spaces $\mathcal{N}$, $\mathcal{P}$). Sometimes we shall also call the implementing endomorphism Θ a weak dilation, but one should keep in mind that Θ may also be used for other purposes.

2.2.5 Converse

Conversely, if Θ is an endomorphism of $\mathcal{B}(\tilde{\mathcal{H}})$, where $\tilde{\mathcal{H}}$ is an arbitrary Hilbert space (infinite-dimensional to get nontrivial results), and if $p_{\mathcal{H}}$ is a projection with range $\mathcal{H} \subset \tilde{\mathcal{H}}$ which is increasing for Θ, i.e. $\Theta(p_{\mathcal{H}}) \geq p_{\mathcal{H}}$, then Θ implements a weak Markov dilation for $Z : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H}), x \mapsto p_{\mathcal{H}}\Theta(xp_{\mathcal{H}})|_{\mathcal{H}}$. This is the point of view in [Bh01]. The restriction to $\mathcal{B}(\hat{\mathcal{H}})$ yields a primary dilation $\hat{J}$, and in ([Bh01], chap.3) it is shown that it is always possible to reduce Θ further to get $\hat{\mathcal{H}}^{min}$ and $\hat{J}^{min}$. Starting with Section 2.3 we will see another approach to look at all these facts.

2.3 Coupling Representation on a Hilbert Space

2.3.1 Dilation of First Order

In this section we want to mix ideas from Sections 1.3, 2.1, 2.2 to construct endomorphisms which are defined as couplings to a shift but which implement weak dilations. Let $Z : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ be a stochastic map. We have seen in 1.3.3 (with S instead of Z) that there exist weak tensor dilations (of first order) corresponding to Stinespring representations of Z. Fix one of them. Then we have a Hilbert space $\mathcal{K}$, a unit vector $\Omega_{\mathcal{K}} \in \mathcal{K}$ and a normal $^*-$homomorphism $J_1 : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H} \otimes \mathcal{K})$ such that $Z(x) = p_{\mathcal{H}} J_1(x)|_{\mathcal{H}}$ for all $x \in \mathcal{B}(\mathcal{H})$. Here $\mathcal{H} \simeq \mathcal{H} \otimes \Omega_{\mathcal{K}} \subset \mathcal{H} \otimes \mathcal{K}$ and $p_{\mathcal{H}}$ is the projection onto $\mathcal{H}$. Using 1.3.3 we also have a Hilbert space $\mathcal{P}$ and an isometry $u_1 : \mathcal{H} \otimes \mathcal{P} \to \mathcal{H} \otimes \mathcal{K}$ with range $J_1(\mathbb{1})(\mathcal{H} \otimes \mathcal{K})$ such that

$$J_1(x) = u_1(x \otimes \mathbb{1})u_1^* = Ad(u_1)(x \otimes \mathbb{1}).$$

Then $Z(x) = (v_1')^*(x \otimes \mathbb{1})v_1'$ with $v_1' := u_1^*|_{\mathcal{H} \otimes \Omega_{\mathcal{K}}}$.
Remark: If one starts with Hilbert spaces $\mathcal{H}$, $\mathcal{K}$, $\mathcal{P}$, a unit vector $\Omega_{\mathcal{K}} \in \mathcal{K}$ and an isometry $u_1 : \mathcal{H} \otimes \mathcal{P} \to \mathcal{H} \otimes \mathcal{K}$ such that $u_1 u_1^* \geq p_{\mathcal{H}}$ (= projection onto $\mathcal{H} \otimes \Omega_{\mathcal{K}} \subset \mathcal{H} \otimes \mathcal{K}$), then we have the setting above with suitable Z.

$$\begin{tikzcd}
\mathcal{H} \arrow[r, "Z(x)"] \arrow[d] & \mathcal{H} \\
\mathcal{H} \otimes \mathcal{K} \arrow[r, "J_1(x)"'] & \mathcal{H} \otimes \mathcal{K} \arrow[u] \\
\mathcal{H} \otimes \mathcal{P} \arrow[r, "x \otimes 1"'] \arrow[u, "u_1"'] & \mathcal{H} \otimes \mathcal{P} \arrow[u, "u_1"']
\end{tikzcd}$$

2.3.2 Weak Markov Dilations as Couplings

We can use the idea of a coupling representation (as in 2.1.6) to construct the time evolution of a weak Markov dilation from J_1. Let $(\mathcal{K}_n, \Omega_n)$ for all $n \in \mathbb{N}$ be copies of $(\mathcal{K}, \Omega_{\mathcal{K}})$ and form the infinite tensor product $\mathcal{K}_{[1,\infty)} := \bigotimes_{i=1}^{\infty} \mathcal{K}_i$ along the given sequence of unit vectors (see [KR83], 11.5.29). We have natural embeddings $\mathcal{K}_{[n,m]} := \bigotimes_{i=n}^{m} \mathcal{K}_i \subset \mathcal{K}_{[1,\infty)}$ if $n \leq m$. Define the following right tensor shift:

$$\begin{aligned} R : \mathcal{B}(\mathcal{K}_{[1,\infty)}) &\to \mathcal{B}(\mathcal{P} \otimes \mathcal{K}_{[2,\infty)}) \\ \tilde{y} &\mapsto \mathbb{1} \otimes \tilde{y}. \end{aligned}$$

Now define $\tilde{\mathcal{H}} := \mathcal{H} \otimes \mathcal{K}_{[1,\infty)}$ and an endomorphism $\tilde{J}$ of $\mathcal{B}(\tilde{\mathcal{H}})$ as follows: If $\mathcal{B}(\tilde{\mathcal{H}}) \ni \tilde{x} = x \otimes \tilde{y} \in \mathcal{B}(\mathcal{H}) \otimes \mathcal{B}(\mathcal{K}_{[1,\infty)})$, then

$$\tilde{J}(\tilde{x}) := \tilde{u}_1 \left(x \otimes R(\tilde{y}) \right) \tilde{u}_1^*.$$

Here $\tilde{u}_1 = u_1 \otimes \mathbb{1}_{[2,\infty)}$, mapping $\mathcal{H} \otimes \mathcal{P} \otimes \mathcal{K}_{[2,\infty)}$ into $\mathcal{H} \otimes \mathcal{K} \otimes \mathcal{K}_{[2,\infty)}$. We may write $\tilde{J} = Ad(\tilde{u}_1)(\mathbb{1} \otimes R)$, or a bit inexact but giving the essentials: $\tilde{J} = J_1 \circ R$. This formula underlines the fact that the range of R is small enough to give meaning to a composition with J_1.

$$\begin{tikzcd}
\mathcal{B}(\mathcal{H} & \otimes & \mathcal{K}_1 \arrow[drr, "R"] & \otimes & \mathcal{K}_2 \arrow[drr, "R"] & \otimes & \ldots) \\
\mathcal{B}(\mathcal{H} \arrow[ur, "J_1"] & \otimes & \mathcal{P} & \otimes & \mathcal{K}_2 & \otimes & \ldots)
\end{tikzcd}$$

It is easily checked that this construction determines an endomorphism $\tilde{J}$ of $\mathcal{B}(\tilde{\mathcal{H}})$ and that, with the embedding $\mathcal{H} \simeq \mathcal{H} \otimes \Omega_{[1,\infty)} \subset \mathcal{H} \otimes \mathcal{K}_{[1,\infty)}$ using the canonical unit vector $\Omega_{[1,\infty)} \in \mathcal{K}_{[1,\infty)}$, we have $Z^n(x) = p_{\mathcal{H}} \tilde{J}^n(x p_{\mathcal{H}})|_{\mathcal{H}}$ for all $x \in \mathcal{B}(\mathcal{H})$, $n \in \mathbb{N}_0$. In other words, $\tilde{J}$ provides us with a weak Markov dilation for Z in the sense of Section 2.2. We shall say that the endomorphism $\tilde{J}$ is given in a **coupling representation**. The coupling between the 'system' $\mathcal{B}(\mathcal{H})$ and the 'heat bath' $\mathcal{B}(\mathcal{K}_{[1,\infty)})$ is specified by the weak tensor dilation (of first order) J_1. Note also that the rank of R and $\tilde{J}$ is $dim \mathcal{P}$.

2.3.3 Discussion

The projection $p_{\mathcal{H}}$ is increasing for $\tilde{J}$. Restricting $\tilde{J}$ to $\hat{\mathcal{H}} = \sup_{n \in \mathbb{N}} \tilde{J}^n(p_{\mathcal{H}})(\tilde{\mathcal{H}})$ yields a primary dilation $\hat{J}$. Because $\tilde{J}(\lim_{n\to\infty} \tilde{J}^n(p_{\mathcal{H}})) = \lim_{n\to\infty} \tilde{J}^n(p_{\mathcal{H}})$ the endomorphism $\hat{J}$ is unital. Using the minimal version $(J_1)^{min}$ as constructed in 1.3.5 we see that the contained minimal dilation $\hat{J}^{min}$ starts with the space $\mathcal{H}_{[0,1]} = (J_1)^{min}(\mathbb{1})(\mathcal{H} \otimes \mathcal{K}_1) \otimes \Omega_{[2,\infty)}$ in the first order, which can be identified with the minimal Stinespring representation space of Z by 1.3.6.

For the minimal version we can make $\mathcal{P}$ smaller and use instead $\mathcal{P}_{v_1'\mathcal{H}}$, with notation from A.3. If J_1 is already minimal (see 1.3.5) then $\hat{\mathcal{H}}^{min} = \hat{\mathcal{H}}$.

If J_1 is unital, i.e. $J_1(\mathbb{1})(\mathcal{H} \otimes \mathcal{K}_1) = \mathcal{H} \otimes \mathcal{K}_1$ (equivalently, by 1.3.3, if $u_1 : \mathcal{H} \otimes \mathcal{P} \to \mathcal{H} \otimes \mathcal{K}$ is unitary), then we get $\hat{\mathcal{H}} = \tilde{\mathcal{H}}$ and the coupling representation directly yields a primary dilation (without restricting). Then $\tilde{J}$ is also unital. In particular, if $\mathcal{H}$ is finite-dimensional and we construct J_1 from the minimal Stinespring representation of Z as in 1.3.3, then $\hat{\mathcal{H}}^{min} = \tilde{\mathcal{H}}$, and the coupling representation above gives us directly a minimal weak Markov dilation of Z.

We see that the concepts of Section 2.2 can be naturally rediscovered here. All the assertions above can be verified directly from the definition of the coupling representation and they will become more transparent in the following sections.

2.4 Cuntz Algebra Representations

2.4.1 Cuntz Algebras

We continue our study of coupling representations of endomorphisms of $\mathcal{B}(\tilde{\mathcal{H}})$ started in the previous section and we want to include the following well-known connection between endomorphisms and Cuntz algebra representations. In the following two lemmas $\hat{\mathcal{H}}$ may be an arbitrary (infinite-dimensional and separable) Hilbert space. Recall that the Cuntz algebra $\mathcal{O}_d$ [Cu77] is generated by $s_1, \ldots, s_d$ satisfying the **Cuntz relations** $s_k^* s_k = \mathbb{1}$ for all k and $\sum_{k=1}^d s_k s_k^* = \mathbb{1}$ (if $d < \infty$). For $d = \infty$ the last condition is replaced by orthogonality of the projections $s_k s_k^*$ and convergence of the sum is only assumed for representations (in the stop-sense).

2.4.2 Cuntz Algebra Representations and Endomorphisms

The connection between endomorphisms of $\mathcal{B}(\hat{\mathcal{H}})$ and Cuntz algebras is contained in the following lemma of M. Laca [La93], compare also [Ar89, BJP96].

Lemma: *A unital $^*-$endomorphism Θ of $\mathcal{B}(\hat{\mathcal{H}})$ with rank $d \geq 2$ is related to a non-degenerate representation $\pi : \mathcal{O}_d \to \mathcal{B}(\hat{\mathcal{H}})$ of the Cuntz algebra $\mathcal{O}_d$ by its minimal Kraus decomposition*

$$\Theta(\tilde{x}) = \sum_{k=1}^{d} s_k \, \tilde{x} \, s_k^*, \quad \tilde{x} \in \mathcal{B}(\hat{\mathcal{H}}).$$

Namely, the operators $s_1, \ldots, s_d \in \mathcal{B}(\hat{\mathcal{H}})$ satisfy the Cuntz relations and generate the representation π. Conversely any non-degenerate representation of a Cuntz algebra gives rise to an endomorphism by the formula above. The *generators* $s_1, \ldots, s_d \in \mathcal{B}(\hat{\mathcal{H}})$ *are uniquely determined by* Θ *up to the canonical action* τ *of the unitary group* $U(d)$ *on the Cuntz algebra* $\mathcal{O}_d$ *given by* $\tau_g(s_k) = \sum_j g_{jk} s_j$ *for* $g = (g_{kj}) \in U(d)$.

2.4.3 Increasing Projections and Markovian Subspaces

Lemma: *Let* $p_{\mathcal{H}}$ *be an orthogonal projection from* $\hat{\mathcal{H}}$ *onto a subspace* $\mathcal{H}$. *The following assertions are equivalent:*

(1) $p_{\mathcal{H}}$ *is* **increasing** *for* $\Theta = \sum_{k=1}^d s_k \cdot s_k^*$, *i.e.* $\Theta(p_{\mathcal{H}}) \geq p_{\mathcal{H}}$.

(2) For all $k = 1, \ldots, d \quad s_k^*(\mathcal{H}) \subset \mathcal{H}$.

Proof: If $s_k^*(\mathcal{H}) \subset \mathcal{H}$ for all $k = 1, \ldots, d$ then

$$\left(\sum s_k\, p_{\mathcal{H}}\, s_k^*\right) p_{\mathcal{H}} = \sum s_k\, (p_{\mathcal{H}}\, s_k^*\, p_{\mathcal{H}}) = \sum s_k\, s_k^*\, p_{\mathcal{H}} = p_{\mathcal{H}}, \quad \text{i.e. } \Theta(p_{\mathcal{H}}) \geq p_{\mathcal{H}}.$$

Conversely if $(\sum s_k\, p_{\mathcal{H}}\, s_k^*) p_{\mathcal{H}} = p_{\mathcal{H}}$ then for all $k = 1, \ldots, d$

$$s_k^*\, p_{\mathcal{H}} = s_k^* \sum s_j\, p_{\mathcal{H}}\, s_j^*\, p_{\mathcal{H}} = p_{\mathcal{H}}\, s_k^*\, p_{\mathcal{H}}, \quad \text{i.e. } s_k^* \mathcal{H} \subset \mathcal{H}. \qquad \square$$

O. Bratteli, P. Jorgensen, A. Kishimoto and R. Werner in [BJKW00] present a detailed investigation of $\{s_k^*\}$-invariant subspaces for Cuntz algebra representations. Combining Section 2.2 with the lemma above we see that these are exactly the subspaces for which the corresponding endomorphism provides us with a weak Markov dilation of its compression. It seems appropiate to call such subspaces **Markovian subspaces** (with respect to the endomorphism). Compare the related notion of Markov partitions for transformations in measure theory, see [Pe89]. The first occurrence of Markovian subspaces, in a different terminology and direction of research, seems to be in a dilation theory of noncommuting tuples of operators by isometries in the spirit of Sz.Nagy-Foias dilation theory (see [Fr82, Po89], see also [DKS01] for new developments in this direction).

2.4.4 Kraus Decompositions

Let us work it out in detail for the coupling representations introduced in Section 2.3. Then we have $\tilde{\mathcal{H}} = \mathcal{H} \otimes \mathcal{K}_{[1,\infty)}$, $\mathcal{K}_{[1,\infty)} = \bigotimes_{i=1}^\infty \mathcal{K}_i$, $\mathcal{K}_i \simeq \mathcal{K}$. Choosing an ONB $\{\epsilon_k\}_{k=1}^d$ of $\mathcal{P}$ the tensor shift $R : \mathcal{B}(\mathcal{K}_{[1,\infty)}) \to \mathcal{B}(\mathcal{P} \otimes \mathcal{K}_{[2,\infty)})$ can be written as $R(\tilde{y}) = \sum_{k=1}^d r_k\, \tilde{y}\, r_k^*$, where $r_k : \mathcal{K}_{[1,\infty)} \to \mathcal{P} \otimes \mathcal{K}_{[2,\infty)}$, $\tilde{\eta} \mapsto \epsilon_k \otimes \tilde{\eta}$. Now let $\tilde{J}$ be an endomorphism of $\mathcal{B}(\tilde{\mathcal{H}})$ given by the coupling representation $\tilde{J} = J_1 \circ R$. If $J_1 : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H} \otimes \mathcal{K}_1)$ is given by $J_1(x) = u_1(x \otimes 1\!\!1)u_1^*$ with an isometry $u_1 : \mathcal{H} \otimes \mathcal{P} \to \mathcal{H} \otimes \mathcal{K}_1$, then $J_1(x) = \sum_{k=1}^d b_k\, x\, b_k^*$ with $b_k := u_1|_{\mathcal{H} \otimes \epsilon_k}$ is the corresponding Kraus decomposition (see A.2.3). We have

$$Z(x) \;=\; (v_1')^*(x \otimes \mathbb{1})v_1' \;=\; \sum_{k=1}^{d} a_k \, x \, a_k^*$$

with $v_1' := u_1^*|_{\mathcal{H}\otimes\Omega_{\mathcal{K}}}$ and $b_k^*|_{\mathcal{H}\otimes\Omega_{\mathcal{K}}} = a_k^*$ for all k. Note that $\sum_{k=1}^d b_k b_k^* = J_1(\mathbb{1})$ and also $b_k^* b_k = \mathbb{1}$ for all k, because u_1 is isometric. Thus we already have a modification of the Cuntz relations for the $\{b_k\}_{k=1}^d$, but these operators act between different Hilbert spaces.

$$\begin{tikzcd}
\mathcal{H}\otimes\epsilon_k & \\
\mathcal{H}\otimes\Omega_{\mathcal{K}} \arrow[u, "a_k^*"'] \arrow[r] & \mathcal{H}\otimes\mathcal{K} \arrow[ul, "b_k^*"']
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
\mathcal{H}\otimes\epsilon_k \arrow[r] \arrow[dr, "b_k"] & \mathcal{H}\otimes\mathcal{P} \arrow[d, "u_1"] \\
 & \mathcal{H}\otimes\mathcal{K}
\end{tikzcd}$$

2.4.5 Coupling Representations of Cuntz Algebras

Composing the two parts analysed in 2.4.4 we get the Cuntz algebra representation associated to the primary part of the coupling representation $\tilde{J} = J_1 \circ R$:

Proposition:

$$\tilde{J}(\tilde{x}) = \sum_{k=1}^{d} t_k \, \tilde{x} \, t_k^*$$

with $t_k = b_k \, r_k$, the composition interpreted in the natural way, explicitly:

$$t_k(\xi \otimes \tilde{\eta}) = u_1(\xi \otimes \epsilon_k) \otimes \tilde{\eta}.$$

Here $\xi \in \mathcal{H}$, $\tilde{\eta} \in \mathcal{K}_{[1,\infty)}$ and on the right side $\tilde{\eta}$ has been shifted one step to the right into $\mathcal{K}_{[2,\infty)}$, the free position then occupied by the image of u_1.

Proof: One may argue that $\tilde{J} = J_1 \circ R$ and $b_j \, r_k = 0$ for $j \neq k$. It is instructive, alternatively, to perform explicitly the following computation. If $\xi \otimes \eta \otimes \tilde{\eta} \in \mathcal{H} \otimes \mathcal{K}_1 \otimes \mathcal{K}_{[2,\infty)}$ and $u_1^*(\xi \otimes \eta) = \sum_{k=1}^d \xi_k \otimes \epsilon_k$ then $t_k^*(\xi \otimes \eta \otimes \tilde{\eta}) = \xi_k \otimes \tilde{\eta} \in \mathcal{H} \otimes \mathcal{K}_{[1,\infty)}$. Thus for $\mathcal{B}(\tilde{\mathcal{H}}) \ni \tilde{x} = x \otimes \tilde{y} \in \mathcal{B}(\mathcal{H}) \otimes \mathcal{B}(\mathcal{K}_{[1,\infty)})$ we get

$$\sum_{k=1}^{d} t_k \, \tilde{x} \, t_k^*(\xi \otimes \eta \otimes \tilde{\eta}) = \sum_{k=1}^{d} t_k(x \, \xi_k \otimes \tilde{y} \, \tilde{\eta})$$
$$= \sum_{k=1}^{d} u_1(x \, \xi_k \otimes \epsilon_k) \otimes \tilde{y} \, \tilde{\eta} = \sum_{k=1}^{d} u_1(x \otimes \mathbb{1})(\xi_k \otimes \epsilon_k) \otimes \tilde{y} \, \tilde{\eta}$$
$$= u_1(x \otimes \mathbb{1})u_1^*(\xi \otimes \eta) \otimes \tilde{y} \, \tilde{\eta} = \tilde{J}(\tilde{x})(\xi \otimes \eta \otimes \tilde{\eta}). \qquad \square$$

If $t_1, \ldots, t_d$ are given on $\tilde{\mathcal{H}}$ by a formula as in the proposition above then we call this a **coupling representation of the Cuntz algebra** $\mathcal{O}_d$. It is easily checked that the canonical $U(d)-$action τ on $\mathcal{O}_d$ (see Lemma 2.4.2), giving another minimal Kraus decomposition $\tilde{J}(\tilde{x}) = \sum_{k=1}^{d} \tau_g(t_k)\, \tilde{x}\, \tau_g(t_k^*)$ for any $g \in U(d)$, corresponds exactly to the possible changes of ONB's in $\mathcal{P}$ and thus occurs very naturally here.

2.4.6 Discussion

An important special case arises if the spaces $\mathcal{K}$ and $\mathcal{P}$ have the same dimension d and can thus be identified ($\mathcal{K} \simeq \mathcal{P}$) and J_1 is unital, i.e. we can think of u_1 as a unitary on $\mathcal{H} \otimes \mathcal{K}$. Compare 1.3.3 and 2.3.3. Then R is an endomorphism of $\mathcal{B}(\mathcal{K}_{[1,\infty)})$ which corresponds to a distinguished Cuntz algebra representation. See Section 2.5 for more details.

A coupling representation may thus be considered as a perturbation of R, specified by the isometry u_1. Recall that any two non-degenerate representations π and π' of $\mathcal{O}_d$ on a common Hilbert space given by $s_1, \ldots, s_d$ and $s_1', \ldots, s_d'$ are related by $s_k' = u\, s_k$ for all k, where u is a unitary on this Hilbert space (see [BJP96]). In Proposition 2.4.5 this relation takes a specific localized form given by the structure of coupling representations.

2.4.7 Invariance Property of Markovian Subspaces

Proposition: $t_k^*|_{\mathcal{H}} = a_k^*$ *for* $k = 1, \ldots, d$. *In particular:* $t_k^* \mathcal{H} \subset \mathcal{H}$.

Proof: $t_k^* \mathcal{H} \subset \mathcal{H}$ follows from the weak Markov property stated in Section 2.3 and Lemma 2.4.3. It also comes out in the following explicit computation (with $\xi_1, \xi_2 \in \mathcal{H},\ \tilde{\eta} \in \mathcal{K}_{[1,\infty)}$):

$$\begin{aligned}
&\langle \xi_1 \otimes \tilde{\eta}, t_k^*(\xi_2 \otimes \Omega_{[1,\infty)}) \rangle = \langle t_k(\xi_1 \otimes \tilde{\eta}), \xi_2 \otimes \Omega_{[1,\infty)} \rangle \\
=\ & \langle u_1(\xi_1 \otimes \epsilon_k) \otimes \tilde{\eta}, \xi_2 \otimes \Omega_{[1,\infty)} \rangle = \langle \xi_1 \otimes \epsilon_k \otimes \tilde{\eta}, u_1^*(\xi_2 \otimes \Omega_1) \otimes \Omega_{[2,\infty)} \rangle \\
=\ & \langle \xi_1 \otimes \epsilon_k \otimes \tilde{\eta}, \sum_{j=1}^{d} a_j^*(\xi_2) \otimes \epsilon_j \otimes \Omega_{[2,\infty)} \rangle \ = \ \langle \xi_1 \otimes \tilde{\eta}, a_k^*(\xi_2) \otimes \Omega_{[1,\infty)} \rangle,
\end{aligned}$$

i.e. $t_k^*|_{\mathcal{H}} = a_k^*$. $\Box$

2.4.8 Comments

In [BJKW00] the authors introduce a stochastic map as a tool to investigate $\{s_k^*\}-$invariant subspaces $\mathcal{H}$ for Cuntz algebra representations π, i.e. Markovian subspaces in our terminology. Proposition 2.4.7 shows that for coupling representations this map is nothing but the stochastic map $Z = \sum a_k \cdot a_k^*$ we started from. The following considerations show that coupling representations are not a specific case, but rather a concrete model of the general case. We use this also to give short and streamlined proofs of some well known facts about weak Markov dilations.

2.4.9 A Stochastic Map from $\mathcal{O}_d$ into $\mathcal{B}(\mathcal{H})$

Lemma: [BJKW00]
Given $a_1, \ldots, a_d \in \mathcal{B}(\mathcal{H})$ there is a stochastic map from $\mathcal{O}_d$ to $\mathcal{B}(\mathcal{H})$ mapping $s_{k_n} \ldots s_{k_1} s_{j_1}^* \ldots s_{j_m}^*$ to $a_{k_n} \ldots a_{k_1} a_{j_1}^* \ldots a_{j_m}^*$ for all families of indices $k_i, j_i \in \{1, \ldots, d\}$ and all $n, m \in \mathbb{N}_0$.

Proof: Realize $\mathcal{H}$ as a Markovian subspace such that $s_k^*|_{\mathcal{H}} = a_k^*$, for example by constructing a coupling representation π of $\mathcal{O}_d$ from $Z = \sum a_k \cdot a_k^*$. Then the map $\mathcal{O}_d \ni z \mapsto p_{\mathcal{H}} \pi(z)|_{\mathcal{H}}$ does the job. See ([BJKW00], 2.2) for a proof which does not use dilation theory. □

2.4.10 Cyclicity

Lemma: *For $\Theta = \sum s_k \cdot s_k^*$ and a Markovian subspace $\mathcal{H} \subset \tilde{\mathcal{H}}$:*

(1) $\hat{\mathcal{H}}_{[0,n]} = \overline{span} \{s_{k_n} \ldots s_{k_1} \xi : \ k_i \in \{1, \ldots, d\}, \ \xi \in \mathcal{H}\}$
(2) $\hat{\mathcal{H}} = \overline{\pi(\mathcal{O}_d)\mathcal{H}}$, i.e. $\mathcal{H}$ is cyclic for $\pi|_{\hat{\mathcal{H}}}$

Proof: By the definition in 2.2.2 the space $\hat{\mathcal{H}}_{[0,n]}$ is the range of $p_{[0,n]} = \Theta^n(p_{\mathcal{H}}) = \sum s_{k_n} \ldots s_{k_1} p_{\mathcal{H}} s_{k_1}^* \ldots s_{k_n}^*$. Thus $\hat{\mathcal{H}}_{[0,n]} \subset \overline{span}$. Conversely, if $\tilde{\xi} \in \overline{span}$, then by using the Cuntz relations one finds that $p_{[0,n]} \tilde{\xi} = \tilde{\xi}$. This proves (1). Further $\hat{\mathcal{H}}$ is the closure of $\bigcup_{n=0}^{\infty} \hat{\mathcal{H}}_{[0,n]}$ which by (1) can be written as $\overline{span}\{s_{k_n} \ldots s_{k_1} \xi : \ k_i \in \{1, \ldots, d\}, \ n \in \mathbb{N}_0, \ \xi \in \mathcal{H}\}$. This is equal to $\overline{\pi(\mathcal{O}_d)\mathcal{H}}$ because $\mathcal{H}$ is s_k^*-invariant for all k. □

2.4.11 Coupling Representation as a General Model

Proposition: *Any primary weak Markov dilation $\hat{\Theta}$ of $Z : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ is conjugate (as a dilation) to the primary part $\hat{J}$ of a coupling representation.*

Conjugacy means that $\hat{\Theta} = u^* \hat{J}(u \cdot u^*) u$ with a unitary u, and by conjugacy as a dilation we mean that additionally it is possible to choose u in such a way that it acts identically on $\mathcal{H}$. Compare ([Bh01], 2.2). As already noted in 2.3.3, if J_1 is unital then the coupling representation $\tilde{J}$ itself is already primary, i.e. $\tilde{J} = \hat{J}$.

Proof: It is shown in ([BJKW00], 5.1), that the operators $a_1, \ldots, a_d$ on a Markovian subspace $\mathcal{H}$ determine $s_1, \ldots, s_d$ up to unitary equivalence on $\overline{\pi(\mathcal{O}_d)\mathcal{H}}$. The unitary acts identically on $\mathcal{H}$. In fact, this is nothing but the uniqueness up to unitary equivalence of the minimal Stinespring representation of the stochastic map defined in Lemma 2.4.9. See ([BJKW00], sect.4) how this can be developed into a kind of commutant lifting theorem. By Lemma 2.4.10(2) $\overline{\pi(\mathcal{O}_d)\mathcal{H}}$ is the space $\hat{\mathcal{H}}$ of the corresponding primary weak Markov dilation. We conclude that any two primary dilations $\hat{J}^{(1)}, \ \hat{J}^{(2)}$ with

$(s_k^{(1)})^*|_{\mathcal{H}} = a_k^* = (s_k^{(2)})^*|_{\mathcal{H}}$ for all k are conjugate (as dilations). We have already shown above that one can start with an arbitrary Kraus decomposition (or Stinespring representation) of Z and construct J_1 and $\tilde{J}$ from it. Thus in every conjugacy class there are also coupling representations. $\square$

2.4.12 Deficiency Index of a Primary Dilation

Proposition: ([Bh01], B.2)
The number d ($= dim\mathcal{P} = rank\tilde{J}$) is a complete invariant for conjugacy classes of primary weak Markov dilations of Z.

Proof: Different d means a different rank of $\tilde{J}$, then conjugacy is not possible. But $J^{(1)} \neq J^{(2)}$ with the same d corresponds to Kraus decompositions

$$Z(x) = \sum_{k=1}^{d} a_k^{(1)}\, x\, (a_k^{(1)})^* = \sum_{k=1}^{d} a_k^{(2)}\, x\, (a_k^{(2)})^*.$$

Then by A.2.6 there exists a unitary matrix $g \in U(d)$ such that $a_k^{(2)} = \sum_j g_{jk}\, a_j^{(1)}$. As shown in 2.4.11 this implies that up to unitary equivalence we also have $s_k^{(2)} = \sum_j g_{jk}\, s_j^{(1)} = \tau_g(s_k^{(1)})$. Using Lemma 2.4.2 we see that the corresponding endomorphisms $\hat{J}^{(1)}$ and $\hat{J}^{(2)}$ are conjugate. $\square$

This result has also been obtained by Bhat in ([Bh01], B.2) by different means. He calls the difference ($rank\,\hat{J} - rank\,Z$) the deficiency index of the primary dilation. The proof given here emphasizes that the deficiency index has its origin in the difference between non-minimal and minimal Kraus decompositions (or Stinespring representations) of Z. For the primary part of a coupling representation we see that conjugacy only depends on v_1' (together with $dim\mathcal{P}$), not on its extension u_1^*. In other words, the arbitrariness involved in this extension as described in 1.3.3 is not important if we consider dilations up to conjugacy. Note also that the equivalence relation relevant here is not the (fine) equivalence of weak tensor dilations of first order considered in Section 1.4, but only the (coarse) unitary equivalence of Stinespring representations.

2.5 Cocycles and Coboundaries

2.5.1 Special Setting

As in the previous sections, let $Z : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ be a stochastic map and $\tilde{J} : \mathcal{B}(\tilde{\mathcal{H}}) \to \mathcal{B}(\tilde{\mathcal{H}})$ the time evolution of a weak Markov dilation given in a coupling representation. Let us further assume that we are in the special situation discussed in 2.4.6, i.e. we have an identification $\mathcal{K} \simeq \mathcal{P}$ and u_1 is a unitary on $\mathcal{H} \otimes \mathcal{K}$. Then J_1 is unital and $\tilde{J} = \hat{J}$. The essentials of the following arguments come out most clearly in this important special case. We shall see in 2.5.7 what modifications are necessary in the general case.

2.5.2 Cocycles

We have $\tilde{J} = Ad(\tilde{u}_1)(\mathbb{1} \otimes R)$ on $\mathcal{B}(\mathcal{H} \otimes \mathcal{K}_{[1,\infty)})$, where

$$R : \mathcal{B}(\mathcal{K}_{[1,\infty)}) \to \mathcal{B}(\mathcal{K}_{[1,\infty)})$$
$$\tilde{y} \mapsto \mathbb{1} \otimes \tilde{y}$$

is an endomorphism of $\mathcal{B}(\mathcal{K}_{[1,\infty)})$. Computing iterates we find

$$\begin{aligned} \tilde{J}^n &= Ad(\tilde{w}_n)(\mathbb{1} \otimes R)^n \\ \text{with} \quad \tilde{w}_n &:= \tilde{u}_1 (\mathbb{1} \otimes R)(\tilde{u}_1) \ldots (\mathbb{1} \otimes R)^{n-1}(\tilde{u}_1). \end{aligned}$$

By analogy with cohomology of groups it is natural to call the family $\big(\tilde{w}_n\big)_{n=1}^{\infty}$ a **unitary cocycle**. It satisfies the cocycle equation

$$\tilde{w}_{n+m} = \tilde{w}_n (\mathbb{1} \otimes R)^n (\tilde{w}_m).$$

See the introduction of [LW00] for a survey how similar notions of cocycles have been used in noncommutative probability theory.

2.5.3 Coboundaries

Let q be a projection in $\mathcal{B}(\tilde{\mathcal{H}})$ fixed by $\mathbb{1} \otimes R$. We say that the unitary cocycle $\big(\tilde{w}_n\big)_{n=1}^{\infty}$ is a **coboundary** (with projection q) if there is an isometry $w \in \mathcal{B}(\tilde{\mathcal{H}})$ with range $q\tilde{\mathcal{H}}$ and such that $\tilde{w}_1 = w^* \, (\mathbb{1} \otimes R)(w)$. Because $ww^* = q$ is fixed by $\mathbb{1} \otimes R$ we get

$$\begin{aligned} \tilde{w}_n &= \tilde{u}_1 \ldots (\mathbb{1} \otimes R)^{n-1}(\tilde{u}_1) = w^* \, (\mathbb{1} \otimes R)(w) \ldots (\mathbb{1} \otimes R)^{n-1}(w^*)(\mathbb{1} \otimes R)^n(w) \\ &= w^* \, (\mathbb{1} \otimes R)^n(w). \end{aligned}$$

Proposition: *$\tilde{J}$ and $(\mathbb{1} \otimes R)|_{q\,\mathcal{B}(\tilde{\mathcal{H}})\,q}$ are conjugate if and only if u_1 can be chosen in such a way that $(\tilde{w}_n)_{n=1}^{\infty}$ is a coboundary (with projection q).*

Proof: If $\tilde{u}_1 = \tilde{w}_1 = w^* \, (\mathbb{1} \otimes R)(w)$, then

$$\tilde{J}(\tilde{x}) = \tilde{u}_1 \, (\mathbb{1} \otimes R)(\tilde{x}) \, \tilde{u}_1^* = w^* \, (\mathbb{1} \otimes R)(w \, \tilde{x} \, w^*) w,$$

proving conjugacy. Conversely we find that $\tilde{u}_1^* w^* \, (\mathbb{1} \otimes R)(w)$ commutes with $(\mathbb{1} \otimes R)(\tilde{x})$ for all $\tilde{x} \in \mathcal{B}(\tilde{\mathcal{H}})$, i.e. $\tilde{u}_1^* w^* \, (\mathbb{1} \otimes R)(w) = y \in \mathcal{B}(\mathcal{K}_1)$. The modified $u_1^\dagger = u_1 y$ generates the same endomorphism $\tilde{J}$ (because $yy^* = \mathbb{1}$) and satisfies $u_1^\dagger = w^* \, (\mathbb{1} \otimes R)(w)$. $\square$

2.5.4 R as a Shift

Let us consider some properties of R on $\mathcal{B}(\mathcal{K}_{[1,\infty)})$. Clearly R satisfies the abstract shift property introduced by R.T. Powers in [Pow88]:

$$\bigcap_{n\geq 0} R^n(\mathcal{B}(\mathcal{K}_{[1,\infty)}) = \mathbb{C}\mathbb{1}.$$

Another terminology for this property is 'pure'. The vector state $\Omega_{[1,\infty)}$ is invariant for R, i.e.

$$\langle \Omega_{[1,\infty)}, \tilde{y}\, \Omega_{[1,\infty)} \rangle = \langle \Omega_{[1,\infty)}, R(\tilde{y})\, \Omega_{[1,\infty)} \rangle.$$

A theorem of Powers [Pow88] states that shifts with invariant vector states are classified by their rank up to conjugacy. Thus R is just a convenient model for this type of endomorphism. In [BJP96] it is called the Haar shift because the corresponding Cuntz algebra representation is related to the Haar wavelet. In our setting the rank is $d = dim\mathcal{K}$.

2.5.5 Invariant Vector States

Consider the case that $Z : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H}),\ x \mapsto (v_1')^*\ x \otimes 1\ v_1'$ has an invariant vector state given by a unit vector $\Omega_\mathcal{H} \in \mathcal{H}$, i.e.

$$\langle \Omega_\mathcal{H}, x\Omega_\mathcal{H} \rangle \ = \ \langle \Omega_\mathcal{H}, Z(x)\Omega_\mathcal{H} \rangle$$

for all $x \in \mathcal{B}(\mathcal{H})$. By A.5.1 there is a unit vector $\Omega_\mathcal{K} \in \mathcal{K}$ such that $v_1'\, \Omega_\mathcal{H} = \Omega_\mathcal{H} \otimes \Omega_\mathcal{K}$. Using $\Omega_\mathcal{K}$ we have $u_1, J_1, \tilde{J}$ as before. From $v_1' = u_1^*|_{\mathcal{H}\otimes\Omega_\mathcal{K}}$ we conclude that $u_1\, \Omega_\mathcal{H} \otimes \Omega_\mathcal{K} = \Omega_\mathcal{H} \otimes \Omega_\mathcal{K}$.

Lemma: *If Z has an invariant vector state given by $\Omega_\mathcal{H} \in \mathcal{H}$, then $\tilde{J}$ has an invariant vector state given by $\Omega_{\tilde{\mathcal{H}}} := \Omega_\mathcal{H} \otimes \Omega_{[1,\infty)}$.*

Proof:

$$\begin{aligned}
\langle \Omega_{\tilde{\mathcal{H}}}, \tilde{J}(x\otimes\tilde{y})\Omega_{\tilde{\mathcal{H}}} \rangle \ &= \ \langle \Omega_\mathcal{H} \otimes \Omega_{[1,\infty)}, \tilde{u}_1\, (x \otimes R(\tilde{y}))\, \tilde{u}_1^*\, \Omega_\mathcal{H} \otimes \Omega_{[1,\infty)} \rangle \\
&= \ \langle u_1^*(\Omega_\mathcal{H} \otimes \Omega_\mathcal{K}) \otimes \Omega_{[2,\infty)}, (x \otimes \mathbb{1} \otimes \tilde{y}) u_1^*(\Omega_\mathcal{H} \otimes \Omega_\mathcal{K}) \otimes \Omega_{[2,\infty)} \rangle \\
&= \ \langle \Omega_\mathcal{H}, x\Omega_\mathcal{H} \rangle\, \langle \Omega_{[1,\infty)}, \tilde{y}\Omega_{[1,\infty)} \rangle \ = \ \langle \Omega_{\tilde{\mathcal{H}}}, x \otimes \tilde{y}\ \Omega_{\tilde{\mathcal{H}}} \rangle. \qquad \square
\end{aligned}$$

2.5.6 Existence of Coboundaries and Ergodicity of *Z*

After these preparations we shall now examine the existence of coboundaries with projection $q := q_{[1,\infty)}$ onto $\mathcal{K}_{[1,\infty)} \simeq \Omega_\mathcal{H} \otimes \mathcal{K}_{[1,\infty)} \subset \tilde{\mathcal{H}}$, which is the natural domain for R to act. We shall also find a formula for computing the isometry w in this case.

Recall that Z is **ergodic** if its only fixed points are multiples of the identity. An invariant state ϕ is called absorbing if $Z^n(x)$ converges to $\phi(x)\mathbb{1}$ for all $x \in \mathcal{B}(\mathcal{H})$ in the weak* sense, or equivalently, if $Z_*^n(x)\psi \to \phi$ for all $\psi \in \mathcal{B}(\mathcal{H})_*$ in the weak sense. Compare A.5.

Theorem: *Let $Z : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ be a stochastic map with an invariant vector state given by $\Omega_\mathcal{H} \in \mathcal{H}$. Further assume that $\mathcal{K} \simeq \mathcal{P}$ and u_1 is a unitary on $\mathcal{H} \otimes \mathcal{K}$ (see 2.5.1 and 2.5.5). The following assertions are equivalent:*

(1) *Z is ergodic.*

(2) *The invariant vector state is absorbing.*

(3) *For all $\xi \in \mathcal{H}$ we have for $n \to \infty$*

$$\| q_{[1,\infty)} \tilde{w}_n^* (\xi \otimes \Omega_{[1,\infty)}) \| \longrightarrow \|\xi\|$$

(4) *For all $\xi \in \mathcal{H}$ we have for $n \to \infty$*

$$\| q_{[1,\infty)} (\mathbb{1} \otimes R)^{n-1}(v_1') \ldots (\mathbb{1} \otimes R)(v_1')\, v_1' (\xi \otimes \Omega_{[1,\infty)}) \| \longrightarrow \|\xi\|$$

(5) *There is an isometry $w \in \mathcal{B}(\tilde{\mathcal{H}})$ with range $\mathcal{K}_{[1,\infty)}$ such that*

$$w = stop - \lim_{n \to \infty} \tilde{w}_n^*$$

(6) *There is an isometry $w \in \mathcal{B}(\tilde{\mathcal{H}})$ with range $\mathcal{K}_{[1,\infty)}$ such that*

$$\tilde{u}_1 = w^* \, (\mathbb{1} \otimes R)(w),$$

i.e. $\left(\tilde{w}_n\right)_{n=1}^{\infty}$ is a coboundary (with projection $q_{[1,\infty)}$).

(7) *There is an isometry $w \in \mathcal{B}(\tilde{\mathcal{H}})$ with range $\mathcal{K}_{[1,\infty)}$ such that*

$$\tilde{J}(\tilde{x}) = w^* \, (\mathbb{1} \otimes R)(w \tilde{x} w^*) \, w \quad \text{for all } \tilde{x} \in \mathcal{B}(\tilde{\mathcal{H}}).$$

(8) *$\tilde{J}$ is conjugate to R.*

Remarks: The stochastic map Z depends only on $v_1' = u_1^*|_{\mathcal{H} \otimes \Omega_\mathcal{K}}$, and also in (4) it becomes explicit that only this restriction of u_1^* is relevant for the decision whether we are in the situation described by the theorem or not. Compare 2.4.12. Note that in (4) the isometry v_1' has to be understood as $v_1' \otimes \mathbb{1}$ but here we omit the ˜ put to u_1 in the same situation. (3) and (4) will get a more intuitive interpretation later when we consider applications to scattering theory, see Sections 2.6-2.8. (5) and (6) can be viewed as mutual inverses, in particular the operators u_1 and w determine each other. As indicated above, if only Z is given then there is some freedom in choosing u_1 and w.

$$\begin{tikzcd}
\mathcal{B}(\tilde{\mathcal{H}}) \arrow[r, "\tilde{J}"] \arrow[d, "Ad(w)"'] & \mathcal{B}(\tilde{\mathcal{H}}) \arrow[d, "Ad(w)"] \\
\mathcal{B}(\mathcal{K}_{[1,\infty)}) \arrow[r, "R"] & \mathcal{B}(\mathcal{K}_{[1,\infty)})
\end{tikzcd}$$

Proof: If Z is ergodic then we can use the commutant lifting result in ([BJKW00], 5.1, compare also 2.4.11) to conclude that the Cuntz algebra representation associated to $\tilde{J}$ is irreducible on $\overline{\pi(\mathcal{O}_d)\mathcal{H}}$. But $\overline{\pi(\mathcal{O}_d)\mathcal{H}} = \hat{\mathcal{H}}$ by Lemma 2.4.10 and $\hat{\mathcal{H}} = \tilde{\mathcal{H}}$ by assumption (see 2.5.1). Therefore in this case $\pi(\mathcal{O}_d)$ is irreducible on $\tilde{\mathcal{H}}$.

By ([BJP96], 3.3) this implies that $\tilde{J}$ is ergodic. Furthermore we have already shown in Lemma 2.5.5 that $\tilde{J}$ has an invariant vector state given by $\Omega_{\tilde{\mathcal{H}}}$. Refining the theorem of Powers cited in 2.5.4 it is also true that an ergodic endomorphism with an invariant vector state is classified by its rank up to conjugacy (see [BJP96], 4.2). Thus $\tilde{J}$ is conjugate to R. We have shown (1) $\Rightarrow$ (8).

To prove (8) $\Rightarrow$ (2) we first show that $\Omega_{[1,\infty)}$ gives an absorbing vector state for R. Weak convergence to this state is immediate if we start with vector states given by $\eta_1 \otimes \ldots \otimes \eta_k \otimes \Omega_{k+1} \otimes \Omega_{k+2} \otimes \ldots \in \mathcal{K}_{[1,\infty)}$ (where the η_i are unit vectors in K). But such vectors are total in $\mathcal{K}_{[1,\infty)}$. Therefore the convergence is valid for all pure states. Taking convex combinations and weak closure completes the argument. Now if $\tilde{J}$ is conjugate to R by (8), it follows that the $\tilde{J}-$invariant vector state $\Omega_{\tilde{\mathcal{H}}}$ must be absorbing for $\tilde{J}$. Because $\Omega_{\tilde{\mathcal{H}}} = \Omega_{\mathcal{H}} \otimes \Omega_{[1,\infty)}$ we conclude by restricting to $\mathcal{H}$ that the vector state given by $\Omega_{\mathcal{H}}$ is absorbing for Z.

For (2) $\Rightarrow$ (1) see A.5.2.

We now prove (2) $\Rightarrow$ (3). Consider the unitary cocycle

$$\tilde{w}_n := \tilde{u}_1(\mathbb{1} \otimes R)(\tilde{u}_1) \ldots (\mathbb{1} \otimes R)^{n-1}(\tilde{u}_1) \quad with\; n \in \mathbb{N}.$$

For $\xi \in \mathcal{H}$ and $x \in \mathcal{B}(\mathcal{H})$ we compute

$$\begin{aligned} &\langle \tilde{w}_n^*(\xi \otimes \Omega_{[1,\infty)}), x \otimes \mathbb{1} \tilde{w}_n^*(\xi \otimes \Omega_{[1,\infty)}) \rangle = \langle \xi \otimes \Omega_{[1,\infty)}, \tilde{w}_n(x \otimes \mathbb{1})\tilde{w}_n^*(\xi \otimes \Omega_{[1,\infty)}) \rangle \\ =\; &\langle \xi \otimes \Omega_{[1,\infty)}, \tilde{J}^n(x \otimes \mathbb{1})(\xi \otimes \Omega_{[1,\infty)}) \rangle = \langle \xi,\, Z^n(x)\xi \rangle. \end{aligned}$$

If $x = p_{\Omega_{\mathcal{H}}}$, the one-dimensional projection onto $\mathbb{C}\,\Omega_{\mathcal{H}}$, we conclude from (2), i.e. the vector state given by $\Omega_{\mathcal{H}}$ is absorbing for Z, that

$$\langle \xi,\, Z^n(p_{\Omega_{\mathcal{H}}})\xi \rangle \;\longrightarrow\; \langle \xi,\, \xi \rangle \,\langle \Omega_{\mathcal{H}},\, p_{\Omega_{\mathcal{H}}} \Omega_{\mathcal{H}} \rangle \;= \|\xi\|^2.$$

Inserting $x = p_{\Omega_{\mathcal{H}}}$ above and using the fact that $q_{[1,\infty)} = p_{\Omega_{\mathcal{H}}} \otimes \mathbb{1}$ we find that

$$\|q_{[1,\infty)} \tilde{w}_n^*(\xi \otimes \Omega_{[1,\infty)})\| \;\longrightarrow\; \|\xi\| \quad \text{for } n \to \infty,$$

which is (3).

(3) and (4) are equivalent because

$$\tilde{w}_n^*|_{\mathcal{H}} = (\mathbb{1} \otimes R)^{n-1}(v_1') \ldots (\mathbb{1} \otimes R)(v_1')\, v_1'.$$

To prove (3) $\Rightarrow$ (5) we have to show that $\big(\tilde{w}_n^*(\tilde{\xi})\big)$ converges for all $\tilde{\xi} \in \tilde{\mathcal{H}}$. It suffices to consider the total set of vectors of the form $\tilde{\xi} = \xi \otimes \eta_1 \otimes \ldots \otimes \eta_k \otimes \Omega_{k+1} \otimes \Omega_{k+2} \ldots \in \mathcal{H} \otimes \mathcal{K}_{[1,\infty)}$. Note that $(\mathbb{1} \otimes R)^j(u_1^*)$ acts nontrivially only on $\mathcal{H}$ and $\mathcal{K}_{j+1}$. Concentrating our attention on the factors $(\mathbb{1} \otimes R)^j(u_1^*)$ with $j \geq k$ we find that it even suffices to prove the convergence of $\big(w_n^*(\xi \otimes \Omega_{[1,\infty)})\big)$ for all $\xi \in \mathcal{H}$. Suppose $m > n$. Then

$$(\tilde{w}_m^* - \tilde{w}_n^*)(\xi \otimes \Omega_{[1,\infty)}) = \Big((\mathbb{1} \otimes R)^{m-1}(\tilde{u}_1^*) \ldots (\mathbb{1} \otimes R)^n(\tilde{u}_1^*) - \mathbb{1}_{\tilde{\mathcal{H}}}\Big)\, \tilde{w}_n^*(\xi \otimes \Omega_{[1,\infty)}).$$

Because $\tilde{w}_n^*(\xi \otimes \Omega_{[1,\infty)}) \in \mathcal{H} \otimes \mathcal{K}_{[1,n]}$ and $\Omega_{\mathcal{H}} \otimes \Omega_{j+1}$ is fixed by $(\mathbb{1} \otimes R)^j(\tilde{u}_1^*)$ we find that

$$\Big((\mathbb{1} \otimes R)^{m-1}(\tilde{u}_1^*) \ldots (\mathbb{1} \otimes R)^n(\tilde{u}_1^*) - \mathbb{1}_{\tilde{\mathcal{H}}}\Big)\, q_{[1,\infty)} \tilde{w}_n^*(\xi \otimes \Omega_{[1,\infty)}) = 0.$$

But the part which is orthogonal to $q_{[1,\infty)}$ tends to zero by (3). Thus $\big(\tilde{w}_n^*(\xi \otimes \Omega_{[1,\infty)})\big)$ is a Cauchy sequence and converges in $\tilde{\mathcal{H}}$. It is already clear that the limit belongs to $\mathcal{K}_{[1,\infty)} \cong \Omega_{\mathcal{H}} \otimes \mathcal{K}_{[1,\infty)} \subset \tilde{\mathcal{H}}$. Defining w as in (5) by $w = stop - \lim_{n \to \infty} \tilde{w}_n^*$ we thus know that the range of w is contained in $\mathcal{K}_{[1,\infty)}$.

It remains to show that $\mathcal{K}_{[1,\infty)}$ is contained in the range of w. Again because $\Omega_{\mathcal{H}} \otimes \Omega_{j+1}$ is fixed by $(\mathbb{1} \otimes R)^j(\tilde{u}_1)$ it is evident that the vector $\tilde{w}_n(\Omega_{\mathcal{H}} \otimes \eta_1 \otimes \ldots \otimes \eta_k \otimes \Omega_{k+1} \otimes \Omega_{k+2} \ldots)$ for $n > k$ does not depend on n. It follows that $\big(\tilde{w}_n(\tilde{\eta})\big)$ is convergent in norm for all $\tilde{\eta} \in \mathcal{K}_{[1,\infty)}$ because this is true on a total subset. The proof of (3) $\Rightarrow$ (5) is finished by applying the following

Lemma: *Let $\big(\tilde{w}_n^*\big)$ be a sequence of unitaries which converges to w in the stop-topology. Then ξ is in the range of w if and only if $\big(\tilde{w}_n(\xi)\big)$ is convergent in norm.*

The lemma can be proved by combining the following elementary facts: $\tilde{w}_n$ converges to w^* in the wop-topology. A weakly convergent sequence of unit vectors is convergent in norm if and only if the limit is a unit vector. ξ is in the range of w if and only if $\|w^*(\xi)\| = \|\xi\|$.

We return to the proof of the theorem. If we start with (5), then by assumption for all $\tilde{\xi} \in \tilde{\mathcal{H}}$ the sequence $\big(\tilde{w}_{n+1}^* \tilde{\xi}\big)$ converges to a vector $w\tilde{\xi} \in \mathcal{K}_{[1,\infty)}$. Again applying the lemma above we see that for all $\tilde{\eta} \in \mathcal{K}_{[1,\infty)}$ we get $\|\tilde{w}_n(\tilde{\eta}) - w^*(\tilde{\eta})\| \to 0$ for $n \to \infty$. We combine this to get

$$\begin{aligned} \tilde{u}_1^* &= (\mathbb{1} \otimes R)(\tilde{u}_1) \ldots (\mathbb{1} \otimes R)^n(\tilde{u}_1)(\mathbb{1} \otimes R)^n(\tilde{u}_1^*) \ldots (\mathbb{1} \otimes R)(\tilde{u}_1^*)\tilde{u}_1^* \\ &= (\mathbb{1} \otimes R)(\tilde{w}_n)\, \tilde{w}_{n+1}^* \overset{stop}{\to} (\mathbb{1} \otimes R)(w^*)\, w \quad \text{for } n \to \infty. \end{aligned}$$

This means that $\tilde{u}_1 = w^*\, (\mathbb{1} \otimes R)(w)$ and shows (5) $\Rightarrow$ (6).

(6) $\Rightarrow$ (7) follows as in Proposition 2.5.3. Finally if we interpret w as a unitary from $\tilde{\mathcal{H}}$ to $\mathcal{K}_{[1,\infty)}$, then (7) tells us that $\tilde{J}$ and R are conjugate, i.e. we have (8). $\square$

2.5.7 General Case

Let us indicate the modifications necessary without the simplifying assumptions $\mathcal{K} \simeq \mathcal{P}$ and u_1 unitary on $\mathcal{H} \otimes \mathcal{K}$ announced in 2.5.1. For a bounded operator

$$\tilde{y} : \bigotimes_1^m \mathcal{P} \otimes \mathcal{K}_{[m+1,\infty)} \to \bigotimes_1^{m'} \mathcal{P} \otimes \mathcal{K}_{[m'+1,\infty)}$$

we define $R_n(\tilde{y}) := \bigotimes_1^n \mathbb{1}_{\mathcal{P}} \otimes \tilde{y}$, such that

$$R_n(\tilde{y}) : \bigotimes_1^{n+m} \mathcal{P} \otimes \mathcal{K}_{[n+m+1,\infty)} \to \bigotimes_1^{n+m'} \mathcal{P} \otimes \mathcal{K}_{[n+m'+1,\infty)}.$$

Thus, strictly speaking, R_n stands for a whole family of amplification maps depending on m and m'. Then with $u_1 : \mathcal{H} \otimes \mathcal{P} \to \mathcal{H} \otimes \mathcal{K}$ isometric and $\tilde{u}_1 = u_1 \otimes \mathbb{1}_{[2,\infty)}$ as in 2.3.1 and 2.3.2, we can further define

$$\tilde{w}_n := \tilde{u}_1 (\mathbb{1} \otimes R_1)(\tilde{u}_1) \ldots (\mathbb{1} \otimes R_{n-1})(\tilde{u}_1) : \mathcal{H} \otimes \bigotimes_1^n \mathcal{P} \otimes \mathcal{K}_{[n+1,\infty)} \to \tilde{\mathcal{H}}.$$

Then we get

$$\tilde{J}^n(\tilde{x}) = \tilde{w}_n\, (\mathbb{1} \otimes R_n)(\tilde{x})\, \tilde{w}_n^*.$$

$(\tilde{w}_n)$ is an isometric cocycle in a somewhat generalized sense, satisfying the cocycle equation $\tilde{w}_{n+m} = \tilde{w}_n (\mathbb{1} \otimes R_n)(\tilde{w}_m)$.

Ignoring $\mathcal{K}_{[n+1,\infty)}$ where $\tilde{w}_n$ acts trivially, we have an isometry $w_n : \mathcal{H} \otimes \bigotimes_1^n \mathcal{P} \to \mathcal{H} \otimes \mathcal{K}_{[1,n]}$. If we identify $\mathcal{H} \otimes \mathcal{K}_{[1,n]} \simeq \mathcal{H} \otimes \mathcal{K}_{[1,n]} \otimes \Omega_{[n+1,\infty)} \subset \tilde{\mathcal{H}}$, then the range of w_n is $\hat{\mathcal{H}}_{[0,n]}$, with notation consistent to that in Section 2.2. In fact, $p_{[0,1]} = \tilde{J}(p_{\mathcal{H}}) = w_1 w_1^*$ and

$$\begin{aligned} p_{[0,n]} \;&=\; \tilde{J}^n(p_{\mathcal{H}}) \;=\; \tilde{J}^{n-1}(w_1 w_1^*) \;=\; w_{n-1}\, (\mathbb{1} \otimes R_{n-1})(w_1 w_1^*)\, w_{n-1}^* \\ &=\; w_{n-1}\, (\mathbb{1} \otimes R_{n-1})(w_1)\, (\mathbb{1} \otimes R_{n-1})(w_1)^*\, w_{n-1}^* \;=\; w_n w_n^* \end{aligned}$$

by the cocycle equation with $m = 1$.

Similarly the recursive procedure for constructing w_n given in 2.2.4 is nothing but a version of the cocycle equation with $m = 1$ where additive increments are made explicit. To see that we write

$$\hat{\mathcal{H}}_{[0,n]} = w_n(\mathcal{H} \otimes \bigotimes_1^n \mathcal{P}) = w_{n-1}\, (\mathbb{1} \otimes R_{n-1})(w_1)(\mathcal{H} \otimes \bigotimes_1^n \mathcal{P}).$$

Now define $\mathcal{N} \subset \mathcal{H} \otimes \mathcal{K}_n$ by

$$(\mathbb{1} \otimes R_{n-1})(w_1)(\mathcal{H} \otimes \bigotimes_1^n \mathcal{P}) \;=\; \big[\mathcal{H} \otimes \bigotimes_1^{n-1} \mathcal{P} \otimes \Omega_n\big] \;\oplus\; \big[\bigotimes_1^{n-1} \mathcal{P} \otimes \mathcal{N}\big].$$

Obviously for different n the space $\mathcal{N}$ has the same size and it is therefore natural to think of the above definition as of an embedding of an abstract Hilbert space $\mathcal{N}$ into $\mathcal{H} \otimes \mathcal{K}_n$. Then

$$\hat{\mathcal{H}}_{[0,n]} = w_{n-1}(\mathcal{H} \otimes \bigotimes_1^{n-1} \mathcal{P} \otimes \Omega_n \ \oplus \ \bigotimes_1^{n-1} \mathcal{P} \otimes \mathcal{N})$$
$$= \hat{\mathcal{H}}_{[0,n-1]} \ \oplus \ w_{n-1}(\bigotimes_1^{n-1} \mathcal{P} \otimes \mathcal{N}),$$

which shows that

$$\hat{\mathcal{H}}_{[0,n]} \ominus \hat{\mathcal{H}}_{[0,n-1]} \simeq \bigotimes_1^{n-1} \mathcal{P} \otimes \mathcal{N}.$$

This is the formula from 2.2.4, where the minimal case has been treated.

Regrouping terms in the tensor product and interpreting the domain $\mathcal{H} \otimes \bigotimes_1^n \mathcal{P} \otimes \mathcal{K}_{[n+1,\infty)}$ of $\tilde{w}_n$ as $\mathcal{H} \otimes \mathcal{K}_{[1,\infty)} \otimes \bigotimes_1^n \mathcal{P}$ (permutation and left shift), we can think of $\tilde{w}_n$ as an operator from $\tilde{\mathcal{H}} \otimes \bigotimes_1^n \mathcal{P}$ into $\tilde{\mathcal{H}}$. Restricting we get $\hat{w}_n : \hat{\mathcal{H}} \otimes \bigotimes_1^n \mathcal{P} \to \hat{\mathcal{H}}$ and $\hat{J}^n(\hat{x}) = \hat{w}_n \, \hat{x} \otimes \mathbb{1} \, \hat{w}_n^*$ as formulas for primary dilations. Again this notation is consistent with 2.2.4, but in this section we continue to work without the regrouping.

If Z has an invariant vector state given by a unit vector $\Omega_\mathcal{H} \in \mathcal{H}$, then by A.5.1 there is a unit vector $\Omega_\mathcal{P} \in \mathcal{P}$ such that $v_1' \Omega_\mathcal{H} = \Omega_\mathcal{H} \otimes \Omega_\mathcal{P}$. Because $v_1' = u_1^*|_{\mathcal{H} \otimes \Omega_\mathcal{K}}$ we have $u_1 \, \Omega_\mathcal{H} \otimes \Omega_\mathcal{P} = \Omega_\mathcal{H} \otimes \Omega_\mathcal{K}$. Using the distinguished unit vector $\Omega_\mathcal{P}$ we can form the infinite tensor product $\mathcal{P}_{[1,\infty)} := \bigotimes_1^\infty \mathcal{P}$ and a right tensor shift on $\mathcal{B}(\bigotimes_1^\infty \mathcal{P})$ denoted by $R_\mathcal{P}$. By $q_{[1,n]}$ we denote the projection from $\mathcal{H} \otimes \bigotimes_1^n \mathcal{P}$ onto $\bigotimes_1^n \mathcal{P} \simeq \Omega_\mathcal{H} \otimes \bigotimes_1^n \mathcal{P} \subset \mathcal{H} \otimes \bigotimes_1^n \mathcal{P}$. Similarly for $n = \infty$.

Restricting to the primary part $\hat{J}$ of the dilation we can now write down the general version of Theorem 2.5.6. Because the arguments in the proof are similar we only give the result.

Theorem: *Let $Z : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ be a stochastic map with an invariant vector state given by $\Omega_\mathcal{H} \in \mathcal{H}$. The following assertions are equivalent:*

(1) *Z is ergodic.*

(2) *The invariant vector state is absorbing.*

(3) *For all $\xi \in \mathcal{H}$ we have for $n \to \infty$*

$$\| q_{[1,n]} \, w_n^* (\xi \otimes \Omega_{[1,n]}) \| \longrightarrow \| \xi \|$$

(4) *For all $\xi \in \mathcal{H}$ we have for $n \to \infty$*

$$\| q_{[1,\infty)} \, (\mathbb{1} \otimes R_\mathcal{P})^{n-1} (v_1') \ldots (\mathbb{1} \otimes R_\mathcal{P})(v_1') \, v_1' (\xi \otimes \Omega_{[1,\infty)}) \| \longrightarrow \| \xi \|$$

(5) *There is a unitary $w : \hat{\mathcal{H}} \to \mathcal{P}_{[1,\infty)}$ such that*

$$w = stop - \lim_{n \to \infty} \ \tilde{w}_n^*$$

(6) *There is a unitary* $w : \hat{\mathcal{H}} \to \mathcal{P}_{[1,\infty)}$ *such that*

$$\tilde{u}_1 = w^* \, (\mathbb{1} \otimes R_1)(w).$$

(7) *There is a unitary* $w : \hat{\mathcal{H}} \to \mathcal{P}_{[1,\infty)}$ *such that*

$$\hat{J}(\hat{x}) = w^* \; R_{\mathcal{P}}(w \hat{x} w^*) \; w \quad \textit{for all } \hat{x} \in \mathcal{B}(\hat{\mathcal{H}}).$$

(8) $\tilde{J}$ *is conjugate to* $R_{\mathcal{P}}$.

In (4) we have to amplify v_1' in the natural way. Beginning with (5) we have to use $\mathcal{P}_{[1,\infty)} \simeq \Omega_{\mathcal{H}} \otimes \mathcal{P}_{[1,\infty)} \subset \mathcal{H} \otimes \mathcal{P}_{[1,\infty)}$. Again (5) provides a way to compute the operator w which gives the conjugacy between $\hat{J}$ and $R_{\mathcal{P}}$ stated in (8). In (6), to interpret w as an argument of $\mathbb{1} \otimes R_1$, we have to take $m = 0$, $m' = \infty$ in the definition of R_1 in the beginning of the section. We have the same w in all assertions and it follows that whenever $\hat{J}$ is conjugate to a shift $R_{\mathcal{P}}$ then the conjugacy is of the type characterized in the theorem.

$$\begin{tikzcd}
\mathcal{B}(\hat{\mathcal{H}}) \arrow[r, "\hat{J}"] \arrow[d, "Ad(w)"'] & \mathcal{B}(\hat{\mathcal{H}}) \arrow[d, "Ad(w)"] \\
\mathcal{B}(\mathcal{P}_{[1,\infty)}) \arrow[r, "R_{\mathcal{P}}"] & \mathcal{B}(\mathcal{P}_{[1,\infty)})
\end{tikzcd}$$

For E_0-semigroups W. Arveson in [Ar89] introduced the notion of product systems, see also [Ar03]. The space $\bigotimes_1^\infty \mathcal{P}$ corresponds to the discrete product system associated to the weak dilation $\hat{J}$ by Bhat in [Bh96]. It is therefore important to recognize that it is exactly the shift $R_{\mathcal{P}}$ on this space which in the ergodic situation with invariant vector state turns out to be conjugate to $\hat{J}$. Note that Theorem 2.5.7 provides a discrete analogue for conjugacies between E_0-semigroups and CCR-shifts, compare [Pow88, Ar89, Bh01, Ar03]. Note also that the conjugacy considered in this section is conjugacy of endomorphisms, not conjugacy of dilations (compare 2.4.11). Here we get examples where quite different Z give rise to conjugate endomorphisms.

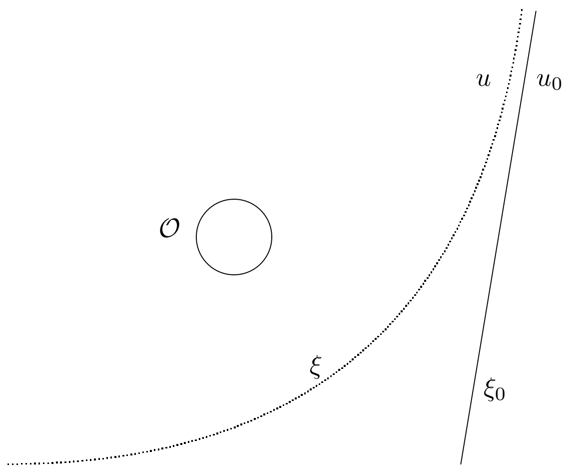
2.6 Kümmerer-Maassen-Scattering Theory

2.6.1 An Application

We want to give an application of the previous results to Kümmerer-Maassen-scattering theory. In this section we prepare this by giving a sketch of this theory. Instead of repeating the contents of [KM00], where the reader can find many more details, we give a motivating account and introduce the basics in such a way that the intended application fits smoothly, see Section 2.7.

2.6.2 Scattering in Physics

In physics a situation like the following gives rise to scattering theory: (see [RS78])



Here the straight line symbolizes a unitary evolution u_0 on a Hilbert space which is interpreted as the free dynamics of a system. But the obstacle $\mathcal{O}$ causes the system to follow instead the perturbed dynamics u. In large distances from $\mathcal{O}$ the difference can be made small, i.e. $\|u^n\,\xi - u_0^n\,\xi_0\| \to 0$ for $n \to \infty$. If this works for all vectors in the Hilbert space then the system is called asymptotically complete. More precisely, we have asymptotic completeness if there exists a so-called Møller operator $\Phi_- := stop - \lim_{n\to\infty} u_0^{-n} u^n$. Then Φ_- intertwines the free and the perturbed dynamics:

$$u_0\,\Phi_- = u_0 \lim_{n\to\infty} u_0^{-n} u^n = \lim_{n\to\infty} u_0^{-(n-1)} u^{n-1}\,u = \Phi_-\,u.$$

If Φ_- is invertible then the free and the perturbed dynamics are unitarily equivalent.

2.6.3 Scattering and Unitary Dilations

A similar situation arises in mathematics in the study of contractions. Let $t : \mathcal{H} \to \mathcal{H}$ be a contraction of a Hilbert space, i.e. $\|t\| \le 1$. One defines a so-called rotation matrix (see [SF70, FF90], see also 3.1.5 for more details):

$$u_1 := \begin{pmatrix} t & (\mathbb{1} - t\,t^*)^{\frac{1}{2}} \\ (\mathbb{1} - t^* t)^{\frac{1}{2}} & -t^* \end{pmatrix}$$

Then $u_1 : \mathcal{H} \oplus \mathcal{H} \to \mathcal{H} \oplus \mathcal{H}$ is unitary and $p_{\mathcal{H}} u_1|_{\mathcal{H}} = t$. From this one gets a unitary dilation of t, i.e. a unitary u on a larger Hilbert space $\tilde{\mathcal{H}}$ satisfying

$p_{\mathcal{H}} u^n|_{\mathcal{H}} = t^n$ for all $n \in \mathbb{N}_0$, which can be constructed as an additive version of a coupling representation: $\tilde{\mathcal{H}} = \bigoplus_{n\in\mathbb{Z}} \mathcal{H}_n$ with copies $\mathcal{H}_n$ of $\mathcal{H}$, and $\mathcal{H}$ identified with $\mathcal{H}_0$ as a subspace of $\tilde{\mathcal{H}}$. Then define $u := u_1\, r$ with u_1 as above acting on $\mathcal{H}_0 \oplus \mathcal{H}_1$ and r the right shift on $\bigoplus_{0\neq n\in\mathbb{Z}} \mathcal{H}_n$ (jumping in one step from $n=-1$ to $n=1$), both extended to $\tilde{\mathcal{H}}$ by identity. It should be clear that the same idea is used here as in the multiplicative coupling representations discussed in 2.1 and 2.3. See also [Kü88a] for a more detailed discussion of that.

We now get a problem similar to scattering if we consider r as free and u as perturbed dynamics. It is easy but instructive to work it out. A typical result is:

There exists a Møller operator $\Phi_- = stop - \lim r^{-n} u^n$ with range $\bigoplus_{0\neq n\in\mathbb{Z}} \mathcal{H}_n$ if and only if $t^n \stackrel{stop}{\rightarrow} 0$.

Such a Møller operator, if it exists, also yields a unitary equivalence of the isometries $u^{-1}|_{\mathcal{H}_{(-\infty,0]}}$ and $r^{-1}|_{\mathcal{H}_{(-\infty,-1]}}$.

Along more refined lines a detailed structure theory of contractions can be developed. References containing further information are [LP67, SF70, Be88, FF90].

2.6.4 Scattering for Markov Processes

The idea of Kümmerer-Maassen-scattering theory is to examine scattering for coupling representations of Markov processes, as described in Section 2.1. Then the Møller operator, if it exists, yields a conjugacy between the Markov dilation α and a Bernoulli shift σ (a tensor shift in the setting of Section 2.1). To write down the Møller operator one uses the two-sided version. We do this with the notation of Section 2.1, with $\tilde{\phi}$ denoting the (two-sided) product state and $\|\cdot\|_{\tilde{\phi}}$ denoting the corresponding Hilbert space norm etc.

Proposition: (see [KM00], 3.3) *The following conditions are equivalent:*

*(1) The Møller operator $\Phi_- = \lim_{n\to\infty} \sigma^{-n}\alpha^n$ exists and its range is the weak closure of $\bigotimes_{0\neq n\in\mathbb{Z}} \mathcal{C}_n$ (***asymptotic completeness***). The limit is understood pointwise with respect to $\|\cdot\|_{\tilde{\phi}}$ or, equivalently, weak*.*

(2) $\lim_{n\to\infty} \|Q\,\alpha^n(a)\|_{\tilde{\phi}} = \|a\|_{\tilde{\phi}}$ for all $a \in \mathcal{A}$. Here Q is the $\tilde{\phi}$-preserving conditional expectation (of tensor type) onto the weak closure of $\bigotimes_{0\neq n\in\mathbb{Z}} \mathcal{C}_n$.

In this case $\sigma\,\Phi_- = \Phi_-\,\alpha$, and Φ_- provides us with a conjugacy as mentioned above.

2.6.5 Asymptotic Completeness

In [KM00] there are further criteria for asymptotic completeness, i.e. property (1) above, but we shall not examine them here. Instead let us point out that an immediate analogue of the criterion in 2.6.3, in terms of the transition operator $T : \mathcal{A} \to \mathcal{A}$ of the Markov process, does not work.

Consider $\tilde{a} \in \mathcal{A} \otimes \mathcal{C}_{[1,\infty)}$. Then if $Q\tilde{a} = \tilde{a}$ we have $\tilde{a} = 1\!\!1 \otimes \tilde{c}$. Then for $P = P_0$, the conditional expectation onto $\mathcal{A}$, we conclude that $P\tilde{a} = \psi_{[1,\infty)}(\tilde{c})1\!\!1 = \tilde{\phi}(\tilde{a})1\!\!1$. If we have asymptotic completeness then by the equivalent property (2) this situation occurs in the limit $n \to \infty$. Thus for $a \in \mathcal{A}$ we get

$$T^n(a) = P\alpha^n(a) \longrightarrow \phi(a)1\!\!1.$$

In other words, the state ϕ is absorbing for T, compare A.5.

However the converse fails: ϕ absorbing for T does not imply asymptotic completeness. Some thought should indeed convince the reader that $T^n(a)$ does not contain enough information about $\alpha^n(a)$ to control $Q\,\alpha^n(a)$ in the way required in 2.6.4(2). Using the analysis of the situation in Section 2.7 we can see that the simplest example for the failure has been constructed in 1.7.2. See the discussion in 2.7.5.

Again, as in 2.6.3, we may note that in the case of asymptotic completeness the Møller operator provides us with a conjugacy between α^{-1} on $\mathcal{A}\otimes\mathcal{C}_{(-\infty,-1]}$ and σ^{-1} on $\mathcal{C}_{(-\infty,-1]}$ (weak closures are required). This shows that the conjugacy revealed by Kümmerer-Maassen-scattering theory is essentially a conjugacy of one-sided dynamical systems. In the following section we shall examine how this is related to the results in Section 2.5.

2.6.6 Further Work

Let us review some further work on these topics. A study of commutative Markov processes with respect to scattering has been completed by T. Lang in [Lan02]. This can be combined with the results of the author outlined in the following sections, see the joint work with B. Kümmerer and T. Lang in [GKL]. Applications to physics, in particular to experiments with micromasers, have been explored by T. Wellens, A. Buchleitner, B. Kümmerer and H. Maassen in [WBKM00]. These developments show clearly that it is worthwhile to take a closer look at the mathematical structure of this kind of scattering theory.

2.7 Restrictions and Extensions

2.7.1 Restrictions of Weak Markov Dilations

In this section we want to discuss how the conjugacy results for Markov processes on different levels, which have been presented in the previous sections, are related to each other.

Suppose we have a coupling representation $\tilde{J}$ for a weak Markov dilation of a stochastic map $Z : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$, as described in 2.3.1. Let us consider the case with $\mathcal{K} \simeq \mathcal{P}$ and $u_1 \in \mathcal{B}(\mathcal{H} \otimes \mathcal{K})$ unitary, see 2.4.6. If there are vector states given by $\Omega_{\mathcal{H}} \in \mathcal{H}$ and $\Omega_{\mathcal{K}} \in \mathcal{K}$ such that $u_1 \Omega_{\mathcal{H}} \otimes \Omega_{\mathcal{K}} = \Omega_{\mathcal{H}} \otimes \Omega_{\mathcal{K}}$, then we have a automorphic weak dilation in the sense of 1.6.3.

Now suppose further that $\mathcal{A} \subset \mathcal{B}(\mathcal{H})$ and $\mathcal{C} \subset \mathcal{B}(\mathcal{K})$ are von Neumann subalgebras such that $Ad(u_1)(\mathcal{A} \otimes \mathcal{C}) = \mathcal{A} \otimes \mathcal{C}$. Restriction of the vector states given by $\Omega_{\mathcal{H}}$ and $\Omega_{\mathcal{K}}$ yields normal states ϕ of $\mathcal{A}$ and ψ of $\mathcal{C}$. Now construct the von Neumann algebras $\mathcal{C}_{[1,\infty)}$ with state $\psi_{[1,\infty)}$ and $\tilde{\mathcal{A}}$ with state $\tilde{\phi}$ as done in 2.1.6. They are subalgebras of $\mathcal{B}(\mathcal{K}_{[1,\infty)})$ and $\mathcal{B}(\tilde{\mathcal{H}})$ in the obvious way. We can also restrict the whole dilation:

$$\begin{aligned} \alpha_1 &:= Ad(u_1)|_{\mathcal{A} \otimes \mathcal{C}}, \\ \sigma &:= R|_{\mathcal{C}_{[1,\infty)}}, \\ \alpha &:= \tilde{J}|_{\tilde{\mathcal{A}}} \end{aligned}$$

It is easy to check that in all cases the restrictions leave the algebra invariant. Except for the fact that the states obtained are not necessarily faithful, we get a noncommutative Markov process in the sense of Kümmerer, compare Section 2.1.

2.7.2 Conjugacy by Restriction

Suppose now that we are in the situation characterized by Theorem 2.5.6, i.e. $Z : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ is ergodic with an invariant vector state given by $\Omega_{\mathcal{H}} \in \mathcal{H}$ and therefore the weak Markov dilation $\tilde{J}$ and the right shift R are conjugate.

We want to analyze the action $Ad(w) = w \cdot w^*$ of the isometry $w \in \mathcal{B}(\tilde{\mathcal{H}})$ constructed in Theorem 2.5.6. Thinking of w as a unitary from $\tilde{\mathcal{H}}$ onto $\mathcal{K}_{[1,\infty)}$ we can interpret $Ad(w)$ as an isomorphism from $\mathcal{B}(\tilde{\mathcal{H}})$ onto $\mathcal{B}(\mathcal{K}_{[1,\infty)})$ and $\Phi_+ := Ad(w)|_{\tilde{\mathcal{A}}}$ maps $\tilde{\mathcal{A}}$ into $\mathcal{B}(\mathcal{K}_{[1,\infty)})$. Moreover we get

Proposition:

(1) $\Phi_+ := Ad(w)|_{\tilde{\mathcal{A}}}$ is an isomorphism from $\tilde{\mathcal{A}}$ onto $\mathcal{C}_{[1,\infty)}$.

(2) $\alpha = \Phi_+^{-1} \, \sigma \, \Phi_+$, i.e. the endomorphism α of $\tilde{\mathcal{A}}$ and the tensor shift σ on $\mathcal{C}_{[1,\infty)}$ are conjugate.

(3) $\psi_{[1,\infty)} \circ \Phi_+ = \tilde{\phi}$, i.e. the conjugacy respects the invariant states.

$$\begin{tikzcd}
\mathcal{B}(\tilde{\mathcal{H}}), \Omega_{\tilde{\mathcal{H}}} \arrow[rrr, "\tilde{J}"] \arrow[ddd, "Ad(w)"'] & & & \mathcal{B}(\tilde{\mathcal{H}}), \Omega_{\tilde{\mathcal{H}}} \arrow[ddd, "Ad(w)"] \\
& \tilde{\mathcal{A}}, \tilde{\phi} \arrow[ul] \arrow[r, "\alpha"] \arrow[d, "\Phi_+"'] & \tilde{\mathcal{A}}, \tilde{\phi} \arrow[ur] \arrow[d, "\Phi_+"] & \\
& \mathcal{C}_{[1,\infty)}, \psi_{[1,\infty)} \arrow[dl] \arrow[r, "\sigma"] & \mathcal{C}_{[1,\infty)}, \psi_{[1,\infty)} \arrow[dr] & \\
\mathcal{B}(\mathcal{K}_{[1,\infty)}), \Omega_{[1,\infty)} \arrow[rrr, "R"] & & & \mathcal{B}(\mathcal{K}_{[1,\infty)}), \Omega_{[1,\infty)}
\end{tikzcd}$$

Proof: As in Section 2.5 consider the cocycle $w_n := \tilde{u}_1(\mathbb{1} \otimes R)(\tilde{u}_1)\ldots(\mathbb{1} \otimes R)^{n-1}(\tilde{u}_1)$. Using $Ad(u_1)(\mathcal{A} \otimes \mathcal{C}) = \mathcal{A} \otimes \mathcal{C}$ we find that $\tilde{\mathcal{A}}$ is globally invariant for $Ad(w_n)$ and $Ad(w_n^*)$. From Theorem 2.5.6(5) we know that $w_n^* \overset{stop}{\to} w$ and the range of w is equal to $\mathcal{K}_{[1,\infty)} = q\tilde{\mathcal{H}}$ (with $q = q_{[1,\infty)}$, the projection from $\tilde{\mathcal{H}}$ onto $\mathcal{K}_{[1,\infty)}$). Thus also $qw_n^* \overset{stop}{\to} w$, $w_n q \overset{stop}{\to} w^*$ (use Lemma 2.5.6), $w_n \overset{wop}{\to} w^*$. The algebra $\mathcal{C}_{[1,\infty)}$ as represented on $\mathcal{K}_{[1,\infty)}$ may be identified with $q\tilde{\mathcal{A}}q$. Therefore we get for all $\tilde{a} \in \tilde{\mathcal{A}}$

$$\mathcal{C}_{[1,\infty)} \ni qw_n^* \tilde{a} w_n q \overset{stop}{\to} w\tilde{a}w^*, \quad \text{i.e. } Ad(w)(\tilde{\mathcal{A}}) \subset \mathcal{C}_{[1,\infty)},$$

$$\tilde{\mathcal{A}} \ni w_n \tilde{a} w_n^* \overset{wop}{\to} w^* \tilde{a} w = w^* q \tilde{a} q w, \quad \text{i.e. } Ad(w^*)(\mathcal{C}_{[1,\infty)}) \subset \tilde{\mathcal{A}}.$$

Here the assumption has been used that $\tilde{\mathcal{A}}$ and $\mathcal{C}_{[1,\infty)}$ are closed in the strong and weak operator topologies. Combining these inclusions we infer that $Ad(w)$ as an isomorphism from $\mathcal{B}(\tilde{\mathcal{H}})$ onto $\mathcal{B}(\mathcal{K}_{[1,\infty)})$ restricts to an isomorphism Φ_+ from $\tilde{\mathcal{A}}$ onto $\mathcal{C}_{[1,\infty)}$. This proves (1). Now (2) is just the restriction of Theorem 2.5.6(7) to the von Neumann subalgebras, and (3) follows from $w\Omega_{\tilde{\mathcal{H}}} = \Omega_{\tilde{\mathcal{H}}} = \Omega_{\mathcal{H}} \otimes \Omega_{[1,\infty)}$. $\square$

2.7.3 Extension of Automorphic Dilations

Conversely assume that we are given a automorphic Markov dilation of $T : (\mathcal{A}, \phi) \to (\mathcal{A}, \phi)$ as in 2.1.6, i.e. an endomorphism $\alpha : (\tilde{\mathcal{A}}, \tilde{\phi}) \to (\tilde{\mathcal{A}}, \tilde{\phi})$ with $P\alpha^n = T^n$ and $\alpha = \alpha_1 \sigma$, where $\alpha_1 : (\mathcal{A} \otimes \mathcal{C}_1, \phi \otimes \psi) \to (\mathcal{A} \otimes \mathcal{C}_1, \phi \otimes \psi)$ is an automorphism and σ is the right shift on $\mathcal{C}_{[1,\infty)}$. It has been shown in 1.6.4 how to obtain on the GNS-Hilbert spaces an associated extension $Ad(u_1)$: $\mathcal{B}(\mathcal{H} \otimes \mathcal{K}) \to \mathcal{B}(\mathcal{H} \otimes \mathcal{K})$ of α_1 and a stochastic map $Z \in \mathcal{Z}(T, \phi)$ defined by $Z(x) = p_{\mathcal{H}} u_1 \; x \otimes \mathbb{1} \; u_1^* p_{\mathcal{H}}$. If we now define the coupling representation $\tilde{J} = Ad(\tilde{u}_1) \circ R$ on $\tilde{\mathcal{H}} = \mathcal{H} \otimes \mathcal{K}_{[1,\infty)}$ then we have extended the automorphic Markov dilation α of T by a automorphic weak Markov dilation $\tilde{J}$ of Z. Proposition 2.7.2 is applicable and yields

Corollary: *If Z is ergodic then α and σ are conjugate (via Φ_+, as in 2.7.2).*

It is remarkable that this criterion for conjugacy of α and σ involves the extended transition operator Z and cannot be formulated in this way if we restrict ourselves to the original Markov process,

2.7.4 Asymptotic Completeness and Ergodicity of Z'

To understand directly the relation to Kümmerer-Maassen-scattering theory as presented in Section 2.6 we proceed as follows. Assume that we are given a (not necessarily automorphic) Markov dilation α of $T : (\mathcal{A}, \phi) \to (\mathcal{A}, \phi)$ as in 2.1.6. Then we have $\alpha = j_1 \sigma$, where $j_1 : (\mathcal{A}, \phi) \to (\mathcal{A} \otimes \mathcal{C}_1, \phi \otimes \psi)$ is a unital injective *−endomorphism and σ is the right shift on $\mathcal{C}_{[1,\infty)}$. Also we have the GNS-spaces as usual. Let $v_1 : \mathcal{H} \to \mathcal{H} \otimes \mathcal{K}_1$ be the isometry associated to j_1, see 1.3.4. Associated to the right shift σ we have

$$
r : \mathcal{K}_{[1,\infty)} \to \mathcal{K}_{[1,\infty)}, \quad \tilde{\eta} \mapsto \Omega_1 \otimes \tilde{\eta}.
$$

Using $\mathcal{H} \simeq \mathcal{H} \otimes \Omega_1 \subset \mathcal{H} \otimes \mathcal{K}_1$ the isometry $v = v_1 \, r : \tilde{\mathcal{H}} \to \tilde{\mathcal{H}}$ can be defined and is associated to α. Explicitly, if $\xi \otimes \tilde{\eta} \in \mathcal{H} \otimes \mathcal{K}_{[1,\infty)}$ then

$$
v(\xi \otimes \tilde{\eta}) = (v_1 \xi) \otimes \tilde{\eta} \in (\mathcal{H} \otimes \mathcal{K}_1) \otimes \mathcal{K}_{[2,\infty)}.
$$

$$
\begin{tikzcd}[column sep=small]
\mathcal{A} \arrow[dd, bend right, "j_1"'] \\
\otimes & & {} \arrow[r, "\sigma"'] & {} \\
\mathcal{C} & \otimes & \mathcal{C} & \otimes & \mathcal{C} & \otimes & \dots
\end{tikzcd}
\qquad
\begin{tikzcd}[column sep=small]
\mathcal{H} \arrow[dd, bend right, "v_1"'] \\
\otimes & & {} \arrow[r, "r"'] & {} \\
\mathcal{K} & \otimes & \mathcal{K} & \otimes & \mathcal{K} & \otimes & \dots
\end{tikzcd}
$$

Recall from 1.5.2 that we also have a stochastic map $Z' \in \mathcal{Z}(T', \phi')$ defined by $Z'(x) = v_1^* \, x \otimes \mathbb{1} \, v_1$.

Theorem: *The following properties are equivalent:*

(a) *Z' is ergodic.*

(b) *asymptotic completeness (see 2.6.4)*

Proof: As noted in 2.6.4, asymptotic completeness is equivalent to the condition

$$
\lim_{n \to \infty} \|Q \, \alpha^n(a)\|_{\tilde{\phi}} = \|a\|_{\tilde{\phi}} \quad \textit{for all } a \in \mathcal{A},
$$

where Q is the conditional expectation from $\tilde{\mathcal{A}}$ onto $\mathcal{C}_{[1,\infty)}$. With $q = q_{[1,\infty)}$, the projection from $\tilde{\mathcal{H}}$ onto $\mathcal{K}_{[1,\infty)}$, this means that

$$
\lim_{n \to \infty} \|q \, v^n a \, \Omega_{\tilde{\mathcal{H}}}\| = \|a \, \Omega_{\mathcal{H}}\| \quad \textit{for all } a \in \mathcal{A}, \textit{ or}
$$

$$
\lim_{n \to \infty} \|q \, v^n \xi \otimes \Omega_{[1,\infty)}\| = \|\xi\| \quad \textit{for all } \xi \in \mathcal{H}.
$$

Using the right shift R on $\mathcal{B}(\mathcal{K}_{[1,\infty)})$ we get

$$v^n = (v_1 r)^n = v_1\,(\mathbb{1}\otimes R)(v_1)\ldots(\mathbb{1}\otimes R)^{n-1}(v_1)\,r^n,$$

and thus

$$\begin{aligned}\|q\,v^n\,\xi\otimes\Omega_{[1,\infty)}\| &= \|q\,v_1(\mathbb{1}\otimes R)(v_1)\ldots(\mathbb{1}\otimes R)^{n-1}(v_1)\,\xi\otimes\Omega_{[1,\infty)}\| \\ &= \|q(\mathbb{1}\otimes R)^{n-1}(v_1)\ldots(\mathbb{1}\otimes R)(v_1)\,v_1\,\xi\otimes\Omega_{[1,\infty)}\|.\end{aligned}$$

For the last equality, note that a permutation of the positions in the tensor product does not change the norm. Applying now Theorem 2.5.7 with $v_1 : \mathcal{H}\to\mathcal{H}\otimes\mathcal{K}$ instead of v_1' and with Z' instead of Z we see that this converges to $\|\xi\|$ for $n\to\infty$ if and only if Z' is ergodic. $\square$

As remarked in 2.6.5, asymptotic completeness yields a conjugacy Φ_- between α^{-1} and σ^{-1} on the negative time axis (if we have a two-sided version of the Markov dilation). In the automorphic case we have already seen in 1.6.4 that the dual extended transition operator Z' corresponds to α^{-1} in the same way as Z corresponds to α. Thus the relation between 2.7.2(3) and 2.7.4 is time inversion.

2.7.5 Discussion

We have noted in 2.6.5 that the transition operator $T : (\mathcal{A},\phi)\to(\mathcal{A},\phi)$ of the Markov process does not suffice to determine asymptotic completeness. Remarkably, Theorem 2.7.4 tells us that the dual extended transition operator Z' does suffice. As shown by the use of Theorem 2.5.7 in the proof of 2.7.4, this may be explained by the fact that the extended weak dilation is much more determined by extended transition as the dilation is by transition. Compare also the results of Section 2.4 in this respect.

We can analyze the situation in more detail: The failure in using T is explained by the fact that the set $\mathcal{Z}(T',\phi')$ of extensions of T' (see Section 1.2) may contain different elements (corresponding to different dilations of T, according to 1.5.4), such that one is ergodic and the other is not. To see that this actually occurs we can use the example in 1.7.2. There we have $Z_1',\ Z_{-1}' : M_2\to M_2$:

$$Z_1'(x) = \frac{1}{2}\begin{pmatrix} x_{11}+x_{12}+x_{21}+x_{22} & 0 \\ 0 & x_{11}+x_{12}+x_{21}+x_{22}\end{pmatrix} = \langle\Omega_{\mathcal{H}}, x\Omega_{\mathcal{H}}\rangle\mathbb{1}$$

Z_1' is ergodic because the vector state given by $\Omega_{\mathcal{H}}$ is absorbing, see A.5.2.

$$Z_{-1}'(x) \quad = \quad \frac{1}{2}\begin{pmatrix} x_{11}+x_{22} & x_{12}+x_{21} \\ x_{12}+x_{21} & x_{11}+x_{22}\end{pmatrix}$$

Z_{-1}' is not ergodic because every matrix of the form $\begin{pmatrix} a & b \\ b & a\end{pmatrix}$ is a fixed point.

These results indicate that the set $\mathcal{Z}(T',\phi')$, or equivalently, equivalence classes of weak dilations of T as in Section 1.4, play an important role in the spatial theory of Markov dilations. Compare also the Introduction for some related general considerations. In particular, Theorem 2.7.4 provides a new criterion for asymptotic completeness and opens new ways to think about Kümmerer-Maassen-scattering theory. In the following section we will find still another interpretation.

2.8 An Interpretation Using Entanglement

2.8.1 Entanglement of Vector States

The notion of entanglement has been introduced by E. Schrödinger [Sch35] to discuss some non-intuitive aspects of states in quantum theory. In the last years there has been renewed interest in it due to attempts to develop quantum information theory, see [NC00]. In its most elementary form we consider Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$ and a unit vector $\chi \in \mathcal{H} \otimes \mathcal{K}$. In quantum theory this describes a pure state of a combined system.

Let p_χ be the one-dimensional projection onto $\mathbb{C}\,\chi$. By $Tr_\mathcal{K}$ we denote the partial trace evaluated in $\mathcal{K}$, i.e.

$$Tr_\mathcal{K} : \mathcal{T}(\mathcal{H} \otimes \mathcal{K}) \to \mathcal{T}(\mathcal{H}), \quad \rho_\mathcal{H} \otimes \rho_\mathcal{K} \mapsto \rho_\mathcal{H}\, Tr(\rho_\mathcal{K}),$$

where Tr is the non-normalized trace. Then

$$\chi^\mathcal{H} \;:=\; Tr_\mathcal{K}(p_\chi) \in \mathcal{T}(\mathcal{H})$$

is called the reduced density matrix corresponding to χ (see [NC00]). It is characterized by the fact that for all elements of the form $x \otimes \mathbb{1} \in \mathcal{B}(\mathcal{H} \otimes \mathcal{K})$, i.e. for observables that can be measured on the system described by $\mathcal{H}$ alone, the expectation values are the same for χ and $\chi^\mathcal{H}$:

$$\langle \chi, x \otimes \mathbb{1}\, \chi \rangle \;=\; Tr(\chi^\mathcal{H}\, x).$$

But in general $\chi^\mathcal{H}$ is not a pure state. If $\chi^\mathcal{H}$ is not pure then the vector state given by χ is called **entangled**.

Note also that if $\{\delta_i\}$ is an ONB of $\mathcal{H}$ and $\chi = \sum_i \delta_i \otimes \chi_i$, then the corresponding matrix entries of $\chi^\mathcal{H}$ are

$$(\chi^\mathcal{H})_{ij} \;=\; \langle \chi_j, \chi_i \rangle,$$

i.e. we get a Gram matrix. The computation is given in A.4.4. It is instructive to think of a Gram matrix as a kind of covariance matrix, which is actually true when the χ_i are realized as centered random variables.

2.8.2 Entanglement of Observables

Consider a tensor product of noncommutative probability spaces (i.e. *−algebras with states, compare 2.1):

$$(\mathcal{A}, \phi) \;=\; (\mathcal{A}_1, \phi_1) \otimes (\mathcal{A}_2, \phi_2).$$

If $(\mathcal{H}, \pi, \Omega)$ arises from the GNS-construction of $(\mathcal{A}, \phi)$, then $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$ and $\Omega = \Omega_1 \otimes \Omega_2$, where the indexed quantities arise from the GNS-construction of $(\mathcal{A}_1, \phi_1)$ and $(\mathcal{A}_2, \phi_2)$. If $a \in \mathcal{A}$ with Hilbert space norm $\|a\|_\phi = \|\pi(a)\Omega\| = 1$, then we can speak of entanglement of $\pi(a)\Omega$. We define

$$a^{\mathcal{H}_1} \;:=\; (\pi(a)\Omega)^{\mathcal{H}_1}$$

and call $a^{\mathcal{H}_1} \in \mathcal{T}(\mathcal{H}_1)$ the **covariance operator** of $a \in \mathcal{A}$.

2.8.3 The Time Evolution of the Covariance Operator

We shall get contact to the previous sections by discussing the following question: How does a Markov process change the covariance operator of an observable with respect to the tensor product decomposition into system and reservoir given by a coupling representation? (Compare 2.1.6.)

Assume that we have, as in 2.1.6 and 2.7.4, the noncommutative probability space $(\tilde{\mathcal{A}}, \tilde{\phi}) = (\mathcal{A}, \phi) \otimes (\mathcal{C}_{[1,\infty)}, \psi_{[1,\infty)})$ and the Markovian time evolution $\alpha = j_1 \sigma$. On the level of GNS-spaces we have $\tilde{\mathcal{H}} = \mathcal{H} \otimes \mathcal{K}_{[1,\infty)}$ and $v = v_1\, r$. As shown in 2.7.4 it acts on $\tilde{\xi} = \xi \otimes \eta \in \mathcal{H} \otimes \mathcal{K}_{[1,\infty)}$ as

$$v\,\tilde{\xi} = v_1 \xi \otimes \eta \in (\mathcal{H} \otimes \mathcal{K}_1) \otimes \mathcal{K}_{[2,\infty)}.$$

In the same way as in quantum mechanics the Schrödinger dynamics on vector states can be lifted to mixed states, we can give an induced dynamics on density matrices which is specified by

$$\mathcal{T}(\mathcal{H} \;\otimes\; \mathcal{K}_{[1,\infty)}) \ni \rho_{\mathcal{H}} \otimes \rho_{[1,\infty)} \mapsto v\, \rho_{\mathcal{H}} \otimes \rho_{[1,\infty)}\, v^*$$
$$=\; v_1 \rho_{\mathcal{H}} v_1^* \;\otimes\; \rho_{[1,\infty)} \in \mathcal{T}(\mathcal{H} \otimes \mathcal{K}_1) \otimes \mathcal{T}(\mathcal{K}_{[2,\infty)}).$$

We denote the partial trace evaluated on $\mathcal{K}_1$ respectively on $\mathcal{K}_{[1,\infty)}$ by Tr_1 respectively $Tr_{[1,\infty)}$.

Theorem: *With* $Z' : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H}),\ x \mapsto v_1^*\ x \otimes \mathbb{1}\ v_1$ *(the dual extended transition operator) we find for* $\tilde{\rho} \in \mathcal{T}(\tilde{\mathcal{H}})$*:*

$$Tr_{[1,\infty)}(\tilde{\rho}) \circ Z' \ = \ Tr_{[1,\infty)}(v\,\tilde{\rho}\,v^*).$$

In particular: If $\tilde{a} \in \tilde{\mathcal{A}}$ *then*

$$\tilde{a}^{\mathcal{H}} \circ Z' = \alpha(\tilde{a})^{\mathcal{H}}.$$

In other words: The preadjoint

$$C := Z'_* : \mathcal{T}(\mathcal{H}) \to \mathcal{T}(\mathcal{H})$$
$$\rho \mapsto Tr_1(v_1\,\rho\,v_1^*)$$

(compare A.4.1) gives the dynamics for the covariance operator $\tilde{a}^{\mathcal{H}}$ *of* $\tilde{a}$*, i.e.*

$$C(\tilde{a}^{\mathcal{H}}) \ = \ \alpha(\tilde{a})^{\mathcal{H}}.$$

Proof: If we have shown the first assertion of the theorem, then the rest follows by inserting for $\tilde{\rho}$ the one-dimensional projection onto $\pi(\tilde{a})\Omega_{\tilde{\mathcal{H}}}$, with $(\tilde{\mathcal{H}}, \pi, \Omega_{\tilde{\mathcal{H}}})$ being the GNS-triple of $(\tilde{\mathcal{A}}, \tilde{\phi})$. Also it suffices to prove the first assertion for $\tilde{\rho} = \rho_{\mathcal{H}} \otimes \rho_{[1,\infty)} \in \mathcal{T}(\mathcal{H}) \otimes \mathcal{T}(\mathcal{K}_{[1,\infty)})$. In the following computation we use the dualities $(\mathcal{T}(\mathcal{H}), \mathcal{B}(\mathcal{H}))$ and $(\mathcal{T}(\mathcal{H} \otimes \mathcal{K}_1), \mathcal{B}(\mathcal{H} \otimes \mathcal{K}_1))$ and the fact that the embedding $i : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H} \otimes \mathcal{K}_1),\ x \mapsto x \otimes \mathbb{1}$ is adjoint to Tr_1. Now for all $a \in \mathcal{B}(\mathcal{H})$ we get

$$\begin{aligned} Tr_{[1,\infty)}(\tilde{\rho}) \circ Z'(x) &= < Tr_{[1,\infty)}(\tilde{\rho}), Z'(x) > \\ &= Tr(\rho_{[1,\infty)}) < \rho_{\mathcal{H}}, v_1^*\ x \otimes \mathbb{1}\ v_1 > \\ &= Tr(\rho_{[1,\infty)}) < v_1\,\rho_{\mathcal{H}}\,v_1^*, i(x) > \\ &= Tr(\rho_{[1,\infty)}) < Tr_1(v_1\,\rho_{\mathcal{H}}\,v_1^*), x > \\ &= < Tr_{[1,\infty)}(v\,\tilde{\rho}\,v^*), x > . \qquad \square \end{aligned}$$

2.8.4 Asymptotic Completeness via Entanglement

Let us indicate how Theorem 2.7.4 may be derived from the just proved Theorem 2.8.3, i.e. we want to use the setting of this section to sketch a proof for the fact that the vector state given by $\Omega_{\mathcal{H}}$ is absorbing for Z' (which is equivalent to ergodicity of Z' by A.5.2) if and only if the coupling representation of the Markov process exhibits asymptotic completeness.

If $\Omega_{\mathcal{H}}$ is absorbing then for $a \in \mathcal{A} \simeq \mathcal{A} \otimes \mathbb{1} \subset \tilde{\mathcal{A}}$ with $\|a\|_\phi = 1$ it follows from Theorem 2.8.3 that $\alpha^n(a)^{\mathcal{H}} = a^{\mathcal{H}} \circ (Z')^n$ converges weakly to $p_{\Omega_{\mathcal{H}}}$, the one-dimensional projection onto $\mathbb{C}\Omega_{\mathcal{H}}$. Choose an ONB $\{\delta_i\}_{i=0}^I$ of $\mathcal{H}$ with $\delta_0 = \Omega_{\mathcal{H}}$. Writing $\alpha^n(a) = \sum_i \delta_i \otimes a_i^{(n)}$ we have the following convergence for $n \to \infty$:

$$\big(\langle a_j^{(n)}, a_i^{(n)} \rangle\big)_{ij} \longrightarrow \begin{pmatrix} 1 & & \\ & 0 & \\ & & \ddots \end{pmatrix}.$$

The conditional expectation Q from $\tilde{\mathcal{A}}$ onto $\mathcal{C}_{[1,\infty)}$ satisfies $Q(\alpha^n(a)) = a_0^{(n)}$. Thus we get

$$\|Q(\alpha^n(a))\|_{\tilde{\phi}} = \|a_0^{(n)}\|_{\tilde{\phi}} = \langle a_0^{(n)}, a_0^{(n)} \rangle^{\frac{1}{2}} \quad \stackrel{n\to\infty}{\longrightarrow} \quad 1 = \|a\|_{\phi},$$

which is the criterion for asymptotic completeness cited in 2.6.4. We are done.

2.8.5 Quantum Information Theory

In view of the derivation in 2.8.4 we can give an intuitive interpretation of asymptotic completeness using entanglement as follows: Asymptotic completeness corresponds to a decay of entanglement between system and reservoir. As noted in 2.7.5 this is more than just decay of correlations between random variables of the process as described in 2.1.5. Classification of different types of correlations and entanglement is an important theme in the evolution of quantum information theory, and it may be interesting to analyze our setting in more detail from this point of view.

The content of Section 2.8 is also discussed in [Go2]. It is shown there that the arguments also work for continuous time parameter and for more general shifts.

3

Adaptedness

In this chapter we develop a framework for the description of stationary processes which (in general) do not fulfil the Markov property. But the following two features are still available. First, there is a filtration generated by the process or, more generally, a filtration to which the process is adapted. Such filtered probability spaces are a well-known tool for the study of stochastic processes in both classical and noncommutative probability theory. Second, and more specifically to stationary processes, there is a time evolution operator of the process. This has already been described in 2.1.3. The main idea of our framework consists in combining these two features.

We define the concept of an adapted endomorphism as an abstraction of a time evolution acting on a filtered probability space. Before giving the definition, we observe that in the more elementary category of Hilbert spaces with isometries a similar idea has been used with great success. Here it amounts to writing an isometry in Hessenberg form and parametrizing it by a choice sequence, which in the most basic situation corresponds to the classical Schur parameters. We refer to [FF90] for this inspiring circle of ideas. Our description of it concentrates on the possibility to write the isometry as an infinite product of 'localized' unitaries, because this is a part of the theory which we later transfer to a probabilistic context.

Then we define 'adapted endomorphism' in the language of category theory. It allows to speak of adaptedness in terms of morphisms. In particular we show that representations as infinite products are still available in rather general situations, and we sketch an application which concerns the step-by-step construction of an adapted endomorphism.

In a first reading it may be better not to spend too much time on these generalities but to look quickly at the application to stationary processes. While filtrations are very familiar there, the concept of an adapted endomorphism has not been studied as such. One more argument in favour of it is based on the observation that for Markov processes we are back to coupling representations which we found to be very useful in Chapter 2. To probe the usefulness of the more general product representations, we again take a spatial

R. Gohm: LNM 1839, pp. 73–111, 2004.

approach and associate to the stationary process the corresponding dynamical system on the GNS-Hilbert space. In an important special case, we get adapted isometries on an infinite tensor product of Hilbert spaces which can be written as an infinite product of 'localized' unitaries. We analyze this case in great detail. In particular, we develop a systematic way to read off properties of the stationary process from the factors of the infinite product. As tools we use stochastic maps which are in fact extended transition operators in the sense of Chapter 1. This opens a completely new field of study, and the range of applicability of this method in analyzing stationary processes is not yet clear and has to be further explored.

Thus sometimes the presentation becomes a bit tentative here. But we can describe at least two computable quantities with a clear probabilistic and dynamical interpretation. The first quantity concerns the computation of one-step nonlinear prediction errors of the process. Compare P. Masani and N. Wiener [MW59] for some historical context of this problem. We derive a formula which is very similar to a well-known formula in linear prediction theory. The second quantity is of interest in the theory of dynamical systems and concerns criteria for the surjectivity of the (one-sided) time evolution. In both cases, this is related to ergodicity properties of some associated extended transition operators, and thus we notice here a surprising connection to the criteria for asymptotic completeness of Kümmerer-Maassen-scattering theory which have been described in Chapter 2.

Some examples for these phenomena are given in Chapter 4, and perhaps it is helpful for the reader to look at these examples from time to time already during his/her study of Chapter 3.

3.1 A Motivation: Hessenberg Form of an Isometry

3.1.1 Powers of an Isometry

Let v be any isometry on a Hilbert space $\tilde{\mathcal{H}}$ and $\xi \in \tilde{\mathcal{H}}$ a unit vector. We want to decompose the Hilbert space $\tilde{\mathcal{H}}$ in such a way that this decomposition helps us to analyze the action of powers of v on ξ. The following approach is well-known (see [FF90], compare also [Ha95, He95] which go further into the direction we want to take). We consider

$$\begin{aligned} \mathcal{H}_0 := \mathcal{H}_{[0,0]} &:= \mathbb{C}\,\xi \\ \mathcal{H}_{[0,n]} &:= \overline{span}\{v^m \mathcal{H}_0 : m = 0, \ldots, n\}. \end{aligned}$$

Then $\mathcal{H}_0 \subset \mathcal{H}_{[0,1]} \subset \mathcal{H}_{[0,2]} \subset \ldots \subset \tilde{\mathcal{H}}$ is an increasing sequence of subspaces. Because we are only interested in actions of powers of v on ξ we shall assume that ξ is cyclic for v, i.e. $\tilde{\mathcal{H}} = \hat{\mathcal{H}} := \overline{span}\{v^m \mathcal{H}_0 : m \in \mathbb{N}_0\}$. If $v^N \xi \notin \mathcal{H}_{[0,N-1]}$ then $dim(\mathcal{H}_{[0,N]} \ominus \mathcal{H}_{[0,N-1]}) = 1$, otherwise $v^n \xi \in \mathcal{H}_{[0,N-1]}$ for all n.

3.1.2 Hessenberg Form and Schur Parameters

Assume for the moment that $v^N \xi \notin \mathcal{H}_{[0,N-1]}$ for all N. Then the above sequence of subspaces is strictly increasing at each step. An ONB $(\delta_n)_{n=0}^\infty$ describing the innovation at each step may be obtained by applying the Gram-Schmidt procedure to $(v^n \xi)_{n=0}^\infty$. Here each δ_n is uniquely determined up to an unimodular complex factor.

What is the matrix of v with respect to the ONB $(\delta_n)_{n=0}^\infty$? By construction it has Hessenberg form, i.e. $v_{ij} = 0$ if $i > j+1$. To get more detailed information one can proceed as follows. Writing p_0 for the orthogonal projection onto $\mathcal{H}_0$, we have with $\delta_0 := \xi \in \mathcal{H}_0$

$$p_0\, v\, \delta_0 = k_1 \delta_0, \quad k_1 \in \mathbb{C},\ |\, k_1 \,| < 1.$$

Then $v\delta_0 = k_1 \delta_0 \oplus d_1 \delta_1$ with $d_1 \in \mathbb{C}$, $|\, k_1 \,|^2 + |\, d_1 \,|^2 = 1$. We can choose the unimodular factor for δ_1 in such a way that $d_1 > 0$. If we now define a unitary $u_1 := \begin{pmatrix} k_1 & d_1 \\ d_1 & -\overline{k_1} \end{pmatrix}$ with respect to δ_0, δ_1, acting identically on the orthogonal complement of $\mathcal{H}_{[0,1]}$, then $v|_{\mathcal{H}_0} = u_1|_{\mathcal{H}_0}$. Defining $v_{[1} := u_1^{-1} v$, we get a product representation $v = u_1 v_{[1}$. Obviously $v_{[1}$ satisfies $v_{[1}\delta_0 = \delta_0$ and we can consider it as an isometry of $\overline{span}\{\delta_n\}_{n=1}^\infty$. Thus we can iterate our construction, i.e. we can do for $v_{[1}$ and δ_1 what we have done for v and δ_0 above. We find $u_2 := \begin{pmatrix} k_2 & d_2 \\ d_2 & -\overline{k_2} \end{pmatrix}$ and $v = u_1\, u_2\, v_{[2}$ etc.

Finally we arrive at an infinite product representation for v:

$$v = stop - \lim_{N \to \infty} u_1 u_2 \dots u_N .$$

This is convergent because u_{n+1} acts identically on $\mathcal{H}_{[0,n-1]}$ for all n. Computing this product yields the entries of the Hessenberg form of v in terms of the k_n and d_n. More explicitly:

$$v = \begin{pmatrix} k_1 & d_1 & & & & \\ d_1 & -\overline{k_1} & & & & \\ & & 1 & & & \\ & & & 1 & & \\ & & & & 1 & \\ & & & & & \dots \end{pmatrix} \times \begin{pmatrix} 1 & & & & & \\ & k_2 & d_2 & & & \\ & d_2 & -\overline{k_2} & & & \\ & & & 1 & & \\ & & & & 1 & \\ & & & & & \dots \end{pmatrix} \times \quad \dots$$

$$= \begin{pmatrix} k_1 & d_1 k_2 & d_1 d_2 k_3 & \dots & d_1 \dots d_{m-1} k_m & \dots \\ d_1 & -\overline{k_1} k_2 & -\overline{k_1} d_2 k_3 & \dots & -\overline{k_1} d_2 \dots d_{m-1} k_m & \dots \\ & d_2 & -\overline{k_2} k_3 & \dots & -\overline{k_2} d_3 \dots d_{m-1} k_m & \dots \\ & & d_3 & \dots & \vdots & \dots \\ & & & & \vdots & \dots \\ & & & & -\overline{k_{m-1}} k_m & \dots \\ & & & & d_m & \dots \\ & & & & & \dots \end{pmatrix} .$$

Here $(k_n)_{n=1}^{\infty}$ is a sequence of complex numbers satisfying $|k_n| < 1$, while the d_n are determined by $d_n > 0$ and $|k_n|^2 + |d_n|^2 = 1$, for all n. The k_n are sometimes called Schur parameters because of their appearance in a continued fraction algorithm of I. Schur, see [FF90, Teu91].

If there exists $N \in \mathbb{N}$ minimal with the property $v^N \xi \in \mathcal{H}_{[0,N-1]}$, then we get $|k_N| = 1$ and the construction above terminates. In this case $\hat{\mathcal{H}} = \mathcal{H}_{[0,N-1]}$ is finite dimensional and we have a finite product representation for v:

$$v = u_1 \dots u_{N-1}.$$

3.1.3 Schur Parameters as Choice Sequences

Schur parameters occur in many different applications, see [FF90, Teu91]. We only mention those which will be reconsidered by us in the setting introduced in Section 3.2.

The sequence $(r_n)_{n=0}^{\infty} \subset \mathbb{C}$ given by $r_n := \langle \delta_0, v^n \delta_0 \rangle$ is positive definite with $r_0 = 1$. Conversely, any such sequence arises in this way for suitably chosen v and δ_0. From this one gets a one-to-one correspondence between positive definite sequences $(r_n)_{n=0}^{\infty} \subset \mathbb{C}$ with $r_0 = 1$ and Schur parameters, i.e. sequences $(k_n)_{n=1}^{N} \subset \mathbb{C}$ with either $N = \infty$ and $|k_N| < 1$ for all n or N finite and $|k_N| < 1$ for $n < N$ and $|k_N| = 1$. See [FF90].

Here we can see why Schur parameters are also called choice sequences: They give a free parametrization of positive definite sequences in the sense that k_n can be chosen independently of the values that occur for smaller subscripts. The correspondence is such that $(r_1, \dots, r_n)$ and $(k_1, \dots, k_n)$ determine each other for all $n \leq N$. Thus they provide a solution to the problem to extend $(r_1, \dots, r_n)$ to positive definite sequences $(r_n)_{n=0}^{\infty}$. They also classify the ways of building an isometry v successively on larger and larger spaces $\mathcal{H}_{[0,n]}$. In particular see ([FF90], II.6) for so-called maximum entropy extensions corresponding to $k_n = 0$. Their analogue in the setting of stochastic processes, namely Markovian extensions, will be given in 4.1.12 and then it is Shannon entropy which is involved, see 4.1.13. To make more precise in what sense this is an analogue we can use an abstract version which is developed in Section 3.2.

3.1.4 Linear Prediction

There are applications to the statistical prediction of time series. Suppose we have observed values $a_0, \dots, a_{n-1} \in \mathbb{R}$ or $\mathbb{C}$ of a time series and we want to estimate the unknown next value a_n. If we use a linear combination of $a_0, \dots, a_{n-1}$ for the estimator, then this is linear prediction one step ahead. N. Wiener and A. Kolmogorov started the theory of optimal linear prediction under the assumption that the time series is a path of a stationary stochastic process. Then 'optimal' can be interpreted statistically as minimizing the expected mean square error.

This is related to our setting as follows. Working in the Hilbert space of square integrable random variables, we can write down the random variables of the stationary process as $\xi, v\xi, \ldots, v^n\xi, \ldots$ Then the optimal linear predictor is given by the orthogonal projection $p_{[0,n-1]}$ onto $\mathcal{H}_{[0,n-1]}$ applied to $v^n\xi$. There are quite efficient algorithms for computing it, for example the Levinson recursion (see [FF90]). We do not describe this here but mention only an easily derived formula for the mean square error of optimal linear prediction in terms of Schur parameters. Reconsider the product representation of v described in 3.1.2. Then

$$\begin{aligned} \|v\,\xi - p_0\, v\,\xi\| &= d_1 \\ \|v^2\,\xi - p_{[0,1]}\, v^2\,\xi\| &= d_1 d_2 \\ &\ldots \end{aligned}$$

in general: $\|v^n\xi - p_{[0,n-1]}v^n\xi\| = d_1 d_2 \ldots d_n$ for $n \leq N$, a product representation for the prediction error. It decreases with n, which is intuitively plausible because we acquire more and more information if we observe the time series for a longer time period and base our prediction on that. In the finite dimensional case $N < \infty$ the N-th optimal one-step prediction is perfect, i.e. has zero error. If $N = \infty$, then we have asymptotically perfect prediction if and only if the infinite product $\prod_{n=1}^{\infty} d_n$ converges to zero or, equivalently, if $\sum_{n=1}^{\infty} |\, k_n\,|^2 = \infty$. We call this the (linear) deterministic case. Complementary, if the error does not decrease below a certain strict positive threshold, we call this the (linear) indeterministic case.

It can be shown that the isometry v is unitary if and only if we are in the deterministic case. Note that because ξ is cyclic for v, the kernel of v^* is at most one dimensional. See [He95] for an explicit computation of this kernel in the indeterministic case.

The notion of cyclicity in the context of prediction theory for stationary processes has been extensively developed by P. Masani and N. Wiener in [MW57]. See further [Con96] for generalizations applicable also to non-stationary processes.

3.1.5 Unitary Dilations and Defect Operators

The theory reviewed above can also be developed if we replace the one-dimensional subspace $\mathcal{H}_0$ by an arbitrary subspace $\mathcal{H}_0$ of $\tilde{\mathcal{H}}$. This is more complicated, the results are similar, but the equivalence of determinism and unitarity of v is not valid in general. Let us give a brief outline, emphasizing product representations. Recall from dilation theory (see [SF70, FF90]) that if $\mathcal{L}, \mathcal{L}'$ are Hilbert spaces and $T : \mathcal{L} \to \mathcal{L}'$ is a contraction, then the rotation matrix

$$R_T := \begin{pmatrix} T & D_{T^*} \\ D_T & -T^* \end{pmatrix} : \quad \mathcal{L} \oplus \mathcal{L}' \to \mathcal{L}' \oplus \mathcal{L},$$

(where $D_T := \sqrt{\mathbb{1} - T^*T}$, $D_{T^*} := \sqrt{\mathbb{1} - TT^*}$ are the defect operators of T) is a first order unitary dilation of T, i.e. R_T is unitary and $p_{\mathcal{L}'} R_T|_{\mathcal{L}} = T$. Defining the defect spaces $\mathcal{D}_T := \overline{D_T \mathcal{L}}$, $\mathcal{D}_{T^*} := \overline{D_{T^*} \mathcal{L}'}$ we have $R_T : \mathcal{L} \oplus \mathcal{D}_{T^*} \to \mathcal{L}' \oplus \mathcal{D}_T$, which is minimal in the sense that it is not possible to get a unitary dilation on smaller spaces by restricting. Minimal first order dilations are unique up to unitary equivalence.

A **choice sequence** $(\Gamma_n)_{n=1}^\infty$ initiated on a Hilbert space $\mathcal{H}_0$ is a sequence of contractions with $\Gamma_1 : \mathcal{H}_0 \to \mathcal{H}_0$, $\Gamma_{n+1} : \mathcal{D}_n \to \mathcal{D}_{n*}$ for all $n \in \mathbb{N}$, where $\mathcal{D}_n$, $\mathcal{D}_{n*}$ are the defect spaces of Γ_n. A choice sequence determines rotation matrices $R_n := R_{\Gamma_n} = \begin{pmatrix} \Gamma_n & D_{n*} \\ D_n & -\Gamma_n^* \end{pmatrix}$ for all n.

3.1.6 Multidimensional Case

With these tools we get the following multidimensional analogue of our earlier considerations:

Proposition ([FF90], XV.2) *Let v be an isometry on a Hilbert space $\tilde{\mathcal{H}}$ and let $\mathcal{H}_0$ be any subspace of $\tilde{\mathcal{H}}$. For all $n \in \mathbb{N}$ define*

$$\mathcal{H}_{[0,n]} := \overline{span}\{v^m \mathcal{H}_0 : m = 0, \dots, n\} =: \mathcal{H}_0 \oplus \mathcal{D}_1 \oplus \dots \mathcal{D}_n.$$

Assume that $\tilde{\mathcal{H}} = \hat{\mathcal{H}} := \overline{span}\{v^m \mathcal{H}_0 : m \in \mathbb{N}_0\}$. Then there is a choice sequence $(\Gamma_n)_{n=1}^\infty$ initiated on $\mathcal{H}_0$ with $\Gamma_1 : \mathcal{H}_0 \to \mathcal{H}_0$, $\Gamma_{n+1} : \mathcal{D}_n \to \mathcal{D}_{n*}$ for all n, such that we have the following block matrix in Hessenberg form for the isometry v:

$$v = \begin{pmatrix} \Gamma_1 & D_{1*}\Gamma_2 & D_{1*}D_{2*}\Gamma_3 & \dots & D_{1*}\dots D_{m-1*}\Gamma_m & \dots \\ D_1 & -\Gamma_1^*\Gamma_2 & -\Gamma_1^* D_{2*}\Gamma_3 & \dots & -\Gamma_1^* D_{2*}\dots D_{m-1*}\Gamma_m & \dots \\ & D_2 & -\Gamma_2^*\Gamma_3 & \dots & -\Gamma_2^* D_{3*}\dots D_{m-1*}\Gamma_m & \dots \\ & & D_3 & \dots & \vdots & \dots \\ & & & & \vdots & \dots \\ & & & & -\Gamma_{m-1}^*\Gamma_m & \dots \\ & & & & D_m & \dots \\ & & & & & \dots \end{pmatrix}.$$

There is a one-to-one correspondence between the set of all positive definite sequences of contractions $(r_n)_{n=0}^\infty \subset \mathcal{B}(\mathcal{H}_0)$ with $r_0 = \mathbb{1}$ and the set of all choice sequences $(\Gamma_n)_{n=1}^\infty$ initiated on $\mathcal{H}_0$. The isometry v is the minimal isometric dilation of $(r_n)_{n=0}^\infty$, i.e. $p_{\mathcal{H}_0} v^n|_{\mathcal{H}_0} = r_n$ for all n.

Concerning product representations of v there is a subtlety neglected in [FF90], XV.2. Consider the following example: $\tilde{\mathcal{H}} := l^2(\mathbb{Z})$, $\mathcal{H}_0 := l^2(-\infty, 0]$, v the right shift. Then

$$\mathcal{H}_0 \neq v\mathcal{H}_0 = l^2(-\infty, 1] = \mathcal{H}_{[0,1]}$$

and thus there cannot be any unitary u_1 on $\mathcal{H}_{[0,1]}$ such that $v|_{\mathcal{H}_0} = u_1|_{\mathcal{H}_0}$. What happened here? If we want to write v as an infinite product then the natural candidate for u_{n+1} is

$$R_{n+1} = \begin{pmatrix} \Gamma_{n+1} & D_{n+1*} \\ D_{n+1} & -\Gamma^*_{n+1} \end{pmatrix} : \mathcal{D}_n \oplus \mathcal{D}_{n+1*} \to \mathcal{D}_{n*} \oplus \mathcal{D}_{n+1}.$$

But we want u_{n+1} to act on $\mathcal{D}_n \oplus \mathcal{D}_{n+1} \subset \tilde{\mathcal{H}}$, when we write down the product representation. This is easily arranged if $\mathcal{D}_n$ and $\mathcal{D}_{n*}$ have the same dimension (for all n): just insert a suitable unitary identification. If $\mathcal{H}_0$ is finite dimensional this is always the case. Therefore we have

Corollary: *If in the proposition the subspace $\mathcal{H}_0$ is finite dimensional, then*

$$v = stop - \lim_{N\to\infty} R_1 R_2 \dots R_N.$$

To get a product representation in the general case we can embed $\mathcal{D}_n$ and $\mathcal{D}_{n*}$ into a Hilbert space $\mathcal{K}_n$ for all n, in such a way that R_{n+1} has a unitary extension $\tilde{R}_{n+1}$ on $\mathcal{K}_n \oplus \mathcal{K}_{n+1}$. The construction of the rotation matrices shows that it is possible to take a copy of $\mathcal{H}_0$ for each $\mathcal{K}_n$. On $\bigoplus_{n=0}^{\infty} \mathcal{K}_n$ we can proceed as in the one dimensional case and we get

Corollary: *In the setting of the proposition define $\mathcal{K}_0 = \mathcal{H}_0$ and $\tilde{\mathcal{H}} := \bigoplus_{m=0}^{\infty} \mathcal{K}_m$, where each $\mathcal{K}_m$ is a copy of $\mathcal{H}_0$, further $\mathcal{K}_{[0,n]} := \bigoplus_{m=0}^{n} \mathcal{K}_m$. There is an embedding of $\hat{\mathcal{H}}$ into $\tilde{\mathcal{H}}$ such that $\mathcal{H}_{[0,n]} \subset \mathcal{K}_{[0,n]}$ for each n and that with the trivial extension $v|_{\tilde{\mathcal{H}} \ominus \hat{\mathcal{H}}} = 1\!\!1|_{\tilde{\mathcal{H}} \ominus \hat{\mathcal{H}}}$ we have*

$$v = stop - \lim_{N\to\infty} \tilde{R}_1 \tilde{R}_2 \dots \tilde{R}_N.$$

Here $\tilde{R}_{n+1}$ is a unitary on $\mathcal{K}_n \oplus \mathcal{K}_{n+1}$, acting trivially elsewhere. It extends R_{n+1} acting on suitable subspaces.

3.2 Adapted Endomorphisms – An Abstract View

3.2.1 Axiomatics

In this section we want to discuss a general framework for investigating adaptedness properties. Ideally it should be close enough to the motivating example reviewed in Section 3.1 to suggest similar results and techniques, but general enough to include adaptedness in commutative and noncommutative probability. Our approach consists in postulating the existence of a time evolution operator and axiomatizing its adaptedness properties. As is often the case, the use of category theory makes it possible to state the most abstract features in an especially clear and short form. Work on more specific features then has to be done in specific categories.

With these aims in mind, we start by defining categories of adapted endomorphisms. No detailed motivation is given in this part. To bridge the gap to the applications given later we then study what this means in concrete categories and how additional properties can be postulated which prove to be useful later. Then we also include some informal discussion which may serve as a motivation for the earlier definitions.

3.2.2 Inductive Limit

For category theory we follow [Mac98]. Consider the category $\omega = \{0 \to 1 \to 2 \to 3 \to \ldots\}$ and another category $\mathcal{O}$ with $\omega-$colimits (= inductive limits). In other words, given a functor $F : \omega \to \mathcal{O}$ and denoting the image of F by the first line in the following diagram, there is an object $\varinjlim F$ in $\mathcal{O}$ (the inductive limit) which together with canonical arrows $\mu_n : F_n \to \varinjlim F$ $(n \geq 0)$ forms a universal cone (see [Mac98], Colimits):

$$
\begin{tikzcd}
F_0 \arrow[r, "i_0"] \arrow[dr, "\mu'_0"] \arrow[ddr, "\mu_0"'] & F_1 \arrow[r, "i_1"] \arrow[d, "\mu'_1"'] & F_2 \arrow[r, "i_2"] \arrow[dl, "\mu'_2"'] \arrow[ddl, "\mu_2"] & \cdots \\
& C & \cdots & \\
& \varinjlim F \arrow[u, "\exists! f"] & &
\end{tikzcd}
$$

(The arrow μ_1 is not drawn.)

3.2.3 Existence and Uniqueness

Lemma: *If for all $n \in \mathbb{N}$ there are arrows $\tilde{f}_n : F_n \to F_n$ such that $\mu_{n+1}\, \tilde{f}_{n+1}\, i_n\, i_{n-1} = \mu_n\, \tilde{f}_n\, i_{n-1}$, then there is a* **unique endomorphism** *$f : \varinjlim F \to \varinjlim F$ satisfying $f\, \mu_n = \mu_{n+1}\, \tilde{f}_{n+1}\, i_n$ for all $n \in \mathbb{N}_0$.*

Proof: Put $C := \varinjlim F$ and $\mu'_n := \mu_{n+1}\, \tilde{f}_{n+1}\, i_n$ in the diagram above and use the universal property. □

$$
\begin{tikzcd}
F_n \arrow[r, "i_n"] \arrow[d, "\mu_n"'] & F_{n+1} \arrow[r, "\tilde{f}_{n+1}"] & F_{n+1} \arrow[d, "\mu_{n+1}"] \\
\varinjlim F \arrow[rr, "f"] & & \varinjlim F
\end{tikzcd}
$$

3.2.4 Natural Transformations

If $\sigma : F \dot{\rightarrow} F$ is a natural transformation, with morphisms $\sigma_n : F_n \to F_n$, then using the properties of natural transformations and the universal property of $\varinjlim F$, it is easy to see that there is a unique arrow $s : \varinjlim F \to \varinjlim F$ such that $s\,\mu_n = \mu_n\,\sigma_n$ for all $n \in \mathbb{N}_0$. If σ is a natural equivalence (i.e. σ_n is invertible in $\mathcal{O}$ for all n) then s is an automorphism of $\varinjlim F$.

3.2.5 The Category of Adapted Endomorphisms

Definition: *Given $\mathcal{O}$, $F : \omega \to \mathcal{O}$ as above. There is a category whose objects are the endomorphisms $f : \varinjlim F \to \varinjlim F$ given by Lemma 3.2.3 for suitable $(\tilde{f}_n)_{n=1}^{\infty}$. Such an f is called an F-***adapted endomorphism***. An arrow $s : f \to g$ (where f, g are F-adapted) of this category is given by $s : \varinjlim F \to \varinjlim F$ arising from a natural transformation σ (as above) and such that the following diagram commutes:*

$$\begin{tikzcd}
\varinjlim F \arrow[r, "s"] \arrow[d, "f"'] & \varinjlim F \arrow[d, "g"] \\
\varinjlim F \arrow[r, "s"] & \varinjlim F
\end{tikzcd}$$

Remarks:

a) The properties of a category are easy to check. If f, g are F-adapted then also sf and gs are F-adapted, and the commuting diagram means an equality $sf = gs$ of F-adapted endomorphisms. Isomorphism of f and g means $g = sfs^{-1}$, i.e. conjugacy by an automorphism s which respects F.

b) Perhaps some readers would have prefered to use 'adapted' for s instead of f. Our choice of terminology is motivated by the fact that time evolutions of adapted processes in probability theory have the form of f, see Section 3.3. Automorphisms s are symmetries of this structure.

c) One may use a finite linear order $\{0 \to 1 \to 2 \to \ldots \to N+1\}$ instead of ω and proceed in a similar way.

d) The sequence $(\tilde{f}_n)$ is not uniquely determined by the F-adapted endomorphism f. In fact, an inspection of Lemma 3.2.3 shows that if $(\tilde{g}_n : F_n \to F_n)$ is such a sequence for the F-adapted endomorphism g and $\tilde{f}_{n+1}\,i_n = \tilde{g}_{n+1}\,i_n$ for all n, then $f = g$. This suggests to impose additional postulates on the morphisms $\tilde{f}_n$.

$$
\begin{tikzcd}
& F_{n+1} \arrow[rr, "\sigma_{n+1}"] \arrow[dd, "\mu_{n+1}"' near start] & & F_{n+1} \arrow[dd, "\mu_{n+1}"'] \\
F_n \arrow[ur, "\tilde{f}_{n+1} i_n"] \arrow[rr, crossing over, "\sigma_n" near start] \arrow[dd, "\mu_n"] & & F_n \arrow[ur, "\tilde{g}_{n+1} i_n"] & \\
& \varinjlim F \arrow[rr, "s" near start] & & \varinjlim F \\
\varinjlim F \arrow[ur, "f"'] \arrow[rr, "s"'] & & \varinjlim F \arrow[ur, "g"'] \arrow[from=uu, crossing over, "\mu_n" near end] &
\end{tikzcd}
$$

3.2.6 Filtrations and Product Representations

Before implementing such additional postulates we restrict the level of generality and give a definition of adaptedness which is more directly applicable for the examples we have in mind. We assume from now on that the category $\mathcal{O}$ is concrete, see [Mac98], which we use informally in the sense that the objects are sets with some structure and the arrows are functions preserving this structure. We assume also that the arrows $i_n : F_n \to F_{n+1}$ and $\mu_n : F_n \to \varinjlim F$ are inclusions giving $F_n \subset F_{n+1}$ and $F_n \subset \varinjlim F$.

Definition: *Assume that $N \in \mathbb{N}$ and $F_0 \subset F_1 \subset \ldots \subset F_{N+1} \subset \tilde{F}$ is an increasing family of objects in a concrete category $\mathcal{O}$. An endomorphism $f : \tilde{F} \to \tilde{F}$ is called* **adapted with product representation (PR)** *for $F_0 \subset F_1 \subset \ldots \subset F_{N+1} \subset \tilde{F}$ if there are automorphisms $\tilde{f}_n : \tilde{F} \to \tilde{F}$, $n = 0, \ldots, N+1$, satisfying*

(0) $\tilde{f}_0 = \mathbb{1}$ (Identity)

(1) $\tilde{f}_{n+1}|_{F_{n+1}}$ is an automorphism of F_{n+1}, for all $n = 0, \ldots, N$

(2) $\tilde{f}_{n+1}|_{F_n} = f|_{F_n}$ for all $n = 0, \ldots, N$

f is adapted with PR for a (countably) infinite increasing family of objects if the axioms are valid for all $N \in \mathbb{N}$.
$(F_n)_{n=0}^{N+1}$ or $(F_n)_{n=0}^{\infty}$ are called a **filtration**.

Such a filtration is a special case of a functor $F : \omega \to \mathcal{O}$ as above. If $\tilde{F} = \varinjlim F$ then one can check that an endomorphism which is adapted with PR is also F-adapted in the sense of Definition 3.2.5. From Definition 3.2.6 we have in addition that the $\tilde{f}_n$ are automorphisms of F_n (compare 3.2.5d) which can be extended to automorphisms of $\tilde{F}$. See 3.2.7e below how this leads to a product representation of f, explaining the terminology.

3.2.7 Elementary Properties

Suppose now that $f : \tilde{F} \to \tilde{F}$ is an adapted endomorphism with PR for $F_0 \subset F_1 \subset \ldots \subset F_{N+1} \subset \tilde{F}$. Let us give some further comments on Definition 3.2.6 and derive some elementary properties.

a) As a motivation for Definition 3.2.6 one should imagine $\tilde{F}$ large and f rather complicated. Then it may be useful to replace f on small parts by more well-behaved morphisms such as the automorphisms $\tilde{f}_n$. If $\{\tilde{g}_n\}$ also corresponds to f (in the sense of Definition 3.2.6) then

$$\tilde{f}_n|_{F_{n-1}} = \tilde{g}_n|_{F_{n-1}} = f|_{F_{n-1}}.$$

But there is nothing in the axioms about the action of $\tilde{f}_n$ on the (set-theoretic) complement of F_n. In a way it doesn't matter because $\tilde{f}_n$ is designed as a tool to study f on F_{n-1}. To avoid unnecessary complications we should choose $\tilde{f}_n$ as simple as possible. For example in many cases it may be possible to choose an identical action on a complementary object. But this depends on the category and therefore we did not include it into Definition 3.2.6.

b) $f|_{F_n}$ is a monomorphism (as a restriction of $\tilde{f}_{n+1}$).

c) $f(F_n) = \tilde{f}_{n+1}(F_n) \subset F_{n+1}$.

This is the adaptedness property familiar from time evolutions in probability theory, see also Section 3.3. Any f satisfying $f(F_n) \subset F_{n+1}$ for all n may be called **adapted in a wide sense**. Definition 3.2.6 is stronger, postulating the existence of certain related morphisms. The advantages are two-fold: First it allows a clear formulation in terms of category theory (see Definition 3.2.5) which suggests developments on a more theoretical side. Second, on a more practical side, the automorphisms $\tilde{f}_n$ in Definition 3.2.6, acting on small parts, suggest concrete computations. We shall see many examples in the following.

d) Using $f(F_n) \subset F_{n+1}$ for all n we find that

$$f^n|_{F_0} = \tilde{f}_n \ldots \tilde{f}_1|_{F_0} = (\tilde{f}_n)^n|_{F_0}$$

and $\bigcup_{n=0}^{N+1} f^n(F_0) \subset F_{n+1}$.

e) Setting $f_{n+1} := \tilde{f}_n^{-1}\tilde{f}_{n+1}$ for $n = 0, \ldots, N$ we have

$$\tilde{f}_{n+1} = f_1 \ldots f_{n+1}.$$

Then $f_1, \ldots, f_{n+1}$ determine $\tilde{f}_1, \ldots, \tilde{f}_{n+1}$ and conversely. Thus for $N \to \infty$ we have a kind of infinite product representing the adapted endomorphism f (at least on all F_n or on $\lim_{\to} F$, if available). This explains the terminology 'product representation' (PR) in Definition 3.2.6. The factors in this product are automorphisms. We refer to them as **factors** of the adapted endomorphism. In general they are not uniquely determined by f. Compare 3.2.5d and Section 3.4. The inductive limit $\varinjlim F$, if available, plays a distinguished role, because on $\varinjlim F$ the adapted endomorphism is uniquely determined by its factors, see Lemma 3.2.3.

f) f_{n+1} fixes F_{n-1} pointwise $(n \geq 1)$.

Proof: $\tilde{f}_n|_{F_{n-1}} = f|_{F_{n-1}} = \tilde{f}_{n+1}|_{F_{n-1}}$. □

3.2.8 Elementary Example

As a first and simple example guiding our intuitive understanding we can look at the following: On the set $\mathbb{N}_0$ of natural numbers the successor map $f : n \mapsto n{+}1$ is adapted with PR for the family of sets $\{0, \ldots, n\},\ n \in \mathbb{N}_0$. Here $\tilde{f}_{n+1}$ is a cyclic shift on $\{0, \ldots, n+1\}$. On the complement of $\{0, \ldots, n+1\}$ we choose it to act identically (which is the simplest choice, compare 3.2.7a).

Then the factor f_{n+1} exchanges n and $n+1$, leaving all other numbers fixed. We see here very clearly what we also want to get for more complicated situations: The factors f_n are rather elementary, and in particular their action is localized with respect to the filtration.

3.2.9 Extending Factors

Going back to the general case let us therefore focus our attention on the factors $f_1, \ldots, f_{N+1}$ of an adapted endomorphism.

Definition: *Let a filtration $F_0 \subset F_1 \subset \ldots \subset F_{N+1} \subset \tilde{F}$ be fixed. An $(N+1)$-tupel of automorphisms $f_1, \ldots, f_{N+1} : \tilde{F} \to \tilde{F}$ is said to be an $(N+1)$-***tupel of factors** *for the filtration if for all $n = 0, \ldots, N$*

(1) $f_1 \ldots f_{n+1}|_{F_{n+1}}$ is an automorphism of F_{n+1}.

(2) $f_{n+1}|_{F_{n-1}} = 1\!\!1|_{F_{n-1}}$.

If only an $N-$tupel of factors $(f_1, \ldots, f_N)$ is given, then any f_{N+1} making $(f_1, \ldots, f_{N+1})$ into an $(N+1)$-tupel of factors is called an **extending factor**.

Lemma: *With respect to a filtration $F_0 \subset F_1 \subset \ldots \subset F_{N+1} \subset \tilde{F}$, automorphisms $f_1, \ldots, f_{N+1} : \tilde{F} \to \tilde{F}$ are factors of an adapted endomorphism $f : \tilde{F} \to \tilde{F}$ with PR if and only if they form an $(N+1)$-tupel of factors according to the definition above.*

Proof: In 3.2.7(e),(f) we have already seen that the factors of an adapted endomorphism with PR satisfy the properties of the above definition. Conversely, if an $(N+1)$-tupel of factors $(f_1, \ldots, f_{N+1})$ is given, then $f := f_1 \ldots f_{N+1}$ is an adapted endomorphism with factors $f_1, \ldots, f_{N+1}$. Indeed, with $\tilde{f}_n := f_1 \ldots f_n$ for all n, one easily verifies the properties given in 3.2.6. □

The lemma shows that adapted endomorphisms with PR also could have been defined in terms of their factors.

3.2.10 The Meaning of Extending Factors

What kind of information about the adapted endomorphism f is contained in a factor f_n?

Lemma: *Let f, g be two endomorphisms adapted with PR for $F_0 \subset F_1 \subset \ldots \subset F_{N+1} \subset \tilde{F}$ which coincide on F_{N-1}. If $f_1, \ldots, f_N, f_{N+1}$ are factors of f then there exists g_{N+1} such that $f_1, \ldots, f_N, g_{N+1}$ are factors of g.*

Proof: If $(\tilde{g}_n)_{n=0}^{N+1}$ correspond to g as in Definition 3.2.6 and $\tilde{g}_{N+1} = \tilde{g}_N \, g^\dagger_{N+1}$ with a factor $g^\dagger_{N+1}$, then we have

$$\tilde{g}_{N+1} = \tilde{f}_N \; (\tilde{f}_N^{-1} \tilde{g}_N g^\dagger_{N+1}).$$

Because $\tilde{g}_{N+1}(F_{N+1}) = F_{N+1}$ and $\tilde{f}_N^{-1} \tilde{g}_N|_{F_{N-1}} = 1\!\!1|_{F_{N-1}}$ by assumption and $g^\dagger_{N+1}|_{F_{N-1}} = 1\!\!1|_{F_{N-1}}$ (as a factor of g), we conclude that

$$g_{N+1} := \tilde{f}_N^{-1} \tilde{g}_N \, g^\dagger_{N+1}$$

is an extending factor for $(f_1, \ldots, f_N)$ (compare Definition 3.2.9). We have $\tilde{g}_{N+1} = \tilde{f}_N \, g_{N+1}$, i.e. $f_1, \ldots, f_N, g_{N+1}$ are factors of g. $\square$

Proposition: *Let f be an endomorphism adapted with PR for $F_0 \subset F_1 \subset \ldots \subset F_{N+1} \subset \tilde{F}$. Fix a corresponding $\tilde{f}_N = f_1 \ldots f_N$ with $\tilde{f}_N(F_N) = F_N$ and $\tilde{f}_N|_{F_{N-1}} = f|_{F_{N-1}}$.*
The extensions of $f|_{F_{N-1}}$ to F_N obtained by considering all endomorphisms g adapted with PR for $F_0 \subset F_1 \subset \ldots \subset F_{N+1} \subset \tilde{F}$ and coinciding with f on F_{N-1} are in one-to-one correspondence to the monomorphisms $g_{N+1}|_{F_N}$, where g_{N+1} is any factor extending $(f_1, \ldots, f_N)$. The correspondence is given by

$$g|_{F_N} \;=\; \tilde{f}_N \, g_{N+1}|_{F_N}.$$

Proof: If g_{N+1} is an extending factor of $(f_1, \ldots, f_N)$, then $g := f_1 \ldots f_N \, g_{N+1}$ is an extension with the required properties. If two such extending factors differ on F_N, say $g_{N+1}|_{F_N} \neq g^\dagger_{N+1}|_{F_N}$, then also the corresponding g and $g^\dagger$ differ on F_N, because factors are monomorphisms. Conversely, if g is given, then the lemma above guarantees the existence of g_{N+1}. $\square$

3.2.11 Constructing Adapted Endomorphisms Step by Step

Proposition 3.2.10 shows that the construction of extending factors solves the problem of constructing an adapted endomorphism step by step, fixing it on larger and larger parts of the given filtration. Note that if for adapted endomorphisms f and g the factors $f_1 = g_1, \ldots, f_N = g_N$ are in common, but $f_{N+1} \neq g_{N+1}$, then $f^n|_{F_0} = g^n|_{F_0}$ for all $0 \leq n \leq N$, but $f^{N+1}|_{F_0}$ and $g^{N+1}|_{F_0}$ may be different.

In this way the analysis of extending factors corresponds to a multitude of classical and new extension problems. In particular, in Section 3.1 we have studied adapted isometries in Hilbert spaces which correspond to extensions of positive definite sequences, and we have seen that there already exists a large amount of theory about it which may be cast into the framework developed in this section.

While it would be possible at this point to study very different categories in this respect, we shall concentrate on probability theory and stochastic processes. While the importance of filtrations is indisputable in this field, the point of view of adapted endomorphisms has not been explored.

3.3 Adapted Endomorphisms and Stationary Processes

3.3.1 Application to Stochastic Processes

We want to apply the concept of an adapted endomorphism, as developed in the previous section, to adapted stochastic processes which may be commutative or noncommutative. There are no theorems proved in this section and we proceed somewhat informal here, but we invent the setting for the things to come in later sections.

3.3.2 Endomorphisms of Probability Spaces

Let $(\tilde{\Omega}, \tilde{\Sigma}, \tilde{\mu})$ be a probability space. A natural notion of **endomorphism** for $(\tilde{\Omega}, \tilde{\Sigma}, \tilde{\mu})$ is a map on the measurable sets which commutes with the operations of the $\sigma-$algebra, leaves the measure $\tilde{\mu}$ invariant and is considered equal to another such map if their difference only concerns sets of $\tilde{\mu}-$measure zero. See Petersen ([Pe89], 1.4C) for a detailed discussion and for a theorem stating that if $(\tilde{\Omega}, \tilde{\Sigma}, \tilde{\mu})$ is a Lebesgue space, then (after removing some sets of $\tilde{\mu}-$measure zero) such an endomorphism arises from a measurable point transformation $\tau : \tilde{\Omega} \to \tilde{\Omega}$ satisfying $\tilde{\mu}(B) = \tilde{\mu}(\tau^{-1}(B))$ for all $B \in \tilde{\Sigma}$. Assume from now on that our probability spaces are Lebesgue, indeed our examples are Lebesgue in all cases.

We can lift such endomorphisms to function spaces: $T_{\tau,p} : L^p(\tilde{\Omega}, \tilde{\Sigma}, \tilde{\mu}) \to L^p(\tilde{\Omega}, \tilde{\Sigma}, \tilde{\mu})$ with $T_{\tau,p}\,\xi(\omega) \;:=\; \xi(\tau(\omega))$. Of particular importance to us are the cases $p = \infty$ and $p = 2$. For $p = \infty$ the transformation $\alpha = T_{\tau,\infty}$ is a $^*-$algebra endomorphism of $L^\infty(\tilde{\Omega}, \tilde{\Sigma}, \tilde{\mu})$ and for $p = 2$ the transformation $v = T_{\tau,2}$ is an isometry on the Hilbert space $L^2(\tilde{\Omega}, \tilde{\Sigma}, \tilde{\mu})$.

Of course, we emphasize these algebraic versions of an endomorphism because they allow a noncommutative generalization in the usual way. As a minimal requirement we can start with a noncommutative probability space $(\tilde{\mathcal{A}}, \tilde{\phi})$ with $\tilde{\mathcal{A}}$ a $^*-$algebra and $\tilde{\phi}$ a state on it. An endomorphism α is then a $^*-$algebra endomorphism of $\tilde{\mathcal{A}}$ satisfying $\tilde{\phi} \circ \alpha = \tilde{\phi}$. Stressing the aspect of noncommutative measure theory one can assume that $\tilde{\mathcal{A}}$ is a von Neumann algebra (with separable predual if needed; this is valid for $L^\infty(\tilde{\Omega}, \tilde{\Sigma}, \tilde{\mu})$ if $(\tilde{\Omega}, \tilde{\Sigma}, \tilde{\mu})$ is Lebesgue). Then an endomorphism is a normal $^*-$endomorphism of $\tilde{\mathcal{A}}$, the state $\tilde{\phi}$ is assumed to be normal (and sometimes also faithful). The GNS-construction for $\tilde{\phi}$ yields a Hilbert space called $\tilde{\mathcal{K}} = L^2(\tilde{\mathcal{A}}, \tilde{\phi})$ (separable if $\tilde{\mathcal{A}}$ has a separable predual) with a cyclic vector $\tilde{\Omega}$ representing $\tilde{\phi}$. Using the invariance we can extend α to an isometry v on $\tilde{\mathcal{K}}$, analogous to the commutative case. It is the associated isometry in the sense of 1.3.4.

3.3.3 Time Evolutions and Adaptedness

Let us now turn to stochastic processes and adaptedness. As a classical stochastic process we consider a sequence $(\xi_n)_{n=0}^\infty$ of random variables on a probability space $(\tilde{\Omega}, \tilde{\Sigma}, \tilde{\mu})$. We think of the subscript n as representing time, but it may represent something else in applications. A **filtration** is given by a sequence of sub-σ-algebras $(\tilde{\Sigma}_n)_{n=0}^\infty$ with $\Sigma_0 := \tilde{\Sigma}_0 \subset \tilde{\Sigma}_1 \subset \tilde{\Sigma}_2 \subset \ldots \subset \tilde{\Sigma}$. We shall assume that $(\tilde{\Sigma}_n)_{n=0}^\infty$ generates $\tilde{\Sigma}$. The process $(\xi_n)_{n=0}^\infty$ is called **adapted** for the filtration $(\tilde{\Sigma}_n)_{n=0}^\infty$ if ξ_n is $\tilde{\Sigma}_n-$measurable for all $n \in \mathbb{N}_0$. In the time interpretation for n the $\sigma-$algebra $\tilde{\Sigma}_n$ represents the information available up to time n, and adaptedness means that ξ_n depends only on events up to time n.

To relate this well-known setting to the framework of Section 3.2 we have to assume the existence of an endomorphism α which represents the time evolution of the process, i.e. $\alpha(\xi_n) = \xi_{n+1}$ for all n. Assume that $\xi_n \in L^\infty(\tilde{\Omega}, \tilde{\Sigma}, \tilde{\mu})$ for all n, which is not a severe restriction because, if this is not the case, then one can look at suitable functions of the random variables carrying the same information. Assuming as above that $\tilde{\mu}$ is invariant for α, the existence of such a time evolution implies that the process $(\xi_n)_{n=0}^\infty$ is stationary, i.e. the correlations between the random variables only depend on time differences. Compare 2.1.3 and see [Kü03] for a careful motivation and discussion of stationarity. It is shown in [Kü03] that essentially the converse is also true: If the process is stationary then there exists a time evolution α with invariant state. Note further that once a useful structure theory for stationary processes has been constructed, nothing prevents us to look also at other states which are not invariant. Note also that from the beginning a more general setting based on quasi-invariant instead of invariant measures would have been possible and may also be cast into the framework of adapted endomorphisms, but in this work we restrict ourselves to stationarity.

All these comments apply to noncommutative stationary processes as well. The noncommutative setting for adaptedness is as follows. Let a noncommutative probability space $\{\tilde{\mathcal{A}}, \tilde{\phi}\}$ be given, together with a process $(\xi_n)_{n=0}^\infty \subset \tilde{\mathcal{A}}$. If one wants to interpret ξ_n as an observable in the sense of quantum theory then one should assume that ξ_n is selfadjoint, but for most of the following this is not needed. A filtration is given by a sequence $(\tilde{\mathcal{A}}_n)_{n=0}^\infty$ of $^*-$subalgebras with $\mathcal{A}_0 := \tilde{\mathcal{A}}_0 \subset \tilde{\mathcal{A}}_1 \subset \tilde{\mathcal{A}}_2 \subset \ldots \subset \tilde{\mathcal{A}}$. This is also called a tower of algebras.

We have written $\tilde{\mathcal{A}}_n$ to distinguish it from the algebra $\mathcal{A}_n$ generated by ξ_n or the algebra $\mathcal{A}_{[0,n]}$ generated by $\xi_0, \dots, \xi_n$. We shall assume that $(\tilde{\mathcal{A}}_n)_{n=0}^{\infty}$ generates $\tilde{\mathcal{A}}$. The process $(\xi_n)_{n=0}^{\infty}$ is adapted for the filtration $(\tilde{\mathcal{A}}_n)_{n=0}^{\infty}$ if $\xi_n \in \tilde{\mathcal{A}}_n$ for all $n \in \mathbb{N}_0$. We assume that there is a time evolution, i.e. an endomorphism α of $\tilde{\mathcal{A}}$ such that $\alpha(\xi_n) = \xi_{n+1}$ for all n and that the state $\tilde{\phi}$ is invariant for α. In other words, α is an endomorphism of $\{\tilde{\mathcal{A}}, \tilde{\phi}\}$ and we deal with stationary processes. The interpretations are also similar to the classical case but the various peculiarities of quantum theory now play their role.

3.3.4 Adapted Endomorphisms

Now the main idea goes as follows: Study adapted stationary processes by using and suitably developing a theory of adapted endomorphisms which occur as time evolutions. One observation is immediate: If α is an endomorphism of $\{\tilde{\mathcal{A}}, \tilde{\phi}\}$ which is adapted to the filtration $(\tilde{\mathcal{A}}_n)_{n=0}^{\infty}$ in the sense of 3.2.5 (or 3.2.6) and if further $\xi_0 \in \mathcal{A}_0$, then $(\xi_n := \alpha^n \xi_0)_{n=0}^{\infty}$ is a stationary process adapted to the filtration. If a stationary process is given in this way then we shall say that it is **adaptedly implemented (with PR)**.

How can we use the additional structures available by adapted endomorphisms to learn something about this process? First note the following: Applying the GNS-construction, we arrive at a filtration of Hilbert spaces $\mathcal{K}_0 := \tilde{\mathcal{K}}_0 \subset \tilde{\mathcal{K}}_1 \subset \tilde{\mathcal{K}}_2 \subset \dots \subset \tilde{\mathcal{K}}$, and the isometry v on $\tilde{\mathcal{K}}$ which extends α is an adapted isometry for this filtration of Hilbert spaces in the sense of 3.2.5 (or 3.2.6). This suggests that part of the work can be done on the level of Hilbert spaces.

The filtrations $(\tilde{\mathcal{K}}_n)_{n=0}^{\infty}$ of Hilbert spaces occurring here are however of a very different character than those considered in Section 3.1 which are built from a Gram-Schmidt procedure. Here the adapted isometry v extends the time evolution α and contains the full algebraic information about the process, not only about some linear span. The size of $\tilde{\mathcal{K}}_n = \tilde{\mathcal{A}}_n \tilde{\Omega}$ typically does not grow linearly but exponentially with n.

3.3.5 Tensor Independence

To start some theoretical investigations it is helpful to look at situations where some simplifying features are present. Taking a hint from classical probability we see that in practice in most cases the filtration is generated by independent variables. In fact, in the time-continuous case the filtrations generated by Brownian motion are the most prominent, and to assume the existence of generating independent variables is just the discrete version of that. If such a factorization is not possible for the canonical filtration $(\mathcal{A}_{[0,n]})_{n=0}^{\infty}$ of the process then we can go to a suitable refinement. In the interpretation of filtrations as specifying available information, such a refinement may also reflect the fact that we can achieve information about the process from other sources than the process itself. All in all, such a setting is well established in probability theory and often provides a background for stochastic integration and martingale representation of processes. See also the remarks about Fock space representations in noncommutative probability given in the Introduction.

Let $(\tilde{\Omega}, \tilde{\Sigma}, \tilde{\mu})$ be a probability space and $(\mathcal{F}_n)_{n=0}^{\infty}$ a sequence of independent sub-σ-algebras. Denote by $\mathcal{F}_{[m,n]}$ the $\sigma-$algebra generated by all $\mathcal{F}_j$ with $m \leq j \leq n$. In particular we have the filtration $(\tilde{\Sigma}_n := \mathcal{F}_{[0,n]})_{n=0}^{\infty}$ as before. For the corresponding algebra $L^\infty(\tilde{\Omega}, \tilde{\Sigma}, \tilde{\mu})$ and the Hilbert space $L^2(\tilde{\Omega}, \tilde{\Sigma}, \tilde{\mu})$ independence translates into a tensor product decomposition. This is our motivation for starting a detailed study of adapted endomorphisms and isometries on tensor products, beginning with Section 3.4.

In the noncommutative setting there are other notions of independence, alternative to tensor independence. See for example [BGS02] for a discussion of this topic. As a first step however we concentrate on tensor independence. The following considerations show that not only classical time series but also the laws of quantum physics attribute particular importance to it.

We have a noncommutative probability space $\{\tilde{\mathcal{A}}, \tilde{\phi}\}$ and a sequence of $^*-$subalgebras $(\mathcal{C}_n)_{n=0}^{\infty}$ such that the filtration $(\tilde{\mathcal{A}}_n := \mathcal{C}_{[0,n]})_{n=0}^{\infty}$ is given by

$$\left(\mathcal{C}_{[0,n]}, \psi_{[0,n]}\right) \quad = \quad (\mathcal{C}_0, \psi_0) \otimes (\mathcal{C}_1, \psi_1) \otimes \ldots \otimes (\mathcal{C}_n, \psi_n)$$

for all n. This notation means that $\mathcal{C}_{[0,n]}$ is a tensor product of algebras and the state $\psi_{[0,n]}$, the restriction of $\tilde{\phi}$ to $\mathcal{C}_{[0,n]}$, is a product state. This is what is meant by **tensor independence**. The embeddings $\mathcal{C}_{[0,n]} \subset \mathcal{C}_{[0,n+1]}$ must be understood as

$$\mathcal{C}_{[0,n]} = \mathcal{C}_0 \otimes \ldots \otimes \mathcal{C}_n \otimes \mathbb{1} \subset \mathcal{C}_0 \otimes \ldots \otimes \mathcal{C}_n \otimes \mathcal{C}_{n+1} = \mathcal{C}_{[0,n+1]}.$$

In the C^*- or W^*-algebraic context we can define $\{\tilde{\mathcal{A}}, \tilde{\phi}\}$ as an infinite tensor product with subalgebras $\mathcal{C}_{[m,n]} = \mathbb{1}_{[0,m-1]} \otimes \mathcal{C}_m \otimes \ldots \otimes \mathcal{C}_n \otimes \mathbb{1}_{[n+1,\infty]}$.
If the process $(\xi_n)_{n=0}^{\infty}$ is adapted, then for all n we have $\xi_n \in \mathcal{C}_{[0,n]} = \mathcal{C}_0 \otimes \ldots \otimes \mathcal{C}_n$. If ξ_n is a selfadjoint element, i.e. an observable in quantum physics, then we can perform measurements and relate it in this way to observed phenomena. While ξ_n represents the observed process at time n, the formula $\xi_n \in \mathcal{C}_0 \otimes \ldots \otimes \mathcal{C}_n$ shows that there may be parts of the information contained in it that can be extracted at earlier times. In other words, it is not assumed that ξ_n is independent from the past $\mathcal{C}_{[0,n-1]}$. Similar to the innovation interpretation of the linear theory given in Section 3.1 we may think of $\mathcal{C}_m$ as representing information available at time m but independent of what has been known at earlier times. For example, at time m with $0 \leq m \leq n$ we can extract some information about ξ_n which just now has become available, by measuring $Q_m(\xi_n)$, the conditional expectation onto $\mathcal{C}_m$ of ξ_n:

$$Q_m(c_0 \otimes c_1 \otimes \ldots) := \psi_0(c_0)\, \psi_1(c_1) \ldots \psi_{m-1}(c_{m-1})\; c_m\; \psi_{m+1}(c_{m+1}) \ldots$$

Tensor independence means that the observables $Q_m(\xi_n)$ commute with each other for different m. Thus according to the laws of quantum physics they are compatible and may be measured without mutual interference.

Similarly, all that can be known up to time $n-1$ about ξ_n must be achieved by measuring $Q_{[0,n-1]}(\xi_n)$, where $Q_{[0,n-1]}$ is the conditional expectation onto $\mathcal{C}_{[0,n-1]}$ of ξ_n:

$$Q_{[0,n-1]}(c_0 \otimes c_1 \otimes \ldots) := c_0 \otimes c_1 \otimes \ldots c_{n-1}\, \psi_n(c_n) \ldots$$

A process $(\xi_n)_{n=0}^{\infty}$ is called predictable (compare [Me91], II.2.4) if ξ_0 is a constant and $\xi_n \in \mathcal{C}_{[0,n-1]}$ for all $n \geq 1$. In fact, this means that it is possible to predict ξ_n one time unit in advance. We will return to prediction in Section 3.5.

As already mentioned, a natural first step for investigating adapted endomorphisms on tensor products consists in looking at the GNS-construction applied to this setting. Because we have a product state we arrive at an adapted isometry on a tensor product of Hilbert spaces. The structure so obtained is the same whether we start from a commutative or from a noncommutative process. It is given explicitly in the following Section 3.4.

3.4 Adapted Isometries on Tensor Products of Hilbert Spaces

3.4.1 Infinite Tensor Products of Hilbert Spaces

Let $(\mathcal{K}_n)_{n=0}^{\infty}$ be a sequence of Hilbert spaces and $(\Omega_n)_{n=0}^{\infty}$ a sequence of unit vectors such that $\Omega_n \in \mathcal{K}_n$ for all n. Then there is an infinite tensor product $\tilde{\mathcal{K}} = \bigotimes_{n=0}^{\infty} \mathcal{K}_n$ along the given sequence of unit vectors (see [KR83], 11.5.29). There is a distinguished unit vector $\tilde{\Omega} = \bigotimes_{n=0}^{\infty} \Omega_n \in \tilde{\mathcal{K}}$. Further we consider the subspaces $\mathcal{K}_{[m,n]} = \bigotimes_{j=m}^{n} \mathcal{K}_j$ (with $m \leq n$) of $\tilde{\mathcal{K}}$, where $\eta \in \mathcal{K}_{[m,n]}$ is identified with $\bigotimes_{j=0}^{m-1} \Omega_j \otimes \eta \otimes \bigotimes_{j=n+1}^{\infty} \Omega_j \in \tilde{\mathcal{K}}$. Then $\tilde{\mathcal{K}}$ is the closure of $\bigcup_{n=0}^{\infty} \mathcal{K}_{[0,n]}$. An operator $x \in \mathcal{B}(\mathcal{K}_{[m,n]})$ is identified with $\mathbb{1}_{[0,m-1]} \otimes x \otimes \mathbb{1}_{[n+1,\infty]} \in \mathcal{B}(\tilde{\mathcal{K}})$.

3.4.2 Probabilistic Interpretation

To make contact to section 3.2 consider the category whose objects are Hilbert spaces with distinguished unit vectors and whose arrows are isometries fixing the distinguished vectors. The filtration considered is $(\mathcal{K}_{[0,n]})_{n=0}^{\infty}$. For example, an isometry $v \in \mathcal{B}(\tilde{\mathcal{K}})$ is adapted in the wide sense if $v\tilde{\Omega} = \tilde{\Omega}$ and $v\mathcal{K}_{[0,n]} \subset \mathcal{K}_{[0,n+1]}$ for all n.

This is exactly what we arrive at if we apply the GNS-construction to the probabilistic setting described in 3.3.5. The vectors Ω_n and $\tilde{\Omega}$ correspond to algebra units and emerge in the GNS-construction as cyclic vectors for the given states. Probabilistic applications are our main motivation for the following.

Note that if we include noncommutative stationary processes, then we can always find such a probabilistic background for a wide sense adapted isometry $v \in \mathcal{B}(\tilde{\mathcal{K}})$. In fact, consider $\mathcal{B}(\tilde{\mathcal{K}})$ as an algebra equipped with the vector state given by $\tilde{\Omega}$ and with the filtration $(\mathcal{B}(\mathcal{K}_{[0,n]}))_{n=0}^{\infty}$. Then $x \mapsto vxv^*$ defines a wide sense adapted algebra endomorphism. Applying the GNS-construction we are back at the starting point, i.e. the adapted isometry v.

3.4.3 Adapted Isometries and Product Representations

Proposition: *If $dim K_n < \infty$ for all n, then any wide sense adapted isometry v is adapted with PR.*

Proof: We have to show that for all $n \in \mathbb{N}_0$ there is a unitary $\tilde{u}_{n+1} \in \mathcal{B}(\mathcal{K}_{[0,n+1]})$ with $\tilde{u}_{n+1}\tilde{\Omega} = \tilde{\Omega}$ such that

$$v|_{\mathcal{K}_{[0,n]}} \quad = \quad \tilde{u}_{n+1}|_{\mathcal{K}_{[0,n]}}.$$

We can choose for $\tilde{u}_{n+1}$ any unitary extension of $v|_{\mathcal{K}_{[0,n]}}$ to $\mathcal{K}_{[0,n+1]}$. Because the $\mathcal{K}_n$ are finite-dimensional so is $\mathcal{K}_{[0,n+1]}$ and such extensions exist by a dimension argument. □

Without the assumption of finite dimensionality it may happen that no unitary extensions $\tilde{u}_n$ exist and we have no PR. But in this case it is possible to get a product representation by enlarging the spaces $\mathcal{K}_n$.

We conclude that it is no essential restriction to assume that the isometry v is adapted with PR. The factors of v (see 3.2.7) are called $(u_n)_{n=1}^{\infty}$. Because $u_{n+1}|_{\mathcal{K}_{[0,n-1]}} = \mathbb{1}|_{\mathcal{K}_{[0,n-1]}}$ (see 3.2.7f), we find that the infinite product converges in the stop-topology:

$$v = stop - \lim_{N\to\infty} \tilde{u}_n = stop - \lim_{N\to\infty} u_1 u_2 \dots u_N.$$

Conversely we can start with a sequence of unitaries $(u_n)_{n=1}^{\infty}$ such that $u_n \in \mathcal{B}(\mathcal{K}_{[0,n]})$, $u_n\tilde{\Omega} = \tilde{\Omega}$ and $u_{n+1}|_{\mathcal{K}_{[0,n-1]}} = \mathbb{1}|_{\mathcal{K}_{[0,n-1]}}$ and define an adapted isometry with PR by the stop-limit above.

3.4.4 Localized Product Representations and Invariants

The product representation would be much simpler if $u_n \in \mathcal{B}(\mathcal{K}_{[n-1,n]})$ for all n which automatically implies $u_{n+1}|_{\mathcal{K}_{[0,n-1]}} = \mathbb{1}|_{\mathcal{K}_{[0,n-1]}}$. As a first step we concentrate our investigation to this special case. The advantage to have a product representation is especially visible then. Let us fix it with the following

Definition: *Let $(u_n)_{n=1}^{\infty}$ be a sequence of unitaries on $\tilde{\mathcal{K}}$ with $u_n\tilde{\Omega} = \tilde{\Omega}$ and $u_n \in \mathcal{B}(\mathcal{K}_{[n-1,n]})$ for all n. Then the isometry v defined by*

$$v = stop - \lim_{N\to\infty} u_1 u_2 \dots u_N$$

is called an adapted isometry with **localized product representation (LPR)**.

Of course the LPR-condition can be defined whenever an adapted endomorphism acts on tensor products, for example for the $^*-$algebra endomorphism α in 3.3.5.

Let an adapted isometry v with LPR be given. We want to examine the question to what extent the sequence $(u_n)_{n=1}^\infty$ is unique. This is important because we are interested in objects or properties which are defined in terms of the local information provided by the unitaries but which nevertheless only depend on the adapted isometry v itself and not on its product representation. Let us call such objects or properties **invariants**.

3.4.5 A Way to Construct Unitaries

Lemma: *Let $\mathcal{G}, \mathcal{G}^\dagger$ be Hilbert spaces and $(\eta_i)_{i\in I} \subset \mathcal{G}$, $(\eta_i^\dagger)_{i\in I} \subset \mathcal{G}^\dagger$ total subsets. If $\langle \eta_i, \eta_j \rangle = \langle \eta_i^\dagger, \eta_j^\dagger \rangle$ for all $i, j \in I$, then there is a unique unitary $w : \mathcal{G} \to \mathcal{G}^\dagger$ such that $w\,\eta_i = \eta_i^\dagger$ for all $i \in I$.*

Proof: If for $\gamma_1, \dots, \gamma_N$ we have $\sum_{n=1}^N \gamma_n \eta_{i_n} = 0$, then also $\sum_{n=1}^N \gamma_n \eta_{i_n}^\dagger = 0$. Thus w is well defined on the linear span and there is a unique unitary extension. □

3.4.6 Technical Preparations

Lemma: *Let $\mathcal{H}_0, \mathcal{H}_1, \mathcal{H}_2$ be Hilbert spaces and u_{01} a unitary on $\mathcal{H}_0 \otimes \mathcal{H}_1$ and u_{12} a unitary on $\mathcal{H}_1 \otimes \mathcal{H}_2$. (We shall also write u_{01} for $u_{01} \otimes \mathbb{1}_{\mathcal{H}_2}$ and u_{12} for $\mathbb{1}_{\mathcal{H}_0} \otimes u_{12}$, to simplify the notation.) Further we fix a closed subspace $\mathcal{G}_2 \subset \mathcal{H}_2$. Let $\mathcal{G}_1 \subset \mathcal{H}_1$ be minimal for the inclusion $u_{12}(\mathcal{H}_1 \otimes \mathcal{G}_2) \subset \mathcal{H}_1 \otimes \mathcal{H}_2$ (see A.3.2).*
Do the same for two other unitaries $u_{01}^\dagger$ and $u_{12}^\dagger$ on the same spaces. In particular we have $\mathcal{G}_1^\dagger \subset \mathcal{H}_1$ minimal for the inclusion $u_{12}^\dagger(\mathcal{H}_1 \otimes \mathcal{G}_2) \subset \mathcal{H}_1 \otimes \mathcal{H}_2$.

If $u_{01}\, u_{12}\, \xi = u_{01}^\dagger\, u_{12}^\dagger\, \xi$ is valid for all $\xi \in \mathcal{H}_0 \otimes \mathcal{H}_1 \otimes \mathcal{G}_2$, then there is a unique unitary $w : \mathcal{G}_1 \to \mathcal{G}_1^\dagger$ (identified with $\mathbb{1}_{\mathcal{H}_0} \otimes w \otimes \mathbb{1}_{\mathcal{H}_2}$, to simplify notation) such that

$$u_{12}^\dagger\, \xi \;=\; w\, u_{12}\, \xi \quad \textit{for all } \xi \in \mathcal{H}_0 \otimes \mathcal{H}_1 \otimes \mathcal{G}_2$$

(i.e. for $\xi \in \mathcal{H}_1 \otimes \mathcal{G}_2$ if u_{12}, $u_{12}^\dagger$ are interpreted in the other way). Further:

$$u_{01}^\dagger \eta^\dagger \;=\; u_{01} w^* \eta^\dagger \quad \textit{for all } \eta^\dagger \in \mathcal{H}_0 \otimes \mathcal{G}_1^\dagger.$$

Proof: Given $\xi = \xi_0 \otimes \xi_{12} \in \mathcal{H}_0 \otimes (\mathcal{H}_1 \otimes \mathcal{G}_2)$ we get for a fixed ONB $(\epsilon_k)_{k \in J}$ of $\mathcal{H}_2$ the expansions

$$u_{12}\,\xi \;=\; \xi_0 \otimes \sum_k \eta_k \otimes \epsilon_k, \quad u_{12}^\dagger\,\xi \;=\; \xi_0 \otimes \sum_k \eta_k^\dagger \otimes \epsilon_k$$

with $\eta_k \in \mathcal{G}_1$ and $\eta_k^\dagger \in \mathcal{G}_1^\dagger$ for all k. Now $u_{01}\,u_{12}\,\xi \;=\; u_{01}^\dagger u_{12}^\dagger \xi$ implies that $u_{01}(\xi_0 \otimes \eta_k) \;=\; u_{01}^\dagger(\xi_0 \otimes \eta_k^\dagger)$ for all k. Thus

$$\begin{aligned} \|\xi_0\|^2 \langle \eta_k, \eta_l \rangle &= \langle \xi_0 \otimes \eta_k, \xi_0 \otimes \eta_l \rangle \\ = \langle u_{01}(\xi_0 \otimes \eta_k), u_{01}(\xi_0 \otimes \eta_l) \rangle &= \langle u_{01}^\dagger(\xi_0 \otimes \eta_k^\dagger), u_{01}^\dagger(\xi_0 \otimes \eta_l^\dagger) \rangle \\ = \langle \xi_0 \otimes \eta_k^\dagger, \xi_0 \otimes \eta_l^\dagger \rangle &= \|\xi_0\|^2 \langle \eta_k^\dagger, \eta_l^\dagger \rangle, \end{aligned}$$

and choosing $\xi_0 \neq 0$ we find that

$$\langle \eta_k, \eta_l \rangle \;=\; \langle \eta_k^\dagger, \eta_l^\dagger \rangle \quad \textit{for all } k, l \in J.$$

Now we consider different ξ and write $\eta_k(\xi_{12})$ instead of η_k. The mapping

$$(\mathcal{H}_1 \otimes \mathcal{G}_2) \times (\mathcal{H}_1 \otimes \mathcal{G}_2) \ni (\xi_{12}, \zeta_{12}) \mapsto \langle \eta_k(\xi_{12}), \eta_l(\zeta_{12}) \rangle$$

is a conjugate-bilinear functional, and by polarization (see [KR83], 2.4) it is determined by the corresponding quadratic form. The same arguments apply to $\eta_k^\dagger$. From this we conclude that

$$\langle \eta_k(\xi_{12}), \eta_l(\zeta_{12}) \rangle \;=\; \langle \eta_k^\dagger(\xi_{12}), \eta_l^\dagger(\zeta_{12}) \rangle$$

for all $k, l \in J$ and $\xi_{12}, \zeta_{12} \in \mathcal{H}_1 \otimes \mathcal{G}_2$. But $\mathcal{G}_1$ is spanned by the $\eta_k(\xi_{12})$ for all $k \in J$ and $\xi_{12} \in \mathcal{H}_1 \otimes \mathcal{G}_2$, see A.3.2(2). Analogous for $\mathcal{G}_1^\dagger$.

Thus by Lemma 3.4.5 we conclude that there exists a unique unitary $w : \mathcal{G}_1 \to \mathcal{G}_1^\dagger$ such that

$$w\,\eta_k(\xi_{12}) \;=\; \eta_k^\dagger(\xi_{12}) \quad \textit{for all } k \in J,\ \xi_{12} \in \mathcal{H}_1 \otimes \mathcal{G}_2.$$

It satisfies $u_{12}^\dagger\,\xi \;=\; w\,u_{12}\,\xi$ for all $\xi \in \mathcal{H}_0 \otimes \mathcal{H}_1 \otimes \mathcal{G}_2$.

Any element $\eta^\dagger \in \mathcal{H}_0 \otimes \mathcal{G}_1^\dagger$ can be approximated by $\sum_k \xi_k \otimes \eta_k^\dagger \in \mathcal{H}_0 \otimes \mathcal{G}_1^\dagger$ with $\eta_k^\dagger$ occurring as above in an expansion derived from some $\xi_{12} \in \mathcal{H}_1 \otimes \mathcal{G}_2$. But we have already seen that $u_{01}^\dagger(\xi_0 \otimes \eta_k^\dagger) \;=\; u_{01}(\xi_0 \otimes \eta_k)$ with $\eta_k = w^* \eta_k^\dagger$. Thus $u_{01}^\dagger\,\eta^\dagger \;=\; u_{01}\,w^* \eta^\dagger$ for all $\eta^\dagger \in \mathcal{H}_0 \otimes \mathcal{G}_1^\dagger$. □

3.4.7 Non-uniqueness of Product Representations: Local Result

If $v = stop - \lim_{N\to\infty} u_1 u_2 \dots u_N$ is an adapted isometry with LPR on $\tilde{\mathcal{K}} = \bigotimes_{n=0}^{\infty} \mathcal{K}_n$, then we have the following **array of subspaces**: For all $N \in \mathbb{N}_0$ define $\mathcal{G}^N_{N+1} := \mathbb{C}\Omega_{N+1} \subset \mathcal{K}_{N+1}$ and then $\mathcal{G}^N_N, \mathcal{G}^N_{N-1}, \dots, \mathcal{G}^N_0$ recursively by the requirement that $\mathcal{G}^N_n \subset \mathcal{K}_n$ is minimal for the inclusion $u_{n+1}(\mathcal{K}_n \otimes \mathcal{G}^N_{n+1}) \subset \mathcal{K}_n \otimes \mathcal{K}_{n+1}$ (see A.3.2).

Proposition:
Let $v = stop - \lim_{N\to\infty} u_1 u_2 \dots u_N$ *and* $v^\dagger = stop - \lim_{N\to\infty} u_1^\dagger u_2^\dagger \dots u_N^\dagger$ *be adapted isometries on* $\tilde{\mathcal{K}} = \bigotimes_{n=0}^{\infty} \mathcal{K}_n$, *both given by an LPR. Fix* $N \in \mathbb{N}_0$. *There are subspaces* $\mathcal{G}^N_{N+1}, \mathcal{G}^N_N, \dots, \mathcal{G}^N_0$ *and* $(\mathcal{G}^N_{N+1})^\dagger, (\mathcal{G}^N_N)^\dagger, \dots, (\mathcal{G}^N_0)^\dagger$ *as defined above.*
Then the following assertions are equivalent:

(1) $v\,|_{\mathcal{K}_{[0,N]}} = v^\dagger\,|_{\mathcal{K}_{[0,N]}}$
(2) *There exist uniquely determined unitaries* $w_m : \mathcal{G}^N_m \to (\mathcal{G}^N_m)^\dagger$ *such that for all* $n = 1, \dots, N+1$ *and all* $\xi^\dagger \in \mathcal{K}_{n-1} \otimes (\mathcal{G}^N_n)^\dagger$ *we have*

$$u_n^\dagger\, \xi^\dagger = w_{n-1} u_n\, w_n^*\, \xi^\dagger$$

(with $w_0 := \mathbb{1}$ *on* $\mathcal{K}_0$ *and* $w_{N+1} := \mathbb{1}$ *on* $\mathbb{C}\Omega_{N+1}$*).*

Note that from $u_n\, \Omega_{n-1} \otimes \Omega_n = \Omega_{n-1} \otimes \Omega_n$ we get $w_n \Omega_n = \Omega_n$ for all n.

Proof: (2) $\Rightarrow$ (1) : If $\xi^\dagger \in \mathcal{K}_{[0,N]}$ ($\simeq \mathcal{K}_{[0,N]} \otimes \mathcal{G}^N_{N+1}$) then

$$\begin{aligned}
v^\dagger \xi^\dagger &= u_1^\dagger \dots u_{N+1}^\dagger\, \xi^\dagger = u_1^\dagger \dots u_N^\dagger\, w_N\, u_{N+1}\, \xi^\dagger \\
&= u_1^\dagger \dots u_{N-1}^\dagger\, (w_{N-1} u_N w_N^*)\, w_N\, u_{N+1}\, \xi^\dagger \\
&\quad \big(\text{because } w_N\, u_{N+1}\, \xi^\dagger \in \mathcal{K}_{[0,N-1]} \otimes (\mathcal{G}^N_N)^\dagger \otimes \mathcal{K}_{N+1}\big) \\
&= u_1^\dagger \dots u_{N-1}^\dagger\, w_{N-1}\, u_N\, u_{N+1}\, \xi^\dagger \\
&\quad \dots \\
&\big(\text{for } n = 1, \dots, N : w_n\, u_{n+1} \dots u_{N+1}\, \xi^\dagger \in \mathcal{K}_{[0,n-1]} \otimes (\mathcal{G}^N_n)^\dagger \otimes \mathcal{K}_{[n+1,N+1]}\big) \\
&= u_1^\dagger\, w_1\, u_2 \dots u_{N+1}\, \xi^\dagger = u_1\, w_1^*\, w_1\, u_2 \dots u_{N+1}\, \xi^\dagger \\
&= u_1 \dots u_{N+1}\, \xi^\dagger = v\, \xi^\dagger .
\end{aligned}$$

(1) $\Rightarrow$ (2) : Given $u_1^\dagger \dots u_{N+1}^\dagger\, \xi^\dagger = u_1 \dots u_{N+1}\, \xi^\dagger$ for all $\xi^\dagger \in \mathcal{K}_{[0,N]}$ ($\simeq \mathcal{K}_{[0,N]} \otimes \mathcal{G}^N_{N+1}$), we can apply Lemma 3.4.6 with $\mathcal{H}_0 = \mathcal{K}_{[0,N-1]}$, $\mathcal{H}_1 = \mathcal{K}_N$, $\mathcal{H}_2 = \mathcal{K}_{N+1}$, $u_{01}^\dagger = u_1^\dagger \dots u_N^\dagger$, $u_{12}^\dagger = u_{N+1}^\dagger$, $u_{01} = u_1 \dots u_N$, $u_{12} = u_{N+1}$. The subspace $\mathcal{G}_2 \subset \mathcal{H}_2$ occurring in the assumptions of Lemma 3.4.6 is replaced by the subspace $\mathcal{G}^N_{N+1} = (\mathcal{G}^N_{N+1})^\dagger = \mathbb{C}\Omega_{N+1} \subset \mathcal{K}_{N+1}$. The subspaces $\mathcal{G}_1$, $\mathcal{G}_1^\dagger$ of Lemma 3.4.6 are also relabelled and are now called $\mathcal{G}^N_N$, $(\mathcal{G}^N_N)^\dagger$, as defined above. Applying Lemma 3.4.6 then gives us a unique unitary $w_N : \mathcal{G}^N_N \to (\mathcal{G}^N_N)^\dagger$ such that $u_{N+1}^\dagger\, \xi^\dagger = w_N\, u_{N+1}\, \xi^\dagger$ for $\xi^\dagger \in \mathcal{K}_N \otimes \mathcal{G}^N_{N+1}$. Equivalently $w_N^*\, u_{N+1}^\dagger\, \xi^\dagger = u_{N+1}\, w_{N+1}^*\, \xi^\dagger$ (because $w_{N+1} = \mathbb{1}$), which is (2)

for $n = N + 1$.
Also by Lemma 3.4.6 we have for all $\eta^\dagger \in \mathcal{K}_{[0,N-1]} \otimes (\mathcal{G}_N^N)^\dagger$:

$$u_1^\dagger \dots u_N^\dagger \, \eta^\dagger \;=\; u_1 \dots u_N \, w_N^* \, \eta^\dagger.$$

Therefore (if $N > 1$) we can apply Lemma 3.4.6 again, now with $\mathcal{H}_0 = \mathcal{K}_{[0,N-2]}$, $\mathcal{H}_1 = \mathcal{K}_{[0,N-1]}$, $\mathcal{H}_2 = \mathcal{K}_N$, $u_{01}^\dagger = u_1^\dagger \dots u_{N-1}^\dagger$, $u_{12}^\dagger = u_N^\dagger$, $u_{01} = u_1 \dots u_{N-1}$, $u_{12} = u_N \, w_N^*$ and with the subspace $(\mathcal{G}_N^N)^\dagger \subset \mathcal{K}_N$ replacing $\mathcal{G}_2$. We find a unitary $w_{N-1} : \mathcal{G}_{N-1}^N \to (\mathcal{G}_{N-1}^N)^\dagger$ such that

$$u_N^\dagger \, \xi^\dagger \;=\; w_{N-1} \, u_N \, w_N^* \, \xi^\dagger \quad \textit{for } \xi^\dagger \in K_{N-1} \otimes (\mathcal{G}_N^N)^\dagger$$

and further

$$u_1^\dagger \dots u_{N-1}^\dagger \, \eta^\dagger \;=\; u_1 \dots u_{N-1} \, w_{N-1}^* \, \eta^\dagger \quad \textit{for } \xi^\dagger \in \mathcal{K}_{[0,N-2]} \otimes (\mathcal{G}_{N-1}^N)^\dagger.$$

Iterating backwards we find all the statements in (2). □

3.4.8 Minimal Subspaces for v

Lemma: *The following equivalent properties define closed* **subspaces** $\mathcal{G}_\mathbf{n} \subset \mathcal{K}_n$ *(for $n \in \mathbb{N}_0$):*

(1) $\mathcal{G}_n \subset \mathcal{K}_n$ is minimal (see A.3.2) for the inclusion $v_{[n} \mathcal{K}_{[n,\infty)} \subset \mathcal{K}_{[n,\infty)}$, where

$$v_{[n} \;=\; \textit{stop} - \lim_{N \to \infty} u_{n+1} \dots u_{n+N}.$$

(2) $\mathcal{G}_n$ is the closure of $\bigcup_{N \geq n} \mathcal{G}_n^N$.

Proof: Note that $\mathcal{G}_n^N$ is increasing if N increases. One can check from the recursive definition in 3.4.7 (and using A.3.2) that for $N \geq n$ the subspace $\mathcal{G}_n^N \subset \mathcal{K}_n$ is minimal for $v_{[n}\mathcal{K}_{[n,N]} \subset \mathcal{K}_{[n,\infty)}$ and it is therefore contained in the minimal space in (1). Conversely note that $\bigcup_{N \geq n} \mathcal{K}_{[n,N]}$ is dense in $\mathcal{K}_{[n,\infty)}$, implying that

$$v_{[n}\mathcal{K}_{[n,\infty)} \subset \left(\textit{closure of } \bigcup_{N \geq n} \mathcal{G}_n^N \right) \otimes \mathcal{K}_{[n+1,\infty)}. \qquad \square$$

Intuitively, by replacing $\mathcal{K}_n$ by $\mathcal{G}_n$ we discard vectors which take no part in the tensor product expansion of the range of v. In particular:

$$v(\mathcal{K}_0 \otimes \mathcal{K}_1 \otimes \dots) \subset \mathcal{G}_0 \otimes \mathcal{G}_1 \otimes \dots.$$

Note that if $v_{[n}$ is unitary on $\mathcal{K}_{[n,\infty)}$ then $\mathcal{G}_n = \mathcal{K}_n$ (and $\mathcal{G}_m = \mathcal{K}_m$ for all $m < n$). However the converse is not true: For example $u_{n+1} = 1\!\!1$ already implies $\mathcal{G}_n = \mathcal{K}_n$. Compare Section 3.5.

3.4.9 Non-uniqueness of Product Representations: Global Result

Theorem:
Let $v = stop - \lim_{N\to\infty} u_1 u_2 \ldots u_N$ and $v^\dagger = stop - \lim_{N\to\infty} u_1^\dagger u_2^\dagger \ldots u_N^\dagger$ be adapted isometries on $\tilde{\mathcal{K}} = \bigotimes_{n=0}^\infty \mathcal{K}_n$, both given by an LPR. The following assertions are equivalent:

(1) $v \;=\; v^\dagger$

(2) *There exists a uniquely determined sequence of unitaries $(w_m)_{m=1}^\infty$ with $w_m : \mathcal{G}_m \to \mathcal{G}_m^\dagger$ and such that for all $n \in \mathbb{N}$ and all $\xi^\dagger \in \mathcal{K}_{n-1} \otimes \mathcal{G}_n^\dagger$ we have*

$$u_n^\dagger \, \xi^\dagger \;=\; w_{n-1}\, u_n\, w_n^*\, \xi^\dagger$$

(with $w_0 := 1\!\!1$ on $\mathcal{K}_0$).

Proof: Note that $v \;=\; v^\dagger$ if and only if $v|_{\mathcal{K}_{[0,N]}} \;=\; v^\dagger|_{\mathcal{K}_{[0,N]}}$ for all N. Applying Proposition 3.4.7 for all N we find that w_n is defined on $\mathcal{G}_n^N$ for all $N \geq n$ and can be extended to the closure $\mathcal{G}_n$. $\square$

3.4.10 Discussion

Let us add some remarks. Comparing with our abstract considerations about the category of adapted endomorphisms after Definition 3.2.5, we have achieved here a much more detailed understanding of our more concretely given adapted isometries. In fact, we have exactly determined the non-uniqueness noticed in 3.2.5d, in terms of the factors. The occurrence of the w_n is a very natural phenomenon. If, for example, the spaces $\mathcal{K}_n$ are finite-dimensional, then $w_n : \mathcal{G}_n \to \mathcal{G}_n^\dagger$ interpreted as a partial isometry can always be extended to a unitary $w_n \in \mathcal{B}(\mathcal{K}_n)$ (otherwise, modifications of $\mathcal{K}_n$ make this possible). Thus the non-uniqueness takes the form of a kind of gauge transformation of a $G-$bundle with G being the unitary group of $\mathcal{K}$, which makes the invariants we are looking for a kind of gauge invariants. Because our adapted isometries are not unitary in general, they 'feel' only restrictions of these transformations, and in the theorem we have determined exactly the relevant subspaces $\mathcal{G}_n$.

We can also look at u_n as a map from $\mathcal{K}_{n-1} \otimes \mathcal{G}_n$ to $\mathcal{G}_{n-1} \otimes \mathcal{K}_n$. In general this is only an isometry and need not be surjective. These isometries contain all information about the adapted isometry v. In fact, the formula

$$v = stop - \lim_{N\to\infty} u_1 u_2 \ldots u_N$$

is well defined if the u_n are interpreted as isometries of this kind. This is not an LPR in the sense of Definition 3.4.4 but it comes nearer to the detailed picture given for the isometries in Section 3.1 by using defect spaces.

$$\begin{array}{ccc} \mathcal{K}_{n-1} \otimes \mathcal{G}_n & \xrightarrow{u_n} & \mathcal{G}_{n-1} \otimes \mathcal{K}_n \\ {\scriptstyle 1\otimes w_n}\downarrow & & \downarrow{\scriptstyle w_{n-1}\otimes 1} \\ \mathcal{K}_{n-1} \otimes \mathcal{G}_n^\dagger & \xrightarrow{u_n^\dagger} & \mathcal{G}_{n-1}^\dagger \otimes \mathcal{K}_n \end{array}$$

Defining $v_{n]} := u_1 \ldots u_n|_{\mathcal{K}_{[0,n-1]} \otimes \mathcal{G}_n}$ we have the decomposition $v = v_{n]}\, v_{[n}$ for the adapted isometry v. For Markov processes in a coupling representation we have seen it before: The formula $v = v_1\, r$ in 2.7.4 is a special case with $n = 1$.

3.4.11 Associated Stochastic Maps

Having Theorem 3.4.9 at our disposal, we are now able to check which objects or properties defined in terms of the sequence $(u_n)_{n=1}^\infty$ are invariants. We now construct a very interesting class of examples. It turns out that certain completely positive maps are good candidates because we can use the slight non-uniqueness of the Stinespring representation ('up to unitary equivalence', see A.2.2) to our advantage. With respect to the probabilistic background they represent extended transitions, in the same spirit as extensively discussed in Chapter 1. It is more intuitive here to consider the action on preduals, i.e. on spaces of trace class operators. In the following we use some notation from A.4.

We proceed as follows: For all $n \in \mathbb{N}$ define isometries v_n and $\bar{v}_n$ as restrictions of u_n:

$$\begin{aligned} v_n : \mathcal{K}_{n-1} \simeq \mathcal{K}_{n-1} \otimes \Omega_n &\to \mathcal{G}_{n-1} \otimes \mathcal{K}_n, \\ \bar{v}_n : \mathcal{G}_n \simeq \Omega_{n-1} \otimes \mathcal{G}_n &\to \mathcal{G}_{n-1} \otimes \mathcal{K}_n. \end{aligned}$$

If a different LPR $(u_n^\dagger)_{n=1}^\infty$ of v is used, then using Theorem 3.4.9 we find that

$$v_n^\dagger = w_{n-1}\, v_n, \quad \bar{v}_n^\dagger = w_{n-1}\, \bar{v}_n\, w_n^*.$$

Now we define (for all n) four **associated completely positive maps**.

$$\begin{aligned} C_n : \mathcal{T}(\mathcal{K}_{n-1}) &\to \mathcal{T}(\mathcal{G}_{n-1}) \\ \rho &\mapsto Tr_n(v_n\, \rho\, v_n^*), \\ D_n : \mathcal{T}(\mathcal{K}_{n-1}) &\to \mathcal{T}(\mathcal{K}_n) \\ \rho &\mapsto Tr_{n-1}(v_n\, \rho\, v_n^*), \\ \bar{C}_n : \mathcal{T}(\mathcal{G}_n) &\to \mathcal{T}(\mathcal{K}_n) \\ \rho &\mapsto Tr_{n-1}(\bar{v}_n\, \rho\, \bar{v}_n^*), \\ \bar{D}_n : \mathcal{T}(\mathcal{G}_n) &\to \mathcal{T}(\mathcal{G}_{n-1}) \\ \rho &\mapsto Tr_n(\bar{v}_n\, \rho\, \bar{v}_n^*). \end{aligned}$$

Note also that $v_n(\mathcal{K}_{n-1}) \subset \mathcal{G}_{n-1}^{n-1}$ and thus $C_n(\mathcal{T}(\mathcal{K}_{n-1})) \subset \mathcal{T}(\mathcal{G}_{n-1}^{n-1})$. Similarly for all $m \geq n$ we have $\bar{v}_n(\mathcal{G}_n^m) \subset \mathcal{G}_{n-1}^m$ and thus $\bar{D}_n(\mathcal{T}(\mathcal{G}_n^m)) \subset \mathcal{T}(\mathcal{G}_{n-1}^m)$. The four maps are trace-preserving and hence they are preadjoints of stochastic maps (see A.4.2).

It may be checked that the operators C_1 and D_n for all n are invariants while the others are not. We can take a hint from Section 3.1 to produce invariants from these operators in a systematic way. Define an infinite block matrix whose entries are completely positive maps:

$$\Lambda_v : \bigoplus_{n=0}^{\infty} \mathcal{T}(\mathcal{K}_n) \to \bigoplus_{n=0}^{\infty} \mathcal{T}(\mathcal{K}_n)$$

$$\Lambda_v = \lim_{N\to\infty} \begin{pmatrix} C_1 & \bar{D}_1 \\ D_1 & \bar{C}_1 \end{pmatrix} \begin{pmatrix} C_2 & \bar{D}_2 \\ D_2 & \bar{C}_2 \end{pmatrix} \cdots \begin{pmatrix} C_N & \bar{D}_N \\ D_N & \bar{C}_N \end{pmatrix}.$$

More explicitly, $\begin{pmatrix} C_N & \bar{D}_N \\ D_N & \bar{C}_N \end{pmatrix}$ stands for the infinite block matrix

$$\begin{pmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ & & C_N & \bar{D}_N & & & \\ & & D_N & \bar{C}_N & & & \\ & & & & 1 & & \\ & & & & & 1 & \\ & & & & & & \ddots \end{pmatrix}$$

and Λ_v is an infinite product of the same type as considered in Section 3.1. The limit exists pointwise because the factors with high indices act identically on positions with low indices. Explicitly we have

$$\Lambda_v = \begin{pmatrix} C_1 & \bar{D}_1 C_2 & \bar{D}_1 \bar{D}_2 C_3 & \ldots & \bar{D}_1 \ldots \bar{D}_{n-1} C_n & \ldots \\ D_1 & \bar{C}_1 C_2 & \bar{C}_1 \bar{D}_2 C_3 & \ldots & \bar{C}_1 \bar{D}_2 \ldots \bar{D}_{n-1} C_n & \ldots \\ & D_2 & \bar{C}_2 C_3 & \ldots & \bar{C}_2 \bar{D}_3 \ldots \bar{D}_{n-1} C_n & \ldots \\ & & D_3 & \ldots & \vdots & \ldots \\ & & & & \vdots & \ldots \\ & & & & \bar{C}_{n-1} C_n & \ldots \\ & & & & D_n & \ldots \\ & & & & & \ldots \end{pmatrix}.$$

If $\rho \in \bigoplus_{n=0}^{\infty} \mathcal{T}(\mathcal{K}_n)$ and $\rho_m = p_m \, \rho \, p_m$ with the projection $p_m : \bigoplus_{n=0}^{\infty} \mathcal{K}_n \to \mathcal{K}_m$, then

$$\Lambda_v \rho = \sum_{n,m} (\Lambda_v)_{nm}(\rho_m) = \sum_{n,m} (\Lambda_v)_{nm}(p_m \, \rho \, p_m),$$

which shows that Λ_v is a completely positive map. The use of the subscript v is justified by the following

Proposition: *The operator Λ_v (and thus the array of entries $(\Lambda_v)_{nm}$) is an invariant.*

Proof: Consider unitaries $(u_n)_{n=1}^{\infty}$, $(u_n^{\dagger})_{n=1}^{\infty}$ with $u_n,\ u_n^{\dagger} \in \mathcal{B}(\mathcal{K}_{[n-1,n]})$ and $(w_n)_{n=0}^{\infty}$, $w_n \in \mathcal{B}(\mathcal{G}_n, \mathcal{G}_n^{\dagger})$, $w_0 = \mathbb{1}$. Define $v_n, v_n^{\dagger}, \ldots$ as above. Consider the following statements:

(1) $\forall n \in \mathbb{N} \quad u_n^{\dagger} = w_{n-1}\, u_n\, w_n^* \quad (\text{on } \mathcal{K}_{n-1} \otimes \mathcal{G}_n^{\dagger})$

(2) $\forall n \in \mathbb{N} \quad v_n^{\dagger} = w_{n-1}\, v_n, \quad \bar{v}_n^{\dagger} = w_{n-1}\, \bar{v}_n\, w_n^*$

(3) $\forall n \in \mathbb{N} \quad C_n^{\dagger} = Ad w_{n-1} \circ C_n, \quad D_n^{\dagger} = D_n,$
$\bar{C}_n^{\dagger} = \bar{C}_n \circ Ad w_n^*, \quad \bar{D}_n^{\dagger} = Ad w_{n-1} \circ \bar{D}_n \circ Ad w_n^*$

(3)′ $\forall n \in \mathbb{N} \quad \begin{pmatrix} C_n^{\dagger} & \bar{D}_n^{\dagger} \\ D_n^{\dagger} & \bar{C}_n^{\dagger} \end{pmatrix} = Ad w_{n-1} \circ \begin{pmatrix} C_n & \bar{D}_n \\ D_n & \bar{C}_n \end{pmatrix} \circ Ad w_n^*$
$\big(\text{on } \mathcal{T}(\mathcal{K}_{n-1}) \oplus \mathcal{T}(\mathcal{G}_n^{\dagger})\ \big)$

(4) $\forall n \in \mathbb{N} \quad \prod_{n=1}^{N} \begin{pmatrix} C_n^{\dagger} & \bar{D}_n^{\dagger} \\ D_n^{\dagger} & \bar{C}_n^{\dagger} \end{pmatrix} = \prod_{n=1}^{N} \begin{pmatrix} C_n & \bar{D}_n \\ D_n & \bar{C}_n \end{pmatrix} \circ Ad w_N^*$
$\big(\text{on } \mathcal{T}(\mathcal{K}_0) \oplus \ldots \oplus \mathcal{T}(\mathcal{K}_{N-1}) \oplus \mathcal{T}(\mathcal{G}_N^{\dagger})\ \big)$

(5) $\lim_{N\to\infty} \prod_{n=1}^{N} \begin{pmatrix} C_n^{\dagger} & \bar{D}_n^{\dagger} \\ D_n^{\dagger} & \bar{C}_n^{\dagger} \end{pmatrix} = \lim_{N\to\infty} \prod_{n=1}^{N} \begin{pmatrix} C_n & \bar{D}_n \\ D_n & \bar{C}_n \end{pmatrix}$

If $v = \lim_{N\to\infty} u_1 u_2 \ldots u_N = \lim_{N\to\infty} u_1^{\dagger} u_2^{\dagger} \ldots u_N^{\dagger}$, then by Theorem 3.4.9 we have (1) for a suitable sequence (w_n). From the definitions it is straightforward to check that

$$(1) \Rightarrow (2) \Rightarrow (3) \Leftrightarrow (3)' \Rightarrow (4) \Rightarrow (5).$$

But (5) tells us that Λ_v is an invariant. □

Properties $(1), \ldots, (5)$ in the proof above also help in discussing the question to what extent the adapted isometry v is determined by Λ_v, i.e. how far Λ_v is from being a complete invariant.

$(2) \Rightarrow (1)$ is not valid in general, because u_n is defined on $\mathcal{K}_{n-1} \otimes \mathcal{G}_n$ while v_n and $v_n^{\dagger}$ only give the restrictions to $\mathcal{K}_{n-1} \otimes \Omega_n$ and $\Omega_{n-1} \otimes \mathcal{G}_n$. It is valid however in an important special case, namely if the adapted isometry is constructed from a stationary adapted process as shown in Section 3.3 and u_n is just the extension of an automorphism α_n from an LPR of an adapted algebra endomorphism. In fact, an automorphism $\alpha_n : \mathcal{C}_{n-1} \otimes \mathcal{C}_n \to \mathcal{C}_{n-1} \otimes \mathcal{C}_n$ is determined by its values on $\mathcal{C}_{n-1} \otimes \mathbb{1}$ and $\mathbb{1} \otimes \mathcal{C}_n$.

$(3) \Rightarrow (2)$ fails if the pair (C_n, D_n) does not determine v_n or $(\bar{C}_n, \bar{D}_n)$ does not determine $\bar{v}_n$. See the example in A.4.2. Similarly $(4) \Rightarrow (3)$ and also $(5) \Rightarrow (4)$ may fail. These failures can all be related to the fact that the $2 \times 2-$matrices $\begin{pmatrix} C_n & \bar{D}_n \\ D_n & \bar{C}_n \end{pmatrix}$ occurring in the infinite product are less well-behaved operators than the unitary $2 \times 2-$matrices in Section 3.1. See however 4.3.4.

For later use (in 4.3.4) we remark that (5) $\Rightarrow$ (4) is valid if all the operators $C_n : \mathcal{T}(\mathcal{K}_{n-1}) \to \mathcal{T}(\mathcal{G}_{n-1})$ are surjective. In fact, we have

$$\begin{pmatrix} C_1 & \bar{D}_1 \\ D_1 & \bar{C}_1 \end{pmatrix} \cdot \begin{pmatrix} C_2 & \bar{D}_2 \\ D_2 & \bar{C}_2 \end{pmatrix} \cdots \begin{pmatrix} C_N & \bar{D}_N \\ D_N & \bar{C}_N \end{pmatrix} \quad = \quad \begin{pmatrix} C_1 & \bar{D}_1 C_2 & \ldots & \bar{D}_1 \ldots \bar{D}_N \\ D_1 & \bar{C}_1 C_2 & \ldots & \bar{C}_1 \bar{D}_2 \ldots \bar{D}_N \\ & & \ldots & \ldots \\ & & D_N & \bar{C}_N \end{pmatrix}$$

and analogous for †. In the matrix for Λ_v the right column occurs with an additional factor C_{N+1} or $C^\dagger_{N+1}$ at each entry. We know that $C^\dagger_{N+1} = Adw_N \circ C_{N+1}$. By (5) the entries of Λ_v are given and if as assumed the range of C_{N+1} is $\mathcal{T}(\mathcal{G}_N)$, then on $\mathcal{T}(\mathcal{K}_0) \oplus \ldots \oplus \mathcal{T}(\mathcal{K}_{N-1}) \oplus \mathcal{T}(\mathcal{G}_N)$

$$\prod_{n=1}^{N} \begin{pmatrix} C^\dagger_n & \bar{D}^\dagger_n \\ D^\dagger_n & \bar{C}^\dagger_n \end{pmatrix} \circ Adw_N \;=\; \prod_{n=1}^{N} \begin{pmatrix} C_n & \bar{D}_n \\ D_n & \bar{C}_n \end{pmatrix},$$

which is (4).

3.4.12 Interpretation of Λ_v

It is possible to give a more concrete interpretation of the entries of Λ_v as follows:

Proposition: *If $\rho_m \in \mathcal{T}(\mathcal{K}_m)$ and $n \le m+1$, then*

$$(\Lambda_v)_{nm}(\rho_m) \;=\; Tr_{k\neq n}\big(v|_{\mathcal{K}_m}\, \rho_m\, (v|_{\mathcal{K}_m})^*\big).$$

Here $Tr_{k\neq n}$ denotes partial trace where evaluation of the trace takes place at all positions except n.

Proof: We have $v|_{\mathcal{K}_m} = \bar{v}_1 \ldots \bar{v}_m v_{m+1}$. Assume that $0 < n < m$. Then

$$\begin{aligned} & Tr_{k\neq n}\big(v|_{\mathcal{K}_m} \rho_m (v|_{\mathcal{K}_m})^*\big) \\ &= Tr_{[0,n-1]}\, Tr_{n+1} \ldots Tr_m\, Tr_{m+1}\big(\bar{v}_1 \ldots \bar{v}_m\, v_{m+1}\, \rho_m v^*_{m+1}\, \bar{v}^*_m \ldots \bar{v}^*_1\big) \\ &= Tr_{n-1}\big(\bar{v}_n\, Tr_{n+1}\big[\bar{v}_{n+1} \ldots Tr_m\big(\bar{v}_m\, Tr_{m+1}(v_{m+1}\, \rho_m\, v^*_{m+1})\, \bar{v}^*_m\big) \ldots \bar{v}^*_{n+1}\big]\bar{v}^*_n\big) \\ &= \bar{C}_n \bar{D}_{n+1} \ldots \bar{D}_m C_{m+1}(\rho_m) \;=\; (\Lambda_v)_{nm}(\rho_m). \end{aligned}$$

The computation for $n = 0$ and $n = m$ is similar and left to the reader. We give the case $n = m+1$ which is particularly important in the sequel (see Section 3.6):

$$Tr_{[0,m]}\big(v|_{\mathcal{K}_m}\, \rho_m\, (v|_{\mathcal{K}_m})^*\big) \;=\; Tr_m(v_{m+1}\, \rho_m\, v^*_{m+1}) \;=\; D_{m+1}(\rho_m). \qquad \square$$

We can also see again here why the invariant Λ_v fails to be complete. It disregards entanglement in the sense that it only deals with $\mathcal{T}(\mathcal{K}_n)$ for different n, while there may be problems about the adapted isometry v which make it necessary to consider elements of $\mathcal{T}(\mathcal{K}_{[n,m]})$. Thus the usefulness of this invariant is limited. Nevertheless we shall see an interesting application using the D_n in Section 3.5.

3.4.13 The Associated Master Equation

The adapted isometry $v = stop - \lim_{N\to\infty} u_1 u_2 \dots u_N$ can be interpreted as a coupling $v = u_1\, v_{[1}$, where $v = stop - \lim_{N\to\infty} u_2 \dots u_N$ is an adapted isometry acting on $\mathcal{K}_{[1,\infty)}$. Compared with the coupling representations discussed in 2.1.6, we have lost the shift property of $v_{[1}$ and thus also the Markov property of the whole dynamics. Nevertheless, if $\mathcal{K}_0$ represents a system under observation then there is some interest in the quantities $(R_n)_{n=0}^{\infty}$ defined by

$$\begin{aligned} R_n : \mathcal{T}(\mathcal{K}_0) &\to \mathcal{T}(\mathcal{K}_0) \\ \rho &\mapsto Tr_{[1,\infty)}(v^n\,\rho\,v^{*n}). \end{aligned}$$

Here R_n is a trace-preserving completely positive map, in particular $R_0 = 1\!\!1$. In the physics literature (see for example [Ca99]) formulas giving the change of $R_n(\rho)$ for varying n are called master equations (usually considered with continuous time parameter). To give a factorization we define $(\hat{C}_n)_{n=1}^{\infty}$ by

$$\hat{C}_n := Tr_n \circ Ad(u_n \dots u_1) : \quad \mathcal{T}(\mathcal{K}_{[0,n]}) \to \mathcal{T}(\mathcal{K}_{[0,n-1]}).$$

These are not invariants, we have

$$\hat{C}_n^{\dagger} = Ad \prod_{m=0}^{n-1} w_m \circ \hat{C}_n \circ Ad \prod_{m=0}^{n} w_m^*.$$

Proposition: $R_n = \hat{C}_1 \dots \hat{C}_n|_{\mathcal{T}(\mathcal{K}_0)}$.

Proof: Note that u_n and u_m commute if $|n-m| \geq 2$. Thus for $\xi \in \mathcal{K}_0$ we get

$$\begin{aligned} v^n \xi &= (u_1 \dots u_n)(u_1 \dots u_{n-1}) \dots u_1 \xi \\ &= u_1(u_2 u_1) \dots (u_{n-1} \dots u_1)(u_n \dots u_1)\xi. \end{aligned}$$

It is enough to consider the one-dimensional projection $\rho = p_\xi$ to get:

$$\begin{aligned} Tr_{[1,\infty)}(v^n\,\rho\,v_n^*) &= Tr_1(u_1\, Tr_2(u_2\, u_1 \dots Tr_n(u_n \dots u_1\, \rho\, u_1^* \dots u_n^*) \dots u_1^*\, u_2^*)\, u_1^*) \\ &= \hat{C}_1 \dots \hat{C}_n(\rho). \qquad \square \end{aligned}$$

We add some remarks. With $v_n = u_n|_{\mathcal{K}_{n-1}}$ we have

$$\hat{C}_n|_{\mathcal{T}(\mathcal{K}_0)} \;=\; Tr_n \circ Ad(v_n \dots v_1) \;=\; C_n \circ Ad(v_{n-1} \dots v_1).$$

In the Markovian case when $v_{[1}$ is a shift r, see 2.7.4 and 3.4.10, then $\hat{C}_n|_{\mathcal{T}(\mathcal{K}_0)} = C_1$ and

$$R_n \;=\; \hat{C}_1 \dots \hat{C}_n|_{\mathcal{T}(\mathcal{K}_0)} \;=\; (C_1)^n.$$

This has been observed earlier, see 2.8.3.

In the general case we see that $(u_n)_{n=1}^\infty$ may be seen as a kind of choice sequence for $(R_n)_{n=1}^\infty$, in the sense that $u_1, \ldots, u_{n-1}$ determine $R_1, \ldots, R_{n-1}$ and then choosing u_n determines R_n. Compare 3.1.3. However the factorization for R_n given in the proposition above is not very satisfactory because the domain of $\hat{C}_n$ becomes larger in an exponential way as n grows. To devise more efficient algorithms to compute (R_n) from (u_n) seems to be a difficult task.

3.5 Nonlinear Prediction Errors

3.5.1 The Problem

Recall the very short sketch of linear prediction theory that we have given in 3.1.4. Using the 'additive' product representations on direct sums defined there we have seen that the n-th one-step linear prediction error can be written as a product $d_1 \ldots d_n$, where the d_m occur in the product representation as the defect operators of a choice sequence. Now in Section 3.4 we have developed 'multiplicative' product representations on tensor products which interpreted for stochastic processes also include nonlinear functions of the variables of the process. We shall see that it is indeed possible to derive a product formula for the n-th one-step nonlinear prediction error, using the maps $D_1, \ldots, D_n$ introduced in 3.4.11. The contents of this section are also discussed in [Go03]. For some general considerations about nonlinear prediction the reader may consult [MW59].

3.5.2 Nonlinear Prediction

We state the one-step nonlinear prediction problem for the setting given in 3.3.5, i.e. we have a process $(\xi_n)_{n=0}^\infty$ adapted for a tensor product filtration $(\mathcal{C}_{[0,n]})_{n=0}^\infty \subset \tilde{\mathcal{A}}$. The algebra $\mathcal{C}_{[0,n-1]}$ represents the information available up to time $n-1$. The optimal one-step nonlinear predictor for ξ_n in the mean square sense is an element of $\mathcal{C}_{[0,n-1]}$ with minimal distance to ξ_n in the Hilbert space norm. The abstract solution to this problem is given by $Q_{[0,n-1]}(\xi_n)$, where $Q_{[0,n-1]}$ is the conditional expectation onto $\mathcal{C}_{[0,n-1]}$, compare 3.3.5. The real problem here is to find formulas and algorithms for this in terms of the data actually given to specify the process. In the quantum mechanical case there are additional problems of interpretation, and the question what actually can be measured becomes more delicate. We shall not enter these problems here but concentrate on the n-th **one-step nonlinear prediction error**

$$f_n := \|\xi_n - Q_{[0,n-1]}(\xi_n)\|,$$

Here $\|\cdot\|$ is the norm of the GNS-Hilbert space and it is therefore natural to probe our spatial approach for computing it.

3.5.3 Technical Preparation

If $v : \mathcal{H} \to \mathcal{G} \otimes \mathcal{P}$ is an isometry then we can define the map

$$D : \mathcal{T}(\mathcal{H}) \to \mathcal{T}(\mathcal{P}), \quad \rho \mapsto Tr_{\mathcal{G}}(v\,\rho\,v^*),$$

which is the preadjoint of a stochastic map, see A.4.2.

Lemma: *Let $\mathcal{H}_0, \mathcal{H}_1, \mathcal{H}_2$ be Hilbert spaces, $u : \mathcal{H}_0 \otimes \mathcal{H}_1 \to \mathcal{H}_0 \otimes \mathcal{H}_1$ unitary and $w : \mathcal{H}_1 \to \mathcal{H}_1 \otimes \mathcal{H}_2$ isometric. To simplify notation we also write u for $u \otimes 1\!\!1_{\mathcal{H}_2}$ and w for $1\!\!1_{\mathcal{H}_0} \otimes w$. If we define an isometry $v := uw : \mathcal{H}_0 \otimes \mathcal{H}_1 \to \mathcal{H}_0 \otimes \mathcal{H}_1 \otimes \mathcal{H}_2$, then*

$$D_v \;=\; D_w \circ Tr_{\mathcal{H}_0} : \mathcal{T}(\mathcal{H}_0 \otimes \mathcal{H}_1) \to \mathcal{T}(\mathcal{H}_2)$$

Proof: Assume $\rho \in \mathcal{T}(\mathcal{H}_0 \otimes \mathcal{H}_1)$. Then

$$\begin{aligned} D_w \circ Tr_{\mathcal{H}_0}(\rho) &= Tr_{\mathcal{H}_1}(w\, Tr_{\mathcal{H}_0}(\rho)\, w^*) \\ = Tr_{\mathcal{H}_0 \otimes \mathcal{H}_1}(w\,\rho\,w^*) &= Tr_{\mathcal{H}_0 \otimes \mathcal{H}_1}(u\,w\,\rho\,w^* u^*) \\ = Tr_{\mathcal{H}_0 \otimes \mathcal{H}_1}(v\,\rho\,v^*) &= D_v(\rho). \qquad \square \end{aligned}$$

3.5.4 A Formula for Nonlinear Prediction Errors

Now we shall derive a product formula for the prediction error in the setting of Section 3.4, i.e. we have an adapted isometry (with LPR) $v = stop - \lim_{N\to\infty} u_1 \ldots u_N$ on a tensor product $\tilde{\mathcal{K}}$. We use $v_m = u_m|_{\mathcal{K}_{m-1}}$ and the maps D_m introduced in 3.4.11. Further we denote by $q_{[0,m]}$ the projection onto $\mathcal{K}_{[0,m]}$.

Lemma: *Assume $\xi, \xi' \in \mathcal{K}_0$. Then for all $n \in \mathbb{N}$*

$$Tr_{[0,n-1]}(|\,v^n\xi'\rangle\langle v^n\xi\,|) \;=\; D_n \ldots D_1(|\,\xi'\rangle\langle\xi\,|) \quad \textit{and}$$

$$\langle q_{[0,n-1]}\, v^n\xi,\, q_{[0,n-1]}\, v^n\xi'\rangle \;=\; \langle \Omega_n, D_n \ldots D_1(|\,\xi'\rangle\langle\xi\,|)\,\Omega_n\rangle$$

Proof: To prove the first part we proceed by induction.
The case $n = 1$ is given in A.4.5 (note that $v|_{\mathcal{K}_0} = v_1$). Now for some $n > 1$ assume that

$$Tr_{[0,n-2]}(|\,v^{n-1}\xi'\rangle\langle v^{n-1}\xi\,|) = D_{n-1} \ldots D_1(|\,\xi'\rangle\langle\xi\,|).$$

We have

$$v^n\xi \;=\; u_1 \ldots u_n v^{n-1}\xi \;=\; u_1 \ldots u_{n-1} v_n v^{n-1}\xi$$

(and the same for ξ'). We can now apply Lemma 3.5.3 with $\mathcal{H}_0 = \mathcal{K}_{[0,n-2]}$, $\mathcal{H}_1 = \mathcal{K}_{n-1}$, $\mathcal{H}_2 = \mathcal{K}_n$, $u = u_1 \ldots u_{n-1}$, $w = v_n$. We get

$$\begin{aligned} Tr_{[0,n-1]}(|\,v^n\xi'\rangle\langle v^n\xi\,|) &= D_{uw}(|\,v^{n-1}\xi'\rangle\langle v^{n-1}\xi\,|) \\ = D_n\big(Tr_{[0,n-2]}(|\,v^{n-1}\xi'\rangle\langle v^{n-1}\xi\,|)\big) &= D_n D_{n-1} \ldots D_1(|\,\xi'\rangle\langle\xi\,|). \end{aligned}$$

The second part follows from the first by a direct computation, compare the proof of A.4.5. □

For $\xi \in \mathcal{K}_0$ the $n-$th (one-step nonlinear) prediction error is given by

$$f_n(\xi) \;:=\; \|v^n \xi \;-\; q_{[0,n-1]} v^n \xi\| \;=\; \left(\|\xi\|^2 - \|q_{[0,n-1]} v^n \xi\|^2\right)^{\frac{1}{2}}.$$

Theorem: *For all unit vectors $\xi \in \mathcal{K}_0$ and $n \in \mathbb{N}$*

$$f_n(\xi)^2 \;+\; \langle \Omega_n, D_n \ldots D_1(p_\xi)\, \Omega_n \rangle \;=\; 1.$$

Here $p_\xi = |\,\xi\rangle\langle\xi\,|$ is the one-dimensional projection onto $\mathbb{C}\xi$.

Proof: This is immediate from the second part of the preceding lemma. □

The formula for prediction errors given in the theorem shows that the operators D_m measure a kind of nonlinear defect in a similar way as the numbers d_m in Section 3.1 are defect operators in the usual sense. In particular our result can be applied to commutative processes and it yields a 'noncommutative' formula also in this case. In this respect it is a nice encouragement for the spatial approach and the concept of extended transition which we are going to elaborate here. An elementary example has been computed in Section 4.2.

3.5.5 Asymptotics

As a direct application of the product formula for the prediction error we want to analyze the behaviour for large time ($n \to \infty$). Let us start with some definitions.

A stationary stochastic process given by $\mathcal{K}_0$ and an adapted isometry v on $\tilde{\mathcal{K}}$ is called **deterministic** with respect to the filtration $(\mathcal{K}_{[0,n]})_{n=0}^{\infty}$ if

$$f_\infty(\xi) := \lim_{n\to\infty} f_n(\xi) = 0 \quad \text{for all } \xi \in \mathcal{K}_0.$$

Of course this should not be confused with the linear concept of determinism mentioned in 3.1.4 which only for Gaussian processes yields the same results.

Let $(\tilde{D}_n)$ be a sequence of maps with $\tilde{D}_n : \mathcal{T}_+^1(\mathcal{K}_0) \to \mathcal{T}_+^1(\mathcal{K}_n)$ for all n. If (Ω_n) is absorbing for all sequences $(\tilde{D}_n(\rho))$ with $\rho \in \mathcal{T}_+^1(\mathcal{K}_0)$ (in the sense of A.5.3, i.e.

$$\lim_{n\to\infty} \left(\tilde{D}_n(\rho) - p_{\Omega_n}\right) = 0$$

weak or with respect to the trace norm), then we call (Ω_n) **absorbing** for $(\tilde{D}_n)$.

Proposition: *For a unit vector $\xi \in \mathcal{K}_0$ the following assertions are equivalent:*

(1) $f_\infty(\xi) = \lim_{n\to\infty} f_n(\xi) = 0$,
(2) $\lim_{n\to\infty} \big(D_n \dots D_1(p_\xi) - p_{\Omega_n}\big) = 0$

A stationary stochastic process given by $\mathcal{K}_0$ and v is deterministic with respect to the filtration $(\mathcal{K}_{[0,n]})_{n=0}^\infty$ if and only if (Ω_n) is absorbing for $(\tilde{D}_n := D_n \dots D_1)$.

Proof: Take the formula for $f_n(\xi)$ in Theorem 3.5.4 and then apply A.3.5 with $\rho_n = D_n \dots D_1(p_\xi)$. The second part then follows by approximating arbitrary density matrices by convex combinations of one-dimensional projections. □

Note that the sequence $(f_n(\xi))$ of prediction errors is in any case a non-increasing sequence of non-negative numbers and thus there is always a limit $f_\infty(\xi) := \lim\limits_{n\to\infty} f_n(\xi)$. This is immediate because the time interval used for prediction increases and there is more and more information available. The proposition gives a criterion for this limit to be zero, i.e. for prediction becoming perfect for $n \to \infty$. This is a nonlinear version of the results discussed in 3.1.4.

3.5.6 Homogeneous Case

We say that an adapted isometry v with LPR is **homogeneous** if all Hilbert spaces $\mathcal{K}_n$ can be identified with a Hilbert space $\mathcal{K}$, all unit vectors $\Omega_n \in \mathcal{K}_n$ with a unit vector $\Omega_\mathcal{K} \in \mathcal{K}$ and all unitaries $u_n \in \mathcal{B}(\mathcal{K}_{[n-1,n]})$ occurring in the LPR $v = \mathit{stop} - \lim_{N\to\infty} u_1 u_2 \dots u_N$ with a unitary $u \in \mathcal{B}(\mathcal{K} \otimes \mathcal{K})$.

In the homogeneous case we can further identify all operators D_m with an operator $D : \mathcal{T}(\mathcal{K}) \to \mathcal{T}(\mathcal{K})$ and the absorption property appearing in Proposition 3.5.5 reduces to the one used in A.5.2, i.e. the state given by $\Omega_\mathcal{K}$ is absorbing for the stochastic map D^*. It is much easier to do computations in the homogeneous case. See A.5.2 for criteria and the example in Section 4.2.

This kind of absorption has also arisen in 2.8.4, and this coincidence yields an interesting correspondence between Markov processes in a coupling representation and homogeneous stationary processes. Given a unitary $u = u_1$ on $\mathcal{K} \otimes \mathcal{K} = \mathcal{K}_0 \otimes \mathcal{K}_1$ we have the isometry $v_1 = u|_{\mathcal{K}\otimes\Omega_\mathcal{K}}$ and the maps $C : \rho \mapsto Tr_1(v_1\, \rho\, v_1^*)$ and $D : \rho \mapsto Tr_0(v_1\, \rho\, v_1^*)$, see A.4 for a detailed analysis. The Markov process with time evolution $Ad\, u_1 \circ \sigma$ (see 2.8.4) is asymptotically complete if and only if $\Omega_\mathcal{K}$ is absorbing for $C = Z'_*$. The homogeneous stationary process specified by u is deterministic if and only if $\Omega_\mathcal{K}$ is absorbing for D. We see that the mathematical structure behind these phenomena is the same. To extend this **C-D-correspondence** to non-homogeneous stationary processes means to consider Markov processes whose transitions vary in time.

3.6 The Adjoint of an Adapted Isometry

3.6.1 Associated Stochastic Maps for the Adjoint

We want to examine the adjoint v^* of an adapted isometry v with LPR given by $v = stop - \lim_{N\to\infty} u_1 u_2 \ldots u_N$. It follows that $v^* = wop - \lim_{N\to\infty} u_N^* \ldots u_1^*$ which in general is neither adapted nor an isometry.

For all $n \in \mathbb{N}$ define isometries v'_n and $\bar{v}'_n$ as restrictions of u_n^*:

$$\begin{aligned} v'_n &: \mathcal{K}_{n-1} \simeq \mathcal{K}_{n-1} \otimes \Omega_n \to \mathcal{K}_{n-1} \otimes \mathcal{K}_n, \\ \bar{v}'_n &: \quad \mathcal{K}_n \simeq \Omega_{n-1} \otimes \mathcal{K}_n \to \mathcal{K}_{n-1} \otimes \mathcal{K}_n. \end{aligned}$$

To identify corresponding invariants, it is useful to determine how these isometries transform under changes of the product representation. If we have another LPR $v = stop - \lim_{N\to\infty} u_1^\dagger u_2^\dagger \ldots u_N^\dagger$ then by Theorem 3.4.9 we get $u_n^\dagger = w_{n-1} u_n w_n^*$ on $\mathcal{K}_{n-1} \otimes \mathcal{G}_n^\dagger$. We use the notation $g_n := 1\!\!1 \otimes p_{\mathcal{G}_n}$, where $1\!\!1$ acts on spaces that will be clear from the context and $p_{\mathcal{G}_n}$ is the projection from $\mathcal{K}_n$ onto $\mathcal{G}_n$. Similarly $g_n^\dagger := 1\!\!1 \otimes p_{\mathcal{G}_n^\dagger}$.

Lemma: *The isometries v'_n and $\bar{v}'_n$ transform as follows:*

$$\begin{aligned} ({v'_n}^\dagger)^*\, g_n^\dagger &= w_{n-1} \big[(v'_n)^*\, g_n\big] w_n^* \\ ((\bar{v}'_n)^\dagger)^*\, g_n^\dagger &= \big[(\bar{v}'_n)^*\, g_n\big] w_n^* \end{aligned}$$

Proof: Writing i_{n-1} for the embedding $\mathcal{K}_{n-1} \to \mathcal{K}_{n-1} \otimes \mathcal{K}_n$ so that $q_{n-1} = i_{n-1}^*$ is the projection from $\mathcal{K}_{n-1} \otimes \mathcal{K}_n$ onto $\mathcal{K}_{n-1}$, we find that $v'_n = u_n^* i_{n-1}$ and $(v'_n)^* = q_{n-1} u_n$. Thus

$$({v'_n}^\dagger)^*\, g_n^\dagger \;=\; q_{n-1}\, u_n^\dagger\, g_n^\dagger \;=\; q_{n-1}\, w_{n-1}\, u_n\, w_n^*\, g_n^\dagger \;=\; w_{n-1}\, q_{n-1}\, u_n\, g_n\, w_n^*,$$

where w_n is interpreted as a partial isometry with initial space $\mathcal{K}_{n-1} \otimes \mathcal{G}_n$ and final space $\mathcal{K}_{n-1} \otimes \mathcal{G}_n^\dagger$. We conclude that

$$({v'_n}^\dagger)^*\, g_n^\dagger \;=\; w_{n-1} \big[(v'_n)^*\, g_n\big] w_n^*.$$

Similarly we have $(\bar{v}'_n)^* \;=\; q_n\, u_n$ and

$$((\bar{v}'_n)^\dagger)^*\, g_n^\dagger \;=\; q_n\, u_n^\dagger\, g_n^\dagger \;=\; q_n\, w_{n-1}\, u_n\, w_n^*\, g_n^\dagger \;=\; q_n\, u_n\, g_n\, w_n^*,$$

$$((\bar{v}'_n)^\dagger)^*\, g_n^\dagger \;=\; \big[(\bar{v}'_n)^*\, g_n\big] w_n^*.$$

□

Analogous to 3.4.11, this suggests to define the following **associated completely positive maps**:

$$\begin{aligned} E_n : \mathcal{T}(\mathcal{G}_{n-1}) &\to \mathcal{T}(\mathcal{K}_{n-1}) \\ \rho &\mapsto Tr_n(g_n\, v_n'\ \rho\ (v_n')^*\, g_n), \end{aligned}$$

$$\begin{aligned} F_n : \mathcal{T}(\mathcal{G}_{n-1}) &\to \mathcal{T}(\mathcal{G}_n) \simeq g_n \mathcal{T}(\mathcal{K}_n) g_n \\ \rho &\mapsto Tr_{n-1}(g_n\, v_n'\, \rho\, (v_n')^*\, g_n) \ = \ g_n\, Tr_{n-1}(v_n'\, \rho\, (v_n')^*)\, g_n, \end{aligned}$$

$$\begin{aligned} \bar{E}_n : \mathcal{T}(\mathcal{K}_n) &\to \mathcal{T}(\mathcal{G}_n) \simeq g_n \mathcal{T}(\mathcal{K}_n) g_n \\ \rho &\mapsto Tr_{n-1}(g_n\, \bar{v}_n'\, \rho\, (\bar{v}_n')^*\, g_n) \ = \ g_n\, Tr_{n-1}(\bar{v}_n'\, \rho\, (\bar{v}_n')^*)\, g_n, \end{aligned}$$

$$\begin{aligned} \bar{F}_n : \mathcal{T}(\mathcal{K}_n) &\to \mathcal{T}(\mathcal{K}_{n-1}) \\ \rho &\mapsto Tr_n(g_n\, \bar{v}_n'\, \rho\, (\bar{v}_n')^*\, g_n). \end{aligned}$$

Inserting the g_n is necessary to control the influence of different product representations, as analyzed above. It has the effect that the trace is not preserved. On positive elements these maps are $Tr-$decreasing. We shall see however that for some purposes the g_n can be ignored and then we shall sometimes use the notation $E_n,\ F_n,\ \bar{E}_n,\ \bar{F}_n$ also for the quantities defined with the g_n omitted.

We can now proceed similar as in 3.4.11. We define

$$\Lambda_{v^*} : \bigoplus_{n=0}^{\infty} \mathcal{T}(\mathcal{K}_n) \to \bigoplus_{n=0}^{\infty} \mathcal{T}(\mathcal{K}_n)$$

$$\Lambda_{v^*} = \lim_{N\to\infty} \begin{pmatrix} E_N & \bar{F}_N \\ F_N & \bar{E}_N \end{pmatrix} \cdot \begin{pmatrix} E_{N-1} & \bar{F}_{N-1} \\ F_{N-1} & \bar{E}_{N-1} \end{pmatrix} \cdots \begin{pmatrix} E_1 & \bar{F}_1 \\ F_1 & \bar{E}_1 \end{pmatrix},$$

where $\begin{pmatrix} E_n & \bar{F}_n \\ F_n & \bar{E}_n \end{pmatrix}$ stands for the infinite block matrix

$$\begin{pmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ & & E_n & \bar{F}_n & & & \\ & & F_n & \bar{E}_n & & & \\ & & & & 1 & & \\ & & & & & 1 & \\ & & & & & & \ddots \end{pmatrix}$$

and the limit exists because each position is nontrivially acted upon only by finitely many factors.

Λ_{v^*} is a completely positive map on $\bigoplus_{n=0}^{\infty} \mathcal{T}(\mathcal{K}_n)$, explicitly:

$$\Lambda_{v^*} = \begin{pmatrix} E_1 & \bar{F}_1 & & & \\ E_2 F_1 & E_2 \bar{E}_1 & \bar{F}_2 & & \\ E_3 F_2 F_1 & E_3 F_2 \bar{E}_1 & E_3 \bar{E}_2 & \bar{F}_3 & \\ \dots & \dots & \dots & \dots & \ddots \end{pmatrix}.$$

Proposition: *The operator Λ_{v^*} (and thus the array of entries $(\Lambda_{v^*})_{nm}$) is an invariant.*

Proof: Using the lemma above we infer

$$\begin{aligned} E_n^\dagger &= E_n \circ Adw_{n-1}^* \\ F_n^\dagger &= Adw_n \circ F_n \circ Adw_{n-1}^* \\ \bar{E}_n^\dagger &= Adw_n \circ \bar{E}_n \\ \bar{F}_n^\dagger &= \bar{F}_n \end{aligned}$$

or equivalently

$$\begin{pmatrix} E_n^\dagger & \bar{F}_n^\dagger \\ F_n^\dagger & \bar{E}_n^\dagger \end{pmatrix} = Adw_n \circ \begin{pmatrix} E_n & \bar{F}_n \\ F_n & \bar{E}_n \end{pmatrix} \circ Adw_{n-1}^*.$$

We conclude that

$$\begin{pmatrix} E_N^\dagger & \bar{F}_N^\dagger \\ F_N^\dagger & \bar{E}_N^\dagger \end{pmatrix} \cdots \begin{pmatrix} E_1^\dagger & \bar{F}_1^\dagger \\ F_1^\dagger & \bar{E}_1^\dagger \end{pmatrix} = Adw_N \circ \begin{pmatrix} E_N & \bar{F}_N \\ F_N & \bar{E}_N \end{pmatrix} \cdots \begin{pmatrix} E_1 & \bar{F}_1 \\ F_1 & \bar{E}_1 \end{pmatrix},$$

and in the limit $N \to \infty$ we have invariance. □

3.6.2 Interpretation of Λ_{v^*}

To understand more clearly the meaning of the invariant Λ_{v^*} we can proceed as follows. We consider the case $n > m$ and leave the small modifications necessary in the other cases to the reader.

First assume that v is unitary. Then for all l we have $\mathcal{G}_l = \mathcal{K}_l,\ g_l = \mathbb{1}$ and $v_{[l}$ is unitary. Thus for $\rho_m \in \mathcal{T}(\mathcal{K}_m)$ we get

$$\begin{aligned} Tr_{k\neq n}(v^* \rho_m v) &= Tr_{k\neq n}(v^*_{[n+1} u^*_{n+1} \dots u^*_m \rho_m u_m \dots u_{n+1} v_{[n+1}) \\ &= Tr_{k\neq n}(u^*_{n+1} \dots u^*_m \rho_m u_m \dots u_{n+1}) \end{aligned}$$

(and by evaluating thev partial traces successively)

$$\begin{aligned} &= Tr_{n+1}\big(u^*_{n+1} Tr_{n-1}\big[u^*_n \dots Tr_m\big(u^*_{m+1} Tr_{m-1}(u^*_m \rho_m u_m) u_{m+1}\big) \dots u_n\big] u_{n+1}\big) \\ &= E_{n+1} F_n \dots F_{m+1} \bar{E}_m(\rho_m) = (\Lambda_{v^*})_{nm}(\rho_m). \end{aligned}$$

In this case all the maps are $Tr-$preserving.

If v is not unitary then also $v_{[n+1}$ is not unitary and the equality above is not valid. From $u_l(\mathcal{K}_{l-1} \otimes \mathcal{G}_l) \subset \mathcal{G}_{l-1} \otimes \mathcal{K}_l$ we find that $g_l\, u_l^*\, g_{l-1}^\perp = 0$ for all l and

$$g_{n+1}\, u_{n+1}^*\, u_n^* \dots u_m^* \;=\; g_{n+1}\, u_{n+1}^*\, g_n\, u_n^* \dots g_m\, u_m^*.$$

Thus in computing a product such as $E_{n+1} F_n \dots F_{m+1} \bar{E}_m$ it is legitimate to drop the g_l from the defining formulas for these maps except the highest one, here g_{n+1}. In particular

$$Tr_{k \neq n}(g_{n+1}\, u_{n+1}^* \dots u_m^*\, \rho_m\, u_m \dots u_{n+1}\, g_{n+1}) \;=\; (\Lambda_{v^*})_{nm}(\rho_m).$$

Consider the decomposition $v = v_{n+1]}\; v_{[n+1}$, see 3.4.10. Here $v_{n+1]} = u_1 \dots u_{n+1}\, g_{n+1}$ and we get

$$Tr_{k \neq n}(v_{n+1]}^*\, \rho_m\, v_{n+1]}) \;=\; (\Lambda_{v^*})_{nm}(\rho_m).$$

Decreasing of the trace reflects the fact that $v_{n+1]}$ is not unitary if $g_{n+1} \neq 1\!\!1$. Because $v_{[n+1}^*\, g_{n+1}^\perp = 0$ we have $g_{n+1}^\perp \tilde{\mathcal{K}} \subset Ker(v_{[n+1}^*)$. Compare 3.4.10.

3.6.3 Invariants Related to the Range of v

We want to find criteria for an adapted isometry to be unitary.

Proposition: *Let $\xi \in \mathcal{K}_0$ be a unit vector. Then for all $n \in \mathbb{N}$*

$$\|p_{v\mathcal{K}_{[0,n-1]}}\, \xi\|^2 \;=\; \langle \Omega_n, F_n \dots F_1(p_\xi)\, \Omega_n \rangle,$$

where $p_{v\mathcal{K}_{[0,n-1]}}$ and p_ξ are the projections onto $v\mathcal{K}_{[0,n-1]}$ and $\mathbb{C}\xi$.

Remark: That the right hand side is an invariant can also be seen from $F_n^\dagger \;=\; Adw_n \circ F_n \circ Adw_{n-1}^*$ together with $w_0 = 1\!\!1,\; w_n \Omega_n = \Omega_n$.

Proof: We denote by $q_{[0,n-1]}$ the projection onto $\mathcal{K}_{[0,n-1]}$. Because $v|_{\mathcal{K}_{[0,n-1]}} = u_1 \dots u_n|_{\mathcal{K}_{[0,n-1]}}$ we find that

$$p_{v\mathcal{K}_{[0,n-1]}} \;=\; v\, q_{[0,n-1]}\, v^* \;=\; u_1 \dots u_n\, q_{[0,n-1]}\, u_n^* \dots u_1^*,$$

$$\|p_{v\mathcal{K}_{[0,n-1]}}\, \xi\| \;=\; \|q_{[0,n-1]}\, u_n^* \dots u_1^*\, \xi\|.$$

Using A.4.5 it follows that

$$\begin{aligned}
&\|q_{[0,n-1]}\, u_n^* \dots u_1^*\, \xi\|^2 \;=\; \langle \Omega_n, Tr_{[0,n-1]}(u_n^* \dots u_1^*\, p_\xi\, u_1 \dots u_n)\, \Omega_n \rangle \\
&= \langle \Omega_n, Tr_{n-1}\big(u_n^*\, Tr_{n-2}(u_{n-1}^* \dots Tr_0(u_1^*\, p_\xi\, u_1) \dots u_{n-1}) u_n\big)\, \Omega_n \rangle \\
&= \langle \Omega_n, F_n \dots F_1(p_\xi)\, \Omega_n \rangle. \qquad \square
\end{aligned}$$

For the last equality of the proof recall from 3.6.1 that in computing products such as $F_n \dots F_1$ all the projections g_m may be omitted except g_n. But $g_n \Omega_n = \Omega_n$, thus for computing the inner product above g_n can be omitted also. In other words: To apply the proposition it is not necessary to know the subspaces

$\mathcal{G}_m$, and for this purpose it is legitimate to drop the projections g_m in the definition of the F_m. For the rest of Section 3.6 this simplification is always valid.

Corollary: *Let $\xi \in \mathcal{K}_0$ be a unit vector. Then*

(a) $\|v^*\xi\|^2 = \lim_{n\to\infty}\langle \Omega_n, F_n \dots F_1(p_\xi)\,\Omega_n\rangle$
(b) $\|v^*\xi\| = \|\xi\| = 1$ *if and only if for* $n \to \infty$

$$F_n \dots F_1(p_\xi) \;-\; p_{\Omega_n} \longrightarrow 0.$$

Here p_{Ω_n} is the projection onto $\mathbb{C}\Omega_n$ and the convergence is weak or with respect to the trace norm (see A.5.2 and A.5.3).

Proof: Note that

$$\|v^*\xi\|^2 \;=\; \|p_{v\tilde{\mathcal{K}}}\xi\|^2 \;=\; \lim_{n\to\infty}\|p_{v\mathcal{K}_{[0,n-1]}}\,\xi\|^2$$

and apply the proposition. Now (b) follows from (a) using A.5.3. □

3.6.4 Criteria for the Unitarity of an Adapted Isometry

To check whether v is unitary or not we have to extend the arguments above for $\tilde{\xi} \in \tilde{\mathcal{K}}$.

Proposition: *For unit vectors $\xi \in \mathcal{K}_{[0,m]}$ we have for all $n > m$*

$$\|p_{v\mathcal{K}_{[0,n-1]}}\,\xi\|^2 \;=\; \langle \Omega_n, F_n \dots F_{m+1}(\rho)\,\Omega_n\rangle$$

with $\rho := Tr_{k\neq m}(u_m^* \dots u_1^*\, p_\xi\, u_1 \dots u_m) \in \mathcal{T}(\mathcal{K}_m)$. Further

$$\|v^*\xi\|^2 \;=\; \|p_{v\tilde{\mathcal{K}}}\,\xi\|^2 \;=\; \lim_{n\to\infty}\langle \Omega_n, F_n \dots F_{m+1}(\rho)\,\Omega_n\rangle$$

and $\|v^*\xi\| = \|\xi\| = 1$ if and only if weak or with respect to the trace norm

$$F_n \dots F_{m+1}(p_\xi) \;-\; p_{\Omega_n} \longrightarrow 0 \quad \textit{for } n \to \infty.$$

Proof: Proceeding similarly as in 3.6.3 we find that

$$\begin{aligned}
&\|q_{[0,n-1]}\,u_n^* \dots u_1^*\,\xi\|^2 \;=\; \langle \Omega_n, Tr_{[0,n-1]}(u_n^* \dots u_1^*\, p_\xi\, u_1 \dots u_n)\,\Omega_n\rangle \\
&= \langle \Omega_n, Tr_{[m,n-1]}\big(u_n^* \dots u_{m+1}^* Tr_{[0,m-1]}(u_m^* \dots u_1^* p_\xi u_1 \dots u_m) u_{m+1} \dots u_n\big)\Omega_n\rangle \\
&= \langle \Omega_n, Tr_{[m,n-1]}(u_n^* \dots u_{m+1}^*\,\rho\, u_{m+1} \dots u_n)\,\Omega_n\rangle \\
&= \langle \Omega_n, F_n \dots F_{m+1}(\rho)\,\Omega_n\rangle.
\end{aligned}$$

The other assertions follow from that as in the proof of Corollary 3.6.3. □

Theorem: *The following assertions are equivalent:*

(a) *v is unitary.*
(b) *For all $m \in \mathbb{N}_0$ and all density matrices $\rho_m \in \mathcal{T}_+^1(\mathcal{K}_m)$*

$$\lim_{n\to\infty} \left[F_n \dots F_{m+1}(\rho_m) \;-\; p_{\Omega_n}\right] = 0$$

weak or with respect to the trace norm.

Proof: The isometry v is unitary if and only if v^* is an isometry, i.e. $\|v^*\tilde{\xi}\| = \|\tilde{\xi}\|$ for all $\tilde{\xi} \in \tilde{\mathcal{K}}$ or, equivalently, for all $\xi \in \mathcal{K}_{[0,m]}$ for all $m \in \mathbb{N}_0$. Now the theorem follows from the preceding proposition: Just note that if ξ varies over all unit vectors in $\mathcal{K}_{[0,m]}$ then $u_m^* \dots u_1^* \,\xi$ does the same, and therefore the corresponding density matrices vary over all elements of $\mathcal{T}_+^1(\mathcal{K}_m)$. □

3.6.5 Homogeneous Case

Recall the definition of homogeneity in 3.5.6. In this situation we can simplify the unitarity criterion above. We use the notation $v_1' \;=\; u^*|_{\mathcal{K}\otimes\Omega_{\mathcal{K}}}$.

Proposition: *If v is homogeneous then the following assertions are equivalent:*

(a) v is unitary.
(b) The vector state given by $\Omega_{\mathcal{K}} \in \mathcal{K}$ is absorbing for the stochastic map $F^* : \mathcal{B}(\mathcal{K}) \to \mathcal{B}(\mathcal{K})$ given by $F^*(y) = (v_1')^* \;\mathbb{1} \otimes y \; v_1'$.

Proof: Because of homogeneity we can also drop the index n for the map F_n and get F (if we also omit the projection g_n in the original definition in 3.6.1, which is legitimate for the considerations here, as remarked earlier). Then F^* is the adjoint map of F, see A.4.2. We also observe that the conditions in Theorem 3.6.4 for different m are equal to each other in the homogeneous case, and in A.5.2 it is shown that the resulting condition is valid if and only if the vector state given by $\Omega_{\mathcal{K}}$ is absorbing. □

Thus we have the interesting result that the problem of unitarity for a homogeneous adapted isometry can be completely reduced to the setting analyzed in A.5.2 (absorbing vector states for stochastic maps) and we can use the equivalences established there to check unitarity. An example is given in 4.3.3.

Note that analogous to the $C-D-$correspondence described in 3.5.6 we also have a **C-F-correspondence** between Markov processes in a coupling representation and homogeneous stationary processes. The remarks in 3.5.6 apply here as well.

4

Examples and Applications

From the previous chapter one may get the impression that a theory of noncommutative stationary processes based on the concept of an adapted endomorphism is rather abstract. Certainly it needs to be supplemented by good examples and applications. In this chapter we take up this task.

It is natural to look first how classical stationary processes fit into the framework. The point of view of adapted endomorphisms is unusual for a classical probabilist. As explained in 3.2.11, the construction of the factors of a product representation corresponds to a step-by-step construction of the adapted endomorphism, i.e. of the time evolution of the process. We call this the problem of stationary extension and give an elementary classification. This seems to be unknown to classical probabilists, confirming the assertion that the point of view is unusual. Then we embed these considerations systematically into the theory of adapted endomorphisms. In the Markovian case we are back at coupling representations and under all stationary extensions the Markovian one is distinguished by a maximal entropy property.

Then we probe our formula for nonlinear prediction, stated and proved in Section 3.5, in the case of classical stationary processes. We get a remarkable connection between classical and noncommutative probability theory because also for classical processes this formula involves completely positive maps on matrices. Concrete evaluation can be done by a reduction to stochastic matrices, but the probabilistic meaning of this is not clear.

Low-dimensional examples provide our starting point for noncommutative processes. In particular, for filtrations based on Clifford algebras we have a functorial construction of adapted endomorphisms which are Bogoljubov transformations. It is interesting to note that for generalized Clifford algebras with generalized commutation relations some simplifying features are preserved, although no functorial construction is available any more. We show how these results of [Go01] fit into the framework. There are additional results in this case, and one may ask how to solve similar problems for more complicated algebras.

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A very interesting noncommutative filtration is obtained by considering tensor products of matrix algebras. There we have the pleasant feature that we automatically get localized product representations for stationary adapted processes. The study of such endomorphisms is a very rich and rather difficult subject. It has been one of the motivations for our general approach to adaptedness to give a careful framework in which this research project can be done in a systematic way. Here we only give a detailed introduction to the basics and discuss how it is connected to other work in operator algebras, in particular to the theory of subfactors of the hyperfinite II_1-factor. Further work refining our approach, for example computations of Jones indices of the occurring subfactors, is contained in [Go3] and is not included here. Of course, much more general towers of algebras and corresponding filtrations can be studied, especially those arising from the basic construction of V.F.R. Jones [Jo83, GHJ89]. This is also not discussed here.

It emerges from these classes of examples that the general theory of Chapter 3 only gives a rather loose framework and that the most interesting part of the work has to take into account the specific properties of the operator algebras. Of course, this is only natural because the amazing diversity of noncommutative stochastic processes arises exactly from these different algebraic structures, and thus this difficulty should be taken as a challenge instead of a drawback.

The last section seems to be a curiosity in some respect, but on the other hand it takes up the important problem of embedding classical processes into noncommutative ones, already touched upon in the Introduction. It is shown that it is not always possible to extend a classical stationary adapted process to all operators on the GNS-spaces such that the extension is again stationary and adapted. This is a non-Markovian phenomenon. A typical example is given by the Fredkin gate, giving one more connection to quantum information theory.

4.1 Commutative Stationarity

4.1.1 Stationarity in Classical Probability

The probabilistic impact of the theory outlined in Chapter 3 becomes clearer by analyzing commutative stationary processes in this respect. While stationary processes are treated in many monographs about classical probability (see for example [Ro74]), the point of view of constructing an adapted endomorphism with PR is new and leads to some considerations of a different flavour. We decided to write them up for processes not only with discrete time index (as always in this work) but also with only finitely many values. This leads to an interesting purely combinatorial structure for which one may seek further applications. The restriction to finitely many values is not essential and may be removed at the price of additional technicalities.

4.1.2 Backward and Forward Transition

Look at the following problem: Let $\xi_0, \xi_1, \ldots, \xi_N$ be random variables with values in $\{1, \ldots, a\}$, where $2 \leq a \in \mathbb{N}$, and with joint distribution μ. By definition μ is a probability measure on the cartesian product $\{1, \ldots, a\}^{\{0,\ldots,N\}}$. We have to introduce some notation: Denote elements of $\{1, \ldots, a\}^{\{0,\ldots,N\}}$ by $(i_0 \ldots i_N)$ where $i_n \in \{1, \ldots, a\}$ for $n = 0, \ldots, N$. We shall use indices $i, k \in \{1, \ldots, a\}$ and multi-indices $j = (j_0 \ldots j_{n-1}) \in \{1, \ldots, a\}^n$ with length $|j| = n$ (where $|j| = 0$ means: omit the index j). $(ij_0 \ldots j_{N-1})$ is abbreviated by (ij).

Cylinder sets are marked by a star $*$ at positions that are not fixed, e.g.: $(*i_1 \ldots i_N) = \bigcup_{i \in \{1,\ldots,a\}} (i\, i_1 \ldots i_N)$. Stars on the right side are sometimes omitted: $(j) = (j*)$. If p, q are probability measures on $\{1, \ldots, a\}$ we denote by $\mathcal{M}(p, q)$ the **set of possible joint distributions** (on $\{1, \ldots, a\}^2$), i.e. for $\lambda \in \mathcal{M}(p, q)$ we have $\sum_i \lambda(i, k) = q(k),\ \sum_k \lambda(i, k) = p(i)$.
We now assume that the measure μ is stationary, i.e.

$$\mu(i_0 \ldots i_{N-1}*) = \mu(*i_0 \ldots i_{N-1})$$

for all indices $i_0, \ldots, i_{N-1}$. This means that the random variables $\xi_0, \xi_1, \ldots, \xi_N$ may be interpreted as a connected section cut from a stationary stochastic process.

For any multi-index $j \in \{1, \ldots, a\}^N$ we consider two associated probability measures on $\{1, \ldots, a\}$, the **backward transition probability measure** p^j and the **forward transition probability measure** q^j:

$$p^j(i) := \frac{\mu(ij)}{\mu(*j)}$$

$$q^j(k) := \frac{\mu(jk)}{\mu(j*)}$$

$\big($ for $N = 0$ this means $p^{\emptyset}(i) = q^{\emptyset}(i) = \mu(i)$ $\big)$.
A probability measure $\tilde{\mu}$ on $\{1, \ldots, a\}^{\{0,\ldots,N+1\}}$ is called a **stationary extension** of μ if $\tilde{\mu}$ is stationary and

$$\tilde{\mu}\,|_{\{1,\ldots,a\}^{\{0,\ldots,N\}}} = \mu.$$

We may think of it as adjoining another random variable ξ_{N+1} in a stationary way.

4.1.3 Classification of Stationary Extensions

The problem we want to consider consists in classifying all stationary extensions of a given stationary measure μ.

Proposition:
Let μ be a stationary probability measure on $\{1,\ldots,a\}^{\{0,\ldots,N\}}$. There is a one-to-one correspondence between the set of stationary extensions $\tilde{\mu}$ of μ and the cartesian product $\Pi_{|j|=N}\mathcal{M}(p^j,q^j)$ which is given by

$$\tilde{\mu}(ijk) = \lambda^j(i,k)\cdot\mu(j*)$$

where $(\lambda^j)_{|j|=N} \in \Pi_{|j|=N}\mathcal{M}(p^j,q^j)$.

Proof: If $\tilde{\mu}$ is a stationary extension then for all j with $|j| = N$ we get

$$\sum_i \tilde{\mu}(ijk) = \tilde{\mu}(*jk) = \tilde{\mu}(jk*) = \mu(jk) = q^j(k)\,\mu(j*)$$
$$\sum_k \tilde{\mu}(ijk) = \tilde{\mu}(ij*) = \mu(ij) = p^j(i)\,\mu(*j) = p^j(i)\,\mu(j*).$$

Conversely if $(\lambda^j)_{|j|=N} \in \Pi_{|j|=N}\mathcal{M}(p^j,q^j)$ is given then

$$\tilde{\mu}(ij*) = \sum_k \tilde{\mu}(ijk) = \sum_k \lambda^j(i,k)\,\mu(j*) = p^j(i)\,\mu(j*) = p^j(i)\,\mu(*j) = \mu(ij),$$

i.e. $\tilde{\mu}$ is an extension of μ and

$$\tilde{\mu}(*jk) = \sum_i \tilde{\mu}(ijk) = \sum_i \lambda^j(i,k)\,\mu(j*) = q^j(k)\,\mu(j*) = \mu(jk) = \tilde{\mu}(jk*),$$

i.e. $\tilde{\mu}$ is stationary. □

If the denominators $\mu(*j)$ or $\mu(j*)$ in the defining formulas for p^j or q^j are zero then in the corresponding path no extension is necessary because it is already a set of measure zero. Therefore we loose nothing if we ignore these cases here and in the sequel.

4.1.4 Transition Operators

As described in 3.3.2, instead of a probability space (Ω,Σ,μ) we can consider the function algebra $\mathcal{A} = L^\infty(\Omega,\Sigma,\mu)$. The measure μ induces the state $\phi = \phi_\mu$. We shall not distinguish in notation between measurable sets in Σ and the corresponding characteristic functions in $\mathcal{A}$. Pairs $(\mathcal{A},\phi)$ are the objects of the category to be studied. As described in 2.1.2, a random variable $\xi : (\tilde{\Omega},\tilde{\Sigma},\tilde{\mu}) \to (\Omega,\Sigma,\mu)$ corresponds to a *-homomorphism

$$j_\xi : L^\infty(\Omega,\Sigma,\mu) \to L^\infty(\tilde{\Omega},\tilde{\Sigma},\tilde{\mu}), \quad f \mapsto f\circ\xi,$$

i.e. to an embedding of a smaller object. j_ξ is also called a random variable and we use the notation $j_\xi : (\mathcal{A}, \phi) \to (\tilde{\mathcal{A}}, \tilde{\phi})$ to indicate that it is a morphism of our category.

A stationary stochastic process is specified by a random variable $j_0 : (\mathcal{A}, \phi) \to (\tilde{\mathcal{A}}, \tilde{\phi})$ and an endomorphism $\alpha : (\tilde{\mathcal{A}}, \tilde{\phi}) \to (\tilde{\mathcal{A}}, \tilde{\phi})$. It is formed by the sequence $(j_n)_{n\geq 0}$ of random variables where $j_n := \alpha^n \circ j_0$. If these random variables have values in $\{1, \ldots, a\}$, i.e. $\mathcal{A} = \mathbb{C}^a$, then we continue to use our notation introduced above for cylinder sets and for the corresponding characteristic functions in $\mathcal{A}$. Without restriction of generality we shall assume in this case that for each canonical basis vector $\delta_i \in \mathbb{C}^a$ we have $\phi(\delta_i) > 0$ (otherwise use $a' < a$).

Joint distributions are replaced by transition operators: Suppose that p, q are probability measures on $\{1, \ldots, a\}$ (and $p(i) > 0$, $q(k) > 0$ for all i,k corresponding to our assumption above). Then to any $\lambda \in \mathcal{M}(p, q)$ we associate the transition operator

$$S : (\mathcal{A} = \mathbb{C}^a, q) \to (\mathcal{B} = \mathbb{C}^a, p),$$

which with respect to the canonical basis $\{\delta_i\}$ of $\mathbb{C}^a$ is given by a stochastic matrix with

$$p(i) \cdot S_{ik} = \lambda(i, k) \quad \text{for all i,k.}$$

Compare Section 1.1 for examples with $a = 2$ and $p = q$. In this way we get a **set $\mathcal{S}(q, p)$ of transition operators** in one-to-one correspondence to the set $\mathcal{M}(p, q)$ of joint distributions.

In the setting of Proposition 4.1.3 to any joint distribution $\lambda^j \in \mathcal{M}(p^j, q^j)$ there is an associated transition operator $S^j \in \mathcal{S}(q^j, p^j)$, and we get

$$S^j_{ik} = \frac{\lambda^j(i, k)}{p^j(i)} = \frac{\tilde{\mu}(ijk)}{\mu(ij)} = q^{ij}(k),$$

i.e. the probability of a transition to k given (ij).

4.1.5 Adapted Endomorphisms

Now recall the definition of adaptedness for this context (compare 3.2.6).

An endomorphism $\alpha : (\tilde{\mathcal{A}}, \tilde{\phi}) \to (\tilde{\mathcal{A}}, \tilde{\phi})$ is adapted with PR for a filtration $\mathcal{A}_0 = \tilde{\mathcal{A}}_0 \subset \tilde{\mathcal{A}}_1 \subset \tilde{\mathcal{A}}_2 \subset \ldots \subset \tilde{\mathcal{A}}$ if there are automorphisms $\tilde{\alpha}_0, \tilde{\alpha}_1, \tilde{\alpha}_2, \ldots$ of $(\tilde{\mathcal{A}}, \tilde{\phi})$ satisfying

(0) $\tilde{\alpha}_0 = 1\!\!1$

(1) $\tilde{\alpha}_{n+1} \tilde{\mathcal{A}}_{n+1} = \tilde{\mathcal{A}}_{n+1}$

(2) $\tilde{\alpha}_{n+1} |_{\tilde{\mathcal{A}}_n} = \alpha |_{\tilde{\mathcal{A}}_n}$.

We can use the terminology and the general theory of Section 3.2 for this category. In particular there exist factors $\alpha_1, \alpha_2, \ldots$, see 3.2.7. For any adapted endomorphism we can consider the stationary process specified by the random variable $j_0 = \mathbb{1}|_{\mathcal{A}_0} : \mathcal{A}_0 \to \tilde{\mathcal{A}}$ and the endomorphism α. A stationary process which is represented in this way has been called adaptedly implemented with PR, see 3.3.4. An adaptedly implemented process is also adapted in the wide sense which is the customary notion of adaptedness in probability theory. In fact, with $\mathcal{A}_n := j_n(\mathcal{A}_0) = \alpha^n \circ j_0(\mathcal{A}_0)$ and $\mathcal{A}_{[0,n]}$ generated by all $\mathcal{A}_m$ with $0 \leq m \leq n$, we have the filtration $(\mathcal{A}_{[0,n]})_{n=0}^{\infty}$ canonically associated to the process, and it follows that $\mathcal{A}_{[0,n]} \subset \tilde{\mathcal{A}}_n$ for all n, compare 3.2.7(d). In fact, the assumption of adapted implementedness with PR is strictly stronger than that, as can be seen by the following example.

4.1.6 Counterexample

The following considerations are a slightly modified version of ideas presented in ([Kü88b], 4.4). We construct a stationary Markov chain with values in $\{1, \ldots, a\}$ in the traditional way:

$$\mathcal{A}_{[0,n]} := \bigotimes_{m=0}^{n} \mathbb{C}^a \cong l^\infty(\{1, \ldots, a\}^{n+1}),$$
$$\mu(ij) := \mu(i)\, t_{ij_0}\, t_{j_0 j_1} \ldots t_{j_{n-2} j_{n-1}},$$

where $\{\mu(i)\}_{i=1}^a$ is a probability distribution which is stationary with respect to a transition matrix $T = (t_{ik})_{i,k=1}^a$. The tensor shift σ implements a Markov chain with transition probabilities $(t_{ik})_{i,k=1}^a$. Now the following problem arises: Is the tensor shift σ adapted with PR for the canonical filtration $\mathcal{A}_0 \subset \mathcal{A}_{[0,1]} \subset \mathcal{A}_{[0,2]} \subset \ldots$?
Every automorphism $\tilde{\alpha}_1$ of $\mathbb{C}^a \otimes \mathbb{C}^a \cong l^\infty(\{1, \ldots, a\}^2)$ is induced by a permutation of $\{1, \ldots, a\}^2$. The adaptedness condition $\tilde{\alpha}_1|_{\mathbb{C}^a \otimes \mathbb{1}} = \sigma|_{\mathbb{C}^a \otimes \mathbb{1}}$ is equivalent to $\tilde{\alpha}_1(i*) = (*i)$ for all $i \in \{1, \ldots, a\}$. This can be done in a $\mu|_{\mathcal{A}_{[0,1]}}$-preserving way if and only if for all $i \in \{1, \ldots, a\}$ the vectors $(\mu(i)\, t_{ik})_{k=1}^a$ and $(\mu(k)\, t_{ki})_{k=1}^a$ are permutations of each other, see ([Kü88b], 4.4) for more details. This is valid in certain cases, for example if $a = 2$ or when the process fulfils the detailed balance condition (which here means that the two vectors are identical), but it is easy to give examples where it is not valid.

This shows that 'adapted in the wide sense' in general does not imply 'adaptedly implemented'. The filtration must also have enough 'space' for the required morphisms.

4.1.7 Stationary Extensions and Extending Factors

Now we show that it is always possible to construct adapted implementations with PR for any stationary process, though not in the traditional way tried in 4.1.6. First we look for conditions to be imposed upon an extending factor α_{N+1} (see 3.2.9) to implement a specified stationary extension. Recall that $\tilde{\alpha}_N = \alpha_1 \ldots \alpha_N$.

Proposition: *Let $(\alpha_1, \ldots, \alpha_N)$ be an N-tupel of factors for the filtration $\mathcal{A}_0 \subset \tilde{\mathcal{A}}_1 \subset \ldots \subset \tilde{\mathcal{A}}_N \subset \tilde{\mathcal{A}}$ (see 3.2.9). Consider the associated random variables $j_0 = 1\!\!1|_{\mathcal{A}_0} : \mathcal{A}_0 \to \tilde{\mathcal{A}}$ and $j_n := (\tilde{\alpha}_N)^n \circ j_0,\ n = 0, \ldots, N$, with their stationary joint distribution μ. Let α_{N+1} be an extending factor for $(\alpha_1, \ldots, \alpha_N)$. The following assertions are equivalent:*

(1) $\alpha := \tilde{\alpha}_N \cdot \alpha_{N+1}$ implements the stationary extension $\tilde{\mu}$ characterized by

$$(\lambda^j)_{|j|=N} \in \Pi_{|j|=N} \mathcal{M}(p^j, q^j).$$

(2) For all multi-indices j with $|j| = N$ consider the conditional probability given by:

$$\phi_j(x) := \frac{\tilde{\phi}(x \cdot (j))}{\tilde{\phi}(j)} \quad \text{for all } x \in \tilde{\mathcal{A}} \cdot (j).$$

Then with the random variables

$$\begin{aligned} i_p^j : (\mathbb{C}^a, p^j) &\to (\tilde{\mathcal{A}} \cdot (j), \phi_j) \\ \delta_i &\mapsto \tilde{\alpha}_N^{-1}(i) \cdot (j) = \tilde{\alpha}_N^{-1}(ij) \\ i_q^j : (\mathbb{C}^a, q^j) &\to (\tilde{\mathcal{A}} \cdot (j), \phi_j) \\ \delta_k &\mapsto (jk) \end{aligned}$$

the following dilation formula is valid:

$$(i_p^j)^* \, \alpha_{N+1} \, i_q^j = S^j,$$

where S^j is the transition operator corresponding to λ^j as discussed above.

Note that if we identify $(\mathbb{C}^a, p^j)$ with its image under i_p^j, then the adjoint $(i_p^j)^*$ is nothing but the ϕ_j-preserving conditional expectation onto this image.

$$\begin{tikzcd} \mathbb{C}^a \arrow[r, "S^j"] \arrow[d, "i_q^j"'] & \mathbb{C}^a \\ \tilde{\mathcal{A}} \arrow[r, "\alpha_{N+1}"] & \tilde{\mathcal{A}} \arrow[u, "(i_p^j)^*"'] \end{tikzcd}$$

Intuitively the proposition states that to implement adaptedly a specified stationary extension you have to find an automorphic dilation (of first order) of the transition operators S^j and to glue it to the given structure in the way indicated by the theorem. This is in accordance with our general philosophy in Section 3.2.9 that extending factors are related to extension problems. In particular the dilations occurring here are similar to the Hilbert space dilations occurring in 3.1.6 which associate to an element of a choice sequence the corresponding rotation matrix.

Proof: Expanding our condensed notation we get

$$\begin{aligned}(ijk) &= (i)\cdot\alpha(j_0)\cdot\alpha^2(j_1)\cdot\ldots\cdot\alpha^N(j_{N-1})\cdot\alpha^{N+1}(k)\\ &= \tilde{\alpha}_N\,[\tilde{\alpha}_N^{-1}(i)\cdot(j)\cdot\alpha_{N+1}(\alpha^N(k))]\\ &= \tilde{\alpha}_N\,[\tilde{\alpha}_N^{-1}(i)\cdot(j)\cdot\alpha_{N+1}(jk)].\end{aligned}$$

Using the correspondence given in Proposition 4.1.3 we infer

$$\begin{aligned}\lambda^j(i,k) &= \frac{1}{\mu(j)}\,\tilde{\mu}(ijk) \;=\; \frac{1}{\tilde{\phi}(j)}\,\tilde{\phi}\big(\tilde{\alpha}_N^{-1}(i)\cdot(j)\cdot\alpha_{N+1}(jk)\big)\\ &= \phi_j\left(i_p^j(\delta_i)\cdot\alpha_{N+1}\,i_q^j(\delta_k)\right)\\ &= p^j\left(\delta_i\cdot(i_p^j)^*\,\alpha_{N+1}\,i_q^j(\delta_k)\right)\end{aligned}$$

and thus

$$(i_p^j)^*\,\alpha_{N+1}\,i_q^j \;=\; S^j.$$

The argument is valid in both directions. □

4.1.8 Elementary Tensor Representations

In 3.3.5 we described the idea to simplify the discussion by using filtrations which are generated by independent variables. We now present such a construction which realizes the dilations introduced in 4.1.7 in a more concrete way.

For a finite dimensional subobject of $(\tilde{\mathcal{A}},\tilde{\phi})$ which is isomorphic to $\mathbb{C}^a$ for some $a\in\mathbb{N}$ the embedding into $(\tilde{\mathcal{A}},\tilde{\phi})$ may be characterized in a short way by indicating which elements of $\tilde{\mathcal{A}}$ correspond to the canonical basis $\{\delta_i\}_{i=1}^a$, i.e. by giving a partition $\mathcal{P}:=\{P_1,\ldots,P_a\}\subset\tilde{\mathcal{A}}$. The $P_i,\ i=1,\ldots,a$, are characteristic functions whose sum is the identity. Again we may assume without restriction of generality that $\tilde{\phi}(P_i)>0$ for all i.

Now consider a family of objects $(\mathcal{C}_0,\psi_0)$, $(\mathcal{C}_1,\psi_1)$, $(\mathcal{C}_2,\psi_2),\ldots$. Using tensor products $\mathcal{C}_{[m,n]}:=\bigotimes_{j=m}^n\mathcal{C}_j$ we have a filtration $\mathcal{C}_0\subset\mathcal{C}_{[0,1]}\subset\mathcal{C}_{[0,2]}\ldots$. The state on $\mathcal{C}_{[0,n]}$ is chosen to be the product state $\psi_{[0,n]}:=\bigotimes_{m=0}^n\psi_m$. We shall write

$$(\mathcal{C}_{[0,n]},\psi_{[0,n]})=\bigotimes_{m=0}^n(\mathcal{C}_m,\psi_m).$$

All this may be embedded into one greater object $(\tilde{\mathcal{A}},\tilde{\phi})$, for example the (possibly infinite) tensor product of all objects involved.

We want to represent stationary processes using this frame. Let a stationary probability measure $\tilde{\mu}$ on $\{1,\ldots,a\}^{\{0,\ldots,N+1\}}$ be given. If for any multiindex j with $|j|\le N+1$ there are partitions $Q^j=\{Q_1^j,\ldots,Q_a^j\}\subset\mathcal{C}_{|j|}$ such that $\psi_{|j|}(Q_k^j)=q^j(k)$ for all k, then we may construct a representation as follows:

$$(k):=Q_k^{\emptyset}\in\mathcal{C}_0$$

and then recursively for any (j) already defined

$$(jk) := (j) \otimes Q^j_k \in \mathcal{C}_{[0,|j|]}.$$

It is easy to check that indeed we always have $\tilde{\phi}(jk) = \tilde{\mu}(jk)$.

This way of representing stationary probability measures may be called an **elementary tensor representation**. If we again denote by $\{\mathcal{A}_{[0,n]}\}_{n=0}^{N+1}$ the canonical filtration of the process constructed in this way, then $\mathcal{A}_{[0,n]}$ is contained in the algebraic span of all partitions Q^j with $|j| \leq n$. In particular we always have $\mathcal{A}_{[0,n]} \subset \mathcal{C}_{[0,n]}$, i.e. adaptedness in the wide sense.

4.1.9 Construction of Extending Factors by Partitions

Now we want to determine when an elementary tensor representation actually provides us with an adapted implementation of the process.

Theorem: *Let a stationary probability measure $\tilde{\mu}$ on $\{1, \ldots, a\}^{\{0,\ldots,N+1\}}$ and a family of objects $(\mathcal{C}_0, \psi_0), (\mathcal{C}_1, \psi_1), \ldots, (\mathcal{C}_{N+1}, \psi_{N+1})$ be given. Form objects $\{(\mathcal{C}_{[0,n]}, \psi_{[0,n]})\}_{n=0}^{N+1} \subset (\tilde{\mathcal{A}}, \tilde{\phi})$ as above.*
We consider all multi-indices j with $|j| \leq N+1$. Suppose that for all such j there are partitions P^j, $Q^j \subset \mathcal{C}_{|j|}$ such that

$$P^j := \{P^j_1, \ldots, P^j_a\}, \qquad \psi_{|j|}(P^j_i) = p^j(i) \quad \textit{for all } i,$$
$$Q^j := \{Q^j_1, \ldots, Q^j_a\}, \qquad \psi_{|j|}(Q^j_k) = q^j(k) \quad \textit{for all } k.$$

$\big($ For $|j| = 0$ we may assume without restriction of generality that $P^\emptyset = Q^\emptyset$ with $\psi_0(P^\emptyset_i) = \psi_0(Q^\emptyset_i) = \tilde{\mu}(i)$ and that $\mathcal{C}_0$ coincides with the algebra specified by this partition. $\big)$
Assume further that there are automorphisms $\alpha_1, \ldots, \alpha_{N+1}$ satisfying

$$\alpha_n \mathcal{C}_{[0,n]} = \mathcal{C}_{[0,n]}$$
$$\alpha_n \,|_{\mathcal{C}_{[n+1,N+1]}} = 1\!\!1 \,|_{\mathcal{C}_{[n+1,N+1]}}$$

(omit the last equation for $n = N+1$) and furthermore

$$\alpha_1(Q^\emptyset_k \otimes P^k_i) = P^\emptyset_i \otimes Q^i_k$$
$$\alpha_{n+1}(x \otimes Q^j_k \otimes P^{jk}_i) = x \otimes P^j_i \otimes Q^{ij}_k$$

for all i,k and (if $n \geq 1$) for all j with $|j| = n$ and all $x \in \mathcal{C}_{[0,n-1]} \cdot (j)$. Here (j) refers to the elementary tensor representation specified by the Q-partitions.
Then the following assertions are valid:

- **(a)** *The elementary tensor representation is adaptedly implemented with PR for the filtration $\mathcal{C}_0 \subset \mathcal{C}_{[0,1]} \subset \ldots \subset \mathcal{C}_{[0,N+1]}$ by the factors $\alpha_1, \ldots, \alpha_{N+1}$.*
- **(b)** *$(j) \otimes P^j_i = \tilde{\alpha}^{-1}_{|j|}(i) \cdot (j) = \tilde{\alpha}^{-1}_{|j|}(ij)$ for all i,j.*

Proof: It is easy to see that the automorphisms $\alpha_1, \ldots, \alpha_{N+1}$ are factors in the sense of 3.2.9, in particular

$$\alpha_{n+1}\,|_{\mathcal{C}_{[0,n-1]}} = 1\!\!1\,|_{\mathcal{C}_{[0,n-1]}}$$

for $n \geq 1$ follows by summing over i,k.

Now we proceed by induction. Assume that the elementary tensor representation in $\mathcal{C}_{[0,N]}$ coincides with the process that is implemented by the factors $\alpha_1, \ldots, \alpha_N$ and that $(j) \otimes P_i^j = \tilde{\alpha}_{|j|}^{-1}(ij)$ is valid for $|j| = N$. This is trivial for $|j| = 0$ with $\tilde{\alpha}_0 = 1\!\!1$. The step $N \to N+1$ can be shown as follows: Inserting $(j) \otimes Q_k^j = (jk)$ and $(j) \otimes P_i^j = \tilde{\alpha}_{|j|}^{-1}(ij)$ into the equation $\alpha_{N+1}(x \otimes Q_k^j \otimes P_i^{jk}) = x \otimes P_i^j \otimes Q_k^{ij}$ (the case $N = 0$ is similar) leads to

$$\alpha_{N+1}((jk) \otimes P_i^{jk}) = \tilde{\alpha}_N^{-1}(ij) \otimes Q_k^{ij}.$$

Multiplication with $\tilde{\alpha}_N$ yields

$$\tilde{\alpha}_{N+1}((jk) \otimes P_i^{jk}) = (ij) \otimes Q_k^{ij} = (ijk).$$

Summing this over i gives

$$\tilde{\alpha}_{N+1}(jk*) = (*jk)$$

showing that it is indeed the tensor representation which is implemented, while an application of $\tilde{\alpha}_{N+1}^{-1}$ yields

$$(jk) \otimes P_i^{jk} = \tilde{\alpha}_{N+1}^{-1}(ijk).$$

This is assertion (b). □

4.1.10 Discussion

Now we shall discuss various aspects and special cases which are implicit in Theorem 4.1.9.

The existence of the partitions P^j, Q^j in $(\mathcal{C}_{|j|}, \psi_{|j|})$ is the main condition which has to be fulfilled by the objects $(\mathcal{C}_{|j|}, \psi_{|j|})$. Note that we can get different looking structures if there is more than one possibility for these embeddings: There is no condition involved concerning the relative position of the partitions corresponding to the same length of j.

According to the definition of the probability measures p^j and q^j we have

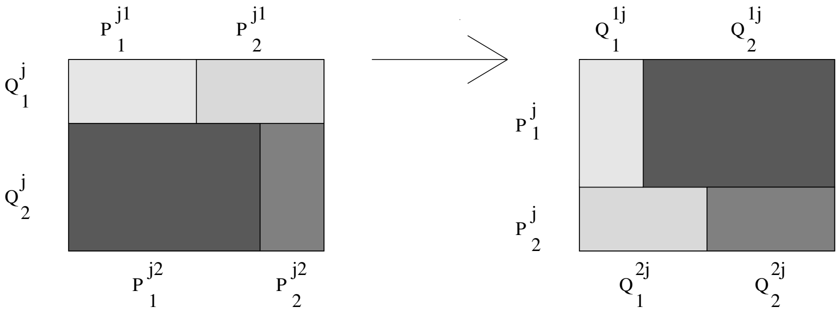
$$\begin{aligned} & q^j(k) \cdot p^{jk}(i) = \mu(ijk \mid *j*) = p^j(i) \cdot q^{ij}(k), \\ \text{i.e.} \qquad & (\psi_{|j|} \otimes \psi_{|j|+1})(Q_k^j \otimes P_i^{jk}) = (\psi_{|j|} \otimes \psi_{|j|+1})(P_i^j \otimes Q_k^{ij}), \end{aligned}$$

which shows that the assumptions on the automorphisms $\alpha_1, \ldots, \alpha_{N+1}$ in the theorem are automatically consistent with the property of state preservation.

For α_{n+1} it must be checked on the object $(\mathcal{C}_n, \psi_n) \otimes (\mathcal{C}_{n+1}, \psi_{n+1})$ whether such an automorphism exists.

We can recognize here the constructive nature of Theorem 4.1.9. If for $|j| \leq N+1$ we have calculated from $\tilde{\mu}$ the probabilities p^j and q^j then we only have to find objects $\mathcal{C}_{|j|}$ which allow the embedding of suitable partitions and the construction of the factors. Then we automatically get an adapted implementation. The 'fitting together' of the automorphisms $\alpha_1, \ldots, \alpha_{N+1}$ is part of the conclusion.

If we choose $(\mathcal{C}_n, \psi_n) := (L^\infty[0,1], \lambda)$ for $n \geq 1$, where λ is Lebesgue measure on $[0,1]$, then it is always possible to satisfy all the assumptions of Theorem 4.1.9. Therefore for any stationary process with values in $\{1, \ldots, a\}$ there is an adapted implementation which can be explicitly constructed using Theorem 4.1.9. Recall also that the assumption that there are only finitely many values was made only to avoid technicalities. The following picture for $a = 2$ and $\alpha_{|j|+1}|_{\mathcal{C}_{[0,|j|+1]} \cdot (j)}$ shows the idea of the construction:



Of course it is often possible to work with smaller objects. If for all i,k and all j of given length the numbers $p^j(i)$, $q^j(k)$ are rational numbers with smallest common denominator $d_{|j|}$ then we can satisfy all assumptions of Theorem 4.1.9 by using the objects

$$(\mathcal{C}_{|j|}, \psi_{|j|}) := (\mathbb{C}^{d_{|j|}}, (\frac{1}{d_{|j|}}, \frac{1}{d_{|j|}}, \ldots, \frac{1}{d_{|j|}}))$$

for $|j| \leq N+1$. Construction of the factors is clearly possible here because all automorphisms are induced by permutations.

To get another interesting special case, assume for all j with $|j| \leq N+1$ that the backward transition probabilities $(p^j(k))_{k=1}^a$ and the forward transition probabilities $(q^j(k))_{k=1}^a$ are permutations of each other. Identifying the corresponding elements of the P- and Q-partitions we can perform the following computation for $n \leq N$ and $|j| = n$ (where π is a permutation of $\{1, \ldots, a\}$):

$$\begin{aligned}
\tilde{\alpha}_{n+1}(jki) &= \tilde{\alpha}_{n+1}((j) \otimes Q^j_k \otimes Q^{jk}_i) \\
&= \tilde{\alpha}_n \alpha_{n+1}((j) \otimes Q^j_k \otimes P^{jk}_{\pi(i)}) \\
&= \tilde{\alpha}_n((j) \otimes P^j_{\pi(i)} \otimes Q^{\pi(i)j}_k) \\
&= \ldots \\
&= \tilde{\alpha}_1(Q^{\emptyset}_{j_0} \otimes P^{j_0}_{\pi(i)} \otimes \ldots \otimes Q^{\pi(i)j}_k) \\
&= P^{\emptyset}_{\pi(i)} \otimes Q^{\pi(i)}_{j_0} \otimes \ldots \otimes Q^{\pi(i)j}_k = (\pi(i)jk).
\end{aligned}$$

We infer that

$$\tilde{\alpha}_{n+1}(\mathcal{A}_{[0,n+1]}) = \mathcal{A}_{[0,n+1]},$$

i.e. in this case we have an adapted implementation with respect to the canonical filtration $\mathcal{A}_0 \subset \mathcal{A}_{[0,1]} \subset \ldots \subset \mathcal{A}_{[0,N+1]}$. This generalizes results about Markov processes in ([Kü88b], 4.4).

The relation between Proposition 4.1.7 and Theorem 4.1.9 becomes transparent by noting that

$$\begin{aligned}
i^j_p(\delta_i) &= (j) \otimes P^j_i = \tilde{\alpha}^{-1}_{|j|}(ij) \\
i^j_q(\delta_k) &= (j) \otimes Q^j_k = (jk).
\end{aligned}$$

Calculating the conditional probability

$$\begin{aligned}
\phi(\alpha_{|j|+1}((j) \otimes Q^j_k) \mid (j) \otimes P^j_i) &= \phi(\alpha_{|j|+1}(\sum_i (j) \otimes Q^j_k \otimes P^{jk}_i) \mid (j) \otimes P^j_i) \\
&= \phi(\sum_i (j) \otimes P^j_i \otimes Q^{ij}_k \mid (j) \otimes P^j_i) \\
&= \psi_{|j|+1}(Q^{ij}_k) = q^{ij}(k) = S^j_{ik}
\end{aligned}$$

yields the dilation property given in Theorem 4.1.7 and shows that in Theorem 4.1.9 we gave a construction realizing these dilations.

There are interesting analogues between this construction and that given in Section 3.1 for Hilbert space isometries which we tried to emphasize by giving a similar presentation. In a way the sets of partitions $\{P^j\}_{|j|=n}$ respectively $\{Q^j\}_{|j|=n}$ here correspond to the defect spaces $\mathcal{D}_{n^*}$ respectively $\mathcal{D}_n$ there.

4.1.11 Two-Valued Processes

If the random variables of a stationary process have only two values, then a more explicit version of Proposition 4.1.3 can be given:
Let $p = (p(1), p(2))$ and $q = (q(1), q(2))$ be probability distributions on $\{1, 2\}$. We assume that $p(1)$, $p(2)$, $q(1)$, $q(2) > 0$.
A $2 \times 2-$matrix S belongs to $\mathcal{S}(q, p)$ if the following conditions are satisfied:

$$
\begin{aligned}
0 \ \leq S_{ik} \leq& \ \ 1 \\
p(1)\, S_{11} + p(2)\, S_{21} =& \ q(1) \qquad (1) \\
p(1)\, S_{12} + p(2)\, S_{22} =& \ q(2) \qquad (2) \\
S_{11} + S_{12} =& \ \ 1 \qquad\quad\ (3) \\
(\Rightarrow \ S_{21} + S_{22} =& \ \ 1)
\end{aligned}
$$

If we choose $0 \leq S_{11} \leq 1$ such that

$$
\begin{aligned}
p(1)\, S_{11} &\leq q(1) \\
p(1)\, (1 - S_{11}) &\leq q(2) \ ,
\end{aligned}
$$

then corresponding values of S_{12}, S_{21} and S_{22} can be calculated using the equalities above. Because these considerations are valid in both directions, we get the following parametrization for $\mathcal{S}(q, p)$:

$$
max\{0, 1 - \frac{q(2)}{p(1)}\} \leq S_{11} \leq min\{1, \frac{q(1)}{p(1)}\}. \qquad (4)
$$

This is an interval of positive length.

Applying Proposition 4.1.3 we find a recursive procedure generating all two-valued stationary processes:

Procedure
Start with any probability distribution $p^{\emptyset} = q^{\emptyset}$ on $\{1, 2\}$.
Then proceed recursively:
If for all multi-indices $j \in \{1, 2\}^N$ (i.e. $|j| = N$) the probability distributions p^j and q^j on $\{1, 2\}$ are known then fix the transition matrices $S^j \in \mathcal{S}(q^j, p^j)$ using the formulas (4),(3),(1),(2) above. Then calculate $p^{j'}$ and $q^{j'}$ for $|j'| = N + 1$ by

$$
\begin{aligned}
q^{ij}(k) &:= S^j_{ik} \\
p^{jk}(i) &:= \frac{p^j(i)}{q^j(k)}\, q^{ij}(k),
\end{aligned}
$$

where $i, k \in \{1, 2\}$.

If $p^j(i) = 0$ (or $q^j(k) = 0$) then the cylinder sets (ij) (or (jk)) are sets of measure zero and no further transition probabilities are needed. Stop the recursion for these paths.

By combining the procedure above with Theorem 4.1.9 many simple and low-dimensional examples of adapted implementations with prescribed probabilistic features can be constructed.

4.1.12 Markov Processes

An important class of stationary stochastic processes is given by stationary Markov processes. In our notation a N-step stationary Markov process with values in $\{1, \ldots, a\}$ may be characterized by the following equivalent properties: For all multi-indices j we have

$$
\begin{aligned}
|j| \geq N \;&\Rightarrow\; \lambda^j \text{ is the product measure of } p^j \text{ and } q^j. \\
|j| \geq N \;&\Rightarrow\; \text{In every row of } S^j \text{ we have } q^j. \\
|j| \geq N \;&\Rightarrow\; q^{ij} = q^j \text{ for all i.} \\
|j| \geq N \;&\Rightarrow\; p^{jk} = p^j \text{ for all k.}
\end{aligned}
$$

Obviously $1-$step Markov is the usual Markov property. The last two properties for $|j| = N$ suggest the following consideration: If we construct an adapted implementation for the N-step Markov process using Theorem 4.1.9 then we may choose $(\mathcal{C}_{N+1}, \psi_{N+1}) = (\mathcal{C}_N, \psi_N)$ and always use the same embedded partitions, i.e. $P^{jk} = P^j$ and $Q^{ij} = Q^j$ for all i,k. This distinguished stationary extension may be called a **Markovian extension**. Now compute the corresponding factor α_{N+1}:

$$
\begin{aligned}
\alpha_{N+1}(\,(j) \,\otimes\, Q_k^j \,\otimes\, P_i^j\,) = \alpha_{N+1}\;(\,(j) \otimes Q_k^j \,\otimes\, P_i^{jk}\,) \\
= (j) \otimes P_i^j \,\otimes\, Q_k^{ij} \\
= (j) \otimes P_i^j \,\otimes\, Q_k^j.
\end{aligned}
$$

Therefore the automorphism α_{N+1} may be realized as a tensor flip of the N-th and (N+1)-th position:

$$\alpha_{N+1} : \mathcal{C}_N \otimes \mathcal{C}_{N+1} \to \mathcal{C}_N \otimes \mathcal{C}_{N+1}, \quad a \otimes b \mapsto b \otimes a.$$

On the other positions α_{N+1} may be chosen to act as identity.

To construct the whole N-step Markov process we have to choose Markovian extensions for all $n \geq N$, i.e. for

$$N \to N+1 \to N+2 \to \ldots$$

If we realize all α_{n+1} for $n \geq N$ as flips then the succession of flips yields a tensor shift σ. Summarizing we get the following adapted implementation of an N-step Markov process:

$$
\begin{aligned}
(\tilde{\mathcal{A}}, \tilde{\phi}) &:= (\mathcal{C}_0, \psi_0) \otimes \ldots \otimes (\mathcal{C}_{N-1}, \psi_{N-1}) \otimes \bigotimes_{n=N}^{\infty} (\mathcal{C}_n, \psi_n) \\
\alpha &:= \alpha_1 \ldots \alpha_N \, \sigma \,,
\end{aligned}
$$

where σ acts as identity on $\bigotimes_{n=0}^{N-1} (\mathcal{C}_n, \psi_n)$ and as a tensor shift on $\bigotimes_{n=N}^{\infty} (\mathcal{C}_n, \psi_n)$.

Clearly this is nothing but a coupling representation as described in Section 2.1.6. In other words, for Markov processes adaptedly implemented elementary tensor representations are coupling representations.

4.1.13 Markovian Extensions and Entropy

Markovian extensions can also be characterized by entropy. First recall some basic definitions:

If p is the probability distribution of a random variable ξ with finitely many values then its **entropy** is defined by

$$H(\xi) := H(p) := -\sum_i p(i) \log p(i).$$

If $\xi_0, \xi_1, \ldots, \xi_{N+1}$ are random variables with values in $\{1, \ldots, a\}$ and with a stationary joint distribution $\tilde{\mu}$ extending μ, then define the **entropy increments**

$$\begin{aligned} h_0 &:= H(\xi_0) \\ h_{n+1} &:= H(\xi_0, \xi_1, \ldots, \xi_{n+1}) - H(\xi_0, \xi_1, \ldots, \xi_n), \end{aligned}$$

in particular $h_{N+1} = H(\tilde{\mu}) - H(\mu)$. For $N \to \infty$ we get $h_N \downarrow h_\infty \geq 0$, where h_∞ is called the entropy of the stationary process. See ([Wa79], chap.4) for a comprehensive treatment of the subject.

For any transition operator $S \in \mathcal{S}(q,p)$, where p,q are probability distributions on $\{1, \ldots, a\}$, we define

$$\begin{aligned} H(p, S) &:= \sum_{i=1}^{a} p(i) \cdot H((S_{ik})_{k=1}^{a}) \\ &= -\sum_{i,k} p(i)\, S_{ik} \log S_{ik}. \end{aligned}$$

h_{N+1} is a functional of the stationary extension $N \to N+1$ from μ to $\tilde{\mu}$, more explicit:

$$\begin{aligned} h_{N+1} &= \sum_{|j|=N+1} \mu(j)\ [-\sum_k q^j(k) \log q^j(k)] \\ &= \sum_{i,|j|=N} \mu(ij)\ [-\sum_k q^{ij}(k) \log q^{ij}(k)] \\ &= \sum_{|j|=N} \mu(j)\ [-\sum_{i,k} p^j(i) q^{ij}(k) \log q^{ij}(k)] \\ &= \sum_{|j|=N} \mu(j)\, H(p^j, S^j). \end{aligned}$$

The entropy functional is concave. This implies

$$\begin{aligned} \sum_i p(i)\, H((S_{ik})_{k=1}^{a}) &\leq H(\sum_i p(i)\, (S_{ik})_{k=1}^{a}) \\ i.e. \quad H(p,S) &\leq H(q). \end{aligned}$$

Equality is attained if and only if each row of S equals q, i.e. if p and q are stochastically independent.

Comparing this to our characterizations of the Markovian extension in 4.1.12 we conclude that the Markovian extension is the unique stationary extension with maximum entropy increment.

4.2 Prediction Errors for Commutative Processes

4.2.1 The Problem

Recall the remark in 3.5.4 that the formula for prediction errors given there yields a noncommutative recipe even for commutative processes. Having established in Section 4.1 that stationary commutative processes indeed can be adaptedly implemented which makes the formula applicable we should take a closer look at it and in particular consider the question whether there is a probabilistic interpretation for it in a more traditional sense. We shall make a surprising observation of this kind in 4.2.4, but there are certainly some questions left open in this respect. The results of this section have also been discussed by the author in [Go03]. Some further work on this structure will be included in [GKL] (where we deal with scattering, but this can be translated to the setting here via the $C-D-$correspondence mentioned in 3.5.6).

4.2.2 Prediction for Finite-Valued Processes

Consider a commutative stationary process which is adaptedly implemented on a tensor product filtration, as discussed in 4.1.9 and 4.1.10. For simplicity let us assume for all $n \in \mathbb{N}_0$ that we have $\mathcal{C}_n = \mathbb{C}^{d_n}$, equipped with a state ψ_n given by a probability measure on $\{1, \ldots, d_n\}$ assigning equal weight to each point. As argued in 4.1.10 this can always be achieved if the forward and backward transition probabilities introduced in 4.1.2 are rational numbers. To apply the results of Section 3.5 we must have an LPR $v = \lim_{N\to\infty} u_1 u_2 \ldots u_N$ for the adapted isometry v which is obtained by extension of the time evolution $\alpha = \lim_{N\to\infty} \alpha_1\alpha_2 \ldots \alpha_N$ on $\tilde{\mathcal{A}} = \bigotimes_{n=0}^{\infty} \mathbb{C}^{d_n}$. The elements of $\tilde{\mathcal{A}}$ are functions on the cartesian product of all $\{1, \ldots, d_n\}$, $n = 0, \ldots, \infty$, which is a space of paths equipped with the product probability measure. These are not paths of the process in question however, we only know that its time evolution α is induced by a measure-preserving transformation $\tilde{\tau}$. To a product representation $\alpha = \lim_{N\to\infty} \alpha_1\alpha_2 \ldots \alpha_N$, where the α_n are constructed as in Theorem 4.1.9, there corresponds a product representation $\tilde{\tau} = \lim_{N\to\infty} \tau_N \ldots \tau_1$, where τ_n is a permutation of $\{1, \ldots, d_0\} \times \ldots \times \{1, \ldots, d_n\}$ which does not change the values at positions $0, \ldots, n-2$. Note that the value $(\tilde{\tau}\omega)_n$ is already determined by $\tau_{n+1}\tau_n \ldots \tau_1\omega$, i.e. the limit is well defined. We observe that the LPR-property (i.e. $u_n \in \mathcal{B}(\mathcal{K}_{[n-1,n]})$ for all n, see 3.4.4) is not automatically fulfilled here: It corresponds to the simplifying assumption that τ_n is a permutation of $\{1,\dots,d_{n-1}\}\times\{1,\dots,d_n\}$. We shall return to this subtle distinction in Section 4.6. In this section we shall from now on assume the LPR-property.

4.2.3 A Guessing Game

Here the prediction problem introduced in Section 3.5 can be formulated as a game. If we are given only $\omega_0,\dots,\omega_{N-1}$ of some path $\omega=(\omega_n)_{n=0}^\infty$ then in general it is not possible to determine $(\tilde\tau^N\omega)_0$. We may try to guess. It depends on $\tilde\tau$ how much uncertainty we have to endure. Indeed the prediction errors show the amounts of errors (in the mean square sense) which are inevitable even with the best strategy. More precisely, let ξ be any (complex-valued) function on $\{1,\dots,d_0\}$. Given certain values $\omega_0,\dots,\omega_{N-1}$ there is a probability distribution $\mu_{\omega_0,\dots,\omega_{N-1}}$ on $\{1,\dots,d_0\}$ for $(\tilde\tau^N\omega)_0$ conditioned by these values. Elementary probability theory shows that the best prediction of $\xi((\tilde\tau^N\omega)_0)$ given $\omega_0,\dots,\omega_{N-1}$ is obtained as expectation of ξ with respect to $\mu_{\omega_0,\dots,\omega_{N-1}}$ with the variance $Var(\xi,\mu_{\omega_0,\dots,\omega_{N-1}})$ as squared error. Then the total mean square error $f_N(\xi)$ is obtained by averaging over all possible $\omega_0,\dots,\omega_{N-1}$:

$$f_N(\xi)^2=(d_0\dots d_{N-1})^{-1}\sum_{\omega_0,\dots,\omega_{N-1}}Var(\xi,\mu_{\omega_0,\dots,\omega_{N-1}}).$$

This justifies our interpretation as a game.

4.2.4 A Combinatorial Formula for Prediction Errors

In Theorem 3.5.4 we derived an alternative expression in terms of a product of completely positive maps $D_n: M_{d_{n-1}}\to M_{d_n}$. (The space of trace class operators $\mathcal{T}(\mathbb{C}^d)$ coincides with M_d.) We have

$$f_N(\xi)^2+\langle\Omega_N, D_N\dots D_1(|\xi\rangle\langle\xi|)\,\Omega_N\rangle=\|\xi\|^2.$$

The unit vector Ω_N arises in the GNS-construction from the state and we have $\Omega_N=(d_N)^{-\frac12}(1,1,\dots,1)\in\mathbb{C}^{d_N}$. We want to write down the operators D_n more explicitly. By definition D_n is derived from the permutation τ_n of $\{1,\dots,d_{n-1}\}\times\{1,\dots,d_n\}$ via the isometry $v_n:\mathbb{C}^{d_{n-1}}\to\mathbb{C}^{d_{n-1}}\otimes\mathbb{C}^{d_n}$ such that for $\rho\in M_{d_{n-1}}$ we have $D_n(\rho)=Tr_{n-1}(v_n\,\rho\,v_n^*)$ (see 3.4.11). We want to calculate the entries of D_n with respect to the canonical basis $\{\delta_i\}_{i=1}^{d_{n-1}}$ of $\mathbb{C}^{d_{n-1}}$ and $\{\epsilon_k\}_{k=1}^{d_n}$ of $\mathbb{C}^{d_n}$. Let us write $i\xrightarrow{k}j$ if the first component of $\tau_n(i,k)$ is j. Then a straightforward computation yields (omitting the index n of D_n for the moment):

Lemma: $v_n\,\delta_j \;=\; (d_n)^{-\frac{1}{2}} \sum_{i \xrightarrow{k} j} \delta_i \otimes \epsilon_k$

$$D_{kk',jj'} \;:=\; \langle \epsilon_k, D(|\,\delta_j\rangle\langle\delta_{j'}\,|)\,\epsilon_{k'}\rangle \;=\; \frac{1}{d_n}\,\sharp\,\{i : i \xrightarrow{k} j \;\; and \;\; i \xrightarrow{k'} j'\},$$

where $\sharp$ counts the number of elements.

Proof: The formula for v_n is immediate if we recall that $v_n = u_n|_{\mathcal{K}_{n-1}}$ and how u_n is given by α_n and τ_n. Comparing with the general formula $v\,\xi = \sum_k a_k^*(\xi) \otimes \epsilon_k$ we conclude that in this case

$$a_k^*(\delta_j) \;=\; (d_n)^{-\frac{1}{2}} \sum_{i \xrightarrow{k} j} \delta_i.$$

Now we can use Lemma A.4.3 to get

$$D_{kk',jj'} \;=\; \langle a_{k'}^*\,\delta_{j'}, a_k^*\,\delta_j\rangle \;=\; \frac{1}{d_n}\,\sharp\,\{i : i \xrightarrow{k} j \;\text{ and }\; i \xrightarrow{k'} j'\}. \qquad \square$$

Some observations about these entries of D_n are immediate: There is a symmetry $D_{kk',jj'} = D_{k'k,j'j}$. Further, fixing k, k' and summing over j, j' always yields $\frac{d_{n-1}}{d_n}$, which proves the surprising fact that $\frac{d_n}{d_{n-1}} D_n$ with respect to the canonical basis gives rise to a stochastic $d_n^2 \times d_{n-1}^2$-matrix. Its entries are a kind of transition probabilities for pairs when applying τ_n, refining the transition probabilities for individuals which are included as $D_{kk,jj} = \frac{1}{d_n}\,\sharp\,\{i : i \xrightarrow{k} j\}$.

Putting all this together we have proved the following combinatorial fact which summarizes the computation of prediction errors in this setting:

Proposition: *If $N \in \mathbb{N}$ and if δ_j is the $j-th$ canonical basis vector in C^{d_0} then*

$$f_N(\delta_j)^2 + \frac{1}{d_N} \sum_{k,k'=1}^{d_N} (D_N \ldots D_1)_{kk',jj} = 1.$$

Here $\frac{d_N}{d_0} \sum_{k,k'=1}^{d_N} (D_N \ldots D_1)_{kk',jj}$ is a column sum of a (row-)stochastic $d_N^2 \times d_0^2$-matrix which is given as a product of (row-)stochastic $d_n^2 \times d_{n-1}^2$-matrices corresponding to $\frac{d_n}{d_{n-1}} D_n$.

4.2.5 Asymptotics

It is not clear how the entries of these stochastic matrices can be interpreted probabilistically in terms of the process and the filtration, but for computational purposes at least this observation is useful, for example in describing the asymptotic theory ($N \to \infty$). See [Se81] for some basic facts about stochastic matrices which we need here, in particular: A set of (column-)indices is called essential for a stochastic matrix if by successive transitions allowed by the matrix it is possible to go from any element of the set to any other, but it is not possible to leave the set. An index not contained in an essential set is called inessential.

Proposition: *For the processes considered in this section the following assertions are equivalent:*

(1) The process is deterministic (see 3.5.5)
(2) All entries of the matrices associated to the products $D_N \dots D_1$ which do not belong to an jj-column ($j \in \{1, \dots, d_0\}$) tend to zero for $N \to \infty$.
If the process is homogeneous ($D_n \simeq D$ for all n, see 3.5.6) then also the follwing property is equivalent:
(3) Indices ij with $i \neq j$ are inessential for the stochastic matrix associated to D.

Proof: Determinism means that $f_N(\delta_j) \to 0$ for all $j \in \{1, \dots, d_0\}$ and $N \to \infty$. By Proposition 4.2.4 this is the case if and only if for all $j \in \{1, \dots, d_0\}$ the expressions $\frac{1}{d_N} \sum_{k,k'=1}^{d_N} (D_N \dots D_1)_{kk',jj}$ tend to 1 for $N \to \infty$. Because $\frac{d_N}{d_0}(D_N \dots D_1)$ is stochastic we find that

$$\frac{1}{d_N} \sum_{k,k'=1} \sum_{j,j'=1} (D_N \dots D_1)_{kk',jj'} \; = \; d_0 .$$

A comparison yields (1) $\Leftrightarrow$ (2). Further it is a general fact that for powers of a single stochastic matrix we have the equivalence (2) $\Leftrightarrow$ (3) (see [Se81], chap.4). □

4.2.6 Example

Especially condition (3) of Proposition 4.2.5 is very easy to check, at least for matrices of moderate size. We give an example:

Choose $d_n = 3$ for all n and consider the homogeneous process generated by the permutation τ of $\{1, 2, 3\}^2$ given by the cycle

$$(11, 12, 13, 23, 22, 21, 31, 32, 33).$$

Using Lemma 4.2.4 we can compute the associated stochastic matrix. The result is shown below (indices ordered as follows: $11, 22, 33, 12, 21, 13, 31, 23, 32$). For example the non-zero entries in the fourth row (with index 12) are obtained from $1 \xrightarrow{1} 1$, $1 \xrightarrow{2} 1$ and $3 \xrightarrow{1} 3$, $3 \xrightarrow{2} 3$ and $2 \xrightarrow{1} 3$, $2 \xrightarrow{2} 2$.

$$\frac{1}{3}\begin{pmatrix} 1&0&2&0&0&0&0&0&0 \\ 1&1&1&0&0&0&0&0&0 \\ 1&2&0&0&0&0&0&0&0 \\ 1&0&1&0&0&0&0&0&1 \\ 1&0&1&0&0&0&0&1&0 \\ 0&0&0&1&0&0&1&0&1 \\ 0&0&0&0&1&1&0&1&0 \\ 0&1&0&1&0&0&1&0&0 \\ 0&1&0&0&1&1&0&0&0 \end{pmatrix}$$

It is easy to check that starting from any index ij we can in at most two steps reach the essential set $\{11, 22, 33\}$. With Proposition 4.2.5(3) it follows that the process is deterministic.

If we want to generalize to the inhomogeneous case then we may use the theory of inhomogeneous products of stochastic matrices (see [Se81], chap.3) to get explicit criteria for determinism.

4.3 Low-Dimensional Examples

4.3.1 Qubits

In this section we want to illustrate results from Chapter 3 for a class of low-dimensional examples: We consider adapted isometries v on $\tilde{\mathcal{K}} = \bigotimes_0^\infty \mathbb{C}^2$ with LPR $v = \lim_{N\to\infty} u_1 \dots u_N$. Fix an ONB $\{\Omega_n = |0\rangle, |1\rangle\}$ for each $\mathcal{K}_n = \mathbb{C}^2$. The unitary $u_n : \mathbb{C}^2 \otimes \mathbb{C}^2 \to \mathbb{C}^2 \otimes \mathbb{C}^2$ is a $4 \times 4-$matrix with respect to $|00\rangle, |10\rangle, |01\rangle, |11\rangle$:

$$u_n \quad = \quad \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & u_{11} & u_{12} & u_{13} \\ 0 & u_{21} & u_{22} & u_{23} \\ 0 & u_{31} & u_{32} & u_{33} \end{pmatrix}$$

For the entries we drop the index n if it is clear from the context, otherwise we write $u_{ij}^{(n)}$. Note that in quantum information theory [NC00], a space $\mathcal{K} = \mathbb{C}^2$ represents a qubit, and then unitaries such as u_n represent logical gates. Our quantum circuits are special in such a way that the gates always act in a certain sequential way.

Let us now proceed in the way established in Chapter 3.

$$\begin{aligned} v_n &: \mathbb{C}^2 \to \mathbb{C}^2 \otimes \mathbb{C}^2, \ |1\rangle \ \mapsto \ u_{11}|10\rangle + u_{21}|01\rangle + u_{31}|11\rangle, \\ \bar{v}_n &: \mathbb{C}^2 \to \mathbb{C}^2 \otimes \mathbb{C}^2, \ |1\rangle \ \mapsto \ u_{12}|10\rangle + u_{22}|01\rangle + u_{32}|11\rangle, \\ v'_n &: \mathbb{C}^2 \to \mathbb{C}^2 \otimes \mathbb{C}^2, \ |1\rangle \ \mapsto \ \bar{u}_{11}|10\rangle + \bar{u}_{12}|01\rangle + \bar{u}_{13}|11\rangle, \\ \bar{v}'_n &: \mathbb{C}^2 \to \mathbb{C}^2 \otimes \mathbb{C}^2, \ |1\rangle \ \mapsto \ \bar{u}_{21}|10\rangle + \bar{u}_{22}|01\rangle + \bar{u}_{23}|11\rangle, \end{aligned}$$

$$v_n \begin{pmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{pmatrix} v_n^* = \begin{pmatrix} \rho_{11} & \bar{u}_{11}\rho_{12} & \bar{u}_{21}\rho_{12} & \bar{u}_{31}\rho_{12} \\ u_{11}\rho_{21} & |u_{11}|^2 \rho_{22} & u_{11}\bar{u}_{21}\rho_{22} & u_{11}\bar{u}_{31}\rho_{22} \\ u_{21}\rho_{21} & u_{21}\bar{u}_{11}\rho_{22} & |u_{21}|^2 \rho_{22} & u_{21}\bar{u}_{31}\rho_{22} \\ u_{31}\rho_{21} & u_{31}\bar{u}_{11}\rho_{22} & u_{31}\bar{u}_{21}\rho_{22} & |u_{31}|^2 \rho_{22} \end{pmatrix},$$

similar for $\bar{v}_n, v'_n, \bar{v}'_n$.

Note that if $| u_{11}^{(n)} |= 1$ for some n then $\mathcal{K}_{[0,n-1]}$ is an invariant subspace for v. In this case a stochastic process with $\mathcal{K}_0$ representing time zero and described by v (see Section 3.3) 'sees' only a finite part of the filtration. Let us neglect this exceptional case in the following and assume that $| u_{11}^{(n)} |< 1$ for all n. Then we also have $\mathcal{G}_n = \mathcal{G}_n^n = \mathcal{K}_n$, see 3.4.7 and 3.4.8.

4.3.2 Associated Stochastic Maps

Using the preparations in 4.3.1 we can compute the operators C_n, D_n, $\bar{C}_n$, $\bar{D}_n$, E_n, F_n, $\bar{E}_n$, $\bar{F}_n$: $M_2 \to M_2$. With $\rho = \begin{pmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{pmatrix} \in M_2$ we get

$$C_n(\rho) = \begin{pmatrix} \rho_{11} + |u_{21}|^2 \rho_{22} & \bar{u}_{11}\,\rho_{12} + u_{21}\,\bar{u}_{31}\,\rho_{22} \\ u_{11}\,\rho_{21} + \bar{u}_{21}\,u_{31}\,\rho_{22} & (|u_{11}|^2 + |u_{31}|^2)\rho_{22} \end{pmatrix}$$

$$D_n(\rho) = \begin{pmatrix} \rho_{11} + |u_{11}|^2 \rho_{22} & \bar{u}_{21}\,\rho_{12} + u_{11}\,\bar{u}_{31}\,\rho_{22} \\ u_{21}\,\rho_{21} + \bar{u}_{11}\,u_{31}\,\rho_{22} & (|u_{21}|^2 + |u_{31}|^2)\rho_{22} \end{pmatrix}$$

$$\bar{C}_n(\rho) = \begin{pmatrix} \rho_{11} + |u_{22}|^2 \rho_{22} & \bar{u}_{12}\,\rho_{12} + u_{22}\,\bar{u}_{32}\,\rho_{22} \\ u_{12}\,\rho_{21} + \bar{u}_{22}\,u_{32}\,\rho_{22} & (|u_{12}|^2 + |u_{32}|^2)\rho_{22} \end{pmatrix}$$

$$\bar{D}_n(\rho) = \begin{pmatrix} \rho_{11} + |u_{12}|^2 \rho_{22} & \bar{u}_{22}\,\rho_{12} + u_{12}\,\bar{u}_{32}\,\rho_{22} \\ u_{22}\,\rho_{21} + \bar{u}_{12}\,u_{32}\,\rho_{22} & (|u_{22}|^2 + |u_{32}|^2)\rho_{22} \end{pmatrix}$$

$$E_n(\rho) = \begin{pmatrix} \rho_{11} + |u_{12}|^2 \rho_{22} & u_{11}\,\rho_{12} + \bar{u}_{12}\,u_{13}\,\rho_{22} \\ \bar{u}_{11}\,\rho_{21} + u_{12}\,\bar{u}_{13}\,\rho_{22} & (|u_{11}|^2 + |u_{13}|^2)\rho_{22} \end{pmatrix}$$

$$F_n(\rho) = \begin{pmatrix} \rho_{11} + |u_{11}|^2 \rho_{22} & u_{12}\,\rho_{12} + \bar{u}_{11}\,u_{13}\,\rho_{22} \\ \bar{u}_{12}\,\rho_{21} + u_{11}\,\bar{u}_{13}\,\rho_{22} & (|u_{12}|^2 + |u_{13}|^2)\rho_{22} \end{pmatrix}$$

$$\bar{E}_n(\rho) = \begin{pmatrix} \rho_{11} + |u_{22}|^2 \rho_{22} & u_{21}\,\rho_{12} + \bar{u}_{22}\,u_{23}\,\rho_{22} \\ \bar{u}_{21}\,\rho_{21} + u_{22}\,\bar{u}_{23}\,\rho_{22} & (|u_{21}|^2 + |u_{23}|^2)\rho_{22} \end{pmatrix}$$

$$\bar{F}_n(\rho) = \begin{pmatrix} \rho_{11} + |u_{21}|^2 \rho_{22} & u_{22}\,\rho_{12} + \bar{u}_{21}\,u_{23}\,\rho_{22} \\ \bar{u}_{22}\,\rho_{21} + u_{21}\,\bar{u}_{23}\,\rho_{22} & (|u_{22}|^2 + |u_{23}|^2)\rho_{22} \end{pmatrix}$$

All these operators have a similar shape. In fact, every Tr-preserving completely positive map on M_2 fixing $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ has this shape with suitable coefficients.

4.3.3 Determinism and Unitarity

Proposition: *With* $|\, u_{11}^{(n)} \,| < 1$ *for all* n*, the following assertions are equivalent:*

(1) v *is deterministic.*
(2) v *is unitary.*
(3) $\sum_{n=1}^{\infty} |\, u_{11}^{(n)} \,|^2 = \infty$.

Proof: Let us write $X_n(\rho) = \begin{pmatrix} \rho_{11} + x_n\, \rho_{22} & * \\ * & (1 - x_n)\, \rho_{22} \end{pmatrix}$ for any of the completely positive maps on M_2 described in 4.3.2. Obviously for $\rho \in \mathcal{T}_+^1(\mathcal{K}_m)$ the following assertions are equivalent:

(a) $\lim_{N\to\infty} X_N \dots X_{m+1}(\rho) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$
(b) $\lim_{N\to\infty} \big(X_N \dots X_{m+1}(\rho)\big)_{11} = 1$
(c) $\lim_{N\to\infty} \big(X_N \dots X_{m+1}(\rho)\big)_{22} = 0$
(d) $\lim_{N\to\infty} (1 - x_N) \dots (1 - x_{m+1})\, \rho_{22} = 0$

If we insert D_n or F_n for X_n then $x_n = |\, u_{11}^{(n)} \,|^2 < 1$. Then (d) is valid for all m and all ρ if and only if the infinite product $\prod_{n=1}^{\infty} (1 - x_n)$ equals zero, which means that $\sum_{n=1}^{\infty} x_n = \infty$. The corresponding assertion (a) for the D_n means that v is deterministic (see Proposition 3.5.5) and the corresponding assertion (a) for the F_n means that v is unitary (see Theorem 3.6.4). □

4.3.4 Complete Invariants

To reconsider the invariants Λ_v and Λ_{v^*} from Sections 3.4 and 3.6, we need the following additive version of 3.4.6.

Lemma: *Let* $A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$, $A' = \begin{pmatrix} A'_{11} & A'_{12} \\ A'_{21} & A'_{22} \end{pmatrix}$ *be block matrices acting on a vector space* $X \oplus Y$ *and* $B = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}$, $B' = \begin{pmatrix} B'_{11} & B'_{12} \\ B'_{21} & B'_{22} \end{pmatrix}$ *acting on a vector space* $Y \oplus Z$*. Extending by the identity, all of these act on* $X \oplus Y \oplus Z$*.*

If $A_{12} + A_{22},\ A'_{12} + A'_{22} : Y \to X \oplus Y$ *are injective and* $B_{11} + B_{12},\ B'_{11} + B'_{12} : Y \oplus Z \to Y$ *are surjective, then the following assertions are equivalent:*

(1) $AB = A'B'$
(2) There is a (unique) bijection $C : Y \to Y$ *such that* $A' = AC^{-1}$ *and* $B' = CB$ *(with* C *extended by the identity).*

Proof: (2) $\Rightarrow$ (1) is immediate. Now assume (1). If for $b_1 \in Y \oplus Z$

$$B(b_1) = y \oplus z_1,$$
$$B'(b_1) = y'_1 \oplus z'_1,$$

then by (1) we have $z_1 = z'_1$ and we would like to define $C(y) := y'_1$. We have to check that this is well defined. If for $b_2 \in Y \oplus Z$

$$B(b_2) = y \oplus z_2,$$
$$B'(b_2) = y'_2 \oplus z'_2,$$

then by (1) we have $z_2 = z'_2$ and we find further that

$$A'(0 \oplus y'_2) \;=\; A(0 \oplus y) \;=\; A'(0 \oplus y'_1).$$

Because $A'_{12} + A'_{22}$ is injective it follows that $y'_2 = y'_1$, i.e. C is well defined. It is defined for all $y \in Y$ because $B_{11} + B_{12}$ is surjective. Reversing the roles of A and A', B and B', we also find the inverse C^{-1}. □

Proposition: *Assume that* $u_{ij}^{(n)} \neq 0$ *for all* $i, j = 1, 2, 3$ *and* $n \in \mathbb{N}$*. Then the pair* $(\Lambda_v, \Lambda_{v^*})$ *is a complete invariant for* v *(see 3.4.11 and 3.6.1).*

Proof: We show that in our special situation the properties $(1), \ldots, (5)$ in the proof of Proposition 3.4.11 are all equivalent. Using the assumption, we can check that $C_n, D_n, \ldots, \bar{F}_n$ are bijections on M_2. In fact, if for example C_n is represented by a $4 \times 4-$matrix with respect to the basis of matrix units of M_2 then the determinant is $\mid u_{11}^{(n)} \mid^2 (1- \mid u_{21}^{(n)} \mid^2) \neq 0$. Thus all entries of Λ_v and Λ_{v^*} are bijections. Now let Λ_v be given. By the remark at the end of 3.4.11 we then know that on $\bigoplus_0^N M_2$ two different product representations are related by

$$\prod_{n=1}^{N} \begin{pmatrix} C_n^\dagger & \bar{D}_n^\dagger \\ D_n^\dagger & \bar{C}_n^\dagger \end{pmatrix} = \prod_{n=1}^{N} \begin{pmatrix} C_n & \bar{D}_n \\ D_n & \bar{C}_n \end{pmatrix} \circ Ad\, w_N^*. \qquad (*)$$

Because all the entries are bijective, we can apply the lemma above for the decomposition

$$\prod_{n=1}^{N-1} \begin{pmatrix} C_n^\dagger & \bar{D}_n^\dagger \\ D_n^\dagger & \bar{C}_n^\dagger \end{pmatrix} \cdot \begin{pmatrix} C_N^\dagger & \bar{D}_N^\dagger \\ D_N^\dagger & \bar{C}_N^\dagger \end{pmatrix} \quad = \quad \prod_{n=1}^{N-1} \begin{pmatrix} C_n & \bar{D}_n \\ D_n & \bar{C}_n \end{pmatrix} \cdot \left[\begin{pmatrix} C_N & \bar{D}_N \\ D_N & \bar{C}_N \end{pmatrix} \circ Ad\, w_N^* \right]$$

and find a bijection C on M_2 (at position $N-1$) such that

$$\prod_{n=1}^{N-1} \begin{pmatrix} C_n^\dagger & \bar{D}_n^\dagger \\ D_n^\dagger & \bar{C}_n^\dagger \end{pmatrix} = \prod_{n=1}^{N-1} \begin{pmatrix} C_n & \bar{D}_n \\ D_n & \bar{C}_n \end{pmatrix} \circ C^{-1},$$
$$\begin{pmatrix} C_N^\dagger & \bar{D}_N^\dagger \\ D_N^\dagger & \bar{C}_N^\dagger \end{pmatrix} = C \circ \begin{pmatrix} C_N & \bar{D}_N \\ D_N & \bar{C}_N \end{pmatrix} \circ Ad\, w_N^*.$$

Comparing with $(*)$ above for $N-1$ instead of N we find that $C^{-1} = Ad\, w_{N-1}^*$. Thus

$$\begin{pmatrix} C_N^\dagger & \bar{D}_N^\dagger \\ D_N^\dagger & \bar{C}_N^\dagger \end{pmatrix} \;=\; Ad\, w_{N-1} \circ \begin{pmatrix} C_N & \bar{D}_N \\ D_N & \bar{C}_N \end{pmatrix} \circ Ad\, w_N^*.$$

Now (C_N, D_N) determines v_N. In fact, inspection of the formulas for C_N, D_N shows that on the diagonals the absolute values $|\,u_{11}^{(N)}\,|$, $|\,u_{21}^{(N)}\,|$, $|\,u_{31}^{(N)}\,|$ are determined, while the off-diagonal elements give the phase relations. Thus from $C_N^\dagger = Ad\, w_{N-1} \circ C_N,\ D_N^\dagger = D_N$ we conclude that $v_N^\dagger = w_{N-1} v_N$. Similarly from $(\bar{C}_N, \bar{D}_N)$ we get $\bar{v}_N^\dagger = w_{N-1} \bar{v}_N w_N^*$. Applying the same argument to Λ_{v^*} instead of Λ_v we get $(v'_N)^\dagger = w_{N-1} v'_N w_N^*$ and $(\bar{v}'_N)^\dagger = \bar{v}'_N w_N^*$. An inspection of the unitary matrix u_N shows that v_N, $\bar{v}_N$, v'_N, $\bar{v}'_N$ determine u_N. Thus $u_N^\dagger = w_{N-1} u_N w_N^*$ (for all N). By Theorem 3.4.9 this determines v. □

4.3.5 Probabilistic Interpretations

As remarked in Section 3.4.2, we can always think of an adapted isometry as being derived from a stationary process if we allow states which are not faithful. For faithful states, a probabilistic interpretation amounts to introducing a product on the Hilbert space $\mathcal{K}_{[n-1,n]} = \mathbb{C}^2 \otimes \mathbb{C}^2$ which makes it a von Neumann algebra. There are only two possibilities, the commutative algebra $\mathbb{C}^4 = \mathbb{C}^2 \otimes \mathbb{C}^2$ and the matrix algebra M_2. The commutative case has been treated in detail in Section 4.1, in particular in 4.1.11, an automorphism of $\mathbb{C}^2 \otimes \mathbb{C}^2$ corresponds to a permutation of 4 points. Examples of the second kind arise in the following Section 4.4.

4.4 Clifford Algebras and Generalizations

4.4.1 Clifford Algebras

Consider the noncommutative probability space $(\tilde{\mathcal{A}}, tr)$ with $\tilde{\mathcal{A}}$ equal to the weak closure of $\bigotimes_{n=0}^\infty M_2$ with respect to the tracial state tr. This von Neumann algebra is isomorphic to the hyperfinite factor of type II_1 (see [Sa71], 4.4.6). In each M_2 we have the Pauli matrices

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

To relate this to the setting of Section 4.3 we think of each M_2 as generated by the subalgebras $span\{\mathbb{1}, \sigma_x\}$ and $span\{\mathbb{1}, \sigma_y\}$. Further, our filtration is

$$span\{\mathbb{1}, \sigma_x\} \otimes \mathbb{1} \otimes \mathbb{1} \ldots \subset M_2 \otimes \mathbb{1} \otimes \mathbb{1} \ldots \subset M_2 \otimes span\{\mathbb{1}, \sigma_x\} \otimes \mathbb{1} \ldots \subset \ldots$$

Using a kind of Jordan-Wigner isomorphism we can think of $\tilde{\mathcal{A}}$ as a Clifford algebra (see [PR94]), and this suggests the following correspondence between certain anti-commuting elements $(e_n)_{n=0}^\infty$ of $\tilde{\mathcal{A}}$ and canonical basis vectors of $\tilde{\mathcal{K}} = \bigotimes_0^\infty \mathbb{C}^2$:

$$
\begin{aligned}
\mathbb{1} \otimes \mathbb{1} \otimes \mathbb{1} \otimes \ldots &\simeq |0\rangle = \tilde{\Omega} \\
e_0 = \sigma_x \otimes \mathbb{1} \otimes \mathbb{1} \otimes \ldots &\simeq |1\rangle \\
e_1 = \sigma_y \otimes \mathbb{1} \otimes \mathbb{1} \otimes \ldots &\simeq |01\rangle \\
e_2 = \sigma_z \otimes \sigma_x \otimes \mathbb{1} \otimes \ldots &\simeq |001\rangle \\
e_3 = \sigma_z \otimes \sigma_y \otimes \mathbb{1} \otimes \ldots &\simeq |0001\rangle \\
e_4 = \sigma_z \otimes \sigma_z \otimes \sigma_x \otimes \ldots &\simeq |00001\rangle \\
&\textit{etc.}
\end{aligned}
$$

Except for $|0\rangle$ itself, zeros on the right are omitted in this notation. The scheme can be completed by letting $e_{i_1} \ldots e_{i_n}$ with $i_1 < \ldots < i_n$ correspond to a vector with 1 exactly at positions $i_1, \ldots, i_n$.
If we now define a unitary u_1 acting on $|0\rangle$, $|1\rangle, |01\rangle$, $|11\rangle$ as

$$
u_1 \quad := \quad \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi & 0 \\ 0 & \sin\phi & \cos\phi & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}
$$

where the $2 \times 2-$matrix $T = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}$ in the middle of u_1 is a rotation of a two-dimensional real plane, then via the correspondence above the unitary u_1 is also an automorphism α_1 of M_2, namely a so-called Bogoljubov transformation of the Clifford algebra generated by σ_x and σ_y. We can proceed similarly on 00, 10, 01, 11 at positions $n-1$ and n to get $\alpha_n \simeq u_n$, and we find an LPR

$$
\alpha = \lim_{N\to\infty} \alpha_1 \ldots \alpha_N \quad \simeq \quad v = \lim_{N\to\infty} u_1 \ldots u_N .
$$

This is a special case of the **Clifford functor** which maps real linear spaces with orthogonal transformations to Clifford algebras with a special class of homomorphisms, the Bogoljubov transformations. See [PR94].

4.4.2 The Clifford Functor

It is instructive to reconsider Proposition 4.3.3 in this example. Suppose the α_n are constructed from T_n as shown above. Then with the Clifford functor the adapted endomorphism α is the image of the orthogonal transformation

$$
T = \lim_{N\to\infty} T_1 \ldots T_N
$$

on $\mathbb{R} \oplus \mathbb{R} \oplus \ldots = l^2_{\mathbb{R}}(\mathbb{N}_0)$.

T is a real version of an (additive) adapted isometry as considered in Section 3.1. If for all n

$$T_n = \begin{pmatrix} \cos\phi_n & -\sin\phi_n \\ \sin\phi_n & \cos\phi_n \end{pmatrix}$$

for some angle ϕ_n, then we find that

$$T \text{ bijective} \quad \Leftrightarrow \quad \sum_{n=1}^{\infty} \cos^2\phi_n = \infty \quad \Leftrightarrow \quad \alpha \text{ bijective}.$$

The equivalence on the left can be derived from the remarks on determinism and unitarity in subsection 3.1.4 applied to T and the equivalence on the right follows by Proposition 4.3.3 applied to α. Of course the equivalence of the left and the right side follows also directly from the Clifford functor.

4.4.3 Generalized Clifford Algebras

The Clifford algebra example above suggests to look what happens for some direct generalizations of Clifford algebras. In fact we have considered such an approach to the subject of adapted endomorphisms in [Go01], before constructing the general theory described in Chapter 3. Due to some special features such as grading and commutation relations, more is known about it than for the general case. Let us briefly introduce the setting. Then by using the results of Chapter 3 we can give more conceptual and transparent proofs for prediction errors and unitarity than those in [Go01]. These proofs are therefore sketched. We also briefly describe some additional results obtained in [Go01]. This introduces some problems that may be interesting also for other adapted endomorphisms.

We consider the same noncommutative probability space $(\tilde{\mathcal{A}}, tr)$ as before but we generalize 4.4.1 by thinking of $\tilde{\mathcal{A}}$ as $\bigotimes_{n=0}^{\infty} M_d$ for some $2 \leq d \in \mathbb{N}$. It can be generated by a sequence of unitaries $(e_n)_{n=0}^{\infty}$ satisfying $e^p = 1\!\!1$ for all n and $e_m e_n = \exp(\frac{2\pi i}{d}) e_n e_m$ for all $m < n$. The filtration successively generated by the sequence can be described abstractly as

$$\mathbb{C}^d \otimes 1\!\!1 \otimes 1\!\!1 \ldots \subset M_d \otimes 1\!\!1 \otimes 1\!\!1 \ldots \subset M_d \otimes \mathbb{C}^d \otimes 1\!\!1 \ldots \subset \ldots$$

The elements e_{n-1} and e_n generate an algebra isomorphic to M_d. For all $n \in \mathbb{N}$ we want to define automorphisms α_n on this M_d such that $\alpha = \lim_{N\to\infty} \alpha_1 \ldots \alpha_N$ is an LPR for an adapted endomorphism α. In ([Go01], sect.1) it is shown that this works if and only if for all n the automorphism α_n is of the form $Ad\, U_n$ with U_n a unitary in the commutative algebra generated by $e^*_{n-1} e_n$. Corresponding to $\{e^j_m\}_{j=0}^{d-1}$ we have an ONB $\{\Omega_m = |0\rangle, |1\rangle, \ldots, |d-1\rangle\}$ of $\mathcal{K}_m \simeq \mathbb{C}^d$, similar as in 4.4.1. Computing the LPR it turns out that

$$v_n\,|j\rangle = \sum_{k=0}^{d-1} \gamma^{(n)}_{jk}\,|j-k, k\rangle$$

with complex numbers $\gamma_{jk}^{(n)}$ satisfying $\sum_{k=0}^{d-1} \mid \gamma_{jk}^{(n)} \mid^2 = 1$ for all j. Now we apply A.4. Comparing with the general formula $v_n\, \xi = \sum_k a_k^*(\xi) \otimes \epsilon_k$ we find that

$$a_k^* \, |j\rangle \quad = \quad \gamma_{jk}^{(n)} \, |j-k\rangle$$

and that the entries of $D_n : M_d \to M_d$ satisfy

$$\begin{aligned} D_{kk',jj'} &= \langle a_{k'}^* \; j', a_k^* \; j \rangle \\ &= \langle \gamma_{j'k'}^{(n)} \; j' - k', \gamma_{jk}^{(n)} \; j - k \rangle. \end{aligned}$$

Note that $D_{kk',jj} = \delta_{kk'} \mid \gamma_{jk}^{(n)} \mid^2$, which shows that D_n maps the diagonal into itself. Introducing the stochastic matrix S_n with $(S_n)_{jk} :=\mid \gamma_{jk}^{(n)} \mid^2$ and computing prediction errors according to Theorem 3.4.9

$$\begin{aligned} f_N(|j\rangle) &= 1 - \langle \Omega_N, D_N \dots D_1(|j\rangle\langle j|)\, \Omega_N \rangle \\ &= 1 - (S_1 \dots S_N)_{j0} \end{aligned}$$

we see that this depends only on the just introduced stochastic matrices, and that the process with $span\{e_0^j\}_{j=0}^{d-1}$ representing the time zero variables and adaptedly implemented by α is deterministic if and only if 0 is absorbing for $(S_1 \dots S_N)_{N=1}^\infty$ in the sense that

$$\lim_{N\to\infty} (S_1 \dots S_N)_{j0} \;=\; 1 \quad \text{for all } j = 0, \dots d-1.$$

If $\max_{j=1,\dots,d-1} | \gamma_{j0}^{(n)} | < 1$ for all n then the canonical filtration of the process coincides with the filtration specified in advance by the sequence $(e_n)_{n=0}^\infty$, see ([Go01], 2.1) Then by using F_n instead of D_n one finds that the criterion for determinism also characterizes the unitarity of the adapted isometry v. In this case $\alpha = \lim_{N\to\infty} \alpha_1 \dots \alpha_N$ is an automorphism of $\tilde{\mathcal{A}}$. It is further possible to characterize when α is an inner automorphism of the factor $\tilde{\mathcal{A}}$ (see [Go01], sect.3) which is an analogue of Blattner's theorem about Clifford algebras (see [Bl58, PR94]).

On the other hand, if α is not surjective then its image is a subfactor of $\tilde{\mathcal{A}}$. In ([Go01], sect.4), with an additional technical condition imposed, its Jones index is computed to be d (in the notation used here). There is also given a sufficient condition for α to be a shift in the sense of Powers.

4.5 Tensor Products of Matrices

4.5.1 Matrix Filtrations

As in section 4.4 our noncommutative probability space is $(\tilde{\mathcal{A}}, tr)$, where $\tilde{\mathcal{A}}$ is the weak closure of $\bigotimes_{n=0}^\infty M_d$ with respect to tr, which is isomorphic to the hyperfinite factor of type II_1. But now we consider the filtration

$$\mathcal{C}_0 = M_d \otimes 1\!\!1 \otimes 1\!\!1 \otimes \dots \subset \mathcal{C}_{[0,1]} = M_d \otimes M_d \otimes 1\!\!1 \otimes \dots \subset \mathcal{C}_{[0,2]} = M_d \otimes M_d \otimes M_d \otimes \dots$$

Again we want to study the adapted endomorphisms.

4.5.2 LPR Is Automatic

Lemma: *If $\gamma : M_n \to M_n \otimes M_d$ is a (unital $^*-$)homomorphism then there exists an automorphism $\alpha : M_n \otimes M_d \to M_n \otimes M_d$ with $\gamma = \alpha|_{M_n \otimes \mathbb{1}}$.*

Proof: Let $\gamma(M_n)'$ be the commutant of $\gamma(M_n)$ in $M_n \otimes M_d$. Because $\gamma(M_n)$ is of type I, there is an isomorphism

$$\lambda : \gamma(M_n) \otimes \gamma(M_n)' \to M_n \otimes M_d, \quad x \otimes x' \mapsto xx',$$

see [Sa71], 1.22.14. By a dimension argument $\gamma(M_n)'$ is isomorphic to M_d. Writing $\gamma' : M_d \to \gamma(M_n)'$ for such an isomorphism we can define $\alpha := \lambda \circ (\gamma \otimes \gamma')$. □

Proposition: *Any wide sense adapted endomorphism α (i.e. $\alpha(\mathcal{C}_{[0,n]}) \subset \mathcal{C}_{[0,n+1]}$ for all n) is adapted with LPR, i.e. for all n there exists an automorphism $\alpha_{n+1} = Ad\, U_{n+1}$ with a unitary $U_{n+1} \in \mathcal{C}_{[n,n+1]}$ such that $\alpha = \lim_{N\to\infty} \alpha_1 \ldots \alpha_N$.*

Proof: Using the lemma we can for all n extend the homomorphism $\alpha|_{\mathcal{C}_{[0,n]}} : \mathcal{C}_{[0,n]} \to \mathcal{C}_{[0,n+1]}$ to an automorphism $\tilde{\alpha}_{n+1}$ of $\mathcal{C}_{[0,n+1]}$. This shows that we can argue analogous as in Proposition 3.4.3 to get adaptedness with PR, but now on the level of algebra morphisms:

$$\alpha = \lim_{n\to\infty} \tilde{\alpha}_{n+1}$$

with $\tilde{\alpha}_{n+1} = Ad\, \tilde{U}_{n+1}$, $\tilde{U}_{n+1} \in \mathcal{C}_{[0,n+1]}$ unitary and $\alpha|_{\mathcal{C}_{[0,n]}} = \tilde{\alpha}_{n+1}|_{\mathcal{C}_{[0,n]}}$. We can write this in terms of factors, i.e.

$$\alpha = \lim_{N\to\infty} \alpha_1 \ldots \alpha_N$$

where $\alpha_{n+1} = Ad\, U_{n+1}$ with a unitary $U_{n+1} \in \mathcal{C}_{[0,n+1]}$. By 3.2.7(f) we have $\alpha_{n+1}|_{\mathcal{C}_{[0,n-1]}} = \mathbb{1}|_{\mathcal{C}_{[0,n-1]}}$. Using now the fact that we have a matrix filtration, this implies that U_{n+1} commutes with $\mathcal{C}_{[0,n-1]}$ and thus $U_{n+1} \in \mathcal{C}_{[n,n+1]} = M_d \otimes M_d$ at positions n and $n+1$. $\big($ As always we identify U_{n+1} with $\mathbb{1}_{[0,n-1]} \otimes U_{n+1} \otimes \mathbb{1}_{[n+2,\infty]}$. $\big)$ But this means that the PR is actually an LPR. □

Of course the associated adapted isometry v is then also adapted with LPR:

$$v = \lim_{N\to\infty} u_1 \ldots u_N.$$

Thus in this setting we have the very satisfactory situation that our theory of LPR's developed in Chapter 3 applies to all wide sense adapted endomorphisms. In detail, we have here $K_n = \mathbb{C}^d \otimes \mathbb{C}^d$ and $\Omega_n = d^{-\frac{1}{2}} \sum_{k=1}^d \delta_k \otimes \delta_k$, where δ_k are the canonical basis vectors in $\mathbb{C}^d$. The n-th copy of M_d in $\tilde{\mathcal{A}}$ acts as $M_d \otimes \mathbb{1}$ on K_n. As shown in Lemma 1.7.4 we have $u_n = U_n \otimes \bar{U}_n$, from which all quantities of the general theory may be computed.

4.5.3 Associated Stochastic Maps

It is particularly interesting to determine when α is not surjective because, as already mentioned in 4.4.3, this gives us recipes to construct subfactors. To use the results from Section 3.6 we have to compute the operators F_n introduced there. Here they map $M_d \otimes M_d$ into $M_d \otimes M_d$. As noted in 3.6.3, the projections g_n used in the definition of F_n can be ignored for our purpose and thus we must compute the operator $F_n = Tr_{n-1}(v'_n \cdot (v'_n)^*)$. In fact, we can use computations made elsewhere. Let us drop the index n for the moment. For the pair (E, F) we can use the results about the pair (C, D) collected in A.4. The operator E is a preadjoint of an extended transition operator $Z(x) = (v')^* \, x \otimes \mathbb{1} \, v'$, compare 1.6.4 and 3.6.1. Therefore we can use the computation in 1.7.4 to get

$$E(\rho) \quad = \quad \sum_{i,j=1}^{d} b_{ij}^* \, \rho \, b_{ij}$$

$$\text{with} \quad b_{ij} \;=\; d^{-1} \sum_{k=1}^{d} U_{ki} \otimes \bar{U}_{kj} \quad \text{for all } i, j.$$

Then by Lemma A.4.3 we find

$$F(\rho) \quad = \quad \sum_{r,s=1}^{d} \check{b}_{rs}^* \, \rho \, \check{b}_{rs}$$

$$\text{with} \quad \big(\check{b}_{rs}\big)_{tu,ij} \;=\; \big(b_{ij}\big)_{tu,rs} \;=\; d^{-1} \sum_{k=1}^{d} (U_{ki})_{tr} (\bar{U}_{kj})_{us}.$$

In other words

$$\check{b}_{rs} \;=\; d^{-1} \sum_{k=1}^{d} V_{kr} \otimes \bar{V}_{ks}$$

with $(V_{kr})_{ti} \;:=\; (U_{ki})_{tr}$.

Less formally, if we think of U as a $d^2 \times d^2-$matrix with $d \times d-$blocks U_{ki} and then exchange the $ir-$column by the $ri-$column for all i, r, then the new $d \times d-$blocks are the V_{kr}. We can also obtain this directly from the fact that F may be identified with E for the automorphism $\kappa \circ Ad\,U$ instead of $Ad\,U$, where κ is the flip on $M_d \otimes M_d$, i.e. $\kappa(a \otimes b) = b \otimes a$. Reintroducing the index n we write $U_{ki}^{(n)}$ and $V_{kr}^{(n)}$.

4.5.4 Non-surjectivity

Of course the fundamental example of a non-surjective adapted endomorphism α is the tensor shift σ. For example for $d = 2$ we have $\sigma = \lim_{N \to \infty} Ad\,(U_1 \dots U_N)$ with

$$U_n \quad = \quad \begin{pmatrix} 1\,0\,0\,0 \\ 0\,0\,1\,0 \\ 0\,1\,0\,0 \\ 0\,0\,0\,1 \end{pmatrix} \qquad \textit{for all } n.$$

There is a large class of non-surjective α containing σ:

Proposition: *Assume that for all* $n \in \mathbb{N}$

$$V_{kr}^{(n)} \;=\; 0 \quad \textit{if } k \neq r.$$

Then $\alpha = \lim_{N\to\infty} Ad\,(U_1 \dots U_N)$ *is not surjective.*

For example in the case $d = 2$ this means that $U = U_n$ has the form

$$\begin{pmatrix} u_{11} & 0 & u_{13} & 0 \\ u_{21} & 0 & u_{23} & 0 \\ 0 & u_{32} & 0 & u_{34} \\ 0 & u_{42} & 0 & u_{44} \end{pmatrix}.$$

Proof: If α is surjective then by Theorem 3.6.4 for all $\rho_m \in \mathcal{T}_+^1(\mathcal{K}_m)$ the sequence $(F_n \dots F_{m+1}(\rho_m) - p_{\Omega_n})$ vanishes for $n \to \infty$. To show non-surjectivity it is therefore enough to find a nontrivial projection p acting on $\mathcal{K} = \mathbb{C}^d \otimes \mathbb{C}^d$ which is a fixed point for all F_n^* simultaneously, compare A.5.2 and A.5.3. This can be done as follows. Dropping the index n and using the assumption and the preparations in 4.5.3 we have for $x \in M_d \otimes M_d$:

$$F^*(x) \;=\; d^{-1} \sum_r (V_{rr} \otimes \bar{V}_{rr})\; x\; (V_{rr}^* \otimes \bar{V}_{rr}^*).$$

Consider vectors $\xi = \eta_1 \otimes \bar{\eta}_2 \; + \; \eta_2 \otimes \bar{\eta}_1 \in \mathbb{C}^d \otimes \mathbb{C}^d = \mathcal{K}$, which except for the componentwise complex conjugation on the right side are symmetric tensors. Then obviously $(V_{rr} \otimes \bar{V}_{rr})\xi$ and $(V_{rr}^* \otimes \bar{V}_{rr}^*)\xi$ are vectors of the same type. Let p be the projection onto the linear span of these vectors. Then p commutes with all $V_{rr} \otimes \bar{V}_{rr}$ and it is thus a fixed point of F^*.

We sketch a second proof using the $C-F-$correspondence. Compare 3.6.5 and 3.5.6.

Because there we have only worked out the homogeneous case, let us assume that all F_n are equal, say $F_n = F$. Then $C - F-$correspondence means to look at a Markov process in a coupling representation for which $C = Z'_*$ coincides with the given F. Using the assumption and comparing with the formulas in 1.7.4 we conclude that the operator U in the coupling representation $Ad\,U \circ \sigma$ has blocks U_{ij} with $U_{ij} = 0$ for $i \neq j$. In other words, $Ad\,U$ maps $M_d \otimes \mathbb{1}$ into $M_d \otimes \mathbb{C}^d$, where $\mathbb{C}^d$ stands for the diagonal algebra. Now in the $C - F-$correspondence the surjectivity of α corresponds to the asymptotic completeness of the Markov process. But asymptotic completeness would produce an embedding of M_d into an infinite tensor product of copies of $\mathbb{C}^d$. This is clearly impossible because the latter algebra is commutative. $\square$

4.5.5 Adapted Endomorphisms in the Literature

It is tempting to conjecture that the various symmetries arising in the theory of subfactors [JS97, EK98] will play a role in more detailed investigations of this kind and that the symmetry considerations in the second proof of Proposition 4.5.4 are only a first and very simple example for that. In fact, the proposition describes a rather trivial case which may also be treated by other means and is included only to demonstrate the applicability of our method in a simple situation.

If we not only want to characterize non-surjectivity but in addition want to compute the Jones indices of the subfactors, then a more detailed analysis is necessary. The problem to compute the index of the range of what we have called an adapted endomorphism was raised in the setting of this section (matrix filtrations) by R. Longo in [Lo94]. See there and [Jo94] for some partial results and [CP96] for some further progress. Studying these references and the work of J. Cuntz on automorphisms of Cuntz algebras in [Cu80, Cu93], one also notices that there are many similarities between adapted endomorphisms and the localized endomorphisms in algebraic quantum field theory. The basic reference for that is [Ha92]

Our criterion 3.6.4 can be used to decide whether the index is one or greater than one. We have taken up the task to calculate index values with our methods in [Go3]. It turns out that it can be done and, moreover, that these considerations lead to an interesting new way of looking at extended transition operators. Thus Proposition 4.5.4 is only a first step and much more can be done here. We do not include these developments in this book and refer to [Go3]. We refer also to ([JS97], chap.4) for a discussion of many related problems about subfactors.

Finally we want to point out that in the important context of matrix filtrations there are various authors who in specific constructions of endomorphisms for various purposes have already used the device which we have called LPR. We do not try to give a complete survey but describe very briefly two important examples.

A. Connes, in ([Co77], 1.6), defined an endomorphism s_d^γ of the hyperfinite II_1-factor by the formula

$$s_d^\gamma \;=\; \lim_{N\to\infty} Ad\, v_\gamma\; \sigma(v_\gamma)\; \ldots\; \sigma^N(v_\gamma),$$

where σ is the right tensor shift on the M_d's and $v_\gamma \in M_d \otimes M_d$ is a unitary defined by

$$v_\gamma \;=\; \begin{pmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & 0 & & \\ & & & \ddots & 1 \\ & & & & 0 \end{pmatrix} \otimes \mathbb{1} \;+\; \begin{pmatrix} & & \\ & & \\ 1 & & \end{pmatrix} \otimes \begin{pmatrix} \bar\gamma & & & \\ & \bar\gamma^2 & & \\ & & \ddots & \\ & & & \bar\gamma^d = 1 \end{pmatrix},$$

where $\gamma \in \mathbb{C}$ is a $d-$th root of unity. He checks by direct computation that s_d^γ is a periodic automorphism. As a main result of the paper, it is later shown that the pair (d, γ) is a complete invariant of cocycle conjugacy for periodic automorphisms of the hyperfinite II_1-factor. In our terminology, the defining formula above for an explicit representative of such an equivalence class is a homogeneous LPR.

As a second example we mention [Jo94], where V.F.R. Jones considers sequences $(U_n(\lambda))_{n=1}^\infty$ of unitaries generating a von Neumann algebra and depending on a parameter λ, such that the following Yang-Baxter-type equations are valid:

(1) $U_n(\lambda)\, U_m(\mu) \;=\; U_m(\mu)\, U_n(\lambda)$ whenever $|n-m| \geq 2$,

(2) $\forall \lambda, \mu\ \exists \nu \quad U_{n+1}(\lambda)\, U_n(\mu)\, U_{n+1}(\nu) \;=\; U_n(\nu)\, U_{n+1}(\mu)\, U_n(\lambda)$.

By (1) the automorphism

$$\alpha_\lambda := \lim_{N\to\infty} Ad\,(U_1(\lambda)\ldots U_N(\lambda))$$

exists and using (2) it can be shown that α_λ and α_μ are almost commuting, see [Jo94] for details. Clearly (1) indicates a generalization of our concept of LPR to more general filtrations and (2) relates the setting to questions arising in statistical mechanics.

4.6 Noncommutative Extension of Adaptedness

4.6.1 Extending Adaptedness

As discussed in the Introduction, one of the aims of the spatial approach is to get a better understanding how commutative processes lie inside noncommutative ones. Here we give some observations about the adaptedness property in this respect. We have seen in Section 4.5 that for tensor products of matrices adapted endomorphisms automatically have an LPR while in the commutative setting analyzed in Section 4.1 this is not automatically the case. Therefore we expect some obstacles in the attempt to extend the latter scheme to the former. In the following we examine the simplest situation where such an obstacle becomes clearly visible.

4.6.2 Three Points of Time

The main observations can already be made in $\mathcal{C}_{[0,2]}$, i.e. with three points of time. Let us write $\mathcal{D} \simeq \mathbb{C}^d$ for the diagonal algebra of M_d. The tracial state tr restricts to a state on $\mathcal{D}$ giving equal weight to each point $1, \ldots, d$. In the following we shall always interpret $\mathbb{C}^d$ as $\mathcal{D}$ and $\mathbb{C}^d \otimes \mathbb{C}^d \otimes \mathbb{C}^d$ as $\mathcal{D} \otimes \mathcal{D} \otimes \mathcal{D} \subset M_d \otimes M_d \otimes M_d$. We say that an automorphism α of $\mathbb{C}^d \otimes \mathbb{C}^d \otimes \mathbb{C}^d$ is adapted with PR if there are permutation matrices $P_1 \in M_d \otimes M_d \otimes \mathbb{1}$

and $P_2 \in \mathcal{D} \otimes M_d \otimes M_d$ such that $\alpha = Ad\, P_1 P_2|_{\mathcal{D}\otimes\mathcal{D}\otimes\mathcal{D}}$. This implies that $\alpha(\mathbb{C}^d \otimes 1\!\!1 \otimes 1\!\!1) \subset \mathbb{C}^d \otimes \mathbb{C}^d \otimes 1\!\!1$, i.e. adaptedness in the wide sense. In fact, the reader should check that the above definition is only a reformulation of the usual notion (see 3.2.6 and Section 4.1), restricted to $\mathcal{C}_{[0,2]}$ and with the special state giving equal weight to each point. As shown in 4.1.7-10 it is just this what can in principle always be achieved in representing adapted commutative stationary processes.

On the other hand, let us say that an automorphism $\tilde{\alpha}$ of $M_d \otimes M_d \otimes M_d$ is adapted if $\tilde{\alpha}(M_d \otimes 1\!\!1 \otimes 1\!\!1) \subset M_d \otimes M_d \otimes 1\!\!1$. Then by 4.5.2 there are unitaries $U_1 \in M_d \otimes M_d \otimes 1\!\!1$ and $U_2 \in 1\!\!1 \otimes M_d \otimes M_d$ such that $\tilde{\alpha} = Ad\, U_1 U_2$, i.e. we have an LPR. Our question can now be made precise: Given α adapted with PR on $\mathbb{C}^d \otimes \mathbb{C}^d \otimes \mathbb{C}^d$, is it possible to find $\tilde{\alpha}$ adapted on $M_d \otimes M_d \otimes M_d$ such that $\alpha = \tilde{\alpha}|_{\mathcal{D}\otimes\mathcal{D}\otimes\mathcal{D}}$? Because in infinite product representations $Ad\, P_1 P_2$ represents the endomorphism correctly only on $\mathbb{C}_d \otimes \mathbb{C}_d \otimes 1\!\!1$ a more important problem is to determine when $\alpha|_{\mathbb{C}^d \otimes \mathbb{C}^d \otimes 1\!\!1} = \tilde{\alpha}|_{\mathcal{D} \otimes \mathcal{D} \otimes 1\!\!1}$.

4.6.3 Criteria for the Extendability of Adaptedness

We have reformulated the definition of adaptedness of α in such a way that we already have a natural candidate for an extension: $Ad\, P_1 P_2$ can also be interpreted as an automorphism β of $M_d \otimes M_d \otimes M_d$.

Theorem: *Let α be adapted with PR $Ad\, P_1 P_2$ on $\mathbb{C}^d \otimes \mathbb{C}^d \otimes \mathbb{C}^d$.*

(a) The following assertions are equivalent:
- *(a1) $\beta(M_d \otimes 1\!\!1 \otimes 1\!\!1) \subset M_d \otimes M_d \otimes 1\!\!1$.*
- *(a2) There exist permutation matrices $P_1^\dagger \in M_d \otimes M_d \otimes 1\!\!1$ and $P_2^\dagger \in 1\!\!1 \otimes M_d \otimes M_d$ such that $P_1 P_2 = P_1^\dagger P_2^\dagger$.*
- *(a3) There exists an adapted endomorphism $\tilde{\alpha}$ of $M_d \otimes M_d \otimes M_d$ such that $\alpha = \tilde{\alpha}|_{\mathcal{D} \otimes \mathcal{D} \otimes \mathcal{D}}$.*

(b) The following assertions are equivalent:
- *(b1) There exists a unitary $Q \in \mathcal{D} \otimes \mathcal{D} \otimes M_d$ such that $\tilde{\alpha} := \beta \circ Ad\, Q$ is adapted on $M_d \otimes M_d \otimes M_d$.*
- *(b3) There exists an adapted endomorphism $\tilde{\alpha}$ of $M_d \otimes M_d \otimes M_d$ such that $\alpha|_{\mathbb{C}^d \otimes \mathbb{C}^d \otimes 1\!\!1} = \tilde{\alpha}|_{\mathcal{D} \otimes \mathcal{D} \otimes 1\!\!1}$.*

If we replace $(a3)$ by $(b3)$ then $(a1)$ has to be replaced by $(b1)$, which explains the numbering.

Proof: We show $(a1) \Rightarrow (a2)$. Using matrix units $\{E_{ij}\}_{i,j=1}^d$ we can write

$$P_2 = \sum_k E_{kk} \otimes R_k \quad \text{with } R_k \in M_d \otimes M_d$$

and we get $P_2\ E_{ij} \otimes 1\!\!1 \otimes 1\!\!1\ P_2^* = E_{ij} \otimes R_i R_j^*$. By (a1) this is an element of $M_d \otimes M_d \otimes 1\!\!1$. In particular $R_1 R_j^* \in M_d \otimes 1\!\!1$, i.e. for all j there exists

$W_j \in M_d \otimes \mathbb{1}$ such that $R_1 = W_j R_j$. Now define $W := \sum_j E_{jj} \otimes W_j$ and $P_1^\dagger := P_1 W^*$, $P_2^\dagger := W P_2$. Then $P_1 P_2 = P_1^\dagger P_2^\dagger$, $P_1^\dagger \in M_d \otimes M_d \otimes \mathbb{1}$ and

$$P_2^\dagger = \sum_k E_{kk} \otimes W_k R_k = \sum_k E_{kk} \otimes R_1 = \mathbb{1} \otimes R_1 \in \mathbb{1} \otimes M_d \otimes M_d.$$

$(a2) \Rightarrow (a3)$ is immediate: Just define $\tilde{\alpha} := Ad\, P_1^\dagger P_2^\dagger$.

To show $(a3) \Rightarrow (a1)$ we compare $\beta(E_{ij} \otimes \mathbb{1} \otimes \mathbb{1})$ and $\tilde{\alpha}(E_{ij} \otimes \mathbb{1} \otimes \mathbb{1})$ for an arbitrary matrix unit E_{ij}. Think of $E_{ij} \otimes \mathbb{1} \otimes \mathbb{1}$ as a $d^3 \times d^3-$matrix with d^2 entries equal to one, the others equal to zero. $\beta = Ad\, P_1 P_2$ moves the non-zero entries to different places. If $\tilde{\alpha} = Ad\, U_1 U_2$ then by assumption

$$(U_1 U_2)(P_1 P_2)^* \in (\mathcal{D} \otimes \mathcal{D} \otimes \mathcal{D})' = \mathcal{D} \otimes \mathcal{D} \otimes \mathcal{D},$$

i.e. there is a unitary diagonal matrix X such that $U_1 U_2 = X P_1 P_2$. We infer that we get $\tilde{\alpha}(E_{ij} \otimes \mathbb{1} \otimes \mathbb{1})$ from $\beta(E_{ij} \otimes \mathbb{1} \otimes \mathbb{1})$ by replacing the ones in the matrix by suitable unimodular complex numbers. Concerning the question whether this is an element in $M_d \otimes M_d \otimes \mathbb{1}$ such a replacement can only make things worse: If $\tilde{\alpha}(E_{ij} \otimes \mathbb{1} \otimes \mathbb{1}) \in M_d \otimes M_d \otimes \mathbb{1}$, which is the case if (a3) is assumed, then necessarily already $\beta(E_{ij} \otimes \mathbb{1} \otimes \mathbb{1}) \in M_d \otimes M_d \otimes \mathbb{1}$. Since we can do this argument with all matrix units we get (a1).

Now starting with (b1) we write $\beta \circ Ad\, Q = Ad\, P_1 P_2 Q$. Check that the argument given for $(a1) \Rightarrow (a2)$ above does not depend on the fact that the occurring matrices are permutation matrices. Using the argument with $P_2 Q$ instead of P_2 we find unitaries $U_1 \in M_d \otimes M_d \otimes \mathbb{1}$ and $U_2 \in \mathbb{1} \otimes M_d \otimes M_d$ (instead of $P_1^\dagger$, $P_2^\dagger$) such that $P_1 P_2 Q = U_1 U_2$. Thus

$$\tilde{\alpha} := \beta \circ Ad\, Q = Ad\, P_1 P_2 Q = Ad\, U_1 U_2$$

is adapted. Further

$$(P_1 P_2)^* (U_1 U_2) = Q \in \mathcal{D} \otimes \mathcal{D} \otimes M_d = (\mathcal{D} \otimes \mathcal{D} \otimes \mathbb{1})',$$

which implies the remaining part of (b3). Conversely, given (b3), write $\tilde{\alpha} = Ad\, U_1 U_2$ and define

$$Q := (P_1 P_2)^* (U_1 U_2) \in (\mathcal{D} \otimes \mathcal{D} \otimes \mathbb{1})' = \mathcal{D} \otimes \mathcal{D} \otimes M_d.$$

Then $\beta \circ Ad\, Q = \tilde{\alpha}$ is adapted, which gives (b1). □

4.6.4 An Example: The Fredkin Gate

The following example shows that the assumptions in 4.6.3a,b are not always satisfied and thus it can happen that there are no adapted extensions. Let α be adapted on $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$ with PR $Ad\, P_1 P_2$ given by $P_1 \in M_d \otimes M_d \otimes \mathbb{1}$ arbitrary and

$$P_2 := \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \otimes 1\!\!1 \otimes 1\!\!1 + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 1 & & & \\ & 0 & 1 & \\ & 1 & 0 & \\ & & & 1 \end{pmatrix} \in \mathcal{D} \otimes M_d \otimes M_d .$$

In analogy to the CNOT-operation ('controlled not') in quantum information theory, this P_2 may be called a controlled flip. It is also known as the Fredkin gate, see [NC00]. We find that

$$Ad\, P_2\Big(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \otimes 1\!\!1 \otimes 1\!\!1 \Big) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 1 & & & \\ & 0 & 1 & \\ & 1 & 0 & \\ & & & 1 \end{pmatrix} \notin M_d \otimes M_d \otimes 1\!\!1,$$

and by 4.6.3a we infer that there is no adapted extension $\tilde{\alpha}$.

There is even no adapted $\tilde{\alpha}$ which coincides with α on $\mathcal{D} \otimes \mathcal{D} \otimes 1\!\!1$. By 4.6.3b it suffices to show that there is no unitary $Q \in \mathcal{D} \otimes \mathcal{D} \otimes M_d$ such that $Ad\, P_2 Q(M_d \otimes 1\!\!1 \otimes 1\!\!1) \subset M_d \otimes M_d \otimes 1\!\!1$. In fact, with $Q = \sum_{i,j=1}^{2} E_{ii} \otimes E_{jj} \otimes Q_{ij}$ and with $E_{12} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ we find that

$$Q\, E_{12} \otimes 1\!\!1 \otimes 1\!\!1\, Q^* = E_{12} \otimes \sum_j (E_{jj} \otimes Q_{1j} Q_{2j}^*)$$

and $P_2 Q\, E_{12} \otimes 1\!\!1 \otimes 1\!\!1\, Q^* P_2^*$

$$= E_{12} \otimes \Big[\sum_j (E_{jj} \otimes Q_{1j} Q_{2j}^*)(E_{11} \otimes E_{11} + E_{12} \otimes E_{21} + E_{21} \otimes E_{12} + E_{22} \otimes E_{22}) \Big].$$

A straightforward computation shows that if $\big[\dots \big]$ belongs to $M_d \otimes 1\!\!1$ then $Q_{1j} Q_{2j}^* = 0$ for $j = 1, 2$. But this is impossible for a unitary Q.

4.6.5 Discussion

Let us summarize the significance of the considerations above for extensions of adapted processes. If we are given a commutative stationary process, adapted with PR to a given filtration, then in general it is not possible to extend it to a process on all operators of the $GNS-$spaces which is stationary and adapted. This failure occurs if the original PR cannot be replaced by an LPR, in the sense made precise in Theorem 4.6.3. Note that no problems of this kind arise for Markov processes given in a coupling representation: A coupling representation is automatically LPR, see 4.1.12, and an adapted and stationary Markovian extension can be written down immediately. We have seen in Chapter 2 how useful such extensions can be, in many respects.

Appendix A

Some Facts about Unital Completely Positive Maps

In this appendix we present some results about unital completely positive maps which are used in the main text at diverse places. Many of them are more or less well-known or elementary (or both), but there seems to be no reference which covers them all and we have taken the opportunity to present them in a way fitting as directly as possible to the needs in this book.

A.1 Stochastic Maps

A.1.1 Terminology

For notions such as **'n-positive'** or **'completely positive'** and basic facts about such maps see for example ([Ta79], IV.3). If $\mathcal{A}$, $\mathcal{B}$ are C^*-algebras and $S : \mathcal{A} \to \mathcal{B}$ is completely positive and unital (i.e. $S(\mathbb{1}) = \mathbb{1}$), then we call S a **stochastic map**. This terminology is not standard, but it is rather natural in view of the fact that for $\mathcal{A}$, $\mathcal{B}$ commutative and finite dimensional we just get stochastic matrices in the usual sense. If $\mathcal{A}$, $\mathcal{B}$ are von Neumann algebras, we include normality into the definition of a stochastic map.

Because these maps are so important there are various names for them in the literature. We mention the term **'operation'** for a slightly more general notion ($S(\mathbb{1}) \leq \mathbb{1}$), which is motivated by the fact that such operations describe changes of systems in quantum physics [Da76, Ho01]. For similar reasons, in quantum information theory [NC00] some authors use **'quantum channel'**.

A.1.2 Kadison-Schwarz Inequality

The important **Kadison-Schwarz inequality** for a stochastic map $S : \mathcal{A} \to \mathcal{B}$ tells us that

$$S(a^*a) \geq S(a)^* S(a) \quad \textit{for all } a \in \mathcal{A}.$$

R. Gohm: LNM 1839, pp. 149–163, 2004.

We can apply it to the following setting: Let $\mathcal{A}$, $\mathcal{B}$ be C^*-algebras, $\phi_\mathcal{A}$ a state on $\mathcal{A}$, $\phi_\mathcal{B}$ a state on $\mathcal{B}$ and $S : \mathcal{A} \to \mathcal{B}$ a stochastic map with $\phi_\mathcal{B} \circ S = \phi_\mathcal{A}$. Then we write $S : (\mathcal{A}, \phi_\mathcal{A}) \to (\mathcal{B}, \phi_\mathcal{B})$. From the Kadison-Schwarz inequality it follows that

$$\|S(a)\|_{\phi_\mathcal{B}} \leq \|a\|_{\phi_\mathcal{A}} \quad \text{for } a \in \mathcal{A},$$

where $\|a\|_{\phi_\mathcal{A}} := \phi_\mathcal{A}(a^*a)^{\frac{1}{2}}$, $\|b\|_{\phi_\mathcal{B}} := \phi_\mathcal{B}(b^*b)^{\frac{1}{2}}$. The GNS-construction for $\mathcal{A}$ and $\mathcal{B}$ provides us with pairs $(\mathcal{G}, \Omega_\mathcal{G})$ and $(\mathcal{H}, \Omega_\mathcal{H})$, i.e. Hilbert spaces with cyclic vectors for the GNS-representations π such that

$$\phi_\mathcal{A}(a) = \langle \Omega_\mathcal{G}, \pi(a)\Omega_\mathcal{G} \rangle \quad \text{and} \quad \phi_\mathcal{B}(b) = \langle \Omega_\mathcal{H}, \pi(b)\Omega_\mathcal{H} \rangle$$

(with $a \in \mathcal{A}$, $b \in \mathcal{B}$). Then we have $\|a\|_{\phi_\mathcal{A}} = \|\pi(a)\Omega_\mathcal{G}\|$ and $\|b\|_{\phi_\mathcal{B}} = \|\pi(b)\Omega_\mathcal{H}\|$.

If $\phi_\mathcal{B}$ is faithful then $\Omega_\mathcal{H}$ is separating for $\mathcal{B}$ and using the Kadison-Schwarz inequality we can check that

$$S_\pi : \pi(\mathcal{A}) \to \pi(\mathcal{B}), \quad \pi(a) \mapsto \pi(S(a))$$

is well-defined and can be extended to a stochastic map acting normally on the weak closures. So at least in this case, we do not loose much and gain simplicity of notation on the other hand if we, from the beginning, work in the following setting (as done for example in Chapter 1):

A.1.3 A Useful Setting

Suppose $\mathcal{A} \subset \mathcal{B}(\mathcal{G})$ and $\mathcal{B} \subset \mathcal{B}(\mathcal{H})$ are von Neumann algebras with cyclic vectors $\Omega_\mathcal{G} \in \mathcal{G}$ and $\Omega_\mathcal{H} \in \mathcal{H}$. Restricting the corresponding vector states to $\mathcal{A}$ and $\mathcal{B}$ we get normal states $\phi_\mathcal{A}$ and $\phi_\mathcal{B}$. Then consider a stochastic map $S : (\mathcal{A}, \phi_\mathcal{A}) \to (\mathcal{B}, \phi_\mathcal{B})$.

A.2 Representation Theorems

A.2.1 W.F. Stinespring and K. Kraus

Like many fundamental concepts complete positivity has a double history in mathematics and in physics. This is documented in the early approaches by W.F. Stinespring [St55] and by K. Kraus [Kr71]. Two basic representation theorems are named after these authors. We state both and indicate the relation between them. More details on most of the following topics can be found in ([Kü88b], chap.1).

A.2.2 Stinespring Representation

Suppose $S : \mathcal{A} \to \mathcal{B} \subset \mathcal{B}(\mathcal{H})$ is a stochastic map. A triple $(\hat{\mathcal{L}}, \pi, v)$ consisting of a Hilbert space $\hat{\mathcal{L}}$, a normal *-representation π of $\mathcal{A}$ on $\hat{\mathcal{L}}$ and an isometry $v : \mathcal{H} \to \hat{\mathcal{L}}$ is called a **Stinespring representation** for S if for all $a \in \mathcal{A}$

$$S(a) \;=\; v^* \, \pi(a) \, v.$$

It is called **minimal** if the subspace $v\mathcal{H}$ is cyclic for $\pi(\mathcal{A})$.

$$\begin{tikzcd} \mathcal{H} \arrow[r, "S(a)"] \arrow[d, "v"'] & \mathcal{H} \\ \hat{\mathcal{L}} \arrow[r, "\pi(a)"] & \hat{\mathcal{L}} \arrow[u, "v^*"'] \end{tikzcd}$$

Stinespring's theorem states: For every stochastic map S there exists a minimal Stinespring representation which is unique up to unitary equivalence.

A.2.3 Kraus Decomposition

Suppose that $\mathcal{A} \subset \mathcal{B}(\mathcal{G})$ and $\mathcal{B} \subset \mathcal{B}(\mathcal{H})$ and $S : \mathcal{A} \to \mathcal{B}$ is a stochastic map. A family $\{a_k\}_{k=1}^d \subset \mathcal{B}(\mathcal{G}, \mathcal{H})$ is called a **Kraus decomposition** of S if for all $a \in \mathcal{A}$

$$S(a) \;=\; \sum_{k=1}^{d} a_k \, a \, a_k^*.$$

A Kraus decomposition is called minimal if the $\{a_k\}$ are linearly independent. From $S(\mathbb{1}) = \mathbb{1}$ one gets $\sum_{k=1}^d a_k \, a_k^* = \mathbb{1}$. If $d = \infty$ the sums should be interpreted as stop-limits. In this case the correct condition for minimality is: For $\{\lambda_k\}_{k=1}^d \subset \mathbb{C}$ from stop-$\sum_{k=1}^d \lambda_k \, a_k^* = 0$ follows $\lambda_k = 0$ for all k. In the following we shall not explicitly mention the case $d = \infty$ and instead remark once and for all that to get the results it will usually be necessary to use the strong operator topology in a suitable way.

Kraus' theorem states that it is always possible to find a minimal Kraus decomposition, and for any two minimal Kraus decompositions $\{a_k\}_{k=1}^{d_1}$ and $\{b_k\}_{k=1}^{d_2}$ we find $d_1 = d_2 =: d^{min}$ and a unitary $d^{min} \times d^{min}-$matrix w such that $b_r^* = \sum_s w_{rs} \, a_s^*$ for all r. This minimal number d^{min} is called the **rank** of S. Another term is 'index' which stresses analogies to the notion of index for continuous semigroups of completely positive maps [Ar97a, Ar03], but this term tends to be a bit overloaded.

A.2.4 Connection between Stinespring and Kraus

Given a von Neumann algebra $\mathcal{A} \subset \mathcal{B}(\mathcal{G})$ the so-called amplification-induction theorem (see [Ta79], IV.5.5) tells us that any normal $^*-$representation of $\mathcal{A}$

is unitarily equivalent to a subrepresentation of $a \mapsto a \otimes \mathbb{1}$ on $\mathcal{G} \otimes \mathcal{P}$, with a Hilbert space $\mathcal{P}$ (of dimension d). If $\mathcal{A} = \mathcal{B}(\mathcal{G})$ then $\mathcal{P}$ can be chosen in such a way that no restriction to a subrepresentation is required.

Applying this to the representation π in A.2.2 we have $v : \mathcal{H} \to \mathcal{G} \otimes \mathcal{P}$. Fixing an ONB $\{\epsilon_k\}_{k=1}^d$ of $\mathcal{P}$, an arbitrary element of $\mathcal{G} \otimes \mathcal{P}$ can be written in the form $\sum_{k=1}^d \xi_k \otimes \epsilon_k$ and for the isometry v we get the formula

$$v\,\xi = \sum_{k=1}^{d} a_k^*(\xi) \otimes \epsilon_k$$

with suitable $a_k \in \mathcal{B}(\mathcal{G}, \mathcal{H})$. This yields a correspondence between the Stinespring representation in A.2.2 and the Kraus decomposition in A.2.3.

$$\begin{tikzcd}
\mathcal{H} \arrow[r, "v"] \arrow[dr, "a_k^*"'] & \mathcal{G} \otimes \mathcal{P} \arrow[d] \\
& \mathcal{G} \simeq \mathcal{G} \otimes \epsilon_k
\end{tikzcd}$$

One can check that minimal Stinespring representations correspond to minimal Kraus decompositions.

Note also that $v : \mathcal{H} \to \mathcal{G} \otimes \mathcal{P}$ defines a stochastic map from $\mathcal{B}(\mathcal{G})$ into $\mathcal{B}(\mathcal{H})$ given by $x \mapsto v^*\ x \otimes \mathbb{1}\ v$ for all $x \in \mathcal{B}(\mathcal{G})$ which is an extension of $S : \mathcal{A} \to \mathcal{B}(\mathcal{H})$. Therefore such extensions always exist. This may be interpreted as a kind of Hahn-Banach theorem for completely positive maps. See ([Ar69], 1.2.3) for the original approach to extensions of completely positive maps and for further results. A recent account is included in [ER00].

A.2.5 Equivalence

Unitary equivalence between two Stinespring representations with subscripts 1 and 2 means explicitly that there is a unitary $w : \hat{\mathcal{L}}_1 \to \hat{\mathcal{L}}_2$ such that $v_2 = w\,v_1$ and $w\,\pi_1(\cdot) = \pi_2(\cdot)\,w$. One always gets the minimal Stinespring representation by restricting an arbitrary one to the minimal representation space, which is equal to the closure of $\pi(\mathcal{A})v\mathcal{H} \subset \hat{\mathcal{L}}$. The projection onto this space belongs to the commutant of $\pi(\mathcal{A})$ and is denoted p'. Thus for any two Stinespring representations of the same stochastic map S there exists a partial isometry $w : \hat{\mathcal{L}}_1 \to \hat{\mathcal{L}}_2$ with $w^*w \geq p_1',\ ww^* \geq p_2',\ v_2 = w\,v_1,\ w\,\pi_1(\cdot) = \pi_2(\cdot)\,w$.

In terms of Kraus decompositions this means that we have a partial isometry w given by a $d_2 \times d_1-$matrix with entries $w_{rs} \in \mathcal{A}' \subset \mathcal{B}(\mathcal{G})$ such that $(a_r^{(2)})^* = \sum_s w_{rs}(a_s^{(1)})^*$ (see [Kü88b], 1.1.8). If $\mathcal{A} = \mathcal{B}(\mathcal{G})$ then these entries are complex numbers.

A.2.6 Non-minimal Decompositions

In particular, if $S : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ with $\{a_k\}_{k=1}^{d^{min}}$ a minimal and $\{b_k\}_{k=1}^d$ an arbitrary Kraus decomposition, then there is an isometric $d \times d^{min}-$matrix

w with complex entries such that

$$\begin{pmatrix} b_1^* \\ \vdots \\ b_d^* \end{pmatrix} = w \begin{pmatrix} a_1^* \\ \vdots \\ a_{d_{min}}^* \end{pmatrix}.$$

See also [Ch75] for a direct computational proof.

Lemma: *Given* $\{b_k\}_{k=1}^d$, $\{c_k\}_{k=1}^d \subset \mathcal{B}(\mathcal{H})$ *(with the same d). The following assertions are equivalent:*

(a) $\sum_{k=1}^d b_k\, x\, b_k^* \;=\; \sum_{k=1}^d c_k\, x\, c_k^*$ *for all* $x \in \mathcal{B}(\mathcal{H})$.
(b) There exists a unitary $d \times d-$*matrix* w *such that*

$$\begin{pmatrix} c_1^* \\ \vdots \\ c_d^* \end{pmatrix} = w \begin{pmatrix} b_1^* \\ \vdots \\ b_d^* \end{pmatrix}.$$

Proof: $(b) \Rightarrow (a)$ follows by direct computation. To show the converse we can use the equation for the minimal decomposition stated above. In fact, we have

$$\begin{pmatrix} b_1^* \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ b_d^* \end{pmatrix} = w^b \begin{pmatrix} a_1^* \\ \vdots \\ a_{d_{min}}^* \\ 0 \\ \vdots \\ 0 \end{pmatrix},$$

$$\begin{pmatrix} c_1^* \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ c_d^* \end{pmatrix} = w^c \begin{pmatrix} a_1^* \\ \vdots \\ a_{d_{min}}^* \\ 0 \\ \vdots \\ 0 \end{pmatrix},$$

where w^b and w^c are unitary $d \times d-$matrices extending the isometry which relates a minimal and the non-minimal decomposition in an arbitrary way. Now we can define $w := w^c\,(w^b)^{-1}$. $\square$

The lemma generalizes the unitary equivalence of minimal decompositions and shows that the number d is a complete invariant for unitary equivalence of Kraus decompositions for given S.

A.3 The Isometry v

A.3.1 A Shifted Point of View

In many parts of the main text the isometry v is the object of interest and the stochastic map for which v occurs in the Stinespring representation is only a tool. Thus let us review the theory from the point of view of the isometry v.

A.3.2 Subspaces Which Are Minimal for an Inclusion

Considering a subset A of a Hilbert space $\mathcal{G} \otimes \mathcal{P}$, the components $\mathcal{G}$ and $\mathcal{P}$ may be unnecessarily large.

Lemma: *The following equivalent conditions characterize a unique closed subspace $\mathcal{G}_A$ of $\mathcal{G}$:*

(1) $\mathcal{G}_A$ is minimal with the property $A \subset \mathcal{G}_A \otimes \mathcal{P}$.

(2) For a basis $\{\epsilon_k\}$ of $\mathcal{P}$ define

$$\mathcal{G}_A := \overline{span}\Big\{\xi_k \in \mathcal{G} : \text{ There exists } a \in A \text{ such that } a = \sum \xi_k \otimes \epsilon_k\Big\}.$$

(3) $\mathcal{G}_A := \Big\{\xi \in \mathcal{G} : \ \langle \xi \otimes \eta, a\rangle = 0 \text{ for all } \eta \in \mathcal{P} \text{ and } a \in A\Big\}^{\perp}.$

We shall say that $\mathcal{G}_A \subset \mathcal{G}$ is **minimal for the inclusion** $A \subset \mathcal{G} \otimes \mathcal{P}$. Conditions (1) and (2) (with *span* instead of $\overline{span}$) make also sense for vector spaces instead of Hilbert spaces. The definition (2) does not depend on the basis. In a similar way we can also define a subspace $\mathcal{P}_A$ of $\mathcal{P}$. Then $A \subset \mathcal{G}_A \otimes \mathcal{P}_A$.

Proof: First we show (1) $\Leftrightarrow$ (2). Using the definition (2) for $\mathcal{G}_A$, it is obvious that $A \subset \mathcal{G}_A \otimes \mathcal{P}$. If $A \subset \mathcal{G}' \otimes \mathcal{P}$ for a closed subspace $\mathcal{G}' \subset \mathcal{G}$ then for all $a \in A$ we have $a = \sum \xi_k \otimes \epsilon_k$ with $\xi_k \in \mathcal{G}'$. Thus $\mathcal{G}_A \subset \mathcal{G}'$.
For (2) $\Leftrightarrow$ (3) we have to show that $\xi \in \mathcal{G}$ is orthogonal to all ξ_k in (2) if and only if $\langle \xi \otimes \eta, a\rangle = 0$ for all $\eta \in \mathcal{P}$ and $a \in A$. If ξ is orthogonal then indeed $\langle \xi \otimes \eta, a\rangle = \langle \xi \otimes \eta, \sum \xi_k \otimes \epsilon_k\rangle = 0$. Conversely if $\langle \xi, \xi_{k_0}\rangle \neq 0$ then if ξ_{k_0} occurs in $a = \sum \xi_k \otimes \epsilon_k$ we get $\langle \xi \otimes \epsilon_{k_0}, a\rangle \neq 0$. □

A.3.3 The Map $a : \mathcal{P} \to \mathcal{B}(\mathcal{G}, \mathcal{H})$

Let $v : \mathcal{H} \to \mathcal{G} \otimes \mathcal{P}$ be an isometry. For $\eta \in \mathcal{P}$ define an operator $a_\eta \in \mathcal{B}(\mathcal{G}, \mathcal{H})$ by

$$a^*_\eta(\xi) \otimes \eta := (\mathbb{1} \otimes p_\eta)\, v\xi$$

for all $\xi \in \mathcal{H}$. Here p_η denotes the one-dimensional projection onto $\mathbb{C}\eta$. Writing $p_\eta = |\,\eta\rangle\langle\eta\,|$ makes clear that $a : \mathcal{P} \to \mathcal{B}(\mathcal{G}, \mathcal{H})$, $\eta \mapsto a_\eta$, is linear. If $\{\epsilon_k\}$ is an ONB of $\mathcal{P}$ then with $a_k := a_{\epsilon_k}$ for all k we are back at the formula $v\,\xi = \sum a^*_k(\xi) \otimes \epsilon_k$ already encountered in A.2.4.

$$\begin{tikzcd} \mathcal{H} \arrow[r, "v"] \arrow[dr, "a_\eta^*"'] & \mathcal{G}\otimes\mathcal{P} \arrow[d] \\ & \mathcal{G} \simeq \mathcal{G}\otimes\eta \end{tikzcd}$$

Using A.3.2 with $A := v\mathcal{H} \subset \mathcal{G}\otimes\mathcal{P}$ we get subspaces $\mathcal{G}_{v\mathcal{H}}$ and $\mathcal{P}_{v\mathcal{H}}$. Then $v\mathcal{H} \subset \mathcal{G}_{v\mathcal{H}} \otimes \mathcal{P}_{v\mathcal{H}}$.

Proposition:

(a) $\eta \in \mathcal{P}_{v\mathcal{H}}^{\perp} \quad \Leftrightarrow \quad a_\eta = 0$.

(b) If $\{\epsilon_r\}$ *is linear independent in* $\mathcal{P}_{v\mathcal{H}}$ *then* $\{a_r := a_{\epsilon_r}\}$ *is linear independent in* $\mathcal{B}(\mathcal{G},\mathcal{H})$.

Proof: By Lemma A.3.2(3) we know that $\eta \in \mathcal{P}_{v\mathcal{H}}^{\perp}$ if and only if for all $\xi_\mathcal{G} \in \mathcal{G}$ and $\xi_\mathcal{H} \in \mathcal{H}$ we have $0 = \langle \xi_\mathcal{G}\otimes\eta,\, v\xi_\mathcal{H}\rangle$. It suffices to consider unit vectors η. Write $v\,\xi_\mathcal{H} = \sum a_k^*(\xi_\mathcal{H})\otimes\epsilon_k$ with an ONB $\{\epsilon_k\}$ of $\mathcal{P}$ containing $\epsilon_{k_0} = \eta$. Then we see that $\langle \xi_\mathcal{G}\otimes\eta, v\xi_\mathcal{H}\rangle = \langle \xi_\mathcal{G}, a_\eta^*(\xi_\mathcal{H})\rangle$. This vanishes for all $\xi_\mathcal{G}\in\mathcal{G}$ and $\xi_\mathcal{H}\in\mathcal{H}$ if and only if $a_\eta = 0$. This proves (a).
To prove (b) assume that $\sum \lambda_r a_r = 0,\ \{\lambda_r\}\subset\mathbb{C}$ with $\lambda_r \neq 0$ only for finitely many r. Then

$$0 = \sum \lambda_r a_r = a_{(\sum \lambda_r\epsilon_r)}$$

and by (a) we get $\sum\lambda_r\epsilon_r \in \mathcal{P}_{v\mathcal{H}}^{\perp}$. By assumption $\epsilon_r \in \mathcal{P}_{v\mathcal{H}}$ for all r. We conclude that $\sum\lambda_r\epsilon_r = 0$ and then by independence that $\lambda_r = 0$ for all r. □

A.3.4 Metric Operator Spaces

It follows from Proposition A.3.3 that

$$a : \mathcal{P}_{v\mathcal{H}} \to a(\mathcal{P}_{v\mathcal{H}}) \subset \mathcal{B}(\mathcal{G},\mathcal{H})$$

is an isomorphism of vector spaces. We can use it to transport the inner product, yielding a Hilbert space of operators $\mathcal{E}\subset\mathcal{B}(\mathcal{G},\mathcal{H})$. The space $\mathcal{E}$ is a metric operator space as defined by Arveson in [Ar97a] (where the case $\mathcal{G}=\mathcal{H}$ is treated):

A **metric operator space** *is a pair* $(\mathcal{E},\langle\cdot,\cdot\rangle)$ *consisting of a complex linear subspace* $\mathcal{E}$ *of* $\mathcal{B}(\mathcal{G},\mathcal{H})$ *together with an inner product with respect to which* $\mathcal{E}$ *is a separable Hilbert space and such that for an ONB* $\{a_k\}$ *of* $\mathcal{E}$ *and any* $\xi\in\mathcal{H}$ *we have*

$$\sum_k \|a_k^*\,\xi\|^2 < \infty.$$

Of course in our case we even get $\sum_k a_k\, a_k^* = \mathbb{1}$.

While our starting point has been the isometry v, Arveson emphasizes the bijective correspondence between metric operator spaces and normal completely positive maps. In our case the map in question is

$$S : \mathcal{B}(\mathcal{G}) \to \mathcal{B}(\mathcal{H}),\ x \mapsto v^*\, x \otimes \mathbb{1}\, v,$$

and the bijective correspondence is an elegant reformulation of the representation theorems of A.2. The point of view of metric operator spaces makes many features look more natural. Note that the minimal space for the Stinespring representation of S is $\mathcal{G} \otimes \mathcal{P}_{v\mathcal{H}} \simeq \mathcal{G} \otimes \mathcal{E}$. Conversely, starting with $\mathcal{G} \otimes \mathcal{E}$ where $\mathcal{E}$ is a metric operator space with $\sum_k a_k\, a_k^* = \mathbb{1}$, we can define the isometry v as the adjoint of the multiplication map

$$\begin{aligned} M : \mathcal{G} \otimes \mathcal{E} &\to \mathcal{H} \\ \xi \otimes a_k &\mapsto a_k(\xi). \end{aligned}$$

In other words, v is a generalized comultiplication. See [Ar97a].

Similarly if we choose an ONB $\{\epsilon_k\}$ of $\mathcal{P}_{v\mathcal{H}}$ then $v\,\xi = \sum_{k=1}^d a_k^*(\xi) \otimes \epsilon_k$, and the corresponding ONB $\{a_k\}$ of $\mathcal{E}$ provides us with a minimal Kraus decomposition $S(a) = \sum a_k\, a\, a_k^*$. The non-uniqueness of minimal Kraus decompositions is due to ONB-changes in $\mathcal{E}$.

A.3.5 Non-minimality Revisited

The point of view of metric operator spaces also helps to understand better the relation between minimal and non-minimal Kraus decompositions.

Suppose $\{b_k\}_{k=1}^d$ are arbitrary elements of a vector space, not necessarily linear independent. There is a canonical way to introduce an inner product on $span\{b_k\}_{k=1}^d$: Start with a Hilbert space $\mathcal{L}$ with ONB $\{e_k\}_{k=1}^d$ and define a linear map γ, which for $d < \infty$ is determined by

$$\begin{aligned} \gamma : \mathcal{L} &\to span\{b_k\}_{k=1}^d \\ e_k &\mapsto b_k. \end{aligned}$$

The restriction of γ to $(Ker\,\gamma)^\perp$ is a bijection and allows us to transport the inner product to $span\{b_k\}_{k=1}^d$. If $d = \infty$ define γ on the non-closed linear span $\check{\mathcal{L}}$ of the e_k as above and use the natural embedding of $\check{\mathcal{L}}\,/\,Ker\,\gamma$ into $(Ker\,\gamma)^\perp \subset \mathcal{L}$ to transport the inner product. In this case it may be necessary to form a completion $\overline{span}$ of $span\{b_k\}_{k=1}^d$ to get a Hilbert space. Then γ can be extended to the whole of $\mathcal{L}$ by continuity. We can characterize the Hilbert space $\overline{span}\{b_k\}_{k=1}^d$ so obtained by the property that the adjoint γ^* is an isometric embedding of $span\{b_k\}_{k=1}^d$ into $\mathcal{L}$.

Exactly this happens if we want to find the inner product of the metric operator space $\mathcal{E}$ of the map $x \mapsto \sum_{k=1}^d b_k\, x\, b_k^*$ with $x \in \mathcal{B}(\mathcal{G})$, $\{b_k\} \subset \mathcal{B}(\mathcal{G}, \mathcal{H})$ not necessarily linear independent. In fact, if $\{a_k\}_{k=1}^{d^{min}}$ is a minimal decomposition, then $\{a_k\}$ is an ONB of $\mathcal{E}$ and from the formula

$$\begin{pmatrix} b_1^* \\ \vdots \\ b_d^* \end{pmatrix} = w \begin{pmatrix} a_1^* \\ \vdots \\ a_{d^{min}}^* \end{pmatrix}.$$

obtained in A.2.6 with an isometric matrix w we can derive an isometric identification of $\overline{span}\{b_k\}_{k=1}^{d}$ and $\mathcal{E}$.

A.4 The Preadjoints *C* and *D*

A.4.1 Preadjoints

Besides the stochastic map $S : \mathcal{B}(\mathcal{G}) \to \mathcal{B}(\mathcal{H})$ there are some further objects associated to an isometry $v : \mathcal{H} \to \mathcal{G} \otimes \mathcal{P}$ which occur in the main text. We have the **preadjoint** $C = S_* : \mathcal{T}(\mathcal{H}) \to \mathcal{T}(\mathcal{G})$ with respect to the **duality** between trace class operators and bounded operators:

$$< C(\rho), x > = < \rho, S(x) > \quad \text{for } \rho \in \mathcal{T}(\mathcal{H}),\ x \in \mathcal{B}(\mathcal{G}).$$

C is a trace-preserving completely positive map. From $S(x) = v^* \, x \otimes \mathbb{1} \, v$ we find the explicit formula

$$C(\rho) = Tr_{\mathcal{P}}(v \, \rho \, v^*),$$

where $Tr_{\mathcal{P}}$ is the **partial trace** evaluated at $\mathcal{P}$, i.e. $Tr_{\mathcal{P}}(x \otimes y) = x \cdot Tr(y)$. (We denote by tr the trace state and by Tr the non-normalized trace.)

A.4.2 The Associated Pair (C,D)

Given v as above, we can consider a pair of trace-preserving completely positive maps:

$$C : \mathcal{T}(\mathcal{H}) \to \mathcal{T}(\mathcal{G}), \quad \rho \mapsto Tr_{\mathcal{P}}(v \, \rho \, v^*),$$
$$D : \mathcal{T}(\mathcal{H}) \to \mathcal{T}(\mathcal{P}), \quad \rho \mapsto Tr_{\mathcal{G}}(v \, \rho \, v^*).$$

The adjoint of D is the stochastic map

$$D^* : \mathcal{B}(\mathcal{P}) \to \mathcal{B}(\mathcal{H}), \quad y \mapsto v^* \, \mathbb{1} \otimes y \, v.$$

Note that in general not even the pair (C, D) determines the isometry v completely. For example take $\mathcal{G} = \mathcal{H} = \mathcal{P} = \mathbb{C}^2$. The unit vectors $\frac{1}{\sqrt{2}}(|11\rangle + |22\rangle)$ and $\frac{1}{\sqrt{2}}(|12\rangle + |21\rangle)$ in $\mathcal{G} \otimes \mathcal{P}$ cannot be distinguished by partial traces. Thus the two isometries v_1 and v_2 determined by

$$\begin{aligned} v_1 \, |1\rangle &= \frac{1}{\sqrt{2}}(|11\rangle + |22\rangle), \;\; v_1 \, |2\rangle &= \frac{1}{\sqrt{2}}(|12\rangle + |21\rangle), \\ v_2 \, |1\rangle &= \frac{1}{\sqrt{2}}(|12\rangle + |21\rangle), \;\; v_2 \, |2\rangle &= \frac{1}{\sqrt{2}}(|11\rangle + |22\rangle) \end{aligned}$$

yield the same pair (C, D).

A.4.3 Coordinates

Given an isometry $v : \mathcal{H} \to \mathcal{G} \otimes \mathcal{P}$ and ONB's $\{\omega_i\} \subset \mathcal{G},\ \{\delta_j\} \subset \mathcal{H},\ \{\epsilon_k\} \subset \mathcal{P}$ we get for $\xi \in \mathcal{H}$

$$v\,\xi = \sum_k a_k^*(\xi) \otimes \epsilon_k = \sum_i \omega_i \otimes \check{a}_i^*(\xi),$$

yielding Kraus decompositions

$$C(\rho) = \sum_k a_k^*\, \rho\, a_k, \qquad D(\rho) = \sum_i \check{a}_i^*\, \rho\, \check{a}_i.$$

Here $a_k \in \mathcal{B}(\mathcal{G}, \mathcal{H})$ and $\check{a}_i \in \mathcal{B}(\mathcal{P}, \mathcal{H})$. It is also possible to represent C and D as matrices with respect to the matrix units corresponding to the ONB's above:

$$C(|\delta_j\rangle\langle\delta_{j'}|) = \sum_{i,i'} C_{ii',jj'}\, |\omega_i\rangle\langle\omega_{i'}|,$$
$$D(|\delta_j\rangle\langle\delta_{j'}|) = \sum_{k,k'} D_{kk',jj'}\, |\epsilon_k\rangle\langle\epsilon_{k'}|.$$

These quantities are related as follows:

Lemma:

$$\begin{aligned} (a_k)_{ji} &= (\check{a}_i)_{jk} \\ C_{ii',jj'} &= \langle \check{a}_{i'}^*\, \delta_{j'}, \check{a}_i^*\, \delta_j \rangle \\ D_{kk',jj'} &= \langle a_{k'}^*\, \delta_{j'}, a_k^*\, \delta_j \rangle \end{aligned}$$

Proof:

$$\begin{aligned} (a_k)_{ji} \;&=\; \langle \delta_j, a_k\, \omega_i \rangle \;=\; \langle a_k^*\, \delta_j, \omega_i \rangle \\ =\; \langle \sum_r a_r^*\, \delta_j \otimes \epsilon_r, \omega_i \otimes \epsilon_k \rangle \;&=\; \langle \sum_s \omega_s \otimes \check{a}_s^*\, \delta_j, \omega_i \otimes \epsilon_k \rangle \\ =\; \langle \check{a}_i^*\, \delta_j, \epsilon_k \rangle \;&=\; \langle \delta_j, \check{a}_i\, \epsilon_k \rangle \;=\; (\check{a}_i)_{jk}. \end{aligned}$$

Geometrically one may think of a three-dimensional array of numbers and of a and $\check{a}$ as two different ways to decompose it by parallel planes.

$$\begin{aligned} C_{ii',jj'} &= \langle \omega_i, C(|\delta_j\rangle\langle\delta_{j'}|)\omega_{i'} \rangle \\ &= \langle \omega_i, Tr_{\mathcal{P}}(|v\,\delta_j\rangle\langle v\,\delta_{j'}|)\omega_{i'} \rangle \\ &= \langle \omega_i, Tr_{\mathcal{P}}(|\sum_r \omega_r \otimes \check{a}_r^*\, \delta_j\rangle\langle\sum_s \omega_s \otimes \check{a}_s^*\, \delta_{j'}|)\omega_{i'} \rangle \\ &= \langle \omega_i, \sum_{r,s} |\omega_r\rangle\langle\omega_s| \langle \check{a}_s^*\, \delta_{j'}, \check{a}_r^*\, \delta_j \rangle \omega_{i'} \rangle \\ &= \langle \check{a}_{i'}^*\, \delta_{j'}, \check{a}_i^*\, \delta_j \rangle. \end{aligned}$$

Similarly for D. □

A.4.4 Formulas for Partial Traces

Let $\chi,\ \chi'$ be vectors in $\mathcal{G} \otimes \mathcal{P}$, $\{\omega_i\}$ an ONB of $\mathcal{G}$ and $\chi = \sum_i \omega_i \otimes \chi_i,\ \chi' = \sum_i \omega_i \otimes \chi'_i$.

Lemma:

$$\big(Tr_{\mathcal{P}}\, |\chi'\rangle\langle\chi|\big)_{ij} = \langle \chi_j, \chi'_i \rangle$$
$$Tr_{\mathcal{G}}\, |\chi'\rangle\langle\chi| = \sum_i |\chi'_i\rangle\langle\chi_i|$$

Proof:

$$\big(Tr_{\mathcal{P}}\, |\chi'\rangle\langle\chi|\big)_{ij} = \langle \omega_i, (Tr_{\mathcal{P}}\, |\chi'\rangle\langle\chi|)\, \omega_j \rangle$$

$$\begin{aligned}
&= \langle \omega_i, Tr_{\mathcal{P}} \left(\sum_r |\omega_r \otimes \chi'_r\rangle \sum_s \langle \omega_s \otimes \chi_s| \right) \omega_j \rangle \\
&= \sum_{r,s} \langle \omega_i, \omega_r \rangle \langle \omega_s, \omega_j \rangle \langle \chi_s, \chi'_r \rangle \\
&= \langle \chi_j, \chi'_i \rangle.
\end{aligned}$$

$$\begin{aligned}
Tr_{\mathcal{G}} |\chi'\rangle\langle\chi| &= Tr_{\mathcal{G}} \big(\sum_r |\omega_r \otimes \chi'_r\rangle \sum_s \langle \omega_s \otimes \chi_s| \big) \\
&= \sum_i |\chi'_i\rangle\langle\chi_i| \qquad \Box
\end{aligned}$$

A.4.5 A Useful Formula for Dilation Theory

Proposition: *Let $\Omega_{\mathcal{P}} \in \mathcal{P}$ be a distinguished unit vector, so that the projection $p_{\mathcal{G}}$ from $\mathcal{G} \otimes \mathcal{P}$ onto $\mathcal{G} \simeq \mathcal{G} \otimes \Omega_{\mathcal{P}} \subset \mathcal{G} \otimes \mathcal{P}$ can be defined. Then for all $\xi,\ \xi' \in \mathcal{H}$ we get*

$$\langle p_{\mathcal{G}}\, v\xi, p_{\mathcal{G}}\, v\xi' \rangle \ = \ \langle \Omega_{\mathcal{P}}, D(|\xi'\rangle\langle\xi|)\, \Omega_{\mathcal{P}} \rangle.$$

Proof: Using Lemma A.4.4 we find for $\chi,\ \chi' \in \mathcal{G} \otimes \mathcal{P}$ that

$$\langle p_{\mathcal{G}}\, \chi, p_{\mathcal{G}}\, \chi' \rangle \ = \ \langle \sum_j \omega_j \otimes \langle \Omega_{\mathcal{P}}, \chi_j \rangle \Omega_{\mathcal{P}}, \sum_i \omega_i \otimes \langle \Omega_{\mathcal{P}}, \chi'_i \rangle \Omega_{\mathcal{P}} \rangle$$

$$\begin{aligned}
&= \ \sum_i \langle \Omega_{\mathcal{P}}, \chi'_i \rangle \langle \chi_i, \Omega_{\mathcal{P}} \rangle \ = \ \langle \Omega_{\mathcal{P}}, \left(\sum_i |\chi'_i\rangle\langle\chi_i| \right) \Omega_{\mathcal{P}} \rangle \\
&= \ \langle \Omega_{\mathcal{P}}, (Tr_{\mathcal{G}} |\chi'\rangle\langle\chi|)\, \Omega_{\mathcal{P}} \rangle.
\end{aligned}$$

Now insert $v\xi = \chi,\ v\xi' = \chi'$ and the definition of D. $\Box$

The proposition shows that the map D plays a role in dilation theory: The isometry v is an isometric dilation (of first order) of the contraction $t := p_{\mathcal{G}}\, v : \mathcal{H} \to \mathcal{G}$. The map D determines the quantities $\langle t\,\xi, t\,\xi' \rangle$, in particular

$$\|t\,\xi\|^2 \;=\; \langle \Omega_{\mathcal{P}}, D(p_\xi)\, \Omega_{\mathcal{P}} \rangle,$$

for all unit vectors $\xi \in \mathcal{H}$, where p_ξ is the one-dimensional projection onto $\mathbb{C}\xi$.

A.5 Absorbing Vector States

A.5.1 Stochastic Maps and Vector States

Proposition: *Assume $v : \mathcal{H} \to \mathcal{G} \otimes \mathcal{P}$ is an isometry and $v\,\xi = \sum_{k=1}^{d} a_k^*(\xi) \otimes \epsilon_k$ with an ONB $\{\epsilon_k\}_{k=1}^d$ of $\mathcal{P}$, so that*

$$S(x) = v^*\, x \otimes \mathbb{1}\, v = \sum_{k=1}^{d} a_k\, x\, a_k^* : \mathcal{B}(\mathcal{G}) \to \mathcal{B}(\mathcal{H})$$

is a stochastic map (see A.2). Further let $\Omega_{\mathcal{G}} \in \mathcal{G}$ and $\Omega_{\mathcal{H}} \in \mathcal{H}$ be unit vectors (compare A.1.3).
Then the following assertions are equivalent:

(1) $\langle \Omega_{\mathcal{G}}, x\, \Omega_{\mathcal{G}} \rangle \;=\; \langle \Omega_{\mathcal{H}}, S(x)\, \Omega_{\mathcal{H}} \rangle$ for all $x \in \mathcal{B}(\mathcal{G})$

(2) There exists a unit vector $\Omega_{\mathcal{P}} \in \mathcal{P}$ such that $v\, \Omega_{\mathcal{H}} = \Omega_{\mathcal{G}} \otimes \Omega_{\mathcal{P}}$.

(3) There exists a function $\omega : \{1, \ldots, d\} \to \mathbb{C}$ such that

$$a_k^*\, \Omega_{\mathcal{H}} = \overline{\omega(k)}\, \Omega_{\mathcal{G}} \quad \textit{for all } k = 1, \ldots, d.$$

Note that if $\mathcal{G} = \mathcal{H}$ and $\Omega_{\mathcal{G}} = \Omega_{\mathcal{H}} =: \Omega$, then the proposition deals with an invariant vector state Ω and (3) means that Ω is a common eigenvector for all $a_k^*,\ k = 1, \ldots, d$.

Proof: Using the formula for v with $\xi = \Omega_{\mathcal{H}}$ we find

$$v\, \Omega_{\mathcal{H}} = \sum_{k=1}^{d} a_k^*(\Omega_{\mathcal{H}}) \otimes \epsilon_k.$$

Now (2) $\Leftrightarrow$ (3) follows by observing that $\Omega_{\mathcal{P}} = \sum_{k=1}^{d} \overline{\omega(k)} \epsilon_k$.
(1) can be written as:

$$\langle \Omega_{\mathcal{G}}, x \Omega_{\mathcal{G}} \rangle \;=\; \langle v\, \Omega_{\mathcal{H}},\; x \otimes \mathbb{1}\; v\, \Omega_{\mathcal{H}} \rangle \quad \textit{for all } x \in \mathcal{B}(\mathcal{H}).$$

Thus (2) $\Rightarrow$ (1) is immediate. For the converse assume that $v\, \Omega_{\mathcal{H}}$ has not the form given in (2). Then inserting $x = p_{\Omega_{\mathcal{G}}}$, the projection onto $\mathbb{C}\Omega_{\mathcal{G}}$, yields

$$|\, \langle v\, \Omega_{\mathcal{H}}, p_{\Omega_{\mathcal{G}}} \otimes \mathbb{1}\; v\, \Omega_{\mathcal{H}} \rangle\, | < 1 \;=\; \langle \Omega_{\mathcal{G}}, p_{\Omega_{\mathcal{G}}} \Omega_{\mathcal{G}} \rangle,$$

contradicting (1). $\square$

A.5.2 Criteria for Absorbing Vector States

Assume that $S : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ is a stochastic map. S is called ergodic if its fixed point space is trivial, i.e. equals $\mathbb{C}1\!\!1$. We denote the space of positive trace class operators with trace one by $\mathcal{T}_1^+(\mathcal{H})$ and call its elements density operators or density matrices. Further let $\Omega_\mathcal{H} \in \mathcal{H}$ be a unit vector. We denote by $p_{\Omega_\mathcal{H}}$ and more general by p_ξ the one-dimensional projection onto the multiples of the vector used as subscript. The vector state given by $\Omega_\mathcal{H}$ is called **absorbing** for S if for all density matrices $\rho \in \mathcal{T}_1^+(\mathcal{H})$ and all $x \in \mathcal{B}(\mathcal{H})$ we have

$$Tr(\rho\, S^n(x)) \longrightarrow \langle \Omega_\mathcal{H}, x\, \Omega_\mathcal{H}\rangle \quad \text{for } n \to \infty.$$

In this case we shall also say that the vector $\Omega_\mathcal{H}$ is **absorbing for the preadjoint** S_*. See the proposition below for direct formulations in terms of S_*.

An absorbing state is always invariant. Intuitively a state is absorbing if the dynamics always approaches it in the long run.

We write $< \cdot, \cdot >$ for the duality between trace class operators and bounded operators. Then the above formula for absorbing can be written as

$$< \rho, S^n(x) > \longrightarrow < p_{\Omega_\mathcal{H}}, x > .$$

By $\| \cdot \|_1$ we denote the canonical norm on trace class operators, i.e. $\|\rho\|_1 = Tr\left[(\rho^*\rho)^{\frac{1}{2}}\right]$.

Proposition: *The following assertions are equivalent:*

(a) S *is ergodic and the vector state given by* $\Omega_\mathcal{H}$ *is invariant.*

(b) The vector state given by $\Omega_\mathcal{H}$ *is absorbing.*

(c) $S^n(p_{\Omega_\mathcal{H}}) \overset{n\to\infty}{\longrightarrow} 1\!\!1$ *stop or (equivalently) weak*.*

(d) $< S_*^n\, p_\xi, p_{\Omega_\mathcal{H}} > \overset{n\to\infty}{\longrightarrow} 1$ *for all unit vectors* $\xi \in \mathcal{H}$.

(e) $\|S_*^n\, \rho - p_{\Omega_\mathcal{H}}\|_1 \overset{n\to\infty}{\longrightarrow} 0$ *for all* $\rho \in \mathcal{T}_1^+(\mathcal{H})$.

Proof: $(b) \Rightarrow (a)$: If $x \in \mathcal{B}(\mathcal{H})$ is a non-trivial fixed point of S, then there exist $\rho_1,\ \rho_2 \in \mathcal{T}_1^+(\mathcal{H})$ such that $Tr(\rho_1\, x) \neq Tr(\rho_2\, x)$. The condition for absorbing cannot be true for this x.
Now assume (a). By invariance

$$\langle \Omega_\mathcal{H}, S(p_{\Omega_\mathcal{H}})\, \Omega_\mathcal{H}\rangle \;=\; \langle \Omega_\mathcal{H}, p_{\Omega_\mathcal{H}}\, \Omega_\mathcal{H}\rangle \;=\; 1.$$

Because $0 \leq S(p_{\Omega_\mathcal{H}}) \leq 1\!\!1$ this implies that $S(p_{\Omega_\mathcal{H}}) \geq p_{\Omega_\mathcal{H}}$. Therefore $S^n(p_{\Omega_\mathcal{H}})$ is increasing. Its stop-limit for $n \to \infty$ is a fixed point situated between $p_{\Omega_\mathcal{H}}$ and $1\!\!1$. By ergodicity it must be $1\!\!1$. This shows (c) with the stop-topology.

Now from (c) with weak*$-$convergence we prove (b), all together giving $(a) \Leftrightarrow (b) \Leftrightarrow (c)$. We have to show that

$$< \rho, S^n(x) > \overset{n\to\infty}{\longrightarrow} < p_{\Omega_\mathcal{H}}, x > \text{for all } \rho \in \mathcal{T}_1^+(\mathcal{H}),\ x \in \mathcal{B}(\mathcal{H}).$$

Now $x = p_{\Omega_{\mathcal{H}}}\, x\, p_{\Omega_{\mathcal{H}}} + p_{\Omega_{\mathcal{H}}}\, x\, p^{\perp}_{\Omega_{\mathcal{H}}} + p^{\perp}_{\Omega_{\mathcal{H}}}\, x\, p_{\Omega_{\mathcal{H}}} + p^{\perp}_{\Omega_{\mathcal{H}}}\, x\, p^{\perp}_{\Omega_{\mathcal{H}}}$. Because $p_{\Omega_{\mathcal{H}}}\, x\, p_{\Omega_{\mathcal{H}}} = \langle p_{\Omega_{\mathcal{H}}}, x \rangle p_{\Omega_{\mathcal{H}}}$ we get from (c):

$$< \rho, S^n(p_{\Omega_{\mathcal{H}}}\, x\, p_{\Omega_{\mathcal{H}}}) > = < p_{\Omega_{\mathcal{H}}}, x >< \rho, S^n(p_{\Omega_{\mathcal{H}}}) > \stackrel{n\to\infty}{\longrightarrow} < p_{\Omega_{\mathcal{H}}}, x >$$

That the other terms tend to zero can be seen by using the Cauchy-Schwarz inequality (see [Ta79], I.9.5) for the states $y \mapsto < \rho, S^n(y) >$. This proves (b).

(c) and (d) are equivalent by duality. Condition (e) is apparently stronger. But (d) also implies (e), as can be seen from [Ta79], III.5.11 or by Lemma A.5.3 below. □

Remarks: Complete positivity is never used here, the proposition is valid for positive unital maps. The implication $(b) \Rightarrow (a)$ also holds for mixed invariant states (with the obvious definition of the absorbing property), but $(a) \Rightarrow (b)$ is not valid in general: Already in the classical Perron-Frobenius theorem about positive matrices there is the phenomenon of periodicity.

Absorbing states for positive semigroups are a well-known subject, both in mathematics and physics, both commutative and noncommutative, compare [AL87, Ar97b, Ar03]. Mathematically, this is a part of ergodic theory. Physically, it means a system's asymptotic approach to an equilibrium. In particular, absorbing vector states can occur when atoms emitting photons return to a ground state.

A.5.3 Absorbing Sequences

Lemma: *Consider sequences $(\mathcal{K}_n)$ of Hilbert spaces, (Ω_n) of unit vectors, (ρ_n) of density matrices such that $\Omega_n \in \mathcal{K}_n$ and $\rho_n \in \mathcal{T}^1_+(\mathcal{K}_n)$ for all n. Then for $n \to \infty$ the following assertions are equivalent:*

(1) $\langle \Omega_n, \rho_n \Omega_n \rangle \to 1$

(2) $\| \rho_n - p_{\Omega_n} \|_1 \to 0$

(3) For all uniformly bounded sequences (x_n) with $x_n \in \mathcal{B}(\mathcal{K}_n)$ for all n: $Tr(\rho_n x_n) - \langle \Omega_n, x_n \Omega_n \rangle \to 0$.

Proof. Because

$$\langle \Omega_n, \rho_n \Omega_n \rangle = Tr(\rho_n\, p_{\Omega_n})$$

we quickly infer (3) ⇒ (1), and (2) ⇒ (3) follows from

$$|\, Tr(\rho_n x_n) - \langle \Omega_n, x_n \Omega_n \rangle \,| \le \| \rho_n - p_{\Omega_n} \|_1 \sup_n \|x_n\|.$$

It remains to prove that (1) ⇒ (2):

Write $\rho_n = \sum_i \alpha_i^{(n)} p_{\epsilon_i^{(n)}}$ with $\alpha_i^{(n)} \ge 0$, $\sum_i \alpha_i^{(n)} = 1$

and $\{\epsilon_i^{(n)}\}$ an ONB of $\mathcal{K}_n$. From (1) we get

$$\langle \Omega_n, \left(\sum_i \alpha_i^{(n)} p_{\epsilon_i^{(n)}} \right) \Omega_n \rangle \ = \ \sum_i \alpha_i^{(n)} \mid \langle \epsilon_i^{(n)}, \Omega_n \rangle \mid^2 \rightarrow 1.$$

If $i = 1$ is an index with $\alpha_1^{(n)} = \max_i \alpha_i^{(n)}$ for all n then because of $\sum_i \alpha_i^{(n)} = 1 = \sum_i \mid \langle \epsilon_i^{(n)}, \Omega_n \rangle \mid^2$ we infer $\alpha_1^{(n)} \rightarrow 1$, i.e. $\sum_{i \neq 1} \alpha_i^{(n)} \rightarrow 0$ and

$$\mid \langle \epsilon_1^{(n)}, \Omega_n \rangle \mid \rightarrow 1, \ \text{i.e.} \ \| p_{\epsilon_1^{(n)}} - p_{\Omega_n} \|_1 \rightarrow 0$$

(by arguing in the two-dimensional subspaces spanned by $\epsilon_1^{(n)}$ and Ω_n). Finally

$$\begin{aligned} \| \rho_n - p_{\Omega_n} \|_1 &= \| \sum_i \alpha_i^{(n)} p_{\epsilon_i^{(n)}} - p_{\Omega_n} \|_1 \\ &\leq \ \| \alpha_1^{(n)} p_{\epsilon_1^{(n)}} - p_{\Omega_n} \|_1 + \| \sum_{i \neq 1} \alpha_i^{(n)} p_{\epsilon_i^{(n)}} \|_1 \\ &\leq | \alpha_1^{(n)} - 1 | + \| p_{\epsilon_1^{(n)}} - p_{\Omega_n} \|_1 + \sum_{i \neq 1} \alpha_i^{(n)} \rightarrow 0. \qquad \square \end{aligned}$$

Generalizing the terminology of A.5.2 we may say that the sequence (Ω_n) of unit vectors is **absorbing** for the sequence (ρ_n) of density matrices if the assertions of the lemma above are valid.

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