CHAPTER 9

Solution of Integral Equations

9.1 Introduction

In this chapter we shall discuss the application of Chebyshev polynomial techniques to the solution of Fredholm (linear) integral equations, which are classified into three kinds taking the following generic forms:

First kind: Given functions $K(x,y)$ and $g(x)$, find a function $f(y)$ on $[a,b]$ such that for all $x \in [a,b]$

$$\int_a^b K(x,y)f(y)\,\mathrm{d}y = g(x); \tag{9.1}$$

Second kind: Given functions $K(x,y)$ and $g(x)$, and a constant λ not providing a solution of (9.3) below, find a function $f(y)$ on $[a,b]$ such that for all $x \in [a,b]$

$$f(x) - \lambda \int_a^b K(x,y)f(y)\,\mathrm{d}y = g(x); \tag{9.2}$$

Third kind: Given a function $K(x,y)$, find values *(eigenvalues)* of the constant λ for which there exists a function *(eigenfunction)* $f(y)$, not vanishing identically on $[a,b]$, such that for all $x \in [a,b]$

$$f(x) - \lambda \int_a^b K(x,y)f(y)\,\mathrm{d}y = 0. \tag{9.3}$$

Equations of these three kinds may be written in more abstract terms as the functional equations

$$\mathcal{K}f = g, \tag{9.4}$$

$$f - \lambda\mathcal{K}f = g, \tag{9.5}$$

$$f - \lambda\mathcal{K}f = 0, \tag{9.6}$$

where $\mathcal{K}$ represents a linear mapping (here an *integral transformation*) from some function space F into itself or possibly (for an equation of the first kind) into another function space G, g represents a given element of F or G as appropriate and f is an element of F to be found.

A detailed account of the theory of integral equations is beyond the scope of this book — we refer the reader to Tricomi (1957), for instance. However, it is broadly true (for most of the kernel functions $K(x, y)$ that one is likely to meet) that equations of the second and third kinds have well-defined and well-behaved solutions. Equations of the first kind are quite another matter—here the problem will very often be ill-posed mathematically in the sense of either having no solution, having infinitely many solutions, or having a solution f that is infinitely sensitive to variations in the function g. It is essential that one reformulates such a problem as a well-posed one by some means, before attempting a numerical solution.

In passing, we should also mention integral equations of Volterra type, which are similar in form to Fredholm equations but with the additional property that $K(x, y) = 0$ for $y > x$, so that

$$\int_a^b K(x, y)f(y)\,\mathrm{d}y$$

is effectively

$$\int_a^x K(x, y)f(y)\,\mathrm{d}y.$$

A Volterra equation of the first kind may often be transformed into one of the second kind by differentiation. Thus

$$\int_a^x K(x, y)f(y)\,\mathrm{d}y = g(x)$$

becomes, on differentiating with respect to x,

$$K(x, x)f(x) + \int_a^x \frac{\partial}{\partial x}K(x, y)f(y)\,\mathrm{d}y = \frac{\mathrm{d}}{\mathrm{d}x}g(x).$$

It is therefore unlikely to suffer from the ill-posedness shown by general Fredholm equations of the first kind. We do not propose to discuss the solution of Volterra equations any further here.

9.2 Fredholm equations of the second kind

In a very early paper, Elliott (1961) studied the use of Chebyshev polynomials for solving non-singular equations of the second kind

$$f(x) - \lambda \int_a^b K(x, y)f(y)\,\mathrm{d}y = g(x), \quad a \le x \le b, \tag{9.7}$$

and this work was later updated by him (Elliott 1979). Here $K(x, y)$ is bounded in $a \le x, y \le b$, and we suppose that λ is not an eigenvalue of (9.3). (If λ were such an eigenvalue, corresponding to the eigenfunction $\phi(y)$, then any solution $f(y)$ of (9.7) would give rise to a multiplicity of solutions of the form $f(y) + \alpha\phi(y)$ where α is an arbitrary constant.)

For simplicity, suppose that $a = -1$ and $b = 1$. Assume that $f(x)$ may be approximated by a finite sum of the form

$$\sideset{}{''}\sum_{j=0}^{N} a_j T_j(x). \tag{9.8}$$

Then we can substitute (9.8) into (9.7) so that the latter becomes the approximate equation

$$\sideset{}{''}\sum_{j=0}^{N} a_j T_j(x) - \lambda \sideset{}{''}\sum_{j=0}^{N} a_j \int_{-1}^{1} K(x,y) T_j(y)\,\mathrm{d}y \sim g(x), \quad -1 \le x \le 1. \tag{9.9}$$

We need to choose the coefficients a_j so that (9.9) is satisfied as well as possible over the interval $-1 \le x \le 1$. A reasonably good way of achieving this is by *collocation* — requiring equation (9.9) to be an exact equality at the $N + 1$ points (the extrema of $T_N(x)$ on the interval)

$$x = y_{i,N} = \cos\frac{i\pi}{N},$$

so that

$$\sideset{}{''}\sum_{j=0}^{N} a_j (P_{ij} - \lambda Q_{ij}) = g(y_{i,N}), \quad i = 0, \ldots, N, \tag{9.10}$$

where

$$P_{ij} = T_j(y_{i,N}), \quad Q_{ij} = \int_{-1}^{1} K(y_{i,N}, y) T_j(y)\,\mathrm{d}y. \tag{9.11}$$

We thus have $N + 1$ linear equations to solve for the $N + 1$ unknowns a_j.

As an alternative to collocation, we may choose the coefficients $b_{i,k}$ so that $K_M(y_{i,N}, y)$ gives a least squares or minimax approximation to $K(y_{i,N}, y)$.

If we cannot evaluate the integrals in (9.11) exactly, we may do so approximately, for instance, by replacing each $K(y_{i,N}, y)$ with a polynomial

$$K_M(y_{i,N}, y) = \sideset{}{''}\sum_{k=0}^{M} b_{i,k} T_k(y) \tag{9.12}$$

for some[1] $M > 0$, with Chebyshev coefficients given by

$$b_{i,k} = \frac{2}{M} \sideset{}{''}\sum_{m=0}^{M} K(y_{i,N}, y_{m,M}) T_k(y_{m,M}), \quad k = 0, \ldots, M, \tag{9.13}$$

[1] There does not need to be any connection between the values of M and N.

where

$$y_{m,M} = \cos\frac{m\pi}{M}, \quad m = 0, \ldots, M.$$

As in the latter part of Section 6.3.2, we can then show that

$$K_M(y_{i,N}, y_{m,M}) = K(y_{i,N}, y_{m,M}), \quad m = 0, \ldots, M,$$

so that, for each i, $K_M(y_{i,N}, y)$ is the polynomial of degree M in y, interpolating $K(y_{i,N}, y)$ at the points $y_{m,M}$.

From (2.43) it is easily shown that

$$\int_{-1}^{1} T_n(x)\,\mathrm{d}x = \begin{cases} \dfrac{-2}{n^2-1}, & n \text{ even}, \\ 0, & n \text{ odd}. \end{cases} \tag{9.14}$$

Hence

$$\begin{aligned} Q_{ij} &\approx \int_{-1}^{1} K_M(y_{i,N}, y) T_j(y)\,\mathrm{d}y \\ &= \sum_{k=0}^{M}{}'' b_{i,k} \int_{-1}^{1} T_k(y) T_j(y)\,\mathrm{d}y \\ &= \tfrac{1}{2} \sum_{k=0}^{M}{}'' b_{i,k} \int_{-1}^{1} \{T_{j+k}(y) + T_{|j-k|}(y)\}\,\mathrm{d}y \\ &= -\sum_{\substack{k=0 \\ j\pm k \text{ even}}}^{M}{}'' b_{i,k} \left\{ \frac{1}{(j+k)^2-1} + \frac{1}{(j-k)^2-1} \right\} \\ &= -2 \sum_{\substack{k=0 \\ j\pm k \text{ even}}}^{M}{}'' b_{i,k} \frac{j^2+k^2-1}{(j^2+k^2-1)^2 - 4j^2k^2}, \end{aligned} \tag{9.15}$$

giving us the approximate integrals we need.

Another interesting approach, based on 'alternating polynomials' (whose equal extrema occur among the given data points), is given by Brutman (1993). It leads to a solution in the form of a sum of Chebyshev polynomials, with error estimates.

9.3 Fredholm equations of the third kind

We can attack integral equations of the third kind in exactly the same way as equations of the second kind, with the difference that we have $g(x) = 0$.

Thus the linear equations (9.10) become

$$\sum_{j=0}^{N}{}'' a_j(P_{ij} - \lambda Q_{ij}) = 0, \quad i = 0, \ldots, N. \tag{9.16}$$

Multiplying each equation by $T_\ell(y_{i,N})$ and carrying out a $\sum''$ summation over i (halving the first and last terms), we obtain (after approximating K by K_M) the equations

$$\frac{MN}{4} a_\ell = \lambda \sum_{j=0}^{N}{}'' a_j \times$$
$$\times \left(\sum_{k=0}^{M}{}'' \sum_{i=0}^{N}{}'' \sum_{m=0}^{M}{}'' T_\ell(y_{i,N}) K(y_{i,N}, y_{m,M}) T_k(y_{m,M}) \int_{-1}^{1} T_k(y) T_j(y)\, \mathrm{d}y \right), \tag{9.17}$$

which is of the form

$$\frac{MN}{4} a_\ell = \lambda \sum_{j=0}^{N} a_j A_{\ell j} \tag{9.18}$$

or, written in terms of vectors and matrices,

$$\frac{MN}{4} \mathbf{a} = \lambda \mathbf{A} \mathbf{a}. \tag{9.19}$$

Once the elements of the $(N+1) \times (N+1)$ matrix $\mathbf{A}$ have been calculated, this is a straightforward (unsymmetric) matrix eigenvalue problem, which may be solved by standard techniques to give approximations to the dominant eigenvalues of the integral equation.

9.4 Fredholm equations of the first kind

Consider now a Fredholm integral equation of the first kind, of the form

$$g(x) = \int_a^b K(x, y) f(y)\, \mathrm{d}y, \quad c \le x \le d. \tag{9.20}$$

We can describe the function $g(x)$ as an integral transform of the function $f(y)$, and we are effectively trying to solve the 'inverse problem' of determining $f(y)$ given $g(x)$.

For certain special kernels $K(x, y)$, a great deal is known. In particular, the choices

$$K(x, y) = \cos xy, \quad K(x, y) = \sin xy \text{ and } K(x, y) = \mathrm{e}^{-xy},$$

with $[a, b] = [0, \infty)$, correspond respectively to the well-known Fourier cosine transform, Fourier sine transform and Laplace transform. We shall not pursue these topics specifically here, but refer the reader to the relevant literature (Erdélyi et al. 1954, for example).

Smooth kernels in general will often lead to inverse problems that are ill-posed in one way or another.

For example, if K is continuous and f is integrable, then it can be shown that $g = \mathcal{K}f$ must be continuous — consequently, if we are given a g that is not continuous then no (integrable) solution f of (9.20) exists.

Uniqueness is another important question to be considered. For example Groetsch (1984) notes that the equation

$$\int_0^\pi x \sin y \, f(y) \; \mathrm{d}y = x$$

has a solution

$$f(y) = \tfrac{1}{2}.$$

However, it has an infinity of further solutions, including

$$f(y) = \tfrac{1}{2} + \sin ny \quad (n = 2, 3, \ldots).$$

An example of a third kind of ill-posedness, given by Bennell (1996), is based on the fact that, if K is absolutely integrable in y for each x, then by the Riemann–Lebesgue theorem,

$$\phi_n(x) \equiv \int_a^b K(x, y) \cos ny \, \mathrm{d}y \to 0 \text{ as } n \to \infty.$$

Hence

$$\int_a^b K(x, y)(f(y) + \alpha \cos ny) \; \mathrm{d}y = g(x) + \alpha\phi_n(x) \to g(x) \text{ as } n \to \infty,$$

where α is an arbitrary positive constant. Thus a small perturbation

$$\delta g(x) = \alpha\phi_n(x)$$

in $g(x)$, converging to a zero limit as $n \to \infty$, can lead to a perturbation

$$\delta f(y) = \alpha \cos ny$$

in $f(y)$ which remains of finite magnitude α for all n. This means that the solution $f(y)$ does not depend continuously on the data $g(x)$, and so the problem is ill-posed.

We thus see that it is not in fact necessarily advantageous for the function K to be smooth. Nevertheless, there are ways of obtaining acceptable numerical solutions to problems such as (9.20). They are based on the technique of *regularisation*, which effectively forces an approximate solution to be appropriately smooth. We return to this topic in Section 9.6 below.

9.5 Singular kernels

A particularly important class of kernels, especially in the context of the study of Chebyshev polynomials in integral equations, comprises the *Hilbert kernel*

$$K(x, y) = \frac{1}{x - y} \tag{9.21}$$

and other related 'Hilbert-type' kernels that behave locally like (9.21) in the neighbourhood of $x = y$.

9.5.1 Hilbert-type kernels and related kernels

If $[a, b] = [-1, 1]$ and

$$K(x, y) = \frac{w(y)}{y - x},$$

where $w(y)$ is one of the weight functions $(1 + y)^\alpha (1 - y)^\beta$ with $\alpha, \beta = \pm\frac{1}{2}$, then there are direct links of the form (9.20) between Chebyshev polynomials of the four kinds (Fromme & Golberg 1981, Mason 1993).

Theorem 9.1

$$\pi U_{n-1}(x) = \int_{-1}^{1} K_1(x, y) T_n(y) \,\mathrm{d}y, \tag{9.22a}$$

$$-\pi T_n(x) = \int_{-1}^{1} K_2(x, y) U_{n-1}(y) \,\mathrm{d}y, \tag{9.22b}$$

$$\pi W_n(x) = \int_{-1}^{1} K_3(x, y) V_n(y) \,\mathrm{d}y, \tag{9.22c}$$

$$-\pi V_n(x) = \int_{-1}^{1} K_4(x, y) W_n(y) \,\mathrm{d}y \tag{9.22d}$$

where

$$K_1(x, y) = \frac{1}{\sqrt{1 - y^2}\,(y - x)},$$

$$K_2(x, y) = \frac{\sqrt{1 - y^2}}{(y - x)},$$

$$K_3(x, y) = \frac{\sqrt{1 + y}}{\sqrt{1 - y}\,(y - x)},$$

$$K_4(x, y) = \frac{\sqrt{1 - y}}{\sqrt{1 + y}\,(y - x)},$$

and each integral is to be interpreted as a Cauchy principal value integral.

Proof: In fact, formulae (9.22a) and (9.22b) correspond under the transformation $x = \cos\theta$ to the trigonometric formulae

$$\int_0^\pi \frac{\cos n\phi}{\cos\phi - \cos\theta}\,\mathrm{d}\phi = \pi\frac{\sin n\theta}{\sin\theta}, \tag{9.23}$$

$$\int_0^\pi \frac{\sin n\phi\sin\phi}{\cos\phi - \cos\theta}\,\mathrm{d}\phi = -\pi\cos n\theta, \tag{9.24}$$

which have already been proved in another chapter (Lemma 8.3). Formulae (9.22c) and (9.22d) follow similarly from Lemma 8.5. ●●

From Theorem 9.1 we may immediately deduce integral relationships between Chebyshev series expansions of functions as follows.

Corollary 9.1A

1. *If* $f(y) \sim \sum_{n=1}^\infty a_n T_n(y)$ *and* $g(x) \sim \pi\sum_{n=1}^\infty a_n U_{n-1}(x)$ *then*

$$g(x) = \int_{-1}^1 \frac{f(y)}{\sqrt{1-y^2}\,(y-x)}\,\mathrm{d}y. \tag{9.25a}$$

2. *If* $f(y) \sim \sum_{n=1}^\infty b_n U_{n-1}(y)$ *and* $g(x) \sim \pi\sum_{n=1}^\infty b_n T_n(x)$ *then*

$$g(x) = -\int_{-1}^1 \frac{\sqrt{1-y^2}\,f(y)}{(y-x)}\,\mathrm{d}y. \tag{9.25b}$$

3. *If* $f(y) \sim \sum_{n=1}^\infty c_n V_n(y)$ *and* $g(x) \sim \pi\sum_{n=1}^\infty c_n W_n(x)$ *then*

$$g(x) = \int_{-1}^1 \frac{\sqrt{1+y}\,f(y)}{\sqrt{1-y}\,(y-x)}\,\mathrm{d}y. \tag{9.25c}$$

4. *If* $f(y) \sim \sum_{n=1}^\infty d_n W_n(y)$ *and* $g(x) \sim \pi\sum_{n=1}^\infty d_n V_n(x)$ *then*

$$g(x) = -\int_{-1}^1 \frac{\sqrt{1-y}\,f(y)}{\sqrt{1+y}\,(y-x)}\,\mathrm{d}y. \tag{9.25d}$$

Note that these expressions do not necessarily provide general solutions to the integral equations (9.25a)–(9.25d), but they simply show that the relevant formal expansions are integral transforms of each other.

These relationships are useful in attacking certain engineering problems. Gladwell & England (1977) use (9.25a) and (9.25b) in elasticity analysis and Fromme & Golberg (1979) use (9.25c), (9.25d) and related properties of V_n and W_n in analysis of the flow of air near the tip of an airfoil.

To proceed to other kernels, we note that by integrating equations (9.22a)–(9.22d) with respect to x, after premultiplying by the appropriate weights, we can deduce the following eigenfunction properties of Chebyshev polynomials for logarithmic kernels. The details are left to the reader (Problem 4).

Theorem 9.2 *The integral equation*

$$\lambda\phi(x) = \int_{-1}^{1} \frac{1}{\sqrt{1-y^2}}\phi(y)K(x,y)\,\mathrm{d}y \tag{9.26}$$

has the following eigensolutions and eigenvalues λ *for the following kernels* K.

1. $K(x,y) = K_5(x,y) = \log|y-x|$;
   $\phi(x) = \phi_n(x) = T_n(x), \quad \lambda = \lambda_n = \pi/n.$

2. $K(x,y) = K_6(x,y) = \log|y-x| - \log\left|1 - xy - \sqrt{(1-x^2)(1-y^2)}\right|$;
   $\phi(x) = \phi_n(x) = \sqrt{1-x^2}\,U_{n-1}(x), \quad \lambda = \lambda_n = \pi/n.$

3. $K(x,y) = K_7(x,y) = \log|y-x| - \log\left|2 + x + y - 2\sqrt{1+x}\sqrt{1+y}\right|$;
   $\phi = \phi_n(x) = \sqrt{1+x}\,V_n(x), \quad \lambda = \lambda_n = \pi/(n+\frac{1}{2}).$

4. $K(x,y) = K_8(x,y) = \log|y-x| - \log\left|2 - x - y - 2\sqrt{1-x}\sqrt{1-y}\right|$;
   $\phi = \phi_n(x) = \sqrt{1-x}\,W_n(x), \quad \lambda = \lambda_n = \pi/(n+\frac{1}{2}).$

Note that each of these four kernels has a (weak) logarithmic singularity at $x = y$. In addition, K_6 has logarithmic singularities at $x = y = \pm 1$, K_7 at $x = y = -1$ and K_8 at $x = y = +1$.

From Theorem 9.2 we may immediately deduce relationships between formal Chebyshev series of the four kinds as follows.

Corollary 9.2A *With the notations of Theorem 9.2, in each of the four cases considered, if*

$$f(y) \sim \sum_{k=1}^{\infty} a_k\phi_k(y) \quad \text{and} \quad g(x) \sim \sum_{k=1}^{\infty} \lambda_k a_k \phi_k(x)$$

then

$$g(x) = \int_{-1}^{1} \frac{1}{\sqrt{1-y^2}} K(x,y) f(y)\,\mathrm{d}y.$$

Thus again four kinds of Chebyshev series may in principle be used to solve (9.26) for $K = K_5,\ K_6,\ K_7,\ K_8$, respectively.

The most useful results in Theorem 9.2 and its corollary are those relating to polynomials of the first kind, where we find from Theorem 9.2 that

$$-\frac{\pi}{n}T_n(x) = \int_{-1}^{1} \frac{1}{\sqrt{1-y^2}} T_n(y)\log|y-x|\,\mathrm{d}y \tag{9.27}$$

and from Corollary 9.2A that, if

$$f(y) \sim \sum_{k=1}^{\infty} a_k T_k(y) \quad \text{and} \quad g(x) \sim \sum_{k=1}^{\infty} -\frac{\pi}{k} a_k T_k(x), \tag{9.28}$$

then

$$g(x) = \int_{-1}^{1} \frac{1}{\sqrt{1-y^2}} \log|y-x|\, f(y)\, \mathrm{d}y. \tag{9.29}$$

Equation (9.29) is usually referred to as *Symm's integral equation*, and clearly a Chebyshev series method is potentially very useful for such problems. We shall discuss this specific problem further in Section 9.5.2.

By differentiating rather than integrating in (9.22a), (9.22b), (9.22c) and (9.22d), we may obtain the further results quoted in Problem 5. The second of these yields the simple equation

$$-n\pi U_{n-1}(x) = \int\!\!\!\!\!\!= _{-1}^{1} \frac{\sqrt{1-y^2}}{(y-x)^2} U_{n-1}(y)\, \mathrm{d}y. \tag{9.30}$$

This integral equation, which has a stronger singularity than (9.22a)–(9.22d), is commonly referred to as a *hypersingular* equation, in which the integral has to be evaluated as a Hadamard finite-part integral (Martin 1991, for example). A rather more general hypersingular integral equation is solved by a Chebyshev method, based on (9.30), in Section 9.7.1 below.

The ability of a Chebyshev series of the first or second kind to handle both Cauchy principal value and hypersingular integral transforms leads us to consider an integral equation that involves both. This can be successfully attacked, and Mason & Venturino (2002) give full details of a Galerkin method, together with both $\mathcal{L}_2$ and $\mathcal{L}_\infty$ error bounds, and convergence proofs.

9.5.2 Symm's integral equation

Consider the integral equation (Symm 1966)

$$G(x) = \mathcal{V}F(x) = \frac{1}{\pi} \int_a^b \log|y-x| F(y)\, \mathrm{d}y, \quad x \in [a,b], \tag{9.31}$$

which is of importance in potential theory.

This equation has a unique solution $F(y)$ (Jorgens 1970) with endpoint singularities of the form $(y-a)^{-\frac{1}{2}}(b-y)^{-\frac{1}{2}}$. In the case $a=-1$, $b=+1$, the required singularity is $(1-y^2)^{-\frac{1}{2}}$, and so we may write

$$F(y) = (1-y^2)^{-\frac{1}{2}} f(y), \quad G(x) = -\pi^{-1} g(x),$$

whereupon (9.31) becomes

$$g(x) = \mathcal{V}^* f(x) = \int_{-1}^{1} \frac{\log |y - x|}{\sqrt{1 - y^2}} f(y)\,\mathrm{d}y \tag{9.32}$$

which is exactly the form (9.29) obtained from Corollary 9.2A.

We noted then (9.29) that if

$$f(y) \sim \sum_{k=1}^{\infty} a_k T_k(y)$$

then

$$g(x) \sim \sum_{k=1}^{\infty} -\frac{\pi}{k} a_k T_k(x)$$

(and vice versa).

Sloan & Stephan (1992), adopt such an idea and furthermore note that

$$\mathcal{V}^* T_0(x) = -\pi \log 2,$$

so that

$$f(y) \sim {\sum_{k=0}^{\infty}}' a_k T_k(y)$$

if

$$g(x) \sim -\tfrac{1}{2} a_0 \pi \log 2 - \sum_{k=1}^{\infty} \frac{\pi}{k} a_k T_k(x).$$

Their method of approximate solution is to write

$$f^*(y) = {\sum_{k=0}^{n-1}}' a_k^* T_k(y) \tag{9.33}$$

and to require that

$$\mathcal{V}^* f^*(x) = g(x)$$

holds at the zeros $x = x_i$ of $T_n(x)$. Then

$$g(x_i) = -\tfrac{1}{2} a_0^* \pi \log 2 - \sum_{k=1}^{n-1} \frac{\pi}{k} a_k^* T_k(x_i), \quad i = 0, \ldots, n-1. \tag{9.34}$$

Using the discrete orthogonality formulae (4.42), we deduce that

$$a_0^* = -\frac{2}{n\pi \log 2} \sum_{i=0}^{n-1} g(x_i), \tag{9.35}$$

$$a_k^* = -\frac{2k}{n\pi} \sum_{i=0}^{n-1} g(x_i) T_k(x_i) \quad (k > 0). \tag{9.36}$$

Thus values of the coefficients $\{a_k^*\}$ are determined explicitly.

The convergence properties of the approximation f^* to f have been established by Sloan & Stephan (1992).

9.6 Regularisation of integral equations

Consider again an integral equation of the first kind, of the form

$$g(x) = \int_a^b K(x,y)f(y)\,\mathrm{d}y, \quad c \le x \le d, \tag{9.37}$$

i.e. $g = \mathcal{K}f$, where $\mathcal{K} : F \to G$ and where the given $g(x)$ may be affected by noise (Bennell & Mason 1989). Such a problem is said to be *well posed* if:

- for each $g \in G$ there exists a solution $f \in F$;
- this solution f is always unique in F;
- f depends continuously on g (i.e., the inverse of $\mathcal{K}$ is continuous).

Unfortunately it is relatively common for an equation of the form (9.37) to be ill posed, so that a method of solution is needed which ensures not only that a computed f is close to being a solution but also that f is an appropriately smooth function. The standard approach is called a *regularisation method*; Tikhonov (1963*b*, 1963*a*) proposed an L_2 approximation which minimises

$$I[f^*] := \int_a^b [\mathcal{K}f^*(x) - g(x)]^2\,\mathrm{d}x + \lambda \int_a^b [p(x)f^*(x)^2 + q(x){f^*}'(x)^2]\,\mathrm{d}x \tag{9.38}$$

where p and q are specified positive weight functions and λ a positive 'smoothing' parameter. The value of λ controls the trade-off between the smoothness of f^* and the fidelity to the data g.

9.6.1 Discrete data with second derivative regularisation

We shall first make two changes to (9.38) on the practical assumptions that we seek a visually smooth (i.e., twice continuously differentiable) solution, and that the data are discrete. We therefore assume that $g(x)$ is known only at n ordinates x_i and then only subject to white noise contamination $\epsilon(x_i)$;

$$g^*(x_i) = \int_a^b K(x_i,y)f(y)\,\mathrm{d}y + \epsilon(x_i) \tag{9.39}$$

where each $\epsilon(x_i) \sim N(0,\sigma^2)$ is drawn from a normal distribution with zero mean and (unknown) variance σ^2. We then approximate f by the $f_\lambda^* \in L_2[a,b]$

that minimises

$$I[f^*] \equiv \frac{1}{n}\sum_{i=1}^{n}\left[\mathcal{K}f^*(x_i) - g^*(x_i)\right]^2 + \lambda\int_a^b \left[f^{*\prime\prime}(y)\right]^2 \,\mathrm{d}y, \tag{9.40}$$

thus replacing the first integral in (9.38) by a discrete sum and the second by one involving the second derivative of f^*.

Ideally, a value λ_{opt} of λ should be chosen (in an outer cycle of iteration) to minimise the true mean-square error

$$R(\lambda) \equiv \frac{1}{n}\sum_{i=1}^{n}\left[\mathcal{K}f^*_\lambda(x_i) - g(x_i)\right]^2. \tag{9.41}$$

This is not directly possible, since the values $g(x_i)$ are unknown. However, Wahba (1977) has shown that a good approximation to λ_{opt} may be obtained by choosing the 'generalised cross-validation' (GCV) estimate λ^*_{opt} that minimises

$$V(\lambda) = \frac{\frac{1}{n}\left\|(\mathbf{I} - \mathbf{A}(\lambda)\mathbf{g}\right\|^2}{\left[\frac{1}{n}\,\mathrm{trace}(\mathbf{I} - \mathbf{A}(\lambda))\right]^2}, \tag{9.42}$$

where

$$\mathcal{K}\mathbf{f}^*_\lambda = \mathbf{A}(\lambda)\mathbf{g}, \tag{9.43}$$

i.e., $\mathbf{A}(\lambda)$ is the matrix which takes the vector of values $g(x_i)$ into $\mathcal{K}f^*_\lambda(x_i)$.

An approximate representation is required for $\mathbf{f}^*_\lambda$. Bennell & Mason (1989) adopt a basis of polynomials orthogonal on $[a, b]$, and more specifically the Chebyshev polynomial sum

$$f^*_\lambda(y) = \sum_{j=0}^{m} a_j T_j(y) \tag{9.44}$$

when $[a, b] = [-1, 1]$.

9.6.2 Details of a smoothing algorithm (second derivative regularisation)

Adopting the representation (9.44), the smoothing term in (9.40) is

$$\int_{-1}^{1}\left[f^{*\prime\prime}_\lambda(y)\right]^2 \mathrm{d}y = \hat{\mathbf{a}}^T\mathbf{B}\hat{\mathbf{a}} \tag{9.45}$$

where $\hat{\mathbf{a}} = (a_2, a_3, \ldots, a_m)^T$ and $\mathbf{B}$ is a matrix with elements

$$B_{ij} = \int_{-1}^{1} P_i''(y)P_j''(y)\,\mathrm{d}y \quad (i, j = 2, \ldots, m). \tag{9.46}$$

The matrix $\mathbf{B}$ is symmetric and positive definite, with a Cholesky decomposition $\mathbf{B} = \mathbf{L}\mathbf{L}^T$, giving

$$\int_{-1}^{1} \left[f_\lambda^{*\prime\prime}(y)\right]^2 \,\mathrm{d}y = \left\|\mathbf{L}^T\hat{\mathbf{a}}\right\|^2.$$

Then, from (9.40),

$$I[f_\lambda^*] = \frac{1}{n}\left\|\mathbf{M}\mathbf{a} - \mathbf{g}^*\right\|^2 + \lambda\left\|\mathbf{L}^T\hat{\mathbf{a}}\right\|^2 \tag{9.47}$$

where $M_{ij} = \int_{-1}^{1} K(x_i, y)T_j(y)\,\mathrm{d}y$ and $\mathbf{a} = (a_1, a_2, \ldots, a_m)^T$.

Bennell & Mason (1989) show that a_0 and a_1 may be eliminated by considering the $\mathbf{QU}$ decomposition of $\mathbf{M}$,

$$\mathbf{M} = \mathbf{QU} = \mathbf{Q}\left[\frac{\mathbf{V}}{\mathbf{0}}\right], \tag{9.48}$$

where $\mathbf{Q}$ is orthogonal, $\mathbf{V}$ is upper triangular of order $m+1$ and then

$$\mathbf{U} = \begin{bmatrix} \mathbf{R}_1 & \mathbf{R}_2 \\ \mathbf{0} & \mathbf{R}_3 \end{bmatrix} \tag{9.49}$$

where $\mathbf{R}_1$ is a 2×2 matrix.

Defining $\tilde{\mathbf{a}} = (a_0, a_1)^T$,

$$\begin{aligned}
\left\|\mathbf{M}\mathbf{a} - \mathbf{g}^*\right\|^2 &= \left\|\mathbf{QU}\mathbf{a} - \mathbf{g}^*\right\|^2 \\
&= \left\|\mathbf{Q}^T(\mathbf{QU}\mathbf{a} - \mathbf{g}^*)\right\|^2 \\
&= \left\|\mathbf{U}\mathbf{a} - \mathbf{e}\right\|^2, \text{ where } \mathbf{e} = \mathbf{Q}^T\mathbf{g}^*, \\
&= \left\|\mathbf{R}_1\tilde{\mathbf{a}} + \mathbf{R}_2\hat{\mathbf{a}} - \tilde{\mathbf{e}}\right\|^2 + \left\|\mathbf{R}_3\hat{\mathbf{a}} - \hat{\mathbf{e}}\right\|^2.
\end{aligned} \tag{9.50}$$

Setting $\tilde{\mathbf{a}} = \mathbf{R}_1^{-1}(\tilde{\mathbf{e}} - \mathbf{R}_2\hat{\mathbf{a}})$,

$$I[f_\lambda^*] = \frac{1}{n}\left\|\mathbf{R}_3\hat{\mathbf{a}} - \hat{\mathbf{e}}\right\|^2 + \lambda\left\|\mathbf{L}^T\hat{\mathbf{a}}\right\|^2. \tag{9.51}$$

The problem of minimising I over $\hat{\mathbf{a}}$ now involves only the independent variables $a_2, \ldots, a_m$, and requires us to solve the equation

$$(\mathbf{H}^T\mathbf{H} + n\lambda\mathbf{I})\mathbf{b} = \mathbf{H}^T\hat{\mathbf{e}} \tag{9.52}$$

where $\mathbf{b} = \mathbf{L}^T\hat{\mathbf{a}}$ and $\mathbf{H} = \mathbf{R}_3(\mathbf{L}^T)^{-1}$.

Hence

$$\mathbf{b} = (\mathbf{H}^T\mathbf{H} + n\lambda\mathbf{I})^{-1}\mathbf{H}^T\hat{\mathbf{e}} \tag{9.53}$$

and it can readily be seen that the GCV matrix is

$$\mathbf{A}(\lambda) = \mathbf{H}(\mathbf{H}^T\mathbf{H} + n\lambda\mathbf{I})^{-1}\mathbf{H}^T. \tag{9.54}$$

The algorithm thus consists of solving the linear system (9.52) for a given λ while minimising $V(\lambda)$ given by (9.42).

Formula (9.42) may be greatly simplified by first determining the singular value decomposition (SVD) of $\mathbf{H}$

$$\mathbf{H} = \mathbf{W}\boldsymbol{\Lambda}\mathbf{X}^T \quad (\mathbf{W},\ \mathbf{X} \text{ orthogonal})$$

where

$$\boldsymbol{\Lambda} = \left[\frac{\boldsymbol{\Delta}}{\mathbf{0}}\right] \quad (\boldsymbol{\Delta} = \operatorname{diag}(d_i)).$$

Then

$$V(\lambda) = \frac{\sum_{k=1}^{m-1}[n\lambda(d_k^2 + n\lambda)^{-1}]^2 z_k^2 + \sum_{k=1}^{m-2} z_k^2}{\sum_{k=1}^{m-1}\lambda(d_k^2 + n\lambda)^{-1} + (n-m-1)n^{-1}} \tag{9.55}$$

where $\mathbf{z} = \mathbf{W}^T\hat{\mathbf{e}}$.

The method has been successfully tested by Bennell & Mason (1989) on a number of problems of the form (9.37), using Chebyshev polynomials. It was noted that there was an optimal choice of the number m of basis functions, beyond which the approximation f_λ^* deteriorated on account of ill-conditioning. In Figures 9.1–9.3, we compare the true solution (dashed curve) with the computed Chebyshev polynomial solution (9.44) (continuous curve) for the function $f(y) = \mathrm{e}^{-y}$ and equation

$$\int_0^\infty \mathrm{e}^{-xy} f(y)\,\mathrm{d}y = \frac{1}{1+x}, \quad 0 < x < \infty,$$

with

- $\epsilon(x) \sim N(0, .005^2)$ and $m = 5$,
- $\epsilon(x) \sim N(0, .01^2)$ and $m = 5$,
- $\epsilon(x) \sim N(0, .01^2)$ and $m = 10$.

No significant improvement was obtained for any other value of m.

9.6.3 A smoothing algorithm with weighted function regularisation

Some simplifications occur in the above algebra if, as proposed by Mason & Venturino (1997), in place of (9.40) we minimise the functional

$$I[f^*] \equiv \frac{1}{n}\sum_{i=1}^{n}[\mathcal{K}f^*(x_i) - g(x_i)]^2 + \lambda\int_a^b w(y)[f^*(y)]^2\,\mathrm{d}y. \tag{9.56}$$

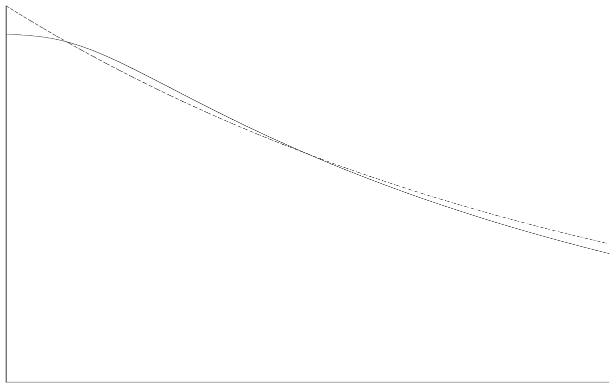

Figure 9.1: Data error $N(0, 0.005^2)$; 5 approximation coefficients

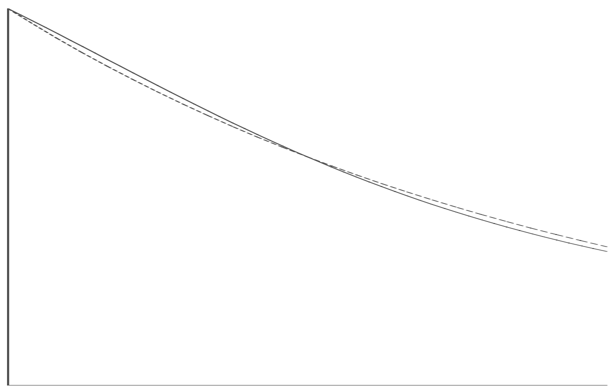

Figure 9.2: Data error $N(0, 0.01^2)$; 5 approximation coefficients

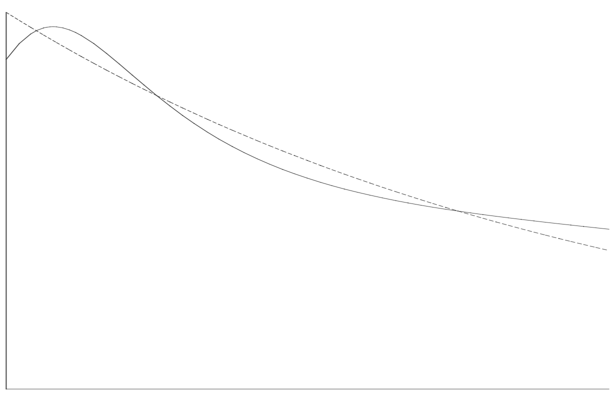

Figure 9.3: Data error $N(0, 0.01^2)$; 10 approximation coefficients

This is closer to the Tikhonov form (9.38) than is (9.40), and involves weaker assumptions about the smoothness of f.

Again we adopt an orthogonal polynomial sum to represent f^*. We choose $w(x)$ to be the weight function corresponding to the orthogonality. In particular, for the first-kind Chebyshev polynomial basis on $[-1, 1]$, and the approximation

$$f_\lambda^*(y) = \sum_{j=0}^{m} a_j T_j(y), \tag{9.57}$$

the weight function is of course $w(x) = 1/\sqrt{1 - x^2}$.

The main changes to the procedure of Section 9.6.2 are that

- in the case of (9.56) we do *not* now need to separate a_0 and a_1 off from the other coefficients a_r, and
- the smoothing matrix corresponding to $\mathbf{B}$ in (9.46) becomes diagonal, so that no $\mathbf{LL}^T$ decomposition is required.

For simplicity, we take the orthonormal basis on $[-1, 1]$, replacing (9.57) with

$$f_\lambda^*(y) = \sum_{j=0}^{m} a_j \phi_j(y) = \sum_{j=0}^{m} a_j \left[T_j(y)/n_j \right], \tag{9.58}$$

where

$$n_j^2 = \begin{cases} 2/\pi, & j > 0; \\ 1/\pi, & j = 0. \end{cases} \tag{9.59}$$

Define two inner products (respectively discrete and continuous);

$$\langle u\,,\, v \rangle_{\mathrm{d}} = \sum_{k=1}^{n} u(x_k) v(x_k); \tag{9.60}$$

$$\langle u\,,\, v \rangle_{\mathrm{c}} = \int_{-1}^{1} w(x) u(x) v(x)\, \mathrm{d}x. \tag{9.61}$$

Then the minimisation of (9.56), for f^* given by (9.58), leads to the system of equations

$$\sum_{j=0}^{m} a_j \langle \mathcal{K}\phi_i\,,\, \mathcal{K}\phi_j \rangle_{\mathrm{d}} - \langle \mathcal{K}\phi_i\,,\, \mathbf{g}^* \rangle_{\mathrm{d}} + n\lambda \sum_{j=0}^{m} a_j \langle \phi_i\,,\, \phi_j \rangle_{\mathrm{c}} = 0, \quad i = 0, \ldots, m. \tag{9.62}$$

Hence

$$(\mathbf{Q}^T\mathbf{Q} + n\lambda\mathbf{I})\mathbf{a} = \mathbf{Q}^T\mathbf{g}^* \tag{9.63}$$

(as a consequence of the orthonormality), where

$$Q_{k,j} = (\mathcal{K}\phi_j)(x_k) \tag{9.64}$$

and $\mathbf{a}$, $\mathbf{g}^*$ are vectors with components a_j and $g^*(x_j)$, respectively.

To determine f^* we need to solve (9.63) for a_j, with λ minimising $V(\lambda)$ as defined in (9.42). The matrix $\mathbf{A}(\lambda)$ in (9.42) is to be such that

$$\mathcal{K}\mathbf{f}^* = \mathbf{A}(\lambda)\mathbf{g}^*. \tag{9.65}$$

Now $\mathcal{K}\mathbf{f}^* = \left\{\sum_j a_j \mathcal{K}\phi_j(x_k)\right\} = \mathbf{Qa}$ and hence, from (9.63)

$$\mathbf{A}(\lambda) = \mathbf{Q}(\mathbf{Q}^T\mathbf{Q} + n\lambda\mathbf{I})^{-1}\mathbf{Q}^T. \tag{9.66}$$

9.6.4 Evaluation of $V(\lambda)$

It remains to clarify the remaining details of the algorithm of Section 9.6.3, and in particular to give an explicit formula for $V(\lambda)$ based on (9.66).

Let

$$\mathbf{Q} = \mathbf{W}\mathbf{\Lambda}\mathbf{X}^T \tag{9.67}$$

be the singular value decomposition of $\mathbf{Q}$, where $\mathbf{W}$ is $n \times n$ orthogonal, $\mathbf{X}$ is $(m+1) \times (m+1)$ orthogonal and

$$\mathbf{\Lambda} = \left[\begin{array}{c} \mathbf{\Delta}_m \\ \hline \mathbf{0} \end{array}\right]_{n\times(m+1)}. \tag{9.68}$$

Define

$$\mathbf{z} = [z_k] = \mathbf{W}^T\mathbf{g}. \tag{9.69}$$

From (9.68),

$$\mathbf{\Lambda}^T\mathbf{\Lambda} = \operatorname{diag}(d_0^2, \ldots, d_m^2). \tag{9.70}$$

It follows that

$$\mathbf{A}(\lambda) = \mathbf{W}\mathbf{B}(\lambda)\mathbf{W}^T \tag{9.71}$$

where

$$\mathbf{B}(\lambda) = \Lambda(\mathbf{\Lambda}^T\mathbf{\Lambda} + n\lambda\mathbf{I})^{-1}\mathbf{\Lambda}^T \tag{9.72}$$

so that $\mathbf{B}(\lambda)$ is the $n \times n$ diagonal matrix with elements

$$B_{kk} = \frac{d_k^2}{d_k^2 + n\lambda} \quad (0 \le k \le m); \qquad B_{kk} = 0 \quad (k > m). \tag{9.73}$$

From (9.71) and (9.72)

$$
\begin{aligned}
\|(\mathbf{I}-\mathbf{A}(\lambda))\mathbf{g}\|^2 &= \|(\mathbf{I}-\mathbf{A}(\lambda))\mathbf{W}\mathbf{z}\|^2 \\
&= \left\|\mathbf{W}^T(\mathbf{I}-\mathbf{A}(\lambda))\mathbf{W}\mathbf{z}\right\|^2 \\
&= \left\|(\mathbf{I}-\mathbf{W}^T\mathbf{A}(\lambda)\mathbf{W})\mathbf{z}\right\|^2 \\
&= \|(\mathbf{I}-\mathbf{B}(\lambda))\mathbf{z}\|^2 \\
&= \sum_{i=0}^{m}\left(\frac{n\lambda}{d_i^2+n\lambda}\right)^2 z_i^2 + \sum_{i=m+1}^{n-1} z_i^2 .
\end{aligned}
$$

Thus

$$
\|(\mathbf{I}-\mathbf{A}(\lambda))\mathbf{g}\|^2 = \sum_{i=0}^{m} n^2 e_i^2 z_i^2 + \sum_{i=m+1}^{n} z_i^2 \tag{9.74}
$$

where

$$
e_i(\lambda) = \frac{\lambda}{d_i^2+n\lambda}. \tag{9.75}
$$

Also

$$
\begin{aligned}
\operatorname{trace}(\mathbf{I}-\mathbf{A}(\lambda)) &= \operatorname{trace}\mathbf{W}^T(\mathbf{I}-\mathbf{A}(\lambda))\mathbf{W} \\
&= \operatorname{trace}(\mathbf{I}-\mathbf{B}(\lambda)) \\
&= \sum_{i=0}^{m}\frac{n\lambda}{d_i^2+n\lambda} + \sum_{i=m+1}^{n-1} 1.
\end{aligned}
$$

Thus

$$
\operatorname{trace}(\mathbf{I}-\mathbf{A}(\lambda)) = \sum_{i=0}^{m} n^2 e_i^2 + (n-m-1). \tag{9.76}
$$

Finally, from (9.75) and (9.76), together with (9.42), it follows that

$$
V(\lambda) = \frac{\displaystyle\sum_{0}^{m} n e_i^2 z_i^2 + \frac{1}{n}\sum_{m+1}^{n} z_i^2}{\left[\displaystyle\sum_{0}^{m} n e_i^2 + \frac{n-m-1}{n}\right]^2}. \tag{9.77}
$$

9.6.5 Other basis functions

It should be pointed out that Chebyshev polynomials are certainly not the only basis functions that could be used in the solution of (9.37) by regularisation. Indeed there is a discussion by Bennell & Mason (1989, Section ii) of three alternative basis functions, each of which yields an efficient algorithmic procedure, namely:

1. a kernel function basis $\{K(x_i, y)\}$,
2. a B-spline basis, and
3. an eigenfunction basis.

Of these, an eigenfunction basis is the most convenient (provided that eigenfunctions are known), whereas a kernel function basis is rarely of practical value. A B-spline basis is of general applicability and possibly comparable to, or slightly more versatile than, the Chebyshev polynomial basis. See Rodriguez & Seatzu (1990) and also Bennell & Mason (1989) for discussion of B-spline algorithms.

9.7 Partial differential equations and boundary integral equation methods

Certain classes of partial differential equations, with suitable boundary conditions, can be transformed into integral equations on the boundary of the domain. This is particularly true for equations related to the Laplace operator. Methods based on the solution of such integral equations are referred to as *boundary integral equation* (BIE) methods (Jaswon & Symm 1977, for instance) or, when they are based on discrete element approximations, as *boundary element methods* (BEM) (Brebbia et al. 1984). Chebyshev polynomials have a part to play in the solution of BIEs, since they lead typically to kernels related to the Hilbert kernel discussed in Section 9.5.1.

We now illustrate the role of Chebyshev polynomials in BIE methods for a particular mixed boundary value problem for Laplace's equation, which leads to a hypersingular boundary integral equation.

9.7.1 A hypersingular integral equation derived from a mixed boundary value problem for Laplace's equation

Derivation

In this section we tackle a 'hard' problem, which relates closely to the hypersingular integral relationship (9.30) satisfied by Chebyshev polynomials of the second kind. The problem and method are taken from Mason & Venturino (1997).

Consider Laplace's equation for $u(x, y)$ in the positive quadrant

$$\Delta u = 0, \quad x, y \geq 0, \tag{9.78}$$

subject to (see Figure 9.4)

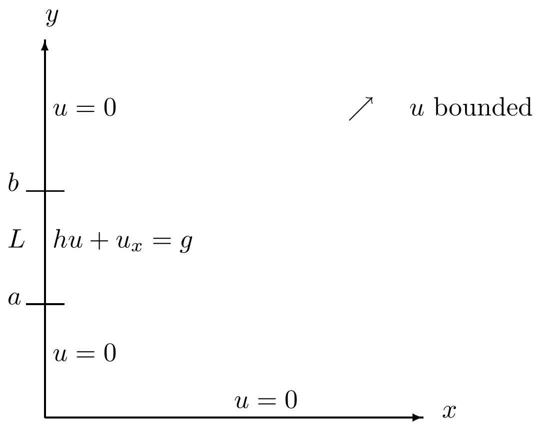

Figure 9.4: Location of the various boundary conditions (9.79)

$$u(x,0) = 0, \quad x \geq 0, \tag{9.79a}$$
$$hu(0,y) + u_x(0,y) = g(y), \quad 0 < a \leq y \leq b, \tag{9.79b}$$
$$u(0,y) = 0, \quad 0 \leq y < a;\; b < y, \tag{9.79c}$$
$$u(x,y) \text{ is bounded}, \quad x, y \to \infty. \tag{9.79d}$$

Thus the boundary conditions are homogeneous apart from a window $L \equiv [a,b]$ of radiation boundary conditions, and the steady-state temperature distribution in the positive quadrant is sought. Here the boundary conditions are 'mixed' in two senses: involving both u and u_x on L and splitting into two different operators on $x = 0$. Such problems are known to lead to Cauchy singular integral equations (Venturino 1986), but in this case a different approach leads to a hypersingular integral equation closely related to (9.30).

By separation of variables in (9.78), using (9.79a) and (9.79d), we find that

$$u(x,y) = \int_0^\infty A(\mu)\sin(\mu y)\exp(-\mu x)\,\mathrm{d}\mu. \tag{9.80}$$

The zero conditions (9.79c) on the complement L^c of L give

$$u(0,y) = \lim_{x\to 0+} \int_0^\infty A(\mu)\sin(\mu y)\exp(-\mu x)\,\mathrm{d}\mu = 0, \quad y \in L^c, \tag{9.81}$$

and differentiation of (9.80) with respect to x in L gives

$$u_x(0,y) = -\lim_{x\to 0+} \int_0^\infty \mu A(\mu)\sin(\mu y)\exp(-\mu x)\,\mathrm{d}\mu = 0, \quad y \in L. \tag{9.82}$$

Substitution of (9.80) and (9.82) into (9.79b) leads to

$$\lim_{x \to 0+} \int_0^\infty (h - \mu) A(\mu) \sin(\mu y) \exp(-\mu x) \, \mathrm{d}\mu = g(y), \quad y \in L. \tag{9.83}$$

Then (9.81) and (9.83) are a pair of dual integral equations for $A(\mu)$, and from which we can deduce u by using (9.80).

To solve (9.81) and (9.83), we define a function $B(y)$ as

$$B(y) := u(0, y) = \int_0^\infty A(\mu) \sin(\mu y) \, \mathrm{d}\mu, \quad y \geq 0. \tag{9.84}$$

Then, from (9.81)

$$B(y) = 0, \quad y \in L^c, \tag{9.85}$$

and, inverting the sine transform (9.84) and using (9.85),

$$\int_L B(t) \sin(st) \, \mathrm{d}t = \tfrac{1}{2} \pi A(s). \tag{9.86}$$

Substituting (9.86) in the integral equation (9.83) gives us

$$hB(y) - \frac{2}{\pi} \int_L I(t) \, \mathrm{d}t = g(t), \quad y \in L \tag{9.87}$$

where

$$\begin{aligned} I(t) &= \lim_{x \to 0+} \int_0^\infty \mu \sin(\mu t) \exp(-\mu x) \, \mathrm{d}\mu \\ &= \tfrac{1}{2} \lim_{x \to 0+} \int_0^\infty \mu \left[\cos \mu(t - y) - \cos \mu(t + y) \right] \exp(-\mu x) \, \mathrm{d}\mu. \end{aligned} \tag{9.88}$$

This simplifies (see Problem 7) to

$$\begin{aligned} I(t) &= \tfrac{1}{2} \lim_{x \to 0+} \left[\frac{x^2 - (t - y)^2}{(x^2 + (t - y)^2)^2} - \frac{x^2 - (t + y)^2}{(x^2 + (t + y)^2)^2} \right] \\ &= -\tfrac{1}{2} \left[\frac{1}{(t - y)^2} - \frac{1}{(t + y)^2} \right]. \end{aligned} \tag{9.89}$$

Substituting (9.89) into (9.87), we obtain the hypersingular integral equation, with strong singularity at $t = y$,

$$hB(y) + \frac{1}{\pi} \int_L B(t) \left[\frac{1}{(t - y)^2} - \frac{1}{(t + y)^2} \right] \mathrm{d}t = g(y), \quad y \in L, \tag{9.90}$$

from which $B(y)$ is to be determined, and hence $A(s)$ from (9.86) and $u(x, y)$ from (9.80).

Method of solution

Continuing to follow Mason & Venturino (1997), equation (9.90) can be rewritten in operator form as

$$\mathcal{A}\phi \equiv (h + \mathcal{H} + \mathcal{K})\phi = f \tag{9.91}$$

where $\mathcal{H}$ is a Hadamard finite-part integral and $\mathcal{K}$ is a compact perturbation, given by

$$(\mathcal{H}\phi)(x) = \int_{-1}^{1} \frac{\phi(s)}{(s-x)^2}\,\mathrm{d}s, \quad -1 < x < 1, \tag{9.92}$$

$$(\mathcal{K}\phi)(x) \equiv \int_{-1}^{1} K(x,s)\phi(s)\,\mathrm{d}s = \int_{-1}^{1} \frac{\phi(s)}{(s+x)^2}\,\mathrm{d}s, \quad -1 < x < 1, \tag{9.93}$$

and

$$f(x) = g\left(\tfrac{1}{2}(b-a)x + \tfrac{1}{2}(b+a)\right). \tag{9.94}$$

It is clear that $\phi(x)$ must vanish at the end points ± 1, since it represents boundary values, and moreover it should possess a square-root singularity (Martin 1991). Hence we write

$$\phi(x) = w(x)y(x), \text{ where } w(x) = \sqrt{1-x^2}. \tag{9.95}$$

We note also that the Hadamard finite-part operator maps second kind Chebyshev polynomials into themselves, as shown by Mason (1993) and Martin (1992) and indicated in (9.30) above; in fact

$$\mathcal{H}(wU_\ell)(x) = -\pi(\ell+1)U_\ell(x), \quad \ell \geq 0. \tag{9.96}$$

Solution of (9.86) in terms of second-kind polynomials is clearly suggested, namely

$$y(x) = \sum_{\ell=0}^{\infty} c_\ell U_\ell(x), \tag{9.97}$$

where the coefficients c_ℓ are to be determined, and we therefore define a weighted inner product

$$\langle u\,,\,v\rangle_w := \int_{-1}^{1} w(t)u(t)v(t)\,\mathrm{d}t$$

and observe that

$$\|U_\ell\|_w^2 = \tfrac{1}{2}\pi, \quad \ell \geq 0. \tag{9.98}$$

We also expand both $f(x)$, the right-hand side of (9.91), and $K(x,t)$ in second-kind polynomials

$$f(x) = \sum_{i=0}^{\infty} f_j U_j(x), \text{ where } f_j = \frac{2}{\pi}\langle f\,,\,U_j\rangle_w\,, \tag{9.99}$$

$$K(x,t) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} K_{ij} U_i(x) U_j(t), \tag{9.100}$$

so that (9.93), (9.97) and (9.98) give

$$\begin{aligned}
\mathcal{K}\phi &= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{\ell=0}^{\infty} c_\ell K_{ij} \int_{-1}^{1} w(t) U_i(x) U_j(t) u_\ell(t)\,\mathrm{d}t \\
&= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{\ell=0}^{\infty} c_\ell K_{ij} U_i(x) \left\|U_j\right\|_w^2 \delta_{j\ell} \\
&= \tfrac{1}{2}\pi \sum_{\ell=0}^{\infty} c_\ell \sum_{i=0}^{\infty} K_{i\ell} U_i(x). \qquad (9.101)
\end{aligned}$$

Substituting (9.95), (9.97), (9.99), (9.101) and (9.99) into (9.91):

$$hw \sum_{\ell=0}^{\infty} c_\ell U_\ell(x) - \pi \sum_{\ell=0}^{\infty} (\ell+1) c_\ell U_\ell(x) + \tfrac{1}{2}\pi \sum_{\ell=0}^{\infty} c_\ell \sum_{i=0}^{\infty} K_{i\ell} U_i(x) = \sum_{j=0}^{\infty} f_j U_j(x). \tag{9.102}$$

Taking the weighted inner product with U_j:

$$\begin{aligned}
h \sum_{\ell=0}^{\infty} c_\ell \left\langle wU_\ell\,,\,U_j\right\rangle_w - \pi \sum_{\ell=0}^{\infty} (\ell+1) c_\ell \left\langle U_\ell\,,\,U_j\right\rangle_w + \\
+ \tfrac{1}{2}\pi \sum_{\ell=0}^{\infty} c_\ell \sum_{i=0}^{\infty} K_{i\ell} \left\langle U_i\,,\,U_j\right\rangle_w = \frac{1}{2}\pi f_j. \qquad (9.103)
\end{aligned}$$

Define

$$b_{jl} := \left\langle wU_\ell\,,\,U_j\right\rangle_w = \int_{-1}^{1} (1-x^2) U_\ell(x) U_j(x)\,\mathrm{d}x. \tag{9.104}$$

Then it can be shown (Problem 8) that

$$b_{j\ell} = \begin{cases} \dfrac{1}{(\ell+j+2)^2-1} - \dfrac{1}{(\ell-j)^2-1}, & j+\ell \text{ even}, \\ 0, & \text{otherwise}. \end{cases} \tag{9.105}$$

Hence, from (9.103),

$$h \sum_{\ell=0}^{\infty} b_{j\ell} c_\ell - \tfrac{1}{2}\pi^2 (j+1) c_j + (\tfrac{1}{2}\pi)^2 \sum_{\ell=0}^{\infty} K_{j\ell} c_\ell = \frac{1}{2}\pi f_j, \quad 0 \le j < \infty. \tag{9.106}$$

Reducing (9.106) to a finite system, to solve for approximate coefficients $\hat{c}_\ell$, we obtain

$$h \sum_{\ell=0}^{N-1} b_{j\ell} \hat{c}_\ell - \tfrac{1}{2}\pi^2 (j+1) \hat{c}_j + (\tfrac{1}{2}\pi)^2 \sum_{\ell=0}^{N-1} K_{j\ell} \hat{c}_\ell = \frac{1}{2}\pi f_j, \quad 0 \le j < N-1. \tag{9.107}$$

Table 9.1: Results for $\mathcal{K} = 0$, $\phi(x) = \sqrt{1-x^2}\exp x$

N	condition number	$\|e_N\|_\infty$
1	1.93	
2	2.81	5×10^{-1}
4	4.17	5×10^{-3}
8	6.54	2×10^{-8}
16	10.69	1.5×10^{-13}

EXAMPLE 9.1: The method is tested by Mason & Venturino (1997) for a slightly different problem, where the well-behaved part $\mathcal{K}$ of the problem is set to zero and the function f is chosen so that $\phi(x) \equiv \sqrt{1-x^2}\exp x$. The condition number of the matrix of the linear system (9.107) defining $\hat{c}_j$ is compared in Table 9.1 with the maximum error $\|e_N\|_\infty$ for various values of N, and it is clear that the conditioning is relatively good and the accuracy achieved is excellent.

Error analysis

A rigorous error analysis has been carried out by Mason & Venturino (1997), but the detail is much too extensive to quote here. However, the conclusion reached was that, if $f \in \mathcal{C}^{p+1}[-1,1]$ and the integral operator $\mathcal{K}$ satisfies certain inequalities, then the method is convergent and

$$\|e_N\|_\infty \leq C.N^{-(p+1)} \tag{9.108}$$

where the constant C depends on the smoothness of K and f but not on N.

For further studies of singular integral equations involving a Cauchy kernel, see Elliott (1989) and Venturino (1992, 1993).

9.8 Problems for Chapter 9

1. Follow through all steps in detail of the proofs of Theorem 9.1 and Corollary 9.1A.

2. Using Corollary 9.1A, find a function $g(x)$ such that

   $$g(x) = -\int_{-1}^{1} \frac{\sqrt{1-y^2}f(y)}{(y-x)}\,\mathrm{d}y$$

   in the cases

(a) $f(y) = 1$;
(b) $f(y) = y^6$;
(c) $f(y) = \mathrm{e}^y$;
(d) $f(y) = \sqrt{1-y^2}$.

3. Using Corollary 9.1A, find a function $g(x)$ such that

$$g(x) = -\int_{-1}^{1} \frac{f(y)}{\sqrt{1-y^2}(y-x)}\,\mathrm{d}y$$

in the cases

(a) $g(x) = \mathrm{e}^x$;
(b) $g(x) = (1+x)^{\frac{1}{2}}(1-x)^{-\frac{1}{2}}$;
(c) $g(x) = x^5$;
(d) $g(x) = 1$.

4. Prove Theorem 9.2 in detail. For instance, the second kernel K_6 in the theorem is derived from

$$K_6(x,y) = \int_{-1}^{x} \frac{K_2(x,y)}{\sqrt{1-x^2}}\,\mathrm{d}x.$$

Setting $x = \cos 2\phi$, $y = \cos 2\psi$ and $\tan\phi = t$, show that K_6 simplifies to

$$\sin 2\psi \int_{\infty}^{t} \frac{\mathrm{d}t}{\sin^2\psi - t^2\cos^2\psi} = \log\left|\frac{\sin(\phi+\psi)}{\sin(\phi-\psi)}\right|.$$

Then, by setting $x = 2u^2-1$, $y = 2v^2-1$ and noting that $\sqrt{1-x^2} = 2u\sqrt{1-u^2}$, $\sqrt{1-y^2} = 2v\sqrt{1-v^2}$, show that $K_6(x,y)$ simplifies to

$$\log|x-y| - \log\left|1 - xy - \sqrt{(1-x^2)(1-y^2)}\right|.$$

5. By differentiating rather than integrating in (9.22a), (9.22b), (9.22c) and (9.22d), and using the properties

$$\begin{aligned}
[\sqrt{1-x^2}\,U_{n-1}(x)]' &= -nT_n(x)/\sqrt{1-x^2},\\
[T_n(x)]' &= nU_{n-1}(x),\\
[\sqrt{1-x}\,W_n(x)]' &= (n+\tfrac{1}{2})V_n(x)/\sqrt{1-x},\\
[\sqrt{1+x}\,V_n(x)]' &= (n+\tfrac{1}{2})W_n(x)/\sqrt{1+x},
\end{aligned}$$

deduce that the integral equation

$$\lambda\phi(x) = \int_{-1}^{1} \frac{1}{\sqrt{1-y^2}}\phi(y)K(x,y)\,\mathrm{d}y$$

has the following eigensolutions ϕ and eigenvalues λ for the following kernels K:

(a) $K(x,y) = K_9(x,y) = \dfrac{\sqrt{1-x^2}}{(y-x)^2} - \dfrac{x}{\sqrt{1-x^2}(y-x)}$;
$\phi = \phi_n(x) = T_{n-1}(x)/\sqrt{1-x^2}$, $\lambda = \lambda_n = -n\pi$.

(b) $K(x,y) = K_{10}(x,y) = \dfrac{1-y^2}{(y-x)^2}$;
$\phi = \phi_n(x) = U_{n-1}(x)$, $\lambda = \lambda_n = -n\pi$.

(c) $K(x,y) = K_{11}(x,y) = \dfrac{\sqrt{(1-x)(1+y)}}{(y-x)^2} - \dfrac{\sqrt{1+y}}{2\sqrt{1-x}(y-x)}$;
$\phi = \phi_n(x) = V_n(x)/\sqrt{1-x}$, $\lambda = \lambda_n = -(n+\frac{1}{2})\pi$.

(d) $K(x,y) = K_{12}(x,y) = \dfrac{\sqrt{(1+x)(1-y)}}{(y-x)^2} + \dfrac{\sqrt{1-y}}{2\sqrt{1+x}(y-x)}$;
$\phi = \phi_n(x) = W_n(x)/\sqrt{1+x}$, $\lambda = \lambda_n = -(n+\frac{1}{2})\pi$.

6. (a) Describe and discuss possible amendments that you might make to the regularisation methods of Section 9.6 in case K has any one of the four singular forms listed in Theorem 9.1. Does the method simplify?

(b) Discuss whether or not it might be better, for a general K, to use one of the Chebyshev polynomials other than $T_j(x)$ in the approximation (9.44).

7. Show that

$$\int_0^\infty \mu \sin \mu t \sin \mu y \exp(-\mu x)\,\mathrm{d}\mu = \frac{x^2-(t-y)^2}{(x^2+(t-y)^2)^2} - \frac{x^2-(t+y)^2}{(x^2+(t+y)^2)^2}.$$

(This completes the determination of $I(t)$, given by (9.88), so as to derive the hypersingular equation (9.90).)

8. Show that

$$\int_{-1}^1 (1-x^2)U_\ell(x)U_j(x)\,\mathrm{d}x = \frac{1}{(\ell+j+2)^2-1} - \frac{1}{(\ell-j)^2-1}$$

for $\ell + j$ even, and that the integral vanishes otherwise. (This is a step required in the derivation of the solution of the hypersingular equation (9.90).)