Chapter 7

Near-Best $\mathcal{L}_\infty$, $\mathcal{L}_1$ and $\mathcal{L}_p$ Approximations

7.1 Near-best $\mathcal{L}_\infty$ (near-minimax) approximations

We have already established in Section 5.5 that partial sums of first kind expansions

$$(S_n^T f)(x) = \sum_{k=0}^{n}{}' c_k U_k(x), \quad c_k = \frac{2}{\pi}\int_{-1}^{1} \frac{f(x)T_k(x)}{\sqrt{1-x^2}}\,\mathrm{d}x \tag{7.1}$$

yield near-minimax approximations within a relative distance of $O(\log n)$ in $\mathcal{C}[-1,1]$. Is this also the case for other kinds of Chebyshev polynomial expansions? The answer is in the affirmative, if we go about the expansion in the right way.

7.1.1 Second-kind expansions in $\mathcal{L}_\infty$

Consider the class $\mathcal{C}_{\pm 1}[-1,1]$ of functions continuous on $[-1,1]$ but constrained to vanish at ± 1. Let $S_n^{(2)} f$ denote the partial sum of the expansion of $f(x)/\sqrt{1-x^2}$ in Chebyshev polynomials of the second kind, $\{U_k(x) : k = 0,1,2,\ldots,n\}$, multiplied by $\sqrt{1-x^2}$. Then

$$(S_n^{(2)} f)(x) = \sqrt{1-x^2}\sum_{k=0}^{n} b_k U_k(x), \quad b_k = \frac{2}{\pi}\int_{-1}^{1} f(x)U_k(x)\,\mathrm{d}x. \tag{7.2}$$

If now we define

$$g(\theta) = \begin{cases} f(\cos\theta) & 0 \le \theta \le \pi \\ -f(\cos\theta) & -\pi \le \theta \le 0 \end{cases}$$

($g(\theta)$ being an odd, continuous and 2π-periodic function since $f(1) = f(-1) = 0$), then we obtain the equivalent Fourier sine series partial sum

$$(S_{n+1}^{FS} g)(\theta) = \sum_{k=0}^{n} b_k \sin(k+1)\theta, \quad b_k = \frac{2}{\pi}\int_0^{\pi} g(\theta)\sin(k+1)\theta\,\mathrm{d}\theta. \tag{7.3}$$

The operator S_{n+1}^{FS} can be identified as the restriction of the Fourier projection S_{n+1}^F to the space $\mathcal{C}_{2\pi,\mathrm{o}}^0$ of continuous functions that are both periodic of period 2π and odd; in fact we have $S_{n+1}^{FS} g = S_{n+1}^F g$ for odd functions g, where

$$(S_{n+1}^F g)(\theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} g(t+\theta)\frac{\sin(n+\frac{3}{2})t}{\sin\frac{1}{2}t}\,\mathrm{d}t. \tag{7.4}$$

If λ_n is the Lebesgue constant defined in (5.71)

$$\lambda_n = \frac{1}{2\pi}\int_{-\pi}^{\pi}\left|\frac{\sin(n+\frac{1}{2})t}{\sin\frac{1}{2}t}\right| \mathrm{d}t$$

and partly tabulated in Table 5.1 on page 126, then, similarly to (5.75), we may show that

$$\left\|S_n^{(2)}\right\|_\infty = \left\|S_{n+1}^{FS}\right\|_\infty \le \lambda_{n+1} \qquad \text{(on the space } \mathcal{C}_{\pm 1}[-1,1]\text{).} \tag{7.5}$$

Therefore $(S_n^{(2)} f)(x)$ is near-minimax within a relative distance λ_{n+1}.

This constant λ_{n+1} is not, however, the best possible, as has been shown by Mason & Elliott (1995) — the argument of Section 5.5.1 falls down because the function

$$\operatorname{sgn}\left(\frac{\sin(n+\frac{3}{2})\theta}{\sin\frac{1}{2}\theta}\right)$$

is even, and cannot therefore be closely approximated by any function in $\mathcal{C}^0_{2\pi,\mathrm{o}}$.

However, g being odd, we may rewrite (7.4) as

$$\begin{aligned}
(S_{n+1}^{FS} g)(\theta) &= \frac{1}{4\pi}\int_{-\pi}^{\pi}\{g(t+\theta)-g(-t-\theta)\}\frac{\sin(n+\frac{3}{2})t}{\sin\frac{1}{2}t}\,\mathrm{d}t \\
&= \frac{1}{4\pi}\int_{-\pi}^{\pi} g(t)\left\{\frac{\sin(n+\frac{3}{2})(t-\theta)}{\sin\frac{1}{2}(t-\theta)} - \frac{\sin(n+\frac{3}{2})(t+\theta)}{\sin\frac{1}{2}(t+\theta)}\right\}\mathrm{d}t \\
&= \frac{1}{4\pi}\int_{-\pi}^{\pi} g(t) K_{n+1}^{FS}(\theta,t)\,\mathrm{d}t.
\end{aligned} \tag{7.6}$$

This kernel $K_{n+1}^{FS}(\theta,t)$ is an odd function of θ and t, and an argument similar to that in Section 5.5.1 can now be used to show that

$$\left\|S_n^{(2)}\right\|_\infty = \left\|S_{n+1}^{FS}\right\|_\infty = \frac{1}{4\pi}\sup_\theta \int_{-\pi}^{\pi}\left|K_{n+1}^{FS}(\theta,t)\right|\,\mathrm{d}t = \lambda_n^{(2)}, \text{ say.} \tag{7.7}$$

Table 7.1: Lower bounds on $\lambda_n^{(2)}$

n	bound	n	bound	n	bound
1	1.327	10	1.953	100	2.836
2	1.467	20	2.207	200	3.114
3	1.571	30	2.362	300	3.278
4	1.653	40	2.474	400	3.394
5	1.721	50	2.561	500	3.484

Mason & Elliott (1995) have actually computed values of $\lambda_n^{(2)}$, which is no straightforward task since the points where the integrand $K_{n+1}^{FS}(\theta,t)$ changes sign are not in general easily determined. For a lower bound to the supremum for each n, however, we may evaluate the integral when $\theta = \pi/(2n+3)$, when the sign changes occur at the precisely-known points $t = 0$, $\pm 3\pi/(2n+3)$, $\pm 5\pi/(2n+3)$, …, $\pm\pi$. This gives the values shown in Table 7.1.

7.1.2 Third-kind expansions in $\mathcal{L}_\infty$

Following Mason & Elliott (1995) again, consider functions f in $\mathcal{C}_{-1}[-1,1]$, continuous on $[-1,1]$ but constrained to vanish at $x = -1$. Then the nth degree projection operator $S_n^{(3)}$, such that $S_n^{(3)} f$ is the partial sum of the expansion of $f(x)\sqrt{2/(1+x)}$ in Chebyshev polynomials of the third kind, $\{V_k(x) : k = 0, 1, 2, \ldots, n\}$, multiplied by $\sqrt{(1+x)/2}$, is defined by

$$\begin{aligned}(S_n^{(3)} f)(x) &= \sqrt{\frac{1+x}{2}} \sum_{k=0}^{n} c_k V_k(x) \\ &= \sum_{k=0}^{n} c_k \cos(k+\tfrac{1}{2})\theta \qquad (7.8)\end{aligned}$$

where $x = \cos\theta$ and

$$\begin{aligned}c_k &= \frac{1}{\pi}\int_{-1}^{1} \sqrt{\frac{2}{1-x}} f(x) V_k(x)\,\mathrm{d}x \\ &= \frac{2}{\pi}\int_{0}^{\pi} g(\theta)\cos(k+\tfrac{1}{2})\theta\,\mathrm{d}\theta \\ &= \frac{1}{2\pi}\int_{-2\pi}^{2\pi} g(\theta)\cos(k+\tfrac{1}{2})\theta\,\mathrm{d}\theta \qquad (7.9)\end{aligned}$$

with g defined as follows:

$$g(\theta) = \begin{cases} f(\cos\theta) & 0 \le \theta \le \pi \\ -g(2\pi-\theta) & \pi \le \theta \le 2\pi \\ g(-\theta) & -2\pi \le \theta \le 0. \end{cases}$$

The function $g(\theta)$ has been defined to be continuous (since $g(\pi) = f(-1) = 0$) and 4π-periodic, and is even about $\theta = 0$ and odd about $\theta = \pi$. Its Fourier expansion (in trigonometric functions of $\frac{1}{2}\theta$) therefore involves only terms in $\cos(2k+1)\frac{\theta}{2} = \cos(k+\frac{1}{2})\theta$ and is of the form (7.8) when truncated. From (7.8) and (7.9),

$$(S_n^{(3)} f)(x) = \frac{1}{2\pi}\int_{-2\pi}^{2\pi} g(t) \sum_{k=0}^{n} \cos(k+\tfrac{1}{2})t \cos(k+\tfrac{1}{2})\theta\,\mathrm{d}t.$$

Table 7.2: Values of $\lambda_n^{(3)}$

n	$\lambda_n^{(3)}$	n	$\lambda_n^{(3)}$	n	$\lambda_n^{(3)}$
1	1.552	10	2.242	100	3.140
2	1.716	20	2.504	200	3.420
3	1.832	30	2.662	300	3.583
4	1.923	40	2.775	400	3.700
5	1.997	50	2.864	500	3.790

We leave it as an exercise to the reader (Problem 1) to deduce that

$$(S_n^{(3)} f)(x) = \frac{1}{\pi} \int_{-\pi}^{\pi} g(t+\theta) \frac{\sin(n+1)t}{\sin \frac{1}{2} t} \, \mathrm{d}t.$$

Thus

$$\left| (S_n^{(3)} f)(x) \right| \leq \|g\|_\infty \lambda_n^{(3)} \tag{7.10}$$

where

$$\lambda_n^{(3)} = \frac{1}{\pi} \int_0^{\pi} \left| \frac{\sin(n+1)t}{\sin \frac{1}{2} t} \right| \mathrm{d}t. \tag{7.11}$$

Hence $\left\| S_n^{(3)} \right\|_\infty \leq \lambda_n^{(3)}$.

Arguing as in Section 5.5.1 as before, we again show that we have an equality

$$\left\| S_n^{(3)} \right\|_\infty = \lambda_n^{(3)}.$$

Numerical values of $\lambda_n^{(3)}$ are shown in Table 7.2, and clearly appear to approach those of λ_n (Table 5.1) asymptotically.

A fuller discussion is given by Mason & Elliott (1995), where it is conjectured that (as for λ_n in (5.77))

$$\lambda_n^{(3)} = \frac{4}{\pi^2} \log n + A_1 + O(1/n)$$

where $A_1 \simeq 1.2703$. (This follows earlier work by Luttman & Rivlin (1965) and by Cheney & Price (1970) on the asymptotic behaviour of λ_n.) Once more, then, we have obtained a near-minimax approximation within a relative distance asymptotic to $4\pi^{-2} \log n$.

For further detailed discussion of Lebesgue functions and constants for interpolation, see Brutman (1997).

7.2 Near-best $\mathcal{L}_1$ approximations

From Section 6.4 we would expect Chebyshev series partial sums to yield near-best $\mathcal{L}_1$ approximations with respect to the weights given in (6.36), namely $w(x) = 1/\sqrt{1-x^2}$, 1, $1/\sqrt{1-x}$, $1/\sqrt{1+x}$, since they already provide best $\mathcal{L}_1$ approximations for a function that is a polynomial of one degree higher. In fact, this can be shown to hold simply by pre-multiplying and post-dividing the functions expanded in Section 7.1 by the additional factor $\sqrt{1-x^2}$. The simplest case to consider here is that of the second-kind polynomials U_n, since the function expanded is then just the original function.

The partial sum of degree n of the second kind, for a continuous function $f(x)$, is defined by the projection

$$P_n^{(2)} : (P_n^{(2)} f)(x) = \sum_{k=0}^{n} b_k U_k(x), \quad b_k = \frac{2}{\pi} \int_{-1}^{1} \sqrt{1-x^2} f(x) U_k(x) \, \mathrm{d}x. \tag{7.12}$$

Defining the function g by

$$g(\theta) = \sin\theta \, f(\cos\theta), \tag{7.13}$$

which is naturally an odd periodic continuous function, we see that

$$b_k = \frac{2}{\pi} \int_0^{\pi} g(\theta) \sin(k+1)\theta \, \mathrm{d}\theta, \tag{7.14}$$

as in (7.3), and $(P_n^{(2)} f)(\cos\theta) = (S_{n+1}^{FS} g)(\theta)$.

Now, treating $f(x)$ as defined on $[-1, 1]$ and $g(\theta)$ as defined on $[-\pi, \pi]$ so that

$$\|g\|_1 = \int_{-\pi}^{\pi} |g(\theta)| \, \mathrm{d}\theta = \int_{-\pi}^{\pi} |\sin\theta \, f(\cos\theta)| \, \mathrm{d}\theta = 2 \int_{-1}^{1} f(x) \, \mathrm{d}x = 2 \|f\|_1 ,$$

we have

$$\begin{aligned}
\left\| P_n^{(2)} f \right\|_1 &= \tfrac{1}{2} \left\| S_{n+1}^{FS} g \right\|_1 \\
&= \tfrac{1}{2} \int_{-\pi}^{\pi} \left| \frac{1}{4\pi} \int_{-\pi}^{\pi} g(t) K_{n+1}^{FS}(\theta, t) \, \mathrm{d}t \right| \, \mathrm{d}\theta \\
&\le \tfrac{1}{2} \int_{-\pi}^{\pi} |g(t)| \, \mathrm{d}t \, \frac{1}{4\pi} \sup_t \int_{-\pi}^{\pi} \left| K_{n+1}^{FS}(\theta, t) \right| \, \mathrm{d}\theta \\
&= \tfrac{1}{2} \|g\|_1 \, \lambda_n^{(2)} \\
&= \|f\|_1 \, \lambda_n^{(2)},
\end{aligned}$$

where $\lambda_n^{(2)}$ is the constant defined in (7.7) above.

Hence

$$\left\| P_n^{(2)} \right\|_1 \le \lambda_n^{(2)}. \tag{7.15}$$

Thus $\lambda_n^{(2)}$ is a bound on $\left\| P_n^{(2)} \right\|_1$, just as it was a bound on $\left\| S_n^{(2)} \right\|_\infty$ in Section 7.1.1, and so $(P_n^{(2)} f)(x)$, given by (7.12), is a near-best $\mathcal{L}_1$ approximation within the relative distance $\lambda_n^{(2)}$ defined in (7.7).

The discussion above is, we believe, novel. Freilich & Mason (1971) established that $\left\| P_n^{(2)} \right\|_1$ is bounded by λ_n, but the new bound (7.15) is smaller by about 0.27.

If we define $P_n^{(1)}$ and $P_n^{(3)}$ to be the corresponding partial sum projections of the first and third kinds,

$$\begin{aligned} (P_n^{(1)} f)(x) &= \frac{1}{\sqrt{1-x^2}} \sideset{}{'}\sum_{k=0}^{n} c_k T_k(x), \\ c_k &= \frac{2}{\pi} \int_{-1}^{1} f(x) T_k(x)\,\mathrm{d}x, \end{aligned} \tag{7.16}$$

$$\begin{aligned} (P_n^{(3)} f)(x) &= \frac{1}{2\sqrt{1-x}} \sum_{k=0}^{n} c_k V_k(x), \\ c_k &= \frac{1}{\pi} \int_{-1}^{1} \sqrt{2(1+x)} f(x) V_k(x)\,\mathrm{d}x, \end{aligned} \tag{7.17}$$

then it is straightforward to show in a similar way (see Problem 2) that

$$\left\| P_n^{(1)} \right\|_1 \le \lambda_n \quad \text{(classical Lebesgue constant)}$$

and

$$\left\| P_n^{(3)} \right\|_1 \le \lambda_n^{(3)} \quad \text{(given by (7.11))}.$$

7.3 Best and near-best $\mathcal{L}_p$ approximations

The minimal $\mathcal{L}_\infty$ and $\mathcal{L}_1$ properties of the weighted Chebyshev polynomials, discussed in Sections 3.3 and 6.4, are in fact special examples of general $\mathcal{L}_p$ minimality properties, which are discussed by Mason & Elliott (1995).

Theorem 7.1 *The monic polynomials $2^{1-n}T_n(z)$, $2^{-n}U_n(z)$, $2^{-n}V_n(z)$, $2^{-n}W_n(z)$ minimise the $\mathcal{L}_p$ norm*

$$\left[\int_{-1}^{1} w(x) \left| P_n(x) \right|^p \,\mathrm{d}x \right]^{\frac{1}{p}} \quad (1 < p < \infty) \tag{7.18}$$

over all monic polynomials $P_n(x)$ *with*

$$w(x) = (1-x)^{\frac{1}{2}(\alpha-1)}(1+x)^{\frac{1}{2}(\beta-1)}$$

for the respective values

$$(\alpha, \beta) = (0,0),\ (p,p),\ (0,p),\ (p,0).$$

The proof of this result depends on the characterisation of the best $\mathcal{L}_p$ approximation according to the following result, which we state without proof.

Lemma 7.2 *The* $\mathcal{L}_p$ *norm (7.18) is minimised if and only if*

$$\int_{-1}^{1} w(x)\,|P_n(x)|^{p-2}\, P_n(x) P_k(x)\,\mathrm{d}x = 0, \quad \forall k < n. \tag{7.19}$$

Proof: (of Theorem 7.1)

We shall concentrate on the first case, that of the first kind polynomials $T_n(x)$, and leave the remaining cases as exercises for the reader (Problem 3).

Define

$$P_n(x) = T_n(x), \quad w(x) = 1/\sqrt{(}1-x^2).$$

Then

$$\int_{-1}^{1} w(x)\,|P_n(x)|^{p-2}\, P_n(x) P_k(x)\,\mathrm{d}x = \int_0^{\pi} |\cos n\theta|^{p-2} \cos n\theta \cos k\theta\,\mathrm{d}\theta.$$

Now, for $0 \le y \le 1$, define

$$C_n(\theta, y) = \begin{cases} 1 & (|\cos n\theta| \le y), \\ 0 & (|\cos n\theta| > y). \end{cases}$$

Then if $y = \cos\eta$ we have $C_n(\theta, y) = 1$ over each range

$$\frac{(r-1)\pi + \eta}{n} \le \theta \le \frac{r\pi - \eta}{n}, \quad r = 1, 2, \ldots, n.$$

Thus, for any integer j with $0 < j < 2n$,

$$\begin{aligned}
\int_0^{\pi} C_n(\theta, y) \cos j\theta\,\mathrm{d}\theta &= \sum_{r=1}^{n} \int_{((r-1)\pi+\eta)/n}^{(r\pi-\eta)/n} \cos j\theta\,\mathrm{d}\theta \\
&= \sum_{r=1}^{n} \frac{1}{j}\left[\sin\frac{j\{(r-1)\pi+\eta\}}{n} - \sin\frac{j\{r\pi-\eta\}}{n}\right] \\
&= \sum_{r=1}^{2n} \frac{1}{j} \sin\frac{j\{(r-1)\pi+\eta\}}{n} \\
&= 0.
\end{aligned}$$

But now, for $0 \le k < n$,

$$\begin{aligned}
&\int_{-1}^{1} w(x)\,|P_n(x)|^{p-2}\,P_n(x)P_k(x)\,\mathrm{d}x \\
&= \int_0^\pi |\cos n\theta|^{p-2} \cos n\theta\ \cos k\theta\,\mathrm{d}\theta \\
&= \int_0^\pi |\cos n\theta|^{p-2}\ \tfrac{1}{2}[\cos(n+k)\theta + \cos(n-k)\theta]\,\mathrm{d}\theta \\
&= \int_0^\pi \left\{ \frac{1}{p-1} \int_0^1 y^{p-1}(1 - C_n(\theta,y))\,\mathrm{d}y \right\} \tfrac{1}{2}[\cos(n+k)\theta + \cos(n-k)\theta]\,\mathrm{d}\theta \\
&= \frac{1}{p-1} \int_0^1 y^{p-1} \left\{ \int_0^\pi (1 - C_n(\theta,y))\ \tfrac{1}{2}[\cos(n+k)\theta + \cos(n-k)\theta]\,\mathrm{d}\theta \right\} \mathrm{d}y \\
&= 0.
\end{aligned}$$

The result then follows from Lemma 7.2. ●●

(An alternative method of proof is to translate into polynomial terms the result on trigonometric polynomials, due to S. N. Bernstein, given in Achieser's book (Achieser 1956, Section 10).)

7.3.1 Complex variable results for elliptic-type regions

It is possible to obtain bounds for norms of projections, and hence measures of near-best $\mathcal{L}_p$ approximation, by using ideas of convexity over a family of $\mathcal{L}_p$ measure spaces for $1 \le p \le \infty$ (Mason 1983*b*, Mason 1983*a*). However, the settings for which there are results have been restricted to ones involving generalised complex Chebyshev series — based on results for Laurent series. Mason & Chalmers (1984) give $\mathcal{L}_p$ results for Fourier, Taylor and Laurent series; moreover Chalmers & Mason (1984) show these to be minimal projections on appropriate analytic function spaces. The settings, involving projection from space X to space Y, where $\mathcal{A}(D)$ denotes the space of functions analytic in D and continuous on $\bar{D}$, are:

1. Chebyshev, first kind: $X = \mathcal{A}(D_\rho)$, where D_ρ is the elliptical domain $\{z : \left|z + \sqrt{z^2 - 1}\right| < \rho\}$; $Y = Y_1 = \Pi_n$ (polynomials of degree n in z);

$$P = G_n$$

   where G_n is the Chebyshev first-kind series projection of $\mathcal{A}(D_\rho)$ into Π_n.

2. Chebyshev, second kind: $X = \{f(z) = \sqrt{z^2 - 1}F(z),\ F \in \mathcal{A}(D_\rho)\}$, $Y = Y_2 = \{f(z) = \sqrt{z^2 - 1}F(z),\ F \in \Pi_n\}$;

$$P = H^*_{n-1} : H^*_{n-1}f = \sqrt{z^2 - 1}H_{n-1}F,$$

where H_n is the Chebyshev second kind series projection of $\mathcal{A}(D_\rho)$ into Π_n.

3. Generalised Chebyshev: $X = \mathcal{A}(\{z : \rho_1 < \left|z + \sqrt{z^2 - 1}\right| < \rho_2\})$ (annulus between two ellipses); $Y = Y_1 \oplus Y_2$;

$$P = J_n = G_n + H^*_{n-1}.$$

Then it is proved by Mason (1983*b*), using convexity arguments, that for each of the three projections above

$$\|P\|_p \le (\sigma_{2n})^{|2p^{-1}-1|} \quad (1 \le p \le \infty) \tag{7.20}$$

where

$$\sigma_n = \frac{1}{n}\int_0^\pi \left|\frac{\sin(n+1)\theta}{\sin\theta}\right| \mathrm{d}\theta.$$

Note that $\sigma_{2n} = \lambda_n$. So the generalised expansion is proved to be as close to minimax as the (separated) first kind one.

For $p = 1$, $p = \infty$, we obtain bounds increasing as $4\pi^{-2}\log n$, while $\|P\|_p \to 1$ as $p \to 2$.

It follows also (Chalmers & Mason 1984) that J_n is a minimal projection; indeed, this appears to be the only such result for Chebyshev series. The component projections G_n and H^*_{n-1} are essentially odd and even respectively, and correspond to the cosine and sine parts of a full Fourier series. In contrast, the projection G_n is not minimal.

The earliest near-best results for $\mathcal{L}_\infty$ and $\mathcal{L}_1$ approximation on elliptic domains appear to be those of Geddes (1978) and Mason (1978). See also Mason & Elliott (1993) for detailed results for all individual cases.

We should also note that it has long been known that

$$\|P\|_p \le C_p \tag{7.21}$$

where C_p is some constant independent of n. Although this is superficially stronger than (7.20) from a theoretical point of view, the bounds (7.20) are certainly small for values of n up to around 500. Moreover, it is known that $C_p \to \infty$ as $p \to \infty$. See Zygmund (1959) for an early derivation of this result, and Mhaskar & Pai (2000) for a recent discussion.

7.4 Problems for Chapter 7

1. Show that

$$\cos(k+\tfrac{1}{2})t\cos(k+\tfrac{1}{2})\theta = \tfrac{1}{2}[\cos(k+\tfrac{1}{2})(t+\theta) + \cos(k+\tfrac{1}{2})(t-\theta)]$$

and that

$$\sum_{k=0}^{n} \cos(k+\tfrac{1}{2})u = \frac{\sin(n+1)u}{2\sin\frac{1}{2}u}.$$

Hence prove that

$$\frac{1}{\pi}\int_{-\pi}^{\pi} g(t)\sum_{k=0}^{n}\cos(k+\tfrac{1}{2})t\cos(k+\tfrac{1}{2})\theta\,\mathrm{d}\theta = \frac{1}{\pi}\int_{0}^{\pi} g(t+\theta)\frac{\sin(n+1)t}{\sin\frac{1}{2}t}\,\mathrm{d}t$$

by showing that the pair of integrals involved are equal.

(This completes the proof of Section 7.1.2, showing that the weighted third-kind expansion has a partial sum which is near-minimax.)

2. Show that $\left\|P_n^{(1)}\right\|_1 \le \lambda_n$ and $\left\|P_n^{(3)}\right\|_1 \le \lambda_n^{(3)}$, where λ_n is the classical Lebesgue constant and $\lambda_n^{(3)}$ is given by (7.11).

3. Prove Theorem 7.1 in the case of polynomials of the second and third kinds.

4. If S_n is a partial sum of a Fourier series

$$(S_n f)(\theta) = \tfrac{1}{2}a_0 + \sum_{k=0}^{n}(a_k\cos k\theta + b_k\sin k\theta),$$

show how this may be written, for suitably defined functions, as a combined first-kind and (weighted) second-kind Chebyshev expansion.

[Hint: $f(\theta) = F(\cos\theta) + \sin\theta\, G(\cos\theta)$ = even part of f + odd part of f.]

5. Consider the Fejér operator $\tilde{F}_n$, which takes the mean of the first n partial sums of the Fourier series.

   (a) Show that $\tilde{F}_n$ is not a projection.

   (b) Show that

   $$(\tilde{F}_n f)(\theta) = \frac{1}{n\pi}\int_0^{2\pi} f(t)\tilde{\sigma}_n(t-\theta)\,\mathrm{d}t$$

   where

   $$\tilde{\sigma}_n(\theta) = \frac{\sin\frac{1}{2}(n+1)\theta\ \sin\frac{1}{2}n\theta}{2\sin\frac{1}{2}\theta}.$$

   (c) Show that $(\tilde{F}_n f)(\theta)$, under the transformation $x = \cos\theta$, becomes a combined third-kind and fourth-kind Chebyshev-Fejér sum, each part being appropriately weighted.

6. Derive the basic result for $p = \infty$, namely $\|P\|_\infty \le \sigma_{2n} = \lambda_n$, for the three projections listed in Section 7.3.1.

7. Derive the corresponding basic results for $p = 1$.

   Would it be possible to obtain a better set of results in this case by using an odd kernel, like that used in (7.6)?

8. Note that $\|P\|_2 = 1$ in Section 7.3.1 and that it is known that $\|P\|_p$ is bounded for any fixed p in the range $1 < p < \infty$. Discuss whether there is a 'better' result than the one quoted.

   (You might like to consider both the practical case $n \leq 500$ and the theoretical case of arbitrarily large n.)

9. Investigate the validity of letting $p \to 1$ in the results of Section 7.3.1, when the interior of the ellipse collapses to the interval $[-1, 1]$.

10. Compute by hand the bounds for $\left\| S_n^{(2)} \right\|_\infty$ in the case $n = 0$.

11. Compute some numerical values of $\lambda_n^{(2)}$ and compare them with the lower bounds given in Table 7.1.