CHAPTER 2

Basic Properties and Formulae

2.1 Introduction

The aim of this chapter is to provide some elementary formulae for the manipulation of Chebyshev polynomials and to summarise the key properties which will be developed in the book. Areas of application will be introduced and discussed in the chapters devoted to them.

2.2 Chebyshev polynomial zeros and extrema

The Chebyshev polynomials of degree $n > 0$ of all four kinds have precisely n zeros and $n + 1$ local extrema in the interval $[-1, 1]$. In the case $n = 5$, this is evident in Figures 1.1, 1.3 and 1.4. Note that $n - 1$ of these extrema are interior to $[-1, 1]$, and are 'true' alternate maxima and minima (in the sense that the gradient vanishes), the other two extrema being at the end points ± 1 (where the gradient is non-zero).

From formula (1.1), the zeros for x in $[-1, 1]$ of $T_n(x)$ must correspond to the zeros for θ in $[0, \pi]$ of $\cos n\theta$, so that

$$n\theta = (k - \tfrac{1}{2})\pi, \quad (k = 1, 2, \ldots, n).$$

Hence, the zeros of $T_n(x)$ are

$$x = x_k = \cos \frac{(k - \frac{1}{2})\pi}{n}, \quad (k = 1, 2, \ldots, n). \tag{2.1}$$

EXAMPLE 2.1: For $n = 3$, the zeros are

$$x = x_1 = \cos \frac{\pi}{6} = \frac{\sqrt{3}}{2}, \quad x_2 = \cos \frac{3\pi}{6} = 0, \quad x_3 = \cos \frac{5\pi}{6} = -\frac{\sqrt{3}}{2}.$$

Note that these zeros are in decreasing order in x (corresponding to increasing θ), and it is sometimes preferable to list them in their natural order as

$$x = \cos \frac{(n - k + \frac{1}{2})\pi}{n}, \quad (k = 1, 2, \ldots, n). \tag{2.2}$$

Note, too, that $x = 0$ is a zero of $T_n(x)$ for all odd n, but not for even n, and that zeros are symmetrically placed in pairs on either side of $x = 0$.

The zeros of $U_n(x)$ (defined by (1.4)) are readily determined in a similar way from the zeros of $\sin(n+1)\theta$ as

$$x = y_k = \cos\frac{k\pi}{(n+1)}, \quad (k = 1, 2, \ldots, n) \tag{2.3}$$

or in their natural order

$$x = \cos\frac{(n-k+1)\pi}{n+1}, \quad (k = 1, 2, \ldots, n). \tag{2.4}$$

One is naturally tempted to extend the set of points (2.3) by including the further values $y_0 = 1$ and $y_{n+1} = -1$, giving the set

$$x = y_k = \cos\frac{k\pi}{(n+1)}, \quad (k = 0, 1, \ldots, n+1). \tag{2.5}$$

These are zeros not of $U_n(x)$, but of the polynomial

$$(1 - x^2)U_n(x). \tag{2.6}$$

However, we shall see that these points are popular as nodes in applications to integration.

The zeros of $V_n(x)$ and $W_n(x)$ (defined by (1.8), (1.9)) correspond to zeros of $\cos(n+\frac{1}{2})\theta$ and $\sin(n+\frac{1}{2})\theta$, respectively. Hence, the zeros of $V_n(x)$ occur at

$$x = \cos\frac{(k-\frac{1}{2})\pi}{n+\frac{1}{2}}, \quad (k = 1, 2, \ldots, n) \tag{2.7}$$

or in their natural order

$$x = \cos\frac{(n-k+\frac{1}{2})\pi}{n+\frac{1}{2}}, \quad (k = 1, 2, \ldots, n), \tag{2.8}$$

while the zeros of $W_n(x)$ occur at

$$x = \cos\frac{k\pi}{n+\frac{1}{2}}, \quad (k = 1, 2, \ldots, n) \tag{2.9}$$

or in their natural order

$$x = \cos\frac{(n-k+1)\pi}{n+\frac{1}{2}}, \quad (k = 1, 2, \ldots, n). \tag{2.10}$$

Note that there are natural extensions of these point sets, by including the value $k = n + 1$ and hence $x = -1$ in (2.7) and the value $k = 0$ and hence $x = 1$ in (2.9). Thus the polynomials

$$(1 + x)V_n(x) \text{ and } (1 - x)W_n(x)$$

have as zeros their natural sets (2.7) for $k = 1, \ldots, n+1$ and (2.9) for $k = 0, 1, \ldots, n$, respectively.

The internal extrema of $T_n(x)$ correspond to the extrema of $\cos n\theta$, namely the zeros of $\sin n\theta$, since

$$\frac{\mathrm{d}}{\mathrm{d}x} T_n(x) = \frac{\mathrm{d}}{\mathrm{d}x} \cos n\theta = \frac{\mathrm{d}}{\mathrm{d}\theta} \cos n\theta \bigg/ \frac{\mathrm{d}x}{\mathrm{d}\theta} = \frac{-n \sin n\theta}{-\sin\theta}.$$

Hence, including those at $x = \pm 1$, the extrema of $T_n(x)$ on $[-1, 1]$ are

$$x = \cos\frac{k\pi}{n}, \quad (k = 0, 1, \ldots, n) \tag{2.11}$$

or in their natural order

$$x = \cos\frac{(n-k)\pi}{n}, \quad (k = 0, 1, \ldots, n). \tag{2.12}$$

These are precisely the zeros of $(1 - x^2)U_{n-1}(x)$, namely the points (2.5) above (with n replaced by $n - 1$). Note that the extrema are all of equal magnitude (unity) and alternate in sign at the points (2.12) between -1 and $+1$, as shown in Figure 1.1.

The extrema of $U_n(x)$, $V_n(x)$, $W_n(x)$ are not in general as readily determined; indeed finding them involves the solution of transcendental equations. For example,

$$\frac{\mathrm{d}}{\mathrm{d}x} U_n(x) = \frac{\mathrm{d}}{\mathrm{d}x} \frac{\sin(n+1)\theta}{\sin\theta} = \frac{-(n+1)\sin\theta\cos(n+1)\theta + \cos\theta\sin(n+1)\theta}{\sin^3\theta}$$

and the extrema therefore correspond to values of θ satisfying the equation

$$\tan(n+1)\theta = (n+1)\tan\theta \neq 0.$$

All that we can say for certain is that the extreme values of $U_n(x)$ have magnitudes which increase monotonically as $|x|$ increases away from 0, until the largest magnitude of $n + 1$ is achieved at $x = \pm 1$.

On the other hand, from the definitions (1.4), (1.8), (1.9), we can show that

$$\begin{aligned} \sqrt{1-x^2}\, U_n(x) &= \sin(n+1)\theta, \\ \sqrt{1+x}\, V_n(x) &= \sqrt{2}\cos(n+\tfrac{1}{2})\theta, \\ \sqrt{1-x}\, W_n(x) &= \sqrt{2}\sin(n+\tfrac{1}{2})\theta; \end{aligned}$$

Hence the extrema of the weighted polynomials $\sqrt{1-x^2}\, U_n(x)$, $\sqrt{1+x}\, V_n(x)$, $\sqrt{1-x}\, W_n(x)$ are explicitly determined and occur, respectively, at

$$x = \cos\frac{(2k+1)\pi}{2(n+1)}, \quad x = \cos\frac{2k\pi}{2n+1}, \quad x = \cos\frac{(2k+1)\pi}{2n+1} \quad (k = 0, 1, \ldots, n).$$

2.3 Relations between Chebyshev polynomials and powers of x

It is useful and convenient in various applications to be able to express Chebyshev polynomials explicitly in terms of powers of x, and vice versa. Such formulae are simplest and easiest to derive in the case of the first kind polynomials $T_n(x)$, and so we concentrate on these.

2.3.1 Powers of x in terms of $\{T_n(x)\}$

The power x^n can be expressed in terms of the Chebyshev polynomials of degrees up to n, but, since these are alternately even and odd, we see at once that we need only include polynomials of alternate degrees, namely $T_n(x)$, $T_{n-2}(x)$, $T_{n-4}(x)$, ... Writing $x = \cos\theta$, we therefore need to express $\cos^n\theta$ in terms of $\cos n\theta$, $\cos(n-2)\theta$, $\cos(n-4)\theta$, ..., and this is readily achieved by using the binomial theorem as follows:

$$\begin{aligned}(\mathrm{e}^{\mathrm{i}\theta} + \mathrm{e}^{-\mathrm{i}\theta})^n &= \mathrm{e}^{\mathrm{i}n\theta} + \binom{n}{1}\mathrm{e}^{\mathrm{i}(n-2)\theta} + \cdots + \binom{n}{n-1}\mathrm{e}^{-\mathrm{i}(n-2)\theta} + \mathrm{e}^{-\mathrm{i}n\theta} \\ &= (\mathrm{e}^{\mathrm{i}n\theta} + \mathrm{e}^{-\mathrm{i}n\theta}) + \binom{n}{1}(\mathrm{e}^{\mathrm{i}(n-2)\theta} + \mathrm{e}^{-\mathrm{i}(n-2)\theta}) + \\ &\qquad + \binom{n}{2}(\mathrm{e}^{\mathrm{i}(n-4)\theta} + \mathrm{e}^{-\mathrm{i}(n-4)\theta}) + \cdots. \end{aligned} \tag{2.13}$$

Here we have paired in brackets the first and last terms, the second and second-to-last terms, and so on. The number of such brackets will be

$$\lfloor n/2 \rfloor + 1$$

where $\lfloor m \rfloor$ denotes the integer part of m. When n is even, the last bracket in (2.13) will contain only the one (middle) term $\mathrm{e}^{0\theta}$ [= 1].

Now using the fact that

$$(\mathrm{e}^{\mathrm{i}\theta} + \mathrm{e}^{-\mathrm{i}\theta})^n = (2\cos\theta)^n = 2^n \cos^n\theta$$

we deduce from (2.13) that

$$2^{n-1}\cos^n\theta = {\sum_{k=0}^{\lfloor n/2 \rfloor}}{}' \binom{n}{k} \cos(n-2k)\theta,$$

where the dash ($\sum'$) denotes that the kth term in the sum is to be halved if n is even and $k = n/2$. Hence, from the definition (1.1) of $T_n(x)$,

$$x^n = 2^{1-n} {\sum_{k=0}^{\lfloor n/2 \rfloor}}{}' \binom{n}{k} T_{n-2k}(x), \tag{2.14}$$

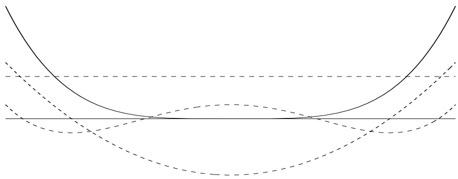

Figure 2.1: x^4 (full curve) and its decomposition into Chebyshev polynomials (broken curves)

where the dash now denotes that the term in $T_0(x)$, if there is one, is to be halved.

EXAMPLE 2.2: Taking $n = 4$ [see Figure 2.1]:

$$\begin{aligned} x^4 &= 2^{-3} \sum_{k=0}^{2}{}' \binom{4}{k} T_{4-2k}(x) \\ &= 2^{-3} \left[T_4(x) + \binom{4}{1} T_2(x) + \tfrac{1}{2} \binom{4}{2} T_0(x) \right] \\ &= \frac{1}{8} T_4(x) + \frac{1}{2} T_2(x) + \frac{3}{8} T_0(x). \end{aligned}$$

2.3.2 $T_n(x)$ in terms of powers of x

It is not quite as simple to derive formulae in the reverse direction. The obvious device to use is de Moivre's Theorem:

$$\cos n\theta + \mathrm{i} \sin n\theta = (\cos\theta + \mathrm{i}\sin\theta)^n.$$

Expanding by the binomial theorem and taking the real part,

$$\cos n\theta = \cos^n\theta - \binom{n}{2} \cos^{n-2}\theta \sin^2\theta + \binom{n}{4} \cos^{n-4}\theta \sin^4\theta + \cdots.$$

If $\sin^2\theta$ is replaced by $1 - \cos^2\theta$ throughout, then a formula is obtained for $\cos n\theta$ in terms of $\cos^n\theta$, $\cos^{n-2}\theta$, $\cos^{n-4}\theta$, …. On transforming to $x = \cos\theta$, this leads to the required formula for $T_n(x)$ in terms of x^n, x^{n-2}, x^{n-4}, ….

We omit the details here, but refer to Rivlin (1974), where the relevant result is obtained in the form

$$T_n(x) = \sum_{k=0}^{\lfloor n/2 \rfloor} \left[(-1)^k \sum_{j=k}^{\lfloor n/2 \rfloor} \binom{n}{2j} \binom{j}{k} \right] x^{n-2k}. \tag{2.15}$$

However, a rather simpler formula is given, for example, by Clenshaw (1962) and Snyder (1966) in the form

$$T_n(x) = \sum_{k=0}^{\lfloor n/2 \rfloor} c_k^{(n)} x^{n-2k} \tag{2.16}$$

where

$$c_k^{(n)} = (-1)^k 2^{n-2k-1} \left[2\binom{n-k}{k} - \binom{n-k-1}{k} \right] \quad (2k < n) \tag{2.17a}$$

and

$$c_k^{(2k)} = (-1)^k \quad (k \geq 0). \tag{2.17b}$$

This formula may be proved by induction, using the three-term recurrence relation (1.3a), and we leave this as an exercise for the reader (Problem 5).

In fact the term in square brackets in (2.17a) may be further simplified, by taking out common ratios, to give

$$c_k^{(n)} = (-1)^k 2^{n-2k-1} \frac{n}{n-k} \binom{n-k}{k}. \tag{2.18}$$

Example 2.3: For $n = 6$ we obtain from (2.17b), (2.18):

$$c_0^{(6)} = 2^5 = 32; \quad c_1^{(6)} = (-1)^1 2^3 \frac{6}{5} \binom{5}{1} = -48;$$

$$c_2^{(6)} = (-1)^2 2^1 \frac{6}{4} \binom{4}{2} = 18; \quad c_3^{(6)} = (-1)^3 2^{-1} \frac{6}{3} \binom{3}{3} = -1.$$

Hence

$$T_6(x) = 32x^6 - 48x^4 + 18x^2 - 1.$$

For an alternative derivation of the results in this section, making use of generating functions, see Chapter 5.

2.3.3 Ratios of coefficients in $T_n(x)$

In applications, recurrence formulae which link pairs of coefficients are often more useful than explicit formulae (such as (2.18) above) for the coefficients themselves since, using such formulae, the whole sequence of coefficients may be assembled rather more simply and efficiently than by working them out one by one.

From (2.18),

$$\begin{aligned} c_k^{(n)} &= (-1)^k 2^{n-2k-1} \frac{n}{k} \frac{(n-k-1)(n-k-2)\cdots(n-2k+1)}{1\cdot 2\cdots(k-1)}; \\ c_{k+1}^{(n)} &= (-1)^{k+1} 2^{n-2k-3} \frac{n}{k+1} \frac{(n-k-2)(n-k-3)\cdots(n-2k-1)}{1\cdot 2\cdots k}. \end{aligned}$$

Hence, on dividing and cancelling common factors,

$$c_{k+1}^{(n)} = -\frac{(n-2k)(n-2k-1)}{4(k+1)(n-k-1)} c_k^{(n)} \tag{2.19}$$

where $c_k^{(n)}$ denotes the coefficient of x^n in $T_n(x)$. Formula (2.19) is valid for $n > 0$ and $k \geq 0$.

2.4 Evaluation of Chebyshev sums, products, integrals and derivatives

A variety of manipulations of Chebyshev polynomials and of sums or series of them can be required in practice. A secret to the efficient and stable execution of these tasks is to avoid rewriting Chebyshev polynomials in terms of powers of x and to operate wherever possible with the Chebyshev polynomials themselves (Clenshaw 1955).

2.4.1 Evaluation of a Chebyshev sum

Suppose that we wish to evaluate the sum

$$S_n = \sum_{r=0}^{n} a_r P_r(x) = a_0 P_0(x) + a_1 P_1(x) + \cdots + a_n P_n(x) \tag{2.20a}$$

where $\{P_r(x)\}$ are Chebyshev polynomials of either the first, second, third or fourth kinds. We may write (2.20a) in vector form as

$$S_n = \mathbf{a}^T \mathbf{p}, \tag{2.20b}$$

where $\mathbf{a}^T$ and $\mathbf{p}$ denote the row- and column-vectors

$$\mathbf{a}^T = (a_0,\ a_1,\ \ldots,\ a_n), \quad \mathbf{p} = \begin{pmatrix} P_0(x) \\ P_1(x) \\ \vdots \\ P_n(x) \end{pmatrix}.$$

In each of the four cases, from (1.3a), (1.6a), (1.12a), (1.12b) above, the recurrence relation between the polynomials takes the same form

$$P_r(x) - 2xP_{r-1}(x) + P_{r-2}(x) = 0,\ r = 2, 3, \ldots. \tag{2.21a}$$

with $P_0(x) = 1$ and, respectively,

$$\begin{gathered} P_1(x) = T_1(x) = x, \quad P_1(x) = U_1(x) = 2x, \\ P_1(x) = V_1(x) = 2x - 1, \quad P_1(x) = W_1(x) = 2x + 1. \end{gathered} \tag{2.21b}$$

Equations (2.21) may be written in matrix notation as

$$\begin{pmatrix} 1 & & & & & & \\ -2x & 1 & & & & & \\ 1 & -2x & 1 & & & & \\ & 1 & -2x & 1 & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & 1 & -2x & 1 & \\ & & & & 1 & -2x & 1 \end{pmatrix} \begin{pmatrix} P_0(x) \\ P_1(x) \\ P_2(x) \\ P_3(x) \\ \vdots \\ P_{n-1}(x) \\ P_n(x) \end{pmatrix} = \begin{pmatrix} 1 \\ X \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix} \tag{2.22a}$$

or (denoting the $(n+1) \times (n+1)$ matrix by $\mathbf{A}$)

$$\mathbf{A}\mathbf{p} = \mathbf{c} \tag{2.22b}$$

where

$$\mathbf{c} = \begin{pmatrix} 1 \\ X \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

with $X = -x,\ 0,\ -1,\ 1$, respectively in the four cases.

Let

$$\mathbf{b}^T = (b_0,\ b_1,\ \ldots,\ b_n)$$

be the row vector satisfying the equation

$$(b_0,\ b_1,\ \ldots,\ b_n)\begin{pmatrix} 1 & & & & & & \\ -2x & 1 & & & & & \\ 1 & -2x & 1 & & & & \\ & 1 & -2x & 1 & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & 1 & -2x & 1 & \\ & & & & 1 & -2x & 1 \end{pmatrix} =$$

$$= (a_0,\ a_1,\ \ldots,\ a_n) \tag{2.23a}$$

or

$$\mathbf{b}^T\mathbf{A} = \mathbf{a}^T. \tag{2.23b}$$

Then we have

$$S_n = \mathbf{a}^T\mathbf{p} = \mathbf{b}^T\mathbf{A}\mathbf{p} = \mathbf{b}^T\mathbf{c} = b_0 + b_1X. \tag{2.24}$$

If we write $b_{n+1} = b_{n+2} = 0$, then the matrix equation (2.23a) can be seen to represent the recurrence relation

$$b_r - 2xb_{r+1} + b_{r+2} = a_r,\ r = 0, 1, \ldots, n. \tag{2.25}$$

We can therefore evaluate S_n by starting with $b_{n+1} = b_{n+2} = 0$ and performing the three-term recurrence (2.25) in the reverse direction,

$$b_r = 2xb_{r+1} - b_{r+2} + a_r,\ r = n, \ldots, 1, 0, \tag{2.26}$$

to obtain b_1 and b_0, and finally evaluating the required result S_n as

$$S_n = b_0 + b_1X. \tag{2.27}$$

For the first-kind polynomials $T_r(x)$, it is more usual to need the modified sum

$$S'_n = \sum_{r=0}^{n}{}' a_rT_r(x) = \tfrac{1}{2}a_0T_0(x) + a_1T_1(x) + \cdots + a_nT_n(x),$$

in which the coefficient of T_0 is halved, in which case (2.27) is replaced (remembering that $X = -x$) by

$$\begin{aligned} S'_n &= S_n - \tfrac{1}{2}a_0 \\ &= (b_0 - b_1x) + \tfrac{1}{2}(b_0 - 2xb_1 + b_2), \end{aligned}$$

or

$$S'_n = \tfrac{1}{2}(b_0 - b_2). \tag{2.28}$$

Note that, for a given x, carrying out the recurrence requires only $O(n)$ multiplications, and hence is as efficient as Horner's rule for evaluating a polynomial as a sum of powers using nested multiplication.

In some applications, which we shall refer to later, it is necessary to evaluate Chebyshev sums of a large number of terms at an equally large number of values of x. While the algorithm described above may certainly be used in such cases, one can often gain dramatically in efficiency by making use of the well-known *fast Fourier transform*, as we shall show later in Section 4.7.1.

Sums of even only or odd only polynomials, such as

$$S_n^{(0)} = \sum_{r=0}^{n}{}' \bar{a}_{2r} T_{2r}(x) \text{ and } S_n^{(1)} = \sum_{r=0}^{n} \bar{a}_{2r+1} T_{2r+1}(x)$$

may of course be evaluated by the above method, setting odd or even coefficients (respectively) to zero. However, the sum may be calculated much more efficiently using only the given even/odd coefficients by using a modified algorithm (Clenshaw 1962) which is given in Problem 7 below.

EXAMPLE 2.4: Consider the case $n = 2$ and $x = 1$ with coefficients

$$a_0 = 1, \quad a_1 = 0.1, \quad a_2 = 0.001.$$

Then from (2.21b) we obtain

$$\begin{aligned} b_3 &= b_4 = 0 \\ b_2 &= a_2 = 0.01 \\ b_1 &= 2b_2 - b_3 + 0.1 = 0.12 \\ b_0 &= 2b_1 - b_2 + 1 = 1.23. \end{aligned}$$

Hence

$$\begin{aligned} \sum_{r=0}^{2}{}' a_r T_r(1) &= \tfrac{1}{2}(b_0 - b_2) = 0.61 \\ \sum_{r=0}^{2} a_r U_r(1) &= b_0 = 1.23 \\ \sum_{r=0}^{2} a_r V_r(1) &= b_0 - b_1 = 1.11 \\ \sum_{r=0}^{2} a_r W_r(1) &= b_0 + b_1 = 1.35. \end{aligned}$$

To verify these formulae, we may set $\theta = 0$ (i.e., $x = 1$) in (1.1), (1.4), (1.8), (1.9), giving

$$T_n(1) = 1, \quad U_n(1) = n + 1, \tag{2.29a}$$

$$V_n(1) = 1, \quad W_n(1) = 2n + 1. \tag{2.29b}$$

Hence

$$\begin{aligned}
{\sum_{r=0}^{2}}' a_r T_r(1) &= \tfrac{1}{2}a_0 + a_1 + a_2 = 0.61 \\
\sum_{r=0}^{2} a_r U_r(1) &= a_0 + 2a_1 + 3a_2 = 1.23 \\
\sum_{r=0}^{2} a_r V_r(1) &= a_0 + a_1 + a_2 = 1.11 \\
\sum_{r=0}^{2} a_r W_r(1) &= a_0 + 3a_1 + 5a_2 = 1.35.
\end{aligned}$$

Incidentally, it is also useful to note that, by setting $\theta = \pi$, $\frac{1}{2}\pi$ in (1.1), (1.4), (1.8), (1.9), we can find further special values of the Chebyshev polynomials at $x = -1$ and $x = 0$, similar to those (2.29) at $x = 1$, namely

$$T_n(-1) = (-1)^n, \quad U_n(-1) = (-1)^n(n+1), \tag{2.30a}$$

$$V_n(-1) = (-1)^n(2n+1), \quad W_n(-1) = (-1)^n, \tag{2.30b}$$

$$T_{2n+1}(0) = U_{2n+1}(0) = 0, \quad T_{2n}(0) = U_{2n}(0) = (-1)^n, \tag{2.30c}$$

$$-V_{2n+1}(0) = W_{2n+1}(0) = (-1)^n, \quad V_{2n}(0) = W_{2n}(0) = (-1)^n. \tag{2.30d}$$

We leave the confirmation of formulae (2.29) and (2.30) as an exercise to the reader (Problem 8 below).

2.4.2 Stability of the evaluation of a Chebyshev sum

It is important to consider the effects of rounding errors when using recurrence relations, and specifically (2.26) above, since it is known that instability can sometimes occur. (By instability, we mean that rounding errors grow unacceptably fast relative to the true solution as the calculation progresses.) Three-term recurrence relations have two families of solutions, and it is possible for contributions from a relatively larger but unwanted solution to appear as rounding errors; so we need to take note of this. A brief discussion is given by Clenshaw (1962); a more detailed discussion is given by Fox & Parker (1968).

In the case of the recurrence (2.26), suppose that each b_s is computed with a local rounding error ϵ_s, which local errors together propagate into errors δ_r in b_r for $r < s$, resulting in an error Δ in S_n or Δ' in S'_n. Writing $\bar{b}_r$ for the computed b_r and $\bar{S}_n$ or $\bar{S}'_n$ for the S_n or S'_n computed without further error from (2.24) or (2.28), then from (2.26) (for fixed x)

$$\bar{b}_r = 2x\bar{b}_{r+1} - \bar{b}_{r+2} + a_r - \epsilon_r \tag{2.31}$$

while

$$b_r - \bar{b}_r = \delta_r. \tag{2.32}$$

Also

$$\begin{aligned} \bar{S}_n &= \bar{b}_0 + \bar{b}_1 X, \\ \bar{S}'_n &= \tfrac{1}{2}(\bar{b}_0 - \bar{b}_2), \end{aligned}$$

and

$$S_n - \bar{S}_n = \Delta, \quad S'_n - \bar{S}'_n = \Delta'.$$

From (2.26), (2.31), (2.32) we deduce that

$$\delta_r = 2x\delta_{r+1} - \delta_{r+2} + \epsilon_r \quad (r < s) \tag{2.33}$$

while

$$\begin{aligned} \Delta &= \delta_0 + \delta_1 X, \\ \Delta' &= \tfrac{1}{2}(\delta_0 - \delta_2). \end{aligned}$$

Now the recurrence (2.33), is identical in form to (2.26), with ϵ_r replacing a_r and δ_r replacing b_r, while obviously $\delta_{n+1} = \delta_{n+2} = 0$. Taking the final steps into account, we deduce that

$$\Delta = \sum_{r=0}^{n} \epsilon_r P_r(x), \tag{2.34}$$

where P_r is T_r, U_r, V_r or W_r, depending on the choice of X, and

$$\Delta' = \sum_{r=0}^{n}{}' \epsilon_r T_r(x). \tag{2.35}$$

Using the well-known inequality

$$\left| \sum_r x_r y_r \right| \leq \left(\sum_r |x_r| \right) \max_r |y_r| ,$$

we deduce the error bounds

$$|\Delta'| \leq \left(\sum_{r=0}^{n}{}' |\epsilon_r| \right) \max_{r=0}^{n} |T_r(x)| \leq \sum_{r=0}^{n}{}' |\epsilon_r| \tag{2.36}$$

and

$$|\Delta| \le \left(\sum_{r=0}^{n} |\epsilon_r|\right) \max_{r=0}^{n} |P_r(x)| \le C_n \sum_{r=0}^{n} |\epsilon_r|, \tag{2.37}$$

where $C_n = 1,\ n+1,\ 2n+1,\ 2n+1$ when P_r is T_r, U_r, V_r or W_r, respectively. (Note that the ϵ_r in these formulae are the absolute, not relative, errors incurred at each step of the calculation.)

2.4.3 Evaluation of a product

It is frequently necessary to be able to multiply Chebyshev polynomials by each other, as well as by factors such as x, $1 - x$ and $1 - x^2$, and to re-express the result in terms of Chebyshev polynomials. Such products are much less readily carried out for second-, third- and fourth-kind polynomials, as a consequence of the denominators in their trigonometric definitions. We therefore emphasise $T_n(x)$ and to a lesser extent $U_n(x)$.

Various formulae are readily obtained by using the substitution $x = \cos\theta$ and trigonometric identities, as follows.

$$T_m(x)T_n(x) = \cos m\theta \cos n\theta = \tfrac{1}{2}(\cos(m+n)\theta + \cos|m-n|\,\theta),$$

giving

$$T_m(x)T_n(x) = \tfrac{1}{2}(T_{m+n}(x) + T_{|m-n|}(x)). \tag{2.38}$$

$$\begin{aligned} xT_n(x) &= \cos\theta \cos n\theta = \tfrac{1}{2}(\cos(n+1)\theta + \cos|n-1|\,\theta), \\ xU_n(x)\sin\theta &= \cos\theta \sin(n+1)\theta = \tfrac{1}{2}(\sin(n+2)\theta + \sin n\theta), \end{aligned}$$

giving

$$xT_n(x) = \tfrac{1}{2}(T_{n+1}(x) + T_{|n-1|}(x)) \tag{2.39}$$

and

$$xU_n(x) = \tfrac{1}{2}(U_{n+1}(x) + U_{n-1}(x)), \tag{2.40}$$

(provided that we interpret $U_{-1}(x)$ as $\sin 0/\sin\theta = 0$).

More generally, we may also obtain expressions for $x^m T_n(x)$ (and similarly $x^m U_n(x)$) for any m, by expressing x^m in terms of Chebyshev polynomials by (2.14) and then using (2.38). (See Problem 4 below.)

In a similar vein,

$$\begin{aligned} (1-x^2)T_n(x) &= \sin^2\theta \cos n\theta = \tfrac{1}{2}(1 - \cos 2\theta)\cos n\theta \\ &= \tfrac{1}{2}\cos n\theta - \tfrac{1}{4}(\cos(n+2)\theta + \cos|n-2|\,\theta), \\ (1-x^2)U_n(x)\sin\theta &= \sin^2\theta \sin(n+1)\theta = \tfrac{1}{2}(1 - \cos 2\theta)\sin(n+1)\theta \\ &= \tfrac{1}{2}\sin(n+1)\theta - \tfrac{1}{4}(\sin(n+3)\theta + \sin(n-1)\theta), \end{aligned}$$

giving

$$(1-x^2)T_n(x) = -\tfrac{1}{4}T_{n+2}(x) + \tfrac{1}{2}T_n(x) - \tfrac{1}{4}T_{|n-2|}(x) \tag{2.41}$$

and

$$(1-x^2)U_n(x) = -\tfrac{1}{4}U_{n+2}(x) + \tfrac{1}{2}U_n(x) - \tfrac{1}{4}U_{n-2}(x) \tag{2.42}$$

where we interpret $U_{-1}(x)$ as 0 again, and $U_{-2}(x)$ as $\sin(-\theta)/\sin\theta = -1$.

Note that the particular cases $n = 0$, $n = 1$ are included in the formulae above, so that, specifically

$$\begin{aligned}
xT_0(x) &= T_1(x),\\
xU_0(x) &= \tfrac{1}{2}U_1(x),\\
(1-x^2)T_0(x) &= \tfrac{1}{2}T_0(x) - \tfrac{1}{2}T_2(x),\\
(1-x^2)T_1(x) &= \tfrac{1}{4}T_1(x) - \tfrac{1}{4}T_3(x),\\
(1-x^2)U_0(x) &= \tfrac{3}{4}U_0(x) - \tfrac{1}{4}U_2(x),\\
(1-x^2)U_1(x) &= \tfrac{1}{2}U_1(x) - \tfrac{1}{4}U_3(x).
\end{aligned}$$

2.4.4 Evaluation of an integral

The indefinite integral of $T_n(x)$ can be expressed in terms of Chebyshev polynomials as follows. By means of the usual substitution $x = \cos\theta$,

$$\begin{aligned}
\int T_n(x)\,\mathrm{d}x &= \int -\cos n\theta \sin\theta\,\mathrm{d}\theta\\
&= -\tfrac{1}{2}\int (\sin(n+1)\theta - \sin(n-1)\theta)\,\mathrm{d}\theta\\
&= \tfrac{1}{2}\left[\frac{\cos(n+1)\theta}{n+1} - \frac{\cos|n-1|\,\theta}{n-1}\right]
\end{aligned}$$

(where the second term in the bracket is to be omitted in the case $n = 1$).

Hence

$$\int T_n(x)\,\mathrm{d}x = \begin{cases} \tfrac{1}{2}\left[\dfrac{T_{n+1}(x)}{n+1} - \dfrac{T_{|n-1|}(x)}{n-1}\right], & n \neq 1;\\ \tfrac{1}{4}T_2(x), & n = 1. \end{cases} \tag{2.43}$$

Clearly this result can be used to integrate the sum

$$S_n(x) = \sum_{r=0}^{n}{}' a_r T_r(x)$$

in the form

$$\begin{aligned} I_{n+1}(x) &= \int S_n(x)\,\mathrm{d}x \\ &= \text{constant } + \tfrac{1}{2}a_0T_1(x) + \tfrac{1}{4}a_1T_2(x) + \sum_{r=2}^{n} \frac{a_r}{2}\left[\frac{T_{r+1}(x)}{r+1} - \frac{T_{r-1}(x)}{r-1}\right] \\ &= {\sum_{r=0}^{n+1}}' A_rT_r(x) \end{aligned} \qquad (2.44)$$

where A_0 is determined from the constant of integration, and

$$A_r = \frac{a_{r-1} - a_{r+1}}{2r}, \quad r > 0, \qquad (2.45)$$

with $a_{n+1} = a_{n+2} = 0$.

EXAMPLE 2.5: Table 2.1 gives 5-decimal values of A_r computed from values of a_r, obtained from an infinite expansion of the function e^x, after each value of a_r had been rounded to 4 decimals (numbers taken from Clenshaw (1962)). Each A_r would be identical to a_r for an exact calculation, but it is interesting to observe that, although there is a possible rounding error of ± 0.00005 in each given a_r, all the computed A_r actually have errors significantly smaller than this.

Table 2.1: Integration of a Chebyshev series

r	a_r	A_r	$\lvert$error in $A_r\rvert$
0	2.53213	—	—
1	1.13032	1.13030	0.00002
2	0.27150	0.27150	0.00000
3	0.04434	0.04433	0.00001
4	0.00547	0.00548	0.00001
5	0.00054	0.00055	0.00001
6	0.00004	0.00004	0.00000

There is an interesting and direct integral relationship between the Chebyshev polynomials of the first and second kinds, namely

$$\int U_n(x)\,\mathrm{d}x = \frac{1}{n+1}T_{n+1}(x) + \text{ constant} \qquad (2.46)$$

(which is easily verified by substituting $x = \cos\theta$). Hence, the sum

$$S_n(x) = \sum_{r=1}^{n} b_r U_{r-1}(x)$$

can be integrated immediately to give

$$\int S_n(x)\,\mathrm{d}x = \sum_{r=1}^{n} \frac{b_r}{r} T_r(x) + \text{ constant.} \tag{2.47}$$

2.4.5 Evaluation of a derivative

The formula for the derivative of $T_n(x)$ in terms of first-kind polynomials is not quite as simple as (2.43). From (2.46) we deduce that

$$\frac{\mathrm{d}}{\mathrm{d}x} T_{n+1}(x) = (n+1)U_n(x), \tag{2.48}$$

so that it is easily expressed in terms of a second-kind polynomial. Then from (1.6b) and (1.7) it follows that

$$\frac{\mathrm{d}}{\mathrm{d}x} T_n(x) = 2n \sideset{}{'}\sum_{\substack{r=0 \\ n-r \text{ odd}}}^{n-1} T_r(x). \tag{2.49}$$

However, the derivative of a finite sum of first-kind Chebyshev polynomials is readily expressible as a sum of such polynomials, by reversing the process used in the integration of (2.44). Given the Chebyshev sum (of degree $n+1$, say)

$$I_{n+1}(x) = \sideset{}{'}\sum_{r=0}^{n+1} A_r T_r(x),$$

then

$$S_n(x) = \frac{\mathrm{d}}{\mathrm{d}x} I_{n+1} = \sideset{}{'}\sum_{r=0}^{n} a_r T_r(x) \tag{2.50}$$

where the coefficients $\{a_r\}$ are derived from the given $\{A_r\}$ by using (2.45) in the form

$$a_{r-1} = a_{r+1} + 2rA_r, \quad (r = n+1, n, \ldots, 1) \tag{2.51a}$$

with

$$a_{n+1} = a_{n+2} = 0. \tag{2.51b}$$

Explicitly, if we prefer, we may say that

$$a_r = \sum_{\substack{k=r+1 \\ k-r \text{ odd}}}^{n+1} 2kA_k. \tag{2.52}$$

EXAMPLE 2.6: Table 2.2 shows 4-decimal values of a_r computed from 4-decimal values of A_r, for the same example as in Table 2.1. Each a_r would be identical to A_r in an exact computation, and we see this time that the derivative $S_n(x)$ is less accurate than the original polynomial $I_{n+1}(x)$ by nearly one decimal place. The contrast between these results is consistent with the principle that, in general, numerical integration is a stable process and numerical differentiation an unstable process. The size of the errors in the latter case can be attributed to the propagation, by (2.51a), of the error inherent in the assumptions (2.51b).

Table 2.2: Differentiation of a Chebyshev series

r	A_r	a_r	$\lvert$error in $a_r\rvert$
0	2.53213	2.5314	0.0007
1	1.13032	1.1300	0.0003
2	0.27150	0.2708	0.0007
3	0.04434	0.0440	0.0003
4	0.00547	0.0050	0.0005
5	0.00054	0.0000	0.0005

There is another relatively simple formula for the derivative of $T_n(x)$, which we can obtain as follows.

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}T_n(x) &= \frac{\mathrm{d}}{\mathrm{d}\theta}\cos n\theta \Big/ \frac{\mathrm{d}}{\mathrm{d}\theta}\cos\theta \\ &= \frac{n\sin n\theta}{\sin\theta} \\ &= \frac{\frac{1}{2}n(\cos(n-1)\theta - \cos(n+1)\theta)}{\sin^2\theta} \\ &= \frac{\frac{1}{2}n(T_{n-1}(x) - T_{n+1}(x))}{1-x^2}.\end{aligned}$$

Thus, for $|x| \neq 1$,

$$\frac{\mathrm{d}}{\mathrm{d}x}T_n(x) = \frac{n}{2}\frac{T_{n-1}(x) - T_{n+1}(x)}{1-x^2}. \tag{2.53}$$

Higher derivatives may be obtained by similar formulae (see Problem 17 for the second derivative).

2.5 Problems for Chapter 2

1. Determine the positions of the zeros of the Chebyshev polynomials of the second and third kinds for the general interval $[a, b]$ of x.

2. From numerical values of the cosine function (from table, calculator or computer), determine the zeros of $T_4(x)$, $U_4(x)$, $V_4(x)$, $W_4(x)$ and the extrema of $T_4(x)$.

3. Show that

   (a) $\frac{1}{2}U_{2k}(x) = \frac{1}{2}T_0(x) + T_2(x) + T_4(x) + \cdots + T_{2k}(x)$;

   (b) $\frac{1}{2}U_{2k+1}(x) = T_1(x) + T_3(x) + \cdots + T_{2k+1}(x)$;

   (c) $xU_{2k+1}(x) = T_0(x) + 2T_2(x) + \cdots + 2T_{2k-2}(x) + T_{2k}(x)$.

   [Hint: In (3a), multiply by $\sin\theta$ and use $2\sin A\cos B = \sin(A-B) + \sin(A+B)$. Use similar ideas in (3b), (3c).]

4. Obtain the expression

$$x^m T_n(x) = 2^{-m}\sum_{r=0}^{m}\binom{m}{r}T_{n-m-2r}(x) \quad (m < n)$$

   (a) by applying the formula (2.39) m times;

   (b) by applying the expression (2.14) for x^m in terms of Chebyshev polynomials and the expression (2.38) for products of Chebyshev polynomials.

5. Prove by induction on n that

$$T_n(x) = \sum_{k=0}^{\lfloor n/2\rfloor} c_k^n x^{n-2k},$$

   where

$$c_k^n = (-1)^k 2^{n-2k-1}\left[2\binom{n-k}{k} - \binom{n-k-1}{k}\right] \quad (n, k > 0)$$

$$c_0^n = 2^{n-1} \quad (n > 0)$$

$$c_0^0 = 1.$$

   [Hint: Assume the formulae are true for $n = N-2$, $N-1$ and hence derive them for $n = N$, using $T_n = 2xT_{n-1} - T_{n-2}$.]

6. Derive formulae for $T_m^*(x)T_n^*(x)$ and $xT_n^*(x)$ in terms of $\{T_r^*(x)\}$, using the ideas of Section 2.4.3.

7. Suppose

$$S_n^{(0)} = \sum_{r=0}^{n}{}' a_{2r}T_{2r}(x), \quad S_n^{(1)} = \sum_{r=0}^{n} a_{2r+1}T_{2r+1}(x)$$

are sums of even-only/odd-only Chebyshev polynomials.

(a) Show that $S_n^{(0)}$ may be efficiently determined by applying the recurrence (2.26) followed by (2.28), with x replaced by $(2x^2-1)$ and a_r replaced by a_{2r};

(b) Show that $S_n^{(1)}$ may be efficiently determined by applying the recurrence (2.26), with x replaced by $(2x^2 - 1)$ and a_r replaced by a_{2r+1}, and then taking

$$S_n^{(1)} = x(b_0 - b_1).$$

[Hint: From (1.14) and (1.15), we have $T_{2r}(x) = T_r(2x^2 - 1)$ and $T_{2r+1}(x) = xV_r(2x^2 - 1)$.]

8. Derive the formulae (2.29a)–(2.30d) for the values of T_n, U_n, V_n, W_n at $x = -1,\ 0,\ 1$, using only the trigonometric definitions of the Chebyshev polynomials.

9. Use the algorithm (2.21b) to evaluate

$$\sum_{r=0}^{3}{}' c_rT_r(x),\ \sum_{r=0}^{3} c_rU_r(x),\ \sum_{r=0}^{3} c_rV_r(x),\ \sum_{r=0}^{3} c_rW_r(x)$$

at $x = -1,\ 0,\ 1$ for $c_0 = 1$, $c_1 = 0.5$, $c_2 = 0.25$, $c_3 = 0.125$. Check your results using correct values of T_r, U_r, V_r, W_r at 0, 1.

10. Illustrate the algorithms (7a), (7b) of Problem 7 by using them to evaluate at $x = -1$

$$\sum_{0}^{2}{}' c_rT_r(x),\ \sum_{0}^{2}{}' c_rT_{2r}(x),\ \sum_{0}^{2}{}' c_rT_{2r+1}(x),$$

where $c_0 = 1$, $c_1 = 0.1$, $c_2 = 0.001$. Check your results using correct values of T_r at $x = -1$.

11. Discuss the stability of the summation formulae for sums of Chebyshev polynomials U_r, V_r, W_r when the size of each sum is

(a) proportional to unity,

(b) proportional to the largest value in $[-1, 1]$ of U_n, V_n, W_n, respectively (where the sums are from $r = 0$ to $r = n$).

12. Show that

  (a) $2(1-x^2)U_{n-2}(x) = T_n(x) - T_{n-2}(x)$;
  (b) $(1+x)V_{n-1}(x) = T_n(x) + T_{n-1}(x)$;
  (c) $(1-x)W_{n-1}(x) = T_n(x) - T_{n-1}(x)$;
  (d) $(1+x)V_m(x)V_n(x) = T_{|m-n|}(x) + T_{m+n+1}(x)$;
  (e) $(1-x)W_m(x)W_n(x) = T_{|m-n|}(x) - T_{m+n+1}(x)$.

13. Show that $T_m(x)U_{n-1}(x) = \frac{1}{2}\{U_{n+m-1}(x) + U_{n-m-1}(x)\}$, and determine an expression for $x^m U_{n-1}(x)$ in terms of $\{U_k\}$ by a similar procedure to that of Problem 4.

14. Show that (ignoring constants of integration)

  (a) $\int (1-x^2)^{-\frac{1}{2}} T_n(x)\,\mathrm{d}x = n^{-1}(1-x^2)^{\frac{1}{2}} U_{n-1}(x)$;
  (b) $\int (1-x)^{-\frac{1}{2}} V_n(x)\,\mathrm{d}x = (n+\frac{1}{2})^{-1}(1-x)^{\frac{1}{2}} W_n(x)$;
  (c) $\int (1+x)^{-\frac{1}{2}} W_n(x)\,\mathrm{d}x = (n+\frac{1}{2})^{-1}(1+x)^{\frac{1}{2}} V_n(x)$.

15. Show that, for $n > 0$,

$$\frac{\mathrm{d}}{\mathrm{d}x}U_n(x) = \frac{(n+2)U_{n-1}(x) - nU_{n+1}(x)}{2(1-x^2)}.$$

16. Using (2.52), show that if

$$\sum_{r=0}^{n+1}{}' A_r T_r(x) = I_{n+1}(x)$$

and

$$\sum_{r=0}^{n-1}{}' a_r T_r(x) = \frac{\mathrm{d}^2}{\mathrm{d}x^2} I_{n+1}(x),$$

then

$$a_r = \sum_{\substack{k=r+2 \\ k-r \text{ even}}}^{n+1} (k-r)k(k+r)A_k.$$

Show further that

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}T_n(x) = \sum_{\substack{r=0 \\ n-r \text{ even}}}^{n-2}{}' (n-r)n(n+r)T_r(x).$$

17. Using (2.53) and (1.3a), prove that, for $n > 1$,

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2} T_n(x) = \frac{n}{4} \frac{(n+1)T_{n-2}(x) - 2nT_n(x) + (n-1)T_{n+2}(x)}{(1-x^2)^2}.$$

18. Show that

$$\sum_{j=0}^{n}{}' T_j(x) T_j(y) = \tfrac{1}{4} \Big\{ W_n \Big(xy + \sqrt{(1-x^2)(1-y^2)} \Big) + {} + W_n \Big(xy - \sqrt{(1-x^2)(1-y^2)} \Big) \Big\}.$$

19. Show that

$$(1-x^2) \sum_{j=0}^{\infty}{}' c_j T_j(x) = \tfrac{1}{4} \sum_{j=0}^{\infty} (c_j - c_{j+2})(T_j(x) - T_{j+2}(x)).$$