Chapter 1

Definitions

1.1 Preliminary remarks

> "Chebyshev polynomials are everywhere dense in numerical analysis."

This remark has been attributed to a number of distinguished mathematicians and numerical analysts. It may be due to Philip Davis, was certainly spoken by George Forsythe, and it is an appealing and apt remark. There is scarcely any area of numerical analysis where Chebyshev polynomials do not drop in like surprise visitors, and indeed there are now a number of subjects in which these polynomials take a significant position in modern developments — including orthogonal polynomials, polynomial approximation, numerical integration, and spectral methods for partial differential equations.

However, there is a different slant that one can give to the quotation above, namely that by studying Chebyshev polynomials one is taken on a journey which leads into all areas of numerical analysis. This has certainly been our personal experience, and it means that the Chebyshev polynomials, far from being an esoteric and narrow subject, provide the student with an opportunity for a broad and unifying introduction to many areas of numerical analysis and mathematics.

1.2 Trigonometric definitions and recurrences

There are several kinds of Chebyshev polynomials. In particular we shall introduce the first and second kind polynomials $T_n(x)$ and $U_n(x)$, as well as a pair of related (Jacobi) polynomials $V_n(x)$ and $W_n(x)$, which we call the 'Chebyshev polynomials of the third and fourth kinds'; in addition we cover the shifted polynomials $T_n^*(x)$, $U_n^*(x)$, $V_n^*(x)$ and $W_n^*(x)$. We shall, however, only make a passing reference to 'Chebyshev's polynomial of a discrete variable', referred to for example in Erdélyi et al. (1953, Section 10.23), since this last polynomial has somewhat different properties from the polynomials on which our main discussion is based.

Some books and many articles use the expression 'Chebyshev polynomial' to refer exclusively to the Chebyshev polynomial $T_n(x)$ of the first kind. Indeed this is by far the most important of the Chebyshev polynomials and, when no other qualification is given, the reader should assume that we too are referring to this polynomial.

Clearly some definition of Chebyshev polynomials is needed right away and, as we shall see as the book progresses, we are spoiled for a choice of definitions. However, what gives the various polynomials their power and relevance is their close relationship with the trigonometric functions 'cosine' and 'sine'. We are all aware of the power of these functions and of their appearance in the description of all kinds of natural phenomena, and this must surely be the key to the versatility of the Chebyshev polynomials. We therefore use as our primary definitions these trigonometric relationships.

1.2.1 The first-kind polynomial T_n

Definition 1.1 *The Chebyshev polynomial $T_n(x)$ of the first kind is a polynomial in x of degree n, defined by the relation*

$$T_n(x) = \cos n\theta \quad \text{when } x = \cos\theta. \tag{1.1}$$

If the range of the variable x is the interval $[-1, 1]$, then the range of the corresponding variable θ can be taken as $[0, \pi]$. These ranges are traversed in opposite directions, since $x = -1$ corresponds to $\theta = \pi$ and $x = 1$ corresponds to $\theta = 0$.

It is well known (as a consequence of de Moivre's Theorem) that $\cos n\theta$ is a polynomial of degree n in $\cos\theta$, and indeed we are familiar with the elementary formulae

$$\begin{gathered}\cos 0\theta = 1, \quad \cos 1\theta = \cos\theta, \quad \cos 2\theta = 2\cos^2\theta - 1, \\ \cos 3\theta = 4\cos^3\theta - 3\cos\theta, \quad \cos 4\theta = 8\cos^4\theta - 8\cos^2\theta + 1, \quad \ldots .\end{gathered}$$

We may immediately deduce from (1.1), that the first few Chebyshev polynomials are

$$\begin{gathered}T_0(x) = 1, \quad T_1(x) = x, \quad T_2(x) = 2x^2 - 1, \\ T_3(x) = 4x^3 - 3x, \quad T_4(x) = 8x^4 - 8x^2 + 1, \quad \ldots .\end{gathered} \tag{1.2}$$

Coefficients of all polynomials $T_n(x)$ up to degree $n = 21$ will be found in Tables C.2a, C.2b in Appendix C.

In practice it is neither convenient nor efficient to work out each $T_n(x)$ from first principles. Rather by combining the trigonometric identity

$$\cos n\theta + \cos(n-2)\theta = 2\cos\theta\ \cos(n-1)\theta$$

with Definition 1.1, we obtain the fundamental recurrence relation

$$T_n(x) = 2xT_{n-1}(x) - T_{n-2}(x), \quad n = 2, 3, \ldots, \tag{1.3a}$$

which together with the initial conditions

$$T_0(x) = 1, \quad T_1(x) = x \tag{1.3b}$$

recursively generates all the polynomials $\{T_n(x)\}$ very efficiently.

It is easy to deduce from (1.3) that the leading coefficient (that of x^n) in $T_n(x)$ for $n > 1$ is double the leading coefficient in $T_{n-1}(x)$ and hence, by induction, is 2^{n-1}.

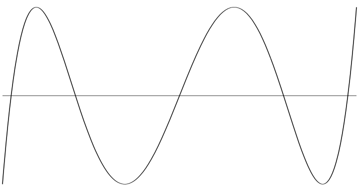

Figure 1.1: $T_5(x)$ on range $[-1, 1]$

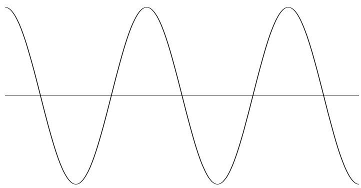

Figure 1.2: $\cos 5\theta$ on range $[0, \pi]$

What does the polynomial $T_n(x)$ look like, and how does a graph in the variable x compare with a graph of $\cos n\theta$ in the variable θ? In Figures 1.1 and 1.2 we show the respective graphs of $T_5(x)$ and $\cos 5\theta$. It will be noted that the shape of $T_5(x)$ on $[-1, 1]$ is very similar to that of $\cos 5\theta$ on $[0, \pi]$, and in particular both oscillate between six extrema of equal magnitudes (unity) and alternating signs. However, there are three key differences — firstly the polynomial $T_5(x)$ corresponds to $\cos 5\theta$ *reversed* (i.e., starting with a value of -1 and finishing with a value of $+1$); secondly the extrema of $T_5(x)$ at the end points $x = \pm 1$ do not correspond to zero gradients (as they do for $\cos 5\theta$) but rather to rapid changes in the polynomial as a function of x; and thirdly the zeros and extrema of $T_5(x)$ are clustered towards the end points ± 1, whereas the zeros and extrema of $\cos 5\theta$ are equally spaced.

The reader will recall that an *even* function $f(x)$ is one for which

$$f(x) = f(-x) \text{ for all } x$$

and an *odd* function $f(x)$ is one for which

$$f(x) = -f(-x) \text{ for all } x.$$

All even powers of x are even functions, and all odd powers of x are odd functions. Equations (1.2) suggest that $T_n(x)$ is an even or odd function, involving only even or odd powers of x, according as n is even or odd. This may be deduced rigorously from (1.3a) by induction, the cases $n = 0$ and $n = 1$ being supplied by the initial conditions (1.3b).

1.2.2 The second-kind polynomial U_n

Definition 1.2 *The Chebyshev polynomial $U_n(x)$ of the second kind is a polynomial of degree n in x defined by*

$$U_n(x) = \sin(n+1)\theta / \sin\theta \quad \text{when } x = \cos\theta. \tag{1.4}$$

The ranges of x and θ are the same as for $T_n(x)$.

Elementary formulae give

$$\sin 1\theta = \sin\theta, \quad \sin 2\theta = 2\sin\theta\,\cos\theta, \quad \sin 3\theta = \sin\theta\,(4\cos^2\theta - 1),$$
$$\sin 4\theta = \sin\theta\,(8\cos^3\theta - 4\cos\theta), \quad \ldots,$$

so that we see that the ratio of sine functions (1.4) is indeed a polynomial in $\cos\theta$, and we may immediately deduce that

$$\begin{gathered} U_0(x) = 1, \quad U_1(x) = 2x, \quad U_2(x) = 4x^2 - 1, \\ U_3(x) = 8x^3 - 4x, \quad \ldots. \end{gathered} \tag{1.5}$$

Coefficients of all polynomials $U_n(x)$ up to degree $n = 21$ will be found in Tables C.3a, C.3b in Appendix C.

By combining the trigonometric identity

$$\sin(n+1)\theta + \sin(n-1)\theta = 2\cos\theta\,\sin n\theta$$

with Definition 1.2, we find that $U_n(x)$ satisfies the recurrence relation

$$U_n(x) = 2xU_{n-1}(x) - U_{n-2}(x), \quad n = 2, 3, \ldots, \tag{1.6a}$$

which together with the initial conditions

$$U_0(x) = 1, \quad U_1(x) = 2x \tag{1.6b}$$

provides an efficient procedure for generating the polynomials.

A similar trigonometric identity

$$\sin(n+1)\theta - \sin(n-1)\theta = 2\sin\theta\,\cos n\theta$$

leads us to a relationship

$$U_n(x) - U_{n-2}(x) = 2T_n(x), \quad n = 2, 3, \ldots, \tag{1.7}$$

between the polynomials of the first and second kinds.

It is easy to deduce from (1.6) that the leading coefficient of x^n in $U_n(x)$ is 2^n.

Note that the recurrence (1.6a) for $\{U_n(x)\}$ is identical in form to the recurrence (1.3a) for $\{T_n(x)\}$. The different initial conditions [(1.6b) and (1.3b)] yield the different polynomial systems.

In Figure 1.3 we show the graph of $U_5(x)$. It oscillates between six extrema, as does $T_5(x)$ in Figure 1.1, but in the present case the extrema have magnitudes which are not equal, but increase monotonically from the centre towards the ends of the range.

From (1.5) it is clear that the second-kind polynomial $U_n(x)$, like the first, is an even or odd function, involving only even or odd powers of x, according as n is even or odd.

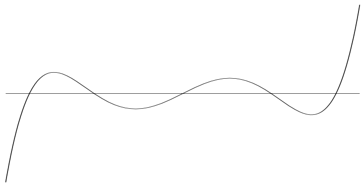

Figure 1.3: $U_5(x)$ on range $[-1, 1]$

1.2.3 The third- and fourth-kind polynomials V_n and W_n (the airfoil polynomials)

Two other families of polynomials V_n and W_n may be constructed, which are related to T_n and U_n, but which have trigonometric definitions involving the half angle $\theta/2$ (where $x = \cos\theta$ as before). These polynomials are sometimes[1] referred to as the 'airfoil polynomials', but Gautschi (1992) rather appropriately named them the 'third- and fourth-kind Chebyshev polynomials'. First we define these polynomials trigonometrically, by a pair of relations parallel to (1.1) and (1.4) above for T_n and U_n. Again the ranges of x and θ are the same as for $T_n(x)$.

Definition 1.3 *The Chebyshev polynomials $V_n(x)$ and $W_n(x)$ of the third and fourth kinds are polynomials of degree n in x defined respectively by*

$$V_n(x) = \cos(n + \tfrac{1}{2})\theta / \cos\tfrac{1}{2}\theta \tag{1.8}$$

and

$$W_n(x) = \sin(n + \tfrac{1}{2})\theta / \sin\tfrac{1}{2}\theta, \tag{1.9}$$

when $x = \cos\theta$.

To justify these definitions, we first observe that $\cos(n + \frac{1}{2})\theta$ is an odd polynomial of degree $2n+1$ in $\cos\frac{1}{2}\theta$. Therefore the right-hand side of (1.8) is an even polynomial of degree $2n$ in $\cos\frac{1}{2}\theta$, which is equivalent to a polynomial of degree n in $\cos^2\frac{1}{2}\theta = \frac{1}{2}(1 + \cos\theta)$ and hence to a polynomial of degree n in $\cos\theta$. Thus $V_n(x)$ is indeed a polynomial of degree n in x. For example

$$V_1(x) = \frac{\cos(1 + \frac{1}{2})\theta}{\cos\frac{1}{2}\theta} = \frac{4\cos^3\frac{1}{2}\theta - 3\cos\frac{1}{2}\theta}{\cos\frac{1}{2}\theta} = 4\cos^2\tfrac{1}{2}\theta - 3 = 2\cos\theta - 1 = 2x - 1.$$

We may readily show that

$$\begin{gathered} V_0(x) = 1, \quad V_1(x) = 2x - 1, \quad V_2(x) = 4x^2 - 2x - 1, \\ V_3(x) = 8x^3 - 4x^2 - 4x + 1, \quad \ldots. \end{gathered} \tag{1.10}$$

[1] See for example, Fromme & Golberg (1981).

Similarly $\sin(n + \frac{1}{2})\theta$ is an odd polynomial of degree $2n + 1$ in $\sin\frac{1}{2}\theta$. Therefore the right-hand side of (1.9) is an even polynomial of degree $2n$ in $\sin\frac{1}{2}\theta$, which is equivalent to a polynomial of degree n in $\sin^2\frac{1}{2}\theta = \frac{1}{2}(1-\cos\theta)$ and hence again to a polynomial of degree n in $\cos\theta$. For example

$$W_1(x) = \frac{\sin(1 + \frac{1}{2})\theta}{\sin\frac{1}{2}\theta} = \frac{3\sin\frac{1}{2}\theta - 4\sin^3\frac{1}{2}\theta}{\sin\frac{1}{2}\theta} = 3-4\sin^2\tfrac{1}{2}\theta = 2\cos\theta+1 = 2x+1.$$

We may readily show that

$$\begin{gathered} W_0(x) = 1, \quad W_1(x) = 2x + 1, \quad W_2(x) = 4x^2 + 2x - 1, \\ W_3(x) = 8x^3 + 4x^2 - 4x - 1, \quad \ldots . \end{gathered} \tag{1.11}$$

The polynomials $V_n(x)$ and $W_n(x)$ are, in fact, rescalings of two particular Jacobi[2] polynomials $P_n^{(\alpha,\beta)}(x)$ with $\alpha = -\frac{1}{2}$, $\beta = \frac{1}{2}$ and vice versa. Explicitly

$$\binom{2n}{n} V_n(x) = 2^{2n} P_n^{(-\frac{1}{2},\frac{1}{2})}(x), \qquad \binom{2n}{n} W_n(x) = 2^{2n} P_n^{(\frac{1}{2},-\frac{1}{2})}(x).$$

Coefficients of all polynomials $V_n(x)$ and $W_n(x)$ up to degree $n = 10$ will be found in Table C.1 in Appendix C.

These polynomials too may be efficiently generated by the use of a recurrence relation. Since

$$\cos(n + \tfrac{1}{2})\theta + \cos(n - 2 + \tfrac{1}{2})\theta = 2\cos\theta\ \cos(n - 1 + \tfrac{1}{2})\theta$$

and

$$\sin(n + \tfrac{1}{2})\theta + \sin(n - 2 + \tfrac{1}{2})\theta = 2\cos\theta\ \sin(n - 1 + \tfrac{1}{2})\theta,$$

it immediately follows that

$$V_n(x) = 2xV_{n-1}(x) - V_{n-2}(x), \quad n = 2, 3, \ldots, \tag{1.12a}$$

and

$$W_n(x) = 2xW_{n-1}(x) - W_{n-2}(x), \quad n = 2, 3, \ldots, \tag{1.12b}$$

with

$$V_0(x) = 1, \quad V_1(x) = 2x - 1 \tag{1.12c}$$

and

$$W_0(x) = 1, \quad W_1(x) = 2x + 1. \tag{1.12d}$$

Thus $V_n(x)$ and $W_n(x)$ share precisely the same recurrence relation as $T_n(x)$ and $U_n(x)$, and their generation differs only in the prescription of the initial condition for $n = 1$.

[2] See Chapter 22 of Abramowitz and Stegun's *Handbook of Mathematical Functions* (1964).

It is immediately clear from (1.12) that both $V_n(x)$ and $W_n(x)$ are polynomials of degree n in x, in which all powers of x are present, and in which the leading coefficients (of x^n) are equal to 2^n.

In Figure1.4 we show graphs of $V_5(x)$ and $W_5(x)$. They are exact inverted mirror images of one another, as will be proved in the next section (1.19).

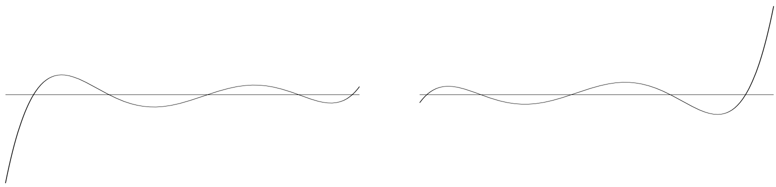

Figure 1.4: $V_5(x)$ and $W_5(x)$ on range $[-1, 1]$

1.2.4 Connections between the four kinds of polynomial

We already have a relationship (1.7) between the polynomials T_n and U_n. It remains to link V_n and W_n to T_n and U_n. This may be done by introducing two auxiliary variables

$$u = [\tfrac{1}{2}(1+x)]^{\frac{1}{2}} = \cos\tfrac{1}{2}\theta, \quad t = [\tfrac{1}{2}(1-x)]^{\frac{1}{2}} = \sin\tfrac{1}{2}\theta. \tag{1.13}$$

Using (1.8) and (1.9) it immediately follows, from the definitions (1.1) and (1.4) of T_n and U_n, that

$$T_n(x) = T_{2n}(u), \quad U_n(x) = \tfrac{1}{2}u^{-1}U_{2n+1}(u), \tag{1.14}$$

$$V_n(x) = u^{-1}T_{2n+1}(u), \quad W_n(x) = U_{2n}(u). \tag{1.15}$$

Thus $T_n(x)$, $U_n(x)$, $V_n(x)$, $W_n(x)$ together form the first- and second-kind polynomials in u, weighted by u^{-1} in the case of odd degrees. Also (1.15) shows that $V_n(x)$ and $W_n(x)$ are directly related, respectively, to the first- and second-kind Chebyshev polynomials, so that the terminology of 'Chebyshev polynomials of the third and fourth kind' is justifiable.

From the discussion above it can be seen that, if we wish to establish properties of V_n and W_n, then we have two main options: we can start from the trigonometric definitions (1.8), (1.9) or we can attempt to exploit properties of T_n and U_n by using the links (1.14)–(1.15).

Note that V_n and W_n are neither even nor odd (unlike T_n and U_n). We have seen that the leading coefficient of x^n is 2^n in both V_n and W_n, as it is in U_n. This suggests a close link with U_n. Indeed if we average the initial conditions (1.12c) and (1.12d) for V_1 and W_1 we obtain the initial condition

(1.6b) for U_1, from which we can show that the average of V_n and W_n satisfies the recurrence (1.6a) subject to (1.6b) and therefore that for all n

$$U_n(x) = \tfrac{1}{2}[V_n(x) + W_n(x)]. \tag{1.16}$$

The last result also follows directly from the trigonometric definitions (1.4), (1.8), (1.9) of U_n, V_n, W_n, since

$$\begin{aligned}\frac{\sin(n+1)\theta}{\sin\theta} &= \frac{\sin(n+\frac{1}{2})\theta\ \cos\frac{1}{2}\theta + \cos(n+\frac{1}{2})\theta\ \sin\frac{1}{2}\theta}{2\sin\frac{1}{2}\theta\ \cos\frac{1}{2}\theta} \\ &= \tfrac{1}{2}\left[\frac{\cos(n+\frac{1}{2})\theta}{\cos\frac{1}{2}\theta} + \frac{\sin(n+\frac{1}{2})\theta}{\sin\frac{1}{2}\theta}\right].\end{aligned}$$

Equation (1.16) is not the only link between the sets $\{V_n\}$, $\{W_n\}$ and $\{U_n\}$. Indeed, by using the trigonometric relations

$$\begin{aligned}2\sin\tfrac{1}{2}\theta\ \cos(n+\tfrac{1}{2})\theta &= \sin(n+1)\theta - \sin n\theta, \\ 2\cos\tfrac{1}{2}\theta\ \sin(n+\tfrac{1}{2})\theta &= \sin(n+1)\theta + \sin n\theta\end{aligned}$$

and dividing through by $\sin\theta$, we can deduce that

$$V_n(x) = U_n(x) - U_{n-1}(x), \tag{1.17}$$

$$W_n(x) = U_n(x) + U_{n-1}(x). \tag{1.18}$$

Thus V_n and W_n may be very simply determined once $\{U_n\}$ are available. Note that (1.17), (1.18) are confirmed in the formulae (1.5), (1.10), (1.11) and are consistent with (1.16) above.

From the evenness/oddness of $U_n(x)$ for n even/odd, we may immediately deduce from (1.17), (1.18) that

$$\begin{array}{ll} W_n(x) = V_n(-x) & (n \text{ even}); \\ W_n(x) = -V_n(-x) & (n \text{ odd}). \end{array} \tag{1.19}$$

This means that the third- and fourth-kind polynomials essentially transform into each other if the range $[-1, 1]$ of x is reversed, and it is therefore sufficient for us to study only one of these kinds of polynomial.

Two further relationships that may be derived from the definitions are

$$V_n(x) + V_{n-1}(x) = W_n(x) - W_{n-1}(x) = 2T_n(x). \tag{1.20}$$

If we were asked for a 'pecking order' of these four Chebyshev polynomials T_n, U_n, V_n and W_n, then we would say that T_n is clearly the most important and versatile. Moreover T_n generally leads to the simplest formulae, whereas results for the other polynomials may involve slight complications. However, all four polynomials have their role. For example, as we shall see, U_n is useful in numerical integration, while V_n and W_n can be useful in situations in which singularities occur at one end point ($+1$ or -1) but not at the other.

1.3 Shifted Chebyshev polynomials

1.3.1 The shifted polynomials T_n^*, U_n^*, V_n^*, W_n^*

Since the range $[0,1]$ is quite often more convenient to use than the range $[-1,1]$, we sometimes map the independent variable x in $[0,1]$ to the variable s in $[-1,1]$ by the transformation

$$s = 2x - 1 \text{ or } x = \tfrac{1}{2}(1+s),$$

and this leads to a shifted Chebyshev polynomial (of the first kind) $T_n^*(x)$ of degree n in x on $[0,1]$ given by

$$T_n^*(x) = T_n(s) = T_n(2x-1). \tag{1.21}$$

Thus we have the polynomials

$$\begin{gathered} T_0^*(x) = 1, \quad T_1^*(x) = 2x - 1, \quad T_2^*(x) = 8x^2 - 8x + 1, \\ T_3^*(x) = 32x^3 - 48x^2 + 18x - 1, \quad \ldots. \end{gathered} \tag{1.22}$$

From (1.21) and (1.3a), we may deduce the recurrence relation for T_n^* in the form

$$T_n^*(x) = 2(2x-1)T_{n-1}^*(x) - T_{n-2}^*(x) \tag{1.23a}$$

with initial conditions

$$T_0^*(x) = 1, \quad T_1^*(x) = 2x - 1. \tag{1.23b}$$

The polynomials $T_n^*(x)$ have a further special property, which derives from (1.1) and (1.21):

$$T_{2n}(x) = \cos 2n\theta = \cos n(2\theta) = T_n(\cos 2\theta) = T_n(2x^2 - 1) = T_n^*(x^2)$$

so that

$$T_{2n}(x) = T_n^*(x^2). \tag{1.24}$$

This property may readily be confirmed for the first few polynomials by comparing the formulae (1.2) and (1.22). Thus $T_n^*(x)$ is precisely $T_{2n}(\sqrt{x})$, a higher degree Chebyshev polynomial in the square root of the argument, and relation (1.24) gives an important link between $\{T_n\}$ and $\{T_n^*\}$ which complements the shift relationship (1.21). Because of this property, Table C.2a in Appendix C, which gives coefficients of the polynomials $T_n(x)$ up to degree $n = 20$ for even n, at the same time gives coefficients of the shifted polynomials $T_n^*(x)$ up to degree $n = 10$.

It is of course possible to define T_n^*, like T_n and U_n, directly by a trigonometric relation. Indeed, if we combine (1.1) and (1.24) we obtain

$$T_n^*(x) = \cos 2n\theta \text{ when } x = \cos^2\theta. \tag{1.25}$$

This relation might alternatively be rewritten, with θ replaced by $\phi/2$, in the form

$$T_n^*(x) = \cos n\phi \text{ when } x = \cos^2 \phi/2 = \tfrac{1}{2}(1 + \cos\phi). \tag{1.26}$$

Indeed the latter formula could be obtained directly from (1.21), by writing

$$T_n(s) = \cos n\phi \text{ when } s = \cos\phi.$$

Note that the shifted Chebyshev polynomial $T_n^*(x)$ is neither even nor odd, and indeed all powers of x from $1 = x^0$ to x^n appear in $T_n^*(x)$. The leading coefficient of x^n in $T_n^*(x)$ for $n > 0$ may be deduced from (1.23a), (1.23b) to be 2^{2n-1}.

Shifted polynomials U_n^*, V_n^*, W_n^* of the second, third and fourth kinds may be defined in precisely analogous ways:

$$U_n^*(x) = U_n(2x-1), \quad V_n^*(x) = V_n(2x-1), \quad W_n^*(x) = W_n(2x-1). \tag{1.27}$$

Links between U_n^*, V_n^*, W_n^* and the unstarred polynomials, analogous to (1.24) above, may readily be established. For example, using (1.4) and (1.27),

$$\begin{aligned} \sin\theta\, U_{2n-1}(x) &= \sin 2n\theta = \sin n(2\theta) = \sin 2\theta\, U_{n-1}(\cos 2\theta) \\ &= 2\sin\theta\cos\theta\, U_{n-1}(2x^2-1) = \sin\theta\,\{2xU_{n-1}^*(x^2)\} \end{aligned}$$

and hence

$$U_{2n-1}(x) = 2xU_{n-1}^*(x^2). \tag{1.28}$$

The corresponding relations for V_n^* and W_n^* are slightly different in that they complement (1.24) and (1.28) by involving T_{2n-1} and U_{2n}. Firstly, using (1.13), (1.15) and (1.27),

$$V_{n-1}^*(u^2) = V_{n-1}(2u^2-1) = V_{n-1}(x) = u^{-1}T_{2n-1}(u)$$

and hence (replacing u by x)

$$T_{2n-1}(x) = xV_{n-1}^*(x^2). \tag{1.29}$$

Similarly,

$$W_{n-1}^*(u^2) = W_{n-1}(2u^2-1) = W_{n-1}(x) = U_{2n}(u)$$

and hence (replacing u by x)

$$U_{2n}(x) = W_n^*(x^2). \tag{1.30}$$

Because of the relationships (1.28)–(1.30), Tables C.3b, C.2b, C.3a in Appendix C, which give coefficients of $T_n(x)$ and $U_n(x)$ up to degree $n = 20$, at the same time give the coefficients of the shifted polynomials $U_n^*(x)$, $V_n^*(x)$, $W_n^*(x)$, respectively, up to degree $n = 10$.

1.3.2 Chebyshev polynomials for the general range $[a, b]$

In the last section, the range $[-1, 1]$ was adjusted to the range $[0, 1]$ for convenience, and this corresponded to the use of the shifted Chebyshev polynomials T_n^*, U_n^*, V_n^*, W_n^* in place of T_n, U_n, V_n, W_n respectively. More generally we may define Chebyshev polynomials appropriate to any given finite range $[a, b]$ of x, by making this range correspond to the range $[-1, 1]$ of a new variable s under the linear transformation

$$s = \frac{2x - (a + b)}{b - a}. \tag{1.31}$$

The Chebyshev polynomials of the first kind appropriate to $[a, b]$ are thus $T_n(s)$, where s is given by (1.31), and similarly the second-, third- and fourth-kind polynomials appropriate to $[a, b]$ are $U_n(s)$, $V_n(s)$, and $W_n(s)$.

EXAMPLE 1.1: The first-kind Chebyshev polynomial of degree three appropriate to the range $[1, 4]$ of x is

$$T_3\left(\frac{2x-5}{3}\right) = 4\left(\frac{2x-5}{3}\right)^3 - 3\left(\frac{2x-5}{3}\right) = \frac{1}{27}(32x^3 - 240x^2 + 546x - 365).$$

Note that in the special case $[a, b] \equiv [0, 1]$, the transformation (1.31) becomes $s = 2x - 1$, and we obtain the shifted Chebyshev polynomials discussed in Section 1.3.1.

Incidentally, the 'Chebyshev Polynomials $S_n(x)$ and $C_n(x)$' tabulated by the National Bureau of Standards (NBS 1952) are no more than mappings of U_n and $2T_n$ to the range $[a, b] \equiv [-2, 2]$. Except for C_0, these polynomials all have unit leading coefficient, but this appears to be their only recommending feature for practical purposes, and they have never caught on.

1.4 Chebyshev polynomials of a complex variable

We have chosen to define the polynomials $T_n(x)$, $U_n(x)$, $V_n(x)$ and $W_n(x)$ with reference to the interval $[-1, 1]$. However, their expressions as sums of powers of x can of course be evaluated for any real x, even though the substitution $x = \cos\theta$ is not possible outside this interval.

For x in the range $[1, \infty)$, we can make the alternative substitution

$$x = \cosh\Theta, \tag{1.32}$$

with Θ in the range $[0, \infty)$, and it is easily verified that precisely the same polynomials (1.2), (1.5), (1.10) and (1.11) are generated by the relations

$$T_n(x) = \cosh n\Theta, \tag{1.33a}$$

$$U_n(x) = \frac{\sinh(n+1)\Theta}{\sinh\Theta}, \tag{1.33b}$$

$$V_n(x) = \frac{\cosh(n+\frac{1}{2})\Theta}{\cosh\frac{1}{2}\Theta}, \tag{1.33c}$$

$$W_n(x) = \frac{\sinh(n+\frac{1}{2})\Theta}{\sinh\frac{1}{2}\Theta}. \tag{1.33d}$$

For x in the range $(-\infty, -1]$ we can make use of the odd or even parity of the Chebyshev polynomials to deduce from (1.33) that, for instance,

$$T_n(x) = (-1)^n \cosh n\Theta$$

where

$$x = -\cosh\Theta.$$

It is easily shown from (1.33) that none of the four kinds of Chebyshev polynomials can have any zeros or turning points in the range $[1, \infty)$. The same applies to the range $(-\infty, -1]$. This will later become obvious, since we shall show in Section 2.2 that T_n, U_n, V_n and W_n each have n real zeros in the interval $[-1, 1]$, and a polynomial of degree n can have at most n zeros in all.

The Chebyshev polynomial $T_n(x)$ can be further extended into (or initially defined as) a polynomial $T_n(z)$ of a complex variable z. Indeed Snyder (1966) and Trefethen (2000) both start from a complex variable in developing their expositions.

1.4.1 Conformal mapping of a circle to and from an ellipse

For convenience, we consider not only the variable z but a related complex variable w such that

$$z = \tfrac{1}{2}(w + w^{-1}). \tag{1.34}$$

Then, if w moves on the circle $|w| = r$ (for $r > 1$) centred at the origin, we have

$$w = r\mathrm{e}^{\mathrm{i}\theta} = r\cos\theta + \mathrm{i}r\sin\theta, \tag{1.35}$$

$$w^{-1} = r^{-1}\mathrm{e}^{-\mathrm{i}\theta} = r^{-1}\cos\theta - \mathrm{i}r^{-1}\sin\theta, \tag{1.36}$$

and so, from (1.34),

$$z = a\cos\theta + \mathrm{i}b\sin\theta \tag{1.37}$$

where

$$a = \tfrac{1}{2}(r + r^{-1}), \quad b = \tfrac{1}{2}(r - r^{-1}). \tag{1.38}$$

Hence z moves on the standard ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \tag{1.39}$$

centred at the origin, with major and minor semi-axes a, b given by (1.38). It is easy to verify from (1.38) that the eccentricity e of this ellipse is such that

$$ae = \sqrt{a^2 - b^2} = 1,$$

and hence the ellipse has foci at $z = \pm 1$.

In the case $r = 1$, where w moves on the unit circle, we have $b = 0$ and the ellipse collapses into the real interval $[-1, 1]$. However, z traverses the interval twice as w moves round the circle: from -1 to 1 as θ moves from $-\pi$ to 0, and from 1 to -1 as θ moves from 0 to π.

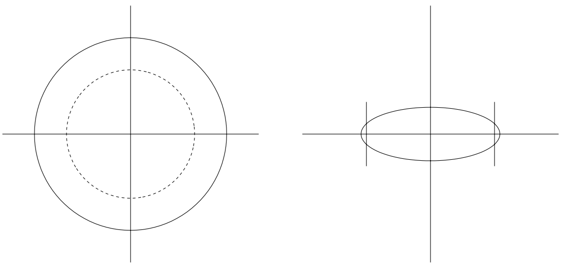

Figure 1.5: The circle $|w| = r = 1.5$ and its image in the z plane

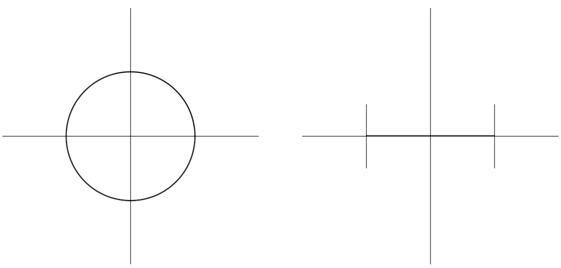

Figure 1.6: The circle $|w| = 1$ and its image in the z plane

The standard circle (1.35) and ellipse (1.39) are shown in Figure 1.5, and the special case $r = 1$ is shown in Figure 1.6. See Henrici (1974–1986) for further discussions of this mapping.

From (1.34) we readily deduce that w satisfies

$$w^2 - 2wz + 1 = 0, \tag{1.40}$$

a quadratic equation with two solutions

$$w = w_1, w_2 = z \pm \sqrt{z^2 - 1}. \tag{1.41}$$

This means that the mapping from w to z is 2 to 1, with branch points at $z = \pm 1$. It is convenient to define the complex square root $\sqrt{z^2 - 1}$ so that it lies in the same quadrant as z (except for z on the real interval $[-1, 1]$, along which the plane must be cut), and to choose the solution

$$w = w_1 = z + \sqrt{z^2 - 1}, \tag{1.42}$$

so that $|w| = |w_1| \geq 1$. Then w depends continuously on z along any path that does not intersect the interval $[-1, 1]$, and it is easy to verify that

$$w_2 = w_1^{-1} = z - \sqrt{z^2 - 1}, \tag{1.43}$$

with $|w_2| \leq 1$.

If w_1 moves on $|w_1| = r$, for $r > 1$, then w_2 moves on $|w_2| = \left|w_1^{-1}\right| = r^{-1} < 1$. Hence both of the concentric circles

$$C_r := \{w : |w| = r\}, \quad C_{1/r} := \left\{w : |w| = r^{-1}\right\}$$

transform into the same ellipse defined by (1.37) or (1.39), namely

$$E_r := \left\{z : \left|z + \sqrt{z^2 - 1}\right| = r\right\}. \tag{1.44}$$

1.4.2 Chebyshev polynomials in z

Defining z by (1.34), we note that if w lies on the unit circle $|w| = 1$ (i.e. C_1), then (1.37) gives

$$z = \cos\theta \tag{1.45}$$

and hence, from (1.42).

$$w = z + \sqrt{z^2 - 1} = \mathrm{e}^{\mathrm{i}\theta}. \tag{1.46}$$

Thus $T_n(z)$ is now a Chebyshev polynomial in a real variable and so by our standard definition (1.1), and (1.45), (1.46),

$$T_n(z) = \cos n\theta = \tfrac{1}{2}(\mathrm{e}^{\mathrm{i}n\theta} + \mathrm{e}^{-\mathrm{i}n\theta}) = \tfrac{1}{2}(w^n + w^{-n}).$$

This leads us immediately to our general definition, for all complex z, namely

$$T_n(z) = \tfrac{1}{2}(w^n + w^{-n}) \tag{1.47}$$

where

$$z = \tfrac{1}{2}(w + w^{-1}). \tag{1.48}$$

Alternatively we may write $T_n(z)$ directly in terms of z, using (1.42) and (1.43), as

$$T_n(z) = \tfrac{1}{2}\{(z + \sqrt{z^2 - 1})^n + (z - \sqrt{z^2 - 1})^n\}. \tag{1.49}$$

If z lies on the ellipse E_r, the locus of (1.48) when $|w| = r > 1$, then it follows from (1.47) that we have the inequality

$$\tfrac{1}{2}(r^n - r^{-n}) \le |T_n(z)| \le \tfrac{1}{2}(r^n + r^{-n}). \tag{1.50}$$

In Fig. 1.7 we show the level curves of the absolute value of $T_5(z)$, and it can easily be seen how these approach an elliptical shape as the value increases.

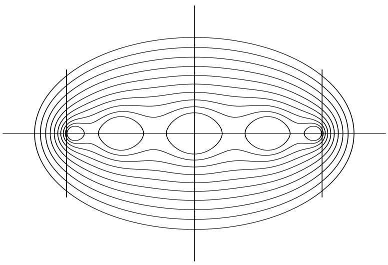

Figure 1.7: Contours of $|T_5(z)|$ in the complex plane

We may similarly extend polynomials of the second kind. If $|w| = 1$, so that $z = \cos\theta$, we have from (1.4),

$$U_{n-1}(z) = \frac{\sin n\theta}{\sin\theta}.$$

Hence, from (1.45) and (1.46), we deduce the general definition

$$U_{n-1}(z) = \frac{w^n - w^{-n}}{w - w^{-1}} \tag{1.51}$$

where again $z = \frac{1}{2}(w + w^{-1})$. Alternatively, writing directly in terms of z,

$$U_{n-1}(z) = \tfrac{1}{2}\frac{(z + \sqrt{z^2 - 1})^n - (z - \sqrt{z^2 - 1})^n}{\sqrt{z^2 - 1}}. \tag{1.52}$$

If z lies on the ellipse (1.44), then it follows directly from (1.51) that

$$\frac{r^n - r^{-n}}{r + r^{-1}} \le |U_{n-1}(z)| \le \frac{r^n + r^{-n}}{r - r^{-1}}; \tag{1.53}$$

however, whereas the bounds (1.50) on $|T_n(z)|$ are attained on the ellipse, the bounds (1.53) on $|U_{n-1}(z)|$ are slightly pessimistic. For a sharp upper bound, we may expand (1.51) into

$$U_{n-1}(z) = w^{n-1} + w^{n-3} + \cdots + w^{3-n} + w^{1-n} \tag{1.54}$$

giving us

$$\begin{aligned} |U_{n-1}(z)| &\le \left|w^{n-1}\right| + \left|w^{n-3}\right| + \cdots + \left|w^{3-n}\right| + \left|w^{1-n}\right| \\ &= r^{n-1} + r^{n-3} + \cdots + r^{3-n} + r^{1-n} \\ &= \frac{r^n - r^{-n}}{r - r^{-1}}, \end{aligned} \tag{1.55}$$

which lies between the two bounds given in (1.53). In Fig. 1.8 we show the level curves of the absolute value of $U_5(z)$.

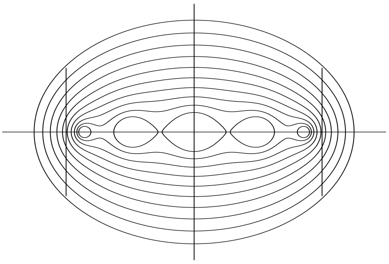

Figure 1.8: Contours of $|U_5(z)|$ in the complex plane

The third- and fourth-kind polynomials of degree n in z may readily be defined in similar fashion (compare (1.51)) by

$$V_n(z) = \frac{w^{n+\frac{1}{2}} + w^{-n-\frac{1}{2}}}{w^{\frac{1}{2}} + w^{-\frac{1}{2}}}, \tag{1.56}$$

$$W_n(z) = \frac{w^{n+\frac{1}{2}} - w^{-n-\frac{1}{2}}}{w^{\frac{1}{2}} - w^{-\frac{1}{2}}} \tag{1.57}$$

where $w^{\frac{1}{2}}$ is consistently defined from w. More precisely, to get round the ambiguities inherent in taking square roots, we may define them by

$$V_n(z) = \frac{w^{n+1} + w^{-n}}{w + 1}, \tag{1.58}$$

$$W_n(z) = \frac{w^{n+1} - w^{-n}}{w - 1} \tag{1.59}$$

It is easily shown, by dividing denominators into numerators, that these give polynomials of degree n in $z = \frac{1}{2}(w + w^{-1})$.

1.4.3 Shabat polynomials

Shabat & Voevodskii (1990) introduced the concept of 'generalised Chebyshev polynomials' (or *Shabat polynomials*), in the context of trees and number theory. The most recent survey paper in this area is that of Shabat & Zvonkin (1994). They are defined as polynomials $P(z)$ with complex coefficients having two critical values A and B such that

$$P'(z) = 0 \Longrightarrow P(z) = A \text{ or } P(z) = B.$$

The prime example of such a polynomial is $T_n(z)$, a first-kind Chebyshev polynomial, for which $A = -1$ and $B = +1$ are the critical values.

1.5 Problems for Chapter 1

1. The equation $x = \cos\theta$ defines infinitely many values of θ corresponding to a given value of x in the range $[-1, 1]$. Show that, whichever value is chosen, the values of $T_n(x)$, $U_n(x)$, $V_n(x)$ and $W_n(x)$ as defined by (1.1), (1.4), (1.8) and (1.9) remain the same.

2. Determine explicitly the Chebyshev polynomials of first and second kinds of degrees 0, 1, 2, 3, 4 appropriate to the range $[-4, 6]$ of x.

3. Prove that

$$T_m(T_n(x)) = T_{mn}(x)$$

   and that

$$U_{m-1}(T_n(x))U_{n-1}(x) = U_{n-1}(T_m(x))U_{m-1}(x) = U_{mn-1}(x).$$

4. Verify that equations (1.33) yield the same polynomials for $x > 1$ as the trigonometric definitions of the Chebyshev polynomials give for $|x| \le 1$.

5. Using the formula

$$z = \tfrac{1}{2}(r + r^{-1})\cos\theta + \tfrac{1}{2}\mathrm{i}(r - r^{-1})\sin\theta, \quad (r > 1)$$

   which defines a point on an ellipse centred at 0 with foci $z = \pm 1$,

   (a) verify that

$$\sqrt{z^2 - 1} = \tfrac{1}{2}(r - r^{-1})\cos\theta + \tfrac{1}{2}\mathrm{i}(r + r^{-1})\sin\theta$$

   and hence

   (b) verify that $\left|z + \sqrt{z^2 - 1}\right| = r$.

6. By expanding by the first row and using the standard three-term recurrence for $T_r(x)$, show that

$$T_n(x) = \begin{vmatrix} 2x & -1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ -1 & 2x & -1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & -1 & 2x & -1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & -1 & 2x & -1 \\ 0 & 0 & 0 & 0 & \cdots & 0 & -1 & x \end{vmatrix} (n \times n \text{ determinant}) .$$

Write down similar expressions for $U_n(x)$, $V_n(x)$ and $W_n(x)$.

7. Given that the four kinds of Chebyshev polynomial each satisfy the same recurrence relation
$$X_n = 2xX_{n-1} - X_{n-2},$$
with $X_0 = 1$ in each case and $X_1 = x,\ 2x,\ 2x+1,\ 2x-1$ for the four respective families, use these relations *only* to establish that
   (a) $V_i(x) + W_i(x) = 2U_i(x)$,
   (b) $V_i(x) - W_i(x) = 2U_{i-1}(x)$,
   (c) $U_i(x) - 2T_i(x) = U_{i-2}(x)$,
   (d) $U_i(x) - T_i(x) = xU_{i-1}(x)$.

8. Derive the same four formulae of Problem 7, this time using only the trigonometric definitions of the Chebyshev polynomials.

9. From the last two results in Problem 7, show that
   (a) $T_i(x) = xU_{i-1}(x) - U_{i-2}(x)$,
   (b) $U_i(x) = 2xU_{i-1}(x) - U_{i-2}(x)$.