Appendix A

Biographical Note

Pafnuty Lvovich Chebyshev was born in Okatovo in the Kaluga region of Russia on 16th May [4th May, Old Style] 1821.

He studied mathematics at Moscow University from 1837 to 1846, then moved to St Petersburg (as it then was and has now again become), where he became an assistant professor at Petersburg University in 1847 and a full professor in 1851, in which post he remained until 1882. It is he who was principally responsible for founding, directing and inspiring the 'Petersburg school' of mathematical research, noted for its emphasis on drawing its problems for study from practical necessities rather than from mere intellectual curiosity. He was elected a foreign associate of the Institut de France in 1874, and a fellow of the Royal Society of London in 1877.

He worked in many fields outside approximation theory, including number theory (the distribution of primes), integration of algebraic functions, geometric theory of hinge mechanisms (the subject which led to his special interest in minimax approximation of functions), the moment problem, quadrature formulae and probability theory (limit theorems).

The Chebyshev polynomials T_n which now bear his name (the symbol 'T' deriving from its continental transliterations as 'Tchebycheff', 'Tschebyscheff' &c.) were first introduced by him in a paper on hinge mechanisms (Chebyshev 1854) presented to the St Petersburg Academy in 1853. They were discussed in more mathematical depth in a second paper (Chebyshev 1859) presented in 1857; see also (Chebyshev 1874). Somewhat surprisingly, in the light of what seems today the obvious connection with Fourier theory, his discussion makes no use of the substitution $x = \cos\theta$.

He died in St Petersburg on 8th December [26th November, Old Style] 1894.

A much more extensive biography, from which these facts were extracted, is to be found in the *Dictionary of Scientific Biography* (Youschkevitch 1981). See also a recent article by Butzer & Jongmans (1999).

Appendix B

Summary of Notations, Definitions and Important Properties

B.1 Miscellaneous notations

$\sum'$	finite or infinite summation with first (T_0) term halved, $\displaystyle\sum_{r=0}^{\infty}{}' a_r T_r = \tfrac{1}{2}a_0T_0 + a_1T_1 + a_2T_2 + \cdots$
$\sum''$	finite summation with first and last terms halved, $\displaystyle\sum_{r=0}^{n}{}'' a_r T_r = \tfrac{1}{2}a_0T_0 + a_1T_1 + \cdots + a_{n-1}T_{n-1} + \tfrac{1}{2}a_nT_n$
$\sum^*$	finite summation with last term halved, $\displaystyle\sum_{r=1}^{n}{}^* a_r P_r = a_1P_1 + \cdots + a_{n-1}P_{n-1} + \tfrac{1}{2}a_nP_n$
$\oint$	integral round a closed contour
$⨍$	Cauchy principal value integral
$\lfloor\cdots\rfloor$	largest integer $\leq \cdots$
$\lVert\cdot\rVert$	a norm (see page 43)
$\langle\cdot\,,\,\cdot\rangle$	an inner product (see pages 72, 97)
$\mathcal{A}(D)$	the linear space of functions analytic on the (complex) domain D and continuous on its closure $\bar{D}$
B_nf	the minimax nth degree polynomial approximation to f on the interval $[-1,1]$
$\mathcal{C}[a,b]$	the linear space of functions continuous on the interval $[a,b]$
$\mathcal{C}^n[a,b]$	the linear space of functions continuous and having n continuous derivatives on the interval $[a,b]$
$\mathcal{C}^0_{2\pi}$	the linear space of continuous periodic functions with period 2π
$\mathcal{C}^0_{2\pi,\mathrm{e}}$	the subspace of $\mathcal{C}^0_{2\pi}$ consisting of even functions only
C_r	the circular contour $\{w : \lvert w\rvert = r\}$ in the complex plane
D_r	the elliptic domain $\{z : 1 \leq \lvert z + \sqrt{z^2-1}\rvert < r\}$
E_r	the elliptic contour $\{z : \lvert z + \sqrt{z^2-1}\rvert = r\}$

	= the image of C_r under $z = \frac{1}{2}(w + w^{-1})$
$J_n f$	the nth degree polynomial interpolating f at $n + 1$ given points
$\mathcal{L}_p[a, b]$	the linear space of functions on $[a, b]$ on which the norm $\|\cdot\|_p$ can be defined
Π_n	the linear space of polynomials of degree n
$S_n^F f$	the nth partial sum of the Fourier expansion of f
$S_n^{FC} f$	the nth partial sum of the Fourier cosine expansion of f
$S_n^{FS} f$	the nth partial sum of the Fourier sine expansion of f
$S_n^T f$	the nth partial sum of the first-kind Chebyshev expansion of f
λ_n	Lebesgue constant (see page 125)
$\omega(\delta)$	the modulus of continuity of a function (see page 119)
∂S	the boundary of the two-dimensional domain S

B.2 The four kinds of Chebyshev polynomial

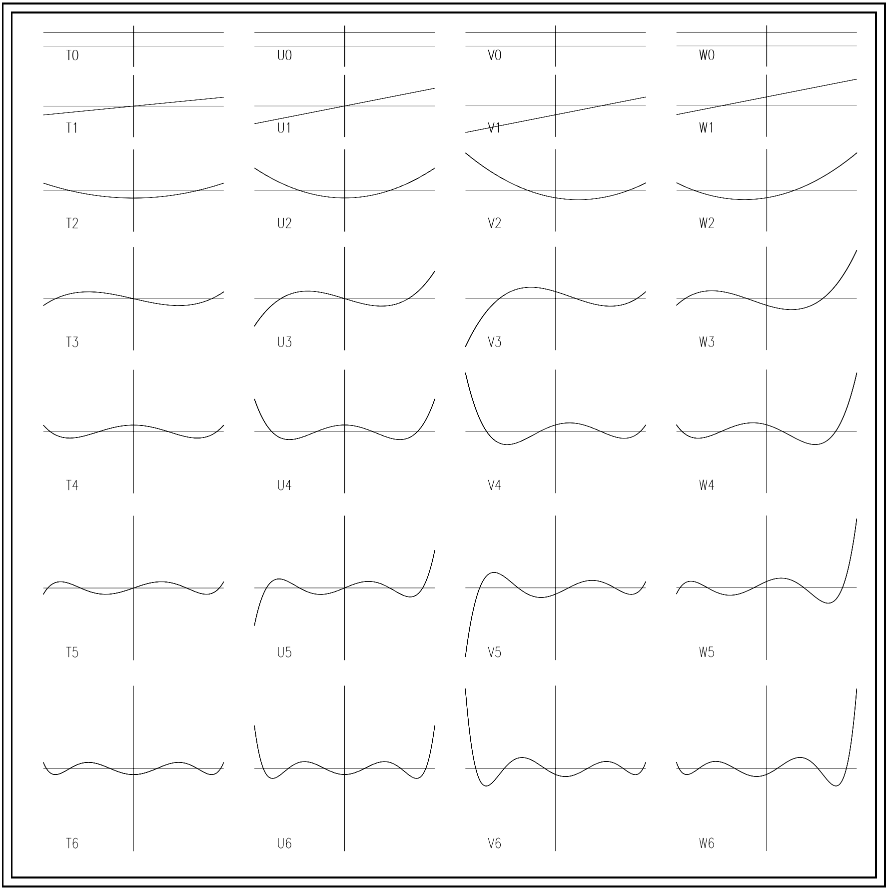

Figure B.1: Plots of the four kinds of Chebyshev polynomial: $T_n(x)$, $U_n(x)$, $V_n(x)$, $W_n(x)$ for values of x in the range $[-1, 1]$ and n running from 0 to 6

Table B.2: Key properties of the four kinds of Chebyshev polynomial

kind	1st	2nd	3rd	4th
$P_n =$	T_n	U_n	V_n	W_n
$P_n(\cos(\theta)) =$	$\cos n\theta$	$\dfrac{\sin(n+1)\theta}{\sin\theta}$	$\dfrac{\cos(n+\frac{1}{2})\theta}{\cos\frac{1}{2}\theta}$	$\dfrac{\sin(n+\frac{1}{2})\theta}{\sin\frac{1}{2}\theta}$
$P_n(\frac{1}{2}(w+w^{-1})) =$	$\frac{1}{2}(w^n+w^{-n})$	$\dfrac{w^{n+1}-w^{-n-1}}{w-w^{-1}}$	$\dfrac{w^{n+\frac{1}{2}}+w^{-n-\frac{1}{2}}}{w^{\frac{1}{2}}+w^{-\frac{1}{2}}}$	$\dfrac{w^{n+\frac{1}{2}}-w^{-n-\frac{1}{2}}}{w^{\frac{1}{2}}-w^{-\frac{1}{2}}}$
$P_0(x) =$	1			
$P_1(x) =$	x	$2x$	$2x-1$	$2x+1$
recurrence	$P_n(x) = 2xP_{n-1}(x) - P_{n-2}(x)$			
x^n coefficient	2^{n-1} $(n>0)$	2^n		
zeros	$x_{k,n} := \cos\dfrac{(k-\frac{1}{2})\pi}{n}$	$\cos\dfrac{k\pi}{n+1}$	$\cos\dfrac{(k-\frac{1}{2})\pi}{n+\frac{1}{2}}$	$\cos\dfrac{k\pi}{n+\frac{1}{2}}$
extrema	$y_{k,n} := \cos\dfrac{k\pi}{n}$	no closed form		
$\lVert P_n\rVert_\infty =$	1	$n+1$	$2n+1$	

Table B.3: Orthogonality properties of the four kinds of Chebyshev polynomial

kind	1st	2nd	3rd	4th
$P_n =$	T_n	U_n	V_n	W_n
weight $w(x) =$	$\dfrac{1}{\sqrt{1-x^2}}$	$\sqrt{1-x^2}$	$\sqrt{\dfrac{1+x}{1-x}}$	$\sqrt{\dfrac{1-x}{1+x}}$
orthogonality	$\begin{aligned} \langle P_m\,,\,P_n\rangle &= \int_{-1}^{1} w(x)P_m(x)P_n(x)\,\mathrm{d}x \\ &= 0 \ (m \neq n) \end{aligned}$			
$\langle P_n\,,\,P_n\rangle =$	$\frac{1}{2}\pi \ \ (n > 0)$	$\frac{1}{2}\pi$	π	
contour orthogonality	$\begin{aligned} \langle P_m\,,\,P_n\rangle &= \oint_{E_r} P_m(z)\overline{P_n(z)}\,\lvert w(z)\,\mathrm{d}z\rvert \\ &= 0 \ (m \neq n) \end{aligned}$ $[E_r = \text{ locus of } \frac{1}{2}(r\mathrm{e}^{\mathrm{i}\theta} + r^{-1}\mathrm{e}^{-\mathrm{i}\theta})]$			
$\langle P_n\,,\,P_n\rangle =$	$\frac{1}{2}\pi(r^{2n} + r^{-2n}) \ \ (n > 0)$	$\frac{1}{2}\pi(r^{2n+2} + r^{-2n-2})$	$\pi(r^{2n+1} + r^{-2n-1})$	

Table B.4: Discrete orthogonality of the four kinds of Chebyshev polynomial

kind	1st	2nd	3rd	4th
$P_n =$	T_n	U_n	V_n	W_n
weight $w(x) =$	$\dfrac{1}{\sqrt{1-x^2}}$	$\sqrt{1-x^2}$	$\sqrt{\dfrac{1+x}{1-x}}$	$\sqrt{\dfrac{1-x}{1+x}}$
abscissae	$x_{k,N+1} = \cos\{(k-\frac{1}{2})\pi/(N+1)\}$			
discrete orthogonality	$\langle P_m\,,\,P_n\rangle = \sum_{k=1}^{N+1} P_m(x_{k,N+1})P_n(x_{k,N+1})w(x_{k,N+1})\sqrt{1-x_{k,N+1}^2} = 0\ (m \neq n \leq N)$			
$\langle P_n\,,\,P_n\rangle =$	$\frac{1}{2}(N+1)\ \ (0 < n \leq N)$	$\frac{1}{2}(N+1)$	$(N+1)$	
abscissae	$y_{k,N} = \cos\{k\pi/N\}$			
discrete orthogonality	$\langle P_m\,,\,P_n\rangle = {\sum_{k=0}^{N}}'' P_m(y_{k,N})P_n(y_{k,N})w(y_{k,N})\sqrt{1-y_{k,N}^2} = 0\ (m \neq n \leq N)$			
$\langle P_n\,,\,P_n\rangle =$	$\frac{1}{2}N\ \ (0 < n < N)$	$\frac{1}{2}N$	N	

Appendix C

Tables of Coefficients

Each of the following five Tables may be used in two ways, to give the coefficients of two different kinds of shifted or unshifted polynomials.

Table C.1: Coefficients of x^k in $V_n(x)$ and of $(-1)^{n+k}x^k$ in $W_n(x)$

$n=$	0	1	2	3	4	5	6	7	8	9	10
$k=0$	1	-1	-1	1	1	-1	-1	1	1	-1	-1
1		2	-2	-4	4	6	-6	-8	8	10	-10
2			4	-4	-12	12	24	-24	-40	40	60
3				8	-8	-32	32	80	-80	-160	160
4					16	-16	-80	80	240	-240	-560
5						32	-32	-192	192	672	-672
6							64	-64	-448	448	1792
7								128	-128	-1024	1024
8									256	-256	-2304
9										512	-512
10											1024

Table C.2a: Coefficients of x^{2k} in $T_{2n}(x)$ and of x^k in $T_n^*(x)$

$n =$	0	1	2	3	4	5	6	7	8	9	10
$k = 0$	1	-1	1	-1	1	-1	1	-1	1	-1	1
1		2	-8	18	-32	50	-72	98	-128	162	-200
2			8	-48	160	-400	840	-1568	2688	-4320	6600
3				32	-256	1120	-3584	9408	-21504	44352	-84480
4					128	-1280	6912	-26880	84480	-228096	549120
5						512	-6144	39424	-180224	658944	-2050048
6							2048	-28672	212992	-1118208	4659200
7								8192	-131072	1105920	-6553600
8									32768	-589824	5570560
9										131072	-2621440
10											524288

Table C.2b: Coefficients of x^{2k+1} in $T_{2n+1}(x)$ and of x^k in $V_n^*(x)$

$n =$	0	1	2	3	4	5	6	7	8	9	10
$k = 0$	1	−3	5	−7	9	−11	13	−15	17	−19	21
1		4	−20	56	−120	220	−364	560	−816	1140	−1540
2			16	−112	432	−1232	2912	−6048	11424	−20064	33264
3				64	−576	2816	−9984	28800	−71808	160512	−329472
4					256	−2816	16640	−70400	239360	−695552	1793792
5						1024	−13312	92160	−452608	1770496	−5870592
6							4096	−61440	487424	−2723840	12042240
7								16384	−278528	2490368	−15597568
8									65536	−1245184	12386304
9										262144	−5505024
10											1048576

Table C.3a: Coefficients of x^{2k} in $U_{2n}(x)$ and of x^k in $W_n^*(x)$

$n =$	0	1	2	3	4	5	6	7	8	9	10
$k = 0$	1	−1	1	−1	1	−1	1	−1	1	−1	1
1		4	−12	24	−40	60	−84	112	−144	180	−220
2			16	−80	240	−560	1120	−2016	3360	−5280	7920
3				64	−448	1792	−5376	13440	−29568	59136	−109824
4					256	−2304	11520	−42240	126720	−329472	768768
5						1024	−11264	67584	−292864	1025024	−3075072
6							4096	−53248	372736	−1863680	7454720
7								16384	−245760	1966080	−11141120
8									65536	−1114112	10027008
9										262144	−4980736
10											1048576

Table C.3b: Coefficients of x^{2k+1} in $U_{2n+1}(x)$ and of x^k in $2U_n^*(x)$

$n =$	0	1	2	3	4	5	6	7	8	9	10
$k = 0$	2	−4	6	−8	10	−12	14	−16	18	−20	22
1		8	−32	80	−160	280	−448	672	−960	1320	−1760
2			32	−192	672	−1792	4032	−8064	14784	−25344	41184
3				128	−1024	4608	−15360	42240	−101376	219648	−439296
4					512	−5120	28160	−112640	366080	−1025024	2562560
5						2048	−24576	159744	−745472	2795520	−8945664
6							8192	−114688	860160	−4587520	19496960
7								32768	−524288	4456448	−26738688
8									131072	−2359296	22413312
9										524288	−10485760
10											2097152