W.I. Fushchych
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Fundamental constants of nucleon-meson dynamics

O. BEDRIJ, W.I. FUSHCHYCH

Запропоновано новий феноменологічний підхід для обчислення констант протону та нейтрону. В основу роботи покладено нестандартну ідею: стала Планка $\hbar$ та швидкість "світла" (мезону) c в нуклон-мезонній динамиці відмінні від цих же констант в квантовій електродинамиці.

In this paper, we are proposing an approach to calculate fundamental physical constants that characterize nucleon-meson dynamics. The approach is based on the referenced papers [1, 2], and on the premise that fundamental constants are reducible to mathematical relations and operations, which can be used to predict, define and calculate other fundamental "natural" unit systems (quanta).

At the present, we have, when compared to available data on quantum electrodynamics (electron-photon dynamics), very limited experimental fundamental constant data for the proton and the neutron. Such constants as the neutron or proton radius, or the Rydberg constant are not adequately defined in nucleon-meson dynamics.

From experiment, we know the mass and the charge of proton and neutron. Other physical characteristics such as nuclear magneton. Compton wavelength of the proton and the neutron are derived quantities, that incorporate $\hbar$ and c constants in the relations. It is presently assumed in physics that electrodynamic constants of $\hbar$ and c are applicable to characterization of nucleon-meson dynamics. Our calculations show that constants $\hbar$ and c for nucleon-meson dynamics are different from the same constants in quantum electrodynamics. This is natural, because the electron emits a photon, while the nucleon emits a meson.

We propose that standard formulas for fundamental characteristics of proton and neutron can be modified to represent the nucleon-meson constants and not electrodynamic constants. Below we show the proposed modifications (Definitions of Quantities are shown in [2]):

	Standard Relationships	**Proposed Relationships**
Compton Wavelength of proton	$\lambda_p = \hbar/m_p c$	$\lambda_p = \hbar_p m_p v_p$
Compton Wavelength of neutron	$\lambda_n = \hbar/m_n c$	$\lambda_n = \hbar_n/m_n v_n$
Proton magneton	$\mu_p = q\hbar/2m_p c$	$\mu_p = q_p \hbar_p/2m_p v_p$
Neutron magneton	$\mu_n = q\hbar/2m_n c$	$\mu_n = q_n \hbar_n/2m_n v_n$
Proton radius		$r_p = \hbar_p \alpha_{fp} m_p v_p$, $\hbar_p \neq \hbar$
Neutron radius		$r_n = \hbar_n \alpha_{fn}/m_n v_n$, $\hbar_n \neq \hbar$

where v_p and v_n are velocities of mesons which are emitted by proton and neutron. In our approach, we assume that:

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1. The physical relationships between quantities are the same for all inertial frames of reference.

2. The scale-symmetry is a fundamental concept in all of physics, including the photon, electron, meson, proton, neutron, etc.: that is, the scale-invariance of the physical relationships between quantities with respect to the scale group.

3. Physical quantities have a fundamental relationship to, an equilibrium frame of reference and that the equilibrium frame of reference is scale invariant [2].

When we consider that the laws of physics are invariant in all inertial frames of reference, and that the scale-symmetry is a fundamental aspect of physical relationships and constants, constant values that deal with quantum electrodynamics (constants that satisfy physical relationships for electron mass, photon, Compton wavelength, etc., ([2] — Table 1), are not applicable for the proton or neutron, which have different masses and hence, different scales of reference.

Earlier [2] we stated that:

$$1 = \left(q_1^{x_1} q_2^{x_2} q_3^{x_3} \cdots q_s^{x_s}\right) / \left(p_1^{j_1} p^{j_2} 2 p_3^{j_3} \cdots p_z^{j_z}\right) \tag{1}$$

or,

$$1 = Y'/KX, \tag{2}$$

where

$$Y' \equiv \left(q_1^{x_1} q_2^{x_2} q_3^{x_3} \cdots p_z^{j_z}\right), \tag{3}$$

$$1/KX \equiv \left(p^{j_1} p^{j_2} p_3^{j_3} \cdots p_z^{j_z}\right), \tag{4}$$

$$(q_s)^0 = 1, \quad (q_s^{-1})^0 = 1,$$

$q_{1'}, q_{2'}, q_{3'}, \ldots, q_{s'}, p_{1'}, p_{2'}, q_{3'}, \ldots, p_{z'}$ are quantities,

$x_{1'}, x_{2'}, x_{3'}, \ldots, x_{s'}, j_{1'}, j_{2'}, j_{3'}, \ldots, j_{z'}$ are real numbers,

K is the slope for line $Y' = KX$,

$j, s, x, z = 1, 2, 3, \ldots.$

We require that the equations (1) and (2) are scale invariant. That is the equations (1) and (2) are invariant with respect to the following transformations:

$$q_1 \to q_1' = aq_1, \quad q_2 \to q_2' = aq_2, \quad q^3 \to q_3' = aq_3, \quad \ldots, \tag{5}$$

$$p_1 \to p_1' = ap_1, \quad p_2 \to p_2' = ap_2, \quad p^3 \to p_3' = ap_3, \quad \ldots, \tag{6}$$

where "a" is a scale transformation parameter, and all physical quantities (q_s and p_z) have to be subjected to transformation. Hence, based on equations (1) and (2), it follows that "1" is always invariant with respect to scale transformations (5) and (6).

Thus, electron, proton, and neutron constants are on the lines:

$$1 = Y'/K_eX, \quad \text{where } K_e \text{ is the slope for electron line,} \tag{7}$$

$$1 = Y'/K_pX, \quad \text{where } K_p \text{ is the slope for proton line,} \tag{8}$$

$$1 = Y'/K_nX, \quad \text{where } K_n \text{ is the slope for neutron line,} \tag{9}$$

Table 1. Fundamental Constants of Proton Dynamics

Symbols	Constants	Relationships of Quantities
V_{op}	$1,075827 \cdot 10^{-36}$	$V_{op} = m/d$
h_p	$2,667688 \cdot 10^{-30}$	$h_p = W/f$
m_p	$1,672623 \cdot 10^{-27}$	$m_p = F/Y$
S_p	$1,440869 \cdot 10^{-22}$	$C_p = q/V$
L_p	$5,635247 \cdot 10^{-18}$	$L_p = \phi/i$
ϕ_p	$3,491143 \cdot 10^{-15}$	$\phi_p = F/H$
S_p	$1,024662 \cdot 10^{-12}$	$S_p = V/E$
W_p	$2,162829 \cdot 10^{-12}$	$W_p = Pt$
λ_p	$4,435318 \cdot 10^{-11}$	$\lambda_p = v/f$
α_{fp}	$1,155117 \cdot 10^{-2}$	$\alpha_{fp} = S/2\lambda$
1	$1,000000 \cdot 10^{0}$	$1 = GR$
$R_{\infty p}$	$1,504171 \cdot 10^{6}$	$R_{\infty p} = \alpha_{fp}^3/S$
D_p	$1,681364 \cdot 10^{7}$	$D_p = q/A$
V_p	$3,595937 \cdot 10^{7}$	$V_p = H/D$
B_p	$3,325110 \cdot 10^{9}$	$B_p = E/v$
H_p	$6,046079 \cdot 10^{14}$	$H_p = i/S$
E_p	$1,195689 \cdot 10^{17}$	$E_p = V/S$
f_p	$8,107560 \cdot 10^{17}$	$f_p = W/h$
ω_p	$3,509387 \cdot 10^{19}$	$\omega_p = (\alpha)^{1/2}$

Table 2. Fundamental Constants of Neutron Dynamics

Symbols	Constants	Relationships of Quantities
V_{on}	$1,077819 \cdot 10^{-36}$	$V_{on} = m/d$
h_n	$2,671749 \cdot 10^{-30}$	$h_n = W/f$
m_n	$1,674929 \cdot 10^{-27}$	$m_n = F/Y$
C_n	$1,442489 \cdot 10^{-22}$	$C_n = q/V$
L_n	$5,640249 \cdot 10^{-18}$	$L_n = \phi/i$
ϕ_n	$3,493739 \cdot 10^{-15}$	$\phi_n = F/H$
S_n	$1,025295 \cdot 10^{-12}$	$S_n = V/E$
W_n	$2,164127 \cdot 10^{-12}$	$W_n = Pt$
λ_n	$4,437681 \cdot 10^{-11}$	$\lambda_n = v/f$
α_{fn}	$1,155214 \cdot 10^{-2}$	$\alpha_{fn} = S/2$
1	$1,000000 \cdot 10^{0}$	$1 = GR$
$R_{\infty n}$	$1,503623 \cdot 10^{5}$	$R_{\infty n} = \alpha_{fp}^3/s$
D_n	$1,680739 \cdot 10^{7}$	$D_n = q/A$
V_n	$3,594539 \cdot 10^{7}$	$V_n = H/D$
B_n	$3,323482 \cdot 10^{9}$	$B_n = E/v$
H_n	$6,041484 \cdot 10^{14}$	$H_n = i/S$
E_n	$1,194639 \cdot 10^{17}$	$E_n = V/S$
f_n	$8,100040 \cdot 10^{17}$	$f_n = W/h$
ω_n	$3,505861 \cdot 10^{19}$	$\omega_n = (\alpha)^{1/2}$

The equations (7)–(9) are straight lines in the $X - Y'$ plane that go through the Absolute frame of reference of 1. Therefore, all electron, proton, and neutron constants are located on straight lines that have fixed slopes of K_e, K_p, and K_n, and a common hidden Absolute frame of reference of 10° or 1. Note, because the lines with slopes K_e, K_p, and K_n go through the center of equilibrium, it requires only one constant and the Absolute frame of reference of 1 to compute another set of constants for a new particle.

We computed constants that characterize proton and neutron, by raising electrons constant values ([2] — Table 1) to a power of the difference between the masses of the proton (and neutron) and the electron [$\ln m_p/m_e = 0,89135$ and $\ln m_n/\ln m_e = 0,89133$]. Some of the calculations are listed in the Tables 1 and 2.

1. Bedrij O., Fundamental constants in quantum electrodynamics, *Dopovidi Ukrainian Academy of Sciences*, 1993, № 3, 40–45.
2. Bedrij O., Scale invariance, unifying principle order and sequence of physical quantities and fundamental constants, *Dopovidi Ukrainian Academy of Sciences*, 1993, № 4, 67–73.

On maximal subalgebras of the rank $n-1$ of the conformal algebra $AC(1,n)$

A.F. BARANNYK, W.I. FUSHCHYCH

Проведено класифікацію максимальних підалгебр рангу $n-1$ алгебри $AC(1,n)$, які належать алгебрі $A\tilde{P}(1,n)$.

Consider the multidimensional eikonal equation

$$\left(\frac{\partial u}{\partial x_0}\right)^2 - \left(\frac{\partial u}{\partial x_1}\right)^2 - \cdots - \left(\frac{\partial u}{\partial x_{n-1}}\right)^2 = 1, \tag{1}$$

where $u = u(x)$ is a scalar function of the variable $x = (x_0, x_1, \ldots, x_{n-1})$, $n \geq 2$. In [1] it was established that the Lie algebra $AC(1,n)$ of the group $C(1,n)$ of the Minkowski $R_{1,n}$ space with the metric $x_0^2 - x_1^2 - \cdots - x_n^2$, where $x_n = u$, is a maximal algebra of the equation (1) invariance. The basis of the algebra $AC(1,n)$ is formed by such vector fields as:

$$P_\alpha = \partial_\alpha, \quad J_{\alpha\beta} = g^{\alpha\gamma}x_\gamma\partial_\beta - g^{\beta\gamma}x_\gamma\partial_\alpha, \quad D = -x^\alpha\partial_\alpha,$$
$$K_\alpha = -2(g^{\alpha\beta}x_\beta)D - (g^{\beta\gamma}x_\beta x_\gamma)\partial_\alpha,$$

where $g_{00} = -g_{11} = \cdots = -g_{nn} = 1$, $g_{\alpha\beta} = 0$, when $\alpha \neq \beta$ $(\alpha, \beta, \gamma = 0, 1, \ldots, n)$. The algebra $AC(1,n)$ contains the Poincaré algebra $AP(1,n)$ which is generated by vector fields P_α, $J_{\alpha\beta}$ and the extended Poincaré algebra $A\tilde{P}(1,n) = AP(1,n) \dotplus \langle D\rangle$.

In order to reduce the equation (1) by subalgebras of the algebra $AC(1,n)$, it is necessary to describe all $C(1,n)$-nonequivalent subalgebras of this algebra. The subalgebras K_1 and K_2 of the algebra $AC(1,n)$ are called as $C(1,n)$-equivalent ones if they have the same invariants with respect to $C(1,n)$-conjugation. Among $C(1,n)$-equivalent algebras there exists one (maximal) subalgebra containing all the other subalgebras. The maximal subalgebras K_1 and K_2 of the algebra $AC(1,n)$ are equivalent if and only if K_1 and K_2 are $C(1,n)$-conjugated.

The maximal subalgebras of the rank n of the algebra $AP(1,n)$ with respect to $P(1,n)$-conjugation are described in [2]. The maximal subalgebras of the rank n of the algebra $A\tilde{P}(1,n)$ with respect to $\tilde{P}(1,n)$-conjugation are described in [3, 4]. The present article is a continuation of researches which were realized in [3, 4]. The full classification of the maximal subalgebras of the rank $n-1$ of the algebra $AC(1,n)$ which are contained in the algebra $A\tilde{P}(1,n)$ has been carried out in the present article. Ansatzes corresponding to these subalgebras reduce the equation (1) to ordinary differential equations.

We will use the notations:

$$M = P_0 + P_n, \quad T = \frac{1}{2}(P_0 - P_n), \quad G_a = J_{0n} - J_{an}, \ a = 1, \ldots, n-1,$$
$$AO[r,s] = \langle J_{ab} \,|\, a, b = r, \ldots, s\rangle, \quad r \leq s,$$

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$$AE[r,s] = \langle P_r, \ldots, P_s \rangle \mathbin{+\!\!\!\supset} AO[r,s], \quad r \le s,$$
$$AE_1[r,s] = \langle G_r, \ldots, G_s \rangle \mathbin{+\!\!\!\supset} AO[r,s], \quad r \le s.$$

If $s > r$ then $AO[r,s] = 0$, $AE[r,s] = 0$ by definition.

Let

$$\Phi(r,s,\gamma) = \langle G_r + \gamma P_r, \ldots, G_s + \gamma P_s \rangle \mathbin{+\!\!\!\supset} AO[r,s], \quad r,s \in \mathbb{N},\ r \le s,\ \gamma \in \mathbb{R}.$$

Let

$$\Gamma_{d,q} = U \mathbin{+\!\!\!\supset} F,$$

where F is the diagonal of $AO[1,d] \oplus AO[d+1,2d] \oplus \cdots \oplus AO[(q-1)d+1,qd]$, and U is the Abelian algebra which has the basis

$$G_1 + \gamma_1 P_1 + \lambda_1 P_{(q-1)d+1},\ \ldots,\ G_d + \gamma_1 P_d + \lambda_1 P_{qd},$$
$$G_{d+1} + \gamma_2 P_{d+1} + \lambda_2 P_{(q-1)d+1},\ \ldots,\ G_{2d} + \gamma_2 P_{2d} + \lambda_2 P_{qd},$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$G_{(q-2)d+1} + \gamma_{q-1} P_{(q-2)d+1} + \lambda_{q-1} P_{(q-1)d+1},\ \ldots,\ G_{(q-1)d} + {}$$
$$+ \gamma_{q-1} P_{(q-1)d} + \lambda_{q-1} P_{qd},$$

where $0 \le \gamma_1 < \gamma_2 < \cdots < \gamma_{q-1}$, $\lambda_1 > 0$, $\lambda_2 > 0$, $\ldots$, $\lambda_{q-1} > 0$.

Results of the work [5] reduce the problem constructing invariants of any subalgebra of the algebra $A\tilde{P}(1,n)$ to the problem of constructing invariants of the irreducible subalgebras of the orthogonal algebra $AO(k)$ for all $k \le n$. The latter problem has no solution in quadratures. Therefore, we shall restrict ourself considering of such subalgebras of the algebra $A\tilde{P}(1,n)$ which projections onto $AO[1,n]$ are subdirect sums on the algebras $AO[r,s]$. Moreover, to find real solutions of the equation (1) it is necessary to exclude from consideration such subalgebras of the algebra $AP(1,n)$ which with respect to equivalence contain $P_0 + P_n$ or P_0. Therefore we prove the following theorems.

Theorem 1. *Let L be the maximal subalgebra of the rank $n-1$ of the algebra $AP(1,n)$. Then L is $C(1,n)$-conjugated with one of the following algebras:*

1) $L_1 = AE[1,n-1]$;
2) $L_2 = AO[1,m] \oplus AE[m+1,n]$, $m = 1,\ldots,n$, $n \ge 2$;
3) $L_3 = AE_1[1,m] \oplus AE[m+1,n-1]$, $m = 1,\ldots,n-1$, $n \ge 2$;
4) $L_4 = AO[1,m] \oplus AE[m+1,n-1] \oplus \langle J_{0n} \rangle$, $m = 1,\ldots,n-1$, $n \ge 3$;
5) $L_5 = AO[0,m] \oplus AE[m+1,n-1]$, $m = 2,\ldots,n-1$, $n \ge 3$;
6) $L_6 = AO[0,m] \oplus AO[m+1,q] \oplus AE[q+1,n-1]$, $m = 2,\ldots,n-1$, $q = m+1,\ldots,n$, $n \ge 3$;
7) $L_7 = \langle G_1 + P_0 - P_n \rangle \oplus AE[2,n-1]$, $n \ge 2$;
8) $L_8 = \Phi(d_0+1,d_1,\gamma_1) \oplus \cdots \oplus \Phi(d_{t-1}+1,m,\gamma_t) \oplus AE[m+1,n-1]$, $m = 1,\ldots,n-1$, $n \ge 3$;
9) $L_9 = \langle J_{0n} + P_1 \rangle \oplus AE[2,n-1]$, $n \ge 2$;
10) $L_{10} = (AE_1[1,m] \oplus \langle J_{0n} + P_{m+1} \rangle) \oplus AE[m+2,n-1]$, $m = 1,\ldots,n-2$, $n \ge 3$;
11) $L_{11} = \langle J_{12} + P_0 \rangle \oplus AE[3,n]$, $n \ge 2$.

Theorem 2. *Let L be the maximal subalgebra of the rank $n-1$ of the algebra $A\tilde{P}(1,n)$ which has a nonzero projection onto $\langle D \rangle$. Then L is $C(1,n)$-conjugated with one of the following algebras:*

1) $L_1 = (AO[0,d] \oplus AO[d+1,m] \oplus AO[m+1,q] \oplus AE[q+1,n]) \subset\!\!\!\!+ \langle D\rangle$, $d = 2,\dots,n-2$, $m = d+1,\dots,n-2$, $q = m+1,\dots,n-1$, $2n \le d+q$, $n \ge 4$;

2) $L_2 = (AO[0,m] \oplus AE[m+1,n-2]) \subset\!\!\!\!+ \langle D+\alpha J_{n-1,n}\rangle$, $m = 2,\dots,n-2$, $n \ge 4$, $\alpha > 0$;

3) $L_3 = (AO[1,m] \oplus AO[m+1,q] \oplus AE[q+1,n]) \subset\!\!\!\!+ \langle D\rangle$, $m = 2,\dots,n-2$, $q = m+2,\dots,n$, $2m \le q$, $n \ge 2$;

4) $L_4 = (AE_1[1,m] \oplus AE[m+1,n-3]) \subset\!\!\!\!+ \langle J_{n-2,n-1} + cJ_{0n}, D+\alpha J_{0n}\rangle$, $m = 1,\dots,n-3$, $n \ge 4$, $c > 0$, $\alpha \ge 0$;

5) $L_5 = (AO[1,m] \oplus AO[m+1,q] \oplus AE[q+1,n-1]) \subset\!\!\!\!+ \langle D, J_{0n}\rangle$, $m = 1,\dots,n-2$, $q = m+1,\dots,n-1$, $2m \le q$, $n \ge 3$;

6) $L_6 = AE[3,n-1] \subset\!\!\!\!+ \langle J_{12} + cJ_{0n}, D+\alpha J_{0n}\rangle$, $c > 0$, $\alpha \ge 0$, $n \ge 3$;

7) $L_7 = (AE_1[1,d] \oplus AO[d+1,m] \oplus AE[m+1,n-1]) \subset\!\!\!\!+ \langle D+\alpha J_{0n}\rangle$, $d = 1,\dots,n-2$, $m = d+1,\dots,n-1$, $n \ge 3$, $\alpha \ge 0$;

8) $L_8 = (AO[1,m] \oplus AE[m+1,n-1]) \subset\!\!\!\!+ \langle D+\alpha J_{0n}\rangle$, $m = 1,\dots,n-1$, $n \ge 2$, $\alpha \ge 0$;

9) $L_9 = (\langle G_1 + 2T\rangle \oplus AO[2,m] \oplus AE[m+1,n-1]) \subset\!\!\!\!+ \langle 2D - J_{0n}\rangle$, $m = 2,\dots,n-1$, $n \ge 3$;

10) $L_{10} = (AE_1[1,d] \oplus AO[d+1,m] \oplus AE[m+1,n-1]) \subset\!\!\!\!+ \langle D + J_{0n} + M\rangle$, $d = 1,\dots,n-2$, $m = d+1,\dots,n-1$, $n \ge 3$;

11) $L_{11} = (AO[1,m] \oplus AE[m+1,n-1]) \subset\!\!\!\!+ \langle D + J_{0n} + M\rangle$, $m = 1,\dots,n-1$, $n \ge 2$;

12) $L_{12} = (AE_1[1,m] \oplus AE[m+1,n-3]) \subset\!\!\!\!+ \langle J_{n-2,n-1} + \alpha M, D + J_{0n} + M\rangle$, $m = 1,\dots,n-3$, $n \ge 4$, $\alpha \ge 0$;

13) $L_{13} = (AE_1[1,m] \oplus AE[m+1,n-3]) \subset\!\!\!\!+ \langle J_{n-2,n-1} + M, D + J_{0n}\rangle$, $m = 1,\dots,n-3$, $n > 4$;

14) $L_{14} = AE[3,n-1] \subset\!\!\!\!+ \langle J_{12} + \alpha M, D + J_{0n} + M\rangle$, $n \ge 3$, $\alpha \ge 0$;

15) $L_{15} = AE[3,n-1] \subset\!\!\!\!+ \langle J_{12} + M, D + J_{0n}\rangle$, $n \le 3$;

16) $L_{16} = (\Gamma_{d,q} \oplus AE[dq+1,n-1] \subset\!\!\!\!+ \langle D - J_{0n}\rangle$, $d \ge 2$, $n \ge 5$;

17) $L_{17} = (\Phi(d_0+1,d_1,\gamma_1) \oplus \Phi(d_1+1,d_2,\gamma_2) \oplus \dots \oplus \Phi(d_{t-1}+1,d_t,\gamma_t) \oplus AO[d_t+1,m] \oplus AE[m+1,n-1]) \subset\!\!\!\!+ \langle D - J_{0n}\rangle$, where $d_0 = 0$, $\gamma_1 < \gamma_2 < \dots < \gamma_t$, $t > 1$, $m = 1,\dots,n-2$, $n \ge 3$;

18) $L_{18} = (\Gamma_{d,q} \oplus \Phi(l_0+1,l_1,\mu_1) \oplus \Phi(l_1+1,l_2,\mu_2) \oplus \dots \oplus \Phi(l_{t-1}+1,l_t,\mu_t) \oplus AE[l_t+1,n-1]) \subset\!\!\!\!+ \langle D - J_{0n}\rangle$, where $\mu_1 < \mu_2 < \dots < \mu_t$, $t \ge 1$, $l_0 = dq$.

L_1–L_{11} and L_1–L_{18} of the theorems 1 and 2 respectively and to carry out a reduction of the equation (1). Consider, for example, the subalgebra L_{17}. The ansatz

$$u^2 = \left[-(x_0 + x_m] + \sum_{i=1}^{t} \frac{1}{x_0 - x_m + \gamma_i}\left(x^2_{d_{i-1}+1} + \cdots x^2_{d_i}\right)\right]\varphi(\omega) - \\ - x^2_{d_t+1} - \cdots - x^2_{m-1}, \quad \omega = x_0 - x_m,$$

corresponds to this subalgebra. This ansatz reduces the equation (1) to equation $\varphi\dot{\varphi} - \varphi = 0$. Using the solution of this equation we find the following solution of the equation (1):

$$u^2 = \left[-(x_0 + x_m] + \sum_{i=1}^{t} \frac{1}{x_0 - x_m + \gamma_i}\left(x^2_{d_{i-1}+1} + \cdots + x^2_{d_i}\right)\right] \times \\ \times (x_0 - x_m + C) - x^2_{d_t+1} - \cdots - x^2_{m-1}.$$

1. Fushchych W.I, Shtelen W.M., The symmetry and some exact solutions of the relativistic eikonal equation, *Lett. Nuovo Cimento*, 1982, **34**, 498–502.

2. Grundland A.M., Harnad L., Winternitz P., Symmetry reduction for nonlinear relativistically invariant equations, *J. Math. Phys.*, 1984, **25**, № 4, 791–806.

3. Фущич В.И., Баранник Л.Ф., Максимальные подалгебры ранга $n-1$ алгебры $AP(1,n)$ и редукция нелинейных волновых уравнений. I, *Укр. мат. журн.*, 1990, **42**, № 11, 1260–1256.

4. Фущич В.И., Баранник Л.Ф., Максимальные подалгебры ранга $n-1$ алгебры $AP(1,n)$ и редукция нелинейных волновых уравнений. II, *Укр. мат. журн.*, 1990, **42**, № 12, 1693–1700.

5. Баранник А.Ф., Баранник Л.Ф., Фущич В.И., Редукция многомерного пуанкаре-инвариантного нелинейного уравнения к двумерным уравнениям, *Укр. мат. журн.*, 1991, **43**, № 10, 1314–1323.

Conditional symmetries of the equations of mathematical physics

W.I. FUSHCHYCH

We briefly present the results of research in conditional symmetries of equations of mathematical and theoretical physics: the Maxwell, D'Alembert, Schrödinger and KdV equations, as well as the equations of heat conduction and acoustics. Exploiting conditional symmetry, we construct a wide class of exact solutions of these equations, which cannot be obtained by the classical method of Sophus Lie.

1. Introduction

The concept and terminology of conditional symmetry and conditional invariance were introduced and developed in the series of articles [1–11] (see also Mathematical Reviews for the years 1983–1993). Later, this concept was exploited by other authors for the construction of solutions of various non-linear equations of mathematical physics. It turned out that nearly all the basic non-linear equations of mathematical physics have non-trivial conditional symmetry [2, 9, 10].

We understand the conditional symmetry of an equation as being a symmetry (local or non-local) of some non-trivial subset of its solution set (the formal definition of the idea of conditional symmetry can be found in Appendix 4 of [2] and in the article [3]). The general definition of conditional symmetry as the symmetry of a subset of the set of solutions is non-constructive and requires further specification: the analytical description of a condition (as an equation) on the solutions of the given equation, which extend or alter the symmetry of the starting equation. Therefore, the basic problem in the investigation of conditional symmetries is that of describing those supplementary equations which increase or change the symmetry of the beginning equation. This is very complex, non-linear problem in general (even in the case of quite simple non-linear equations), which can often be significantly more complicated than constructing solutions of the equation at hand. It is thus meaningful to talk of the conditional symmetry of some class of equations.

Non-trivial conditional symmetries of a PDE (partial differential equation) allows us to obtain in explicit form such solutions which can not be found by using the symmetries of the whole set of solutions of the given PDE. Moreover, conditional symmetries increase significantly the class of PDEs for which we can construct ansatzes which reduce these equations to (systems of) ODEs (ordinary differential equations). As a rule, the reduced equations one obtains from conditional symmetries are significantly simpler than those found by reduction using symmetries of the full set of solutions. This allows us to construct exact solutions of the reduced equations.

Looking back, we can say today, that many mathematicians, mechanicians and physicists, such as Euler, D'Alembert, Poincaré, Volterra, Whittaker, Bateman, implicitly used conditional symmetries for the construction of exact solutions of the linear

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wave equation. Some well-known solutions of this equation can not be obtained by using only Lie symmetries of the full solution set.

2. Conditional symmetry of Maxwell's equation

We shall first consider the first pair of Maxwell's equations

$$\frac{\partial \boldsymbol{E}}{\partial t} = \operatorname{rot} \boldsymbol{H}, \quad \frac{\partial \boldsymbol{H}}{\partial t} = -\operatorname{rot} \boldsymbol{E}. \tag{1}$$

The maximal invariance algebra (in the sense of Lie) of these equations is studied in [2]. The basis elements of this algebra $\langle \partial_0, \partial_a, J_{ab}, D \rangle$ are

$$\begin{gathered} \partial_0 = \frac{\partial}{\partial x^0}, \quad \partial_a = \frac{\partial}{\partial x^a}, \quad J_{ab} = x_a\partial_b - x_b\partial_a + s_{ab}, \quad a, b = 1, 2, 3, \\ D = x^\mu \partial_\mu + \text{const}, \end{gathered} \tag{2}$$

s_{ab} are 6×6 matrices realizing a representation of the group $O(3)$. Thus the system (1) is invariant under the four-dimensional translations ∂_μ, the rotations J_{ab} and scale transformations D, but it is not invariant under the Lorentz boosts

$$J_{0a} = x_0\partial_a - x_a\partial_0 + s_{0a}, \quad x_0 = t, \tag{3}$$

the matrices $\langle s_{0a}, s_{ab} \rangle$ realizing a representation of the Lorentz group $O(1, 3)$.

Theorem 1 ([2] 1983, [15] 1987). *The system (1) is conditionally invariant under the Lorentz boosts (3) if and only if the solutions of (1) satisfy the conditions*

$$\operatorname{div} \boldsymbol{E} = 0, \quad \operatorname{div} \boldsymbol{H} = 0. \tag{4}$$

It is evident from this theorem, that the concept of conditional invariance of a PDE is natural, and leads us, by purely group-theoretic means, to the fundamental, overdetermined system of Maxwell's equations.

3. Conditional symmetry of the wave equation

We now examine the non-linear D'Alembert equation

$$\Box u = F(u), \quad u = u(x_0, x_1, x_2, x_3), \tag{5}$$

$F(u)$ being an arbitrary, smooth function. Equation (5) has conformal symmetry $C(1, 3)$ if and only if $F = \lambda u^3$ or $F = 0$ (see for instance [8, 10]). This is the maximal symmetry of all of the solution set of equation (5). For an arbitrary function, (5) admits only the symmetry groups $P(1, 3)$.

Theorem 2 ([5], 1985). *Equation (5), with $F = 0$ is conditionally invariant under the infinite-dimensional algebra with basis elements*

$$X = \xi^\mu(x, u)\frac{\partial}{\partial x^\mu} + \eta(x, u)\frac{\partial}{\partial u}, \tag{6}$$

$$\xi^\mu(x, u) = c^{00}(u)x^\mu + c^{\mu\nu}(u)x_\nu + d^\mu(u), \quad \eta(x, u) = \eta(u), \tag{7}$$

where $c^{00}(u)$, $c^{\mu\nu}(u)$, $d^\mu(u)$, $\eta(u)$ are arbitrary functions of u, if one imposes the condition

$$\frac{\partial u}{\partial x_\mu}\frac{\partial u}{\partial x^\mu} = 0. \tag{8}$$

In this way, the eikonal equation (8), significantly increases the symmetry of the starting equation (5). The system of equations (5), (8), with $F=0$, is consistent.

Theorem 3 ([10, 15], 1988, 1989). *The equation (5) is conditionally invariant under the conformal group, if*

$$F = \frac{3\lambda}{u+c}, \tag{9}$$

$$\frac{\partial u}{\partial x_\mu}\frac{\partial u}{\partial x^\mu} = \lambda, \tag{10}$$

where λ, c are arbitrary constants. The operators of conformal symmetry are

$$\begin{aligned} &K_\mu = 2x_\mu D - (x_\alpha x^\alpha - u^2)\frac{\partial}{\partial x^\mu}, \quad \mu = 0,1,2,3,\\ &D = x^\mu\frac{\partial}{\partial x^\mu} + u\frac{\partial}{\partial u}. \end{aligned} \tag{11}$$

Remark. It is important to note, that the operators (11) differ principally from the conformal operators for equation (5), when $F=0$ or $F=\lambda u^3$. In those cases, the conformal operators are

$$\hat{K}_\mu = 2x_\mu D - x_\alpha x^\alpha\frac{\partial}{\partial x^\mu}, \quad D = x^\mu\frac{\partial}{\partial x^\mu}. \tag{12}$$

The operators (11) are *non-linear*, whereas those in (12) are linear.

Thus the wave equation (5), (9), with non-linear condition (10), has a symmetry possessed by neither the solution set for the linear equation, nor that for the nonlinear equation.

4. Criteria for conditional symmetry

Let us consider some PDE

$$\begin{aligned} &L(x, u_{(1)}, u_{(2)}, \ldots, u_{(n)}) = 0,\\ &u_{(1)} = (u_0, u_1, \ldots, u_n), \quad u_{(2)} = (u_{01}, u_{02}, \ldots, u_{nn}), \quad \ldots,\\ &u_\mu = \frac{\partial u}{\partial x^\mu}, \quad u_{\mu\nu} = \frac{\partial^2 u}{\partial x^\mu \partial x^\nu}, \quad \ldots. \end{aligned} \tag{13}$$

Definition 1 (S. Lie, 1884). *Equation (13) is invariant with respect to the operator (6) if*

$$X_s L = \lambda L, \tag{14}$$

where X_s is the s-th prolongation of (6), and $\lambda = \lambda(x,u)$ is an arbitrary function.

Let us denote by the symbol

$$Q = \langle Q_1, Q_2, \ldots, Q_r\rangle \tag{15}$$

some set of operators which does not belong to the invariance algebra (IA) of equation (13).

Definition 2 ([2], 1987). *Equation(13) is said to be conditionally invariant under the operators Q from (15), if there exists a supplementary condition on the solutions of (13) of the form*

$$L_1(x, u, u_{(1)}, \dots, u_{(n)}) = 0 \tag{16}$$

such that (13) together with (16) is invariant under the Q.

Thus one has the following conditions

$$Q_s L = \lambda_0 L + \lambda_1 L_1, \tag{17}$$

$$Q_s L_1 = \lambda_2 L + \lambda_3 L_1 \tag{18}$$

or

$$Q_s L\Big|_{\substack{L=0\\ L_1=0}} = 0, \quad Q_s L_1\Big|_{\substack{L=0\\ L_1=0}} = 0. \tag{19}$$

An important class of supplementary conditions (16) is that for which the equation $L_1 = 0$ is a quasi-linear equation of first order

$$L_1(x, u, u_{(1)}) \equiv Qu = 0, \tag{20}$$

$$Q = y^\mu(x, u)\frac{\partial}{\partial x^\mu} + z(x, u)\frac{\partial}{\partial u} \tag{21}$$

with y^μ, z being smooth functions. In this case, we shall say that (13) is Q-conditionally invariant.

In this way, the problem of finding the conditional symmetry of (13) reduces to the solution of the equations (17), (18). The conditions (16), (20) can be considered as equations for the construction of ansatzes for the starting equation (13). The problem of calculating the conditional symmetry is far more complicated than the usual method of Lie for finding the symmetry of the full solution set. In the case of conditional symmetries, the defining equations are, as a rule, non-linear equations which can be solved in only some cases. Fortunately, for most of the equations of non-linear mathematical physics, one can construct partial solutions of the defining equations.

5. A list of equations with non-trivial conditional symmetry

Conditional symmetries began to be exploited only quite recently, and the first publications appeared only in 1983 [1, 2]. Now, the number of articles in this are is increasing rapidly with each year, and therefore it is difficult to give a complete list (for 1992) of important equations of mathematical physics possessing conditional symmetry. So I shall only give those equations which we have studied and which are interested from our Kievan point of view. We have put in brackets the year(s) when the conditional symmetry of the given equation was found. More detailed information about ansatzes and solutions of the above equations are to be found in the original articles, a list of which are given in [2, 9, 11].

$$1. \quad u_0 + u_{11} = F(u) = \begin{cases} \lambda u(u^2 - 1), \\ \lambda(u^3 - 3u + 2), \\ \lambda u^3, \\ \lambda u(u^3 + 1). \end{cases} \qquad (1988,\ 1990)$$

2. $iu_0 + \Delta u + F(|u|)u = 0,$
$F(|u|) = \lambda_1 |u|^{4/r} + \lambda_2 |u|^{-4/r}, \quad F(|u|) = \lambda_3 \ln(u^* u),$ (1990)
$\lambda_1,\ \lambda_2,\ r$ arbitrary, real; λ_3 arbitrary, complex.

3. $u_{00} = u\Delta u, \quad u_{00} = c(x, u, u_{(1)})\Delta u.$ (1987, 1988)

4. $u_{01} - (F(u)u_1)_1 - u_{22} - u_{33} = 0.$ (1990)

5. $u_0 + \nabla(F(u)\nabla u) = 0.$ (1988)

6. $u_0 + F(u)u_1^k + u_{111} = 0.$ (1991)

7. $u_0 + (\varphi(u))_{11} + \frac{N}{x_1}(\varphi(u))_1 = F(u),$
$u_0 + u_{11} + \frac{3}{2x_1}u_1 = \lambda u^3,$ (1992)
$u_0 + uu_{11} + \frac{N}{x_1}uu_1 = \lambda u + \lambda_2.$

8. $\boldsymbol{u}_0 + (\boldsymbol{u}\nabla)\boldsymbol{u} = -\frac{1}{\rho}\nabla p,$
$\rho_0 + \operatorname{div}(\rho\boldsymbol{u}) = 0, \quad p = f(\rho), \quad p = \frac{1}{2}\rho^2.$ (1992)

9. $\gamma^\mu \partial_\mu \Psi + F(\bar{\Psi}\Psi)\Psi = 0.$ (1989)

10. $(1 - u_\alpha u^\alpha)\Box u + u^\mu u^\nu u_{\mu\nu} = 0.$ (1989)

6. Conditional symmetry and exact solutions of KdV type equations

To illustrate the constructive nature of conditional symmetries, we shall examine the equation

$$u_0 + F(u)u_1^k + u_{111} = 0, \tag{22}$$

where $F(u)$ is a smooth function, $k \neq 0$ is an arbitrary, real parameter. When $F(u) = u$, $k = 1$, equation (22) coincides with the standard KdV equation.

Theorem 4 ([11], 1991). *Equation (22) is Q-conditionally invariant with respect to the following operators*

$$Q = x_0^r \partial_1 + H(x, u)\partial_u \tag{23}$$

with r an arbitrary, real parameter, in the following cases

1. $F(u) = \lambda_1 u^{(2-k)/k} + \lambda_2 u^{(1-k)/2}, \quad H(x, u) = \left(\frac{\lambda_1 k}{2}\right)^{-1/k} u^{1/2};$ (24)

2. $F(u) = (\lambda_1 \ln u)^{1-k}, \quad H(x, u) = (k\lambda_1)^{-1/k};$ (25)

3. $F(u) = (\lambda_1 \arcsin u + \lambda_2)(1 - u^2)^{(1-k)/2},$
$H(x, u) = (k\lambda_1)^{-1/k}(1 + u^2)^{1/2};$ (26)

$$4. \quad F(u) = (\lambda_1 \sinh^{-1} u + \lambda_2)(1+u^2)^{(1-k)/2}, \quad H(x,u) = (k\lambda_1)^{-1/k}(1+u^2)^{1/2}; \tag{27}$$

$$5. \quad F(u) = \lambda_1 u, \quad H(x,u) = (k\lambda_1)^{-1/k}, \tag{28}$$

where $r = 1/k$, $k \neq 0$, λ_1, λ_2 are arbitrary, real parameters.

Exploiting the operator of conditional symmetry (23), one can construct ansatzes for the solutions of equation (22), some of which I now exhibit.

The ansatz

$$u = \left(\frac{x_1}{2}\left(\frac{k\lambda_1 x_0}{2}\right)^{-1/k} + \varphi(x_0)\right)^2$$

gives the solution

$$u = \left(\frac{x_1}{2}\left(\frac{k\lambda_1 x_0}{2}\right)^{-1/k} + \lambda x_0^{-1/k} - \lambda_2/\lambda_1\right)^2$$

when $F(u)$ is as in (24). The ansatz

$$u = \exp\left(\varphi(x_0) + (k\lambda_1 x_0)^{-1/k} x_1\right)$$

gives the solution

$$u = \exp\left(-\frac{k(k\lambda_1)^{-3/k}}{k-2} x_0^{1-3/k} + \lambda x_0^{-1/k} + (k\lambda_1 x_0)^{-1/k} x_1 - \lambda_2/\lambda_1\right)$$

when $F(u)$ is as in (25) with $k \neq 2$. The ansatz

$$u = \sin\left(\varphi(x_0) + (k\lambda_1 x_0)^{-1/k} x_1\right)$$

gives the solution

$$u = \sin\left(\frac{k(k\lambda_1)^{-3/k}}{k-2} x_0^{1-3/k} + \lambda x_0^{-1/k} + (k\lambda_1 x_0)^{-1/k} x_1 - \lambda_2/\lambda_1\right)$$

for $k \neq 2$.

Theorem 5 ([12], 1990). *The equation*

$$u_{01} - (F(u)u_1)_1 - u_{22} - u_{33} = 0 \tag{29}$$

is invariant under under the infinite-dimensional algebra

$$X = a_i(u) R_i, \quad i = 1, \ldots, 12, \tag{30}$$

where $a_i(u)$ are arbitrary, smooth functions, if one adds to (29) the condition

$$u_0 u_1 - F(u) u_1^2 - u_2^2 - u_3^2 = 0. \tag{31}$$

The operators R_i are given as follows:

$$R_{\mu+1} = \partial_\mu, \quad \mu = 0, \ldots, 3, \quad R_5 = x_3\partial_2 - x_2\partial_3, \quad R_6 = x_2\partial_1 + 2x_0\partial_2,$$
$$R_7 = x_3\partial_1 + 2x_0\partial_3, \quad R_8 = x^\mu \partial_\mu,$$

$$R_9 = x_0\partial_0 + 2x_1\partial_1 + 3x_2\partial_2 + 3x_3\partial_3 - 2\frac{F(u)}{\dot F(u)}\partial_u, \quad R_{10} = \dot F(u)x_0\partial_1 - \partial_u,$$
$$R_{11} = x_2\partial_0 + 2(x_1 + F(u)x_0)\partial_2, \quad R_{12} = x_3\partial_0 + 2(x_1 + F(u)x_0)\partial_3.$$

7. Antireduction

In [10], we have begun work on antireduction. By the term antireduction of a PDE we understand the finding of such ansatzes which transform the given PDE into a system of equations for some (unknown, and to be found) functions. In this process, the number of independent variables may remain the same, or be reduced (dimensional reduction), but the number of dependent variables increases. As a rule, one usually exploits the converse of this, that is, one reduces to a system with fewer dependent variables (reduction of components). To illustrate the effectiveness of antireduction, we consider the equation for short waves in gas dynamics

$$2u_{01} - 2(x_1 + u_1)u_{11} + u_{22} + 2\lambda u_1 = 0. \tag{32}$$

We impose the condition

$$\left[u_{111}x_1^{3/2}\right]_1 = 0 \tag{33}$$

on (32). The general solution of (33) is

$$u = x_1^{3/2}\varphi^1 + x_1^2\varphi^2 + x_1\varphi^3 + \varphi^4 \tag{34}$$

with $\varphi^i = \varphi^i(x_0, x_2)$, $i = 1, 2, 3, 4$ being arbitrary functions. Using (34) as an ansatz, equation is reduced to a system with two independent variables

$$\begin{gathered}\varphi^3 = 0, \quad \varphi^1_{22} = 0, \quad \varphi^2_{22} = 0, \quad \varphi^4_{22} = \frac{9}{4}(\varphi^2)^2, \\ \varphi^1_0 = \varphi^1\left(3\varphi^2 + \frac{1}{2} - \lambda\right), \quad \varphi^2_0 = 2\varphi^2_2 - \varphi_2(1-\lambda).\end{gathered} \tag{35}$$

Solving the system (35), we found exact solutions of the starting equation (32) [11].

The above results are only a sample of those already obtained. They illustrate the very fruitful nature of conditional symmetry and conditional invariance, and I hope that I have been able to demonstrate that there are new aspects to this concept which are yet to be exploited fully.

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1. Fushchych W., On symmetry and partial solutions of some multi-dimensional equations of mathematical physics, in Algebraic-theoretical methods in the problems of mathematical physics, Kiev, Institute of mathematics, Ukrainian Academy of Sciences, 1983, 4–23.
2. Fushchych W., Nikitin A., Symmetries of Maxwell's equations, Kiev, Naukova Dumka, 1983; Dordrecht, Kluwer, 1987.

3. Fushchych W., How can one extend the symmetry of differential equations? in Symmetry and solutions of non-linear equations in mathematical physics, Kiev, Institute of mathematics, Ukrainian Academy of Sciences, 1987, 4–16.

4. Fushchych W., Tsyfra I., On reduction and solutions of the nonlinear wave-equation with broken symmetry, *J. Phys. A*, 1987, **20**, L45–L48.

5. Shulha M., Symmetry and some exact solutions of the D'Alembert equation with a nonlinear condition, in Symmetry and solutions of equations of mathematical physics, Kiev, Institute of Mathematics, Ukrainian Academy of Sciences, 1985, 36–38.

6. Fushchych W., Serov M., Chopyk W., Conditional invariance and non-linear heat equations, *Proc. Ukr. Acad. Sci., Ser. A*, 1988, № 9, 17–21.

7. Fushchych W., Serov M., Conditional invariance and exact solutions of a non-linear equation of acoustics, *Proc. Ukr. Acad. Sci., Ser. A*, 1988, № 10, 27–31.

8. Fushchych W., Shtelen W., Serov M., Symmetry analysis and exact solutions of nonlinear equations of mathematical physics, Kiev, Naukova dumka, 1989; Dordrecht, Kluwer, 1993.

9. Fushchych W., Chopyk W., Conditional invariance of a non-linear Schrödinger equation, *Proc. Ukr. Acad. Sci., Ser. A*, 1990, № 4, 30–33.

10. Fushchych W., Conditional symmetry of the equations of mathematical physics, *Ukr. Math. J.*, 1991, **43**, № 11, 1456–1470.

11. Fushchych W., Serov M., Amerov T., On the conditional symmetry of the generalized Korteweg-de Vries equation, *Proc. Ukr. Acad. Sci., Ser. A*, 1991, № 12, 28–30.

12. Fushchych W., Chopyk W., Myronyuk P., Conditional invariance and exact solutions of three-dimensional non-linear equations in acoustics, *Proc. Ukr. Acad. Sci. Ser. A*, 1990, № 9, 25–28.

13. Fushchych W., Shtelen W., Serov M., Popowych R., Conditional symmetry of the linear heat equation, *Proc. Ukr. Acad. Sci., Ser. A*, 1992, № 12, 20–26.

14. Fushchych W., Repeta W., Exact solutions of the equations of gas dynamics and non-linear acoustics, *Proc. Ukr. Acad. Sci., Ser. A*, 1991, № 8, 35–42.

15. Fushchych W., Zhdanov R., Symmetry and exact solutions of non-linear spinor equations, *Phys. Rep.*, 1989, **46**, 325–365.

Invariants of one-parameter subgroups of the conformal group $C(1,n)$

W.I. FUSHCHYCH, L.F. BARANNIK, V.I. LAGNO

Досліжена структура інваріантів одної з основних груп симетрії математичної фізики — конформної групи $C(1,n)$ у просторі Мінковського $R(1,n)$ для $n \geq 3$. Зокрема, побудовані повні системи інваріантів однопараметричних підгруп групи $C(1,n)$.

The conformal group $C(1,n)$ of transformations in the Minkovsky space $R(1,n)$ takes the central place among the invariance Lie groups of mathematical physics [1]. Our interest in the functional invariants of one-parameter subgroups has been motivated by their application in finding solutions of diferential equations. In Ref. [2] substitution (ansatz)

$$u(x) = f(x)\varphi(\omega) + g(x) \tag{1}$$

for construction of exact solutions of multi-dimensional equations is proposed. In ansatz (1) the unknown function $\varphi(\omega)$ depends on the complete system of functional invariants $\omega_1, \omega_2, \ldots, \omega_m$ of the one-parameter subgroups of the invariance Lie group of the given equation.

In this paper complete systems of functional invariants of one-parameter subgroups of the conformal group $C(1,n)$ ($n \geq 3$) are obtained. It should be noted that analogous problem for Poincaré groups $P(1,n)$ and $P(2,n)$ in Refs. [3] and [4] is determined.

It is known that to each local Lie group corresponds its Lie algebra, in particular, to group $C(1,n)$ corresponds Lie algebra $AC(1,n)$. Generators P_α, J_{0a}, J_{ab}, D, K_α ($\alpha = \overline{0,n}$, $a,b = \overline{1,n}$ generate the basis of Lie algebra $AC(1,n)$. We shall consider the Lie algebra $AC(1,n)$ as algebra of differential operators determined in the space of scalar functions $u(x)$ ($x \in R(1,n)$):

$$\begin{gathered} P_\alpha = -\partial_\alpha = -\frac{\partial}{\partial x_\alpha}, \quad J_{\alpha\beta} = x_\alpha \partial_\beta - x_\beta \partial_\alpha, \quad x_\alpha = g_{\alpha\beta} x^\beta, \\ g_{\alpha\beta} = (1, -1, \ldots, -1) \times \delta_{\alpha\beta}, \quad D = -x^\alpha \partial_\alpha, \quad K_\alpha = 2x_\alpha D + s^2 \partial_\alpha \\ (s^2 \equiv x^\alpha x_\alpha = x_0^2 - x_1^2 - \ldots - x_n^2), \quad \alpha, \beta = \overline{0,n}. \end{gathered} \tag{2}$$

It should be noted that the Lie algebra $AC(1,n)$ (2) is an invariance algebra of many differential equations [1].

The function $F(x)$ ($x \in R(1,n)$) is the invariant of the on-parameter subgroups of the group $C(1,n)$ if and only if it is the solution of the differential equation

$$Lu = 0, \tag{3}$$

where L is the corresponding one-dimensional subalgebra of the Lie algebra $AC(1,n)$ (2) (see, for example, [5]).

Consequently, the problem of finding of invariants of one-parameter subgroups, of the group $C(1,n)$ is reduced to finding the system of functionally-independent

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solutions of equation (3). Such systems of functionally-independent solutions of equation (3) will be called complete systems of invariants (CSI) of the corresponding one-dimensional subalgebras of the algebra $AC(1,n)$.

Lie algebra $AP(1,n) = \langle P_\alpha, J_{\alpha\beta} \,|\, \alpha = \overline{0,n},\ \beta = \overline{1,n}\rangle$ is subalgebra the algebra $AC(1,n)$. CSI of one-dimensional subalgebras of the algebra $AP(1,n)$ are constructed in [3]. Consequently, we shall describe CSI of the one-dimensional subalgebras of the factor algebra $AC(1,n)/AP(1,n)$.

One-dimensional subalgebras of the algebra $AC(1,n)/AP(1,n)$ are such algebras [3, 6]:

$$\begin{aligned}
&L_1 = \langle D + \alpha J_{0n}\rangle\ (0 < \alpha \le 1); \quad L_2 = \langle D + J_{0n} + M\rangle;\\
&L_3 = \langle X_t + \alpha D + \beta J_{0n}\rangle\ (\alpha \ge \beta \ge 0,\ \alpha \ne 0);\\
&L_4 = \langle X_t + \alpha(D + J_{0n} + M)\rangle\ (\alpha > 0);\\
&L_5 = \langle J_{12} + \beta_1 J_{34} + \cdots + \beta_{\frac{n}{2}-1} J_{n-1,n} + \gamma D\rangle\\
&(n \equiv 0 \pmod 2),\ \gamma > 0,\ 0 \le \beta_1 \le \cdots \le \beta_{\frac{n}{2}-1} \le 1);\\
&L_6 = \langle S + T\rangle; \quad L_7 = \langle S + T \pm M\rangle;\\
&L_8 = \langle X_t + \alpha(S + T)\rangle\ (\alpha > 0); \quad L_9 = \langle S + T + \alpha Z\rangle\ (\alpha > 0);\\
&L_{10} = \langle X_t + \alpha(S + T) \pm M\rangle\ (\alpha > 0);\\
&L_{11} = \langle X_t + S + T + G_1 + P_2\rangle; \quad L_{12} = \langle X_t + \alpha(S + T) + \beta Z\rangle\ (\alpha, \beta > 0);\\
&L_{13} = \langle P_0 + K_0\rangle; \quad L_{14} = \langle \alpha(P_0 + K_0) + J_{12}\rangle\ (\alpha > 0);\\
&L_{15} = \langle \alpha(P_0 + K_0) + J_{12} + \beta_1 J_{34} + \cdots + \beta_s J_{2s+1,2s+2}\rangle\\
&(\alpha > 0;\ 0 < \beta_1 \le \ldots \le \beta_s \le 1;\ s = 1, 2, \ldots, [(n-2)/2]);\\
&L_{16} = \langle \alpha(P_0 + L_0) + J_{12} + \gamma_1 J_{34} + \cdots + \gamma_{\frac{n-3}{2}} J_{n-2,n-1} + \gamma_{\frac{n-1}{2}}(K_n - P_n)\rangle\\
&(\alpha > 0,\ 0 < \gamma_1 \le \cdots \le \gamma_{\frac{n-1}{2}} \le 1);\\
&L_{17} = \langle J_{12} + \gamma_1 J_{34} + \cdots + \gamma_{\frac{n-1}{2}}(K_n - P_n)\rangle\\
&(0 < \gamma_1 \le \cdots \le +\gamma_{\frac{n-1}{2}} \le 1),
\end{aligned} \tag{4}$$

where

$$\begin{aligned}
&X_t = \alpha_1 J_{12} + \alpha_2 J_{34} + \cdots + \alpha_t J_{2t-1,2t}\\
&(\alpha_1 = 1,\ 0 \le \alpha_2 \le \cdots \le \alpha_t \text{ if } t \ne 1;\ t = 1, 2, \ldots, [(n-1)/2]),\\
&M = P_0 + P_n, \quad T = \tfrac{1}{2}(P_0 - P_n), \quad S = \tfrac{1}{2}(K_0 + K_n), \quad G_1 = J_{01} - J_{1n},
\end{aligned}$$

$Z = J_{0n} - D$. In algebras L_{16} and L_{17} value n is an odd number.

Let $y = y(x) = x_0 + x_n$, $z = z(x) = x_0 - x_n$, $h_a = h_a(x) = x_{2a-1}^2 + x_{2a}^2$, $\varphi_a = \varphi_a(x) = \operatorname{arctg} \frac{x_{2a}}{x_{2a-1}}$, $\psi = \psi(x) = x_1^2 + x_2^2 + \cdots + x_{n-1}^2$.

Record $L : f_1(x), f_2(x), \ldots, f_s(x)$ designates that functions $f_1(x), f_2(x), \ldots, f_s(x)$ form CSI of the algebra L.

Theorem. *Following functions are CSI of one-dimensional subalgebras of the algebra $AC(1,n)/AP(1,n)$:*

$$L_1: \quad z x_1^{-1-\alpha},\ y x_1^{\alpha-1},\ x_2 x_1^{-1},\ x_3 x_1^{-1},\ \ldots,\ x_{n-1} x_1^{-1};$$

L_2: $y-\ln|z|,\ y-2\ln|x_1|,\ x_2x_1^{-1},\ x_3x_1^{-1},\ \ldots,\ x_{n-1}x_1^{-1};$

L_3: $z^{\alpha}x_{2t+1}^{-\alpha-\beta},\ y^{\alpha}x_{2t+1}^{\beta-\alpha},\ \alpha\varphi_1-\alpha_1\ln|x_{2t+1}|,\ \alpha\varphi_2-\alpha_2\ln|x_{2t+1}|,\ \ldots,$
$\alpha\varphi_t-\alpha_t\ln|x_{2t+1}|,\ h_1x_{2t+1}^{-2},\ h_2x_{2t+1}^{-2},\ \ldots,\ h_tx_{2t+1}^{-2},\ x_{2t+2}x_{2t+1}^{-1},$
$x_{2t+3}x_{2t+1}^{-1},\ \ldots,\ x_{n-1}x_{2t+1}^{-1};$

L_4: $zx_{2t+1}^{-2},\ y-2\ln|x_{2t+1}|,\ \alpha\varphi_1-\alpha_1\ln|x_{2t+1}|,\ \alpha\varphi_2-\alpha_2\ln|x_{2t+1}|,\ \ldots,$
$\alpha\varphi_t\ln|x_{2t+1}|,\ h_1x_{2t+1}^{-2},\ h_2x_{2t+1}^{-2},\ \ldots,\ h_tx_{2t+1}^{-2}, h_2x_{2t+1}^{-2},\ \ldots,\ h_tx_{2t+1}^{-2},$
$x_{2t+2}x_{2t+1}^{-1},\ x_{2t+3}x_{2t+1}^{-1},\ \ldots,\ x_{n-1}x_{2t+1}^{-1};$

L_5: $\ln h_1-2\gamma\varphi_1,\ \beta_1\ln h_1-2\gamma\varphi_2,\ \ldots,\ \beta_{\frac{n}{2}-1}\ln h_1-2\gamma\varphi_{\frac{n}{2}}\ln h_1-2\gamma\varphi_{\frac{n}{2}},$
$x_0^2h_1^{-1},\ h_2h_1^{-1},\ \ldots,\ h_3h_1^{-1},\ \ldots,\ h_{\frac{n}{2}}h_1^{-1};$

L_6: $(1+z^2)x_1^{-2},\ y-z\psi(1+z^2)^{-1},\ x_2x_1^{-1},\ x_3x_1^{-1},\ \ldots,\ x_{n-1}x_1^{-1};$

L_7: $y\pm 2\,\mathrm{arctg}\,z-z(1+z^2)^{-1}\psi,\ (1+z^2)x_1^{-2},\ x_2x_1^{-1},$
$x_3x_1^{-1},\ \ldots,\ x_{n-1}x_1^{-1};$

L_8: $y-z(1+z^2)^{-1}\psi,\ (1+z^2)h_1^{-1},\ \alpha\varphi_1-\alpha_1\,\mathrm{arctg}\,z,\ x_{2t+1}^2h_1^{-1},\ h_2h_1^{-1},$
$h_3h_1^{-1},\ \ldots,\ h_th_1^{1},\ \alpha_2\varphi_1-\alpha_1\varphi_2,\ \alpha_3\varphi_1-\alpha_1\varphi_3,\ \ldots,$
$\alpha_t\varphi_1-\alpha_1\varphi_t,\ x_{2t+2}x_{2t+1}^{-1},\ x_{2t+3}x_{2t+1}^{-1},\ \ldots,\ x_{n-1}x_{2t+1}^{-1};$

L_9: $2\alpha\,\mathrm{arctg}\,z+\ln(x_1^2(1+z^2)^{-1}),\ 2\alpha\,\mathrm{arctg}\,z+\ln(y+z\psi(1+z^2)^{-1}),$
$x_2x_1^{-1},\ x_3x_1^{-1},\ \ldots,\ x_{n-1}x_1^{-1};$

L_{10}: $\alpha y\pm 2\,\mathrm{arctg}\,z-\alpha z(1+z^2)^{-1}\psi,\ \alpha\varphi_1-\alpha_1\,\mathrm{arctg}\,z,\ (1+z^2)h_1^{-1},$
$x_{2t+1}^2h_1^{-1},\ h_2h_1^{-1},\ h_3h_1^{-1},\ \ldots,\ h_th_1^{-1},\ x_{2t+2}x_{2t+1}^{-1},\ x_{2t+3}x_{2t+1}^{-1},\ \ldots,$
$x_{n-1}x_{2t+1}^{-1},\ \alpha_2\varphi_1-\alpha_1\varphi_2,\ \alpha_3\varphi_1-\alpha_1\varphi_3,\ \ldots,\ \alpha_t\varphi_1-\alpha_1\varphi_t;$

L_{11}: $(1+z^2)x_{2t+1}^{-2},\ (x_1+zx_2)(1+z^2)^{-1},\ \varphi_2-\alpha_2\,\mathrm{arctg}\,z,$
$y+2(x_1+zx_2)(1+z^2)^{-1}\,\mathrm{arctg}\,z-z\psi(1+z^2)^{-1},$
$\mathrm{arctg}\,z-(x_2-zx_1)(1+z^2)^{-1},\ h_2x_{2t+1}^{-2},\ h_3x_{2t+1}^{-2},\ \ldots,\ h_tx_{2t+1}^{-2},$
$x_{2t+2}x_{2t+1}^{-1},\ x_{2t+3}x_{2t+1}^{-1},\ \ldots,\ x_{n-1}x_{2t+1}^{-1};$

L_{12}: $2\beta\,\mathrm{arctg}\,z+\alpha\ln|y-z\psi(1+z^2)^{-1}|,\ 2\beta\,\mathrm{arctg}\,z+\alpha\ln h_1(1+z^2)^{-1},$
$\alpha\varphi_1-\alpha_1\,\mathrm{arctg}\,z,\ \alpha_2\varphi_1-\alpha_1\varphi_2,\ \alpha_3\varphi_1-\alpha_1\varphi_3,\ \ldots,\ \alpha_t\varphi_1-\alpha_1\varphi_t,$
$h_2h_1^{-1},\ h_3h_1^{-1},\ \ldots,\ h_th_1^{-1},\ x_{2t+2}x_{2t+1}^{-1},\ x_{2t+3}x_{2t+1}^{-1},\ \ldots,\ x_{n-1}x_{2t+1}^{-1};$

L_{13}: $(yz-\psi+1)x_1^{-1},\ x_2x_1^{-1},\ x_3x_1^{-1},\ \ldots,\ x_nx_1^{-1};$

L_{14}: $2\alpha\varphi_1-\mathrm{arctg}((yz-\psi-1)(2x_0)^{-1}),\ (yz-\psi+1)x_3^{-1},$
$h_1x_3^{-1},\ x_4x_3^{-1},\ x_5x_3^{-1},\ \ldots,\ x_nx_3^{-1};$

L_{15}: $2\alpha\varphi_1-\mathrm{arctg}((yz-\psi-1)(2x_0)^{-1}),\ (yz-\psi+1)^2h_1^{-1},\ \beta_1\varphi_1-\varphi_2,$
$\beta_2\varphi_1-\varphi_3,\ \ldots,\ \beta_s\varphi_1-\varphi_{s+1},\ h_2h_1^{-1},\ h_3h_1^{-1},\ \ldots,\ h_{s+1}h_1^{-1},$
$x_{2s+3}^2h_1^{-1},\ x_{2s+4}x_{2s+3}^{-1},\ x_{2s+5}x_{2s+3}^{-1},\ \ldots,\ x_nx_{2s+3}^{-1};$

L_{16}: $2\alpha\varphi_1-\mathrm{arctg}((yz-\psi-1)(2x_0)^{-1}),$
$2\gamma_{\frac{n-1}{2}}\varphi_1-\mathrm{arctg}((yz-\psi+1)(2x_n)^{-1}),\ ((yz-\psi-1)^2+4x_0^2)(4h_1)^{-1},$
$h_2h_1^{-1},\ h_3h_1^{-1},\ \ldots,\ h_{\frac{n-1}{2}}h_1^{-1},\ \gamma_1\varphi_1-\varphi_2,\ \gamma_2\varphi_1-\varphi_3,\ \ldots,$
$\gamma_{\frac{n-3}{2}}\varphi_1-\varphi_{\frac{n-1}{2}};$

L_{17}: $2\gamma_{\frac{n-1}{2}}\varphi_1-\mathrm{arctg}((yz-\psi+1)(2x_n)^{-1}),\ (yz-\psi-1)x_0^{-1},\ \gamma_1\varphi_1-\varphi_2,$
$\alpha_2\varphi_1-\varphi_3,\ \ldots,\ \gamma_{\frac{n-3}{2}}\varphi_1-\varphi_{\frac{n-1}{2}},\ h_2h_1^{-1},\ h_3h_1^{-1},\ \ldots,\ h_{\frac{n-1}{2}}h_1^{-1},\ h_1x_0^{-2}.$

Values of numerical parameters are given in expression (4).

In order to prove the theorem it is sufficient to verify that each CSI satisfies the equation (3).

1. Фущич В.И., Штелень В.М., Серов Н.И., Симметрийный анализ и точные решения нелинейных уравнений математической физики, Киев, Наук. думка, 1989, 336 с.
2. Фущич В.И., Симметрия в задачах математической физики, в сб. Теоретико-алгебраические исследования в математической физике, Киев, Ин-т математики АН УССР, 1981, 6–28.
3. Фущич В.И., Баранник Л.Ф., Баранник А.Ф., Подгрупповой анализ групп Галилея, Пуанкаре и редукция нелинейных уравнений, Киев, Наук. думка, 1991, 304 с.
4. Баранник Л.Ф., Лагно В.И., Однопараметрические подгруппы обобщенной группы Пуанкаре $P(1,n)$ и их инварианты, *Укр. мат. журн.*, 1989, **41**, № 9, 1169–1172.
5. Овсянников Л.В., Групповой анализ дифференциальных уравнений, М., Наука, 1978, 400 с.
6. Баранник Л.Ф., Фущич В.И., О непрерывных подгруппах конформной группы пространства Минковского $R(1,n)$, Препринт № 88.34, Киев, Ин-т математики АН УССР, 1988, 48 с.

Системи нелінійних еволюційних рівнянь другого порядку, інваріантні відносно алгебри Галілея та її розширень

В.І. ФУЩИЧ, Р.М. ЧЕРНІГА

New classes of systems of nonlinear evolution equations are constructed which are invariant in regard to the Galilei algebra and its extentions (including operators of scale and projective transformations). New nonlinear generalisation of the Schödinger equation are proposed which retain Galilean symmetry of the linear equation.

Нижче розглядаються системи нелінійних двовимірних параболічних рівнянь вигляду

$$\lambda_1\psi_t^{(1)} = A^{11}\psi_{xx}^{(1)} + A^{12}\psi_{xx}^{(2)} + B^{(1)}, \quad \lambda_2\psi_t^{(2)} = A^{21}\psi_{xx}^{(1)} + A^{22}\psi_{xx}^{(2)} + B^{(2)}, \tag{1}$$

де $A^{nm} = A^{nm}(\psi^{(1)}, \psi^{(2)}, \psi_x^{(1)}, \psi_x^{(2)})$, $B^{(n)} = B^{(n)}(\psi^{(1)}, \psi^{(2)}, \psi_x^{(1)}, \psi_x^{(2)})$ — довільні комплексні або дійсні функції, неперервно диференційовані за всіма змінними, $\lambda_n \in \mathbb{C}$, $\psi_t^{(n)} = \frac{\partial\psi^{(n)}}{\partial t}$, $\psi_x^{(n)} = \frac{\partial\psi^{(n)}}{\partial x}$, $\psi_{xx}^{(n)} = \frac{\partial^2\psi^{(n)}}{\partial x^2}$, $\psi^{(n)} = \psi^{(n)}(t,x)$ — шукані комплексні або дійсні функції, індекси n і m скрізь набувають значень 1, 2.

Система рівнянь (1) узагальнює практично всі відомі двовимірні системи еволюційних рівнянь другого порядку, якими описуються найрізноманітніші процеси у фізиці, хімії, біології (досить згадати процеси тепломасопереносу, фільтрації двофазної рідини, дифузії при хімічних реакціях, руху популяції в природі тощо) [1].

У випадку комплексних функцій $\psi = \psi^{(1)} = \psi^{*(2)}$, $C = A^{11} = A^{*22}$, $D = A^{12} = A^{*21}$, $B = B^{(1)} = B^{*(2)}$, $\lambda_1 = \lambda_2^* = i$ система рівнянь (1) перетворюється на пару комплексно спряжених рівнянь, які інтерпретуватимемо як клас нелінійних узагальнень рівняння Шредінгера, а саме:

$$i\psi_t = C\psi_{xx} + D\psi_{xx}^* + B, \tag{2a}$$

$$-i\psi_t^* = C^*\psi_{xx}^* + D^*\psi_{xx}^* + B^* \tag{2b}$$

(нижче комплексно спряжені рівняння (2b) скрізь опущено). Очевидно, що при $C = k \in \mathbb{R}$, $D = B = 0$ рівняння (2a) перетворюється на класичне рівняння Шредінгера з нульовим потенціалом

$$i\psi_t = k\psi_{xx}, \quad 0 \neq k \in \mathbb{R}. \tag{3}$$

Шляхом відповідного вибору функції $B(\psi, \psi^*, \psi_x, \psi_x^*)$ можна одержати найрізноманітніші нелінійні узагальнення рівняння (3), які зустрічаються в літературі [2, 4].

Відомо, що лінійне рівняння (3) інваріантне відносно узагальненої алгебри Галілея $AG_2(1,1)$ з базовими операторами [3, 4]

$$P_t = \partial_t, \quad P_x = \partial_x, \tag{4a}$$

Доповіді АН України, 1993, № 8, С. 44–51.

$$G_x = t\partial_x - \frac{x}{2k}J, \quad J = i(\psi\partial_\psi - \psi^*\partial_{\psi^*}), \tag{4b}$$

$$D = 2t\partial_t + x\partial_x + \alpha(\psi\partial_\psi + \psi^*\partial_{\psi^*}), \tag{4c}$$

$$\Pi = tD - t^2\partial_t - \frac{x^2}{4k}J, \quad \alpha = -\frac{1}{2}, \tag{4d}$$

де $\partial_t = \frac{\partial}{\partial t}$, $\partial_x = \frac{\partial}{\partial x}$, $\partial_\psi = \frac{\partial}{\partial \psi}$, $\partial_{\psi^*} = \frac{\partial}{\partial \psi^*}$.

Алгебру, утворену операторами (4a), (4b), називають алгеброю Галілея $AG(1,1)$, а її розширення за допомогою оператора (4c) позначимо $AG_1(1,1)$. Відзначимо, що симетрія багатовимірних систем рівнянь (1) при $A^{nm} = A^{nm}(\psi^{(1)}, \psi^{(2)})$, $A^{12} = A^{21} = 0$ досліджена в роботі [4]. Проте для математичного моделювання деяких процесів необхідно вимагати, щоб $A_{12} \neq 0$, $A^{21} \neq 0$ (див., наприклад, [2, 5]). З іншого боку, в останні роки запропоновано деякі нелінійні рівняння Шредінгера [6, 7], які, згідно з висновками роботи [4], не зберігають галілеївську симетрію лінійного рівняння (3). Це підкреслює необхідність побудови систем рівнянь вигляду (1), інваріантних відносно ланцюжка алгебр $AG(1,1) \subset AG_1(1,1) \subset AG_2(1,1)$.

Розглянемо алгебру Галілея з зображенням (4a) і

$$G_x = t\partial_x - \frac{x}{2}J_\lambda, \quad J_\lambda = \lambda_1\psi^{(1)}\partial_{\psi^{(1)}} + \lambda_2\psi^{(2)}\partial_{\psi^{(2)}}. \tag{5}$$

Теорема 1. *Система нелінійних рівнянь (1) інваріантна відносно алгебри Галілея з зображенням (4a), (5) тоді і тільки тоді, коли вона має вигляд*

1) у випадку $\lambda_1\lambda_2 \neq 0$

$$\begin{aligned} \lambda_1\psi^{(1)}_t &= g^{11}\left(\psi^{(1)}_{xx} - \frac{(\psi^{(1)}_x)^2}{\psi^{(1)}}\right) + g^{12}\frac{\psi^{(1)}}{\psi^{(2)}}\left(\psi^{(2)}_{xx} - \frac{(\psi^{(2)}_x)^2}{\psi^{(2)}}\right) + \\ &\quad + \psi^{(1)}\left(f^{(1)} + \left(\frac{\psi^{(1)}_x}{\psi^{(1)}}\right)^2\right), \\ \lambda_2\psi^{(2)}_t &= g^{21}\frac{\psi^{(2)}}{\psi^{(1)}}\left(\psi^{(1)}_{xx} - \frac{(\psi^{(1)}_x)^2}{\psi^{(1)}}\right) + g^{22}\left(\psi^{(2)}_{xx} - \frac{(\psi^{(2)}_x)^2}{\psi^{(2)}}\right) + \\ &\quad + \psi^{(2)}\left(f^{(2)} + \left(\frac{\psi^{(2)}_x}{\psi^{(2)}}\right)^2\right), \end{aligned} \tag{6}$$

де $g^{nm} = g^{nm}(v, v_x)$, $f^{(n)} = f^{(n)}(v, v_x)$ — довільні функції,

$$v = (\psi^{(1)})^{\lambda_2}(\psi^{(2)})^{-\lambda_1}, \quad v_x = \frac{\partial v}{\partial x} \equiv \left(\lambda_2\frac{\psi^{(1)}_x}{\psi^{(1)}} - \lambda_1\frac{\psi^{(2)}_x}{\psi^{(2)}}\right)v;$$

2) у випадку $\lambda_1 = 0$, $\lambda_2 = \lambda \neq 0$

$$\begin{aligned} 0 &= g^{11}\psi^{(1)}_{xx} + g^{12}\frac{\psi^{(1)}}{\psi^{(2)}}\left(\psi^{(2)}_{xx} - \frac{(\psi^{(2)}_x)^2}{\psi^{(2)}}\right) + \psi^{(1)}f^{(1)}, \\ \lambda\psi^{(2)}_t &= g^{21}\frac{\psi^{(2)}}{\psi^{(1)}}\psi^{(1)}_{xx} + g^{22}\left(\psi^{(2)}_{xx} - \frac{(\psi^{(2)}_x)^2}{\psi^{(2)}}\right) + \psi^{(2)}\left(f^{(2)} + \left(\frac{\psi^{(2)}_x}{\psi^{(2)}}\right)^2\right), \end{aligned} \tag{7}$$

де $g^{nm} = g^{nm}(\psi^{(1)}, \psi^{(1)}_x)$, $f^{(n)} = f^{(n)}(\psi^{(1)}, \psi^{(1)}_x)$ *— довільні функції.*

Доведення теореми 1, як і теорем 2, 3, грунтується на класичній схемі Лі, реалізація якої для знаходження галілей-інваріантних систем наведена в роботі [8]. Оскільки викладки досить громіздкі, ми їх опускаємо.

Наслідок 1. *В класі нелінійних рівнянь (2) алгебру* $AG(1,1)$ *(4a)–(4b) лінійного рівняння Шредінгера (3) зберігають тільки такі:*

$$\frac{i}{k}\psi_t = g^{(1)}\left(\psi_{xx} - \frac{(\psi_x)^2}{\psi}\right) + g^{(2)}\frac{\psi}{\psi^*}\left(\psi^*_{xx} - \frac{(\psi^*_x)^2}{\psi^*}\right) + \psi\left(f + \left(\frac{\psi_x}{\psi}\right)^2\right), \quad (8)$$

де $g^{(n)} = g^{(n)}(|\psi|, |\psi|_x)$, $f = f(|\psi|, |\psi|_x)$ *— довільні функції,* $|\psi| = \sqrt{\psi\psi^*}$, $|\psi|_x = \frac{\partial|\psi|}{\partial x}$.

Легко помітити, що рівняння (8) при $g^{(1)} = 1$, $g^{(2)} = 0$ і $f = a|\psi|^\beta$, $\beta \in \mathbb{R}$ перетворюється на відоме рівняння Шредінгера зі степеневою нелінійністю, а при $g^{(1)} = 0$, $g^{(2)} = -1$, $f = -4|\psi|^2_x|\psi|^{-6} + a|\psi|^2$, $a \in \mathbb{C}$ — на рівняння

$$\frac{i}{k}\psi_t = -\frac{\psi}{\psi^*}\psi^*_{xx} + a\psi|\psi|^2 - 2\psi\frac{|\psi_x|^2}{|\psi|^2}, \quad (9)$$

яке за структурою нагадує рівняння [6, 7]

$$i\psi_t = c_1\psi_{xx} + a\psi|\psi|^2 - c\psi\frac{|\psi_x|^2}{|\psi|^2}, \quad c_1, c \in \mathbb{C}. \quad (10)$$

Відзначимо, що рівняння (10), на відміну від (9), не інваріантне відносно алгебри Галілея, оскільки воно не належить класу (8).

Розглянемо алгебри $AG_1(1,1)$ і $AG_2(1,1)$, які є розширеннями алгебри Галілея $AG(1,1)$ (4a), (5), за допомогою операторів

$$D = 2t\partial_t + x\partial_x + I_\alpha, \quad (11a)$$

$$\Pi = t^2\partial_t + tx\partial_x - \frac{x^2}{4}J_\lambda + tI_\alpha, \quad (11b)$$

де $I_\alpha = \alpha_1\psi^{(1)}\partial_{\psi^{(1)}} + \alpha_2\psi^{(2)}\partial_{\psi^{(2)}}$, $\alpha_n \in \mathbb{C}$ (для рівняння Шредінгера (3) $\alpha_1 = \alpha_2 = \alpha$, $\lambda_1 = \lambda_2^* = \frac{i}{k}$). Виявляється, що класифікація систем рівнянь, інваріантних відносно алгебр $AG_1(1,1)$ і $AG_2(1,1)$, суттєво залежить від значення визначника $\delta = \begin{vmatrix} \alpha_1 & \alpha_2 \\ \lambda_1 & \lambda_2 \end{vmatrix}$, який, зокрема, при $\lambda_1 = 0$, $\lambda_2 = \lambda \neq 0$ дорівнює $\lambda\alpha_2$.

Теорема 2. *Система нелінійних рівнянь (1) інваріантна відносно алгебри* $AG_1(1,1)$ *з базовими операторами (4a), (5), (11a) тоді і тільки тоді, коли вона має вигляд*

1) у випадку $\lambda_1\lambda_2 \neq 0$

$$\lambda_1\psi_t^{(1)} = h^{11}\left(\psi_{xx}^{(1)} - \frac{(\psi_x^{(1)})^2}{\psi^{(1)}}\right) + h^{12}\frac{\psi^{(1)}}{\psi^{(2)}}\left(\psi_{xx}^{(2)} - \frac{(\psi_x^{(2)})^2}{\psi^{(2)}}\right) + $$
$$+ \psi^{(1)}\left(v^{-\frac{2}{\delta}}W^{(1)} + \left(\frac{\psi_x^{(1)}}{\psi^{(1)}}\right)^2\right),$$
$$\lambda_2\psi_t^{(2)} = h^{21}\frac{\psi^{(2)}}{\psi^{(1)}}\left(\psi_{xx}^{(1)} - \frac{(\psi_x^{(1)})^2}{\psi^{(1)}}\right) + h^{22}\left(\psi_{xx}^{(2)} - \frac{(\psi_x^{(2)})^2}{\psi^{(2)}}\right) + $$
$$+ \psi^{(2)}\left(v^{-\frac{2}{\delta}}W^{(2)} + \left(\frac{\psi_x^{(2)}}{\psi^{(2)}}\right)^2\right),$$

де $h^{nm} = h^{nm}(\theta)$, $W^{(n)} = W^{(n)}(\theta)$ — довільні функції,

$$\theta = v_x v^{\frac{1}{\delta}-1} \equiv \left(\lambda_2\frac{\psi_x^{(1)}}{\psi^{(1)}} - \lambda_1\frac{\psi_x^{(2)}}{\psi^{(2)}}\right),$$

якщо $\delta \neq 0$ або, якщо $\delta = 0$ — вигляд

$$\lambda_1\psi_t^{(1)} = h^{11}\left(\psi_{xx}^{(1)} - \frac{(\psi_x^{(1)})^2}{\psi^{(1)}}\right) + h^{12}\frac{\psi^{(1)}}{\psi^{(2)}}\left(\psi_{xx}^{(2)} - \frac{(\psi_x^{(2)})^2}{\psi^{(2)}}\right) + $$
$$+ \psi^{(1)}\left(v_x^2W^{(1)} + \left(\frac{\psi_x^{(1)}}{\psi^{(1)}}\right)^2\right),$$
$$\lambda_2\psi_t^{(2)} = h^{21}\frac{\psi^{(2)}}{\psi^{(1)}}\left(\psi_{xx}^{(1)} - \frac{(\psi_x^{(1)})^2}{\psi^{(1)}}\right) + h^{22}\left(\psi_{xx}^{(2)} - \frac{(\psi_x^{(2)})^2}{\psi^{(2)}}\right) + $$
$$+ \psi^{(2)}\left(v_x^2W^{(2)} + \left(\frac{\psi_x^{(2)}}{\psi^{(2)}}\right)^2\right),$$

де $h^{nm} = h^{nm}(v)$, $W^{(n)} = W^{(n)}(v)$ — довільні функції, (v, v_x — див. теорему 1).

2) у випадку $\lambda_1 = 0$, $\lambda_2 = \lambda \neq 0$

$$0 = h^{11}\psi_{xx}^{(1)} + h^{12}\frac{\psi^{(1)}}{\psi^{(2)}}\left(\psi_{xx}^{(2)} - \frac{(\psi_x^{(2)})^2}{\psi^{(2)}}\right) + (\psi^{(1)})^{1-2/\alpha_1}W^{(1)},$$
$$\lambda\psi_t^{(2)} = h^{21}\frac{\psi^{(2)}}{\psi^{(1)}}\psi_{xx}^{(1)} + h^{22}\left(\psi_{xx}^{(2)} - \frac{(\psi_x^{(2)})^2}{\psi^{(2)}}\right) + $$
$$+ \psi^{(2)}\left((\psi^{(1)})^{-\frac{2}{\alpha_1}}W^{(2)} + \left(\frac{\psi_x^{(2)}}{\psi^{(2)}}\right)^2\right),$$

де $h^{nm} = h^{nm}(\theta_0)$, $W^{(n)} = W^{(n)}(\theta_0)$ *— довільні функції,* $\theta_0 = \psi_x^{(1)}(\psi^{(1)})^{\frac{1}{\alpha_1}-1}$, *якщо* $\alpha_1 \neq 0$ *або, якщо* $\alpha_1 = 0$ *— вигляд*

$$0 = h^{11}\psi_{xx}^{(1)} + h^{12}\frac{\psi^{(1)}}{\psi^{(2)}}\left(\psi_{xx}^{(2)} - \frac{(\psi_x^{(2)})^2}{\psi^{(2)}}\right) + \psi^{(1)}(\psi_x^{(1)})^1 W^{(1)},$$

$$\lambda\psi_t^{(2)} = h^{21}\frac{\psi^{(2)}}{\psi^{(1)}}\psi_{xx}^{(1)} + h^{22}\left(\psi_{xx}^{(2)} - \frac{(\psi_x^{(2)})^2}{\psi^{(2)}}\right) + \\ + \psi^{(2)}\left((\psi_x^{(1)})^2 W^{(2)} + \left(\frac{\psi_x^{(2)}}{\psi^{(2)}}\right)^2\right),$$

де $h^{nm} = h^{nm}(\psi^{(1)})$, $W^{(n)} = W^{(n)}(\psi^{(1)})$ *— довільні функції.*

Наслідок 2. *В класі нелінійних рівнянь (2) алгебру* $AG_1(1,1)$ *(4a)–(4b), (11a) лінійного рівняння Шредінгера (3) зберігають тільки такі:*

$$\frac{i}{k}\psi_t = h^{(1)}\left(\psi_{xx} - \frac{(\psi_x)^2}{\psi}\right) + h^{(2)}\frac{\psi}{\psi^*}\left(\psi_{xx}^* - \frac{(\psi_x^*)^2}{\psi^*}\right) + \\ + \psi\left(|\psi|^{-\frac{2}{\alpha}}W + \left(\frac{\psi_x}{\psi}\right)^2\right), \tag{12}$$

де $h^{(n)} = h^{(n)}\left(|\psi|_x|\psi|^{\frac{1}{\alpha}-1}\right)$, $W = W\left(|\psi|_x|\psi|^{\frac{1}{\alpha}-1}\right)$ *— довільні функції,* $\alpha = \alpha_1 = \alpha_2$ *— параметр в операторі* I_α, $\alpha \neq 0$.

Теорема 3. *Система нелінійних рівнянь (1) інваріантна відносно узагальненої алгебри Галілея* $AG_2(1,1)$ *з базовими операторами (4a), (5), (11) тоді і тільки тоді, коли вона має вигляд*

1) у випадку $\lambda_1\lambda_2 \neq 0$

$$\lambda_1\psi_t^{(1)} = h^{11}\left(\psi_{xx}^{(1)} - \frac{(\psi_x^{(1)})^2}{\psi^{(1)}}\right) - \frac{\lambda_1}{\lambda_2}(h^{11} + 2\alpha_1)\frac{\psi^{(1)}}{\psi^{(2)}}\left(\psi_{xx}^{(2)} - \frac{(\psi_x^{(2)})^2}{\psi^{(2)}}\right) + \\ + \psi^{(1)}\left(v^{-\frac{2}{\delta}}W^{(1)} + \left(\frac{\psi_x^{(1)}}{\psi^{(1)}}\right)^2\right),$$

$$\lambda_2\psi_t^{(2)} = -\frac{\lambda_2}{\lambda_1}(h^{22} + 2\alpha_2)\frac{\psi^{(2)}}{\psi^{(1)}}\left(\psi_{xx}^{(1)} - \frac{(\psi_x^{(1)})^2}{\psi^{(1)}}\right) + h^{22}\left(\psi_{xx}^{(2)} - \frac{(\psi_x^{(2)})^2}{\psi^{(2)}}\right) + \\ + \psi^{(2)}\left(v^{-\frac{2}{\delta}}W^{(2)} + \left(\frac{\psi_x^{(2)}}{\psi^{(2)}}\right)^2\right),$$

де $h^{nn} = h^{nn}(\theta)$, $W^{(n)} = W^{(n)}(\theta)$ *— довільні функції (*v, θ *— див. теореми 1, 2),* *якщо* $\delta \neq 0$ *або, якщо* $\delta = 0$ *— вигляд*

$$\lambda_1 \psi_t^{(1)} = h^{11}\left(\psi_{xx}^{(1)} - \frac{(\psi_x^{(1)})^2}{\psi^{(1)}}\right) - \frac{\lambda_1}{\lambda_2}(h^{11} + 2\alpha_1)\frac{\psi^{(1)}}{\psi^{(2)}}\left(\psi_{xx}^{(2)} - \frac{(\psi_x^{(2)})^2}{\psi^{(2)}}\right) + $$
$$+ \psi^{(1)}\left(v_x^2 W^{(1)} + \left(\frac{\psi_x^{(1)}}{\psi^{(1)}}\right)^2\right),$$
$$\lambda_2 \psi_t^{(2)} = -\frac{\lambda_2}{\lambda_1}(h^{22} + 2\alpha_2)\frac{\psi^{(2)}}{\psi^{(1)}}\left(\psi_{xx}^{(1)} - \frac{(\psi_x^{(1)})^2}{\psi^{(1)}}\right) + h^{22}\left(\psi_{xx}^{(2)} - \frac{(\psi_x^{(2)})^2}{\psi^{(2)}}\right) + $$
$$+ \psi^{(2)}\left(v_x^2 W^{(2)} + \left(\frac{\psi_x^{(2)}}{\psi^{(2)}}\right)^2\right),$$

де $h^{nn} = h^{nn}(v)$, $W^{(n)} = W^{(n)}(v)$ *— довільні функції;*

2) у випадку $\lambda_1 = 0$, $\lambda_2 = \lambda \neq 0$

$$0 = h^{11}\psi_{xx}^{(1)} + (\psi^{(1)})^{1-2/\alpha_1} W^{(1)}, \tag{13a}$$

$$\lambda \psi_t^{(2)} = h^{22}\frac{\psi^{(2)}}{\psi^{(1)}}\psi_{xx}^{(1)} - 2\alpha_2 \psi_{xx}^{(2)}(1 + 2\alpha_2)\frac{(\psi_x^{(2)})^2}{\psi^{(2)}} + \psi^{(2)}(\psi^{(1)})^{-2/\alpha_1} W^{(2)}, \tag{13b}$$

де $h^{nn} = h^{nn}(\theta_0)$, $W^{(n)} = W^{(n)}(\theta_0)$ *— довільні функції* (θ_0 *— див. теорему 2), якщо* $\alpha_1 \neq 0$ *або, якщо* $\alpha_1 = 0$

$$0 = h^{11}\psi_{xx}^{(1)} + \psi^{(1)}(\psi_x^{(1)})^2 W^{(1)}, \tag{14a}$$

$$\lambda \psi_t^{(2)} = h^{22}\frac{\psi^{(2)}}{\psi^{(1)}}\psi_{xx}^{(1)} - 2\alpha_2 \psi_{xx}^{(2)} + (1 + 2\alpha_2)\frac{(\psi_x^{(2)})^2}{\psi^{(2)}} + \psi^{(2)}(\psi_x^{(1)})^2 W^{(2)}, \tag{14b}$$

де $h^{nn} = h^{nn}(\psi^{(1)})$, $W^{(n)} = W^{(n)}(\psi^{(1)})$ *— довільні функції.*

Варто зауважити, що для $AG_2(1,1)$-інваріантних систем рівнянь (13), (14) значно простіше вирішується питання їх інтегрування. Дійсно, завдяки відсутності шуканої функції $\psi^{(2)}$ у перших рівняннях цих систем, вони перетворюються на звичайні диференціальні рівняння другого порядку. Загальний розв'язок рівняння (14a) для довільних функцій h^{11}, $W^{(1)}$ у неявному вигляді записується так:

$$\int \left[\exp \int W^{(1)}/h^{11} d\psi^{(1)}\right] d\psi^{(1)} + e^{-a(t)}\psi^{(1)} = x + b(t),$$

Проінтегрувавши ці рівняння і підставивши знайдені розв'язки $\psi^{(1)}(t,x)$ в (13b) і (14b), одержимо рівняння для знаходження функції $\psi^{(2)}$, які лінеаризуються заміною $\Phi = (\psi^{(2)})^{-1/2\alpha_2}$, $\alpha_2 \neq 0$.

Таким чином, для відшукання функції $\Phi(t,x)$ в обох випадках одержуємо лінійне рівняння вигляду

$$\lambda \frac{\partial \Phi}{\partial t} = -2\alpha_2 \frac{\partial^2 \Phi}{\partial x^2} + \Phi F(t,x),$$

для інтегрування якого можна вибрати той чи інший класичний метод залежно від конкретного значення функції

$$F(t,x) = \begin{cases} (\psi^{(1)})^{-2/\alpha_1}\left(W^{(2)} - W^{(1)}\dfrac{h^{22}}{h^{11}}\right), & \psi^{(1)}(t,x) \ \text{— розв'язок (13a)}, \\ (\psi_x^{(1)})^2\left(W^{(2)} - W^{(1)}\dfrac{h^{22}}{h^{11}}\right), & \psi^{(1)}(t,x) \ \text{— розв'язок (14a)}. \end{cases}$$

Наслідок 3. *В класі нелінійних рівнянь (2) алгебру* $AG_2(1,1)$ *лінійного рівняння Шредінгера (3) зберігають тільки такі рівняння:*

$$\frac{i}{k}\psi_t = h\left(\psi_{xx} - \frac{(\psi_x)^2}{\psi}\right) + (h-1)\frac{\psi}{\psi^*}\left(\psi^*_{xx} - \frac{(\psi^*_x)^2}{\psi^*}\right) + \\ + \psi\left(|\psi|^4 W + \left(\frac{\psi_x}{\psi}\right)^2\right), \tag{15}$$

де $h = h\left(|\psi|_x|\psi|^{-3}\right)$, $W = W\left(|\psi|_x|\psi|^{-3}\right)$ *— довільні функції.*

У випадку $h = 1$, $W = 0$ рівняння (15) переходить у лінійне рівняння (3), а при $h = 1$ — у рівняння [4, 8]

$$i\psi_t = k\psi_{xx} + \psi|\psi|^4 W,$$

яке в свою чергу при $W = \lambda_1 + \lambda_2|\psi|_x|\psi|^{-3}$, $\lambda_1, \lambda_2 \in \mathbb{C}$ зводиться до [10]

$$i\psi_t = k\psi_{xx} + \lambda_1\psi|\psi|^4 + \lambda_2\psi|\psi||\psi|_x. \tag{15'}$$

Зауважимо, що при $k = -1$, $\lambda_1 = -1$, $\lambda_2 = -4$ рівняння (15′) відоме в літературі як рівняння Екгауса [11], проте, наскільки нам відомо, досі ніхто не вказував на такий факт:

Наслідок 4. *Рівняння Екгауса* $i\psi_t + \psi_{xx} + \psi|\psi|^4 + 4\psi|\psi||\psi|_x = 0$ *зберігає симетрію лінійного рівняння Шредінгера, тобто алгебру з базовими операторами (4).*

Разом з тим клас рівнянь (15) містить таке нетривіальне узагальнення рівняння (3), що зберігає його симетрію, як

$$i\psi_t = -k\frac{\psi}{\psi^*}\psi^*_{xx} + k\psi\left(\left(\frac{\psi_x}{\psi}\right)^2 + \left(\frac{\psi^*_x}{\psi^*}\right)^2\right). \tag{16}$$

Зазначимо, що в рівнянні (16) потенціал

$$V = \left(\frac{\psi_x}{\psi}\right)^2 + \left(\frac{\psi^*_x}{\psi^*}\right)^2 = 2\left(2|\psi|_x^2 - |\psi_x|^2\right)/|\psi|^2$$

— дійсна функція.

При $h = 0$, $W = -4\left(|\psi|_x|\psi|^{-3}\right)^2 + a$ рівняння (15) зводиться до рівняння

$$\frac{i}{k}\psi_t = -\frac{\psi}{\psi^*}\psi^*_{xx} + a\psi|\psi|^4 - 2\psi\frac{|\psi_x|^2}{|\psi|^2}. \tag{17}$$

Рівняння (17) поряд з (9), (10) логічно інтерпретувати як нелінійні рівняння Шредінгера. Проте на противагу рівнянням (9), (10), рівняння (17) повністю зберігає симетрію, тобто алгебру інваріантності $AG_2(1,1)$ лінійного рівняння (3).

Виявляється, що висліди, одержані в теоремах 1, 2, 3, неможливо тривіальним чином узагальнити на багатовимірний випадок. Про це свідчить

Теорема 4. *В класі нелінійних $(n+1)$-вимірних еволюційних систем*

$$\lambda_k\psi_t^{(k)} = A_{ab}^{k1}(\psi^{(1)},\psi^{(2)},\underset{1}{\psi}^{(1)},\underset{1}{\psi}^{(2)})\psi_{ab}^{(1)} + A_{ab}^{k2}(\psi^{(1)},\psi^{(2)},\underset{1}{\psi}^{(1)},\underset{1}{\psi}^{(2)})\psi_{ab}^{(2)} + \\ + B^{(k)}(\psi^{(1)},\psi^{(2)},\underset{1}{\psi}^{(1)},\underset{1}{\psi}^{(2)}), \tag{18}$$

де $k=1,2$, $\psi^{(k)}=\psi^{(k)}(t,x_1,\ldots,x_n)$, $\underset{1}{\psi}^{(k)}=(\psi_1^{(k)},\ldots,\psi_n^{(k)})$, $\psi_a^{(k)}=\frac{\psi^{(k)}}{\partial x_a}$, $\psi_{ab}^{(k)}=\frac{\partial^2\psi^{(k)}}{\partial x_a\partial x_b}$, $A_{ab}^{kk_1}$, $B^{(k)}$ — достатньо гладкі функції від $2(n+1)$ аргументів, інваріантними відносно алгебри Галілея $AG(1,n)$ [4] є тільки системи вигляду

$$\lambda_k\psi_t^{(k)} = A_0^{kk}\left(\Delta\psi^{(k)} - \frac{\psi_a^{(k)}\psi_a^{(k)}}{\psi^{(k)}}\right) + \frac{\psi^{(k)}}{\psi^{(k_1)}}A_0^{kk_1}\left(\Delta\psi^{(k_1)} - \frac{\psi_a^{(k_1)}\psi_a^{(k_1)}}{\psi^{(k_1)}}\right) + \\ + A^{kk}\frac{\partial v}{\partial x_a}\frac{\partial v}{\partial x_b}\left(\psi_{ab}^{(k)} - \frac{\psi_a^{(k)}\psi_b^{(k)}}{\psi^{(k)}}\right) + \\ + \frac{\psi^{(k)}}{\psi^{(k_1)}}A^{kk_1}\frac{\partial v}{\partial x_a}\frac{\partial v}{\partial x_b}\left(\psi_{ab}^{(k_1)} - \frac{\psi_a^{(k_1)}\psi_b^{(k_1)}}{\psi^{(k_1)}}\right) + \\ + \psi^{(k)}B_0^{(k)} + \frac{\psi_a^{(k)}\psi_a^{(k)}}{\psi^{(k)}}, \quad k,k_1=1,2, \quad k\neq k_1, \tag{19}$$

де $A_0^{k_1k_2}$, $A^{k_1k_2}$, $B_0^{(k)}$ — довільні функції аргументів $v=(\psi^{(1)})^{\lambda_2}(\psi^{(2)})^{-\lambda_1}$, $\theta=\frac{\partial v}{\partial x_a}\frac{\partial v}{\partial x_a}$. За індексами a, b, що повторюються, слід підсумовувати від 1 до n, $k_2=1,2$.

Наслідок 5. *У випадку, коли система рівнянь (19) являє собою пару комплексно спряжених рівнянь, одержуємо клас нелінійних узагальнень рівняння Шредінгера*

$$i\psi_t = A^{(1)}\left(\Delta\psi - \frac{\psi_a\psi_a}{\psi}\right) + \frac{\psi}{\psi^*}A^{(2)}\left(\Delta\psi^* - \frac{\psi_a^*\psi_a^*}{\psi^*}\right) + \psi B + \frac{\psi_a\psi_a}{\psi} + \\ + A^{(3)}(\psi\psi^*)_a(\psi\psi^*)_b\left(\psi_{ab} - \frac{\psi_a\psi_b}{\psi}\right) + \frac{\psi}{\psi^*}A^{(4)}(\psi\psi^*)_a(\psi\psi^*)_b\left(\psi_{ab}^* - \frac{\psi_a^*\psi_b^*}{\psi^*}\right), \tag{20}$$

які зберігають алгебру $AG(1,n)$ лінійного $(n+1)$-вимірного рівняння Шредінгера.

У рівняннях (20) $A^{(j)}$, $j=1,2,3,4$, B — довільні функції від двох аргументів $\psi\psi^*$, $(\psi\psi^*)_a(\psi\psi^*)_a$, $(\psi\psi^*)_a\equiv\frac{\partial(\psi\psi^*)}{\partial x_a}$. Зокрема, клас рівнянь (20) містить такі нетривіальні узагальнення рівняння Шредінгера, які не мають аналогів у класі двовимірних рівнянь (2), як

$$i\psi_t = k\Delta\psi + \lambda|\psi|^\alpha|\psi|_a|\psi|_b\left(\psi_{ab} - \frac{\psi_a\psi_b}{\psi}\right),$$

$$i\psi_t = k\Delta\psi + \lambda\frac{|\psi|_a|\psi|_b}{|\psi|_{a_1}|\psi|_{a_1}}\left(\psi_{ab} - \frac{\psi_a\psi_b}{\psi}\right),$$

$$i\psi_t = k\Delta\psi + \lambda|\psi|_a|\psi|_b\left(\psi_{ab} - \frac{\psi_a\psi_b}{\psi} + \frac{\psi}{\psi^*}\left(\psi^*_{ab} - \frac{\psi^*_a\psi^*_b}{\psi^*}\right)\right),$$

де індекси $a, a_1, b = 1, 2, \ldots, n$, $|\psi|^2_a \equiv (\psi\psi^*)_a$, $\lambda, \alpha \in \mathbb{C}$, $k \in \mathbb{R}$.

Серед класу рівнянь (20) вдається виділити підклас рівнянь вигляду

$$i\psi_t = \Delta\psi + A_0\left[\Delta\psi - \frac{\psi_a\psi_a}{\psi} + \frac{\psi}{\psi^*}\left(\Delta\psi^* - \frac{\psi^*_a\psi^*_a}{\psi^*}\right)\right] + \psi|\psi|^{4/n}B_0 + + \frac{|\psi|_a|\psi|_b}{|\psi|^{2+4/n}}A\left[\psi_{ab} - \frac{\psi_a\psi_b}{\psi} + \frac{\psi}{\psi^*}\left(\psi^*_{ab} - \frac{\psi^*_a\psi^*_b}{\psi^*}\right)\right], \tag{21}$$

які зберігають симетрію $AG_2(1,n)$ лінійного $(n+1)$-вимірного рівняння Шредінгера. В рівнянні (21) A_0, A і B_0 — довільні функції від аргумента $|\psi|_a|\psi|_a|\psi|^{-2-4/n}$, який є диференціальним інваріантом узагальненої алгебри Галілея $AG_2(1,n)$ [4]. Наведемо кілька прикладів $AG_2(1,n)$-інваріантних узагальнень рівняння Шредінгера, які не мають аналогів у класі двовимірних рівнянь (2), а саме:

$$i\psi_t = k\Delta\psi + \lambda\frac{|\psi|_a|\psi|_b}{|\psi|_{a_1}|\psi|_{a_1}}\frac{|\psi|^2_{ab}}{|\psi|^2}\psi,$$

$$i\psi_t = \Delta\psi + \alpha\frac{\Delta|\psi|^2}{|\psi|^2}\psi + \lambda\frac{|\psi|_a|\psi|_b}{|\psi|^{4+4/n}}|\psi|^2_{ab}\psi, \tag{22}$$

$$i\psi_t = -k\frac{\psi}{\psi^*}\Delta\psi^* + k\left(\frac{\psi_a\psi_a}{\psi^2} + \frac{\psi^*_a\psi^*_a}{\psi^{*2}}\right)\psi + \lambda\frac{|\psi|_a|\psi|_b}{|\psi|_{a_1}|\psi|_{a_1}}\frac{|\psi|^2_{ab}}{|\psi|^2}\psi.$$

Для побудови рівнянь (22) було використано тотожності

$$|\psi|_{ab} = \frac{\partial^2(\psi\psi^*)}{\partial x_a\partial x_b},$$

$$\Delta\psi - \frac{\psi_a\psi_a}{\psi} + \frac{\psi}{\psi^*}\left(\Delta\psi^* - \frac{\psi^*_a\psi^*_a}{\psi^*}\right) \equiv \frac{\Delta|\psi|^2 - 4|\psi|_a|\psi|_a}{|\psi|^2}\psi,$$

$$\left[\psi_{ab} - \frac{\psi_a\psi_b}{\psi} + \frac{\psi}{\psi^*}\left(\psi^*_{ab} - \frac{\psi^*_a\psi^*_b}{\psi^*}\right)\right] \equiv \left[|\psi|_a|\psi|_b|\psi|^2_{ab} - 4(|\psi|_a|\psi|_a)^2\right]\frac{\psi}{|\psi|^2}.$$

На закінчення відзначимо, що наслідки 1–3 можна одержати шляхом конструювання рівнянь (8), (12), (13) за допомогою диференціальних інваріантів алгебр $AG(1,1)$, $AG_1(1,1)$, $AG_2(1,1)$, повний набір яких описано в роботі [9].

Детальнішому розгляду деяких конкретних еволюційних систем, описаних у теоремах 1–3, у випадку дійсних функцій $\psi^{(1)}$, $\psi^{(2)}$ буде присвячена окрема публікація.

1. Хенри Д., Геометрическая теория полулинейных параболических уравнений, М., Мир, 1985, 376 с.
2. Dodd R.K., Fordy A.P., Prolongation structures of complex quasi-polinomial evolution equations, *J. Phys. A: Math. Gen.*, 1984, **17**, № 16, 3249–3266.
3. Niederer U., The maximal kinematical invariance group of the free Schrödinger equation, *Helv. Phys. Acta*, 1972, **45**, № 5, 808–816.

4. Фущич В.И., Чернига Р.М., Галилей-инвариантные нелинейные уравнения шредингеровского типа и их точные решения I, II, *Укр. мат. журн.*, 1989, **41**, 1349–4357, 1687–1694.

5. Кричлоу Г.Б., Современная разработка нефтяных месторождений — проблемы моделирования, М., Недра, 1979, 303 с.

6. Malomed B. A., Stenflo L., Modulational instabilities and soliton solutions of a generalized nonlinear Schrödinger equation, *J. Phys. A: Math. Gen.*, 1991, **24**, L1149–L1153.

7. Doebner H.-D., Goldin G.A., On a general nonlinear Schrödinger equation admitting diffusion currents, *Phys. Lett. A*, 1992, **162**, 397–401.

8. Фущич В.И., Чернига Р.М., О точных решениях двух многомерных нелинейных уравнений шредингеровского типа, Препринт № 86.85, Киев, Ин-т математики АН Украины, 1986, 44 с.

9. Фущич В.I., Єгорченко I.А., Диференцiальнi iнварiанти алгебри Галiлея, *Доп. АН України, Сер. А*, 1989, № 4, 29–32.

10. Kundu A., Landau–Lifshitz and higher-order nonlinear systems gauge generated from nonlinear Schrödinger type equations, *J. Math. Phys.*, 1984, **25**, № 12, 3433–3438.

11. Calogero F., De Lillo S., Cauchy problems on the semiline and on a finite interval for the Eckhaus equation, *Inverse Problems*, 1988, **4**, L33–L37.

Умовна симетрія та нові зображення алгебри Галілея для нелінійних рівнянь параболічного типу

В.І. ФУЩИЧ, В.І. ЧОПИК

An effective method for finding conditional symmetry operators is constructed for a class of Galilei noninvariant parabolic equations. The obtained operators form a basis of the Galilei algebra. The additional conditions, under which the extension of a symmetry is possible, are obtained. For the equations under consideration, the anti-reduction is carried out and some exact solutions are found by using the conditional Galilei-invariance of its differential consequences.

Для класу Галілей-неінваріангних рівнянь параболічного типу запропоновано конструктивний метод знаходження операторів умовної симетрії, які утворюють базис алгебри Галілея. Описані додаткові умови, при яких можливе розширення симетрії. Проведено антиредукцію, а також знайдені деякі точні розв'язки розглядуваного нелінійного рівняння, виходячи з умовної галілей-інваріантності його диференціальних наслідків.

Вступ. В роботі [1] вказано на такий парадоксальний факт: серед нелінійних рівнянь теплопровідності

$$u_0 + \partial_a(f_1(x,u)u_a) = f_2(x,u), \tag{1}$$

де

$$u_0 = \partial/\partial x_0,\ x_0 \equiv t,\ \partial_a = \partial/\partial x_a,\ u_a = \partial/\partial x_a,\ x = (x_1, \ldots, x_n),\ a = \overline{1,n},$$

які широко застосовуються в різних областях математики та фізики, немає жодного рівняння, для якого б виконувався принцип відносності Галілея. З симетрійної точки зору це значить, що рівняння (1) ні при яких f_1, f_2 таких, що $f_1 \neq \mathrm{const}$, $f_2 \neq \mathrm{const}$ одночасно, не допускає алгебри Галілея. Але виявляється, що з множини розв'язків рівняння (1) можна виділити підмножини, які залишаються інваріантними при перетвореннях Галілея. Природньо постає питання знаходження цих розв'язків. Ці підмножини розв'язків можна знаходити, використовуючи оператори умовної симетрії рівняння [2], а також, як буде показано нижче, оператори умовної галілей-інваріантності його диференційних наслідків.

Існує конструктивний метод для знаходження операторів Q-умовної симетрії [2]. Основним недоліком цих операторів є те, що вони не утворюють алгебри Лі. В роботах [3, 4] побудовано деякі оператори умовної симетрії для рівнянь типу (1), які утворюють алгебру разом з базисними операторами симетрії Лі, цього рівняння. Зауважимо, що алгоритмічного методу знаходження операторів умовної симетрії, які утворювали б алгебру Лі, до цього часу не існує.

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В даній роботі на прикладі нелінійного рівняння

$$(\partial_0 + u_a\partial_a)h(x,u) - \Delta u = F(u), \tag{2}$$

де

$$h_u \neq 0, \quad \Delta = \partial^2/\partial x_a\partial x_a, \quad a = \overline{1,n},$$

n — число просторових змінних, запропоновано конструктивний метод знаходження алгебри умовної інваріантності. Цей метод грунтується на вимозі того, щоб оператори симетрії Лі рівняння разом з операторами умовної симетрії утворювали базис алгебри Галілея. Причому розглядаються різні зображення алгебри Галілея, які допускає рівняння типу (2). Завдяки цьому вдається описати додаткові умови, при яких можливі такі розширення симетрії.

Процес знаходження алгебри умовної інваріантності ми розіб'ємо на кілька етапів: перший етап — виділення з класу рівнянь (2) рівняння, для якого можливе розширення симетрії до алгебри Галілея; другий етап — знаходження зображення алгебри Галілея, інваріантність відносно якого ми будемо вимагати від виділеного нами рівняння; третій етап — знаходження умов, при яких наше рівняння умовно галілей-інваріантне.

Очевидно, що наша робота буде мати зміст лише у тому випадку, коли одержана перевизначена система рівнянь буде сумісною. У цій статті питання сумісності ми окремо досліджувати не будемо, але наведемо деякі нетривіальні розв'язки одержаних перевизначених систем.

1. Інваріантність відносно перетворень типу Галілея. Опишемо всі дійсні функції h, F такі, при яких рівняння (2) інваріантне відносно операторів

$$X_a = f(x_0)\partial_a + g(x_0)x_a\partial_u, \quad \partial_u = \partial/\partial u, \tag{3}$$

які породжують такі скінченні перетворення:

$$x_0 \to x_0' = x_0, \quad x_a \to x_a' = x_a + v_a f(x_0),$$
$$u \to u' = u + (x_a\nu_a + (1/2)v^2 f(x_0))g(x_0).$$

Зауважимо, що ці перетворення при $f(x_0) = x_0$ співпадають з перетвореннями Галілея.

За формулами Лі (див., наприклад, [2]) знайдемо друге продовження операторів (3):

$$\overset{(2)}{X} = X + (g'x_a - f'u_a)\partial_{u_0} + g\partial_{u_a}, \quad \text{де} \quad \partial_{u_0} = \partial/\partial_{u_0}, \quad \partial_{u_a} = \partial/\partial_{u_a},$$

і подіємо ним на рівняння (2). В результаті одержимо такі умови:

$$h_{uu}gx_a + h_{ua}f = 0, \quad a = \overline{1,n}, \tag{4}$$

$$-f'h_u + 2gh_u + fh_{aa} + gx_a h_{au} = 0, \tag{5}$$

$$g'h_u x_a + h_a g - F'gx_a = 0. \tag{6}$$

Для знаходження розв'язку системи рівнянь (4)–(6) розглянемо два таких випадки: $h_u = \mathrm{const}$ та $h_u \neq \mathrm{const}$.

Випадок 1: $h_u = \text{const}$. Не зменшуючи загальності, можна прийняти, що $h_u = 1$. У цьому випадку рівняння (4) виконується тотожньо. Оскільки $h = h(u, x)$, а функції f і g залежать тільки від x_0 то рівняння (5) розпадається на такі рівняння:

$$h_{aa} = \beta, \quad f' - 2g = \beta f, \quad \beta \in \mathbb{R}. \tag{7}$$

Диференціюючи рівняння (6) по u, одержуємо умову на F:

$$F'' = 0 \;\Leftrightarrow\; F = \alpha u + \alpha_1, \quad \{\alpha, \alpha_1\} \subset \mathbb{R}. \tag{8}$$

Враховуючи (7), (8), після диференціювання рівняння (6) по x_a маємо

$$g' + (\beta - \alpha) g = 0 \;\Rightarrow\; g = \lambda \exp\{(\alpha - \beta) x_0\}. \tag{9}$$

Підставляючи (9) в (7), одержуємо умову на $f(x_0)$:

$$f' - \beta f = 2\lambda \exp\{(\alpha - \beta) x_0\}.$$

Розв'язок останнього рівняння знаходиться у вигляді $f = A(x_0) \exp\{\beta x_0\}$, де $A(x_0)$ задовольняє рівняння

$$A' = 2\lambda \exp\{(\alpha - \beta) x_0\}.$$

Інтегруючи це рівняння, маємо

$$A = \begin{cases} \dfrac{2\lambda}{\alpha - 2\beta} \exp\{(\alpha - 2\beta) x_0\} + \lambda_1 & \text{при} \quad \alpha \neq 2\beta, \\ 2\lambda x_0 + \lambda_1 & \text{при} \quad \alpha = 2\beta. \end{cases}$$

Остаточно одержуємо

$$f = \begin{cases} \dfrac{2\lambda}{\alpha - 2\beta} \exp\{(\alpha - 2\beta) x_0\} + \lambda_1 \exp\{\beta x_0\} & \text{при} \quad \alpha \neq 2\beta, \\ 2\lambda x_0 \exp\{\beta x_0\} + \lambda_1 \exp\{\beta x_0\} & \text{при} \quad \alpha = 2\beta. \end{cases} \tag{10}$$

Підсумовуючи попередні результати, одержуємо, що рівняння

$$u_0 + u_a u_a + \beta x_a u_a - \Delta u = \alpha u, \quad \alpha \in \mathbb{R}, \tag{11}$$

допускає оператори X_a, що визначаються (3), (9), (10) (використовуючи заміну $u = u' - \alpha_1/\alpha$, константу α_1 в (8) можна прирівняти до нуля).

Зауваження 1. Рівняння (11) при $\beta = \alpha/2 - 1$ співпадає з рівнянням ренорм-групи (RG) Вільсона [5, 6]. Симетрія Лі цього рівняння знайдена в [7]. Підкреслимо, що випадок $\alpha = 2\beta$ не виконується для рівняння RG Вільсона.

2. При $\beta = 0$ рівняння (11) заміною

$$u = \ln v, \tag{12}$$

зводиться до нелінійного рівняння теплопровідності

$$v_0 - \Delta v = \alpha v \ln v. \tag{13}$$

Симетрійні властивості цього рівняння досліджені в роботі [4]. Далі ми покажемо, що рівняння (13) може допускати ще одне зображення алгебри Галілея.

Знайдемо максимальну алгебру інваріантності (MAI) рівняння (11).

Теорема 1. *MAI рівняння (11) задається таким набором базисних операторів:*

$$\begin{aligned} 1)\quad & P_0 = \partial_0, \quad X_a^{(1)} = \exp\{\beta x_0\}\partial_a \quad J_{ab} = x_a\partial_b - x_b\partial_a, \quad M = \exp\{\alpha x_0\}\partial_u, \\ & X_a^{(2)} = \exp\{(\alpha-\beta)x_0\}\left\{\frac{2}{\alpha - 2\beta}\partial_a + x_a\partial_u\right\} \quad \text{при} \quad \alpha \neq 2\beta; \end{aligned} \tag{14}$$

$$\begin{aligned} 2)\quad & P_0, \quad X_a^{(1)} = \exp\{\beta x_0\}\partial_a, \quad J_{ab}, \quad M = \exp\{2\beta x_0\}\partial_u, \\ & X_a^{(2)} = \exp\{\beta x_0\}\{2x_0\partial_a + x_a\partial_u\} \quad \text{при} \quad \alpha = 2\beta. \end{aligned} \tag{15}$$

Доведення теореми проводиться за схемою Лі [2]. Оператори $X_a^{(1)}$, $X_a^{(2)}$ одержуються із X_a при $\lambda = 0$ та $\lambda_1 = 0$ відповідно. Базисні оператори (14) задовольняють такі комутаційні співвідношення:

$$\begin{aligned} & [P_0, X_a^{(1)}] = \beta X_a^{(1)}, \quad [P_0, M] = \alpha M, \\ & [P_0, X_a^{(2)}] = \begin{cases} cX_a^{(2)} & \text{при} \quad \alpha \neq \beta, \ \alpha \neq 2\beta, \\ 0 \quad \text{при} \quad \alpha = 2\beta, \end{cases} \\ & [P_0, J_{ab}] = [X_a^{(1)}, M] = [J_{ab}, M] = [X_a^{(1)}, X_b^{(1)}] = [X_a^{(2)}, X_b^{(2)}] = 0, \\ & [X_a^{(1)}, X_b^{(2)}] = \delta_{ab}M, \quad [X_a^{(1)}, J_{bc}] = \delta_{ab}X_c^{(1)} - \delta_{ab}X_b^{(1)}, \\ & [X_a^{(2)}, J_{bc}] = \delta_{ab}X_c^{(2)} - \delta_{ac}X_b^{(2)}, \\ & [J_{ab}, J_{cd}] = \delta_{ac}J_{bd} + \delta_{bd}J_{ac} - \delta_{bc}J_{ad} - \delta_{ad}J_{bc}, \quad c \in \mathbb{R}. \end{aligned}$$

З цих співвідношень випливає, що у кожному з випадків: 1) $\beta = 0$; 2) $\alpha = \beta$; 3) $\alpha \neq \beta$; $\alpha \neq 2\beta$; рівнянню (11) відповідає інша алгебра Лі. Жодна з цих алгебр не є алгеброю Галілея (про алгебру Галілея див. [2, 8]).

Для випадку, коли $\alpha = 2\beta$, оператори (15) задовольняють співвідношення

$$\begin{aligned} & [P_0, X_a^{(1)}] = \beta X_a^{(1)}, \quad [P_0, X_a^{(2)}] = \beta X_a^{(2)} + 2X_a^{(1)}, \quad [P_0, M] = 2\beta M, \\ & [P_0, J_{ab}] = [X_a^{(1)}, M] = [J_{ab}, M] = [X_a^{(1)}, X_b^{(1)}] = [X_a^{(2)}, X_b^{(2)}] = 0, \\ & [X_a^{(1)}, X_b^{(2)}] = \delta_{ab}M, \quad [X_a^{(1)}, J_{bc}] = \delta_{ab}X_c^{(1)} - \delta_{ac}X_b^{(1)}, \\ & [X_a^{(2)}, J_{bc}] = \delta_{ab}X_c^{(2)} - \delta_{ac}X_b^{(2)}, \\ & [J_{ab}, J_{cd}] = \delta_{ac}J_{bd} + \delta_{bd}J_{ac} - \delta_{bc}J_{ad} - \delta_{ad}J_{bc}. \end{aligned} \tag{16}$$

З (16) випливає, що при $\beta = 0$ алгебра, що породжується операторами (15), задовольняє стандартні комутаційні співвідношення алгебри Галілея $AG(1,n)$. Але цей випадок не викликає зацікавлення, оскільки при $\alpha = 2\beta = 0$ рівняння (11) заміною (12) зводиться до лінійного рівняння теплопровідності (13) ($\alpha = 0$). Надалі при $\alpha = 2\beta$ ми розглядатимемо тільки випадок, коли $\beta \neq 0$. У цьому випадку з (16) випливає, що рівняння (11) також не допускає алгебри Галілея.

Випадок 2: $h_u \neq \text{const}$. Для цього випадку розв'язок системи (4)–(6) задається так:

$$h_u = d_1(u - \lambda x^2/2), \quad h_a = \lambda d_1 x_a(5\lambda x^2 - u) + d_2, \quad F' = \alpha, \quad d_k \in \mathbb{R},$$

а функції f і g із (3) мають вигляд

$$f = d\exp\{\lambda x_0\}, \quad g = \lambda d\exp\{\lambda x_0\}, \quad d \in \mathbb{R}.$$

Тобто рівняння

$$h_u(u_0 + u_a u_a) + h_a u_a - \Delta u = \alpha u$$

інваріантне відносно перетворень типу Галілея, що породжуються (3). Надалі ми обмежимося розглядом рівняння (11).

2. Знаходження потрібних представлень алгебри Галілея. З комутаційних співвідношень для (14), (15) випливає, що всі оператори, за винятком P_0 задовольняють комутаційні співвідношення алгебри Галілея. В роботі [4] показано, що для алгебри Лі рівняння (13) (частковий випадок рівняння (11)) можна вказати такий оператор P_t, що

$$[P_t, X_a^{(1)}] = [P_t, M] = [P_t, J_{ab}] = 0, \quad [P_t, X_a^{(2)}] = c_1 X_a^{(1)}. \tag{17}$$

Легко бачити, що при виконанні (17) оператори P_t, $X_a^{(1)}$, $X_a^{(2)}$, M, J_{ab} утворюють базис алгебри Галілея, яку ми позначатимемо $AG^{(1)}(1,n)$. Нижче ми узагальнимо цей результат для рівняння (11) та вкажемо нове зображення алгебри Галілея $AG^{(1)}(1,n)$, що визначається співвідношеннями, відмінними від (17).

Випадок 1. Знайдемо явний вигляд оператора P_t такого, що виконуються співвідношення (17). Шукатимемо оператор P_t у вигляді

$$P_t = \xi^0 \partial_0 + \xi^a \partial_a + \eta \partial_u, \quad \xi^\mu = \xi^\mu(x_0, x, u), \quad \eta = \eta(x_0, x, u), \quad \mu = \overline{0, n}. \tag{18}$$

З умови $[P_t, X_a^{(1)}] = 0$ одержимо такі умови на коефіцієнтні функції оператора P_t:

$$\beta \xi^0 = \xi_a^a, \quad \xi_a^0 = \xi_a^b = \eta_a = 0, \quad a \neq b. \tag{19}$$

Аналогічно одержуємо такі умови на функції ξ^μ, η:

$$\beta \xi^0 = \eta_u, \quad \xi^a = x_a \xi_a^a, \quad \xi_u^0 = \xi_u^a = 0, \quad (\alpha - \beta)\xi^0 x_a + \xi^a - x_a \eta_u = 0; \tag{20}$$

а також

$$\begin{aligned} &2\exp\{(\alpha - 2\beta)x_0\}((\alpha - \beta)\xi^0 - \xi_a^a)/(\alpha - 2\beta) = c_1 \quad \text{при} \quad \alpha \neq 2\beta, \\ &2(\beta x_0 + 1)\xi^0 - 2x_0 \xi_a^a = c_1 \quad \text{при} \quad \alpha = 2\beta, \quad c_1 \in \mathbb{R}. \end{aligned} \tag{21}$$

Розв'язуючи умови (19)–(21), знаходимо явний вигляд оператора P_t. Для випадків, коли $\alpha \neq 2\beta$ та $\alpha = 2\beta$, цей оператор можна записати у єдиному вигляді:

$$P_t = \exp\{(2\beta - \alpha)x_0\}(\partial_0 + \beta x_a \partial_a + (\alpha u + f(x_0))\partial_u). \tag{22}$$

В результаті ми довели таке твердження.

Лема 1. *Оператори P_t, $X_a^{(1)}$, M, J_{ab}, $X_a^{(2)}$, що мають вигляд (14), (22) при $\alpha \neq 2\beta$ та (15), (22) при $\alpha = 2\beta$, задають базис алгебри Галілея $AG^{(1)}(1,n)$, що визначається комутаційними співвідношеннями (16), (17).*

Випадок 2. Нехай оператор T такий, що виконуються

$$[T, X_a^{(2)}] = [T, M] = [T, J_{ab}] = 0, \quad [T, X_a^{(1)}] = c_1 X_a^{(2)}. \tag{23}$$

Легко переконатися, що при виконанні (23) оператори T, $X_a^{(1)}$, $X_a^{(2)}$, M, J_{ab} утворюватимуть базис алгебри Галілея, яку ми надалі позначатимемо $AG^{(2)}(1,n)$.

Оператор T шукатимемо у вигляді (18). З (23) одержимо такі умови на функції ξ^μ, η:

$$\xi^0_a = \xi^b_a = \xi^0_u = \xi^a_u = 0, \quad a \neq b, \quad \xi^a = x_a \xi^a_a, \tag{24}$$

а також

$$\begin{aligned} &\xi^a_a = -c_2 2 \exp\{(\alpha-\beta)x_0\}/(\alpha - 2\beta), \quad (\alpha-\beta)\xi^0 = \xi^a_a, \\ &(\alpha-\beta)x_a\xi^0 + \xi^a - 2\eta_a/(\alpha-2\beta) - x_a\eta_u = 0, \quad \alpha\xi^0 = \eta_u, \\ &\eta_a = -c_2 \exp\{(\alpha-\beta)x_0\} x_a \quad \text{при} \quad \alpha \neq 2\beta, \end{aligned} \tag{25}$$

$$\begin{aligned} &\eta_a = -c_2 x_a, \quad \beta x_0 \xi^0 + \xi^0 - x_0 \xi^a_a = 0, \\ &2\beta\xi^0 = \eta_u, \quad \beta\xi^0 - \xi^a_a = 2c_2 x_0, \\ &\beta x_a \xi^0 + \xi^a - 2x_0\eta_a - x_a\eta_u = 0 \quad \text{при} \quad \alpha = 2\beta. \end{aligned} \tag{26}$$

З умов (24), (25) одержуємо явний вигляд T при $\alpha \neq 2\beta$:

$$T = \exp\{(\alpha-\beta)x_0\}\left(\partial_0 + (\alpha-\beta)x_a\partial_a + \left(\alpha u + (\alpha-2\beta)^2\frac{x^2}{4} + f_1(x_0)\right)\partial_u\right). \tag{27}$$

З (24), (26) випливає

$$T = x_0^2\partial_0 + (\beta x_0 + 1)x_0 x_a \partial_a + \left(\frac{x^2}{4} + 2\beta x_0^2 u + f_2(x_0)\right)\partial_u \quad \text{при} \quad \alpha = 2\beta. \tag{28}$$

Лема 2. *Оператори T, $X^{(1)}_a$, M, J_{ab}, $X^{(2)}_a$, що мають вигляд (14), (21) при $\alpha \neq 2\beta$ та (15), (28) при $\alpha = 2\beta$, реалізують представлення алгебри Галілея $AG^{(2)}(1,n)$. Алгебра $AG^{(2)}(1,n)$ задається комутаційними співвідношеннями (16), (23).*

3. Умовна галілей-інваріантність рівняння (11).

Випадок 1. Вимагатимемо інваріантність рівняння (11) відносно алгебри Галілея $AG^{(2)}(1,n)$. Для цього подіємо другим продовженням оператора (22):

$$\begin{aligned} \overset{(2)}{P}_t = P_t + \exp\{(2\beta-\alpha)x_0\}[\{f' + (2\beta-\alpha)(\alpha u + f) - \beta(2\beta-\alpha)x_a u_a + \\ + 2(\alpha-\beta)u_0\}\partial_{u_0} + (\alpha-\beta)u_a\}\partial_{u_a} + (\alpha-2\beta)u_{aa}\partial_{u_{aa}})] \end{aligned}$$

на рівняння (11). Одержуємо:

$$\begin{aligned} \overset{(2)}{P}_t\{u_0 + u_a u_a + \beta x_a u_a - \Delta u - \alpha u\} = 2(\alpha-\beta)\exp\{(2\beta-\alpha)x_0\} \times \\ \times \{u_0 + u_a u_a + \beta x_a u_a - \Delta u - \alpha u\} + \exp\{(2\beta-\alpha)x_0\} \times \\ \times [\alpha\Delta u + 2(\beta-\alpha)f + f']. \end{aligned} \tag{29}$$

З (29) випливає, що при $\alpha = 0$, $2\beta f + f' = 0$ рівняння (11) інваріантне (у сенсі Лі) відносно $AG^{(1)}(1,n)$. Але при $\alpha = 0$ рівняння (11) заміною (12) зводиться до лінійного рівняння

$$v_0 + \beta x_a v_a - \Delta v = 0,$$

симетрія якого нескінченна [7]. Надалі ми обмежимось розглядом таких α, що $\alpha \neq 0$.

Подіємо $\overset{(2)}{P}_t$ на рівняння

$$\alpha\Delta u + 2(\beta - \alpha)f + f' = 0. \tag{30}$$

Одержимо

$$\begin{aligned} &\overset{(2)}{P}_t\{\alpha\Delta u + 2(\beta - \alpha)f + f'\} = \\ &\qquad = (\alpha - 2\beta)\exp\{(\alpha - 2\beta)x_0\}\{\alpha\Delta u + 2(\beta - \alpha)f + f'\}, \end{aligned} \tag{31}$$

якщо функція $f_0(x)$ задовольняє рівняння:

$$f'' + (4\beta - 3\alpha)f' + 2(\alpha - \beta)f = 0. \tag{32}$$

Згідно з критерієм умовної інваріантності (див., наприклад, [2, 3]), з (29), (31) випливає, що рівняння (11) умовно інваріантне відносно оператора P_t (22), якщо f задовольняє (32), тобто

$$f = c_1\exp\{(\alpha - 2\beta)x_0\} + c_2\exp\{(\alpha - \beta)x_0\}, \tag{33}$$

причому додаткова умова (30) з урахуванням (33) має вигляд

$$\Delta u = c_1\exp\{(\alpha - 2\beta)x_0\}. \tag{34}$$

Теорема 2. *Рівняння (11) умовно інваріантне відносно алгебри Галілея $AG^{(1)}(1,n) = \langle P_t, X_a^{(1)}, M, J_{ab}, X_a^{(2)}\rangle$, де оператор P_t задається (22), (33), а оператори $X_a^{(1)}$, M, J_{ab}, $X_a^{(2)}$ мають вигляд (14) при $\alpha \neq 2\beta$ та (15) при $\alpha = 2\beta$. Додаткова умова має вигляд (34).*

Для доведення теореми нам залишається перевірити інваріантність додаткові умови (34) відносно операторів $X_a^{(1)}$, M, J_{ab}, $X_a^{(2)}$. Легко переконатись, що ці оператори є абсолютними інваріантами для цього рівняння. Цей факт підтверджує справедливість теореми.

Зауваження 3. Якщо вимагати додатково інваріантність відносно оператора P_0, то з (34) випливає умова $c_1 = 0$. В роботі [4] знайдено МАІ перевизначеної системи рівнянь (11), (34) при $\beta = c_1 = 0$. Загальний розв'язок цієї системи має вигляд

$$u = (d_0 + d_a x_a - d^2\alpha^{-1}\exp\{\alpha x_0\})\exp\{\alpha x_0\},$$

де

$$d^2 = d_a d_a, \quad d_\mu \in \mathbb{R}, \quad a = \overline{1,n}, \quad \mu = \overline{0,n}.$$

Випадок 2. Вимагатимемо інваріантність рівняння (11) відносно алгебри Галілея $AG^{(2)}(1,n)$ при $\alpha \neq 2\beta$. Для цього подіємо другим продовженням оператора T (27) на рівняння (11). Одержимо

$$\begin{aligned} &\overset{(2)}{T}\{u_0 + u_a u_a + \beta x_a u_a - \Delta u - \alpha u\} = \\ &\qquad = \exp\{(\alpha - \beta)x_0\}[\beta\{u_0 + u_a u_a + \beta x_a u_a - \Delta u - \alpha u\} + L_1], \end{aligned}$$

де

$$L_1 = \beta u_a u_a + \beta(2\beta-\alpha)x_a u_a - (\beta-\alpha)\Delta u + \\ + (\beta/4)(\alpha-2\beta)^2 x^2 - \beta f_1 + f_1' - (n/2)(\alpha-2\beta)^2.$$

Справедлива рівність

$$\overset{(2)}{T}\{L_1\} = \exp\{(\alpha-\beta)x_0\}[2\beta L_1 + L_2], \tag{35}$$

причому

$$L_2 = \alpha(\beta-\alpha)\Delta u + f_1^{''} - \beta f_1' + 2\beta^2 f_1 + (n/2)(\alpha-2\beta)^2(\alpha+\beta). \tag{36}$$

З (35), (36) випливає, що рівняння $L_1 = 0$ інваріантне (у сенсі Лі) відносно оператора T лише у таких випадках:

$$1)\quad \alpha=\beta,\quad f_1'' - \beta f_1' + 2\beta^2 f_1 + (n/2)(\alpha-2\beta)^2(\alpha+\beta) = 0; \tag{37}$$

$$2)\quad \beta=0,\quad f'' + \alpha f_1' = 0. \tag{38}$$

Для випадку (38) справедлива теорема.

Теорема 3. *Рівняння (11) при $\alpha \neq 2\beta$ та $\beta = 0$ умовно інваріантне відносно алгебри Галілея $AG^{(2)}(1,n) = \langle T, X_a^{(1)}, M, J_{ab}, X_a^{(2)}\rangle$, де*

$$T = \exp\{\alpha x_0\}\left(\partial_0 + \alpha x_a\partial_a + \left(\alpha u + \alpha^2\frac{x^2}{4} + f_1(x_0)\right)\partial_u\right),\quad X_a^{(1)} = \partial_a, \\ J_{ab} = x_a\partial_b - x_b\partial_a,\quad M = \exp\{\alpha x_0\}\partial_u,\quad X_a^{(2)} = \exp\{\alpha x_0\}\{(2/\alpha)\partial_a + x_a\partial_u\}, \tag{39}$$

причому додаткова умова має вигляд

$$L_1 = \alpha\Delta u + f_1' - (n/2)\alpha^2 = 0, \tag{40}$$

де функція f_1 задовольняє лінійне рівняння (38).

При $f_1 = \mathrm{const}$ рівняння (11), (40) інваріантні відносно алгебри $AG^{(2)}(1,n)$ (39), доповненої оператором P_0.

Наслідок. *Нелінійне рівняння теплопровідності (13) умовно інваріантне відносно таких алгебр:*

$$1)\quad AG^{(2)}(1,n) = \langle T, X_a^{(1)}, M, J_{ab}, X_a^{(2)}\rangle,$$

де

$$T = \exp\{\alpha x_0\}\left(\partial_0 + \alpha x_a\partial_a + \left(\alpha\ln v + \alpha^2\frac{x^2}{4} + f_1(x_0)\right)v\partial_v\right), \\ J_{ab} = x_a\partial_b - x_b\partial_a,\quad P_a = \partial_a,\quad M = \exp\{\alpha x_0\}v\partial_v, \\ X_a^{(2)} = \exp\{\alpha x_0\}\{(2/\alpha)\partial_a + x_a v\partial_v\} \tag{41}$$

при

$$f_1 = d_1\exp\{-\alpha x_0\} + d_2,\quad \{d_1, d_2\} \subset \mathbb{R};$$

$$2)\quad \langle AG^{(2)}(1,n), P_0\rangle \quad \text{при}\quad f_1 = d_2,$$

причому додаткова умова має вигляд

$$\alpha(v\Delta v - v_a v_a) - (\alpha^2 n/2 + d_1 \exp\{-\alpha x_0\} + d_2)v^2 = 0. \quad (42)$$

Зауважимо, що формули (41), (42) одержуються відповідно з (39), (40) заміною (12).

Зауваження 4. Ми довели, що у випадку 1 рівняння (11) умовно інваріантне відносно оператора T при додатковій умові $L_1 = 0$. Але при цьому рівняння $L_1 = 0$ не допускає операторів $X_a^{(1)}$. Тому у випадку (37) рівняння (11) не є умовно інваріантним відносно алгебри $AG^{(2)}(1, n)$.

Подіємо другим продовженням $\overset{(2)}{T}$ на рівняння $L_2 = 0$, де L_2 визначено в (36). В результаті одержимо

$$\overset{(2)}{T}\{L_2\} = \exp\{(\alpha - \beta)x_0\}[(2\beta - \alpha)L_2 + F(x_0)],$$

де

$$F(x_0) = f_1''' + (\alpha - 2\beta)f'' + \beta(4\beta - \alpha)f_1' + 2\beta^2(\alpha - 2\beta)^2 f_1 + n(\alpha - 2\beta)^2\beta^2.$$

Звідси робимо висновок, що рівняння (11) при додаткових умовах $L_1 = 0$, $L_2 = 0$ та умові на функцію f_1: $F(x_0) = 0$, інваріантне відносно оператора T. Але, як і у випадку (37), рівняння (11) при цьому не допускає алгебри $AG^{(2)}(1, n)$.

Випадок 3. Вимагатимемо інваріантність рівняння (11) відносно алгебри Галілея $AG^{(2)}(1, n)$ при $\alpha = 2\beta$. Для цього подіємо другим продовженням оператора T (28) на рівняння (11). Одержимо:

$$\overset{(2)}{T}\{u_0 + u_a u_a + \beta x_a u_a - \Delta u - \alpha u\} = \\ = 2x_0(\beta x_0 - 1)\{u_0 + u_a u_a + \beta x_a u_a - \Delta u - \alpha u\} + L_1,$$

де

$$L_1 = 2\beta x_0^2/\Delta u - 2\beta f_2 + f_2' - n/2. \quad (43)$$

Справедливе рівняння

$$\overset{(2)}{T}\{L_1\} = x_0^2(f_2'' - 2\beta f_2' + \beta n).$$

Згідно з критерієм умовної інваріантності одержуємо, що рівняння (11) при додатковій умові

$$L_1 = 2\beta x_0^2 \Delta u - 2\beta/f_2 + f_2' - n/2 = 0 \quad (44)$$

інваріантне відносно оператора T (28), якщо виконується

$$f_2'' - 2\beta f_2' + \beta n = 0. \quad (45)$$

Загальний розв'язок останнього рівняння задається так:

$$f_2(x_0) = d\exp\{2\beta x_0\} + nx_0/2 + c, \quad d \in \mathbb{R}, \quad c \in \mathbb{R}.$$

Теорема 4. *Рівняння (11) при* $\alpha = 2\beta$ *умовно інваріантне відносно алгебри Галілея* $AG^{(2)}(1,n) = \langle T, X_a^{(1)}, M, J_{ab}, X_a^{(2)}\rangle$*, де*

$$T = x_0^2\partial_0 + (\beta x_0 - 1)x_0x_a\partial_a + \left(\frac{x^2}{4} + 2\beta x_0^2 u + nx_0/2 + c\right)\partial_u, \tag{46}$$

якщо виконується додаткова умова $2x_0^2\Delta u - nx_0 - 2c = 0$.

Зауваження 5. Оператор T (46) при $\beta = c = 0$ співпадає з стандартним оператором проективних перетворень [2]. У цьому випадку рівняння (11) інваріантне відносно оператора (46) у розумінні Лі.

Аналогом оператора масштабних перетворень для рівняння (11) при $\alpha = 2\beta \neq 0$ є оператор

$$D = x_0\partial_0 + (2\beta x_0 + 1)x_a\partial_a + (4\beta x_0 + f_2')\partial_u. \tag{47}$$

Подіявши другим продовженням оператора (47) на рівняння (11), будемо мати

$$\overset{(2)}{D}\{u_0 + u_au_a + \beta x_au_a - \Delta u - \alpha u\} = \\ = 2(2\beta x_0 - 1)\{u_0 + u_au_a + \beta x_au_a - \Delta u - \alpha u\} + L_1,$$

де

$$L_1 = 4\beta x_0\Delta u - 2\beta f_2' + f_2''. \tag{48}$$

Виконується рівність $\overset{(2)}{D}\{L_1\} = 2x_0(f_2''' - 2\beta f_2'')$. Таким чином, ми показали, що рівняння (11), (48) при

$$f_2''' - 2\beta f_2'' = 0 \tag{49}$$

інваріантне відносно оператора D що має вигляд (47).

Теорема 5. *Рівняння (11) при* $\alpha = 2\beta$ *умовно інваріантне відносно розширеної алгебри Галілея* $AG^{(2)}(1,n) = \langle T, X_a^{(1)}, M, J_{ab}, X_a^{(2)}, D\rangle$*, де оператори* T *і* D *задаються (46), (47), при виконанні додаткових умов (44), (45) та умови третього порядку*

$$\partial_0(\Delta u) = 0. \tag{50}$$

Для доведення теореми достатньо перевірити, що при диференціюванні по x_0 додаткова умова (44) співпадатиме з (48), (50), а умова на функцію f_2 (45) визначатиме умову (49).

4. Висновки. З використанням запропонованого методу знаходження алгебр умовної інваріантності ми досягли таких результатів:

1) для нелінійного рівняння (11) знайдено дві різних алгебри Галілея $AG^{(1)}(1,n)$ та $AG^{(2)}(1,n)$, які є його алгебрами умовної інваріантності (див. теореми 2–4). Причому кожна з цих алгебр допускає по два різних зображення (леми 1, 2);

2) знайдено додаткові умови, при яких наше рівняння умовно галілей-інваріантне (див. (34), (40), (44), (50));

3) для редукції рівняння (11) можна скористатися підалгебрами алгебр $AG^{(1)}(1,n)$ та $AG^{(2)}(1,n)$, оскільки структура алгебр Галілея досить добре вивчена [8];

4) розв'язки рівняння можна нетривіальним чином розмножувати по операторах умовної симетрії при умові, що ці розв'язки задовольняють відповідні додаткові умови.

5. Антиредукція нелінійного рівняння (11). Розглянемо такий анзац:

$$u = \varphi_0(x_0) + x_a\varphi_a(x_0) + x^2/4x_0, \tag{51}$$

де $\varphi_\mu(x_0)$ — довільні функції від x_0, $x^2 = x_a x_a$, $a = \overline{1,n}$. Після підстановки (51) в рівняння (11) одержимо

$$\begin{aligned}&\varphi_0' + \varphi_a\varphi_a - \alpha\varphi_0 - (n/2)x_0 + x_a(\beta\varphi_a + \varphi_a' + \\ &\qquad + \varphi_a/x_0 - \alpha\varphi_a) + x^2(2\beta - \alpha)/4x_0 = 0.\end{aligned}$$

З останнього рівняння ми можемо зробити такий висновок: анзац (51) редукує рівняння (11) для функції u від $(n+1)$ змінної x_μ при $\alpha = 2\beta$ до системи звичайних диференціальних рівнянь

$$\begin{aligned}&\varphi_a' + \varphi_a/x_0 - \beta\varphi_a = 0,\\ &\varphi_0' + \varphi_a\varphi_a - 2\beta\varphi_0 - n/2x_0 = 0,\end{aligned} \tag{52}$$

для $(n+1)$ функції φ_μ від x_0. Тобто анзац (51) здійснює при $\alpha = 2\beta$ антиредукцію [9] рівняння (11) до системи (52), що складається із $(n+1)$ рівнянь.

Система рівнянь (52), на відміну від рівняння (11), легко інтегрується. Загальний розв'язок системи (52) має вигляд

$$\varphi_a = d_a \exp\{\beta x_0\}/x_0, \quad \varphi_0 = \exp\{2\beta x_0\}(nF(x_0)/2 + d^2/x_0), \tag{53}$$

де

$$F(x_0) = \int (x_0 \exp\{2\beta x_0\})^{-1} dx_0, \quad d^2 = d_a d_a, \quad d_a \in \mathbb{R}, \quad a = \overline{1,n}.$$

Підстановка (53) в анзац (51) задає нам багатопараметричну сім'ю розв'язків нелінійного рівняння (11).

Анзац (51) породжується набором таких операторів:

$$G_a = 2x_0\partial_a + \partial_{u_a}, \quad u_a = \partial u/\partial x_a. \tag{54}$$

Нелокальні оператори G_a по змінних x_a породжують стандартні перетворення Галілея $x_a \to x_a' = x_a + v_a x_0$, де v_a — групові параметри. Але, як ми вище довели, рівняння (11) не допускає цих перетворень (як у розумінні Лі, так і в термінах умовної інваріантності).

Оскільки анзац (51), побудований по операторах (54), редукує рівняння (11), то виникає питання про зв'язок цього рівняння з нелокальними операторами Галілея (54). Справедлива така теорема.

Теорема 6. *Диференціальні наслідки рівняння (11) умовно інваріантні відносно нелокальних операторів G_a (54), причому додаткова умова має вигляд $G_a u = 0$.*

Доведення теореми проведемо для випадку $n = 1$. Продиференціюемо рівняння (11) по x_1 та проведемо у ньому нелокальну заміну

$$V^1 = u_1. \tag{55}$$

Одержимо рівняння

$$V_0^1 + \beta x_1 V_1^1 + 2V^1 V_1^1 - V_{11}^1 + (\beta - \alpha)V^1 = 0, \qquad (56)$$

де

$$V^1 = V^1(x_0, x_1), \quad V_0^1 = \partial V^1/\partial x_0, \quad V_1^1 = \partial V^1/\partial x_1.$$

Після використання (55) оператор G_1 набуває вигляду

$$G_1 = 2x_0\partial_1 + \partial_{V^1}. \qquad (57)$$

Діючи другим продовженням оператора G_1 (57) на (56), одержуємо

$$\overset{(2)}{G}_1\{V_0^1 + \beta x_1 V_1^1 + 2V^1 V_1^1 - V_{11}^1 + (\beta - \alpha)V^1\} = \beta G_1 V^1 \quad \text{при} \quad \alpha = 2\beta.$$

Згідно з критерієм умовної інваріантності [2, 3] рівняння (56) Q-умовно інваріантне відносно оператора Галілея (57) при $\alpha = 2\beta$. Враховуючи, що ми провели заміну (55), переконаємося у справедливості теореми 6 при $n = 1$.

Підсумовуючи попередні результати, ми можемо зробити такий важливий висновок: *для редукції та знаходження точних розв'язків нелінійних рівнянь важливо знати симетрійні властивості не тільки самого рівняння, а й симетрію його диференціальних наслідків.*

1. Фущич В.И., Как расширить симметрию дифференциальных уравнений? в сб. Симметрия и решения нелинейных уравнений математической физики, Киев, Ин-т математики АН УССР, 1987, 4–16.
2. Фущич В.И., Штелень В.М., Серов Н.И., Симметрийный анализ и точные решения нелинейных уравнений математической физики, Киев, Наук. думка, 1989, 336 с.
3. Фущич В.И., Серов Н.И., Чопик В.И., Условная инвариантность и нелинейные уравнения теплопроводности, *Докл. АН УССР, Сер. А*, 1988, № 9, 17–20.
4. Myronyuk P., Chopyk V., Conditional Galilei-invariance of multidimensional nonlinear heat equation, in Symmetry Analysis of Equations of Mathematical Physics, Kiev, Inst. Math. Ukr. Acad. Sci., 1992, 66–68.
5. Tokar V.I., A new renormalization scheme in the Landau–Ginzburg–Wilson model, *Phys. Lett. A*, 1984, **104**, № 3, 135–139.
6. Filippov A.E., Breus S.A., On the physical branch of the exact (local) RG equation, *Phys. Lett. A*, 1991, **158**, № 6. 300–306.
7. Shtelen W., Spichak S., Lie and Q-condtoonal symmetry and exact solutions of local Wilson renormalization group equation, in Symmetry Analysis of Equations of Mathematical Physics, Kiev, Inst. Math. Ukr. Acad Sci., 1992, 50–54.
8. Фущич В.И., Баранник Л.Ф., Баранник А.Ф., Подгрупповой анализ групп Галилея, Пуанкаре и редукция нелинейных уравнений, Киев, Наук. думка, 1991, 299 с.
9. Фущич В.И., Серов Н.И., Репета В.К., Условная симметрия, редукция и точные решения нелинейного волнового уравнения, *Докл. АН Украины, Сер. А*, 1991, № 5, 29–34.

Симметрія та неліївська редукція нелінійного рівняння Шредінгера

В.І. ФУЩИЧ, В.І. ЧОПИК

Описані нелінійні рівняння типу Шредінгера, інваріантні відносно розширених груп Галілея. Вивчена умовна симетрія таких рівнянь і проведена їх редукція, побудовані класи точних розв'язків.

1. Вступ. Розглянемо нелінійне рівняння Шредінгера

$$L_1(u) \equiv Su - uF(u, u^*) = 0, \tag{1}$$

де $S = i\partial/\partial x_0 + \lambda\Delta$, $x_0 \equiv t$, $\Delta = \partial^2/\partial x_1^2 + \cdots + \partial^2/\partial x_n^2$, $i^2 = -1$, $\lambda \in \mathbb{R}$, n — число просторових змінних.

Як відомо, рівняння (1) інваріантне відносно алгебри Галілея $AG(1,n)$ тоді і тільки тоді, коли $F = F(uu^*)$. Базисні оператори алгебри $AG(1,n)$ мають вигляд

$$\begin{gathered} P_0 = \partial/\partial x_0, \quad P_a = \partial/\partial x_a, \quad J_{ab} = x_a P_b - x_b P_a, \quad a, b = \overline{1,n}, \\ Q = i(u\partial/\partial u - u^*\partial/\partial u^*), \quad G_a = x_0 P_a + (1/2\lambda) x_a Q. \end{gathered} \tag{2}$$

Інших представлень алгебри Галілея рівняння (1) не допускає. В [1] описані всі нелінійні рівняння типу (1), інваріантні відносно таких розширень алгебри Галілея $AG(1,n)$:

$$1)\ AG_1(1,n) = \langle AG(1,n), D\rangle, \tag{3}$$

де оператор масштабних перетворень D має вигляд

$$D = x_0^2 P_0 + x_a P_a + kI, \quad I = (u\partial/\partial u + u^*\partial/\partial u^*), \quad k \in \mathbb{R};$$

$$2)\ AG_2(1,n) = \langle AG_1(1,n), A\rangle, \tag{4}$$

де оператор проективних перетворень A має вигляд

$$A = x_0^2 P_0 + x_0 x_a P_a + \boldsymbol{x}^2 (4\lambda)^{-1} Q + \frac{n}{2} x_0 I, \quad \boldsymbol{x}^2 = x_a x_a, \quad a = \overline{1,n}.$$

Узагальнена алгебра Галілея $AG_2(1,n)$, доповнена оператором I, являється максимальною алгеброю інваріантності вільного рівняння Шредінгера (1) ($F = 0$).

Однак, в [1] не досліджене таке важливе питання: чи існують рівняння типу (1), які були б інваріантні відносно алгебри (2) та інших її розширень?

У даній роботі дано ствердну відповідь на це питання. Зокрема, доведено, що рівняння Шредінгера з логарифмічною нелінійністю $u\ln(uu^*)$ допускає два різних розширення алгебри $AG(1,n)$. Вивчена умовна симетрія рівнянь типу (1). Показано, що рівняння (1) з нелінійністю $u\ln(uu^{*-1}) + uF(uu^*)$ умовно інваріантне відносно алгебри Галілея у нестандартному представленні. Здійснена неліївська редукція та побудовані класи точних розв'язків розглядуваних рівнянь.

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2. Симетрія Лі рівняння (1). Інформація про ліівську симетрію рівняння (1) міститься в наступних твердженнях.

Теорема 1 [1]. *Рівняння (1)* $(F \neq 0)$ *інваріантне відносно алгебр* $AG_1(1,n)$ *(3) та* $AG_2(1,n)$ *(4) тоді і тільки тоді, коли*

$$F = \lambda_1 |u|^{-2/k}, \quad \lambda_1 \in \mathbb{C}, \quad |u| = (uu^*)^{1/2},$$
$$F = \lambda_2 |u|^{4/n}, \quad \lambda_2 \in \mathbb{C},$$

відповідно.

Теорема 2. *Серед рівнянь класу (1) тільки рівняння з нелінійністю*

$$F = \lambda_3 \ln(uu^*), \quad \lambda_3 \in \mathbb{C}, \quad \lambda_3 = b + ib_1 \tag{5}$$

інваріантне відносно алгебр [2]:

$$1) \quad AG_3(1,n) = \langle AG(1,n), B\rangle \quad \text{при} \quad b_1 = 0, \quad \text{де} \quad B = I - 2bx_0Q; \tag{6}$$

$$2) \quad AG_4(1,n) = \langle AG(1,n), C\rangle \quad \text{при} \quad b_1 \neq 0, \quad \text{де} \quad C = \exp\{2b_1x_0\}(I + i(b/b_1)Q). \tag{7}$$

Зауваження 1. При $b = 0$ рівняння (1), (5) інваріантне відносно алгебри $AG_4(1,n) = \langle AG(1,n), C^{(1)}\rangle$, де

$$C^{(1)} = \exp\{2b_1x_0\}I. \tag{8}$$

Оператор $C^{(1)}$ одержується з (7) при $b = 0$.

Теорема 3. *Рівняння (1) інваріантне відносно таких алгебр:*

1) $A_1 = \langle P_0, P_a, J_{ab}, I\rangle$, *коли* $F = F(uu^{*-1})$;

2) $A_2 = \langle P_0, P_a, J_{ab}, C^{(1)}\rangle$, *коли* $F = ib_1 \ln(uu^{*-1}) + F_1(uu^{*-1})$, *а оператор* $C^{(1)}$ *має вигляд (8)*;

3) $A_3 = \langle P_0, P_a, J_{ab}, Q^{(1)}, G_a^{(1)}\rangle$, *де*
$$Q^{(1)} = \exp\{2\beta x_0\}Q, \quad G_a^{(1)} = \exp\{2\beta x_0\}(P_a + (\beta/\lambda)x_a)Q, \tag{9}$$
коли $F = -i\beta \ln(uu^{*-1}) + F_2(uu^*)$, $\beta \in \mathbb{R}$;

4) $A_4 = \langle P_0, P_a, J_{ab}, Q^{(1)}, G_a^{(1)}, I\rangle$, *коли* $F = -i\beta \ln(uu^{*-1})$, $\beta \in \mathbb{R}$, $\beta \neq 0$; (10)

5) $A_5 = \langle P_0, P_a, J_{ab}, Q^{(1)}, G_a^{(1)}, \beta I + \beta_1 Q\rangle$, *коли* $F = \beta_1 \ln(uu^*) - i\beta \ln(uu^{*-1})$, $\beta, \beta_1 \in \mathbb{R}$;

6) $A_6 = \langle P_0, P_a, J_{ab}, Q^{(1)}, G_a^{(1)}, C^{(1)}\rangle$, *коли* $F = ib_1 \ln(uu^*) - i\beta \ln(uu^{*-1})$, $\beta, b_1 \in \mathbb{R}$;

7) $A_7 = \langle P_0, P_a, J_{ab}, I, D^{(1)}\rangle$, $D^{(1)} = 2x_0P_0 + x_aP_a + dQ$, $d \in \mathbb{R}$, $d \neq 0$, *коли* $F = \lambda_4(uu^{*-1})^{i/d}$, $\lambda_4 \in \mathbb{C}$;

8) $A_8 = \langle P_0, P_a, J_{ab}, kI + dQ\rangle, \quad k, d \in \mathbb{R}, \; k \neq 0, \; d \neq 0,$
коли $F = F(u^{\alpha}u^{*\alpha^*}), \quad \alpha = \alpha_1 - i\alpha_2, \; k\alpha_1 + d\alpha_2 = 0;$

9) $A_9 = \langle P_0, P_a, J_{ab}, I, D^{(2)}\rangle, \; D^{(2)} = 2x_0P_0 + x_aP_a + kI + dQ, \; k, d \neq 0,$
коли $F = F(u^{\alpha}u^{*\alpha^*})(uu^*)^{-1}, \quad \alpha = \alpha_1 - i\alpha_2, \; k\alpha_1 + d\alpha_2 = 0,$

де k, d, α_1, α_2 — довільні дійсні параметри.

Наслідок 1. *З теореми 2 випливає, що рівняння*

$$iu_0 + \lambda\Delta u = \lambda_3 \ln(uu^*)u, \quad \lambda_3 = b + ib_1 \tag{11}$$

іваріантне відносно таких скінченних перетворень:

$$x_0 \to x_0' = x_0, \quad x_a \to x_a' = x_a, \quad a = \overline{1, n},$$

а також:

$$\begin{aligned} &1) \quad u \to u' = \exp\{\theta_1(1 - 2ibx_0)\}u \quad \textit{при } b_1 = 0;\\ &2) \quad u \to u' = \exp\{\theta_2 \exp\{2b_1x_0(1 - i(b/b_1)\}\}u \quad \textit{при } b_1 \neq 0;\\ &3) \quad u \to u' = \exp\{\theta_3 \exp\{2b_1x_0\}\}u \quad \textit{при } b \neq 0, \; b_1 \neq 0, \end{aligned} \tag{12}$$

де θ_1, θ_2, θ_3 — групові параметри.

Наслідок 2. *З комутаційних співвідношень для оператора B: $[B, P_0] = c_1Q$, $[B, P_a] = [B, J_{ab}] = [B, Q] = [B, G_a] = 0$, $c_1 \in \mathbb{R}$; та оператора C: $[C, P_0] = c_2C$, $[C, P_a] = [C, J_{ab}] = [C, Q] = [C, G_a] = 0$, $c_2 \in \mathbb{R}$, випливає, що алгебри $AG_3(1, n)$ та $AG_4(1, n)$ різні. Тобто рівняння Шредінгера з логарифмічною нелінійністю (5) допускає два різних розширення алгебри Галілея $AG(1, n)$.*

Зауваження 2. Рівняння (5) при $\lambda_3 \in \mathbb{R}$ ($b_1 = 0$) співпадає з рівнянням, запропонованим у роботі [3]. В цій роботі вказані перетворення (12) (за винятком оператора B, що їх породжує). Це рівняння використовується в ядерній фізиці для опису нуклонів та альфа-частинок. Дослідженню цього рівняння присвячені також роботи [2, 4].

Рівняння

$$iu_0 + \lambda\Delta u = -i\beta \ln(uu^{*-1}) + F_2(uu^*), \quad \beta \in \mathbb{R}, \tag{13}$$

широко використовується в математичній фізиці і його називають фазовим рівнянням Шредінгера [4, 5].

Наслідок 3. *З комутаційних співвідношень для алгебри A_3 (9):*

$$\begin{aligned} &[P_0, P_a] = [P_0, J_{ab}] = [P_a, Q^{(1)}] = [J_{ab}, Q^{(1)}] = [G_a^{(1)}, G_b^{(1)}] = [P_a, P_b] = 0,\\ &[P_0, Q^{(1)}] = c_1Q^{(1)}, \quad [P_0, G_a^{(1)}] = c_2G^{(1)}, \quad [P_a, J_{bc}] = \delta_{ab}P_c - \delta_{ac}P_b,\\ &[G_a^{(1)}, J_{bc}] = \delta_{ab}G_c^{(1)} - \delta_{ac}G_b^{(1)}, \quad c_1, c_2 \in \mathbb{R} \end{aligned}$$

випливає, що базисні оператори цієї алгебри не утворюють алгебри Галілея. Оператори $G_a^{(1)}$ породжують такі скінченні перетворення:

$$\begin{aligned} &x_0 \to x_0' = x_0, \quad x_a \to x_a' = \exp\{2\beta x_0\}\theta_a + x_a,\\ &u \to u' = u\exp\{i[(\beta/2\lambda)\exp\{4\beta x_0\}\theta^2 + \exp\{2\beta x_0\}x_a\theta_a]\}, \end{aligned}$$

де θ_a — групові параметри, $\theta^2 = \theta_a\theta_a$.

3. Умовна симетрія. Розглянемо рівняння класу (1)

$$L_1(u) \equiv Su - uF(uu^*) = 0, \tag{14}$$

інваріантне відносно алгебри Галілея $AG(1,n)$ (2). Відповідь на питання про існування операторів умовної симетрії рівняння (14) випливає з наступних теорем.

Теорема 4. *Рівняння (14) умовно інваріантне відносно таких алгебр:*

1) $A_{10} = \langle AG(1,n), Q^{(2)}\rangle$, $Q^{(2)} = x_a P_a - i\ln(uu^{*-1})Q$, *якщо* $F = -F^*$, *і виконується додаткова умова*

$$L_2(u) \equiv \Delta|u| = 0, \quad |u| = (uu^*)^{1/2}; \tag{15}$$

2) $A_{11} = \langle A_{10}, C^{(1)}\rangle$, *якщо* $F = ib_1\ln(uu^*)$, $b_1 \in \mathbb{R}$, $C^{(1)}$ *має вигляд (8) і виконується (15);*

3) $A_{12} = \langle AG(1,n), Q^{(3)}\rangle$, $Q^{(3)} = x_0P_0 + x_aP_a - (i/2)\ln(uu^{*-1})Q$, *якщо функція* F *приймає дійсні значення (тобто* $F = F^*$*) і виконується додаткова умова (15);*

4) $A_{13} = \langle A_{12}, B\rangle$, *якщо* $F = b\ln(uu^*)$, $b \in \mathbb{R}$, *оператор* B *має вигляд (6) і виконується додаткова умова (15);*

5) $A_{14} = \langle AG(1,n), Q^{(4)}\rangle$, $Q^{(4)} = x_0P_0 + (i/2)\ln(uu^{*-1})Q$ *і виконуються умови* $F^* = F$, $L_2(u) \equiv V_0 + \lambda V_aV_a = 0$, $2V = -i\ln(uu^{*-1})$.

Теорема 5 [6]. *Рівняння (14) при*

$$F = \alpha_1|u|^{2r^{-1}} + \alpha_2|u|^{-2r^{-1}}, \quad r, \alpha_1, \alpha_2 \in \mathbb{R}, \quad r \neq 0, \tag{16}$$

умовно інваріантне відносно оператора

$$Q^{(5)} = x_aP_a + rI - i\ln(uu^{*-1})Q, \tag{17}$$

якщо $L_2(u) \equiv \Delta|u| - \alpha_3|u|^{(r-2)/r} = 0$, $\alpha_3 = \alpha_2\lambda^{-1}$.

Наслідок 4. *Рівняння Шредінгера (14) з нелінійністю (16) умовно інваріантне відносно алгебри* $AG_5(1,n) = \langle AG_1(1,n), Q^{(5)}\rangle$, *якщо виконується одна з умов:*

$$\alpha_1 = 0, \quad r = k \tag{18}$$

або

$$\alpha_2 = 0, \quad r = -k. \tag{19}$$

Наслідок 5. *Рівняння (14), (16) умовно інваріантне відносно алгебри* $AG_6(1,n) = \langle AG_2(1,n), Q^{(5)}\rangle$ *при виконанні однієї з умов (18), (19) та умови, що* $k = -n/2$.

Структура алгебри $AG_6(1,n)$ вивчена в роботах [7, 8].

Наслідок 6. *Оператор* $Q^{(5)}$ *породжує такі скінченні перетворення:*

$$x_0 \to x_0' = x_0, \quad x_a \to x_a' = \exp(\theta)x_a,$$
$$u \to u' = \exp(r\theta)\exp\{\exp(2\theta)\}(uu^{*-1})^{1/2}|u|,$$

θ — *груповий параметр.*

Теорема 6. *Рівняння (14) з нелінійністю*

$$F = i\alpha_1|u|^{-r^{-1}} + \alpha_2|u|^{-(1+\beta)r^{-1}} + \alpha_3|u|^{-(1-\beta)r^{-1}}, \quad \alpha_j, \beta, r \in \mathbb{R}, \quad j = \overline{1,3}, \tag{20}$$

умовно інваріантне відносно оператора

$$Q^{(6)} = 2x_0P_0 + (1+\beta)x_aP_a + i\beta\ln(uu^{*-1})Q + rI, \quad r \neq 0,$$

якщо $\Delta|u| = \alpha_4|u|^{1-(1+\beta)r^{-1}}$, $\alpha_4 = \alpha_2\lambda^{-1}$.

Наслідок 7. *Оператор* $Q^{(6)}$ *породжує такі скінченні перетворення:*

$$x_0 \to x_0' = \exp(2\theta)x_0, \quad x_a \to x_a' = \exp((1+\beta)\theta)x_a,$$
$$u \to u' = \exp(r\theta)\exp\{\exp(-2\beta\theta)\}(uu^{*-1})^{1/2}|u|,$$

θ *— груповий параметр.*

4. Умовна галілей-інваріантність фазового рівняння Шредінгера. З комутаційних співвідношень для алгебри A_3 (9) (див. наслідок 3) випливає, що всі оператори цієї алгебри, за винятком оператора P_0, задовольняють комутаційні співвідношення алгебри Галілея [1].

Твердження. *Якщо оператор* $P_0^{(1)}$ *має вигляд*

$$P_0^{(1)} = \exp\{-2\beta x_0\}(P_0 - i\beta\ln(uu^{*-1})Q), \tag{21}$$

то оператори $P_0^{(1)}$, P_a, J_{ab}, $Q^{(1)}$, $G_a^{(1)}$ *утворюють базис алгебри Галілея, яку позначатимемо* $AG^{(1)}(1,n)$.

Справедливість цього твердження випливає з комутаційних співвідношень для оператора $P_0^{(1)}$:

$$[P_0^{(1)}, P_a] = [P_0^{(1)}, J_{ab}] = [P_0^{(1)}, Q^{(1)}] = 0, \quad [P_0^{(1)}, G_a^{(1)}] = cP_a, \quad c \in \mathbb{R}.$$

Вимагатимемо інваріантність фазового рівняння Шредінгера

$$Su + i\beta u\ln(uu^{*-1}) = 0 \tag{22}$$

відносно оператора $P_0^{(1)}$. Результат сформулюємо у вигляді такої теореми.

Теорема 7. *Фазове рівняння Шредінгера (22) умовно інваріантне відносно алгебри* $A_{15} = \langle AG^{(1)}(1,n), P_0, Q^{(2)}, A^{(1)}, I\rangle$, *де*

$$Q^{(2)} = x_aP_a - i\ln(uu^{*-1})Q,$$
$$A^{(1)} = \exp\{2\beta x_0\}(P_0/\beta + 2x_aP_a - i\ln(uu^{*-1})Q + (\beta/\lambda)\boldsymbol{x}^2Q - nI),$$

якщо виконається додаткова умова

$$L_2 = V_0 + \lambda V_aV_a - 2\beta V = 0, \quad 2V = -i\ln(uu^{*-1}), \quad \beta \neq 0. \tag{23}$$

Оператори $P_0^{(1)}$, $Q^{(2)}$ *породжують такі скінченні перетворення:*

$$x_0 \to x_0' = (2\beta)^{-1}\ln(2\beta\theta_1 + \exp\{2\beta x_0\}), \quad x_a \to x_a' = \exp\{\theta_2\}x_a,$$
$$u \to u' = \exp\left\{2i\beta\exp\left\{\frac{\theta_1 + \exp(2\beta x_0)(2\theta_2 + \ln((4i\beta)^{-1}\ln(uu^{*-1})))}{2\beta\theta_1 + \exp(2\beta x_0)}\right\}\right\}|u|,$$

де θ_1, θ_2 *— групові параметри.*

Наслідок 8. *Алгебра* A_{15} *ізоморфна алгебрі умовної інваріантності вільного рівняння Шредінгера [6]. Тобто оператори* $P_0^{(1)}$, P_a, J_{ab}, P_0, $Q^{(2)}$, $Q^{(1)}$, $G_a^{(1)}$,

$A^{(1)}$, I реалiзують нове представлення алгебри $AG_6(1,n)$, доповненої оператором I.

Дослiдимо симетрiйнi властивостi рiвняння (13) при додатковiй умовi (23).

Теорема 8. *Рiвняння (13) умовно iнварiантне вiдносно таких алгебр Галiлея:*

1) $A_{16} = \langle AG^{(1)}(1,n), P_0\rangle$, якщо функцiя $F(uu^*)$ дiйсна;

2) $A_{17} = \langle A_{16}, D^{(1)}\rangle$, якщо $F = \lambda_1|u|^{-2/k}$, $\lambda_1, k \in \mathbb{R}$, $k \neq 0$, де $D^{(1)} = Q^{(2)} + 2P_0 + kI$;

3) $A_{18} = \langle A_{17}, A^{(1)}\rangle$, якщо $F = \lambda_2|u|^{4/n}$, $\lambda_2 \in \mathbb{R}$, $k = -n/2$.

Додаткова умова має вигляд (23).

Алгебра A_{18} iзоморфна алгебрi $AG_6(1,n)$. Це випливає з комутацiйних спiввiдношень для цих алгебр [6–8].

Таким чином, доведено, що фазове рiвняння Шредiнгера умовно iнварiантне вiдносно алгебри Галiлея у нестандартному представленнi.

Сформулюємо ще одну теорему про умовну iнварiантнiсть рiвняння (13).

Теорема 9. *Рiвняння (13) умовно iнварiантне вiдносно таких алгебр:*

1) $A_{19} = \langle A_3, Q^{(2)}\rangle$, якщо функцiя $iF(uu^*)$ дiйсна;

2) $A_{20} = \langle A_6, Q^{(2)}\rangle$, якщо $F = ib_2\ln(uu^*)$, $b_2 \in \mathbb{R}$, при додатковiй умовi на модуль функцiї u (15).

5. Доведення теорем. Повне доведення наведених теорем досить громiздке, тому ми вкажемо тiльки основнi етапи його, опускаючи деталi.

Позначимо через X довiльний оператор з алгебри iнварiантностi рiвняння (1). Для доведення теорем 1–3 необхiдно скористатися алгоритмом Лi.

1. Побудувати за формулами Лi друге продовження $\overset{(2)}{X}$ операторiв (див., наприклад, [1]).

2. Подiяти операторами другого продовження $\overset{(2)}{X}$ на многовид (1) i знайти диференцiальне рiвняння для функцiї $F(u,u^*)$. Розв'язавши це рiвняння, одержимо явний вигляд функцiй $F(u,u^*)$, при яких рiвняння (1) має ту чи iншу симетрiю.

Для доведення теорем 4–9 потрiбно використати критерiй умовної iнварiантностi. В розглядуваному випадку цей критерiй має вигляд [1, 6]

$$\overset{(2)}{X}L_1 = g_{11}L_1 + g_{12}L_2 = \overset{(2)}{X}L_1\Bigg|_{\substack{L_1=0\\L_2=0}} = 0, \tag{24}$$

$$\overset{(2)}{X}L_2 = g_{21}L_1 + g_{22}L_2 = \overset{(2)}{X}L_2\Bigg|_{\substack{L_1=0\\L_2=0}} = 0, \tag{25}$$

де g_{11}, g_{12}, g_{21}, g_{22} — взагалi кажучи, деякi оператори. Розв'язавши систему (24), (25), одержимо умови на u i u^* при яких рiвняння (1) iнварiантне вiдносно оператора X.

Наведемо доведення теореми 5 про умовну iнварiантнiсть рiвняння (14) вiдносно оператора $Q^{(5)} = X$.

Дiючи оператором $\overset{(2)}{X}$ на многовид $L_1(u)$, одержимо

$$\overset{(2)}{X}L_1(u) = 2L_1 - 4\lambda\Delta|u||u|^{-1} + 2F - r(uF_u + u^*F_{u^*}), \tag{26}$$

де $F_u = \partial F/\partial u$, $F_{u^*} = \partial F/\partial u^*$. Отже,

$$L_2(u) = -4\lambda\Delta|u||u|^{-1} + 2F - r(uF_u + u^*F_{u^*}). \tag{27}$$

Діючи оператором $\overset{(2)}{X}$ на многовид $L_2(u)$, одержимо

$$\overset{(2)}{X} L_2 = -2L_2 + 4F - r^2(uF_u + u^*F_{u^*} + u^2F_{uu} + u^{*2}F_{u^*u^*} + 2uu^*F_{uu^*}). \tag{28}$$

З (26), (29) випливає, що рівняння (14) інваріантне відносно $Q^{(5)}$ при додатковій умові $L_2(u) = 0$, при чому нелінійність $F(|u|)$ повинна задовольняти умову

$$4F - r^2(uF_u + u^*F_{u^*} + u^2F_{uu} + u^{*2}F_{u^*u^*} + 2uu^*F_{uu^*}) = 0,$$

де $F_{uu} = \partial^2F/\partial u^2$, $F_{u^*u^*} = \partial^2F/\partial u^{*2}$, $F_{uu^*} = \partial^2F/\partial u\partial u^*$. Всі інші теореми про умовну симетрію доводяться по наведеній схемі.

6. Неліївська редукція рівняння (14). Будемо шукати розв'язки чотиривимірного рівняння (14) з нелінійністю (16) у вигляді [6]

$$u(x) = u(x_0, x_1, x_2, x_3) = f_1(x)\varphi_1(\omega)\exp\{if_2(x)\varphi_2(\omega)\}. \tag{29}$$

Підставивши (29) у рівняння (14), (16), одержимо

$$\begin{aligned}
&f_{1_0}\varphi_1 + f_1\varphi_{1_\omega}\omega_0 + 2\lambda f_{1_a}f_{2_a}\varphi_1\varphi_2 + \lambda f_1f_2\omega_a\omega_a(2\varphi_{1_\omega}\varphi_{2_\omega} + \varphi_1\varphi_{2_{\omega\omega}}) + \\
&\quad + 2\lambda f_{1_a}f_2\varphi_1\varphi_{2_\omega}\omega_a + 2\lambda f_1f_{2_a}\varphi_{1_\omega}\varphi_2\omega_a + \lambda f_1\Delta f_2\varphi_1\varphi_2 + \\
&\quad + \lambda f_1f_2\varphi_1\varphi_{2_\omega}\omega_{aa} + 2\lambda f_1f_{2_a}\varphi_1\varphi_{2_\omega}\omega_a = \operatorname{Im} F, \\
&\lambda\Delta f_1\varphi_1 + \lambda f_1\varphi_{1_\omega}\omega_{aa} + \lambda f_1\varphi_{1_{\omega\omega}}\omega_a\omega_a + 2\lambda f_{1_\omega}\omega_0 - f_1f_{2_0}\varphi_1\varphi_2 - \\
&\quad - f_1f_2\varphi_1\varphi_{2_\omega}\omega_0 - \lambda f_1f_{2_a}f_{2_a}\varphi_1\varphi_2^2 - \lambda f_1f_2^2\varphi_{2_\omega}^2\omega_a\omega_a - \\
&\quad - 2\lambda f_1f_2f_{2_a}\varphi_1\varphi_2\varphi_{2_\omega}\omega_a = \operatorname{Re} F(f_1\varphi_1),
\end{aligned} \tag{30}$$

де

$$\begin{aligned}
&f_{j_\mu} = \partial f_j/\partial x_\mu, \quad \varphi_{j_\omega} = \partial\varphi_j/\partial\omega, \quad \omega_\mu = \partial\omega/\partial x_\mu, \\
&\omega = (\omega_1, \omega_2, \omega_3), \quad \mu = \overline{0,3}, \quad j = 1, 2.
\end{aligned}$$

Дійсні функції $f_1(x)$, $f_2(x)$ повинні бути так визначені, щоб з (30) випливала система рівнянь для функцій $\varphi_1(\omega)$, $\varphi_2(\omega)$, в яку входять тільки змінні $\omega = (\omega_1, \omega_2, \omega_3)$. Тому функції f_1, f_2, ω_1, ω_2, ω_3 повинні задовольняти деяку систему рівнянь, яку будемо називати *умовами редукції*. Більш детально про метод редукції див. [9]. Отже, проблема редукції чотиривимірного рівняння (14), (16) зводиться до розв'язання складної системи нелінійних рівнянь (умов редукції). Так, рівняння (14), (16) редукується до системи ЗДР

$$\begin{aligned}
&\theta_1\varphi_1 + \theta_2\dot\varphi_1 + \lambda\theta_3\varphi_1\varphi_2 + \lambda\theta_4(2\dot\varphi_1\dot\varphi_2 + \varphi_1\ddot\varphi_2) + \\
&\quad + \lambda\theta_5\varphi_1\dot\varphi_2 + 2\lambda\theta_6(\dot\varphi_1\varphi_2 + \varphi_1\dot\varphi_2) = 0, \\
&\theta_7\varphi_1 + \theta_8\dot\varphi_1 + \theta_9\ddot\varphi_1 = \alpha_2\varphi_1^{(r-2)/r}, \\
&\theta_{10}\varphi_2 + \theta_{11}\dot\varphi_2 + \lambda\theta_{12}\varphi_2^2 + \lambda\theta_{13}\dot\varphi_2^2 + 2\lambda\theta_{14}\varphi_2\dot\varphi_2 = \alpha_1\varphi_1^{2/r}, \\
&\dot\varphi_j = \partial\varphi_j/\partial\omega, \qquad \ddot\varphi_j = \partial^2\varphi_j/\partial\omega^2, \quad j = 1, 2,
\end{aligned}$$

якщо фукнції $\theta_1, \ldots, \theta_{14}$ задовольняють такі умови редукції:

$$
\begin{gathered}
f_{1_0} = h(x)\theta_1(\omega), \quad \Delta\omega + 2f_1^{-1} f_{1_a}\omega_a = \theta_9(\omega) f_1^{-2/r}, \quad f_{1_\omega} = h(x)\theta_2(\omega), \\
f_2 = \theta_{10}(\omega) f_1^{2/r}, \quad 2f_{1_a} f_{2_a} + f_1\Delta f_2 = h(x)\theta_3(\omega), \quad f_2\omega_0 = \theta_{11}(\omega) f_1^{2/r}, \\
f_1 f_2 \omega_a\omega_a = h(x)\theta_4(\omega), \quad f_{2_a} f_{2_a} = \theta_{12}(\omega) f_1^{2/r}, \quad f_{1_a} f_2\omega_a = h(x)\theta_5(\omega), \\
f_{2_a}\omega_a\omega_a = \theta_{13}(\omega) f_1^{2/r}, \quad f_1 f_{2_a}\omega_a = h(x)\theta_6(\omega), \quad f_2 f_{2_a}\omega_a = \theta_{14}(\omega) f_1^{2/r}, \\
\Delta f_1 = \theta_7(\omega) f_1^{(r-2)/r}, \quad \omega_a\omega_a = \theta_8(\omega) f_1^{(r-2)/r},
\end{gathered}
\tag{31}
$$

де $h(x)$ — довільна функція від x, функції $\theta_1, \ldots, \theta_{14}$ залежать від $\omega = \omega(x)$.

Побудувати загальний розв'язок умов редукції (31), напевно, неможливо, але знайти частинні розв'язки не так важко. Далі наведемо деякі частинні розв'язки умов редукції (31) і відповідні редуковані системи ЗДР для функцій $\varphi_1(\omega)$, $\varphi_2(\omega)$.

1. Функції $f_1 = x_1^{-1/2}$, $f_2 = x_1^2$, $\omega = x_0$ задовольняють систему (31). Редукована система ЗДР має вигляд

$$
\dot{\varphi}_2 + 4\lambda\varphi_2^2 + \frac{4\alpha_1\alpha_2}{3\lambda} = 0, \quad \varphi_1 = \left(\frac{3\lambda}{4\alpha_2}\right), \quad \alpha_2 \neq 0, \quad \lambda\alpha_2 > 0. \tag{32}
$$

Загальний розв'язок рівняння (32) задається виразом

$$
\varphi_2 = \begin{cases}
\dfrac{\sqrt{\alpha_1\alpha_2}}{\sqrt{3}\lambda} \operatorname{tg}\left(c - \dfrac{4\sqrt{\alpha_1\alpha_2}}{\sqrt{3}}\right) & \text{при } \alpha_1\alpha_2 > 0, \\
\dfrac{2}{\lambda} \dfrac{\sqrt{-\alpha_1\alpha_2}}{\sqrt{3}\left(1 - c\exp\left\{-4\sqrt{3}^{-1}\sqrt{-\alpha_1\alpha_2}\omega\right\}\right)} - \sqrt{-\alpha_1\alpha_2} & \text{при } \alpha_1\alpha_2 < 0, \\
\dfrac{1}{4\lambda\omega + c} & \text{при } \alpha_1 = 0.
\end{cases} \tag{33}
$$

Таким чином, формули (29), (32), (33) визначають однопараметричну сім'ю розв'язків нелінійного рівняння Шредінгера (14), (16). Використовуючи симетрію $AG(1,n)$ рівняння (14), за цим розв'язком можна побудувати [1] багатопараметричну сім'ю розв'язків рівняння (14), (16).

2. Функції $f_1 = x_1$, $f_2 = x_1^2$, $\omega = x_0$, $r = 1$, $\alpha_2 = 0$ задовольняють систему (31). Редукована система ЗДР для φ_1 і φ_2 має вигляд

$$
\dot{\varphi}_1 + 6\lambda\varphi_1\varphi_2 = 0, \quad \dot{\varphi}_2 + 4\lambda\varphi_2^2 + \alpha_1\varphi_1^2 = 0. \tag{34}
$$

Останні еквівалентні системі

$$
\varphi_2 = -\frac{1}{6\lambda}\dot{t}, \quad t = \ln\varphi_1(\omega), \quad \ddot{t} - \frac{2}{3}\dot{t}^2 - 6\lambda\alpha_1\exp(2t) = 0. \tag{35}
$$

В (35) зробимо заміну

$$
\dot{t}^2 = y(t). \tag{36}
$$

При цьому система редукованих рівнянь (34) набуває вигляду

$$
\varphi_2 = -\frac{1}{6\lambda}\sqrt{y}, \quad \dot{y} - \frac{4}{3}y - 12\lambda\alpha_1\exp(2t) = 0,
$$

звідки маємо

$$\varphi_2 = -\frac{1}{6\lambda}\sqrt{y}, \quad y = 18\lambda\alpha_1 \exp(2t) + c\exp\left(\frac{4}{3}t\right), \quad c = \mathrm{const}. \tag{37}$$

З системи (37) випливає

$$\varphi_2 = -\frac{\exp\left(\frac{2}{3}t\right)}{6\lambda}\sqrt{c_2 + 18\lambda\alpha_1 \exp\left(\frac{2}{3}t\right)},$$
$$t = \frac{3}{2}\ln\left(\left(\sqrt{2\lambda\alpha_1}\omega + c_1\right)^2 - \frac{c_2}{18\lambda\alpha_1}\right), \quad \alpha_1 \neq 0, \quad c_1, c_2 = \mathrm{const}.$$

Остаточно маємо такі рівняння:

$$\begin{aligned} &\varphi_2 = -\frac{1}{6\lambda}\varphi_1^{2/3}\sqrt{c_2 + 18\lambda\alpha_1\varphi_1^{2/3}}, \\ &\varphi_1 = \left(\left(\sqrt{2\lambda\alpha_1}\omega + c_1\right)^2 - \frac{c_2}{18\lambda\alpha_1}\right)^{3/2}, \quad c_1, c_2 = \mathrm{const}. \end{aligned} \tag{38}$$

Таким чином, при підстановці φ_1, φ_2 з (38) в (29) одержуємо точний розв'язок нелінійного рівняння (14), (16).

3. При $f_1 = (\boldsymbol{x}^2)^{r/2}$, $f_2 = \boldsymbol{x}^2 \equiv x_1^2 + x_2^2 + x_3^2$, $\omega = x_0$ система редукованих рівнянь набуває вигляду

$$\dot\varphi_1 + 10\lambda\varphi_1\varphi_2 = 0, \quad \dot\varphi_2 + 4\lambda\varphi_2^2 + \alpha_1\varphi_1^2 = 0,$$

якщо $\alpha_2 = 0$, $r = 1$.

Якщо $\alpha_2 \neq 0$, $r = -3/2$, то анзац (29) редукує (14), (16) до ЗДР

$$\dot\varphi_2 + 4\lambda\varphi_2^2 + \frac{4\alpha_1\alpha_2}{15\lambda} = 0, \quad \varphi_1 = \left(\frac{15\lambda}{4\alpha_2}\right)^{3/4}.$$

4. Функції $f_1 = (x_1^2 + x_2^2)^{r/2}$, $f_2 = x_1^2 + x_2^2$, $\omega = \sqrt{2}\,\mathrm{arctg}(x_2/x_1) - x_0$ задовольняють систему (31). Система редукованих рівнянь має вигляд:

$$\begin{aligned} &2\lambda\varphi_1\ddot\varphi_2 + 4\lambda\dot\varphi_1\dot\varphi_2 - \dot\varphi_1 + 4\lambda(r+1)\varphi_1\varphi_2 = 0, \\ &2\lambda\dot\varphi_2^2 - \dot\varphi_2 + 4\lambda\varphi_2^2 + \alpha_1\varphi_1^{2/r} = 0, \quad 2\lambda\ddot\varphi_1 - r^2\lambda\varphi_1 - \alpha_2\varphi_1^{-2/r} = 0. \end{aligned} \tag{39}$$

5. Функції $f_1 = (x_1^2 + x_2^2)^{r/2}$, $f_2 = x_1^2 + x_2^2$, $\omega = (x_1^2 + x_2^2)\exp\{2\alpha\,\mathrm{arctg}(x_2/x_1) - \sqrt{2}x_0\}$, $\alpha \geq 0$, задовольняють систему (31). Редуковані рівняння мають вигляд

$$\begin{aligned} &2(1+\alpha^2)\omega^2\varphi_1\ddot\varphi_2 + 4(1+\alpha^2)\omega^2\dot\varphi_1\dot\varphi_2 + 4\omega\dot\varphi_1\varphi_2 + \\ &\quad + \left(5 - 2\alpha^2 + 2r - \frac{1}{\sqrt{2}\lambda}\right)\omega\dot\varphi_2\varphi_1 + (1+2r)\varphi_1\varphi_2 = 0, \\ &4(1+\alpha)\omega^2\dot\varphi_2^2 + 8\omega\dot\varphi_2\varphi_2 - \sqrt{2}\omega\dot\varphi_2 + 4\lambda\varphi_2^2 + \alpha_2\varphi_1^{2/r} = 0, \\ &4\alpha^2\omega\ddot\varphi_1\varphi_1 - 4\alpha^2\omega\dot\varphi_1^2 + 4(1+\alpha^2)\omega^2\dot\varphi_1^2 + 4(1+r+\alpha^2)\omega\dot\varphi_1\varphi_1 - \alpha_2\varphi_1^{(2-r)/r} = 0. \end{aligned} \tag{40}$$

Зауважимо, що системи редукованих рівнянь (39), (40) перевизначені. Тому, природньо, виникає питання сумісності систем ЗДР (39), (40). Для системи (39) при

$r=-1$ можна вказати такі константи c_1 і c_2, що $\varphi_1=c_1$, $\varphi_2=c_2$. Тоді анзац (29) задаватиме точний розв'язок системи нелінійних рівнянь (14), (16).

Для знаходження розв' язків нелінійного рівняння

$$Su=i\alpha_1|u|^{-r^{-1}}+\alpha_2|u|^{-2r^{-1}} \tag{41}$$

(частковий випадок рівняння (14), (20)) скористаємось анзацом [10]

$$u=f_1(x)\varphi_1(\omega)\exp\{i(f_2(x)\varphi_2(\omega)+g(x))\}. \tag{42}$$

Вкажемо деякі набори функцій $f_1(x)$, $f_2(x)$, $g(x)$, $\omega(x)$, які задовольнятимуть умови редукції рівняння (41) та відповідні редуковані рівняння.

6. Набір функцій $f_1=x_0^r$, $f_2=x_0^{-1}$, $g=x_3^2/4\lambda x_0$, $\omega=x_2$ задовольняє умови редукції рівняння (41). Відповідна система редукованих рівнянь матиме вигляд

$$\begin{gathered}\lambda\varphi_1\ddot\varphi_2+(r+1/2)\varphi_1+2\lambda\varphi_1\dot\varphi_2=\alpha_1\varphi_1^{(r-1)/r},\\ \lambda\dot\varphi_2^2-\varphi_2+\alpha_2\varphi_1^{-2/r}=0,\quad \ddot\varphi_1=0.\end{gathered} \tag{43}$$

При підстановці частинного розв'язку системи (43)

$$\varphi_1=\left(\frac{r+1}{\alpha_1}\right)^{-r},\quad \alpha_1\neq0,\quad \varphi_2=\left(\frac{\omega+c}{4\lambda}\right)^2+\alpha_2\left(\frac{r+1}{\alpha_1}\right)^2$$

в анзац (42) одержуємо однопараметричну сім'ю розв'язків рівняння (41).

7. При $f_1=x_0^r$, $f_2=x_0^{-1}$, $g=x_3^2/4\lambda x_0$, $\omega=(x_1^2+x_2^2)^{1/2}$ анзац (42) редукує рівняння (41) до системи ЗДР

$$\begin{gathered}\lambda\varphi_1\ddot\varphi_1+\lambda\omega^{-1}\varphi_1\dot\varphi_2+2\lambda\dot\varphi_1\dot\varphi_2-\lambda\varphi_1\dot\varphi_2+r=\alpha_1\varphi_1^{(r-1)/r},\\ \lambda\dot\varphi_2-\varphi_2+\alpha_2\varphi_1^{-2/r}=0,\quad \ddot\varphi_1-\dot\varphi_1+\omega^{-1}\dot\varphi_1=0.\end{gathered}$$

8. При $f_1=x_0^r$, $f_2=x_0^{-1}$, $g=(x_1^2+x_2^2)/4\lambda x_0$, $\omega=x_3$ система редукованих рівнянь буде мати вигляд

$$\begin{gathered}\lambda\varphi_1\ddot\varphi_1+2\lambda\dot\varphi_1\dot\varphi_2+(r+1)\varphi_1=\alpha_1\varphi_1^{(r-1)/r},\\ \lambda\dot\varphi_2-\varphi_2+\alpha_2\varphi_1^{-2/r}=0,\quad \ddot\varphi_1=0.\end{gathered}$$

Для редукції рівняння (14) з нелінійністю, що задовольняє умову $F=-F^*$, скористаємось анзацом

$$u=\varphi_1(\omega)\exp\{i(f(x)\varphi_2(\omega)+g(x))\}. \tag{44}$$

Якщо виписати відповідні умови редукції, то функції

$$f=\frac{x_1^2+x_2^2}{x_0^2},\quad g=0,\quad \omega=\ln\ln(uu^{*-1})^{1/2i}-\ln(x_0(x_1^2+x_2^2))$$

задовольнятимуть ці умови. Система ЗДР матиме вигляд

$$\varphi_2(1-4\lambda\varphi_2)=\dot\varphi_2(1-4\lambda\varphi_2),\quad 2\varphi_2(2\varphi_1-\dot\varphi_1)=F(\varphi_1)\varphi_1. \tag{45}$$

Для рівняння Шредінгера з логарифмічною нелінійністю

$$F(uu^*)=ib_1\ln(uu^*),\quad b_1\in\mathbb{R}, \tag{46}$$

частинний розв'язок системи рівнянь (45)

$$\varphi_2 = \frac{1}{4\lambda}, \quad \varphi_1 = \exp\left\{\frac{1 - c\exp(-2\lambda b_1\omega)}{\lambda b_1}\right\}, \quad b_1 \neq 0, \quad c = \mathrm{const},$$

при підстановці в анзац (44) задаватиме точний розв'язок (14), (46).

9. Анзац (44), де $f = x_0^{-1}$, $g = 0$, $\omega = x_3$, редукує рівняння (14) з дійсною нелінійністю ($F = F^*$) до ЗДР

$$\varphi_1\ddot{\varphi}_2 + 2\varphi_1\varphi_2 = 0, \quad \lambda\dot{\varphi}_2^2 - \varphi_2 = 0, \quad \lambda\ddot{\varphi}_1 = F(\varphi_1)\varphi_1. \tag{47}$$

З (47) випливає, що $\varphi_1 = c_1(\omega + c_2)^{-1/2}$, $\varphi_2 = ((\omega + c_2)/4\lambda)^2$ при $F = (3/4)\lambda|u|^8$.

10. Анзац (44), де $f = x_0^{-1}$, $g = 0$, $\omega = (x_1^2 + x_2^2)^{1/2}$, редукує рівняння (14) з дійсною нелінійністю до ЗДР

$$\omega\varphi_1\ddot{\varphi}_2 + 2\omega^2\dot{\varphi}_1\dot{\varphi}_2 - \varphi_1\dot{\varphi}_2 = 0, \quad \lambda\dot{\varphi}_2^2 - \varphi_2 = 0, \quad \lambda\ddot{\varphi}_1 - \omega^{-1}\dot{\varphi}_1 = F(\varphi_1)\varphi_1.$$

7. Анзац для фазового рівняння Шредінгера. Розв'язки рівняння (22) будемо шукати у вигляді (29). Для знаходження явного вигляду функцій $f_1(x)$, $f_2(x)$, $\omega(x)$, скористаємось тим, що фазове рівняння Шредінгера умовно інваріантне відносно алгебри A_{15} (див. теорему 7). Наведемо деякі приклади неліївської редукції рівняння (22) до системи ЗДР.

1) По підалгебрі корозмірності 1 $\langle Q^{(2)} + kI, J_{ab}\rangle$ можна побудувати анзац (29), де

$$f_1 = (\boldsymbol{x}^2)^{k/2}, \quad f_2 = \exp\{2\beta x_0\}\boldsymbol{x}^2, \quad \omega = \exp\{2\beta x_0\}, \tag{48}$$

який редукує фазове рівняння Шредінгера (22) до системи ЗДР

$$\begin{aligned} &\beta\varphi_1' + \lambda(2k + n)\varphi_1\varphi_2 = 0, \\ &\beta\varphi_2' + 2\lambda\varphi_2^2 = 0, \quad (k^2 + kn - 2k)(\boldsymbol{x}^2)^{-1} = 0. \end{aligned} \tag{49}$$

Очевидно, ця система сумісна тільки тоді, коли $k = 0$, та $k + n - 2 = 0$. Загальний розв'язок системи редукованих рівнянь (49) має вигляд

$$\varphi_1 = c_2(2\lambda\omega + c_1)^{-(2k+n)\beta/2}, \quad \varphi_2 = \frac{\beta}{2\lambda\omega + c_1}, \quad c_1, c_2 \in \mathbb{R}.$$

Підстановка φ_1, φ_2 в анзац (29), (48) дає такий розв'язок нелінійного рівняння (22):

$$u = (\boldsymbol{x}^2)^{k/2}c_2(c_1 + 2\lambda\exp\{2\beta x_0\})^{-(2k+n)\beta/2}\exp\left\{i\boldsymbol{x}^2\frac{\beta\exp\{2\beta x_0\}}{2\lambda\exp\{2\beta x_0\} + c_1}\right\},$$

де k задовольняє $k(k + n - 2) = 0$, n — число просторових змінних.

2) Анзац (29), де

$$f_1 = (x_1^2 + x_2^2)^{k/2}, \quad f_2 = (x_1^2 + x_2^2)\exp\{2\beta x_0\}, \quad \omega = \operatorname{arctg}\frac{x_2}{x_1} - \exp\{2\beta x_0\}, \tag{50}$$

редукує рівняння (22) до системи ЗДР

$$\lambda\varphi_2''\varphi_1 + 2\lambda\varphi_1'\varphi_2' - \beta\varphi_1' + 2\lambda\varphi_1\varphi_2(1 + k) = 0, \tag{51a}$$

$$\lambda\varphi_2' + 2\lambda\varphi_2^2 - \beta\varphi_2' = 0, \tag{51b}$$

$$2\varphi_1'' + k^2\varphi_1 = 0. \tag{51c}$$

З рівняння (51c) при $k = 0$ одержуємо $\varphi_1 = c_1\omega + c_2$, $c_1, c_2 \in \mathbb{R}$. Загальна розв'язок рівняння (51в) має вигляд $\varphi_2 = \frac{\lambda-\beta}{2\lambda\omega+c_3}$, $c_3 \in \mathbb{R}$. Підстановка цих значень функцій φ_1 і φ_2 в рівняння (51a) приводить до таких умов: 1) $c_2 = c_3 = 0$, $\lambda = 2\beta$; 2) $c_3 = 2c_2/c_1$, $\lambda = 2\beta = 1$.

Таким чином, загальний розв'язок перевизначеної системи рівнянь (51a)–(51c) при $k = 0$ приймає такі значення:

1. $\varphi_1 = c_1\omega, \quad \varphi_2 = (4\omega)^{-1} \quad$ при $\lambda = 2\beta$;
2. $\varphi_1 = c_1\omega + c_2, \quad \varphi_2 = \dfrac{1}{4\omega + 4c_2/c_1} \quad$ при $\lambda = 2\beta = 1, \quad c_1 \neq 0$.

Підстановка цих значень $\varphi_1(\omega)$, $\varphi_2(\omega)$ в анзац (29), (50) задаватиме точний розв'язок (22).

Отже, виходячи з умовної інваріантності фазового рівняння Шредінгера відносно алгебри A_{15}, можна проводити неліївську редукцію і знаходити точні нетривіальні розв'язки цього нелінійного рівняння.

Зауваження 4. Симетрійним аналогом фазового рівняння Шредінгера (22) для випадку, коли функція u дійсна, є таке нелінійне рівняння теплопровідності

$$u_0 + \lambda\Delta u = \beta u \ln u, \quad \lambda, \beta \in \mathbb{R}.$$

В роботі [11] вказано додаткову умову, при якій це рівняння умовно інваріантне відносно двох різних представлень розширеної алгебри Галілея $AG_1(1, n)$. Зауважимо, що вдається знайти загальний розв'язок одержаної перевизначеної системи рівнянь. Він має вигляд

$$u = \exp\{(\alpha_a x_a - \lambda\beta^{-1}\alpha^2 \exp\{\beta x_0\} + \alpha_0)\} \exp\{\beta x_0\},$$
$$\alpha^2 = \alpha_a\alpha_a, \quad a = \overline{1, n}, \quad \alpha_\mu \in \mathbb{R}.$$

(В цитованій роботі вказано лише частковий розв'язок даної системи.)

8. Розділення змінних для нелінійного рівняння (11). Розв'язки галілей-інваріантних рівнянь типу (14) будемо шукати у вигляді

$$u = f(x_0, \boldsymbol{x})\varphi_1(\omega^1)\varphi_2(\omega^2), \quad \omega^k = \omega^k(x_0, \boldsymbol{x}), \quad k = 1, 2. \tag{52}$$

Опишемо всі функції $F(uu^*)$, f, ω^1, ω^2 такі, щоб анзац (52) зводив рівняння (14) до системи рівнянь

$$\Phi^k(\omega^k, \varphi_k, \varphi_k', \varphi_k'') = 0, \quad k = 1, 2, \tag{53}$$

де φ_k — нові комплекснозначні функції, кожна з яких залежить від однієї змінної ω^k, $\varphi_k' = \partial\varphi_k/\partial\omega^k$, $\varphi_k'' = \partial^2\varphi_k/\partial(\omega^k)^2$.

Підставляючи (52) в (14), одержуємо

$$\left\{i\frac{f_0}{f} + \lambda\frac{\Delta f}{f}\right\} + \frac{\varphi_k'}{\varphi_k}\left\{i\omega_0^k + 2\lambda\frac{f_a}{f}\omega_a^k + \lambda\Delta\omega^k\right\} +$$
$$+ 2\lambda\frac{\varphi_1'\varphi_2'}{\varphi_1\varphi_2}\omega_a^1\omega_a^2 + \lambda\frac{\varphi_k''}{\varphi_k}\omega_a^k\omega_a^k = F(ff^*\varphi_1\varphi_1^*\varphi_2\varphi_2^*)$$

(тут по індексах k, що повторюються, проводиться сумування). З останнього рівняння випливає наступна теорема.

Теорема 10. *Для того щоб анзац (52) зводив рівняння (14) до системи рівнянь (53), необхідно, щоб функція $F(uu^*)$ задовольняла (5), а також виконувалися умови:*

$$
\begin{gathered}
if_0 + \lambda\Delta f - \lambda_3 f \ln(ff^*) = f(R^1(\omega^1) + R^2(\omega^2)), \\
i\omega_0^k + 2\lambda\frac{f_a}{f}\omega_a^k + \lambda\Delta\omega^k = G^k(\omega^k), \quad \omega_a^1\omega_a^2 = 0, \quad \lambda\omega_a^k\omega_a^k = H^k(\omega^k).
\end{gathered} \tag{54}
$$

При виконанні умов теореми 10 рівняння (14) розщеплюється на таких два рівняння:

$$
R^k(\omega^k)\varphi_k + G^k(\omega^k)\varphi_k' + H^k(\omega^k)\varphi_k'' = \lambda_3\varphi_k(\varphi_k\varphi_k^*),
$$

де індекс k приймає значення $k = 1, 2$.

Розглянемо випадок, коли

$$
n = 3, \quad f = f(x_0, x_3), \quad \omega^k = \omega^k(x_0, x_k) \tag{55}
$$

і функція f задовольняє рівняння Шредінгера з логарифмічною нелінійністю (11).

Наслідок 9. *Анзац (52), (55) розщеплює рівняння (11) до системи рівнянь*

$$
G^k(\omega^k)\varphi_k' + H^k(\omega^k)\varphi_k'' = \lambda_3\varphi_k(\varphi_k\varphi_k^*), \tag{56}
$$

де ω^k задовольняють систему

$$
i\omega_0^k + \lambda\Delta\omega^k = G^k(\omega^k), \quad \lambda\omega_k^k\omega_k^k = H^k(\omega^k), \quad k = 1, 2.
$$

Для часткового випадку $\omega^k = x_k$ система (56) зводиться до рівнянь $\lambda\varphi_k'' = \lambda_3\varphi_k(\varphi_k\varphi_k^*)$, $\lambda_3 = b + ib_1$.

1. Фущич В.И., Штелень В.М., Серов Н.И., Симметрийный анализ и точные решения нелинейных уравнений математической физики, Киев, Наук., думка, 1989, 336 с.
2. Chopyk V.I., Symmetry and reduction of multi-dimensional Schrödinger equation with the logarithmic nonlinearity, in Symmetry analysis of equations of mathematical physics, Kiev, Institute of Mathematics, 1992, 54–62.
3. Bialynicki-Birula I., Mucielski J., Nonlinear wave mechanics, *Ann. Phys.*, 1976, **100**, № 1–2, 62–93.
4. Schuch D., Chung K.-M., Hartman H., Nonlinear Schrödinger-type equation for the description of dissipative systems. I. Derivation of the nonlinear field equation and one-dimensional examples, *J. Math. Phys.*, 1983, **24**, № 6, 1652–1660.
5. Brüll L., Lange H., The Schrödinger–Langevin equation: special solutions and nonexistence of solitary waves, *J. Math. Phys.*, 1984, **25**, № 4, 786–790.
6. Фущич В.І., Чопик В.І., Умовна інваріантність нелінійного рівняння Шредінгера, *Допов. АН УРСР, Сер. А*, 1990, № 4, 30–33.
7. Фущич В.И., Баранник Л.Ф., Баранник А.Ф., Подгрупповой анализ групп Галилея, Пуанкаре и редукция нелинейных уравнений, Киев, Наук. думка, 1991, 299 с.
8. Burdet G., Patera J., Perrin M., Winternitz P., The optical group and its subgroups, *J. Math. Phys.*, 1978, **19**, № 8, 1758–1786.
9. Clarkson P., Dimensional reductions and exact solutions of a generalized nonlinear Schrödinger equation, *Nonlinearity*, 1992, **5**, 453–472.
10. Чопик В., Неліївська редукція нелінійного рівняння Шредінгера, *Укр. мат. журн.*, 1991, **43**, № 11, 1504–1509.
11. Myronyuk P., Chopyk V., Conditional Galilei-invariance of multi-dimensional heat equation, in Symmetry analysis of equations of mathematical physics, Kiev, Institute of Mathematics, 1992, 66–68.

Symmetry analysis and ansatzes for the Schrödinger equations with the logarithmic nonlinearity

W.I. FUSHCHYCH, V.I. CHOPYK

Symmetry properties of the Schrödinger equations with the nonlinearity $u\ln(uu^*)$ are investigated. It is shown that these equations are invariant with respect to various extensions of the Galilei algebra $AG(1,n)$. The conditional symmetry of these nonlinear Schrödinger equations are investigated. Lie, non-Lie dimensional reduction and reduction by number of dependent variables carried out. The exact solutions of these equations are constructed.

1. Introduction. Let us consider the Schödinger equations with the logarithmic nonlinearity:

$$Su \equiv bu\ln(uu^*), \quad b \in \mathbb{R} \tag{1}$$

and

$$Su \equiv (\lambda_1 + i\lambda_2)u\ln(uu^*), \quad \lambda_2 \neq 0, \tag{2}$$

where $S = i\frac{\partial}{\partial x_0} + \lambda\Delta$, $x_0 \equiv t$, $\Delta = \frac{\partial^2}{\partial x_a \partial x_a}$, $a = \overline{1,n}$, $\lambda, \lambda_i \in \mathbb{R}$, n is the number of space variables.

For the case when b is a real constant the equation (1) is equivalent to the equation suggested by I. Bialynicki-Birula and J. Mycielski [1]. The equation (1) is investigated by many authors using different methods (see e.q. [2, 3]). For this case the equation of continuity:

$$\begin{gathered}\frac{\partial\rho}{\partial x_0} + \operatorname{div}\boldsymbol{j} = 0,\\ \rho = (uu^*), \quad \boldsymbol{j} = (j_1, j_2, \dots, j_n), \quad j_a = -i\lambda\left(u^*\frac{\partial u}{\partial x_a} - u\frac{\partial u^*}{\partial x_a}\right), \quad a = \overline{1,n}\end{gathered} \tag{3}$$

is satisfied.

For the case when $\lambda_2 \neq 0$ the equation of continuity (3) is not satisfied and the formula:

$$\frac{\partial\rho}{\partial x_0} + \operatorname{div}\boldsymbol{j} = \lambda_2\rho\ln\rho$$

can be considered instead of condition (3).

For the equation (2) the conditions:

$$\frac{\partial\rho}{\partial x_0}j_a + \frac{\partial}{\partial x_b}T_{ab} = 0,$$

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where T_{ab} is the stress tensor, $a, b = \overline{1, n}$, are not satisfied (in contrast with the case of the equation (1) [1]).

It will be shown further, that symmetry properties of the equations (1) and (2) are essentially different.

2. Lie symmetry. It is well-known that the equations (1), (2) are invariant under the Galilei algebra $AG(1, n)$ generates by operators:

$$\begin{aligned} &P_0 = \frac{\partial}{\partial x_0}, \quad P_a = \frac{\partial}{\partial x_a}, \quad J_{ab} = x_a P_b - x_b P_a, \\ &Q = i\left(u\frac{\partial}{\partial u} - u^*\frac{\partial}{\partial u^*}\right), \quad G_a = x_0 P_a + \frac{x_a}{2\lambda} Q. \end{aligned} \tag{4}$$

However, it appears that the Lie symmetry of the Schrödinger equations with logarithmic nonlinearity are not exhausted by the algebra (4).

Theorem 1. *The equation (1) is invariant with respect to the algebra:*

$$AG_3(1, n) = \langle AG(1, n), B\rangle, \tag{5}$$

where $B = I - 2bx_0 Q$, $I = u\frac{\partial}{\partial u} + u^*\frac{\partial}{\partial u^*}$.

Theorem 2. *The equation (2) is invariant with respect to the algebra:*

$$AG_4(1, n) = \langle AG(1, n), C\rangle, \tag{6}$$

where $C = \exp\{2\lambda_2 x_0\}\left(I - \frac{\lambda_1}{\lambda_2}Q\right)$, *when* $\lambda_2 \neq 0$.

The above theorems can be proved using the Lie algorithm [4, 5].

The operator C generates the following finite transformations [6]:

$$\begin{aligned} &x_0 \to x_0' = x_0, \quad x_a \to x_a' = x_a, \\ &u \to u' = \exp\left\{\theta\left(1 - i\frac{\lambda_1}{\lambda_2}\right)\exp(2\lambda_2 x_0)\right\} u, \end{aligned} \tag{7}$$

where θ is a group parameter.

Under transformations (7), the equation (2) becomes:

$$\exp\left\{-\theta\left(1 - i\frac{\lambda_1}{\lambda_2}\right)\exp(2\lambda_2 x_0)\right\}[Su' - (\lambda_1 + i\lambda_2)u' \ln(u' u'^*)].$$

This shows that the equation (2) is invariant with respect to the operator C.

Note 1. Solutions of the equation (1) can be generated by means of transformations [1]:

$$x_0 \to x_0' = x_0, \quad x_a \to x_a' = x_a, \quad u \to u' = \exp\{\theta(1 - 2ibx_0)\}u$$

which are generated by the operator B.

From the commutation relations for the operator C

$$[C, P_0] = d_1 C, \quad [C, P_a] = [C, J_{ab}] = [C, Q] = [C, G_a] = 0$$

and for the operator B

$$[B, P_0] = d_2 Q, \quad [B, P_a] = [B, J_{ab}] = [B, Q] = [B, G_a] = 0, \quad d_1, d_2 \in \mathbb{R}$$

it follows that the algebras $AG_3(1, n)$ and $AG_4(1, n)$ differ.

3. Lie reduction by number of independent variables. In this paper we systematically use symmetry properties of equations (1) and (2) to find their exact solutions. The method of finding exact solutions of differential equations is based on Lie's ideas of invariant solutions and it is described in full detail in [4, 5].

In this section we describe the some ansatzes of codimension 1 and 2

$$u = f(x_0, \boldsymbol{x})\rho(\omega_1, \omega_2)\exp\{g(x_0, \boldsymbol{x}) + \varphi(\omega_1, \omega_2)\},$$

where the functions f, g and new variables $\omega_i = \omega_i(x_0, \boldsymbol{x})$ are determined by means of operators of subalgebras of $AG_3(1,n)$ and $AG_4(1,n)$.

Let us consider some subalgebras of $AG_3(1,n)$, which reduce the equation (1) to system of differential equations with one and two independent variables.

1) $\langle B + \alpha P_0, J_{ab}\rangle$, $\alpha \neq 0$. The ansatz and corresponding systems of reduced equations has the form:

$$u = \exp\left\{\frac{x_0}{\alpha}\right\}\rho(\omega)\exp\left\{i\left[-\frac{b}{\alpha}x_0^2 + \varphi(\omega)\right]\right\}, \quad \alpha \neq 0, \quad \alpha \in \mathbb{R}, \tag{8}$$

where $\omega = (\boldsymbol{x}^2)^{1/2}$, $\boldsymbol{x}^2 = x_1^2 + \cdots + x_n^2$ and

$$\begin{aligned}
&\frac{1}{\alpha}\rho + 2\lambda\dot\rho\dot\varphi + \lambda\rho\ddot\varphi + \lambda\rho\frac{n-1}{\omega}\dot\varphi = 0,\\
&\lambda\ddot\rho + \lambda(n-1)\omega^{-1}\rho - \lambda\rho\dot\varphi^2 = 2b\rho\ln\rho,
\end{aligned}$$

where $\dot\rho = \frac{\partial\rho}{\partial\omega}$, $\dot\varphi = \frac{\partial\varphi}{\partial\omega}$, $\ddot\rho = \frac{\partial^2\rho}{\partial\omega^2}$, $\ddot\varphi = \frac{\partial^2\varphi}{\partial\omega^2}$.

2) $\langle B + \alpha P_0, J_{12} + \beta P_3\rangle$, $\alpha, \beta \neq 0$, $\alpha, \beta \in \mathbb{R}$

$$u = \exp\left\{\frac{x_0}{\alpha}\right\}\rho(\omega_1, \omega_2)\exp\left\{i\left[-\frac{b}{\alpha}x_0^2 + \varphi(\omega_1, \omega_2)\right]\right\}, \quad \alpha \neq 0, \quad \alpha \in \mathbb{R}, \tag{9}$$

where

$$\omega_1 = (x_1^2 + x_2^2)^{1/2}, \quad \omega_2 = \operatorname{arctg}\frac{x_2}{x_1} - \frac{x_3}{\beta}.$$

The system of reduced equations has the form (for the case $n = 3$)

$$\begin{aligned}
&\alpha^{-1}\rho + 2\lambda\rho_1\varphi_1 + 2\lambda\rho_2\varphi_2(\omega_1^{-2} + \beta^{-2}) + \lambda\rho\varphi_{11} +\\
&\quad + \lambda\rho\varphi_{22}(\omega_1^{-2} + \beta^{-2}) + \lambda\rho\omega_1^{-1}\varphi_1 = 0,\\
&\lambda\rho_{11} + \lambda\rho_{22}(\omega_1^{-2} + \beta^{-2}) + \lambda\rho\omega_1^{-1} - \lambda\rho\varphi_1^2 + \rho\varphi_2^2(\omega_1^{-2} + \beta^{-2}) = 2b\rho\ln\rho.
\end{aligned}$$

3) The ansatz

$$u = \exp\left\{\frac{x_0}{\alpha}\right\}\rho(\omega_1, \omega_2)\exp\left\{i\left[\frac{x_0x_1}{\alpha} - \frac{b}{\alpha}x_0^2 - \frac{x_0^3}{6\lambda\alpha} + \varphi(\omega_1, \omega_2)\right]\right\} \tag{10}$$

when $n = 3$ reduces equation (1) to the system:

$$\begin{aligned}
&2\lambda\rho_1\varphi_1 + 2\lambda\rho_2\varphi_2 + \alpha^{-1}\rho + \lambda\rho(\omega_2^{-1}\varphi_2 + \varphi_{11} + \varphi_{22}) = 0,\\
&\lambda\rho_{11} + \lambda\rho_{22} + \lambda\omega_2^{-1}\rho_2 - \lambda\rho(\varphi_1^2 + \varphi_2^2) = 2b\rho\ln\rho - (2\lambda\alpha)^{-1}\rho\omega_1,
\end{aligned} \tag{11}$$

where

$$\omega_1 = \frac{\lambda x_0^2}{\alpha} - x_1, \quad \omega_2 = (x_2^2 + x_3^2)^{1/2}, \quad \alpha \neq 0,$$
$$\rho_i = \frac{\partial \rho}{\partial \omega_i}, \quad \varphi_i = \frac{\partial \varphi}{\partial \omega_i}, \quad i = 1, 2.$$

Solving the system of reduced equations (11) one can following partial solution of the equation (1)

$$u = \exp\left\{ \frac{x_0^2}{8\lambda\alpha b} + \frac{x_0}{\alpha} - d_1^a x_a + c_1 + \right. \\ \left. + i\left[-\frac{x_0}{6\lambda\alpha} - \frac{2b}{\alpha} x_0^2 + \frac{x_0}{\alpha} d_2^a x_a + d_3^a x_a + c_2 \right] \right\}, \tag{12}$$

where $d_k^a, c_i, \alpha \in \mathbb{R}$, $k = 1, 2, 3$, $a = \overline{1, n}$ and d_k^a satisfy the following conditions:

$$d_1^a d_1^a = \frac{1}{8b^2\lambda^2\alpha}, \quad d_1^a d_2^a = \frac{1}{8\lambda^2 b}, \quad d_1^a d_3^a = -\frac{1}{2\lambda\alpha},$$
$$d_2^a d_2^a = \frac{\alpha}{4\lambda^2}, \quad d_2^a d_3^a = -\frac{b}{\lambda}, \quad d_3^a d_3^a = \frac{1}{16\lambda\alpha b^2} - 2bc_1.$$

It is easy to see that the exact solution (12) of the nonlinear equation (1) is non-analytical by b.

Note 2. The ansatzes (7)–(9) follows from the fact that the equation (1) is invariant to the operator B.

Let us adduce some examples of reduction of equation (2).

Example 1. $\langle C + \alpha P_3, J_{12} \rangle$. The ansatz

$$u = \exp\left\{ \frac{1}{\alpha} \exp(2\lambda_2 x_0) x_3 \right\} \rho(\omega_1, \omega_2) \times \\ \times \exp\left\{ i\left[-\frac{\lambda_1}{\alpha\lambda_2} \exp(2\lambda_2 x_0) x_3 + \varphi(\omega_1, \omega_2) \right] \right\}, \quad \alpha \neq 0, \tag{13}$$

where $\omega_1 = x_0$, $\omega_2 = (x_1^2 + x_2^2)^{1/2}$ reduces equation (2) (when $n = 3$) to the system:

$$\rho_1 + 2\lambda\rho_2\varphi_2 + \lambda\rho(\omega_2^{-1}\rho_2 + \varphi_{22}) = 2\lambda_2\rho\ln\rho,$$
$$\alpha^{-2}\lambda\exp(4\lambda_2\omega_1)(1 - \lambda_1^2\lambda_2^{-2})\rho + \lambda\rho_{22} + \lambda\omega_2^{-1}\rho_2 - \rho\varphi_1 - \lambda\rho\varphi_2^2 = 2\lambda_1\rho\ln\rho.$$

Example 2. The ansatz

$$u = \exp\left\{ \frac{1}{2\lambda_2\alpha} \exp(2\lambda_2 x_0) \right\} \rho(\omega) \exp\left\{ i\left[-\frac{\lambda_1}{2\alpha\lambda_2^2} \exp(2\lambda_2 x_0) + \varphi(\omega) \right] \right\}, \tag{14}$$

where $\omega = (\boldsymbol{x}^2)^{1/2}$ reduces (2) when $\lambda_2 \neq 0$ to the system of ODE:

$$2\lambda\dot\rho\dot\varphi + \lambda\rho\ddot\varphi + \lambda\rho\omega^{-1}(n-1)\dot\varphi = 2\lambda_2\rho\ln\rho,$$
$$\lambda\ddot\rho + \lambda(n-1)\omega^{-1}\rho + \lambda\rho\dot\varphi^2 = 2\lambda_1\rho\ln\rho.$$

Example 3. The ansatz

$$
\begin{aligned}
u = \exp\left\{\operatorname{arctg}\frac{x_2}{x_1}\exp(2\lambda_2 x_0)\right\}\rho(\omega_1,\omega_2)\times \\
\times\exp\left\{i\left[\frac{\lambda_1}{\lambda_2}\exp(2\lambda_2 x_0)\operatorname{arctg}\frac{x_2}{x_1}+\frac{x_3^2+\cdots+x_n^2}{4\lambda x_0}+\varphi(\omega_1,\omega_2)\right]\right\},
\end{aligned}
\tag{15}
$$

where $\omega_1 = x_0$, $\omega_2 = (x_1^2+x_2^2)^{1/2}$ reduces the equation (2) (when $n \geq 2$) to the system:

$$
\begin{aligned}
&\rho_1 + 2\lambda\exp(4\lambda_2\omega_1)\frac{\lambda_1}{\lambda_2}\rho\omega_2^{-2} + \omega_2^2\varphi_2\rho_2 + 2\lambda\rho\omega_2^{-1}\varphi_2 + \lambda\rho\varphi_{22} + \frac{n-2}{2\omega_1}\rho = \\
&\qquad = 2\lambda_2\rho\ln\rho, \\
&\lambda\exp(4\lambda_2\omega_1)\omega_2^{-2}\rho(1-\lambda_1^2\lambda_2^{-2}) + 2\lambda\rho_2(1+\omega_2^{-1}) - \rho\varphi_1 - \lambda\rho\varphi_2^2 = 2\lambda_1\rho\ln\rho.
\end{aligned}
$$

Example 4. The ansatz

$$
\begin{aligned}
u = \exp(\exp(2\lambda_2 x_0)x_1\rho(x_0)\times \\
\times\exp\left\{i\left[\frac{\lambda_1}{\lambda_2}x_1\exp(2\lambda_2 x_0)+\frac{x_2^2+\cdots+x_n^2}{4\lambda x_0}+\varphi(x_0)\right]\right\},
\end{aligned}
\tag{16}
$$

reduces the equation (2) when $\lambda_2 \neq 0$ to the system:

$$
\begin{aligned}
&\dot\rho - 2\lambda\lambda_1\lambda_2^{-1}\rho\exp(4\lambda_2 x_0) + \frac{n-1}{2x_0} = 2\lambda_2\rho\ln\rho, \\
&\dot\varphi = \lambda\exp(4\lambda_2 x_0) + \lambda\lambda_1\lambda_2^{-1}\exp(2\lambda_2 x_0) - 2\lambda_1\ln\rho.
\end{aligned}
\tag{17}
$$

The system of equations (17) by means of the change of variables $\rho = \exp\phi$ is reduced to a linear system of ODE which has the general solution of the form

$$
\begin{aligned}
&\phi = \frac{\lambda\lambda_1}{\lambda_2^2}\exp(4\lambda_2 x_0) - \exp(2\lambda_2 x_0)\left(d_1 + \frac{n-1}{2}F(2\lambda_2)\right), \quad d_1\in\mathbb{R}, \\
&\varphi = \frac{\lambda}{4\lambda_2}\exp(4\lambda_2 x_0)\left(1-\frac{2\lambda_1^2}{\lambda_2^2}\right) + \frac{\lambda_1}{2\lambda_2}(\lambda+2d_1)\exp(2\lambda_2 x_0) + \\
&\qquad + \lambda_1(n-1)\int F(2\lambda_2)\exp(2\lambda_2 x_0)dx,
\end{aligned}
\tag{18}
$$

where

$$
F(\theta) = \int\exp(-\theta x_0)\frac{dx_0}{x_0}.
$$

The substitution of (18) into the ansatz (16) gives the following solution of the equation (2) when $\lambda_2 \neq 0$ for $n = 1$

$$
\begin{aligned}
u = \exp\left\{(x_1-d_1)\exp(2\lambda_2 x_0) + \frac{\lambda\lambda_1}{\lambda_2^2}\exp(4\lambda_2 x_0) + \right. \\
\left. + i\left[\left(\frac{\lambda\lambda_1+2\lambda_1\lambda_2 d_1}{2\lambda_2^2} - \frac{\lambda_1}{\lambda_2}x_1\right)\exp(2\lambda_2 x_0) + \frac{\lambda\lambda_2^2-2\lambda\lambda_1^2}{4\lambda_2^3}\exp(2\lambda_2 x_0)\right]\right\}.
\end{aligned}
$$

Note 3. The ansatzes (13)–(16) are obtained from the fact that the equation (2) is invariant with respect to the algebra $AG_4(1,n)$ (as distinct from the equation (1)).

4. Component-wise reduction. The reduction by number of dependent variables of the equations (1), (2) is possible because of invariance these equations respectively to the operators B and C.

1) For reduction of the equation (1) by operator B it is necessary to change or variables:

$$W = F(x_0, \boldsymbol{x}) - i(4bx_0)^{-1}\ln(u/u^*), \quad V = \ln|u| - i(4bx_0)^{-1}\ln(u/u^*), \tag{19}$$

where F is a some real function.

Then the change of variables (19) is constructed, the equation (1) has the form:

$$\begin{gathered} F_0 - W_0 + V_0 + 4\lambda bx_0(F_a - W_a + V_a)(W_a + F_a) + 2\lambda bx_0(\Delta W - \Delta F) = 0, \\ \lambda(F_a - W_a + V_a)(F_a - W_a + V_a) + \lambda(\Delta F - \Delta W + \Delta V) - \\ - 2bx_0(W_0 - F_0) - 4\lambda b^2x_0^2(W_a - F_a)(W_a - F_a) = 2bV, \end{gathered}$$

where $F_\mu = \frac{\partial F}{\partial x_\mu}$, $W_\mu = \frac{\partial W}{\partial x_\mu}$, $V_\mu = \frac{\partial V}{\partial x_\mu}$, $\Delta = \frac{\partial^2}{\partial x_a \partial x_a}$ and the operator B has the form $B = \frac{\partial}{\partial W}$.

The reduction of the equation (1) by operator B is equivalent to the condition $W = 0$.

Thus, we can find the solutions of the equation (1) in the form:

$$u = \exp\{V(x_0, \boldsymbol{x}) + (1 - 2ibx_0)F(x_0, \boldsymbol{x})\}, \tag{20}$$

where functions V and F satisfy the system:

$$\begin{gathered} F_0 + V_0 - 4\lambda bx_0(F_a + V_a)F_a - 2\lambda bx_0\Delta F = 0, \\ \lambda(F_a + V_a)(F_a + V_a) + \lambda(\Delta F + V) + 2bx_0(F_0 - 2\lambda bx_0F_aF_a) = 0. \end{gathered} \tag{21}$$

Case 1. The functions V and F satisfy the conditions:

$$F = f_1(x_0), \quad V = f_2(x_0) + \varphi(\omega), \quad \omega = \omega(\boldsymbol{x}). \tag{22}$$

Substitution of the expression (22) into (21) yields the ODE

$$(\ddot{\varphi} + \dot{\varphi}^2)\theta_1(\omega) + \dot{\varphi}\theta_2(\omega) = 2b\lambda^{-1}\varphi, \tag{23}$$

where

$$\omega_a\omega_a = \theta_1(\omega), \quad \Delta\omega = \theta_2(\omega), \tag{24}$$

and

$$f_1 = c_2 - c_1x_0^{-1}, \quad f_2 = c_1x_0^{-1}, \quad c_1, c_2 \in \mathbb{R}. \tag{25}$$

Note 4. The necessary conditions of compatibility and the general solution of system (24) construct in papers [7, 8].

For the partially case $\omega = \alpha_a x_a$, $\alpha_a\alpha_a = 1$, $\alpha_a \in \mathbb{R}$, $a = \overline{1, n}$, the equation (23) has the form:

$$\ddot{\varphi} + \dot{\varphi}^2 = 2b\lambda^{-1}\varphi. \tag{26}$$

This equation by means of change of variables

$$\dot{\varphi}^2 = \Phi(\varphi)$$

is reduced to a linear equation:

$$\dot{\Phi}(\varphi) + 2\Phi(\varphi) = 4b\lambda^{-1}\varphi.$$

The last equation can be easily integrated and the result is as follows:

$$\int \left[\varphi + c\exp(-2\varphi) - \frac{1}{2}\right]^{-1/2} d\varphi = (2b\lambda^{-1})^{1/2} d\omega, \quad c \in \mathbb{R}. \tag{27}$$

When $c = 0$ we get from (27) the following solution of (26):

$$\varphi = \frac{b}{2\lambda}(\omega + c_3)^2 + \frac{1}{2}, \quad c_3 \in \mathbb{R}. \tag{28}$$

Summarizing results (20), (22), (25), (28) we write down the exact solution of equation (1):

$$u = \exp\left\{\frac{b}{2\lambda}(\alpha_a x_a + c_3)^2 + c_2 + \frac{1}{2} - 2ib(c_2 x_0 - c_1)\right\},$$

where $c_i \in \mathbb{R}$, $i = 1, 2, 3$, $\alpha_a \alpha_a = 1$.

Case 2. $V = 0$ and F satisfy the overdetermined system:

$$F_0 - 4\lambda b x_0 F_a F_a - 2\lambda b x_0 \Delta F = 0,$$
$$\lambda F_a F_a + \lambda \Delta F + 2b x_0 F_0 - 4\lambda b^2 x_0^2 F_a F_a = 0.$$

For this case the ansatz (2) has the form:

$$u = \exp\{1 - 2ibx_0)F(x_0, \boldsymbol{x})\}. \tag{29}$$

Consequence. *The ansatz (29) gives the solutions of the equation (1) if the real function F satisfy:*

$$F_t - \lambda b F_a F_a = 0, \quad F_t + \lambda b \Delta F = 0, \quad t = x_0^2. \tag{30}$$

The system (30) have non-trivial symmetry properties:

Theorem 3. *The overdetermined system (30) is invariant with respect to the extended Galilei algebra having basis elements:*

$$P_t = \frac{\partial}{\partial t}, \quad t = x_0^2, \quad P_a, \quad J_{ab}, \quad P_{n+1} = \frac{\partial}{\partial F},$$
$$G_a^{(1)} = FP_a - x_a(2\lambda b)^{-1} P_t, \quad D^{(1)} = 2t\partial_t + x_a P_a.$$

Note 5. The operator $G_a^{(1)}$ generates the transformation:

$$t \to t' = t - (2\lambda b)^{-1}\theta_a x_a - (4\lambda b)^{-1}\theta_a^2, \quad x_b \to x_b' = x_b,$$
$$x_a \to x_a' = x_a + \theta_a F, \quad F \to F' = F,$$

where θ_a is a group parameter.

2) For reduction of the equation (2) ($\lambda_1 \neq 0$) by operator C it is necessary to change of variables:

$$W = F(x_0, \boldsymbol{x}, \omega), \quad \omega = \frac{1}{2}\exp(-2\lambda_2 x_0)(\ln|u| - (2i\lambda_1)^{-1}\lambda_2 \ln(u/u^*)),$$
$$V = \lambda_1 \ln|u| - i\frac{1}{2}\lambda_2 \ln(u/u^*). \tag{31}$$

Substituting (31) for the partially case $F_\omega = 1$ into the equation (2) we get:

$$\begin{aligned}
&(2\lambda_1)^{-1}V_0 - \exp(2\lambda_2 x_0)(F_0 - W_0) + 2\lambda[(2\lambda)^{-1}V_a - \\
&\quad - \exp(2\lambda_2 x_0)(F_a - W_a)((2\lambda_2)^{-1}V_a + \lambda_1(\lambda_2)^{-1}\exp(2\lambda_2 x_0)(F_a - W_a)] + \\
&\quad + (2\lambda_2)^{-1}\lambda[\Delta V + 2\lambda_1 \exp(2\lambda_2 x_0)(\Delta F - \Delta W)] - (\lambda_1)^{-1}\lambda_2 V = 0, \\
&\lambda[(2\lambda_1)^{-1}V_a - \exp(2\lambda_2 x_0)(F_a - W_a)(2\lambda_1)^{-1}V_a - \exp(2\lambda_2 x_0)(F_a - W_a)] + \\
&\quad + \lambda[(2\lambda_1)^{-1}\Delta V - \exp(2\lambda_2 x_0)(\Delta F - \Delta W)] - (2\lambda_2)^{-1}[V_0 + \\
&\quad + 2\lambda_1 \exp(2\lambda_2 x_0)(F_0 - W_0)] - \lambda(4\lambda_2^2)^{-1}[V_a + 2\lambda_1 \exp(2\lambda_2 x_0)(F_a - W_a)] \times \\
&\quad \times [V_a + 2\lambda_2 \exp(2\lambda_2 x_0)(F_a - W_a)] - V = 0,
\end{aligned}$$

and the operator C has the form:

$$C = \frac{\partial}{\partial W}. \tag{32}$$

From (31), (32) follows that the solutions of the equation (2) (with $\lambda_1, \lambda_2 \neq 0$) we can find in the form:

$$u = \exp\left\{(2\lambda_1\lambda_2)^{-1}V(\lambda_2 + i\lambda_1) - (\lambda_1)^{-1}F\exp(2\lambda_2 x_0)(\lambda_2 - i\lambda_1)\right\},$$

where the real functions V and F satisfy the system:

$$\begin{aligned}
&(2\lambda_1)^{-1}V_0 - \exp(2\lambda_2 x_0)F_0 + 2\lambda[(2\lambda_1)^{-1}V_a - \exp(2\lambda_2 x_0)(F_a - (2\lambda_2)^{-1}V_a) + \\
&\quad + \lambda_1(\lambda_2)^{-1}\exp(2\lambda_2 x_0)F_a] + \\
&\quad + (2\lambda_2)^{-1}\lambda[\Delta V + 2\lambda_1\exp(2\lambda_2 x_0)\Delta F] - (\lambda_1)^{-1}\lambda_2 V = 0, \\
&\lambda[(2\lambda_1)^{-1}V_a - \exp(2\lambda_2 x_0)F_a] + \lambda[(2\lambda_1)^{-1}\Delta V - \exp(2\lambda_2 x_0)\Delta F - \\
&\quad - (2\lambda_2)^{-1}[V_0 + 2\lambda_1\exp(2\lambda_2 x_0)F_0] - \\
&\quad - \lambda(4\lambda_2^2)^{-1}[V_a + 2\lambda_1\exp(2\lambda_2 x_0)F_a]^2 - V = 0.
\end{aligned} \tag{33}$$

Case 1: $V = 0$. For this case the ansatz

$$u = \exp\left\{-(\lambda_1)^{-1}F\exp(2\lambda_2 x_0)(\lambda_2 - i\lambda_1)\right\}$$

reduces the equation (2) when $\lambda_1 \neq 0$ to the system:

$$\begin{aligned}
&F_0 + \lambda\lambda_1(\lambda_2)^{-1}\Delta F = 0, \\
&F_0 + \lambda\lambda_1(\lambda_2)^{-1}\exp(2\lambda_2 x_0)F_aF_a = 0.
\end{aligned}$$

Case 2: $F = 0$. For this case the ansatz

$$u = \exp\left\{(2\lambda_1\lambda_2)^{-1}V(\lambda_2 + i\lambda_1)\right\}$$

reduces the equation (2) with $\lambda_1 \neq 0$ to the overdetermined system:

$$\begin{aligned}
&V_0 + \lambda(\lambda_2)^{-1}V_aV_a + \lambda\lambda_1(\lambda_2)^{-1}\Delta V - 2\lambda_2 V = 0, \\
&V_0 + \lambda(\lambda_1^2 - \lambda_2^2)(2\lambda_1\lambda_2)^{-1}V_aV_a - \lambda\lambda_2(\lambda_1)^{-1}\Delta V + 2\lambda_2 V = 0.
\end{aligned} \tag{34}$$

For the partially case $\lambda_1^2 = \lambda_2^2$ the system (34) has the form:

$$V_0 + \lambda(2\lambda_2)^{-1}V_aV_a = 0, \quad V_0 + \lambda\Delta V \mp 2\lambda_2 V = 0,$$

and for the partially case $3\lambda_1^2 = \lambda_2^2$ this system has the form:

$$V_0 + \lambda(2\lambda_2)^{-1}V_aV_a - \lambda_2 V = 0, \quad \sqrt{3}V_0 \mp \lambda\Delta V = 0.$$

5. Conditional symmetry. The symmetry of the equations (1), (2) can be extended essentially, if we put a certain additional condition on its solutions (see [4, 9, 10]). As to Schrödinger equations with the logarithmic nonlinearity one of such additional conditions is vanishing of the interior potential [11] that is equivalent to the following condition:

$$\Delta|u| = 0, \quad |u| = (uu^*)^{1/2}. \tag{35}$$

Theorem 4. *The equation (1) is conditionally invariant with respect to the following algebras:*

$$1)\ AG_5(1,n) = \langle AG_3(1,n), Q_1\rangle,$$

where

$$Q_1 = x_0P_0 + x_aP_a - \frac{i}{2}\ln(uu^{*-1})Q$$

with additional condition (35);

$$2)\ AG_5(1,n) = \langle AG(1,n), Q_2\rangle,$$

where the operator Q_2 is of the form [9]:

$$Q_2 = \frac{i}{2}\ln(uu^{*-1})Q + x_0P_0$$

if the module of the function u satisfies the condition

$$\lambda\Delta|u| = 2b|u|\ln|u|. \tag{36}$$

Note 6. The operator Q_1 generates the following finite transformations:

$$x_0 \to x_0' = \theta_1 x_0, \quad x_a \to x_a' = \theta_1 x_a, \quad u \to u' = |u|(uu^{*-1})^{1/2\theta_1},$$

and the operator Q_2 generates the following transformations:

$$x_0 \to x_0' = \theta_2 x_0, \quad x_a \to x_a' = x_a, \quad u \to u' = |u|(uu^{*-1})^{-1/2\theta_2},$$

where θ_1 and θ_2 are group parameters.

Theorem 5. *The equation (2) is conditional invariant with respect to the algebra:*

$$AG_7(1,n) = \langle AG_4(1,n), Q_3\rangle,$$

where

$$Q_3 = Q_1 - Q_2 = x_aP_a - i\ln(uu^{*-1})Q,$$

and the operator C is of the form $C = \exp(2\lambda_2 x_0)I$. The additional condition has the form (34).

Note 7. The operator Q_3 generates the transformations:

$$x_0 \to x_0' = x_0, \quad x_a \to x_a' = \theta_3 x_a, \quad u \to u' = |u|(uu^{*-1})^{\theta_3}, \quad a = \overline{1,n}.$$

The following theorems can be proved by means of conditional invariance algorithm (see e.g. [5, 10]).

So we can see that the additional conditions (34) and (35) expand the symmetry of the equations (1), (2).

6. Applications: non-Lie reduction. In this section we consider some non-Lie ansatzes for the equations (1), (2) which cannot, be obtained by means of classical Lie approach. The examples of non-Lie reduction of the Schrödinger equations with degree nonlinearity are adduced in [12, 13].

1) The ansatz

$$u = x_0^2\rho(\omega_1,\omega_2)\exp\{i[\alpha_a x_a - 4bx_0\ln x_0 + x_0\varphi(\omega_1,\omega_2)]\},$$
$$\omega_1 = \frac{x_1}{x_2},\quad \omega_2 = \frac{x_2}{x_0},\quad \alpha_a\in\mathbb{R},\quad a=\overline{1,n} \tag{37}$$

reduces the equation (1) to the system:

$$2\rho - \omega_1\rho_1 - \omega_2\rho_2 + 2\lambda\rho_1\varphi_1 + 2\lambda\rho_2\varphi_2 + \lambda\rho(\varphi_{11}+\varphi_{22}) = 0,$$
$$\rho_{11}+\rho_{22} = 0, \tag{38}$$
$$\lambda\varphi_1^2 + \lambda\varphi_2^2 - \omega_1\varphi_1 - \omega_2\varphi_2 = 4b - \lambda\alpha_a\alpha_a - 2b\ln\rho,\quad a=3,\dots,n,\quad \alpha_a\in\mathbb{R}.$$

2) The ansatz

$$u = x_0^2\rho(\omega_1,\omega_2)\exp\left\{i\left[\frac{x_1^2}{4\lambda x_0} - 4bx_0\ln x_0 + x_0\varphi(\omega_1,\omega_2)\right]\right\}, \tag{39}$$

where

$$\omega_1 = \frac{x_1}{x_0} - \operatorname{arctg}\frac{x_3}{x_2},\quad \omega_2 = \frac{x_2^2+x_3^2}{x_0}$$

reduces the equation (1) (when $n=3$) to the system:

$$2\rho - \omega_1\rho_1(1+\omega_2^{-2}) - \omega_2\rho_2 + \rho_2\varphi_2 + \rho\omega_2\varphi_2 + \rho\varphi_{11}(1+\omega_2^{-2}) + \rho\varphi_{22} = 0,$$
$$\rho_{11}(1+\omega_2^{-2}) + \omega_2^2\rho_{22} + \omega_2\rho_2 = 0, \tag{40}$$
$$\lambda(1+\omega_2^{-2})\varphi_1^2 + \lambda\varphi_2^2 - \omega_2\varphi + \varphi - 4b + 2b\ln\rho = 0.$$

3) The ansatz

$$u = x_0^2\rho(\omega_1,\omega_2)\exp\left\{i\left[\frac{x_1^2}{4\lambda x_0} - 4bx_0\ln x_0 + x_0\varphi(\omega_1,\omega_2)\right]\right\},$$
$$\omega_1 = \frac{x_2}{x_0},\quad \omega_2 = \frac{x_3}{x_0} \tag{41}$$

reduces the equation (1) to the system (when $n=3$):

$$2\rho - \omega_1\rho_1 - \omega_2\rho_2 + 2\lambda\rho_1\varphi_1 + 2\lambda\rho_2\varphi_2 + \frac{1}{2}\rho + 2\lambda\rho(\varphi_{11}+\varphi_{22}) = 0,$$
$$\rho_{11}+\rho_{22} = 0, \tag{42}$$
$$\lambda\varphi_1^2 + \lambda\varphi_2^2 - \omega_1\varphi_1 - \omega_2\varphi_2 + \varphi - 4b + 2b\ln\rho = 0.$$

Note 8. The ansatzes (37), (39), (41) are obtained as a consequence of conditional invariance of the equation (1) respect to the algebra $AG_5(1,n)$.

4) The ansatz

$$u = \exp\left\{\frac{2}{\alpha}\exp(2\lambda_2 x_0)\right\}\rho(\omega)\exp\left\{i\left[\exp\left(\frac{2x_0}{\alpha}\right)\varphi(\omega)\right]\right\},$$
$$\omega = (\boldsymbol{x}^2)^{1/2}\exp\left\{-\frac{x_0}{\alpha}\right\}, \quad \alpha \neq 0, \tag{43}$$

reduces equation (2) with $\lambda_1 = 0$, $\lambda_2 \neq 0$ to the system ODE:

$$\rho\ddot{\varphi} + \dot{\rho}\dot{\varphi} + (n-1)\omega^{-1}\rho\dot{\varphi} + \alpha^{-1}\omega\dot{\rho} = 2\lambda_2\rho\ln\rho,$$
$$\ddot{\rho} + (n-1)\omega^{-1}\dot{\rho} = 0, \tag{44}$$
$$\lambda\alpha\dot{\varphi}^2 - \omega\dot{\varphi} + 2\varphi = 0.$$

The systems of reduced equations (38), (40), (42), (44) are overdetermined. Therefore it is necessary to consider their compatibility.

1. Bialynicki-Birula I., Mycielski J., Nonlinear wave mechanics, *Annals of Phys. (N.Y.)*, 1970, **100**, № 1/2, 62–93.
2. Schuch D., Ghung, K.-M., Hartman H., Nonlinear Schrödinger-type field equation for the description of dissipative systems. I. Derivation of the nonlinear field equation and one-dimensional examples, *J. Math. Phys.*, 1983, **24**, № 6, 1652–1660.
3. Nassar A., New method for the solution of the logarithmic nonlinear Schrödinger equation via stochastic mechanics, *Phys. Rev. A*, 1986, **33**, № 5, 3502–3505.
4. Fushchych V.I., Shtelen V.M., Serov N.I., Symmetry analysis and exact solutions of nonlinear equations of mathematical physics, Kiev, Naukova Dumka, 1989, 336 p.
5. Olver P., Applications of Lie groups to differential equations, N.Y., Springer, 1986, 497 p.
6. Chopyk V.I., Symmetry and reduction of multi-dimensional Schrödinger equation with the logarithmic nonlinearity, in Symmetry Analysis of Equations of Mathematical Physics, Kiev, Inst. of Mathematics of Ukr. Acad. Sci., 1992, 54–62.
7. Collins G.B., Complex potential equations. A technique for solution, *Math. Proc. Gambr. Phyl. Soc.*, 1976, **80**, № 1, 165–171.
8. Fushchych W.I., Zhdanov R.Z., Revenko I.V., General solutions of the nonlinear wave equation and eikonal equation, *Ukr. Math. J.*, 1991, **43**, 1471–1486 (in Russian).
9. Fushchych W., Ghopyk V., Conditional invariance of the nonlinear Schrödinger equations, *Dokl. Ukr. Acad. Sci.*, 1988, № 9, 17–20 (in Ukrainian).
10. Fushchych W.I., Conditional symmetry of equations of nonlinear mathematical physics, in Symmetry Analysis of Equations of Mathematical Physics, Kiev, Inst. of Mathematics of Ukr. Acad. Sci., 1992, 7–27.
11. Wilhelm H.F., Hydrodynamic model of quantum mechanics, *Phys. Rev. D*, 1970, **1**, № 8, 2278–2285.
12. Chopyk V., Non-Lie reduction of the nonlinear Schrödinger equations, *Ukr. Math. J.*, 1991, **43**, № 11, 1504–1509 (in Ukrainian).
13. Fushchych W., Yegorchenko I., Non-Lie ansatzes and conditional symmetry of the nonlinear Schrödinger equation, *Ukr. Math. J.*, 1991, **43**, № 12, 1620–1628 (in Russian).

Symmetry and exact solutions of multidimensional nonlinear Fokker–Planck equation

W.I. FUSHCHYCH, V.I. CHOPYK, V.P. CHERKASENKO

Розглядається нелінійне рівняння Фоккера–Планка. За рахунок накладання на коефіцієнти функції нелінійної додаткової умови вдалось значно розширити симетрію рівняння Фоккера–Планка. Досліджено умовну симетрію, проведено редукцію та знайдено деякі точні розв'язки цього рівняння.

1. Let us consider equation

$$\frac{\partial p}{\partial t} = -\sum_{k=1}^{n} \frac{\partial}{\partial x_k}(A_k\rho) + \frac{1}{2}\sum_{i,k=1}^{n} \frac{\partial^2}{\partial x_i \partial x_k}(B_{ik}\rho) + F(\rho), \tag{1}$$

where $\rho(t,\vec{x})$, $\vec{x} = (x_1,\dots,x_n)$, $A_k(t,\vec{x})$, $B_{ik}(t,\vec{x})$, $F(\rho)$ are smooth real functions. If $F(\rho) = 0$, (1) coincides with classical linear Fokker–Planck equation (FPE), which finds broad application in the theory of Markov processes. In this case [1] ρ is the conditional probability density, $\vec{A} = (A_1, A_2, \dots A_n)$ is a drift velocity vector, B_{ik} are elements of diffusion matrix $B(t,\vec{x}) = \|B_{ik}\|_{i,k=1}^{n}$.

In the cases, when (1) (for $F(\rho) = 0$) is equivalent to the linear heat equation, it is possible to use effectively group-theoretical analysis methods to construct solutions of the linear FPE [2]. In other cases equation (1) for fixed A_k and B_{ik}, as a rule, has no nontrivial symmetry. Thus, it is impossible to apply to it symmetry methods [3].

In [4] a new interpretation for the equations like (1) was proposed, it opens wide possibilities for application of group-theoretical methods. The idea consists in complementing (1) with equations for coefficient functions A_k and B_{ik}. That is we add to (1) some system of equations for A_k and B_{ik}, (1) turns out then to be a nonlinear system (even if $F(\rho) = 0$). Such an extended system, as we show, can have a nontrivial symmetry which is used to construct exact solutions of equation (1).

Therefore our paper is based on the idea of nonlinear extension of equation (1).

2. We require the components of the vector $\vec{A}$ to satisfy conditions having the form of Euler's equation for the ideal liquid

$$\frac{\partial A_k}{\partial t} + A_l \frac{\partial A_k}{\partial x_l} = F_k(\rho). \tag{2}$$

For the potential flow when $A_k = \dfrac{\partial \varphi}{\partial x_k}$, $F_k = \frac{\partial F_1(\rho)}{\partial x_k}$, and $B_{ik} = D^{ik}$ $(D = \mathrm{const} \geq 0)$ equations (1) and (2) can be written as

$$\rho_0 + \rho_a\varphi_a + \rho\Delta\varphi - \frac{D}{2}\Delta\rho = F(\rho), \tag{3}$$

Доповіді АН України, 1993, № 2, С. 32–42.

$$\varphi_0 + \frac{1}{2}\varphi_a\varphi_a = F_1(\rho), \tag{4}$$

where $\rho_0 = \frac{\partial\rho}{\partial x_0}$, $x_0 \equiv t$, $\varphi_a = \frac{\partial\varphi}{\partial x_a}$, $a = \overline{1,n}$. Thus we tend to investigate symmetry properties and to construct families of solutions for (3), (4).

3. We assume D to be nonvanishing.

Theorem 1. *Equation (3) for $D > 0$ is invariant under the following algebras:*

1) $A_1 = \langle P_0, J_{ab}, X_a, Y, P_a\rangle$, *where*

$$P_0 = \frac{\partial}{\partial x_0}, \quad P_a = \frac{\partial}{\partial x_a}, \quad J_{ab} = x_aP_b - x_bP_a, \quad \{a,b\} = \overline{1,n},$$
$$X_a = g_a(x_0)P_a + g'_a(x_0)x_a\frac{\partial}{\partial\varphi}, \quad Y = h(x_0)\frac{\partial}{\partial\varphi},$$

(g_a, h are arbitrary smooth functions) for an arbitrary $F(\rho)$;

2) $A_2 = \langle A_1, D\rangle$, *where the operator of scale transformations D has the form*

$$D = 2x_0P_0 + x_aP_a - \frac{2}{k}I,$$

where $I = \rho\frac{\partial}{\partial\rho}$, for $F = \lambda\rho^{k+1}$, $k \neq 0$;

3) $A_3 = \langle A_2, A\rangle$, *where the operator of projective transformations A has the form*

$$A = x_0^2P_0 + x_0x_aP_a + \frac{\vec{x}^{\,2}}{2}\frac{\partial}{\partial\varphi} - nx_0I, \quad \text{if} \quad F = \lambda\rho^{\frac{2}{n}+1},$$

where $\vec{x}^{\,2} = x_1^2 + x_2^2 + \cdots + x_n^2$;

4) $A_4 = \langle A_1, S\rangle$, *where*

$$S = f(x_0)P_0 + \frac{1}{2}f'(x_0)x_aP_a + \frac{1}{4}f''(x_0)\vec{x}^{\,2}\frac{\partial}{\partial\varphi} - \frac{n}{2}f'(x_0)I$$

(f is an arbitrary smooth function), if $F = 0$;

5) $A_5 = \langle A_1, C_0\rangle$, *where*

$$C_0 = \exp\{\lambda x_0\}I, \quad \text{if} \quad F = \lambda\ln p.$$

Proof of this and following theorems can be made using Lie's algorithm (see, e.g. [5, 6]).

Remark 1. Algebra A_4 coincides with A_3, if we require condition $f''' = 0$ to be satisfied.

Remark 2. In the case $D = 0$ equation (1) turns into Liouville's equation. The question on the symmetry of (4) (if $F = 0$) then can be answered by the following theorem [7].

Theorem 2. *Equation (4) with $D = F = 0$ is invariant under infinitely-dimensional algebra which is generated by the operator*

$$X = 2f_1(x_0)P_0 + f'_1(x_0)x_aP_a + f_2(x_0)\left\{x_aP_a + 2\varphi\frac{\partial}{\partial\varphi}\right\} +$$
$$+ (f''_1 + f'_2)\left\{\frac{\vec{x}^{\,2}}{2}\frac{\partial}{\partial\varphi} - \frac{n}{2}x_0I\right\} + f_{3a}P_a + f'_{3a}(x_0)x_a\frac{\partial}{\partial\varphi} + dI + c_{ab}J_{ab}.$$

where $f_1'''(x_0) + f_2''(x_0) = 0$, $c_{ab} = -c_{ba}$, $\{d, c_{ab}\} \subset \mathbb{R}$, $f_1(x_0)$, $f_2(x_0)$, $f_{3a}(x_0)$, $a = \overline{1,4}$, $f_4(x_0)$ *are an arbitrary smooth functions. Operators* X_a *lead to the following finite transformations:*

$$x'_a = x_a + g_a(x_0)\theta, \quad \varphi' = \varphi + \dot{g}_a(x_0)x_a\theta + \frac{1}{2}g_a(x_0)\dot{g}_a(x_0)\theta^2,$$

$\rho' = \rho$, $x'_0 = x_0$, $x'_b = x_b$, *where* $\dot{g}_a = \frac{dg_a}{dx_0}$, θ *is a group parameter.*

4. Let us now require the condition (4) on φ to be satisfied.

Theorem 3. *The system of equations (3), (4) for* $D \neq 0$ *and arbitrary* F, F_1 *is invariant under the algebra*

1) $AG(1,n) = \langle P_0, P_a, J_{ab}, G_a, Q\rangle$, *where* $G_a = x_0P_a + x_aQ$, $Q = \frac{\partial}{\partial\varphi}$ *and additionally is invariant under the following algebras:*

2) $AG_1(1,n) = \langle AG(1,n), D\rangle$, *if* $F = \lambda\rho^{k+1}$, $F_1 = \lambda_1\rho^k$, $k \neq 0$;

3) $AG_2(1,n) = \langle AG_1(1,n), A\rangle$, *if* $F = \lambda\rho^{\frac{2}{n}+1}$, $F_1 = \lambda_1\rho^{\frac{2}{n}}$;

4) $AG_3(1,n) = \langle AG_1(1,n), B\rangle$, *where the operator* B *has the form* $B = I + \lambda_1 x_0 Q$, *if* $F_1 = \lambda_1 \ln\rho$, $F = 0$;

5) $AG_4(1,n) = \langle AG(1,n), C\rangle$, *where* $C = \exp\{\lambda x_0\}\left(\frac{\lambda_1}{\lambda}Q + I\right)$, *if* $F = \lambda\rho\ln\rho$, $F_1 = \lambda_1\ln\rho$, $\lambda \neq 0$;

6) $AG_5(1,n) = \langle AG_2(1,n), I\rangle$, *if* $F = F_1 = 0$, *where* λ_i *are arbitrary real constants,* $i = 1, 2$.

Remark 3. Operator C with $\lambda_1 = 0$ coincides with C_0.

Remark 4. In the case $D = 0$ the system (3), (4) is employed in quanturn mechanics and is called there "the classical approximation of the Schrödinger equations" [8]. Its symmetry was investigated in [7].

5. Conditional symmetry. The system (3), (4) has conditional symmetry. The condition which allows to enlarge symmetry of this system has the form

$$\Delta\rho = F_2(\rho). \tag{5}$$

Then the system of equations (3), (4), (5) is equivalent to the following system:

$$\begin{aligned} &\rho_0 + \rho_a\varphi_a + \rho\Delta\varphi = F(\rho), \\ &\varphi_0 + \frac{1}{2}\varphi_a\varphi_a = F_1(\rho), \\ &\Delta\rho = F_2(\rho). \end{aligned} \tag{6}$$

Theorem 4. *The system of equations (6) for arbitrary* F, F_1, F_2 *is invariant under the algebra* $AG(1,n)$ *and additionally under the following algebras:*

1) $AG_6(1,n) = \langle AG(1,n), Q_1\rangle$, *where* $Q_1 = x_aP_a + 2\varphi Q$ *if* F *is arbitrary and* $F_1 = F_2 = 0$;

2) $AG_7(1,n) = \langle AG(1,n), Q_2\rangle$, *where* $Q_2 = x_0P_0 - \varphi Q$ *for an arbitrary* F_2 *and* $F = F_1 = 0$;

3) $AG_8(1,n) = \langle AG(1,n), Q_1 + Q_2\rangle$ *for an arbitrary* F_1 *and* $F = F_2 = 0$;

4) $AG_9(1,n) = \langle AG_1(1,n), Q_3\rangle$, *where the operator* Q_3 *has the form:* $Q_3 = x_aP_a + 2\varphi Q - \frac{2}{k}I$, *if* $F = 0$, $F_1 = \lambda_1\rho^{-k}$, $F_2 = 0$, $k \neq 0$;

5) $AG_{10}(1,n) = \langle AG_1(1,n), Q_2\rangle$, *if* $F = F_1 = 0$, $F_2 = \lambda_2\rho^{k+1}$, $k \neq 0$;

6) $AG_{11}(1,n) = \langle AG_1(1,n), Q_1\rangle$, *if* $F = \lambda\rho^{k+1}$, $F_1 = F_2 = 0$, $k \neq 0$;

7) $AG_{12}(1,n) = \langle AG(1,n), Q_3\rangle$, *if* $F = 0$, $F_1 = \lambda_1\rho^{-k}$, $F_2 = \lambda\rho^{k+1}$, $k \neq 0$;

8) $AG_{13}(1,n) = \langle AG_1(1,n), Q_4\rangle$, *where the operator* Q_4 *has the form:* $Q_4 = x_0P_0 - \varphi Q - \frac{2}{k}I$, *if* $F = \lambda\rho^{\frac{k+2}{2}}$, $F_1 = \lambda_1\rho^k$, $F_2 = 0$, $k \neq 0$;

9) $AG_1(1,n)$ *if* $F = \lambda\rho^{k+1}$, $F_1 = \lambda_1\rho^k$, $F_2 = \lambda_2\rho^{k+1}$, $k \neq 0$;

10) $AG_{14}(1,n) = \langle AG(1,n), Q_3 + mQ_4\rangle$, $m \in \mathbb{R}$, *if* $F = \lambda\rho^{\frac{mk+2}{2}}$, $F_1 = \lambda_1\rho^{mk-k}$, $F_2 = \lambda_2\rho^{k+1}$, $k \neq 0$;

11) $AG_2(1,n)$, *if* $F = \lambda\rho^{\frac{2+n}{n}}$, $F_1 = \lambda_1\rho^{\frac{2}{n}}$, $F_2 = \lambda_2\rho^{\frac{2+n}{n}}$;

12) $AG_{15}(1,n) = \langle AG_2(1,n), Q_1\rangle$, *if* $F = \lambda_2\rho^{\frac{2+n}{n}}$, $F_1 = F_2 = 0$;

13) $AG_{16}(1,n) = \langle AG_2(1,n), Q_2\rangle$, *if* $F = F_1 = 0$, $F_2 = \lambda_2\rho^{\frac{2+n}{n}}$;

14) $AG_{17}(1,n) = \langle AG_2(1,n), Q_1 + Q_2\rangle$, *if* $F_1 = \lambda_1\rho^{\frac{2}{n}}$, $F = F_2 = 0$;

15) $AG_{18}(1,n) = \langle AG_2(1,n), Q_1, Q_2\rangle$, *if* $F = F_1 = F_2 = 0$;

16) $AG_{19}(1,n) = \langle AG_8(1,n), B\rangle$, *if* $F = F_2 = 0$, $F_1 = \lambda_1\ln\rho$;

17) $AG_4(1,n) = \langle AG(1,n), C\rangle$, *if* $F = \lambda\rho\ln\rho$, $F_2 = 0$;

18) $AG_{20}(1,n) = \langle AG_6(1,n), C_0\rangle$, *if* $F = \lambda\rho\ln\rho$, $F_1 = F_2 = 0$, $\lambda \neq 0$.

Remark 5. It follows from the commutation equalities that some of above mentioned algebras coincide (for instance, $AG(1,n)$ and $AG_{12}(1,n)$, $AG_7(1,n)$ and $AG_{13}(1,n)$).

6. Reduction of the system (3), (4). Using the operators mentioned in Theorems 3 and 4 we have constructed ansatzes and have obtained corresponding reduced systems of equations. Some of them are adduced below (for the case of three spatial variables, $n = 3$):

1) Ansatz $\rho = \exp\left\{\frac{2x_0}{\alpha}\right\}\Phi(\omega_1,\omega_2)$, $\alpha \in \mathbb{R}$, $\alpha \neq 0$,

$$\varphi = \frac{\lambda_1}{2\alpha}x_0^2 - \frac{x_0^3}{3\alpha^2} + \frac{x_0x_1}{\alpha} + g(\omega_1,\omega_2), \quad \omega_1 = \frac{x_0^2}{\alpha} - x_1, \quad \omega_2 = (\vec{x}^{\,2})^{\frac{1}{2}},$$

reduces (3), (4), if $F = 0$, $F_1 = \lambda_1\ln\rho$ to the following system:

$$\left(\frac{2}{\alpha} + g_{11} + g_{22}\right)\Phi + g_1\Phi_1 + g_2\omega_2^{-1}\Phi + \frac{D}{2}(\omega_2^{-1} + \Phi_{11} + \Phi_{22}) = 0,$$

$$g_1^2 + g_2^2 = \lambda\ln\Phi + \frac{2}{\alpha}\omega_1, \quad \text{where } g_i = \frac{\partial g}{\partial\omega_i}, \quad \Phi_i = \frac{\partial\Phi}{\partial\omega_i}, \quad i = 1,2.$$

2) Ansatz $\rho = \exp\left\{\frac{2x_0}{\alpha}\right\}\Phi(\omega)$, $\omega = (\vec{x}^{\,2})^{\frac{1}{2}}$, $\alpha \neq 0$

$$\varphi = \frac{\lambda_1}{\alpha}x_0^2 + g(\omega), \quad \text{with } F = 0, \ F_1 = \lambda_1\ln\rho$$

reduces (3), (4) to the system:

$$\left(\frac{2}{\alpha} + \frac{n-1}{\omega}g' + g''\right)\Phi + g'\Phi' + \frac{D}{2}\left(\Phi'' + \frac{n-1}{\omega}\Phi'\right) = 0,$$

$$(g')^2 = 2\lambda_1\ln\Phi, \quad \text{where } g' = \frac{dg}{d\omega}, \quad \Phi' = \frac{d\Phi}{d\omega}.$$

3) $F = 0$, $F_1 = \lambda_1\ln\rho$,

$$\rho = \exp\left\{\frac{2x_0}{\alpha}\right\}\Phi(\omega,\omega_2), \quad \omega_1 = (x_1^2 + x_2^2)^{\frac{1}{2}}, \quad \omega_2 = \operatorname{arctg}\frac{x_2}{x_1} - \frac{x_3}{\beta},$$

$$\varphi = \frac{\lambda_1}{2\alpha}x_0^2 + g(\omega_1,\omega_2), \quad \{\alpha,\beta\} \neq 0,$$

$$\left(\frac{2}{\alpha}+g_1\omega_1^{-1}+g_{11}\right)\Phi+g_1\Phi_1+g_2\Phi_2(\omega_1^{-2}+\beta^{-2})+g_{22}\Phi(\omega_1^{-2}+\beta^{-2})+$$
$$+\frac{D}{2}(\Phi_{11}+\omega_1^{-1}\Phi_1+\Phi_{22}(\omega_1^{-2}+\beta^{-2}))=0,$$
$$g_1^2+g_2^2(\omega_1^{-2}+\beta^{-2})=\frac{\lambda_1}{2}\ln\Phi.$$

4) Ansatz

$$\rho=\exp\left\{\frac{2}{\lambda\alpha}\exp(\lambda x_0)\right\}\Phi(x_3),\quad \varphi=2g(x_3),\quad \{\lambda,\alpha\}\neq 0,\quad \alpha\in\mathbb{R},$$

reduces (3), (4), if $F=\lambda\rho\ln\rho$, $F_1=0$, to the following system:

$$\Phi''+\frac{2\lambda}{D}\Phi\ln\Phi=0,$$
$$g'=0,\quad D\neq 0.$$

5) Ansatz $\rho=\exp\{2x_1\exp(\lambda x_0)\}\Phi(x_0)$,

$$\varphi=\frac{\lambda_1}{\lambda}x_1\exp\{\lambda x_0\}+\frac{x_2^2+x_3^2}{2x_0}+g(x_0),\quad \lambda\neq 0$$

reduces (3), (4), if $F=\lambda\rho\ln\rho$, $F_1=\lambda_1\ln\rho$, to the system ODE:

$$\Phi'+2x_0^{-1}\Phi+2\left(D+\frac{\lambda_1}{\lambda}\exp\right)\exp\{2\lambda x_0\}\Phi=\lambda\Phi\ln\Phi,$$
$$g'+\frac{\lambda_1^2}{2\lambda^2}\exp\{2\lambda x_0\}=\lambda_1\ln\Phi.$$

6) Ansatz $\rho=\exp\left\{\frac{2}{\alpha}x_3\exp(\lambda x_0)\right\}\Phi(\omega_1,\omega_2)$, $\{\alpha,\lambda\}\neq 0$, $\alpha\in\mathbb{R}$,

$$\varphi=\frac{\lambda_1}{\lambda}\exp\{\lambda x_0\}x_3+g(\omega_1,\omega_2),\quad \omega_1=x_0,\quad \omega_2=(x_1^2+x_2^2)^{\frac{1}{2}},$$

reduces (3), (4), if $F=\lambda\rho\ln\rho$, $F_1=\lambda_1\ln\rho$, to the following system:

$$\Phi_1+g_2\Phi_2+\left(\omega_2^{-1}g_2+\frac{\lambda_1}{\lambda\alpha^2}\exp\{2\lambda\omega_1\}\right)\Phi+\frac{D}{2}(\Phi_{22}+\omega_2^{-1}\Phi_2)=\lambda\Phi\ln\Phi,$$
$$2g_1+g_2^2=\lambda_1\ln\Phi-\frac{\lambda_1^2}{2\lambda^2\alpha^2}\exp\{2\lambda\omega_1\}.$$

7) $F=\lambda\rho\ln\rho$, $F_1=\lambda_1\ln\rho$,

$$\rho=\exp\left\{2\,\mathrm{arctg}\,\frac{x_2}{x_1}\exp(\lambda x_0)\right\}\Phi(\omega_1,\omega_2),\quad \omega_1=x_0,\quad \omega_2=(x_1^2+x_2^2)^{\frac{1}{2}},$$
$$\varphi=\frac{\lambda_1}{\lambda}\exp(\lambda x_0)\,\mathrm{arctg}\,\frac{x_2}{x_1}+\frac{x_3^2}{2x_0}+g(\omega_1,\omega_2),\quad \lambda\neq 0,$$
$$\Phi_1+\omega_2^{-2}\left(g_2+\left(D+\frac{2\lambda_1}{\lambda}\right)\exp\{2\lambda\omega_1\}\right)\Phi+g_2\Phi_2+\frac{D}{2}\Phi_{22}+$$
$$+\frac{D}{2}\omega_2^{-1}\Phi_2+g_{22}\Phi=\lambda\Phi\ln\Phi,$$
$$2g_1+g_2^2=\lambda_1\ln\Phi-\left(\frac{\lambda_1}{\lambda}\omega_2^{-1}\exp\{\lambda\omega_1\}\right)^2.$$

8) $F=\lambda\rho\ln\rho$, $F_1=\lambda_1\ln\rho$,

$$\rho=\exp\left\{\frac{2}{\lambda\alpha}\exp(\lambda x_0)\right\}\Phi(\omega),\quad \varphi=\frac{\lambda_1}{\lambda\alpha}\exp(\lambda x_0)+g(\omega),\quad \omega=(\vec{x}^{\,2})^{\frac12},$$

$$g'\Phi'+\Phi\left(g''+\frac{n-1}{\omega}g'\right)+\frac{D}{2}\left(\Phi''+\frac{n-1}{\omega}\Phi'\right)=\lambda\Phi\ln\Phi,$$
$$(g')^2=\lambda_1\ln\Phi.$$

9) Ansatz $\rho=\Phi(\omega)$, $\varphi=x_0x_3-\frac{x_0^3}{3}+g(\omega)$, $\omega=x_3-\frac{x_0^2}{2}$ for arbitrary $F(\rho)$, $F_1(\rho)$ reduces (3), (4) to the following system:

$$g'\Phi'+g''\Phi+\frac{D}{2}\Phi''=F(\Phi),$$
$$(g')^2+2\omega=F_1(\Phi).$$

10) $\rho=\Phi(\omega_1,\omega_2)$, $\varphi=\frac{x_3^2}{2x_0}+g(\omega_1,\omega_2)$, $\omega_1=x_0$, $\omega_2=(x_1^2+x_2^2)^{\frac12}$, $F(\rho)$, $F_1(\rho)$ are arbitrary functions,

$$\Phi_1+g_2\Phi_2+\Phi(g_{22}+\omega_2^{-1}g_2)+\frac{D}{2}(\Phi_{22}+\omega_2^{-1}\Phi_2)=F(\Phi)-\omega_1^{-1}\Phi,$$
$$g_1+g_2^2=F_1(\Phi).$$

11) $\rho=\Phi(\omega)$, $\varphi=g(\omega)-x_0-\sqrt{2}\,\mathrm{arctg}\,\frac{x_2}{x_1}$, $\omega=(x_1^2+x_2^2)^{\frac12}$, $F(\rho)$, $F_1(\rho)$ are arbitrary functions,

$$g'(\Phi'+\omega^{-1}\Phi)+g''\Phi+\frac{D}{2}(\Phi''+\omega^{-1}\Phi')=F(\Phi),$$
$$(g')^2=2F_1(\Phi).$$

12) $\rho=\Phi(x_0)$, $\varphi=\frac{x_3^2}{2x_0}+g(x_0)$,

$$\Phi'+\omega^{-1}\Phi=F(\Phi),$$
$$(g')=F_1(\Phi),$$

13) $\rho=\Phi(x_3)$, $\varphi=g(x_3)-x_0$,

$$g'\Phi'+g''\Phi+\frac{D}{2}\Phi''=F(\Phi),$$
$$2+(g')^2=2F_1(\Phi).$$

14) Ansatz $\rho=x_0^m\Phi(\omega)$, $\varphi=\frac{x_1^2+\cdots+x_k^2}{2x_0}+g(\omega)$, where $\omega=\frac{x_{k+1}^2+\cdots+x_{k+l}^2}{2x_0}$, $0\le k\le 2$, $1\le l\le 3-k$ reduces (3), (4), if $F=\lambda\rho^{\frac{m-1}{m}}$, $F_1=\lambda\rho^{-\frac1m}$, to the system ODE:

$$(k+m)\Phi+\Phi'(2\omega g'-\omega)+\Phi(lg'+2\omega g'')+D(l\Phi'+2\omega\Phi'')=\lambda\Phi^{\frac{m-1}{m}},$$
$$(g')^2-g'=\lambda_1\omega^{-1}\Phi^{-\frac1m}.$$

15) Ansatz $\rho = x_0^2\Phi(\omega_1;\omega_2)$, $\omega_1 = \frac{x_1}{x_0}$, $\omega_2 = \frac{x_2}{x_0}$, $\varphi = 2\lambda_1 x_0\ln x_0 + \alpha x_3 + x_0 g(\omega_1,\omega_2)$, $\alpha\in\mathbb{R}$ reduces system (6), if $F=0$, $F_1 = \lambda_1\ln\rho$ to the following system:

$$\begin{aligned}
&2\lambda_1 + g - \omega_1 g_1 - \omega_2 g_2 + \frac{\alpha^2}{2} = \lambda_1\ln\Phi,\\
&\Phi_{11} + \Phi_{22} = 0,\\
&2\Phi - \omega_1\Phi_1 - \omega_2\Phi_2 + g_1\Phi_1 + g_2\Phi_2 + (g_{11}+g_{22})\Phi = 0.
\end{aligned}\tag{7}$$

16) $F=0$, $F_1 = \lambda_1\ln\rho$, $F_2 = 0$,

$$\begin{aligned}
&\rho = x_0^2\Phi(\omega_1,\omega_2), \quad \varphi = \frac{x_1^2}{2x_0} + \lambda_1 x_0\ln x_0 + x_0 g(\omega_1,\omega_2),\\
&\omega_1 = \frac{x_1}{x_0} - \operatorname{arctg}\frac{x_3}{x_2}, \quad \omega_2 = x_0^{-1}(x_2^2+x_3^2)^{\frac{1}{2}},
\end{aligned}$$

$$\begin{aligned}
&\lambda_1 + g - \omega_2 g_2 + \frac{1}{2}g_1^2(1+\omega_2^{-2}) + g_2^2 = \frac{\lambda_1}{2}\ln\Phi,\\
&\Phi_{11}(1+\omega_2^{-2}) + \Phi_{22}\omega_2^2 + \Phi_2\omega_2 = 0,\\
&3\Phi - \omega_2\Phi_2 + \omega_1(1+\omega_2^{-2})\Phi_1 + g_2\Phi_2 + g_{11}(1+\omega_2^{-2})\Phi + (g_{22}+\omega_2 g_2)\Phi = 0.
\end{aligned}\tag{8}$$

17) Ansatz $\rho = x_0^2\Phi(\omega_1,\omega_2)$, $\omega_1 = \frac{x_2}{x_0}$, $\omega_2 = \frac{x_3}{x_0}$, $\varphi = \frac{x_1^2}{2x_0} + \lambda_1 x_0\ln x_0 + x_0 g(\omega_1,\omega_2)$ reduces system (6), if $F=0$, $F_1 = \lambda_1\ln\rho$, to the following system:

$$\begin{aligned}
&\lambda_1 + g - \omega_1 g_1 - \omega_2 g_2 + \frac{1}{2}(g_1^2+g_2^2) = \frac{\lambda_1}{2}\ln\Phi,\\
&\Phi_{11} + \Phi_{22} = 0,\\
&3\Phi + g_1\Phi_1 + g_2\Phi_2 + (g_{11}+g_{22})\Phi - \omega_1\Phi_1 - \omega_2\Phi_2 = 0.
\end{aligned}\tag{9}$$

18) Ansatz $\rho = \exp\{\exp(\lambda x_0)\ln x_1\}\Phi(\omega_1,\omega_2)$, $\varphi = \frac{x_2^2}{2x_0} + x_1^2 g(\omega_1,\omega_2)$, $\omega_1 = x_0$, $\omega_2 = \frac{x_2}{x_1} - \frac{x_0x_3}{x_1}$ reduces system (6), if $F = \lambda\rho\ln\rho$, $F_1 = 0$, to the system:

$$\begin{aligned}
&2g_1 + 2\omega_2 g_2(\omega_1^{-1}-2) + g^2 + g_2^2(1+\omega_1^2+\omega_2^2) = 0,\\
&\exp(\lambda\omega_1)\Phi + \frac{1}{2}\Phi_{22}(1+\omega_1^2+\omega_2^2) + \frac{1}{2}\omega_2\Phi_2(1-4\exp(\lambda\omega_1)) +\\
&\quad + 2\Phi\exp(2\lambda\omega_1) = 0,\\
&\Phi_1 + \Phi_2(\omega_1^{-1} - 4\omega_2 g) + g\Phi(2+4\exp(\lambda\omega_1)) - 2(1+\exp(\lambda\omega_1))\omega_2 g_2\Phi +\\
&\quad + (g_{22}\Phi + g_2\Phi_2)(1+\omega_1^2+\omega_2^2) + \omega_1^{-1}\Phi = \lambda\Phi\ln\Phi.
\end{aligned}\tag{10}$$

19) Ansatz $\rho = \exp\left\{\frac{2}{\alpha}\exp(\lambda x_0)\right\}\Phi(\omega)$, $\alpha\neq 0$,

$$\varphi = \exp\left\{\frac{2x_0}{\alpha}\right\}g(\omega), \quad \omega = (\vec{x}^{\,2})^{\frac{1}{2}}\exp\left\{-\frac{x_0}{\alpha}\right\}, \quad F = \lambda\rho\ln\rho, \quad F_1 = 0$$

reduces (6) to the system ODE:

$$\begin{aligned}
&\Phi' g' + g'\frac{n-1}{\omega}\Phi - \frac{1}{\alpha}\omega\Phi + \Phi g'' = \lambda\Phi\ln\Phi,\\
&\Phi'' + \frac{n-1}{\omega}\Phi' = 0,\\
&\frac{2}{\alpha}g - \frac{1}{\alpha}\omega g' + \frac{1}{2}(g')^2 = 0.
\end{aligned}\tag{11}$$

Remark 6. Systems of reduced equations (7)–(11) are overdetermined.

7. Exact solutions of nonlinear Fokker–Planck equations. Below we list some exact solutions of the FPEs in case of three spatial variables.

1) Equation (1) has a solution $\rho = x_1^{-3}$, if $A_k = \frac{x_k}{x_0}$, $F(\rho) = -6D\rho^{\frac{5}{3}}$;

2) $\rho = (x_1^2 + x_2^2)^{-\frac{3}{2}}, \quad A_k = \dfrac{x_k}{x_0}, \quad F(\rho) = -\dfrac{9}{2}D\rho^{\frac{5}{3}};$

3) $\rho = x_1^{-2}, \quad A_1 = \dfrac{x_1}{x_0}, \quad A_2 = \dfrac{x_2}{x_0}, \quad A_3 = 0, \quad F(\rho) = -3D\rho^2;$

4) $\rho = x_1^{-1}, \quad A_k = \delta^{1k}\dfrac{x_k}{x_0}, \quad F(\rho) = -D\rho^3;$

5) $\rho = (x^2)^{-\frac{3}{2}}, \quad A_k = \dfrac{x_k}{x_0}, \quad F(\rho) = -3D\rho^{\frac{5}{3}};$

6) $\rho = (x_1^2 + x_2^2)^{-\frac{3}{2}}, \quad A_1 = \dfrac{x_2}{x_0}, \quad A_2 = \dfrac{x_2}{x_0}, \quad A_3 = 0, \quad F = -2D\rho^3;$

7) $$\rho = \exp\left\{\frac{2}{\alpha}x_0 + \frac{D^2}{\lambda}\left(c \pm \frac{\lambda}{D^2}\vec{x}^{\,2}\right)^2\right\}, \quad \{\alpha, \lambda, D\} \neq 0, \quad \alpha \in \mathbb{R},$$
$$A_k = -D\left(c \pm \frac{\lambda}{D^2}(\vec{x}^{\,2})^{\frac{1}{2}}\right)x_k(\vec{x}^{\,2})^{-\frac{1}{2}}, \quad F = \frac{2}{\alpha}\rho, \quad c \in \mathbb{R};$$

8) $$\rho = \exp\left\{\frac{2}{\alpha}x_0 + \frac{2}{D}y(\omega_1) + \frac{2}{D}z(\omega_2)\right\}, \quad A_k = \frac{\partial\varphi}{\partial x_k}, \quad F = \frac{2}{\alpha}\rho,$$
$$\varphi = \frac{\lambda}{2\alpha}x_0^2 - \frac{x_0^3}{3\alpha^2} + \frac{x_0 x_1}{\alpha} + y(\omega_1) + z(\omega_2),$$
$$\omega_1 = \frac{x_0^2}{2\alpha} - x_1, \quad \omega_2 = (x_2^2 + x_3^2)^{\frac{1}{2}},$$

where $z = \dfrac{D}{2\lambda}\left(c_3 + \dfrac{2}{D}\omega_2\right)^2$ and y can be determined implicitly via relations

$$\pm D\left(\frac{2\lambda}{D}y + \frac{2}{\alpha}\omega_1\right)^{\frac{1}{2}} + \frac{D^2}{\lambda\alpha}\ln\left|\lambda\left(\frac{2\lambda}{D}y + \frac{2}{\alpha}\omega_1\right)^{\frac{1}{2}} \pm \frac{D}{\alpha}\right| = c - \lambda\omega_1, \quad \{\lambda, \alpha\} \neq 0;$$

9) $$\rho = \exp\left\{\frac{2}{D}y(\omega_1) + \frac{2}{D}z(\omega_2)\right\}, \quad A_k = \frac{\partial\varphi}{\partial x_k}, \quad F = 0,$$
$$\varphi = x_0 x_3 - \frac{x_0^3}{3} + y(\omega_1) + z(\omega_2), \quad \omega_1 = x_3 - \frac{x_0^2}{2}, \quad \omega_2 = (x_1^2 + x_2^2)^{\frac{1}{2}};$$

10) $$\rho = \exp\left\{\frac{2x_1}{x_0} - (3 + 2\lambda)\ln x_0 + 2c\right\},$$
$$A_k = \lambda\delta^{1k} + \frac{x_k}{x_0}, \quad c \in \mathbb{R}, \quad F = 0;$$

11) $$\rho = \exp\{(2x_3 - x_0^2)^2\}, \quad A_k = (x_0 + 1)\delta^{3k},$$
$$F = \rho(2\sqrt{2}\ln^{\frac{1}{2}}\rho - 4D\ln\rho - 2D);$$

12) $$\rho = \exp\left\{\frac{2}{\alpha}x_3\exp(\lambda x_0) - \exp(\lambda x_0)\Phi_{-\lambda}(x_0)\right\}, \quad \alpha \in \mathbb{R},$$
$$A_1 = \frac{x_1}{x_0}, \quad A_2 = \frac{x_2}{x_0}, \quad A_3 = 0, \quad F = \lambda\rho\ln\rho,$$

and $\Phi_{-\lambda}(x_0)$ can be determined via relation:

$$\Phi_\gamma(x_0) = \int \frac{\exp(\gamma x_0)}{x_0} dx_0; \qquad (12)$$

13) $$\rho = \exp\left\{2\exp(\lambda x_0)\operatorname{arctg}\frac{x_2}{x_1} - \exp(\lambda x_0)\Phi_{-\lambda}(x_0)\right\},$$
$$A_1 = \frac{x_1}{x_0}, \quad A_2 = \frac{x_2}{x_0}, \quad A_3 = 0, \quad F(p) = \lambda\rho\ln\rho,$$

where $\Phi_{-\lambda}(x_0)$ is determined in (12);

14) $$\rho = \exp\left\{2x_1\exp(\lambda x_0) - \exp(\lambda x_0)\Phi_{-\lambda}(x_0) - 2\left(\frac{c}{\lambda^2} - \frac{D}{\lambda}\right)\exp(2\lambda x_0)\right\},$$
$$A_1 = \frac{c}{\lambda}\exp(\lambda x_0), \quad A_2 = \frac{x_2}{x_0}, \quad A_3 = 0, \quad F = \lambda\rho\ln\rho, \quad c \in \mathbb{R}; \quad \lambda \neq 0;$$

15) $$\rho = \exp\left\{2x_1\exp(\lambda x_0) - 2\exp(\lambda x_0)\Phi_{-\lambda}(x_0) - 2\left(\frac{c}{\lambda^2} - \frac{D}{\lambda}\right)\exp(2\lambda x_0)\right\},$$
$$A_1 = \frac{c}{\lambda}\exp(\lambda x_0), \quad A_2 = \frac{x_2}{x_0}, \quad A_3 = \frac{x_3}{x_0}, \quad F = \lambda\rho\ln\rho; \quad \lambda \neq 0;$$

16) $$\rho = \exp\left\{\frac{2}{\lambda\alpha}\exp(\lambda x_0) \pm 2\left(c + 2\left(-\frac{\lambda}{D}x_3\right)^{\frac{1}{2}}\right)^2\right\}, \quad \{\lambda, \alpha\} \neq 0,$$
$$A_k = 0, \quad F = \lambda\rho\ln\rho, \quad \{c, \alpha\} \subset \mathbb{R}.$$

1. Гардинер К.В., Стохастические методы в естественных науках, М., Мир, 1986, 526 с.
2. Shtelen W.M., Stogny V.I., Symmetry properties of one- and two-dimensional Fokker–Planck equations, *J. Phys. A: Math. Gen.*, 1989, **22**, № 13, L539–L543.
3. Wolf F., Lie algebraic solutions of linear Fokker–Planck equation, *Lett. Nuovo Cim.*, 1988, **29**, № 2, 305–307.
4. Фущич В.И., О некоторых новых волновых уравнениях математической физики, в сб. Теоретико-алгебраический анализ уравнений математической физики, Киев, Ин-т математики АН УССР, 1990, 8–12.
5. Olver P., Applications of Lie groups to differential equations, New York, Springer, 1986, 497 p.
6. Фущич В.И., Штелень В.М., Серов Н.И., Симметрийный анализ и точные решения уравнений математической физики, Киев, Наукова думка, 1989, 336 с.
7. Чопик В.И., Симметрия одной системы уравнений квантовой механики, в сб. Симметрийный анализ и решения уравнений математической физики, Киев, Ин-т математики АН УССР, 1988, 79–81.
8. Мессиа А., Квантовая механика, в 2-х т., М., Наука, 1978, Т.1, 470 с.

On nonlinear representation of the conformal algebra $AC(2,2)$

W.I. FUSHCHYCH, V.I. LAGNO, R.Z. ZHDANOV

Одержано вичерпний опис нееквівалентних представлень алгебри Пуанкаре $AP(2,2)$ та конформної алгебри $AC(2,2)$ у класі диференціальних операторів першого порядку. Встановлено, що існують лише два нееквівалентних представлення алгебри $AP(2,2)$ одне з яких є нелінійним. Це представлення допускає розширення до представлення повної конформної алгебри $AC(2,2)$. Розглянуто деякі узагальнення.

The central problem to be solved in the framework of the classical Lie approach to the partial differential equation (PDE) study

$$F(x,u,\underset{1}{u},\underset{2}{u},\ldots)=0 \tag{1}$$

is the construction of its maximal symmetry group. But the inverse problem of symmetry analysis of PDE-description of equations invariant under given transformation group is not of less importance. For example, relativistic field theory motion equations have to satisfy the Lorentz–Poincaré–Einstein relativity principle. It means that considered equations must int under the Poincaré group $P(1,3)$. Consequently, to study relativistically-invariant equations one has to study representations of the group $P(1,3)$ (see e.g. [1]).

There exists vast literature on the representations of the generalized Poincaré groups $P(n,m)$, $n,m\in\mathbb{N}$ but only a few papers are devoted to nonlinear representations [2, 3].

In the present paper we adduce results on description of unequivalent representations of the generalized Poincaré group $P(2,2)$ and its extention — conformal group $C(2,2)$ acting as transformation groups in the space $V=M(2,2)\times\mathbb{R}^1$, where $M(2,2)$ is the Minkowski space with the metric tensor

$$g_{\alpha\beta}=\begin{cases}\ \ 1, & \alpha=\beta=1,2;\\ -1, & \alpha=\beta=3,4;\\ \ \ 0, & \alpha\neq\beta.\end{cases}$$

Lie algebra of the above conformal transformation group (called conformal algebra $AC(2,2)$) has the basis elements of the form

$$Q=\sum_{a=1}^{4}\xi_a(x,u)\frac{\partial}{\partial x_a}+\eta(x,u)\frac{\partial}{\partial u} \tag{2}$$

that satisfy the following commutational relations:

$$\begin{aligned}&[P_\alpha,P_\beta]=0,\quad [P_\alpha,J_{\beta\gamma}]=g_{\alpha\beta}P_\gamma-g_{\alpha\gamma}P_\beta,\\ &[J_{\alpha\beta},J_{\gamma\beta}]=g_{\alpha\delta}J_{\beta\gamma}+g_{\beta\gamma}J_{\alpha\delta}-g_{\alpha\gamma}J_{\beta\delta}-g_{\beta\delta}J_{\alpha\gamma},\\ &[D,J_{\alpha\beta}]=0,\quad [P_\alpha,D]=P_\alpha,\quad [K_\alpha,J_{\beta\gamma}]=g_{\alpha\beta}K_\gamma-g_{\alpha\gamma}K_\beta,\end{aligned} \tag{3}$$

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$$[P_\alpha, K_\beta] = 2(g_{\alpha\beta}D - J_{\alpha\beta}), \quad [D, K_\alpha] = K_\alpha, \quad [K_\alpha, K_\beta] = 0.$$

Here $\alpha, \beta, \gamma, \delta = \overline{1,4}$.

Let us note that operators P_α, $J_{\beta\gamma}$ form generalized Poincaré algebra $AP(2,2)$ which is a subalgebra of the conformal algebra.

Definition 1. Set of operators P_α, $J_{\beta\gamma}$, D, K_α of the form (2) satisfying the commutational relations (3) is called a representation of the conformal algebra $AC(2,2)$.

Definition 2. Representation of the algebra $AC(2,2)$ is called linear if coefficients of its basis operators (2) satisfy the conditions

$$\xi_\alpha = \xi_\alpha(x), \quad \eta = a(x)u. \tag{4}$$

If conditions (4) are not satisfied, representation is called nonlinear.

It is well-known that commutational relations are not altered by the change of variables

$$x'_\alpha = f_a(x,u), \quad u' = g(x,u). \tag{5}$$

That is why two representations $\{P_\alpha, J_{\beta\gamma}, D, K_\alpha\}$ and $\{P'_\alpha, J'_{\beta\gamma}, D', K'_\alpha\}$, are called equivalent provided they are connected by the relations (5).

Theorem 1. *There exist only two unequivalent representations of the Poincaré algebra $AP(2,2)$:*

$$1.\quad P_\alpha = \partial_\alpha, \quad J_{\beta\gamma} = g_{\beta\delta}x_\delta\partial_\gamma - g_{\gamma\delta}x_\delta\partial_\beta, \tag{6}$$

$$\begin{aligned} 2.\quad & P_\alpha = \partial_\alpha, \quad J_{12} = -x_2\partial_1 + x_1\partial_2 + \partial_u, \\ & J_{13} = x_3\partial_1 + x_1\partial_3 + \cos u\partial_u, \\ & J_{14} = x_4\partial_1 + x_1\partial_4 - \varepsilon\sin u\partial_u, \\ & J_{23} = x_3\partial_2 + x_1\partial_3 + \sin u\partial_u, \\ & J_{24} = x_4\partial_2 + x_2\partial_4 + \varepsilon\cos u\partial_u, \\ & J_{34} = x_4\partial_3 - x_3\partial_4 + \varepsilon\partial_u, \quad \varepsilon = \pm 1. \end{aligned} \tag{7}$$

Here $\partial_\alpha = \partial/\partial x_\alpha$, $\partial_u = \partial/\partial u$; $\alpha, \beta, \gamma, \delta = \overline{1,4}$, the summation over the repeated indices from 1 to 4 is understood.

Because of the lack of the space we adduce only a sketch of the proof.

Since operators P_α, $\alpha = \overline{1,4}$ commute, there exists a change of variables (5) reducing these to the form $P_\alpha \to P_\alpha$, $\alpha = \overline{1,4}$ [4]. From the commutational relations $[P_\alpha, J_{\beta\gamma}] = g_{\alpha\beta}P_\gamma - g_{\alpha\gamma}P_\beta$ it follows that operators $J_{\beta\gamma}$ are of the form $J_{\beta\gamma} = g_{\beta\delta}x_\delta\partial_\gamma - g_{\gamma\delta}x_\delta\partial_\beta + \xi_{\beta\gamma\delta}(u)\partial_\delta + \eta_{\beta\gamma}(u)\partial_u$, where $\xi_{\beta\gamma\delta}$, $\eta_{\beta\gamma}$ are some smooth functions, $\beta, \gamma, \delta = \overline{1,4}$.

Substituting the obtained result into the third equality from (3) we get a system of nonlinear ordinary differential equations. On solving it we arrive at the formulae (6), (7).

Thus, there exists up to the equivalence relation (5) only one nonlinear representation of the algebra $AP(2,2)$. Applying the Lie method one can prove that the only first-order PDE admitting algebra (7) is the eikonal equation

$$u_{x_1}^2 + u_{x_2}^2 - u_{x_3}^2 - u_{x_4}^2 = 0.$$

Using results of subalgebraic analysis of the algebra $AP(2,2)$ obtained in [5], one can construct broad classes of exact solutions of the nonlinear PDE (8) by symmetry reduction procedure.

Theorem 2. *There exist only three unequivalent representations of the conformal algebra $AC(2,2)$:*

1. P_α, $J_{\beta\gamma}$ *are of the form (6),*
$$D = x_\alpha \partial_\alpha + \varphi(u)\partial_u, \qquad K_\alpha = 2g_{\alpha\beta}x_\beta D - (g_{\beta\gamma}x_\beta x_\gamma)\partial_\alpha, \tag{8}$$

2. P_α, $J_{\beta\gamma}$ *are of the form (6),*
$$D = x_\alpha \partial_\alpha + u\partial_u, \qquad K_\alpha = 2g_{\alpha\beta}x_\beta D - (g_{\beta\gamma}x_\beta x_\gamma)\partial_\alpha \pm u^2\partial_\alpha, \tag{9}$$

3. P_α, $J_{\beta\gamma}$ *are of the form (7),* $D = x_\alpha\partial_\alpha$,
$$\begin{aligned}
K_1 &= 2x_1 D - (g_{\beta\gamma}x_\beta x_\gamma)\partial_1 + 2(x_2 + x_3\cos u - \varepsilon x_4 \sin u)\partial_u,\\
K_2 &= 2x_2 D - (g_{\beta\gamma}x_\beta x_\gamma)\partial_2 + 2(-x_1 + x_3\sin u + \varepsilon x_4 \cos u)\partial_u,\\
K_3 &= -2x_3 D - (g_{\beta\gamma}x_\beta x_\gamma)\partial_3 + 2(\varepsilon x_4 - x_1\cos u - x_2 \sin u)\partial_u,\\
K_4 &= -2x_4 D - (g_{\beta\gamma}x_\beta x_\gamma)\partial_4 + 2(-\varepsilon x_3 + \varepsilon x_1\sin u - \varepsilon x_2 \cos u)\partial_u.
\end{aligned} \tag{10}$$

Representation of the form (9) is realized on the set of solutions of the nonlinear wave equation

$$g_{\alpha\beta}u_{x\alpha}x_\beta = \lambda u^3, \quad \lambda \in \mathbb{R}^1$$

under $\varphi(u) = -\frac{3}{2}u$.

As shown in [6] the system of nonlinear PDE

$$g_{\alpha\beta}u_{x_\alpha x_\beta} = \pm 3u^{-3}, \quad g_{\alpha\beta}u_{x_\alpha x_\beta} = \pm 1$$

is invariant under the conformal algebra having basis operators (10).

A detailed study of the second-order PDE admitting conformal the algebra with basis operators (7), (11) will be the topic of our future papers.

In conclusion, we adduce some generalizations of the above assertions.

Theorem 3. *An arbitrary representation of the generalized Poincaré $AP(n,m)$ with* $\max\{n,m\} \geq 3$ *in the class of the operators (2) is equivalent to the standard representation*

$$P_\alpha = \partial_\alpha, \quad J_{\beta\gamma} = \tilde{g}_{\beta\delta}x_\delta\partial_\gamma - \tilde{g}_{\gamma\delta}x_\delta\partial_\beta, \tag{11}$$

where $\tilde{g}_{\alpha\beta}$ is the metric tensor of the pseudo-Euclidean space $M(n,m)$, $\alpha,\beta,\gamma,\delta =$ $1,2,3,\ldots,n+m$.

Consequently, only the algebras $AP(1,1)$ [2], $AP(1,2)$, $AP(2,1)$ [3] and $AP(2,2)$ have the nonlinear representation.

Theorem 4. *An arbitrary representation of the conformal group $C(n,m)$ with* $\max\{n,m\} \geq 3$ *is equivalent either to (9) or to (10) (where one must replace tensor $g_{\alpha\beta}$ by $\tilde{g}_{\alpha\beta}$).*

1. Гельфанд Я.М., Минлос Р.А., Шапиро З.Я., Представления группы вращений и группы Лоренца, М., Физматгиз, 1958, 368 с.

2. Rideau G., Winternitz P., Nonlinear equations invariant under the Poincaré, similitude and conformal groups in two-dimensional space-time, *J. Math. Phys.*, 1990, **31**, № 9, 1095–1105.

3. Yegorchenko I.A., Nonlinear representation of the Poincaré algebra and invariant equations, in Symmetry Analysis of Equations of Mathematical Physics, Kiev, Institute of Mathematics, 1992, 62–65.

4. Fushchych W.I., Zhdanov R.Z., Conditional symmetry and reduction of partial differential equations, *Ukr. Math. J.*, 1992, **44**, № 7, 970–982.

5. Баранник Л.Ф., Лагно В.И., Фущич В.И., Подалгебры Пуанкаре $AP(2,3)$ и симметрийная редукция нелинейного ультрагиперболического уравнения Д'Аламбера, *Укр. мат. журн.*, 1988, **40**, № 4, 411–416.

6. Fushchych W.I., Zhdanov R.Z., Yegorchenko I.A., On the reduction of the nonlinear multidimensional wave equations and compatibility of the d'Alambert–Hamilton system, *J. Math. Anal. Appl.*, 1991, **161**, № 2, 352–360.

Нелокальные анзацы и решения нелинейной системы уравнений теплопроводности

В.И. ФУЩИЧ, Н.И. СЕРОВ, Т.К. АМЕРОВ

Нелинейная система теплопроводности нелокальной подстановкой сведена к скалярному нелинейному уравнению теплопроводности. Лиевская и условная инвариантность скалярного уравнения использована для нахождения нелокальных анзацев, которые редуцируют исходную систему к системам обыкновенных дифференциальных уравнений

Рассмотрим систему нелинейных уравнений

$$u_0 = f(v)u_{11}, \qquad v_0 = u_{11}, \tag{1}$$

где $u = u(x)$, $v = v(x)$, $x = (x_0, x_1) \in \mathbb{R}^2$, $v_0 = \partial v/\partial x_0$, $u_0 = \partial u/\partial x_0$, $u_{11} = \partial^2 u/\partial x_1^2$, которая часто встречается в теории тепломассопереноса. Нелокальная замена

$$u = w_0, \quad v = w_{11} \tag{2}$$

сводит систему (1) к одному уравнению

$$w_{00} = f(w_{11})w_{110}. \tag{3}$$

Проинтегрировав (3) по x_0, будем иметь

$$w_0 = F(w_{11}), \tag{4}$$

где F — первообразная функции f. Дважды продифференцировав (4) по x_1, получим

$$w_{001} = \partial_1(f(w_{11})w_{111}). \tag{5}$$

После замены

$$w_{11} = z \tag{6}$$

имеем уравнение

$$z_0 = \partial_1(f(z)z_1). \tag{7}$$

Таким образом, система уравнений (1) свелась к нелинейному уравнению диффузии (7).

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Лиевская симметрия уравнения (7) исчерпывающе изучена Л.В. Овсянниковым [1], а условная симметрия (7) исследована в [2, 3]. В настоящей статье приведены лиевские анзацы, редуцирующие уравнение (7) к обыкновенным дифференциальным уравнениям (ОДУ). Путем преобразования лиевских и некоторых нелиевских анзацев, посредством замен (2) и (6), описаны нелокальные анзацы, редуцирующие систему (1) к системе ОДУ. Построены семейства точных решений системы (1).

С использованием лиевской симметрии получены следующие неэквивалентные анзацы для уравнения (7).

А. $f(z)$ — произвольная гладкая функция:

$$\begin{aligned} &z=\varphi(\omega), \quad \omega=x_1x_0^{-1/2};\\ &z=\varphi(\omega), \quad \omega=\lambda_0x_0+\lambda_1x_1. \end{aligned} \tag{8}$$

Б. $f(z)=e^z$:

$$\begin{aligned} &z=\varphi(\omega)+(2+\lambda^{-1})\ln x_1, \quad \omega=x_1x_0^{\lambda};\\ &z=\varphi(\omega)+\lambda^{-1}x_1, \quad \omega=x_1+\lambda\ln x_0;\\ &z=\varphi(\omega)-\ln x_0, \quad \omega=x_1;\\ &z=\varphi(\omega)+2\ln x_1, \quad \omega=x_1e^{\lambda x_0};\\ &z=\varphi(\omega)+\ln x_1, \quad \omega=x_0. \end{aligned} \tag{9}$$

С. $f(z)=z^k$, k — произвольная постоянная, отличная от нуля:

$$\begin{aligned} &z=\varphi(\omega)x_0^{-\frac{1}{k}(2\lambda+1)}, \quad \omega=x_1x_0^{\lambda};\\ &z=\varphi(\omega)x_0^{-1/k}, \quad \omega=x_1+\lambda_1\ln x_0;\\ &z=\varphi(\omega)e^{-\frac{2\lambda}{k}x_0}, \quad \omega=x_1e^{\lambda x_0};\\ &z=\varphi(\omega)x_1^{2/k}, \quad \omega=x_0 \end{aligned} \tag{10}$$

Д. $f(z)=z^{-4/3}$:

$$\begin{aligned} &z=\varphi(\omega)(x_1^2+\lambda_1)^{-3/2}, \quad \omega=x_0;\\ &z=\varphi(\omega)x_1^{-3/2}, \quad \omega=x_0;\\ &z=\varphi(\omega)(x_1^2+\alpha^2)^{-3/2}, \quad \omega=x_0+\lambda\operatorname{arctg}\frac{x_1}{\alpha};\\ &z=\varphi(\omega)(x_1^2-\alpha^2)^{-3/2}, \quad \omega=x_0+\lambda\operatorname{Arth}\frac{x_1}{\alpha};\\ &z=\varphi(\omega)x_1^{-3}, \quad \omega=\lambda x_0+x_1^{-1};\\ &z=\varphi(\omega)e^{\frac{3}{2}\lambda x_0}, \quad \omega=x_1e^{\lambda x_0};\\ &z=\varphi(\omega)x_0^{3/4}(x_1^2+\alpha^2)^{-\frac{3}{2}}, \quad \omega=x_0e^{\lambda\operatorname{arctg}\frac{x_1}{\alpha}};\\ &z=\varphi(\omega)x_0^{3/4}(x_1^2-\alpha^2)^{-\frac{3}{2}}, \quad \omega=x_0e^{\lambda\operatorname{Arth}\frac{x_1}{\alpha}};\\ &z=\varphi(\omega)x_0^{3/4}x_1^{-3}, \quad \omega=\lambda\ln x_0+x_1^{-1};\\ &z=\varphi(\omega)x_0^{\frac{3}{2}(\lambda+\frac{1}{2})}, \quad \omega=x_1x_0^{\lambda}, \end{aligned} \tag{11}$$

где $\lambda_0=\mathrm{const}$, $\lambda_1=\mathrm{const}$, $\lambda_0=\mathrm{const}\neq 0$, $\alpha=\mathrm{const}\neq 0$. Некоторые из анзацев (8)–(11) посредством преобразований (6) и (2) преобразуются в нелокальные анзацы для системы (1).

А. $f(v)$ — произвольная гладкая функция:

1) $u = \varphi^1(\omega) - \dfrac{\omega}{2}\dot\varphi^1(\omega) + x_1\varphi^2(x_0) + \varphi^3(x_0),$
$v = \ddot\varphi^1(\omega), \quad \omega = x_1 x_0^{-1/2};$

2) $u = \dfrac{\lambda_0}{\lambda_1^2}\varphi^1(\omega) + x_1\varphi^2(x_0) + \varphi^3(x_0),$
$v = \dot\varphi^1(\omega), \quad \omega = \lambda_0 x_0 + \lambda_1 x_1;$

3) $u = \dfrac{\dot\varphi^1(x_0)}{2}x_1^2 + x_1\varphi^2(x_0) + \varphi^3(x_0),$
$v = \varphi^1(x_0).$

Б. $f(v) = e^v$:

4) $u = -2\lambda x_0^{-2\lambda-1}\varphi^1(\omega) + \lambda x_1 x_0^{-\lambda-1}\dot\varphi^1(\omega) + \varphi^2(x_0)x_1 + \varphi^3(x_0),$
$v = (2+\lambda^{-1})\ln x_1 + \ddot\varphi^1(\omega), \quad \omega = x_1 x_0^{\lambda};$

5) $u = \lambda\varphi^1(\omega)x_0^{-1} + \varphi^2(x_0)x_1 + \varphi^3(x_0),$
$v = x_1\lambda^{-1} + \dot\varphi^1(\omega), \quad \omega = x_1 + \lambda\ln x_0;$

6) $u = -\dfrac{1}{2x_0}x_1^2 + x_1\varphi^2(x_0) + \varphi^3(x_0),$
$v = -\ln x_0 + \varphi^1(x_1);$

7) $u = -2\lambda e^{-2\lambda x_0}\varphi^1(\omega) + \lambda x_1 e^{-\lambda x_0}\dot\varphi^1(\omega) + x_1\varphi^2(x_0) + \varphi^3(x_0),$
$v = 2\ln x_1 + \ddot\varphi^1(\omega), \quad \omega = x_1 e^{\lambda x_0};$

8) $u = \dfrac{x_1^2}{2}\dot\varphi^1(x_0) + x_1\varphi^2(x_0) + \varphi^2(x_0) + \varphi^3(x_0),$
$v = \ln x_1 + \varphi^1(x_0).$

С. $f = v^k$, k — произвольная постоянная ($k \neq 0$):

9) $u = \left(-\dfrac{1}{k}(2\lambda+1) - 2\lambda\right) x_0^{-(2\lambda+1)\left(\frac{1}{k}+1\right)}\varphi^1(\omega) +$
$+ x_1\lambda x_0^{-\frac{1}{k}(2\lambda+1)-\lambda-1}\dot\varphi^1(\omega) + x_1\varphi^2(x_0) + \varphi^3(x_0),$
$v = x_0^{-\frac{1}{k}(2\lambda+1)}\ddot\varphi^1(\omega), \quad \omega = x_1 x_0^{\lambda};$

10) $u = x_0^{-(1/k)-1}\left(-\dfrac{1}{k}\varphi^1(\omega) + \lambda\dot\varphi^1(\omega)\right) + x_1\varphi^2(x_0) + \varphi^3(x_0),$
$v = x_0^{-1/k}\ddot\varphi(\omega), \quad \omega = x_1 + \lambda\ln x_0;$

11) $u = -\dfrac{1}{k}x_0^{-(1/k)-1}\varphi^1(x_1) + \varphi^2(x_0)x_1 + \varphi^3(x_0),$
$v = x_0^{-1/k}\ddot\varphi^1(x_1);$

12) $u = -2\lambda\left(\dfrac{1}{k}+1\right)e^{2\lambda((1/k)+1)x_0}\varphi^1(\omega) +$
$+ \lambda x_1 e^{-\lambda((2/k)+1)x_0}\dot\varphi^1(\omega) + x_1\varphi^2(x_0) + \varphi^3(x_0),$
$v = e^{-\frac{2\lambda}{k}}\ddot\varphi^1(\omega), \quad \omega = x_1 e^{\lambda x_0};$

13) $u = -\dot\varphi^1(x_0)\ln x_1 + x_1\varphi^2(x_0) + \varphi^3(x_0),$
$v = \varphi^1(x_0)x_1^{-2}, \quad k = -1;$

14) $u = \dot{\varphi}^1(x_0)[x_1 \ln x_1 - x_1] + \varphi^2(x_0)x_1 + \varphi^3(x_0),$
$v = \varphi^1(x_0)x_1^{-1}, \quad k = -2;$

15) $u = \dot{\varphi}^1(x_0)\dfrac{k^2}{(2+k)(2+2k)}x_1^{\frac{2+2k}{k}} + x_1\varphi^2(x_0) + \varphi^3(x_0),$
$v = \varphi^1(x_0)x_1^{2/k}, \quad k \neq 0; -1; -2.$

Д. $f(v) = v^{-4/3}$:

16) $u = \dfrac{1}{\lambda^2}(x_1^2 + \lambda^2)^{1/2}\dot{\varphi}(x_0) + x_1\varphi^2(x_0) + \varphi^3(x_0),$
$v = (x_1^2 + \lambda^2)^{-3/2}\varphi^1(x_0);$

17) $u = -\lambda^{-2}(x_1^2 - \lambda^2)^{1/2}\dot{\varphi}(x_0) + x_1\varphi^2(x_0) + \varphi^3(x_0),$
$v = (x_1^2 - \lambda^2)^{-3/2}\varphi^1(x_0);$

18) $u = \dfrac{1}{2x_1}\dot{\varphi}^1(x_0) + x_1\varphi^2(x_0) + \varphi^3(x_0),$
$v = x_1^{-3}\varphi^1(x_0);$

19) $u = -4x^{1/2}\dot{\varphi}^1(x_0) + x_1\varphi^2(x_0) + \varphi^3(x_0),$
$v = x_1^{-3/2}\varphi^1(x_0);$

20) $u = \dfrac{x_1^2}{2}\dot{\varphi}^1(x_0) + x_1\varphi^2(x_0) + \varphi^3(x_0),$
$v = \varphi^1(x_0);$

21) $u = \lambda x_1\varphi^1(\omega) + x_1\varphi^2(x_0) + \varphi^3(x_0),$
$v = x_1^{-3}\dot{\varphi}^1(\omega), \quad \omega = \lambda x_0 + \dfrac{1}{x_1};$

22) $u = -\dfrac{\lambda}{2}e^{-\frac{\lambda}{2}x_0}\varphi^1(\omega) + \lambda x_1 e^{\frac{\lambda}{2}x_0}\dot{\varphi}^1(\omega) + x_1\varphi^2(x_0) + \varphi^3(x_0),$
$v = e^{\frac{3}{2}\lambda x_0}\ddot{\varphi}^1(\omega), \quad \omega = x_1 e^{\lambda x_0};$

23) $u = x_1\varphi^2(x_0) + \varphi^3(x_0),$
$v = \varphi^1(x_1);$

24) $u = x_1 x_0^{-1/4}\left[\dfrac{3}{4}\varphi^1(\omega) + \lambda\dot{\varphi}^1(\omega)\right] + x_1\varphi^2(x_0) + \varphi^3(x_0),$
$v = x_0^{3/4}x_1^{-3}\ddot{\varphi}^1(\omega), \quad \omega = \lambda \ln x_0 + \dfrac{1}{x_1};$

25) $u = \left(-\dfrac{\lambda}{2} + \dfrac{3}{4}\right)x_0^{-\frac{\lambda}{2}-\frac{1}{4}}\varphi^1(\omega) + \lambda x_1 x_0^{\frac{\lambda}{2}-\frac{1}{4}}\dot{\varphi}^1(\omega) + x_1\varphi^2(x_0) + \varphi^3(x_0),$
$v = x_0^{\frac{3}{2}(\lambda+\frac{1}{2})}\ddot{\varphi}^1(\omega), \quad \omega = x_1 x_0^{\lambda};$

26) $u = x_0^{-1/4}\left[\dfrac{3}{4}\varphi^1(\omega) + \lambda\dot{\varphi}^1(\omega)\right] + \varphi^2(x_0)x_1 + \varphi^3(x_0),$
$v = x_0^{3/4}\ddot{\varphi}^1(\omega), \quad \omega = \lambda \ln x_0 + x_1;$

27) $u = \dfrac{3}{4}x_0^{-1/4}\varphi^1(x_1) + x_1\varphi^2(x_0) + \varphi^3(x_0),$
$v = \ddot{\varphi}^1(x_1)x_0^{3/4},$

где $\lambda_0=\mathrm{const}$, $\lambda_1=\mathrm{const}$, $\lambda_0=\mathrm{const}\neq 0$; φ^1, φ^2, φ^3 — произвольные гладкие функции; $\dot\varphi^1(\omega)=d\varphi^1/d\omega$, $\ddot\varphi^1(\omega)=d^2\varphi^1/d\omega^2$. Выписанные выше нелиевские анзацы редуцируют (1) к следующим системам ОДУ:

$$
\begin{aligned}
1)\quad & \frac{1}{4}\omega^2\ddot\varphi^1(\omega)-\frac{1}{4}\omega\dot\varphi^1(\omega)+\frac{1}{2}f(\ddot\varphi^1(\omega))\omega\dddot\varphi^1(\omega)+c_1\omega+c_2=0,\\
& \dot\varphi^2(x_0)=c_1x_0^{-3/2},\\
& \dot\varphi^3(x_0)=c_2x_0^{-1};\\
2)\quad & \frac{\lambda_0^2}{\lambda_1^2}\dot\varphi^1(\omega)-f(\dot\varphi(\omega))\lambda_0\ddot\varphi^1(\omega)+c_1\omega+c_2=0,\\
& \dot\varphi^2(x_0)=\lambda_1c_1,\\
& \dot\varphi^3(x_0)=\lambda_0c_1x_0+c_2;\\
3)\quad & \ddot\varphi^1(x_0)=0,\\
& \dot\varphi^2(x_0)=0,\\
& \dot\varphi^3(x_0)=f(\varphi^1(x_0))\dot\varphi^1(x_0);\\
4)\quad & 2\lambda(2\lambda+1)\varphi^1(\omega)-(3\lambda^2+\lambda)\omega\dot\varphi^1(\omega)+\lambda^2\omega^2\ddot\varphi^1(\omega)-\\
& \quad -\lambda\omega^{3+\frac{1}{\lambda}}e^{\ddot\varphi^1(\omega)}\dddot\varphi^1(\omega)+c_1\omega+c_2=0,\\
& \dot\varphi^2(x_0)=c_1x_0^{-\lambda-2},\\
& \dot\varphi^3(x_0)=c_2x_0^{-2\lambda-2};\\
5)\quad & \lambda^2\dot\varphi^1(\omega)-\lambda\varphi^1(\omega))-\lambda e^{\dot\varphi^1(\omega)+(\omega/\lambda)}\ddot\varphi^1(\omega)+c_1\omega+c_2=0,\\
& \dot\varphi^2(x_0)=c_1x_0^{-2},\\
& \dot\varphi^3(x_0)=c_2x_0^{-2}+\lambda\ln x_0\dot\varphi^2(x_0);\\
6)\quad & \frac{x_1^2}{2}+e^{\varphi^1(x_1)}+c_1x_1+c_2=0,\\
& \dot\varphi^2(x_0)=c_1x_0^{-2},\\
& \dot\varphi^3(x_0)=c_2x_0^{-2};\\
7)\quad & 4\lambda^2\varphi^1(\omega)-3\lambda^2\omega\varphi^1(\omega)+\lambda^2\ddot\varphi^1(\omega)-\lambda e^{\ddot\varphi^1(\omega)}\dddot\varphi^1(\omega)+c_1\omega+c_2=0,\\
& \dot\varphi^2(x_0)=c_1e^{-\lambda x_0},\\
& \dot\varphi^3(x_0)=c_3e^{-2\lambda x_0};\\
8)\quad & \ddot\varphi^1(x_0)=0,\\
& \dot\varphi^2(x_0)=e^{\varphi^1(x_0)}\dot\varphi^1(x_0),\\
& \dot\varphi^3(x_0)=0;\\
9)\quad & \left(-\frac{1}{k}(2\lambda+1)-2\lambda\right)\left(-\frac{1}{k}(2\lambda+1)-2\lambda-1\right)\varphi^1(\omega)-\\
& \quad -\lambda\left(\frac{2}{k}(2\lambda+1)+3\lambda+1\right)\omega\dot\varphi^1(\omega)+\lambda^2\omega^2\ddot\varphi^1(\omega)-\\
& \quad -[\ddot\varphi^1(\omega)]^k\left[-\frac{1}{k}(2\lambda+1)\ddot\varphi^1(\omega)+\lambda\omega\dddot\varphi^1(\omega)\right]+c_1\omega+c_2=0,\\
& \dot\varphi^2(x_0)=c_1x_0^{-\frac{1}{k}(2\lambda+1)-\lambda-2},\\
& \dot\varphi^3(x_0)=c_2x_0^{-\frac{1}{k}(2\lambda+1)-2\lambda-2};
\end{aligned}
$$

10) $\frac{1}{k}\left(\frac{1}{k}+1\right)\varphi^1(\omega)-\lambda\left(\frac{2}{k}+1\right)\dot{\varphi}^1(\omega)+\lambda^2\ddot{\varphi}^1(\omega)-$
$-[\ddot{\varphi}^1(\omega)]^k\left(-\frac{1}{k}\ddot{\varphi}^1(\omega)+\lambda\dddot{\varphi}^1(\omega)\right)+c_1\omega+c_2=0,$
$\lambda\ln x_0\dot{\varphi}^2(x_0)=-c_1,$
$\dot{\varphi}^3(x_0)=c_2;$

11) $\frac{1}{k}\left(\frac{1}{k}+1\right)\varphi^1(x_1)+\frac{1}{k}(\ddot{\varphi}^1(\omega))^{k+1}+c_1x_1+c_2=0,$
$x_0^{(1/k)+2}\dot{\varphi}^2(x_0)=c_1,$
$x_0^{(1/k)+2}\dot{\varphi}^3(x_0)=c_2;$

12) $4\lambda^2\left(\frac{1}{k}+1\right)^2\varphi^1(\omega)-\left(\frac{4}{k}+3)\lambda^2\omega\dot{\varphi}^1(\omega\right)+\lambda^2\ddot{\varphi}^1(\omega)-$
$-[\ddot{\varphi}^1(\omega)]^k\left(-\frac{2\lambda}{k}\ddot{\varphi}^1(\omega)+\lambda\dddot{\varphi}^1(\omega)\right)+c_1\omega+c_2=0,$
$\dot{\varphi}^2(x_0)=c_1e^{-\lambda\left(\frac{2}{k}+1\right)x_0},$
$\dot{\varphi}^3(x_0)=c_2e^{-2\lambda\left(\frac{2}{k}+1\right)x_0};$

13) $\ddot{\varphi}^1(x_0)=0,$
$\dot{\varphi}^2(x_0)=0,$
$\dot{\varphi}^3(x_0)=\dot{\varphi}^1(x_0)[\varphi^1(x_0)]^{-2};$

14) $\ddot{\varphi}^1(x_0)=0,$
$\dot{\varphi}^2(x_0)=\dot{\varphi}^1(x_0)[\varphi^1(x_0)]^{-2},$
$\dot{\varphi}^3(x_0)=0;$

15) $\ddot{\varphi}^1(x_0)\frac{k^2}{(2+k)(2+2k)}=[\varphi^1(x_0)]^k\dot{\varphi}^1(x_0),$
$\dot{\varphi}^2(x_0)=0,$
$\dot{\varphi}^3(x_0)=0;$

16) $\frac{\ddot{\varphi}^1(x_0)}{\lambda^2}=[\varphi^1(x_0)]^{-4/3}\dot{\varphi}^1(x_0),$
$\dot{\varphi}^2(x_0)=0,$
$\dot{\varphi}^3(x_0)=0;$

17) $-\lambda^{-2}\ddot{\varphi}^1(x_0)=[\varphi^1(x_0)]^{-4/3}\dot{\varphi}^1(x_0),$
$\dot{\varphi}^2(x_0)=0,$
$\dot{\varphi}^3(x_0)=0;$

18) $\ddot{\varphi}^1(x_0)=0,$
$\dot{\varphi}^2(x_0)=[\varphi^1(x_0)]^{-4/3}\dot{\varphi}^1(x_0),$
$\dot{\varphi}^3(x_0)=0;$

19) $-4\ddot{\varphi}^1(x_0)=[\varphi^1(x_0)]^{-4/3}\dot{\varphi}^1(x_0),$
$\dot{\varphi}^2(x_0)=0,$
$\dot{\varphi}^3(x_0)=0;$

20) $\ddot\varphi^1(x_0) = 2[\varphi^1(x_0)]^{-4/3}\dot\varphi^1(x_0),$
$\dot\varphi^2(x_0) = 0,$
$\dot\varphi^3(x_0) = 0;$

21) $\lambda^2\dot\varphi^1(\omega) - \lambda[\dot\varphi^1(\omega)]^{-4/3}\ddot\varphi^1(\omega) + c_1\omega + c_2 = 0,$
$\dot\varphi^2(x_0) = -\lambda\dot\varphi^3(x_0)x_0 + c_2,$
$\dot\varphi^3(x_0) = c_1;$

22) $$\lambda\omega\dddot\varphi^1(\omega)[\ddot\varphi^1(\omega)]^{-4/3} + \frac{3}{2}\lambda[\ddot\varphi^1(\omega)]^{-1/3} - \lambda^2\omega^2\ddot\varphi^1(\omega) - \\ - \varphi^1(\omega) - c_1\omega - c_2 = 0,$$
$\dot\varphi^2(x_0) = c_1 e^{\frac{\lambda}{2}x_0},$
$\dot\varphi^3(x_0) = c_2 e^{-\frac{\lambda}{2}x_0};$

23) $\dot\varphi^2(x_0) = 0,$
$\dot\varphi^3(x_0) = 0;$

24) $$-\frac{3}{16}\varphi^1(\omega) + \frac{\lambda}{2}\dot\varphi^1(\omega) + \lambda\ddot\varphi^1(\omega) - (\ddot\varphi^1(\omega))^{-4/3}\left[\frac{3}{4}\ddot\varphi^1(\omega) + \lambda\dddot\varphi^1(\omega)\right] + \\ + c_1\omega + c_2 = 0,$$
$x_0^{5/4}\dot\varphi^3(x_0) = c_1,$
$x_0^{5/4}[\dot\varphi^2(x_0) - \lambda\ln x_0\dot\varphi^3(x_0)] = c_2;$

25) $$\left(\frac{\lambda}{2} - \frac{3}{4}\right)\left(\frac{\lambda}{2} + \frac{1}{4}\right)\varphi^1(\omega) + \frac{\lambda}{2}\omega\dot\varphi^1(\omega) + \lambda^2\omega^2\ddot\varphi^1(\omega) + \\ + \left(\frac{3}{2}\lambda + \frac{3}{4}\right)(\ddot\varphi^1(\omega))^{-1/3} - \lambda\omega[\ddot\varphi^1(\omega)]^{-4/3}\dddot\varphi^1(\omega) + c_1\omega + c_2 = 0,$$
$\dot\varphi^2(x_0) = c_1 x_0^{(\lambda/2)-(5/4)},$
$\dot\varphi^3(x_0) = c_2 x_0^{-(\lambda/2)-(5/4)};$

26) $$-\frac{3}{16}\varphi^1(\omega) + \frac{\lambda}{2}\dot\varphi^1(\omega) + \lambda^2\ddot\varphi^1(\omega) - [\ddot\varphi^1(\omega)]^{4/3}\left[\frac{3}{4}\ddot\varphi^1(\omega) + \lambda\dddot\varphi^1(\omega)\right] + \\ + c_1\omega + c_2 = 0,$$
$\dot\varphi^2(x_0) = c_1 x_0^{-5/4},$
$\dot\varphi^3(x_0) = c_2 x_0^{-5/4} + \lambda\ln x_0\dot\varphi^2(x_0);$

27) $[\ddot\varphi^1(x_1)]^{-1/3} + \frac{1}{4}\varphi^1(x_1) + c_1x_1 + c_2 = 0,$
$x_0^{5/4}\dot\varphi^2(x_0) = \frac{3}{4}c_1,$
$x_0^{5/4}\dot\varphi^3(x_0) = \frac{3}{4}c_2,$

где c_1, c_2 — произвольные постоянные.

Если проинтегрировать приведенные выше уравнения и подставить их решения в соответствующие анзацы, то получим решения системы (1). Приведем некоторые из них:

A. $u = \frac{c_1}{2}x_1^2 + c_3x_1 + \int f(c_1x_0 + c_2)dx_0 + c_4, \quad v = c_1x_0 + c_2;$

Б. $u = -\dfrac{1}{2x_0}x_1^2 + x_1\left(-\dfrac{c_1}{x_0} + c_3\right) - c_2x_0^{-1} + c_4,$

$v = -\ln x_0 + \ln\left(-\dfrac{x_1^2}{2} - c_1x_1 - c_2\right);$

$u = \dfrac{x_1^2}{2}(c_1x_0 + c_2) + x_1(e^{c_1x_0+c_2} + c_3) + c_4,$

$v = \ln x_1 + c_1x_0 + c_2;$

С. $u = -c_1\ln x_1 + c_3x_1 - (c_1x_0 + c_2)^{-1} + c_4,$

$v = (c_1x_0 + c_2)x_1^{-2}, \quad k = -1;$

$u = c_1[x_1\ln x_1 - x_1] + x_1[-(c_1x_0 + c_2)^{-1} + c_3] + c_4,$

$v = (c_1x_0 + c_2)x_1^{-1}, \quad k = -2;$

$u = \dfrac{1}{k+1}\left(-x_0\dfrac{(2+k)(2+2k)}{k(k+1)} + c_1\right)^{-\frac{1}{k}-1} x_1^{\frac{2+2k}{k}} + x_1c_2 + c_3,$

$v = \left[-x_0\dfrac{(2+k)(2+2k)}{k(k+1)} + c_1\right]^{-\frac{1}{k}-1} x_1^{\frac{2}{k}}, \quad k \neq 0, -1, -2;$

Д. $u = (x_1^2 + \lambda^2)^{1/2}3(-4\lambda^2x_0 + c_1)^{-1/4} + c_2x_1 + c_3,$

$v = (x_1^2 + \lambda^2)^{-3/2}(-4\lambda^2x_0 + c_1)^{3/4};$

$u = -3(x_1^2 - \lambda^2)^{1/2}(4\lambda^2x_0 + c_0)^{-1/4} + x_1c_2 + c_3,$

$v = (x_1^2 - \lambda^2)^{-3/2}(4\lambda^2x_0 + c_1)^{3/4};$

$u = \dfrac{c_1}{2x_1} + x_1[-3(c_1x_0 + c_2)^{-1/3} + c_3] + c_4,$

$v = x_1^{-3}(c_1x_0 + c_2);$

$u = -x_1^{1/2}48(16x_0 + c_1)^{-1/4} + x_1c_2 + c_3,$

$v = x_1^{-3/2}(16x_0 + c_1)^{3/4};$

$u = -3x_1^2(-8x_0 + c_1)^{-1/4} + x_1c_2 + c_3,$

$v = (-8x_0 + c_1)^{3/4};$

$u = c_1x_1 + c_2,$

$\nu = \varphi^1(x_1), \quad \varphi^1$ — произвольная гладкая функция;

$u = 3x_0^{-1/4}(\sqrt{x_1} - c_1x_1 - c_2) + x_1[-3c_1x_0^{-1/4} + c_4] + [-3c_2x_0^{-1/4} + c_5],$

$v = x_0^{3/4}(-x_1^{-3/2}),$

где $c_i = \text{const}$, $i = \overline{1,5}$.

В работе [2] для уравнения (7) приведены условно инвариантные анзацы, используя которые, можно построить нелокальные анзацы для системы (1). Приведем два таких анзаца для случая, когда $f(v) = \lambda e^v$.

По анзацам для уравнения (7):

$$\begin{aligned} &\text{а)} \quad z = \ln(\varphi(x_0) - x_1) + \ln(\varphi(x_0) + x_1) - \ln(2\lambda x_0); \\ &\text{б)} \quad z = 2\ln(\varphi(x_0) + x_1) - \ln(-2\lambda x_0) \end{aligned} \tag{12}$$

находим анзацы для системы (1):

а) $u = \dot\varphi^1[(\varphi^1 - x_1)\ln(\varphi^1 - x_1) + (\varphi^1 + x_1)\ln(\varphi^1 + x_1) + \varphi^1] -$

$$
\begin{aligned}
& -\frac{x_1^2}{2x_0} + \dot{x}_1\varphi^2(x_0) + \varphi^3(x_0), \\
& v = \ln(\varphi^1 - x_1) + \ln(\varphi^1 + x_1) - \ln 2\lambda x_0, \\
& \text{где} \ \ \varphi^1 = \varphi^1(x_0); \qquad (13) \\
\text{б)} \quad & u = 2\dot{\varphi}^1(x_1 + \varphi^1)[\ln(x_1 + \varphi^1) - 1] - \frac{x_1^2}{2x_0} + x_1\varphi^2 + \varphi^3, \\
& v = 2\ln(x_1 + \varphi^1) - \ln(-2\lambda x_0),
\end{aligned}
$$

которые редуцируют систему (1) к следующим системам ОДУ:

$$
\begin{aligned}
\text{а)} \quad & \dot{\varphi}^1 = 0, \quad \dot{\varphi}^2 = -\varphi^1 x_0^{-2}, \quad \dot{\varphi}^3 = \frac{-(\varphi^1)^2}{2x_0^2}; \\
\text{б)} \quad & \dot{\varphi}^1 = 0, \quad \dot{\varphi}^2 = \varphi^1 x_0^{-2}, \quad \dot{\varphi}^3 = \frac{(\varphi^1)^2}{2x_0^2}.
\end{aligned}
$$

Решив редуцированные системы, по формулам (13) найдем точные решения системы (1):

$$
\begin{aligned}
\text{а)} \quad & u = -\frac{c_1^2 + x_1^2}{2x_0^2} + x_1c_2 + c_3 + \frac{c_1}{x_0}x_1, \\
& v = \ln\frac{c_1^2 - x_1^2}{2\lambda x_0}; \\
\text{б)} \quad & u = -\frac{x_1^2}{2x_0} + \left(c_2 - \frac{c_1}{x_0}\right)x_1 - \frac{c_1^2}{2x_0} + c_3, \\
& v = 2\ln(x_1 + c_1) - \ln(-2\lambda x_0).
\end{aligned}
$$

Итак, приведенные результаты говорят о том, что нелинейные уравнения обладают скрытыми нелокальными симметриями, которые к настоящему времени совершенно не изучены и не использованы для их интегрирования.

1. Овсянников Л.В., Групповые свойства уравнения нелинейной теплопроводности, *Доклады АН СССР, Сер. А*, 1959, **125**, № 3, 492–495.
2. Фущич В.И., Серов Н.И., Амеров Т.К., Условная инвариантность нелинейного уравнения теплопроводности, *Докл. АН УССР, Сер. А*, 1990, № 11, 15–18.
3. Фущич В.И., Штелень В.М., Серов Н.И., Симметрнйный анализ и точные решения нелинейных уравнений математической физики, Киев, Наук. думка, 1989, 336 с.

The conditional invariance and exact solutions of the nonlinear diffusion equation

W.I. FUSHCYCH, N.I. SEROV, L.A. TULUPOVA

Исследована условная инвариантность нелинейного уравнения диффузии. Операторы условной инвариантности использованы для построения анзацев, редуцирующих данное уравнение к обыкновенным дифференциальным уравнениям. Найдены некоторые точные решения исходного уравнения.

Let us consider the nonlinear diffusion equation

$$H(u)u_0 + u_{11} = F(u), \tag{1}$$

where $u = u(x) \in \mathbb{R}_1$, $x = (x_0, x_1) \in \mathbb{R}_2$, $u_0 = \frac{\partial u}{\partial x_0}$, $u_{11} = \frac{\partial^2 u}{\partial x_1^2}$, $H(u)$ and $F(u)$ are arbitrary smooth functions.

Usually the equation (1) is investigated in the equivalent form

$$u_0 + \partial_1(f(u)u_1) = g(u). \tag{2}$$

In this way, for example, in papers [1, 2] Lie invariance of this equation was investigated.

The present paper is a continuation of the works [3, 4], where the Q-conditional invariance of the equation (1) was studied when $H(u) \equiv 1$ and $H(u) = u^{-1}$, $F(u) = 0$. In this paper Q-conditional invariance of the equation (1) is studied when $H(u)$ and $F(u)$ are arbitrary functions. Using obtained operators of Q-conditional invariance exact solutions of the given equation are found.

Let

$$Q = A(x,u)\partial_0 + B(x,u)\partial_1 + C(x,u)\partial_u, \tag{3}$$

where A, B, C are smooth functions, be a differential operator of the first order, acting on the manifold $(x, u) \in \mathbb{R}_3$.

The following theorem is proved analogously, as in [5].

Theorem 1. *The equation (1) is Q-conditionally invariant under the operator (3), if the functions A, B, C satisfy the following conditions:*

Case 1. $A = 1$.

$$\begin{gathered} B_{uu} = 0, \quad C_{uu} = 2(B_{1u} + HBB_u), \\ 3B_uF = 2(C_{1u} + HB_uC) - (HB_0 + B_{11} + 2HBB_1 + H_uBC), \\ CF_u - (C_u - 2B_1)F = HC_0 + C_{11} + 2HCB_1 + H_uC^2; \end{gathered} \tag{4}$$

Case 2. $A = 0$, $B = 1$.

$$CF_u - C_uF = HC_0 + C_{11} + 2CC_{1u} + C^2C_{uu} + \frac{H_u}{H}C(F - C_1 - CC_u). \tag{5}$$

Доповіді АН України, 1993, № 4, С. 37–40.

In formulae (4), (5) and everywhere below a subscript means differentiation with respect to corresponding argument.

Theorem 2. *The equation (1) is Q-conditionally invariant under the operator*

$$Q = \partial_0 + u\partial_1 + C(u)\partial_u, \tag{6}$$

if it has form

$$\left(3\lambda_1 + \frac{\lambda_2}{u}\right) u_0 + u_{11} = \left(2\lambda_1 + \frac{\lambda_2}{u}\right) P_3(u), \tag{7}$$

where $P_3(u) = \lambda_1 u^3 + \lambda_2 u^2 + \lambda_3 u + \lambda_4$ *is arbitrary third-order polynomial of* u*,* λ_k *are arbitrary constants,* $k = \overline{1,4}$*. In this case* $C(u) = P_3(u)$*.*

Proof. Substituting $B = u$, $C = C(u)$ into (4), we have

$$C_{uu} = 2uH, \quad F = \frac{1}{3}(2H - uH_u)C, \quad uH_{uu} + 2H_u = 0.$$

Whence it appears that

$$H = 3\lambda_1 + \frac{\lambda_2}{u}, \quad C = P_3(u), \quad F = \left(2\lambda_1 + \frac{\lambda_2}{u}\right) P_3(u),$$

The theorem is proved.

We use the operator (6) for finding solutions of the equation (7). The ansatz obtained with the help of the operator (6) has the form

$$x_1 - \int \frac{udu}{P_3(u)} = \varphi(\omega), \quad \omega = x_0 - \int \frac{du}{P_3(u)}. \tag{8}$$

The ansatz (8) reduces the equation (7) to the ordinary differential equation (ODE)

$$\ddot{\varphi} + P_3(\dot{\varphi}) = 0. \tag{9}$$

Integration of the equation (9) depends on a form of the roots of the polynomial P_3. There are seven essentially different cases. We give one example of each case.

1) $P_3(u) = (u-1)^3, \qquad (\varphi - \omega)^2 = 2\omega,$

$$u = 1 + \frac{x_1 - x_0}{x_0 - \frac{1}{2}(x_1 - x_0)^2};$$

2) $P_3(u) = (u+1)(u-1)^2, \quad \mathrm{th}\,(\varphi - \omega) - 1 = \dfrac{1}{\varphi + \omega},$

$$\frac{\left(x_0 + x_1 + \frac{1}{u-1}\right) u - 1}{x_0 + x_1 + \frac{1}{u-1} - u} = \mathrm{th}\,(x_0 - x_1);$$

3) $P_3(u) = (u-2)(u^2-1), \quad \exp 3(\varphi - \omega) - 3\exp(\varphi + \omega) + 2 = 0,$

$$u = -\frac{\exp 3(x_1 - x_0) + 3\exp(x_1 + x_0) - 4}{\exp 3(x_1 - x_0) - 3\exp(x_1 + x_0) + 2};$$

4) $P_3(u) = (u-1)(u^2+2u+2)$,
$3\cos(2\varphi - 2\omega) + 4\sin(2\varphi - 2\omega) + 5 = 2\exp(-4\varphi - 6\omega)$,
$$u = \frac{\exp(-3x_0 - 2x_1) + 3\sin(x_0 - x_1) - \cos(x_0 - x_1)}{\exp(-3x_0 - 2x_1) + 2\sin(x_0 - x_1) - \cos(x_0 - x_1)};$$

5) $P_3(u) = (u-1)^2, \quad \varphi = \omega + \ln\omega, \quad u = 1 + \exp(x_1 - x_0);$

6) $P_3(u) = u^2 - 1, \quad \varphi = \ln \operatorname{ch}\omega, \quad u = \dfrac{\operatorname{ch} x_0 - \exp x_1}{\operatorname{sh} x_0};$

7) $P_3(u) = u^2 + 1, \quad \varphi = \ln\cos\omega, \quad u = \dfrac{-\cos x_0 + \exp x_1}{\sin x_0}.$

Theorem 3. *The equation*

$$u_0 + uu_{11} = \lambda_1 u + \lambda_2, \quad (\lambda_1, \lambda_2 = \text{const}) \tag{10}$$

is Q-conditionally invariant under the operator

$$Q = \partial_0 + \frac{u}{x_1}\partial_1 + (\lambda_1 u + \lambda_2)\partial_u. \tag{11}$$

Proof. If we find a prolongation of the operator (11) and act on the equation (10), then we have

$$\tilde{Q}(u_0 + uu_{11} - \lambda_1 u - \lambda_2) = \left(\frac{2u}{x_1^2} - \frac{3u_1}{x_1} + 2\lambda_1 + \frac{\lambda_2}{u}\right) \times$$
$$\times (u_0 + uu_{11} - \lambda_1 u - \lambda_2) - \left(\frac{2u}{x_1^2} - \frac{2u_1}{x_1} + \lambda_1 + \frac{\lambda_2}{u}\right) \times$$
$$\times \left(u_0 + \frac{uu_1}{x_1} - \lambda_1 u - \lambda_2\right),$$

i.e.

$$\tilde{Q}S = \alpha S + \beta Q u,$$

The theorem is proved.

The ansatz

$$\lambda_1 v + \lambda_2 = e^{\lambda_1 x_0}\varphi(\omega), \quad \omega = u - \lambda_2 x_0 - \lambda_1\frac{x_1^2}{2}, \tag{12}$$

obtained with the help of the operator (11) reduces the equation (10) to the following ODE

$$\ddot{\varphi} = 0. \tag{13}$$

Solving the equation (13) and using the ansatz (12), we find the solution of the equation (10):

$$\lambda_1 u + \lambda_2 = e^{\lambda_1 x_0}\left[c_1\left(u - \lambda_2 x_0 - \lambda_1\frac{x_1^2}{2}\right) + c^2\right]. \tag{14}$$

Now we give some more results on the Q-conditional invariance of the equation (1). The results are written in the following order — an equation (1), a corresponded operator, an ansatz, a reduced equation, a solution of the equation (1).

1) $(\lambda_1 u^2+\lambda_2)u_0+u_{11}=\lambda_2 u^3, \quad Q=\lambda_2 x_1^2\partial_0+3x_1\partial_1+3u\partial_u,$

$u=x_1\psi(\omega), \quad \omega=x_0-\dfrac{\lambda_2}{6}x_1^2, \qquad \lambda_1\varphi^2\varphi+\dfrac{\lambda_2^2}{9}\ddot\varphi=\lambda_3\varphi^3,$

2) $(\lambda_1 u^2+\lambda_2)u_0+u_{11}=\lambda_3 u^3+2u,$

$Q=\lambda_2(1+\cos 2x_1)\partial_0-3\sin 2x_1\partial_1+6u\partial_u, \quad u=\operatorname{ctg} x_1\varphi(\omega),$

$\omega=\dfrac{3}{\lambda_2}x_0+\ln\sin x_1, \quad \ddot\varphi-3\dot\varphi+2\varphi+3\dfrac{\lambda_1}{\lambda_2}\varphi^2\dot\varphi=\lambda_3\varphi^3,$

3) $(\lambda_1 u^2+\lambda_2)u_0+u_{11}=\lambda_3 u^3-2u, \quad Q=\lambda_2\partial_0+3\operatorname{th} x_1\partial_1-\dfrac{3u}{\operatorname{ch}^2 x_1}\partial_u,$

$u=\operatorname{cth} x_1\varphi(\omega), \quad \omega=\dfrac{3}{\lambda_2}x_0-\ln\operatorname{sh} x_1, \quad \ddot\varphi+3\dot\varphi+2\varphi+3\dfrac{\lambda_1}{\lambda_2}\varphi^2\dot\varphi=\lambda_3\varphi^3,$

4) $e^u u_0+u_{11}=e^u,$

$a)$ $Q=x_1\partial_1-2\partial_u, \quad u=\varphi(x_0)-2\ln x_1, \quad e^\varphi\dot\varphi+2=e^\varphi, \quad u=\ln\dfrac{e^{x_0}+2}{x_1^2},$

$b)$ $Q=\partial_1+\operatorname{tg}\dfrac{x_1}{2}\partial_u, \quad u=\varphi(x_0)-2\ln\cos\dfrac{x_1}{2}, \quad e^\varphi\dot\varphi+\dfrac12=e^\varphi,$

$u=\ln\dfrac{e^{x_0}+\frac12}{\cos^2\frac{x_1}{2}},$

$c)$ $Q=\partial_1+\operatorname{th}\dfrac{x_1}{2}\partial_u, \quad u=\varphi(x_0)-2\ln\operatorname{ch}\dfrac{x_1}{2}, \quad e^\varphi\dot\varphi-\dfrac12=e^\varphi,$

$u=\ln\dfrac{e^{x_0}-\frac12}{\cos^2\frac{x_1}{2}},$

5) $\lambda u u_0+u_{11}=\lambda u^2,$

$a)$ $Q=\partial_0+\left(u+\dfrac{1}{\lambda x_1^2}\right)\partial_u, \quad u=\dfrac{1}{\lambda x_1^2}+e^{x_0}\varphi(x_1),$

$x_1^2\ddot\varphi-6(2\lambda-1)\varphi=0,$

$b)$ $Q=\partial_0-\left(u-\dfrac{1}{\lambda}W(x_1)\right)\partial_u, \quad u=\dfrac{1}{\lambda}W+e^{x_0}\varphi(x_1), \quad \ddot\varphi=W\varphi,$

$u=\dfrac{1}{\lambda}W(x_1)+e^{x_0}\lambda(x_1),$

where $W(x)$ is the Weierstrass function, $\lambda(x)$ is the Lame function.

1. Овсянников Л.В., Групповой анализ дифференциальных уравнений, М., Наука, 1978, 400 с.

2. Дородницын В.А., Князева И.В., Свищевский С.Р., Групповые свойства уравнени теплопроводности с источником в двумерном и трехмерном случаях, *Дифференц. уравнения*, 1983, **19**, № 7, 1215–1224.

3. Фущич В.И., Условная симметрия уравнений нелинейной математической физики, *Укр. мат. журн.*, 1991, **43**, № 11, 1456–1470.

4. Фущич В.І., Сєров М.І., Умовна інваріантність і точні розв'язки нелінійного рівняння акустики, *Доп. АН УРСР, Сер. А*, 1988, № 1, 28–32.

5. Серов Н.И., Условная инвариантность и точные решения нелинейного уравнения теплопроводности, *Укр. мат. журн.*, 1990, **42**, № 10, 1370–1376.

A new conformal-invariant non-linear spinor equation

W. FUSHCHYCH, W. SHTELEN, P. BASARAB-HORWATH

We propose a new model for a spinor particle, based on a non-linear Dirac equation. We invoke group invariance and use symmetry reduction in order to obtain a multi-parameter family of exact solutions of the proposed equation.

1. Introduction

Since the discovery of the electron, many people have proposed and discussed the hypothesis that the mass of the electron is generated by an electromagnetic field, which the electron produces itself, so that the electron can be thought of as localized electromagnetic energy. In other words, this means that the electron is described by a non-linear dynamical system (see, for instance [1, 2] for these ideas). We propose a realization of this old and interesting physical idea in the framework of the classical theory of spinor fields. For the electron, we propose the following Lorentz-invariant spinor equation

$$(i\gamma\partial - m(u, v, \bar{\Psi}\Psi, j_\mu j^\mu))\Psi = 0, \tag{1.1}$$

where

$$\gamma\partial = \gamma^\mu\partial_\mu, \quad \mu = 0, 1, 2, 3$$

and the γ^μ are the Dirac matrices

$$\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \gamma^a = \begin{pmatrix} 0 & \sigma^a \\ -\sigma^a & 0 \end{pmatrix}, \quad a = 1, 2, 3,$$

where the σ^a are the 2×2 Pauli matrices

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

$$u = -\frac{1}{2}F_{\mu\nu}F^{\mu\nu}, \quad v = -\frac{1}{4}F_{\mu\nu}\tilde{F}^{\mu\nu},$$

where $F^{\mu\nu}$ is an antisymmetric tensor and

$$\partial_\mu F^{\mu\nu} = j^\nu, \quad \partial_\mu\tilde{F}^{\mu\nu} = 0$$

with

$$\tilde{F}_{\mu\nu} = \frac{1}{2}\varepsilon_{\mu\nu\alpha\beta}F^{\alpha\beta}$$

and $\varepsilon_{\mu\nu\alpha\beta}$ is the antisymmetric Kronecker symbol.

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The electromagnetic field which the electron itself produces satisfies Maxwell's equations:

$$\partial_\nu F^{\mu\nu} = j\nu' \quad \text{with} \quad j\nu = e\bar{\Psi}\gamma^\nu\Psi, \tag{1.2}$$

where e is the charge of the electron.

We can interpret (1.1) as follows: the mass, m, of an electron is generated by the electromagnetic field $F^{\mu\nu}$, and its own spinor field Ψ. In the usual Dirac equation, m is a parameter which does not depend on the electromagnetic and spinor fields. Equation (1.1), in contrast to the standard Dirac equation, is a complicated non-linear equation, and as a result one has the following problem: how does one find at least some non-trivial solutions of such an equation?

For the case of m depending only on the spinor field, some classes of exact solutions of (1.1) have been found [5, 6, 10, 11]. In order to construct solutions of (1.1), (1.2), we first examine the symmetries of this system, and then we give some families of exact solutions. The system (1.1), (1.2) is non-linear even for $m = \text{const}$, and can be thought of as a first modification of the Dirac equation in our approach.

2. Symmetries

In the spinor equation (1.1), (1.2), we shall consider the fields $F^{\mu\nu}$, $\bar{\Psi}$, Ψ as independent, and we shall look for symmetry operators of that system in the form

$$X = \xi_\mu \frac{\partial}{\partial x^\mu} + \eta^{(1)}_\mu \frac{\partial}{\partial \Psi_\mu} + \eta^{(2)}_\mu \frac{\partial}{\bar{\Psi}_\mu} + \eta^{(3)}_{\mu\nu} \frac{\partial}{\partial F^{\mu\nu}},$$

where the coefficients are functions of x, Ψ, $\bar{\Psi}$, $F^{\mu\nu}$. In finding these symmetry operators, we use the method of Lie [4, 8, 9]. Indeed, after a painstaking calculation, we obtain the following:

Theorem 1. *The maximal point symmetry algebra of the system of (1.1), (1.2), with* $m = \text{const}$*, has as basis the following vector fields:*

$$\partial_\mu = \partial/\partial x^\mu, \quad J_{\mu\nu} = x_\mu\partial_\nu - x_\nu\partial_\mu + (\sigma_{\mu\nu}\Psi)^\rho \frac{\partial}{\partial \Psi^\rho} + F^{\mu\rho}\frac{\partial}{\partial F^{\nu\rho}} - F^{\nu\rho}\frac{\partial}{\partial F^{\mu\rho}}, \tag{2.1}$$

$$D = \Psi^\mu \frac{\partial}{\partial \Psi^\mu} + F^{\mu\nu}\frac{\partial}{\partial F^{\mu\nu}}, \tag{2.2}$$

$$P = P^{\mu\nu}\frac{\partial}{\partial F^{\mu\nu}}, \tag{2.3}$$

where $\partial_\mu P^{\mu\nu} = 0$, $\partial_\mu \tilde{P}^{\mu\nu} = 0$ *and*

$$\sigma_{\mu\nu} = -\frac{i}{4}[\gamma^\mu, \gamma^\nu].$$

Remark 1. The operator D generates scale transformations in the space of the field variables Ψ^m, $F^{\mu\nu}$, not in Minkowski space $\mathbb{R}(1,3)$. The operators $\langle \partial_\mu, J_{\mu\nu}, D\rangle$, generate the extended Poincaré algebra [4].

If we assume dependence of the mass on the Lorentz-invariant quantities u, v, defined in (1.1), (1.2), we retain invariance under the Poincaré group, but not always under the extended Poincaré group. In fact, we have the following result:

Theorem 2. *The system (1.1), (1.2), where m is a function of the invariants u, v defined in (2.1), (2.2), is invariant under the algebra generated by (1.3), (1.4) if and only if*

$$m = \begin{cases} m\left(\dfrac{v}{u}\right), & u \neq 0, \\ m = \text{const}, & u = 0 \end{cases}$$

Remark 2. Theorem 2 implies that there exists a wide class of non-linear systems of the form (1.1), (1.2), which are invariant with respect to the extended Poincaré algebra. This is so when we assume that the mass depends only on the electromagnetic field.

3. Conformally invariant equations

In this paragraph, we shall describe equations of the form (1.1), (1.2), which are invariant under the conformal group, under the assumption that the mass has the following dependence on the fields:

$$m = \lambda_1 F_1(u, v) + \lambda_2(\bar{\Psi}\Psi)^k. \tag{3.1}$$

The conformal group, $C(1,3)$ is well-known (see for instance [4], [5]). It consists of the Poincaré group together with the following non-linear transformations:

$$x'_\mu = \frac{x_\mu - c_\mu x^2}{\sigma}, \tag{3.2}$$

$$\Psi'(x') = \sigma(1 - (\gamma c)(\gamma x))\Psi(x), \tag{3.3}$$

$$\begin{aligned} F'_{\mu\nu}(x') = \sigma^2 F_{\mu\nu} + 2\sigma\{x^\beta[(2(cx) - 1)(c_\mu F_{\beta\nu} - c_\nu F_{\beta\mu}) - \\ - c^2(x_\mu F_{\beta\nu} - x_\nu F_{\beta\mu})] + c^\alpha[x_\alpha F_{\alpha\nu} - x_\nu F_{\alpha\mu} - \\ - x^2(c_\mu F_{\alpha\nu} - c_\nu F_{\alpha\mu})] + 2(c_\mu x_\nu - c_\nu x_\mu)F_{\alpha\beta}c^\alpha x^\beta\}, \end{aligned} \tag{3.4}$$

$$x'_\mu = e^\theta x_\mu, \tag{3.5}$$

$$\Psi'(x') = e^{-\frac{3}{2}\theta}\Psi(x), \tag{3.6}$$

$$F'_{\mu\nu}(x') = e^{-2\theta}F_{\mu\nu}, \tag{3.7}$$

where the primes denote transformed quantities, θ and c_μ are arbitrary real constants, $cx = c_\mu x^\mu$, $c^2 = c_\mu c^\mu$, $x^2 = x_\mu x^\mu$.

Applying Lie's method for calculating symmetry operators, one can prove the following result:

Theorem 3. *The system of equations (1.1), (1.2), with mass given by (3.1), is invariant under the conformal group if and only if* $k = \frac{1}{3}$ *and*

$$F_1(u, v) = \begin{cases} u^{\frac{1}{4}} F\left(\dfrac{v}{u}\right), & u \neq 0, \\ v^{\frac{1}{4}}, & u = 0, \end{cases} \tag{3.8}$$

where F is an arbitrary, smooth function.

One can easily verify that (1.1), (1.2), with mass defined by (3.1), is indeed invariant under the scale transformations (2.4)–(2.6). Substituting these into the equations yields

$$[i\gamma\partial - \lambda_1 F_1(e^{-4\theta}u, e^{-4\theta}v) - \lambda_2 e^{\theta(1-3k)}(\bar{\Psi}\Psi)^k]\Psi = 0,$$
$$\partial_\nu F^{\mu\nu} = e\bar{\Psi}\gamma^\mu\Psi, \quad \partial_\nu \tilde{F}^{\mu\nu} = 0.$$

The condition of invariance then gives

$$e^\theta F_1(e^{-4\theta}u, e^{-4\theta}v) = F_1(u,v), \quad \theta(1-3k) = 0$$

which immediately implies $k = \frac{1}{3}$, and, differentiating with respect to θ, that F_1 satisfies the equation

$$4u\frac{\partial F_1}{\partial u} + 4v\frac{\partial F_1}{\partial v} = F_1.$$

The general solution of this equation is easily shown to be that given by (3.8). Conformal invariance follows by using the transformations

$$\bar{\Psi}\Psi \mapsto \sigma^3\bar{\Psi}\Psi, \quad u \mapsto \sigma^4 u, \quad v \mapsto \sigma^4 v.$$

Remark 3. Requiring conformal invariance narrows quite considerably the class of admissible systems (1.1), (1.2). Fixing the function $F\left(\frac{u}{v}\right)$, we obtain different conformally-invariant equations for a spinor particle.

4. Exact solutions

We shall construct a class of exact solutions for the simplest conformally-invariant system (1.1), (1.2), namely for the case $F = 1$, so that our system becomes

$$\left(i\gamma\partial - \lambda_1 u^{\frac{1}{4}} - \lambda_2(\bar{\Psi}\Psi)^{\frac{1}{3}}\right)\Psi = 0,$$
$$\partial_\nu F^{\mu\nu} = e\bar{\Psi}\gamma^\mu\Psi, \quad \partial_\nu \tilde{F}^{\mu\nu} = 0. \qquad (4.1)$$

We shall look for solutions of this system by the method of reduction [4], that is we reduce the system of partial differential equations to systems of ordinary differential equations. For these, we use the following ansatzes [4, 5, 6, 7, 10, 11]:

$$\Psi(x) = \varphi(\omega), \quad F^{\mu\nu}(x) = f^{\mu\nu}(\omega), \qquad (4.2)$$

where $\varphi(\omega)$ is a four-component vector, $f^{\mu\nu}(\omega)$ an antisymmetric tensor, $\omega = \beta x$, with β a constant vector satisfying $\beta^2 = 1$. Substituting (4.2) into (4.1), we obtain the reduced system of ordinary differential equations

$$i(\gamma\beta)\dot{\varphi} - \left(\lambda_1 z^{\frac{1}{4}} + \lambda_2(\bar{\varphi}\varphi)^{\frac{1}{3}}\right) = 0,$$
$$\beta_\nu \dot{f}^{\mu\nu} = e\bar{\varphi}\gamma^\mu\varphi, \quad \beta_\nu f^{\mu\nu} = 0 \qquad (4.3)$$

with $z = -\frac{1}{2}f_{\mu\nu}f^{\mu\nu}$ and the dot denotes differentiation with respect to the argument ω. Since $f^{\mu\nu}$ is anisymmetric, it follows that $\beta_\mu\beta_\nu\dot{f}^{\mu\nu} = 0$, so that the second equation in (4.3) yields $\bar{\varphi}(\gamma\beta)\varphi = 0$. Using the relation

$$\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}$$

and the fact that β is chosen so that $\beta^2 = 1$, it is easy to show that $(\gamma\beta)(\gamma\beta) = 1$. Multiplying the first equation of (4.3) on the left by $\bar{\varphi}(\gamma\beta)$ we then obtain

$$\bar{\varphi}\dot{\varphi} = 0.$$

We therefore find that φ satisfies

$$\bar{\varphi}\varphi = \text{const}, \quad \bar{\varphi}(\gamma\beta)\varphi = 0. \tag{4.4}$$

These equations imply that we should look for solutions φ in the form

$$\varphi = \exp(i(\gamma\beta)g(\omega))\chi, \tag{4.5}$$

where $g(\omega)$ is a function we must find and χ is a constant vector which satisfies $\bar{\chi}(\gamma\beta)\chi = 0$. Since $(\gamma\beta)^2 = 1$, it follows that

$$\varphi = [\cos(g(\omega)) - i(\gamma\beta)\sin(g(\omega))]\chi, \tag{4.6}$$

$$\bar{\varphi}\gamma_\mu\varphi = \alpha^\mu \cos(2g(\omega)) + c^\mu \sin(2g(\omega)), \tag{4.7}$$

$$\alpha^\mu = \bar{\chi}\gamma^\mu\chi, \quad c^\mu = \frac{i}{2}\bar{\chi}[(\gamma\beta), \gamma^\mu]\chi. \tag{4.8}$$

Clearly, $\alpha\beta = 0$. Equation (4.3) together with (4.6), (4.7), (4.8), can be written as

$$\begin{aligned} &\dot{g} = \lambda_1 z^{\frac{1}{4}} + \lambda_2(\bar{\chi}\chi)^{\frac{1}{3}}, \\ &\beta_\nu \dot{f}^{\mu\nu} = e(\alpha^\mu \cos(2g) + c^\mu \sin(2g)), \\ &\beta_\nu \dot{\tilde{f}}^{\mu\nu} = 0. \end{aligned} \tag{4.9}$$

We now seek solutions of (4.9) of the form

$$\begin{aligned} &g(\omega) = \kappa\omega, \\ &f^{\mu\nu} = \varepsilon[(\alpha^\mu\beta^\nu - \alpha^\nu\beta^\mu)\sin(2\kappa\omega) - (c^\mu\beta^\nu - c^\nu\beta^\mu)\cos(2\kappa\omega)], \end{aligned} \tag{4.10}$$

where κ, ε are constants. Without loss of generality, we assume $\alpha^2 = c^2 = -1$, since we have $\beta^2 = 1$, $\alpha\beta = \beta c = \alpha c = 0$. With these conventions, (4.9) and (4.10) give

$$\kappa = \lambda_1\sqrt{\varepsilon} + \lambda_2(\bar{\chi}\chi)^{\frac{1}{3}}, \quad e = 2\varepsilon\kappa. \tag{4.11}$$

Let us now consider solutions of (4.11). The first case is when $\lambda_1 \neq 0$, $\lambda_2 = 0$. Then

$$\varepsilon = \left(\frac{e}{2\lambda_1}\right)^{\frac{2}{3}}, \quad \kappa = \left(\frac{e\lambda_1^2}{2}\right)^{\frac{1}{3}}. \tag{4.12}$$

The second case is $\lambda_1 = 0$, $\lambda_2 \neq 0$, which gives

$$\kappa = \lambda_2(\bar{\chi}\chi)^{\frac{1}{3}}, \quad \varepsilon = \frac{e}{2\lambda_2(\bar{\chi}\chi)^{\frac{1}{3}}}. \tag{4.13}$$

Finally, when $\lambda_1 \neq 0$, $\lambda_2 \neq 0$ equation (4.12) becomes the cubic equation

$$y^3 + py + q = 0, \quad \varepsilon = \frac{e}{2\kappa}, \tag{4.14}$$

where

$$y = \sqrt{\kappa} = \sqrt[3]{-\frac{q}{2} + \sqrt{Q}} + \sqrt[3]{-\frac{q}{2} - \sqrt{Q}}, \quad Q = \frac{e\lambda_1^2}{8} - \frac{\lambda_2^2(\bar{\chi}\chi)}{27}.$$

In this way we obtain exact solutions of the system (4.1), (4.2) in the following form

$$\psi(x) = \exp(-i\kappa(\gamma\beta)\omega)\chi, \quad \omega = \beta x, \tag{4.15}$$

$$F^{\mu\nu} = \frac{e}{2\kappa}[(\alpha^\mu\beta^\nu - \alpha^\nu\beta^\mu)\sin(2\kappa\omega) - (c^\mu\beta^\nu - c^\nu\beta^\mu)\cos(2\kappa\omega)], \tag{4.16}$$

$$\alpha^\mu = \bar{\chi}\gamma^\mu\chi, \quad c^\mu = \frac{i}{2}\bar{\chi}[(\gamma\beta), \gamma^\mu]\chi,$$

$$\beta^2 = 1, \quad \alpha^2 = c^2 = -1, \quad \alpha\beta = \alpha c = \beta c = 0.$$

For conformally invariant solutions of (4.1) we exploit the ansatzes [6, 7]

$$\begin{aligned} &\psi(x) = \frac{\gamma x}{(x^2)^2}\varphi(\omega), \quad \omega = \frac{\beta x}{x^2}, \quad \beta^2 = 1, \\ &F^{\mu\nu} = \frac{f^{\mu\nu}(\omega)}{x^2} - \frac{2x_\rho[x^\mu f^{\rho\nu}(\omega) - x^\nu f^{\rho\nu}(\omega)]}{(x^2)^3}. \end{aligned} \tag{4.17}$$

Combining (4.1) and (4.17) yields the system of ordinary differential equations

$$\begin{aligned} &-i(\gamma\beta)\dot{\varphi} = \lambda_1 z^{\frac{1}{4}} + \lambda_2(\bar{\varphi}\varphi)^{\frac{1}{3}}, \\ &\beta_\nu \dot{f}^{\mu\nu} = -e\bar{\varphi}\gamma^\mu\varphi, \\ &\beta_\nu \dot{\tilde{f}}^{\mu\nu} = 0 \end{aligned} \tag{4.18}$$

with $z = -\frac{1}{2}f_{\mu\nu}f^{\mu\nu}$, which is formally similar to (4.3). Using this fact, we can write down the following solutions of (4.1), (4.17):

$$\Psi(x) = \frac{\gamma x}{(x^2)^2}\exp(i\kappa(\gamma\beta)\omega)\chi, \quad \omega = \frac{\beta x}{x^2}, \tag{4.19}$$

$$\begin{aligned} F^{\mu\nu} = \frac{e}{2\kappa(x^2)^2}\Big\{&\Big[\beta^\mu\alpha^\nu - \beta^\mu\alpha^\nu) + 2(\alpha^\mu x^\nu - \alpha^\nu x^\mu)\omega + \\ &+ 2\frac{\alpha x}{x^2}(x^\mu\beta^\nu - x^\nu\beta^\mu)\Big]\sin(2\kappa\omega) + \Big[(c^\mu\beta^\nu - c^\nu\beta^\mu) + \\ &+ 2\omega(x^\mu c^\nu - x^\nu c^\mu) - 2\frac{cx}{x^2}(x^\mu\beta^\nu - x^\nu\beta^\mu)\Big]\cos(2\kappa\omega)\Big\}, \end{aligned} \tag{4.20}$$

where

$$\alpha^\mu = \bar{\chi}\gamma^\mu\chi, \quad c^\mu = \frac{i}{2}\bar{\chi}[(\gamma\beta), \gamma^\mu]\chi,$$
$$\beta^2 = 1, \quad \alpha^2 = c^2 = -1, \quad \alpha\beta = \alpha c = \beta c = 0.$$

The solutions found show that the system (1.1), (1.2) is consistent, at least in certain cases of the mass function. Furthermore, we can calculate the mass corresponding to these solutions:

$$m = \lambda_1 u^{\frac{1}{4}} + \lambda_2(\bar{\psi}\psi)^{\frac{1}{3}} = \frac{\kappa}{x^2}.$$

5. Conclusion

We have shown that there exists a consistent non-linear dynamical model for a classical spinor particle, in which the mass is generated by an electromagnetic field and a spinor field, which the particle itself creates. The proposed model (3.1) is conformally-invariant, as is the class of solutions we obtain, For these solutions, we have also found an explicit form for the Lorentz-invariant mass. The question of quantizing the model (3.1) will be taken up in future papers.

1. Yukawa H. (ed.), Nonlinear field theory, *Progress of Theoretical Physics, Supplements*, 1959, V.9, 1–128.
2. Heisenberg W., Introduction to the unified theory of elementary particles. Collected works, Series B, Berlin, Springer-Verlag, 1984.
3. Bedrij O., Fushchych W., On the electromagnetic structure of elementary particle masses, *Proceedings of the Ukrainian Academy of Sciences*, 1991, № 2, 38–40.
4. Fushchych W., Shtelen W., Serov M., Symmetry and exact solutions of equationsof nonlinear mathematical physics, Dordrecht, Kluwer, 1993.
5. Fushchych W., Nikitin A., Symmetries of Maxwell's equations, Dordrecht, D. Reidel, 1987.
6. Fushchych W., Shtelen W., On some exact solutions of the nonlinear Dirac equation, *J. Phys. A*, 1983, **16**, 271–277.
7. Fushchych W., Shtelen W., On some exact solutions of the nonlinear equations of quantum electrodynamics, *Phys. Lett. B*, 1983, V.128, 215–217.
8. Olver P., Applications of Lie groups to differential equations, New York, Springer, 1986.
9. Ovsjannikov L.V., Group analysis of differential equations, New York, Academic Press, 1982.
10. Fushchych W., Zhdanov R., Symmetry and exact solutions of non-linear spinor equations, *Physics Reports*, 1989, V.172, 123–174.
11. Fushchych W., Zhdanov R., Non-linear spinor equations: symmetry and exact solutions, Kiev, Naukova Dumka, 1992 (in Russian).

Generation of solutions for nonlinear equations via the Euler–Amperé transformation

W.I. FUSHCHYCH, V.A. TYCHYNIN

За допомогою контактного перетворення Ейлера–Ампера одержано формули розмноження розв'язків. Побудовано класи ДРЧП другого порядку, які інваріантні відносно цього перетворення.

The invariance of DE under a nonlocal transformation of variables allows us to generate its solutions from the known ones. The reducing of a nonlinear PDE to a linear equation makes it possible to construct for it the formula of a nonlinear superposition of solutions. In the present paper the solutions generating formulae are obtained via the Euler–Amperé contact transformation. Classes of Euler–Amperé invariant PDEs are constructed. The efficiency of the obtained formulae is illustrated in several of examples.

1. Nonlocal invariance and the solutions generating formula. Let us consider the Euler–Amperé transformation in the space $\mathbb{R}(1,n-1)$ of n independent variables [1, 2]:

$$\begin{aligned}
&u = y_a v_a - v, \quad x_0 = y_0, \quad x_a = v_a, \\
&v_\mu = \partial_\mu v = \frac{\partial v}{\partial y_\mu}, \quad a,b = \overline{1,n-1}, \quad \delta \equiv \det(v_{ab}) \neq 0, \quad \mu,\nu = \overline{1,n-1}.
\end{aligned} \tag{1}$$

The first and second order derivatives are changing as

$$\begin{aligned}
&u_0 = -v_0, \quad u_a = y_a, \\
&v_{00} = -\det{}^{-1}(v_{ab})\det(v_{\mu\nu}), \quad u_{0a} = -\det{}^{-1}(v_{ab}) v_{0b} a_{ba}(v_{cd}), \\
&u_{ab} = -\det{}^{-1}(v_{cd}) a_{ab}(v_{cd}) \quad (a,b,c,d = \overline{1,n-1}).
\end{aligned} \tag{2}$$

Hereafter the summation over repeated Greek indices is understood in the space $\mathbb{R}(1,n-1)$ with the metric $g_{\mu\nu} = \mathrm{diag}\,(1,-1,\dots,-1)$ and over repeated Latin indices it is understood in the space $\mathbb{R}(0,n-1)$ with the metric $g_{ab} = \mathrm{diag}\,(1,1,\dots,1)$, $u_{\mu\nu} = \partial_{\mu\nu} u = \frac{\partial^2 u}{\partial x_\mu \partial x_\nu}$, $\det(u_{ab}) = a_{00}(u_{ab})$. $a_{\lambda\sigma}(u_{\mu\nu})$, $a_{ab}(u_{cd})$ are the cofactors to the elements $u_{\lambda\sigma}$ and u_{ab} respectively, $\lambda,\sigma = \overline{0,n-1}$.

Following expressions are absolute differential invariants of order ≤ 2 with respect to (1) due to its involutivity:

$$\begin{aligned}
&f^0(x_0), \quad f^1(x_a,u_a), \quad f^2(x_0,-u_0), \quad f^3(u, x_a u_a - u), \\
&f^4(u_{00}, -\det{}^{-1}(u_{ab})\det(u_{\mu\nu})), \quad f^5(u_{0a}, -\det{}^{-1}(u_{ab}) u_{0b} a_{ba}(u_{cd})), \\
&f^6(u_{ab}, -\det{}^{-1}(u_{cd}) a_{ab}(u_{cd})).
\end{aligned} \tag{3}$$

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Here f^0 is an arbitrary smooth function, and f^k, $k = 1, 6$ are arbitrary smooth and symmetric on arguments functions:

$$f^k(x, z) = f^k(z, x).$$

Let us construct by means of the expressions (3) the absolutely invariant under transformation (1) second order PDE

$$F(\{f^\sigma\}) = 0 \quad (\sigma = \overline{0, 6}). \tag{4}$$

$F(\cdot)$ is an arbitrary smooth function. Such equations are contained in the class (4):

$$u_0 - u_a u_a + x^2 = 0, \quad x^2 = x_a x_a; \tag{5.1}$$

$$\lambda u_0 - \Delta u - \det{}^{-1}(u_{cd})\,\mathrm{Slid}\,(u_{cd}) = 0; \tag{5.2}$$

$$u_{00} - \det{}^{-1}(u_{cd})\det(u_{\mu\nu}) = 0; \tag{5.3}$$

$$\lambda u_0 - \det{}^m(u_{cd}) + (-1)^{m\cdot n}\det{}^{-m\cdot n}(u_{cd})\det[a_{ab}(u_{cd})] = 0; \tag{5.4}$$

$$\lambda u_0^{2h} + \varphi(x_c)u_a u_a + \varphi(u_c)x^2 = 0; \tag{5.5}$$

$$\lambda u_0^{2h} + \varphi(u_c)u_a u_a + \varphi(x_c)x^2 = 0; \tag{5.6}$$

$$\lambda u_0 - \varphi(x_c, u)\Delta - \det{}^{-1}(u_{cd})\varphi(u_c, x_a u_a - u)\cdot \mathrm{Slid}\,(u_{cd}) = 0. \tag{5.7}$$

One can continue this list of equations (5) in the obvious manner. Δ is the Laplacian,

$$\mathrm{Slid}\,(u_{cd}) \stackrel{\mathrm{def}}{=} g_{ab}a_{ab}(u_{cd}),$$

$\varphi(x, z)$ is an arbitrary smooth function, λ is an arbitrary parameter, m, h are real numbers.

Let $\overset{(1)}{u}(x_0, x)$ be a known partial solution of Eq. (4). For constructing new solution $\overset{(2)}{u}(x_0, x)$ of this Eq. (4) we rewrite the formula (1) in parametric form, replacing x_a for parameters τ^a, $a = \overline{1, n-1}$. Substitute $\overset{(1)}{u}(x_0, \tau)$, $\overset{(1)}{u}_a(x_0, \tau)$ to (1). So, as a result, we obtain the formula

$$\begin{aligned} &\overset{(2)}{u}(x_0, x) = \tau^a \overset{(1)}{u}_a(x_0, \tau) - \overset{(1)}{u}(x_0, \tau) = \tau^a x_a - \overset{(1)}{u}(x_0, \tau), \\ &x_a = \overset{(1)}{u}_a(x_0, \tau), \quad a = \overline{1, n-1}. \end{aligned} \tag{6}$$

Here $x = (x_1, x_2, \ldots, x_{n-1})$, $\tau = (\tau^1, \tau^2, \ldots, \tau^{n-1})$. The formula (6) allows us to construct efficiently the new solutions of nonlinear equations (5) by resolving the last system (6) with respect to parameters τ.

Example 1. Let us consider the equation

$$u_0 u_{11} - u_{11}^2 + 1 = 0, \tag{7}$$

which is invariant under the transformation (1). The function

$$\overset{(1)}{u}(x_0, x_1) = \varphi(\omega), \quad \omega = \alpha_0 x_0 + \alpha_1 x_1$$

is a solution of Eq. (7), when φ satisfies the first order ODE

$$q + 2(k\dot\varphi)^{-1}\sqrt{(k\dot\varphi)^2+1} + 4k\omega + c_1 = 0,$$
$$q \equiv \ln\left|k\dot\varphi - \sqrt{(k\dot\varphi)^2+1}\right| - \ln\left|k\dot\varphi + \sqrt{(k\dot\varphi)^2+1}\right|. \tag{8}$$

Here $k = \frac{1}{2}\alpha_0$, $\alpha_1 = 1$, $\alpha_1 = 1$, α_0, c_1 are arbitrary constants. In this case the generating of new solutions is realized according to the formulae

$$\overset{(2)}{u}(x_0, x_1) = \tau\cdot\varphi(\omega) - \dot\varphi(\omega), \quad x_1 = \dot\varphi(\omega), \quad \omega = 2kx_0 + \tau. \tag{9}$$

From the second equation of the system (9) we get

$$\tau = [\dot\varphi]^{-1}(x_1) - 2kx_0.$$

Here $[\dot\varphi]^{-1}(x)$ is the inverse function to $\dot\varphi(x)$. Note, that

$$[\dot\varphi]^{-1}(x_1) = \omega, \quad \dot\varphi(\omega) = \dot\varphi(\omega) = \dot\varphi([\dot\varphi]^{-1}(x_1)) = x_1.$$

Then from the first equation of the system (9) we obtain

$$\overset{(2)}{u}(x_0, x_1) = \tau\cdot x_1 + \varphi(2kx_0 + \tau). \tag{10}$$

Here $\varphi(2kx_0+\tau)$ is a solution of ODE (8) of argument $2kx_0+\tau$. Due to equality $x_1 = \overset{(1)}{u}(x_0,\tau) = \dot\varphi(2kx_0+\tau)$ we get τ from the correlation (8)

$$\tau = -\left\{g^* + 2(kx_1)^{-1}\sqrt{(kx_1)^2+1} + 2kx_0 + (4k)^{-1}\cdot c_1\right\},$$
$$g^* \equiv \ln\left|kx_1 - \sqrt{(kx_1)^2+1}\right| - \ln\left|kx_1 + \sqrt{(kx_1)^2+1}\right|. \tag{11}$$

Thus, the solution $u(x_0, x_1)$ is determined by the parametric system of equations

$$\overset{(2)}{u}(x_0,x_1) = \varphi\left(-q^* - 2(kx_1)^{-1}\sqrt{(kx_1)^2+1} - (4k)^{-1}\cdot c_1\right) -$$
$$- x_1\left\{g^* + 2(kx_1)^{-1}\sqrt{(kx_1)^2+1} + 2kx_0 + (4k)^{-1}\cdot c_1\right\}, \tag{12}$$

$$q + 2(k\dot\varphi)^{-1}\sqrt{(k\dot\varphi)^2+1} + 4k\omega + c_1 = 0, \quad \varphi = \varphi(\omega). \tag{13}$$

Example 2. The equation

$$(u_0 - \Delta_{(2)}u)(u_{11}u_{22} - u_{12}^2) - \Delta_{(2)}u = 0 \tag{14}$$

is (1)-invariant, when the condition $u_{11}u_{22} - u_{12}^2 \neq 0$ is satisfied. The partial solution of Eq. (13) is

$$\overset{(1)}{u} = \ln r^2, \quad r^2 = x_1^2 + x_2^2.$$

Let us replace x_a, $a = 1, 2$ in $\overset{(1)}{u}$ for parameters τ^a

$$\overset{(1)}{u} = \ln\rho^2, \quad \rho^2 = (\tau^1)^2 + (\tau^2)^2.$$

and substitute this result into to the formula (6). We obtain

$$\overset{(2)}{u}(x_0,x_1,x_2)=2-\ln\rho^2,\quad x_a=2\tau^a\rho^{-2},\quad a=1,2. \tag{15}$$

Let us express τ^1, τ^2 through x_1, x_2 from the last two conditions of the system (15)

$$\tau^a=2x_ar^{-2},\quad a=1,2. \tag{16}$$

Substituting τ from (16) into the first equation of the system (15), we get the solution $\overset{(2)}{u}$:

$$\overset{(2)}{u}=2(1-\ln 2)+\ln r^2.$$

2. Nonlocal linearization and nonlinear superposition formula. Let us apply the transformation (1) to a general second order linear PDE

$$b^{\mu\nu}(y_0,y)v_{\mu\nu}+b^{\mu}(y_0,y)v_\mu+b(y_0,y)v+c(y_0,y)=0. \tag{17}$$

$y=(y_1,y_2,\ldots,y_{n-1})$, $b^{\mu\nu}=b^{\nu\mu}$, b^μ, b, c are arbitrary smooth functions of y_0, y. As a result we get the nonlinear equation

$$\begin{aligned}&\{b^{00}(x_0,\underset{1}{u})\det(u_{\mu\nu})-2b^{0a}(x_0,\underset{1}{u})u_{0b}a_{ba}(u_{cd})+\\&\quad+b^{ab}(x_0,\underset{1}{u})a_{ab}(u_{cd})\}\det{}^{-1}(u_{cd})+b^0(x_0,\underset{1}{u})\cdot u_0+\\&\quad+b^a(x_0,\underset{1}{u})x_a-b(x_0,\underset{1}{u})[x_au_a-u]-c(x_0,\underset{1}{u})=0.\end{aligned} \tag{18}$$

Here $\underset{1}{u}=(u_1,u_2,\ldots,u_{n-1})$, $a,b=\overline{1,n-1}$. Eq. (18) possesses the solutions superposition property, which arises from the superposition of solutions of the linear equation (17)

$$\overset{(3)}{v}(y_0,y)=\overset{(1)}{v}(y_0,y)+\overset{(2)}{v}(y_0,y).$$

Let $\overset{(k)}{u}$, $k=1,2$ be known solutions of Eq. (18) and $\overset{(3)}{u}(x_0,x)$ be a new solution of the same equation. Let us express $\overset{(3)}{u}$ through $\overset{(1)}{u}$ and $\overset{(2)}{u}$. Making use of Euler–Amperé transformation (1), we get

$$\begin{aligned}&\overset{(3)}{u}(x_0,x)=y_a\overset{(3)}{v}_a-\overset{(3)}{v}=y_a(\overset{(1)}{v}_a+\overset{(2)}{v}_a)-\overset{(1)}{v}-\overset{(2)}{v},\\&x_a=\overset{(3)}{v}_a=\overset{(1)}{v}_a+\overset{(2)}{v}_a,\quad x_0=y_0.\end{aligned} \tag{19}$$

One can express $\overset{(1)}{v}$ and $\overset{(2)}{v}$ via $\overset{(1)}{u}$ and $\overset{(2)}{u}$ accordingly, where x are replaced for parameters $\tau=(\tau^1,\tau^2,\ldots,\tau^{n-1})$ in the first and $\theta=(\theta^1,\theta^2,\ldots,\theta^{n-1})$ in the second ones:

$$\begin{aligned}&\overset{(k)}{v}=\overset{(k)}{\tau}{}^a\,\overset{(k)}{u}_a-\overset{(k)}{u},\quad k=1,2,\quad y_0=x_0=\overset{(k)}{\tau}{}^0,\quad \overset{(1)}{\tau}\equiv\tau,\\&y_a=\overset{(k)}{u}_a(\overset{(k)}{\tau}{}^0,\overset{(k)}{\tau}),\quad \overset{(2)}{\tau}\equiv\theta,\quad \overset{(k)}{v}_a=\overset{(k)}{\tau}{}^a.\end{aligned} \tag{20}$$

Substituting the relations (20) into (19) we obtain the solutions superposition formula for Eq. (18)

$$\overset{(3)}{u}(x_0,x) = \overset{(1)}{u}(x_0,\tau) + \overset{(2)}{u}(x_0,x-\tau), \quad \overset{(1)}{u}_a(x_0,\tau) = \overset{(2)}{u}_a(x_0,\theta), \quad \theta = x-\tau. \tag{21}$$

Here the second equation of the system (19) is used essentially $x_a = \tau^a + \theta^a$ for eliminating parameters θ^a in the formula (21) and the designation $\overset{(2)}{u}_a(x_0,\theta) \equiv \overset{(2)}{u}_{\theta^a}(x_0,\theta)$ is adopted as well.

Example 3. Let us use as initial partial solutions

$$\overset{(1)}{u}(x_0,x_1) = x_0 - \frac{1}{2}x_1^2, \quad \overset{(2)}{u}(x_0,x_1) = k[1+x_1-x_0]^{\frac{3}{2}}, \quad k = -\frac{2}{3}\sqrt{2}$$

of the Euler–Amperé-linearizable equation

$$u_0u_{11} + 1 = 0, \quad u_{11} \neq 0. \tag{22}$$

Replacing the argument x_1 for parameter τ in $\overset{(1)}{u}$ and for parameter $\theta = x_1 - \tau$ in $\overset{(2)}{u}$ and making use of the formula (21), we obtain a new solution of Eq. (22)

$$\overset{(3)}{u}(x_0,x_1) = x_0 - h - \frac{2}{3}\sqrt{2}h^{\frac{3}{2}}, \quad h \equiv 2 + x_1 - x_0 \pm \sqrt{2(x_1-x_0)+3}. \tag{23}$$

Example 4. The nonlinear heat conduction equation

$$\begin{aligned} &u_0\det(u_{ab}) + \Delta_{(2)}u = 0,\\ &\Delta_{(2)} \equiv \partial_1^2 + \partial_2^2, \quad \det(u_{ab}) \neq 0 \end{aligned} \tag{24}$$

admits the linearization under the transformation (1) to the equation

$$v_0 - \Delta_{(2)}v = 0.$$

This Eq. (24) possesses the partial solution in parametric form

$$\begin{aligned} &\overset{(1)}{u}(x_0,x_1,x_2) = \theta x_1^{-1}r^2 + 2x_0x_1\theta^{-1}, \quad r^2 = x_1^2 + x_2^2,\\ &x_1(8\pi x_0^2) = \pm\theta\exp\{-\theta^2r^2(8x_0x_1^2)^{-1}\}. \end{aligned} \tag{25}$$

Let the second solution of Eq. (24) take the form

$$\overset{(2)}{u}(x_0,x_1,x_2) = x_1^2 - x_2^2. \tag{26}$$

Making use of the formula (21) we obtain the new solution $\overset{(3)}{u}$:

$$\begin{aligned} &\overset{(3)}{u}(x_0,x_1,x_2) = \theta(x_1-\theta)^{-2}\left(x_1 - \frac{1}{2}\theta\right)(r^2+\theta^2-2x_1\theta) +\\ &\qquad + 2x_0\theta^{-1}\left(x_1-\frac{1}{2}\theta\right) + \frac{1}{4}\theta^2 - x_2^2(x_1-\theta)^{-2}\left(x_1-\frac{1}{2}\theta\right)^2, \\ &8\pi x_0^2\left(x_1-\frac{1}{2}\theta\right) = \pm\theta\exp\left\{-\theta^2\frac{(r^2+\theta^2-2x_1\theta)}{4x_0(x_1-\theta)^2}\right\}, \end{aligned} \tag{27}$$

θ is the parameter to be eliminated.

1. Гурса Е., Інтегрування рівнянь з частинними похідними першого порядку, Київ, Рад. шк., 1941, 415 с.
2. Фущич В.И., Тычинин В.А., Жданов Р.З., Нелокальная линеаризация и точные решения некоторых уравнений Монжа–Ампера, Дирака, Препринт № 85.88, Киев, Институт математики АН УССР, 1985, 28 с.

Hodograph transformations and generating of solutions for nonlinear differential equations

W.I. FUSHCHYCH, V.A. TYCHYNIN

Перетворення годографа однієї скалярної функції в $\mathbb{R}(1,1)$ та $\mathbb{R}(1,3)$, а також двох скалярних функцій в $\mathbb{R}(1,1)$ використані для розмноження розв'язків нелінійних рівнянь; побудовані класи годограф-інваріантних рівнянь другого порядку.

The results of using the hodograph transformations for solution of applied problems are well-known. One can find them for example in [1, 2, 3]. We note also the paper [4], in which a number of invariants for hodograph transformation as well as hodograph-invariant equations were constructed.

1. Hodograph-invariant and -linearizable equations in $\mathbb{R}(1,1)$. Let us consider the hodograph transformation for one scalar function ($M = 1$) of two independent variables $x = (x_0, x_1)$, $n = 2$:

$$\begin{aligned} &u(x) = y_1, \quad x_0 = y_0, \quad x_1 = v(y), \\ &\delta = v_1 = \partial_1 v = \frac{\partial v}{\partial y_1} \neq 0, \quad y = (y_0, y_1). \end{aligned} \tag{1}$$

Differential prolongations of the transformation (1) generate such expressions for the first and second order derivatives:

$$u_1 = v_1^{-1}, \quad u_0 = -v_0 v_1^{-1}, \tag{2}$$

$$\begin{aligned} &u_{11} = -v_1^{-3} v_{11}, \quad u_{10} = -v_1^{-3}(v_1 v_{10} - v_0 v_{11}), \\ &u_{00} = -v_1^{-3}[v_0^2 v_{11} - 2v_0 v_1 v_{10} + v_1^2 v_{00}]. \end{aligned} \tag{3}$$

It is clear that (1) is an involutory transformation. This allows to write a set of differential expressions of order ≤ 2, which are absolutely invariant under the transformation (1):

$$\begin{aligned} &f^0(x_0), \quad f^1(x_1, u), \quad f^2(u_1, u_1^{-1}), \quad f^3(u_0, -u_0 u_1^{-1}), \quad f^4(u_{11}, -u_1^{-3} u_{11}), \\ &f^5(u_{10}, -u_1^{-3}(u_1 u_{10} - u_0 u_{11})), \quad f^6(u_{00}, -u_1^{-3}[u_0^2 u_{11} - 2u_0 u_1 u_{10} + u_1^2 u_{00}]). \end{aligned} \tag{4}$$

Here f^0 is an arbitrary smooth function, f^i, $i = \overline{1,6}$ are arbitrary functions symmetric on arguments, i.e. $f^i(x,z) = f^i(z,x)$. So, the second order PDE invariant under the transformation (1) has the form

$$F(\{f^\sigma\}) = 0, \quad \{f^\sigma\} = \{f^0, f^1, \ldots, f^6\}, \quad \sigma = \overline{0,6}, \tag{5}$$

F is an arbitrary smooth function.

Such well-known equations are contained in the class (5):

1. $u_0^2 - u_1^2 - 1 = 0$ — the eikonal equation; (6)

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2. $u_{11} - u_{00}[u_0^2 u_{11} - 2u_0 u_1 u_{10} + u_1^2 u_{00}] = 0$ — the Born–Infeld equation; (7)

3. $u_{00} u_{11} - u_{10}^2 = 0$ — the Monge–Amperé equation; (8)

4. $u_0 = f(u_1) u_{11}, \quad f(u_1) = f(u_1^{-1}) u_1^{-2}$ — the nonlinear heat equation [5]. (9)

Particularly, such equation as

$$u_0 - u_1^{-1} u_{11} = 0 \tag{10}$$

is contained in the last class (9).

Let $\overset{(1)}{u}(x_0, x_1)$ be a known solution of Eq. (5). To construct a new solution $\overset{(2)}{u}(x_0, x_1)$ let us write the first solution replacing in it an argument x_1 for parameter τ: $\overset{(1)}{u}(x_0, \tau)$ and substitute it to the hodograph transformation formula (1). So, we obtain the solutions generating formula for Eq. (5).

$$\overset{(2)}{u}(x_0, x_1) = \tau, \quad x_1 = \overset{(1)}{u}(x_0, \tau). \tag{11}$$

Let us now describe some class of (1)-linearizable equations. Making use of formulae (1) to transform general linear second order PDE

$$b^{\mu\nu}(y) v_{\mu\nu} + b^{\mu}(y) v_{\mu} + b(y) v + c(y) = 0, \quad y = (y_0, y_1), \quad \mu, \nu = 0, 1, \tag{12}$$

we obtain

$$\begin{aligned} & b^{00}(x_0, u) u_1^{-3} (u_0^2 u_{11} - 2u_0 u_1 u_{10} + u_1^2 u_{00}) - \\ & \quad - 2b^{10}(x_0, u) u_1^{-3} (u_1 u_{10} - u_0 u_{11}) + b^{11}(x_0, u) u_1^{-3} u_{11} + \\ & \quad + b^0(x_0, u) u_1^{-1} u_0 + b^1(x_0, u) u_1^{-1} - b(x_0, u) x_1 - c(x_0, u) = 0. \end{aligned} \tag{13}$$

$b^{\mu\nu}$, b^{μ}, c are arbitrary smooth functions, $b^{10} = b^{01}$. Summation over repeated indices is understood in the space $\mathbb{R}(1,1)$ with the metric $g_{\mu\nu} = \mathrm{diag}\,(1, -1)$. The repeated use of this transformation to Eq. (12) turn us again to the Eq. (11).

For any equation of the class (12) the principle of nonlinear superposition is satisfied

$$\overset{(3)}{u}(x_0, x_1) = \overset{(1)}{u}(x_0, \tau), \quad \overset{(1)}{u}(x_0, x_1) = \overset{(2)}{u}(x_0, x_1 - \tau), \tag{14}$$

Here $\overset{(k)}{u}(x_0, x_1)$, $k = 1, 2$ are known solutions of Eq. (12), $\overset{(3)}{u}(x_0, x_1)$ is a new solution of this equation. Parameter τ must be eliminated due to second equality of the system (13). For example, such equations important for applications are contained in this class (12):

$$\begin{aligned} & u_0 - u_1^{-2} u_{11} = 0, \quad u_0 u_{11} - u_1 u_{10} = 0, \\ & u_0^2 u_{11} - 2u_0 u_1 u_{10} + u_1^2 u_{00} = 0, \quad u_0 - c(x_0, u) u_1 = 0. \end{aligned}$$

Let us consider now an example of constructing new solutions from two known ones by means of solutions superposition formula (13).

Example 1. A nonlinear heat equation

$$u_0 - u_1^{-2} u_{11} = 0$$

is reduced to the linear equation

$$v_0 - v_{11} = 0 \tag{15}$$

Therefore, the formula (13) is true for (14). The functions

$$\overset{(1)}{u} = x_1, \quad \overset{(2)}{u} = \sqrt{x_1 - 2x_0} \tag{16}$$

are both partial solutions of Eq. (14). We construct a new solution $\overset{(3)}{u}$ of this Eq. (14) via $\overset{(1)}{u}$ and $\overset{(2)}{u}$. It has the form

$$\overset{(3)}{u}(x_0, x_1) = -\frac{1}{2} \pm \sqrt{\frac{1}{4} + x_1 - 2x_0}, \tag{17}$$

2. Hodograph-invariant and -linearizable equations in $\mathbb{R}(1,3)$. The hodograph transformation of a scalar function $u(x)$ of four independent variables $x = (x_0, x_1, x_2, x_3)$ has the form

$$v(x) = y_1, \quad x_1 = v(y), \quad x_\theta = y_\theta, \quad \theta = 0, 2, 3. \tag{18}$$

Prolongation formulae for (18) are obtained via calculations [6, 7]:

$$\begin{gathered} u_1 = v_1^{-1}, \quad u_\theta = -v_1^{-1} v_\theta, \quad u_{11} = -v_1^{-3} v_{11}, \\ u_{1\theta} = -v_1^{-3}(v_1 v_{1\theta} - v_\theta v_{11}), \quad v_{\theta\theta} = -v_1^{-3}(v_1^2 v_{\theta\theta} - 2v_\theta v_1 v_{1\theta} + v_\theta^2 v_{11}), \\ u_{\theta\gamma} = -v_1^{-3}[v_1(v_1 v_{\theta\gamma} - v_\gamma v_{1\theta}) - v_\theta(v_1 v_{1\gamma} - v_\gamma v_{11})]. \end{gathered} \tag{19}$$

Here $\theta, \gamma = 0, 2, 3$, $\theta \neq \gamma$. Making use of involutivity of the transformation (18) we list for it a such set of absolute differential invariant expressions of order ≤ 2:

$$\begin{gathered} f^0(x_0, x_2, x_3), \quad f^1(x_1, u), \quad f^2(u_1, u_1^{-1}), \quad f^3(u_\theta, -u_1^{-1} u_\theta), \\ f^4(u_{11}, -u_1^{-3} u_{11}), \quad f^5(u_{1\theta}, -u_1^{-3}(u_1 u_{1\theta} - u_\theta u_{11})), \\ f^6(u_{\theta\theta}, -u_1^{-3}(u_1^2 u_{\theta\theta} - 2u_1 u_\theta u_{1\theta} + u_\theta^2 u_{11})). \\ f^7(u_{\theta\gamma}, -u_1^{-3}[u_1(u_1 u_{\gamma\theta} - u_\gamma u_{1\theta}) - u_\theta(u_1 u_{1\gamma} - u_\gamma u_{11})]). \end{gathered} \tag{20}$$

There is no summation over θ here, as before, f^0 is an arbitrary smooth function, f^j, $j = \overline{1,7}$ are arbitrary symmetric.

An equation invariant under transformation (18) has the form

$$F(\{f^\lambda\}) = 0 \quad (\lambda = \overline{0,7}). \tag{21}$$

The solutions generating formula has the same form as (10)

$$\overset{(2)}{u}(x_0, x_1, x_2, x_3) = \tau, \quad x_1 = \overset{(1)}{u}(x_0, \tau, x_2, x_3). \tag{22}$$

Here $\overset{(1)}{u}(x)$ is a known solution of Eq. (21), $\overset{(2)}{u}(x)$ is its new solution. The following well-known equations are contained in this class (21):

1. $u_0^2 - u_a u_a - 1 = 0$, $a = \overline{1,3}$, the eikonal equation;
2. $(1 - u_\nu u^\nu)\Box u - u^\mu u^\nu u_{\mu\nu} = 0$, $\mu, \nu = \overline{0,3}$, the Born–Infeld equation [8];
3. $\det(u_{\mu\nu}) = 0$ the Monge–Amperé equation.

Here summation over repeated indices is understood in the space $\mathbb{R}(1,3)$ with the metric $g_{\mu\nu} = \mathrm{diag}\,(1,-1,-1,-1)$.

$$\Box u = \partial_\mu \partial^\mu u = u_{00} - u_{11} - u_{22} - u_{33}$$

is the d'Alembert operator,

$$u_a u_a = u_1^2 + u_2^2 + u_2^2 + u_3^2 = (\nabla u)^2.$$

The class of hodograph-linearizable equations in $\mathbb{R}(1,3)$ is constructed analogously as above. Making use of transformation (18) for linear equation (11), written in $\mathbb{R}(1,3)$, we get

$$\begin{aligned} &b^{11}(x_\delta,u)u_1^{-3}u_{11} + b^{\theta\theta}(x_\delta,u)u_1^{-3}(u_1^2u_{\theta\theta} - 2u_1u_\theta u_{10} + u_\theta^2 u_{11}) + \\ &\quad + b^{\gamma\theta}(x_\delta,u)u_1^{-3}[u_1(u_1u_{\gamma\theta} - u_\gamma u_{10}) - u_\theta(u_1u_{1\gamma} - u_\gamma u_{11})] + \\ &\quad + b^1(x_\delta,u)u_1^{-1}u_\theta - b(x_\delta,u)x_1 - c(x_\delta,u) = 0, \quad x_\delta = (x_0,x_2,x_3). \end{aligned} \tag{23}$$

Here $\delta,\theta = 0,2,3$ and summation over θ is understood in the space $\mathbb{R}(1,2)$ with metric $\tilde{g}_{\theta\gamma} = \mathrm{diag}\,(1,-1,-1)$.

Note, that multidimensional nonlinear heat equation

$$u_0 - u_1^{-2}(1+u_2^2+u_3^2)u_{11} - u_{22} - u_{33} + 2u_1^{-1}(u_2u_{12} + u_3u_{13}) = 0 \tag{24}$$

reduces due to transformation (18) to linear equation $v_0 = \Delta_{(3)}v$, where $\Delta_{(3)} \equiv \partial_1^2 + \partial_2^2 + \partial_3^2$ is the Laplace operator.

So, the solutions superposition formula for the equations (23) and (24) is

$$\overset{(3)}{u}(x_0,x_1,x_2,x_3) = \overset{(1)}{u}(x_0,\tau,x_2,x_3), \tag{25}$$

$$\overset{(1)}{u}(x_0,\tau,x_2,x_3) = \overset{(2)}{u}(x_0,x_1-\tau,x_2,x_3). \tag{26}$$

Example 2. Let partial solutions of Eq. (24)

$$\overset{(1)}{u} = x_0 - x_2 - x_3 - \ln\frac{x_1 - c_2}{c_1}, \quad \overset{(2)}{u} = \left[\frac{9}{4}c_3^2(x_1-c_4)^2 - x_2^2 - x_3^2\right]^{\frac{1}{2}}$$

be initial for generating a new solution $\overset{(3)}{u}$. Then this new solution of Eq. (24) is determined via (25), (26) by the equality

$$\begin{aligned} &\overset{(3)}{u}{}^2(x) + x_2^2 + x_3^2 = c_3\big[x_1 - c_2 - c_1\exp\{x_0 - x_2 - x_3 - \overset{(3)}{u}(x)\}\big]^2, \\ &c_3 = \frac{9}{4}c_3^2, \quad c_2 = c_4 + c_2. \end{aligned} \tag{27}$$

Thus, the formula (27) gives us a new solution of Eq. (24) in the implicite form.

3. Hodograph-invariant and -linearizable systems of PDE in $\mathbb{R}(1,1)$. Let us consider two functions $u^\mu(x_0,x_1)$, $\mu = 0,1$ of independent variables x_0, x_1. The hodograph transformation in this case, as is known [2], has the form

$$\begin{aligned} &u^0(x_0,x_1) = y_0, \quad u^1(x_0,x_1) = y_1, \quad x_0 = v^0(y_0,y_1), \quad x_1 = v^1(y_0,y_1), \\ &\delta = u_1^1u_0^0 - u_0^1u_1^0 \neq 0, \quad \delta^* = v_1^1v_0^0 - v_0^1v_1^0 \neq 0. \end{aligned} \tag{28}$$

The first and second order derevatives are changing as

$$u_1^1 = \delta^{*-1} v_0^0, \quad u_0^1 = -\delta^{*-1} v_0^1, \quad u_1^0 = -\delta^{*-1} v_1^0, \quad u_0^0 = \delta^{*-1} v_1^1, \tag{29}$$

$$\begin{aligned}
u_{11}^1 &= -\delta^{*-3} \cdot [(v_0^0)^2(v_0^1 v_{11}^0 - v_0^0 v_{11}^1) + (v_1^0)^2(v_0^1 v_{00}^0 - v_0^0 v_{00}^1) - \\
&\quad - 2v_1^0 v_0^0 (u_0^1 v_{10}^0 - v_0^0 v_{10}^1)], \\
u_{00}^1 &= -\delta^{*-3} \cdot [(v_0^1)^2(v_0^1 v_{11}^0 - v_0^0 v_{11}^1 + (v_1^1)^2(v_0^1 v_{00}^0 - v_0^0 v_{00}^1) - \\
&\quad - 2v_0^1 v_1^1 (v_0^1 v_{10}^0 - v_0^0 v_{10}^1)], \\
u_{10}^1 &= \delta^{*-3} \cdot [v_0^0 v_0^1 (v_0^1 v_{11}^0 - v_0^0 v_{11}^1) + v_1^0 v_1^1 (v_0^1 v_{00}^0 - v_0^0 v_{00}^1) - \\
&\quad - (v_0^1 v_{10}^0 - v_1^0 v_{10}^1)(v_1^1 v_0^0 + v_0^1 v_1^0)], \\
u_{11}^0 &= -\delta^{*-3}[(v_0^0)^2(v_1^0 v_{11}^1 - v_1^1 v_{11}^0) + (v_1^0)^2(v_1^0 v_{00}^1 - v_1^1 v_{00}^0) - \\
&\quad - 2v_1^0 v_0^0 (v_1^0 v_{10}^1 - v_1^1 v_{10}^0)], \\
u_{00}^0 &= -\delta^{*-3}[(v_0^1)^2(v_1^0 v_{11}^1 - v_1^1 v_{11}^0) + (v_1^1)^2(v_1^0 v_{00}^1 - v_1^1 v_{00}^0) - \\
&\quad - 2v_1^1 v_0^1 (v_1^0 v_{10}^1 - v_1^1 v_{10}^0)], \\
u_{10}^0 &= -\delta^{*-3}[v_0^0 v_0^1 (v_1^0 v_{11}^1 - v_1^1 v_{11}^0) + v_1^0 v_1^1 (v_1^0 v_{00}^1 - v_1^1 v_{00}^0) - \\
&\quad - (v_1^0 v_{10}^1 - v_1^1 v_{10}^0)(v_1^1 v_0^0 + v_0^1 v_1^0)].
\end{aligned} \tag{30}$$

Let us now construct the absolute differential invariants with respect to (28)–(30) of order ≤ 2. Making use of involutivity of this transformation we get

$$f^1(x_\mu, u^\mu), \quad \mu = 0, 1, \quad f^2(u_\mu^\mu, \delta u_\nu^\nu), \quad \mu \neq \nu, \quad \mu, \nu = 0, 1,$$

there is no summation over repeated indices here,

$$\begin{aligned}
&f^3(u_\nu^\mu, -\delta^{-1} u_\nu^\mu), \quad \mu \neq \nu, \quad \mu, \nu = 0, 1; \\
&f^4(u_{11}^1, -\delta^{-3}[(u_0^0)^2(u_0^1 v_{11}^0 - u_0^0 u_{11}^1) + (u_1^0)^2(u_0^1 u_{00}^0 - u_0^0 u_{00}^1) - \\
&\qquad - 2u_1^0 u_0^0 (u_0^1 u_{10}^0 - u_0^0 u_{10}^1)]), \\
&f^5(u_{00}^1, -\delta^{-3} \cdot [(u_0^1)^2(u_0^1 u_{11}^0 - u_0^0 u_{11}^1) + (u_1^1)^2(u_0^1 u_{00}^0 - u_0^0 u_{00}^1) - \\
&\qquad - 2u_0^1 u_1^1 (u_0^1 v_{10}^0 - u_0^0 u_{10}^1)]), \\
&f^6(u_{10}^1, -\delta^{-3} \cdot [u_0^0 u_0^1 (u_0^1 u_{11}^0 - u_0^0 u_{11}^1) + u_1^0 u_1^1 (u_0^1 u_{00}^0 - u_0^0 u_{00}^1 - \\
&\qquad - (u_0^1 u_{10}^0 - u_0^0 u_{10}^1)(u_1^1 u_0^0 + u_0^1 u_1^0)]), \\
&f^7(u_{11}^0, -\delta^{-3} \cdot [(u_0^0)^2(u_1^0 u_{11}^1 - u_1^1 u_{11}^0) + (u_1^0)^2(u_1^0 u_{00}^1 - u_1^1 u_{00}^0) - \\
&\qquad - 2u_1^0 u_0^0 (u_1^0 u_{10}^1 - u_1^1 u_{10}^0)]), \\
&f^8(u_{00}^0, -\delta^{-3}[(u_0^1)^2(u_1^0 u_{11}^1 - u_1^1 u_{11}^0) + (u_1^1)^2(u_1^0 u_{00}^1 - u_1^1 u_{00}^0) - \\
&\qquad - 2u_1^1 u_0^1 (u_1^0 u_{10}^1 - u_1^1 u_{10}^0)]), \\
&f^9(u_{10}^0, -\delta^{-3}[u_0^0 u_0^1 (u_1^0 u_{11}^1 - u_1^1 u_{11}^0) + u_1^0 u_1^1 (u_1^0 u_{00}^1 - u_1^1 u_{00}^0) - \\
&\qquad - (u_1^0 u_{10}^1 - u_1^1 u_{10}^0)(u_1^1 u_0^0 + u_0^1 u_1^0)]).
\end{aligned} \tag{31}$$

All functions f^k, $k = \overline{1, 9}$ are arbitrary smooth and symmetric.

So, we now are able to construct the hodograph-invariant system of second order PDEs

$$F^\sigma(\{f^k\}) = 0, \quad k = \overline{1, 9}, \quad \sigma = 1, 2, \ldots, N. \tag{32}$$

We construct a new solution $\overset{(2)}{u} = (\overset{(2)}{u}{}^0, \overset{(2)}{u}{}^1)$ of system (32) via known solution $\overset{(1)}{u} = (\overset{(1)}{u}{}^0, \overset{(1)}{u}{}^1)$ according to the formula

$$\overset{(2)}{u}(x) = \tau, \quad x = \overset{(1)}{u}(\tau). \tag{33}$$

Here $x = (x_0, x_1)$, $\tau = (\tau^0, \tau^1)$, τ^μ are parameters to be eliminated out of system (33).

Example 3. Let us consider the simplest hodograph-invariant system of first order PDE

$$u_0^1 - u_1^0 = 0, \quad u_1^1 - u_0^0 = 0. \tag{34}$$

It is easily to verify, that pair of functions

$$\overset{(1)}{u}{}^0 = 2x_0x_1 + c, \quad \overset{(1)}{u}{}^1 - x_0^2 + x_1^2$$

is the solution of system (34). Making use of formula (33) one obtain the new solution of this system

$$\begin{aligned} \overset{(2)}{u}{}^1 &= \pm\frac{1}{\sqrt{2}}\left[x_1 \pm \sqrt{x_1^2 + (x_0 - c)^2}\right]^{\frac{1}{2}}, \\ \overset{(2)}{u}{}^0 &= \pm\frac{x_0 - c}{\sqrt{2}}\left[x_1 \pm \sqrt{x_1^2 + (x_0 - c)^2}\right]^{-\frac{1}{2}}. \end{aligned} \tag{35}$$

Let us consider the linear system of first order PDEs

$$b_\mu^{\sigma\nu}(y)v_\mu^\nu + b^{\sigma\nu}(y)v^\nu + c^\sigma(y) = 0. \tag{36}$$

Here $b_\mu^{\sigma\nu}$, $b^{\sigma\nu}$, c^σ are arbitrary smooth functions of $y = (y_0, y_1)$, summation over repeated indices is understood in the space with metric $g_{\mu\nu}^* = \operatorname{diag}(1, 1)$. This system (36) under transformation (28) reduces into system of nonlinear PDEs

$$\begin{aligned} &b^{\sigma 0}(u)\delta^{-1}u_1^1 - b_1^{\sigma 0}(u)\delta^{-1}u_1^0 - b_0^{\sigma 1}(u)\delta^{-1}u_0^1 + \\ &\quad + b_1^{\sigma 1}(u)\delta^{-1}u_0^0 + b^{\sigma 0}(u)x_0 + b^{\sigma 1}(u)x_1 + c^\sigma(u) = 0. \end{aligned} \tag{37}$$

The solutions superposition formula for the system (37) has the form

$$\begin{aligned} &\overset{(3)}{u}{}^0(x_0, x_1) = \overset{(1)}{u}{}^0(\tau^0, \tau^1), \quad \overset{(1)}{u}{}^0(\tau^0, \tau^1) = \overset{(2)}{u}{}^0(x_0 - \tau^0, x_1 - \tau^1), \\ &\overset{(3)}{u}{}^1(x_0, x_1) = \overset{(1)}{u}{}^1(\tau^0, \tau^1), \quad \overset{(1)}{u}{}^1(\tau^0, \tau^1) = \overset{(2)}{u}{}^1(x_0 - \tau^0, x_1 - \tau^1). \end{aligned} \tag{38}$$

Making use of designations $u = (u^0, u^1)$, $x = (x_0, x_1)$, $\tau = (\tau^0, \tau^1)$, one can rewrite the formula (38) in another way:

$$\overset{(3)}{u}(x) = \overset{(1)}{u}(\tau), \quad \overset{(1)}{u}(\tau) = \overset{(2)}{u}(x - \tau). \tag{38a}$$

Example 4. It is obviously, that two pairs of functions

$$\begin{aligned} &\overset{(1)}{u} = \frac{1}{2}x_0, \quad \overset{(1)}{\rho} = (2\lambda)^{-1}\sqrt{\frac{1}{4}x_0^2 - x_1}, \\ &\overset{(2)}{u} = x_0^{-1}\left[\frac{1}{2}c_1 + x_1\right], \quad \overset{(2)}{\rho} = (2\lambda x_0)^{-1}c_0 \end{aligned} \tag{39}$$

give two partial solutions of the system

$$\begin{aligned} &u_0 + uu_1 + 4\lambda^2\rho\rho_1 = 0, \\ &\rho_0 + u_1\rho + u\rho_1 = 0. \end{aligned} \tag{40}$$

Let us apply the formula (38) to construct a new solution $\overset{(3)}{u}$, $\overset{(3)}{\rho}$ via (39). Finally we get

$$\overset{(3)}{u}{}^2(x_0,x_1) - c_2^2(x_0 - 2\overset{(3)}{u}(x_0,x_1))^{-2} - x_0\overset{(3)}{u}(x_0,x_1) + x_1 + \frac{1}{2}c_1 = 0,$$
$$\overset{(3)}{\rho}(x_0,x_1) = (2\lambda)^{-1}\left[x_0\overset{(3)}{u}(x_0,x_1) - \overset{(3)}{u}{}^2(x_0,x_1) - x_1 - \frac{1}{2}c_1\right]^{\frac{1}{2}}.$$

1. Forsyth A.R., Theory of differential equations, New York, Dover Publication, 1959, Vol. 5, 478 p.; Vol. 6, 596 p.
2. Ames W.F., Nonlinear partial differential equations in engineering, New York, Academic Press, 1965, Vol. 1, 511 p.; 1972, Vol. 2, 301 p.
3. Курант Р., Уравнения в частных производных, М., Мир, 1964, 830 с.
4. Фущич В.И., Серов Н.И., Негрупповая симметрия некоторых нелинейных волновых уравнений, *Докл. АН УССР*, 1991, № 9, 45–49.
5. Fushchych W.I., Serov N.I., Tychynin V.A., Amerov T.K., On nonlocal symmetries of nonlinear heat equation, *Докл. АН Украины, Сер. А*, 1992, № 11, 27–33.
6. Фущич В.И., Тычинин В.А., О линеаризации некоторых нелинейных уравнений с помощью нелокальных преобразований, Препринт № 82.33, Киев, Ин-т математики АН УССР, 1982, 53 с.
7. Фущич В.И., Тычинин В.А., Жданов Р.З., Нелокальная линеаризация и точные решения некоторых уравнений Монжа–Ампера, Дирака, Препринт № 85.88, Киев, Ин-т математики АН УССР, 1985, 28 с.
8. Тычинин В.А., Нелокальная линеаризация и точные решения уравнения Борна–Инфельда и некоторых его обобщений, в сб. Теоретико-групповые исследования уравнений математической физики, Киев, Ин-т математики АН УССР, 1986, 54–60.

New conditionally invariant solutions for non-linear d'Alembert equation

W.I. FUSHCHYCH, I.A. YEGORCHENKO

We describe all ansatzes of a specific form that reduce the non-linear d'Alembert equation. In this way we obtain some new solutions of the equation with a polynomial non-linearity.

1. Introduction. Let us consider a non-linear d'Alembert equation of the form

$$\Box u = \lambda u^k, \tag{1}$$

where $u = u(x_0, x_1, x_2, x_3)$ is a real function; $k \neq 1$, λ are parameters,

$$\Box u \equiv \frac{\partial^2 u}{\partial x_0^2} - \frac{\partial^2 u}{\partial x_1^2} - \frac{\partial^2 u}{\partial x_2^2} - \frac{\partial^2 u}{\partial x_3^2}.$$

Equation (1) is invariant under the Poincaré algebra $AP(1,3) \Subset D$ with the following basis operators:

$$\begin{gathered} \partial_0, \quad \partial_a, \quad J_{0a} = x_0\partial_a + x_a\partial_0, \quad J_{ab} = x_a\partial_b - x_b\partial_a, \\ D = x_0\partial_0 + x_a\partial_a + \frac{2}{1-k}u\partial_u, \end{gathered} \tag{2}$$

when k is arbitrary, $k \neq 1$. Here $a, b = 1, 2, 3$, and we imply summation over the repeated indices from 1 to 3. We shall not consider here the special case $k = 3$ when equation (1) is invariant under the conformal algebra.

All similarity solutions for equation (1) are adduced in [1, 2]. The similarity ansatzes corresponding to three-dimensional subalgebras of the algebra (2) have the form

$$u = f(x)\varphi(\omega), \tag{3}$$

where $f(x)$ is some function, $\omega = \omega(x)$ is a new invariant variable.

In this paper we try to search for a wider class of solutions than similar ones by means of the ansatz (3). Some ansatzes of this form were described in [3]. An example of such ansatz is

$$u = (x^2)^{-1/2}\varphi(\alpha x), \tag{4}$$

where $x^2 = x_0^2 - x_a x_a$, $\alpha_0^2 - \alpha_a\alpha_a = 0$.

The substitution (3) reduces equation (1) to an ordinary differential equation of the functions f and ω satisfy the following set of equations:

$$\begin{gathered} \Box f = f^k S(\omega), \\ 2f_\mu\omega_\mu + f\Box(\omega) = f^k T(\omega), \quad \omega_\mu\omega_\mu = R(\omega)f^{k-1}. \end{gathered} \tag{5}$$

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Here $f_\mu \equiv \frac{\partial f}{\partial x_\mu}$, the summation over the repeated Greek indices is as follows: $f_\mu\omega_\mu \equiv f_0\omega_0 - f_a\omega_a$, $a = 1, 2, 3$; S, T, R are some functions; T and R do not vanish simultaneously.

Further we shall consider the system (5) for the case $\omega_\mu\omega_\mu = 0$.

2. New ansatzes for the d'Alembert equation (1). We succeeded to find all solutions of the system (5) for $\omega = \alpha x$, $\alpha^2 = 0$. In this case the system (5) reduces to the equations

$$\Box f = f^k S(\alpha x), \qquad 2f_\mu\alpha_\mu = f^k T(\alpha x).$$

Its solutions have the following form:

$$f = [h(\omega, \beta x, \gamma x) + \delta x)]^{\frac{1}{1-k}}, \tag{6}$$

where the parameters α_μ, β_μ, γ_μ, δ_μ satisfy the relations $\alpha\beta = \alpha\gamma = \delta^2 = \beta\gamma = 0$, $\alpha\delta = -\beta^2 = -\gamma^2 = 1$.

$$h = \frac{1}{2}\frac{(\beta x)^2(\omega + B_1) + 2B_3^2(\beta x)(\gamma x) + (\gamma x)^2(\omega + B_2)}{(\omega + B_1)(\omega + B_2) - B_3^2}, \tag{7}$$

$$h = \frac{(\beta x)^2}{2\omega + B_1}, \tag{8}$$

$$h = B_1\beta x + B_2 + \frac{B_1^2}{2}\omega. \tag{9}$$

Here B_1, B_2, B_3 are some constants. If $B_1 = B_2$, $B_3 = 0$ we get an ansatz that is equivalent to (4).

3. Operators of conditional symmetry for equation (1). The notion of conditional symmetry had been defined in [2, 4–6]. This approach enabled to construct wide classes of exact solutions for nonlinear partial differential equations of mathematical physics (see [2, 4–6, 8]). In this paper we do not search specially for operators of conditional symmetry but for ansatzes of the form (3) explicitly.

The following statement describes the operators of conditional invariance corresponding to ansatzes of the form (3) with $\omega = \alpha x$, $\alpha^2 = 0$, f being of the form (6), (7).

Theorem 1. *Equation (1) with the additional conditions*

$$\begin{aligned} L_1 &= f\beta_\mu u_\mu - \beta_\mu f_\mu u = 0, \\ L_2 &= f\gamma_\mu u_\mu - \gamma_\mu f_\mu u = 0, \\ L_3 &= 2\delta_\mu u_\mu(1-k) - f^{k-1}u = 0 \end{aligned} \tag{10}$$

is invariant under operators:

$$\begin{aligned} Q_1 &= r(x)(f\beta_\mu\partial x_\mu - \beta_\mu f_\mu u\partial u) = 0, \\ Q_2 &= r(x)(f\gamma_\mu\partial x_\mu - \gamma_\mu f_\mu u\partial u = 0, \\ Q_3 &= r(x)(2\delta_\mu\partial x_\mu - \tfrac{1}{1-k}f^{k-1}u\partial u = 0, \end{aligned} \tag{11}$$

where $r(x)$ is an arbitrary non-zero function, f satisfies the equations

$$f_\mu\delta_\mu = \frac{1}{1-k}f^k, \qquad \Box f = f^k S(\omega), \tag{12}$$

where S is some function.

The above theorem can be proved by means of the Lie algorithm (see e.g. [7]).

Note 1. The same ansatzes may also be obtained from the Lie symmetry operators.

4. Exact solutions of equation (1). The ansatz (3) with $\omega = \alpha x$, $\alpha^2 = 0$, f of the form (6), (7) reduces equation (1) to the following ordinary differential equation:

$$\varphi' \frac{2}{1-k} + S(\omega)\varphi = \lambda \varphi^k, \tag{13}$$

$S(\omega)$ being of the form

$$S(\omega) = -\frac{1}{1-k} \frac{\omega + B_1 + B_2}{(\omega + B_1)(\omega + B_2) - B_3^2},$$

Equation (13) for arbitrary constants B_1, B_2, B_3, $k \neq 1$ can be solved in quadratures:

$$\varphi = \sqrt{\theta} \left[\frac{\lambda(1-k)^2}{2} \int \theta(\omega)^{\frac{k-1}{2}} d\omega \right]^{\frac{1}{1-k}}, \tag{14}$$

where $\theta = (\omega + B_1)(\omega + B_2) - B_3^2$.

Substituting (14) into (3) with f of the form (6), (7), we can obtain a class of solutions for the non-linear d'Alembert equation (1).

5. Compatibility and solutions of the system (5) with $\omega_\mu \omega_\mu = 0$. In this case $R(\omega) = 0$, so $T(\omega)$ must not vanish. We can take $T(\omega) = \frac{2}{1-k}$ and obtain the system

$$f_\mu \omega_\mu + \frac{1}{2} f \Box \omega = \frac{1}{1-k} f^k, \qquad \Box f = f^k S(\omega). \tag{15}$$

If $\Box \omega = 0$, then from the first equation of (15)

$$f = \left[h(\omega, \theta^1, \theta^2) + \theta^3) \right]^{\frac{1}{1-k}}, \tag{16}$$

where θ^1, θ^2, θ^3 are functions on x,

$$\begin{aligned} &\theta^1_\mu \theta^1_\mu = \theta^2_\mu \theta^2_\mu = -1, \\ &\theta^1_\mu \omega_\mu = \theta^2_\mu \omega_\mu = \theta^3_\mu \theta^3_\mu = \theta^1_\mu \theta^2_\mu = 0, \\ &\theta^1_\mu \theta^3_\mu = 1. \end{aligned} \tag{17}$$

With the substitution (16) the second equation (15) reduces to the form

$$\Phi_{\theta^1\theta^1} + \Phi_{\theta^2\theta^2} = \hat{S}(\omega), \qquad 2\Phi_\omega - \Phi^2_{\theta^1} - \Phi^2_{\theta^2} = 0. \tag{18}$$

The compatibility and solutions of the system of Laplace and Hamilton–Jacobi equations were considered in detail in [8]. The system (18) is compatible iff

$$\hat{S}(\omega) = \frac{\rho'}{\rho}, \quad \text{where} \quad \rho''' = 0.$$

If we take for the solutions of the system (17)

$$\theta^1 = \beta x, \quad \theta^2 = \gamma x, \quad \theta^3 = \delta x,$$

where β_μ, γ_μ, δ_μ are parameters satisfying (6), we shall get the solutions (6), (7)–(9) of the system (15).

Note 2. A system similar to (15) arose in [8] while searching for ansatzes of the form $u = \exp(if(x))\varphi(\omega)$ for a nonlinear Schrödinger equation $2iu_t + u_{aa} - uF(|u|) = 0$. It is known [9] that complex n-dimensional non-linear d'Alembert equation can be reduced by similarity methods to $(n-1)$-dimensional Schrödinger equation.

Note 3. The ansatz (3), (6), (7) can be used to get solutions also for complex non-linear d'Alembert equation, the function φ being complex-valued.

For the equation

$$\Box u = \lambda u (uu^*)^{\frac{k-1}{2}},$$

we get the reduced equation

$$2\varphi' - \frac{\rho'}{\rho}\varphi = \lambda(1-k)(\varphi\varphi^*)^{\frac{k-1}{2}},$$

where $\rho = (\omega + B_1)(\omega + B_2) - B_3^2$.

From the reduced equation we can find φ:

$$\varphi = \sqrt{\rho}\left[\frac{\lambda(1-k)^2}{2}\int \rho^{\frac{k-1}{2}} d\omega\right]^{\frac{1}{1-k}} \exp i\sigma,$$

where σ is an arbitrary constant.

1. Fushchych W.I., Serov N.I., *J. Phys. A*, 1983, **16**, 3645–3656.
2. Fushchych W.I., Shtelen W.M. and Serov N.I., Symmetry analysis and exact solutions of nonlinear equations of mathematical physics, Kyiv, Naukova Dumka, 1989 (in Russian); Dordrecht, Kluwer Publishers, 1993 (in English).
3. Fushchych W.I., Yegorchenko I.A., *J. Phys. A*, 1989, **22**, 2643–2652.
4. Fushchych W.I., In the book Symmetry and Solutions of Nonlinear Equations of Mathematical Physics, Kiev, Institute of Mathematics, 1987, 4–16.
5. Fushchych W.I., Serov N.I., In the book Symmetry and Solutions of Equations of Mathematical Physics, Kiev, Institute of Mathematics, 1988, 96–103.
6. Fushchych W.I., *Ukr. Math. Zhurn.*, 1991, **43**, 1456–1470.
7. Ovsyannikov L.V., Group analysis of differential equations, New York, Academic Press, 1982.
8. Fushchych W.I., Yegorchenko I.A., *Ukr. Math. Zhurn.*, 1991, **43**, 1620–1628.
9. Fushchych W.I., Barannik L.F. and Barannik A.F., Subgroup analysis of the Galilei and Poincaré group and reduction of nonlinear equations, Kiev, Naukova Dumka, 1991.

Anti-reduction of the nonlinear wave equation

W.I. FUSHCHYCH, R.Z. ZHDANOV

Ми запропонували конструктивний метод зведення рівняння з частинними похідними до декількох рівнянь з меншим числом незалежних змінних. Застосувавши цей підхід до багатовимірного нелінійного хвильового рівняння, ми побудували низку принципово нових анзаців, які редукують його до двох звичайних диференціальних рівнянь.

The wide class of solutions of the multi-dimensional wave equation

$$\Box u \equiv u_{x_0x_0} - \Delta_3 u = F(u) \tag{1}$$

can be obtained by means of the following ansatz [1–3]:

$$u = \varphi(\omega), \tag{2}$$

where φ is an arbitrary smooth function and $\omega = \omega(x)$ is the absolute invariant of some three-dimensional subgroup of the Poincaré group $P(1,3)$. As a result, one gets ordinary differential equation (ODE) for a function $\varphi(\omega)$. That is why, the term “reduction” is used: a number of dependent and independent variables is decreased.

On the other hand, there are examples of ansatzes reducing one nonlinear partial differential equation (PDE) to two or even to three equations [4]. Such procedure leads to an increase of the number of dependent variables and is called an “anti-reduction” [4].

In the present paper we suggest a regular approach to the anti-reduction of the nonlinear differential equation (1).

Consider the ansatz

$$u(x) = f(x, \varphi_1(\omega_1), \varphi_2(\omega_2), \ldots, \varphi_N(\omega_N)) \tag{3}$$

and the following ordinary differential equations:

$$\ddot{\varphi}_i = R_i(\omega_i, \varphi_i, \dot{\varphi}_i), \quad i = \overline{1, N}, \tag{4}$$

where f, R_i are smooth enough functions, $\omega_i = \omega_i(x) \in C^2(\mathbb{R}^n, \mathbb{R}^1)$, $i = \overline{1,N}$. If substitution of (3) into Eq. (1) with subsequent exclusion of the second derivatives $\ddot{\varphi}^i$, $i = \overline{1,N}$ according to (4) yields an identity with respect to variables $\dot{\varphi}^i$, φ_i, $i = \overline{1,N}$ then we say that the anti-reduction of nonlinear PDE (1) to N ODE takes place.

In fact, the above definition contains an algorithm of the anti-reduction. We are going to realize it, provided $N = 2$.

Theorem. *The equation (1) with a logarithmic nonlinearity*

$$\Box u = \lambda u \ln u, \quad \lambda \in \mathbb{R}^1 \tag{5}$$

Доповіді АН України, 1993, № 11, С. 37–41.

is the only nonlinear wave equation belonging to the class of PDE (1) that admits anti-reduction to two second-order ODE and that is more the ansatz (2) has the form

$$u(x) = a(x)\varphi_1(\omega_1)\varphi_2(\omega_2), \tag{6}$$

where $a(x)$, $\omega_1(x)$, $\omega_2(x)$ are smooth functions satisfying the system of PDE

$$\begin{aligned} &1) \quad \omega_{1x_\mu}\omega_{2x_\mu} = 0,\\ &2) \quad a\Box\omega_i + 2a_{x\mu}\omega_{ix_\mu} = 0, \quad i=\overline{1,2},\\ &3) \quad \omega_{ix_\mu}\omega_{ix_\mu} = Q_i(\omega_i), \quad i=\overline{1,2},\\ &4) \quad \Box a = \lambda \ln a. \end{aligned} \tag{7}$$

Here Q_i are arbitrary smooth functions, $h_{x_\mu}g_{x_\mu} = h_{x_0}g_{x_0} - \sum\limits_{a=1}^{3} h_{x_a}g_{x_a}$.

Omitting intermidiate computations, we adduce main steps of the proof.

Substituting (3) with $N = 2$ into Eq. (1), we get

$$\begin{aligned} &f_{x_\mu x_\mu} + \sum_{i=1}^{2}\{f_{\varphi_i}(\ddot\varphi_i\omega_{ix_\mu}\omega_{ix_\mu} + \dot\varphi_i\Box\omega_i) + f_{\varphi_i\varphi_i}\dot\varphi_i^2\omega_{ix_\mu}\omega_{ix_\mu} + 2f_{\varphi_i x_\mu}\omega_{ix_\mu}\varphi_i\} +\\ &\qquad + 2f_{\varphi_1\varphi_2}\dot\varphi_1\dot\varphi_2\omega_{1x_\mu}\omega_{2x_\mu} = F(f(x,\varphi_1,\varphi_2)). \end{aligned}$$

Replacing $\ddot\varphi_i$ by $R_i(\omega_i,\varphi_i,\dot\varphi_i)$ and splitting the obtained equality with respect to $\dot\varphi_1$, $\dot\varphi_2$, we have

$$\begin{aligned} &R_i = A_i(\omega_i,\varphi_i)\dot\varphi_i^2 + B_i(\omega_i,\varphi_i)\dot\varphi_i + C_i(\omega_i,\varphi_i), \quad i=\overline{1,2},\\ &\omega_{1x_\mu}\omega_{2x_\mu}f_{\varphi_1\varphi_2} = 0. \end{aligned}$$

Since the equality $f_{\varphi_1\varphi_2} = 0$ leads to the case $F_{uu} = 0$, we can put $f_{\varphi_1\varphi_2} \neq 0$ whence $\omega_{1x_\mu}\omega_{2x_\mu} = 0$.

By force of the above facts we get

$$\begin{aligned} &1) \quad f_{\varphi_i\varphi_i} + A_{if_{\varphi_i}} = 0, \quad i=\overline{1,2},\\ &2) \quad f_{\varphi_i}(B_i\omega_{ix_\mu}\omega_{ix_\mu} + \Box\omega_i) + 2f_{\varphi_i x_\mu}\omega_{ix_\mu} = 0,\\ &3) \quad f_{x_\mu x_\mu} + \sum_{i=1}^{2} C_i f_{\varphi_i}\omega_{ix_\mu}\omega_{ix_\mu} = F(f),\\ &4) \quad \omega_{1x_\mu}\omega_{2x_\mu} = 0. \end{aligned} \tag{8}$$

From the first two equations of the system (8) it follows that

$$f = H_1(\omega_1,\varphi_1)H_2(\omega_2,\varphi_2)a(x) + b(x),$$

where H_i, $a(x)$, $b(x)$ are arbitrary smooth functions.

By redefining functions $\varphi_i : \varphi_i \to \tilde\varphi_i H_i(\omega_i,\varphi_i)$, $i = 1, 2$, we may choose

$$f = a(x)\varphi_1(\omega_1)\varphi_2(\omega_2) + b(x), \tag{9}$$

whence $A_1 = A_2 = 0$.

From the Eq. 2 of the system (8) by force of (9) it follows that $B_i = B_i(\omega_i)$, $i = 1, 2$. Consequently, by redefining functions ω_i

$$\omega_i \to \tilde{\omega}_i = W_i(\omega_i), \quad i = \overline{1,2},$$

we may choose $B_1 = B_2 = 0$. As a result, the system (8) is read

$$\begin{aligned} &1) \quad \omega_{1x_\mu}\omega_{2x_\mu} = 0, \\ &2) \quad a\Box\omega_i + 2a_{x_\mu}\omega_{ix_\mu} = 0, \quad i = \overline{1,2}, \\ &3) \quad (\Box a)\varphi_1\varphi_2 + \Box b + a[C_1(\omega_1, \varphi_1)\varphi_2\omega_{1x_\mu}\omega_{1x_\mu} + \\ &\qquad + C_2(\omega_2, \varphi_2)\varphi_1\omega_{2x_\mu}\omega_{2x_\mu}] = F(a\varphi_1\varphi_2 + b). \end{aligned} \tag{10}$$

The only thing left is to split Eq. 3 from (10) with respect to variables φ_1, φ_2. Dividing Eq. 3 into $\varphi_1\varphi_2$ and differentiating it with respect to variables $\varphi_1\varphi_2$ we get $\{(\varphi_1\varphi_2)^{-1}[F(a\varphi_1\varphi_2 + b) - \Box b]\}_{\varphi_1\varphi_2} = 0$, whence

$$a^2x^2\frac{d^2F}{d\omega^2} - ax\frac{dF}{d\omega} + F = \Box b, \quad x = \varphi_1\varphi_2, \quad \omega = ax + b. \tag{11}$$

Differentiation of (11) with respect to x yields

$$ax\frac{d^3F}{d\omega^3} + \frac{d^2F}{d\omega^2} = 0.$$

Since we are interested in a nonlinear case, the inequality $\ddot{F} \neq 0$ holds. Hence, it follows that

$$\dddot{F}(\ddot{F})^{-1} = -(ax)^{-1}$$

or

$$\ddot{F}(\dddot{F})^{-1} = -\omega + b.$$

Differentiating the above equality with respect to ω we obtain nonlinear ODE for $F(\omega)$: $\ddot{F}\,\ddddot{F} - 2(\dddot{F})^2 = 0$, which general solution reads $F(\omega) = \alpha_1^{-2}(\alpha_1\omega + \alpha_2)\ln(\alpha_1\omega + \alpha_2) + \alpha_3\omega + \alpha_4$ and what is more $b = -\alpha_2\alpha_1^{-1}$ (without loss of generality we may put $b = \alpha_2 = 0$).

In the above formulae α_1, α_2, α_3, α_4 are arbitrary real constants, $\alpha_1 \neq 0$.

Substitution of the expression for F

$$F = \lambda_1\omega\ln\omega + \lambda_1\omega + \lambda_3 \tag{12}$$

into Eq. 3 from the system (10) yields

$$\begin{aligned} &\omega_{ix_\mu}\omega_{ix_\mu} = Q_i(\omega_i), \quad i = \overline{1,2}, \\ &C_i = \lambda_1 Q_i^{-1}(\omega_i)\varphi_i\ln\varphi_i, \quad i = \overline{1,2}, \\ &\Box a = \lambda_1 a\ln a + \lambda_2 a, \quad \lambda_3 = 0. \end{aligned}$$

Since in Eq. (12) $\lambda_1 \neq 0$, we can rescale the function $\omega \to k\omega$ in such a way that $F(\omega)$ takes the form $F = \lambda_1\omega\ln\omega$. The theorem is proved.

Note. A classical example of the anti-reduction of mathematical physics equations is the procedure of separation of variables. But the method of separation of variables can be effectively applied to linear second-order PDEs only, whereas the anti-reduction procedure is evidently applicable to nonlinear differential equations.

Thus each solution of the system (7) after being substituted into ansatz (6) reduces the nonlinear PDE (5) to two second-order QDEs

$$Q_i(\omega_i)\dot{\varphi}_i = \lambda\varphi_i \ln\varphi_i, \quad i = \overline{1,2}.$$

Let us write down some particular solutions of Eqs. (7) under $a = 1$.

1. $\omega_1 = \ln(x_0^2 - x_3^2), \quad \omega_2 = \ln(x_1^2 + x_2^2);$
2. $\omega_1 = \ln(x_0^2 - x_3^2), \quad \omega_2 = x_1;$
3. $\omega_1 = x_0, \quad \omega_2 = \ln(x_1^2 + x_2^2);$
4. $\omega_1 = \ln(x_1^2 + x_2^2), \quad \omega_2 = x_3;$
5. $\omega_1 = x_0, \quad \omega_2 = x_1;$
6. $\omega_1 = (x_0^2 - x_1^2 - x_2^2)^{-1/2}, \quad \omega = x_3;$
7. $\omega_1 = x_0, \quad \omega_2 = (x_1^2 + x_2^2 + x_3^2)^{-1/2};$
8. $\omega_1 = x_1\cos\omega_1 + x_2\sin\omega_1 + \omega_2, \quad \omega_2 = x_1\sin\omega_1 - x_2\cos\omega_1 + \omega_3.$

In the above formulae ω_1, ω_2, ω_3 are arbitrary smooth functions on $x_0 + x_3$.

Let us emphasize that the above ansatzes can not be obtained within the framework of the classical Lie approach (see, e.g. [5, 6]), because the maximal symmetry group admitted by Eq. (5) is the Poincaré group $P(1,3)$ [2] and the general form of Poincaré-invariant ansatz is given by the formula (2).

1. Фущич В.И., Штелень В.М., Серов Н.И., Симметрийный анализ и точные решения нелинейных уравнений математической физики, Киев, Наук, думка, 1989, 336 с.
2. Patera J., Sharp R.T., Winternitz P., Zassenhaus H., Subgroups of the Poincaré group and their invariants, *J. Math. Phys.*, 1976, **17**, 977–985.
3. Fushchych W.I., Zhdanov R.Z., Yegorchenko I.A., On the reduction of the nonlinear multi-dimensional wave equations and compatibility of the d'Alembert–Hamilton system, *J. Math. Anal. Appl.*, 1991, **161**, 352–360.
4. Фущич В.И., Условная симметрия уравнений нелинейной математической физики, *Укр. мат. журн.*, 1991, **43**, № 11, 1456–1470.
5. Овсянников Л.В., Групповой анализ дифференциальных уравнений, М., Наука, 1978, 400 с.
6. Olver P., Applications of Lie groups to differential equations, New York, Springer, 1986, 497 p.

On the new approach to variable separation in the wave equation with potential

W.I. FUSHCHYCH, R.Z. ZHDANOV, I.V. REVENKO

Пропонується конструктивний підхід до розв'язання проблем розділення змінних для двомірного хвильового рівняння $u_{tt} - u_{xx} = V(x)u$. У рамках цього підходу описані усі потенціали, що допускають, розділення змінних і вказані відповідні системи координат.

A problem of variable separation (VS) in the wave equation

$$u_{x_0x_0} - u_{x_1x_1} + V(x_1)u = 0 \tag{1}$$

as considered in [1–3], consists of two problems. The first one is to describe all functions $V(x_1)$ providing VS in (1) in, at least, two inequivalent coordinate systems. The second one is to describe all coordinate-systems such that equation (1) admits VS for a given potential $V(x_1)$. Surprisingly enough, the both problems are not completely solved yet.

Our approach to the problem of VS in the wave equation (1) is based on the idea of its reduction to two ordinary differential equations

$$\ddot{\varphi}_i = A_i(\omega_i, \lambda)\varphi_i + B_i(\omega_i, \lambda)\varphi_i, \quad i = 1, 2 \tag{2}$$

with the use of ansatz of special structure [4–6]

$$u = A(x_0, x_1)\varphi_1(\omega_1(x_0, x_1))\varphi_2(\omega_2(x_0, x_1)). \tag{3}$$

In the formulas (2), (3) A_1, A_2, B_1, B_2, A, ω_1, ω_2 are sufficiently smooth real functions, $\lambda \in \mathbb{R}^1$ is some parameter, no summation over i is carried out.

The formulas of the form (3) can be found in the classical works Euler, d'Alembert, Batemen and by some other contemporary mathematicians (see, for example, the review by Koornwinder [7]).

Definition. *We say, that equation (1) admits VS in the coordinates ω_1, ω_2 if substitution of the ansatz (3) into (1) with subsequent exclusion of the second derivatives, $\ddot{\varphi}_1$, $\ddot{\varphi}_2$ according to formulas (2) turns it into zero identically with respect to the variables $\dot{\varphi}_1$, $\dot{\varphi}_2$, φ_1, φ_2, λ.*

Substituting ansatz (3) into differential equation (1), expressing functions $\ddot{\varphi}_i$ in terms of $\dot{\varphi}_i$, φ_i, $i = 1, 2$ and splitting the obtained expression with respect to the independent variables $\dot{\varphi}_1\dot{\varphi}_2$, $\dot{\varphi}_1\varphi_2$, $\varphi_1\dot{\varphi}_2$, $\varphi_1\varphi_2$ we get the following system of nonlinear partial differential equations:

$$\begin{aligned}
&1)\quad A\Box\omega_1 + 2A_{x_\mu}\omega_{1x^\mu} + AA_1\omega_{1x_\mu}\omega_{1x^\mu} = 0,\\
&2)\quad A\Box\omega_2 + 2A_{x_\mu}\omega_{2x^\mu} + AA_2\omega_{2x_\mu}\omega_{2x^\mu} = 0,\\
&3)\quad \Box A + A(B_1\omega_{1x_\mu}\omega_{1x^\mu} + B_2\omega_{2x_\mu}\omega_{2x^\mu}) + AV(x_1) = 0,\\
&4)\quad \omega_{1x_\mu}\omega_{2x^\mu} = 0.
\end{aligned} \tag{4}$$

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Hereafter, the summation over the repeated Greek indices is under-stood in the Minkovski space $M(1,1)$ with a metric tensor $g_{\mu\nu} = \mathrm{diag}\,(1,-1)$.

Thus to describe all potentials $V(x_1)$ and coordinate systems ω_1, ω_2 providing VS in (1) one has to solve nonlinear system (4). At first glance such an approach seems to have poor prospects: to solve linear equation (1) it is necessary to integrate rather complicated system of nonlinear partial differential equations (4). But system (4) is overdetermined one. This fact has enabled us to construct its general solution in explicit form. Let us emphasize that the same is true when reducing nonlinear wave equation to the ordinary differential equation [5, 6] .

It is not difficult to show that from the forth equation of system (4) it follows that

$$(\omega_{1x_\mu}\omega_{1x^\mu})(\omega_{2x_\mu}\omega_{2x^\mu}) \neq 0. \tag{5}$$

Differentiating equations 1), 2) from (4) and using (5) we have

$$A_{1\lambda} = A_{2\lambda} = 0.$$

Consequently, the relation $B_{1\lambda}B_{2\lambda} \neq 0$ holds. Differentiating equation (3) with respect to λ, we get

$$B_{1\lambda}\omega_{1x_\mu}\omega_{1x^\mu} + B_{2\lambda}\omega_{2x_\mu}\omega_{2x^\mu} = 0$$

or

$$\frac{B_{1\lambda}}{B_{2\lambda}} = -\frac{\omega_{2x_\mu}\omega_{2x^\mu}}{\omega_{1x_\mu}\omega_{1x^\mu}}.$$

Differentiating the above equality with respect to λ, we obtain

$$\frac{B_{1\lambda\lambda}}{B_{1\lambda}} = \frac{B_{2\lambda\lambda}}{B_{2\lambda}}. \tag{6}$$

Since function B_i depends on the variable ω_i and the functions ω_1, ω_2 are independent, it follows from (6) that

$$B_{i\lambda\lambda} = \varkappa(\lambda)B_{i\lambda}, \quad i = 1, 2.$$

Integration of the above ordinary differential equations yields

$$B_i = \Lambda(\lambda)f_i(\omega_i) + g_i(\omega_i), \quad i = 1, 2.$$

After redefining the parameter λ, we have

$$B_i = \lambda f_i(\omega_i) + g_i(\omega_i), \quad i = 1, 2. \tag{7}$$

Substituting (7) into equation (3) and splitting the obtained equality with respect to λ, we come to the following partial differential equations:

$$\begin{aligned} &3a) \quad \Box A + A(g_1\omega_{1x_\mu}\omega_{1x^\mu} + g_2\omega_{2x_\mu}\omega_{2x^\mu}) + V(x_1)A = 0, \\ &3b) \quad f_1\omega_{1x_\mu}\omega_{1x^\mu} + f_2\omega_{2x_\mu}\omega_{2x^\mu} = 0. \end{aligned} \tag{8}$$

Before integrating overdetermined system of nonlinear equations (4), (8), make an important remark. It is evident, that the ansatz structure does not change with the transformation of the form

$$A \to Ah_1(\omega_1)h_2(\omega_2), \quad \omega_i \to \Phi_1(\omega_i), \quad i = 1, 2. \tag{9}$$

That is why, solutions of system (4), (8) connected by relations (9) are considered as equivalent.

Making the change (9) in equations 1), 2), 3b) by the appropriate, choice of functions h_i, Φ_i one can obtain $f_1 = f_2 = 1$, $A_1 = A_2 = 0$. Consequently, functions ω_1, ω_2 satisfy the equations

$$\omega_{1x_\mu}\omega_{2x^\mu} = 0, \quad \omega_{1x_\mu}\omega_{1x^\mu} + \omega_{2x_\mu}\omega_{2x^\mu} = 0.$$

whence

$$(\omega_1 \pm \omega_2)x_\mu(\omega_1 \pm \omega_2)x_\mu = 0.$$

Integrating the above equations we get

$$\omega_1 = f(\xi) + g(\eta), \quad \omega_2 = f(\xi) - g(\eta), \tag{10}$$

where f, g are arbitrary smooth functions, $\xi = \frac{1}{2}(x_1 + x_0)$, $\eta = \frac{1}{2}(x_1 - x_0)$.

Substitution of the formulas (10) into equations 1), 2) from (4) yields the following equations for a function $A(x_0, x_1)$

$$(\ln A)_{x_0} = 0, \quad (\ln A)_{x_1} = 0,$$

whence $A = 1$.

At last, substituting the obtained results into the equation 3b) from (8) we come to a conclusion that the problem of integration of system (4), (8) is reduced to solution of the functional-differential equation

$$V(x_1) = [g_1(f+g) - g_2(f-g)]\frac{df}{d\xi}\frac{dg}{d\eta}. \tag{11}$$

And what is more, solution with separated variables (3) reads

$$u = \varphi_1(f(\xi) + g(\eta))\varphi_2(f(\xi) - g(\eta)). \tag{12}$$

To integrate (11) it is convenient to make the hodograph transformation

$$\xi = P(f), \quad \eta = R(g) \tag{13}$$

with $\dot{P} \not\equiv 0$, $\dot{R} \not\equiv 0$, equation (11) taking the form

$$g_1(f+g) - g_2(f-g) = \dot{P}(f)\dot{R}(g)\dot{R}(g)V(P+R). \tag{14}$$

Evidently, equality (14) is equivalent to the following relation:

$$(\partial_f^2 - \partial_g^2)[\dot{P}(f)\dot{R}(g)V(P+R)] = 0$$

or

$$(\dddot{P}\dot{R} - \dddot{R}\dot{P})V + 3\dot{P}\dot{R}(\ddot{P} - \ddot{R})\dot{V} + \dot{P}\dot{R}(\dot{P}^2 - \ddot{R}^2)\ddot{V} = 0. \tag{15}$$

Without going into details of integration of equation (15) we give the final results.

Theorem 1. *The general solution of (15) is given by one of the following formulas:*

1. $V = V(x_1)$ *is an arbitrary function*, $\dot{R} = \dot{P} = \alpha$;
2. $V = m(x_1 + C)\dot{p}^2 = \alpha P + \beta$, $\dot{R}^2 = \alpha R + \gamma$;

$$\begin{aligned} 3.\quad & V = m(x_1 + C)^{-2}, \quad P = F(f), \quad R = G(g), \\ & \dot{F}^2 = \alpha F^4 + \beta F^3 + \gamma F^2 + \delta F + \rho, \\ & \dot{G}^2 = \alpha G^4 - \beta G^3 + \gamma G^2 - \delta G + \rho; \end{aligned} \tag{16}$$

$$4.\quad V = m \sin^{-2}(x_1 + C), \quad P = \operatorname{arctg} F(f), \quad R = \operatorname{arctg} G(g),$$

where F, G are determined by (16);

$$5.\quad V = m \operatorname{sh}^{-2}(x_1 + C), \quad P = \operatorname{arth} F(f), \quad R = \operatorname{arth} G(g),$$

where F, G are determined by (16);

$$6.\quad V = m \operatorname{ch}^{-2}(x_1 + C), \quad P = \operatorname{arctg} F(f), \quad R = \operatorname{arth} G(g),$$

where F, G are determined by (16);

$$\begin{aligned} 7.\quad & V = m \exp(-\alpha x_1), \quad \dot{P}^2 = \alpha e^{2P} + \beta e^P + \gamma, \quad \dot{R}^2 = \alpha e^{2R} + \delta e^R + \rho; \\ 8.\quad & V = \cos^{-2}(x_1 + C)[m_1 + m_2 \sin(x_1 + C)], \\ & \dot{P}^2 = \alpha^2 \sin 2P + \beta^2, \quad \dot{R}^2 = \alpha^2 \sin 2R + \beta^2; \\ 9.\quad & V = \operatorname{ch}^{-2}(x_1 + C)[m_1 + m_2 \operatorname{sh}(x_1 + C)], \\ & \dot{P}^2 = \alpha \operatorname{sh} 2P + \beta \operatorname{ch} 2P - \gamma^2, \quad \dot{R}^2 = \alpha \operatorname{sh} 2R - \beta \operatorname{ch} 2R - \gamma^2; \\ 10.\quad & V = \operatorname{sh}^{-2}(x_1 + C)[m_1 + m_2 \operatorname{ch}(x_1 + C)], \\ & \dot{P}^2 = \alpha \operatorname{sh} 2P + \beta \operatorname{ch} 2P - \gamma^2, \quad \dot{R}^2 = -\alpha \operatorname{sh} 2R + \beta \operatorname{ch} 2R - \gamma^2; \\ 11.\quad & V = m_1 \exp C_1 x_1 + m_2 \exp 2C x_1, \quad \ddot{P} = \alpha \dot{P}^2 + \beta, \quad \ddot{R} = \alpha \dot{R}^2 + \beta; \\ 12.\quad & V = m_1 + m_2(x_1 + C)^{-2}, \quad \dot{P}^2 = \alpha P^2 + \beta P + \gamma, \quad \dot{R}^2 = \alpha R^2 - \beta R + \gamma; \\ 13.\quad & V = m, \quad \dot{P}^2 = \alpha P^2 + \beta_1 P + \gamma_1, \quad \dot{R}^2 = \alpha R^2 + \beta_2 R + \gamma_2. \end{aligned}$$

Here α, β_i, γ_i, σ, ρ, m, m_1, m_2, C are arbitrary real constants.

Thus, Theorem 1 gives the complete solution of the problem of VS in wave equation (1).

Note 1. Equation (1) with potentials $V = m \sin^{-2} x_1$, $V = m \operatorname{ch}^2 x_1$, $V = m \operatorname{sh}^{-2} x_1$ is reduced to equation (1) with the potential $V = m x_1^{-2}$ by the changes of variables

$$\begin{aligned} & \frac{1}{2}(y_1 \pm y_0) = \operatorname{arctg} \frac{1}{2}(x_1 \pm x_0), \\ & \frac{1}{2}(y_1 \pm y_0) = \operatorname{arcth} \frac{1}{2}(x_1 \pm x_0), \\ & \frac{1}{2}(y_1 + y_0) = \operatorname{arctg} \frac{1}{2}(x_1 + x_0), \\ & \frac{1}{2}(y_1 - y_0) = \operatorname{arcth} \frac{1}{2}(x_1 - x_0). \end{aligned}$$

Note 2. Equation (1) with the potential $V = m \exp C x_1$ is reduced to the Klein–Gordon–Fock equation $\Box u + mu = 0$ with the change of variables

$$\frac{1}{2}(y_1 \pm y_0) = C^{-1} \exp \frac{C}{2}(x_1 \pm x_0).$$

It is evident that the equation (1) admits VS in Cartesian coordinates $\omega_1 = x_0$, $\omega_2 = x_1$ under arbitrary function $V(x_1)$. That is why the most interesting potentials are such that there exist new coordinate systems providing VS. From the Theorem 1 and Notes 1, 2 it follows that equations (1) admitting VS in, at least, two inequivalent coordinate systems, are locally equivalent to one of the following wave equations

$$\begin{aligned}
&1.\quad \Box u + mu = 0,\\
&2.\quad \Box u + mx_1 u = 0,\\
&3.\quad \Box u + mx_1^{-2} u = 0,\\
&4.\quad \Box u + (m_1 + m_2 x_1^{-2})u = 0,\\
&5.\quad \Box u + (m_1 + m_2 \sin x_1)\cos^{-2} x_1 u = 0,\\
&6.\quad \Box u + (m_1 + m_2 \operatorname{sh} x_1)\operatorname{ch}^{-2} x_1 u = 0,\\
&7.\quad \Box u + (m_1 + m_2 \operatorname{ch} x_1)\operatorname{sh}^{-2} x_1 u = 0,\\
&8.\quad \Box u + (m_1 + m_2 e^{x_1})e^{x_1} u = 0.
\end{aligned} \tag{17}$$

A detailed analysis of the coordinate systems providing VS in equation (17) will be carried out in our future work.

In conclusion, we note that the equation (1) is intimately connected with the wave equation

$$v_{tt} - C^2(x)v_{xx} = 0. \tag{18}$$

This connection is given by the formula

$$v(t,x) = \sqrt{C(x)}\,u\left(t, \int \frac{dx}{C(x)}\right). \tag{19}$$

Applying the Theorem 1 and the formula (19) it is not difficult to carry out VS in partial differential equation (18).

Besides, Lorentz-invariant wave equation

$$u_{y_0 y_0} - u_{y_1 y_1} + U(y_0^2 - y_1^2)u = 0 \tag{20}$$

can also be reduced to the form (1), where $U(\tau) = \frac{1}{4\tau}V(\ln\tau)$ by the change of variables

$$y_0 = e^{x_1/2}\operatorname{ch} x_0, \quad y_1 = e^{x_1/2}\operatorname{sh} x_0.$$

That is why, one can at once, point out all potentials $U = U(\tau)$, $\tau = x_0^2 - x_1^2$ providing VS in the wave equation (20):

$$\begin{aligned}
&U = m\tau^{-1}\ln\tau, \quad U = m\tau^{-1}(\ln\tau)^{-2}, \quad U = m_1\tau^{-1} + m_2\tau^{-1}(\ln\tau)^{-2},\\
&U = m\tau^{-1}, \quad U = \tau^{-1}(m_1 + m_2\sin\ln\tau)(\cos\ln\tau)^{-2},\\
&U = \tau^{-1}(m_1 + m_2\operatorname{sh}\ln\tau)(\operatorname{ch}\ln\tau)^{-2},\\
&U = \tau^{-1}(m_1 + m_2\operatorname{ch}\ln\tau)(\operatorname{sh}\ln\tau)^{-2}, \quad U = m_1 + m_2\tau.
\end{aligned}$$

1. Miller W., Symmetry and separation of variables, London, Addison Wesley, 1977, 280 p.

2. Шаповалов В.Н., Симметрия и разделение переменных в линейных дифференциальных уравнениях второго порядка, *Дифференциальные уравнения*, 1980, **16**, № 10, 1864–1874.

3. Багров В.Г., Гитман Д.М., Тернов Д.М., О точных, решениях релятивистских волновых уравнениях, Новосибирск, Наука, 1982, 143 с.

4. Фущич В.И., Штелень В.М., Серов Н.И., Симметрийный анализ и точные решения нелинейных уравнений математической физики, Киев, Наук. думка, 1989, 336 с.

5. Fushchych W.I., Zhdanov R.Z., On some new exact solutions of the nonlinear d'Alembert–Hamilton system, *Phys. Lett. A*, 1989, **141**, № 3–4, 113–115.

6. Fushchych W.I., Zhdanov R.Z., Yegorchenko I.A., On the reduction of the nonlinear multi-dimensional wave equations and compatibility of the d'Alembert–Hamilton system, *J. Math. Anal. Appl.*, 1991, **161**, № 2, 352–360.

7. Koornwinder T.H., A priecise definition of separation of variables, *Lecture Notes in Math.*, 1980, **810**, 240–263.

Orthogonal and non-orthogonal separation of variables in the wave equation $u_{tt} - u_{xx} + V(x)u = 0$

R.Z. ZHDANOV, I.V. REVENKO, W.I. FUSHCHYCH

We develop a direct approach to the separation of variables in partial differential equations. Within the framework of this approach, the problem of the separation of variables in the wave equation with time-independent potential reduces to solving an overdetermined system of nonlinear differential equations. We have succeeded in constructing its general solution and, as a result, all potentials $V(x)$ permitting variable separation have been found. For each of them we have constructed all inequivalent coordinate systems providing separability of the equation under study. It should be noted that the above approach yields both orthogonal and non-orthogonal systems of coordinates.

1. Introduction

Separation of variables (SV) in two- and three-dimensional Laplace, Helmholtz, d'Alembert and Klein–Gordon–Fock equations has been carried out in classical works by Bocher [1], Darboux [2], Eisenhart [3], Stepvanov [4], Olevsky [5], and Kalnins and Miller (see [6] and references therein). Nevertheless, a complete solution to the problem of sv in a two-dimensional wave equation with time-independent potential

$$(\Box + V(x))u \equiv u_{tt} - u_{xx} + V(x)u = 0 \tag{1}$$

has not been obtained yet. In (1) $u = u(t,x) \in C^2(\mathbb{R}^2, \mathbb{R}^1)$, $V(x) \in C(\mathbb{R}^1, \mathbb{R}^1)$.

Equations belonging to the class (1) are widely used in modern mathematical physics and can be related to other important linear and nonlinear partial differential equations (PDE). First, we mention the Lorentz-invariant wave equation

$$u_{y_0 y_0} - u_{y_1 y_1} + U(y_0^2 - y_1^2)u = 0. \tag{2}$$

The above equation can be reduced to the form (1) with the change of variables [7]

$$t = \exp(y_1/2)\cosh y_0, \quad x = \exp(y_1/2)\sinh y_0$$

and what is more, potentials $V(\tau)$, $U(\tau)$ are connected by the following relation:

$$U(\tau) = (4\tau)^{-1}V(\tau).$$

Another related equation is the hyperbolic type equation

$$v_{x_0 x_0} - c^2(x_1)v_{x_1 x_1} = 0 \tag{3}$$

that is widely used in various areas of mathematical physics.

Equation (3) is reduced to the form (1) by the change of variables

$$u(t,x) = [c(x_0)]^{-1/2} v(x_0, x_1), \quad t = x_0, \quad x = \int [c(x_1)]^{-1} dx_1$$

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and what is more

$$V(x) = -c^{3/2}(x_1)[c^{1/2}(x_1)], \tag{4}$$

where $x = \int [c(x_1)]^{-1} dx_1$.

The third related equation is the nonlinear wave equation

$$W_{tt} - [c^{-2}(W)W_x]_x = 0. \tag{5}$$

By substitution $W = R_x$, equation (5) is reduced to the form

$$R_{tt} - c^{-2}(R_x)R_{xx} = 0.$$

Applying to the above equation the Legendre transformation

$$x_0 = R_t, \quad x_1 = R_x, \quad v_{x_0} = t, \quad v_{x_1} = x, \quad v = tR_t + xR_x - R,$$

we obtain (3). Consequently, the method of SV in the linear equation (1) makes it possible to construct exact solutions of the nonlinear wave equation (5).

Let us also mention the Euler–Poisson–Darboux equation

$$v_{tt} - v_{xx} - x^{-1}v_x + m^2x^{-2}v = 0 \tag{6}$$

that is reduced to an equation of the form (1)

$$u_{tt} - u_{xx} + (m^2 - 1/4)x^{-2}u = 0$$

by the change of dependent variable $v(t,x) = x^{-1/2}u(t,x)$.

For the solution of (1) with separated variables $\omega_1(t,x)$, $\omega_2(t,x)$, we use the ansatz

$$u(t,x) = Q(t,x)\varphi_1(\omega_1)\varphi_2(\omega_2) \tag{7}$$

which reduces PDE (1) to two ordinary differential equations (ODE) for functions φ_1, φ_2.

There exist three possibilities for SV in (1). The first is to separate it into two second-order ODE. The second possibility is to separate (1) into first-order and second-order ODE, and the third possibility is to separate (1) into two first-order ODE. In the present paper we shall investigate in detail the first two possibilities. The third possibility requires special separate consideration and will be the topic of future publications.

Consider the following ODE:

$$\ddot{\varphi}_i = A_i(\omega_i,\lambda)\dot{\varphi}_i + B_i(\omega_i,\lambda)\varphi_i, \quad i = 1,2, \tag{8}$$

where $A_i, B_i \subset C^2(\mathbb{R}^1 \times \Lambda, \mathbb{R}^1)$ are some unknown functions, $\lambda \in \Lambda \subset \mathbb{R}^1$ is a real parameter (separation constant).

Definition 1 [7, 8]. *Equation (1) separates into two ODE if substitution of the ansatz (7) into (1) with subsequent exclusion of the second derivatives $\ddot{\varphi}_1$, $\ddot{\varphi}_2$ according to (8) yields an identity with respect to the variables $\dot{\varphi}_i$, φ_i, λ (considered as independent).*

On the basis of the above definition one can formulate a constructive procedure of SV in (1), suggested for the first time in [7]. At the first step, one has to substitute expression (7) into (1) and to express the second derivatives $\ddot{\varphi}_1$, $\ddot{\varphi}_2$ via functions $\dot{\varphi}_i$, φ_i

according to (8). At the second step, the equality obtained is split with respect to the independent variables $\dot{\varphi}_i$, φ_i, λ. As a result, one obtains an over-determinated system of partial differential equations for functions Q, ω_1 and ω_2 with undefined coefficients. The general solution of this system gives rise to all systems of coordinates providing separability of (1).

Definition 2. *Equation (1) separates into first- and second-order ODE*

$$\begin{aligned} &\dot{\varphi}_1 = A(\omega_1, \lambda)\varphi_1, \\ &\ddot{\varphi}_2 = B_1(\omega_2, \lambda)\dot{\varphi}_2 + B_2(\omega_2, \lambda)\varphi_2 \end{aligned} \tag{9}$$

if substitution of the ansatz (7) into (1) with subsequent exclusion of derivatives $\dot{\varphi}_1$, $\ddot{\varphi}_2$ *according to (9) yields an identity with respect to the variables* φ_1, $\dot{\varphi}_2$, φ_2, λ *(considered as independent).*

Let us emphasize that the above approach to SV in (1) has much in common with the non-Lie method of reduction of nonlinear PDE suggested in [9–11]. It is also important to note that the idea to represent solutions of linear differential equations in the "separated" form (7) goes as far as the classical works by Fourier and Euler (for a modern exposition of the problem of SV, see Miller [12] and Koornwinder [13]).

2. Orthogonal separation of variables in equation (1)

It is evident that (1) admits SV in Cartesian coordinates $\omega_1 = t$, $\omega_2 = x$ under arbitrary $V = V(x)$.

Definition 3. *Equation (1) admits non-trivial SV if there exist at least one coordinate system* $\omega_1(t, x)$, $\omega_2(t, x)$ *different from the Cartesian system providing its separability.*

Next, if one makes in (1) the following transformations:

$$\begin{aligned} &t \to C_1 t, \quad x \to C_1 x, \\ &t \to t, \quad x \to x + C_2, \quad C_i \in \mathbb{R}^1 \end{aligned} \tag{10}$$

then the class of equations (1) transforms into itself and what is more

$$\begin{aligned} &V(x) \to V'(x) = C_1^2 V(C_1 x), \\ &V(x) \to V'(x) = V(x + C_2). \end{aligned} \tag{10a}$$

That is why potentials $V(x)$ and $V'(x)$, connected by one of the above relations, are considered as equivalent ones.

When separating variables in (1) one has to solve an intermediate problem of description of all inequivalent potentials such that the equation admits non-trivial SV (classification problem). The next step is to obtain a complete description of the coordinate systems providing SV in (1) with these potentials.

First, we adduce the principal results on separation of (1) into two second-order ODE and then give an outline of the proof of the corresponding theorems.

Theorem 1. *Equation (1) admits non-trivial SV in the sense of Definition 1 iff the function* $V(x)$ *is given, up to equivalence relations (10a), by one of the following formulae:*

(1) $V = mx$;

(2) $V = mx^{-2}$;

$$\begin{array}{ll}
(3) & V = m\sin^{-2}x; \\
(4) & V = m\sinh^{-2}x; \\
(5) & V = m\cosh^{-2}x; \\
(6) & V = m\exp x; \\
(7) & V = \cos^{-2}x(m_1 + m_2\sin x); \\
(8) & V = \cosh^{-2}x(m_1 + m_2\sinh x); \\
(9) & V = \sinh^{-2}x(m_1 + m_2\cosh x); \\
(10) & V = m_1\exp x + m_2\exp 2x; \\
(11) & V = m_1 + m_2 x^{-2}; \\
(12) & V = m.
\end{array} \tag{11}$$

Here m, m_1, m_2 are arbitrary real parameters, $m_2 \neq 0$.

Note 1. Equation (1) having the potential (6) from (11) is transformed with the change of variables [7]

$$x' = \exp(x/2)\cosh t, \quad t' = \exp(x/2)\sinh t$$

into (1) with $V(x) = m$.

Note 2. Equations (1) having the potentials (3), (4), (5) from (11) are transformed into (1) with $V(x) = mx^{-2}$ by means of changes of variables [7]

$$\begin{array}{ll}
x' = \tan\xi + \tan\eta, & t' = \tan\xi - \tan\eta, \\
x' = \tanh\xi + \tanh\eta, & t' = \tanh\xi - \tanh\eta, \\
x' = \coth\xi + \tanh\eta, & t' = \coth\xi - \tanh\eta.
\end{array}$$

Hereafter $\xi = \frac{1}{2}(x + t)$, $\eta = \frac{1}{2}(x - t)$ are cone variables.

By virtue of the above remarks, the validity of the assertion follows from Theorem 1.

Theorem 2. *Provided equation (1) admits non-trivial SV in the sense of Definition 1, it is locally equivalent to one of the following equations:*

$$\begin{array}{ll}
(1) & \Box u + mxu = 0; \\
(2) & \Box u + mx^{-2}u = 0; \\
(3) & \Box u + \cos^{-2}x(m_1 + m_2\sin x)u = 0; \\
(4) & \Box u + \cosh^{-2}x(m_1 + m_2\sinh x) = 0; \\
(5) & \Box u + \sinh^{-2}x(m_1 + m_2\cosh x) = 0; \\
(6) & \Box u + \exp x(m_1 + m_2\exp x)u = 0; \\
(7) & \Box u + (m_1 + m_2 x^{-2})u = 0; \\
(8) & \Box u + mu = 0.
\end{array} \tag{12}$$

Thus, there exist eight inequivalent types of equations of the form (1) admitting non-trivial SV.

It is well known that there are 11 coordinate systems providing separability of the Klein–Gordon–Fock equation $\Box u + mu = 0$ into two second-order ODE [6]. Besides that, in [14] it was established that the Euler–Poisson–Darboux equation (6), which is equivalent to the second equation of (12), separates in nine coordinate systems. That is why cases $V(x) = m$ and $V(x) = mx^{-2}$ are not considered here.

As is shown below, the general form of solution with separated variables of (12) is as follows:

$$u(t,x) = \varphi_1(\omega_1(t,x))\varphi_2(\omega_2(t,x)), \tag{13}$$

where $\varphi_1(\omega_1)$, $\varphi_2(\omega_2)$ are arbitrary solutions of the separated ODE

$$\ddot{\varphi}_i = (\lambda + g_i(\omega_i))\varphi_i, \quad i = 1, 2 \tag{14}$$

and explicit forms of the functions $\omega_i(t,x)$, $g_i(\omega_i)$ are given below.

Theorem 3. *Equation* $\Box u + mxu = 0$ *separates in two coordinate systems*

$$\begin{aligned}
&(1) \quad \omega_1 = t \qquad \omega_2 = x, \quad g_1 = 0, \quad g_2 = m\omega_2; \\
&(2) \quad \omega_1 = (x+t)^{1/2} + (x-t)^{1/2}, \quad \omega_2 = (x+t)^{1/2} - (x-t)^{1/2}, \\
&\qquad g_1 = -\frac{1}{4}m\omega_1^4, \quad g_2 = -\frac{1}{4}m\omega_2^4.
\end{aligned} \tag{15}$$

Theorem 4. *Equation* $\Box u + \sin^{-2}x(m_1 + m_2\cos x)u = 0$ *separates in four coordinate systems*

$$\begin{aligned}
&(1) \quad \omega_1 = t, \quad \omega_2 = x, \quad g_1 = 0, \quad g_2 = \cos^{-2}\omega_2(m_1 + m_2\sin\omega_2); \\
&(2) \quad \left\{ \begin{matrix} x \\ t \end{matrix} \right\} = \arctan\sinh(\omega_1+\omega_2) \pm \arctan\sinh(\omega_1-\omega_2), \\
&\qquad g_1 = (m_1+m_2)\sinh^{-2}\omega_1, \quad g_2 = -(m_1-m_2)\cosh^{-2}\omega_2; \\
&(3) \quad \left\{ \begin{matrix} x \\ t \end{matrix} \right\} = \arctan\frac{\mathrm{sn}\,(\omega_1+\omega_2)}{\mathrm{cn}\,(\omega_1+\omega_2)} \pm \arctan\frac{\mathrm{sn}\,(\omega_1-\omega_2)}{\mathrm{cn}\,(\omega_1-\omega_2)}, \\
&\qquad g_1 = m_1\,\mathrm{dn}^2\omega_1\,\mathrm{cn}^{-2}\omega_1\,\mathrm{sn}^{-2}\omega_1 + m_2[\mathrm{cn}^{-2}\omega_1 - \mathrm{dn}^2\omega_1\,\mathrm{cn}^{-2}\omega_1], \\
&\qquad g_2 = m_1k^4\,\mathrm{sn}^2\omega_2\,\mathrm{cn}^2\omega_2\,\mathrm{dn}^{-2}\omega_2 + m_2k^2[\mathrm{cn}^2\omega_2\,\mathrm{dn}^{-2}\omega_2 - \mathrm{sn}^2\omega_2]; \\
&(4) \quad \left\{ \begin{matrix} x \\ t \end{matrix} \right\} = \arctan\left(\frac{k}{k'}\right)^{1/2}\mathrm{cn}\,(\omega_1+\omega_2) \pm \arctan\left(\frac{k}{k'}\right)^{1/2}\mathrm{cn}\,(\omega_1-\omega_2), \\
&\qquad g_1 = m_1[\mathrm{dn}^2\omega_1\,\mathrm{cn}^{-2}\omega_1 + k^2\,\mathrm{sn}^2\omega_1] + m_2[(k')^2\,\mathrm{cn}^{-2}\omega_1 + k^2\,\mathrm{cn}^2\omega_1], \\
&\qquad g_2 = m_1[\mathrm{dn}^2\omega_2\,\mathrm{cn}^{-2}\omega_2 + k^2\,\mathrm{sn}^2\omega_2] + m_2[(k')^2\,\mathrm{cn}^{-2}\omega_2 + k^2\,\mathrm{cn}^2\omega_2].
\end{aligned} \tag{16}$$

In the above formulae (16) k, $k' = (1-k^2)^{1/2}$ *are the moduli of corresponding elliptic Jacobi functions, and* k *is an arbitrary constant satisfying the inequality* $0 < k < 1$.

Theorem 5. *Equation* $\Box u + \cosh^{-2}x(m_1 + m_2\sinh x)u = 0$ *separates in four coordinate systems*

$$\begin{aligned}
&(1) \quad \omega_1 = t, \quad \omega_2 = x, \quad g_1 = 0, \quad g_2 = \cosh^{-2}\omega_2(m_1 + m_2\sinh\omega_2); \\
&(2) \quad \left\{ \begin{matrix} t \\ x \end{matrix} \right\} = -\ln\left[\left(\frac{k'}{k}\right)^{1/2}\mathrm{cn}\,(\omega_1+\omega_2)\right] \pm \ln\left[\left(\frac{k'}{k}\right)^{1/2}\mathrm{cn}\,(\omega_1-\omega_2)\right], \\
&\qquad g_1 = m_1(k')^2\,\mathrm{dn}^{-2}2\omega_1 + m_2\,\mathrm{cn}\,2\omega_1\,\mathrm{dn}^{-2}2\omega_1, \\
&\qquad g_2 = m_1(k')^2\,\mathrm{dn}^{-2}2\omega_2 + m_2\,\mathrm{cn}\,2\omega_2\,\mathrm{dn}^{-2}2\omega_2;
\end{aligned}$$

$$
\begin{aligned}
&(3)\quad \left\{\begin{matrix} x \\ t \end{matrix}\right\} = -\ln\sinh\frac{1}{2}(\omega_1+\omega_2) \pm \ln\cosh\frac{1}{2}(\omega_1-\omega_2),\\
&\qquad g_1 = \cosh^{-2}\omega_1(m_1 - m_2\sinh\omega_1), \quad g_2 = \cosh^{-2}\omega_2(m_1 - m_2\sinh\omega_2);\\
&(4)\quad \left\{\begin{matrix} x \\ t \end{matrix}\right\} = \ln\frac{\operatorname{sn}\frac{1}{2}(\omega_1+\omega_2)}{\operatorname{cn}\frac{1}{2}(\omega_1+\omega_2)} \pm \ln\,\operatorname{dn}\frac{1}{2}(\omega_1+\omega_2),\\
&\qquad g_1 = -m_1k^2\operatorname{sn}^2\omega_1 + k^2m_2\operatorname{sn}\omega_1\operatorname{cn}\omega_1,\\
&\qquad g_2 = -m_1k^2\operatorname{sn}^2\omega_2 + k^2m_2\operatorname{sn}\omega_2\operatorname{cn}\omega_2.
\end{aligned}
\tag{17}
$$

Here k, $k' = (1-k^2)^{1/2}$ are the moduli of corresponding elliptic functions, $0 \le k \le 1$.

Theorem 6. *Equation $\Box u + \sinh^{-2}x(m_1 + m_2\cosh x)u = 0$ separates in eleven coordinate systems:*

$$
\begin{aligned}
&(1)\quad \omega_1 = t, \quad \omega_2 = x, \quad g_1 = 0, \quad g_2 = \sinh^{-2}\omega_2(m_1 + m_2\cosh\omega_2);\\
&(2)\quad \left\{\begin{matrix} x \\ t \end{matrix}\right\} = -\ln\frac{1}{2}(\omega_1+\omega_2) \pm \ln\frac{1}{2}(\omega_1-\omega_2),\\
&\qquad g_1 = (m_1 - m_2)\omega_1^{-2}, \quad g_2 = (m_1 + m_2)\omega_2^{-2};\\
&(3)\quad \left\{\begin{matrix} x \\ t \end{matrix}\right\} = -\ln\sin\frac{1}{2}(\omega_1+\omega_2) \pm \ln\sin\frac{1}{2}(\omega_1-\omega_2),\\
&\qquad g_1 = (m_1 - m_2)\sin^{-2}\omega_1, \quad g_2 = (m_1 + m_2)\sin^{-2}\omega_2;\\
&(4)\quad \left\{\begin{matrix} t \\ x \end{matrix}\right\} = -\ln\sinh\frac{1}{2}(\omega_1+\omega_2) \pm \ln\sinh\frac{1}{2}(\omega_1-\omega_2),\\
&\qquad g_1 = \sinh^{-2}\omega_1(m_1 + m_2)\cosh\omega_1), \quad g_2 = \sinh^{-2}\omega_2(m_1 - m_2\cosh\omega_2);\\
&(5)\quad \left\{\begin{matrix} t \\ x \end{matrix}\right\} = -\ln\cosh\frac{1}{2}(\omega_1+\omega_2) \pm \ln\cosh\frac{1}{2}(\omega_1-\omega_2),\\
&\qquad g_1 = \sinh^{-2}\omega_1(m_1 - m_2\cosh\omega_1), \quad g_2 = \sinh^{-2}\omega_2(m_1 - m_2\cosh\omega_2);\\
&(6)\quad \left\{\begin{matrix} x \\ t \end{matrix}\right\} = \ln\tanh\frac{1}{2}(\omega_1+\omega_2) \pm \ln\tanh\frac{1}{2}(\omega_1-\omega_2),\\
&\qquad g_1 = \cosh^{-2}\omega_1(m_1 - m_2), \quad g_2 = -\cosh^{-2}\omega_2(m_1 + m_2);\\
&(7)\quad \left\{\begin{matrix} x \\ t \end{matrix}\right\} = \ln\tan\frac{1}{2}(\omega_1+\omega_2) \pm \ln\tan\frac{1}{2}(\omega_1-\omega_2),\\
&\qquad g_1 = \cos^{-2}\omega_1(m_1 + m_2), \quad g_2 = \cos^{-2}\omega_2(m_1 - m_2);\\
&(8)\quad \left\{\begin{matrix} x \\ t \end{matrix}\right\} = \operatorname{arctanh}\operatorname{cn}(\omega_1+\omega_2) \pm \operatorname{arctanh}\operatorname{cn}(\omega_1-\omega_2),\\
&\qquad g_1 = (m_1 + m_2)\operatorname{dn}^2\omega_1\operatorname{cn}^{-2}\omega_1 + (m_1 - m_2)k^2\operatorname{sn}^2\omega_1,\\
&\qquad g_2 = (m_1 - m_2)\operatorname{dn}^2\omega_2\operatorname{cn}^{-2}\omega_2 + (m_1 + m_2)k^2\operatorname{sn}^2\omega_2;\\
&(9)\quad \left\{\begin{matrix} x \\ t \end{matrix}\right\} = \operatorname{arctanh}\operatorname{dn}(\omega_1+\omega_2) \pm \operatorname{arctanh}\operatorname{dn}(\omega_1-\omega_2),\\
&\qquad g_1 = (m_1 + m_2)k^2\operatorname{cn}^2\omega_1\operatorname{dn}^{-2}\omega_1 + (m - m_2)k^2\operatorname{sn}^2\omega_1,\\
&\qquad g_2 = (m_1 - m_2)k^2\operatorname{cn}^2\omega_2\operatorname{cn}^{-2}\omega_2 + (m_1 + m_2)k^2\operatorname{sn}^2\omega_2;\\
&(10)\quad \left\{\begin{matrix} x \\ t \end{matrix}\right\} = \operatorname{arctanh}\operatorname{sn}(\omega_1+\omega_2) \pm \operatorname{arctanh}\operatorname{sn}(\omega_1-\omega_2),
\end{aligned}
\tag{18}
$$

$$g_1 = (m_1 + m_2)\,\mathrm{sn}^{-2}\omega_1 + (m_1 - m_2)k^2\,\mathrm{sn}^2\omega_1,$$
$$g_2 = (m_1 + m_2)k^2\,\mathrm{cn}^2\omega_2\,\mathrm{dn}^{-2}\omega_2 + (m_1 - m_2)k^2\,\mathrm{dn}^2\omega_2\,\mathrm{cn}^{-2}\omega_2;$$

(11) $$\left\{ \begin{matrix} x \\ t \end{matrix} \right\} = \pm \ln\,\mathrm{cn}\,(\omega_1 + \omega_2) \pm \ln\,\mathrm{cn}\,(\omega_1 - \omega_2),$$
$$g_1 = -m_1\,\mathrm{sn}^{-2}\omega_1 - m_2\,\mathrm{cn}\,\omega_1\,\mathrm{sn}^{-2}\omega_1,$$
$$g_2 = -m_1\,\mathrm{sn}^{-2}\omega_2 - m_2\,\mathrm{cn}\,\omega_2\,\mathrm{sn}^{-2}\omega_2.$$

Here k are the moduli of corresponding elliptic functions, $0 < k < 1$.

Theorem 7. *Equation* $\Box u + \exp x(m_1 + m_2 \exp x)u = 0$ *separates in six coordinate systems:*

$$(1)\quad \omega_1 = t, \quad \omega_2 = x, \quad g_1 = 0, \quad g_2 = \exp\omega_2(m_1 + m_2\exp\omega_2);$$

$$(2)\quad \left\{ \begin{matrix} x \\ t \end{matrix} \right\} = -\ln\cos(\omega_1 + \omega_2) \pm \ln\cos(\omega_1 - \omega_2),$$
$$g_1 = -2m_1\cos 2\omega_1 - \frac{1}{2}m_2\cos 4\omega_1,$$
$$g_2 = -2m_1\cos 2\omega_2 - \frac{1}{2}m_2\cos 4\omega_2;$$

$$(3)\quad \left\{ \begin{matrix} x \\ t \end{matrix} \right\} = \ln\sinh(\omega_1 + \omega_2) \pm \ln\sinh(\omega_1 - \omega_2),$$
$$g_1 = -2m_1\cosh 2\omega_1 - \frac{1}{2}m_2\cosh 4\omega_1,$$
$$g_2 = -2m_1\cosh 2\omega_2 - \frac{1}{2}m_2\cosh 4\omega_2;$$

$$(4)\quad \left\{ \begin{matrix} x \\ t \end{matrix} \right\} = \ln\cosh(\omega_1 + \omega_2) \pm \ln\cosh(\omega_1 - \omega_2), \qquad (19)$$
$$g_1 = -2m_1\cosh 2\omega_1 - \frac{1}{2}m_2\cosh 4\omega_1,$$
$$g_2 = -2m_1\cosh 2\omega_2 - \frac{1}{2}m_2\cosh 4\omega_2;$$

$$(5)\quad \left\{ \begin{matrix} x \\ t \end{matrix} \right\} = \ln\cosh(\omega_1 + \omega_2) \pm \ln\sinh(\omega_1 - \omega_2),$$
$$g_1 = -2m_1\sinh 2\omega_1 - \frac{1}{2}m_2\cosh 4\omega_1,$$
$$g_2 = -2m_1\sinh 2\omega_2 - \frac{1}{2}m_2\cosh 4\omega_2;$$

$$(6)\quad \left\{ \begin{matrix} x \\ t \end{matrix} \right\} = \ln(\omega_1 + \omega_2) \pm \ln(\omega_1 - \omega_2),$$
$$g_1 = 2m_1 + 2m_2\omega_1^2, \quad g_2 = -2m_1 + 2m_2\omega_2^2.$$

Theorem 8. *Equation* $\Box u + (m_1 + m_2 x^{-2})u = 0$ *separates in six coordinate systems:*

$$(1)\quad \omega_1 = t, \quad \omega_2 = x, \quad g_1 = 0, \quad g_2 = m_1 + m_2\omega_2^{-2};$$

$$(2)\quad \left\{ \begin{matrix} x \\ t \end{matrix} \right\} = \exp(\omega_1 + \omega_2) \pm \exp(\omega_1 - \omega_2),$$
$$g_1 = 4m_1\exp 2\omega_1, \quad g_2 = m_2\cosh^{-2}\omega_2;$$

(3) $\left\{ \begin{matrix} x \\ t \end{matrix} \right\} = \sin(\omega_1 + \omega_2) \pm \sin(\omega_1 - \omega_2),$

$g_1 = 2m_1 \cos 2\omega_1 + m_2 \sin^{-2} \omega_1, \qquad g_2 = -2m_1 \cos 2\omega_2 + m_2 \cos^{-2} \omega_2;$

(4) $\left\{ \begin{matrix} x \\ t \end{matrix} \right\} = \sinh(\omega_1 + \omega_2) \pm \sinh(\omega_1 - \omega_2),$

$g_1 = 2m_1 \sinh 2\omega_1 + m_2 \sinh^{-2} \omega_1,$

$g_2 = -2m_1 \sinh 2\omega_2 - m_2 \sinh^{-2} \omega_2;$ (20)

(5) $\left\{ \begin{matrix} x \\ t \end{matrix} \right\} = \cosh(\omega_1 + \omega_2) \pm \cosh(\omega_1 - \omega_2),$

$g_1 = 2m_1 \cosh 2\omega_1 - m_2 \cosh^{-2} \omega_1, \quad g_2 = 2m_1 \cosh 2\omega_2 - m_2 \cosh^{-2} \omega_2;$

(6) $\left\{ \begin{matrix} x \\ t \end{matrix} \right\} = (\omega_1 + \omega_2)^2 \pm (\omega_1 - \omega_2)^2,$

$g_1 = -16m_1\omega_1^2 + m_2\omega_1^{-2}, \quad g_2 = -16m_1\omega_2^2 + m_2\omega_2^{-2}.$

We now give a sketch of the proof of the above assertions. Substituting ansatz (7) into (1), expressing functions $\ddot{\varphi}_i$ via functions $\dot{\varphi}_1$, φ_i by means of equalities (8) and splitting the equation obtained with respect to independent variables $\dot{\varphi}_i$, φ_i we obtain the following system of nonlinear PDE:

(1) $Q\Box\omega_i + 2(Q_t\omega_{it} - Q_x\omega_{ix}) + QA_i(\omega_i, \lambda)(\omega_{it}^2 - \omega_{ix}^2) = 0, \quad i = 1, 2;$ (21)

(2) $\Box Q + Q[B_1(\omega_1, \lambda)(\omega_{1t}^2 - \omega_{1x}^2) + B_2(\omega_2, \lambda)(\omega_{2t}^2 - \omega_{2x}^2)] + QV(x) = 0;$ (22)

(3) $\omega_{1t}\omega_{2t} - \omega_{1x}\omega_{2x} = 0.$ (23)

Here $\Box = \partial_t^2 - \partial_x^2$.

Thus, to separate variables in the linear PDE (1) one has to construct the general solution of the system of nonlinear equations (21)–(23). The same assertion holds true for any general linear differential equation, i.e. the problem of SV is an essentially nonlinear one.

It is not difficult to become convinced of the fact that, from (23), it follows that

$$(\omega_{1t}^2 - \omega_{1x}^2)(\omega_{2t}^2 - \omega_{2x}^2) \neq 0. \tag{24}$$

Differentiating (21) with respect to λ and using (24) we obtain

$$A_{1\lambda} = A_{2\lambda} = 0,$$

whence $B_{1\lambda}B_{2\lambda} \neq 0$. Differentiating (22) with respect to λ, we have

$$B_{1\lambda}(\omega_{1t}^2 - \omega_{1x}^2) + B_{2\lambda}(\omega_{2t}^2 - \omega_{2x}^2) = 0$$

or

$$\frac{B_{1\lambda}}{B_{2\lambda}} = -\frac{\omega_{2t}^2 - \omega_{2x}^2}{\omega_{1t}^2 - \omega_{1x}^2}.$$

Differentiation of the above equality with respect to λ yields

$$B_{1\lambda\lambda}B_{2\lambda} - B_{1\lambda}B_{2\lambda\lambda} = 0$$

or

$$\frac{B_{1\lambda\lambda}}{B_{1\lambda}} = \frac{B_{2\lambda\lambda}}{B_{2\lambda}}.$$

Since functions $B_1 = B_1(\omega_1)$, $B_2 = B_2(\omega_2)$ are independent, there exists a function $\varkappa(\lambda)$ such that

$$B_{i\lambda\lambda} = \varkappa(\lambda) B_{i\lambda}, \quad i = 1, 2.$$

Integrating the above differential equation with respect to λ we obtain

$$B_i(\omega_i) = \Lambda(\lambda) f_i(\omega_i) + g_i(\omega_i), \quad i = 1, 2,$$

where f_i, g_i are arbitrary smooth functions.

On redefining the parameter $\lambda \to \Lambda(\lambda)$, we have

$$B_i(\omega_i) = \lambda f_i(\omega_i) + g_i(\omega_i). \tag{25}$$

Substitution of (25) into (22) with subsequent splitting with respect to λ yields the following equations:

$$\Box Q + Q[g_1(\omega_{1t}^2 - \omega_{1x}^2) + g_2(\omega_{1t}^2 - \omega_{1x}^2)] + V(x)Q = 0, \tag{26}$$

$$f_1(\omega_{1t}^2 - \omega_{1x}^2) + f_2(\omega_{2t}^2 - \omega_{2x}^2) = 0. \tag{27}$$

Thus, system (21)–(23) is equivalent to the system of equations (21), (23), (26), (27). Before integrating, we make a remark: it is evident that the structure of ansatz (7) is not altered by transformation

$$Q \to Q' = Q h_1(\omega_1) h_2(\omega_2), \quad \omega_i \to \omega_i' = R_i(\omega_i), \quad i = 1, 2, \tag{28}$$

where h_i, R_i are smooth-enough functions. This is why solutions of the system under study connected by relations (28) are considered to be equivalent.

Choosing the functions h_i, R_i in a proper way, we can put in (21) and (27)

$$f_1 = f_2 = 1, \quad A_1 = A_2 = 0.$$

Consequently, functions ω_1, ω_2 satisfy equations of the form

$$\omega_{1t}\omega_{2t} - \omega_{1x}\omega_{2x} = 0, \quad \omega_{1t}^2 - \omega_{1x}^2 + \omega_{2t}^2 - \omega_{2x}^2 = 0,$$

whence

$$(\omega_1 \pm \omega_2)_t^2 - (\omega_1 \pm \omega_2)_x^2 = 0.$$

Integrating the above equations, we obtain

$$\omega_1 = F(\xi) + G(\eta), \quad \omega_2 = F(\xi) - G(\eta), \tag{29}$$

where $F, G \subset C^2(\mathbb{R}^1, \mathbb{R}^1)$ are arbitrary functions, $\xi = (x+t)/2$, $\eta = (x-t)/2$.

Substitution of (29) into (21) with $A_1 = A_2 = 0$ yields the following equations:

$$(\ln Q)_t = 0, \quad (\ln Q)_x = 0,$$

whence $Q = 1$.

Finally, substituting the results obtained into (26), we have

$$V(x) = [g_1(F+G) - g_2(F-G)]\frac{dF}{d\xi}\frac{dG}{d\eta}. \tag{30}$$

Thus, the problem of integrating an over-determined system of nonlinear differential equations (21)–(23) is reduced to integration of the functional-differential equation (30).

Let us summarize the results obtained. The general form of solution of (1) with separated variables is as follows

$$u = \varphi(F(\xi)+G(\eta))\varphi_2(F(\xi)-G(\eta)) \tag{31}$$

where φ_i are arbitrary solutions of (14), functions $F(\xi)$, $G(\eta)$, $g_1(F+G)$, $g_2(F-G)$ being determined by (30).

To integrate Eq. (31) we make the hodograph transformation

$$\xi = P(F), \quad \eta = R(G), \tag{32}$$

where $\dot P \not\equiv 0$, $\dot R \not\equiv 0$.

After making the transformation (32), we obtain

$$g_1(F+G) - g_2(F-G) = \dot P(F)\dot R(G)V(P+R). \tag{33}$$

Evidently, equation (33) is equivalent to the following equation:

$$(\partial_F^2 - \partial_G^2)[\dot P(F)\dot R(G)V(P+R)] = 0$$

or

$$(\dddot P\,\dot P^{-1} - \dddot R\,\dot R^{-1})V + 3(\ddot P - \ddot R)\dot V + (\dot P^2 - \dot R^2)\ddot V = 0. \tag{34}$$

Thus, to integrate (30) it is enough to construct all functions $P(F)$, $R(G)$, $V(P+R)$ satisfying (34) and to substitute them into (33).

In [8] we have proved the following assertion:

Lemma. *The general solution of (34) determined up to transformation (10) is given by one of the following formulae:*

(1) $V = V(x)$ *is an arbitrary function,* $\dot P = \alpha$, $\dot R = \alpha$;

(2) $V = mx$, $\dot P^2 = \alpha P + \beta$, $\dot R^2 = \alpha R + \gamma$;

(3) $V = mx^{-2}$, $P = Q_1(F)$, $R = Q_2(G)$,

$$\begin{aligned} \dot Q_1^2 &= \alpha Q_1^4 + \beta Q_1^3 + \gamma Q_1^2 + \delta Q_1 + \rho, \\ \dot Q_2^2 &= \alpha Q_2^4 - \beta Q_2^3 + \gamma Q_2^2 - \delta Q_2 + \rho; \end{aligned} \tag{35}$$

(4) $V = m\sinh^{-2}x$, $P = \operatorname{arctanh} Q_1(F)$, $R = \tan Q_2(G)$

and Q_1, Q_2 *are determined by (35);*

(5) $V = m\sinh^{-2}x$, $P = \operatorname{arctanh} Q_1(F)$, $R = \operatorname{arctanh} Q_2(G)$

and Q_1, Q_2 *are determined by (35);*

(6) $V = m\cosh^{-2}x$, $P = \operatorname{arccoth} Q_1(F)$, $R = \operatorname{arctanh} Q_2(G)$

and Q_1, Q_2 are determined by (35);

(7) $V = m \exp x$,
$\dot{P}^2 = \alpha \exp 2P + \beta \exp P + \gamma, \quad \dot{R}^2 = \alpha \exp 2R + \delta \exp R + \rho;$

(8) $V = \cos^{-2} x(m_1 + m_2 \sin x)$,
$\dot{P}^2 = \alpha \sin 2P + \beta \cos 2P + \gamma, \quad \dot{R}^2 = \alpha \sin 2R + \beta \cos 2R + \gamma;$

(9) $V = \cosh^{-2} x(m_1 + m_2 \sinh x)$,
$\dot{P}^2 = \alpha \sinh 2P + \beta \cosh 2P + \gamma, \quad \dot{R}^2 = \alpha \sinh 2R - \beta \cosh 2R + \gamma;$

(10) $V = \sinh^{-2} x(m_1 + m_2 \cosh x)$,
$\dot{P}^2 = \alpha \sinh 2P + \beta \cosh 2P + \gamma, \quad \dot{R}^2 = -\alpha \sinh 2R + \beta \cosh 2R + \gamma;$

(11) $V = (m_1 + m_2 \exp x) \exp x$,
$\ddot{P} = -\dot{P}^2 + \beta, \quad \ddot{R} = -\dot{R}^2 + \beta;$

(12) $V = m_1 + m_2 x^{-2}$,
$\dot{P}^2 = \alpha P^2 + \beta P + \gamma, \quad \ddot{R}^2 = \alpha R^2 - \beta R + \gamma,$

(13) $V = m$,
$\dot{P}^2 = \alpha P^2 + \beta P + \gamma, \quad \dot{R}^2 = \alpha R^2 + \delta R + \rho.$

Here α, β, γ, δ, ρ, m_1, m_2, m are arbitrary real parameters; $x = \xi + \eta = P + R$.

Theorems 1 and 2 are direct consequences of the above Lemma. To prove Theorems 3–8 one has to integrate the ODE for $P(F)$, $R(G)$ and substitute the expressions obtained into formulae (32)

$$\frac{1}{2}(x + t) = P(F) \equiv P((\omega_1 + \omega_2)/2), \quad \frac{1}{2}(x - t) = R(G) \equiv R((\omega_1 - \omega_2)/2)$$

and into (33).

Thus, the problem of separation of the wave equation (1) into two second-order differential equations is completely solved.

Since all coordinate systems ω_1, ω_2 satisfy equation (23), we have orthogonal separation of variables. To obtain non-orthogonal coordinate systems providing separability of (1) one has to carry out SV following Definition 2.

3. Non-orthogonal separation of variables in equation (1)

Utilizing the SV procedure in (1) determined by Definition 2, we come to the following assertions (corresponding computations are omitted).

Theorem 9. *Equation (1) admits SV in the sense of Definition 2 iff it is locally-equivalent to one of the following equations:*

(1) $\Box u + mu = 0;$

(2) $\Box u + mx^{-2} u = 0,$

where m is an arbitrary real constant.

Theorem 10. *Equation* $\Box u + mu = 0$ *separates in two coordinate systems*

$$
\begin{aligned}
&(1)\quad \omega_1 = \xi, \quad \omega_2 = \xi + \eta,\\
&\qquad \dot\varphi_1 = -\lambda\varphi_1, \quad \ddot\varphi_2 = \lambda\dot\varphi_2 + m\varphi_2;\\
&(2)\quad \omega_1 = \xi, \quad \omega_2 = \ln\xi + \ln\eta,\\
&\qquad \dot\varphi_1 = -\lambda\omega_1^{-1}\varphi_1, \quad \ddot\varphi_2 = \lambda\dot\varphi_2 + m\exp(\omega_2)\varphi_2.
\end{aligned}
$$

Theorem 11. *Equation* $\Box u + mx^{-2}u = 0$ *separates in eight coordinate systems*

$$
\begin{aligned}
&(1)\quad \omega_1 = \xi, \quad \omega_2 = \xi + \eta,\\
&\qquad \dot\varphi_1 = -\lambda\varphi_1, \quad \ddot\varphi_2 = \lambda\dot\varphi_2 + m\omega_2^{-2}\varphi_2;\\
&(2)\quad \omega_1 = \xi, \quad \omega_2 = \arctan\xi + \arctan\eta,\\
&\qquad \dot\varphi_1 = -\lambda(1+\omega_1^2)\varphi_1, \quad \ddot\varphi_2 = \lambda\dot\varphi_2 + m\sin^{-2}\omega_2\varphi_2;\\
&(3)\quad \omega_1 = \xi, \quad \omega_2 = \operatorname{arctanh}\xi + \operatorname{arctanh}\eta,\\
&\qquad \dot\varphi_1 = -\lambda(1-\omega_1^2)^{-1}\varphi_1, \quad \ddot\varphi_2 = \lambda\dot\varphi_2 + m\sinh^{-2}\omega_2\varphi_2;\\
&(4)\quad \omega_1 = \xi, \quad \omega_2 = \operatorname{arccoth}\xi + \operatorname{arccoth},\eta,\\
&\qquad \dot\varphi_1 = \lambda(1-\omega_1^2)^{-1}\varphi_1, \quad \ddot\varphi_2 = \lambda\dot\varphi_2 + m\sinh^{-2}\omega_2\varphi_2;\\
&(5)\quad \omega_1 = \xi, \quad \omega_2 = \operatorname{arctanh}\xi + \operatorname{arctanh}\eta,\\
&\qquad \dot\varphi_1 = -\lambda(1-\omega_1^2)^{-1}\varphi_1, \quad \ddot\varphi_2 = \lambda\dot\varphi_2 - m\cosh^{-2}\omega_2\varphi_2,\\
&(6)\quad \omega_1 = \xi, \quad \omega_2 = \operatorname{arccoth}\xi + \operatorname{arccoth}\eta,\\
&\qquad \dot\varphi_1 = \lambda(1-\omega_1^2)^{-1}\varphi_1, \quad \ddot\varphi_2 = \lambda\dot\varphi_2 - m\cosh^{-2}\omega_2\varphi_2;\\
&(7)\quad \omega_1 = \xi, \quad \omega_2 = \frac{1}{2}(\ln\xi - \ln\eta),\\
&\qquad \dot\varphi_1 = -\lambda(2\omega_1)^{-1}\varphi_1, \quad \ddot\varphi_2 = \lambda\dot\varphi_2 - m\cosh^{-2}\omega_2\varphi_2;\\
&(8)\quad \omega_1 = \xi, \quad \omega_2 = \xi^{-1} + \eta^{-1},\\
&\qquad \dot\varphi_1 = \lambda\omega_1^{-2}\varphi_1, \quad \ddot\varphi_2 = \lambda\dot\varphi_2 + m\omega_2^{-2}\varphi_2.
\end{aligned}
$$

In the above formulae λ *is a separation constant,* $\xi = \frac{1}{2}(x+t)$*,* $\eta = \frac{1}{2}(x-t)$.

As a direct check shows, the above coordinate systems do not satisfy (23). Consequently, they are non-orthogonal.

4. Conclusion

Let us say a few words about the intrinsic characterization of SV in (1). It is well known that the solution of the second-order linear PDE with separated variables is a joint eigenfunction of mutually-commuting symmetry operators of the equation under study (for more detail, see [13, 14]). Below, we construct the second-order symmetry operator of (1) such that solution with separated variables is its eigenfunction and parameter λ is an eigenvalue.

Making in (1) the change of variables (29), we obtain

$$u_{\omega_1\omega_1} - u_{\omega_2\omega_2} = V(\xi+\eta)[\dot F(\xi)\dot G(\eta)]^{-1}u.$$

Provided (1) admits SV, by virtue of (33) there exist functions $g_1(F+G)$, $g_2(F-G)$ such that

$$V(\xi+\eta)[\dot F(\xi)\dot G(\eta)]^{-1} = g_1(F+G) - g_2(F-G).$$

Since $F+G=\omega_1, F-G=\omega_2$, equation (36) takes the form

$$u_{\omega_1\omega_1} - u_{\omega_2\omega_2} = [g_1(\omega_1) - g_2(\omega_2)]u$$

or

$$Xu = 0, \quad X = \partial^2_{\omega_1} - \partial^2_{\omega_2} - g_1(\omega_1) + g_2(\omega_2).$$

Clearly, the operators $Q_i = \partial^2_{\omega_i} - g_i(\omega_i)$, $i=1,2$ commute with the operator X, i.e. they are symmetry operators of (1) and, what is more, the relations

$$Q_i u = Q_i \varphi_1(\omega_1)\varphi_2(\omega_2) = \lambda\varphi_1(\omega_1)\varphi_2(\omega_2) = \lambda u, \quad i=1,2$$

hold.

It should be noted that V.N. Shapovalov carried out classification of potentials $V(x)$ such that (1) admitted a non-trivial second-order symmetry operator [15] but he lost cases (4) and (9) from Theorem 1.

It was shown by Osborne and Stuart [16] that the method of SV could be applied to nonlinear PDE. In [8] we suggested a regular approach to SV in nonlinear partial differential equations. In future publications we intend to apply this approach to separate variables in the nonlinear wave equation $u_{tt} - u_{xx} = F(u)$.

1. Bocher M., Die Reihentwickelungen der Potentialtheorie Leipzig, Teubner, 1894.
2. Darboux G., Lecons sur les Systemes Orhogonaux et les Coordonnees Curvilignes, Paris, Hermann, 1910.
3. Eisenhart L.P., *Ann. Math.*, 1934, **35**, 284.
4. Stepanov V.V., *Matemat. Sbornik*, 1942, **11(53)**, 204.
5. Olevsky M.N., *Matemat. Sbornik*, 1950, **27(69)**, 379.
6. Kalnins E., Miller W., *J. Math. Phys.*, 1974, **15**, 1025.
7. Zhdanov R.Z, Revenko I.V. and Fushchych W.I., *Doklady AN Ukrainy, Ser. A*, 1993, № 1, 27.
8. Zhdanov R.Z., Revenko I.V., Fushchych W.I., *Ukrain. Mat. Zh.*, to appear.
9. Fushchych W.I., Serov N.I., *J. Phys. A: Math. Gen.*, 1983, **16**, 3645.
10. Fushchych W.I., Zhdanov R.Z., *Phys. Lett. A*, 1989, **141**, 113.
11. Fushchych W.I., Zhdanov R.Z., Yegorchenko I.A., *Math. Anal. Appl.*, 1991, **161**, 352.
12. Miller W., Symmetry and separation of variables, Addison-Wesley, 1977.
13. Koornwinder T.H., *Lecture Notes in Math.*, 1980, **810**, 240.
14. Kalnins E., Miller W., *J. Math. Phys.*, 1976, **17**, 369.
15. Shapovalov V.N., *Izv. Vuzov. Fizika*, 1969, **9**, 64.
16. Osborn A., Stuart A., *J. Math. Phys.*, 1978, **19**, 1573.

Редукция многомерного уравнения Даламбера к двумерным уравнениям

А.Ф. БАРАННИК, Л.Ф. БАРАННИК, В.И. ФУЩИЧ

We give a classification of the maximal subalgebras of rank $n-1$ for the extended Poincaré algebra $A\tilde{P}(1,n)$, which is realized on the set of solutions of the d'Alembert equation $\Box u + \lambda u^k = 0$. These subalgebras are used for constructing the anzatses reducing this equation to differential equations with two invariant variables.

Проведена класифікація максимальних підалгебр рангу $n-1$ розширеної алгебри Пуанкаре $A\tilde{P}(1,n)$, яка реалізується на множині розв'язків рівняння Даламбера $\Box u + \lambda u^k = 0$. Одержані підалгебри використано для побудови анзаців, що редукують це рівняння до диференціальних рівнянь з двома інваріантними змінними.

1. Введение. В настоящей статье изучается редукция нелинейного уравнения Даламбера

$$\Box u + \lambda u^k = 0 \qquad (1)$$

к двумерным уравнениям. Здесь $u = u(x)$ — скалярная функция переменной $u = (x_0, x_1, \ldots, x_n)$, k — произвольное вещественное число, отличное от 1, а

$$\Box u = \frac{\partial^2 u}{\partial x_0^2} - \frac{\partial^2 u}{\partial x_1^2} - \cdots - \frac{\partial^2 u}{\partial x_n^2}.$$

В [1, 2] установлено, что алгеброй инвариантности уравнения (1) является алгебра Ли $A\tilde{P}(1,n)$ расширенной группы Пуанкаре $\tilde{P}(1,n)$, базис которой образуют такие векторные поля:

$$J_{0a} = x_0\partial_a + x_a\partial_0, \quad J_{ab} = x_b\partial_a - x_a\partial_b, \quad P_\mu = \partial_\mu,$$
$$D = -x^\mu\partial_\mu + \frac{2}{k-1}u\partial_u,$$

где

$$\partial_\mu = \frac{\partial}{\partial x_\mu}, \quad \partial_u = \frac{\partial}{\partial u}, \quad a,b = 1,\ldots,n, \quad \mu = 0,1,\ldots,n.$$

Это позволяет использовать подалгебры алгебры $A\tilde{P}(1,n)$ для проведения редукции уравнения (1) к диференциальным уравнениям с меньшим числом переменных.

Если $\omega_1(x),\ldots,\omega_m(x),\omega_{m+1}(x,u)$ — полная система инвариантов некоторой подалгебры L алгебры $A\tilde{P}(1,n)$, то анзац

$$\omega_{m+1} = \varphi(\omega_1,\ldots,\omega_m) \qquad (2)$$

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преобразует уравнение (1) в уравнение, содержащее только $\varphi, \omega_1, \ldots, \omega_m$ и производные от φ по $\omega_1, \ldots, \omega_m$. Число m связано с рангом r алгебры L соотношением $m = n + 1 - r$.

Случай $m = 1$ исследован в [1–6]. Случай $m = 2$ для $n \le 3$ рассматривался в [1–3], а для произвольного n такое исследование с привлечением подалгебр алгебры Ли $AP(1,n)$ группы Пуанкаре $P(1,n)$ проведено в [7]. Поэтому для завершения изучения случая $m = 2$ необходимо выполнить редукцию по тем подалгебрам ранга $n-1$ алгебры $A\tilde{P}(1,n)$, которые имеют ненулевую проекцию на $\langle D\rangle$.

В данной статье с точностью до $\tilde{P}(1,n)$-эквивалентности найдены все максимальные подалгебры ранга $n-1$ алгебры $A\tilde{P}(1,n)$, имеющие ненулевую проекцию на $\langle D\rangle$, для каждой из них построен анзац (2), посредством которого проведена редукция уравнения (1) к дифференциальному уравнению с двумя переменными.

2. Основные обозначения и некоторые общие замечания. Базисные элементы алгебры $A\tilde{P}(1,n)$ связаны следующими коммутационными соотношениями:

$$[J_{\alpha\beta}, J_{\gamma\delta}] = g_{\alpha\delta}J_{\beta\gamma} + g_{\beta\gamma}J_{\alpha\delta} - g_{\alpha\gamma}J_{\beta\delta} - g_{\beta\delta}J_{\alpha\gamma}.$$
$$[P_\alpha, J_{\beta\gamma}] = g_{\alpha\beta}P_\gamma - g_{\alpha\gamma}P_\beta, \quad [P_\alpha, P_\beta] = 0, \quad [D, J_{\alpha\beta}] = 0, \quad [D, P_\alpha] = P_\alpha.$$

где $g_{00} = -g_{11} = \cdots = -g_{nn} = 1$, $g_{\alpha\beta} = 0$ при $\alpha \ne \beta$, $\alpha, \beta, \gamma, \delta = 0, 1, \ldots, n$. Алгебра $A\tilde{P}(1,n)$ содержит алгебру Пуанкаре $AP(1,n)$, порожденную $J_{\alpha\beta}$ и P_α, ортогональную алгебру $AO(n) = \langle J_{ab} \,|\, a, b = 1, \ldots, n\rangle$, псевдоортогональную алгебру $AO(1,n) = \langle J_{\alpha\beta} \,|\, \alpha, \beta = 0, 1, \ldots, n\rangle$, коммутативный идеал $V = \langle P_0, P_1, \ldots, P_n\rangle$. Важной подалгеброй алгебры $A\tilde{P}(1,n)$ является нормализатор $\mathfrak{N}$ изотропного пространства $\langle P_0 + P_n\rangle$ в алгебре $A\tilde{P}(1,n)$. Нетрудно получить

$$\mathfrak{N} = \langle M, T, P_1, \ldots, P_{n-1}, G_1, \ldots, G_{n-1}\rangle \subset\!\!\!\!+ \ (AO(n-1) \oplus \langle D, J_{0n}\rangle),$$

где

$$M = P_0 + P_n, \quad T = (P_0 - P_n)/2, \quad G_a = J_{0a} - J_{an}, \quad a = 1, \ldots, n-1,$$
$$AO(n-1) = \langle J_{ab} \,|\, a, b = 1, \ldots, n-1\rangle.$$

Алгебра $\mathfrak{N}$ содержит расширенную изохронную алгебру Галилея

$$A\tilde{G}(0, n-1) = \langle M, P_1, \ldots, P_{n-1}, G_1, \ldots, G_{n-1}\rangle \subset\!\!\!\!+ \ AO(n-1).$$

В работе будут использованы еще и такие обозначения:

$$AO[r,s] = \langle J_{ab} \,|\, a, b = r, \ldots, s\rangle, \quad r \le s;$$
$$AE[r,s] = \langle P_r \ldots, P_s\rangle, \subset\!\!\!\!+ \ AO(r,s), \quad r \le s;$$
$$AE_1[r,s] = \langle G_r \ldots, G_s\rangle, \subset\!\!\!\!+ \ AO(r,s), \quad r \le s;$$
$$V[1, n-1] = \langle G_1, \ldots, G_{n-1}\rangle, \quad W[1, n-1] = \langle P_1, \ldots, P_{n-1}\rangle;$$

π — проектирование $\mathfrak{N}$ на $\langle D, J_{0n}\rangle$. Если $s > r$, то, по определению, $AO[r,s] = 0$, $AE[r,s] = 0$.

Для проведения редукции уравнения (1) по подалгебрам алгебры $A\tilde{P}(1,n)$ необходимо описать подалгебры этой алгебры с точностью до $\tilde{P}(1,n)$-эквивалентности. Две подалгебры K_1, K_2 алгебры $A\tilde{P}(1,n)$ называются $\tilde{P}(1,n)$-эквивалентными, если с точностью до $\tilde{P}(1,n)$-сопряженности они имеют одни и те же инварианты.

Среди подалгебр, имеющих одну и ту же полную систему инвариантов, существует одна (максимальная) подалгебра, содержащая все остальные подалгебры. Будем называть ее i-максимальной подалгеброй алгебры $A\tilde{P}(1,n)$. Две i-максимальные подалгебры K_1 и K_2 алгебры $A\tilde{P}(1,n)$ эквивалентны тогда и только тогда, когда K_1 и K_2 $\tilde{P}(1,n)$-сопряжены. Таким образом, для нахождения с точностью до $\tilde{P}(1,n)$-сопряженности всех анзацев, редуцирующих уравнение (1) к диференциальным уравнениям с двумя инвариантными переменными, требуется описать i-максимальные подалгебры ранга $n-1$ алгебры $A\tilde{P}(1,n)$ с точностью до $\tilde{P}(1,n)$-сопряженности.

При доказательстве излагаемых результатов нам понадобится следующая лемма.

Лемма 1. *Пусть L — максимальная подалгебра алгебры $A\tilde{P}(1,n)$, K_1 — подалгебра L. Тогда в L существует подалгебра K_2, удовлетворяющая таким условиям:*

1) *K_2 — i-максимальная подалгебра алгебры $A\tilde{P}(1,n)$;*
2) *$K_1 \subset K_2$;*
3) *K_1 и K_2 имеют одни и те же инварианты.*

Доказательство. Пусть $\omega_1,\dots,\omega_s$ — полная система инвариантов подалгебры K_1. Обозначим через K_2 i-максимальную подалгебру алгебры $A\tilde{P}(1,n)$, имеющую полную систему инвариантов $\omega_1,\dots,\omega_s$. Докажем, что $K_2 \subset L$. Действительно, пусть f — произвольный инвариант алгебры L. Так как $K_1 \subset L$, то f является инвариантом подалгебры K_1 и в силу теоремы об универсальном инварианте получаем $f = f(\omega_1,\dots,\omega_s)$. Так как L — i-максимальная подалгебра алгебры $A\tilde{P}(1,n)$, то отсюда вытекает, что $K_2 \subset L$. Лемма доказана.

Из результатов, изложенных в п. 3 [7], следует, что задача построения инвариантов произвольной подалгебры алгебры $A\tilde{P}(1,n)$ сводится к задаче построения инвариантов неприводимых подалгебр ортогональной алгебры $AO(k)$ для всех $k \le n$. Последняя же задача в общем случае, видимо, неразрешима в квадратурах. В связи с этим ограничимся рассмотрением только тех подалгебр алгебры $AP(1,n)$, проекции которых на $AO(1,n)$ являются под прямыми суммами алгебр вида $AO[r,s]$.

3. Максимальные подалгебры ранга $n-1$, не содержащие P_0 и P_0+P_n. В следующих ниже леммах L обозначает максимальную подалгебру алгебры $A\tilde{P}(1,n)$, имеющую ненулевую проекцию на $\langle D\rangle$ и не содержащую P_0 и P_0+P_n.

Лемма 2. *Если проекция L, на $AO(1,n)$ не имеет в пространстве V инвариантных изотропных подпространств, то L сопряжена с одной из таких алгебр:*

$$\begin{aligned}
&L_1 = (AO[0,d] \oplus AO[d+1,m] \oplus AO[m+1,q] \oplus AE[q+1,n]) \subset\!\!\!\!+ \langle D\rangle,\\
&\qquad d=2,\dots,n-2,\ m=d+1,\dots,n-2,\ q=m+1,\dots,n-1,\\
&\qquad 2n \le d+q,\ n\ge 4;\\
&L_2 = (AO[0,m] \oplus AE[m+1,n-2]) \subset\!\!\!\!+ \langle D+\alpha J_{n-1,n}\rangle,\\
&\qquad m=2,\dots,n-2,\ n\ge 4,\ \alpha>0;\\
&L_3 = (AO[1,m] \oplus AO[m+1,q] \oplus AE[q+1,n]) \subset\!\!\!\!+ \langle D\rangle,\\
&\qquad m=2,\dots,n-2,\ q=m+2,\dots,n,\ 2m\le q,\ n\ge 2.
\end{aligned}$$

Доказательство. Если $D \in L$ то $L = K \subset\!\!\!+ \langle D \rangle$, где K — максимальная подалгебра ранга $n-2$ алгебры $AP(1,n)$, относящаяся к классу 0 и являющаяся расщепляемой. Отсюда на основании теоремы 4 [7] получаем, что L сопряжена с L_1 или L_3. Если $D \notin L$, то в силу леммы 1 $L = N \subset\!\!\!+ \langle D + \alpha J_{n-1,n} \rangle$, $\alpha > 0$ где N — максимальная подалгебра ранга $n-2$ алгебры $AP(1,n-2)$. В силу [4] алгебра N сопряжена с $AO[0,m] \oplus AE[m+1,n-2]$, $2 \le m \le n-2, n \ge 4$. Лемма доказана.

Лемма 3. *Если $L \subset \mathfrak{N}$ и $\pi(L) = \langle D, J_{0n} \rangle$, то L сопряжена с одной из таких алгебр:*

$$L_4 = (AE[1,m] \oplus AE[m+1,n-3]) \subset\!\!\!+ \langle J_{n-2,n-1} + cJ_{0n}, D + \alpha J_{0n} \rangle,$$
$$m = 1,\dots,n-3,\ n \ge 4,\ c > 0,\ \alpha \ge 0;$$
$$L_5 = (AO[1,m] \oplus AO[m+1,q] \oplus AE[q+1,n-1]) \subset\!\!\!+ \langle D, J_{0n} \rangle,$$
$$m = 1,\dots,n-2,\ q = m+1,\dots,n-1,\ 2m \le q,\ n \ge 3;$$
$$L_6 = AE[3,n-1] \subset\!\!\!+ \langle J_{12} + cJ_{0n}, D + \alpha J_{0n} \rangle,\quad c > 0,\ \alpha \ge 0,\ n \le 3.$$

Доказательство. Согласно геореме IV.3.4 [6] алгебра L сопряжена с алгеброй $(U_1 + U_2) \subset\!\!\!+ F$, где $U_1 \subset V[1,n-1], U_2 \subset W[1,n-1]$, а $F \subset AO(n-1) \oplus \langle D, J_{0n} \rangle$. В силу леммы 1 $L = K \subset\!\!\!+ \langle D + X_1, J_{0n} + X_2 \rangle$, где $X_1, X_2 \in AO(n-1)$, а K — максимальная подалгебра ранга $n-3$ алгебры $A\tilde{G}(0,n-1)$. Поскольку алгебра $AE_1[1,m] \subset\!\!\!+ \langle J_{0n} \rangle$, имеющая ранг $m+1$, сопряжена с подалгеброй алгебры $AO[0,m+1]$, не являющейся в ней максимальной, то по лемме 1 получаем, что $X_2 \ne 0$ при $U_1 \ne 0$. В этом случае согласно предложению 2 [6] алгебра L сопряжена с L_4. Если $U_1 = 0$, то на основании этого же предложения 2 алгебра L сопряжена с L_5 или L_6. Лемма доказана.

Лемма 4. *Пусть $L \subset \mathfrak{N}$, $\pi(L) = \langle D + \alpha J_{0n} \rangle$ и L — расщепляемая алгебра при $\alpha = \pm 1$. Тогда L сопряжена с одной из таких алгебр:*

$$L_7 = (AE_1[1,d] \oplus AO[d+1,m] \oplus AE[m+1,n-1]) \subset\!\!\!+ \langle D + \alpha J_{0n} \rangle,$$
$$d = 1,\dots,n-2,\ m = d+1,\dots,n-1,\ n \ge 4,\ \alpha \ge 0;$$
$$L_8 = (AO[1,m] \oplus AE[m+1,n-1]) \subset\!\!\!+ \langle D + \alpha J_{0n} \rangle,$$
$$m = 1,\dots,n-1,\ n \ge 2,\ \alpha \ge 0;$$
$$L_9 = (\langle G_1 + 2T \rangle \oplus AO[2,m] \oplus AE[m+1,n-1]) \subset\!\!\!+ \langle 2D - J_{0n} \rangle,$$
$$m = 2,\dots,n-1,\ n \ge 3.$$

Доказательство. Если $\alpha \notin \{0, \pm 1, -1/2\}$, то в силу теоремы IV.3.4 [6] алгебра L сопряжена с алгеброй $(U_1 + U_2) \subset\!\!\!+ F$, где $U_1 \subset V[1,n-1], U_2 \subset W[1,n-1]$, а $F \subset AO(n-1) \oplus \langle D + \alpha J_{0n} \rangle$. По лемме 1 $L = K \subset\!\!\!+ \langle D + \alpha J_{0n} + X \rangle$, где $X \in AO(n-1)$, а K — максимальная подалгебра ранга $n-2$ алгебры $A\tilde{G}(0,n-1)$. Согласно теореме 1 [7] алгебра K совпадает с точностью до сопряженности с одной из алгебр

$$AE_1[1,d] \oplus AO[d+1,m] \oplus AE[m+1,n-1],$$
$$1 \le d \le n-2,\ d+1 \le m \le n-1,\ n \ge 3;$$
$$AO[1,m] \oplus AE[m+1,n-1],\quad 1 \le m \le n-1,\ n \ge 2.$$

Так как $[X,K] \subset K$, то X принадлежит проекции L на $AO(n-1)$, а значит, можно предполагать, что $X = 0$, и мы получаем алгебры L_7, L_8.

Пусть $\alpha = -1/2$. Тогда согласно теореме IV.3.4 [6] алгебра L сопряжена с алгеброй $(U_1 + U_2) \niplus F$, где $U_1 \subset V[1, n-1] + \langle T \rangle$ (как пространство), $U_2 \subset W[1, n-1]$, а F является подалгеброй алгебры $AO(n-1) \oplus \langle 2D - J_{0n} \rangle$. Если проекция U_1 на $\langle T \rangle$ равна 0, то L совпадает с L_7 или L_8. Допустим, что проекция U_1 на $\langle T \rangle$ совпадает с $\langle T \rangle$. Если $G_1 + 2T, G_2 \in L$, то L содержит $[G_1 + 2T, G_2] = 2P_2$, $[P_2, G_2] = M$, что противоречит предположению относительно L. Отсюда вытекает $U_1 = \langle G_1 + 2T \rangle$. Если $U_2 \neq 0$, то по теореме Витта $U_2 = \langle P_{m+1}, \dots, P_n \rangle$, $n \geq 2$, а значит, L сопряжена с алгеброй L_9.

В силу теоремы 1V.3.4 [6] случаи, когда $\alpha = \pm 1$ и L — расщепляемая алгебра, не отличаются от рассмотренного случая $\alpha \notin \{0, \pm 1, -1/2\}$. Лемма доказана.

Лемма 5. *Пусть L — нерасщепляемая подалгебра алгебры $\mathfrak{N}$ и $\pi(L) = \langle D + J_{0n} \rangle$. Тогда L сопряжена с одной из таких алгебр:*

$$
\begin{aligned}
&L_{10} = (AE_1[1, d] \oplus AO[d+1, m] \oplus AE[m+1, n-1]) \niplus \langle D + J_{0n} + M \rangle,\\
&\qquad d = 1, \dots, n-2,\ m = d+1, \dots, n-1,\ n \geq 3;\\
&L_{11} = (AO[1, m] \oplus AE[m+1, n-1]) \niplus \langle D + J_{0n} + M \rangle,\\
&\qquad m = 1, \dots, n-1,\ n \geq 2;\\
&L_{12} = (AE_1[1, m] \oplus AE[m+1, n-3]) \niplus \langle J_{n-2,n-1} + \alpha M, D + J_{0n} + M \rangle,\\
&\qquad m = 1, \dots, n-3,\ n \geq 4,\ \alpha \geq 0;\\
&L_{13} = (AE_1[1, m] \oplus AE[m+1, n-3]) \niplus \langle J_{n-2,n-1} + M, D + J_{0n} \rangle,\\
&\qquad m = 1, \dots, n-3,\ n \geq 4;\\
&L_{14} = AE[3, n-1] \niplus \langle J_{12} + \alpha M, D + J_{0n} + M \rangle, \quad \alpha \geq 0,\ n \geq 3;\\
&L_{15} = AE[3, n-1] \niplus \langle J_{12} + M, D + J_{0n} \rangle, \quad n \geq 3.
\end{aligned}
$$

Доказательство. Согласно теореме IV.3.4 [6] алгебра L, сопряжена с алгеброй $(U_1 + U_2) \niplus F$, где $U_1 \subset V[1, n-1]$, $U_2 \subset W[1, n-1]$, а $L \subset AO(n-1) \oplus \langle D + J_{0n}, M \rangle$. Легко убедиться, что алгебры $\langle G_{2j-1}, G_{2j}, G_{2j-1,2j} \rangle$ и $\langle G_{2j-1}, G_{2j} \rangle$ являются эквивалентными. Поэтому если $D + J_{0n} + \gamma M + \delta J_{n-2,n-1} \in L$ и $\delta \neq 0$, то по лемме 1 $U_1 \subset V[1, n-3]$ и $U_2 \subset W[1, n-3]$. В этом случае $J_{n-2,n-1} + \beta M \in L$, а следовательно, алгебра L, сопряжена с $K \niplus \langle J_{n-2,n-1} + \beta M, D + J_{0n} + \gamma M \rangle$, где K — максимальная подалгебра ранга $n-3$ алгебры $A\tilde{G}(0, n-3)$. Нетрудно получить, что с точностью до сопряженности K совпадает с $AE_1[1, m] \oplus AE[m+1, n-3]$, $1 \leq m \leq n-3$, $n \geq 4$, или $AE[1, n-3]$, $n \leq 3$, а потому L сопряжена с одной из алгебр L_j, $j = 12, \dots, 15$. В оставшихся случаях в силу предложения 2 [7] алгебра L сопряжена с L_{10} или L_{11}. Лемма доказана.

Для выделения оставшихся алгебр нам необходимы дополнительные обозначения. Для любых двух натуральных чисел r и s, $r \leq s$, положим

$$\Phi(r, s, \gamma) = \langle G_r + \gamma_1 P_r, \dots, G_s + \gamma P_s \rangle \niplus AO[r, s], \quad \gamma \in \mathbb{R}.$$

Пусть, далее, $\Gamma_{d,q} = U \niplus F$, где F — диагональ в $AO[1, d] \oplus AO[d+1, 2d] \oplus \dots \oplus AO[(q-1)d+1, qd]$, а U — коммутативная алгебра, имеющая базис

$$
\begin{aligned}
&G_1 + \gamma_1 P_1 + \lambda_1 P_{(q-1)d+1},\ \dots,\ G_d + \gamma_1 P_d + \lambda_1 P_{qd},\\
&G_{d+1} + \gamma_2 P_{d+1} + \lambda_2 P_{(q-1)d+1}, \dots, G_{2d} + \gamma_2 P_{2d} + \lambda_1 P_{qd},\\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots
\end{aligned}
$$

$$G_{(q-2)d+1} + \gamma_{q-1} P_{(q-2)d+1} + \lambda_{q-1} P_{(q-1)d+1}, \ \dots,$$
$$G_{(q-1)d} + \gamma_{q-1} P_{(q-1)d} + \lambda_{q-1} P_{qd},$$
$$0 \le \gamma_1 < \gamma_2 < \dots < \gamma_{q-1}, \ \lambda_1 > 0, \ \lambda_2 > 0, \ \dots, \ \lambda_{q-1} > 0.$$

Лемма 6. *Если L — нерасщепляемая подалгебра алгебры $\mathfrak{N}$ и $\pi(L) = \langle D - J_{0n} \rangle$, то L сопряжена с алгеброй L', для которой $\pi(L') = \langle D + J_{0n} \rangle$, или с одной из алгебр*

$$L_{16} = (\Gamma_{d,q} \oplus AE[dq+1, n-1]) \subset\!\!\!\!+ \ \langle D - J_{0n} \rangle, \quad d = 2, \ n \ge 5;$$
$$L_{17} = (\Phi(d_0+1, d_1, \gamma_1) \oplus \Phi(d_1+1, d_2, \gamma_2) \oplus \ldots \oplus \Phi(d_{t-1}+1, d_t, \gamma_t) \oplus$$
$$\oplus AO[d_t+1, m] \oplus AE[m+1, n-1]) \subset\!\!\!\!+ \ \langle D - J_{0n} \rangle,$$

где $d_0 = 0$, $\gamma_1 < \gamma_2 < \dots < \gamma_t$, $t > 1$, $m = 1, \dots, n-2$, $n \ge 3$;

$$L_{18} = (\Gamma_{d,q} \oplus \Phi(dq+1, l_1, \mu_1) \oplus \Phi(l_1+1, l_2, \mu_2) \oplus \cdots$$
$$\cdots \oplus \Phi(l_{t-1}+1, l_t, \mu_t) \oplus AE[l_t+1, n-1]) \subset\!\!\!\!+ \ \langle D - J_{0n} \rangle,$$

где $\mu_1 < \mu_2 < \dots < \mu_t$, $t \ge 1$, $l_0 = dq$.

Доказательство. В силу теоремы IV.3.4 [6] алгебра L, сопряжена с алгеброй $U \subset\!\!\!\!+ F$, где $U \subset V[1, n-1] + W[1, n-1]$ (как пространство), а $F \subset AO(n-1) \oplus \langle D - J_{0n}, T \rangle$, причем если проекция L, на $\langle T \rangle$ является ненулевой, то $[T, U] \subset U$. Поскольку $[T, G_a] = 2P_a$, $[P_a, G_a] = M$ и $M \notin L$, то в последнем случае проекция U на $V[1, n-1]$ является нулевой. Но тогда применим $O[1, n]$-автоморфизм алгебры $A\tilde{P}(1, n)$, соответствующий матрице $\mathrm{diag}\,[-1, 1, \dots, 1]$, который преобразует $\langle -J_{0n}, M \rangle$. Поэтому можно предполагать, что проекция L на $\langle T \rangle$ является нулевой.

Согласно лемме 1 $L = K \subset\!\!\!\!+ \ \langle D - J_{0n} + X \rangle$, где $X \in AO(n-1)$, а K — максимальная подалгебра ранга $n-2$ алгебры $A\tilde{G}(0, n-1)$. Последнее обстоятельство позволяет воспользоваться перечнем таких подалгебр, приведенным в теореме 1 [7]. Учитывая, что $[D - J_{0n}, K] \subset K$, нетрудно получить, что L, сопряжена с одной из алгебр L_{16}, L_{17}, L_{18}. Лемма доказана.

Теорема. *Максимальные подалгебры ранга $n-1$ алгебры $A\tilde{P}(1, n)$, имеющие ненулевую проекцию на $\langle D \rangle$ и не содержащие P_0 и $P_0 + P_n$, исчерпываются с точностью до $\tilde{P}(1, n)$-сопряженности алгебрами $L_1, \dots, L_{18}$, описанными в леммах 2–6.*

4. Редукция по подалгебрам, не содержащим P_0 и $P_0 + P_n$. Подалгебра L_j, $j = 1, \dots, 18$, имеет полную систему инвариантов вида $\omega_1(x)$, $\omega_2(x)$, $uf(x)^{-1}$. Поэтому для каждой из этих подалгебр анзац (2) удобно представить в виде [1, 2, 6]

$$u = f(x) \varphi(\omega_1, \omega_2), \tag{3}$$

где φ — неизвестная функция. Легко видеть, что

$$\Box u = \varphi \Box f + 2\varphi_1 (\nabla f \nabla \omega_1) + 2\varphi_2 (\nabla f \nabla \omega_2) + f\{\varphi_{11} (\nabla \omega_1 \nabla \omega_1) + \\ + 2\varphi_{12} (\nabla \omega_1 \nabla \omega_2) + \varphi_{22} (\nabla \omega_2 \nabla \omega_2) + \varphi_1 \Box \omega_1 + \varphi_2 \Box \omega_2\}, \tag{4}$$

где

$$\varphi_{11} = \frac{\partial^2 \varphi}{\partial \omega_1^2}, \quad \varphi_{12} = \frac{\partial^2 \varphi}{\partial \omega_1 \partial \omega_2}, \quad \varphi_{22} = \frac{\partial^2 \varphi}{\partial \omega_2^2}, \quad \varphi_1 = \frac{\partial \varphi}{\partial \omega_1}, \quad \varphi_2 = \frac{\partial \varphi}{\partial \omega_2},$$

$$\nabla g \nabla h = \frac{\partial g}{\partial x_0}\frac{\partial h}{\partial x_0} - \frac{\partial g}{\partial x_1}\frac{\partial h}{\partial x_1} - \cdots - \frac{\partial g}{\partial x_n}\frac{\partial h}{\partial x_n}.$$

Редукцию уравнения (1) удобно проводить с использованием формулы (4), поскольку она сводит нахождение $\Box u$ к менее громоздким вычислениям производных от функций f, ω_1, ω_2.

Редуцированное уравнение, соответствующее анзацу (3), имеет вид

$$a_{11}(x)\varphi_{11} + a_{12}(x)\varphi_{12} + a_{22}(x)\varphi_{22} + b_1(x)\varphi_1 + b_2(x)\varphi_2 + c(x, \varphi) = 0.$$

Ниже для каждой подалгебры L_j, $j = 1, \ldots, 18$ указываем соответствующие ей функции f, ω_1, ω_2, a_{11}, a_{12}, a_{22}, b_1, b_2, c.

1. Алгебра L_1:

$$f(x) = (x_{m+1}^2 + x_q^2)^{1/(1-k)},$$
$$\omega_1 = \frac{x_0^2 - x_1^2 - \cdots - x_d^2}{x_{m+1}^2 + \cdots + x_q^2}, \quad \omega_2 = \frac{x_{d+1}^2 + \cdots + x_m^2}{x_{m+1}^2 + \cdots + x_q^2},$$
$$a_{11} = 4\omega_1(1 - \omega_1), \quad a_{12} = -8\omega_1\omega_2, \quad a_{22} = -4\omega_2(1 + \omega_2),$$
$$b_1 = 2(d+1) + \frac{2q - 2m - (2q - 2m - 8)k}{1 - k}\omega_1,$$
$$b_2 = 2(d - m) + \frac{2q - 2m - (2q - 2m - 8)k}{1 - k}\omega_2,$$
$$c = \frac{2(m - q) + (2q - 2m - 4)k}{(1 - k)^2}\varphi + \lambda\varphi^k.$$

2. Алгебра L_2:

$$f(x) = (x_{n-1}^2 + x_n^2)^{1/(1-k)},$$
$$\omega_1 = \alpha \ln(x_{n-1}^2 + x_n^2) - 2 \operatorname{arctg} \frac{x_n}{x_{n-1}}, \quad \omega_2 = \frac{x_0^2 - x_1^2 - \cdots - x_m^2}{x_{n-1}^2 + x_n^2},$$
$$a_{11} = 4(\alpha^2 + 1), \quad a_{12} = -8\alpha\omega_2, \quad a_{22} = 4\omega_2(\omega_2 - 1),$$
$$b_1 = \frac{8\alpha}{1 - k}, \quad b_2 = \frac{4(k+1)}{k - 1}\omega_2 - 2m - 2, \quad c = \frac{4}{(1 - k)^2}\varphi - \lambda\varphi^k.$$

3. Алгебра L_3:

$$f(x) = (x_1^2 + \cdots + x_m^2)^{1/(1-k)}, \quad \omega_1 = \frac{x_1^2 + \cdots + x_m^2}{x_0^2},$$
$$\omega_2 = \frac{x_{m+1}^2 + \cdots + x_q^2}{x_0^2}, \quad a_{11} = 4\omega_1^2(\omega_2 - 1), \quad a_{12} = 8\omega_1^2\omega_2,$$
$$a_{22} = 4\omega_2^2(\omega_2 - 1), \quad b_1 = -\frac{8\omega_1}{1 - k} + (6\omega_1 - 2m)\omega_1,$$
$$b_2 = (6\omega_2 - 2q + 2m)\omega_2, \quad c = \frac{-2m + 2(m - 2)k}{(1 - k)^2}\varphi + \lambda\varphi^k.$$

4. Алгебра L_4:

$$f(x) = (x_{n-2}^2 + x_{n-1}^2)^{1/(1-k)},$$

$$\omega_1 = 2\ln(x_0 - x_n) - (1+\alpha)\ln(x_{n-2}^2 + x_{n-1}^2) - 2c\,\mathrm{arctg}\,\frac{x_{n-1}}{x_{n-2}},$$
$$\omega_2 = \frac{x_0^2 - x_1^2 - \cdots - x_m^2 - x_n^2}{x_{n-2}^2 + x_{n-1}^2}, \quad a_{11} = 4[c^2 + (1+\alpha)^2],$$
$$a_{12} = 8[(1+\alpha)\omega_2 - 1], \quad a_{22} = 4\omega_2(\omega_2 - 1), \quad b_1 = \frac{8(1+\alpha)}{k-1},$$
$$b_2 = \frac{4(k+1)\omega_2 - (2m+4)(k-1)}{k-1}, \quad c = \frac{4}{(k-1)^2}\varphi - \lambda\varphi^k.$$

5. Алгебра L_5:

$$f(x) = (x_0^2 - x_n^2)^{1/(1-k)}, \quad \omega_1 = \frac{x_1^2 + \cdots + x_m^2}{x_0^2 - x_n^2}, \quad \omega_2 = \frac{x_{m+1}^2 + \cdots + x_q^2}{x_0^2 - x_n^2}.$$
$$a_{11} = 4\omega_1(\omega_1 - 1), \quad a_{12} = 8\omega_1\omega_2, \quad a_{22} = 4\omega_2(\omega_2 - 1),$$
$$b_1 = \frac{4(k+1)}{k-1}\omega_1 - 2m, \quad b_2 = \frac{4(k+1)}{k-1}\omega_2 - 2q + 2m, \quad c = \frac{4}{(1-k)^2}\varphi + \lambda\varphi^k.$$

6. Алгебра L_6:

$$f(x) = (x_1^2 + x_2^2)^{1/(1-k)}, \quad \omega_1 = \frac{x_0^2 - x_n^2}{x_1^2 + x_2^2},$$
$$\omega_2 = (1+\alpha)\ln(x_1^2 + x_2^2) - 2\ln(x_0 - x_n) + 2c\,\mathrm{arctg}\,\frac{x_2}{x_1},$$
$$a_{11} = 4\omega_1(1 - \omega_1), \quad a_{12} = 8(\omega_1 - 1), \quad a_{22} = -4[(1+\alpha)^2 + c^2],$$
$$b_1 = 4 + \frac{4(1+k)}{1-k}\omega_1, \quad b_2 = -\frac{8(1+\alpha)}{1-k}, \quad c = \frac{4}{(1-k)^2}\varphi + \lambda\varphi^k.$$

7. Алгебра L_7:

$$f(x) = (x_{d+1}^2 + \cdots + x_m^2)^{1/(1-k)}, \quad \omega_1 = \frac{x_0^2 - x_1^2 - \cdots - x_d^2 - x_n^2}{x_{d+1}^2 + \ldots + x_m^2},$$
$$\omega_2 = 2\ln(x_0 - x_n) - (1+\alpha)\ln(x_{d+1}^2 + \cdots + x_m^2), \quad a_{11} = 4\omega_1(1 - \omega_1),$$
$$a_{12} = 8[1 - (1+\alpha)\omega_1], \quad a_{22} = -4(1+\alpha)^2,$$
$$b_1 = 2(d+2) + 2(m-d-4)\omega_1 + \frac{8\omega_1}{1-k},$$
$$c = -\frac{2(m-d)(1+k)}{(1-k)^2}\varphi + \lambda\varphi^k, \quad b_2 = 2(1+\alpha)(m-d-2) + \frac{8(1+\alpha)}{1-k}.$$

8. Алгебра L_8:

$$f(x) = (x_0^2 - x_n^2)^{1/(1-k)}, \quad \omega_1 = \frac{x_0^2 - x_n^2}{x_1^2 + \cdots + x_m^2},$$
$$\omega_2 = 2\ln(x_0 - x_n) - (1+\alpha)\ln(x_1^2 + \cdots + x_m^2), \quad a_{11} = 4\omega_1^2(1 - \omega_1),$$
$$a_{12} = 8\omega_1[1 - (1+\alpha)\omega_1], \quad a_{22} = -4(1+\alpha)^2\omega_1,$$
$$b_1 = \frac{4(3-k)\omega_1}{1-k} + (2m-8)\omega_1^2, \quad b_2 = \frac{8}{1-k} + 2(1+\alpha)(m+2)\omega_1,$$
$$c = \frac{4}{(1-k)^2}\varphi + \lambda\varphi^k.$$

9. Алгебра L_9:

$$f(x)=(x_2^2+\cdots+x_m^2)^{1/(1-k)},\quad \omega_1=\frac{(x_0-x_n)^2-4x_1}{(x_2^2+\cdots+x_m^2)^{1/2}},$$
$$\omega_2=3[\ln(x_0-x_n)^2-4x_1]-2\ln[6(x_0+x_n)-6x_1(x_0-x_n)-(x_0-x_n)^3],$$
$$a_{11}=16+\omega_1^2,\quad a_{12}=\frac{96}{\omega_1},\quad a_{22}=\frac{144(1-e^{\omega_2})}{\omega_1^2},\quad b_1=\frac{(m-4)k-m}{1-k}\omega_1,$$
$$b_2=-\frac{48+72e^{\omega_2}}{\omega_1^2},\quad c=\frac{(6-2m)k+2m-2}{(1-k)^2}\varphi-\lambda\varphi^k.$$

10. Алгебра L_{10}:

$$f(x)=(x_{d+1}^2+\cdots+x_m^2)^{1/(1-k)},\quad \omega_1=\frac{x_{d+1}^2+\cdots+x_m^2}{x_0-x_n},$$
$$\omega_2=\frac{x_0^2-x_1^2-\cdots-x_d^2-x_n^2}{x_0-x_n}+\ln(x_0-x_n),\quad a_{11}=-4\omega_1^2,\quad a_{12}=-4\omega_1^2,$$
$$a_{22}=-4\omega_1,\quad b_1=-\left[2(m-d)+\frac{8}{1-k}\right]\omega_1,\quad b_2=2d\omega_1,$$
$$c=\frac{2(d-m)+2(m-d-2)k}{(1-k)^2}\varphi+\lambda\varphi^k.$$

11. Алгебра L_{11}:

$$f(x)=(x_1^2+\cdots+x_m^2)^{1/(1-k)},\quad \omega_1=\frac{x_1^2+\cdots+x_m^2}{x_0-x_n},$$
$$\omega_2=x_0+x_n+\ln(x_0-x_n),\quad a_{11}=-4\omega_1^2,\quad a_{12}=-4\omega_1^2,\quad a_{22}=4\omega_1,$$
$$b_1=\frac{4(-2-m+mk)}{1-k}\omega_1,\quad b_2=0,\quad c=\frac{-2m+2k(m-2)}{(1-k)^2}\varphi+\lambda\varphi^k.$$

12. Алгебра L_{12}:

$$f(x)=(x_{n-2}^2+x_{n-1}^2)^{1/(1-k)},\quad \omega_1=\frac{x_{n-2}^2+x_{n-1}^2}{x_0-x_n},$$
$$\omega_2=\frac{x_0^2-x_1^2-\cdots-x_m^2-x_n^2}{x_0-x_n}+\ln(x_0-x_n),\quad a_{11}=4\omega_1^2,\quad a_{12}=4\omega_1^2,$$
$$a_{22}=-4\omega_1,\quad b_1=\frac{4(3-k)\omega_1}{1-k},\quad b_2=-2m\omega_1,\quad c=\frac{4}{(1-k)^2}\varphi-\lambda\varphi^k.$$

13. Алгебра L_{13}:

$$f(x)=(x_{n-2}^2+x_{n-1}^2)^{1/(1-k)},\quad \omega_1=\frac{x_{n-2}^2+x_{n-1}^2}{x_0-x_n},$$
$$\omega_2=\frac{x_0^2-x_1^2-\cdots-x_m^2-x_n^2}{x_0-x_n}+2\operatorname{arctg}\frac{x_{n-1}}{x_{n-2}},\quad a_{11}=-4\omega_1^2,\quad a_{12}=-4\omega_1^2,$$
$$a_{22}=-4\omega_2-4,\quad b_1=\frac{4(-3+k)}{1-k}\omega_1,\quad b_2=0,\quad c=-\frac{4}{(1-k)^2}\varphi+\lambda\varphi^k.$$

14. Алгебра L_{14}:

$$f(x)=(x_1^2+x_2^2)^{1/(1-k)},\quad \omega_1=\frac{x_1^2+x_2^2}{x_0-x_n},$$

$$\omega_2 = x_0 + x_n + \ln(x_0 - x_n) + 2\alpha \operatorname{arctg} \frac{x_2}{x_1}, \quad a_{11} = 4\omega_1^2, \quad a_{12} = 4\omega_1^2,$$
$$a_{22} = 4(\alpha^2 - \omega_1), \quad b_1 = \frac{4(3-k)}{1-k}\omega_1, \quad b_2 = 0, \quad c = \frac{4}{(1-k)^2}\varphi - \lambda\varphi^k.$$

15. Алгебра L_{15}:

$$f(x) = (x_1^2 + x_2^2)^{1/(1-k)}, \quad \omega_1 = \frac{x_1^2 + x_2^2}{x_0 - x_n},$$
$$\omega_2 = x_0 + x_n + 2 \operatorname{arctg} \frac{x_2}{x_1}, \quad a_{11} = 4\omega_1^2, \quad a_{12} = 4\omega_1^2,$$
$$a_{22} = 4, \quad b_1 = \frac{4(3-k)\omega_1}{1-k}\omega_1, \quad b_2 = 0, \quad c = \frac{4}{(1-k)^2}\varphi - \lambda\varphi^k.$$

16. Алгебра L_{16}:

$$f(x) = \left\{-(x_0 + x_n) + \sum_{i=1}^{q-1} \frac{1}{x_0 - x_n + \gamma_i}(x_{(i-1)d+1}^2 + \cdots + x_{id}^2)\right\}^{1/(1-k)},$$
$$\omega_1 = x_0 - x_n, \quad \omega_2 = \frac{y_1^2 + y_2^2 + \cdots + y_d^2}{\{f(x)\}^{1/(1-k)}}.$$

где

$$y_j = \sum_{i=1}^{q-1} \frac{\lambda_i x_{(i-1)d+j}}{x_0 - x_n + \gamma_i} - x_{(q-1)d+j}, \quad j = 1, \ldots, d,$$
$$a_{11} = 0, \quad a_{12} = 4, \quad a_{22} = 4\left\{1 + \sum_{i=1}^{q-1} \frac{\lambda_i^2}{(\omega_1 + \gamma_i)^2}\right\}\omega_2, \quad b_1 = \frac{4}{1-k},$$
$$b_2 = 2d\left[1 - \sum_{i=1}^{q-1} \frac{\omega_2(\omega_1 + \gamma_i) - \lambda_i^2}{(\omega_1 + \gamma_i)^2}\right], \quad c = \frac{2d}{1-k}\sum_{i=1}^{q-1} \frac{1}{\omega_1 + \gamma_i}\varphi - \lambda\varphi^k.$$

17. Алгебра L_{17}:

$$f(x) = \left(x_{d_t+1}^2 + \cdots + x_m^2\right)^{1/(1-k)}, \quad \omega_1 = x_0 - x_n,$$
$$\omega_2 = \frac{-(x_0 + x_n) + \sum\limits_{i=1}^{t} \frac{1}{x_0 - x_n + \gamma_i}(x_{d_{i+1}+1}^2 + \cdots + x_{d_i}^2)}{x_{d_t+1}^2 + \cdots + x_m^2}, \quad a_{11} = 0, \quad a_{12} = 4,$$
$$a_{22} = 4\omega_2^2, \quad b_1 = 0, \quad b_2 = \sum_{i=1}^{t} \frac{4(d_i - d_{i-1})}{\omega_1 + \gamma_i} - \frac{2(m - d_t) - 2(m - d_1 - 4)k}{1-k}\omega_2,$$
$$c = \frac{2(m - d_t) - 2(m - d_t + 2)k}{(1-k)^2}\varphi - \lambda\varphi^k.$$

18. Алгебра L_{18}:

$$f(x) = \left\{\sum_{i=1}^{q-1} \frac{1}{x_0 - x_n + \gamma_i}(x_{(i-1)d+1}^2 + \ldots + x_{id}^2) + \right.$$

$$+\sum_{i=1}^{t}\frac{1}{x_0-x_n+\mu_r}(x_{l_{r-1}}^2+\ldots+x_{l_r}^2)-(x_0-x_n)\Biggr\}^{1/(1-k)},$$

$$\omega_1=x_0-x_n,\quad \omega_2=\frac{y_1^2+y_2^2+\cdots+y_d^2}{f(x)^{1/(1-k)}},$$

где

$$y_j=\sum_{i=1}^{q-1}\frac{\lambda_i x_{(i-1)d+j}}{x_0-x_n+\gamma_i}-x_{(q-1)d+j},\quad j=1,\ldots,d,$$

$$a_{11}=0,\quad a_{12}=4,\quad a_{22}=\left[1+\sum_{i=1}^{q-1}\frac{\lambda_i^2}{(\omega_1+\gamma_i)^2}\right]\omega_2,\quad b_1=\frac{4}{1-k},$$

$$b_2=2d+2d\sum_{i=1}^{q-1}\frac{\lambda_i^2-\omega_2(\omega_1+\gamma_i)}{(\omega_1+\gamma_i)^2}-2\sum_{r=1}^{t}\frac{(l_r-l_{r-1})\omega_2}{\omega_1+\mu_r},$$

$$c=\left[\frac{2d}{1-k}\sum_{i=1}^{q-1}\frac{1}{\omega_i+\gamma_i}+\frac{2}{1-k}\sum_{i=1}^{t}\frac{l_r-l_{r-1}}{\omega_1+\mu_r}\right]\varphi-\lambda\varphi^k.$$

5. Редукция по подалгебрам, содержащим P_0 или P_0+P_n. В настоящем пункте проведем редукцию уравнения (1) к дифференциальным уравнениям с двумя инвариантными переменными, используя подалгебры алгебры $A\tilde{P}(1,n)$, содержащие P_0 или P_0+P_n.

Пусть L — некоторая подалгебра алгебры $A\tilde{P}(1,n)$. Если $P_0\in L$, то любое решение $u=u(x)$ уравнения (1), инвариантное относительно L, не зависит от x_0 и потому является решением уравнения

$$-\Box u+\lambda u^k=0 \tag{5}$$

евклидовом пространстве E_n, где

$$\Box u=\frac{\partial^2 u}{\partial x_1^2}+\cdots+\frac{\partial^2 u}{\partial x_n^2}. \tag{6}$$

Уравнение (5) инвариантно относительно расширенной алгебры Евклида $A\tilde{E}(n)=\langle P_1,\ldots,P_n,J_{12},\ldots J_{n-1,n}\rangle \mathbin{+\!\!\!\!\supset} \langle D_1\rangle$, генераторы которой имеют вид

$$P_a=\partial_a,\quad J_{ab}=x_b\partial_a-x_a\partial_b,\quad D_1=-x^a\partial_a+\frac{2}{k-1}u\partial_u,\quad a,b=1,\ldots,n.$$

Аналогично, если $P_0+P_n\in L$, то любое решение уравнения (1), инвариантно относительно L, имеет вид $u=u(x_0-x_n,x_1,\ldots,x_{n-1})$ и потому является решением уравнения (5) в евклидовом пространстве E_{n-1}. Поэтому в рассматриваемых случаях для редукции уравнения (5), а значит, и уравнения (1) к двумерным уравнениям достаточно классифицировать максимальные подалгебры ранга $n-2$ алгебры $A\tilde{E}(n)$, имеющие ненулевую проекцию на $\langle D_1\rangle$.

Теорема 2. *Максимальные подалгебры ранга $n-2$ алгебры $A\tilde{E}(n)$ имеющие ненулевую проекцию на $\langle D_1\rangle$, исчерпываются с точностью до $\tilde{E}(n)$-сопряженности следующими алгебрами:*

$$L_{19}=(AO[1,d]\oplus AO[d+1,m]\oplus AO[m+1,q]\oplus AE[q+1,n]) \mathbin{+\!\!\!\!\supset} \langle D_1\rangle,$$
$$d=1,\ldots,n-2,\ q=d+1,\ldots,n-1,\ m=q+1,\ldots,n,\ n\geq 3;$$

$$L_{20} = (AO[1,m] \oplus AE[m+1,n-2]) \mathbin{\supset\!\!\!+} \langle D_1 + \alpha J_{n-1,n}\rangle,$$
$$m = 1,\ldots,n-2,\ n\geq 3,\ \alpha>0;$$
$$L_{21} = (K \oplus AE[m+1,n-2]) \mathbin{\supset\!\!\!+} \langle D_1 + \alpha J\rangle,$$

где K *— диагональ в* $AO[1,d]\oplus AO[d+1,2d]$, $J = \sum\limits_{a=1}^{d} J_{a,a+d}$, $d = 2,\ldots,[n/2]$, $m = 2d+1,\ldots,2[n/2]$, $\alpha\geq 0$.

Доказательство аналогично доказательству теоремы 1.

Подалгебрам L_{20}, L_{21} соответствуют следующие анзацы и редуцированные уравнения.

1. Алгебра L_{20}:

$$f(x) = (x_1^2+\cdots+x_d^2)^{1/(1-k)},\quad \omega_1 = \frac{x_1^2+\cdots+x_d^2}{x_{m+1}^2+\cdots+x_q^2},$$
$$\omega_2 = \frac{x_{d+1}^2+\cdots+x_m^2}{x_{m+1}^2+\cdots+x_q^2},\quad a_{11} = 4\omega_1(1-\omega_1),\quad a_{12} = -8\omega_1\omega_2,$$
$$a_{22} = -4\omega_2(1+\omega_2),\quad b_1 = 2(d+1) + \frac{2q-2m-(2q-2m-8)k}{1-k}\omega_1,$$
$$b_2 = 2(d-m) + \frac{2q-2m-(2q-2m-8)k}{1-k}\omega_2,$$
$$c = \frac{2(m-q)+(2q-2m-4)k}{(1-k)^2}\varphi + \lambda\varphi^k.$$

2. Алгебра L_{21}:

$$f(x) = (x_{n-1}^2+x_n^2)^{1/(1-k)},\quad \omega_1 = \alpha\ln(x_{n-1}^2+x_n^2) - 2\,\mathrm{arctg}\,\frac{x_n}{x_{n-1}},$$
$$\omega_2 = \frac{x_1^2+\cdots+x_m^2}{x_{n-1}^2+x_n^2},\quad a_{11} = 4(\alpha^2+1),\quad a_{12} = -8\alpha\omega_2,\quad a_{22} = -4\omega_2(\omega_2-1),$$
$$b_1 = \frac{8\alpha}{1-k},\quad b_2 = \frac{4(k+1)}{k-1}\omega_2 - 2m - 2,\quad c = \frac{4}{(1-k)^2}\varphi - \lambda\varphi^k.$$

1. Fushchych W.I., Serov N.I., The symmetry and some exact solution of the nonlinear multi-dimensional Liouville, d'Alembert and eikonal equations, *J. Phys. A.: Math. Gen.*, 1983, **16**, № 15, 3645–3656.
2. Fushchych W., Sthelen W., Serov N., Symmetry analysis and exact solutions of equations of nonlinear mathematical physics, Dordrecht, Kluwer Publ., 1993.
3. Grundland A.M., Harnad J., Winternitz P., Symmetry reduction for nonlinear relativistically equations, *J. Math. Phys.*, 1984, **25**, № 4, 791–806.
4. Фущич В.И., Баранник А.Ф., Максимальные подалгебры ранга $n-1$ алгебры $AP(1,n)$ и редукция нелинейных волновых уравнений. I, *Укр. мат. журн.*, 1990, **42**, № 11, 1552–1559.
5. Фущич В.И., Баранник А.Ф., Максимальные подалгебры ранга $n-1$ алгебры $AP(1,n)$ и редукция нелинейных волновых уравнений, II, *Укр. мат. журн.*, 1990, **42**, № 12, 1693–1700.
6. Фущич В.И., Баранник Л.Ф., Баранник А.Ф., Подгрупповой анализ групп Галилея, Пуанкаре и редукция нелинейных уравнений, Киев, Наук. думка, 1991, 304 с.
7. Баранник А.Ф., Баранник Л.Ф., Фущич В.И., Редукция многомерного пуанкаре-инвариантного нелинейного уравнения к двумерным уравнениям, *Укр. мат. журн.*, 1991, **43**, № 10, 1311–1323.

Q-symmetry generators and exact solutions for nonlinear heat conduction

N. EULER, A. KÖHLER, W.I. FUSHCHYCH

We investigate conditional invariance by considering Q-symmetry generators of the nonlinear heat equation $\partial u/\partial x_0 - \lambda\partial^2 u/\partial x_1^2 = f(u)$, where λ is a real constant and f an arbitrary differentiable function. With the obtained Q-generators we construct exact solutions by the use of similarity ansatze and reductions to ordinary differential equations. A generalization to m-space dimensions is performed.

1. Introduction

Most nonlinear partial differential equations are not integrable and cannot be treated via the inverse scattering transform, nor its generalization. Such equations are mostly treated by numerical methods. Interesting qualitative and quantitative features are however often missed in this manner and it is of great value to be able to obtain exact analytic solutions of nonintegrable equations. The application of Lie transformation groups, whereby a transformation is obtained that leaves the differential equation invariant, is useful in finding exact solutions (see [1–8]). If an equation is invariant under some Lie transformation group, the equation is said to have a symmetry. It is known that the integrability and the existence of symmetries is connected. This was studied in connection with the Painlevé test (see [1–3]). Many important nonintegrable partial differential equations have no significant symmetries. In this article we consider conditional symmetries of partial differential equations as introduced in [9–13]. We make use of these conditional symmetries to obtain exact solutions. The following equation is studied:

$$\frac{\partial u}{\partial x_0} - \lambda\frac{\partial^2 u}{\partial x_1^2} = f(u), \tag{1}$$

where x_0 indicates time, λ is a real constant, and f an arbitrary differentiable function. For nonlinear functions f this equation plays an important role in nonlinear heat transfer processes.

Before we consider conditional symmetries of (1), let us briefly describe the classical Lie approach and introduce our notation [1]. We are concerned with a partial differential equation of order r with $m+1$ independent variables $(x_0, x_1, \ldots, x_m)$ and one field variable u, i.e. an equation of the form

$$F\left(x_0, \ldots, x_m, u, \frac{\partial_u}{\partial x_0}, \ldots, \frac{\partial^r u}{\partial x_{j_1}\cdots\partial x_{j_r}}\right) = 0, \tag{2}$$

where $0 \le j_1 \le j_2 \le \cdots \le j_r \le m$, $j = 0, \ldots, m$. The submanifold $\mathbb{R}^r$ of the r-jet bundle $J^r(M,1)$ is determined by the constrained equation

$$F(x_0, \ldots, x_m, u, u_0, \ldots, u_{j_1\cdots j_r}) = 0, \tag{3}$$

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where the dimension of the differential manifold M is m. A Lie transformation group that leaves (3) invariant is generated by a Lie (point) symmetry generator Z, defined by

$$Z = \sum_{j=0}^{m} \xi_j(x_0, \ldots, x_m, u)\frac{\partial}{\partial x_j} + \eta(x_0, \ldots, x_m, u)\frac{\partial}{\partial u}. \tag{4}$$

Z_{v} is the associated vertical form of (4) on $J^1(M,1)$, defined by

$$Z_{\mathrm{v}} = \left(\eta - \sum_{j=0}^{m} \xi_j u_j\right)\frac{\partial}{\partial u}, \tag{5}$$

where $Z_{\mathrm{v}}\rfloor\theta = Z\rfloor\theta$. Here θ is a differential 1-form, called the contact form on $J^1(M,1)$, defined by

$$\theta = du - \sum_{j=0}^{m} u_j dx_j$$

with $js^*\theta = 0$. Here js^* denotes the pull-back map. Equation (3) is called invariant under the prolonged Lie symmetry generator $\bar{Z}_{\mathrm{v}}$ if

$$L_{\bar{Z}_{\mathrm{v}}} F \,\hat{=}\, 0, \tag{6}$$

where $\hat{=}$ indicates the restriction to solutions of (3) and its prolongations. L denotes the Lie derivative. $\bar{Z}_{\mathrm{v}}$ is found by prolonging the vertical generator Z_{v}, i.e.,

$$\bar{Z}_{\mathrm{v}} = U\frac{\partial}{\partial u} = \sum_{j=0}^{m} D_j(U)\frac{\partial}{\partial u_j} + \cdots + \sum_{j_1,\ldots,j_r=0}^{m} D_{j_1\cdots j_r}(U)\frac{\partial}{\partial u_{j_1\cdots j_r}} + \cdots,$$

where

$$U = \eta - \sum_{j=0}^{m} \xi_j u_j \equiv Z_{\mathrm{v}}\rfloor\theta$$

and D_j is the total derivative operator. A similarity ansatz for (2) is obtained by solving the linear partial differential equation

$$js^*(Z_{\mathrm{v}}\rfloor\theta) = 0, \tag{7}$$

with Z_{v} an associated vertical Lie symmetry generator for the equation. This ansatz will reduce the dimension of (2) by one. The solution of the reduced equation is known as a similarity solution of (2). Thus, the existence of a symmetry provides us with a similarity ansatz and a possible exact solution can be calculated. The converse is however not and true, i.e., any exact solution of a partial differential equation is not associated with a symmetry of the equation. For such solutions one can introduce conditional symmetries, i.e. symmetries that leave the equation invariant under some additional condition.

2. Q-symmetry generators

Following [9–13] we give the definition for conditional invariance of (2).

Definition. *Equation (3) is called Q-conditionally invariant if*

$$L_{\bar{Q}_{\rm v}} F \,\hat{\hat{=}}\, 0 \tag{8}$$

under the condition

$$Q_{\rm v} \rfloor \theta = 0. \tag{9}$$

Q is called the Q-symmetry generator and $\bar{Q}_{\rm v}$ the prolonged vertical Q-symmetry generator.

Here $\hat{\hat{=}}$ indicates the restriction to solution of $F = 0$ and (9) together with their prolongations. Q is considered in the form of a Lie symmetry generator.

Let us now study (1) by the used of the above definition. We are interested only in nonlinear functions f. From the definition it follows that the Lie derivative (8), for the equation

$$F \equiv u_0 - \lambda u_{11} - f(u) = 0 \tag{10}$$

under the condition

$$Q_{\rm v} \rfloor \theta \equiv \eta - \xi_0 u_0 - \xi_1 u_1 = 0, \tag{11}$$

has to be studied. Let us consider the Q-symmetry generator in the form

$$Q = c\frac{\partial}{\partial x_0} + \xi_1(u)\frac{\partial}{\partial x_1} + \eta(u)\frac{\partial}{\partial u}, \tag{12}$$

where c is an arbitrary real constant. We can state the following

Theorem 1. *The generator*

$$Q = k_1\frac{\partial}{\partial x_1} + \eta(u)\frac{\partial}{\partial u} \tag{13}$$

is a Q-symmetry generator for (1) if an only if

$$f(u) = \eta(u)\left(-\frac{\lambda}{k_1^2}\frac{d\eta}{du} + c_1\right), \tag{14}$$

where η is an arbitrary differentiate function of u and k_1, c_1 are arbitrary real constants.

Proof. By applying the Lie derivative (8) and condition (9), with generator (13), we obtain the following determining equations using computer algebra [15, 16]:

$$c^3\lambda\frac{d^2\xi_1}{du^2} = 0, \quad -3c\lambda\frac{d^2\xi_1}{du^2}\eta^2 + (3cf + 2\eta)\frac{d\xi_1}{du}\xi_1^2 + 2c\lambda\frac{d^2\eta}{du^2}\xi_1\eta = 0,$$
$$c\left(3c\lambda\frac{d^2\xi_1}{du^2}\eta - 2\frac{d\xi_1}{du}\xi_1^2 - c\lambda\frac{d^2\eta}{du^2}\xi_1\right) = 0$$

and

$$\lambda\frac{d^2\xi_1}{du^2}\eta^3 - 3f\frac{d\xi_1}{du}\xi_1^2\eta + f\frac{d\eta}{du}\xi_1^3 - \lambda\frac{d^2\eta}{du^2}\xi_1\eta^2 - \frac{df}{du}\xi_1^3\eta = 0.$$

For $c \neq 0$ the general solution of the above four equations gives only a linear function $f(u) = a_1 u + a_2$, where a_1 and a_2 are arbitrary real constants. For $c = 0$ the general solution

$$\xi_1(u) = k_1, \quad f(u) = \eta(u)\left(-\frac{\lambda}{k_1^2}\frac{d\eta}{du} + c_1\right),$$

follows. ■

Consider the Q-symmetry generator in the form

$$Q = \xi_0(u)\frac{\partial}{\partial x_0} + c\frac{\partial}{\partial x_1} + \eta(u)\frac{\partial}{\partial u}, \tag{15}$$

where c is an arbitrary constant. We can state the following

Theorem 2. *The generator*

$$Q = \frac{k_2}{u+k_1}\frac{\partial}{\partial x_0} + c\frac{\partial}{\partial x_1} + \left[-\frac{2}{3}\frac{c^2}{\lambda k_2}\left(\frac{1}{2}u^2 + k_1 u\right) - \frac{k_3}{u+k_1} + k_4\right]\frac{\partial}{\partial u} \tag{16}$$

is a Q-symmetry generator for (1) if an only if

$$f(u) = \frac{2}{3}\frac{\eta}{\xi_0}, \tag{17}$$

where

$$\xi_0 = \frac{k_2}{u+k_1}$$

and

$$\eta = -\frac{2}{3}\frac{c^2}{\lambda k_2}\left(\frac{1}{2}u^2 + k_1 u\right) - \frac{k_3}{u+k_1} + k_4.$$

Here $k_1, \ldots, k_4$ are arbitrary real constants.

Proof. Applying the Lie derivative (8) and condition (9), with generator (15), the determining equations are given by

$$c(3f\xi_0 - 2\eta) = 0,$$
$$\lambda\frac{d^2\xi_0}{du^2}\xi_0\eta - 2\lambda\left(\frac{d\xi_0}{du}\right)^2\eta + 2\lambda\frac{d\xi_0}{du}\frac{d\eta}{du}\xi_0 + 2c^2\frac{d\xi_0}{du} - \lambda\frac{d^2\eta}{du^2}\xi_0^2 = 0,$$
$$c\left[-\frac{d^2\xi_0}{du^2}\xi_0 + 2\left(\frac{d\xi_0}{du}\right)^2\right] = 0$$

and

$$-f\frac{d\xi_0}{du}\eta + f\frac{d\eta}{du}\xi_0 - \frac{df}{du}\xi_0\eta = 0.$$

For $c = 0$ only linear functions f are obtained for the general solution of the above system. If $c \neq 0$, f follows from the first determining equation and the condition on η is in the form of a linear second order equation, namely

$$\frac{d^2\eta}{du^2} + \frac{2}{u+k_1}\frac{d\eta}{du} + \frac{2c^2}{\lambda k^2} = 0.$$

The general solution, given in theorem 2, follows. ■

For the nonlinear equation

$$\frac{\partial u}{\partial x_0} + \frac{\partial^2 u}{\partial x_1^2} = au^k, \tag{18}$$

with a and k arbitrary real constants, and $k \neq 1$, we can state the following

Theorem 3. *The generator*

$$Q = \frac{\partial}{\partial x_0} + \xi_1(x_0, x_1)\frac{\partial}{\partial x_1} + \alpha(x_0, x_1)u\frac{\partial}{\partial u}, \tag{19}$$

is a Q-symmetry generator for (18) if and only if the following conditions on ξ_1 and α are satisfied:

$$\frac{\partial \alpha}{\partial x_0} + \frac{\partial^2 \alpha}{\partial x_1^2} = (k-1)\alpha^2, \tag{20}$$

$$\frac{\partial \xi_1}{\partial x_0} - (k-1)\alpha\xi_1 = \frac{k+3}{2}\frac{\partial \alpha}{\partial x_1} \tag{21}$$

and

$$\frac{\partial \xi_1}{\partial x_1} = \frac{1-k}{2}\alpha. \tag{22}$$

The proof follows directly from the invariance condition (8) together with (9). Note that the above condition on ξ_1 reduces to the following third order ordinary differential equation:

$$\frac{2}{k-1}\frac{d^3\xi_1}{dx_1^3} + \xi_1\frac{d^2\xi_1}{dx_1^2} = 0,$$

which can be transformed to the Abel equation of the second kind

$$xy\frac{dy}{dx} + y^2 + \left(7x + \frac{k-1}{2}\right)y + 6x^2 + (k-1)x = 0$$

where

$$\frac{\xi_1}{x_1} = P(\xi_1), \quad P(\xi_1) = \xi_1^2 x(z), \quad z = \ln(\xi_1), \quad \frac{dx}{dz} = y(x).$$

With other special ansätze for ξ_0, ξ_1 and η we obtain the following results for (1) with $\lambda = -1$.

Theorem 4. 1. *The generator*

$$Q = 2\sqrt{x_0}\frac{\partial}{\partial x_1} + f(u)\frac{\partial}{\partial u} \tag{23}$$

is a Q-symmetry generator for (1) ($\lambda = -1$)*, if and only if f satisfies the equation*

$$f\frac{d^2 f}{du^2} = 2. \tag{24}$$

The general solution of (24) is given by

$$\pm \int \frac{df}{\sqrt{4 \ln f + \tilde{k}_1}} = u + \tilde{k}_2,$$

where $\tilde{k}_1$ *and* $\tilde{k}_2$ *are integrating constants.*

2. *The generator*

$$Q = x_1 \frac{\partial}{\partial x_1} + f(u) \frac{\partial}{\partial u} \tag{25}$$

is a Q-symmetry generator for (1) $(\lambda = -1)$, *if and only if* f *satisfies the equation*

$$f \frac{d^2 f}{du^2} = 2 \left(\frac{df}{du} - 1 \right). \tag{26}$$

The general solution of (26) is given by

$$f(u) = \pm \sqrt{(w-1) \exp(w)/\tilde{k}_1},$$

where w *is obtained from*

$$\pm \int \sqrt{\tilde{k}_1^{-1} (w-1)^{-1} \exp(w)} dw = u + \tilde{k}_2.$$

Here $\tilde{k}_1$ *and* $\tilde{k}_2$ *are integrating constants.*

The proof follows by applying the invariance condition (8) together with (9).

Let us make some remarks on Q-symmetries. The determining equations for Q-generators are nonlinear over-determined systems of differential equations. This is in contrast to Lie symmetry generators where the determining equations are linear differential equations. It is obvious that every Lie symmetry of an equation is also a Q-symmetry but that the converse is not true, so that the above Q-symmetries do not generate Lie transformation groups that leave the equation invariant. If we multiply a Q-symmetry (or Lie symmetry) of a particular equation by an arbitrary function, we again find a Q-symmetry for that equation.

3. Q-similarity solutions

Let us now make use of theorems 1 to 4 to construct exact solutions of (1). The similarity ansatz is obtained by solving the linear partial differential equation

$$js^*(Q \rfloor \theta) \equiv \xi_0 \frac{\partial u}{\partial x_0} + \xi_1 \frac{\partial u}{\partial x_1} - \eta = 0. \tag{27}$$

We seek the general solution of (27) in the form

$$\psi(x_0, x_1, u) = \phi\{\omega[x_0, x_1, u(x_0, x_1)]\},$$

where ψ is an arbitrary function of its arguments and ϕ is an arbitrary function of the similarity variable ω. We call solutions, obtained by Q-symmetries, the Q-similarity solutions.

Let us consider the following cases:

Case 1(a): Consider the equation

$$\frac{\partial u}{\partial x_0} - \lambda \frac{\partial^2 u}{\partial x_1^2} = -\frac{\lambda}{k_1^2}\exp(2u) + k_2 \exp(u). \tag{28}$$

This corresponds to $\eta = \exp(u)$ for the Q-symmetry given in theorem 1. By solving (27) for the Q-symmetry

$$Q = k_1 \frac{\partial}{\partial x_1} + \exp(u)\frac{\partial}{\partial u}$$

we obtain the similarity ansatz

$$u = -\ln\left(\phi(\omega) - \frac{x_1}{k_1}\right) \quad \text{and} \quad \omega = x_0.$$

On insertion into (28) we obtain the reduced equation

$$\frac{d\phi}{d\omega} - k_2 = 0. \tag{29}$$

An exact solution of (28) is thus given by

$$u(x_0, x_1) = -\ln\left(-\frac{x_1}{k_1} + k_2 x_0 + k_3\right),$$

where k_1, k_2, k_3 are arbitrary real constants.

Case 1(b): Consider the equation

$$\frac{\partial u}{\partial x_0} - \lambda \frac{\partial^2 u}{\partial x_1^2} = -(b_5 u^5 + b_3 u^3 + b_1 u). \tag{30}$$

This corresponds to $\eta = a_3 u^3 + a_1 u$ for the Q-symmetry given in theorem 1 with $k_1 = 1$ and

$$a_1 = \frac{1}{6\lambda}\left(b_3\sqrt{\frac{3\lambda}{b_5}} + \sqrt{\frac{3\lambda b_3^2}{b_5} - 12 b_1 \lambda}\right), \quad a_3 = \sqrt{\frac{b_5}{3\lambda}},$$

$$c_1 = -\frac{1}{3}\sqrt{\frac{3\lambda b_3^2}{b_5}} + \frac{2}{3}\left(\sqrt{\frac{3 b_3^2 \lambda}{b_5} - 12 b_1 \lambda}\right).$$

By solving (27) for the Q-symmetry

$$Q = k_1 \frac{\partial}{\partial x_1} + (a_3 u^3 + a_1 u)\frac{\partial}{\partial u},$$

we obtain the similarity ansatz

$$u = \frac{\sqrt{a_1}}{\phi(\omega)}\exp(a_1 x_1/c)\sqrt{1 - a_3 \phi^2(\omega)\exp(2 a_1 x_1/c)} \quad \text{and} \quad \omega = x_0.$$

The reduced equation is given by

$$\frac{d\phi}{d\omega} - a_1 c_1 \phi = 0, \tag{31}$$

with the general solution

$$\phi(\omega) = \tilde{c}\exp(a_1 c_1 \omega).$$

Here $\tilde{c}$ is an arbitrary real constant. An exact solution of (30) is thus given by

$$u = \frac{\tilde{c}\sqrt{a_1}\exp[a_1(x_1 + c_1 x_0)]}{\sqrt{1 - a_3\tilde{c}^2\exp[2a_1(x_1 + c_1 x_0)]}}.$$

Case 2: Consider the equation

$$\frac{\partial u}{\partial x_0} - \lambda\frac{\partial^2 u}{\partial x_1^2} = -b_3 u^3 - b_2 u^2 + b_1 u + b_0. \tag{32}$$

This corresponds to $\eta(u) = q_3 u^2 + q_2 u + q_1$ for the Q-symmetry in theorem 2. Here

$$q_3 = -c\sqrt{\frac{b_3}{2\lambda}}, \quad q_2 = -\frac{c}{3}\sqrt{\frac{2}{\lambda b_3}}b_2, \quad q_1 = \frac{c}{\sqrt{\lambda b_3}}\left(\frac{b_1}{\sqrt{2}} + \frac{\sqrt{2}b_2^2}{9b_3}\right).$$

The real constants b_1, b_2, b_3 are related to the constants k_1, k_2, k_3 and k_4 in the Q-symmetry given in theorem 2 by the relations

$$k_1 = \frac{b_2}{3b_3}, \quad k_2 = \frac{c}{3}\sqrt{\frac{2}{\lambda b_3}}, \quad k_3 = 0, \quad k_4 = \frac{b_1 c}{\sqrt{2\lambda b_3}} + \frac{2cb_2^2}{9\sqrt{2\lambda b_3}b_3},$$

where

$$b_0 = \frac{b_1 b_2}{3b_3} + \frac{2b_2^3}{27b_3^2}.$$

In terms of b_2 and b_3, ξ_0 is given by

$$\xi_0(u) = \frac{c\sqrt{2b_3}}{\sqrt{\lambda}(3b_3 u + b_2)}.$$

In order to solve (27) for the above given ξ_0 and η we must solve the equation

$$\frac{d^2 y}{d\varepsilon^2} - q_2\frac{dy}{d\varepsilon} + q_3 q_1 y = 0,$$

where

$$u = -\frac{1}{q_3}\frac{d}{d\varepsilon}(\ln y).$$

ε is the group parameter. Thus, there are three cases to be studied:

$$q_2^2 - 4q_1 q_3 = 0, \quad q_2^2 - 4q_1 q_3 < 0, \quad q_2^2 - 4q_1 q_3 > 0.$$

Case 2(a): Consider $q_2^2 - 4q_1 q_3 = 0$, i.e.,

$$b_1 = -\frac{1}{3}\frac{b_2^2}{b_3}.$$

The similarity ansatz is given by

$$u = \frac{b_2[x_1 - \phi(\omega)] - 3\sqrt{2b_3\lambda}}{3b_3[\phi(\omega) - x_1]} \quad \text{and} \quad \omega = x_0 - \frac{3b_3}{(3ub_3 + b_2)^2}.$$

The reduced equation then takes the form

$$\frac{d^2\phi}{d\omega^2} - \frac{1}{3\lambda}\left(\frac{d\phi}{d\omega}\right)^3 = 0, \tag{33}$$

which has the general solution

$$\phi(\omega) = -3\lambda\sqrt{-\frac{2}{3\lambda}\omega + \tilde{c}_1} + \tilde{c}_2.$$

Here $\tilde{c}_1$, $\tilde{c}_2$ are arbitrary real constants. Solving for u, an exact solution of (32) takes the form

$$u = \frac{6\sqrt{2b_3\lambda}(x_1 - \tilde{c}_2) - b_2(x_1^2 - 2\tilde{c}_2x_1 - 9\lambda^2\tilde{c}_1 + 6\lambda x_0 + \tilde{c}_2^2)}{3b_3(x_1^2 - 2\tilde{c}_2x_1 - 9\lambda^2\tilde{c}_1 + 6\lambda x_0 + \tilde{c}_2^2)}.$$

Case 2(b): Consider $q_2^2 - 4q_1q_3 < 0$, i.e.

$$b_1 + \frac{b_2^2}{3b_3} < 0.$$

The similarity ansatz is then given by

$$u = -\frac{\beta}{q_3}\frac{1}{\tan\{\beta[x_1 - \phi(\omega)]/c\}} - \frac{\alpha}{q_3},$$
$$\omega = x_0 + \frac{c^2}{3\lambda\beta^2}\ln\left\{\left[1 + \frac{\beta^2}{(q_3u + \alpha)^2}\right]^{-1/2}\right\},$$

where

$$\alpha = \frac{q_2}{2}, \quad \beta = \frac{1}{2}\sqrt{4q_1q_3 - q_2^2}.$$

The reduced equation takes the form

$$A\frac{d^2\phi}{d\omega^2} + B\frac{d\phi}{d\omega} + C\left(\frac{d\phi}{d\omega}\right)^3 = 0, \tag{34}$$

where

$$A = 6b_3(81b_1^4b_3^4 + 108b_1^3b_2^2b_3^3 + 54b_1^2b_2^4b_3^2 + 12b_1b_2^6b_3 + b_2^8),$$
$$B = 9b_1b_3(81b_1^4b_3^4 + 135b_1^3b_2^2b_3^3 + 90b_1^2b_2^4b_3^2 + 30b_1b_2^6b_3 + 5b_2^8) + 3b_1^{10},$$

and

$$C = -\frac{1}{3}A.$$

The general solution of (34) is

$$\phi(\omega) = \frac{A}{B}\sqrt{\frac{B}{C}}\arctan\sqrt{\exp(2B\omega/A) - C\tilde{c}_1/A} + \tilde{c}_2,$$

where $\tilde{c}_1$, $\tilde{c}_2$ are arbitrary real constants. An implicit solution of (32) can then be given in the form

$$\frac{A}{B}\sqrt{\frac{B}{C}}\arctan\sqrt{\exp\left(\frac{2Bx_0}{A}\right)\left[1+\frac{\beta^2}{(q_3u+\alpha)^2}\right]^{-c^2B/(2A\beta^2\lambda)}-\frac{C\tilde{c}_1}{A}}+$$
$$+\frac{c}{\beta}\arctan\left(-\frac{\beta}{q_3u}\right)=x_1-\tilde{c}_2.$$

Case 2(c): Consider $q_2^2-4q_1q_1>0$, i.e.

$$b_1+\frac{b_2^2}{3b_3}>0.$$

The similarity ansatz is then given by

$$u=\frac{\phi(\omega)\exp(A_1+A_2)(q_2+\sqrt{\Delta})-4q_3\sqrt{\Delta}b_2}{12q_3\sqrt{\Delta}b_3-2q_3\phi(\omega)\exp(A_1+A_2)}$$

and

$$\omega=-\frac{2uq_3+q_2-\sqrt{\Delta}}{2uq_3+q_2+\sqrt{\Delta}}\exp(-x_1\sqrt{\Delta}/c),$$

where

$$A_1=\frac{x_0\sqrt{\Delta}}{cq_3\sqrt{2b_3}}\left(-b_2q_2q_3+3b_3q_1q_3+\frac{b_2^2q_3^2}{3b_3}\right),$$
$$A_2=\frac{x_1}{c}\left(\frac{q_2}{2}+\frac{\sqrt{\Delta}}{2}-\frac{b_2q_3}{3b_3}\right)$$

and

$$\Delta=\frac{2c^2}{\lambda}\left(\frac{b_2^2}{3b_3}+b_1\right).$$

The reduced equation is given by

$$\frac{d^2\phi}{d\omega^2}=0 \tag{35}$$

so that the two nontrivial exact solution of (32) take the form

$$u=[\pm 2\{6S_1S_2\sqrt{2S_2}c\tilde{c}_1\exp[(3\sqrt{2S_2}x_1b_3+S_1S_2x_0)/(2S_1b_3)]+$$
$$+3S_2\lambda\tilde{c}_1^2\exp[(\sqrt{2S_2}x_1b_3+S_1S_2x_0)/(S_1b_3)]+$$
$$+18S_2^2b_3c^2\exp[(2\sqrt{2S_2}x_1/S_1]\}^{1/2}+$$
$$+2\sqrt{\lambda}\tilde{c}_2(b_2-\sqrt{3S_2})\exp[(3\sqrt{2S_2}x_1b_3+S_1S_2x_0)/(2S_1b_3)]-$$
$$-2\sqrt{6S_2b_3}c(2b_2-\sqrt{3S_2})\exp[\sqrt{2S_2}x_1/S_1]-$$
$$-2\sqrt{\lambda}b_2\tilde{c}_1\exp[(\sqrt{2S_2}x_1b_3+S_1S_2x_0)/(2S_1b_3)]\times$$
$$\times\frac{1}{6b_3}[2\sqrt{6S_2b_3}c\exp[\sqrt{2S_2}x_1/S_1]+\sqrt{\lambda}\tilde{c}_1\times$$
$$\times\exp[(\sqrt{2S_2}x_1b_3+S_1S_2x_0)/(2S_1b_3)]-$$
$$-\sqrt{\lambda}\tilde{c}_1\exp[(3\sqrt{2S_2}x_1b_3+S_1S_2x_0)/(2S_1b_3)]^{-1},$$

where

$$S_1 = \sqrt{3\lambda b_3}, \quad S_2 = 3b_1 b_3 + b_2^2.$$

Note. The case $\lambda = -1$ and $f(u) = \tilde{b}_3 u^3 + \tilde{b}_1 u + \tilde{b}_0$, i.e. the equation

$$\frac{\partial u}{\partial x_0} + \frac{\partial^2 u}{\partial x_1^2} = \tilde{b}_3 u^3 + \tilde{b}_1 u + \tilde{b}_0 \tag{36}$$

has been studied by Fushchych et al. [14]. This case can be obtained from theorem 2 by considering

$$k_1 = 0, \quad k_2 = \frac{c}{3}\sqrt{\frac{2}{\tilde{b}_3}}, \quad k_3 = -\frac{c}{2}\tilde{b}_0\sqrt{\frac{2}{\tilde{b}_3}}, \quad k_4 = \frac{c}{2}\tilde{b}_1\sqrt{\frac{2}{\tilde{b}_3}},$$

i.e.

$$Q = \frac{\partial}{\partial x_0} + \frac{3}{2}\sqrt{2\tilde{b}_3}u\frac{\partial}{\partial x_1} + \frac{3}{2}(\tilde{b}_3 u^3 + \tilde{b}_1 u + \tilde{b}_0)\frac{\partial}{\partial u}.$$

Case 3: Consider the equation

$$\frac{\partial}{\partial x_0} + \frac{\partial^2 u}{\partial x_1^2} = au^3. \tag{37}$$

From theorem 3, with $\alpha = x_1^{-2}$, it follows that

$$Q = x_1^2\frac{\partial}{\partial x_0} + 3x_1\frac{\partial}{\partial x_1} + 3u\frac{\partial}{\partial u}.$$

The similarity ansatz is then given by

$$u = x_1\phi(\omega), \quad \omega = x_0 - \frac{1}{6}x_1^2$$

so that the reduced equation takes the form

$$\frac{d^2\phi}{d\omega^2} = 9a\phi^3. \tag{38}$$

The general solution, in terms of an elliptic integral, is given by

$$\int_0^{\phi} \frac{d\tau}{\sqrt{c_1 + \tau^4}} = \frac{3}{2}\sqrt{3a}(\omega + c_2)$$

so that a solution of (37) can be given in the form

$$\int_0^{u/x_1} \frac{d\tau}{\sqrt{c_1 + \tau^4}} = \frac{3}{2}\sqrt{2a}\left(x_0 - \frac{1}{6}x_1^2 + c_2\right).$$

Case 4(a): From theorem 4.1 we obtain the implicit ansatz

$$\frac{df}{du} = \phi(x_0) + \frac{x_1}{\sqrt{x_0}}$$

for (1) where f satisfies (26) and $\lambda = -1$. The reduced equation takes the following form

$$\frac{d\phi}{dx_0} + \frac{1}{2x_0}\phi - 2 = 0 \tag{39}$$

so that an exact solution of the nonlinear partial differential equation is

$$\frac{df}{du} = \frac{x_1}{\sqrt{x_0}} + \frac{4}{3}x_0.$$

Case 4(b): From theorem 4.2 we obtain the implicit ansatz

$$\frac{df}{du} = x_1^2\phi(x_0) + 1$$

for (1) where f satisfies (26) and $\lambda = -1$. The reduced equation takes the following form

$$\frac{d\phi}{dx_0} - 2\phi + 2\phi^2 = 0 \tag{40}$$

so that an exact solution of the nonlinear partial differential ansatz equation is given by

$$\frac{df}{du} = \frac{x_1^2}{1 + c_1\exp(-2x_0)} + 1$$

4. Generalization to $m+1$ dimensions

For a generalization for m space dimensions we consider the equations

$$\frac{\partial u}{\partial x_0} + \frac{1}{2n}\Delta u = au^3, \tag{41}$$

$$\frac{\partial u}{\partial x_0} + \frac{1}{2n}\Delta u = f(u), \tag{42}$$

where a and n are real constants, f satisfies

$$f\frac{d^2 f}{du^2} = 2, \quad \text{and} \quad \Delta \equiv \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{2x_m^2}.$$

An exact solution for (41) is found to be

$$u = \frac{2\boldsymbol{\beta}\cdot\boldsymbol{x}}{3ax_0 - (\boldsymbol{\beta}\cdot\boldsymbol{x})^2},$$

where $\boldsymbol{\beta}\cdot\boldsymbol{x} = \sum\limits_{j=1}^{m}\beta_j x_j$, etc., with β_j arbitrary real constants.

This solution is obtained from the Q-symmetries

$$Q_j = 2\rho_j\frac{\partial}{\partial x_0} + 3au\frac{\partial}{\partial x_j} + 3a\rho_j u^3\frac{\partial}{\partial u},$$

where $\boldsymbol{\rho}^2 = an$ and $j = 1, \ldots, m$. This leads to the ansatz

$$u = \frac{2}{\phi(\omega) - 2\boldsymbol{\beta}\cdot\boldsymbol{x}}, \quad \omega = -\frac{1}{u^2} - 2ax_0$$

from which the reduced equation

$$2\frac{d^2\phi}{d\omega^2} = \left(\frac{d\phi}{d\omega}\right)^3$$

follows. From the Q-symmetry

$$Q_j = \alpha_j(\boldsymbol{\alpha}\cdot\boldsymbol{x})^2\frac{\partial}{\partial x_0} + 3\boldsymbol{\alpha}\cdot\boldsymbol{x}\frac{\partial}{\partial x_j} + 3\alpha_j u\frac{\partial}{\partial u}$$

with $\boldsymbol{\alpha}^2 = 1$ and $j = 1, \ldots, m$, the ansatz

$$u = \boldsymbol{\alpha}\cdot\boldsymbol{x}\phi(\omega), \quad \omega = x_0 - \frac{1}{6}(\boldsymbol{\alpha}\cdot\boldsymbol{x})^2$$

reduced (41) to the equation

$$\frac{d^2\phi}{d\omega^2} = 9a\phi^3. \tag{43}$$

An exact solution for (41) is then given by

$$\int_0^{u/(\boldsymbol{\alpha}\cdot\boldsymbol{x})}\frac{d\tau}{\sqrt{c_1+\tau^4}} = \frac{3}{2}\sqrt{2a}\left[x_0 - \frac{1}{6}(\boldsymbol{\alpha}\cdot\boldsymbol{x})^2 + c_2\right].$$

For (42) we obtain the Q-symmetries

$$Q_j = 2\sqrt{x_0}\frac{\partial}{\partial x_0} + \gamma_j f(u)\frac{\partial}{\partial u},$$

where $\boldsymbol{\gamma}^2 = 2n$ and $j = 1, \ldots, m$. This leads to the implicit ansatz

$$\frac{df}{du} = \frac{\boldsymbol{\gamma}\cdot\boldsymbol{x}}{\sqrt{x_0}} + \phi(x_0),$$

and the reduces equation

$$\frac{d\phi}{dx_0} + \frac{\phi}{2x_0} - 2 = 0. \tag{44}$$

An exact solution of (42) is then given by

$$\frac{df}{du} = \frac{\boldsymbol{\gamma}\cdot\boldsymbol{x} + c_1}{\sqrt{x_0}} + \frac{4}{3}(x_0).$$

5. Concluding remarks

From the above results it is clear that the study of Q-symmetries provides a useful method for obtaining exact solutions for nonlinear partial differential equations. Note that all the reduced equations: (31), (33)–(35), (38)–(40), (43), (44) that were obtained by Q-symmetry reductions are integrable and we were able to solve these reduced equations in general.

Generalized Q-symmetries, in the form of Q-Bäcklund symmetries, defined by

$$Q_{\mathrm{B}} = g(x_0, \ldots, x_m, u, u_0, \ldots, u_{j_1\cdots j_q})\frac{\partial}{\partial u}, \tag{45}$$

can be considered for eq. (2). Here $q > r$. This will extend the number of exact solutions for (2). We could find no Q-Bäcklund symmetry for eq. (1) with nonlinear f. In [1, 17] an example is given to demonstrate the method by which one can obtain exact solutions with Lie–Bäcklund generators. Note that, in the case of Lie–Bäcklund or Q-Bäcklund symmetries for (2), one can, in general, not find the general solution of the equation

$$js^*(Q_B \rfloor \theta) = 0. \tag{46}$$

This is due to the fact that (46) is usually more complicated, in that it has a higher order of derivatives and of nonlinearity, than (2). By, however, considering linear combinations of symmetries in the contraction (46), one can combine (2) and (46) to eliminate some derivatives or non-linearities (see [1, 17]).

In the study of conditional symmetries one can consider additional differential equations as conditions for (2), and then study the symmetry properties of the combined two equations. However, one then has to consider the compatibility problem between (2) and the additional equation. This approach was studied, and exact solutions were obtained, for the multi-dimensional d'Alembert equation [18, 19] and some nonlinear equations of acoustics [20].

1. Euler N., Steeb W.-H., Continuous symmetries, Lie algebras and differential equations, Mannheim, B.I. Wissenschaftsverlag, 1992.
2. Euler N., Shul'ga M.W., Steeb W.H., *J. Phys. A: Math. Gen.*, 1993, to appear.
3. Steeb W.-H., Euler N., Nonlinear evolution equations and Painlevé test, Singaporte, World Scientific Publishing, 1988.
4. Ovsiannikov L.V., Group analysis of differential equations, translation from Russian (edited by W.F. Ames), New York, Academic Press, 1982.
5. Olver P.J., Applications of Lie groups to differential equations, New York, Springer, 1986.
6. Fushchych W.I., Nikitin, A.G., Symmetries of Maxwell equations, Dordrecht, D. Reidel, 1987.
7. Bluman G.W., Kumei S., Symmetries and differential equations, *Applied Mathematical Sciences*, Vol. 81, New York, Springer, 1989.
8. Fushchych W.I., Shtelen W.M., Serov N.I., Symmetry analysis and exact solutions of equations of nonlinear mathematical physics, Kluwer Academic Publishers, 1993.
9. Fushchych W.I., in Symmetry and Solutions of Nonlinear Equations of Mathematical Physics, Kyiv, Inst. Math., 1987, 4–16 (in Russian).
10. Fushchych W.I., *Ukr. Mat. Zh.*, 1987, **39**, 116.
11. Fushchych W.I., Serov N.I., Chopyk V.I., *Dokl. Akad. Nauk Ukr. SSR, Ser. A*, 1988, № 9, 17.
12. Fushchych W.I., Serov N.I., *Dokl. Akad. Nauk Ukr. SSR, Ser. A*, 1988, № 10, 27.
13. Fushchych W.I., Serov, N.I., in Symmetry and Solutions of Equations of Mathematical Physics, Kyiv, Inst. Math., 1989, 95 (in Russian).
14. Fushchych W.I., Serov N.I., *Dokl. Akad. Nauk Ukr. SSR, Ser. A*, 1990, № 7, 24.
15. Steeb W.-H., Lewien D., Algorithms and computations with REDUCE, Mannheim, B.I. Wissenschaftsverlag, 1992.
16. Hehl F.W., Winkelmann V., Meyer H., Computer Algebra: Ein Kompakturs über die Anwendung von REDUCE, Berlin, Springer, 1992.
17. Euler N., Steeb W.-H., Mulser P., *J. Phys. Soc. Jpn.*, 1991, **60**, 1132.
18. Shul'ga M.W., in Group-Theoretical Investigations of the Equations of Mathematical Physics, Kyiv, Inst. Math, 1985 (in Russian).
19. Fushchych W.I., Zhdanov R.Z., *Phys. Rep.*, 1989, **46**, 325.
20. Fushchych W.I., Chopyk V.I., Myronyuk P.P., *Dokl. Akad. Nauk Ukr. SSR, Ser. A*, 1990, № 9, 25.

Галілей-інваріантні системи нелінійних рівнянь типу Гамільтона–Якобі та реакції-дифузії

В.І. ФУЩИЧ, Р.М. ЧЕРНІГА

All systems of $(n+1)$-dimensional evolutional second-order equations invariant under chain of algebras $AG(1,n)$, $AG_1(1,n)$, $AG_2(1,n)$ are described. The obtained results are illustrated by the examples of reaction-diffusion equations and Hamilton–Jacobi type systems.

1. Відомо, що система $(n+1)$-вимірних рівнянь дифузії (теплопровідності)

$$\lambda_1 U_t = \Delta U, \tag{1.a}$$

$$\lambda_2 V_t = \Delta U, \tag{1.b}$$

де $U(t,x)$, $V(t,x)$ — шукані дійсні функції, $U_t = \frac{\partial U}{\partial t}$, $V_t = \frac{\partial V}{\partial t}$, $x = (x_1, \ldots, x_n)$, інваріантна відносно узагальненої алгебри Галілея $AG_2(1,n)$ з базою

$$P_t = \partial_t, \quad P_a = \partial_a, \tag{2.a}$$

$$Q_\lambda, \quad G_a = tP_a - \frac{1}{2}x_a Q_\lambda, \quad J_{ab} = x_a P_b - x_b P_a, \tag{2.b}$$

$$D = 2tP_t + x_a P_a + I_\alpha, \tag{2.c}$$

$$\Pi = t^2 P_t + t x_a P_a - \frac{|x|^2}{4} Q_\lambda + t I_\alpha, \quad \alpha_k = -\frac{n}{2}. \tag{2.d}$$

У співвідношеннях (2) і скрізь далі $I_\alpha = \alpha_1 U\partial_U + \alpha_2 V\partial_V$, $Q_\lambda = \lambda_1 U\partial_U + \lambda_2 V\partial_V$, $\partial_U = \frac{\partial}{\partial U}$, $\partial_V = \frac{\partial}{\partial V}$, $\partial_t = \frac{\partial}{\partial t}$, $\partial_a = \frac{\partial}{\partial x_a}$, $\alpha_k, \lambda_k \in \mathbb{R}^1$, а за індексами a і b, що повторюються, передбачається сумування від 1 до n; $k = 1, 2$.

Алгебра утворена операторами (2a)–(2b) називається алгеброю Галілея, а її розширення за допомогою оператора (2c) позначимо $AG_1(1,n)$.

Очевидно, що одиничні оператори Q_λ і I_α є лінійно залежними лише у випадку $\delta = \begin{vmatrix} \alpha_1 & \alpha_2 \\ \lambda_1 & \lambda_2 \end{vmatrix} = 0$. У зв'язку з цим одержуємо два випадки принципово різних представлень алгебр $AG_1(1,n)$ та $AG_2(1,n)$ при $\delta = 0$ і $\delta \neq 0$, чого не було у випадку одного рівняння дифузії (інваріантність нелінійного рівняння дифузії відносно низки підалгебр алгебри $AG_2(1,n)$ досліджена в [1]).

Зазначимо, що у випадку,коли система рівнянь (1) є парою комплексно спряжених рівнянь Шрьодінгера, тобто $U = V^*$, $\lambda_1 = \lambda_2^* = i$, оператори Q_λ і I_α лінійно незалежні. Це приводить до того, що нелінійні узагальнення рівняння Шрьодінгера, які повністю зберігають його симетрію [2], принципово відрізняються від нелінійних узагальнень системи рівнянь дифузії (1) при $\delta = 0$.

Доповіді НАН України, 1994, № 3, P. 31–37.

Розглянемо систему щонайможливіших квазілінійних узагальнень системи рівнянь (СР) дифузії (1) вигляду

$$\lambda_1 U_t = A_{ab}U_{ab} + C_{ab}V_{ab} + B_1, \tag{3.a}$$

$$\lambda_2 V_t = D_{ab}U_{ab} + E_{ab}V_{ab} + B_2, \tag{3.b}$$

де A_{ab}, C_{ab}, D_{ab}, E_{ab}, B_1, B_2 — довільні дійсні неперервно диференційовані функції від $(2n+2)$ змінних U, V, $U_1,\ldots,U_n$, $V_1,\ldots,V_n$. Індекси $a = 1,\ldots,n$ та $b = 1,\ldots,n$ біля функцій U і V означають диференціювання за x_a та x_b.

СР (3) узагальнює практично всі відомі нелінійні системи еволюційних рівнянь першого і другого порядків, якими описуються найрізноманітніші процеси у фізиці, хімії, біології (досить згадати процеси тепломасопереносу, фільтрації двофазної рідини, дифузії при хімічних реакціях, динаміки руху популяцій, тощо) [3, 4].

У пропонованій роботі описані всі системи еволюційних рівнянь (3), які інваріантні відносно ланцюжка алгебр $AG(1,n) \subset AG_1(1,n) \subset AG_2(1,n)$, та проілюстровано одержані висліди на прикладах СР реакції-дифузії та систем типу Гамільтона–Якобі.

2. В алгебру симетрій системи рівнянь дифузії (1) входять оператори G_a, $a = 1,\ldots,n$, які є дифереційним виразженням справедливості принципу відносності Галілея для них. Також відомо [1], що оператори Галілея тісно пов'язані з фундаментальним розв'язком рівняння дифузії. У цьому зв'язку логічним виглядає пошук у класі систем рівнянь (3) галілей-інваріантних нелінійних узагальнень системи (1).

Теорема 1. *Система нелінійних рівнянь (3) інваріантна відносно алгебри Галілея з представленням (2a), (2b) тоді і тільки тоді,коли вона має вигляд*

$$\begin{aligned}
\lambda_1 U_t &= \Delta U + U[A_1\Delta(\ln U) + C_1\Delta(\ln V) + B_1] + \\
&\quad + U_{\omega_a\omega_b}[A_2(\ln U)_{ab} + C_2(\ln V)_{ab}], \\
\lambda_2 V_t &= \Delta V + V[D_1\Delta(\ln U) + E_1\Delta(\ln V) + B_2] + \\
&\quad + V_{\omega_a\omega_b}[D_2(\ln U)_{ab} + E_2(\ln V)_{ab}],
\end{aligned} \tag{4}$$

де $\omega_a = \frac{\partial\omega}{\partial x_a} \equiv (\lambda_2 U_a/U - \lambda_1 V_a/V)\omega$, $\omega = U^{\lambda_2}\cdot V^{-\lambda_1}$, $(\ln U)_{ab} = \frac{\partial^2(\ln U)}{\partial x_a\partial x_b}$, $(\ln V)_{ab} = \frac{\partial^2(\ln V)}{\partial x_a\partial x_b}$, *а* A_k, B_k, C_k, D_k, E_k *— довільні функції від абсолютних інваріантів* $AG(1,n)$ ω *і* $\theta = \omega_a\omega_a$, $k = 1,2$.

Доведення теореми, як і наступних теорем, грунтується на класичній схемі Лі, реалізація якої для знаходження ґалілей-інваріантних систем наведена в [5]. Оскільки викладки досить громіздкі, то тут вони опущені.

Зазначимо, що у випадку $\lambda_1 = 0$, тобто перше рівняння системи (3) еліптичне, абсолютні інваріанти алгебри Ґалілея значно спрощуються, а саме $\omega = U$, $\theta = U_aU_a$.

При побудові СР вигляду (3) які володіють $AG_1(1,n)$- та $AG_2(1,n)$-інваріантністю, структура таких систем суттєво залежить від визначника δ.

Теорема 2. *Нелінійна СР (3) інваріантна відносно алгебри* $AG_1(1,n)$ *з базовими операторами (2a)–(2c) тоді і тільки тоді коли вона має вигляд*

1. Випадок $\delta \neq 0$

$$
\begin{aligned}
\lambda_1 U_t &= \Delta U + U[A_1(\hat{\theta})\Delta(\ln U) + A_2(\hat{\theta})\Delta(\ln V) + \omega^{-2/\delta}B_1(\hat{\theta})] + \\
&\quad + U\omega^{2/\delta-2}\omega_a\omega_b[C_1(\hat{\theta})(\ln U)_{ab} + C_2(\hat{\theta})(\ln V)_{ab}], \\
\lambda_2 V_t &= \Delta V + V[D_1(\hat{\theta})\Delta(\ln U) + D_2(\hat{\theta})\Delta(\ln V) + \omega^{-2/\delta}B_2(\hat{\theta})] + \\
&\quad + V\omega^{2/\delta-2}\omega_a\omega_b[E_1(\hat{\theta})(\ln U)_{ab} + E_2(\hat{\theta})(\ln V)_{ab}].
\end{aligned}
\tag{5}
$$

2. Випадок $\delta = 0$

$$
\begin{aligned}
\lambda_1 U_t &= \Delta U + U[A_1(\omega)\Delta(\ln U) + A_2(\omega)\Delta(\ln V) + \omega_a\omega_a B_1(\omega)] + \\
&\quad + U\frac{\omega_a\omega_b}{\omega_{a_1}\omega_{a_1}}[C_1(\omega)(\ln U)_{ab} + C_2(\omega)(\ln V)_{ab}], \\
\lambda_2 U_t &= \Delta V + V[D_1(\omega)\Delta(\ln U) + D_2(\omega)\Delta(\ln V) + \omega_a\omega_a B_2(\omega)] + \\
&\quad + V\frac{\omega_a\omega_b}{\omega_{a_1}\omega_{a_1}}[E_1(\omega)(\ln U)_{ab} + E_2(\omega)(\ln V)_{ab}],
\end{aligned}
\tag{6}
$$

де A_k, B_k, C_k, D_k, E_k *— довільні функції,* $k = 1, 2$, $\hat{\theta} = \omega_a\omega_a\omega^{2/\delta-2}$ *— абсолютний диференційний інваріант першого порядку алгебри (див. теорему 1).*

У випадку виродження першого рівняння СР (3) в еліптичне ($\lambda_1 = 0$) абсолютні інваріанти в системах (5), (6) спрощуються і $\hat{\theta} = U_a U_a U^{2/\alpha_1-2}$ при $\delta \neq 0$, $\omega = U$ при $\delta = 0$.

Теорема 3. *У класі СР (3) алгебру інваріантності* $AG_2(1, n)$ *рівнянь (3) зберігають тільки такі, які мають вигляд*

1. Випадок $\delta \neq 0$

$$
\begin{aligned}
\lambda_1 U_t &= \hat{\alpha}\Delta U + UA(\hat{\theta})[\lambda_2\Delta(\ln U) - \lambda_1\Delta(\ln V)] + U\omega^{-2/\delta}B_1(\hat{\theta}) + \\
&\quad + (1-\hat{\alpha}_1)\frac{U_aU_a}{U} + U\omega^{2/\delta-2}\omega_a\omega_b C(\hat{\theta})[\lambda_2(\ln U)_{ab} - \lambda_1(\ln V)_{ab}], \\
\lambda_1 V_t &= \hat{\alpha}_2\Delta V + VD(\hat{\theta})[\lambda_2\Delta(\ln U) - \lambda_1\Delta(\ln V)] + V\omega^{-2/\delta}B_2(\hat{\theta}) + \\
&\quad + (1-\hat{\alpha}_2)\frac{V_aV_a}{V} + V\omega^{2/\delta-2}\omega_a\omega_b E(\hat{\theta})[\lambda_2(\ln U)_{ab} - \lambda_1(\ln V)_{ab}].
\end{aligned}
\tag{7}
$$

2. Випадок $\delta = 0$

$$
\begin{aligned}
\lambda_1 U_t &= \hat{\alpha}_1\Delta U + UA(\omega)[\lambda_2\Delta(\ln U) - \lambda_1\Delta(\ln V)] + U\omega_a\omega_a B_1(\omega) + \\
&\quad + (1-\hat{\alpha}_1)\frac{U_aU_a}{U} + U\frac{\omega_a\omega_b}{\omega_{\alpha_1}\omega_{a_1}}C(\omega)[\lambda_2(\ln U)_{ab} - \lambda_1(\ln V)_{ab}], \\
\lambda_2 V_t &= \hat{\alpha}_2\Delta V + VD(\omega)[\lambda_2\Delta(\ln U) - \lambda_1\Delta(\ln V)] + V\omega_a\omega_a B_2(\omega) + \\
&\quad + (1-\hat{\alpha}_2)\frac{V_aV_a}{V} + V\frac{\omega_a\omega_b}{\omega_{a_1}\omega_{a_1}}E(\omega)[\lambda_2(\ln U)_{ab} - \lambda_1(\ln V)_{ab}],
\end{aligned}
\tag{8}
$$

де A, B_1, B_2, C, D, E *— довільні функції,* $\hat{\alpha}_k = -2\alpha_k/n$, $k = 1, 2$ *(*α_k *— див. оператор* I_α*),* $a_1 = 1, 2, \ldots, n$.

Можна помітити, що у випадку $\alpha_1 \cdot \alpha_2 \neq 0$ СР (7) і (8) локальною заміною $U \to U^{\hat{\alpha}_1}$, $V \to V^{\hat{\alpha}_2}$ зводяться до систем такої ж структури, але з $\hat{\alpha}_k = 1$, тобто $\hat{\alpha}_k = -n/2$. Випадок $\alpha_1 = \alpha_2 = 0$ — особливий і нище буде розглянутий.

Одержані класи $AG_2(1,n)$-інваріантних СР (7) і (8) містять, зокрема, такі нелінійні узагальнення СР (1) як

$$\lambda_1 U_t = \Delta U + e_1 U\Delta(\ln\omega) + e_2 U\omega^{2/\delta-2}\omega_a\omega_b(\ln\omega)_{ab},$$
$$\lambda_2 V_t = \Delta V + e_3 V\Delta(\ln\omega) + e_4 V\omega^{2/\delta-2}\omega_a\omega_b(\ln\omega)_{ab}, \quad \delta\neq 0$$

та

$$\lambda_1 U_t = \Delta U + e_1 U\Delta(\ln\omega) + e_2 U\omega_a\omega_a\omega^{\beta_2},$$
$$\lambda_2 V_t = \Delta V + e_3 V\Delta(\ln\omega) + e_4 V\omega_a\omega_a\omega^{\beta_2}, \quad \delta = 0,$$

де $e_1, e_2, e_3, e_4, \beta_1, \beta_2 \in \mathbb{R}$, $(\ln\omega)_{ab} = \lambda_2(\ln U)_{ab} - \lambda_1(\ln V)_{ab}$.

У випадку виродження першого рівняння системи (3) в еліптичне ($\lambda_1 = 0$) $AG_2(1,n)$-інваріантними є тільки СР вигляду

$$\begin{aligned}
0 &= A_1(\hat\theta)\Delta U + A_2(\hat\theta)\frac{U_aU_b}{U_{a_1}U_{a_1}}U_{ab} + U^{1-2/\alpha_1}B_1(\hat\theta) + \\
&\quad + UC(\hat\theta)\left[\Delta(\ln V) - \frac{U_aU_b}{U_{a_1}U_{a_1}}(\ln V)_{ab}\right], \quad \hat\theta = U_aU_aU^{2/\alpha_1-2}, \\
\lambda_2 V_t &= \hat\alpha_2\Delta V + \frac{V}{U}\left(D_1(\hat\theta)\Delta U + D_2(\hat\theta)\frac{U_aU_b}{U_{a_1}U_{a_1}}U_{ab}\right) + \\
&\quad + VU^{-2/\alpha_1}B_2(\hat\theta) + (1-\hat\alpha_2)\frac{V_aV_a}{V} + VE(\hat\theta)\left[\Delta(\ln V) - \frac{U_aU_b}{U_{a_1}U_{a_1}}(\ln V)_{ab}\right],
\end{aligned} \tag{9}$$

якщо $\delta\neq 0$ та

$$\begin{aligned}
0 &= A_1(U)\Delta U + A_2(U)\frac{U_aU_b}{U_{a_1}U_{a_1}}U_{ab} + U_aU_aB(U) + \\
&\quad + C(U)\left[\Delta(\ln V) - \frac{U_aU_b}{U_{a_1}U_{a_1}}(\ln V)_{ab}\right], \\
\lambda_2 V_t &= \hat\alpha_2\Delta V + V\left(D_1(U)\Delta U + D_2(U)\frac{U_aU_b}{U_{a_1}U_{a_1}}U_{ab}\right) + \\
&\quad + VU_aU_aB_2(U) + (1-\hat\alpha_2)\frac{V_aV_a}{V} + VE(U)\left[\Delta(\ln V) - \frac{U_aU_b}{U_{a_1}U_{a_1}}(\ln V)_{ab}\right],
\end{aligned} \tag{10}$$

якщо $\delta = 0$. У формулах (9), (10) A_k, B_k, D_k, E, C — довільні функції, $\hat\alpha_2 = -2\alpha_2/n$.

В роботі [6] показано,що інтегрування двовимірних СР вигляду (9), (10) зводиться до інтегрування лінійного рівняння дифузії з джерелом.

Зауваження 1. Одержані вище теореми 1–3 справедливі і для випадку СР (3) з комплексними функціями, тому вони є нетривіальним узагальненням вислідів роботи [6] на багатовимірний випадок.

Зауваження 2. Для запису всіх побудованих вище галілей-інваріантних СР можна скористатися тотожностями

$$(\ln U)_{ab} = \frac{U_{ab}}{U} - \frac{U_aU_b}{U^2}, \quad (\ln V)_{ab} = \frac{V_{ab}}{V} - \frac{V_aV_b}{V^2}.$$

Такий запис, очевидно, буде корисним при фізичному інтерпретуванні одержаних СР.

3. Звернемо увагу на те, що локальна заміна $U = M(\hat{U})$, $V = N(\hat{V})$, де M, N — довільні диференційовані функції, зводить будь-яку СР з симетрією $AG(1,n)$, $AG_1(1,n)$ чи $AG_2(1,n)$ до локально-еквівалентної системи з такою ж симетрією, але з іншим представленням операторів Q_λ і I_α, а саме:

$$\hat{Q}_\lambda = \lambda_1 M \left(\frac{dM}{d\hat{U}}\right)^{-1} \frac{\partial}{d\hat{U}} + \lambda_2 N \left(\frac{dN}{d\hat{V}}\right)^{-1} \frac{\partial}{\partial\hat{V}},$$
$$\hat{I}_\alpha = \alpha_1 M \left(\frac{dM}{d\hat{U}}\right)^{-1} \frac{\partial}{d\hat{U}} + \alpha_2 N \left(\frac{dN}{d\hat{V}}\right)^{-1} \frac{\partial}{\partial\hat{V}}.$$

Зокрема, у випадку $M = \exp U$, $N = \exp V$ одержуємо

$$\hat{Q}_\lambda = \lambda_1 \frac{\partial}{\partial\hat{U}} + \lambda_2 \frac{\partial}{\partial\hat{V}}, \quad \hat{I}_\alpha = \alpha_1 \frac{\partial}{\partial\hat{U}} + \alpha_2 \frac{\partial}{\partial\hat{V}}. \tag{11}$$

В цьому випадку клас СР, інваріантних відносно алгебри $AG_2(1,n)$ з представленням (2), (11), у випадку $\delta = 0$ має вигляд

$$\begin{aligned}
\lambda_1 \hat{U}_t &= \alpha_1 \Delta\hat{U} + A(\hat{\omega})(\lambda_2\Delta\hat{U} - \lambda_1\Delta\hat{V}) + \hat{U}_a\hat{U}_a + \hat{\omega}_a\hat{\omega}_a B_1(\hat{\omega}) + \\
&\quad + C(\hat{\omega})\frac{\hat{\omega}_a\hat{\omega}_b}{\hat{\omega}_{a_1}\hat{\omega}_{a_1}}(\lambda_2\hat{U}_{ab} - \lambda_1\hat{V}_{ab}), \\
\lambda_2 \hat{V}_t &= \alpha_2 \Delta\hat{V} + D(\hat{\omega})(\lambda_2\Delta\hat{U} - \lambda_1\Delta\hat{V}) + \hat{V}_a\hat{V}_a + \hat{\omega}_a\hat{\omega}_a B_2(\hat{\omega}) + \\
&\quad + E(\hat{\omega})\frac{\hat{\omega}_a\hat{\omega}_b}{\hat{\omega}_{a_1}\hat{\omega}_{a_1}}(\lambda_2\hat{U}_{ab} - \lambda_1\hat{V}_{ab}),
\end{aligned} \tag{12}$$

де $\hat{\omega} = \lambda_2 U - \lambda_1 V$, $\hat{\omega}_a = \lambda_2 U_a - \lambda_1 V_a$.

У випадку $\alpha_1 = \alpha_2 = 0$, $A = B = C = D = E = 0$ СР (12) зводиться до систем вигляду (нижче знак $\hat{}$ опущено)

$$\begin{aligned}
\lambda_1 U_t &= U_a U_a + B_1(\omega)\omega_a\omega_a, \\
\lambda_2 V_t &= V_a V_a + B_2(\omega)\omega_a\omega_a, \quad \lambda_1 \cdot \lambda_2 \neq 0.
\end{aligned} \tag{13}$$

СР (13) природньо назвати узагальненням незачепленої СР Гамільтона–Якобі (Г–Я)

$$\begin{aligned}
\lambda_1 U_t &= U_a U_a, \\
\lambda_2 V_t &= V_a V_a.
\end{aligned} \tag{14}$$

На відміну від симетрії одного рівняння Г–Я [7, 8] локальна симетрія СР (14) вичерпується алгеброю $AG_2(1,n)$ (2), (11) при $\alpha_1 = \alpha_2 = 0$ з додатковими операторами

$$P_V = \partial_V, \quad D_1 = -t\partial_t + U\partial_U + V\partial_V. \tag{15}$$

Таким чином, СР (13) вичерпуються всі нелінійні узагальнення вигляду

$$\begin{aligned}
\lambda_1 U_t &= U_a U_a + B_1(U, V, U_1, \ldots, U_n, V_1, \ldots, V_n), \\
\lambda_2 V_t &= V_a V_a + B_2(U, V, U_1, \ldots, U_n, V_1, \ldots, V_n)
\end{aligned} \tag{16}$$

системи Г–Я, які зберігають її симетрію $AG_2(1,n)$.

У випадку $B_1 = 0$ симетрія СР (13) розширюється за допомогою операторів

$$\hat{P}_V = B_2^{-\gamma}\partial_V, \quad \gamma = \lambda_1^2/(1+\lambda_1^2),$$
$$\hat{D}_1 = -t\partial_t + U\partial_U + \left(\frac{\lambda_2}{\lambda_1}U - B_2^{-\gamma}\int B_2^{\gamma}d\omega\right)\partial_V.$$

Аналогічні додаткові оператори з'являються в алгебрі симетрій СР (13) і при деяких конкретних функціях $B_1B_2 \neq 0$. Зокрема, для $B_1 = -1/\lambda_1^2$ одержуємо $AG_2(1,n)$-інваріантну систему

$$U_t = -\frac{\lambda_1}{\lambda_2^2}V_aV_a + \frac{2}{\lambda_2}U_aV_a,$$
$$V_t = -\frac{\lambda_2}{\lambda_1^2}U_aU_a + \frac{2}{\lambda_1}U_aV_a$$

з додатковими операторами (15).

Виявляється, серед нелінійних узагальнень системи Г–Я (13) існує СР з унікальними симетрійними властивостями, а саме, при $B_1 = B_2 = -1/\lambda_1^2$ (нижче $\lambda_1 = 1$, $\lambda_2 = \lambda$).

Теорема 4. *Максимальна (в розумінні Лі) алгебра інваріантності СР*

$$U_t = U_aU_a,$$
$$V_t = -\lambda U_aU_a + 2U_aV_a \tag{17}$$

породжується базовими операторами

$$P_t, \quad P_a, \quad J_{ab}, \quad Q_\lambda = \lambda\partial_U - \partial_V, \quad X = W(\lambda U - V)\partial_V,$$
$$G_a = tP_a - \frac{1}{2}x_aQ_\lambda, \quad D = 2tP_t + x_aP_a,$$
$$\Pi = t^2P_t + tx_aP_a - \frac{|x|^2}{4}Q_\lambda, \quad G_a^1 = UP_a - \frac{1}{2}x_aP_t, \tag{18}$$
$$D_1 = 2U\partial_U + x_aP_a, \quad \Pi_1 = U^2\partial_U + Ux_aP_a - \frac{|x|^2}{4}P_t,$$
$$K_a = x_atP_t - \left(\frac{1}{2}|x|^2 + 2tU\right)P_a + x_ax_bP_b + x_aUQ_\lambda,$$

де W — довільна диференційована функція.

Зазначимо, що наявність в алгебрі інваріантності СР (17) оператора X з довільною функцією W є природньою, оскільки друге рівняння системи лінійне відносно функції V. Значно цікавішим є те, що СР (17) можна вважати узагальненням класичного рівняння Г–Я на випадок двох шуканих функцій, адже оператори (18) при $W = 1$ породжують таку ж алгебру, що й рівняння Г–Я. Вважаємо,що це дуже важливий факт, оскільки тривіальне узагальнення згаданого рівняння до СР (14) не зберігає симетрію рівняння Г–Я.

4. Розглянемо нелінійну систему еволюційних рівнянь вигляду

$$\lambda_1U_t = \Delta U + f(U,V),$$
$$\lambda_2V_t = \Delta V + g(U,V), \tag{19}$$

де f, g — довільні диференційовані функції. Системи рівнянь реакції дифузії вигляду (21) в останній час інтенсивно досліджуються (див., напр. [3, 9]). Як випливає з теорем 1–3, клас СР (19) містить системи з широкою симетрією. Зокрема, інваріантними відносно алгебри Галілея будуть всі СР рівнянь вигляду

$$\begin{aligned} &\lambda_1 U_t = \Delta U + U f(\omega), \\ &\lambda_2 V_t = \Delta V + V g(\omega), \quad \omega = U^{\lambda_2} \cdot V^{-\lambda_1}. \end{aligned} \tag{20}$$

У випадку $f = \beta_1 \omega^{-2/\delta}$, $g = \beta_2 \omega^{-2/\delta}$, $\beta_k \in \mathbb{R}$ матимемо інваріантність відносно алгебри $AG_1(1,n)$. Нарешті при $\delta = \frac{n}{2}(\lambda_1 - \lambda_2)$, тобто $\alpha_1 = \alpha = -n/2$, одержуємо СР

$$\begin{aligned} &\lambda_1 U_t = \Delta U + \beta_1 U^{1+\lambda_2\gamma} V^{-\lambda_1\gamma}, \\ &\lambda_2 V_t = \Delta V + \beta_2 U^{\lambda_2\gamma} V^{1-\lambda_1\gamma}, \end{aligned}$$

де $\gamma = 4/n/(\lambda_2 - \lambda_1)$, $\lambda_1 \neq \lambda_2$, $\beta_k \in \mathbb{R}$, яка зберігає $AG_2(1,n)$ симетрію лінійної СР (1).

Зауваження 3. Дифузійна СР (18) при $\lambda_1 = -\lambda_2 = \lambda$ заміною

$$U = Y + Z, \quad V = Y - Z, \quad Y = Y(t,x), \quad Z = Z(t,x) \tag{21}$$

зводиться до СР

$$\begin{aligned} &\lambda \frac{\partial Y}{\partial t} = \Delta Z + \hat{f}(Y,Z), \\ &\lambda \frac{\partial Z}{\partial t} = \Delta Y + \hat{g}(Y,Z), \end{aligned}$$

інваріантність якої відносно ланцюжка алгебр $AG(1,n) \subset AG_1(1,n) \subset AG_2(1,n)$ з одиничним оператором $Q_\lambda = \lambda\left(Y\frac{\partial}{\partial Z} + Z\frac{\partial}{\partial Y}\right)$ описується шляхом застосування заміни (21) до СР вигляду (18) з відповідною симетрією.

На закінчення наведемо ще одну цікаву систему рівнянь вигляду (19), а саме

$$\begin{aligned} &\lambda U_t = \Delta U + \beta_1 U^2/V, \\ &\lambda V_t = \Delta V + \beta_2 U, \quad \beta_1 \neq \beta_2, \quad \beta_k \in \mathbb{R}. \end{aligned} \tag{22}$$

Максимальна алгебра інваріантності СР (22) є узагальненою алгеброю Галілея з базовими операторами (2a), (2b) та

$$\begin{aligned} &D = 2tP_t + x_a P_a - 2U\partial_U - \left(\frac{n}{2} + \frac{\beta_2}{\beta_1 - \beta_2}\right) Q_\lambda, \\ &\Pi = tD - t^2 P_t - \frac{|x|^2}{4} Q_\lambda - \frac{\lambda}{\beta_1 - \beta_2} V_{\partial_U}. \end{aligned} \tag{23}$$

Між іншим, серед СР вигляду (18) у випадку $\lambda_1 = \lambda_2 = \lambda$ не існує $AG_2(1,n)$-інваріантних зі стандартним представленням (2). Проективний оператор (23) породжує групу скінченних перетворень

$$\begin{aligned} &t' = t/(1-pt), \quad x'_a = x_a/(1-pt), \quad p \in \mathbb{R}, \\ &\begin{pmatrix} U' \\ V' \end{pmatrix} = (1-pt)^{2+\frac{n}{2}+\hat{\beta}} \exp\left(-\frac{\lambda p|x|^2}{4(1-pt)}\right) A(t) \begin{pmatrix} U \\ V \end{pmatrix}, \quad \hat{\beta} = \frac{\beta_2}{\beta_1 - \beta_2}, \end{aligned}$$

де матриця $A(t) = \begin{pmatrix} 1 & \frac{\lambda p}{(1-pt)(\beta_2-\beta_1)} \\ 0 & (1-pt)^{-2} \end{pmatrix}$ не діагональна, а в перетвореннях генерованих оператором (2d) аналогічна матриця діагональна [5].

Деякі класи точних розв'язків СР (22) одержано в роботі [10].

1. Fushchych W.I., Cherniha R.M., The Galilean relativistic principle and nonlinear partial differential equations, *J. Phys. A.: Math and Gen.*, 1985, **18**, № 18, 3491–3503.
2. Фущич В.И., Чернига Р.М., Галилей-инвариантные нелинейные уравнения шредингеровского типа и их точные решения. I, *Укр. мат. журн.*, 1989, **41**, № 10, 1349–1357.
3. Aris R., The theory of reaction and diffusion in permeable catalysts, Oxford, 1975, 300 p.
4. Вільгельмссон Г. (Wilhelmsson H.), Коливання та встановлення рівноваги за умов взаємозв'язку температури та густини у термоядерних плазмах, *Укр. фіз. журн.*, 1993, **38**, № 1, 44–53.
5. Фущич В.И., Чернига Р.М., О точных решениях двух многомерных уравнений шредингеровского типа, Препринт № 87.16, Киев, Ин-т математики, 1987, 44 с.
6. Фущич В.І., Черніга Р.М., Системи нелінійних еволюційних рівнянь другого порядку, інваріантні відносно алгебри Галілея та її розширень, *Доповіді АН України*, 1993, № 8, 44–51.
7. Boyer C.P., Penafiel M.N. Conformal symmetry of the Hamilton–Jacobi equation and quantization, *Nuovo Cim. B*, 1976, **31**, № 2, 195–210.
8. Фущич В.И., Штелень В.М., Серов Н.И., Симметрийный анализ и точные решения нелинейных уравнений математической физики, Киев, Наук. думка, 1989, 336 с.
9. Fife P.C., Mathematical aspects of reacting and diffusing systems, Berlin, Springer, 1979, 285 p.
10. Чернига Р.М., О точных решениях одной нелинейной системы диффузионного типа, в сб. Симметрийный анализ и решения уравнений математической физики, Киев, Ин-т математики, 1988, 49–53.

Симетрійна редукція і деякі точні розв'язки рівняння Монжа–Ампера

В.І. ФУЩИЧ, В.М. ФЕДОРЧУК, О.С. ЛЕЙБОВ

For the Monge–Amperé equation the ansatzes which reduce this equation to differential equation with a less number of the independent variables have been constructed. Some exact solutions of the equation under investigation have been found.

У роботах [1, 2] вивчена симетрія і на основі спеціальних анзаців побудовані класи точних розв'язків багатовимірного рівняння Монжа–Ампера.

Дана робота присвячена вивченню рівняння вигляду

$$\det(u_{\mu\nu}) = 0, \tag{1}$$

де $u = u(x)$, $x = (x_0, x_1, x_2, x_3) \in \mathbb{R}_4$, $u_{\mu\nu} \equiv \frac{\partial^2 u}{\partial x_\mu \partial_{x_\nu}}$, $\mu, \nu = 0, 1, 2, 3$. Результати робіт [1, 2] дають змогу, зокрема, зробити висновок про те, що група інваріантності рівняння (1) містить як підгрупу узагальнену групу Пуанкаре $P(1,4)$ — групу поворотів та зсувів п'ятивимірного простору Мінковського. Для дослідження рівняння (1) використано підгрупову структуру [3–7] групи $P(1,4)$. На основі неспряжених підгруп групи $P(1,4)$ побудовані анзаци, які редукують рівняння (1) да диференціальних рівнянь із меншою кількістю незалежних змінних, проведена відповідна симетрійна редукція. На основі розв'язків редукованих рівнянь побудовані деякі класи точних розв'язків рівняння Монжа–Ампера.

Нижче виписані анзаци, які редукують рівняння (1) до звичайних диференціальних рівнянь (ЗДР), одержані ЗДР та розв'язки рівняння Монжа–Ампера

1.1. $u^2 = \varphi^2(\omega) - x_1^2 - x_2^2 - x_3^2, \quad \omega = x_0, \quad \varphi'' = 0,$
$u^2 = (c_1 x_0 + c_2)^2 - x_1^2 - x_2^2 - x_3^2.$

1.2. $u^2 = -\varphi^2(\omega) + x_0^2, \quad \omega = (x_1^2 + x_2^2 + x_3^2)^{1/2}, \quad \varphi'' = 0,$
$u^2 = x_0^2 - (c_1(x_1^2 + x_2^2 + x_3^2)^{1/2} + c_2)^2.$

1.3. $u^2 = \varphi^2(\omega) + x_0^2 - x_1^2 - x_2^2, \quad \omega = x_3, \quad \varphi'' = 0,$
$u^2 = x_0^2 - x_1^2 - x_2^2 + (c_1 x_3 + c_2)^2.$

1.4. $u^2 = \varphi^2(\omega) + x_0^2 - x_3^2, \quad \omega = (x_1^2 + x_2^2)^{1/2}, \quad \varphi'' = 0,$
$u^2 = x_0^2 - x_3^2 + (c_1(x_1^2 + x_2^2)^{1/2} + c_2)^2.$

1.5. $u^2 = \varphi(\omega), \quad \omega = (x_1^2 + x_2^2 + x_3^2 - x_0^2)^{1/2}, \quad \varphi'' = 0,$
$u^2 = c_1(x_1^2 + x_2^2 + x_3^2 - x_0^2)^{1/2} + c_2.$

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1.6. $u^2 = \varphi^2(\omega) + x_0^2 - x_1^2 - x_2^2, \quad \omega = \dfrac{c}{\alpha}x_3 + \ln(x_0 + u),$

$$\varphi''\varphi^2 - \varphi''\varphi'\varphi - \varphi'^3 = 0, \quad u = k_1 \exp\left(\frac{c}{\alpha}x_3 + k_2\right) \pm$$

$$\pm\left[\left(k_1 \exp\left(\frac{c}{\alpha}x_3 + k_2\right) + x_0\right)^2 - x_1^2 - x_2^2\right]^{1/2}, \quad k_1, k_2, c, \alpha = \mathrm{const},$$

$c, \alpha > 0.$

1.7. $u^2 = -\varphi^2(\omega) + x_0^2 - x_1^2 - x_2^2, \quad \omega = x_3 + \alpha\ln(x_0 + u), \quad \alpha > 0,$

$$\frac{1}{\alpha}\varphi''\varphi^2 - \varphi''\varphi'\varphi - \varphi'^3 = 0, \quad u = k_1 \exp\left(\frac{x_3}{\alpha} + k_2\right) \pm$$

$$\pm\left[\left(k_1 \exp\left(\frac{x_3}{\alpha} + k_2\right) - x_0\right)^2 - x_1^2 - x_2^2\right]^{1/2}, \quad k_1, k_2 = \mathrm{const}.$$

Анзаци (1.1)–(1.7) можна записати у такому вигляді:

$$h(u) = f(x)\varphi(\omega) + g(x), \tag{2}$$

де $h(u)$, $f(x)$, $g(x)$ — задані функції; $\varphi(\omega)$ — невідома функція; $\omega = \omega(x, u)$ — одновимірні інваріанти підгруп групи $P(1, 4)$.

2.1 $2x_0\omega = -\varphi(\omega) + x_1^2 + x_2^2 - x_3^2, \quad \omega = x_0 + u, \quad \dfrac{1}{2}\omega^2\varphi'' - \omega\varphi' + \varphi = 0,$

$$u = -x_0\left(1 + \frac{1}{c_2} \pm \frac{1}{2c_2}[(2x_0 + c_1)^2 + 4c_2(x_1^2 + x_2^2 + x_3^2)]^{1/2} - \frac{c_1}{2c_2}\right).$$

2.2 $\dfrac{\alpha x_3^2}{2\omega} - 2\omega x_0 = \varphi(\omega) - x_1^2 - x_2^2 - x_3^2 + \alpha x_0, \quad \omega = x_0 + u, \quad \alpha > 0,$

$$(2\omega + \alpha)^2\varphi'' - 4(2\omega + \alpha)\varphi' - 8\varphi = 0,$$

$$x_1^2 + x_2^2 - \frac{2(x_0 + u) + \alpha}{x_0 + u}\left[(x_0 + u)x_0 - \frac{x_3^2}{2}\right] =$$

$$= \left(x_0 + u + \frac{\alpha}{2}\right)(c_1 + c_2(x_0 + u)).$$

2.3 $\dfrac{\alpha x_3^2}{2\omega} + 2\omega x_0 = -\varphi(\omega) + x_1^2 + x_2^2 + x_3^2 + \alpha x_0, \quad \omega = x_0 + u, \quad \alpha > 0,$

$$(2\omega - \alpha)^2\varphi'' - 4(2\omega - \alpha)\varphi' + 8\varphi = 0,$$

$$x_1^2 + x_2^2 - \frac{2(x_0 + u) - \alpha}{x_0 + u}\left[(x_0 + u)x_0 - \frac{x_3^2}{2}\right] =$$

$$= \left(x_0 + u - \frac{\alpha}{2}\right)(c_1 + c_2(x_0 + u)).$$

2.4 $2(\omega^2 - \omega)(\omega - x_0) + \omega(x_1^2 + x_2^2 + x_3^2) = 2\varphi(\omega) + x_1^2 + x_2^2, \quad \omega = x_0 + u,$

$$\omega^2(\omega - 1)^2\varphi'' - 2\omega(2\omega^2 - 3\omega + 1)\varphi' + 2(3\omega^2 - 3\omega + 1)\varphi = 0,$$

$$\frac{(x_0 + u)x_3^2}{2} - (1 - (x_0 + u))\left[u(x_0 + u) + \frac{x_1^2}{2} + \frac{x_2^2}{2}\right] =$$

$$= (x_0 + u)(x_0 + u - 1)(c_1(x_0 + u) + c_2).$$

2.5 $2(\omega^2 + \omega)(\omega - x_0) + \omega(x_1^2 + x_2^2 + x_3^2) = 2\varphi(\omega) - (x_1^2 + x_2^2), \quad \omega = x_0 + u,$

$$\omega^2(\omega + 1)^2\varphi'' - 2\omega(2\omega^2 + 3\omega + 1)\varphi' + 2(3\omega^2 + 3\omega + 1)\varphi = 0,$$

$$\frac{(x_0+u)x_3^2}{2} - (x_0+u+1)\left[u(x_0+u) + \frac{x_1^2}{2} + \frac{x_2^2}{2}\right] =$$
$$= (x_0+u)(x_0+u+1)(c_1(x_0+u)+c_2).$$

Анзаци (2.1)–(2.5) запишемо таким чином:

$$h(\omega, x) = f(x)\varphi(\omega) + g(x), \tag{3}$$

де $h(\omega, x)$, $f(x)$, $g(x)$ — задані функції; $\varphi(\omega)$ — невідома функція; $\omega = \omega(x, u)$ — одновимірні інваріанти підгруп групи $P(1,4)$. Анзаци (2.1)–(2.5) редукують рівняння Монжа–Ампера до лінійних ЗДР.

Випишемо анзаци, які редукують рівняння (1) до двовимірних ди ференціальних рівнянь з частинними похідними, і відповідні їм редуковані рівняння.

3.1 $u^2 = \varphi^2(\omega_1, \omega_2) + x_0^2 - x_3^2, \quad \omega_1 = x_1, \quad \omega_2 = x_2, \quad \det\varphi = 0,$
$\det\varphi \equiv \begin{vmatrix} \varphi_{11} & \varphi_{12} \\ \varphi_{21} & \varphi_{22} \end{vmatrix}, \quad \varphi_{ij} \equiv \dfrac{\partial^2\varphi}{\partial\omega_i\partial\omega_j}, \quad i,j = 1,2.$

3.2 $u^2 = \varphi^2(\omega_1, \omega_2) - x_3^2, \quad \omega_1 = x_0, \quad \omega_2 = (x_1^2 + x_2^2)^{1/2},$
$\varphi_2 \det\varphi = 0, \quad \varphi_i \equiv \dfrac{\partial\varphi}{\partial\omega_i}, \quad i = 1,2.$

3.3 $u^2 = -\varphi^2(\omega_1, \omega_2) + x_0^2, \quad \omega_1 = (x_1^2 + x_2^2)^{1/2},$
$\omega_2 = \dfrac{x_3}{\alpha} + \arcsin\dfrac{x_2}{\sqrt{x_1^2 + x_2^2}}, \quad \alpha > 0, \quad \varphi(\omega_1^3\varphi_1 \det\varphi - \varphi_2^2\varphi_{22}) = 0.$

Анзаци (3.1)–(3.3) можна записати у вигляді (2), де $\omega = \omega(x) = (\omega_1(x), \omega_2(x))$ — двовимірні інваріанти підгруп групи $P(1,4)$.

4.1 $2x_0\omega_1 = -\varphi(\omega_1, \omega_2) + x_1^2 + x_2^2, \quad \omega_1 = x_0 + u, \quad \omega_2 = x_3,$
$\omega_1^2 \det\varphi + 2\omega_1\varphi_2\varphi_{12} + 2(\varphi - \omega_1\varphi_1)\varphi_{22} - \varphi_2^2 = 0.$

4.2 $2x_0\omega_1 = -\varphi(\omega_1, \omega_2) + x_3^2, \quad \omega_1 = x_0 + u, \quad \omega_2 = (x_1^2 + x_2^2)^{1/2},$
$\varphi_2[\omega_1^2 \det\varphi + 2\omega_1\varphi_2\varphi_{12} + 2(\varphi - \omega_1\varphi_1)\varphi_{22} - \varphi_2^2] = 0.$

4.3 $\dfrac{\alpha}{\mu}\operatorname{arch}\dfrac{x_0}{\omega} - \arcsin\dfrac{x_1}{\omega_1} = \varphi(\omega_1, \omega_2) - \dfrac{x_3}{\mu}, \quad \omega_1 = (x_1^2 + x_2^2)^{1/2},$
$\omega_2 = (x_0^2 - u^2)^{1/2}, \quad \varphi_1\varphi_2 \det\varphi + \dfrac{\alpha^2}{\mu^2\omega_2^3}\varphi_1\varphi_{11} - \dfrac{1}{\omega_1^3}\varphi_2\varphi_{22} -$
$- \dfrac{\alpha^2}{\mu^2\omega_2^3\omega_2^3} = 0, \quad \alpha, \mu \in \mathbb{R}, \quad \alpha, \mu > 0.$

4.4 $\dfrac{1}{3\mu^2}(2(\omega_2 - \mu x_3))^{1/2}(\mu x_3 + 2\omega_2) - \arcsin\dfrac{x_2}{\omega_1} = \varphi(\omega_1, \omega_2) - x_0,$
$\omega_1 = (x_1^2 + x_2^2)^{1/2}, \quad \omega_2 = \mu x_3 + \dfrac{(x_0+u)^2}{2},$
$\mu^4\omega_1^3\varphi_1\varphi_2 \cdot \det\varphi - \omega_1^3\varphi_1\varphi_{11} - \mu^4\varphi_2\varphi_{22} + 1 = 0, \quad \mu > 0.$

4.5 $\dfrac{1}{3}(2\omega_2 + x_3)(2(\omega_2 - x_3))^{1/2} = \varphi(\omega_1, \omega_2) - x_0, \quad \omega_1 = (x_1^2 + x_2^2)^{1/2},$
$\omega_2 = x_3 + \dfrac{(x_0+u)^2}{2}, \quad \varphi_1(\varphi_2 \det\varphi - \varphi_{11}) = 0.$

Анзаци (4.1)–(4.5) можна записати у вигляді (3), де $\omega = \omega(x,u) = (\omega_1(x,u), \omega_2(x,u))$ — інваріанти підгруп групи $P(1,4)$.

$$\sin\frac{x_2}{\omega_1} = \varphi(\omega_1,\omega_2) + \frac{1}{d}\ln(x_0+u), \quad \omega_1 = (x_1^2+x_2^2)^{1/2},$$
$$\omega_2 = (x_3^2+u^2-x_0^2)^{1/2}, \quad d>0, \quad \varphi_2\left[\frac{1}{d\omega_2}\left(\frac{1}{d}\varphi_1 - \frac{1}{\omega_1}\right)\det\varphi - \right.$$
$$-\frac{1}{d\omega_2}\varphi_2\left(3\varphi_1\varphi_2 - d\omega_2\varphi_1\varphi_2^2 + \frac{1}{d\omega_2}\varphi_1 - \frac{1}{\omega_1\omega_2}\right)\varphi_{11} -$$
$$-\frac{1}{d^2\omega_2}\left(d\varphi_1^3 + \frac{2}{\omega_1^2}\varphi_1 + \frac{1}{\omega_1^3}\right)\varphi_{22} + \frac{4}{d\omega_2}\varphi_2\left(\varphi_1^2 + \frac{1}{\omega_1^2}\right)\varphi_{12} +$$
$$\left. + \varphi_2\left(\frac{3}{d\omega_1^3\omega_2}\varphi_2 + \frac{1}{\omega_1^3}\varphi_2^2 + \frac{1}{\omega_2^2}\varphi_1^3 + \frac{2}{d\omega_1^2\omega_2^2}\varphi_1 + \frac{1}{d^2\omega_1^3\omega_2^2}\right)\right] = 0.$$

Інші анзаци редукують рівняння (1) до тривимірних диференціальних рівнянь з частинними похідними вигляду

$$A\det\varphi = B_1M_{11} + B_2M_{22} + B_3M_{33} + 2B_4M_{12} + 2B_5M_{13} + 2B_6M_{23} + P,$$

де

$$\det\varphi \equiv \begin{vmatrix} \varphi_{11} & \varphi_{12} & \varphi_{13} \\ \varphi_{21} & \varphi_{22} & \varphi_{23} \\ \varphi_{31} & \varphi_{32} & \varphi_{33} \end{vmatrix}, \quad M_{11} \equiv \begin{vmatrix} \varphi_{22} & \varphi_{23} \\ \varphi_{32} & \varphi_{33} \end{vmatrix}, \quad M_{22} \equiv \begin{vmatrix} \varphi_{11} & \varphi_{13} \\ \varphi_{31} & \varphi_{33} \end{vmatrix},$$
$$\varphi_{ij} \equiv \frac{\partial^2\varphi}{\partial\omega_i\partial\omega_j}, \quad i,j=1,2,3, \quad M_{33} \equiv \begin{vmatrix} \varphi_{11} & \varphi_{12} \\ \varphi_{21} & \varphi_{22} \end{vmatrix}, \quad M_{12} \equiv \begin{vmatrix} \varphi_{21} & \varphi_{23} \\ \varphi_{31} & \varphi_{33} \end{vmatrix},$$
$$M_{13} \equiv \begin{vmatrix} \varphi_{21} & \varphi_{22} \\ \varphi_{31} & \varphi_{32} \end{vmatrix}, \quad M_{23} \equiv \begin{vmatrix} \varphi_{11} & \varphi_{12} \\ \varphi_{31} & \varphi_{32} \end{vmatrix},$$

Вигляд коефіцієнтів A, $B_1,\ldots,B_6$, P залежить від розглядуваного анзацу. Нижче випишемо анзаци і відповідні їм коефіцієнти редукованого рівняння.

5.1 $u = \varphi(\omega_1,\omega_2,\omega_3), \quad \omega_1 = x_0, \quad \omega_2 = (x_1^2+x_2^2)^{1/2}, \quad \omega_3 = x_3,$
$A = \varphi_2, \quad B_i = 0, \quad i=1,\ldots,6, \quad P = 0, \quad \varphi_i \equiv \dfrac{\partial\varphi}{\partial\omega_i}, \quad i=1,2,3.$

5.2 $u^2 = -\varphi^2(\omega_1,\omega_2,\omega_3) + x_0^2, \quad \omega_1 = x_1, \quad \omega_2 = x_2, \quad \omega_3 = x_3,$
$A = 1, \quad B_i = 0, \quad i=1,\ldots,6, \quad P = 0.$

Анзаци 5.1 і 5.2 можна записати у вигляді (2), де $\omega = \omega(x) = (\omega_1(x),\omega_2(x),\omega_3(x))$ — тривимірні інваріанти підгруп групи $P(1,4)$.

6.1 $2x_0\omega_3 = -\varphi(\omega_1,\omega_2,\omega_3) + x_3^2, \quad \omega_1 = x_1, \quad \omega_2 = x_2, \quad \omega_3 = x_0+u,$
$A = \omega_3^2, \quad B_1 = B_2 = 0, \quad B_3 = 2(\omega_3\varphi_3 - \varphi), \quad B_4 = 0; \quad B_5 = \omega_3\varphi_1,$
$B_6 = -\omega_3\varphi_2, \quad P = 0.$

6.2 $\dfrac{1}{e}\arcsin\dfrac{x_2}{\omega_1} + \dfrac{1}{2}\arcsin\dfrac{x_3}{\omega_1} = \varphi(\omega_1,\omega_2,\omega_3), \quad \omega_1 = (x_1^2+x_2^2)^{1/2},$
$\omega_2 = (x_3^2+u^2)^{1/2}, \quad \omega_3 = x_0, \quad A = \omega_1^2\omega_2^2(e^2\omega_1\varphi_1 + 4\omega_2\varphi_2),$

$B_1 = -\omega_2^2(4e^2\omega_1^3\omega_2\varphi_1^3\varphi_2 + 8\omega_1\omega_2\varphi_1\varphi_2 - 1),$
$B_2 = -\omega_1^2(4e^2\omega_1\omega_2^3\varphi_1\varphi_2^3 + 2e^2\omega_1\omega_2\varphi_1\varphi_2 - 1), \quad B_3 = -4e^2\omega_1^3\omega_2^4\varphi_1\varphi_2\varphi_3^2,$
$B_4 = \omega_1\omega_2(e^2\omega_1^2\varphi_1^2\omega_2^2\varphi_2^2 + 4e^2\omega_1^2\omega_2^2\varphi_1^2\varphi_2^2 + 1),$
$B_5 = -4\omega_1\omega_2^3(e^2\omega_1^2\omega_1^2 + 1)\varphi_2\varphi_3, \quad B_6 = e^2\omega_1^3\omega_2\varphi_1\varphi_3(4\omega_2^2\varphi_2^2 + 1),$
$P = e^2\omega_1^3\varphi_1\varphi_3^2\varphi_{11} + 4\omega_2^3\varphi_2\varphi_3^2\varphi_{22} + (e^2\omega_1^3\varphi_1^3 + 2\omega_1\varphi_1 + 4\omega_2^3\varphi_2^3 +$
$+2\omega_2\varphi_2)\varphi_{33} - 2\omega_1(e^2\omega_1^2\varphi_1^2 + 1)\varphi_3\varphi_{13} -$
$- 2\omega_2(4\omega_2^2\varphi_2^2 + 1)\varphi_3\varphi_{23} - \varphi_3^2, \quad e \neq 0.$

6.3 $\arcsin\dfrac{x_2}{\omega_1} - \dfrac{1}{e}\operatorname{arch}\dfrac{x_0}{\omega_2} = \varphi(\omega_1,\omega_2,\omega_3), \quad \omega_1 = (x_1^2 + x_2^2)^{1/2},$
$\omega_2 = (x_0^2 - u^2)^{1/2}, \quad \omega_3 = x_3, \quad A = \omega_1^2\omega_2^2(\omega_1\varphi_1 - e^2\omega_2\varphi_2),$
$B_1 = \omega_2^2(e^2\omega_1^3\omega_2\varphi_1^3\varphi_2 + 2\omega_1\omega_2 e^2\varphi_1\varphi_2 + 1),$
$B_2 = \omega_1^2(e^2\omega_1\omega_2^3\varphi_1\varphi_2^3 - 2\omega_1\omega_2\varphi_1\varphi_2 + 1),$
$B_3 = e^2\omega_1^3\omega_2^3\varphi_1\varphi_2\varphi_3^2, \quad B_4 = \omega_1\omega_2(\omega_1^2\varphi_1^2 - e^2\omega_2^2\varphi_2^2 - e^2\omega_1^2\omega_2^2\varphi_1^2\varphi_2^2 + 1),$
$B_5 = e^2\omega_1\omega_2^3(\omega_1^2\varphi_1^2 + 1)\varphi_2\varphi_3, \quad B_6 = \omega_1^3\omega_2(e^2\omega_2^2\varphi_2^2 + 1)\varphi_1\varphi_3,$
$P = \omega_1^3\varphi_1\varphi_3^2\varphi_{11} - e^2\omega_2^3\varphi_2\varphi_3^2\varphi_{22} + (\omega_1^3\varphi_1^3 + 2\omega_1\varphi_1 - e^2\omega_2^3\varphi_2^3 +$
$+ 2\omega_2\varphi_2)\varphi_{33} - 2\omega_1(\omega_1^2\varphi_1^2 + 1)\varphi_3\varphi_{13} -$
$- 2\omega_2(e^2\omega_2^2\varphi_2^2 + 1)\varphi_3\varphi_{23} - \varphi_3^2, \quad e > 0.$

6.4 $\arcsin\dfrac{x_2}{\omega_1} + \dfrac{x_3}{\varepsilon\omega_2} = \varphi(\omega_1,\omega_2,\omega_3), \quad \omega_1 = (x_1^2 + x_2^2)^{1/2},$
$\omega_2 = x_0 + u, \quad \omega_3 = x_3^2 - 2x_0(x_0 + u), \quad \varepsilon = \pm 1,$
$$A = 4\left(\varphi_1 + \varphi_3\frac{2\omega_2^2}{\omega_1}\right), \quad B_1 = -4\left(2\omega_2^2\varphi_1^3\varphi_3 + \frac{4\omega_2^2}{\omega_1^2}\varphi_1\varphi_3 - \frac{1}{\omega_1^3}\right),$$
$$B_2 = -4\left(2\omega_2^2\varphi_1\varphi_2^2\varphi_3 + \frac{2}{\omega_2}\varphi_1\varphi_2 - \frac{2\omega_3}{\omega_2^3}\varphi_1\varphi_3 - \frac{4\omega_3}{\omega_1}\varphi_3^2 - \frac{1}{\omega_1\omega_2^2}\right),$$
$$B_3 = -8\omega_2^2\varphi_1\varphi_3^3, \quad B_4 = 4\left(2\omega_2^2\varphi_1^2\varphi_2\varphi_3 + \frac{2\omega_2^2}{\omega_1^2}\varphi_2\varphi_3 + \frac{1}{\omega_2}\varphi_1^2 + \frac{1}{\omega_1^2\omega_2}\right),$$
$$B_5 = -8\omega_2^2\left(\varphi_1^2 + \frac{1}{\omega_1^2}\right)\varphi_3^2, \quad B_6 = 8\varphi_3\left(\frac{\omega_2}{\omega_1}\varphi_3 + \frac{1}{\omega_2}\varphi_1 - \omega_2^2\varphi_1\varphi_2\varphi_3\right),$$
$$P = 8\varphi_3^2\left(2\omega_3\varphi_1\varphi_3^2 + 2\omega_2\varphi_1\varphi_2\varphi_3 + \frac{2}{\omega_2^2}\varphi_1\varphi_3 + \frac{1}{\omega_1}\varphi_3\right)\varphi_{11} +$$
$$+ \frac{8\omega_2^2}{\omega_1^3}\varphi_3^3\varphi_{22} + 4\left(\frac{2\omega_2^2}{\omega_1^3}\varphi_2^2\varphi_3 - \frac{2\omega_3}{\omega_1^3\omega_2^2}\varphi_3 + \frac{2}{\omega_1^3\omega_2}\varphi_2 + \frac{1}{\omega_2^2}\varphi_1^3 + \frac{2}{\omega_1^2\omega_2^2}\varphi_1 +\right.$$
$$\left. + \frac{8\omega_3}{\omega_1^2}\varphi_1\varphi_3^2 + 4\omega_3\varphi_1^3\varphi_3^2\right)\varphi_{33} - 16\omega_2\left(\varphi_1^2 + \frac{1}{\omega_1^2}\right)\varphi_3^3\varphi_{12} - 16\left(\frac{\omega_2}{\omega_1^2}\varphi_2\varphi_3 +\right.$$
$$\left. + \frac{2\omega_3}{\omega_1^2}\varphi_3^2 + \frac{1}{\omega_1^2\omega_2^2} + \frac{1}{\omega_2^2}\varphi_1^2 + \frac{\omega_3}{\varepsilon\omega_2}\varphi_1^2\varphi_3^2\right)\varphi_3\varphi_{13} -$$
$$- 16\omega_2\left(\frac{\omega_2}{\omega_1^3}\varphi_2 - \frac{2}{\omega_1}\varphi_1 - \varphi_1^3\right)\varphi_3^2\varphi_{23} -$$
$$- 8\left(\frac{2\omega_2}{\omega_1^3}\varphi_2\varphi_3 + \frac{2\omega_3}{\omega_1^3}\varphi_3^2 - \frac{2}{\omega_1^2}\varphi_1\varphi_3 - \varphi_1^3\varphi_3 + \frac{1}{\omega_1^3\varphi_2^2}\right)\varphi_3^2.$$

6.5 $\operatorname{arch}\dfrac{x_0}{\omega_3} = \varphi(\omega_1,\omega_2,\omega_3) - \dfrac{c}{\alpha}x_3, \quad \omega_1 = (x_1^2 + x_2^2)^{1/2},$

$$\omega_2 = \frac{x_3}{\alpha} + \arcsin\frac{x_2}{\sqrt{x_1^2+x_2^2}}, \quad \omega_3 = (x_0^2 - u^2)^{1/2}, \quad c > 0,\ \alpha \in \mathbb{R},\ \alpha > 0,$$

$$A = \frac{1}{\omega_3}\varphi_1 - \frac{c^2}{\omega_1}\varphi_3, \quad B_1 = \varphi_1^3\varphi_3 + \frac{2c}{\omega_1^2}\varphi_1\varphi_2\varphi_3 + \frac{1}{\omega_1^3\omega_3}\varphi_2^2,$$

$$B_2 = \varphi_1\varphi_2^2\varphi_3 + c^2, \quad B_3 = \varphi_1\varphi_3^3 - \frac{2}{\omega_3^2}\varphi_1\varphi_3 + \frac{c^2}{\omega_1\omega_3^3},$$

$$B_4 = \left(\frac{c^2}{\omega_1^2}\varphi_2 - \frac{c}{\omega_1^2\omega_3}\varphi_2^2 - \varphi_1^2\varphi_2 + c\varphi_1^2\right)\varphi_3,$$

$$B_5 = -\frac{1}{\omega_3^2}\varphi_1^2 - \frac{c}{\omega_1^2\omega_3^2}\varphi^2 + \varphi_1^2\varphi_3^2 + \frac{c}{\omega_1^2}\varphi^2\varphi_3^2,$$

$$B_6 = \left(c\varphi_3^2 + \frac{1}{\omega_3^2}\varphi_2 - \varphi_2\varphi_3^2 - \frac{c}{\omega_3^2}\right)\varphi_1,$$

$$P = \frac{1}{\omega_3^2}(\varphi_2 - c)^2\varphi_1\varphi_{11} - \left(\frac{1}{\omega_1^3}\varphi_2^2\varphi_3^3 - \frac{2}{\omega_1^3\omega_3^2}\varphi_2^2\varphi_3 - \frac{1}{\omega_3^3}\varphi_1^3 - \right.$$

$$\left. - \frac{2c}{\omega_1^2\omega_3^3}\varphi_1\varphi_2\right)\varphi_{22} - \frac{1}{\omega_1^3}(\varphi_2 - c)^2\varphi_2^2\varphi_3\varphi_{33} - \frac{2}{\omega_3^3}(\varphi_2 - c) \times$$

$$\times \left(\frac{1}{\omega_1^2} - \varphi_3^2\right)\varphi_{12} - \frac{2}{\omega_1^3}(\varphi_2 - c)\left(\frac{1}{\omega_3^2} - \varphi_3^2\right)\varphi_2^2\varphi_{23} - \frac{1}{\omega_1^3\omega_3^3}(\varphi_2 - c)^2\varphi_2^2.$$

6.6 $$\frac{1}{3\alpha}\left(\frac{2}{\alpha}\omega_3 + x_3\right)(2(\omega_3 - \alpha x_3))^{1/2} = \varphi(\omega_1,\omega_2,\omega_3)) - x_0,$$

$$\omega_1 = (x_1^2 + x_2^2)^{1/2}, \quad \omega_2 = \arcsin\frac{x_2}{\sqrt{x_1^2+x_2^2}} - \frac{x_0 + u}{\alpha},$$

$$\omega_3 = \alpha x_3 + \frac{(x_0+u)^2}{2}, \quad \alpha > 0, \quad A = \omega_1^2(\omega_1\varphi_1 + \alpha^2\varphi_3),$$

$$B_1 = \varphi_2^2, \quad B_2 = -\alpha^2\omega_1^3\varphi_1\varphi_3, \quad B_3 = \frac{\omega_1^2}{\alpha^2}, \quad B_4 = -\alpha^2\omega_1\varphi_2\varphi_3,$$

$$B_5 = \frac{\omega_1}{\alpha}\varphi_2, \quad B_6 = \frac{\omega_1^3}{\alpha}\varphi_1,$$

$$P = \frac{\omega_1^3}{\alpha^2}\varphi_1\varphi_{11} + \alpha^2\varphi_2^2\varphi_3\varphi_{33} + \frac{2\omega_1}{\alpha^2}\varphi_2\varphi_{12} - \frac{2}{\alpha}\varphi_2^2\varphi_{23} - \frac{1}{\alpha^2}\varphi_2^2.$$

6.7 $$\arcsin\frac{x_3}{\omega_3} = \varphi(\omega_1,\omega_2,\omega_3) - \frac{x_0}{\alpha}, \quad \omega_1 = (x_1^2 + x_2^2)^{1/2},$$

$$\omega_2 = \frac{x_0}{\alpha} - \arcsin\frac{x_2}{\sqrt{x_1^2+x_2^2}}, \quad \omega_3 = (x_3^2 + u^2)^{1/2}, \quad \alpha \in \mathbb{R},\ \alpha > 0,$$

$$A = \frac{1}{\omega_1}\varphi_3 + \frac{1}{\omega_3}\varphi_1, \quad B_1 = \frac{1}{\omega_1^3\omega_3}\varphi_2^2 + \frac{1}{\omega_1^2}\varphi_1\varphi_2^2\varphi_3 - \frac{2}{\omega_1^2}\varphi_1\varphi_2\varphi_3 - \varphi_1^3\varphi_3,$$

$$B_2 = -(\varphi_2 - 1)^2\varphi_1\varphi_3, \quad B_3 = \frac{1}{\omega_1\omega_3^3} - \frac{2}{\omega_3^2}\varphi_1\varphi_3 - \varphi_1\varphi_3^3,$$

$$B_4 = \left(\frac{1}{\omega_1^2}\varphi_2^2 - \frac{1}{\omega_1^2}\varphi_2 + \varphi_1^2\varphi_2 - \varphi_1^2\right)\varphi_3,$$

$$B_5 = -\varphi_1^2\varphi_3^2 - \frac{1}{\omega_3^2}\varphi_1^2 - \frac{1}{\omega_1^2}\varphi_2\varphi_3^2 - \frac{1}{\omega_1^2\omega_3^2}\varphi_2,$$

$$B_6 = \left(\frac{1}{\omega_3^2}\varphi_2 - \frac{1}{\omega_3^2}\varphi_2\varphi_3^2 - \varphi_3^2\right)\varphi_1,$$

$$P = \frac{1}{\omega_3^3}(\varphi_2 - 1)^2\varphi_1\varphi_{11} + \left(\frac{1}{\omega_1^3}\varphi_2^2\varphi_3^2 + \frac{2}{\omega_1^3\omega_3^2}\varphi_2^2\varphi_3 + \frac{1}{\omega_3^3}\varphi_1^3 + \right.$$
$$\left. + \frac{2}{\omega_1^2\omega_3^3}\varphi_1\varphi_2\right)\varphi_{22} + \frac{1}{\omega_1^3}(\varphi_2 - 1)^2\varphi_2^2\varphi_3\varphi_{33} - \frac{2}{\omega_3^3}(\varphi_2 - 1)\times$$
$$\times\left(\varphi_1^2 + \frac{1}{\omega_1^2}\varphi_2\right)\varphi_{12} - \frac{2}{\omega_1^3}\varphi_2^2(\varphi_2 - 1)\left(\varphi_3^2 + \frac{1}{\omega_3^2}\right)\varphi_{23} - \frac{1}{\omega_1^3\omega_3^3}(\varphi_2 - 1)^2\varphi_2^2.$$

6.8 $$\frac{1}{c}\arcsin\frac{x_3}{\omega_3} = \varphi(\omega_1, \omega_2, \omega_3) - \frac{x_0}{\alpha}, \quad \omega_1 = (x_1^2 + x_2^2)^{1/2},$$
$$\omega_2 = \frac{x_0}{\alpha} - \arcsin\frac{x_2}{\sqrt{x_1^2 + x_2^2}}, \quad \omega_3 = (x_3^2 + u^2)^{1/2},$$
$$0 < c < 1,\ \alpha \in \mathbb{R},\ \alpha > 0, \quad A = \frac{1}{\omega_1}\varphi_3 + \frac{1}{c^2\omega_3}\varphi_1,$$
$$B_1 = \frac{1}{c^2\omega_1^3\omega_3}\varphi_2^2 + \frac{1}{\omega_1^2}\varphi_1\varphi_2^2\varphi_3 - \frac{2}{\omega_1^2}\varphi_1\varphi_2\varphi_3 - \varphi_1^3\varphi_3,$$
$$B_2 = -(\varphi_2 - 1)^2\varphi_1\varphi_3, \quad B_3 = \frac{1}{c^2\omega_1\omega_3^3} - \frac{2}{c^2\omega_3^2}\varphi_1\varphi_3 - \varphi_1\varphi_3^3,$$
$$B_4 = \left(\frac{1}{\omega_1^2}\varphi_2^2 - \frac{1}{\omega_1^2}\varphi_2 + \varphi_1^2\varphi_2 - \varphi_1^2\right)\varphi_3,$$
$$B_5 = -\varphi_1^2\varphi_3^2 - \frac{1}{c^2\omega_3^2}\varphi_1^2 - \frac{1}{\omega_1^2}\varphi_2\varphi_3^2 - \frac{1}{c^2\omega_1^2\omega_3^2}\varphi_2,$$
$$B_6 = \left(\frac{1}{c^2\omega_3^2}\varphi_2 - \frac{1}{c^2\omega_3^2} + \varphi_2\varphi_3^2 - \varphi_3^2\right)\varphi_1, \quad P = \frac{1}{c^2\omega_3^3}(\varphi_2 - 1)^2\varphi_1\varphi_{11} +$$
$$+ \left(\frac{1}{\omega_1^3}\varphi_2^2\varphi_3^2 + \frac{2}{c^2\omega_1^3\omega_3^2}\varphi_2^2\varphi_3 + \frac{1}{c^2\omega_3^3}\varphi_1^3 + \frac{2}{c^2\omega_1^2\omega_3^3}\varphi_1\varphi_2\right)\varphi_{22} +$$
$$+ \frac{1}{\omega_1^3}(\varphi_2 - 1)^2\varphi_2^2\varphi_3\varphi_{33} - \frac{2}{c^2\omega_3^3}(\varphi_2 - 1)\left(\varphi_1^2 + \frac{1}{\omega_1^2}\varphi_2\right)\varphi_{12} -$$
$$- \frac{2}{\omega_1^3}\varphi_2^2(\varphi_2 - 1)\left(\varphi_3^2 + \frac{1}{c^2\omega_3^2}\right)\varphi_{23} - \frac{1}{c^2\omega_1^3\omega_3^3}(\varphi_2 - 1)^2\varphi_2^2.$$

Анзаци (6.1)–(6.8) можна записати у вигляді (3), де $\omega = \omega(x, u) = (\omega_1(x, u), \omega_2(x, u), \omega_3(x, u))$ — інваріанти підгруп групи $P(1, 4)$.

1. Фущич В.И., Серов Н.И., Симметрия и некоторые точные решения многомерного уравнения Монжа–Ампера, *Докл. АН СССР*, 1983, **273**, № 3, 543–546.
2. Fushchych W., Shteten W., Serov N., Symmetry analysis and exact solutions of equations of nonlinear mathematics physics, Dordrecht, Kluwer Academic Publishers, 1993, 435 p.
3. Федорчук В.М., Непрерывные подгруппы неоднородной группы де Ситтера $P(1, 4)$, Препринт № 78.18, Киев, Ин-т математики АН УССР, 1978, 36 с.
4. Федорчук В.М., Расщепляющиеся подалгебры алгебры Ли обобщенной группы Пуанкаре $P(1, 4)$, 1979, *Укр. мат. журн.*, **31**, № 6, 717–722.
5. Федорчук В.М., Нерасщепляющиеся подалгебры алгебры Ли обобщенной группы Пуанкаре $P(1, 4)$, *Укр. мат. журн.*, 1981, **33**, № 5, 696–700.
6. Федорчук В.М., Фущич В.И., О подгруппах обобщенной группы Пуанкаре, в кн. Теоретико-групповые методы в физике, Т. 1, М., Наука, 1980, 61–66.
7. Fushchych W.I., Barannik A.F., Barannik L.F., Fedorchuk V.M., Continious subgroups of the Poincaré group $P(1, 4)$, *J. Phys. A: Math. Gen.*, 1985, **18**, № 14, 2893–2899.

Symmetry reduction and exact solutions of the Navier–Stokes equations

W.I. FUSHCHYCH, R.O. POPOVYCH

Ansatzes for the Navier–Stokes field are described. These ansatzes reduce the Navier–Stokes equations to system of differential equations in three, two, and one independent variables. The large sets of exact solutions of the Navier–Stokes equations are constructed.

1 Introduction

The Navier–Stokes equations (NSEs)

$$\begin{aligned}&\vec{u}_t + (\vec{u}\cdot\vec{\nabla})\vec{u} - \triangle\vec{u} + \vec{\nabla}p = \vec{0},\\ &\operatorname{div}\vec{u} = 0\end{aligned} \tag{1.1}$$

which describe the motion of an incompressible viscous fluid are the basic equations of modern hydrodynamics. In (1.1) and below $\vec{u} = \{u^a(t,\vec{x})\}$ denotes the velocity field of a fluid, $p = p(t,\vec{x})$ denotes the pressure, $\vec{x} = \{x_a\}$, $\partial_t = \partial/\partial t$, $\partial_a = \partial/\partial x_a$, $\vec{\nabla} = \{\partial_a\}$, $\triangle = \vec{\nabla}\cdot\vec{\nabla}$ is the Laplacian, the kinematic coefficient of viscosity and fluid density are set equal to unity. Repead indices denote summation whereby we consider the indices a, b to take on values in $\{1,2,3\}$ and the indices i, j to take on values in $\{1,2\}$.

The problem of finding exact solutions of non-linear equations (1.1) is an important but rather complicated one. There are some ways to solve it. Considerable progress in this field can be achieved by means of making use of a symmetry approach. Equations (1.1) have non-trivial symmetry properties. It was known long ago [37, 2] that they are invariant under the eleven-parametric extended Galilei group. Let us denote it by $G_1(1,3)$. This group includes the Galilei group and scale transformations. The Lie algebra $AG_1(1,3)$ of $G_1(1,3)$ is generated by the operators

$$P_0,\quad J_{ab},\quad D,\quad P_a,\quad G_a,$$

where

$$\begin{aligned}&P_0 = \partial_t,\quad D = 2t\partial_t + x_a\partial_a - u^a\partial_{u^a} - 2p\partial_p,\\ &J_{ab} = x_a\partial_b - x_b\partial_a + u^a\partial_{u^b} - u^b\partial_{u^a},\quad a\neq b,\\ &G_a = t\partial_a + \partial_{u^a},\quad P_a = \partial_a.\end{aligned}$$

Relatively recently it was found by means of the Lie method [8, 5, 26] that the maximal Lie invariance algebra (MIA) of the NSEs (1.1) is the infinite-dimensional algebra $A(NS)$ with the basis elements

$$\partial_t,\quad D,\quad J_{ab},\quad R(\vec{m}),\quad Z(\chi), \tag{1.2}$$

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where

$$R(\vec{m}) = R(\vec{m}(t)) = m^a(t)\partial_a + m^a_t(t)\partial_{u^a} - m^a_{tt}(t)x_a\partial_p, \tag{1.3}$$

$$Z(\chi) = Z(\chi(t)) = \chi(t)\partial_p, \tag{1.4}$$

$m^a = m^a(t)$ and $\chi = \chi(t)$ are arbitrary smooth functions of t (degree of their smoothness is discussed in Note A.1).

The algebra $AG_1(1,3)$ is a subalgebra of $A(NS)$. Indeed, setting $m^a = \delta_{ab}$, where b is fixed, we obtain $R(\vec{m}) = \partial_b$, and if $m^a = \delta_{ab}t$ then $R(\vec{m}) = G_b$. Here δ_{ab} is the Kronecker symbol ($\delta_{ab} = 1$ if $a = b$, $\delta_{ab} = 0$ if $a \neq b$).

Operators (1.2) generate the following invariance transformations of system (1.1):

$$\begin{aligned}
&\partial_t: && \tilde{\vec{u}}(t,\vec{x}) = \vec{u}(t+\varepsilon,\vec{x}), \quad \tilde{p}(t,\vec{x}) = p(t+\varepsilon,\vec{x}) \\
& && \text{(translations with respect to } t\text{)}, \\
&J_{ab}: && \tilde{\vec{u}}(t,\vec{x}) = B\vec{u}(t,B^T\vec{x}), \quad \tilde{p}(t,\vec{x}) = p(t,B^T\vec{x}) \\
& && \text{(space rotations)}, \\
&D: && \tilde{\vec{u}}(t,\vec{x}) = e^{\varepsilon}\vec{u}(e^{2\varepsilon}t, e^{\varepsilon}\vec{x}), \quad \tilde{p}(t,\vec{x}) = e^{2\varepsilon}p(e^{2\varepsilon}t, e^{\varepsilon}\vec{x}) \\
& && \text{(scale transformations)}, \\
&R(\vec{m}): && \tilde{\vec{u}}(t,\vec{x}) = \vec{u}(t,\vec{x}-\vec{m}(t)) + \vec{m}_t(t), \\
& && \tilde{p}(t,\vec{x}) = p(t,\vec{x}-\vec{m}(t)) - \vec{m}_{tt}\cdot\vec{x} - \tfrac{1}{2}\vec{m}\cdot\vec{m}_{tt} \\
& && \text{(these transformations include the space translations} \\
& && \text{and the Galilei transformations)}, \\
&Z(\chi): && \tilde{\vec{u}}(t,\vec{x}) = \vec{u}(t,\vec{x}), \quad \tilde{p}(t,\vec{x}) = p(t,\vec{x}) + \chi(t).
\end{aligned} \tag{1.5}$$

Here $\varepsilon \in \mathbb{R}$, $B = \{\beta_{ab}\} \in O(3)$, i.e. $BB^T = \{\delta_{ab}\}$, B^T is the transposed matrix.

Besides continuous transformations (1.5) the NSEs admit discrete transformations of the form

$$\begin{aligned}
&\tilde{t} = t, \quad \tilde{x}_a = x_a,\ a \neq b, \quad \tilde{x}_b = -x_b, \\
&\tilde{p} = p, \quad \tilde{u}^a = u^a,\ a \neq b, \quad \tilde{u}^b = -u^b,
\end{aligned} \tag{1.6}$$

where b is fixed. Invariance under transformations (1.5) and (1.6) means that $(\tilde{\vec{u}},\tilde{p})$ is a solution of (1.1) if $(\vec{u},p)$ is a solution of (1.1).

A complete review of exact solutions found for the NSEs before 1963 is contained in [1]. We should like also to mark more modern reviews [16, 7, 36] despite their subjects slightly differ from subjects of our investigations. To find exact solutions of (1.1), symmetry approach in explicit form was used in [2, 31, 32, 6, 20, 21, 4, 17, 15, 12, 10, 11, 30]. This article is a continuation and a extention of our works [15, 12, 10, 11, 30]. In it we make symmetry reduction of the NSEs to systems of PDEs in three and two independent variables and to systems of ODEs, using subalgebraic structure of $A(NS)$. We investigate symmetry properties of the reduced systems of PDEs and construct exact solutions of the reduced systems of ODEs when it is possible. As a result, large classes of exact solutions of the NSEs are obtained.

The reduction problem for the NSEs is to describe ansatzes of the form [9]:

$$u^a = f^{ab}(t,\vec{x})v^b(\omega) + g^a(t,\vec{x}), \quad p = f^0(t,\vec{x})q(\omega) + g^0(t,\vec{x}) \tag{1.7}$$

that reduce system (1.1) in four independent variables to systems of differential equations in the functions v^a and q depending on the variables $\omega = \{\omega_n\}$ $(n = \overline{1,N})$, where N takes on a fixed value from the set $\{1,2,3\}$. In formulas (1.7) f^{ab}, g^a, f^0, g^0, and ω_n are smooth functions to be described. In such a general formulation the reduction problem is too complex to solve. But using Lie symmetry, some ansatzes (1.7) reducing the NSEs can be obtained. According to the Lie method, first a complete set of $A(NS)$-inequivalent subalgebras of dimension $M = 4 - N$ is to be constructed. For $N = 3$, $N = 2$, and $N = 1$ such sets are given in Subsections A.2, A.3, and A.4, correspondingly. Knowing subalgebraic structure of $A(NS)$, one can find explicit forms for the functions f^{ab}, g^a, f^0, g^0, and ω_n and obtain reduced systems in the functions v^k and q. This is made in Section 2 ($N = 3$), Section 3 ($N = 2$) and Section 4 ($N = 1$). Moreover, in Subsection 2.3 symmetry properties of all reduced systems of PDEs in three independent variables are investigated, and in Subsection 4.3 exact solutions of the reduced systems of ODEs are constructed. Symmetry properties and exact solutions of some reduced systems of PDEs in two independent variables are discussed in Sections 5 and 6. In Section 7 we make symmetry reduction of a some reduced system of PDEs in three independent variables.

In conclusion of the section, for convenience, we give some abbreviations, notations, and default rules used in this article.

Abbreviations:

the NSEs: the Navier–Stokes equations

the MIA: the maximal Lie invariance algebra (of either a some equation or a some system of equations)

a ODE: a ordinary differential equation

a PDE: a partial differential equation

Notations:

$C^\infty((t_0,t_1),\mathbb{R})$**:** the set of infinite-differentiable functions from (t_0,t_1) into $\mathbb{R}$, where $-\infty \le t_0 < t_1 \le +\infty$

$C^\infty((t_0,t_1),\mathbb{R}^3)$**:** the set of infinite-differentiable vector-functions from (t_0,t_1) into $\mathbb{R}^3$, where $-\infty \le t_0 < t_1 \le +\infty$

$\partial_t = \partial/\partial_t$, $\partial_a = \partial/\partial_{x_a}$, $\partial_{u^a} = \partial/\partial_{u^a}$, ...

Default rules:

Repead indices denote summation whereby we consider the indices a, b to take on values in $\{1,2,3\}$ and the indices i, j to take on values in $\{1,2\}$.

All theorems on the MIAs of PDEs are proved by means of the standard Lie algorithm.

Subscripts of functions denote differentiation.

2 Reduction of the Navier–Stokes equations to systems of PDEs in three independent variables

2.1 Ansatzes of codimension one

In this subsection we give ansatzes that reduce the NSEs to systems of PDEs in three independent variables. The ansatzes are constructed with the subalgebraic analysis of $A(NS)$ (see Subsection A.2) by means of the method discribed in Section B.

1. $$\begin{aligned} u^1 &= |t|^{-1/2}(v^1\cos\tau - v^2\sin\tau) + \tfrac{1}{2}x_1t^{-1} - \varkappa x_2t^{-1},\\ u^2 &= |t|^{-1/2}(v^1\sin\tau + v^2\cos\tau) + \tfrac{1}{2}x_2t^{-1} + \varkappa x_1t^{-1},\\ u^3 &= |t|^{-1/2}v^3 + \tfrac{1}{2}x_3t^{-1},\\ p &= |t|^{-1}q + \tfrac{1}{2}\varkappa^2t^{-2}r^2 + \tfrac{1}{8}t^{-2}x_ax_a, \end{aligned} \tag{2.1}$$

where

$$y_1 = |t|^{-1/2}(x_1\cos\tau + x_2\sin\tau),\quad y_2 = |t|^{-1/2}(-x_1\sin\tau + x_2\cos\tau),$$
$$y_3 = |t|^{-1/2}x_3,\quad \varkappa \ge 0,\quad \tau = \varkappa\ln|t|.$$

Here and below $v^a = v^a(y_1,y_2,y_3)$, $q = q(y_1,y_2,y_3)$, $r = (x_1^2 + x_2^2)^{1/2}$.

2. $$\begin{aligned} u^1 &= v^1\cos\varkappa t - v^2\sin\varkappa t - \varkappa x_2,\\ u^2 &= v^1\sin\varkappa t + v^2\cos\varkappa t + \varkappa x_1,\\ u^3 &= v^3,\\ p &= q + \tfrac{1}{2}\varkappa^2r^2, \end{aligned} \tag{2.2}$$

where

$$y_1 = x_1\cos\varkappa t + x_2\sin\varkappa t,\quad y_2 = -x_1\sin\varkappa t + x_2\cos\varkappa t,$$
$$y_3 = x_3,\quad \varkappa\in\{0;1\}.$$

3. $$\begin{aligned} u^1 &= x_1r^{-1}v^1 - x_2r^{-1}v^2 + x_1r^{-2},\\ u^2 &= x_2r^{-1}v^1 + x_1r^{-1}v^2 + x_2r^{-2},\\ u^3 &= v^3 + \eta(t)r^{-1}v^2 + \eta_t(t)\arctan x_2/x_1,\\ p &= q - \tfrac{1}{2}\eta_{tt}(t)(\eta(t))^{-1}x_3^2 - \tfrac{1}{2}r^{-2} + \chi(t)\arctan x_2/x_1, \end{aligned} \tag{2.3}$$

where

$$y_1 = t,\quad y_2 = r,\quad y_3 = x_3 - \eta(t)\arctan x_2/x_1,\quad \eta,\chi\in C^\infty((t_0,t_1),\mathbb{R}).$$

Note 2.1 The expression for the pressure p from ansatz (2.3) is indeterminate in the points $t\in(t_0,t_1)$ where $\eta(t)=0$. If there are such points t, we will consider ansatz (2.3) on the intervals (t_0^n,t_1^n) that are contained in the interval (t_0,t_1) and that satisfy one of the conditions:

a) $\eta(t)\ne 0\quad \forall\, t\in(t_0^n,t_1^n)$;

b) $\eta(t) = 0\quad \forall\, t\in(t_0^n,t_1^n)$.

In the last case we consider $\eta_{tt}/\eta := 0$.

$$4.\quad \vec{u} = v^i\vec{n}^i + (\vec{m}\cdot\vec{m})^{-1}v^3\vec{m} + (\vec{m}\cdot\vec{m})^{-1}(\vec{m}\cdot\vec{x})\vec{m}_t - y_i\vec{n}^i_t,$$

$$p = q - \tfrac{3}{2}(\vec{m}\cdot\vec{m})^{-1}((\vec{m}_t\cdot\vec{n}^i)y_i)^2 - (\vec{m}\cdot\vec{m})^{-1}(\vec{m}_{tt}\cdot\vec{x})(\vec{m}\cdot\vec{x}) + \qquad (2.4)$$

$$+ \tfrac{1}{2}(\vec{m}_{tt}\cdot\vec{m})(\vec{m}\cdot\vec{m})^{-2}(\vec{m}\cdot\vec{x})^2,$$

where

$$y_i = \vec{n}^i\cdot\vec{x}, \quad y_3 = t, \quad \vec{m}, \vec{n}^i \in C^\infty((t_0,t_1),\mathbb{R}^3).$$

$$\vec{n}^i\cdot\vec{m} = \vec{n}^1\cdot\vec{n}^2 = \vec{n}^1_t\cdot\vec{n}^2 = 0, \quad |\vec{n}^i| = 1. \qquad (2.5)$$

Note 2.2 There exist vector-functions $\vec{n}^i$ which satisfy conditions (2.5). They can be constructed in the following way: let us fix the vector-functions $\vec{k}^i = \vec{k}^i(t)$ such that $\vec{k}^i\cdot\vec{m} = \vec{k}^1\cdot\vec{k}^2 = 0$, $|\vec{k}^i| = 1$, and set

$$\begin{aligned} \vec{n}^1 &= \vec{k}^1\cos\psi(t) - \vec{k}^2\sin\psi(t),\\ \vec{n}^2 &= \vec{k}^1\sin\psi(t) + \vec{k}^2\cos\psi(t). \end{aligned} \qquad (2.6)$$

Then $\vec{n}^1_t\cdot\vec{n}^2 = \vec{k}^1_t\cdot\vec{k}^2 - \psi_t = 0$ if $\psi = \int(\vec{k}^1_t\cdot\vec{k}^2)dt$.

2.2 Reduced systems

1–2. Substituting ansatzes (2.1) and (2.2) into the NSEs (1.1), we obtain reduced systems of PDEs with the same general form

$$\begin{aligned} &v^a v^1_a - v^1_{aa} + q_1 + \gamma_1 v^2 = 0,\\ &v^a v^2_a - v^2_{aa} + q_2 - \gamma_1 v^1 = 0,\\ &v^a v^3_a - v^3_{aa} + q_3 = 0,\\ &v^a_a = \gamma_2. \end{aligned} \qquad (2.7)$$

Hereafter subscripts 1, 2, and 3 of functions denote differentiation with respect to y_1, y_2, and y_3, accordingly. The constants γ_i take the values

1. $\gamma_1 = -2\varkappa, \quad \gamma_2 = -\frac{3}{2} \quad$ if $\quad t>0, \quad \gamma_1 = 2\varkappa, \quad \gamma_2 = \frac{3}{2} \quad$ if $\quad t<0.$
2. $\gamma_1 = -2\varkappa, \quad \gamma_2 = 0.$

For ansatzes (2.3) and (2.4) the reduced equations have the form

$$\begin{aligned} 3.\quad &v^1_1 + v^1v^1_2 + v^3v^1_3 - y_2^{-1}v^2v^2 - \left(v^1_{22} + (1+\eta^2y_2^{-2})v^1_{33}\right) - 2\eta y_2^{-2}v^2_3 + q_2 = 0,\\ &v^2_1 + v^1v^2_2 + v^3v^2_3 + y_2^{-1}v^1v^2 - \left(v^2_{22} + (1+\eta^2y_2^{-2})v^2_{33}\right) +\\ &+ 2\eta y_2^{-2}v^1_3 + 2y_2^{-2}v^2 - \eta y_2^{-1}q_3 + \chi y_2^{-1} = 0,\\ &v^3_1 + v^1v^3_2 + v^3v^3_3 - \left(v^3_{22} + (1+\eta^2y_2^{-2})v^3_{33}\right) - 2\eta^2y_2^{-3}v^1_3 + 2\eta_1y_2^{-1}v^2 +\\ &+ 2\eta y_2^{-1}(y_2^{-1}v^2)_2 + (1+\eta^2y_2^{-2})q_3 - \eta_{11}\eta^{-1}y_3 - \chi\eta y_2^{-2} = 0,\\ &y_2^{-1}v^1 + v^1_2 + v^3_3 = 0. \end{aligned} \qquad (2.8)$$

$$\begin{aligned} 4.\quad &v^i_3 + v^jv^i_j - v^i_{jj} + q_i + \rho^i(y_3)v^3 = 0,\\ &v^3_3 + v^jv^3_j - v^3_{jj} = 0,\\ &v^i_i + \rho^3(y_3) = 0, \end{aligned} \qquad (2.9)$$

where

$$\begin{aligned}\rho^i &= \rho^i(y_3) = 2(\vec{m}\cdot\vec{m})^{-1}(\vec{m}_t\cdot\vec{n}^i),\\ \rho^3 &= \rho^3(y_3) = (\vec{m}\cdot\vec{m})^{-1}(\vec{m}_t\cdot\vec{m}).\end{aligned} \tag{2.10}$$

2.3 Symmetry of reduced systems

Let us study symmetry properties of systems (2.7), (2.8), and (2.9). All results of this subsection are obtained by means of the standard Lie algorithm [28, 27]. First, let us consider system (2.7).

Theorem 2.1 *The MIA of system (2.7) is the algebra*

$$\begin{aligned}&a)\quad \langle\partial_a,\partial_q,J^1_{12}\rangle \quad \textit{if}\quad \gamma_1\neq 0;\\ &b)\quad \langle\partial_a,\partial_q,J^1_{ab}\rangle \quad \textit{if}\quad \gamma_1=0,\ \gamma_2\neq 0;\\ &c)\quad \langle\partial_a,\partial_q,J^1_{ab},D^1_1\rangle \quad \textit{if}\quad \gamma_1=\gamma_2=0.\end{aligned}$$

Here $J^1_{ab}=y_a\partial_b-y_b\partial_a+v^a\partial_{v^b}-v^b\partial_{v^a}$, $D^1_1=y_a\partial_a-v^a\partial_{v^a}-2q\partial_q$.

Note 2.3 All Lie symmetry operators of (2.7) are induced by operators from $A(NS)$: The operators J^1_{ab} and D^1_1 are induced by J_{ab} and D. The operators $c_a\partial_a$ ($c_a=\mathrm{const}$) and ∂_q are induced by either

$$R(|t|^{1/2}(c_1\cos\tau-c_2\sin\tau,c_1\sin\tau+c_2\cos\tau,c_3)),\quad Z(|t|^{-1}),$$

where $\tau=\varkappa\ln|t|$, for ansats (2.1) or

$$R(c_1\cos\varkappa t-c_2\sin\varkappa t,c_1\sin\varkappa t+c_2\cos\varkappa t,c_3),\quad Z(1)$$

for ansatz (2.2), respectively. Therefore, Lie reductions of system (2.7) give only solutions that can be obtained by reducing the NSEs with two- and three-dimensional subalgebras of $A(NS)$.

Let us continue to system (2.8). We denote $A^{\max}$ as the MIA of (2.8). Studying symmetry properties of (2.8), one has to consider the following cases:

A. $\eta,\chi\equiv 0$. Then

$$A^{\max}=\langle\partial^1,D^1_2,R_1(\psi(y_1)),Z^1(\lambda(y_1))\rangle,$$

where

$$\begin{aligned}&D^1_2=2y_1\partial_1+y_2\partial_2+y_3\partial_3-v^a\partial_{v^a}-2q\partial_q,\\ &R_1(\psi(y_1))=\psi\partial_3+\psi_1\partial_{v^3}-\psi_{11}y_3\partial_q,\quad Z^1(\lambda(y_1))=\lambda(y_1)\partial_q.\end{aligned}$$

Here and below $\psi=\psi(y_1)$ and $\lambda=\lambda(y_1)$ are arbitrary smooth functions of $y_1=t$.

B. $\eta\equiv 0$, $\chi\not\equiv 0$. In this case an extension of $A^{\max}$ exists for $\chi=(C_1y_1+C_2)^{-1}$, where $C_1,C_2=\mathrm{const}$. Let $C_1\neq 0$. We can make C_2 vanish by means of equivalence transformation (A.6), i.e., $\chi=Cy_1^{-1}$, where $C=\mathrm{const}$. Then

$$A^{\max}=\langle D^1_2,R_1(\psi(y_1)),Z^1(\lambda(y_1))\rangle.$$

If $C_1 = 0$, $\chi = C = \mathrm{const}$ and

$$A^{\max} = \langle \partial_1, R_1(\psi(y_1)), Z^1(\lambda(y_1))\rangle.$$

For other values of χ, i.e., when $\chi_{11}\chi \neq \chi_1\chi_1$,

$$A^{\max} = \langle R_1(\psi(y_1)), Z^1(\lambda(y_1))\rangle.$$

C. $\eta \neq 0$. By means of equivalence transformation (A.6) we make $\chi = 0$. In this case an extension of $A^{\max}$ exists for $\eta = \pm|C_1y_1 + C_2|^{1/2}$, where $C_1, C_2 = \mathrm{const}$. Let $C_1 \neq 0$. We can make C_2 vanish by means of equivalence transformation (A.6), i.e., $\eta = C|y_1|^{1/2}$, where $C = \mathrm{const}$. Then

$$A^{\max} = \langle D^1_2, R_2(|y_1|^{1/2}), R_2(|y_1|^{1/2}\ln|y_1|), Z^1(\lambda(y_1))\rangle,$$

where $R_2(\psi(y_1)) = \psi\partial_3 + \psi_1\partial_{v^3}$. If $C_1 = 0$, i.e., $\eta = C = \mathrm{const}$,

$$A^{\max} = \langle \partial^1, \partial_3, y_1\partial_3 + \partial_{v^3} Z^1(\lambda(y_1))\rangle.$$

For other values of η, i.e., when $(\eta^2)_{11} \neq 0$,

$$A^{\max} = \langle R_2(\eta(y_1)), R_2(\eta(y_1)\textstyle\int(\eta(y_1))^{-2}dy_1), Z^1(\lambda(y_1))\rangle.$$

Note 2.4 In all cases considered above the Lie symmetry operators of (2.8) are induced by operators from $A(NS)$: The operators ∂_1, D^1_2, and $Z^1(\lambda(y_1))$ are induced by ∂_t, D, and $Z(\lambda(t))$, respectively. The operator $R(0,0,\psi(t))$ induces the operator $R_1(\psi(y_1))$ for $\eta \equiv 0$ and the operator $R_2(\psi(y_1))$ (if $\psi_{11}\eta - \psi\eta_{11} = 0$) for $\eta \neq 0$. Therefore, the Lie reduction of system (2.8) gives only solutions that can be obtained by reducing the NSEs with two- and three-dimentional subalgebras of $A(NS)$.

When $\eta = \chi = 0$, system (2.8) describes axially symmetric motion of a fluid and can be transformed into a system of two equations for a stream function Ψ^1 and a function Ψ^2 that are determined by

$$\Psi^1_3 = y_2v^1, \quad \Psi^1_2 = -y_2v^3, \quad \Psi^2 = y_2v^2.$$

The transformed system was studied by L.V. Kapitanskiy [20, 21].

Consider system (2.9). Let us introduce the notations

$$\begin{aligned}
&t = y_3, \quad \rho = \rho(t) = \textstyle\int \rho^3(t)dt,\\
&R_3(\psi^1(t), \psi^2(t)) = \psi^i\partial_{y_i} + \psi^i_t\partial_{v^i} - \psi^i_{tt}y_i\partial_q,\\
&Z^1(\lambda(t)) = \lambda(t)\partial_q, \quad S = \partial_{v^3} - \rho^i(t)y_i\partial_q,\\
&E(\chi(t)) = 2\chi\partial_t + \chi_t y_i\partial_{y_i} + (\chi_{tt}y_i - \chi_t v^i)\partial_{v^i} - (2\chi_t q + \tfrac{1}{2}\chi_{ttt}y_jy_j)\partial_q,\\
&J^1_{12} = y_1\partial_2 - y_2\partial_1 + v^1\partial_{v^2} - v^2\partial_{v^1}.
\end{aligned}$$

Theorem 2.2 *The MIA of(2.9) is the algebra*

1) $\langle R_3(\psi^1(t),\ \psi^2(t)),\ Z^1(\lambda(t)),\ S,\ E(\chi^1(t)),\ E(\chi^2(t)),\ v^3\partial_{v^3},\ J^1_{12}\rangle$,

where $\chi^1 = e^{-\rho(t)}\int e^{\rho(t)}dt$ *and* $\chi^2 = e^{-\rho(t)}$, *if* $\rho^i = 0$;

2) $\langle R_3(\psi^1(t),\ \psi^2(t)),\ Z^1(\lambda(t)),\ S,\ E(\chi(t)) + 2a_1v^3\partial_{v^3} + 2a_2J^1_{12}\rangle$,

where a_1, a_2, and a_3 are fixed constants, $\chi = e^{-\rho(t)}\left(\int e^{\rho(t)}dt + a_3\right)$, *if*

$$\rho^1 = e^{\frac{3}{2}\rho}\hat\rho^{-\frac{3}{2}-a_1}\big(C_1\cos(a_2\ln\hat\rho) - C_2\sin(a_2\ln\hat\rho)\big),$$
$$\rho^2 = e^{\frac{3}{2}\rho}\hat\rho^{-\frac{3}{2}-a_1}\big(C_1\sin(a_2\ln\hat\rho) + C_2\cos(a_2\ln\hat\rho)\big)$$

with $\hat\rho = \hat\rho(t) = |\int e^{\rho(t)}dt + a_3|$, $C_1, C_2 = \mathrm{const}$, $(C_1, C_2) \neq (0,0)$;

3) $\langle R_3(\psi^1(t), \psi^2(t)),\ Z^1(\lambda(t)),\ S,\ E(\chi(t)) + 2a_1v^3\partial_{v^3} + 2a_2J^1_{12}\rangle$,

where a_1 and a_2 are fixed constants, $\chi = e^{-\rho(t)}$, *if*

$$\rho^1 = e^{\frac{3}{2}\rho - a_1\hat\rho}\big(C_1\cos(a_2\hat\rho) - C_2\sin(a_2\hat\rho)\big),$$
$$\rho^2 = e^{\frac{3}{2}\rho - a_1\hat\rho}\big(C_1\sin(a_2\hat\rho) + C_2\cos(a_2\hat\rho)\big)$$

with $\hat\rho = \hat\rho(t) = \int e^{\rho(t)}dt$, $C_1, C_2 = \mathrm{const}$, $(C_1, C_2) \neq (0,0)$;

4) $\langle R_3(\psi^1(t),\ \psi^2(t)),\ Z^1(\lambda(t)),\ S\rangle$

in all other cases.

Here $\psi^i = \psi^i(t)$, $\lambda = \lambda(t)$ *are arbitrary smooth function of* $t = y_3$.

Note 2.5 If functions ρ^b are determined by (2.10), then $e^{\rho(t)} = C|\vec m(t)|$, where $C = \mathrm{const}$, and the condition $\rho^i = 0$ implies that $\vec m = |\vec m(t)|\vec e$, where $\vec e = \mathrm{const}$ and $|\vec e| = 1$.

Note 2.6 The vector-functions $\vec n^i$ from Note 2.2 are determined up to the transformation

$$\vec n^1 = \vec n^1\cos\delta - \vec n^2\sin\delta, \quad \vec n^2 = \vec n^1\sin\delta + \vec n^2\cos\delta,$$

where $\delta = \mathrm{const}$. Therefore, δ can be chosen such that $C_2 = 0$ (then $C_1 \neq 0$).

Note 2.7 The operators $R_3(\psi^1, \psi^2) + \alpha S$ and $Z^1(\lambda)$ are induced by $R(\vec l) + Z(\chi)$ and $Z(\lambda)$, respectively. Here $\vec l = \psi^i\vec n^i + \psi^3\vec m$, $\psi^3_t(\vec m\cdot\vec m) + 2\psi^i(\vec n^i_t\cdot\vec m) = \alpha$,

$$\chi - \tfrac{3}{2}(\vec m\cdot\vec m)^{-1}((\vec m_t\cdot\vec n^i)\psi^i)^2 - \tfrac{1}{2}(\vec m_{tt}\cdot\vec n^i)\psi^3\psi^i + \tfrac{1}{2}(\vec l_{tt}\cdot\vec n^i)\psi^i = 0.$$

If $\vec m = |\vec m|\vec e$, where $\vec e = \mathrm{const}$ and $|\vec e| = 1$, the operator J^1_{12} is induced by $e^1J_{23} + e^2J_{31} + e^3J_{12}$.

For

$$\vec m = \beta_3e^{\sigma t}(\beta_2\cos\tau, \beta_2\sin\tau, \beta_1)^T$$

with $\tau = \varkappa t + \delta$ and $\beta_a = \mathrm{const}$, where $\beta_1^2 + \beta_2^2 = 1$, the operator $\partial_t + \varkappa J_{12}$ induces the operator $\partial_{y_3} - \beta_1\varkappa J^1_{12} + \sigma v^3\partial_{v^3}$ if the following vector-functions $\vec n^i$ are chosen:

$$\vec n^1 = \vec k^1\cos\beta_1\tau + \vec k^2\sin\beta_1\tau, \quad \vec n^2 = -\vec k^1\sin\beta_1\tau + \vec k^2\cos\beta_1\tau, \tag{2.11}$$

where $\vec k^1 = (-\sin\tau, \cos\tau, 0)^T$ and $\vec k^2 = (\beta_1\cos\tau, \beta_1\sin\tau, -\beta_2)^T$.

For

$$\vec m = \beta_3|t + \beta_4|^{\sigma+1/2}(\beta_2\cos\tau, \beta_2\sin\tau, \beta_1)^T$$

with $\tau = \varkappa \ln|t+\beta_4| + \delta$ and $\beta_a, \beta_4 = \text{const}$, where $\beta_1^2 + \beta_2^2 = 1$, the operator $D + 2\beta_4\partial_t + 2\varkappa J_{12}$ induces the operator

$$D_3^1 + 2\beta_4\partial_{y_3} - 2\beta_1\varkappa J_{12}^1 + 2\sigma v^3\partial_{v^3},$$

where $D_3^1 = y_i\partial_{y_i} + 2y_3\partial_{y_3} - v^i\partial_{v^i} - 2q\partial_q$, if the vector-functions $\vec{n}^i$ are chosen in form (2.11). In all other cases the basis elements of the MIA of (2.9) are not induced by operators from $A(NS)$.

Note 2.8 The invariance algebras of systems of form (2.9) with different parameter-functions $\rho^3 = \rho^3(t)$ and $\tilde{\rho}^3 = \tilde{\rho}^3(t)$ are similar . It suggests that there exists a local transformation of variables which make ρ^3 vanish. So, let us transform variables in the following way:

$$\begin{aligned}
&\tilde{y}_i = y_i e^{\frac{1}{2}\rho(t)}, \quad \tilde{y}_3 = \textstyle\int e^{\rho(t)}dt,\\
&\tilde{v}^i = \left(v^i + \tfrac{1}{2}y_i\rho^3(t)\right)e^{-\frac{1}{2}\rho(t)}, \quad \tilde{v}^3 = v^3,\\
&\tilde{q} = qe^{-\rho(t)} + \tfrac{1}{8}y_iy_i\left((\rho^3(t)^2) - 2\rho^3_t(t)\right)e^{-\rho(t)}.
\end{aligned} \tag{2.12}$$

As a result, we obtain the system

$$\begin{aligned}
&\tilde{v}^i_3 + \tilde{v}^j\tilde{v}^i_j - \tilde{v}^i_{jj} + \tilde{q}_i + \tilde{\rho}^i(\tilde{y}_3)\tilde{v}^3 = 0,\\
&\tilde{v}^3_3 + \tilde{v}^j v^3_j - \tilde{v}^3_{jj} = 0,\\
&\tilde{v}^i_i = 0
\end{aligned}$$

for the functions $\tilde{v}^a = \tilde{v}^a(\tilde{y}_1, \tilde{y}_2, \tilde{y}_3)$ and $\tilde{q} = \tilde{q}(\tilde{y}_1, \tilde{y}_2, \tilde{y}_3)$. Here subscripts 1, 2, and 3 denote differentiation with respect to $\tilde{y}_1$, $\tilde{y}_2$, and $\tilde{y}_3$, accordingly. Also $\tilde{\rho}^i(\tilde{y}_3) = \rho^i(t)e^{-\frac{3}{2}\rho(t)}$.

3 Reduction of the Navier–Stokes equations to systems of PDEs in two independent variables

3.1 Ansatzes of codimension two

In this subsection we give ansatzes that reduce the NSEs to systems of PDEs in two independent variables. The ansatzes are constructed with the subalgebrical analysis of $A(NS)$ (see Subsection A.3) by means of the method discribed in Section B.

$$\begin{aligned}
1.\quad &u^1 = (rR)^{-1}((x_1 - \varkappa x_2)w^1 - x_2w^2 + x_1x_3r^{-1}w^3),\\
&u^2 = (rR)^{-1}((x_2 + \varkappa x_1)w^1 + x_1w^2 + x_2x_3r^{-1}w^3),\\
&u^3 = x_3(rR)^{-1}w^1 - R^{-1}w^3,\\
&p = R^{-2}s,
\end{aligned} \tag{3.1}$$

where $z_1 = \arctan x_2/x_1 - \varkappa \ln R$, $z_2 = \arctan r/x_3$, $\varkappa \geq 0$.

Here and below $w^a = w^a(z_1, z_2)$, $s = s(z_1, z_2)$, $r = (x_1^2 + x_2^2)^{1/2}$, $R = (x_1^2 + x_2^2 + x_3^2)^{1/2}$, $\varkappa$, ε, σ, μ, and ν are real constants.

$$\begin{aligned} 2.\quad u^1 &= |t|^{-1/2} r^{-1}(x_1 w^1 - x_2 w^2) + \tfrac{1}{2} t^{-1} x_1 + x_1 r^{-2}, \\ u^2 &= |t|^{-1/2} r^{-1}(x_2 w^1 + x_1 w^2) + \tfrac{1}{2} t^{-1} x_2 + x_2 r^{-2}, \\ u^3 &= |t|^{-1/2} w^3 + \varkappa r^{-1} w^2 + \tfrac{1}{2} t^{-1} x_3, \\ p &= |t|^{-1} s - \tfrac{1}{2} r^{-2} + \tfrac{1}{8} t^{-2} R^2 + \varepsilon |t|^{-1} \arctan x_2/x_1, \end{aligned} \tag{3.2}$$

where $z_1 = |t|^{-1/2} r$, $z_2 = |t|^{-1/2} x_3 - \varkappa \arctan x_2/x_1$, $\varkappa \geq 0$, $\varepsilon \geq 0$.

$$\begin{aligned} 3.\quad u^1 &= r^{-1}(x_1 w^1 - x_2 w^2) + x_1 r^{-2}, \\ u^2 &= r^{-1}(x_2 w^1 + x_1 w^2) + x_2 r^{-2}, \\ u^3 &= w^3 + \varkappa r^{-1} w^2, \\ p &= s - \tfrac{1}{2} r^{-2} + \varepsilon \arctan x_2/x_1, \end{aligned} \tag{3.3}$$

where $z_1 = r$, $z_2 = x_3 - \varkappa \arctan x_2/x_1$, $\varkappa \in \{0; 1\}$, $\varepsilon \geq 0$ if $\varkappa = 1$ and $\varepsilon \in \{0; 1\}$ if $\varkappa = 0$.

$$\begin{aligned} 4.\quad u^1 &= |t|^{-1/2}(\mu w^1 + \nu w^3)\cos\tau - |t|^{-1/2} w^2 \sin\tau + \\ &\quad + \nu\xi t^{-1}\cos\tau + \tfrac{1}{2} t^{-1} x_1 - \varkappa t^{-1} x_2, \\ u^2 &= |t|^{-1/2}(\mu w^1 + \nu w^3)\sin\tau + |t|^{-1/2} w^2 \cos\tau + \\ &\quad + \nu\xi t^{-1}\sin\tau + \tfrac{1}{2} t^{-1} x_2 + \varkappa t^{-1} x_1, \\ u^3 &= |t|^{-1/2}(-\nu w^1 + \mu w^3) + \mu\xi t^{-1} + \tfrac{1}{2} t^{-1} x_3, \\ p &= |t|^{-1} s - \tfrac{1}{2} t^{-2}\xi^2 + \tfrac{1}{8} t^{-2} R^2 + \tfrac{1}{2}\varkappa^2 t^{-2} r^2 + \\ &\quad + \varepsilon |t|^{-3/2}(\nu x_1 \cos\tau + \nu x_2 \sin\tau + \mu x_3), \end{aligned} \tag{3.4}$$

where

$$\begin{aligned} & z_1 = |t|^{-1/2}(\mu x_1 \cos\tau + \mu x_2 \sin\tau - \nu x_3), \\ & z_2 = |t|^{-1/2}(x_2 \cos\tau - x_1 \sin\tau), \\ & \xi = \sigma(\nu x_1 \cos\tau + \nu x_2 \sin\tau + \mu x_3) + 2\varkappa\nu(x_2 \cos\tau - x_1 \sin\tau), \\ & \tau = \varkappa \ln|t|, \quad \varkappa > 0, \quad \mu \geq 0, \quad \nu \geq 0, \quad \mu^2 + \nu^2 = 1, \quad \sigma\varepsilon = 0, \quad \varepsilon \geq 0. \end{aligned}$$

$$\begin{aligned} 5.\quad u^1 &= |t|^{-1/2} w^1 + \tfrac{1}{2} t^{-1} x_1, \\ u^2 &= |t|^{-1/2} w^2 + \tfrac{1}{2} t^{-1} x_2, \\ u^3 &= |t|^{-1/2} w^3 + (\sigma + \tfrac{1}{2}) t^{-1} x_3, \\ p &= |t|^{-1} s - \tfrac{1}{2}\sigma^2 t^{-2} x_3^2 + \tfrac{1}{8} t^{-2} R^2 + \varepsilon |t|^{-3/2} x_3, \end{aligned} \tag{3.5}$$

where

$$z_1 = |t|^{-1/2} x_1, \quad z_2 = |t|^{-1/2} x_2, \quad \sigma\varepsilon = 0, \quad \varepsilon \geq 0.$$

$$6.\quad \begin{aligned} &u^1 = (\mu w^1 + \nu w^3)\cos t - w^2 \sin t + \nu\xi \cos t - x_2,\\ &u^2 = (\mu w^1 + \nu w^3)\sin t + w^2 \cos t + \nu\xi \sin t + x_1,\\ &u^3 = (-\nu w^1 + \mu w^3) + \mu\xi,\\ &p = s - \tfrac{1}{2}\xi^2 + \tfrac{1}{2}r^2 + \varepsilon(\nu x_1 \cos t + \nu x_2 \sin t + \mu x_3), \end{aligned} \tag{3.6}$$

where

$$\begin{aligned} &z_1 = (\mu x_1 \cos t + \mu x_2 \sin t - \nu x_3),\\ &z_2 = (x_2 \cos t - x_1 \sin t),\\ &\xi = \sigma(\nu x_1 \cos t + \nu x_2 \sin t + \mu x_3) + 2\nu(x_2 \cos t - x_1 \sin t),\\ &\mu \geq 0, \quad \nu \geq 0, \quad \mu^2 + \nu^2 = 1, \quad \sigma\varepsilon = 0, \quad \varepsilon \geq 0. \end{aligned}$$

$$7.\quad \begin{aligned} &u^1 = w^1, \quad u^2 = w^2, \quad u^3 = w^3 + \sigma x_3,\\ &p = s - \tfrac{1}{2}\sigma^2 x_3^2 + \varepsilon x_3, \end{aligned} \tag{3.7}$$

where

$$z_1 = x_1, \quad z_2 = x_2, \quad \sigma\varepsilon = 0, \quad \varepsilon \in \{0; 1\}.$$

$$8.\quad \begin{aligned} &u^1 = x_1 w^1 - x_2 r^{-2}(w^2 - \chi(t)),\\ &u^2 = x_2 w^1 + x_1 r^{-2}(w^2 - \chi(t)),\\ &u^3 = (\rho(t))^{-1}(w^3 + \rho_t(t)x_3 + \varepsilon \arctan x_2/x_1),\\ &p = s - \tfrac{1}{2}\rho_{tt}(t)(\rho(t))^{-1}x_3^2 + \chi_t(t) \arctan x_2/x_1, \end{aligned} \tag{3.8}$$

where

$$z_1 = t, \quad z_2 = r, \quad \varepsilon \in \{0; 1\}, \quad \chi, \rho \in C^\infty((t_0, t_1), \mathbb{R}).$$

$$9.\quad \begin{aligned} &\vec{u} = \vec{w} + \lambda^{-1}(\vec{n}^i \cdot \vec{x})\vec{m}^i_t - \lambda^{-1}(\vec{k} \cdot \vec{x})\vec{k}_t,\\ &p = s - \tfrac{1}{2}\lambda^{-1}(\vec{m}^i_{tt} \cdot \vec{x})(\vec{n}^i \cdot \vec{x}) - \tfrac{1}{2}\lambda^{-2}(m^i_{tt} \cdot \vec{k})(\vec{n}^i \cdot \vec{x})(\vec{k} \cdot \vec{x}), \end{aligned} \tag{3.9}$$

where

$$\begin{aligned} &z_1 = t, \quad z_2 = (\vec{k} \cdot \vec{x}), \quad \vec{m}^i \in C^\infty((t_0, t_1), \mathbb{R}^3),\\ &\vec{m}^1_{tt} \cdot \vec{m}^2 - \vec{m}^1 \cdot \vec{m}^2_{tt} = 0, \quad \vec{k} = \vec{m}^1 \times \vec{m}^2, \quad \vec{n}^1 = \vec{m}^2 \times \vec{k},\\ &\vec{n}^2 = \vec{k} \times \vec{m}^1, \quad \lambda = \lambda(t) = \vec{k} \cdot \vec{k} \neq 0 \quad \forall\, t \in (t_0, t_1). \end{aligned}$$

3.2 Reduced systems

Substituting ansatzes (3.1)–(3.9) into the NSEs (1.1), we obtain the following systems of reduced equations:

$$1.\quad \begin{aligned} &w^2 w^1_1 + w^3 w^1_2 - w^1 w^3 \cot z_2 - (w^1)^2 - (w^2 + \varkappa w^1)^2 \sin^2 z_2 -\\ &- (w^3)^2 - \big((\varkappa^2 + \sin^{-2} z_2)w^1_{11} + w^1_{22} - \varkappa w^1_1 - 2w^3_2 - 2w^2_1 -\\ &- 2w^1\big) \sin z_2 + w^1_2 \cos z_2 - w^1 \sin^{-1} z_2 - (2s + \varkappa s_1) \sin^2 z_2 = 0, \end{aligned}$$

$$
\begin{aligned}
& w^2w^2_1 + w^3w^2_2 + w^3(w^2 + 2\varkappa w^1)\cot z_2 - \\
& - \varkappa\big((w^1)^2 + (w^3)^2 + (w^2 + \varkappa w^1)^2\sin^2 z_2\big) - \\
& - \big((\varkappa^2 + \sin^{-2} z_2)w^2_{11} + w^2_{22} + 3\varkappa w^2_1 + 2\varkappa(w^3_2 + \varkappa w^1_1 + w^1)\big)\sin z_2 + \\
& + (2w^1_1 + 2w^3_1\cot z_2 - w^2 - 2\varkappa w^1)\sin^{-1} z_2 - \\
& - (w^2_2 + 2\varkappa w^1_2)\cos z_2 + 2\varkappa s\sin^2 z_2 + (1 + \varkappa^2\sin^2 z_2)s_1 = 0, \\
& w^2w^3_1 + w^3w^3_2 - (w^3)^2\cot z_2 - (w^2 + \varkappa w^1)^2\sin z_2\cos z_2 - \\
& - \big((\varkappa^2 + \sin^{-2} z_2)w^3_{11} + w^3_{22} + \varkappa w^3_1 + 2w^1_2\big)\sin z_2 + \\
& + (2w^1 + w^3_2 + w^2_1 + \varkappa w^1_1)\cos z_2 + s_2\sin^2 z_2 = 0, \\
& w^1 + w^2_1 + w^3_2 = 0.
\end{aligned}
\tag{3.10}
$$

Hereafter numeration of the reduced systems corresponds to that of the ansatzes in Subsection 3.1. Subscripts 1 and 2 denote differentiation with respect to the variables z_1 and z_2, accordingly.

2–3.
$$
\begin{aligned}
& w^1w^1_1 + w^3w^1_2 - z_1^{-1}w^2w^2 - \big(w^1_{11} + (1 + \varkappa^2 z_1^{-2})w^1_{22}\big) - \\
& - 2\varkappa z_1^{-2}w^2_2 + s_1 = 0, \\
& w^1w^2_1 + w^3w^2_2 + z_1^{-1}w^1w^2 - \big(w^2_{11} + (1 + \varkappa^2 z_1^{-2})w^2_{22}\big) + \\
& + 2\varkappa z_1^{-2}w^1_2 + 2z_1^{-2}w^2 - \varkappa z_1^{-1}s_2 + \varepsilon z_1^{-1} = 0, \\
& w^1w^3_1 + w^3w^3_2 - 2\varkappa z_1^{-2}w^1w^2 - \big(w^3_{11} + (1 + \varkappa^2 z_1^{-2})w^3_{22}\big) + \\
& + 2\varkappa(z_1^{-2}w^2)_1 - 2\varkappa^2 z_1^{-3}w^1_2 + (1 + \varkappa^2 z_1^{-2})s_2 - \varepsilon\varkappa z_1^{-2} = 0, \\
& w^1_1 + w^3_2 + z_1^{-1}w^1 + \gamma = 0,
\end{aligned}
\tag{3.11}
$$

where $\gamma = \pm 3/2$ for ansatz (3.2) and $\gamma = 0$ for ansatz (3.3). Here and below the upper and lower sign in the symbols "$\pm$" and "$\mp$" are associated with $t > 0$ and $t < 0$, respectively.

4–7. For ansatzes (3.4)–(3.7) the reduced equations can be written in the form

$$
\begin{aligned}
& w^iw^1_i - w^1_{ii} + s_1 + \alpha_2 w^2 = 0, \\
& w^iw^2_i - w^2_{ii} + s_2 - \alpha_2 w^1 + \alpha_1 w^3 = 0, \\
& w^iw^3_i - w^3_{ii} + \alpha_4 w^3 + \alpha_5 = 0, \\
& w^i_i = \alpha_3
\end{aligned}
\tag{3.12}
$$

where the constants α_n $(n = \overline{1,5})$, take on the values

4. $\alpha_1 = \pm 2\varkappa\nu$, $\alpha_2 = \mp 2\varkappa\mu$, $\alpha_3 = \mp(\sigma + 3/2)$, $\alpha_4 = \pm\sigma$, $\alpha_5 = \varepsilon$.
5. $\alpha_1 = 0$, $\alpha_2 = 0$, $\alpha_3 = \mp(\sigma + 3/2)$, $\alpha_4 = \pm\sigma$, $\alpha_5 = \varepsilon$.
6. $\alpha_1 = 2\nu$, $\alpha_2 = -2\mu$, $\alpha_3 = -\sigma$, $\alpha_4 = \sigma$, $\alpha_5 = \varepsilon$.
7. $\alpha_1 = 0$, $\alpha_2 = 0$, $\alpha_3 = -\sigma$, $\alpha_4 = \sigma$, $\alpha_5 = \varepsilon$.

8.
$$
\begin{aligned}
& w^1_1 + (w^1)^2 - z_2^{-4}(w^2 - \chi)^2 + z_2w^1w^1_2 - w^1_{22} - \\
& - 3z_2w^1_2 + z_2^{-1}s_2 = 0,
\end{aligned}
\tag{3.13}
$$

$$
w^2_1 + z_2w^1w^2_2 - w^2_{22} + z_2^{-1}w^2_2 = 0,
\tag{3.14}
$$

$$
w^3_1 + z_2w^1w^3_2 - w^3_{22} - z_2^{-1}w^3_2 + z_2^{-2}(w^2 - \chi) = 0,
\tag{3.15}
$$

$$2w^1 + z_2 w^1_2 + \rho_1/\rho = 0. \tag{3.16}$$

9. $$\vec{w}_1 - \lambda\vec{w}_{22} + s_2\vec{k} + \lambda^{-1}(\vec{n}^i\cdot\vec{w})\vec{m}^i_t + z_2\vec{e} = \vec{0}, \tag{3.17}$$

$$\vec{k}\cdot\vec{w}_2 = 0, \tag{3.18}$$

where $y_1 = t$ and

$$\vec{e} = \vec{e}(t) = 2\lambda^{-2}(\vec{m}^1_t\cdot\vec{m}^2 - \vec{m}^1\cdot\vec{m}^2_t)\vec{k}_t\times\vec{k} + \lambda^{-2}(2\vec{k}_t\cdot\vec{k}_t - \vec{k}_{tt}\cdot\vec{k}).$$

Let us study symmetry properties of reduced systems (3.10) and (3.11).

Theorem 3.1 *The MIA of (3.10) is given by the algebra* $\langle\partial_1\rangle$.

Theorem 3.2 *The MIA of (3.11) is given by the following algebras:*

a) $\langle\partial_2, \partial_s, D^2_1 = z_i\partial_i - w^a\partial_{w^a} - 2s\partial_s\rangle$ *if* $\gamma = \varkappa = \varepsilon = 0$;

b) $\langle\partial_2, \partial_s\rangle$ *if* $(\gamma, \varkappa, \varepsilon) \neq (0,0,0)$.

All the Lie symmetry operators of systems (3.10) and (3.11) are induced by elements of $A(NS)$. So, for system (3.10) the operator ∂_1 is induced by J_{12}. For system (3.11), when $\gamma = 0$ ($\gamma = \pm 3/2$), the operators D^2_1, ∂_2, and ∂_s (∂_2 and ∂_s) are induced by D, $R(0,0,1)$, and $Z(1)$ ($R(0,0,|t|^{-1/2})$ and $Z(|t|^{-1})$), accordingly. Therefore, the Lie reductions of systems (3.10) and (3.11) give only solutions that can be obtained by reducing the NSEs with three-dimensional subalgebras of $A(NS)$ immediately to ODEs.

Investigation of reduced systems (3.13)–(3.16), (3.17)–(3.18), and (3.12) is given in Sections 5 and 6.

4 Reduction of the Navier–Stokes equations to ordinary differential equations

4.1 Ansatzes of codimension three

By means of subalgebraic analysis of $A(NS)$ (see Subsection A.3) and the method described in Section B one can obtain the following ansatzes that reduce the NSEs to ODEs:

1. $$\begin{aligned} u^1 &= x_1R^{-2}\varphi^1 - x_2(Rr)^{-1}\varphi^2 + x_1x_3r^{-1}R^{-2}\varphi^3,\\ u^2 &= x_2R^{-2}\varphi^1 + x_1(Rr)^{-1}\varphi^2 + x_2x_3r^{-1}R^{-2}\varphi^3,\\ u^3 &= x_3R^{-2}\varphi^1 - rR^{-2}\varphi^3,\\ p &= R^{-2}h, \end{aligned} \tag{4.1}$$

where $\omega = \arctan r/x_3$. Here and below $\varphi^a = \varphi^a(\omega)$, $h = h(\omega)$, $r = (x_1^2 + x_2^2)^{1/2}$, $R = (x_1^2 + x_2^2 + x_3^2)^{1/2}$.

2. $$\begin{aligned} &u^1 = r^{-2}(x_1\varphi^1 - x_2\varphi^2), \quad u^2 = r^{-2}(x_2\varphi^1 + x_1\varphi^2),\\ &u^3 = r^{-1}\varphi^3, \quad p = r^{-2}h, \end{aligned} \tag{4.2}$$

where $\omega = \arctan x_2/x_1 - \varkappa \ln r$, $\varkappa \geq 0$.

$$\begin{aligned}
3.\quad & u^1 = x_1|t|^{-1}\varphi^1 - x_2r^{-2}\varphi^2 + \tfrac{1}{2}x_1t^{-1},\\
& u^2 = x_2|t|^{-1}\varphi^1 + x_1r^{-2}\varphi^2 + \tfrac{1}{2}x_2t^{-1},\\
& u^3 = |t|^{-1/2}\varphi^3 + (\sigma + \tfrac{1}{2})x_3t^{-1} + \nu|t|^{1/2}t^{-1}\arctan x_2/x_1,\\
& p = |t|^{-1}h + \tfrac{1}{8}t^{-2}R^2 - \tfrac{1}{2}\sigma^2x_3^2t^{-2} + \\
& \qquad + \varepsilon_1|t|^{-1}\arctan x_2/x_1 + \varepsilon_2x_3|t|^{-3/2},
\end{aligned} \tag{4.3}$$

where $\omega = |t|^{-1/2}r$, $\nu\sigma = 0$, $\varepsilon_2\sigma = 0$, $\varepsilon_1 \geq 0$, $\nu \geq 0$.

$$\begin{aligned}
4.\quad & u^1 = x_1\varphi^1 - x_2r^{-2}\varphi^2,\\
& u^2 = x_2\varphi^1 + x_1r^{-2}\varphi^2,\\
& u^3 = \varphi^3 + \sigma x_3 + \nu \arctan x_2/x_1,\\
& p = h - \tfrac{1}{2}\sigma^2x_3^2 + \varepsilon_1 \arctan x_2/x_1 + \varepsilon_2x_3,
\end{aligned} \tag{4.4}$$

where $\omega = r$, $\nu\sigma = 0$, $\varepsilon_2\sigma = 0$, and for $\sigma = 0$ one of the conditions

$$\nu = 1,\ \varepsilon_1 \geq 0; \quad \nu = 0,\ \varepsilon_1 = 1,\ \varepsilon_2 \geq 0; \quad \nu = \varepsilon_1 = 0,\ \varepsilon_2 \in \{0; 1\}$$

is satisfied.

Two ansatzes are described better in the following way:

5. The expressions for u^a and p are determined by (2.1), where

$$\begin{aligned}
& v^1 = a_1\varphi^1 + a_2\varphi^3 + b_{1i}\omega_i,\\
& v^2 = \varphi^2 + b_{2i}\omega_i,\\
& v^3 = a_2\varphi^1 - a_1\varphi^3 + b_{3i}\omega_i,\\
& p = h + c_{1i}\omega_i + c_{2i}\omega\omega_i + \tfrac{1}{2}d_{ij}\omega_i\omega_j.
\end{aligned} \tag{4.5}$$

In formulas (4.5) we use the following definitions:

$$\begin{aligned}
& \omega_1 = a_1y_1 + a_2y_3, \quad \omega_2 = y_2, \quad \omega = \omega_3 = a_2y_1 - a_1y_3;\\
& a_i = \text{const}, \quad a_1^2 + a_2^2 = 1; \quad a_2 = 0 \text{ if } \gamma_1 = 0;\\
& \gamma_1 = -2\varkappa,\ \gamma_2 = -\tfrac{3}{2} \quad \text{if} \quad t > 0 \quad \text{and} \quad \gamma_1 = 2\varkappa,\ \gamma_2 = \tfrac{3}{2} \quad \text{if} \quad t < 0.
\end{aligned}$$

b_{ai}, B_i, c_{ij}, and d_{ij} are real constants that satisfy the equations

$$\begin{aligned}
& b_{1i} = a_1B_i, \quad b_{3i} = a_2B_i, \quad c_{2i} + a_2\gamma_1b_{2i} = 0,\\
& b_{21}B_i + b_{22}b_{2i} - \gamma_1a_1B_i + d_{2i} = 0,\\
& B_1B_i + B_2b_{2i} + \gamma_1a_1B_i + d_{1i} = 0,\\
& (B_1 + b_{22})(B_2 + a_1\gamma_1 - b_{21}) = 0.
\end{aligned} \tag{4.6}$$

6. The expressions for u^a and p have form (2.2), where v^a and q are determined by (4.5), (4.6), and $\gamma_1 = -2\varkappa$, $\gamma_2 = 0$.

Note 4.1 Formulas (4.5) and (4.6) determine an ansatz for system (2.7), where equations (4.6) are the necessary and sufficient condition to reduce system (2.7) by means of an ansatz of form (4.5).

$$
\begin{aligned}
7.\quad & u^1=\varphi^1\cos x_3/\eta^3-\varphi^2\sin x_3/\eta^3+x_1\theta^1(t)+x_2\theta^2(t),\\
& u^2=\varphi^1\sin x_3/\eta^3+\varphi^2\cos x_3/\eta^3-x_1\theta^2(t)+x_2\theta^1(t),\\
& u^3=\varphi^3+\eta^3_t(\eta^3)^{-1}x_3,\\
& p=h-\tfrac{1}{2}\eta^3_{tt}(\eta^3)^{-1}x_3^2-\tfrac{1}{2}\eta^j_{tt}\eta^j(\eta^i\eta^i)^{-1}r^2,
\end{aligned}
\tag{4.7}
$$

where $\omega=t$,

$$
\eta^a\in C^\infty((t_0,t_1),\mathbb{R}),\quad \eta^3\neq 0,\quad \eta^i\eta^i\neq 0,\quad \eta^1_t\eta^2-\eta^1\eta^2_t\in\{0;\tfrac{1}{2}\},
$$
$$
\theta^1=\eta^i_t\eta^i(\eta^j\eta^j)^{-1},\quad \theta^2=(\eta^1_t\eta^2-\eta^1\eta^2_t)(\eta^j\eta^j)^{-1}.
$$

$$
\begin{aligned}
8.\quad & \vec u=\vec\varphi+\lambda^{-1}(\vec n^a\cdot\vec x)\vec m^a_t,\\
& p=h-\lambda^{-1}(\vec m^a_{tt}\cdot\vec x)(\vec n^a\cdot\vec x)+\tfrac{1}{2}\lambda^{-2}(\vec m^b_{tt}\cdot\vec m^a)(\vec n^a\cdot\vec x)(\vec n^b\cdot\vec x),
\end{aligned}
\tag{4.8}
$$

where $\omega=t$, $\vec m^a\in C^\infty((t_0,t_1),\mathbb{R})$, $\vec m^a_{tt}\cdot\vec m^b-\vec m^a\cdot\vec m^b_{tt}=0$,

$$
\lambda=\lambda(t)=(\vec m^1\times\vec m^2)\cdot\vec m^3\neq 0\quad \forall t\in(t_0,t_1),
$$
$$
\vec n^1=\vec m^2\times\vec m^3,\quad \vec n^2=\vec m^3\times\vec m^1,\quad \vec n^3=\vec m^1\times\vec m^2.
$$

4.2 Reduced systems

Substituting the ansatzes 1–8 into the NSEs (1.1), we obtain the following systems of ODE in the functions φ^a and h:

$$
\begin{aligned}
1.\quad & \varphi^3\varphi^1_\omega-\varphi^a\varphi^a-\varphi^1_{\omega\omega}-\varphi^1_\omega\cot\omega-2h=0,\\
& \varphi^3\varphi^2_\omega+\varphi^2\varphi^3\cot\omega-\varphi^2_{\omega\omega}-\varphi^2_\omega\cot\omega+\varphi^2\sin^{-2}\omega=0,\\
& \varphi^3\varphi^3_\omega-\varphi^2\varphi^2\cot\omega-\varphi^3_{\omega\omega}-\varphi^3_\omega\cot\omega+\varphi^3\sin^{-2}\omega-2\varphi^1_\omega+h_\omega=0,\\
& \varphi^1+\varphi^3_\omega+\varphi^3\cot\omega=0.
\end{aligned}
\tag{4.9}
$$

$$
\begin{aligned}
2.\quad & (\varphi^2-\varkappa\varphi^1)\varphi^1_\omega-(1+\varkappa^2)\varphi^1_{\omega\omega}-\varphi^1\varphi^1-\varphi^2\varphi^2-\varkappa h_\omega-2h=0,\\
& (\varphi^2-\varkappa\varphi^1)\varphi^2_\omega-(1+\varkappa^2)\varphi^2_{\omega\omega}-2(\varkappa\varphi^2_\omega+\varphi^1_\omega)+h_\omega=0,\\
& (\varphi^2-\varkappa\varphi^1)\varphi^3_\omega-(1+\varkappa^2)\varphi^3_{\omega\omega}-\varphi^1\varphi^3-\varphi^3-2\varkappa\varphi^3_\omega=0,\\
& \varphi^2_\omega-\varkappa\varphi^1_\omega=0.
\end{aligned}
\tag{4.10}
$$

$$
\begin{aligned}
3\text{–}4.\quad & \varphi^1\varphi^1-\omega^{-4}\varphi^2\varphi^2+\omega\varphi^1\varphi^1_\omega-\varphi^1_{\omega\omega}-3\omega^{-1}\varphi^1_\omega+\omega^{-1}h_\omega=0,\\
& \omega\varphi^1\varphi^2_\omega-\varphi^2_{\omega\omega}+\omega^{-1}\varphi^2_\omega+\varepsilon_1=0,\\
& \omega\varphi^1\varphi^3_\omega+\sigma_1\varphi^3+\nu\omega^{-2}\varphi^2-\varphi^3_{\omega\omega}-\omega^{-1}\varphi^3_\omega+\varepsilon_2=0,\\
& 2\varphi^1+\omega\varphi^1_\omega+\sigma_2=0,
\end{aligned}
\tag{4.11}
$$

where

3. $\sigma_1=\sigma,\qquad \sigma_2=(\sigma+\tfrac{3}{2})\qquad$ if $\quad t>0,$
   $\sigma_1=-\sigma,\qquad \sigma_2=-(\sigma+\tfrac{3}{2})\qquad$ if $\quad t<0.$
4. $\sigma_1=\sigma_2=\sigma.$

$$
\begin{aligned}
5\text{–}6.\quad & \varphi^3\varphi^1_\omega-\varphi^1_{\omega\omega}-\mu_{1i}\varphi^i+c_{11}+c_{21}\omega=0,\\
& \varphi^3\varphi^2_\omega-\varphi^2_{\omega\omega}-\mu_{2i}\varphi^i+c_{12}+c_{22}\omega+\gamma_2a_2\varphi^3=0,\\
& \varphi^3\varphi^3_\omega-\varphi^3_{\omega\omega}+\gamma_1a_2\varphi^2+h_\omega=0,\\
& \varphi^3_\omega=\sigma,
\end{aligned}
\tag{4.12}
$$

where $\mu_{11}=-B_1$, $\mu_{12}=-B_2-\gamma_1a_1$, $\mu_{21}=-b_{21}+\gamma_1a_1$, $\mu_{22}=-b_{22}$, $\sigma=\gamma_1-B_1-b_{22}$.

$$\begin{aligned}
7.\quad &\varphi^1_\omega+\theta^1\varphi^1+\theta^2\varphi^2-(\eta^3)^{-1}\varphi^3\varphi^2+(\eta^3)^{-2}\varphi^1=0,\\
&\varphi^2_\omega-\theta^2\varphi^1+\theta^1\varphi^2+(\eta^3)^{-1}\varphi^3\varphi^1+(\eta^3)^{-2}\varphi^2=0,\\
&\varphi^3_\omega+\eta^3_t(\eta^3)^{-1}\varphi^3=0,\\
&2\theta^1+\eta^3_t(\eta^3)^{-1}=0.
\end{aligned} \tag{4.13}$$

$$\begin{aligned}
8.\quad &\vec\varphi_\omega+\lambda^{-1}(\vec n^b\cdot\vec\varphi)\vec m^b_t=0,\\
&\vec n^a\cdot\vec m^a_t=0.
\end{aligned} \tag{4.14}$$

4.3 Exact solutions of the reduced systems

1. Ansatz (4.1) and system (4.9) determine the class of solutions of the NSEs (1.1) that are called the steady axially symmetric conically similar flows of a viscous fluid in hydrodynamics. This class of solutions was studied in a number of works (for example, see references in [16]). For $\varphi^2=0$ it was shown, by N.A. Slezkin [34], that system (4.9) is reduced to a Riccati equation. The general solution of this equation was expressed in terms of hypergeometric functions. Later similar calculations were made by V.I. Yatseev [38] and H.B. Squire [35]. The particular case in the class of solutions with $\varphi^2=0$ is formed by the Landau jets [24]. For swirling flows, where $\varphi^2\neq 0$, the order of system (4.9) can be reduced too. For example [33], an arbitrary solution of (4.9) satisfies the equation

$$\varphi^2\varphi^2\sin^2\omega-\sin\omega(\Phi_\omega\sin^{-1}\omega)_\omega+2\Phi_\omega\cot\omega+2\Phi=\text{const},$$

where $\Phi=(\varphi^3_\omega-\frac12\varphi^3\varphi^3)\sin^2\omega-\varphi^3\cos\omega\sin\omega$, and the Yatseev results [38] are completely extended to the case $\varphi^2\sin\omega=\text{const}$.

2. System (4.10) implies that

$$\begin{aligned}
&\varphi^2=\varkappa\varphi^1+C_1,\\
&h=\varkappa(1+\varkappa^2)\varphi^1_\omega+(2\varkappa^2+2-\varkappa C_1)\varphi^1+C_2,\\
&(1+\varkappa^2)\varphi^1_{\omega\omega}+(4\varkappa-C_1)\varphi^1_\omega+\varphi^1\varphi^1+4\varphi^1+\\
&\ +(1+\varkappa^2)^{-1}(C_1^2+2C_2)=0,\\
&(1+\varkappa^2)\varphi^3_{\omega\omega}-(C_1-2\varkappa)\varphi^3_\omega+(1+\varphi^1)\varphi^3=0.
\end{aligned} \tag{4.15}$$

If $\varphi^3=0$, the solution determined by ansatz (4.10) and formulas (4.15) coincides with the Hamel solution [18, 23]. In Section 6 we consider system (6.14) which is more general than system (4.10).

3–4. Let us integrate the last equation of system (4.11), i.e.,

$$\varphi^1=C_1\omega^{-2}-\tfrac12\sigma_2. \tag{4.16}$$

Taking into account the integration result, the other equations of system (4.11) can be written in the form

$$h_\omega=\omega^{-3}\varphi^2\varphi^2+C_1^2\omega^{-3}-\tfrac14\sigma_2^2\omega,$$

$$\varphi^2_{\omega\omega}-((C_1+1)\omega^{-1}-\tfrac12\sigma_2\omega)\varphi^2_\omega=\varepsilon_1,$$

$$\varphi^3_{\omega\omega} - ((C_1-1)\omega^{-1} - \tfrac{1}{2}\sigma_2\omega)\varphi^3_\omega - \sigma_1\varphi^3 = \nu\omega^{-2}\varphi^2 + \varepsilon_2. \tag{4.17}$$

Therefore,

$$h = \textstyle\int \omega^{-3}\varphi^2\varphi^2 d\omega - \tfrac{1}{2}C_1^2\omega^{-2} - \tfrac{1}{8}\sigma_2^2\omega^2, \tag{4.18}$$

$$\begin{aligned}\varphi^2 = {}& C_2 + C_3 \textstyle\int |\omega|^{C_1+1} e^{-\frac{1}{4}\sigma_2\omega^2} d\omega + \\ & + \varepsilon_1 \textstyle\int |\omega|^{C_1+1} e^{-\frac{1}{4}\sigma_2\omega^2} \left(\int |\omega|^{-C_1-1} e^{\frac{1}{4}\sigma_2\omega^2} d\omega\right) d\omega.\end{aligned} \tag{4.19}$$

If $\sigma_1 = 0$, it follows that

$$\begin{aligned}\varphi^3 = {}& C_4 + C_5 \textstyle\int |\omega|^{C_1-1} e^{-\frac{1}{4}\sigma_2\omega^2} d\omega + \\ & + \textstyle\int |\omega|^{C_1-1} e^{-\frac{1}{4}\sigma_2\omega^2} \left(\int |\omega|^{-C_1+1} e^{\frac{1}{4}\sigma_2\omega^2} (\varepsilon_2 + \nu\omega^{-2}\varphi^2) d\omega\right) d\omega.\end{aligned} \tag{4.20}$$

Let $\sigma_1 \neq 0$ (and, therefore, $\nu = 0$). Then, if $\sigma_2 \neq 0$, the general solution of equation (4.17) is expressed in terms of Whittaker functions:

$$\varphi^3 = |\omega|^{\frac{1}{2}C_1 - 1} e^{-\frac{1}{8}\sigma_2\omega^2} W(-\sigma_1\sigma_2^{-1} + \tfrac{1}{4}C_1 - \tfrac{1}{2}, \tfrac{1}{4}C_1, \tfrac{1}{4}\sigma_2\omega^2),$$

where $W(\varkappa, \mu, \tau)$ is the general solution of the Whittaker equation

$$4\tau^2 W_{\tau\tau} = (\tau^2 - 4\varkappa\tau + 4\mu^2 - 1)W. \tag{4.21}$$

If $\sigma_2 = 0$, the general solution of equation (4.16) is expressed in terms of Bessel functions:

$$\varphi^3 = |\omega|^{\frac{1}{2}C_1} Z_{\frac{1}{2}C_1}\big((-\sigma_1)^{1/2}\omega\big),$$

where $Z_\nu(\tau)$ is the general solution of the Bessel equation

$$\tau^2 Z_{\tau\tau} + \tau Z_\tau + (\tau^2 - \nu^2) Z = 0. \tag{4.22}$$

Note 4.2 If $\sigma_2 = 0$, all quadratures in formulas (4.18)–(4.20) are easily integrated. For example,

$$\varphi^2 = \begin{cases} C_2 + C_3 \ln|\omega| + \frac{1}{4}\varepsilon_1\omega^2 & \text{if} \quad C_1 = -2, \\ C_2 + C_3\frac{1}{2}\omega^2 + \frac{1}{2}\varepsilon_1\omega^2(\ln\omega - \frac{1}{2}) & \text{if} \quad C_1 = 0, \\ C_2 + C_3(C_1+2)^{-1}|\omega|^{C_1+2} - \frac{1}{2}\varepsilon_1 C_1^{-1}\omega^2 & \text{if} \quad C_1 \neq -2, 0. \end{cases}$$

5–6. Let $\sigma = 0$. Then the last equation of system (4.12) implies that $\varphi^3 = C_0 =$ const. The other equations of system (4.12) can be written in the form

$$\begin{aligned} & h = -\gamma_1 a_2 \textstyle\int \varphi^2(\omega) d\omega, \\ & \varphi^i_{\omega\omega} - C_0\varphi^i_\omega + \mu_{ij}\varphi^j = \nu_{1i} + \nu_{2i}\omega, \end{aligned} \tag{4.23}$$

where $\nu_{11} = c_{11}$, $\nu_{21} = c_{21}$, $\nu_{12} = c_{12} + \gamma_2 a_2 C_0$, $\nu_{22} = c_{22}$. System (4.23) is a linear nonhomogeneous system of ODEs with constant coefficients. The form of its general solution depends on the Jordan form of the matrix $\mathrm{M} = \{\mu_{ij}\}$. Now let us transform the dependent variables

$$\varphi^i = e_{ij}\psi^j,$$

where the constants e_{ij} are determined by means of the system of linear algebraic equations

$$e_{ij}\tilde{\mu}_{jk} = \mu_{ij}e_{jk} \quad (i,j,k=1,2)$$

with the condition $\det\{e_{ij}\} \neq 0$. Here $\tilde{\mathrm{M}} = \{\tilde{\mu}_{ij}\}$ is the real Jordan form of the matrix M. The new unknown functions ψ^i have to satisfy the following system

$$\psi^i_{\omega\omega} - C_0\psi^i_\omega + \tilde{\mu}_{ij}\psi^j = \tilde{\nu}_{1i} + \tilde{\nu}_{2i}\omega, \tag{4.24}$$

where $\nu_{1i} = e_{ij}\tilde{\nu}_{1j}$, $\nu_{2i} = e_{ij}\tilde{\nu}_{2j}$. Depending on the form of $\tilde{\mathrm{M}}$, we consider the following cases:

A. $\det\tilde{\mathrm{M}} = 0$ (this is equivalent to the condition $\det \mathrm{M} = 0$).

i. $\tilde{M} = \begin{pmatrix} 0 & \varepsilon \\ 0 & 0 \end{pmatrix}$, where $\varepsilon \in \{0;1\}$. Then

$$\begin{aligned}
\psi^2 &= C_1 + C_2e^{C_0\omega} - \tfrac{1}{2}\tilde{\nu}_{22}C_0^{-1}\omega^2 - (\tilde{\nu}_{12} - \tilde{\nu}_{22}C_0^{-1})C_0^{-1}\omega,\\
\psi^1 &= C_3 + C_4e^{C_0\omega} - \tfrac{1}{2}\tilde{\nu}_{21}C_0^{-1}\omega^2 - (\tilde{\nu}_{11} - \tilde{\nu}_{21}C_0^{-1})C_0^{-1}\omega +\\
&\quad + \varepsilon\Big(-\tfrac{1}{6}\tilde{\nu}_{22}C_0^{-2}\omega^3 - \tfrac{1}{2}(\tilde{\nu}_{12} - 2\tilde{\nu}_{22}C_0^{-1})C_0^{-2}\omega^2 +\\
&\quad + \big(C_1 + (\tilde{\nu}_{21} - 2\tilde{\nu}_{22}C_0^{-1})C_0^{-2}\big)C_0^{-1}\omega - C_2C_0^{-1}\omega e^{C_0\omega}\Big)
\end{aligned} \tag{4.25}$$

for $C_0 \neq 0$, and

$$\begin{aligned}
\psi^2 &= C_1 + C_2\omega + \tfrac{1}{6}\tilde{\nu}_{22}\omega^3 + \tfrac{1}{2}\tilde{\nu}_{12}\omega^2,\\
\psi^1 &= C_3 + C_4\omega + \tfrac{1}{6}(\tilde{\nu}_{21} - C_2)\omega^3 + \tfrac{1}{2}(\tilde{\nu}_{11} - C_1)\omega^2 - \tfrac{1}{120}\tilde{\nu}_{22}\omega^5 - \tfrac{1}{24}\tilde{\nu}_{12}\omega^4
\end{aligned} \tag{4.26}$$

for $C_0 = 0$.

ii. $\tilde{\mathrm{M}} = \begin{pmatrix} \varkappa_1 & 0 \\ 0 & 0 \end{pmatrix}$, where $\varkappa_1 \in \mathbb{R}\backslash\{0\}$. Then the form of ψ^2 is given either by formula (4.25) for $C_0 \neq 0$ or by formula (4.26) for $C_0 = 0$. The form of ψ^1 is given by formula (4.28) (see below).

B. $\det\tilde{\mathrm{M}} \neq 0$ (this is equivalent to the condition $\det\mathrm{M} \neq 0$).

i. $\tilde{\mathrm{M}} = \begin{pmatrix} \varkappa_1 & 0 \\ 0 & \varkappa_2 \end{pmatrix}$, where $\varkappa_i \in \mathbb{R}\backslash\{0\}$. Then

$$\psi^2 = \tilde{\nu}_{22}\varkappa_2^{-1}\omega + (\tilde{\nu}_{12} - C_0\tilde{\nu}_{22}\varkappa_2^{-1})\varkappa_2^{-1} + C_1\theta^{21}(\omega) + C_2\theta^{22}(\omega), \tag{4.27}$$

$$\psi^1 = \tilde{\nu}_{21}\varkappa_1^{-1}\omega + (\tilde{\nu}_{11} - C_0\tilde{\nu}_{21}\varkappa_1^{-1})\varkappa_1^{-1} + C_3\theta^{11}(\omega) + C_4\theta^{12}(\omega), \tag{4.28}$$

where

$$\theta^{i1}(\omega) = \exp\big(\tfrac{1}{2}(C_0 - \sqrt{D_i})\omega\big), \quad \theta^{i2}(\omega) = \exp\big(\tfrac{1}{2}(C_0 + \sqrt{D_i})\omega\big)$$

if $D_i = C_0^2 - 4\varkappa_i > 0$,

$$\theta^{i1}(\omega) = e^{\frac{1}{2}C_0\omega}\cos\big(\tfrac{1}{2}\sqrt{-D_i}\omega\big), \quad \theta^{i2}(\omega) = e^{\frac{1}{2}C_0\omega}\sin\big(\tfrac{1}{2}\sqrt{-D_i}\omega\big)$$

if $D_i < 0$,

$$\theta^{i1}(\omega) = e^{\frac{1}{2}C_0\omega}, \qquad \theta^{i2}(\omega) = \omega e^{\frac{1}{2}C_0\omega}$$

if $D_i = 0$.

ii. $\tilde{\mathrm{M}}=\begin{pmatrix}\varkappa_2 & 1\\ 0 & \varkappa_2\end{pmatrix}$, where $\varkappa_2\in\mathbb{R}\backslash\{0\}$. Then the form of ψ^2 is given by formula (4.27), and

$$\psi^1=\big(\tilde\nu_{11}-(\tilde\nu_{12}-C_0\tilde\nu_{22}\varkappa_2^{-1})\varkappa_2^{-1}-C_0(\tilde\nu_{21}-\tilde\nu_{22}\varkappa_2^{-1})\varkappa_2^{-1}\big)\varkappa_2^{-1}+ \\ +(\tilde\nu_{21}-\tilde\nu_{22}\varkappa_2^{-1})\varkappa_2^{-1}\omega+C_3\theta^{21}(\omega)+C_4\theta^{22}(\omega)-C_i\eta^i(\omega),$$

where

$$\eta^j(\omega)=D_2^{-1}\omega\big(2\theta^{2j}_\omega(\omega)-C_0\theta^{2j}(\omega)\big)\qquad \text{if}\quad D_2\neq0,$$
$$\eta^1(\omega)=\tfrac12\omega^2e^{\frac12C_0\omega},\quad \eta^2(\omega)=\tfrac16\omega^3e^{\frac12C_0\omega}\quad \text{if}\quad D_2=0.$$

iii. $\tilde{\mathrm{M}}=\begin{pmatrix}\varkappa_1 & -\varkappa_2\\ \varkappa_2 & \varkappa_1\end{pmatrix}$, where $\varkappa_i\in\mathbb{R},\ \ \varkappa_2\neq0$. Then

$$\psi^1=(\varkappa_i\varkappa_i)^{-1}(\tilde\nu_{21}\varkappa_1+\tilde\nu_{22}\varkappa_2)\omega+(\varkappa_i\varkappa_i)^{-1}(\tilde\nu_{11}\varkappa_1+\tilde\nu_{12}\varkappa_2)- \\ -C_0(\varkappa_i\varkappa_i)^{-2}\big(\tilde\nu_{21}(\varkappa_2^2-\varkappa_1^2)-\tilde\nu_{22}2\varkappa_1\varkappa_2\big)+C_n\theta^{1n}(\omega),$$
$$\psi^2=(\varkappa_i\varkappa_i)^{-1}(-\tilde\nu_{21}\varkappa_2+\tilde\nu_{22}\varkappa_1)\omega+(\varkappa_i\varkappa_i)^{-1}(-\tilde\nu_{11}\varkappa_2+\tilde\nu_{12}\varkappa_1)- \\ -C_0(\varkappa_i\varkappa_i)^{-2}\big(\tilde\nu_{21}2\varkappa_1\varkappa_2+\tilde\nu_{22}(\varkappa_2^2-\varkappa_1^2)\big)+C_n\theta^{2n}(\omega),$$

where $n=\overline{1,4}$,

$$\gamma=\sqrt{(C_0^2-4\varkappa_1)^2+(4\varkappa_2)^2},$$
$$\beta_1=\tfrac14\sqrt{2(\gamma+C_0^2-4\varkappa_1)},\quad \beta_2=\tfrac14\tfrac{|\varkappa_2|}{\varkappa_2}\sqrt{2(\gamma-C_0^2+4\varkappa_1)},$$
$$\theta^{11}(\omega)=\theta^{22}(\omega)=\exp\big((\tfrac12C_0-\beta_1)\omega\big)\cos\beta_2\omega,$$
$$-\theta^{21}(\omega)=\theta^{12}(\omega)=\exp\big((\tfrac12C_0-\beta_1)\omega\big)\sin\beta_2\omega,$$
$$\theta^{13}(\omega)=\theta^{24}(\omega)=\exp\big((\tfrac12C_0+\beta_1)\omega\big)\cos\beta_2\omega,$$
$$\theta^{23}(\omega)=-\theta^{14}(\omega)=\exp\big((\tfrac12C_0+\beta_1)\omega\big)\sin\beta_2\omega.$$

If $\sigma\neq0$, the last equation of system (4.12) implies that $\psi^3=\sigma\omega$ (translating ω, the integration constant can be made to vanish). The other equations of system (4.12) can be written in the form

$$h=-\gamma_1a_2\int\varphi^2(\omega)d\omega-\tfrac12\sigma^2\omega^2, \\ \varphi^i_{\omega\omega}-\sigma\omega\varphi^i_\omega+\mu_{ij}\varphi^j=\nu_{1i}+\nu_{2i}\omega, \tag{4.29}$$

where $\nu_{11}=c_{11}$, $\ \nu_{21}=c_{21}$, $\nu_{12}=c_{12}$, $\nu_{22}=c_{22}+\gamma_2a_2\sigma$. The form of the general solution of system (4.29) depends on the Jordan form of the matrix $\mathrm{M}=\{\mu_{ij}\}$. Now, let us transform the dependent variables

$$\varphi^i=e_{ij}\psi^j,$$

where the constants e_{ij} are determined by means of the system of linear algebraic equations

$$e_{ij}\tilde\mu_{jk}=\mu_{ij}e_{jk}\quad (i,j,k=1,2)$$

with the condition $\det\{e_{ij}\}\neq0$. Here $\tilde{\mathrm{M}}=\{\tilde\mu_{ij}\}$ is the real Jordan form of the matrix M. The new unknown functions ψ^i have to satisfy the following system

$$\psi^i_{\omega\omega}-\sigma\omega\psi^i_\omega+\tilde\mu_{ij}\psi^j=\tilde\nu_{1i}+\tilde\nu_{2i}\omega, \tag{4.30}$$

where $\nu_{1i} = e_{ij}\tilde{\nu}_{1j}$, $\nu_{2i} = e_{ij}\tilde{\nu}_{2j}$. Depending on the form of $\tilde{\mathrm{M}}$, we consider the following cases:

A. $\det\tilde{\mathrm{M}} = 0$ (this is equivalent to the condition $\det\mathrm{M} = 0$).

i. $\tilde{M} = \begin{pmatrix} 0 & \varepsilon \\ 0 & 0 \end{pmatrix}$, where $\varepsilon \in \{0; 1\}$. Then

$$\psi^2 = C_1 + C_2\int e^{\frac{1}{2}\sigma\omega^2}d\omega - \sigma^{-1}\tilde{\nu}_{22}\omega + \tilde{\nu}_{12}\int e^{\frac{1}{2}\sigma\omega^2}\Bigl(\int e^{-\frac{1}{2}\sigma\omega^2}d\omega\Bigr)d\omega, \tag{4.31}$$

$$\psi^1 = C_3 + C_4\int e^{\frac{1}{2}\sigma\omega^2}d\omega - \sigma^{-1}\tilde{\nu}_{21}\omega + \int e^{\frac{1}{2}\sigma\omega^2}\Bigl(\int e^{-\frac{1}{2}\sigma\omega^2}(\tilde{\nu}_{11} - \varepsilon\psi^2)d\omega\Bigr)d\omega.$$

ii. $\tilde{\mathrm{M}} = \begin{pmatrix} \sigma & 0 \\ 0 & 0 \end{pmatrix}$. Then the form of ψ^2 is given by formula (4.31), and

$$\begin{aligned}\psi^1 = {}& C_3\omega + C_4\bigl(\omega\int e^{\frac{1}{2}\sigma\omega^2}d\omega - \sigma^{-1}e^{\frac{1}{2}\sigma\omega^2}\bigr) + \sigma^{-1}\tilde{\nu}_{11} + \\ & + \sigma^{-1}\tilde{\nu}_{21}\bigl(\sigma\omega\int e^{\frac{1}{2}\sigma\omega^2}\lambda^1(\omega)d\omega - e^{\frac{1}{2}\sigma\omega^2}\lambda^1(\omega)\bigr),\end{aligned}$$

where $\lambda^1(\omega) = \int e^{-\frac{1}{2}\sigma\omega^2}d\omega$.

iii. $\tilde{\mathrm{M}} = \begin{pmatrix} \varkappa_1 & 0 \\ 0 & 0 \end{pmatrix}$, where $\varkappa_1 \in \mathbb{R}\backslash\{0; \sigma\}$. Then ψ^2 is determited by (4.31), and the form of ψ^1 is given by (4.33) (see below).

B. $\det\tilde{\mathrm{M}} \neq 0$, $\det\{\tilde{\mu}_{ij} - \sigma\delta_{ij}\} = 0$ (this is equivalent to the conditions $\det\mathrm{M} \neq 0$, $\det\{\mu_{ij} - \sigma\delta_{ij}\} = 0$; here δ_{ij} is the Kronecker symbol).

i. $\tilde{\mathrm{M}} = \begin{pmatrix} \sigma & \varepsilon \\ 0 & \sigma \end{pmatrix}$, where $\varepsilon \in \{0; 1\}$. Then

$$\begin{aligned}\psi^2 = {}& C_1\omega + C_2\bigl(\omega\int e^{\frac{1}{2}\sigma\omega^2}d\omega - \sigma^{-1}e^{\frac{1}{2}\sigma\omega^2}\bigr) + \sigma^{-1}\tilde{\nu}_{12} + vspace1mm \\ & + \sigma^{-1}\tilde{\nu}_{22}\bigl(\sigma\omega\int e^{\frac{1}{2}\sigma\omega^2}\lambda^1(\omega)d\omega - e^{\frac{1}{2}\sigma\omega^2}\lambda^1(\omega)\bigr),\end{aligned} \tag{4.32}$$

$$\begin{aligned}\psi^1 = {}& C_3\omega + C_4\bigl(\omega\int e^{\frac{1}{2}\sigma\omega^2}d\omega - \sigma^{-1}e^{\frac{1}{2}\sigma\omega^2}\bigr) + \sigma^{-1}\tilde{\nu}_{11} + \\ & + \sigma\omega\int e^{\frac{1}{2}\sigma\omega^2}\lambda^2(\omega)d\omega - e^{\frac{1}{2}\sigma\omega^2}\lambda^2(\omega) + \sigma^{-1}(\tilde{\nu}_{21}\omega - \varepsilon\psi^2),\end{aligned}$$

where $\lambda^1(\omega) = \int e^{-\frac{1}{2}\sigma\omega^2}d\omega$, $\lambda^2(\omega) = \sigma^{-1}\int e^{-\frac{1}{2}\sigma\omega^2}(\tilde{\nu}_{21} - \varepsilon\psi^2_\omega)d\omega$.

ii. $\tilde{\mathrm{M}} = \begin{pmatrix} \varkappa_1 & 0 \\ 0 & \sigma \end{pmatrix}$, where $\varkappa_1 \in \mathbb{R}\backslash\{0; \sigma\}$. In this case ψ^2 is determined by (4.32), and the form of ψ^1 is given by (4.33) (see below).

C. $\det\tilde{\mathrm{M}} \neq 0$, $\det\{\tilde{\mu}_{ij} - \sigma\delta_{ij}\} \neq 0$ (this is equivalent to the condition $\det\mathrm{M} \neq 0$, $\det\{\mu_{ij} - \sigma\delta_{ij}\} \neq 0$: here δ_{ij} is the Kronecker symbol).

i. $\tilde{\mathrm{M}} = \begin{pmatrix} \varkappa_1 & 0 \\ 0 & \varkappa_2 \end{pmatrix}$, where $\varkappa_i \in \mathbb{R}\backslash\{0; \sigma\}$. Then

$$\begin{aligned}\psi^1 = {}& \varkappa_1^{-1}\tilde{\nu}_{11} + (\varkappa_1 - \sigma)^{-1}\tilde{\nu}_{21}\omega + |\omega|^{-1/2}e^{\frac{1}{4}\sigma\omega^2} \times \\ & \times\Bigl(C_3M\bigl(\tfrac{1}{2}\varkappa_1\sigma^{-1} + \tfrac{1}{4}, \tfrac{1}{4}, \tfrac{1}{2}\sigma\omega^2\bigr) + C_4M\bigl(\tfrac{1}{2}\varkappa_1\sigma^{-1} + \tfrac{1}{4}, -\tfrac{1}{4}, \tfrac{1}{2}\sigma\omega^2\bigr)\Bigr),\end{aligned} \tag{4.33}$$

$$\psi^2 = \varkappa_2^{-1}\tilde\nu_{12} + (\varkappa_2-\sigma)^{-1}\tilde\nu_{22}\omega + |\omega|^{-1/2}e^{\frac14\sigma\omega^2}\times \\ \times\Big(C_1M\big(\tfrac12\varkappa_2\sigma^{-1}+\tfrac14,\tfrac14,\tfrac12\sigma\omega^2\big) + C_2M\big(\tfrac12\varkappa_2\sigma^{-1}+\tfrac14,-\tfrac14,\tfrac12\sigma\omega^2\big)\Big), \tag{4.34}$$

where $M(\varkappa,\mu,\tau)$ is the Whittaker function:

$$M(\varkappa,\mu,\tau) = \tau^{\frac12+\mu}e^{-\frac12\tau}{}_1F_1\big(\tfrac12+\mu-\varkappa, 2\mu+1,\tau\big), \tag{4.35}$$

and ${}_1F_1(a,b,\tau)$ is the degenerate hypergeometric function defined by means of the series:

$${}_1F_1(a,b,\tau) = 1 + \sum_{n=1}^{\infty}\frac{a(a+1)\dots(a+n-1)}{b(b+1)\dots(b+n-1)}\frac{\tau^n}{n!},$$

$b\neq 0,-1,-2,\dots$.

ii. $\tilde{\mathrm M} = \begin{pmatrix}\varkappa_1 & -\varkappa_2\\ \varkappa_2 & \varkappa_1\end{pmatrix}$, where $\varkappa_i\in\mathbb R$, $\varkappa_2\neq 0$. Then

$$\psi^1 = (\varkappa_j\varkappa_j)^{-1}(\varkappa_1\tilde\nu_{11}+\varkappa_2\tilde\nu_{12}) + ((\varkappa_1-\sigma)^2+\varkappa_2^2)^{-1}((\varkappa_1-\sigma)\tilde\nu_{21}+\varkappa_2\tilde\nu_{22})\omega + \\ + C_1\mathrm{Re}\,\eta^1(\omega) - C_2\mathrm{Im}\,\eta^1(\omega) + C_3\mathrm{Re}\,\eta^2(\omega) - C_4\mathrm{Im}\,\eta^2(\omega),$$

$$\psi^2 = (\varkappa_j\varkappa_j)^{-1}(-\varkappa_2\tilde\nu_{11}+\varkappa_1\tilde\nu_{12}) + \\ + ((\varkappa_1-\sigma)^2+\varkappa_2^2)^{-1}(-\varkappa_2\tilde\nu_{21}+(\varkappa_1-\sigma)\tilde\nu_{22})\omega + \\ + C_1\mathrm{Im}\,\eta^1(\omega) + C_2\mathrm{Re}\,\eta^1(\omega) + C_3\mathrm{Im}\,\eta^2(\omega) + C_4\mathrm{Re}\,\eta^2(\omega),$$

where

$$\eta^1(\omega) = M\big(\tfrac12(\varkappa_1+\varkappa_2\mathrm i)\sigma^{-1}+\tfrac14,\tfrac14,\tfrac12\sigma\omega^2\big), \\ \eta^2(\omega) = M\big(\tfrac12(\varkappa_1+\varkappa_2\mathrm i)\sigma^{-1}+\tfrac14,-\tfrac14,\tfrac12\sigma\omega^2\big), \quad \mathrm i^2=-1.$$

iii. $\tilde{\mathrm M} = \begin{pmatrix}\varkappa_2 & 1\\ 0 & \varkappa_2\end{pmatrix}$, where $\varkappa_2\in\mathbb R\backslash\{0;\sigma\}$. Here the form of ψ^2 is given by (4.34), and

$$\psi^1 = (\tilde\nu_{11}-\tilde\nu_{12}\varkappa_2^{-1})\varkappa_2^{-1} + \big(\tilde\nu_{21}-\tilde\nu_{22}(\varkappa_2-\sigma)^{-1}\big)(\varkappa_2-\sigma)^{-1}\omega + \\ + |\omega|^{-1/2}e^{\frac14\sigma\omega^2}\Big(C_3\theta^1(\tau)+C_4\theta^2(\tau) - \sigma^{-1}\theta^1(\tau)\textstyle\int\tau^{-1}\theta^2(\tau)C_i\theta^i(\tau)d\tau + \\ + \sigma^{-1}\theta^2(\tau)\textstyle\int\tau^{-1}\theta^1(\tau)C_i\theta^i(\tau)d\tau\Big),$$

where $\tau=\frac12\sigma\omega^2$,

$$\theta^1(\tau) = M\big(\tfrac12\varkappa_2\sigma^{-1}+\tfrac14,\tfrac14,\tau\big),\quad \theta^2(\tau) = M\big(\tfrac12\varkappa_2\sigma^{-1}+\tfrac14,-\tfrac14,\tau\big).$$

Note 4.3 The general solution of the equation

$$\psi_{\omega\omega} - \sigma\omega\psi_\omega - (n+1)\sigma\psi = 0,$$

where n is an integer and $n\geq 0$, is determined by the formula

$$\psi = \left(\frac{d^n}{d\omega^n}e^{\frac12\sigma\omega^2}\right)\left(C_1 + C_2\int e^{\frac12\sigma\omega^2}\left(\frac{d^n}{d\omega^n}e^{\frac12\sigma\omega^2}\right)^{-2}d\omega\right).$$

Note 4.4 If function ψ satisfies the equation

$$\psi_{\omega\omega} - \sigma\omega\psi_\omega + \varkappa\psi = 0 \quad (\varkappa \neq -\sigma),$$

then $\int \psi(\omega)d\omega = (\varkappa + \sigma)^{-1}(\sigma\omega\psi - \psi_\omega) + C_1$.

7. The last equation of system (4.13) is the compatibility condition of the NSEs (1.1) and ansatz (4.7). Integrating this equation, we obtain that

$$\eta^3 = C_0(\eta^i\eta^i)^{-1}, \quad C_0 \neq 0.$$

As $\varphi^3_\omega = -\eta^3_\omega(\eta^3)^{-1}\varphi^3 = 2\theta^1\varphi^3$, $\varphi^3 = C_3\eta^i\eta^i$. Then system (4.13) is reduced to the equations

$$\begin{aligned}\varphi^1_\omega &= \chi^1(\omega)\varphi^1 - \chi^2(\omega)\varphi^2,\\ \varphi^2_\omega &= \chi^2(\omega)\varphi^1 + \chi^1(\omega)\varphi^2,\end{aligned} \tag{4.36}$$

where $\chi^1 = -C_0^{-2}(\eta^i\eta^i)^2 - \theta^1$ and $\chi^2 = \theta^2 - C_3C_0^{-1}(\eta^i\eta^i)^2$. System (4.36) implies that

$$\begin{aligned}\varphi^1 &= \exp\left(\textstyle\int \chi^1(\omega)d\omega\right)\Big(C_1\cos\left(\textstyle\int \chi^2(\omega)d\omega\right) - C_2\sin\left(\textstyle\int \chi^2(\omega)d\omega\right)\Big),\\ \varphi^2 &= \exp\left(\textstyle\int \chi^1(\omega)d\omega\right)\Big(C_1\sin\left(\textstyle\int \chi^2(\omega)d\omega\right) + C_2\cos\left(\textstyle\int \chi^2(\omega)d\omega\right)\Big).\end{aligned}$$

8. Let us apply the trasformation generated by the operator $R(\vec{k}(t))$, where

$$\vec{k}_t = \lambda^{-1}(\vec{n}^b\cdot\vec{k})\vec{m}^b_t - \vec{\varphi},$$

to ansatz (4.8). As a result we obtain an ansatz of the same form, where the functions $\vec{\varphi}$ and h are replaced by the new functions $\tilde{\vec{\varphi}}$ and $\tilde{h}$:

$$\begin{aligned}\tilde{\vec{\varphi}} &= \vec{\varphi} - \lambda^{-1}(\vec{n}^a\cdot\vec{k})\vec{m}^a_t + \vec{k}_t = 0,\\ \tilde{h} &= h - \lambda^{-1}(\vec{m}^a_{tt}\cdot\vec{k})(\vec{n}^a\cdot\vec{k}) + \tfrac{1}{2}\lambda^{-2}(\vec{m}^b_{tt}\cdot\vec{m}^a)(\vec{n}^a\cdot\vec{k})(\vec{n}^b\cdot\vec{k}).\end{aligned}$$

Let us make $\tilde{h}$ vanish by means of the transformation generated by the operator $Z(-\tilde{h}(t))$. Therefore, the functions φ^a and h can be considered to vanish. The equation $(\vec{n}^a\cdot\vec{m}^a_t) = 0$ is the compatibility condition of ansatz (4.8) and the NSEs (1.1).

Note 4.5 The solutions of the NSEs obtained by means of ansatzes 5–8 are equivalent to either solutions (5.1) or solutions (5.5).

5 Reduction of the Navier–Stokes equations to linear systems of PDEs

Let us show that non-linear systems 8 and 9, from Subsection 3.2, are reduced to linear systems of PDEs.

5.1 Investigation of system (3.17)–(3.18)

Consider system 9 from Subsection 3.2, i.e., equations (3.17) and (3.18). Equation (3.18) integrates with respect to z_2 to the following expression:

$$\vec{k}\cdot\vec{w}=\psi(t).$$

Here $\psi=\psi(t)$ is an arbitrary smooth function of $z_1=t$. Let us make the transformation from the symmetry group of the NSEs:

$$\tilde{\vec{u}}(t,\vec{x})=\vec{u}(t,\vec{x}-\vec{l})+\vec{l}_t(t),$$
$$\tilde{p}(t,\vec{x})=p(t,\vec{x}-\vec{l})-\vec{l}_{tt}(t)\cdot\vec{x},$$

where $\vec{l}_{tt}\cdot\vec{m}^i-\vec{l}\cdot\vec{m}^i_{tt}=0$ and

$$\vec{k}\cdot\big(\vec{l}_t-\lambda^{-1}(\vec{n}^i\cdot\vec{l})m^i_t+\lambda^{-1}(\vec{k}\cdot\vec{l})\vec{k}_t\big)+\psi=0.$$

This transformation does not modify ansatz (3.9), but it makes the function $\psi(t)$ vanish, i.e., $\vec{k}\cdot\tilde{\vec{w}}=0$. Therefore, without loss of generality we may assume, at once, that $\vec{k}\cdot\vec{w}=0$.

Let $f^i=f^i(z_1,z_2)=\vec{m}^i\cdot\vec{w}$. Since $\vec{m}^1_{tt}\cdot\vec{m}^2-\vec{m}^1\cdot\vec{m}^2_{tt}=0$, it follows that $\vec{m}^1_t\cdot\vec{m}^2-\vec{m}^1\cdot\vec{m}^2_t=C=\text{const}$. Let us multiply the scalar equation (3.17) by $\vec{m}^i$ and $\vec{k}$. As a result we obtain the linear system of PDEs with variable coefficients in the functions f^i and s:

$$f^i_1-\lambda f^i_{22}+C\lambda^{-1}\big((\vec{m}^i\cdot\vec{m}^2)f^1-(\vec{m}^i\cdot\vec{m}^1)f^2\big)-2C\lambda^{-2}\big((\vec{k}\times\vec{k}_t)\cdot\vec{m}^i\big)z_2=0,$$
$$s_2=2\lambda^{-2}(\vec{n}^i\cdot\vec{k}_t)f^i+\lambda^{-2}(\vec{k}_{tt}\cdot\vec{k}-2\vec{k}_t\cdot\vec{k}_t)z_2.$$

Consider two possible cases.

A. Let $C=0$. Then there exist functions $g^i=g^i(\tau,\omega)$, where $\tau=\int\lambda(t)dt$ and $\omega=z_2$, such that $f^i=g^i_\omega$ and $g^i_\tau-g^i_{\omega\omega}=0$. Therefore,

$$\begin{aligned}&\vec{u}=\lambda^{-1}(g^i_\omega(\tau,\omega)+\vec{m}^i_t\cdot\vec{x})\vec{n}^i-\lambda^{-1}(\vec{k}_t\cdot\vec{x})k,\\&p=2\lambda^{-2}(\vec{n}^i\cdot\vec{k}_t)g^i(\tau,\omega)+\tfrac12\lambda^{-2}(\vec{k}_{tt}\cdot\vec{k}-2\vec{k}_t\cdot\vec{k}_t)\omega^2-\\&\qquad-\tfrac12\lambda^{-1}(\vec{n}^i\cdot\vec{x})(\vec{m}^i_{tt}\cdot\vec{x})-\tfrac12\lambda^{-2}(\vec{k}\cdot\vec{m}^i_{tt})(\vec{n}^i\cdot\vec{x})(\vec{k}\cdot\vec{x}),\end{aligned}\tag{5.1}$$

where $\vec{m}^1_t\cdot\vec{m}^2-\vec{m}^1\cdot\vec{m}^2_t=0$, $\vec{k}=\vec{m}^1\times\vec{m}^2$, $\vec{n}^1=\vec{m}^2\times\vec{k}$, $\vec{n}^2=\vec{k}\times\vec{m}^1$, $\lambda=|\vec{k}|^2$, $\omega=\vec{k}\cdot\vec{x}$, $\tau=\int\lambda(t)dt$, and $g^i_\tau-g^i_{\omega\omega}=0$.

For example, if $\vec{m}=(\eta^1(t),0,0)$ and $\vec{n}=(0,\eta^2(t),0)$ with $\eta^i(t)\neq0$, it follows that

$$\begin{aligned}&u^1=(\eta^1)^{-1}(f^1+\eta^1_tx_1),\quad u^2=(\eta^2)^{-1}(f^2+\eta^2_tx_2),\quad u^3=-(\eta^1\eta^2)_t(\eta^1\eta^2)^{-1}x_3,\\&p=-\tfrac12\eta^1_{tt}(\eta^1)^{-1}x_1^2-\tfrac12\eta^2_{tt}(\eta^2)^{-1}x_2^2+\\&\qquad+\Big(\tfrac12(\eta^1\eta^2)_{tt}(\eta^1\eta^2)^{-1}-\big((\eta^1\eta^2)_t(\eta^1\eta^2)^{-1}\big)^2\Big)x_3^2,\end{aligned}$$

where $f^i=f^i(\tau,\omega)$, $f^i_\tau-f^i_{\omega\omega}=0$, $\tau=\int(\eta^1\eta^2)^2dt$, and $\omega=\eta^1\eta^2x_3$. If $\vec{m}^1=(\eta^1(t),\eta^2(t),0)$ and $\vec{m}^2=(0,0,\eta^3(t))$ with $\eta^3(t)\neq0$ and $\eta^i(t)\eta^i(t)\neq0$, we obtain that

$$
\begin{aligned}
&u^1 = (\eta^i\eta^i)^{-1}\Big\{\eta^1(g_\omega + \eta^i_t x_i) - \eta^2\big(\eta^3_t(\eta^3)^{-2}\omega + \eta^2_t x_1 - \eta^1_t x_2\big)\Big\},\\
&u^2 = (\eta^i\eta^i)^{-1}\Big\{\eta^2(g_\omega + \eta^i_t x_i) + \eta^1\big(\eta^3_t(\eta^3)^{-2}\omega + \eta^2_t x_1 - \eta^1_t x_2\big)\Big\},\\
&u^3 = (\eta^3)^{-1}(f + \eta^3_t x_3),\\
&p = 2(\eta^3)^{-1}(\eta^1\eta^2_t - \eta^1_t\eta^2)(\eta^i\eta^i)^{-2}g + \tfrac{1}{2}\lambda^{-1}\times\\
&\qquad\times\Big\{\lambda^{-1}\big((\eta^3_{tt}\eta^3 - 2\eta^3_t\eta^3_t)\eta^i\eta^i - 2\eta^3\eta^3_t\eta^i\eta^i_t - 2(\eta^3)^2\eta^i_t\eta^i_t\big)\omega^2 +\\
&\qquad + (\eta^3)^2\big((\eta^2\eta^2_{tt} - \eta^1\eta^1_{tt})(x_1^2 - x_2^2) - 2(\eta^1_{tt}\eta^2 + \eta^1\eta^2_{tt})x_1x_2\big) - \eta^i\eta^i\eta^3\eta^3_{tt}x_3^2\Big\}.
\end{aligned}
$$

Here $f = f(\tau,\omega)$, $f_\tau - f_{\omega\omega} = 0$, $g = g(\tau,\omega)$, $g_\tau - g_{\omega\omega} = 0$, $\tau = \int(\eta^3)^2\eta^i\eta^i dt$, $\omega = \eta^3(\eta^2 x_1 - \eta^1 x_2)$, and $\lambda = (\eta^3)^2\eta^i\eta^i$.

Note 5.1 The equation

$$\vec m^1_t\cdot\vec m^2 - \vec m^1\cdot\vec m^2_t = 0 \tag{5.2}$$

can easily be solved in the following way: Let us fix arbitrary smooth vector-functions $\vec m^1, \vec l\in C^\infty((t_0,t_1),\mathbb{R}^3)$ such that $\vec m^1(t)\neq\vec 0$, $\vec l(t)\neq\vec 0$, and $\vec m^1(t)\cdot\vec l(t) = 0$ for all $t\in(t_0,t_1)$. Then the vector-function $\vec m^2 = \vec m^2(t)$ is taken in the form

$$\vec m^2(t) = \rho(t)\vec m^1 + \vec l(t). \tag{5.3}$$

Equation (5.2) implies

$$\rho(t) = \int(\vec m^1\cdot\vec m^1)^{-1}(\vec m^1_t\cdot\vec l - \vec m^1\cdot\vec l_t)dt. \tag{5.4}$$

B. Let $C\neq 0$. By means of the transformation $\vec m^i\to a_{ij}\vec m^j$, where $a_{ij} = \mathrm{const}$ and $\det\{a_{ij}\} = C$, we make $C = 1$. Then we obtain the following solution of the NSEs (1.1)

$$
\begin{aligned}
&\vec u = \lambda^{-1}\Big(\theta^{ij}(t)g^j_\omega(\tau,\omega) + \theta^{i0}(t)\omega + \vec m^i_t\cdot\vec x - \lambda^{-1}((\vec k\times\vec m^i)\cdot\vec x)\Big)\vec n^i - \lambda^{-1}(\vec k_t\cdot\vec x)\vec k,\\
&p = 2\lambda^{-2}(\vec n^i\cdot\vec k_t)(\theta^{ij}(t)g^i(\tau,\omega) + \tfrac{1}{2}\theta^{i0}(t)\omega^2) + \tfrac{1}{2}\lambda^{-2}(\vec k_{tt}\cdot\vec k - 2\vec k_t\cdot\vec k_t)\omega^2 -\\
&\qquad - \tfrac{1}{2}\lambda^{-1}(\vec n^i\cdot\vec x)(\vec m^i_{tt}\cdot\vec x) - \tfrac{1}{2}\lambda^{-2}(\vec k\cdot\vec m^i_{tt})(\vec n^i\cdot\vec x)(\vec k\cdot\vec x).
\end{aligned}
\tag{5.5}
$$

Here $\vec m^1_t\cdot\vec m^2 - \vec m^1\cdot\vec m^2_t = 1$, $\vec k = \vec m^1\times\vec m^2$, $\vec n^1 = \vec m^2\times\vec k$, $\vec n^2 = \vec k\times\vec m^1$, $\lambda = |\vec k|^2$, $\omega = \vec k\cdot\vec x$, $\tau = \int\lambda(t)dt$, and $g^i_\tau - g^i_{\omega\omega} = 0$. $(\theta^{1i}(t),\theta^{2i}(t))$ $(i = 1,2)$ are linearly independent solutions of the system

$$\theta^i_t + \lambda^{-1}(\vec m^i\cdot\vec m^2)\theta^1 - \lambda^{-1}(\vec m^i\cdot\vec m^1)\theta^2 = 0, \tag{5.6}$$

and $(\theta^{10}(t),\theta^{20}(t))$ is a particular solution of the nonhomogeneous system

$$\theta^i_t + \lambda^{-1}(\vec m^i\cdot\vec m^2)\theta^1 - \lambda^{-1}(\vec m^i\cdot\vec m^1)\theta^2 = 2\lambda^{-2}\big((\vec k\times\vec k_t)\cdot\vec m^i\big). \tag{5.7}$$

For example, if $\vec m^1 = (\eta\cos\psi, \eta\sin\psi, 0)$ and $\vec m^2 = (-\eta\sin\psi, \eta\cos\psi, 0)$, where $\eta = \eta(t)\neq 0$ and $\psi = -\frac{1}{2}\int(\eta)^{-2}dt$ (therefore, $\vec m^1_t\cdot\vec m^2 - \vec m^1\cdot\vec m^2_t = 1$), we obtain

$$
\begin{aligned}
&u^1 = \eta^{-1}\big(f^1\cos\psi - f^2\sin\psi + \eta_t x_1 - \tfrac{1}{2}\eta^{-1}x_2\big),\\
&u^2 = \eta^{-1}\big(f^1\sin\psi + f^2\cos\psi + \eta_t x_2 + \tfrac{1}{2}\eta^{-1}x_1\big),\\
&u^3 = -2\eta_t\eta^{-1}x_3,\\
&p = (\eta_{tt}\eta - 3\eta_t\eta_t)\eta^{-2}x_3^2 - \tfrac{1}{2}(\eta_{tt}\eta^{-1} - \tfrac{1}{4}\eta^{-4})x_ix_i.
\end{aligned}
$$

Here $f^i = f^i(\tau,\omega)$, $f^i_\tau - f^i_{\omega\omega} = 0$, $\tau = \int (\eta)^4 dt$, and $\omega = (\eta)^2 x_3$.

Note 5.2 As in the case $C = 0$, the solutions of the equation

$$\vec{m}^1_t \cdot \vec{m}^2 - \vec{m}^1 \cdot \vec{m}^2_t = 1 \tag{5.8}$$

can be sought in form (5.3). As a result we obtain that

$$\rho(t) = \int |\vec{m}^1|^{-2} (\vec{m}^1_t \cdot \vec{l} - \vec{m}^1 \cdot \vec{l}_t - 1) dt. \tag{5.9}$$

Note 5.3 System (5.6) can be reduced to a second-order homogeneous differential equation either in θ^1, i.e.,

$$\left(\lambda |\vec{m}^1|^{-2} \theta^1_t\right)_t + \left(\left((\vec{m}^1 \cdot \vec{m}^2)|\vec{m}^1|^{-2}\right)_t + |\vec{m}^1|^{-2}\right)\theta^1 = 0 \tag{5.10}$$

(then $\theta^2 = |\vec{m}^1|^{-2}(\lambda\theta^1_t + (\vec{m}^1 \cdot \vec{m}^2)\theta^1))$, or in θ^2, i.e.,

$$\left(\lambda |\vec{m}^2|^{-2} \theta^2_t\right)_t + \left(-\left((\vec{m}^1 \cdot \vec{m}^2)|\vec{m}^2|^{-2}\right)_t + |\vec{m}^2|^{-2}\right)\theta^2 = 0 \tag{5.11}$$

(then $\theta^1 = |\vec{m}^2|^{-2}(-\lambda\theta^2_t + (\vec{m}^1 \cdot \vec{m}^2)\theta^2))$. Under the notation of Note 5.1 equation (5.10) has the form:

$$((\vec{l} \cdot \vec{l})\theta^1_t)_t + |\vec{m}^1|^{-2}(\vec{m}^1_t \cdot \vec{l} - \vec{m}^1 \cdot \vec{l}_t)\theta^1 = 0. \tag{5.12}$$

The vector-functions $\vec{m}^1$ and $\vec{l}$ are chosen in such a way that one can find a fundamental set of solutions for equation (5.12). For example, let $\vec{m} \times \vec{m}_t \neq 0\ \forall t \in (t_0, t_1)$. Let us introduce the notation $\vec{m} := \vec{m}^1$ and put $\vec{l} = \eta(t)\vec{m} \times \vec{m}_t$, where $\eta \in C^\infty((t_0,t_1),\mathbb{R})$, $\eta(t) \neq 0\ \forall t \in (t_0, t_1)$. Then

$$\begin{aligned}
&\vec{m} \cdot \vec{l} = 0, \quad \vec{m}_t \cdot \vec{l} - \vec{m} \cdot \vec{l}_t = 0, \quad \vec{m}^2 = -\left(\textstyle\int |\vec{m}|^{-2} dt\right)\vec{m} + \eta\vec{m} \times \vec{m}_t,\\
&\vec{k} = \eta\vec{m} \times (\vec{m} \times \vec{m}_t), \quad \lambda = (\eta)^2 |\vec{m}|^2 |\vec{m} \times \vec{m}_t|^{-2},\\
&\vec{n}^2 = \eta |\vec{m}|^2 \vec{m} \times \vec{m}_t, \quad \vec{n}^1 = \left(\textstyle\int |\vec{m}|^{-2} dt\right)\vec{n}^2 + (\eta)^2 |\vec{m} \times \vec{m}_t|^{-2}\vec{m},\\
&\theta^{11}(t) = \textstyle\int (\eta)^{-2} |\vec{m} \times \vec{m}_t|^{-2} dt, \quad \theta^{21}(t) = 1 - \theta^{11} \int |\vec{m}|^{-2} dt,\\
&\theta^{12}(t) = 1, \quad \theta^{22}(t) = -\textstyle\int |\vec{m}|^{-2} dt,\\
&\theta^{10}(t) = 2\textstyle\int \left(((\vec{m} \times \vec{m}_t) \cdot \vec{m}_{tt})|\vec{m} \times \vec{m}_t|^{-2} + \int \eta^{-1} |\vec{m}|^{-4} dt\right)\eta^{-2} |\vec{m} \times \vec{m}_t|^{-2} dt,\\
&\theta^{20}(t) = -\theta^{10}(t) \textstyle\int |\vec{m}|^{-2} dt + 2\int \eta^{-1} |\vec{m}|^{-4} dt.
\end{aligned}$$

Consider the following cases: $\vec{m} \times \vec{m}_t \equiv \vec{0}$, i.e., $\vec{m} = \chi(t)\vec{a}$, where $\chi(t) \in C^\infty((t_0,t_1),\mathbb{R})$, $\chi(t) \neq 0\ \forall t \in (t_0, t_1)$, $\vec{a} = \mathrm{const}$, and $|\vec{a}| = 1$. Let us put

$$\vec{l}(t) = \eta^1(t)\vec{b} + \eta^2(t)\vec{c},$$

where $\eta^1, \eta^2 \in C^\infty((t_0,t_1),\mathbb{R})$, $(\eta^1(t), \eta^2(t)) \neq (0,0)\ \forall t \in (t_0, t_1)$, $\vec{b} = \mathrm{const}$, $|\vec{b}| = 1$, $\vec{a} \cdot \vec{b} = 0$, and $\vec{c} = \vec{a} \times \vec{b}$. Then

$$\begin{aligned}
&\vec{m}^2 = -\left(\chi \textstyle\int \chi^{-2} dt\right)\vec{a} + \eta^1\vec{b} + \eta^2\vec{c}, \quad \vec{k} = \chi\eta^1\vec{c} - \chi\eta^2\vec{b},\\
&\lambda = (\chi)^2 \eta^i\eta^i, \quad \vec{n}^2 = (\chi)^2(\eta^1\vec{b} + \eta^2\vec{c}), \quad \vec{n}^1 = \left(\textstyle\int \chi^{-2} dt\right)\vec{n}^2 + \chi\eta^i\eta^i\vec{a},\\
&\theta^{11} = \textstyle\int (\eta^i\eta^i)^{-1} dt, \quad \theta^{21} = 1 - \theta^{11} \int \chi^{-2} dt, \quad \theta^{12} = 1, \quad \theta^{22} = -\int \chi^{-2} dt,\\
&\theta^{10} = 2\textstyle\int (\eta^2_t\eta^1 - \eta^2\eta^1_t)\chi^{-1}(\eta^i\eta^i)^{-1} dt, \quad \theta^{20} = -\theta^{10} \int \chi^{-2} dt.
\end{aligned}$$

Note 5.4 In formulas (5.1) and (5.5) solutions of the NSEs (1.1) are expressed in terms of solutions of the decomposed system of two linear one-dimensional heat equations (LOHEs) that have the form:

$$g^i_\tau = g^i_{\omega\omega}. \tag{5.13}$$

The Lie symmetry of the LOHE are known. Large sets of its exact solutions were constructed [27, 3]. The Q-conditional symmetries of LOHE were investigated in [14]. Moreover, being decomposed system (5.13) admits transformations of the form

$$\begin{aligned}
&\tilde g^1(\tau',\omega') = F^1(\tau,\omega,g^1(\tau,\omega)), \quad \tau' = G^1(\tau,\omega), \quad \omega' = H^1(\tau,\omega),\\
&\tilde g^2(\tau'',\omega'') = F^2(\tau,\omega,g^2(\tau,\omega)), \quad \tau'' = G^2(\tau,\omega), \quad \omega'' = H^2(\tau,\omega),
\end{aligned}$$

where $(G^1,H^1) \neq (G^2,H^2)$, i.e. the independent variables can be transformed in the functions g^1 and g^2 in different ways. A similar statement is true for system (5.19)–(5.20) (see below) if $\varepsilon = 0$.

Note 5.5 It can be proved that an arbitrary Navier–Stokes field $(\vec u, p)$, where

$$\vec u = \vec w(t,\omega) + (\vec k^i(t)\cdot\vec x)\vec l^i(t)$$

with $\vec k^i, \vec l^i \in C^\infty((t_0,t_1),\mathbb{R}^3)$, $\vec k^1 \times \vec k^2 \neq 0$, and $\omega = (\vec k^1 \times \vec k^2)\cdot\vec x$, is equivalent to either a solution from family (5.1) or a solution from family (5.5). The equivalence transformation is generated by $R(\vec m)$ and $Z(\chi)$.

5.2 Investigation of system (3.13)–(3.16)

Consider system 8 from Subsection 3.2, i.e., equations (3.13)–(3.16). Equation (3.16) immediately gives

$$w^1 = -\tfrac{1}{2}\rho_t\rho^{-1} + (\eta-1)z_2^{-2}, \tag{5.14}$$

where $\eta = \eta(t)$ is an arbitrary smooth function of $z_1 = t$. Substituting (5.14) into remaining equations (5.13)–(5.15), we get

$$q_2 = \tfrac{1}{2}\big((\rho_t\rho^{-1})_t - \tfrac{1}{2}(\rho_t\rho^{-1})^2\big)z_2 - \eta_t z_2^{-1} - (\eta-1)^2 z_2^{-3} + (w^2-\chi)^2 z_2^{-3}, \tag{5.15}$$

$$w^2_1 - w^2_{22} + \big(\eta z_2^{-1} - \tfrac{1}{2}\rho_t\rho^{-1}z_2\big)w^2_2 = 0, \tag{5.16}$$

$$w^3_1 - w^3_{22} + \big(\eta z_2^{-1} - \tfrac{1}{2}\rho_t\rho^{-1}z_2\big)w^3_2 + \varepsilon(w^2-\chi)z_2^{-2} = 0. \tag{5.17}$$

Recall that $\rho = \rho(t)$ and $\chi = \chi(t)$ are arbitrary smooth functions of t; $\varepsilon \in \{0;1\}$. After the change of the independent variables

$$\tau = \textstyle\int |\rho(t)|dt, \quad z = |\rho(t)|^{1/2} z_2 \tag{5.18}$$

in equations (5.16) and (5.17), we obtain a linear system of a simpler form:

$$w^2_\tau - w^2_{zz} + \hat\eta(\tau)z^{-1}w^2_z = 0, \tag{5.19}$$

$$w^3_\tau - w^3_{zz} + (\hat\eta(\tau)-2)z^{-1}w^3_z + \varepsilon(w^2-\hat\chi(\tau))z^{-2} = 0, \tag{5.20}$$

where $\hat\eta(\tau)=\eta(t)$ and $\hat\chi(\tau)=\chi(t)$. Equation (5.15) implies

$$\begin{aligned}q=\tfrac14\big((\rho_t\rho^{-1})_t-\tfrac12(\rho_t\rho^{-1})^2\big)z_2^2-\eta_t\ln|z_2|-\\ -\tfrac12(\eta-1)^2z_2^{-2}+\textstyle\int(w^2(\tau,z)-\hat\chi(\tau))^2z_2^{-3}dz_2.\end{aligned}\tag{5.21}$$

Formulas (5.14), (5.18)–(5.21), and ansatz (3.8) determine a solution of the NSEs (1.1).

If $\varepsilon=0$ system (5.19)–(5.20) is decomposed and consists of two translational linear equations of the general form

$$f_\tau+\tilde\eta(\tau)z^{-1}f_z-f_{zz}=0,\tag{5.22}$$

where $\tilde\eta=\hat\eta$ ($\tilde\eta=\hat\eta-2$) for equation (5.19) ((5.20)). Tilde over η is omitted below. Let us investigate symmetry properties of equation (5.22) and construct some of its exact solutions.

Theorem 5.1 *The MIA of (5.22) is given by the following algebras*

a) $L_1=\langle f\partial_f,\ g(\tau,z)\partial_f\rangle$ *if* $\eta(\tau)\neq\text{const}$;

b) $L_2=\langle\partial_\tau,\ \hat D,\ \Pi,\ f\partial_f,\ g(\tau,z)\partial_f\rangle$ *if* $\eta(\tau)=\text{const},\ \eta\notin\{0;-2\}$;

c) $L_3=\langle\partial_\tau,\ \hat D,\ \Pi,\ \partial_z+\frac12\eta z^{-1}f\partial_f,\ G=2\tau\partial_\tau-(z-\eta z^{-1}\tau)f\partial_f,\ f\partial_f,$ $g(\tau,z)\partial_f\rangle$ *if* $\eta\in\{0;-2\}$.

Here $\hat D=2\tau\partial_\tau+z\partial_z$, $\Pi=4\tau^2\partial_\tau+4\tau z\partial_z-(z^2+2(1-\eta)\tau)f\partial_f$; $g=g(\tau,z)$ *is an arbitrary solution of (5.22).*

When $\eta=0$, equation (5.22) is the heat equation, and, when $\eta=-2$, it is reduced to the heat equation by means of the change $\tilde f=zf$.

For the case $\eta=\text{const}$ equation (5.22) can be reduced by inequivalent one-dimensional subalgebras of L_2. We construct the following solutions:

For the subalgebra $\langle\partial_\tau+af\partial_f\rangle$, where $a\in\{-1;0;1\}$, it follows that

$$\begin{aligned}&f=e^{-\tau}z^\nu(C_1J_\nu(z)+C_2Y_\nu(z)) \quad\text{if}\quad a=-1,\\ &f=e^{\tau}z^\nu(C_1I_\nu(z)+C_2K_\nu(z)) \quad\text{if}\quad a=1,\\ &f=C_1z^{\eta+1}+C_2 \quad\text{if}\quad a=0\quad\text{and}\quad \eta\neq-1,\\ &f=C_1\ln z+C_2 \quad\text{if}\quad a=0\quad\text{and}\quad \eta=-1.\end{aligned}$$

Here J_ν and Y_ν are the Bessel functions of a real variable, whereas I_ν and K_ν are the Bessel functions of an imaginary variable, and $\nu=\frac12(\eta+1)$.

For the subalgebra $\langle\hat D+2af\partial_f\rangle$, where $a\in\mathbb{R}$, it follows that

$$f=|\tau|^ae^{-\frac12\omega}|\omega|^{\frac12(\eta-1)}W\big(\tfrac14(\eta-1)-a,\tfrac14(\eta+1),\omega\big)$$

with $\omega=\frac14z^2\tau^{-1}$. Here $W(\varkappa,\mu,\omega)$ is the general solution of the Whittaker equation

$$4\omega^2W_{\omega\omega}=(\omega^2-4\varkappa\omega+4\mu^2-1)W.$$

For the subalgebra $\langle\partial_\tau+\Pi+af\partial_f\rangle$, where $a\in\mathbb{R}$, it follows that

$$f=(4\tau^2+1)^{\frac14(\eta-1)}\exp(-\tau\omega+\tfrac12a\arctan2\tau)\varphi(\omega)$$

with $\omega = z^2(4\tau^2+1)^{-1}$. The function φ is a solution of the equation

$$4\omega\varphi_{\omega\omega} + 2(1-\eta)\varphi_\omega + (\omega - a)\varphi = 0.$$

For example if $a = 0$, then $\varphi(\omega) = \omega^\mu\Big(C_1 J_\mu(\tfrac{1}{2}\omega) + C_2 Y_\mu(\tfrac{1}{2}\omega)\Big)$, where $\mu = \frac{1}{4}(\eta+1)$.

Consider equation (5.22), where η is an arbitrary smooth function of τ.

Theorem 5.2 *Equation (5.22) is Q-conditional invariant under the operators*

$$Q^1 = \partial_\tau + g^1(\tau,z)\partial_z + \big(g^2(\tau,z)f + g^3(\tau,z)\big)\partial_f \tag{5.23}$$

if and only if

$$\begin{aligned} &g^1_\tau - \eta z^{-1} g^1_z + \eta z^{-2} g^1 - g^1_{zz} + 2g^1_z g^1 - \eta_\tau z^{-1} + 2g^2_z = 0,\\ &g^k_\tau + \eta z^{-1} g^k_z - g^k_{zz} + 2g^1_z g^k = 0, \quad k = 2,3, \end{aligned} \tag{5.24}$$

and

$$Q^2 = \partial_z + B(\tau,z,f)\partial_f \tag{5.25}$$

if and only if

$$B_\tau - \eta z^{-2} B + \eta z^{-1} B_z - B_{zz} - 2BB_{zf} - B^2 B_{ff} = 0. \tag{5.26}$$

An arbitrary operator of Q-conditional symmetry of equation (5.22) is equivalent to either an operator of form (5.23) or an operator of form (5.25).

Theorem 5.2 is proved by means of the method described in [13].

Note 5.6 It can be shown (in a way analogous to one in [13]) that system (5.24) is reduced to the decomposed linear system

$$f^a_\tau + \eta z^{-1} f^a_z - f^a_{zz} = 0 \tag{5.27}$$

by means of the following non-local transformation

$$\begin{aligned} g^1 &= -\frac{f^1_{zz} f^2 - f^1 f^2_{zz}}{f^1_z f^2 - f^1 f^2_z} + \eta z^{-1},\\ g^2 &= -\frac{f^1_{zz} f^2_z - f^1_z f^2_{zz}}{f^1_z f^2 - f^1 f^2_z},\\ g^3 &= f^3_{zz} - \eta z^{-1} f^3_z + g^1 f^3_z - g^2 f^3. \end{aligned} \tag{5.28}$$

Equation (5.26) is reduced, by means of the change

$$B = -\Phi_\tau/\Phi_f, \quad \Phi = \Phi(\tau,z,f)$$

and the hodograph transformation

$$y_0 = \tau, \quad y_1 = z, \quad y_2 = \Phi, \quad \Psi = f,$$

to the following equation in the function $\Psi = \Psi(y_0,y_1,y_2)$:

$$\Psi_{y_0} + \eta(y_0) y_1^{-1} \Psi_{y_1} - \Psi_{y_1 y_1} = 0.$$

Therefore, unlike Lie symmetries Q-conditional symmetries of (5.22) are more extended for an arbitrary smooth function $\eta=\eta(\tau)$. Thus, Theorem 5.2 implies that equation (5.22) is Q-conditional invariant under the operators

$$\partial_z,\quad X=\partial_\tau+(\eta-1)z^{-1}\partial_z,\quad G=(2\tau+C)\partial_z-zf\partial_f$$

with $C=\text{const}$. Reducing equation (5.22) by means of the operator G, we obtain the following solution:

$$f=C_2\bigl(z^2-2\textstyle\int(\eta(\tau)-1)d\tau\bigr)+C_1. \tag{5.29}$$

In generalizing this we can construct solutions of the form

$$f=\sum_{k=0}^{N}T^k(\tau)z^{2k}, \tag{5.30}$$

where the coefficients $T^k=T^k(\tau)$ $(k=\overline{0,N})$ satisfy the system of ODEs:

$$T^k_\tau+(2k+2)(\eta(\tau)-2k-1)T^{k+1}=0,\quad k=\overline{0,N-1},\quad T^N_\tau=0. \tag{5.31}$$

Equation (5.31) is easily integrated for arbitrary $N\in\mathbb{N}$. For example if $N=2$, it follows that

$$f=C_3\Bigl\{z^4-4z^2\textstyle\int(\eta(\tau)-3)d\tau+8\int\Bigl((\eta(\tau)-1)\int(\eta(\tau)-3)d\tau\Bigr)d\tau\Bigr\}+$$
$$+C_2\Bigl\{z^2-2\textstyle\int(\eta(\tau)-1)d\tau\Bigr\}+C_1.$$

An explicit form for solution (5.30) with $N=1$ is given by (5.29).

Generalizing the solution

$$f=C_0\exp\{-z^2(4\tau+2C)^{-1}+\textstyle\int(\eta(\tau)-1)(2\tau+C)^{-1}d\tau\} \tag{5.32}$$

obtained by means of reduction of (5.22) by the operator G, we can construct solutions of the general form

$$f=\sum_{k=0}^{N}S^k(\tau)\bigl(z(2\tau+C)^{-1}\bigr)^{2k}\times$$
$$\times\exp\Bigl\{-z^2(4\tau+2C)^{-1}+\textstyle\int(\eta(\tau)-1)(2\tau+C)^{-1}d\tau\Bigr\}, \tag{5.33}$$

where the coefficients $S^k=S^k(\tau)$ $(k=\overline{0,N})$ satisfy the system of ODEs:

$$S^k_\tau+(2k+2)(\eta(\tau)-2k-1)(2\tau+C)^{-2}S^{k+1}=0,$$
$$k=\overline{0,N-1},\quad S^N_\tau=0. \tag{5.34}$$

For example if $N=1$, then

$$f=\Bigl\{C_1\Bigl(z^2(2\tau+C)^{-2}-2\textstyle\int(\eta(\tau)-1)(2\tau+C)^{-2}d\tau\Bigr)+C_0\Bigr\}\times$$
$$\times\exp\Bigl\{-z^2(4\tau+2C)^{-1}+\textstyle\int(\eta(\tau)-1)(2\tau+C)^{-1}d\tau\Bigr\}.$$

Here we do not present results for arbitrary N as they are very cumbersome.

Putting $g^2 = g^3 = 0$ in system (5.24), we obtain one equation in the function g^1:

$$g^1_\tau - \eta z^{-1} g^1_z + \eta z^{-2} g^1 - g^1_{zz} + 2 g^1_z g^1 - \eta_\tau z^{-1} = 0.$$

It follows that $g^1 = -g_z/g + (\eta - 1)/z$, where $g = g(\tau, z)$ is a solution of the equation

$$g_\tau + (\eta - 2) z^{-1} g_z - g_{zz} = 0. \tag{5.35}$$

Q-conditional symmetry of (5.22) under the operator

$$Q = \partial_\tau + \big(-g_z/g + (\eta - 1)/z\big)\partial_z \tag{5.36}$$

gives rise to the following

Theorem 5.3 *If g is a solution of equation (5.35) and*

$$\begin{aligned} f(\tau, z) = {} & \textstyle\int_{z_0}^{z} z' g(\tau, z') dz' + \\ & + \textstyle\int_{\tau_0}^{\tau} \Big(z_0 g_z(\tau', z_0) - (\eta(\tau') - 1) g(\tau', z_0) \Big) d\tau', \end{aligned} \tag{5.37}$$

where (τ_0, z_0) is a fixed point, then f is a solution of equation (5.22).

Proof. Equation (5.35) implies

$$(zg)_\tau = (z g_z - (\eta - 1) g)_z$$

Therefore, $f_z = zg, \quad f_\tau = z g_z - (\eta - 1) g$ and

$$f_\tau + \eta z^{-1} f_z - f_{zz} = z g_z - (\eta - 1) g + \eta g - (zg)_z = 0. \qquad \text{QED.}$$

The converse of Theorem 5.3 is the following obvious

Theorem 5.4 *If f is a solution of (5.22), the function*

$$g = z^{-1} f_z \tag{5.38}$$

satisfies (5.35).

Theorems 5.3 and 5.4 imply that, when $\eta = 2n$ ($n \in \mathbb{Z}$), solutions of (5.22) can be constructed from known solutions of the heat equation by means of applying either formula (5.37) (for $n > 0$) or formula (5.38) (for $n < 0$) $|n|$ times.

Let us investigate symmetry properties and construct some exact solutions of system (5.19)–(5.20) for $\varepsilon = 1$, i.e., the system

$$w^1_\tau - w^1_{zz} + \hat\eta(\tau) z^{-1} w^1_z = 0, \tag{5.39}$$

$$w^2_\tau - w^2_{zz} + (\hat\eta(\tau) - 2) z^{-1} w^2_z + (w^1 - \hat\chi(\tau)) z^{-2} = 0. \tag{5.40}$$

If (w^1, w^2) is a solution of system (5.39)–(5.40), then $(w^1, w^2 + g)$ (where $g = g(\tau, z)$) is also a solution of (5.39)–(5.40) if and only if the function g satisfies the following equation

$$g_\tau - g_{zz} + (\hat\eta(\tau) - 2) z^{-1} g_z = 0 \tag{5.41}$$

System (5.39)–(5.40), for some $\hat\chi=\hat\chi(\tau)$, has particular solutions of the form

$$w^1=\sum_{k=0}^{N}T^k(\tau)z^{2k},\quad w^2=\sum_{k=0}^{N-1}S^k(\tau)z^{2k},$$

where $T^0(\tau)=\hat\chi(\tau)$. For example, if $\hat\chi(\tau)=-2C_1\int(\hat\eta(\tau)-1)d\tau+C_2$ and $N=1$, then

$$w^1=C_1\bigl(z^2-2\int(\hat\eta(\tau)-1)d\tau\bigr)+C_2,\quad w^2=-C_1\tau.$$

Let $\hat\chi(\tau)=0$.

Theorem 5.5 *The MIA of system (5.39)–(5.40) with $\hat\chi(\tau)=0$ is given by the following algebras*

a) $\langle w^i\partial_{w^i},\ \tilde w^i(\tau,z)\partial_{w^i}\rangle$ *if* $\hat\eta(\tau)\neq\mathrm{const}$;
b) $\langle 2\tau\partial_\tau+z\partial_z,\ \partial_\tau,\ w^i\partial_{w^i},\ \tilde w^i(\tau,z)\partial_{w^i}\rangle$ *if* $\hat\eta(\tau)=\mathrm{const},\ \hat\eta\neq0$;
c) $\langle 2\tau\partial_\tau+z\partial_z,\ \partial_\tau,\ w^1z^{-1}\partial_{w^2},\ w^i\partial_{w^i},\ \tilde w^i(\tau,z)\partial_{w^i}\rangle$ *if* $\hat\eta\equiv0$.

Here $(\tilde w^1,\tilde w^2)$ is an arbitrary solution of (5.39)–(5.40) with $\hat\chi(\tau)=0$.

For the case $\hat\chi(\tau)=0$ and $\hat\eta(\tau)=\mathrm{const}$ system (5.39)–(5.40) can be reduced by inequivalent one-dimensional subalgebras of its MIA. We obtain the following solutions:

For the subalgebra $\langle\partial_\tau\rangle$ it follows that

$$w^1=C_1\ln z+C_2,$$
$$w^2=\tfrac14C_1(\ln^2z-\ln z)+\tfrac12C_2\ln z+C_3z^{-2}+C_4$$

if $\hat\eta=-1$;

$$w^1=C_1z^2+C_2,$$
$$w^2=\tfrac14C_1z^2+\tfrac12C_2\ln^2z+C_3\ln z+C_4$$

if $\hat\eta=1$;

$$w^1=C_1z^{\hat\eta+1}+C_2,$$
$$w^2=\tfrac12C_1(\hat\eta+1)^{-1}z^{\hat\eta+1}+C_2(\hat\eta-1)^{-1}\ln z+C_3z^{\hat\eta-1}+C_4$$

if $\hat\eta\not\in\{-1;1\}$.

For the subalgebra $\langle\partial_\tau-w^i\partial_{w^i}\rangle$ it follows that

$$w^1=e^{-\tau}z^{\frac12(\hat\eta+1)}\psi^1(z),\quad w^2=e^{-\tau}z^{\frac12(\hat\eta-1)}\psi^2(z),$$

where the functions ψ^1 and ψ^2 satisfy the system

$$z^2\psi^1_{zz}+z\psi^1_z+\bigl(z^2-\tfrac14(\hat\eta+1)^2\bigr)\psi^1=0,\tag{5.42}$$

$$z^2\psi^2_{zz}+z\psi^2_z+\bigl(z^2-\tfrac14(\hat\eta-1)^2\bigr)\psi^2=z\psi^1.\tag{5.43}$$

The general solution of system (5.42)–(5.43) can be expressed by quadratures in terms of the Bessel functions of a real variable $J_\nu(z)$ and $Y_\nu(z)$:

$$\psi^1 = C_1 J_{\nu+1}(z) + C_2 Y_{\nu+1}(z),$$
$$\psi^2 = C_3 J_\nu(z) + C_4 Y_\nu(z) + \tfrac{\pi}{2} Y_\nu(z) \int J_\nu(z)\psi^1(z)dz - \tfrac{\pi}{2} J_\nu(z) \int Y_\nu(z)\psi^1(z)dz$$

with $\nu = \frac{1}{2}(\hat\eta - 1)$;

For the subalgebra $\langle \partial_\tau + w^i \partial_{w^i}\rangle$ it follows that

$$w^1 = e^\tau z^{\frac{1}{2}(\hat\eta+1)}\psi^1(z), \quad w^2 = e^\tau z^{\frac{1}{2}(\hat\eta-1)}\psi^2(z),$$

where the functions ψ^1 and ψ^2 satisfy the system

$$z^2\psi^1_{zz} + z\psi^1_z - \left(z^2 + \tfrac{1}{4}(\hat\eta+1)^2\right)\psi^1 = 0, \tag{5.44}$$

$$z^2\psi^2_{zz} + z\psi^2_z - \left(z^2 + \tfrac{1}{4}(\hat\eta-1)^2\right)\psi^2 = z\psi^1. \tag{5.45}$$

The general solution of system (5.44)–(5.45) can be expressed by quadratures in terms of the Bessel functions of an imaginary variable $I_\nu(z)$ and $K_\nu(z)$:

$$\psi^1 = C_1 I_{\nu+1}(z) + C_2 K_{\nu+1}(z),$$
$$\psi^2 = C_3 I_\nu(z) + C_4 K_\nu(z) + K_\nu(z) \int I_\nu(z)\psi^1(z)dz - I_\nu(z) \int K_\nu(z)\psi^1(z)dz$$

with $\nu = \frac{1}{2}(\hat\eta - 1)$.

For the subalgebra $\langle 2\tau\partial_\tau + z\partial_z + aw^i\partial_{w^i}\rangle$ it follows that

$$w^1 = |\tau|^a e^{-\frac{1}{2}\omega}|\omega|^{\frac{1}{4}(\hat\eta-1)}\psi^1(\omega), \quad w^2 = |\tau|^a e^{-\frac{1}{2}\omega}|\omega|^{\frac{1}{4}(\hat\eta-3)}\psi^2(\omega)$$

with $\omega = \frac{1}{4}z^2\tau^{-1}$, where the functions ψ^1 and ψ^2 satisfy the system

$$4\omega^2\psi^1_{\omega\omega} = \Big(\omega^2 + \big(a - \tfrac{1}{4}(\hat\eta-1)\big)\omega + \tfrac{1}{4}(\hat\eta+1)^2 - 1\Big)\psi^1, \tag{5.46}$$

$$4\omega^2\psi^2_{\omega\omega} = \Big(\omega^2 + \big(a - \tfrac{1}{4}(\hat\eta-3)\big)\omega + \tfrac{1}{4}(\hat\eta-1)^2 - 1\Big)\psi^2 + 2|\omega|^{1/2}\psi^1. \tag{5.47}$$

The general solution of system (5.46)–(5.47) can be expressed by quadratures in terms of the Whittaker functions.

6 Symmetry properties and exact solutions of system (3.12)

As was mentioned in Section 3, ansatzes (3.4)–(3.7) reduce the NSEs (1.1) to the systems of PDEs of a similar structure that have the general form (see (3.12)):

$$\begin{aligned}
&w^i w^1_i - w^1_{ii} + s_1 + \alpha_2 w^2 = 0,\\
&w^i w^2_i - w^2_{ii} + s_2 - \alpha_2 w^1 + \alpha_1 w^3 = 0,\\
&w^i w^3_i - w^3_{ii} + \alpha_4 w^3 + \alpha_5 = 0,\\
&w^i_i = \alpha_3,
\end{aligned} \tag{6.1}$$

where α_n $(n = \overline{1,5})$ are real parameters.

Setting $\alpha_k = 0$ ($k = \overline{2,5}$) in (6.1), we obtain equations describing a plane convective flow that is brought about by nonhomogeneous heating of boudaries [25]. In this case w^i are the coordinates of the flow velocity vector, w^3 is the flow temperature, s is the pressure, the Grasshoff number λ is equal to $-\alpha_1$, and the Prandtl number σ is equal to 1. Some similarity solutions of these equations were constructed in [22]. The particular case of system (6.1) for $\alpha_1 = \alpha_2 = \alpha_4 = \alpha_5 = 0$ and $\alpha_3 = 1$ was considered in [31].

In this section we study symmetry properties of system (6.1) and construct large sets of its exact solutions.

Theorem 6.1 *The MIA of (6.1) is the algebra*

1. $E_1 = \langle \partial_1,\ \partial_2,\ \partial_s \rangle$ *if* $\alpha_1 \neq 0,\ \alpha_4 \neq 0$.
2. $E_2 = \langle \partial_1,\ \partial_2,\ \partial_s,\ \partial_{w^3} - \alpha_1 z_2 \partial_s \rangle$ *if* $\alpha_1 \neq 0,\ \alpha_4 = 0,\ (\alpha_1, \alpha_2, \alpha_5) \neq (0,0,0)$.
3. $E_3 = \langle \partial_1,\ \partial_2,\ \partial_s,\ \partial_{w^3} - \alpha_1 z_2 \partial_s,\ \tilde{D} - 3w^3\partial_{w^3} \rangle$ *if* $\alpha_1 \neq 0,\ \alpha_k = 0,\ k = \overline{2,5}$.
4. $E_4 = \langle \partial_1,\ \partial_2,\ \partial_s,\ J,\ (w^3 + \alpha_5/\alpha_4)\partial_{w^3} \rangle$ *if* $\alpha_1 = 0,\ \alpha_4 \neq 0$.
5. $E_5 = \langle \partial_1,\ \partial_2,\ \partial_s,\ J,\ \partial_{w^3} \rangle$ *if* $\alpha_1 = \alpha_4 = 0,\ (\alpha_2, \alpha_3) \neq (0,0),\ \alpha_5 \neq 0$.
6. $E_6 = \langle \partial_1,\ \partial_2,\ \partial_s,\ J,\ \partial_{w^3},\ w^3\partial_{w^3} \rangle$ *if* $\alpha_1 = \alpha_4 = \alpha_5 = 0,\ (\alpha_2, \alpha_3) \neq (0,0)$.
7. $E_7 = \langle \partial_1,\ \partial_2,\ \partial_s,\ J,\ \partial_{w^3},\ \tilde{D} + 2w^3\partial_{w^3} \rangle$ *if* $\alpha_5 \neq 0,\ \alpha_l = 0,\ l = \overline{1,4}$.
8. $E_8 = \langle \partial_1,\ \partial_2,\ \partial_s,\ J,\ \partial_{w^3},\ \tilde{D}, w^3\partial_{w^3} \rangle$ *if* $\alpha_n = 0,\ n = \overline{1,5}$.

Here $\tilde{D} = z_i\partial_i - w^i\partial_{w^i} - 2s\partial_s$*,* $J = z_1\partial_2 - z_2\partial_1 + w^1\partial_{w^2} - w^2\partial_{w^1}$*,* $\partial_i = \partial_{z_i}$*.*

Note 6.1 The bases of the algebras E_6 and E_8 contain the operator $w^3\partial_{w^3}$ that is not induced by elements of $A(NS)$.

Note 6.2 If $\alpha_4 \neq 0$, the constant α_5 can be made to vanish by means of local transformation

$$\tilde{w}^3 = w^3 + \alpha_5/\alpha_4, \quad \tilde{s} = s - \alpha_1\alpha_5\alpha_4^{-1} z_2, \tag{6.2}$$

where the independent variables and the functions w^i are not transformed. Therefore, we consider below that $\alpha_5 = 0$ if $\alpha_4 \neq 0$.

Note 6.3 Making the non-local transformation

$$\tilde{s} = s + \alpha_2\Psi, \tag{6.3}$$

where $\Psi_1 = w^2$, $\Psi_2 = -w^1$ (such a function Ψ exists in view of the last equation of (6.1)), in system (6.1) with $\alpha_3 = 0$, we obtain a system of form (6.1) with $\tilde{\alpha}_3 = \tilde{\alpha}_2 = 0$. In some cases ($\alpha_1 \neq 0$, $\alpha_3 = \alpha_4 = \alpha_5 = 0$, $\alpha_2 \neq 0$; $\alpha_1 = \alpha_3 = \alpha_4 = 0$, $\alpha_2 \neq 0$) transformation (6.3) allows the symmetry of (6.1) to be extended and non-Lie solutions to be constructed. Moreover, it means that in the cases listed above system (6.1) is invariant under the non-local transformation

$$\hat{z}_i = e^{\varepsilon} z_i, \quad \hat{w}^i = e^{-\varepsilon} w^i, \quad \hat{w}^3 = e^{\delta\varepsilon} w^3, \quad \hat{s} = e^{-2\varepsilon} s + \alpha_2(e^{-2\varepsilon} - 1)\Psi,$$

where

$$\delta = -3 \quad \text{if} \quad \alpha_3 = \alpha_4 = \alpha_5 = 0, \quad \alpha_1, \alpha_2 \neq 0;$$
$$\delta = 2 \quad \text{if} \quad \alpha_1 = \alpha_3 = \alpha_4 = 0, \quad \alpha_2, \alpha_5 \neq 0;$$
$$\delta = 0 \quad \text{if} \quad \alpha_1 = \alpha_3 = \alpha_4 = \alpha_5 = 0, \quad \alpha_2 \neq 0.$$

Let us consider an ansatz of the form:

$$\begin{aligned}
&w^1 = a_1\varphi^1 - a_2\varphi^3 + b_1\omega_2,\\
&w^2 = a_2\varphi^1 + a_1\varphi^3 + b_2\omega_2,\\
&w^3 = \varphi^2 + b_3\omega_2,\\
&s = h + d_1\omega_2 + d_2\omega_1\omega_2 + \tfrac{1}{2}d_3\omega_2^2,
\end{aligned}\tag{6.4}$$

where $a_1^2 + a_2^2 = 1$, $\omega = \omega_1 = a_1z_2 - a_2z_1$, $\omega_2 = a_1z_1 + a_2z_2$, $B, b_a, d_a = \text{const}$,

$$\begin{aligned}
&b_i = Ba_i, \quad b_3(B+\alpha_4) = 0,\\
&d_2 = \alpha_2 B - \alpha_1 b_3 a_1, \quad d_3 = -B^2 - \alpha_1 b_3 a_2,
\end{aligned}\tag{6.5}$$

Here and below $\varphi^a = \varphi^a(\omega)$ and $h = h(\omega)$. Indeed, formulas (6.4) and (6.5) determine a whole set of ansatzes for system (6.1). This set contains both Lie ansatzes, constructed by means of subalgebras of the form

$$\langle a_1\partial_1 + a_2\partial_2 + a_3(\partial_{w^3} - \alpha_1 z_2\partial_s) + a_4\partial_s\rangle, \tag{6.6}$$

and non-Lie ansatzes. Equation (6.5) is the necessary and sufficient condition to reduce (6.1) by means of an ansatz of form (6.3). As a result of reduction we obtain the following system of ODEs:

$$\begin{aligned}
&\varphi^3\varphi^1_\omega - \varphi^1_{\omega\omega} + \mu_{1j}\varphi^j + d_1 + d_2\omega + \alpha_2\varphi^3 = 0,\\
&\varphi^3\varphi^2_\omega - \varphi^2_{\omega\omega} + \mu_{2j}\varphi^j + \alpha_5 = 0,\\
&\varphi^3\varphi^3_\omega - \varphi^3_{\omega\omega} + h_\omega - \alpha_2\varphi^1 + \alpha_1 a_1\varphi^2 = 0,\\
&\varphi^3_\omega = \sigma,
\end{aligned}\tag{6.7}$$

where $\mu_{11} = -B$, $\mu_{12} = -\alpha_1 a_2$, $\mu_{21} = -b_3$, $\mu_{22} = -\alpha_4$, $\sigma = \alpha_3 - B$. If $\sigma = 0$, system (6.7) implies that

$$\begin{aligned}
&\varphi^3 = C_0 = \text{const},\\
&h = \alpha_2\textstyle\int \varphi^1(\omega)d\omega - \alpha_1 a_1 \int \varphi^2(\omega)d\omega,
\end{aligned}$$

and the functions φ^i satisfy system (4.23), where $\nu_{11} = d_1 + \alpha_2 C_0$, $\nu_{21} = d_2$, $\nu_{12} = \alpha_5$, $\nu_{22} = 0$. If $\sigma \neq 0$, then $\varphi^3 = \sigma\omega$ (translating ω, the integration constant can be made to vanish),

$$h = -\tfrac{1}{2}\sigma^2\omega^2 + \alpha_2\textstyle\int \varphi^1(\omega)d\omega - \alpha_1 a_1\int \varphi^2(\omega)d\omega,$$

and the functions satisfy system (4.29), where $\nu_{11} = d_1$, $\nu_{21} = d_2 + \alpha_2\sigma$, $\nu_{12} = \alpha_5$, $\nu_{22} = 0$.

Note 6.4 Step-by-step reduction of the NSEs (1.1) by means of ansatzes (3.4)–(3.7) and (6.4) is equivalent to a particular case of immediate reduction of the NSEs (1.1) to ODEs by means of ansatzes 5 and 6 from Subsection 4.1.

Table 1. Complete sets of inequivalent one-dimensional subalgebras of the algebras $E_1 - E_8$ (a and a_l ($l = \overline{1,4}$) are real constants)

Algebra	Subalgebras	Values of parameters
E_1	$\langle a_1\partial_1 + a_2\partial_2 + a_3\partial_s\rangle$, $\langle\partial_s\rangle$	$a_1^2 + a_2^2 = 1$
E_2	$\langle a_1\partial_1 + a_2\partial_2 + a_3(\partial_{w^3} - \alpha_1 z_2\partial_s)\rangle$, $\langle\partial_1 + a_4\partial_s\rangle$, $\langle\partial_{w^3} - \alpha_1 z_2\partial_s\rangle$, $\langle\partial_s\rangle$	$a_1^2 + a_2^2 = 1$, $a_4 \neq 0$
E_3	$\langle a_1\partial_1 + a_2\partial_2 + a_3(\partial_{w^3} - \alpha_1 z_2\partial_s)\rangle$, $\langle\partial_1 + a_4\partial_s\rangle$, $\langle\tilde{D} - 3w^3\partial_{w^3}\rangle$, $\langle\partial_{w^3} - \alpha_1 z_2\partial_s\rangle$, $\langle\partial_s\rangle$	$a_1^2 + a_2^2 = 1$, $a_3 \in \{-1; 0; 1\}$, $a_4 \in \{-1; 1\}$
E_4	$\langle J + a_1\partial_s + a_2 w^3\partial_{w^3}\rangle$, $\langle\partial_2 + a_1\partial_s + a_2 w^3\partial_{w^3}\rangle$, $\langle w^3\partial_{w^3} + a_1\partial_s\rangle$, $\langle\partial_s\rangle$	
E_5	$\langle J + a_1\partial_s + a_2\partial_{w^3}\rangle$, $\langle\partial_2 + a_1\partial_s + a_2\partial_{w^3}\rangle$, $\langle\partial_{w^3} + a_1\partial_s >, < \partial_s\rangle$	
E_6	$\langle J + a_1\partial_s + a_2 w^3\partial_{w^3}\rangle$, $\langle\partial_2 + a_1\partial_s + a_2 w^3\partial_{w^3}\rangle$, $\langle J + a_1\partial_s + a_3\partial_{w^3}\rangle$, $\langle\partial_2 + a_1\partial_s + a_3\partial_{w^3}\rangle$, $\langle w^3\partial_{w^3} + a_1\partial_s\rangle$, $\langle\partial_{w^3} + a_1\partial_s\rangle$, $\langle\partial_s\rangle$	$a_2 \neq 0$, $a_3 \in \{-1; 0; 1\}$
E_7	$\langle\tilde{D} + aJ + 2w^3\partial_{w^3}\rangle$, $\langle J + a_1\partial_s + a_2\partial_{w^3}\rangle$, $\langle\partial_2 + a_1\partial_s + a_2\partial_{w^3}\rangle, \langle\partial_{w^3} + a_2\partial_s\rangle$, $\langle\partial_s\rangle$	$a_2 \in \{-1; 0; 1\}$, $a_1 \in \{-1; 0; 1\}$ if $a_2 = 0$
E_8	$\langle\tilde{D} + aJ + a_3 w^3\partial_{w^3}\rangle$, $\langle\tilde{D} + aJ + a_3\partial_{w^3}\rangle$, $\langle J + a_1\partial_s + a_4 w^3\partial_{w^3}\rangle$, $\langle\partial_2 + a_1\partial_s + a_4 w^3\partial_{w^3}\rangle$, $\langle J + a_1\partial_s + a_2\partial_{w^3}\rangle$, $\langle\partial_2 + a_1\partial_s + a_2\partial_{w^3}\rangle$, $\langle w^3\partial_{w^3} + a_1\partial_s\rangle$, $\langle\partial_{w^3} + a_1\partial_s\rangle$, $\langle\partial_s\rangle$	$a_i \in \{-1; 0; 1\}$, $a_4 \neq 0$

Now let us choose such algebras, among the algebras from Table 1, that can be used to reduce system (6.1) and do not belong to the set of algebras (6.6). By means of the chosen algebras we construct ansatzes that are tabulated in the form of Table 2.

Table 2. Ansatzes reducing system (6.1) $(r = (z_1^2 + z_2^2)^{1/2})$

N	Values of α_n	Algebra	Invariant variable	Ansatz
1	$\alpha_1 \neq 0$, $\alpha_k = 0$, $k = \overline{2,5}$	$\langle \tilde{D} - 3w^3\partial_{w^3} \rangle$	$\omega = \arctan\frac{z_2}{z_1}$	$w^1 = r^{-2}(z_1\varphi^1 - z_2\varphi^2)$, $w^2 = r^{-2}(z_2\varphi^1 + z_1\varphi^2)$, $w^3 = r^{-3}\varphi^3$, $s = r^{-2}h$
2	$\alpha_1 = 0$, $\alpha_5 = 0$	$\langle \partial_2 + a_1\partial_s + a_2 w^3\partial_{w^3} \rangle$, $a_2 \neq 0$	$\omega = z_1$	$w^1 = \varphi^1$, $w^2 = \varphi^2$, $w^3 = \varphi^3 e^{a_2 z_2}$, $s = h + a_1 z_2$
3	$\alpha_1 = 0$, $\alpha_4 = 0$	$\langle J + a_1\partial_s + a_2\partial_{w^3} \rangle$	$\omega = r$	$w^1 = z_1\varphi^1 - z_2 r^{-2}\varphi^2$, $w^2 = z_2\varphi^1 + z_1 r^{-2}\varphi^2$, $w^3 = \varphi^3 + a_2 \arctan\frac{z_2}{z_1}$, $s = h + a_1 \arctan\frac{z_2}{z_1}$
4	$\alpha_1 = 0$, $\alpha_5 = 0$	$\langle J + a_1\partial_s + a_2 w_3\partial_{w^3} \rangle$ $a_2 \neq 0$ if $\alpha_4 = 0$	$\omega = r$	$w^1 = z_1\varphi^1 - z_2 r^{-2}\varphi^2$, $w^2 = z_2\varphi^1 + z_1 r^{-2}\varphi^2$, $w^3 = \varphi^3 e^{a_2 \arctan\frac{z_2}{z_1}}$, $s = h + a_1 \arctan\frac{z_2}{z_1}$
5	$\alpha_5 \neq 0$, $\alpha_l = 0$, $l = \overline{1,4}$	$\langle \tilde{D} + aJ + 2w^3\partial_{w^3} \rangle$	$\omega = \arctan\frac{z_2}{z_1} - -a\ln r$	$w^1 = r^{-2}(z_1\varphi^1 - z_2\varphi^2)$, $w^2 = r^{-2}(z_2\varphi^1 + z_1\varphi^2)$, $w^3 = r^2\varphi^3$, $s = r^{-2}h$
6	$\alpha_n = 0$, $n = \overline{1,5}$	$\langle \tilde{D} + aJ + a_1\partial_{w^3} \rangle$	$\omega = \arctan\frac{z_2}{z_1} - -a\ln r$	$w^1 = r^{-2}(z_1\varphi^1 - z_2\varphi^2)$, $w^2 = r^{-2}(z_2\varphi^1 + z_1\varphi^2)$, $w^3 = \varphi^3 + a_1 \ln r$, $s = r^{-2}h$
7	$\alpha_n = 0$, $n = \overline{1,5}$	$\langle \tilde{D} + aJ + a_1 w^3\partial_{w^3} \rangle$, $a_1 \neq 0$	$\omega = \arctan\frac{z_2}{z_1} - -a\ln r$	$w^1 = r^{-2}(z_1\varphi^1 - z_2\varphi^2)$, $w^2 = r^{-2}(z_2\varphi^1 + z_1\varphi^2)$, $w^3 = r^{a_1}\varphi^3$, $s = r^{-2}h$

Substituting the ansatzes from Table 2 into system (6.1), we obtain the reduced systems of ODEs in the functions φ^a and h:

$$\begin{aligned} 1.\quad & \varphi^2\varphi^1_\omega - \varphi^1_{\omega\omega} - \varphi^1\varphi^1 - \varphi^2\varphi^2 - 2h + \alpha_1\varphi^3\sin\omega + 2\varphi^2_\omega = 0,\\ & \varphi^2\varphi^2_\omega - \varphi^2_{\omega\omega} + h_\omega - 2\varphi^1_\omega + \alpha_1\varphi^3\cos\omega = 0,\\ & \varphi^2\varphi^3_\omega - \varphi^3_{\omega\omega} - 3\varphi^1\varphi^3 - 9\varphi^3 = 0,\\ & \varphi^2_\omega = 0. \end{aligned} \tag{6.8}$$

$$\begin{aligned} 2.\quad & \varphi^1\varphi^1_\omega - \varphi^1_{\omega\omega} + \alpha_2\varphi^2 + h_\omega = 0,\\ & \varphi^1\varphi^2_\omega - \varphi^2_{\omega\omega} - \alpha_2\varphi^1 + a_1 = 0,\\ & \varphi^1\varphi^3_\omega - \varphi^3_{\omega\omega} + (a_2\varphi^2 + \alpha_4 - a_2^2)\varphi^3 = 0,\\ & \varphi^1_\omega = \alpha_3. \end{aligned} \tag{6.9}$$

$$\begin{aligned} 3.\quad & \omega\varphi^1\varphi^1_\omega - \varphi^1_{\omega\omega} + \varphi^1\varphi^1 - \omega^{-4}\varphi^2\varphi^2 - 3\omega^{-1}\varphi^1_\omega + \alpha_2\omega^{-2}\varphi^2 + \omega^{-1}h_\omega = 0,\\ & \omega\varphi^1\varphi^2_\omega - \varphi^2_{\omega\omega} + \omega^{-1}\varphi^2_\omega - \alpha_2\omega^2\varphi^1 + a_1 = 0,\\ & \omega\varphi^1\varphi^3_\omega - \varphi^3_{\omega\omega} + a_2\omega^{-2}\varphi^2 - \omega^{-1}\varphi^3_\omega + \alpha_5 = 0,\\ & 2\varphi^1 + \omega\varphi^1_\omega = \alpha_3. \end{aligned} \tag{6.10}$$

$$\begin{aligned} 4.\quad & \omega\varphi^1\varphi^1_\omega - \varphi^1_{\omega\omega} + \varphi^1\varphi^1 - \omega^{-4}\varphi^2\varphi^2 - 3\omega^{-1}\varphi^1_\omega + \alpha_2\omega^{-2}\varphi^2 + \omega^{-1}h_\omega = 0,\\ & \omega\varphi^1\varphi^2_\omega - \varphi^2_{\omega\omega} + \omega^{-1}\varphi^2_\omega - \alpha_2\omega^2\varphi^1 + a_1 = 0,\\ & \omega\varphi^1\varphi^3_\omega - \varphi^3_{\omega\omega} + a_2\omega^{-2}\varphi^2\varphi^3 - \omega^{-1}\varphi^3_\omega + (\alpha_4 - a_2^2\omega^{-2})\varphi^3 = 0,\\ & 2\varphi^1 + \omega\varphi^1_\omega = \alpha_3. \end{aligned} \tag{6.11}$$

$$\begin{aligned} 5.\quad & (\varphi^2 - a\varphi^1)\varphi^1_\omega - (1+a^2)\varphi^1_{\omega\omega} - \varphi^1\varphi^1 - \varphi^2\varphi^2 - ah_\omega - 2h = 0,\\ & (\varphi^2 - a\varphi^1)\varphi^2_\omega - (1+a^2)\varphi^2_{\omega\omega} - 2(a\varphi^2_\omega + \varphi^1_\omega) + h_\omega = 0,\\ & (\varphi^2 - a\varphi^1)\varphi^3_\omega - (1+a^2)\varphi^3_{\omega\omega} + 2\varphi^1\varphi^3 - 4\varphi^3 + 4a\varphi^3_\omega + \alpha_5 = 0,\\ & \varphi^2_\omega - a\varphi^1_\omega = 0. \end{aligned} \tag{6.12}$$

$$\begin{aligned} 6.\quad & (\varphi^2 - a\varphi^1)\varphi^1_\omega - (1+a^2)\varphi^1_{\omega\omega} - \varphi^1\varphi^1 - \varphi^2\varphi^2 - ah_\omega - 2h = 0,\\ & (\varphi^2 - a\varphi^1)\varphi^2_\omega - (1+a^2)\varphi^2_{\omega\omega} - 2(a\varphi^2_\omega + \varphi^1_\omega) + h_\omega = 0,\\ & (\varphi^2 - a\varphi^1)\varphi^3_\omega - (1+a^2)\varphi^3_{\omega\omega} + a_1\varphi^1 = 0,\\ & \varphi^2_\omega - a\varphi^1_\omega = 0. \end{aligned} \tag{6.13}$$

$$\begin{aligned} 7.\quad & (\varphi^2 - a\varphi^1)\varphi^1_\omega - (1+a^2)\varphi^1_{\omega\omega} - \varphi^1\varphi^1 - \varphi^2\varphi^2 - ah_\omega - 2h = 0,\\ & (\varphi^2 - a\varphi^1)\varphi^2_\omega - (1+a^2)\varphi^2_{\omega\omega} - 2(a\varphi^2_\omega + \varphi^1_\omega) + h_\omega = 0,\\ & (\varphi^2 - a\varphi^1)\varphi^3_\omega - (1+a^2)\varphi^3_{\omega\omega} + a_1\varphi^1\varphi^3 - a_1^2\varphi^3 + 2aa_1\varphi^3_\omega = 0,\\ & \varphi^2_\omega - a\varphi^1_\omega = 0. \end{aligned} \tag{6.14}$$

Numeration of reduced systems (6.8)–(6.14) corresponds to that of the ansatzes in Table 2. Let us integrate systems (6.8)–(6.14) in such cases when it is possible. Below, in this section, $C_k = \mathrm{const}$ ($k = \overline{1,6}$).

1. We failed to integrate system (6.8) in the general case, but we managed to find the following particular solutions:

a) $\varphi^1 = -6\wp(\omega + C_3, \frac{1}{3}(4 - 2C_1), C_2) - 2,$
$\varphi^2 = \varphi^3 = 0, \quad h = 2\varphi^1 + C_1;$

b) $\varphi^1 = -6C_1^2 e^{2C_1\omega}\wp(e^{C_1\omega} + C_3, 0, C_2) + 3C_1^2 - 2,$
$\varphi^2 = 5C_1, \quad \varphi^3 = 0,$
$h = -12C_1^2 e^{2C_1\omega}\wp(e^{C_1\omega} + C_3, 0, C_2) - 2 - \frac{13}{2}C_1^2 - \frac{9}{4}C_1^4;$

c) $\varphi^1 = C_1, \quad \varphi^2 = C_2, \quad \varphi^3 = 0, \quad h = -\frac{1}{2}(C_1^2 + C_2^2).$

Here $\wp(\tau, \varkappa_1, \varkappa_2)$ is the Weierstrass function that satisfies the equation (see [19]):

$$(\wp_\tau)^2 = 4\wp^3 - \varkappa_1\wp - \varkappa_2. \tag{6.15}$$

2. If $\alpha_3 = 0$, the last equation of (6.9) implies that $\varphi^1 = C_1$. It follows from the other equations of (6.9) that

$$\varphi^2 = C_3 + C_2 e^{C_1\omega} - (a_1 C_1^{-1} - \alpha_2)\omega,$$
$$h = C_6 - \alpha_2 C_3\omega - \alpha_2 C_2 C_1^{-1} e^{C_1\omega} + \tfrac{1}{2}\alpha_2(a_1 C_1^{-1} - \alpha_2)\omega^2$$

if $C_1 \neq 0$, and

$$\varphi^2 = C_3 + C_2\omega + \tfrac{1}{2}a_1\omega^2,$$
$$h = C_6 - \alpha_2 C_3\omega - \tfrac{1}{2}\alpha_2 C_2\omega^2 - \tfrac{1}{6}\alpha_2 a_1\omega^3$$

if $C_1 = 0$. The function φ^3 satisfies the equation

$$\varphi^3_{\omega\omega} - C_1\varphi^3_\omega + (a_2^2 - \alpha_4 - a_2\varphi^2)\varphi^3 = 0. \tag{6.16}$$

We solve equation (6.16) for the following cases:

A. $C_2 = a_1 - \alpha_2 C_1 = 0$:

$$\varphi^3 = \begin{cases} e^{\frac{1}{2}C_1\omega}\big(C_4 e^{\mu^{1/2}\omega} + C_5 e^{-\mu^{1/2}\omega}\big), & \mu > 0, \\ e^{\frac{1}{2}C_1\omega}\big(C_4 + C_5\omega\big), & \mu = 0, \\ e^{\frac{1}{2}C_1\omega}\big(C_4\cos((-\mu)^{1/2}\omega) + C_5\sin((-\mu)^{1/2}\omega)\big), & \mu < 0, \end{cases}$$

where $\mu = \frac{1}{4}C_1^2 - a_2^2 + \alpha_4 + a_2 C_3$.

B. $C_1 = a_1 = 0$, $C_2 \neq 0$ ([19]):

$$\varphi^3 = \xi^{1/2} Z_{1/3}\big(\tfrac{2}{3}(-a_2 C_2)^{1/2}\xi^{3/2}\big),$$

where $\xi = \omega + (C_3 a_2 - a_2^2 - \alpha_4)/(a_2 C_2)$. Here $Z_\nu(\tau)$ is the general solution of the Bessel equation (4.22).

C. $C_1 = 0$, $a_1 \neq 0$ ([19]):

$$\varphi^3 = (\omega + C_2 a_1^{-1})^{-1/2} W\big(\nu, \tfrac{1}{4}, (\tfrac{1}{2}a_1 a_2)^{-1/2}(\omega + C_2 a_1^{-1})^2\big),$$

where $\nu = \frac{1}{4}(\frac{1}{2}a_1 a_2)^{-1/2}\big(a_2^2 - \alpha_4 - a_2 C_3 + \frac{1}{2}a_2 C_3^2 a_1^{-1}\big)$. Here $W(\varkappa, \mu, \tau)$ is the general solution of the Whittaker equation (4.21).

D. $C_1 \neq 0$, $C_2 \neq 0$, $a_1 - \alpha_2 C_1 = 0$ ([19]):

$$\varphi^3 = e^{\frac{1}{2}C_1\omega} Z_\nu\big(2C_1^{-1}(-a_2C_2)^{1/2}e^{\frac{1}{2}C_1\omega}\big),$$

where $\nu = C_1^{-1}\big(C_1^2 + 4(\alpha_4 + a_2C_3 - a_2^2)\big)^{1/2}$. Here $Z_\nu(\tau)$ is the general solution of the Bessel equation (4.22).

E. $C_1 \neq 0$, $a_1 - \alpha_2 C_1 \neq 0$, $C_2 = 0$ ([19]):

$$\varphi^3 = e^{\frac{1}{2}C_1\omega}\xi^{1/2}Z_{1/3}\Big(\tfrac{2}{3}\big(a_2(a_1C_1^{-1} - \alpha_2)\big)^{1/2}\xi^{3/2}\Big),$$

where $\xi = \omega + \big(a_2^2 - \frac{1}{4}C_1^2 - C_3a_2 - \alpha_4\big)/\big(a_2(a_1C_1^{-1} - \alpha_2)\big)$. Here $Z_\nu(\tau)$ is the general solution of the Bessel equation (4.22).

If $\alpha_3 \neq 0$, then $\varphi^1 = \alpha_3\omega$ (translating ω, the integration constant can be made to vanish),

$$\varphi^2 = C_1 + C_2\int e^{\frac{1}{2}\alpha_3\omega^2}d\omega + a_1\int e^{\frac{1}{2}\alpha_3\omega^2}\Big(\int e^{-\frac{1}{2}\alpha_3\omega^2}d\omega\Big)d\omega + \alpha_2\omega,$$

$$h = C_3 - \tfrac{1}{2}(\alpha_2^2 + \alpha_3^2)\omega^2 - \alpha_2C_1\omega - \alpha_2C_2\Big(\omega\int e^{\frac{1}{2}\alpha_3\omega^2}d\omega - \alpha_3^{-1}e^{\frac{1}{2}\alpha_3\omega^2}\Big) -$$
$$- \alpha_2a_1\Big(\omega\int e^{\frac{1}{2}\alpha_3\omega^2}\big(\int e^{-\frac{1}{2}\alpha_3\omega^2}d\omega\big)d\omega - \alpha_3^{-1}e^{\frac{1}{2}\alpha_3\omega^2}\int e^{-\frac{1}{2}\alpha_3\omega^2}d\omega + \alpha_3^{-1}\omega\Big),$$

and the function φ^3 satisfies the equation

$$\varphi^3_{\omega\omega} - \alpha_3\omega\varphi^3_\omega + (a_2^2 - \alpha_4 - a_2\varphi^2)\varphi^3 = 0. \tag{6.17}$$

We managed to find a solution of (6.17) only for the case $a_1 = C_2 = 0$, i.e.,

$$\varphi^3 = e^{\frac{1}{4}\alpha_3\omega^2}V\big(\alpha_3^{1/2}(\omega + 2a_2\alpha_2\alpha_3^{-2}), \nu\big),$$

where $\nu = 4\alpha_3^{-1}\big(\alpha_4 + a_2C_1 - a_2^2(\alpha_2^2\alpha_3^{-2} + 1)\big)$. Here $V(\tau,\nu)$ is the general solution of the Weber equation

$$4V_{\tau\tau} = (\tau^2 + \nu)V. \tag{6.18}$$

3. The general solution of system (6.10) has the form:

$$\varphi^1 = C_1\omega^{-2} + \tfrac{1}{2}\alpha_3, \tag{6.19}$$

$$\varphi^2 = C_2 + C_3\int\omega^{C_1+1}e^{\frac{1}{4}\alpha_3\omega^2}d\omega - \tfrac{1}{2}\alpha_2\omega^2 + \\ + a_1\int\omega^{C_1+1}e^{\frac{1}{4}\alpha_3\omega^2}\Big(\int\omega^{-C_1-1}e^{-\frac{1}{4}\alpha_3\omega^2}d\omega\Big)d\omega, \tag{6.20}$$

$$\varphi^3 = C_4 + C_5\int\omega^{C_1-1}e^{\frac{1}{4}\alpha_3\omega^2}d\omega + \\ + \int\omega^{C_1-1}e^{\frac{1}{4}\alpha_3\omega^2}\Big(\int\omega^{1-C_1}e^{-\frac{1}{4}\alpha_3\omega^2}(\alpha_5 + a_2\omega^{-2}\varphi^2)d\omega\Big)d\omega,$$

$$h = C_6 - \tfrac{1}{8}\alpha_3^2\omega^2 - \tfrac{1}{2}C_1^2\omega^{-2} + \int(\varphi^2(\omega))^2\omega^{-3}d\omega - \alpha_2\int\omega^{-1}\varphi^2(\omega)d\omega. \tag{6.21}$$

4. System (6.11) implies that the functions φ^i and h are determined by (6.19)–(6.21), and the function φ^3 satisfies the equation

$$\varphi^3_{\omega\omega} - \big((C_1-1)\omega^{-1} + \tfrac{1}{2}\alpha_3\omega\big)\varphi^3_\omega + \big(a_2\omega^{-2}(a_2 - \varphi^2) - \alpha_4\big)\varphi^3 = 0. \tag{6.22}$$

We managed to solve equation (6.22) in following cases:

A. $C_3 = a_1 = 0,\ \alpha_3 \neq 0$:

$$\varphi^3 = \omega^{\frac{1}{2}C_1 - 1} e^{\frac{1}{8}\alpha_3\omega^2} W(\varkappa, \mu, \tfrac{1}{4}\alpha_3\omega^2),$$

where $\varkappa = \frac{1}{4}\big(2 - C_1 - (4\alpha_4 + 2\alpha_2 a_2)\alpha_3^{-1}\big)$, $\mu = \frac{1}{4}(C_1^2 - 4a_2^2 + 4a_2C_2)^{1/2}$. Here $W(\varkappa, \mu, \tau)$ is the general solution of the Whittaker equation (4.21).

Let $\alpha_3 = 0$, then

$$\varphi^2 = \begin{cases} C_2 + C_3 \ln\omega + \frac{1}{4}(a_1 + 2\alpha_2)\omega^2, & C_1 = -2, \\ C_2 + \frac{1}{2}C_3\omega^2 + \frac{1}{2}a_1\omega^2(\ln\omega - \frac{1}{2}), & C_1 = 0, \\ C_2 + C_3(C_1+2)^{-1}\omega^{C_1+2} - \frac{1}{2}C_1^{-1}(a_1 - \alpha_2 C_1)\omega^2, & C_1 \neq 0, -2. \end{cases}$$

B. $C_3 = a_1 - \alpha_2 C_1 = 0$:

$$\varphi^3 = \begin{cases} \omega^{\frac{1}{2}C_1} Z_\nu(\mu^{1/2}\omega), & \mu \neq 0, \\ \omega^{\frac{1}{2}C_1}(C_5\omega^\nu + C_6\omega^{-\nu}), & \mu = 0, \nu \neq 0, \\ \omega^{\frac{1}{2}C_1}(C_5 + C_6\ln\omega), & \mu = 0, \nu = 0, \end{cases} \qquad (6.23)$$

where $\mu = -\alpha_4$, $\nu = \frac{1}{2}(C_1^2 - 4a_2^2 + 4a_2C_2)^{1/2}$. Here and below $Z_\nu(\tau)$ is the general solution of the Bessel equation (4.22).

C. $C_3 = 0,\ C_1 \neq 0$: φ^3 is determined by (6.23), where

$$\mu = \tfrac{1}{2}a_2C_1^{-1}(a_1 - \alpha_2C_1) - \alpha_4, \quad \nu = \tfrac{1}{2}(C_1^2 - 4a_2^2 + 4a_2C_2)^{1/2}.$$

D. $C_1 = a_1 = 0$: φ^3 is determined by (6.23), where

$$\mu = -\tfrac{1}{2}a_2C_3 - \alpha_4, \quad \nu = (-a_2^2 + a_2C_2)^{1/2}.$$

E. $C_3 \neq 0,\ C_1 \notin \{0; -2\},\ a_2(a_1 - \alpha_2C_1) - 2\alpha_4C_1 = 0$:

$$\varphi^3 = \omega^{\frac{1}{2}C_1} Z_\nu(\mu\omega^{1+\frac{1}{2}C_1}),$$

where $\mu = 2C_3^{1/2}(C_1+2)^{-3/2}$, $\nu = (C_1+2)^{-1}(C_1^2 - 4a_2^2 + 4a_2C_2)^{1/2}$.

F. $C_1 = -2,\ C_3 \neq 0,\ a_2(a_1 + 2\alpha_2) + 4\alpha_4 = 0$ ([19]):

$$\varphi^3 = \omega^{-1}\xi^{1/2} Z_{1/3}(\tfrac{2}{3}C_3^{1/2}\xi^{3/2}),$$

where $\xi = \ln\omega + C_3^{-1}(a_2^2 - a_2C_2 - 1)$.

G. $C_1 = 2,\ C_3 < 0,\ 1 - a_2^2 + a_2C_2 \geq 0$:

$$\varphi^3 = W(\varkappa, \mu, \tfrac{1}{2}(-C_3)^{1/2}\omega^2),$$

where $\varkappa = \frac{1}{8}(-C_3)^{-1/2}(-4\alpha_4 + a_2^2 - 2\alpha_2a_2)$, $\mu = \frac{1}{2}(1 - a_2^2 + a_2C_2)^{1/2}$. Here $W(\varkappa, \mu, \tau)$ is the general solution of the Whittaker equation (4.21).

5–7. Identical corollaries of system (6.12), (6.13), and (6.14) are the equations

$$\varphi^2 = a\varphi^1 + C_1, \qquad (6.24)$$

$$h = a(1 + a^2)\varphi^1_\omega + (2 + 2a^2 - aC_1)\varphi^1 + C_2, \qquad (6.25)$$

$$(1+a^2)\varphi^1_{\omega\omega} + (4a - C_1)\varphi^1_\omega + \varphi^1\varphi^1 + 4\varphi^1 + (1+a^2)^{-1}(C_1^2 + 2C_2) = 0. \quad (6.26)$$

We found the following solutions of (6.26):

A. If $(1+a^2)^{-1}(C_1^2+2C_2) < 4$:

$$\varphi^1 = \big(4 - (1+a^2)^{-1}(C_1^2+2C_2)\big)^{1/2} - 2. \quad (6.27)$$

B. If $C_1 = 4a$:

$$\varphi^1 = -6\wp\left(\frac{\omega}{(1+a^2)^{1/2}} + C_4,\ \frac{4}{3} - \frac{(C_1^2+2C_2)}{3(1+a^2)},\ C_3\right) - 2. \quad (6.28)$$

Here and below $\wp(\tau, \varkappa_1, \varkappa_2)$ is the Weierstrass function satisfying equation (6.15). If $C_2 = 2 - 6a^2$ and $C_3 = 0$, a particular case of (6.28) is the function

$$\varphi^1 = -6(1+a^2)\omega^2 - 2 \quad (6.29)$$

(the constant C_4 is considered to vanish).

C. If ${}_1 \neq 4a$, $(1+a^2)^{-1}(C_1^2+2C_2) - 4 = -9\mu^4$:

$$\varphi^1 = -6\mu^2 e^{-2\xi}\wp(e^{-\xi} + C_4, 0, C_3) + 3\mu^2 - 2, \quad (6.30)$$

where $\xi = (1+a^2)^{-1/2}\mu\omega$, $\mu = \frac{1}{5}(4a - C_1)(1+a^2)^{-1/2}$. If $C_3 = 0$, a paticular case of (6.30) is the function

$$\varphi^1 = -6\mu^2 e^{-2\xi}(e^{-\xi} + C_4)^{-2} + 3\mu^2 - 2, \quad (6.31)$$

where the constant C_4 is considered not to vanish.

The function φ^3 has to be found for systems (6.12), (6.13), and (6.14) individually.

5. The function φ^3 satisfy the equation

$$(1+a^2)\varphi^3_{\omega\omega} - (C_1 + 4a)\varphi^3_\omega - (2\varphi^1 - 4)\varphi^3 - \alpha_5 = 0.$$

If φ^1 is determined by (6.27), we obtain

$$\varphi^3 = \exp\big(\tfrac{1}{2}(1+a^2)^{-1}(C_1+4a)\omega\big)\times$$
$$\times\left\{\begin{array}{ll} C_5\exp(\nu^{1/2}\omega) + C_6\exp(-\nu^{1/2}\omega), & \nu > 0 \\ C_5\cos((-\nu)^{1/2}\omega) + C_6\sin((-\nu)^{1/2}\omega), & \nu < 0 \\ C_5 + C_6\omega, & \nu = 0 \end{array}\right\} +$$
$$+\left\{\begin{array}{lll} -\alpha_5(2\varphi^1 - 4)^{-1}, & 2\varphi^1 - 4 \neq 0 & \\ -\alpha_5(4a + C_1)^{-1}\omega, & 2\varphi^1 - 4 = 0, & C_1 + 4a \neq 0 \\ \frac{1}{2}\alpha_5(1+a^2)^{-1}\omega^2, & 2\varphi^1 - 4 = 0, & C_1 + 4a = 0 \end{array}\right\},$$

where $\nu = \frac{1}{4}(1+a^2)^{-2}(C_1+4a)^2 - (1+a^2)^{-1}(4 - 2\varphi^1)$.

6. In this case φ^3 satisfy the equation

$$(1+a^2)\varphi^3_{\omega\omega} - C_1\varphi^3_\omega = a_1\varphi^1.$$

Therefore,

$$\varphi^3 = C_5 + C_6 \exp\big((1+a^2)^{-1}C_1\omega\big) + a_1 C_1^{-1}\Big(\int \varphi^1(\omega)d\omega + \\ + \exp\big((1+a^2)^{-1}C_1\omega\big)\int \exp\big(-(1+a^2)^{-1}C_1\omega\big)\varphi^1(\omega)d\omega\Big)$$

for $C_1 \neq 0$, and

$$\varphi^3 = C_5 + C_6\omega + a_1(1+a^2)^{-1}\big(\omega\int\varphi^1(\omega)d\omega - \int\omega\varphi^1(\omega)d\omega\big)$$

for $C_1 = 0$.

7. The function φ^3 satisfy the equation

$$(1+a^2)\varphi^3_{\omega\omega} - (C_1+2a_1a)\varphi^3_\omega + (a_1^2 - a_1\varphi^1)\varphi^3 = 0. \tag{6.32}$$

A. If φ^1 is determined by (6.27), it follows that

$$\varphi^3 = \exp\big(\tfrac{1}{2}(1+a^2)^{-1}(C_1+2a_1a)\omega\big)\times \\ \times\left\{\begin{array}{ll} C_5\exp(\nu^{1/2}\omega) + C_6\exp(-\nu^{1/2}\omega), & \nu>0 \\ C_5\cos((-\nu)^{1/2}\omega) + C_6\sin((-\nu)^{1/2}\omega), & \nu<0 \\ C_5 + C_6\omega, & \nu=0\end{array}\right\},$$

where $\nu = \frac{1}{4}(1+a^2)^{-2}(C_1+2a_1a)^2 - (1+a^2)^{-1}(a_1^2 - a_1\varphi^1)$.

B. If $C_1 = 4a$, that is, φ^1 is determined by (6.27), we obtain

$$\varphi^3 = \exp\big(a(a_1+2)(1+a^2)^{-1}\omega\big)\theta(\tau),$$

where $\tau = (1+a^2)^{-1/2}\omega + C_4$. Here the function $\theta = \theta(\tau)$ is the general solution of of the following Lame equation ([19]):

$$\theta_{\tau\tau} + \big(6a_1\wp(\tau) + a_1^2 + 2a_1 - a^2(2+a_1)^2(1+a^2)^{-1}\big)\theta = 0$$

with the Weierstrass function

$$\wp(\tau) = \wp\Big(\tau, \tfrac{1}{3}\big(4 - (1+a^2)^{-1}(C_1^2 + 2C_2)\big), C_3\Big).$$

Consider the particular case when $C_2 = 2 - 6a^2$ and $C_3 = 0$ additionally, i.e., φ^1 can be given in form (6.29). Depending on the values of a and a_1, we obtain the following expression for φ^3:

Case 1. $a_1 \neq -2$, $a_1 \neq 2a^2$:

$$\varphi^3 = |\omega|^{1/2} exp\left(\frac{a(2+a_1)}{1+a^2}\omega\right) Z_\nu\left(\frac{\big((2+a_1)(a_1-2a^2)\big)^{1/2}}{1+a^2}\omega\right),$$

where $\nu = (\frac{1}{4} - 6a_1)^{1/2}$.

Case 2. $a_1 = -2$: $\varphi^3 = C_5\omega^4 + C_6\omega^{-3}$.

Case 3. $a_1 = 2a^2$:

Case 3.1. $48a^2 < 1$: $\varphi^3 = |\omega|^{1/2}e^{2a\omega}\big(C_5\omega^\sigma + C_6\omega^{-\sigma}\big)$, where $\sigma = \frac{1}{2}\sqrt{1-48a^2}$.

Case 3.2. $48a^2 = 1$, that is, $a = \pm\frac{1}{12}\sqrt{3}$: $\varphi^3 = |\omega|^{1/2}(C_5 + C_6\ln\omega)$.

Case 3.3. $48a^2 > 1$: $\varphi^3 = |\omega|^{1/2}e^{2a\omega}\big(C_5\cos(\gamma\ln\omega) + C_6\sin(\gamma\ln\omega)\big)$, where $\gamma = \frac{1}{2}\sqrt{48a^2-1}$.

C. Let the conditions

$$C_1 \neq 4a, \quad (1+a^2)^{-1}(C_1^2 + 2C_2) - 4 = -9\mu^4$$

be satisfied, that is, let φ^1 be determined by (6.30). Transforming the variables in equation (6.32) by the formulas:

$$\varphi^3 = \tau^{-1/2}\exp\Big(\tfrac{1}{2}(C_1 + 2aa_1)(1+a^2)^{-1}\omega\Big)\theta(\tau),$$
$$\tau = \exp\big(-\mu(1+a^2)^{-1/2}\omega\big),$$

we obtain the following equation in the function $\theta = \theta(\tau)$:

$$\tau^2\theta_{\tau\tau} + \big(6a_1\tau^2\wp(\tau + C_4, 0, C_3) + \sigma\big)\theta = 0, \tag{6.33}$$

where $\sigma = \mu^{-2}\big(a_1^2 + 2a_1 - \frac{1}{4}(1+a^2)^{-1}(C_1^2 + 2aa_1)^2\big) - 3a_1 + \frac{1}{4}$. If $\sigma = 0$, equation (6.33) is a Lame equation.

In the particular case when φ^1 is determined by (6.31), equation (6.33) has the form:

$$\tau^2(\tau + C_4)^2\theta_{\tau\tau} + \big(6a_1\tau^2 + \sigma(\tau + C_4)^2\big)\theta = 0. \tag{6.34}$$

By means of the following transformation of variables:

$$\theta = |\xi|^{\nu_1}|\xi - 1|^{\nu_2}\psi(\xi), \quad \xi = -C_4^{-1}\tau,$$

where $\nu_1(\nu_1 - 1) + \sigma = 0$ and $\nu_2(\nu_2 - 1) + 6a_1 = 0$, equation (6.34) is reduced to a hypergeometric equation of the form (see [19]):

$$\xi(\xi - 1)\psi_{\xi\xi} + \big(2(\nu_1 + \nu_2)\xi - 2\nu_1\big)\psi_\xi + 2\nu_1\nu_2\psi = 0.$$

If $\sigma = 0$, equation (6.34) implies that

$$(\tau + C_4)^2\theta_{\tau\tau} + 6a_1\theta = 0.$$

Therefore,

$$\theta = C_5|\tau + C_4|^{1/2-\nu} + C_6|\tau + C_4|^{1/2+\nu}$$

if $a_1 < \frac{1}{24}$, where $\nu = (\frac{1}{4} - 6a_1)^{1/2}$,

$$\theta = |\tau + C_4|^{1/2}\big(C_5 + C_6\ln|\tau + C_4|\big)$$

if $a_1 = \frac{1}{24}$, and

$$\theta = |\tau + C_4|^{1/2}\big(C_5\cos(\nu\ln|\tau + C_4|) + C_6\sin(\nu\ln|\tau + C_4|)\big)$$

if $a_1 > \frac{1}{24}$, where $\nu = (6a_1 - \frac{1}{4})^{1/2}$.

7 Exact solutions of system (2.9)

Among the reduced systems from Section 2, only particular cases of system (2.9) have Lie symmetry operators that are not induced by elements from $A(NS)$. Therefore, Lie reductions of the other systems from Section 2 give only solutions that can be obtained by means of reducing the NSEs with two- and three-dimensional subalgebras of $A(NS)$.

Here we consider system (2.9) with ρ^i vanishing. As mentioned in Note 2.5, in this case the vector-function $\vec{m}$ has the form $\vec{m} = \eta(t)\vec{e}$, where $\vec{e} = \text{const}$, $|\vec{e}| = 1$, and $\eta = \eta(t) = |\vec{m}(t)| \neq 0$. Without loss of generality we can assume that $\vec{e} = (0,0,1)$, i.e.,

$$\vec{m} = (0, 0, \eta(t)).$$

For such vector $\vec{m}$, conditions (2.5) are satisfied by the following vector $\vec{n}^i$:

$$\vec{n}^1 = (1,0,0), \quad \vec{n}^2 = (0,1,0).$$

Therefore, ansatz (2.4) and system (2.9) can be written, respectively, in the forms:

$$\begin{aligned} &u^1 = v^1, \quad u^2 = v^2, \quad u^3 = \big(\eta(t)\big)^{-1}\big(v^3 + \eta_t(t)x_3\big), \\ &p = q - \tfrac{1}{2}\eta_{tt}(t)\big(\eta(t)\big)^{-1}x_3^2, \end{aligned} \tag{7.1}$$

where $v = v(y_1, y_2, y_3)$, $q = q(y_1, y_2, y_3)$, $y_i = x_i$, $y_3 = t$, and

$$\begin{aligned} &v^i_t + v^j v^i_j - v^i_{jj} + q_i = 0, \\ &v^3_t + v^j v^3_j - v^3_{jj} = 0, \\ &v^i_i + \rho^3 = 0, \end{aligned} \tag{7.2}$$

where $\rho^3 = \rho^3(t) = \eta_t/\eta$.

It was shown in Note 2.8 that there exists a local transformation which make ρ^3 vanish. Therefore, we can consider system (7.2) only with ρ^3 vanishing and extend the obtained results in the case $\rho^3 \neq 0$ by means of transformation (2.12). However it will be sometimes convenient to investigate, at once, system (7.2) with an arbitrary function ρ^3.

The MIA of (7.2) with $\rho^3 = 0$ is given by the algebra

$$B = \langle R_3(\bar{\psi}),\ Z^1(\lambda),\ D^1_3,\ \partial_t,\ J^1_{12},\ \partial_{v^3},\ v^3\partial_{v^3}\rangle$$

(see notations in Subsection 2.1). We construct complete sets of inequivalent one-dimensional subalgebras of B and choose such algebras, among these subalgebras, that can be used to reduce system (7.2) and do not lie in the linear span of the operators $R_3(\bar{\psi})$, $Z^1(\lambda)$, J^1_{12}, i.e., the operators which are induced by operators from $A(NS)$ for arbitrary ρ^3. As a result we obtain the following algebras (more exactly, the following classes of algebras):

The one-dimentional subalgebras:

1. $B^1_1 = \langle D^1_3 + 2\varkappa J^1_{12} + 2\gamma v^3\partial_{v^3} + 2\beta\partial_{v^3}\rangle$, where $\gamma\beta = 0$.
2. $B^1_2 = \langle \partial_t + \varkappa J^1_{12} + \gamma v^3\partial_{v^3} + \beta\partial_{v^3}\rangle$, where $\gamma\beta = 0$, $\varkappa \in \{0; 1\}$.
3. $B^1_3 = \langle J^1_{12} + \gamma v^3\partial_{v^3} + Z^1(\lambda(t))\rangle$, where $\gamma \neq 0$, $\lambda \in C^\infty((t_0, t_1), \mathbb{R})$.
4. $B^1_4 = \langle R_3(\bar{\psi}(t)) + \gamma v^3\partial_{v^3}\rangle$, where $\gamma \neq 0$,
   $\bar{\psi}(t) = (\psi^1(t), \psi^2(t)) \neq (0,0)\ \forall\, t \in (t_0, t_1)$, $\psi^i \in C^\infty((t_0, t_1), \mathbb{R})$.

The two-dimentional subalgebras:

1. $B^2_1 = \langle \partial_t + \beta_2\partial_{v^3},\ D^1_3 + \varkappa J^1_{12} + \gamma v^3\partial_{v^3} + \beta_1\partial_{v^3}\rangle$,
   where $\gamma\beta_1 = 0, (\gamma-2)\beta_2 = 0$.
2. $B^2_2 = \langle D^1_3 + 2\gamma_1 v^3\partial_{v^3} + 2\beta_1\partial_{v^3},\ J^1_{12} + \gamma_2 v^3\partial_{v^3} + \beta_2\partial_{v^3} + Z^1(\varepsilon|t|^{-1})\rangle$,
   where $\gamma_1\beta_1 = 0,\ \gamma_2\beta_2 = 0,\ \gamma_1\beta_2 - \gamma_2\beta_1 = 0$.
3. $B^2_3 = \langle D^1_3 + 2\varkappa J^1_{12} + 2\gamma_1 v^3\partial_{v^3} + 2\beta_1\partial_{v^3},\ R_3(|t|^{\sigma+1/2}\cos\tau, |t|^{\sigma+1/2}\sin\tau) +$
   $+ \gamma_2 v^3\partial_{v^3} + \beta_2\partial_{v^3} + Z^1(\varepsilon|t|^{\sigma-1})\rangle$, where $\tau = \varkappa\ln|t|$,
   $(\gamma_1+\sigma)\beta_1 - \gamma_2\beta_1 = 0,\ \sigma\gamma_2 = 0,\ \varepsilon\sigma = 0$.
4. $B^2_4 = \langle \partial_t + \gamma_1 v^3\partial_{v^3} + \beta_1\partial_{v^3},\ J^1_{12} + \gamma_2 v^3\partial_{v^3} + \beta_2\partial_{v^3} + Z^1(\varepsilon)\rangle$,
   where $\gamma_1\beta_1 = 0,\ \gamma_2\beta_2 = 0,\ \gamma_1\beta_2 - \gamma_2\beta_1 = 0$.
5. $B^2_5 = \langle \partial_t + \varkappa J^1_{12} + \gamma_1 v^3\partial_{v^3} + \beta_1\partial_{v^3},\ R_3(e^{\sigma t}\cos\varkappa t, e^{\sigma t}\sin\varkappa t) +$
   $+ Z^1(\varepsilon e^{\sigma t}) + \gamma_2 v^3\partial_{v^3} + \beta_2\partial_{v^3}\rangle$, where $(\gamma_1+\sigma)\beta_1 - \gamma_2\beta_1 = 0$,
   $\sigma\gamma_2 = 0,\ \varepsilon\sigma = 0$.
6. $B^2_6 = \langle R_3(\bar\psi^1) + \gamma v^3\partial_{v^3},\ R_3(\bar\psi^2)\rangle$, where $\bar\psi^i = (\psi^{i1}(t), \psi^{i2}(t)) \neq (0,0)$
   $\forall\, t \in (t_0, t_1),\ \psi^{ij} \in C^\infty((t_0,t_1),\mathbb{R}),\ \bar\psi^1_{tt}\cdot\bar\psi^2 - \bar\psi^1\cdot\bar\psi^2_{tt} = 0,\ \gamma \neq 0$.
   Hereafter $\bar\psi^1\cdot\bar\psi^2 := \psi^{1i}\psi^{2i}$.

Let us reduce system (7.2) to systems of PDEs in two independent variables. With the algebras B^1_1–B^1_4 we can construct the following complete set of Lie ansatzes of codimension 1 for system (7.2) with $\rho^3 = 0$:

1.
$$\begin{aligned} &v^1 = |t|^{-1/2}(w^1\cos\tau - w^2\sin\tau) + \tfrac12 y_1 t^{-1} - \varkappa y_2 t^{-1},\\ &v^2 = |t|^{-1/2}(w^1\sin\tau + w^2\cos\tau) + \tfrac12 y_2 t^{-1} + \varkappa y_1 t^{-1},\\ &v^3 = |t|^{\gamma}w^3 + \beta\ln|t|,\\ &q = |t|^{-1}s + \tfrac12(\varkappa^2 + \tfrac14)t^{-2}r^2, \end{aligned} \tag{7.3}$$

where $\tau = \varkappa\ln|t|$, $\gamma\beta = 0$,

$$z_1 = |t|^{-1/2}(y_1\cos\tau + y_2\sin\tau), \quad z_2 = |t|^{-1/2}(-y_1\sin\tau + y_2\cos\tau).$$

Here and below $w^a = w^a(z_1,z_2)$, $s = s(z_1,z_2)$, $r = (y_1^2 + y_2^2)^{1/2}$.

2.
$$\begin{aligned} &v^1 = w^1\cos\varkappa t - w^2\sin\varkappa t - \varkappa y_2,\\ &v^2 = w^1\sin\varkappa t + w^2\cos\varkappa t + \varkappa y_1,\\ &v^3 = w^3 e^{\gamma t} + \beta t,\\ &q = s + \tfrac12\varkappa^2 r^2, \end{aligned} \tag{7.4}$$

where $\varkappa \in \{0;1\}$, $\gamma\beta = 0$,

$$z_1 = y_1\cos\varkappa t + y_2\sin\varkappa t, \quad z_2 = -y_1\sin\varkappa t + y_2\cos\varkappa t.$$

3.
$$\begin{aligned} &v^1 = y_1 r^{-1}w^3 - y_2 r^{-2}w^1 - \gamma y_2 r^{-2},\\ &v^2 = y_2 r^{-1}w^3 + y_1 r^{-2}w^1 + \gamma y_1 r^{-2},\\ &v^3 = w^2 e^{\gamma\arctan y_2/y_1},\\ &q = s + \lambda(t)\arctan y_2/y_1, \end{aligned} \tag{7.5}$$

where ${}_1 = t$, $z_2 = r$, $\gamma \neq 0$, $\lambda \in C^\infty((t_0,t_1),\mathbb{R})$.

$$4.\quad \begin{aligned} &\bar v = (\bar\psi\cdot\bar\psi)^{-1}\Big((w^1+\gamma)\bar\psi + w^3\bar\theta + (\bar\psi\cdot\bar y)\bar\psi_t - z_2\bar\theta_t\Big)\\ &v^3 = w^2\exp\big(\gamma(\bar\psi\cdot\bar\psi)^{-1}(\bar\psi\cdot\bar y)\big)\\ &q = s - (\bar\psi\cdot\bar\psi)^{-1}(\bar\psi_{tt}\cdot\bar y)(\bar\psi\cdot\bar y) + \tfrac12(\bar\psi\cdot\bar\psi)^{-2}(\bar\psi_{tt}\cdot\bar\psi)(\bar\psi\cdot\bar y)^2,\end{aligned} \tag{7.6}$$

where $z_1 = t$, $z_2 = (\bar\theta\cdot\bar y)$, $\gamma\neq 0$, $\bar v = (v^1,v^2)$, $\bar y = (y_1,y_2)$, $\psi^i \in C^\infty((t_0,t_1),\mathbb{R})$, $\bar\theta = (-\psi^2,\psi^1)$.

Substituting ansatzes (7.3) and (7.4) into system (7.2) with $\rho^3 = 0$, we obtain a reduced system of the form (6.1), where

$$\begin{aligned}&\alpha_1 = 0,\quad \alpha_2 = -1,\quad \alpha_3 = -2\varkappa,\quad \alpha_4 = \gamma,\quad \alpha_5 = \beta \quad\text{if}\quad t>0 \quad\text{and}\\ &\alpha_1 = 0,\quad \alpha_2 = 1,\quad \alpha_3 = 2\varkappa,\quad \alpha_4 = -\gamma,\quad \alpha_5 = -\beta \quad\text{if}\quad t<0\end{aligned}$$

for ansatz (7.3) and

$$\alpha_1 = 0,\quad \alpha_2 = 0,\quad \alpha_3 = -2\varkappa,\quad \alpha_4 = \gamma,\quad \alpha_5 = \beta$$

for ansatz (7.4). System (6.1) is investigated in Section 6 in detail.

Because the form of ansatzes (7.3) is not changed after transformation (2.12), it is convinient to substitute their into a system of form (7.2) with an arbitrary function ρ^3. As a result of substituting, we obtain the following reduced systems:

$$3.\quad \begin{aligned}&w^3_1 + w^3w^3_2 - z_2^{-3}(w^1+\gamma)^2 - (w^3_{22} + z_2^{-1}w^3_2 - z_2^{-2}w^3) + s_2 = 0,\\ &w^1_1 + w^3w^1_2 - w^1_{22} + z_2^{-1}w^1_2 + \lambda = 0,\\ &w^2_1 + w^3w^2_2 - w^2_{22} - z_2^{-1}w^2_2 + \gamma z_2^{-2}w^1w^2 = 0,\\ &w^3_2 + z_2^{-1}w^3 = -\eta_1/\eta.\end{aligned} \tag{7.7}$$

$$4.\quad \begin{aligned}&w^1_1 + w^3w^1_2 - (\bar\psi\cdot\bar\psi)w^1_{22} = 0,\\ &w^3_1 + w^3w^3_2 - (\bar\psi\cdot\bar\psi)w^3_{22} + (\bar\psi\cdot\bar\psi)s_2 + 2(w^1+\gamma)(\bar\psi\cdot\bar\theta)(\bar\psi\cdot\bar\psi)^{-1} -\\ &\qquad - 2(\bar\psi_t\cdot\bar\psi)(\bar\psi\cdot\bar\psi)^{-1}w^3 + (2\bar\psi_t\cdot\bar\psi_t - \bar\psi_{tt}\cdot\bar\psi)(\bar\psi\cdot\bar\psi)^{-1}z_2 = 0,\\ &w^2_1 + w^3w^2_2 - (\bar\psi\cdot\bar\psi)w^2_{22} + \gamma(\bar\psi\cdot\bar\psi)^{-1}\big(w^1 + (\bar\psi_t\cdot\bar\theta)(\bar\psi\cdot\bar\psi)^{-1}z_2\big)w^2 = 0,\\ &w^3_2 + \eta_t/\eta = 0.\end{aligned} \tag{7.8}$$

Unlike systems 8 and 9 from Subsection 3.2, systems (7.7) and (7.8) are not reduced to linear systems of PDEs.

Let us investigate system (7.7). The last equation of (7.7) immediately gives

$$\begin{aligned}&(w^3_2 + z_2^{-1}w^3)_2 = w^3_{22} + z_2^{-1}w^3_2 - z_2^{-2}w^3 = 0,\\ &w^3 = (\chi-1)z_2^{-1} - \tfrac12\eta_t\eta^{-1}z_2,\end{aligned} \tag{7.9}$$

where $\chi = \chi(t)$ is an arbitrary differentiable function of $t = z_2$. Then it follows from the first equation of (7.7) that

$$s = \int z_2^{-3}(w^1+\gamma)^2dz_2 - \tfrac12(\chi-1)^2z_2^{-2} + \tfrac14 z_2^2\Big((\eta_t/\eta)_t - \tfrac12(\eta_t/\eta)^2\Big) - \chi_t\ln|z_2|.$$

Substituting (7.9) into the remaining equations of (7.7), we get

$$w^1_1 - w^1_{22} + \big(\chi z_2^{-1} - \tfrac{1}{2}\eta_t\eta^{-1}z_2\big)w^1_2 + \lambda = 0,$$
$$w^2_1 - w^2_{22} + \big((\chi-2)z_2^{-1} - \tfrac{1}{2}\eta_t\eta^{-1}z_2\big)w^2_2 + \gamma z_2^{-2}w^1w^2 = 0. \tag{7.10}$$

By means of changing the independent variables

$$\tau = \textstyle\int |\eta(t)|dt, \quad z = |\eta(t)|^{1/2}z_2, \tag{7.11}$$

system (7.10) can be transformed to a system of a simpler form:

$$w^1_\tau - w^1_{zz} + \hat\chi z^{-1}w^2_z + \hat\lambda|\hat\eta|^{-1} = 0,$$
$$w^2_\tau - w^2_{zz} + (\hat\chi-2)z^{-1}w^2_z + \gamma z^{-2}w^1w^2 = 0, \tag{7.12}$$

where $\hat\eta(\tau) = \eta(t)$, $\hat\chi(\tau) = \chi(t)$, and $\hat\lambda(\tau) = \lambda(t)$.

If $\lambda(t) = -2C\eta(t)(\chi(t)-1)$ for some fixed constant C, particular solutions of (7.10) are functions

$$w^1 = C\eta(t)z_2^2, \quad w^2 = f(z_1,z_2)\exp\big(\gamma C\textstyle\int\eta(t)dt\big),$$

where f is an arbitrary solution of the following equation

$$f_1 - f_{22} + \big((\chi-2)z_2^{-1} - \tfrac{1}{2}\eta_t\eta^{-1}z_2\big)f_2 = 0. \tag{7.13}$$

In the variables from (7.11), equation (7.13) has form (5.22) with $\tilde\eta(\tau) = \chi(t)-2$.

In the case $\lambda(t) = 8C(\chi(t)-1)\eta(t)\int\eta(t)(\chi(t)-3)dt$ ($C=\mathrm{const}$), particular solutions of (7.10) are functions

$$w^1 = C\Big((\eta(t))^2z_2^4 - 4z_2^2\eta(t)\textstyle\int\eta(t)(\chi(t)-3)dt\Big),$$
$$w^2 = f(z_1,z_2)\exp\big(\tfrac{1}{2}(\gamma C)^{1/2}\eta(t)z_2^2 + \xi(t)\big),$$

where $\xi(t) = -(\gamma C)^{1/2}\int\eta(t)(\chi(t)-3)dt + 4\gamma C\int\eta(t)\big(\int\eta(t)(\chi(t)-3)dt\big)dt$ and f is an arbitrary solution of the following equation

$$f_1 - f_{22} + \big((\chi-2)z_2^{-1} - (\tfrac{1}{2}\eta_t\eta^{-1} + 2(\gamma C)^{1/2})z_2\big)f_2 = 0. \tag{7.14}$$

After the change of the independent variables

$$\tau = \textstyle\int|\eta(t)|\exp\big(4(\gamma C)^{1/2}\int\eta(t)dt\big)dt, \quad z = |\eta(t)|^{1/2}\exp\big(2(\gamma C)^{1/2}\int\eta(t)dt\big)z_2$$

in (7.14), we obtain equation (5.22) with $\tilde\eta(\tau) = \chi(t)-2$ again.

Let us continue to system (7.8). The last equation of (7.8) integrates with respect to z_2 to the following expression: $w^3 = -\eta_t\eta^{-1}z_2 + \chi$. Here $\chi = \chi(t)$ is an differentiable function of $z_1 = y_3 = t$. Let us make the transformation from the symmetry group of (7.2):

$$\tilde{\bar v}(t,\bar y) = \bar v(t,\bar y - \bar\xi(t)) + \bar\xi_t(t), \quad \tilde v^3 = v^3, \quad \tilde q(t,\bar y) = q(t,\bar y - \bar\xi(t)) - \bar\xi_{tt}(t)\cdot\bar y,$$

where $\bar\xi_{tt}\cdot\bar\psi - \bar\xi\cdot\bar\psi_{tt} = 0$ and

$$\bar\xi_t\cdot\bar\theta + \chi + \eta_t\eta^{-1}(\bar\xi\cdot\bar\theta) - |\bar\psi|^{-2}(\bar\xi\cdot\bar\psi)(\bar\psi_t\cdot\bar\theta) + |\bar\psi|^{-2}(\bar\xi\cdot\bar\theta)(\bar\theta_t\cdot\bar\theta) = 0.$$

Hereafter $|\bar\psi|^2=\bar\psi\cdot\bar\psi$. This transformation does not modify ansatz (7.6), but it makes the function χ vanish, i.e., $\tilde w^3=-\eta_t\eta^{-1}z_2$. Therefore, without loss of generality we may assume, at once, that $w^3=-\eta_t\eta^{-1}z_2$.

Substituting the expression for w^3 in the other equations of (7.8), we obtain that

$$s=z_2^2|\bar\psi|^{-2}\Big(\big(\tfrac12\bar\psi_{tt}\cdot\bar\psi-\bar\psi_t\cdot\bar\psi_t-(\bar\psi_t\cdot\bar\psi)\eta_t\eta^{-1}\big)|\bar\psi|^{-2}+\tfrac12\eta_{tt}\eta^{-1}-(\eta_t)^2\eta^{-2}\Big)-\\ -2(\bar\psi_t\cdot\bar\theta)|\bar\psi|^{-2}\textstyle\int w^1(z_1,z_2)dz_2,$$

$$\begin{aligned}&w^1_1-\eta_1\eta^{-1}z_2w^1_2-|\bar\psi|^2w^1_{22}=0,\\&w^2_1-\eta_1\eta^{-1}z_2w^2_2-|\bar\psi|^2w^2_{22}+\gamma|\bar\psi|^{-2}\big(2(\bar\psi_t\cdot\bar\theta)|\bar\psi|^{-2}z_2+w^1\big)w^2=0.\end{aligned}\tag{7.15}$$

The change of the independent variables

$$\tau=\textstyle\int(\eta(t)|\bar\psi|)^2dt,\quad z=\eta(t)z_2$$

reduces system (7.15) to the following form:

$$\begin{aligned}&w^1_\tau-w^1_{zz}=0,\\&w^2_\tau-w^2_{zz}+\gamma|\hat{\bar\psi}|^{-4}\hat\eta^{-2}\big(2(\hat{\bar\psi}_t\cdot\hat{\bar\theta})\hat\eta z+w^1\big)w^2=0,\end{aligned}\tag{7.16}$$

where $\hat{\bar\psi}(\tau)=\bar\psi(t)$, $\hat{\bar\theta}(\tau)=\bar\theta(t)$, $\hat\eta(\tau)=\eta(t)$.

Particular solutions of (7.15) are the functions

$$\begin{aligned}&w^1=C_1+C_2\eta(t)z_2+C_3\big(\tfrac12(\eta(t)z_2)^2+\textstyle\int(\eta(t)|\bar\psi|)^2dt\big),\\&w^2=f(t,z_2)\exp\big(\xi^2(t)z_2^2+\xi^1(t)z_2+\xi^0(t)\big),\end{aligned}$$

where $(\xi^2(t),\xi^1(t),\xi^0(t))$ is a particular solution of the system of ODEs:

$$\begin{aligned}&\xi^2_t-2\eta_t\eta^{-1}\xi^2-4|\bar\psi|^2(\xi^2)^2+\tfrac12C_3\gamma\eta^2|\bar\psi|^{-2}=0,\\&\xi^1_t-\eta_t\eta^{-1}\xi^1-4|\bar\psi|^2\xi^2\xi^1+2\gamma(\bar\psi_t\cdot\bar\theta)|\bar\psi|^{-4}+C_2\gamma\eta|\bar\psi|^{-2}=0,\\&\xi^0_t-2|\bar\psi|^2\xi^2-|\bar\psi|^2(\xi^1)^2+\gamma\big(C_1+C_3\textstyle\int(\eta(t)|\bar\psi|)^2dt\big)|\bar\psi|^{-2}=0,\end{aligned}$$

and f is an arbitrary solution of the following equation

$$f_1-|\bar\psi|^2f_{22}+\big((\eta_t\eta^{-1}+4|\bar\psi|^2\xi^2)z_2+2|\bar\psi|^2\xi^1\big)f_2=0.\tag{7.17}$$

Equation (7.17) is reduced by means of a local transformation of the independent variables to the heat equation.

Consider the Lie reductions of system (7.2) to systems of ODEs. The second basis operator of the each algebra B^2_k, $k=\overline{1,5}$ induces, for the reduced system obtained from system (7.2) by means of the first basis operator, either a Lie symmetry operator from Table 2 or a operator giving a ansatz of form (6.4). Therefore, the Lie reduction of system (7.2) with the algebras $B^2_1-B^2_5$ gives only solutions that can be constructed for system (7.2) by means of reducing with the algebras B^1_1 and B^1_2 to system (6.1).

With the algebra B^2_6 we obtain an ansatz and a reduced system of the following forms:

$$\begin{aligned}&\bar v=\bar\phi+\lambda^{-1}(\bar\theta^i\cdot\bar y)\bar\psi^i_t,\quad v^3=\phi^3\exp\big(\gamma\lambda(\bar\theta^1\cdot\bar y)\big),\\&s=h-\tfrac12\lambda^{-1}(\bar\psi^i_{tt}\cdot\bar y)(\bar\theta^i\cdot\bar y),\end{aligned}\tag{7.18}$$

where $\phi^a = \phi^a(\omega)$, $h = h(\omega)$, $\omega = t$, $\lambda = \psi^{11}\psi^{22} - \psi^{12}\psi^{21} = \bar{\psi}^1 \cdot \bar{\theta}^1 = \bar{\psi}^2 \cdot \bar{\theta}^2$, $\bar{\theta}^1 = (\psi^{22}, -\psi^{21})$, $\bar{\theta}^2 = (-\psi^{12}, \psi^{11})$, and

$$\begin{aligned}&\bar{\phi}_t + \lambda^{-1}(\bar{\theta}^i \cdot \bar{\phi})\bar{\psi}^i_t = 0, \quad \phi^3_t + \left(\gamma\lambda^{-1}(\bar{\theta}^1 \cdot \bar{\phi}) - \gamma^2\lambda^{-2}(\bar{\theta}^1 \cdot \bar{\theta}^1)\right)\phi^3 = 0,\\ &\lambda^{-1}(\bar{\theta}^i \cdot \bar{\psi}^i_t) + \eta_t\eta^{-1} = 0.\end{aligned} \tag{7.19}$$

Let us make the transformation from the symmetry group of system (7.2):

$$\tilde{\bar{v}}(t, \bar{y}) = \bar{v}(t, \bar{y} - \bar{\xi}) + \bar{\xi}_t, \quad \tilde{v}^3(t, \bar{y}) = v^3(t, \bar{y} - \bar{\xi}), \quad \tilde{s}(t, \bar{y}) = s(t, \bar{y} - \bar{\xi}) - \bar{\xi}_{tt} \cdot \bar{y},$$

where

$$\bar{\xi}_t + \lambda^{-1}(\bar{\theta}^i \cdot \bar{\xi})\bar{\psi}^i_t + \bar{\phi} = 0. \tag{7.20}$$

It follows from (7.20) that $\bar{\xi}_{tt} = \lambda^{-1}(\bar{\theta}^i \cdot \bar{\xi})\bar{\psi}^i_{tt}$, i.e., $\bar{\theta}^i_{tt} \cdot \bar{\xi} - \bar{\theta}^i \cdot \bar{\xi}_{tt} = 0$. Therefore, this trasformation does not modify ansatz (7.18), but it makes the functions ϕ^i vanish. And without loss of generality we may assume, at once, that $\phi^i \equiv 0$. Then

$$\phi^3 = C \exp\Bigl(\int \left(\gamma\lambda^{-1}|\theta|\right)^2 dt\Bigr), \quad C = \text{const}.$$

The last equation of system (7.19) is the compatibility condition of system (7.2) and ansatz (7.18).

8 Conclusion

In this article we reduced the NSEs to systems of PDEs in three and two independent variables and systems of ODEs by means of the Lie method. Then, we investigated symmetry properties of the reduced systems of PDEs and made Lie reductions of systems which admitted non-trivial symmetry operators, i.e., operators that are not induced by operators from $A(NS)$. Some of the systems in two independent variables were reduced to linear systems of either two one-dimensional heat equations or two translational equations. We also managed to find exact solutions for most of the reduced systems of ODEs.

Now, let us give some remaining problems. Firstly, we failed, for the present, to describe the non-Lie ansatzes of form (1.6) that reduce the NSEs. (These ansatzes include, for example, the well-known ansatzes for the Karman swirling flows (see bibliography in [16]). One can also consider non-local ansatzes for the Navier–Stokes field, i.e., ansatzes containing derivatives of new unknown functions.

Second problem is to study non-Lie (i.e., non-local, conditional, and Q-conditional) symmetries of the NSEs [13].

And finally, it would be interesting to investigate compatability and to construct exact solutions of overdetermined systems that are obtained from the NSEs by means of different additional conditions. Usually one uses the condition where the nonlinearity has a simple form, for example, the potential form (see review [36]), i.e., $\mathrm{rot}((\vec{u} \cdot \vec{\nabla})\vec{u}) = \vec{0}$ (the NS fields satisfying this condition is called the generalized Beltrami flows). We managed to describe the general solution of the NSEs with the additional condition where the convective terms vanish [29, 30]. But one can give other conditions, for example,

$$\triangle\vec{u} = \vec{0}, \quad \vec{u}_t + (\vec{u} \cdot \vec{\nabla})\vec{u} = \vec{0},$$

and so on.

We will consider the problems above elsewhere.

Appendix

A Inequivalent one-, two-, and three-dimensional subalgebras of $A(NS)$

To find complete sets of inequivalent subalgebras of $A(NS)$, we use the method given, for example, in [27, 28]. Let us describe it briefly.

1. We find the commutation relations between the basis elements of $A(NS)$.

2. For arbitrary basis elements V, W^0 of $A(NS)$ and each $\varepsilon \in \mathbb{R}$ we calculate the adjoint action

$$W(\varepsilon) = \mathrm{Ad}(\varepsilon V)W^0 = \mathrm{Ad}(\exp(\varepsilon V))W^0$$

of the element $\exp(\varepsilon V)$ from the one-parameter group generated by the operator V on W^0. This calculation can be made in two ways: either by means of summing the Lie series

$$W(\varepsilon) = \sum_{n=0}^{\infty} \frac{\varepsilon^n}{n!}\{V^n, W^0\} = W^0 + \frac{\varepsilon}{1!}[V, W^0] + \frac{\varepsilon^2}{2!}[V, [V, W^0]] + \cdots, \qquad \text{(A.1)}$$

where $\{V^0, W^0\} = W^0$, $\{V^n, W^0\} = [V, \{V^{n-1}, W^0\}]$, or directly by means of solving the initial value problem

$$\frac{dW(\varepsilon)}{d\varepsilon} = [V, W(\varepsilon)], \quad W(0) = W^0. \qquad \text{(A.2)}$$

3. We take a subalgebra of a general form with a fixed dimension. Taking into account that the subalgebra is closed under the Lie bracket, we try to simplify it by means of adjoint actions as much as possible.

A.1 The commutation relations and the adjoint representation of the algebra $A(NS)$

Basis elements (1.2) of $A(NS)$ satisfy the following commutation relations:

$$\begin{aligned}
&[J_{12}, J_{23}] = -J_{31}, \quad [J_{23}, J_{31}] = -J_{12}, \quad [J_{31}, J_{12}] = -J_{23},\\
&[\partial_t, J_{ab}] = [D, J_{ab}] = 0, \quad [\partial_t, D] = 2\partial_t,\\
&[\partial_t, R(\vec{m})] = R(\vec{m}_t), \quad [D, R(\vec{m})] = R(2t\vec{m}_t - \vec{m}),\\
&[\partial_t, Z(\chi)] = Z(\chi_t), \quad [D, Z(\chi)] = Z(2t\chi_t + 2\chi),\\
&[R(\vec{m}), R(\vec{n})] = Z(\vec{m}_{tt} \cdot \vec{n} - \vec{m} \cdot \vec{n}_{tt}), \quad [J_{ab}, R(\vec{m})] = R(\vec{\tilde{m}}),\\
&[J_{ab}, Z(\chi)] = [Z(\chi), R(\vec{m})] = [Z(\chi), Z(\eta)] = 0,
\end{aligned} \qquad \text{(A.3)}$$

where $\tilde{m}^a = m^b$, $\tilde{m}^b = -m^a$, $\tilde{m}^c = 0$, $a \neq b \neq c \neq a$.

Note A.1 Relations (A.3) imply that the set of operators (1.2) generates an algebra when, for example, the parameter-functions m^a and χ belong to $C^\infty((t_0, t_1), \mathbb{R})$ ($C_0^\infty((t_0, t_1), \mathbb{R})$, $A((t_0, t_1), \mathbb{R})$), i.e., the set of infinite-differentiable (infinite-differentiable finite, real analytic) functions from (t_0, t_1) in $\mathbb{R}$, where $-\infty \leq t_0 < t_1 \leq +\infty$.

But the NSEs (1.1) admit operators (1.3) and (1.4) with parameter-functions of a less degree of smoothness. Moreover, the minimal degree of their smoothness depends on the smoothness that is demanded for the solutions of the NSEs (1.1). Thus, if $u^a \in C^2((t_0,t_1)\times\Omega,\mathbb{R})$ and $p\in C^1((t_0,t_1)\times\Omega,\mathbb{R})$, where Ω is a domain in $\mathbb{R}^3$, then it is sufficient that $m^a\in C^3((t_0,t_1),\mathbb{R})$ and $\chi\in C^1((t_0,t_1),\mathbb{R})$. Therefore, one can consider the "pseudoalgebra" generated by operators (1.2). The prefix "pseudo-" means that in this set of operators the commutation operation is not determined for all pairs of its elements, and the algebra axioms are satisfied only by elements, where they are defined. It is better to indicate the functional classes that are sets of values for the parameters m^a and χ in the notation of the algebra $A(NS)$. But below, for simplicity, we fix these classes, taking $m^a,\chi\in C^\infty((t_0,t_1),\mathbb{R})$, and keep the notation of the algebra generated by operators (1.2) in the form $A(NS)$. However, all calculations will be made in such a way that they can be translated for the case of a less degree of smoothness.

Most of the adjoint actions are calculated simply as sums of their Lie series. Thus,

$$
\begin{gathered}
\mathrm{Ad}(\varepsilon\partial_t)D = D+2\varepsilon\partial_t,\quad \mathrm{Ad}(\varepsilon D)\partial_t = e^{-2\varepsilon}\partial_t,\\
\mathrm{Ad}(\varepsilon Z(\chi))\partial_t=\partial_t-\varepsilon Z(\chi_t),\quad \mathrm{Ad}(\varepsilon Z(\chi))D = D-\varepsilon Z(2t\chi_t+2\chi),\\
\mathrm{Ad}(\varepsilon R(\vec m))\partial_t = \partial_t-\varepsilon R(\vec m_t)-\tfrac12\varepsilon^2 Z(\vec m_t\cdot\vec m_{tt}-\vec m\cdot\vec m_{ttt}),\\
\mathrm{Ad}(\varepsilon R(\vec m))D = D-\varepsilon R(2t\vec m_t-\vec m)-\\
-\tfrac12\varepsilon^2 Z(2t\vec m_t\cdot\vec m_{tt}-2t\vec m\cdot\vec m_{ttt}-4\vec m\cdot\vec m_{tt}),\\
\mathrm{Ad}(\varepsilon R(\vec m))J_{ab} = J_{ab}-\varepsilon R(\vec{\tilde m})+\varepsilon^2 Z(m^a m^b_{tt}-m^a_{tt}m^b),\\
\mathrm{Ad}(\varepsilon R(\vec m))R(\vec n) = R(\vec n)+\varepsilon Z(\vec m_{tt}\cdot\vec n-\vec m\cdot\vec n_{tt}),\quad \mathrm{Ad}(\varepsilon J_{ab})R(\vec m)=R(\vec{\hat m}),\\
\mathrm{Ad}(\varepsilon J_{ab})J_{cd} = J_{cd}\cos\varepsilon+[J_{ab},J_{cd}]\sin\varepsilon\quad \big((a,b)\neq(c,d)\neq(b,a)\big),
\end{gathered}
\tag{A.4}
$$

where

$$
\begin{gathered}
\tilde m^a = m^b,\quad \tilde m^b=-m^a,\quad \tilde m^c=0,\quad a\neq b\neq c\neq a,\\
\hat m^d = m^d\cos\varepsilon+\tilde m^d\sin\varepsilon,\quad \hat m^c=m^c,\quad a\neq b\neq c\neq a,\quad d\in\{a;b\}.
\end{gathered}
$$

Four adjoint actions are better found by means of integrating a system of form (A.2). As a result we obtain that

$$
\begin{gathered}
\mathrm{Ad}(\varepsilon\partial_t)Z(\chi(t)) = Z(\chi(t+\varepsilon)),\quad \mathrm{Ad}(\varepsilon D)Z(\chi(t)) = Z(e^{2\varepsilon}\chi(te^{2\varepsilon})),\\
\mathrm{Ad}(\varepsilon\partial_t)R(\vec m(t)) = R(\vec m(t+\varepsilon)),\quad \mathrm{Ad}(\varepsilon D)R(\vec m(t)) = R(e^{-\varepsilon}\vec m(te^{2\varepsilon})).
\end{gathered}
\tag{A.5}
$$

Cases where adjoint actions coincide with the identical mapping are omitted.

Note A.2 If $Z(\chi(t))\in A(NS)[C^\infty((t_0,t_1),\mathbb{R})]$ with $-\infty<t_0$ or $t_1<+\infty$, the adjoint representation $\mathrm{Ad}(\varepsilon\partial_t)$ ($\mathrm{Ad}(\varepsilon D)$) gives an equivalence relation between the operators $Z(\chi(t))$ and $Z(\chi(t+\varepsilon))$ ($Z(\chi(t))$ and $Z(e^{2\varepsilon}\chi(te^{2\varepsilon}))$) that belong to the different algebras

$$
\begin{gathered}
A(NS)[C^\infty((t_0,t_1),\mathbb{R})]\quad\text{and}\quad A(NS)[C^\infty((t_0-\varepsilon,t_1-\varepsilon),\mathbb{R})]\\
(A(NS)[C^\infty((t_0,t_1),\mathbb{R})]\quad\text{and}\quad A(NS)[C^\infty((t_0e^{-2\varepsilon},t_1e^{-2\varepsilon}),\mathbb{R})])
\end{gathered}
$$

respectively. An analogous statement is true for the operator $R(\vec m)$. Equivalence of subalgebras in Theorems A.1 and A.2 is also meant in this sense.

Note A.3 Besides the adjoint representations of operators (1.2) we make use of discrete transformation (1.6) for classifying the subalgebras of $A(NS)$,

To prove the theorem of this section, the following obvious lemma is used.

Lemma A.1 *Let* $N \in \mathbb{N}$.

A. *If* $\chi \in C^N((t_0,t_1),\mathbb{R})$, *then* $\exists\,\eta \in C^N((t_0,t_1),\mathbb{R}):\ 2t\eta_t + 2\eta = \chi$.

B. *If* $\chi \in C^N((t_0,t_1),\mathbb{R})$, *then* $\exists\,\eta \in C^N((t_0,t_1),\mathbb{R}):\ 2t\eta_t - \eta = \chi$.

C. *If* $m^i \in C^N((t_0,t_1),\mathbb{R})$ *and* $a \in \mathbb{R}$, *then* $\exists\, l^i \in C^N((t_0,t_1),\mathbb{R}):$
$2tl^1_t - l^1 + al^2 = m^1, \quad 2tl^2_t - l^2 - al^1 = m^2.$

A.2 One-dimensional subalgebras

Theorem A.1 *A complete set of $A(NS)$-inequivalent one-dimensional subalgebras of $A(NS)$ is exhausted by the following algebras:*

1. $A^1_1(\varkappa) = \langle D + 2\varkappa J_{12}\rangle$, *where* $\varkappa \geq 0$.

2. $A^1_2(\varkappa) = \langle \partial_t + \varkappa J_{12}\rangle$, *where* $\varkappa \in \{0;1\}$.

3. $A^1_3(\eta,\chi) = \langle J_{12} + R(0,0,\eta(t)) + Z(\chi(t))\rangle$ *with smooth functions* η *and* χ. *Algebras* $A^1_3(\eta,\chi)$ *and* $A^1_3(\tilde\eta,\tilde\chi)$ *are equivalent if* $\exists\,\varepsilon,\delta \in \mathbb{R}$, $\exists\,\lambda \in C^\infty((t_0,t_1),\mathbb{R})$:

$$\tilde\eta(\tilde t) = e^{-\varepsilon}\eta(t), \quad \tilde\chi(\tilde t) = e^{2\varepsilon}(\chi(t) + \lambda_{tt}(t)\eta(t) - \lambda(t)\eta_{tt}(t)), \tag{A.6}$$

where $\tilde t = te^{-2\varepsilon} + \delta$.

4. $A^1_4(\vec m,\chi) = \langle R(\vec m(t)) + Z(\chi(t))\rangle$ *with smooth functions* $\vec m$ *and* χ: $(\vec m,\chi) \not\equiv (\vec 0, 0)$. *Algebras* $A^1_4(\vec m,\chi)$ *and* $A^1_4(\tilde{\vec m},\tilde\chi)$ *are equivalent if* $\exists\,\varepsilon,\delta \in \mathbb{R}$, $\exists\, C \neq 0$, $\exists\, B \in O(3)$, $\exists\,\vec l \in C^\infty((t_0,t_1),\mathbb{R}^3)$:

$$\tilde{\vec m}(\tilde t) = Ce^{-\varepsilon}B\vec m(t), \quad \tilde\chi(\tilde t) = Ce^{2\varepsilon}\big(\chi(t) + \vec l_{tt}(t)\cdot\vec m(t) - \vec m_{tt}(t)\cdot\vec l(t)\big), \tag{A.7}$$

where $\tilde t = te^{-2\varepsilon} + \delta$.

Proof. Consider an arbitrary one-dimensional subalgebra generated by

$$V = a_1D + a_2\partial_t + a_3J_{12} + a_4J_{23} + a_5J_{31} + R(\vec m) + Z(\chi).$$

The coefficients a_4 and a_5 are omitted below since they always can be made to vanish by means of the adjoint representations $\mathrm{Ad}(\varepsilon_1 J_{12})$ and $\mathrm{Ad}(\varepsilon_2 J_{31})$.

If $a_1 \neq 0$ we get $\tilde a_1 = 1$ by means of a change of basis. Next, step-by-step we make a_2, $\vec m$, and χ vanish by means of the adjoint representations $\mathrm{Ad}(-\frac12 a_2a_1^{-1}\partial_t)$, $\mathrm{Ad}(R(\vec l))$, and $\mathrm{Ad}(Z(\chi))$, where

$$\vec l \in C^\infty((t_0 + \tfrac12 a_2a_1^{-1}, t_1 + \tfrac12 a_2a_1^{-1}),\mathbb{R}^3),$$
$$\eta \in C^\infty((t_0 + \tfrac12 a_2a_1^{-1}, t_1 + \tfrac12 a_2a_1^{-1}),\mathbb{R}),$$

and $\vec l$, η are solutions of the equations

$$2t\vec l_t - \vec l + a_3a_1^{-1}(l^2,-l^1,0)^T = \tilde{\vec m}, \quad 2t\eta_t + 2\eta = \hat\chi + \frac12(\vec l_{tt}\cdot\tilde{\vec m} - \vec l\cdot\tilde{\vec m}_{tt})$$

with $\hat{\vec{m}}(t) = a_1^{-1}\vec{m}(t - \frac{1}{2}a_2a_1^{-1})$ and $\hat{\chi}(t) = a_1^{-1}\chi(t - \frac{1}{2}a_2a_1^{-1})$. Such $\vec{l}$ and η exist in virtue of Lemma A.1. As a result we obtain the algebra $A_1^1(\varkappa)$, where $2\varkappa = a_3a_1^{-1}$. In case $\varkappa < 0$ additionally one has to apply transformation (1.6) with $b = 1$.

If $a_1 = 0$ and $a_2 \neq 0$, we make $\tilde{a}_2 = 1$ by means of a change of basis. Next, step-by-step we make $\vec{m}$ and χ vanish by means of the adjoint representations $\mathrm{Ad}(R(\vec{l}))$ and $\mathrm{Ad}(Z(\chi))$, where $\vec{l} \in C^\infty((t_0,t_1),\mathbb{R}^3)$, $\eta \in C^\infty((t_0,t_1),\mathbb{R})$, and

$$a_2\vec{l}_t + a_3(l^2, -l^1, 0)^T = \vec{m}, \quad a_2\eta_t = \chi + \frac{1}{2}(\vec{l}_{tt}\cdot\vec{m} - \vec{l}\cdot\vec{m}_{tt}).$$

If $a_3 = 0$ we obtain the algebra $A_2^1(0)$ at once. If $a_3 \neq 0$, using the adjoint representation $\mathrm{Ad}(\varepsilon D)$ and transformation (1.6) (in case of need), we obtain the algebra $A_2^1(1)$.

If $a_1 = a_2 = 0$ and $a_3 \neq 0$, after a change of basis and applying the adjoint representation $\mathrm{Ad}(R(-a_3^{-1}m^2, a_3^{-1}m^1, 0))$ we get the algebra $A_3^1(\eta, \tilde{\chi})$, where $\eta = a_3^{-1}m^3$ and $\tilde{\chi} = a_3^{-1}\chi + a_3^{-2}(m^1_{tt}m^2 - m^1m^2_{tt})$. Equivalence relation (A.6) is generated by the adjoint representations $\mathrm{Ad}(\varepsilon D)$, $\mathrm{Ad}(\delta\partial_t)$, and $\mathrm{Ad}(R(0,0,\lambda))$.

If $a_1 = a_2 = a_3 = 0$, at once we get the algebra $A_4^1(\vec{m}, \chi)$. Equivalence relation (A.7) is generated by the adjoint representations $\mathrm{Ad}(\varepsilon D)$, $\mathrm{Ad}(\delta\partial_t)$, $\mathrm{Ad}(R(\vec{l}))$, and $\mathrm{Ad}(\varepsilon_{ab}J_{ab})$.

A.3 Two-dimensional subalgebras

Theorem A.2 *A complete set of $A(NS)$-inequivalent two-dimensional subalgebras of $A(NS)$ is exhausted by the following algebras:*

1. $A_1^2(\varkappa) = \langle \partial_t,\ D + \varkappa J_{12}\rangle$, *where* $\varkappa \geq 0$.

2. $A_2^2(\varkappa, \varepsilon) = \langle D,\ J_{12} + R(0,0,\varkappa|t|^{1/2}) + Z(\varepsilon t^{-1})\rangle$, *where* $\varkappa \geq 0$, $\varepsilon \geq 0$.

3. $A_3^2(\varkappa, \varepsilon) = \langle \partial_t,\ J_{12} + R(0,0,\varkappa) + Z(\varepsilon)\rangle$, *where* $\varkappa \in \{0;1\}$, $\varepsilon \geq 0$ *if* $\varkappa = 1$ *and* $\varepsilon \in \{0;1\}$ *if* $\varkappa = 0$.

4. $A_4^2(\sigma, \varkappa, \mu, \nu, \varepsilon) = \langle D + 2\varkappa J_{12},\ R\big(|t|^{\sigma+1/2}(\nu\cos\tau,\ \nu\sin\tau,\ \mu)\big) + Z(\varepsilon|t|^{\sigma-1})\rangle$, *where* $\tau = \varkappa\ln|t|$, $\varkappa > 0$, $\mu \geq 0$, $\nu \geq 0$, $\mu^2 + \nu^2 = 1$, $\varepsilon\sigma = 0$, *and* $\varepsilon \geq 0$.

5. $A_5^2(\sigma, \varepsilon) = \langle D,\ R(0,0,|t|^{\sigma+1/2}) + Z(\varepsilon|t|^{\sigma-1})\rangle$, *where* $\varepsilon\sigma = 0$ *and* $\varepsilon \geq 0$.

6. $A_6^2(\sigma, \mu, \nu, \varepsilon) = \langle \partial_t + J_{12},\ R(\nu e^{\sigma t}\cos t, \nu e^{\sigma t}\sin t, \mu e^{\sigma t}) + Z(\varepsilon e^{\sigma t})\rangle$, *where* $\mu \geq 0$, $\nu \geq 0$, $\mu^2 + \nu^2 = 1$, $\varepsilon\sigma = 0$, *and* $\varepsilon \geq 0$.

7. $A_7^2(\sigma, \varepsilon) = \langle \partial_t,\ R(0,0,e^{\sigma t}) + Z(\varepsilon e^{\sigma t})\rangle$, *where* $\sigma \in \{-1;0;1\}$, $\varepsilon\sigma = 0$, *and* $\varepsilon \geq 0$.

8. $A_8^2(\lambda, \psi^1, \rho, \psi^2) = \langle J_{12} + R(0,0,\lambda) + Z(\psi^1),\ R(0,0,\rho) + Z(\psi^2)\rangle$ *with smooth functions (of t) λ, ρ, and ψ^i: $(\rho, \psi^2) \not\equiv (0,0)$ and $\lambda_{tt}\rho - \lambda\rho_{tt} \equiv 0$. Algebras $A_8^2(\lambda, \psi^1, \rho, \psi^2)$ and $A_8^2(\tilde{\lambda}, \tilde{\psi}^1, \tilde{\rho}, \tilde{\psi}^2)$ are equivalent if $\exists C_1 \neq 0$, $\exists \varepsilon, \delta, C_2 \in \mathbb{R}$, $\exists \theta \in C^\infty((t_0,t_1),\mathbb{R})$:*

$$\begin{aligned}
&\tilde{\lambda}(\tilde{t}) = e^{\varepsilon}(\lambda(t) + C_2\rho(t)), \quad \tilde{\rho}(\tilde{t}) = C_1e^{-\varepsilon}\rho(t),\\
&\tilde{\psi}^1(\tilde{t}) = e^{2\varepsilon}\big(\psi^1(t) + \theta_{tt}(t)\lambda(t) - \theta(t)\lambda_{tt}(t) +\\
&\qquad + C_2(\psi^2(t) + \theta_{tt}(t)\rho(t) - \theta(t)\rho_{tt}(t))\big),\\
&\tilde{\psi}^2(\tilde{t}) = C_1e^{2\varepsilon}(\psi^2(t) + \theta_{tt}(t)\rho(t) - \theta(t)\rho_{tt}(t)),
\end{aligned} \tag{A.8}$$

where $\tilde{t} = te^{-2\varepsilon} + \delta$.

9. $A_9^2(\vec{m}^1,\chi^1,\vec{m}^2,\chi^2) = \langle R(\vec{m}^1(t)) + Z(\chi^1(t)),\ R(\vec{m}^2(t)) + Z(\chi^2(t))\rangle$ *with smooth functions* $\vec{m}^i$ *and* χ^i*:*

$$\vec{m}^1_{tt}\cdot\vec{m}^2 - \vec{m}^1\cdot\vec{m}^2_{tt} = 0, \quad \operatorname{rank}\big((\vec{m}^1,\chi^1),(\vec{m}^2,\chi^2)\big) = 2.$$

Algebras $A_9^2(\vec{m}^1,\chi^1,\vec{m}^2,\chi^2)$ *and* $A_9^2(\tilde{\vec{m}}^1,\tilde{\chi}^1,\tilde{\vec{m}}^2,\tilde{\chi}^2)$ *are equivalent if* $\exists\,\varepsilon,\delta\in\mathbb{R}$, $\exists\,\{a_{ij}\}_{i,j=1,2} : \det\{a_{ij}\}\neq 0$, $\exists\,B\in O(3)$, $\exists\,\vec{l}\in C^\infty((t_0,t_1),\mathbb{R}^3)$*:*

$$\begin{aligned}&\tilde{\vec{m}}^i(\tilde{t}) = e^{-\varepsilon}a_{ij}B\vec{m}^j(t),\\ &\tilde{\chi}^i(\tilde{t}) = e^{2\varepsilon}a_{ij}\big(\chi^j(t) + \vec{l}_{tt}(t)\cdot\vec{m}^j(t) - \vec{l}(t)\cdot\vec{m}^j_{tt}(t)\big),\end{aligned} \tag{A.9}$$

where $\tilde{t} = te^{-2\varepsilon}+\delta$.

10. $A_{10}^2(\varkappa,\sigma) = \langle D+\varkappa J_{12},\ Z(|t|^\sigma)\rangle$, *where* $\varkappa\geq 0$, $\sigma\in\mathbb{R}$.

11. $A_{11}^2(\sigma) = \langle \partial_t + J_{12},\ Z(e^{\sigma t})\rangle$, *where* $\sigma\in\mathbb{R}$.

12. $A_{12}^2(\sigma) = \langle \partial_t,\ Z(e^{\sigma t})\rangle$, *where* $\sigma\in\{-1;0;1\}$.

The proof of Theorem A.2 is analogous to that of Theorem A.1. Let us take an arbitrary two-dimensional subalgebra generated by two linearly independent operators of the form

$$V^i = a_1^iD + a_2^i\partial_t + a_3^iJ_{12} + a_4^iJ_{23} + a_5^iJ_{31} + R(\vec{m}^i) + Z(\chi^i),$$

where $a_n^i = \text{const}$ $(n=\overline{1,5})$ and $[V^1,V^2]\in\langle V^1,V^2\rangle$. Considering the different possible cases we try to simplify V^i by means of adjoint representation as much as possible. Here we do not present the proof of Theorem A.2 as it is too cumbersome.

A.4 Three-dimensional subalgebras

We also constructed a complete set of $A(NS)$-inequivalent three-dimensional subalgebras. It contains 52 classes of algebras. By means of 22 classes from this set one can obtain ansatzes of codimension three for the Navier–Stokes field. Here we only give 8 superclasses that arise from unification of some of these classes:

1. $A_1^3 = \langle D,\ \partial_t,\ J_{12}\rangle$.

2. $A_2^3 = \langle D+\varkappa J_{12},\ \partial_t,\ R(0,0,1)\rangle$, where $\varkappa\geq 0$. Here and below $\varkappa$, σ, ε_1, ε_2, μ, ν, and a_{ij} are real constants.

3. $A_3^3(\sigma,\nu,\varepsilon_1,\varepsilon_2) = \langle D,\ J_{12}+\nu\big(R(0,0,|t|^{1/2}\ln|t|) + Z(\varepsilon_2|t|^{-1}\ln|t|)\big) + Z(\varepsilon_1|t|^{-1}),$ $R(0,0,|t|^{\sigma+1/2}) + Z(\varepsilon_2|t|^{\sigma-1})\rangle$, where $\nu\sigma=0$, $\varepsilon_1\geq 0$, $\nu\geq 0$, and $\sigma\varepsilon_2 = 0$.

4. $A_4^3(\sigma,\nu,\varepsilon_1,\varepsilon_2) = \langle \partial_t,\ J_{12}+Z(\varepsilon_1)+\nu\big(R(0,0,t)+Z(\varepsilon_2t)\big),\ R(0,0,e^{\sigma t})+Z(\varepsilon_2e^{\sigma t})\rangle$, where $\nu\sigma=0$, $\sigma\varepsilon_2=0$, and, if $\sigma=0$, the constants ν, ε_1, and ε_2 satisfy one of the following conditions:

$$\nu=1,\ \varepsilon_1\geq 0; \quad \nu=0,\ \varepsilon_1=1,\ \varepsilon_2\geq 0; \quad \nu=\varepsilon_1=0,\ \varepsilon_2\in\{0;1\}.$$

5. $A_5^3(\varkappa,\vec{m}^1,\vec{m}^2,\chi^1,\chi^2) = \langle D+2\varkappa J_{12},\ R(\vec{m}^1)+Z(\chi^1),\ R(\vec{m}^2)+Z(\chi^2)\rangle$, where $\varkappa\geq 0$, $\operatorname{rank}(\vec{m}^1,\vec{m}^2)=2$,

$$\begin{aligned}&t\vec{m}^i_t - \tfrac{1}{2}\vec{m}^i + \varkappa(m^{i2},-m^{i1},0)^T = a_{ij}\vec{m}^j,\\ &t\chi^i_t + \chi^i = a_{ij}\chi^j, \quad a_{ij} = \text{const},\end{aligned}$$

$$(a_{11}+a_{22})\big(a_{21}\vec m^1\cdot\vec m^1+(a_{22}-a_{11})\vec m^1\cdot\vec m^2-a_{12}\vec m^2\cdot\vec m^2+ \\ +2\varkappa(m^{12}m^{21}-m^{11}m^{22})\big)=0. \tag{A.10}$$

This superclass contains eight inequivalent classes of subalgebras that can be obtained from it by means of a change of basis and the adjoint actions

$$\mathrm{Ad}(\delta_1 D),\quad \mathrm{Ad}(\delta_2 J_{12}),\quad \mathrm{Ad}(R(\vec n)+Z(\eta)),$$
$$\big(\mathrm{Ad}(\delta D),\quad \mathrm{Ad}(\varepsilon_{ab}J_{ab}),\quad \mathrm{Ad}(R(\vec n)+Z(\eta))\big)$$

if $\varkappa>0$ ($\varkappa=0$) respectively. Here the functions $\vec n$ and η satisfy the following equations:

$$t\vec n_t-\tfrac12\vec n+\varkappa(n^2,-n^1,0)^T=b_i\vec m^i,$$
$$t\eta_t+\eta=b_i\chi_i+\tfrac12 t(\vec n_{ttt}\cdot\vec n-\vec n_{tt}\cdot\vec n_t)+\vec n_{tt}\cdot\vec n+\varkappa(n^1n^2_{tt}-n^1_{tt}n^2).$$

6. $A^3_6(\varkappa,\vec m^1,\vec m^2,\chi^1,\chi^2)=\langle\partial_t+\varkappa J_{12},\ R(\vec m^1)+Z(\chi^1),\ R(\vec m^2)+Z(\chi^2)\rangle$, where $\varkappa\in\{0;1\}$, $\mathrm{rank}(\vec m^1,\vec m^2)=2$,

$$\vec m^i_t-\varkappa(m^{i2},-m^{i1},0)^T=a_{ij}\vec m^j,\quad t\chi^i_t=a_{ij}\chi^j,$$

and a_{ij} are constants satisfying (A.10). This superclass contains eight inequivalent classes of subalgebras that can be obtained from it by means of a change of basis and the adjoint actions

$$\mathrm{Ad}(\delta_1\partial_t),\quad \mathrm{Ad}(\delta_2 J_{12}),\quad \mathrm{Ad}(R(\vec n)+Z(\eta)),$$
$$\big(\mathrm{Ad}(\delta_1\partial_t),\quad \mathrm{Ad}(\delta_2 D),\quad \mathrm{Ad}(\varepsilon_{ab}J_{ab}),\quad \mathrm{Ad}(R(\vec n)+Z(\eta))\big)$$

if $\varkappa=1$ ($\varkappa=0$) respectively. Here the functions $\vec n$ and η satisfy the following equations:

$$\vec n_t+\varkappa(n^2,-n^1,0)^T=b_i\vec m^i,$$
$$\eta_t=b_i\chi_i+\tfrac12(\vec n_{ttt}\cdot\vec n-\vec n_{tt}\cdot\vec n_t)+\varkappa(n^1n^2_{tt}-n^1_{tt}n^2).$$

7. $A^3_7(\eta^1,\eta^2,\eta^3,\chi)=\langle J_{12}+R(0,0,\eta^3),\ R(\eta^1,\eta^2,0),\ R(-\eta^2,\eta^1,0)\rangle$, where

$$\eta^a\in C^\infty((t_0,t_1),\mathbb R),\quad \eta^1_{tt}\eta^2-\eta^1\eta^2_{tt}\equiv0,\quad \eta^i\eta^i\not\equiv0,\quad \eta^3\neq0.$$

Algebras $A^3_7(\eta^1,\eta^2,\eta^3)$ and $A^3_7(\tilde\eta^1,\tilde\eta^2,\tilde\eta^3)$ are equivalent if $\exists\,\delta_a\in\mathbb R$, $\exists\,\delta_4\neq0$:

$$\begin{aligned}&\tilde\eta^1(\tilde t)=\delta_4(\eta^1(t)\cos\delta_3-\eta^2(t)\sin\delta_3),\\&\tilde\eta^2(\tilde t)=\delta_4(\eta^1(t)\sin\delta_3+\eta^2(t)\cos\delta_3),\\&\tilde\eta^3(\tilde t)=e^{-\delta_1}\eta^3(t),\end{aligned}\tag{A.11}$$

where $\tilde t=te^{-2\delta_1}+\delta_2$.

8. $A^3_8(\vec m^1,\vec m^2,\vec m^3)=\langle R(\vec m^1),\ R(\vec m^2),\ R(\vec m^3)\rangle$, where

$$\vec m^a\in C^\infty((t_0,t_1),\mathbb R^3),\quad \mathrm{rank}(\vec m^1,\vec m^2,\vec m^3)=3,\quad \vec m^a_{tt}\cdot\vec m^b-\vec m^a\cdot\vec m^b_{tt}=0.$$

Algebras $A^3_8(\vec m^1,\vec m^2,\vec m^3)$ and $A^3_8(\vec{\tilde m}^1,\vec{\tilde m}^2,\vec{\tilde m}^3)$ are equivalent if $\exists\,\delta_i\in\mathbb R^3$, $\exists\,B\in O(3)$, $\exists\,\{d_{ab}\}:\det\{d_{ab}\}\neq0$ such that

$$\vec{\tilde m}^a(\tilde t)=d_{ab}B\vec m^b(t),\tag{A.12}$$

where $\tilde t=te^{-2\delta_1}+\delta_2$.

B On construction of ansatzes for the Navier–Stokes field by means of the Lie method

The general method for constructing a complete set of inequivalent Lie ansatzes of a system of PDEs are well known and described, for example, in [27, 28]. However, in some cases when the symmetry operators of the system have a special form, this method can be modified [9]. Thus, in the case of the NSEs, coefficients of an arbitrary operator

$$Q = \xi^0\partial_t + \xi^a\partial_a + \eta^a\partial_{u^a} + \eta^0\partial_p$$

from $A(NS)$ satisfy the following conditions:

$$\begin{aligned} &\xi^0 = \xi^0(t,\vec{x}), \quad \xi^a = \xi^a(t,\vec{x}), \quad \eta^a = \eta^{ab}(t,\vec{x})u^b + \eta^{a0}(t,\vec{x}), \\ &\eta^0 = \eta^{01}(t,\vec{x})p + \eta^{00}(t,\vec{x}). \end{aligned} \tag{B.1}$$

(The coefficients ξ^a, ξ^0, η^a, and η^0 also satisfy stronger conditions than (B.1). For example if $Q \in A(NS)$, then $\xi^0 = \xi^0(t)$, $\eta^{ab} = \text{const}$, and so on. But conditions (B.1) are sufficient to simplify the general method.) Therefore, ansatzes for the Navier–Stokes field can be constructing in the following way:

1. We fix a M-dimensional subalgebra of $A(NS)$ with the basis elements

$$Q^m = \xi^{m0}\partial_t + \xi^{ma}\partial_a + (\eta^{mab}u^b + \eta^{ma0})\partial_{u^a} + (\eta^{m01}p + \eta^{m00})\partial_p, \tag{B.2}$$

where $M \in \{1;2;3\}$, $m = \overline{1,M}$, and

$$\text{rank}\{(\xi^{m0},\xi^{m1},\xi^{m2},\xi^{m3}),\ m = \overline{1,M}\} = M. \tag{B.3}$$

To construct a complete set of inequivalent Lie ansatzes of codimension M for the Navier–Stokes field, we have to use the set of M-dimensional subalgebras from Section A. Condition (B.3) is neeeded for the existance of ansatzes connected with this subalgebra.

2. We find the invariant independent variables $\omega_n = \omega_n(t,\vec{x})$, $n = \overline{1,N}$, where $N = 4 - M$, as a set of functionally independent solutions of the following system:

$$L^m\omega = Q^m\omega = \xi^{m0}\partial_t\omega + \xi^{ma}\partial_a\omega = 0, \quad m = \overline{1,M}, \tag{B.4}$$

where $L^m := \xi^{m0}\partial_t + \xi^{ma}\partial_a$.

3. We present the Navier–Stokes field in the form:

$$u^a = f^{ab}(t,\vec{x})v^b(\bar\omega) + g^a(t,\vec{x}), \quad p = f^0(t,\vec{x})q(\bar\omega) + g^0(t,\vec{x}), \tag{B.5}$$

where v^a and q are new unknown functions of $\bar\omega = \{\omega_n,\ n = \overline{1,N}\}$. Acting on representation (B.5) with the operators Q^m, we obtain the following equations on functions f^{ab}, g^a, f^0, and g^0:

$$\begin{aligned} &L^m f^{ab} = \eta^{mac}f^{cb}, \quad L^m g^a = \eta^{mab}g^b + \eta^{ma0}, \quad c = \overline{1,3}, \\ &L^m f^0 = \eta^{m01}f^0, \quad L^m g^0 = \eta^{m01}g^0 + \eta^{m00}. \end{aligned} \tag{B.6}$$

If the set of functions f^{ab}, f^0, g^a, and g^0 is a particular solution of (B.6) and satisfies the conditions $\text{rank}\{(f^{1b},f^{2,b},f^{3b}),\ b = \overline{1,3}\} = 3$ and $f^0 \neq 0$, formulas (B.5) give an ansatz for the Navier–Stokes field.

The ansatz connected with the fixed subalgebra is not determined in an unique manner. Thus, if

$$\begin{gathered}\tilde\omega_l=\tilde\omega_l(\bar\omega),\quad \det\left\{\frac{\partial\tilde\omega_l}{\partial\omega_n}\right\}_{l,n=\overline{1,N}}\neq 0,\\ \tilde f^{ab}(t,\vec x)=f^{ac}(t,\vec x)F^{cb}(\bar\omega),\quad \tilde g^a(t,\vec x)=g^a(t,\vec x)+f^{ac}(t,\vec x)G^c(\bar\omega),\\ \tilde f^0(t,\vec x)=f^0(t,\vec x)F^0(\bar\omega),\quad \tilde g^0(t,\vec x)=g^0(t,\vec x)+f^0(t,\vec x)G^0(\bar\omega),\end{gathered}\tag{B.7}$$

the formulas

$$u^a=\tilde f^{ab}(t,\vec x)\tilde v^b(\bar{\tilde\omega})+\tilde g^a(t,\vec x),\quad p=\tilde f^0(t,\vec x)q(\bar{\tilde\omega})+\tilde g^0(t,\vec x)\tag{B.8}$$

give an ansatz which is equivalent to ansatz (B.5). The reduced system of PDEs on the functions $\tilde v^a$ and $\tilde q$ is obtained from the system on v^a and q by means of a local transformation. Our problem is to find or “to guess”, at once, such an ansatz that the corresponding reduced system has a simple and convenient form for our investigation. Otherwise, we can obtain a very complicated reduced system which will be not convenient for investigation and we can not simplify it.

Consider a simple example.

Let $M=1$ and let us give the algebra $\langle\partial_t+\varkappa J_{12}\rangle$, where $\varkappa\in\{0;1\}$. For this algebra, the invariant independent variables $y_a=y_a(t,\vec x)$ are functionally independent solutions of the equation $Ly=0$ (see (B.4)), where

$$L:=\partial_t+\varkappa(x_1\partial_{x_2}-x_2\partial_{x_1}).\tag{B.9}$$

There exists an infinite set of choices for the variables y_a. For example, we can give the following expressions for y_a:

$$y_1=\arctan\frac{x_1}{x_2}-\varkappa t,\quad y_2=(x_1^2+x_2^2)^{1/2},\quad y_3=x_3.$$

However choosing y_a in such a way, for $\varkappa\neq 0$ we obtain a reduced system which strongly differs from the “natural” reduced system for $\varkappa=0$ (the NSEs for steady flows of a viscous fluid in Cartesian coordinates). It is better to choose the following variables y_a:

$$y_1=x_1\cos\varkappa t+x_2\sin\varkappa t,\quad y_2=-x_1\sin\varkappa t+x_2\cos\varkappa t,\quad y_3=x_3.$$

The vector-functions $\vec f^{\,b}=(f^{1b},f^{2b},f^{3b})$, $b=\overline{1,3}$, should be linearly independent solutions of the system

$$Lf^1=-\varkappa f^2,\quad Lf^2=\varkappa f^1,\quad Lf^3=0$$

and the function f^0 should satisfy the equation $Lf^0=0$ and the condition $f^0\neq 0$. Here the operator L is defined by (B.9). We give the following values of these functions:

$$\vec f^{\,1}=(\cos\varkappa t,\sin\varkappa t,0),\quad \vec f^{\,2}=(-\sin\varkappa t,\cos\varkappa t,0),\quad \vec f^{\,3}=(0,0,1),\quad f^0=1.$$

The functions g^a and g^0 are solutions of the equations

$$Lg^1=-\varkappa g^2,\quad Lg^2=\varkappa g^1,\quad Lg^3=0,\quad Lg^0=0.$$

We can make, for example, g^a and g^0 vanish. Then the corresponding ansatz has the form:

$$u^1 = \tilde{v}^1 \cos \varkappa t - \tilde{v}^2 \sin \varkappa t, \quad u^2 = \tilde{v}^1 \sin \varkappa t + \tilde{v}^2 \cos \varkappa t, \quad u^3 = \tilde{v}^3, \quad p = \tilde{q}, \tag{B.10}$$

where $\tilde{v}^a = \tilde{v}^a(y_1, y_2, y_3)$ and $\tilde{q} = \tilde{q}(y_1, y_2, y_3)$ are the new unknown functions. Substituting ansatz (B.10) into the NSEs, we obtain the following reduced system:

$$\begin{aligned}
&\tilde{v}^a \tilde{v}^1_a - \tilde{v}^1_{aa} + \tilde{q}_1 + \varkappa y_2 \tilde{v}^1_1 - \varkappa y_1 \tilde{v}^1_2 - \varkappa \tilde{v}^2 = 0,\\
&\tilde{v}^a \tilde{v}^2_a - \tilde{v}^2_{aa} + \tilde{q}_2 + \varkappa y_2 \tilde{v}^2_1 - \varkappa y_1 \tilde{v}^2_2 + \varkappa \tilde{v}^1 = 0,\\
&\tilde{v}^a \tilde{v}^3_a - \tilde{v}^3_{aa} + \tilde{q}_3 + \varkappa y_2 \tilde{v}^3_1 - \varkappa y_1 \tilde{v}^3_2 = 0,\\
&\tilde{v}^a_a = 0.
\end{aligned} \tag{B.11}$$

Here subscripts 1,2, and 3 of functions in (B.11) denote differentiation with respect to y_1, y_2, and y_3 accordingly. System (B.11), having variable coefficients, can be simplified by means of the local transformation

$$\tilde{v}^1 = v^1 - \varkappa y_2, \quad \tilde{v}^2 = v^2 + \varkappa y_1, \quad \tilde{v}^3 = v^3, \quad \tilde{q} = q + \tfrac{1}{2}(y_1^2 + y_2^2). \tag{B.12}$$

Ansatz (B.10) and system (B.11) are transformed under (B.12) into ansatz (2.2) and system (2.7), where

$$g^1 = -\varkappa x_2, \quad g^2 = \varkappa x_1, \quad g_3 = 0, \quad g^0 = \tfrac{1}{2}\varkappa^2(x_1^2 + x_2^2), \tag{B.13}$$

$\gamma_1 = -2\varkappa$, and $\gamma_2 = 0$. Therefore, we can give the values of g^a and g^0 from (B.13) and obtain ansatz (2.2) and system (2.7) at once.

The above is a good example how a reduced system can be simplified by means of modifying (complicating) an ansatz corresponding to it. Thus, system (2.7) is simpler than system (B.11) and ansatz (2.2) is more complicated than ansatz (B.10).

Finally, let us make several short notes about constructing other ansatzes for the Navier–Stokes field.

Ansatz corresponding to the algebra $A^1_4(\vec{m}, \chi)$ (see Subsection A.2) can be constructed only for such t that $\vec{m}(t) \neq \vec{0}$. For these values of t, the parameter-function χ can be made to vanish by means of equivalence transformations (A.7).

Ansatz corresponding to the algebra $A^2_8(\lambda, \psi^1, \rho, \psi^2)$ (see Subsection A.3) can be constructed only for such t that $\rho(t) \neq 0$. For these values of t, the parameter-function ψ^2 can be made to vanish by means of equivalence transformations (A.8). Moreover, it can be considered that $\lambda_t \rho - \lambda \rho_t \in \{0; 1\}$. The algebra obtained finally is denoted by $A^2_8(\lambda, \chi, \rho, 0)$.

Ansatz corresponding to the algebra $A^2_9(\vec{m}^1, \chi^1, \vec{m}^2, \chi^2)$ (see Subsection A.3) can be constructed only for such t that $\operatorname{rank}(\vec{m}^1, \vec{m}^2) = 2$. For these values of t, the parameter-functions χ^i can be made to vanish by means of equivalence transformations (A.9).

The algebras $A^2_{10}(\varkappa, \sigma)$, $A^2_{11}(\sigma)$, and $A^2_{12}(\sigma)$ can not be used to construct ansatzes by means of the Lie algorithm.

In view of equivalence transformation (A.11), the functions η^i in the algebra $A^3_7(\eta^1, \eta^2, \eta^3)$ (see Subsection A.4) can be considered to satisfy the following condition:

$$\eta^1_t \eta^2 - \eta^1 \eta^2_t \in \{0; \tfrac{1}{2}\}.$$

1. Berker R., Integration des equations du mouvement d'un fluid visqueux incompressible, in Handbuch der Physik, Vol. 8/2, Berlin, Springer, 1963, 1–384.
2. Birkgoff G., Hydrodynamics, Princeton, Princeton, University Press, Princeton, 1960.
3. Bluman G.W., Cole J.D., The general similarity solutions of the heat equation, *J. Math. Mech.*, 1969, **18**, № 11, 1025.
4. Boisvert R.E., Ames W.F., Srivastava U.N., Group properties and new solutions of Navier–Stokes equations, *J. Eng. Math.*, 1983, **17**, 203–221.
5. Bytev V.O., Group properties of the Navier–Stokes equations, *Chislennye Metody Mehaniki Sploshnoy Sredy*, 1972, V.3, 13–17.
6. Bytev V.O., Invariant solutions of the Navier–Stokes equations, *PTMF J. Appl. Mech. Tech.Phys.*, 1972, № 6, 56–64.
7. Craik A.D.D., Criminale W.O., Evolution of wavelike disturbance in shear flows: a class of exact solutions of the Navier–Stokes equations, *Proc. Roy. Soc. London A*, 1986, **406**, 13–29.
8. Danilov Yu.A., Group properties of the Maxwell and Navier–Stokes equations, Preprint, Moscow, Kurchatov In-t of Atomic Energy, Acad. Sci. USSR, Moscow, 1967.
9. Fushchych W.I., Symmetry in the problems of mathematical physics, in Theoretic-Algebraic Investigations in Mathematical Physics, Kyiv, Institute of Mathematics, 1981, 6–28.
10. Fushchych W.I., Popovych R.O., Symmetry reduction of the Navier–Stokes equations to linear two-dimensional systems of equations, *Dopovidi Acad. Sci. Ukrainy*, 1992, № 8, 29–37.
11. Fushchych W.I., Popovych R.O., Symmetry reduction and exact solution of the Navier–Stokes equations, Preprint 93.7, Kyiv, Institute of Mathematics Acad. Sci. Ukraine, 1993.
12. Fushchych W.I., Shtelen W.M., Popovych R.O., On reduction of the Navier–Stokes equations to linear heat equations, *Dopovidi Acad. Sci. Ukrainy*, 1992, № 2, 23–30.
13. Fushchych W.I., Shtelen W.M., Serov M.I., Symmetry analysis and exact solutions equations of non-linear mathematical physics, Dordrecht, Kluwer Academic Publisher, 1992.
14. Fushchych W.I., Shtelen W.M., Serov M.I., Popovych R.O., Q-conditional symmetry of the linear heat equation, *Dopovidi Acad. Sci. Ukrainy*, 1992, № 12, 29–37.
15. Fushchych W.I., Shtelen W.M., Slavutskiy S.L., Reduction and exact solutions of the Navier–Stokes equations, *J. Phys. A: Math. Gen.*, 1991, **24**, № 4, 971–984.
16. Goldshtik M.A., Shtern V.N., Yavorskiy N.I., Viscous flows with paradoxical properties, Novosibirsk, Nauka, 1989.
17. Grauel A., Steeb W.-H., Similarity of the Euler equation and the Navier–Stokes equation in two space dimensions, *Int. J. Theor. Phys.*, 1985, **24**, № 3, 255–265.
18. Hamel G., Spiralförmige bewegungen zäher flüssigkeiten, *Jahresbericht der deutschen Mathematiker Vereinigung*, 1916, **25**, 34–60.
19. Kamke E., Differentialgleichungen Lösungsmethoden und Lösungen, Chelsea, New York, 1971.
20. Kapitanskiy L.V., Group analysis of the Navier–Stokes and Euler equations in the presence of rotational symmetry and new exact solutions of these equations, *Dokl. Acad. Sci. USSR*, 1978, **243**, № 4, 901–904.
21. Kapitanskiy L.V., Group analysis of the Navier–Stokes equations in the presence of rotational symmetry and somenew exact solutions, *Zapiski Nauchnogo Sem. LOMI*, 1979, **84**, 89–107.
22. Katkov V.L., Exact solutions of some problems of convection, *Prikl. Matem. and Mechan.*, 1968, **32**, № 3, 482–486.
23. Kochin N.E., Kibel I.A., Roze N.B., Theoretical hydrodynamics, V.2, Moscow, Fysmatgis, 1963.
24. Landau L.D., On a new exact solution of the Navier–Stokes equations, *Dokl. Acad. Sci. USSR*, 1944, **28**, № 7, 299–301.
25. Landau L.D., Lifshits E.M., Hydrodynamics, Vol. 6, Moscow, Nauka, 1988.
26. Lloyd S.P., The infinitesimal group of the Navier–Stokes equations, *Acta Mech.*, 1981, **38**, № 1–2, 85–98.
27. Olver P., Applications of Lie groups to differential equations, New-York, Springer-Verlag, 1989.
28. Ovsiannikov L.V., Group analysis of differential equations, New York, Acadamic Press, 1978.

29. Popovych R.O., On the general solution of the Navier–Stokes equations with an additional condition, *Dopovidi Acad. Sci. Ukrainy*, 1992, № 4, 21–24.

30. Popovych R.O., Symmetry reduction and exact solution of the Navier–Stokes equations, PhD thesis, Kyiv, Institute of Mathematics Acad. Sci. Ukraine, 1992.

31. Pukhnachev V.V., Group properties of the equations of Navier–Stokes in the planar case, *PMTF J. Appl. Mech. Tech. Phys.*, 1960, № 1, 83–90.

32. Pukhnachev V.V., Invariant solutions of the Navier–Stokes equations describing motion with a free boundary, *Dokl. Acad. Sci. USSR*, 1972, **202**, № 2, 302–305.

33. Sedov L.I., Methods of similarity and dimensionality in mechanics, Moscow, Nauka, 1987.

34. Slezkin N.A., On a case of integrebility of the complete differential equations of viscous fluid motion, *Uchonye Zapiski MGU*, 1934, **2**, 89–91.

35. Squire H.B., Some viscous fluid flow problem 1. jet emerging from a hole in a plane wall, *Philos. Mag.*, 1952, **43**, № 7, 942–945.

36. Wang C.Y., Exact solutions of the Navier–Stokes equations-the generalized beltrami flows, review and extention, *Acta Mech.*, 1990, **81**, № 1–2, 69–74.

37. Wilzynski E.J., An application of group theory to hydrodynamics, *Trans. Amer. Math. Soc.*, 1900, **1**, 339–352.

38. Yatseev V.I., On a class of exact solutions of equations of viscous fluid motion, *Zurn. Experim. Teor. Phys.*, 1950, **20**, № 11, 1031–1034.

Ansätze of codimension one for the Navier–Stokes field and reduction of the Navier–Stokes equation

W.I. FUSHCHYCH, R.O. POPOVYCH, G.V. POPOVYCH

Використовуючи максимальну в сенсі Лі (нескінченновимірну) алгебру інваріантності рівнянь Нав'є–Стокса, побудований повний набір нееквівалентних лієвських анзаців корозмірності один для поля Нав'є–Стокса. З їх допомогою проведено редукцію рівнянь Нав'є–Стокса до систем ДРЧП з трьома незалежними змінними. Вивчені симетрійні властивості редукованих систем.

Finding exact solutions of the Navier–Stokes equations (NSEs) for an incompressible viscous fluid is an actual problem of mathematical physics and hydrodynamics. There are some ways to solve this problem. One of them is a usage of symmetry analysis [1–8]. In this article we construct a complete set of inequivalent ansatze of codimension one for the Navier–Stokes field. Using them, we reduce the NSEs to systems of partial differential equations in three independent variables and study their symmetry properties.

It is known that the NSEs

$$\vec{u}_t + (\vec{u}\cdot\vec{\nabla})\vec{u} - \Delta\vec{u} + \vec{\nabla}p = \vec{0}, \quad \operatorname{div}\vec{u} = 0 \tag{1}$$

are invariant under the infinite dimensional algebra A(NS) with basic elements

$$\begin{gathered}\partial_t = \partial/\partial t, \quad D = 2t\partial_t + x_a\partial_a - u^a\partial_{u^a} - 2p\partial_p,\\ J_{ab} = x_a\partial_b - x_b\partial_a + u^a\partial_{u^b} - u^b\partial_{u^a}, \quad a\neq b,\\ R(\vec{m}(t)) = m^a(t)\partial_a + m^a_t(t)\partial_{u^a} - m^a_{tt}(t)x_a\partial_p, \quad Z(\chi(t)) = \chi(t)\partial_p.\end{gathered} \tag{2}$$

Here and from now on $\vec{u} = \vec{u}(t,\vec{x}) = \{u^a\}$ is the velocity field of a fluid, $p = p(t,\vec{x})$ is the pressure, $\vec{x} = \{x_a\}$, $\partial_t = \partial/\partial_t$, $\partial_a = \partial/\partial x_a$, $\vec{\nabla} = \{\partial_a\}$, $\Delta = \vec{\nabla}\cdot\vec{\nabla}$, $m^a = m^a(t)$, $\chi = \chi(t)$ are arbitrary smooth functions of t (for example, from $C^\infty((t_0,t_1),\mathbb{R})$), $a,b = \overline{1,3}$, $i,j = 1,2$, repetition of an index signifies a sum.

The set of operators (2) determines the maximal, in the sense of Lie, invariance algebra of the NSEs [9–11].

Theorem 1. *A complete set of A(NS)-inequivalent one-dimensional subalgebras of A(NS) is exhausted by such algebras:*

1) $A^1_1(\varkappa) = \langle D + 2\varkappa J_{12}\rangle, \quad \varkappa \geq 0;$
2) $A^1_2(\varkappa) = \langle \partial_t + \varkappa J_{12}\rangle, \quad \varkappa \in \{0;1\};$
3) $A^1_3(\eta,\chi) = \langle J_{12} + R(0,0,\eta(t)) + Z(\chi(t))\rangle,$

where algebras $A^1_3(\eta,\chi)$ and $A^1_3(\tilde\eta,\tilde\chi)$ are equivalent if $\exists\,\varepsilon,\delta\in\mathbb{R}$, $\exists\,\lambda\in C^\infty((t_0,t_1),\mathbb{R})$:

$$(\tilde\eta,\tilde\chi)(t) = (e^{-\varepsilon}\eta, e^{2\varepsilon}(\chi + \ddot\lambda\eta - \ddot\eta\lambda))(te^{2\varepsilon}+\delta); \tag{3}$$

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$$4)\quad A_4^1(\vec{m},\chi)=\langle R(\vec{m})+Z(\chi)\rangle,\quad (\vec{m},\chi)\neq(\vec{0},0),$$

where algebras $A_4^1(\vec{m},\chi)$ *and* $A_4^1(\tilde{\vec{m}},\tilde{\chi})$ *are equivalent if* $\exists\,\varepsilon,\delta\in\mathbb{R}$, $\exists\,c\neq 0$, $\exists\,B\in O(3)$, $\exists\,\vec{l}\in C^\infty((t_0,t_1),\mathbb{R}^3)$*:*

$$(\tilde{\vec{m}},\chi)(t)=(ce^{-\varepsilon}B\vec{m},\quad ce^{2\varepsilon}(\chi+\dot{\vec{l}}\cdot\vec{m}-\dot{\vec{m}}\cdot\vec{l}))(te^{2\varepsilon}+\delta). \tag{4}$$

Theorem 1 is proved by the method described in [12, 13].

With the algebras A_1^1–A_3^1 from theorem 1 and with the algebra A_4^1 (if some additional demands are satisfied) one can construct such a set of inequivalent ansätze of codimension one for the Navier–Stokes field:

$$\begin{aligned}1.\quad & u^1=|t|^{-1/2}(v^1\cos\tau-v^2\sin\tau)+\tfrac{1}{2}x_1t^{-1}-\varkappa x_2t^{-1},\\ & u^2=|t|^{-1/2}(v^1\sin\tau+v^2\cos\tau)+\tfrac{1}{2}x_2t^{-1}+\varkappa x_1t^{-1},\\ & u^3=|t|^{-1/2}v^3+\tfrac{1}{2}x_3t^{-1},\\ & p=t^{-1}q+\tfrac{1}{8}t^{-2}x_ax_a+\tfrac{1}{2}\varkappa^2t^{-2}r^2,\end{aligned} \tag{5}$$

where

$$\tau=\varkappa\ln|t|,\quad r=(x_1^2+x_2^2)^{1/2},\quad y_1=|t|^{-1/2}(x_1\cos\tau+x_2\sin\tau),$$
$$y_2=|t|^{-1/2}(-x_1\sin\tau+x_2\cos\tau),\quad y_3=|t|^{-1/2}x_3;$$

here and from now on $v^a=v^a(y_1,y_2,y_3)$, $q=q(y_1,y_2,y_3)$, numeration of ansätze corresponds to that of algebras in theorem 1.

$$\begin{aligned}2.\quad & u^1=v^1\cos\varkappa t-v^2\sin\varkappa t-\varkappa x_2,\\ & u^2=v^1\sin\varkappa t+v^2\cos\varkappa t+\varkappa x_1,\\ & u^3=v^3,\quad p=q+\tfrac{1}{2}\varkappa^2r^2,\end{aligned} \tag{6}$$

where $y_1=x_1\cos\varkappa t+x_2\sin\varkappa t$, $y_2=-x_1\sin\varkappa t+x_2\cos\varkappa t$, $y_3=x_3$.

$$\begin{aligned}3.\quad & u^1=x_1r^{-1}v^1-x_2r^{-1}v^2+x_1r^{-2},\\ & u^2=x_2r^{-1}v^1+x_1r^{-1}v^2+x_2r^{-2},\\ & u^3=v^3+\eta(t)r^{-1}v^2+\dot\eta(t)\,\mathrm{arctg}\,x_2/x_1,\\ & p=q-\tfrac{1}{2}\ddot\eta(t)(\eta(t))^{-1}x_3^2-\tfrac{1}{2}r^{-2}+\chi(t)\,\mathrm{arctg}\,x_2/x_1,\end{aligned} \tag{7}$$

where $y_1=t$, $y_2=r$, $y_3=x_3-\eta(t)\,\mathrm{arctg}\,x_2/x_1$.

Remark 1. The expression for the pressure p from the ansatz (7) is indeterminate in points $t\in\{t_0,t_1\}$, where $\eta(t)=0$. If there are such points t, we will consider the ansatz (7) in intervals (t_0^n,t_1^n) that are contained by the interval (t_0,t_1) and for which one from the conditions

$$a)\quad \forall\,t\in(t_0^n,t_1^n):\ \eta(t)\neq 0;$$
$$b)\quad \eta(t)\equiv 0 \text{ in } (t_0^n,t_1^n)$$

is satisfied. In the last case we consider that $\ddot\eta/\eta:=0$.

4. With the algebra $A_4^1(\vec{m},\chi)$, an ansatz can be constructed only for such a t wherefor $\vec{m}(t)\neq\vec{0}$. If this condition is satisfied, it follows from (2) that the algebra

$A_4^1(\vec m,\chi)$ is equivalent to the algebra $A_4^1(\vec m,0)$. An ansatz constructed with the algebra $A_4^1(\vec m,0)$ is

$$\begin{array}{l}\vec u = v^i\vec n^i + (\vec m\cdot\vec m)^{-1}v^3\vec m + (\vec m\cdot\vec x)(\vec m\cdot\vec m)^{-1}\dot{\vec m} - y_i\dot{\vec n}^i,\\ p = q - \frac32(\vec m\cdot\vec m)^{-1}((\dot{\vec m}\cdot\vec n^i)y_i)^2 - (\vec m\cdot\vec m)^{-1}(\ddot{\vec m}\cdot\vec x)(\vec m\cdot\vec x) +\\ \qquad + (\dot{\vec m}\cdot\vec m)(\vec m\cdot\vec m)^{-2}(\vec m\cdot\vec x)^2,\end{array} \tag{8}$$

where $y_i = \vec n^i\cdot\vec x$, $y_3 = t$,

$$\vec n^i = \vec n^i(t),\quad \vec n^i\cdot\vec m = \vec n^1\cdot\vec n^2 = 0,\quad |\vec n^i| = 1,\quad \dot{\vec n}^{\,1}\cdot\vec n^2 = 0. \tag{9}$$

Remark 2. Vector-functions $\vec n^i$ satisfying conditions (9) exist. They can be constructed in such a way: let us fix vector-functions $k^i = k^i(t)$ for which $\vec k^i\cdot\vec m = \vec k^1\cdot\vec k^2 = 0, |\vec k^i| = 1$ and set

$$\vec n^1 = \vec k^1\cos\psi(t) - \vec k^2\sin\psi(t),\quad \vec n^2 = \vec k^1\sin\psi(t) + \vec k^2\cos\psi(t). \tag{10}$$

Then $\dot{\vec n}^{\,1}\cdot\vec n^2 = \dot{\vec k}^{\,1}\cdot\vec k^2 - \dot\psi = 0$ if $\int(\dot{\vec k}^{\,1}\cdot\vec k^2)dt$.

Substituting the ansätze (5), (6) to the NSEs (1), we obtain reduced systems of PDEs that have the same general form

$$\begin{array}{l} v^a v^1_a - v^1_{aa} + q_1 + \gamma_1 v^2 = 0,\\ v^a v^2_a - v^2_{aa} + q_2 - \gamma_1 v^1 = 0,\\ v^a v^3_a - v^3_{aa} + q_3 = 0,\\ v^a_a = \gamma_2,\end{array} \tag{11}$$

where the constant γ_i, takes the values

1. $\gamma_1 = -2\varkappa,\quad \gamma_2 = -\frac32,\quad$ if $\quad t>0,\quad \gamma_1 = 2\varkappa,\quad \gamma_2 = \frac32,\quad$ if $\quad t<0.$
2. $\gamma_1 = -2\varkappa,\quad \gamma_2 = 0.$

For the ansätze (7), (8) reduced equations have the form

$$\begin{array}{ll} 3. & v^1_1 + v^1v^1_2 + v^3v^1_3 - y_2^{-1}v^2v^2 - [v^1_{22} + (1+\eta^2y_2^{-2})v^1_{33} + 2\eta y_2^{-2}v^2_3] + q_2 = 0,\\ & v^2_1 + v^1v^2_2 + v^3v^2_3 + y_2^{-1}v^1v^2 - [v^2_{22} + (1+\eta^2y_2^{-2})v^2_{33} - 2\eta y_2^{-2}v^1_3] +\\ & \qquad + 2y_2^{-2}v^2 - \eta y_2^{-1}q_3 + \chi y_2^{-1} = 0,\\ & v^3_1 + v^1v^3_2 + v^2v^3_3 - [v^3_{22} + (1+\eta^2y_2^{-2})v^3_{33}] - 2\eta^2y_2^{-3}v^1_3 + 2\dot\eta y_2^{-1}v^2 +\\ & \qquad + 2\eta y_2^{-1}(y_2^{-1}v^2)_2 + (1+\eta^2y_2^{-2})q_3 - \ddot\eta\eta^{-1}y_3 - \chi\eta y_2^{-2} = 0,\\ & y_2^{-1}v^1 + v^1_2 + v^3_3 = 0.\end{array} \tag{12}$$

$$\begin{array}{ll} 4. & v^i_3 + v^iv^i_j - v^i_{jj} + q_i + \rho^i(y_3)v^3 = 0,\\ & v^3_3 + v^jv^3_j - v^3_{jj} = 0,\\ & v^i_j + \rho^3(y_3) = 0,\end{array} \tag{13}$$

where

$$\rho^i = \rho^i(y_3) = 2(\vec m\cdot\vec m)^{-1}(\dot{\vec m}\cdot\vec n^i),\quad \rho^3 = \rho^3(\dot y_3) = (\vec m\cdot\vec m)^{-1}(\dot{\vec m}\cdot\vec m). \tag{14}$$

Let us study symmetry properties of the systems (11)–(13). All following results are obtained with the standard Lie algorithm [11, 12]. At first consider the sustem (11).

Theorem 2. *The maximal, in the sense of Lie, invariance algebra of (11) is the algebra*

$$\begin{array}{ll} a) & \langle \partial_a, \partial_q, J^1_{12}\rangle \quad \textit{if} \quad \gamma_1 \neq 0; \\ b) & \langle \partial_a, \partial_q, J^1_{ab}\rangle \quad \textit{if} \quad \gamma_1 = 0, \quad \gamma_2 \neq 0; \\ c) & \langle \partial_a, \partial_q, J^1_{ab}, D^1\rangle \quad \textit{if} \quad \gamma_1 = \gamma_2 = 0. \end{array}$$

Here

$$J^1_{ab} = y_a\partial_b - y_b\partial_a + v^a\partial_{v^b} - v^b\partial_{v^a}, \quad D^1 = y_a\partial_a - v^a\partial_{v^a} - 2q\partial_q.$$

All Lie symmetry operators of (11) are induced by operators from A(NS). Namely, the operators J^1_{ab}, D^1 are induced by J_{ab}, D and the operators $c_a\partial_a$ ($c_a = \text{const}$), ∂_q is done by

$$R(|t|^{1/2}(c_1\cos\tau - c_2\sin\tau, c_1\sin\tau + c_2\cos\tau, c_3)), \quad Z(|t|^{-1})$$

for the anzatz (5) and by

$$R(c_1\cos\varkappa t - c_2\sin\varkappa t, c_1\sin\varkappa t + c_2\cos\varkappa t, c_3), \quad Z(1)$$

for the anzatz (6) respectively. Therefore, Lie reduction of the system (11) gives only solutions that can be obtained by reducing the NSEs with two- and three-dimensional subalgebras of A(NS). Let us proceed to the system (12). Let $A^{\max}$ be the maximal, in the sense of Lie, invariance algebra of (12). Studying symmetry properties of (12), one has to consider the following cases.

A. $\eta, \chi \equiv 0$. Then

$$A^{\max} = \langle \partial_1, D^1_2, R_1(\psi(y_1)), Z^1(\lambda(y_1))\rangle,$$

where $D^1_2 = 2y_1\partial_1 + y_2\partial_2 + y_3\partial_3 - v^a\partial_{v^a} - 2q\partial_q$, $Z^1(\lambda(y_1)) = \lambda(y_1)\partial_q$, $R_1(\psi(y_1)) = \psi\partial_3 + \psi_1\partial_{v^2} - \psi_{11}y_3\partial_q$; here and from now on $\psi = \psi(y_1)$, $\lambda = \lambda(y_1)$ are arbitrary smooth functions of $y_1 = t$.

B. $\eta \equiv 0$, $\chi \neq 0$. In this case expansion of $A^{\max}$ is for $\chi = (C_1y_1 + C_2)^{-1}$, where $C_1, C_2 = \text{const}$. Let $C_1 \neq 0$. It can be done with the equivalence transformation (3) so that the constant C_2 will vanish, i.e. $\chi = Cy^{-1}$ where $C = \text{const}$. Then

$$A^{\max} = \langle D^1_2, R_1(\psi(y_1)), Z^1(\lambda(y_1))\rangle.$$

If $C_1 = 0$, $\chi = C = \text{const}$ and

$$A^{\max} = \langle \partial_1, R_1(\psi(y_1)), Z^1(\lambda(y_1))\rangle.$$

For other values of χ, i.e. when $\chi_{11}\chi \neq \chi_1\chi_1$,

$$A^{\max} = \langle R_1(\psi(y_1)), Z^1(\lambda(y_1))\rangle.$$

C. $\eta \neq 0$. With the equivalence transformation (3), we do $\chi = 0$. In this case expansion of $A^{\max}$ is for $\eta = \pm|C_1y_1 + C_2|^{1/2}$, where $C_1, C_2 = \text{const}$. Let $C_1 \neq 0$.

It can be done with the equivalence transformation (3) so that the constant C_2 will vanish, i.e. $\eta = C|y_1|^{1/2}$, where $C = \text{const}$. Then

$$A^{\max} = \langle D_2^1, Z^1(\lambda(y_1)), R_2(|y_1|^{1/2}), R_2(|y_1|^{1/2}\ln|y_1|)\rangle,$$

where $R_2(\psi(y_1)) = \psi\partial_3 + \psi_1\partial_{v^3}$. If $C_1 = 0$, i.e. $\eta = C = \text{const}$;

$$A^{\max} = \langle \partial_1, Z^1(\lambda(y_1)), \partial_3, y_1\partial_3 + \partial_{v^3}\rangle.$$

For other values of η, i.e. when $(\eta^2)_{11} \neq 0$,

$$A^{\max} = \langle Z^1(\lambda(y_1)), R_2(\eta(y_1)), R_2\left(\eta(y_1)\int(\eta(y_1))^{-2}dy_1\right)\rangle.$$

In all cases considered above, Lie symmetry operators of (12) are induced by operators from A(NS). Namely, the operators ∂_1, D_2^1, $Z^1(\lambda(y_1))$ are induced by ∂_t, D, $Z(\lambda(t))$ respectively. In case $\eta \equiv 0$ the operator $R_1(\psi(y_1))$ and in case $\eta \neq 0$ the operator $R_1(\psi(y_1))$ where $\psi\ddot{\eta} - \psi\ddot{\eta} = 0$ are done by $R(0,0,\psi(t))$. Therefore, Lie reduction of the system (12) gives only solutions that can be obtained by reducing the NSEs with two- and three-dimensional subalgebras of A(NS).

When $\eta = \chi = 0$ the system (12) describes axisymmetric motion of a fluid and can be transformed into a system of two equations for a stream function Ψ^1 and a function Ψ^2 that are determined by

$$\Psi^1_3 = y_2v^1, \quad \Psi^1_2 = -y_2v^3, \quad \Psi^2 = y_2v^2.$$

The transformed system has been studied by L.V. Kapitanskiy [8].

Consider the system (13). Let us introduce the notations

$$\begin{aligned}
&t = y_3, \quad \tilde{\rho} = \int\rho^3(t)dt, \quad R_3(\psi^1(t),\psi^2(t)) = \psi^i\partial_i + \psi^i_t\partial_{v^i} - \psi^i_{tt}y_i\partial_q,\\
&Z^1(\lambda(t)) = \lambda(t)\partial_q, \quad S = \partial_{v^3} - \rho^i(t)y_i\partial_q,\\
&E(\chi(t)) = 2_\chi\partial_t + \chi_t y_i\partial y_i + (\chi_{tt}y_i - \chi_t v^i)\partial_{v^i} - \left(2_{\chi_t q} + \frac{1}{2}\chi_{ttt}y_jy_j\right)\partial_q,\\
&J^1_{12} = y_1\partial_2 - y_2\partial_1 + v^1\partial_{v^2} - v^2\partial_{v^1}.
\end{aligned}$$

Theorem 3. *The maximal, in the sense of Lie, invariance algebra of (13) is the algebra*

$$1)\quad \langle R_3(\psi^1(t),\,\psi^2(t)), Z^1(\lambda(t)), S, E(\chi^1(t)), E(\chi^2(t)), v^3\partial_{v^3}, J^1_{12}\rangle,$$

where $\chi^1 = e^{-\tilde{\rho}(t)}\int e^{\tilde{\rho}(t)}dt$, $\chi^2 = e^{-\tilde{\rho}(t)}$, *if* $\rho^1 = \rho^2 = 0$,

$$2)\quad \langle R_3(\psi^1(t),\,\psi^2(t)), Z^1(\lambda(t)), S, E(\chi(t)) + 2a_1v^3\partial_{v^3} + 2a_2J^1_{12}\rangle,$$

where a_1, a_2, a_3 *are fixed constants,* $\chi = e^{-\tilde{\rho}(t)}\left(\int e^{\tilde{\rho}(t)}dt + a_3\right)$ *if*

$$\begin{aligned}
\rho^1 &= e^{\frac{3}{2}\tilde{\rho}(t)}(\rho(t))^{-\frac{3}{2}-a_1}(C_1\cos(a_2\ln\hat{\rho}(t)) - C_2\sin(a_2\ln\hat{\rho}(t))),\\
\rho^2 &= e^{\frac{3}{2}\tilde{\rho}(t)}(\rho(t))^{-\frac{3}{2}-a_1}(C_1\sin(a_2\ln\hat{\rho}(t)) + C_2\cos(a_2\ln\hat{\rho}(t))),
\end{aligned} \tag{15}$$

where $\hat{\rho}(t) = \left|\int e^{\tilde{\rho}(t)}dt + a_3\right|$, $C_1, C_2 = \text{const}$, $(C_1, C_2) \neq (0,0)$;

$$3)\quad \langle R_3(\psi^1(t),\psi^2(t)), Z^1(\lambda(t)), S, E(\chi(t)) + 2a_1v^3\partial_{v^3} + 2a_2J^1_{12}\rangle,$$

where a_1, a_2 *are fixed constants,* $\chi = e^{-\tilde\rho(t)}$ *if*

$$\rho^1 = e^{\frac{3}{2}\tilde\rho(t)-a_1\hat\rho(t)}(C_1\cos(a_2\hat\rho(t)) - C_2\sin(a_2\hat\rho(t))),$$
$$\rho^2 = e^{\frac{3}{2}\tilde\rho(t)-a_1\hat\rho(t)}(C_1\sin(a_2\hat\rho(t)) + C_2\cos(a_2\hat\rho(t))),$$

where $\hat\rho(t) = \int e^{\tilde\rho(t)}dt$, $C_1, C_2 = \mathrm{const}$, $(C_1, C_2) \neq (0,0)$*;*

4) $\langle R_3(\psi^1(t),\,\psi^2(t)), Z^1(\lambda(t)), S\rangle$ *in all other cases.*

Here $\psi^i = \psi^i(t)$, $\lambda = \lambda(t)$ *are arbitrary smooth functions of* $t = y_3$*.*

Remark 3. If functions $\rho^b = \rho^b(t)$ are determined by (14), $e^{\tilde\rho(t)} = C|\vec m(t)|$, where $C = \mathrm{const}$ and it follows from the condition $\rho^1 = \rho^2 = 0$ that $\vec m = |\vec m(t)|\vec e$, where $|\vec e| = 1$, $\vec e = \mathrm{const}$.

Remark 4. Vector-functions $\vec n^i$ from remark 2 are determined up to the transformation

$$\vec n^1 = \vec n^1\cos\delta - \vec n^2\sin\delta, \quad \vec n^2 = \vec n^1\sin\delta + \vec n^2\cos\delta,$$

where $\delta = \mathrm{const}$. Therefore, choosing δ, we can do so that $C_2 = 0$ (then $C_1 \neq 0$).

The operators $R_3(\psi^1, \psi^2) + \alpha S$, $Z^1(\lambda)$ are induced by $R(\vec l) + Z(\chi)$, $Z(\lambda)$ respectively, where $\vec l = \psi^i\vec n^i + \psi^3\vec m$, $\psi^3_t(\vec m\cdot\vec m) + 2\psi^i(\vec n^i_t\cdot\vec m) = \alpha$, $\chi - \frac{3}{2}(\vec m\cdot\vec m)^{-1}(\psi^i(\vec m_t\cdot\vec n^i))^2 - \frac{1}{2}\psi^3\psi^i(\vec m_{tt}\cdot\vec n^i) + \frac{1}{2}\psi^i(\vec l_{tt}\cdot\vec n^i) = 0$,

If $\vec m = |\vec m(t)|\vec e$, where $\vec e = \mathrm{const}$, $|\vec e| = 1$, the operator J^1_{12} is induced by $e^1J_{23} + eJ_{31} + e^3J_{12}$. For

$$\vec m = \beta_3 e^{\sigma t}(\beta_2\cos\tau, \beta_2\sin\tau, \beta_1)^T, \quad \beta_1^2 + \beta_2^2 = 1, \quad \tau = \varkappa t + \delta,$$

the operator $\partial_t + \varkappa J_{12}$ induces the operator $\partial_{y_3} - \beta_1\varkappa J_{12} + \sigma v^3\partial_{v^2}$ if such vector-functions $\vec n^i$ are chosen:

$$\vec n^1 = \vec k^1\cos\beta_1\tau + \vec k^2\sin\beta_1\tau, \quad \vec n^2 = -\vec k^1\sin\beta_1\tau + \vec k^2\cos\beta_1\tau, \tag{16}$$

where $\vec k^1 = (-\sin\tau, \cos\tau, 0)^T$, $\vec k^2 = (\beta_1\cos\tau, \beta_1\sin\tau, -\beta_2)^T$. For

$$\vec m = \beta_3|t+\beta_4|^{\sigma+1/2}(\beta_2\cos\tau, \beta_2\sin\tau, \beta_1)^T, \quad \beta_1^2 + \beta_2^2 = 1,$$
$$\tau = \varkappa\ln|t+\beta_4| + \delta,$$

the operator $D + 2\beta_4\partial_t + 2\varkappa J_{12}$ induces the operator

$$D^1_3 + 2\beta_4\partial_{y_3} - 2\beta\varkappa J_{12} + 2\sigma v^3\partial_{v^3}$$

if vector-functions $\vec n^i$ are chosen in the form (15). In all other cases the basis elements of the maximal, in the sense of Lie, invariance algebra of (13) are not induced by operators from A(NS).

Remark 5. The invariance algebra of a system of the form (13) with a parameter-function $\rho^3 = \rho^3(t)$ is like one with a different parameter-function $\tilde\rho^3 = \rho^3(t)$. It suggest an idea that there is a local transformation of variables with which one can make ρ^3 to vanish. Indeed, let us transform variables in the way

$$\tilde y_i = y_i e^{\frac{1}{2}\tilde\rho(t)}, \quad \tilde y_3 = \int e^{\tilde\rho(t)}dt, \quad \tilde v^i = \left(v^i + \frac{1}{2}y_i\rho^3(t)\right)e^{-\frac{1}{2}\tilde\rho(t)}, \quad \tilde v^3 = v^3,$$
$$\tilde q = qe^{-\tilde\rho(t)} + \frac{1}{8}y_iy_i[(\rho^3(t))^2 - 2\dot\rho^3(t)]e^{-\tilde\rho(t)}.$$

As a result, we obtain the system

$$\tilde v^i_3 + \tilde v^j \tilde v^i_{jj} + \tilde q_i + \tilde\rho^i(\tilde y_3)\tilde v^3 = 0,$$
$$\tilde v^3_3 + \tilde v^j \tilde v^3_j - \tilde v^3_{jj} = 0,$$
$$\tilde v^i_j = 0,$$

for functions $\tilde v^a = \tilde v^a(\tilde y_1, \tilde y_2, \tilde y_3)$, $\tilde q = \tilde q(\tilde y_1, \tilde y_2, \tilde y_3)$, where $\tilde\rho^i(\tilde y_3) = \rho^i(t)e^{-\frac{3}{2}\tilde\rho(t)}$, subscripts 1, 2, 3 mean differentiation with respect to $\tilde y_1$, $\tilde y_2$, $\tilde y_3$ accordingly.

1. Fushchych W.I., Popovych R.O., Symmetry reduction of the Navier–Stokes equations to linear two-dimensional systems of equations, *Доповіді АН України*, 1992, № 8, 29–37.
2. Fushchych W.I., Shtelen W.M., Slavutsky S.L., Reduction and exact solutions of the Navier–Stokes equations, *J. Phys. A: Math. Gen.*, 1991, **24**, № 4, 971–984.
3. Boisvert R.E., Ames W.F., Srivastava U.N., Group properties and new solutions of Navier–Stokes equations, *J. Engineering Math.*, 1983, **17**, 203–221.
4. Фущич В.І., Попович Р.О., Симетрійна редукція і точні розв'язки рівнянь Нав'є–Стокса, Препринт № 93.7, Київ, Ін-т математики, 1993, 56 с.
5. Пухначев В.В., Групповые свойства уравнений Навье–Стокса в плоском случае, *Журн. прикл. механики и техн. физики*, 1960, № 1, 83—90.
6. Пухначев В.В., Инвариантные решения уравнений Навье–Стокса, описывающие движения со свободной границей, *Докл. АН СССР*, 1972, **202**, № 2, 302—305.
7. Бытев В.О., Инвариантные решения уравнений Навье–Стокса, *Журн. прикл. механики и техн. физики*, 1972, № 6, 56–64.
8. Капитанский Л.В., Групповой анализ уравнений Навье–Стокса при наличии вращательной симметрии и некоторые новые точные решения, *Зап. науч. семинара ЛОМИ*, 1979, **84**, 89—107.
9. Данилов Ю.А., Групповые свойства уравнений Максвелла и Навье–Стокса, Препринт, Москва, Ин-т атомной енергии им. И.В. Курчатова АН СССР, 1967, 15 с.
10. Бытев В.О., Групповые свойства уравнений Навье–Стокса, *Численные методы механики сплошной среды* (Новосибирск, Вычисл. центр СО АН СССР), 1972, **3**, № 4, 13–17.
11. Lloyd S.P., The infinitesimal group of the Navier–Stokes equations, *Acta Mech.*, 1981, **38**, № 1–2, 85–98.
12. Овсянников Л.В., Групповой анализ дифференциальных уравнений, Москва, Наука, 1978, 400 с.
13. Олвер П., Приложения групп Ли к дифференциальным уравнениям, Москва, Мир, 1980, 639 с.

Antireduction and exact solutions of nonlinear heat equations

W.I. FUSHCHYCH, R.Z. ZHDANOV

We construct a number of ansatzes that reduce one-dimensional nonlinear heat equations to systems of ordinary differential equations. Integrating these, we obtain new exact solution of nonlinear heat equations with various nonlinearities.

By the term antireduction for a partial differential equation (PDE) we mean the construction of an ansatz which transforms the PDE to a system of differential equations for several unknown differentiable functions. As a rule, such procedure reduces the PDE under consideration to a system of PDE with fewer numbers of independent variables and greater number of dependent variables [1–4].

Antireduction of the nonlinear acoustics equation

$$u_{x_0x_1} - (u_{x_1}u)_{x_1} - u_{x_2x_2} - u_{x_3x_3} = 0 \tag{1}$$

is carried out in the paper [2] with the use of the ansatz

$$u = \frac{1}{3}x_1\varphi_1(x_0, x_2, x_3) - \frac{1}{6}x_1^2\varphi_2(x_0, x_2, x_3) + \varphi_3(x_0, x_2, x_3). \tag{2}$$

In [3] antireduction of the equation for short waves in gas dynamics

$$2u_{x_0x_1} - 2(2x_1 + u_{x_1})u_{x_1x_1} + u_{x_2x_2} + 2\lambda u_{x_1} = 0 \tag{3}$$

is carried out via the following ansatz:

$$u = x_1\varphi_1 + x_1^2\varphi_2 + x_1^{3/2}\varphi_3 + \varphi_4, \qquad \varphi_i = \varphi_i(x_0, x_2). \tag{4}$$

Ansatzes (2), (4) reduce equations (1), (3) to system of PDE for three and four functions, respectively.

In the present paper we adduce some new results on antireduction for the nonlinear heat equations of the form

$$u_t = \big(a(u)u_x\big)_x + F(u). \tag{5}$$

The antireduction of equation (5) is performed by means of the ansatz

$$h\big(t, x, u, \varphi_1(\omega), \varphi_2(\omega),\ \ldots,\ \varphi_N(\omega)\big) = 0 \tag{6}$$

where $\omega = \omega(t, x, u)$ is a new independent variable. Ansatz (6) reduces equation (5) to a system of ordinary differential equations (ODE) for the unknown functions $\varphi_i(\omega)$, $i = \overline{1, N}$.

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Below we list, without derivation, explicit forms of $a(u)$ and $F(u)$, such that equation (5) admits an antireduction of the form (6). For each case the reduced ODE are given.

1. $a(u) = \ddot\theta(u)\theta(u), \quad F(u) = \big(\lambda_1 + \lambda_2\dot\theta(u)\big)\big(\ddot\theta(u)\big)^{-1},$
$\dot\theta(u) = \varphi_1(t) + \varphi_2(t)x, \quad \dot\varphi_1 = (\lambda_2 + \varphi_2^2)\varphi_1 + \lambda_1, \quad \dot\varphi_2 = (\lambda_2 + \varphi_2^2)\varphi_2;$
2. $a(u) = u\dot\theta(u), \quad F(u) = \big(\lambda_1 + \lambda_2\theta(u)\big)\big(\dot\theta(u)\big)^{-1},$
$\theta(u) = \varphi_1(t) + \varphi_2(t)x, \quad \dot\varphi_1 = \lambda_2\varphi_1 + \varphi_2^2 + \lambda_1, \quad \dot\varphi_2 = \lambda_2\varphi_2;$
3. $a(u) = \dot\theta(u), \quad F(u) = \big(\lambda_1 + \lambda_2\theta(u)\big)\big(\dot\theta(u)\big)^{-1},$
$\theta(u) = \varphi_1(t) + \varphi_2(t)x, \quad \dot\varphi_1 = \lambda_2\varphi_1 + \lambda_1, \quad \dot\varphi_2 = \lambda_2\varphi_2;$
4. $a(u) = \lambda u^k, \quad F(u) = \lambda_1 u + \lambda_2 u^{1-k}, \quad u^k = \varphi_1(t) + \varphi_2(t)x + \varphi_3(t)x^2,$
$\dot\varphi_1 = 2\lambda\varphi_1\varphi_3 + \lambda k^{-1}\varphi_2^2 + k\lambda_2, \quad \dot\varphi_2 = 2\lambda(1 + 2k^{-1})\varphi_2\varphi_3 + k\lambda_1\varphi_2,$
$\dot\varphi_3 = 2\lambda(1 + 2k^{-1})\varphi_3^2 + k\lambda_1\varphi_3;$
5. $a(u) = \lambda e^u, \quad F(u) = \lambda_1 + \lambda_2 e^{-u}, \quad e^u = \varphi_1(t) + \varphi_2(t)x + \varphi_3(t)x^2,$
$\dot\varphi_1 = 2\lambda\varphi_1\varphi_3 + \lambda_1\varphi_1 + \lambda_2, \quad \dot\varphi_2 = 2\lambda\varphi_2\varphi_3 + \lambda_1\varphi_2, \quad \dot\varphi_3 = 2\lambda\varphi_3^2 + \lambda_1\varphi_3;$
6. $a(u) = \lambda u^{-3/2}, \quad F(u) = \lambda_1 u + \lambda_2 u^{5/2},$
$u^{-3/2} = \varphi_1(t) + \varphi_2(t)x + \varphi_3(t)x^2 + \varphi_4(t)x^3,$
$\dot\varphi_1 = 2\lambda\varphi_1\varphi_3 - \frac{2}{3}\lambda\varphi_2^2 - \frac{3}{2}\lambda_1\varphi_1 - \frac{3}{2}\lambda_2,$
$\dot\varphi_2 = -\frac{2}{3}\lambda\varphi_2\varphi_3 + 6\lambda\varphi_1\varphi_4 - \frac{3}{2}\lambda_1\varphi_2,$
$\dot\varphi_3 = -\frac{2}{3}\lambda\varphi_3^2 + 2\lambda\varphi_2\varphi_4 - \frac{3}{2}\lambda_1\varphi_3, \quad \dot\varphi_4 = -\frac{3}{2}\lambda_1\varphi_4;$
7. $a(u) = 1, \quad F(u) = (\alpha + \beta\ln u)u, \quad \ln u = \varphi_1(t) + \varphi_2(t)x,$
$\dot\varphi_1 = \beta\varphi_1 + \varphi_2^2 + \alpha, \quad \dot\varphi_2 = \alpha\varphi_2;$
8. $a(u) = 1, \quad F(u) = \big(\alpha + \beta\ln u - \gamma^2(\ln u)^2\big)u, \quad \ln u = \varphi_1(t) + \varphi_2(t)e^{\gamma x},$
$\dot\varphi_1 = \alpha + \beta\varphi_1 - \gamma^2\varphi_1^2, \quad \dot\varphi_2 = (\beta + \gamma^2 - 2\gamma^2\varphi_1)\varphi_2;$
9. $a(u) = 1, \quad F(u) = -u(1 + \ln u^2)\big(\alpha + \beta(\ln u)^{-1/2}\big),$
$\int^{\ln\, u}\Big(2\alpha\tau + 4\beta\tau^{1/2} + \varphi_2(t)\Big)^{-1/2} d\tau = x + \varphi_1(t),$
$\dot\varphi_1 = 0, \quad \dot\varphi_2 = 4\beta^2 - 2\alpha\varphi_2;$
10. $a(u) = 1, \quad F(u) = -2(u^3 + \alpha u^2 + \beta u),$
(a) $\alpha = \beta = 0$
$u = \big(\varphi_1(t) + 2\varphi_2(t)x\big)\big(1 + \varphi_1(t)x + \varphi_2(t)x^2\big)^{-1},$
$\dot\varphi_1 = -6\varphi_1\varphi_2, \quad \dot\varphi_2 = -6\varphi_2^2;$
(b) $\alpha^2 = 4\beta \neq 0$
$u = \left(-\frac{\alpha}{2}\varphi_1(t) + \left(1 - \frac{\alpha}{2}x\right)\varphi_2(t)\right)\left(e^{\alpha x/2} + \varphi_1(t) + \varphi_2(t)x\right)^{-1},$
$\dot\varphi_1 = -\frac{\alpha^2}{4}\varphi_1 - \alpha\varphi_2, \quad \dot\varphi_2 = -\frac{\alpha^2}{4}\varphi_2;$
(c) $\alpha^2 > 4\beta$
$u = \big((A + B)\varphi_1(t)e^{Bx} + (A - B)\varphi_2(t)e^{-Bx}\big) \times$

$$\times\left(e^{-Ax}+\varphi_1(t)e^{Bx}+\varphi_2(t)e^{-Bx}\right)^{-1},$$
$$A=-\frac{\alpha}{2},\quad B=\frac{1}{2}(\alpha^2-4\beta)^{1/2},$$
$$\dot\varphi_1=\left(\frac{\alpha^2}{2}-3\beta-\frac{\alpha}{2}(\alpha^2-4\beta)^{1/2}\right)\varphi_1,$$
$$\dot\varphi_2=\left(\frac{\alpha^2}{2}-3\beta+\frac{\alpha}{2}(\alpha^2-4\beta)^{1/2}\right)\varphi_2;$$

(d) $\alpha^2<4\beta$

$$u=\big(\varphi_1(t)(A\cos Bx-B\sin Bx)+\varphi_2(t)(A\sin Bx+$$
$$+B\cos Bx)\big)\left(e^{-Ax}+\varphi_1(t)\cos Bx+\varphi_2(t)\sin Bx\right)^{-1},$$
$$\dot\varphi_1=\left(\frac{\alpha^2}{2}-3\beta\right)\varphi_1-\frac{\alpha}{2}(4\beta-\alpha^2)^{1/2}\varphi_2,$$
$$\dot\varphi_2=\left(\frac{\alpha^2}{2}-3\beta\right)\varphi_2+\frac{\alpha}{2}(4\beta-\alpha^2)^{1/2}\varphi_1.$$

In the above formulae $\theta=\theta(u)\in C^2(\mathbb{R}^1,\mathbb{R}^1)$ is an arbitrary function; λ, λ_1, λ_2, α, β, γ are arbitrary real constants; overdot means differentiation with respect to the corresponding argument.

Most of above adduced system of ODE can be integrated. As a result, one obtains a number of new exact solutions of the nonlinear heat equation (5). Detailed study of reduced systems of ODE and construction of exact solutions of equation (5) will be a topic of our future paper. Here we present some exact solutions of the nonlinear heat equation

$$u_t=u_{xx}+F(u)$$

obtained with the help of ansatzes 7–10 which are listed above.

1) $F(u)=\big(\alpha+\beta\ln u-\gamma^2(\ln u)^2\big)u$,

(a) $\Delta=\beta^2+4\alpha\gamma^2>0$

$$u=C\left(\cos\frac{\Delta^{1/2}t}{2}\right)^{-2}e^{\gamma x+\gamma^2t}+\frac{1}{2\gamma^2}\left(\beta-\Delta^{1/2}\mathrm{tg}\,\frac{\Delta^{1/2}t}{2}\right);$$

(b) $\Delta=-\beta^2-4\alpha\gamma^2>0$

$$u=C\left(\mathrm{ch}\,\frac{\Delta^{1/2}t}{2}\right)^{-2}e^{\gamma x+\gamma^2t}+\frac{1}{2\gamma^2}\left(\beta+\Delta^{1/2}\mathrm{th}\,\frac{\Delta^{1/2}t}{2}\right);$$

(c) $\Delta=\beta^2+4\alpha\gamma^2=0$

$$u=Ct^{-2}e^{\gamma x+\gamma^2t}+\frac{1}{2\gamma^2t}(\beta t+2);$$

2) $F(u)=-u(1+\ln u^2)\big(\alpha+\beta(\ln u)^{-1/2}\big)$,

(a) $\alpha\neq0$

$$\int^{\ln u}\left(2\alpha\tau+4\beta\tau^{1/2}+Ce^{-2\alpha t}+2\beta^2\alpha^{-1}\right)^{-1/2}d\tau=x;$$

(b) $\alpha=0$

$$\int^{\ln u}\left(4\beta\tau^{1/2}+4\beta^2t\right)^{-1/2}d\tau=x;$$

3) $F(u) = -2u(u^2 + \alpha u + \beta)$,

(a) $\alpha^2 = 4\beta$

$$u = \left(1 - \frac{\alpha}{2}(x - \alpha t)\right)\left(x - \alpha t + C\exp\left(\frac{\alpha}{2}\left(x + \frac{\alpha t}{2}\right)\right)\right)^{-1};$$

(b) $\alpha^2 > 4\beta$

$$u = \Big((A+B)C_1\exp\big((A+B)(x-\alpha t)\big) + (A-B)C_2 \times$$
$$\times \exp\big((A-B)(x-\alpha t)\big)\Big)\Big(\exp(3\beta t) + C_1\exp\big((A+B)(x-\alpha t)\big) +$$
$$+ C_2\exp\big((A-B)(x-\alpha t)\big)\Big)^{-1},$$
$$A = -\frac{\alpha}{2}, \quad B = \frac{1}{2}(\alpha^2 - 4\beta)^{1/2};$$

(c) $\alpha^2 < 4\beta$

$$u = \big((\alpha AC_1 - BC_2)\cos B(x-\alpha t) + (AC_2 + BC_1)\times$$
$$\times \sin B(x-\alpha t)\big)\Big(\exp\big(3\beta t - A(x-\alpha t)\big) +$$
$$+ C_1\cos B(x-\alpha t) + C_2\sin B(x-\alpha t)\Big)^{-1},$$
$$A = -\frac{\alpha}{2}, \quad B = \frac{1}{2}(4\beta - \alpha^2)^{1/2}.$$

In the above formulae C, C_1, C_2 are arbitrary constants.

It is worth noting that the above solutions can not be obtained with the use of the classical Lie symmetry reduction technique [6]. That is why they are essentially new. Another impotant feature is that solutions 3(a) and 3(c) are soliton-like solutions. Consequently, nonlinear heat equation with cubic nonlinearity admits soliton-like solutions.

1. Fushchych W.I., Conditional symmetry of equations of nonlinear mathematical physics, *Ukr. Math. J.*, 1991, **43**, № 11, 1456–1470.
2. Fushchych W.I., Mironyuk P.I., Conditional symmetry and exact solutions of nonlinear acoustics equation, *Dopovidi of the Ukrainian Academy of Sciences*, 1991, № 6, 23–29.
3. Fushchych W.I., Repeta V., Exact solutions of equations of gas dynamics and nonlinear acoustics, *Dopovidi of the Ukrainian Academy of Sciences*, 1991, № 8, 35–38.
4. Fushchych W.I., Conditional symmetry of equations of mathematical physics, in Proceedings of the International Workshop "Modern Group Analysis", Editors N. Ibragimov, M. Torrisi and A. Valenti, Kluwer Academic Publishers, 1993, 231–240.
5. Fushchych W.I., Zhdanov R.Z., Antireduction of the nonlinear wave equations, *Dopovidi of the Ukrainian Academy of Sciences*, 1993, № 11, 37–41.
6. Fushchych W., Shtelen W., Serov N., Symmetry analysis and exact solutions of equations of nonlinear mathematical physics, Kluwer Academic Publishers, 1993.

Conditional symmetry and anti-reduction of nonlinear heat equation

W.I. FUSHCHYCH, R.Z. ZHDANOV

Введено поняття Q-умовної інваріантності диференціальних рівнянь відносно векторних полів Лі–Беклунда. Це дозволило провести редукцію ряду нелінійних рівнянь теплопровідності до систем звичайних диференціальних рівнянь (нагадаємо, що при редукції з допомогою векторних полів Лі в результаті одержуємо одне редуковане рівняння).

The key idea making it possible to solve a linear heat equation

$$u_t = u_{xx}$$

by the method of separation of variables is reduction of it to two ordinary differential equations (ODE) with the help of the special ansatz (see, e.g. [1, 2])

$$u(t,x) = R(t,x)\varphi_1(\omega_1(t,x)\varphi_2(\omega_2(t,x)).$$

Unfortunately, the method of separation of variables can not be applied to nonlinear second-order partial differential equations. However, some progress is possible if we apply the anti-reduction procedure. The main idea is the same with the one of the method of separation of variables. Namely, we look for a solution of a nonlinear differential equation using the special ansatz reducing it to several equations which have a less number of independent variables [3]. With application to the nonlinear heat equation

$$u_t = [a(u)u_x]_x + F(u) \tag{1}$$

it means that its solution is searched for (1) in the form

$$G(t,x,u,\varphi_0(\omega),\ldots,\varphi_N(\omega)) = 0, \tag{2}$$

where $\omega=\omega(t,x)$ is a new independent variable, $\varphi_0(\omega),\varphi_1(\omega),\ldots,\varphi_N(\omega)$ are smooth functions satisfying some system of ODE.

The principal difficulty of the anti-reduction procedure is the proper choice of the ansatz (2). In the present paper we construct a number of ansatzes reducing nonlinear heat equations of the form (1) to ODE. These ansatzes are obtained by using Q-conditional invariance of the equation under study with respect to some Lie–Bäcklund vector field (the definition of Q-conditional invariance with respect to the Lie vector field was suggested in [4]).

Definition. *We say that Eq. (1) is Q-conditionally invariant under the Lie–Bäcklund vector field*

$$Q = \eta\frac{\partial}{\partial u} + (D_t\eta)\frac{\partial}{\partial u_t} + (D_x\eta)\frac{\partial}{\partial u_x} + \cdots \tag{3}$$

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if there exist such a finite-order differential operator

$$X = R_0 + R_1 D_t + R_2 D_x + R_3 D_t^2 + \cdots$$

and the function R that the equality

$$Q(u_t - au_{xx} - \dot{a}u_x^2 - F) = X(u_t - au_{xx} - \dot{a}u_x^2 - F) + R\eta \tag{4}$$

holds.

In the above formulae (3), (4) D_t, D_x are total differentiation operators and η, R, R_0, R_1, ..., are functions on t, x, u, u_x, u_{xx}, ...

Roughly speaking, Eq. (1) is Q-conditionally invariant with respect to the vector field (3) if the system

$$\begin{cases} Eq.\ (1), \\ \eta(t, x, u, u_x, u_{xx}, \ldots) = 0 \end{cases}$$

is invariant under the vector field (3) in a usual sense. That is why, to study Q-conditional invariance of Eq. (1) one can apply the standard infinitesimal algorithm [5]. But the system of determining equations for η is nonlinear (let us remind that in the classical Lie approach determining equations are always linear).

We look for conditional symmetry operator (3) with

$$\eta = D_{xg}^2(u), \quad g(u) \in C^2(\mathbb{R}^1, \mathbb{R}^1). \tag{5}$$

Lemma. *Eq. (1) is Q-conditionally invariant with respect to the Lie–Bäcklund vector field (3), (5) if the functions $a(u)$, $F(u)$, $g(u)$ are given by one of the following formulae:*

$$1)\ a(u) = \ddot{\theta}(u)\theta(u), \quad F(u) = (\lambda_1 + \lambda_2\dot{\theta}(u))(\ddot{\theta}(u))^{-1}, \quad g(u) = \dot{\theta}(u); \tag{6a}$$

$$2)\ a(u) = u\dot{\theta}(u), \quad F(u) = (\lambda_1 + \lambda_2\theta(u))(\dot{\theta}(u))^{-1}, \quad g(u) = \dot{\theta}(u); \tag{6b}$$

$$3)\ a(u) = \dot{\theta}(u), \quad F(u) = (\lambda_1 + \lambda_2\theta(u))(\dot{\theta}(u))^{-1}, \quad g(u) = \dot{\theta}(u). \tag{6c}$$

In (6) λ_1, λ_2 are arbitrary constants, $\theta \in C^3(\mathbb{R}^1, \mathbb{R}^1)$ is an arbitrary function.

The proof of the lemma is rather tedious, therefore it is omitted. We restrict ourselves by proving that Eq. (1) with $a(u)$, $F(u)$ from (6a)

$$u_t = [\theta(u)\ddot{\theta}(u)u_x]_x + (\lambda_1 + \lambda_2\dot{\theta}(u))(\ddot{\theta}(u))^{-1} \tag{7}$$

is Q-conditionally invariant with respect to the Lie–Bäcklund vector field (3) under $\eta = \ddot{\theta}(u)u_{xx} + \dddot{\theta}(u)u_x^2$.

Consider an over-determined system

$$u_t = (\theta\ddot{\theta}u_x)_x + (\lambda_1 + \lambda_2\dot{\theta})\ddot{\theta}^{-1},$$
$$\eta \equiv \ddot{\theta}u_{xx} + \dddot{\theta}u^2x = 0.$$

Introducing a new independent variable $v = \dot{\theta}(u)$, we get

$$v_t = \theta(u)\ddot{\theta}(u)v_{xx} + vv_x^2 + \lambda_1 + \lambda_2 v,$$
$$v_{xx} = 0$$

or, equivalently,

$$\begin{aligned} &v_t = v v_x^2 + \lambda_1 + \lambda_2 v, \\ &v_{xx} = 0. \end{aligned} \tag{8}$$

The Lie–Bäcklund vector field $Q = (\ddot\theta u_{xx} + \dddot\theta u_x^2)\frac{\partial}{\partial u} + \cdots$ takes the form

$$\tilde Q = v_{xx}\frac{\partial}{\partial v} + v_{txx}\frac{\partial}{\partial v_t} + v_{xxx}\frac{\partial}{\partial v_x} + \cdots \tag{9}$$

Acting by the operator (9) on the first equation from (8) we get

$$\tilde Q(v_t - v v_x^2 - \lambda_1 - \lambda_2 v) = D_x^2(v_t - v v_x^2 - \lambda_1 - \lambda_2 v) + (4v_x^2 + 2v v_{xx})v_{xx}.$$

Hence, it follows that system (9) is Q-conditionally invariant under the Lie–Bäcklund vector field (9).

To construct solution invariant under the Lie–Bäcklund vector field (3), (5) one has to solve an equation $\eta \equiv D_x^2 g(u) = 0$. General solution of the above equation reads

$$g(u) = \varphi_0(t) + x\varphi_1(t), \tag{10}$$

where φ_0, φ_1 are arbitrary smooth functions. Replacing in (10) $g(u)$ by $\theta(u)$ we get ansatz for Eq. (7) invariant with respect to the Lie–Bäcklund vector field (3) with

$$\dot\theta(u) = \varphi_0(t) + x\varphi_1(t). \tag{11}$$

Substitution of (11) into Eq.(7) yields the system of two ODE for $\varphi_0(t)$, $\varphi_1(t)$

$$\dot\varphi_0 = (\lambda_2 + \varphi_1^2)\varphi_0 + \lambda_1, \quad \dot\varphi_1 = (\lambda_2 + \varphi_1^2)\varphi_1,$$

which general solution has the form

$$\begin{aligned} &\varphi_0 = -\frac{\lambda_1}{\lambda_2} + \frac{\lambda_1}{\lambda_2}(e^{2\lambda_2 t} - 1)^{-1/2}\,\mathrm{arctg}(e^{-2\lambda_2 t} - 1)^{1/2}, \\ &\varphi_1 = \lambda_2^{1/2} e^{\lambda_2 t}(1 - e^{2\lambda_2 t})^{-1/2}. \end{aligned} \tag{12}$$

Substituting the obtained formulae into (11) we get the exact solution of the nonlinear heat equation (7). Since the maximal in Lie's sense invariance group of Eq. (7) is the two-parameter group of translations with respect to t, x, solution (11), (12) can not be obtained by the symmetry reduction procedure. Consequently, it is essentially new.

In the same way we construct Q-conditionally invariant ansatzes for Eqs. (5), (6b) and (5), (6c). They are of the form

$$\theta(u) = \varphi_0(t) + \varphi_1(t)x. \tag{13}$$

Substituting (13) into the corresponding nonlinear equations we get the following systems of ODE:

$$\dot\varphi_0 = \lambda_2\varphi_0 + \varphi_1^2 + \lambda_1, \quad \dot\varphi_1 = \lambda_2\varphi_1$$

and

$$\dot\varphi_0 = \lambda_2\varphi_0 + \lambda_1, \quad \dot\varphi_1 = \lambda_2\varphi_1.$$

Provided the functions $a(u)$, $F(u)$ take a more specific form, it is possible to construct ansatzes reducing Eq. (1) to three, four and even five ODE. Corresponding results are listed below

$$\begin{aligned}
&1)\quad a(u)=\lambda u^k,\quad F(u)=\lambda_1 u+\lambda_2 u^{1-k},\quad u^k=\varphi_0(t)+\varphi_1(t)x+\varphi_2(t)x^2;\\
&2)\quad a(u)=\lambda e^u,\quad F(u)=\lambda_1+\lambda_2 e^{-u},\quad e^u=\varphi_0(t)+\varphi_1(t)x+\varphi_2(t)x^2;\\
&3)\quad a(u)=\lambda u^{-3/2},\quad F(u)=\lambda_1 u+\lambda_2 u^{5/2},\\
&\qquad u^{-3/2}=\varphi_0(t)+\varphi_1(t)x+\varphi_2(t)x^2+\varphi_3(t)x^3;\\
&4)\quad a(u)=\lambda u^{-4/3},\quad F(u)=\lambda_1 u+\lambda_2 u^{7/3},\\
&\qquad u^{-4/3}=\varphi_0(t)+\varphi_1(t)x+\varphi_2(t)x^2+\varphi_3(t)x^3+\varphi_4(t)x^4.
\end{aligned}\tag{14}$$

(the formulae 4 from the above list were obtained by Galaktionov [6]).

Here λ, λ_1, λ_2 are arbitrary real constants; $\varphi_0(t)$, $\varphi_1(t)$, $\ldots$, $\varphi_4(t)$ are arbitrary smooth functions.

It is interesting to note that the cases 1, 2, 4 exhaust all possible nonlinearities $a(u)$ such that invariance group of Eq. (1) is wider than two-parameter translation group [7].

Besides the above mentioned cases, we established that Eq. (1) with $a(u)=1$, $F(u)=\frac{2}{3}\left(C_3+C_2u-\frac{1}{3}C_1^2u^3\right)$, $C_i\in\mathbb{R}^1$ is Q-conditionally invariant under the Lie–Bäcklund vector field (3) with $\eta=u_{xx}-C_1uu_x-\frac{1}{3}\left(C_3+C_2u-\frac{1}{3}C_1^2u^3\right)$. This fact can also be used for antyreduction of the corresponding nonlinear heat equation.

In conclusion let us mention another important point. It is well-known that Eq. (1) admits the Lie–Bäcklund vector field only if it is equivalent to the linear heat equation or to the Burgers equation [8]. Consequently, the conception of conditional invariance widens essentially our possibilities to use a non-Lie symmetry to solve nonlinear partial differential equations.

1. Miller W., Symmetry and separation of variables, Massachusetst, Addison–Wesley, 1977, 288 p.
2. Fushchych W.I., Zhdanov R.Z., Revenko I.V., On the new approach to variable separation in the wave equation with potential, *Докл. АН Украины*, 1993, № 1, 27–32.
3. Fushchych W.I., Serov M.I., Repeta V.K., Conditional symmetry, reduction and exact solutions of nonlinear wave equation, *Докл. АН Украины*, 1991, № 5, 29–34.
4. Fushchych W.I., Nikitin A.G., Symmetries of Maxwell's equations, Dordrecht, D. Reidel Publ., 1987, 212 p.
5. Ovsyannikov L.V., Group analysis of differential equations, M., Наука, 1978, 400 p.
6. Galaktionov V.A., Invariant subspaces and new explicit solutions to evolution equations with quadratic nonlinearities, Report № AM–91–11, School of Math. University of Bristol, 1991.
7. Dorodnitsyn V.A., Knyazeua I.V., Svirshchevsky S.R., Group properties of the heat equation with source in the two-and three-dimensional cases, *Дифференц. уравнения*, 1983, **19**, № 9, 1215–1223.
8. Ibragimov N.Kh., Transformation groups in mathematical physics, M., Наука, 1983, 280 p.

On linear and non-linear representations of the generalized Poincaré groups in the class of Lie vector fields

W.I. FUSHCHYCH, R.Z. ZHDANOV, V.I. LAHNO

We study representations of the generalized Poincaré group and its extensions in the class of Lie vector fields acting in a space of $n+m$ independent and one dependent variables. We prove that an arbitrary representation of the group $P(n,m)$ with $\max\{n,m\} \geq 3$ is equivalent to the standard one, while the conformal group $C(n,m)$ has non-trivial nonlinear representations. Besides that, we investigate in detail representations of the Poincaré group $P(2,2)$, extended Poincaré groups $\widetilde{P}(1,2)$, $\widetilde{P}(2,2)$, and conformal groups $C(1,2)$, $C(2,2)$ and obtain their linear and nonlinear representations.

1 Introduction

The central problem to be solved within the framework of the classical Lie approach to investigation of the partial differential equation (PDE)

$$F(x, u, \underset{1}{u}, \underset{2}{u}, \ldots, \underset{r}{u}) = 0, \tag{1}$$

where symbol $\underset{k}{u}$ denotes a set of k-th order derivatives of the function $u = u(x)$, is to compute its maximal symmetry group. Sophus Lie developed the universal infinitesimal algorithm which reduced the above problem to solving some linear over-determined system of PDE (see, e.g. [1–3]). The said method enables us to solve the inverse problem of symmetry analysis of differential equations — description of equations invariant under given transformation group. This problem is of great importance of mathematical and theoretical physics. For example, in relativistic field theory motion equations have to obey the Lorentz–Poincaré–Einstein relativity principle. It means that equations considered should be invariant under the Poincaré group $P(1,3)$. That is why, there exists a deep connection between the theory of relativistically-invariant wave equations and representations of the Poincaré group [4–6].

There exists a vast literature on representations of the generalized Poincaré group $P(n,m)$ [6], $n, m \in \mathbb{N}$ but only a few papers are devoted to a study of nonlinear representations. It should be noted that nonlinear representations of the Poincaré and conformal groups often occur as realizations of symmetry groups of nonlinear PDE such as eikonal, Born–Infeld and Monge–Amperé equations (see [3] and references therein). On sets of solutions of some nonlinear heat equations nonlinear representations of the Galilei group are realized [3]. So, nonlinear representations of the transformations groups are intimately connected with nonlinear PDE, and systematic study of these is of great importance.

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In the present paper we obtain the complete description of the Poincaré group $P(n,m)$ (called for bravity the Poincaré group) and of its extensions — the extended Poincaré group $\widetilde{P}(n,m)$ and conformal group $C(n,m)$ acting as Lie transformation groups in the space $\mathbb{R}(n,m)\times\mathbb{R}^1$, where $\mathbb{R}(n,m)$ is the pseudo-Euclidean space with the metric tensor

$$g_{\alpha\beta}=\begin{cases} 1, & \alpha=\beta=\overline{1,n},\\ -1, & \alpha=\beta=\overline{n+1,n+m},\\ 0, & \alpha\neq\beta.\end{cases}$$

The paper is organized as follows. In Section 2 we give all necessary notations and definitions. In Section 3 we investigate representations of groups $P(n,m)$, $\widetilde{P}(n,m)$, $C(n,m)$ with $\max\{n,m\}\geq 3$ and prove, in particular, that each representation of the Poincaré group $P(n,m)$ with $\max\{n,m\}\geq 3$ is equivalent to the standard linear representation. In Section 3 we study representations of the above groups with $\max\{n,m\}<3$ and show that groups $\widetilde{P}(1,2)$, $C(1,2)$, $P(2,2)$, $\widetilde{P}(2,2)$, $C(2,2)$ have nontrivial nonlinear representations. It should be noted that nonlinear representations of the groups $P(1,1)$, $\widetilde{P}(1,1)$, $C(1,1)$ were constructed in [9] and of the group $P(1,2)$ — in [10].

2 Notations and definitions

Saying about a representation of the Poincaré group $P(n,m)$ in the class of Lie transformation groups we mean the transformation group

$$x'_\mu=f_\mu(x,u,a),\quad \mu=\overline{1,n+m},\quad u'=g(x,u,a), \tag{2}$$

where $a=\{a_N,\ N=1,2,\dots,n+m+C^2_{n+m}\}$ are group parameters preserving the quadratic form $S(x)=g_{\alpha\beta}x_\alpha x_\beta$. Here and below summation over the repeated indices is understood.

It is common knowledge that a problem of description of inequivalent representations of the Lie transformation group (2) can be reduced to a study of inequivalent representations of its Lie algebra [1, 2, 12].

Definition 1. *Set of $n+m+C^2_{n+m}$ differential operators P_μ, $J_{\alpha\beta}=-j_{\beta\alpha}$, $\mu,\alpha,\beta=\overline{1,n+m}$ of the form*

$$Q=\xi_\mu(x,u)\partial_\mu+\eta(x,u)\partial_u \tag{3}$$

satisfying the commutational relations

$$\begin{aligned}&[P_\alpha,P_\beta]=0,\quad [P_\alpha,P_{\beta\gamma}]=g_{\alpha\beta}P_\gamma-g_\alpha\gamma P_\beta,\\ &[J_{\alpha\beta},J_{\mu\nu}]=g_{\alpha\nu}J_{\beta\mu}+g_{\beta\mu}J_{\alpha\nu}-g_{\alpha\mu}J_{\beta\nu}-g_{\beta\nu}J_{\alpha\mu}\end{aligned} \tag{4}$$

is called a representation of the Poincaré algebra $AP(n,m)$ in the class of Lie vector fields.

In the above formulae

$$\partial_\mu=\frac{\partial}{\partial x_\mu},\quad \partial_u=\frac{\partial}{\partial u},\quad [Q_1,Q_2]=Q_1Q_2-Q_2Q_1,\quad \alpha,\beta,\gamma,\mu,\nu=\overline{1,n+m}.$$

Definition 2. *Set of* $1+n+m+C^2_{n+m}$ *differential operators* P_μ, $J_{\alpha\beta}=-J_{\beta\alpha}$, D $(\mu,\alpha,\beta=\overline{1,n+m})$ *of the form (3) satisfying the commutational relations (4) and*

$$[D,J_{\alpha\beta}]=0,\quad [P_\alpha,D]=P_\alpha\quad (\alpha,\beta=\overline{1,n+m}) \tag{5}$$

is called a representation of the extended Poincaré algebra $A\widetilde{P}(n,m)$ *in the class of Lie vector fields.*

Using the Lie theorem [1, 2] one can construct the $(1+n+m+C^2_{n+m})$-parameter Lie transformation group corresponding to the Lie algebra $\{P_\mu,J_{\alpha\beta},D\}$. This transformation group is called a representation of the extended Poincaré group $\widetilde{P}(n,m)$.

Definition 3. *Set of* $1+2(n+m)+C^2_{n+m}$ *differential operators* P_μ, $J_{\alpha\beta}=-J_{\beta\alpha}$, D, K_μ $(\mu,\alpha,\beta=\overline{1,n+m})$ *of the form (3) satisfying the commutational relations (4), (5) and*

$$\begin{gathered}[K_\alpha,K_\beta]=0,\quad [K_\alpha,J_{\beta\gamma}]=g_{\alpha\beta}K_\gamma-g_{\alpha\gamma}K_\beta,\\ [P_\alpha,K_\beta]=2(g_{\alpha\beta}D-J_{\alpha\beta}),\quad [D,K_\alpha]=K_\alpha,\end{gathered} \tag{6}$$

is called a representation of the conformal algebra $AC(n,m)$ *in the class of Lie vector fields.*

$(1+2(n+m)+C^2_{n+m})$*-parameter transformation group corresponding to the Lie algebra* $\{P_\mu,J_{\alpha\beta},D,K_\mu\}$ *is called a representation of the conformal group* $C(n,m)$.

Definition 4. *Representation of the Lie transformation group (2) is called linear if functions* f_μ, g *satisfy conditions* $f_\mu=f_\mu(x,a)$ $(\mu=\overline{1,n+m})$, $g=\widetilde{g}(x,a)u$. *If these conditions are not satisfied, representation is called nonlinear.*

Definition 5. *Representation of the Lie algebra in the class of Lie vector fields (3) is called linear if coefficients of its basis elements satisfy the conditions*

$$\xi_\alpha=\xi_\alpha(x),\quad \alpha=\overline{1,n+m},\quad \eta=\widetilde{\eta}(x)u, \tag{7}$$

otherwise it is called nonlinear.

Using the Lie equations [1, 2] it is easy to establish that if a Lie algebra has a nonlinear representation, its Lie group also has a nonlinear representation and vice versa.

Since commutational relations (4)–(6) are not altered by the change of variables

$$x'_\alpha=F_\alpha(x,u),\quad u'=G(x,u), \tag{8}$$

two representations $\{P_\alpha,J_{\alpha\beta},D,K_\alpha\}$ and $\{P'_\alpha,J'_{\alpha\beta},D',K'_\alpha\}$ are called equivalent provided they are connected by relations (8).

3 Representations of the algebras $AP(n,m)$, $A\widetilde{P}(n,m)$, $AC(n,m)$ with $\max\{n,m\}\geq 3$

Theorem 1. *Arbitrary representation of the Poincaré algebra* $AP(n,m)$ *with* $\max\{n,m\}\geq 3$ *in the class of Lie vector fields is equivalent to the standard representation*

$$P_\alpha=\partial_\alpha,\quad J_{\alpha\beta}=g_{\alpha\gamma}x_\gamma\partial_\beta-g_{\beta\gamma}x_\gamma\partial_\alpha\quad (\alpha,\beta=\overline{1,n+m}). \tag{9}$$

Proof. By force of the fact that operators P_α commute, there exists the change of variables (8) reducing these to the form $P_\alpha = \partial_\alpha$, $\alpha = \overline{1, n+m}$ (a rather simple proof of this assertion can be found in [1, 3]). Substituting operators $P_\alpha = \partial_\alpha$, $J_{\alpha\beta} = \xi_{\alpha\beta\gamma}(x,u)\partial_\gamma + \eta_{\alpha\beta}(x,u)\partial_u$ into relations $[P_\alpha, J_{\beta\gamma}] = g_{\alpha\beta}P_\gamma - g_{\alpha\gamma}P_\beta$ and equating coefficients at the linearly-independent operators ∂_α, ∂_u we get a system of PDE for unknown functions $\xi_{\alpha\beta\gamma}$, $\eta_{\alpha\beta}$

$$\xi_{\alpha\beta\gamma x_\mu} = g_{\mu\alpha}g_{\gamma\beta} - g_{\mu\beta}g_{\gamma\alpha}, \quad \eta_{\alpha\beta x_\mu} = 0, \quad \alpha,\beta,\gamma,\mu = \overline{1, n+m},$$

whence

$$\xi_{\alpha\beta\gamma} = x_\alpha g_{\gamma\beta} - x_\beta g_{\gamma\alpha} + F_{\alpha\beta\gamma}(u), \quad \eta_{\alpha\beta} = G_{\alpha\beta}(u).$$

Here $F_{\alpha\beta\gamma} = -F_{\beta\alpha\gamma}$, $G_{\alpha\beta} = -G_{\beta\alpha}$ are arbitrary smooth functions, $\alpha,\beta,\gamma = \overline{1, n+m}$.

Consider the third commutational relation from (4) under $1 \le \alpha,\beta,\mu,\nu \le n$, $\beta = \mu$. Equating coefficients at the operator ∂_u, we get the system of nonlinear ordinary differential equations for $G_{\mu\nu}(u)$

$$G_{\alpha\nu} = G_{\alpha\beta}\dot G_{\beta\nu} - G_{\beta\nu}\dot G_{\alpha\beta} \tag{11a}$$

(no summation over β), where a dot means differentiation with respect to u.

Since (11a) holds under arbitrary $\alpha,\beta,\nu = \overline{1, n}$, we can redenote subscripts in order to obtain the following equations

$$G_{\beta\nu} = G_{\beta\alpha}\dot G_{\alpha\nu} - G_{\alpha\nu}\dot G_{\beta\alpha}, \tag{11b}$$

$$G_{\alpha\beta} = G_{\alpha\nu}\dot G_{\nu\beta} - G_{\nu\beta}\dot G_{\alpha\nu} \tag{11c}$$

(no summation over α and ν).

Multiplying (11a) by $G_{\alpha\nu}$, (11b) by $G_{\beta\nu}$, (11c) by $G_{\alpha\beta}$ and summing we get

$$G^2_{\alpha\mu} + G^2_{\beta\mu} + G^2_{\alpha\beta} = 0,$$

whence $G_{\alpha\nu} = G_{\beta\gamma} = G_{\alpha\beta} = 0$.

Since α, β, ν are arbitrary indices satisfying the restriction $1 \le \alpha,\beta,\nu \le n$, we conclude that $G_{\alpha\beta} = 0$ for all $\alpha,\beta = 1,2,\ldots,n$.

Furthermore, from commutational relations for operators $J_{\alpha\beta}$, $\alpha,\beta = \overline{1, n}$ we get the homogeneous system of linear algebraic equations for functions $F_{\alpha\beta\gamma}(u)$, which general solution reads

$$F_{\alpha\beta\gamma} = F_\alpha(u)g_{\beta\gamma} - F_\beta(u)g_{\alpha\gamma}, \quad \alpha,\beta,\gamma = \overline{1, n},$$

where $F_\alpha(u)$ are arbitrary smooth functions.

Consequently, the most general form of operators P_μ, $J_{\alpha\beta}$ with $1 \le \alpha,\beta \le n$ satisfying (4) is equivalent to the following:

$$P_\mu = \partial_\mu, \quad J_{\alpha\beta} = \big(x_\alpha + F_\alpha(u)\big)\partial_\beta - \big(x_\beta + F_\beta(u)\big)\partial_\alpha.$$

Making in the above operators the change of variables

$$x'_\mu = x_\mu + F_\mu(u), \quad \mu = \overline{1, n}, \quad x'_A = x_A, \quad A = \overline{n+1, n+m}, \quad u' = 0$$

and omitting primes we arrive at the formulae (9) with $1 \le \alpha,\beta \le n$.

Consider the commutator of operators $J_{\alpha\beta}$, $J_{\alpha A}$ under $1 \le \alpha, \beta \le n$, $n+1 \le A \le n+m$

$$\begin{aligned}\left[J_{\alpha\beta}, J_{\alpha A}\right] = [x_\alpha \partial_\beta - x_\beta \partial_\alpha, g_{\alpha\gamma} x_\gamma \partial_A - g_{A\gamma} x_\gamma \partial_\alpha + \\ + F_{\alpha A\gamma}(u)\partial_\gamma + G_{\alpha A}(u)\partial_u] = x_A \partial_\beta - x_\beta \partial_A.\end{aligned} \tag{12a}$$

On the other hand, by force of commutational relations (4) an equality

$$[J_{\alpha\beta}, J_{\alpha A}] = J_{\beta A} \tag{12b}$$

holds. Comparing right-hand sides of (12a) and (12b) we come to conclusion that $F_{\alpha A\gamma} = 0$, $G_{\alpha A} = 0$. Consequently, operators $J_{\alpha A} = -J_{A\alpha}$ with $\alpha = \overline{1,n}$, $A = \overline{n+1, n+m}$ have the form (9).

Analogously, computing the commutator of operators $J_{\alpha A}$, J_{AB} under $1 \le \alpha \le n$, $n+1 \le A, B \le n+m$ and taking into account commutational relations (4) we get $F_{AB\gamma} = 0$, $A, B = \overline{n+1, n+m}$, $\gamma = \overline{1,n}$. Consequently, operators J_{AB} are of the form

$$J_{AB} = x_B \partial_A - x_A \partial_B + G_{AB}(u)\partial_u, \quad A, B = \overline{n+1, n+m}.$$

At last, substituting the results obtained into commutational relations

$$[J_{\alpha A}, J_{\alpha B}] = -J_{AB}$$

(no summation over α), where $\alpha = \overline{1,n}$, $A, B = \overline{n+1, n+m}$, we get

$$G_{AB} = 0, \quad A, B = \overline{n+1, n+m}.$$

Thus, we have proved that there exists the change of variables (8) reducing an arbitrary representation of the Poincaré algebra $AP(n,m)$ with $\max\{n,m\} \ge 3$ to the standard representation (9). Theorem is proved.

Note 1. Poincaré algebra $AP(n,m)$ contains as a subalgebra the Euclid algebra $AE(n)$ with basis elements P_α, $J_{\alpha\beta}$, $\alpha, \beta = \overline{1,n}$. When proving the above theorem we have established that arbitrary representations of the algebra $AE(n)$ with $n \ge 3$ in the class of Lie vector fields are equivalent to the standard representation

$$P_\mu = \partial_\mu, \quad J_{\alpha\beta} = x_\alpha \partial_\beta - x_\beta \partial_\alpha, \quad \mu, \alpha, \beta = \overline{1,n}.$$

Theorem 2. *Arbitrary representation of the extended Poincaré algebra $A\widetilde{P}(n,m)$ with $\max\{n,m\} \ge 3$ in the class of Lie vector fields is equivalent to the following representation:*

$$P_\alpha = \partial_\alpha, \quad J_{\alpha\beta} = g_{\alpha\gamma} x_\gamma \partial_\beta - g_{\beta\gamma} x_\gamma \partial_\alpha, \quad D = x_\alpha \partial_\alpha + \varepsilon u \partial_u, \tag{13}$$

where $\varepsilon = 0, 1$; $\alpha, \beta, \gamma = \overline{1, n+m}$.

Proof. From theorem 1 it follows that a representation of the Poincaré algebra $AP(n,m) = \langle P_\mu, J_{\alpha\beta}\rangle$ can always be reduced to the form (9). To find the explicit form of the dilatation operator $D = \xi_\mu(x,u)\partial_\mu + \eta(x,u)\partial_u$ we use the commutational relations $[P_\alpha, D] = P_\alpha$. Equating coefficients at linearly-independent operators ∂_μ, ∂_u, we get

$$\xi_{\mu x_\alpha} = \delta_{\mu\alpha}, \quad \eta_{x_\alpha} = 0,$$

where $\delta_{\mu\alpha}$ is a Kronecker symbol; $\mu, \alpha = \overline{1, n+m}$.

Integrating the above equations we have

$$\xi_\mu = x_\mu + F_\mu(u), \quad \eta = G(u),$$

where $P_\mu(u)$, $G(u)$ are arbitrary smooth functions.

Using commutational relations $[J_{\mu\nu}, D] = 0$ we arrive at the following equalities:

$$g_{\mu\gamma} F_\gamma \partial_\nu - g_{\nu\gamma} F_\gamma \partial_\mu = 0; \quad \mu, \nu = \overline{1, n+m},$$

whence $F_\gamma = 0$, $\gamma = \overline{1, n+m}$.

Thus, the most general form of the operator D is the following:

$$D = x_\mu \partial_\mu + G(u) \partial_u.$$

Provided $G(u) = 0$, we get the formulae (13) under $\varepsilon = 0$. If $G(u) = 0$, then after making the change of variables

$$x'_\mu = x_\mu, \quad \mu = \overline{1, n+m}, \quad u' = \int \bigl(G(u)\bigr)^{-1} du$$

we obtain the formulae (8) under $\varepsilon = 1$. Theorem is proved.

Theorem 3. *Arbitrary representation of the conformal algebra $AC(n,m)$ with $\max\{n,m\} \geq 3$ in the class of Lie vector fields is equivalent to one of the following representations:*

1) *operators P_μ, $J_{\alpha\beta}$, D are given by (13), and operators K_α have the form*

$$K_\alpha = 2 g_{\alpha\beta} x_\beta D - (g_{\mu\nu} x_\mu x_\nu) \partial_\alpha; \tag{14}$$

2) *operators P_μ, $J_{\alpha\beta}$, D are given by (13) with $\varepsilon = 1$, and operators K_α have the form*

$$K_\alpha = 2 g_{\alpha\beta} x_\beta D - (g_{\mu\nu} x_\mu x_\nu \pm u^2) \partial_\alpha. \tag{15}$$

Proof. From theorem 2 it follows that the basis of the algebra $A\widetilde{P}(n,m)$ up to the change of variables (8) can be chosen in the form (13).

From the commutational relations for operators $P_\alpha = \partial_\alpha$ and $K_\beta = \xi_{\beta\mu}(x,u)\partial_\mu + \eta_\beta(x,u)\partial_u$ we get the following system of PDE:

$$\xi_{\beta\mu x_\alpha} = 2 g_{\alpha\beta} x_\mu - 2 g_{\alpha\nu} x_\nu \delta_{\beta\mu} + 2 g_{\beta\nu} x_\nu \delta_{\mu\alpha}, \quad \eta_{\beta x_\alpha} = 2\varepsilon g_{\beta\alpha} u.$$

Integrating these we have

$$\xi_{\beta\mu} = 2 g_{\beta\nu} x_\nu x_\mu - g_{\alpha\nu} x_\alpha x_\nu \delta_{\beta\mu} + F_{\beta\mu}(u), \quad \eta_\beta = 2\varepsilon x_\beta u + G_\beta(u),$$

where $F_{\beta\mu}$, g_β are arbitrary smooth functions, $\alpha, \beta, \mu, \nu = \overline{1, n+m}$.

Next, we make use of commutational relations $[D, K_\alpha] = K_\alpha$. Direct computation shows that the following equalities hold

$$\begin{aligned}
[D, K_\alpha] = {} & [x_\mu \partial_\mu + \varepsilon u \partial_u, 2 g_{\alpha\beta} x_\beta (x_\mu \partial_\mu + \varepsilon u \partial_u) - g_{\mu\nu} x_\mu x_\nu \partial_\alpha + \\
& + F_{\alpha\beta}(u) \partial_\beta + G_\alpha(u) \partial_u] = 2 g_{\alpha\beta} x_\beta (x_\mu \partial_\mu + \varepsilon u \partial_u) - \\
& - (g_{\mu\nu} x_\mu x_\nu) \partial_\alpha + (\varepsilon u F_{\alpha\beta u} - F_{\alpha\beta}) \partial_\beta + \varepsilon (u G_{\alpha u} - G_\alpha) \partial_u.
\end{aligned}$$

Comparison of the right-hand sides of the above equalities yields the system of PDE

$$2F_{\alpha\beta} = \varepsilon u F_{\alpha\beta u}, \quad G_\alpha = \varepsilon(uG_{\alpha u} - G_\alpha), \quad \alpha, \beta = \overline{1, n+m}. \tag{16}$$

In the following, we will consider the cases $\varepsilon = 0$ and $\varepsilon = 1$ separately.

Case 1, $\varepsilon = 0$. Then it follows from (16) that $F_{\alpha\beta} = 0$, $G_\alpha = 0$, $\alpha, \beta = \overline{1, n+m}$, i.e. operators K_μ are given by (14) with $\varepsilon = 0$. It is not difficult to verify that the rest of commutational relations (6) also holds.

Case 2, $\varepsilon = 1$. Integrating the equations (16) we get

$$F_{\alpha\beta} = C_{\alpha\beta} u^2, \quad G_\alpha = C_\alpha u^2,$$

where $C_{\alpha\beta}$, C_α are arbitrary real constants.

Next, from the commutational relations for K_α, $J_{\mu\nu}$ it follows that

$$C_{\alpha\beta} = C\delta_{\alpha\beta}, \quad C_\alpha = 0,$$

where C is an arbitrary constant, $\alpha, \beta = \overline{1, n+m}$.

Thus, operators K_μ have the form

$$K_\mu = 2g_{\mu\nu} x_\nu D - (g_{\alpha\beta} x_\alpha x_\beta)\partial_\mu + Cu^2 \partial_\mu. \tag{17}$$

Easy check shows that the operators (17) commute, whence it follows that all commutational relations of the conformal algebra hold.

If in (17) $C = 0$, then we have the case (14) with $\varepsilon = 1$. If $C \neq 0$, then after rescaling the dependent variable $u' = u|c|^{1/2}$ we obtain the operators (15). Theorem is proved.

Note 2. Nonlinear representations of the conformal algebra given by (13) with $\varepsilon = 1$ and (15) are realized on the set of solutions of the eikonal equations [3, 14]

$$g_{\mu\nu} u_{x_\mu} u_{x_\nu} \pm 1 = 0$$

and on the set of solutions of d'Alembert–eikonal system [15]

$$g_{\mu\nu} u_{x_\mu} u_{x_\nu} \pm 1 = 0, \quad g_{\mu\nu} u_{x_\mu x_\nu} \pm (n+m-1)u^{-1} = 0.$$

Thus, the Poincaré group $P(n,m)$ with $\max\{n,m\} \geq 3$ has no truly nonlinear representations. The only hope to obtain nonlinear representations of the Poincaré group is to study the case when $\max\{n,m\} < 3$.

4 Representations of the algebras $AP(n,m)$, $A\widetilde{P}(n,m)$, $AC(n,m)$ with $\max\{n,m\} < 3$

Representations of algebras $AP(1,1)$, $A\widetilde{P}(1,1)$, $AC(1,1)$ in the class of Lie vector fields were completely described by Rideau and Winternitz [9]. They have established, in particular, that the Poincaré algebra $AP(1,1)$ has no nonequivalent representations distinct from the standard one (9), while algebras $A\widetilde{P}(1,1)$, $AC(1,1)$ admit nonlinear representations. In the paper [10] nonlinear representations of the Poincaré algebra $AP(1,2)$

$$P_\mu = \partial_\mu, \quad J_{12} = x_1\partial_2 + x_2\partial_1 + \partial_u,$$
$$J_{13} = x_1\partial_3 + x_3\partial_1 + \cos u\partial_u, \quad J_{23} = x_2\partial_3 - x_3\partial_2 - \sin u\partial_u, \tag{18}$$

were constructed and besides that, it was proved that an arbitrary representation of the algebra $AP(1,2)$ in the class of Lie vector fields is equivalent either to the standard representation or to (18).

In the paper [11] we have constructed nonlinear representations of the algebras $AP(2,2)$ and $AC(2,2)$.

Theorem 4. *Arbitrary representation of the Poincaré algebra $AP(2,2)$ in the class of Lie vector fields is equivalent to the following representation:*

$$P_\mu = \partial_\mu, \quad \mu = \overline{1,4},$$
$$J_{12} = x_1\partial_2 - x_2\partial_1 + \varepsilon\partial_u, \quad J_{13} = x_3\partial_1 + x_1\partial_3 + \varepsilon\cos u\partial_u,$$
$$J_{14} = x_4\partial_1 + x_1\partial_4 \mp \varepsilon\sin u\partial_u, \quad J_{23} = x_3\partial_2 + x_2\partial_3 + \varepsilon\sin u\partial_u, \tag{19}$$
$$J_{24} = x_4\partial_2 + x_2\partial_4 \pm \varepsilon\cos u\partial_u, \quad J_{34} = x_4\partial_3 - x_3\partial_4 \pm \varepsilon\partial_u,$$

where $\varepsilon = 0, 1$.

Proof. When, proving the theorem 1, we have established that the operators P_μ, $J_{\alpha\beta}$ can be reduced to the form

$$P_\mu = \partial_\mu, \quad J_{\mu\nu} = g_{\mu\alpha}x_\alpha\partial_\nu - g_{\nu\alpha}x_\alpha\partial_\mu + F_{\mu\nu\alpha}(u)\partial_\alpha + G_{\mu\nu}(u)\partial_u, \tag{19a}$$

where $F_{\mu\nu\alpha} = -F_{\nu\mu\alpha}$, $G_{\mu\nu} = -G_{\nu\mu}$ are arbitrary smooth functions; $\mu,\nu,\alpha = \overline{1,4}$.

Consider the triplet of operators J_{12}, J_{13}, J_{23}. From commutational relations (4) we obtain the following system of nonlinear ordinary differential equations for functions G_{12}, G_{13}, G_{23}:

$$G_{23} = G_{13}\dot G_{12} - G_{12}\dot G_{13}, \quad G_{13} = G_{12}\dot G_{23} - G_{23}\dot G_{12},$$
$$G_{12} = G_{13}\dot G_{23} - G_{23}\dot G_{13}, \tag{20}$$

(a dot means differentiations with respect to u).

Multiplying the first equation of the system (20) by G_{23}, the second — by G_{13} and the third — by G_{12} and summing we get an equality

$$G_{12}^2 = G_{13}^2 + G_{23}^2. \tag{21}$$

In the following one has to consider cases $G_{12} \neq 0$ and $G_{12} = 0$ separately.

Case 1, $G_{12} \neq 0$. General solution of the algebraic equation (21) reads

$$G_{12} = f(u), \quad G_{13} = f(u)\cos g(u), \quad G_{23} = f(u)\sin g(u), \tag{22}$$

where $f(u)$, $g(u)$ are arbitrary smooth functions.

Substitution of (22) into (20) yields $\dot g f^2 = f$. Since $f(u) = g_{12} \neq 0$, the equality $\dot g = f^{-1}$ holds. Consequently, the general solution of the system (20) is of the form

$$G_{12} = \dot g^{-1}, \quad G_{13} = \dot g^{-1}\cos g, \quad G_{23} = \dot g^{-1}\sin g,$$

where $g = g(u)$ is an arbitrary smooth function.

On making the change of variables

$$x'_\alpha = x_\alpha, \quad \alpha = \overline{1,4}, \quad u' = g(u),$$

which does not alter the structure of operators P_μ, $J_{\mu\nu}$ (19a), we reduce operators J_{12}, J_{23}, J_{13} to the form

$$\begin{aligned} &J_{12} = x_1\partial_2 - x_2\partial_1 + \partial_u + \widetilde{F}_{12\alpha}(u)\partial_\alpha, \\ &J_{23} = x_3\partial_2 + x_2\partial_3 + (\sin u)\partial_u + \widetilde{F}_{23\alpha}(u)\partial_\alpha, \\ &J_{13} = x_3\partial_1 + x_1\partial_3 + (\cos u)\partial_u + \widetilde{F}_{13\alpha}(u)\partial_\alpha, \end{aligned} \tag{23}$$

where $\widetilde{F}_{12\alpha}$, $\widetilde{F}_{23\alpha}$, $\widetilde{F}_{13\alpha}$, $\alpha = \overline{1,4}$ are arbitrary smooth functions.

Substitution of (23) into (4) yields the system of linear ordinary differentional equations, which for general solution reads

$$\begin{aligned} &\widetilde{F}_{121} = \dot{V} + W, \quad \widetilde{F}_{122} = \dot{W} - V, \quad \widetilde{F}_{123} = \dot{Q}, \quad \widetilde{F}_{131} = \dot{V}\cos u - Q, \\ &\widetilde{F}_{132} = \dot{W}\cos u, \quad \widetilde{F}_{133} = \dot{Q}\cos u - V, \quad \widetilde{F}_{231} = \dot{V}\sin u, \quad \widetilde{F}_{232} = \dot{W}\sin u - Q, \\ &\widetilde{F}_{233} = \dot{Q}\sin u - W, \quad \widetilde{F}_{124} = R, \quad \widetilde{F}_{134} = R\cos u - C_1\sin u, \\ &\widetilde{F}_{234} = R\sin u + C_1\cos u. \end{aligned}$$

Here V, W, Q, R are arbitrary smooth functions on u, C_1 is an arbitrary constant.

The change of variables

$$\begin{aligned} &x'_1 = x_1 - V(u), \quad x'_2 = x_2 - W(u), \\ &x'_3 = x_3 - Q(u), \quad x'_4 = x_4 - \int R(u)du, \quad u' = u \end{aligned}$$

reduce operators J_{12}, J_{23}, J_{13} to the form

$$\begin{aligned} &J_{12} = x_1\partial_2 - x_2\partial_1 + \partial_u, \\ &J_{13} = x_3\partial_1 + x_1\partial_3 - C_1\sin u\partial_u + \cos u\partial_u, \\ &J_{23} = x_3\partial_2 + x_2\partial_3 + C_1\sin u\partial_u + \sin u\partial_u, \end{aligned} \tag{24}$$

the rest of basis elements of the algebra $AP(2,2)$ having the form (19a).

Computing commutational relations (4) for operators J_{ab}; $\alpha,\beta = \overline{1,4}$ given by formulae (19a) with $\mu = \overline{1,3}$, $\nu = 4$ and (24) we obtain system of equations for unknown functions $F_{\mu_4\alpha}$, G_{μ_4}; $\alpha = \overline{1,4}$; $\mu = \overline{1,3}$. General solution of the system reads

$$\begin{aligned} &G_{14} = \mp\sin u, \quad G_{24} = \pm\cos u, \quad G_{34} = \pm 1, \quad C_1 = 0, \\ &F_{141} = F_{242} = F_{343} = C_2, \quad F_{\alpha 4\beta} = 0, \quad \alpha = \beta, \end{aligned}$$

where C_2 is an arbitrary constant.

Substituting the result obtain into the formulae (19a) and making the change of variables

$$x'_\alpha = x_\alpha, \quad \alpha = \overline{1,3}; \quad x'_4 = x_4 + C_2; \quad u' = u$$

we conclude that operators $J_{\alpha 4}$, $\alpha = \overline{1,3}$ are given by (19) with $\varepsilon = 1$.

Case 2, $G_{12} = 0$. In this case from (21) it follows that $G_{12} = G_{13} = G_{23} = 0$. Computing commutators of operators J_{12}, J_{14} and J_{12}, J_{24} we get $G_{14} = G_{24}$. Next, computing commutator of operators J_{13}, J_{23} we came to conclusion that $G_{34} = 0$.

Substitution of operators $J_{\mu\nu}$ from (19a) with $G_{\mu\eta} = 0$, $\mu, \nu = \overline{1,4}$ into commutational relations (4) yields a homogenerous system of linear algebraic equations for functions $F_{\mu\nu\alpha}$. Its general solution can be represented in the form

$$F_{\mu\nu\alpha} = F_\mu(u)g_{\nu\alpha} - F_\nu(u)g_{\mu\alpha}, \quad \mu, \nu, \alpha = \overline{1,4},$$

where $F_\mu(u)$ are arbitrary smooth functions.

Consequently, operators (19a) take the form

$$P_\mu = \partial_\mu, \quad J_{\alpha\beta} = g_{\alpha\gamma}(x_\gamma + F_\gamma(u))\partial_\beta - g_{\beta\gamma}(x_\gamma + F_\gamma(u))\partial_\alpha.$$

Making in the above operators the change of variables $x'_\mu = x_\mu + F_\mu(u)$, $\mu = \overline{1,4}$, $u' = u$ we arrive at formulae (19) with $\varepsilon = 0$. Theorem is proved.

Theorem 5. *Arbitrary representations of the extended Poincaré algebra $A\widetilde{P}(2,2)$ in the class of Lie vector fields is equalent to one of the following representations:*

1) *P_μ, $J_{\alpha\beta}$ are of the form (19) with $\varepsilon = 1$, $D = x_\mu\partial_\mu$;*
2) *P_μ, $J_{\alpha\beta}$ are of the form (19) with $\varepsilon = 0$, $D = x_\mu\partial_\mu + \varepsilon_1 u\partial_u$, $\varepsilon_1 = 0, 1$.*

Theorem 6. *Arbitrary representation of the conformal algebra $AC(2,2)$ in thew class of Lie vector field is equivalent to one of the following representations:*

1) *P_μ, $J_{\alpha\beta}$ are of the form (19) with $\varepsilon = 0$,*

$$D = x_\alpha\partial_\alpha + \varepsilon_1 u\partial_u, \quad \varepsilon_1 = 0, 1,$$
$$K_\alpha = 2g_{\alpha\beta}x_\beta D - (g_{\mu\nu}x_\mu x_\nu)\partial_\alpha;$$

2) *P_μ, $J_{\alpha\beta}$ are of the form (19) with $\varepsilon = 0$,*

$$D = x_\alpha\partial_\alpha + u\partial_u,$$
$$K_\alpha = 2g_{\alpha\beta}x_\beta D - (g_{\mu\nu}x_\mu x_\nu \pm u^2)\partial_\alpha;$$

3) *P_α, $J_{\mu\nu}$ are of the form (19) with $\varepsilon = 1$,*

$$D = x_\alpha\partial_\alpha,$$
$$K_1 = 2x_1D - (g_{\mu\nu}x_\mu x_\nu)\partial_1 + 2(x_2 + x_3\cos u \mp x_4\sin u)\partial_u,$$
$$K_2 = 2x_2D - (g_{\mu\nu}x_\mu x_\nu)\partial_2 + 2(-x_1 + x_3\sin u \pm x_4\cos u)\partial_u,$$
$$K_3 = -2x_3D - (g_{\mu\nu}x_\mu x_\nu)\partial_3 + 2(\pm x_4 + x_1\cos u - x_2\sin u)\partial_u,$$
$$K_4 = -2x_4D - (g_{\mu\nu}x_\mu x_\nu)\partial_4 + 2(\mp x_4 \pm x_1\sin u \mp x_2\cos u)\partial_u,$$

where $\mu, \alpha, \beta, \nu = 1, 2, 3, 4$.

Proofs of the theorems 5 and 6 are similar to the proofs of the theorems 2, 3 that is why they are omitted.

In conclusion of the Section we adduce all nonequivalent representations of the extended Poincaré algebra $A\widetilde{P}(1,2)$ [10]

1) P_μ, $J_{\alpha\beta}$ are of the form (9),

$$D = x_\mu\partial_\mu + \varepsilon u\partial_u, \quad \varepsilon = 0, 1;$$

2) P_μ, $J_{\alpha\beta}$ are of the form (18),

$$D = x_\mu\partial_\mu$$

and the conformal algebra $AC(1,2)$ [10]

1) P_μ, $J_{\alpha\beta}$ are of the form (9),

$$D = x_\mu\partial_\mu + \varepsilon u\partial_u, \quad \varepsilon = 0, 1,$$
$$K_\alpha = 2g_{\alpha\beta}x_\beta D - (g_{\mu\nu}x_\mu x_\nu)\partial_\alpha;$$

2) P_μ, $J_{\alpha\beta}$ are of the form (9),

$$D = x_\mu\partial_\mu + u\partial_u,$$
$$K_\alpha = 2g_{\alpha\beta}x_\beta D - (g_{\mu\nu}x_\mu x_\nu \pm u^2)\partial_\alpha;$$

3) P_μ, $J_{\alpha\beta}$ are of the form (18),

$$D = x_\mu\partial_\mu,$$
$$K_1 = 2x_1 D - (g_{\mu\nu}x_\mu x_\nu)\partial_1 + 2(x_2 + x_3\cos u)\partial_u,$$
$$K_2 = -2x_2 D - (g_{\mu\nu}x_\mu x_\nu)\partial_2 + 2(-x_1 + x_3\sin u)\partial_u,$$
$$K_3 = -2x_3 D - (g_{\mu\nu}x_\mu x_\nu)\partial_3 - 2(x_1\cos u + x_2\sin u)\partial_u.$$

Here $\mu, \alpha, \beta, \nu = 1, 2, 3$.

5 Conclusion

Thus, we have obtained the complete description of nonequivalent representations of the generalized Poincaré group $P(n,m)$ by operators of the form (3). This fact makes a problem of constructing Poincaré-invariant equations of the form (1) purely algorithmic. To obtain all nonequivalent Poincaré-invariant equations on the order N, one has to construct complete sets of functionally-independent differential invariants of the order N for each nonequivalent representation [1, 2].

For example, each $P(n,m)$-invariant first-order PDE with $\max\{n,m\} \geq 3$ can be reduced by appropriate change of variables (2) to the eikonal equation

$$g_{\mu\nu}u_{x_\mu}u_{x_\nu} = F(u), \tag{25}$$

where $F(u)$ is an arbitrary smooth function.

Equation (26) with an arbitrary $F(u)$ is invariant under the algebra $AP(n,m)$ having the basis elements (9). Provided $F(u) = 0$, $n = m = 2$, it admits also the Poincaré algebra with the basis elements (19) [11].

Another interesting example is provided by $P(1,n)$-invariant PDE ($n \geq 3$). In [16] a complete basis of functionally-independent differential invariants of the order 2 of the algebra $AP(1,n)$ with the basis elements (9) has been constructed. Since each representation of the algebra $AP(1,n)$ with $n \geq 3$ is equivalent to (9), the above mentioned result gives the exhaustive description of Poincaré-invariant equations (1) in the Minkowski space $\mathbb{R}(1,n)$.

It would be of interest to apply the technique developed in [15] to construct PDE of the order higher than 1 which are invariant under the Poincaré algebra $AP(2,2)$ with the basis elements (19).

In the present papers we have studied representations of the Poincaré algebra in spaces with one dependent variable. But no less important is to investigate nonlinear representations of the Poincaré algebra in spaces with more number of dependent variables [17]. Linear representations of such a kind are realized on sets of solutions of the complex d'Alembert, of Maxwell, and of Dirac equations. If nonlinear representations in question would be obtained, one could construct principially new Poincaré-invariant mathematical models for describing real physical processes.

We intend to study the above mentioned problems in our future publications. Besides that, we will construct nonlinear representations of the Galilei group $G(1,n)$, which plays in Galilean relativistic quantum mechanics the same role as the Poincaré group in relativistic field theory.

1. Ovsiannikov L.V., Group analysis of differential equations, New York, Academic Press, 1982.
2. Olver P., Applications of Lie group to differential equations, Berlin, Springer, 1986.
3. Fushchych W.I., Shtelen W.M., Serov N.I., Symmetry analysis and exact solutions of equations of nonlinear mathematical physics, Dordrecht, Kluwer Academic Publ., 1993.
4. Wigner E., Unitary representations of the inhomogenerous Lorentz group, *Ann. Math.*, 1939, **40**, 149–204.
5. Gelfand I.M., Minlos A., Shapiro Z.Ya., Representations of the rotation and Lorentz groups, Moscow, Fizmatgiz, 1958.
6. Fushchych W.I., Nikitin A.G., Symmetry of equations of quantum mechanics, Moscow, Nauka, 1990; New York, Allerton Press, 1994.
7. Rideau G., An analytic nonlinear representation of the Poincaré group, *Lett. Math. Phys.*, 1985, **9**, 337–352.
8. Bertrand J., Rideau G., An analytic nonlinear representation of the Poincaré group: II. The case of helicities $\pm 1/2$, *Lett. Math. Phys.*, 1985, **10**, 325–331.
9. Rideau G., Winternitz P., Nonlinear equations invariant under the Poincaré, similitude and conformal groups in two-dimensional space-time, *J. Math. Phys.*, 1990, **31**, 1095–1105.
10. Yehorchenko I.A., Nonlinear representation of the Poincaré algebra and invariant equations, in Symmetry Analysis of Equations of Mathematical Physics, Kyiv, Institute of Mathematics, 1992, 62–66.
11. Fushchych W.I., Zhdanov R.Z., Lahno V.I., On nonlinear representation of the conformal algebra $AC(2,2)$, *Dokl. AN Ukrainy*, 1993, № 9, 44–47.
12. Fushchych W.I., Barannik L.F., Barannik A.F., Subgroup analysis of the Galilei and Poincaré groups and reduction of nonlinear equations, Kyiv, Naukova Dumka, 1991.
13. Fushchych W.I., Zhdanov R.Z., Conditional symmetry and reduction of partial differential equations, *Ukr. Math. J.*, 1992, **44**, 970–982.
14. Fushchych W.I., Shtelen W.M., The symmetry and some exact solutions of the relativistic eikonal equation, *Lett. Nuovo Cim.*, 1982, **34**, 498–502.
15. Fushchych W.I., Zhdanov R.Z., Yehorchenko I.A., On the reduction of the nonlinear multi-dimensional wave equations and compatibility of the d'Alembert–Hamilton system, *J. Math. Anal. Appl.*, 1991, **161**, 352–360.
16. Fushchych W.I., Yehorchenko I.A., Second-order differential invariants of the rotation group $O(n)$ and of its extensions: $E(n)$, $P(1,n)$, *Acta Appl. Math.*, 1992, **28**, 69–92.
17. Fushchych W.I., Tsyfra I.M., Boyko V.M., Nonlinear representations for Poincaré and Galilei algebras and nonlinear equations for electromagnetic fields, *J. Nonlinear Math. Phys.*, 1994, **1**, № 2, 210–221.

Nonlocal ansatzes for nonlinear wave equation

W.I. FUSHCHYCH, I.M. TSYFRA

Запропоновано нелокальні анзаци, що редукують нелінійні хвильові рівняння до системи хвильових рівнянь з меншим числом незалежних змінних. Показано, що ці анзаци можна одержати, використовуючи оператори нелокальної симетрії рівняння.

1. In the present paper we suggest a nonlocal ansatz

$$\frac{\partial u}{\partial x_\mu} = a_{\mu\nu}(x,u)\varphi_\nu(\omega) + h_\mu(x,u), \quad \mu,\nu = 0,1,2,3, \tag{1}$$

for reduction of the second order nonlinear differential equation

$$g_{\mu\nu}(x,u)\frac{\partial^2 u}{\partial x^\mu \partial x^\nu} + F\left(x,u,\frac{\partial u}{\partial x^\mu}\right) = 0 \tag{2}$$

to the system of equations for some functions $\varphi_\nu(\omega)$, $\omega = (\omega_1,\omega_2,\omega_3)$. The functions $a_{\mu\nu}(x,u)$, $h_\mu(x,u)$ are determined from the condition that the equation (2) is reduced to the system of equations for $\varphi_\nu(\omega)$ (for more detail about the reduction method see [1, 2]).

To illustrate the efficiency of the ansatz (1) we consider two nonlinear two-dimensional equations of type (2)

$$u_{12} = u_1 F_1(u_1 - u), \tag{3}$$

$$u_{00} = F_2(u_{11}), \tag{4}$$

where $u_{\mu\nu} \equiv \frac{\partial^2 u}{\partial x_\mu \partial x_\nu}$, $u_\mu = \frac{\partial u}{\partial x_\mu}$, F_1, F_2 are smooth functions.

2. For equation (3) we shall search for ansatz (1) in the form

$$\frac{\partial u}{\partial x_1} = \varphi_1(\omega) + h_1(x,u), \quad \frac{\partial u}{\partial x_2} = \varphi_2(\omega) + h_2(x,u), \tag{5}$$

h_1, h_2, ω has to be determined in the way that functions φ_1, φ_2 satisfy the system of the ordinary differential equations with a new independent variable ω [1, 2]. Substituting (5) into (3) and using the compatibility condition $u_{12} \equiv u_{21}$, we obtain

$$\begin{aligned}
&\frac{\partial h_1}{\partial x_2} - \frac{\partial h_1}{\partial u}(\varphi_2 + h_2) + \frac{\partial \varphi_1}{\partial \omega}\frac{\partial \omega}{\partial x_2} = (\varphi_1 + h_1)[\varphi_1 + h_1 - u],\\
&\frac{\partial h_2}{\partial x_1} + \frac{\partial h_2}{\partial u}(\varphi_1 + h_1) + \frac{\partial \varphi_2}{\partial \omega}\frac{\partial \omega}{\partial x_1} = (\varphi_1 + h_1)[\varphi_1 + h_1 - u],\\
&\frac{\partial h_1}{\partial x_2} + \frac{\partial h_1}{\partial u}h_2 = h_1 F_1[h_1 + \varphi_1 - u],
\end{aligned} \tag{6}$$

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$$\varphi_2\frac{\partial h_1}{\partial u}+\frac{\partial \varphi_1}{\partial \omega}\frac{\partial \omega}{\partial x_2}=\varphi_1 F_1[h_1+\varphi_1-u]=R_1(\omega),$$
$$\frac{\partial h_2}{\partial x_1}+\frac{\partial h_2}{\partial u}h_1=h_1F_1[h_1+\varphi_1-u], \tag{7}$$
$$\varphi_1\frac{\partial h_2}{\partial u}+\frac{\partial \varphi_2}{\partial \omega}\frac{\partial \omega}{\partial x_1}=\varphi_1 F_1[h_1+\varphi_1-u]=R_2(\omega),$$

where $R_1(\omega)$, $R_2(\omega)$ are unknown functions. System (7) is a condition on functions h_1, h_2, $\omega(x_1,x_2)$ guaranteeing that system (6) depends on the ω only, i.e., ansatz (5) reduces partial differential equation to a system of ordinary differential equations for functions φ_1 and φ_2. Hence, in order to describe ansatzes of type (5) it is necessary to solve nonlinear system (7). Here, we get a particular solution of system (7) only, namely

$$h_1=u,\quad h_2=F_1[\varphi_1(x_2)]u,\quad \omega=x_2. \tag{8}$$

It is easy to verify that solution (8) satisfies system (7) and in this case reduced system (6) takes the form

$$\varphi_2(x_2)+\frac{\partial\varphi_1}{\partial x_2}=\varphi_1(x_2)F_1[\varphi_1(x_2)]. \tag{9}$$

Having integrated the system

$$\frac{\partial u}{\partial x_1}=u+\varphi_1(x_2),\quad \frac{\partial u}{\partial x_2}=F_1[\varphi_1(x_2)]u+\varphi_1(x_2)F_1[\varphi_1(x_2)]-\frac{\partial\varphi_1}{\partial x_2} \tag{10}$$

one can obtain particular solutions of equation (3). The solution of equation (10) is given by the formula

$$u=-\varphi_1(x_2)+ce^{x_1+\int F_1(\varphi_1(x_2))dx_2}, \tag{11}$$

where $\varphi_1(x_2)$ is an arbitrary smooth function and C is an arbitrary constant.

3. Now we suggest the method of construction of ansatzes (1), based on a nonlocal symmetry of the equation

$$\frac{\partial^2 u}{\partial x_1\partial x_2}=F\left(u,\frac{\partial u}{\partial x_1},\frac{\partial u}{\partial x_2}\right). \tag{12}$$

We consider the first order system

$$V^1_2+V^1_3V^2=F(x_3,V^1,V^2), \tag{13}$$

$$V^1_2+V^2_3V^1=F(x_3,V^1,V^2), \tag{14}$$

corresponding to the equation (12), where $V^k_i\equiv\frac{\partial V^k}{\partial x_i}$, $x_3\equiv u$, $\frac{\partial u}{\partial x_i}\equiv V^i$.

The problem of construction of all ansatzes from the class (1) for equation (12) is equivalent to the problem of finding all operators of the Q-conditional symmetry [1, 2, 5].

Theorem 1. *The system (13), (14) is Q-conditionally invariant under the operators* $Q_1 = \partial_{x_1}$, $Q_2 = \partial_{x_3} + \eta^1\partial_{V_1} + \eta^2\partial_{V_2}$ *if and only if the functions* η^1, η^2 *satisfy the following equation*

$$\begin{aligned} &\eta^1_{V_2} = 0, \quad \eta^1_{x_2} = \eta^1_{x_1} = \eta^2_{x_1} = 0, \quad \eta^2_{V^1} = \frac{F}{V^1}, \\ &\eta^1_{x_2} - \eta^1_{V^1}F = F'_{x_3} + \eta^1 F'_{V_1} + \frac{F}{V^1}F'_{V_2} - \eta^1\frac{F}{V^1} - \eta^1_{x_3}V^2. \end{aligned} \tag{15}$$

The correctness of Theorem 1 is easily verified with the help of the infinitesimal criterion of the Q-conditional invariance [1, 5]. Thus, arbitrary setting $\eta^1(x_3, V^1)$ as a function on x_3, V^1 we get classes of nonlinearities $F(x^3, V^1, V^2)$ with which equation (12) admits operators $\{Q_1, Q_2\}$. In the case of equation (3) η^1, η^2 are as follows: $\eta^1 = 1$, $\eta^2 = F_1(V^1 - x_3)$, $F = V^1F_1(V^1 - x_3)$. It should be noted that Q_2 is not a prolongation of Lie operator, but it is the nonlocal symmetry operator of the equation (12). Operators $\{Q_1, Q_2\}$ lead to the ansatz (10).

Then we consider the equation

$$u_{00} = F_2(u_{11}), \tag{16}$$

where F_2 is an arbitrary smooth function. Using the invariance of equation (16) under the operators ∂x_0, ∂x_1, ∂_u, $x_1\partial_u$, $x_2\partial_u$ we write it in the form of the following system

$$V^0_1 = V^1_0, \quad V^2_0 = V^1_1, \quad V^0 = F_2(V^2), \tag{17}$$

where $u_{00} \equiv V^0$, $u_{01} \equiv V^1$, $u_{11} \equiv V^2$.

Theorem 2. *The system (17) is invariant with respect to the continuous group of transformations with the infinitesimal operator*

$$Q = \xi^0\partial_{x_0} + \xi^1\partial_{x_1} \tag{18}$$

if ξ^0, ξ^1 *are a solution of the system of equations*

$$\begin{aligned} &\frac{\partial\xi^0}{\partial x_0} = \frac{\partial\xi^0}{\partial x_1} = \frac{\partial\xi^1}{\partial x_0} = \frac{\partial\xi^1}{\partial x_1} = 0, \\ &\frac{\partial\xi^0}{\partial V^1} = \frac{\partial\xi^1}{\partial V^0}F'_2(V^2) + \frac{\partial\xi^1}{\partial V^2}, \\ &\frac{\partial\xi^1}{\partial V^1} = \frac{\partial\xi^0}{\partial V^0}F'_2(V^2) + \frac{\partial\xi^0}{\partial V^2}. \end{aligned} \tag{19}$$

The finite transformations

$$x'_0 = x_0 + a\xi^0, \quad x'_1 = x_1 + a\xi^1 \tag{20}$$

correspond to the operator (18). Formulae (18), (19) give the operator of the nonlocal symmetry of equation (16). With the help of this operator, one can construct nonlocal ansatzes reducing the equation (16) to the system of three ordinary differential equations for three unknown functions. The analogous procedure has been called an ati-reduction in [6].

Furthermore, the finite transformations (20) can be used for generating new solutions. The transformations (20) are more general than contact ones since ξ^0, ξ^1 are the functions on u_{00}, u_{01}, u_{11}.

For example, we shall take $F_2(u_{11}) = \sin u_{11}$. In this case one of the solutions of system (19) is

$$\xi^0 = \frac{c}{2}(V^1)^2 - C\cos V^2 + D, \quad \xi^1 = cV^1\sin V^2 + DV^1, \tag{21}$$

where $C, D = \mathrm{const}$. We start from a solution of the equation $u = \frac{x_0x_1}{2} - \sin x_0$. Then

$$V^0 = \sin x_0, \quad V^1 = x_1, \quad V^2 = x_0. \tag{22}$$

Using the transformations (20) we obtain the system

$$\begin{aligned} &V^0 = \sin\left[x_0 + a\left(\frac{c}{2}(V^1)^2 - C\cos V^2 + D\right)\right], \\ &V^1 = x_1 + a[CV^1\sin V^2 + DV^1], \\ &V^2 = x_0 + a\left[\frac{c}{2}(V^1)^2 - C\cos V^2 + D\right]. \end{aligned} \tag{23}$$

Thus, in order to find new solutions of equations (16) it is necessary to solve the overdetermined but compatible system

$$\begin{aligned} &V_{00} = \sin\left[x_0 + a\left(\frac{c}{2}(u_{01})^2 - C\cos u_{11} + D\right)\right], \\ &u_{01} = x_1 + a[cu_{01}\sin u_{11} + Du_{01}], \\ &u_{11} = x_0 + a\left[\frac{c}{2}(u_{01})^2 - C\cos u_{11} + D\right]. \end{aligned} \tag{24}$$

The maximal local invariance group of equation (16) is the 7-parameter group. The basic elements of the corresponding algebra are

$$\begin{aligned} &P_0 = \partial_{x_0}, \quad P_1 = \partial_{x_1}, \quad P_2 = \partial_u, \quad D = x_0\partial_{x_0} + x_1\partial_{x_1} + 2u\partial_u, \\ &Q_1 = x_1\partial_u, \quad Q_2 = x_2\partial_u, \quad Q_3 = x_1x_2\partial_u. \end{aligned} \tag{25}$$

It can be shown, that the system (24) has no solutions invariant under the operator $\alpha_0P_0 + \alpha_1P_1 + \alpha_2P_2 + dD + \beta_1Q_1 + \beta_2Q_2 + \beta_3Q_3$, where α_0, α_1, α_2, d, β_1, β_2, β_3 are arbitrary constants. Therefore, no solution of system (24) is invariant one for equation (16).

Further, we consider the equation

$$[F(u)]_1 = u_{22}. \tag{26}$$

We write the equation (26) in the form of the system

$$F(u) = \theta_{x_2}, \quad u_{x_2} = \theta_{x_1}. \tag{27}$$

Theorem 3. *The system (27) is invariant with respect to the one-parameter Lie group generated by an operator of the form*

$$Q = -x_1\partial_{x_1} + \theta\partial_{x_2} + u\partial_u \tag{28}$$

if $F = \frac{1}{\ln u}$.

Operator (28) is a nonlocal symmetry operator of equation (26). We use It to construct the nonlocal ansatz and exact solutions of the equation

$$\left(\frac{1}{\ln u}\right)_1 = u_{22}. \tag{29}$$

The ansatz

$$u = \frac{f(\theta)}{x_1}, \quad \theta = \frac{x_2}{\varphi(\theta) - \ln x_1} \tag{30}$$

corresponds to operator (28), where $f(\theta)$, $\varphi(\theta)$ are unknown functions.

Substituting (30) into (27) we obtain the reduced system of ordinary differential equations

$$\theta\varphi' + \varphi = \ln f, \quad f' = \theta. \tag{31}$$

The solution of system (31) has the form

$$\varphi = \frac{\theta^2}{2} \ln \frac{\theta^2 + 2c}{2} - \frac{\theta^2}{2} + C\ln(\theta^2 + 2c) + c_1, \quad f = \frac{\theta^2}{2} + c. \tag{32}$$

Using the formula (30) and the substitution $\frac{1}{\ln u} = z$ we obtain the solution of the equation

$$z_1 + \left(\frac{1}{z^2} e^{\frac{1}{z}} z_2\right)_2 = 0 \tag{33}$$

namely

$$e^{\frac{1}{z}} = \frac{\theta^2}{2} + c,$$
$$\theta = \frac{x^2}{\frac{\theta^2}{2} \ln \frac{\theta^2+2c}{2} - \frac{\theta^2}{2} + C\ln(\theta^2 + 2c) - \ln x_1 + c_1}. \tag{34}$$

Formulae (34) give the solution of the nonlinear diffusion equation (33). In conclusion, we emphasize that the finite transformations

$$x_1' = e^{-a}x_1, \quad x_2' = x_2 + a\theta, \quad z' = \frac{z}{1 + az}, \quad \theta' = 0 \tag{35}$$

can be used for the nonlocal generating of solutions of equation (33), since the system (27) admits the operator (28) in Lie sense. In this case the formula of generating solutions takes the form

$$z'' = \frac{z'(e^{-a}x_1, x_2 + a\theta)}{1 - az'(e^{-a}x_1, x_2 + a\theta)}, \tag{36}$$

where θ is the solution of the system

$$\theta_{x_1} = -z'_{x_2}\left(\frac{1}{(z')^2} e^{\frac{1}{z'}}\right), \quad \theta_{x_2} = z', \tag{37}$$

z' is the initial solution and z'' is the new solution of the equation (33), a is an arbitrary constant.

Suggested approach can be effectively applied for the nonlocal generating of solutions of equations which are invariant with respect to the group of contact transformations.

1. Fushchych W., Shlelen W., Serov N., Symmetry analysis and exact solutions of nonlinear equations of mathematical physics, Reidel, Kluwer Publ. Comp., 1993, 450 p.

2. Fuschchych W.I., Zhdanov R.Z., Nonlinear spinor equations: symmetry and exact solutions, Киев, Наук. думка, 1992, 288 p.

3. Fushchych W.I., How to expand symmetry of differential equations? in Symmetry and Solutions of Nonlinear Equations of Mathematical Physics, Киев, Ин-т математики, 1987, 4–16.

4. Fushchych W.I., On symmetry and exact solutions of many-dimentional nonlinear wave equations, *Укр. мат. журн.*, 1987, **39**, № 1, 116–123.

5. Fushchych W.I., Tsifra I.M., On a reduction and solutions of nonlinear wave equations with broken symmetry, *J. Phys. A*, 1987, **20**, L45–L48.

6. Fushchych W.I., Serov M.I., Repeta V.C., Conditional symmetry reduction and exact solutions of nonlinear wave equations, *Доповіді АН України*, 1991, № 5, 29–34.

Nonlinear representations for Poincaré and Galilei algebras and nonlinear equations for electromagnetic fields

W.I. FUSHCHYCH, I.M. TSYFRA, V.M. BOYKO

We construct nonlinear representations of the Poincaré, Galilei, and conformal algebras on a set of the vector-functions $\Psi = (\vec{E}, \vec{H})$. A nonlinear complex equation of Euler type for the electromagnetic field is proposed. The invariance algebra of this equation is found.

1. Introduction

It is well known that the linear representations of the Poincaré algebra $AP(1,3)$ and conformal algebra $AC(1,3)$, with the basis elements

$$P_\mu = ig^{\mu\nu}\partial_\nu, \quad J_{\mu\nu} = x_\mu P_\nu - x_\nu P_\mu + S_{\mu\nu}, \tag{1}$$

$$D = x_\nu P^\nu - 2i, \tag{2}$$

$$K_\mu = 2x_\mu D - (x_\nu x^\nu) P_\mu + 2x^\nu S_{\mu\nu}, \tag{3}$$

is realized on the set of solutions of the Maxwell equations for the electromagnetic field in vacuum (see e.g. [1, 2])

$$\frac{\partial \vec{E}}{\partial t} = \operatorname{rot} \vec{H}, \quad \frac{\partial \vec{H}}{\partial t} = -\operatorname{rot} \vec{E}, \tag{4}$$

$$\operatorname{div} \vec{E} = 0, \quad \operatorname{div} \vec{H} = 0. \tag{5}$$

Here $S_{\mu\nu}$ realize the representation $D(0,1) \oplus D(1,0)$ of the Lorentz group.

Operators (1)–(3) satisfy the following commutation relations:

$$[P_\mu, P_\nu] = 0, \quad [P_\mu, J_{\alpha\beta}] = i(g_{\mu\alpha} P_\beta - g_{\mu\beta} P_\alpha), \tag{6}$$

$$[J_{\alpha\beta}, J_{\mu\nu}] = i(g_{\beta\mu} J_{\alpha\nu} + g_{\alpha\nu} J_{\beta\mu} - g_{\alpha\mu} J_{\beta\nu} - g_{\beta\nu} J_{\alpha\mu}), \tag{7}$$

$$[D, P_\mu] = -iP_\mu, \quad [D, J_{\mu\nu}] = 0, \tag{8}$$

$$[K_\mu, P_\alpha] = i(2J_{\alpha\mu} - 2g_{\mu\alpha} D), \quad [K_\mu, J_{\alpha\beta}] = i(g_{\mu\nu} K_\beta - g_{\mu\beta} K_\alpha), \tag{9}$$

$$[K_\mu, D] = -iK_\mu, \quad [K_\mu, K_\nu] = 0, \quad \mu, \nu, \alpha, \beta = 0, 1, 2, 3. \tag{10}$$

In this paper the nonlinear representations of the Poincaré, Galilei, and conformal algebras for the electromagnetic field $\vec{E}$, $\vec{H}$ are constructed. In particular, we prove that the continuity equation for the electromagnetic field is not invariant under the Lorentz group if the velocity of the electromagnetic field is taken in accordance with

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the Poynting definition. Conditional symmetry of the continuity equation is studied. The complex Euler equation for the electromagnetic field is introduced. The symmetry of this equation is investigated.

2. Formulation of the main results

The operators, realizing the nonlinear representations of the Poincaré algebras $AP(1,3) = \langle P_\mu, J_{\mu\nu}\rangle$, $AP_1(1,3) = \langle P_\mu, J_{\mu\nu}, D\rangle$, and conformal algebra $AC(1,3) = \langle P_\mu, J_{\mu\nu}, D, K_\mu\rangle$, have the structure

$$P_\mu = \partial_{x_\mu}, \tag{11}$$

$$J_{kl} = x_k\partial_{x_l} - x_l\partial_{x_k} + S_{kl}, \tag{12}$$

$$J_{0k} = x_0\partial_{x_k} + x_k\partial_{x_0} + S_{0k}, \quad k,l = 1,2,3, \tag{13}$$

$$D = x_\mu\partial_{x_\mu}, \tag{14}$$

$$K_0 = x_0^2\partial_{x_0} + x_0x_k\partial_{x_k} + (x_k - x_0E^k)\partial_{E^k} - x_0H^k\partial_{H^k}, \tag{15}$$

$$\begin{aligned} K_l = {} & x_0x_l\partial_{x_0} + x_lx_k\partial_{x_k} + [x_kE^l - x_0(E^lE^k - H^lH^k)]\partial_{E^k} + \\ & + [x_kH^l - x_0(H^lE^k + E^lH^k)]\partial_{H^k}, \end{aligned} \tag{16}$$

where

$$S_{kl} = E^k\partial_{E^l} - E^l\partial_{E^k} + H^k\partial_{H^l} - H^l\partial_{H^k},$$
$$S_{0k} = \partial_{E^k} - (E^kE^l - H^kH^l)\partial_{E^l} - (E^kH^l + H^kE^l)\partial_{H^l}.$$

The operators, realizing the nonlinear representations of the Galilei algebras $AG^{(2)}(1,3) = \langle P_\mu, J_{kl}, G_k^{(2)}\rangle$, $AG_1^{(2)}(1,3) = \langle P_\mu, J_{kl}, G_k^{(2)}, D\rangle$ have the form:

$$P_\mu = \partial_{x_\mu}, \quad J_{kl} = x_k\partial_{x_l} - x_l\partial_{x_k} + S_{kl}, \tag{17}$$

$$G_k^2 = x_k\partial_{x_0} - (E^kE^l - H^kH^l)\partial_{E^l} - (E^kH^l + H^kE^l)\partial_{H^l}, \tag{18}$$

$$D = x_0\partial_{x_0} + 2x_k\partial_{x_k} + E^k\partial_{E^k} + H^k\partial_{H^k}. \tag{19}$$

We see by direct verification that all represented operators satisfy the commutation relations of the algebras $AP(1,3)$, $AC(1,3)$, $AG(1,3)$.

3. Construction of nonlinear representations

In order to construct the nonlinear representations of Euclid-, Poincaré-, and Galilei groups and their extensions the following idea was proposed in [2, 3]: to use nonlinear equations invariant under these groups; it is necessary to find (point out, guess) the equations, which admit symmetry operators having a nonlinear structure. Such equation for the scalar field $u(x_0, x_1, x_2, x_3)$ is the eikonal equation

$$\frac{\partial u}{\partial x_\mu}\frac{\partial u}{\partial x^\mu} = 0, \quad \mu = 0,1,2,3 \tag{20}$$

which is invariant under the conformal algebra $AC(1,3)$ with the nonlinear operator K_μ [2, 3].

The nonlinear Euler equation for an ideal fluid

$$\frac{\partial v_k}{\partial t} + v_l \frac{\partial v_k}{\partial x_l} = 0, \quad k = 1, 2, 3 \tag{21}$$

which is invariant under nonlinear representation of the $AP(1,3)$ algebra, with basis elements

$$P_\mu = \partial_{x_\mu}, \quad J_{kl} = x_k \partial_{x_l} - x_l \partial_{x_k} + v_k \partial_{v_l} - v_l \partial_{v_k}, \tag{22}$$

$$J_{0k} = x_k \partial_0 + x_0 \partial_{x_k} + \partial_{v_k} - v_k v_l \partial_{v_l}, \tag{23}$$

was proposed in [3] to construct the nonlinear representation for the vector field. Note that equation (21) is also invariant with respect to the Galilei algebra $AG(1,3)$ with the basis elements

$$P_\mu = \partial_{x_\mu}, \quad J_{kl} = x_k \partial_{x_l} - x_l \partial_{x_k} + v_k \partial_{v_l} - v_l \partial_{v_k}, \quad G_a = x_0 \partial_{x_a} + \partial_{v_a}. \tag{24}$$

As mentioned in [2, 3] both the Lorentz–Poincaré–Einstein and Galilean principles of relativity are valid for system (21). We use the following nonlinear system of equations [4]

$$\frac{\partial E^k}{\partial x_0} + H^l \frac{\partial E^k}{\partial x_l} = 0, \quad \frac{\partial H^k}{\partial x_0} + E^l \frac{\partial H^k}{\partial x_l} = 0, \tag{25}$$

for constructing a nonlinear representation of the $AP(1,3)$ and $AG(1,3)$ algebras for the electromagnetic field. To construct the basis elements of the $AP(1,3)$ and $AG(1,3)$ algebras in explicit form we investigate the symmetry of system (25). We search for the symmetry operators of equations (25) in the form:

$$X = \xi^\mu \partial_{x_\mu} + \eta^l \partial_{E^l} + \beta^l \partial_{H^l}, \tag{26}$$

where $\xi^\mu = \xi^\mu(x, \vec{E}, \vec{H})$, $\eta^l = \eta^l(x, \vec{E}, \vec{H})$, $\beta^l = \beta^l(x, \vec{E}, \vec{H})$.

Theorem 1. *The maximal invariance algebra of system (25) in the class of operators (26) is the 20-dimensional algebra, whose basis elements are given by the formulas*

$$P_\mu = \partial x_\mu, \tag{27}$$

$$J^{(1)}_{kl} = x_k \partial_{x_l} - x_l \partial_{x_k} + E^k \partial_{E^l} - E^l \partial_{E^k} + H^k \partial_{H^l} - H^l \partial_{H^k}, \tag{28}$$

$$J^{(2)}_{kl} = x_k \partial_{x_l} + x_l \partial_{x_k} + E^k \partial_{E^l} + E^l \partial_{E^k} + H^k \partial_{H^l} + H^l \partial_{H^k}, \tag{29}$$

$$G^{(1)}_a = x_0 \partial_{x_a} + \partial_{E^a} + \partial_{H^a}, \tag{30}$$

$$G^{(2)}_a = x_a \partial_{x_0} - E^a E^k \partial_{E^k} - H^a H^k \partial_{H^k}, \tag{31}$$

$$D_0 = x_0 \partial_{x_0} - E^l \partial_{E^l} - H^l \partial_{H^l}, \tag{32}$$

$$D_1 = x_1 \partial_{x_1} + E^1 \partial_{E^1} + H^1 \partial_{H^1}, \tag{33}$$

$$D_2 = x_2 \partial_{x_2} + E^2 \partial_{E^2} + H^2 \partial_{H^2}, \tag{34}$$

$$D_3 = x_3 \partial_{x_3} + E^3 \partial_{E^3} + H^3 \partial_{H^3}. \tag{35}$$

Proof. To prove theorem 1 we use Lie's algorithm. The condition of invariance of the system $L(\vec{E},\vec{H})$, i.e. (25), with respect to operator X has the form

$$\underset{1}{X} L\Big|_{L=0} = 0, \tag{36}$$

where

$$\underset{1}{X} = X + [D_\alpha(\eta^l) - E^l_j D_\alpha(\xi^j)]\partial_{E^l_\alpha} + [D_\alpha(\beta^l) - H^l_j D_\alpha(\xi^j)]\partial_{H^l_\alpha},$$
$$E^l_\alpha = \frac{\partial E^l}{\partial x_\alpha}, \quad H^l_\alpha = \frac{\partial H^l}{\partial x_\alpha}, \quad l = 1,2,3; \quad \alpha = 0,1,2,3$$

is the prolonged operator. From the invariance condition (36) we obtain the system of equations which determine the coefficient functions ξ^μ, η^l, β^l of the operator (26):

$$\begin{aligned}
&\eta^l_k = 0, \quad \eta^l_0 = 0, \quad \beta^l_k = 0, \quad \beta^l_0 = 0, \quad \xi^\mu_{\alpha\nu} = 0, \quad \xi^\mu_{E^a} = 0, \quad \xi^\mu_{H^a} = 0,\\
&\eta^k = -E^k\xi^0_0 + \xi^k_0 + E^a\xi^k_a - E^aE^k\xi^0_a,\\
&\beta^k = -H^k\xi^0_0 + \xi^k_0 + H^a\xi^k_a - H^aH^k\xi^0_a,
\end{aligned} \tag{37}$$

where

$$\eta^l_k = \frac{\partial\eta^l}{\partial x_k}, \quad \eta^l_0 = \frac{\partial\eta^l}{\partial x_0}, \quad \xi^\mu_{E^a} = \frac{\partial\xi^\mu}{\partial E^a}, \quad \xi^\mu_{\alpha\nu} = \frac{\partial^2\xi^\mu}{\partial x_\alpha\partial x_\nu}.$$

Having found the general solution of system (37), we get the explicit form of all the linear independent symmetry operators of system (25), which have the structure (27)–(35). Operators of Lorentz rotations J_{0k} is given by the linear combination of the Galilean operators $G^{(1)}_k$ and $G^{(2)}_k$:

$$J_{0k} = G^{(1)}_k + G^{(2)}_k. \tag{38}$$

All the following statements, given here without proofs, can be proved in analogy with the above-mentioned scheme.

4. The finite transformations and invariants

We present some finite transformations which are generated by the operators J_{0k}:

$$\begin{aligned}
J_{01}: \quad & x_0 \to x_0' = x_0 \operatorname{ch}\theta_1 + x_1 \operatorname{sh}\theta_1,\\
& x_1 \to x_1' = x_1 \operatorname{ch}\theta_1 + x_0 \operatorname{sh}\theta_1,\\
& x_2 \to x_2' = x_2, \quad x_3 \to x_3' = x_3,
\end{aligned} \tag{39}$$

$$\begin{aligned}
&E^1 \to E^{1'} = \frac{E^1 \operatorname{ch}\theta_1 + \operatorname{sh}\theta_1}{E^1 \operatorname{sh}\theta_1 + \operatorname{ch}\theta_1}, \quad H^1 \to H^{1'} = \frac{H^1 \operatorname{ch}\theta_1 + \operatorname{sh}\theta_1}{H^1 \operatorname{sh}\theta_1 + \operatorname{ch}\theta_1},\\
&E^2 \to E^{2'} = \frac{E^2}{E^1 \operatorname{sh}\theta_1 + \operatorname{ch}\theta_1}, \quad H^2 \to H^{2'} = \frac{H^2}{H^1 \operatorname{sh}\theta_1 + \operatorname{ch}\theta_1},\\
&E^3 \to E^{3'} = \frac{E^3}{E^1 \operatorname{sh}\theta_1 + \operatorname{ch}\theta_1}, \quad H^3 \to H^{3'} = \frac{H^3}{H^1 \operatorname{sh}\theta_1 + \operatorname{ch}\theta_1}.
\end{aligned} \tag{40}$$

The operators J_{02}, J_{03} generate analogous transformations. θ_1 is the real group parameter of the geometric Lorentz transformation. Operators $G_k^{(2)}$ generate the following transformations:

$$G_1^{(2)}: \quad x_0 \to x_0' = x_0 + \theta_1 x_1, \quad x_k \to x_k' = x_k,$$
$$E^k \to E^{k'} = \frac{E^k}{1+\theta_1 E^1}, \quad H^k \to H^{k'} = \frac{H^k}{1+\theta_1 H^1}.$$

Analogous transformations are generated by the operators $G_2^{(2)}$, $G_3^{(2)}$. Operators $G_k^{(1)}$ generate the following transformations:

$$\begin{aligned} G_1^{(1)}: \quad & x_0 \to x_0' = x_0, \quad x_1 \to x_1' = x_1 + x_0\theta_1, \\ & x_2 \to x_2' = x_2, \quad x_3 \to x_3' = x_3, \\ & E^1 \to E^{1'} = E^1 + \theta_1, \quad H^1 \to H^{1'} = H^1 + \theta_1, \\ & E^2 \to E^{2'} = E^2, \quad E^3 \to E^{3'} = E^3, \\ & H^2 \to H^{2'} = H^2, \quad H^3 \to H^{3'} = H^3. \end{aligned}$$

The operators $G_2^{(1)}$, $G_3^{(1)}$ generate analogous transformations.

It is easy to verify that

$$I_1 = \frac{\left(1 - \vec{E}\vec{H}\right)^2}{\left(1-\vec{E}^2\right)\left(1-\vec{H}^2\right)}, \quad \vec{E}^2 \neq 1, \quad \vec{H}^2 \neq 1 \tag{41}$$

is invariant with respect to the nonlinear transformations of the Poincaré group which are generated by representations (28), (38).

The invariant of the Galilei group which is generated by representations (28), (31) has the form:

$$I_2 = \frac{\vec{E}^2\vec{H}^2}{\left(\vec{E}\vec{H}\right)^2}, \tag{42}$$

whereas the Galilei group which is generated by representations (28), (30) has the invariant

$$I_3 = (\vec{E} - \vec{H})^2. \tag{43}$$

5. Complex Euler equation for the electromagnetic field

Let us consider the system of equations

$$\frac{\partial \Sigma^k}{\partial x_0} + \Sigma^l \frac{\partial \Sigma^k}{\partial x_l} = 0, \quad \Sigma^k = E^k + iH^k. \tag{44}$$

The complex system (44) is equivalent to the real system of equations for $\vec{E}$ and $\vec{H}$

$$\begin{aligned} &\frac{\partial E^k}{\partial x_0} + E^l \frac{\partial E^k}{\partial x_l} - H^l \frac{\partial H^k}{\partial x_l} = 0, \\ &\frac{\partial H^k}{\partial x_0} + H^l \frac{\partial E^k}{\partial x_l} + E^l \frac{\partial H^k}{\partial x_l} = 0. \end{aligned} \tag{45}$$

The following statement has been proved with the help of Lie's algorithm.

Theorem 2. *The maximal invariance algebra of the system (45) is the 24-dimensional Lie algebra whose basis elements are given by the formulas*

$$
\begin{aligned}
&P_\mu = \partial_{x_\mu},\\
&J^{(1)}_{kl} = x_k\partial_{x_l} - x_l\partial_{x_k} + E^k\partial_{E^l} - E^l\partial_{E^k} + H^k\partial_{H^l} - H^l\partial_{H^k},\\
&J^{(2)}_{kl} = x_k\partial_{x_l} + x_l\partial_{x_k} + E^k\partial_{E^l} + E^l\partial_{E^k} + H^k\partial_{H^l} + H^l\partial_{H^k},\\
&G^{(1)}_a = x_0\partial_{x_a} + \partial_{E^a},\\
&G^{(2)}_a = x_a\partial_{x_0} - (E^aE^k - H^aH^k)\partial_{E^a} - (E^aH^k + H^aE^k)\partial_{H^k},\\
&D_0 = x_0\partial_{x_0} - E^k\partial_{E^k} - H^k\partial_{H^k},\\
&D_a = x_a\partial_{x_a} + E^a\partial_{E^a} + H^a\partial_{H^a} \quad \text{(no sum over } a),\\
&K_0 = x_0^2\partial_{x_0} + x_0x_k\partial_{x_k} + (x_k - x_0E^k)\partial_{E^k} - x_0H^k\partial_{H^k},\\
&K_a = x_0x_a\partial_{x_0} + x_ax_k\partial_{x_k} + [x_kE^a - x_0(E^aE^k - H^aH^k)]\partial_{E^k} +\\
&\qquad + [x_kH^a - x_0(H^aE^k + E^aH^k)]\partial_{H^k}.
\end{aligned}
\tag{46}
$$

The algebra, engendered by the operators (46), include the Galilei algebras $AG^{(1)}(1,3)$, $AG^{(2)}(1,3)$ and Poincaré algebra $AP(1,3)$, and conformal algebra $AC(1,3)$ as subalgebras. Operators $G^{(2)}_a$ generate the linear geometrical transformations in $\mathbb{R}(1,3)$

$$
x_0 \to x_0' = x_0 + \theta_a x_a \quad \text{(no sum over } a), \quad x_l \to x_l', \tag{47}
$$

as well as the nonlinear transformations of the fields

$$
\begin{aligned}
&E^l + iH^l \to E^{l'} + iH^{l'} = \frac{E^l + iH^l}{1 + \theta_a(E^a + iH^a)} \quad \text{(no sum over } a),\\
&E^l - iH^l \to E^{l'} - iH^{l'} = \frac{E^l - iH^l}{1 + \theta_a(E^a - iH^a)}.
\end{aligned}
\tag{48}
$$

The invariant of the group $G^{(2)}(1,3)$ is

$$
I_4 = \frac{\left(\vec{E}^2 - \vec{H}^2\right) + 4\left(\vec{E}\vec{H}\right)^2}{\left(\vec{E}^2 + \vec{H}^2\right)^2}. \tag{49}
$$

Operators J_{0k} generate the linear transformations in $\mathbb{R}(1,3)$

$$
\begin{aligned}
&x_0 \to x_0' = x_0 \operatorname{ch}\theta_k + x_0 \operatorname{sh}\theta_k,\\
&x_k \to x_k' = x_k \operatorname{ch}\theta_k + x_0 \operatorname{sh}\theta_k \quad \text{(no sum over } k),\\
&x_l \to x_l' = x_l, \quad \text{if } l \neq k,
\end{aligned}
\tag{50}
$$

as well as the nonlinear transformations of the fields

$$
\begin{aligned}
&E^k + iH^k \to E^{k'} + iH^{k'} = \frac{(E^k + iH^k)\operatorname{ch}\theta_k + \operatorname{sh}\theta_k}{(E^k + iH^k)\operatorname{sh}\theta_k + \operatorname{ch}\theta_k},\\
&E^k - iH^k \to E^{k'} - iH^{k'} = \frac{(E^k - iH^k)\operatorname{ch}\theta_k + \operatorname{sh}\theta_k}{(E^k - iH^k)\operatorname{sh}\theta_k + \operatorname{ch}\theta_k}.
\end{aligned}
$$

If $l \neq k$, then

$$E^l + iH^l \to E^{l'} + iH^{l'} = \frac{E^l + iH^l}{(E^k + iH^k)\operatorname{sh}\theta_k + \operatorname{ch}\theta_k},$$
$$E^l - iH^l \to E^{l'} - iH^{l'} = \frac{E^l - iH^l}{(E^k - iH^k)\operatorname{sh}\theta_k + \operatorname{ch}\theta_k} \quad \text{(no sum over } k). \tag{51}$$

The invariant of group $P(1,3)$ is

$$I_5 = \frac{1 - 2\left[(\vec{E}^2 - \vec{H}^2) - \frac{1}{2}(\vec{E}^2 - \vec{H}^2)^2 - 2(\vec{E}\vec{H})^2\right]}{\left[1 - (\vec{E}^2 + \vec{H}^2)\right]^2}, \quad \vec{E}^2 + \vec{H}^2 \neq 1. \tag{52}$$

The operator K_0 generates the following nonlinear transformations in $\mathbb{R}(1,3)$ and linear transformations of the fields

$$\begin{aligned} &x_\mu \to x'_\mu = \frac{x_\mu}{1 - \theta_0 x_0}, \\ &E^k \to E^{k'} = E^k + \theta_0(x_k - x_0 E^k), \\ &H^k \to H^{k'} = H^k(1 - \theta_0 x_0). \end{aligned} \tag{53}$$

The operators K_a generate nonlinear transformations in both $\mathbb{R}(1,3)$ and of the fields

$$x_0 \to x'_0 = \frac{x_0}{1 - x_a\theta_a}, \quad x_a \to x'_a = \frac{x_a}{1 - x_a\theta_a}.$$

If $k \neq a$, then

$$\begin{aligned} &x_k \to x'_k = \frac{x_k}{1 - x_a\theta_a}, \\ &E^a + iH^a \to E^{a'} + iH^{a'} = \frac{E^a + iH^a}{1 + \theta_a[x_0(E^a + iH^a) - x_a]}, \\ &E^a - iH^a \to E^{a'} - iH^{a'} = \frac{E^a - iH^a}{1 + \theta_a[x_0(E^a - iH^a) - x_a]}. \end{aligned}$$

If $k \neq a$, then

$$\begin{aligned} &E^k + iH^k \to E^{k'} + iH^{k'} = \frac{E^k + iH^k + \theta_a(E^a + iH^a)x_k}{1 + \theta_a[x_0(E^a + iH^a) - x_a]}, \\ &E^k - iH^k \to E^{k'} - iH^{k'} = \frac{E^k - iH^k + \theta_a(E^a - iH^a)x_k}{1 + \theta_a[x_0(E^a - iH^a) - x_a]} \quad \text{(no sum over } a). \end{aligned} \tag{54}$$

Note 1. Setting $\vec{\Sigma} = a\vec{E} + ib\vec{H}$, where a, b are arbitrary functions of the invariants $\vec{E}^2$, $\vec{H}^2$, $\vec{E}\vec{H}$, we obtain more general form of the equation (44). The equation

$$\frac{\partial \Sigma^k}{\partial x_0} + \Sigma^l \frac{\partial \Sigma^k}{\partial x_l} = F(\vec{E}\vec{H}, \vec{E}^2, \vec{H}^2)\Sigma^k$$

is invariant only under some subalgebras of algebra (46) depending on the choice of function F.

Note 2. If we analyse the symmetry of the following equations

$$\begin{aligned}&\left(\frac{\partial}{\partial x_0}+E^l\frac{\partial}{\partial x_l}+H^l\frac{\partial}{\partial x_l}\right)E^k=0,\\&\left(\frac{\partial}{\partial x_0}+E^l\frac{\partial}{\partial x_l}+H^l\frac{\partial}{\partial x_l}\right)H^k=0;\end{aligned}\qquad(*)$$

or

$$\begin{aligned}&\frac{\partial E^k}{\partial x_0}=\pm\left(E^l\frac{\partial}{\partial x_l}+H^l\frac{\partial}{\partial x_l}\right)H^k,\\&\frac{\partial H^k}{\partial x_0}=\pm\left(E^l\frac{\partial}{\partial x_l}+H^l\frac{\partial}{\partial x_l}\right)E^k,\end{aligned}\qquad(**)$$

we obtain concrete examples of nonlinear representations for the Poincaré and Galilei algebras. This problem will be considered in a future paper.

6. Symmetry of the continuity equation and the Poynting vector

Let us consider the continuity equation for the electromagnetic field

$$L(\vec{E},\vec{H})\equiv\frac{\partial\rho}{\partial x_0}+\operatorname{div}\rho\vec{v}=0. \qquad (55)$$

According to the Poynting definition ρ and ρv^k have the forms

$$\rho=\frac{1}{2}(\vec{E}^2+\vec{H}^2),\quad \rho v^k=\varepsilon_{kln}E^lH^n. \qquad (56)$$

Theorem 3. *The nonlinear system (55), (56) is not invariant under the Lorentz algebra, with basis elements:*

$$\begin{aligned}&J_{kl}=x_k\partial_{x_l}-x_l\partial_{x_k}+E^k\partial_{E^l}-E^l\partial_{E^k}+H^k\partial_{H^l}-H^l\partial_{H^k},\\&J_{0k}=x_k\partial_{x_0}+x_0\partial_{x_k}+\varepsilon_{kln}(E^l\partial_{H^n}-H^l\partial_{E^n}),\quad k,l,n=1,2,3.\end{aligned}\qquad(57)$$

To prove theorem 3 it is necessary to substitute ρ and ρv^k, from formulas (56), to equation (55) and to apply Lie's algorithm, i.e., it is necessary to verify that the invariance condition

$$\underset{1}{J}_{\mu\nu}\left(L(\vec{E},\vec{H})\right)\Big|_{L=0}\equiv 0 \qquad (58)$$

is not satisfied, where $\underset{1}{J}_{\mu\nu}$ is the first prolongation of the operator $J_{\mu\nu}$.

Theorem 4. *The continuity equation (55), (56) is conditionally invariant with respect to the operators $J_{\mu\nu}$, given in (57) if and only if $\vec{E}$, $\vec{H}$ satisfy the Maxwell equation (4), (5).*

Thus the continuity equation, which is the mathematical expression of the conservation law of the electromagnetic field energy and impulse is not Lorentz-invariant if $\vec{E}$, $\vec{H}$ does not satisfy the Maxwell equation. A more detailed discussion on conditional symmetries can be found in [1, 2].

The following statement can be proved in the case when

$$\rho=F^0(\vec{E},\vec{H})\quad\text{and}\quad \rho v^k=F^k(\vec{E},\vec{H}), \qquad (59)$$

where F^0, F^k are arbitrary smooth functions $F^0\not\equiv 0$, $F^k\not\equiv 0$.

Theorem 5. *The continuity equation (55), (59) is invariant with respect to the classic geometrical Lorentz transformations if and only if*

$$r(B) = 4, \tag{60}$$

where $r(B)$ is the rank of the Jacobi matrix of functions F^μ.

In conclusion we present some statements about the symmetry of the following systems:

$$\begin{gathered}\frac{\partial \vec{E}}{\partial x_0} = \operatorname{rot} \vec{H} + \vec{F}_1(\vec{E},\vec{H}), \quad \frac{\partial \vec{H}}{\partial x_0} = -\operatorname{rot} \vec{E} + \vec{F}_2(\vec{E},\vec{H}),\\ \operatorname{div} \vec{E} = R_1(\vec{E},\vec{H}), \quad \operatorname{div} \vec{H} = R_2(\vec{E},\vec{H}),\end{gathered} \tag{61}$$

$$\begin{gathered}\frac{\partial (R\vec{E})}{\partial x_0} = \operatorname{rot} (R\vec{H}), \quad \frac{\partial N\vec{H}}{\partial x_0} = -\operatorname{rot} (N\vec{E}),\\ \operatorname{div} (R\vec{E}) = 0, \quad \operatorname{div} (N\vec{H}) = 0.\end{gathered} \tag{62}$$

Here

$$R = R(W_1, W_2), \quad N = N(W_1, W_2), \quad W_1 = \vec{E}^2 - \vec{H}^2, \quad W_2 = \vec{E}\vec{H}.$$

$$\operatorname{div} (R\vec{E} + N\vec{H}) = 0. \tag{63}$$

Theorem 6. *The system of equations (61) is invariant under the Lorentz algebra with the basis elements (57) if and only if*

$$\vec{F}_1 \equiv \vec{F}_2 \equiv 0, \quad R_1 \equiv R_2 \equiv 0.$$

Theorem 7. *The system of equations (62) is invariant under the Lorentz algebra (57) if R and N are arbitrary functions of the invariants $W_1 = \vec{E}^2 - \vec{H}^2$, $W_2 = \vec{E}\vec{H}$.*

Theorem 8. *The equation (63) is invariant under the Lorentz algebra with the basis elements (57) if and only if $\vec{E}$, $\vec{H}$ satisfy the system of equations*

$$\frac{\partial (R\vec{E} + N\vec{H})}{\partial x_0} = \operatorname{rot} (R\vec{H} - N\vec{E}).$$

Thus it is established that, besides the generally recognized linear representation of the Lorentz group discovered by Henry Poincaré in 1905 [5], there exists the nonlinear representation constructed by using the nonlinear equations of hydrodynamical type [4]. It is obvious that for instance the linear superposition principle does not hold for a non-Maxwell electrodynamic theory based on the equation (25) or (45).

The nonlinear representations for the algebras $AG(1,3)$, $A\tilde{P}(1,2)$, $A\tilde{P}(2,2)$, $AC(1,2)$, $AC(2,2)$ for a scalar field have been considered in [6], $AP(1,1)$ in [7], and $AP(1,2)$ in [8].

1. Fushchych W., Nikitin A., Symmetries of Maxwell's equations, Dordrecht, Reidel Publ. Comp., 1987.
2. Fushchych W., Shtelen W., Serov N., Symmetry analysis and exact solutions of equations of nonlinear mathematical physics, Dordrecht, Kluwer Publishers, 1993.
3. Fushchych W., On symmetry and exact solutions of the multidimensional equations of mathematical physic, in Algebraic-Theoretical Studies in Mathematical Physics Problems, Kyiv, Institute of Mathematics, 1983, 4–23.

4. Fushchych W., New nonlinear equations for electromagnetic field having the velocity different from c, *Dopovidi of the Ukrainian Academy of Sciences,* 1982, № 4, 24–28.
5. Poincaré H., On the dynamics of the electron, *Comptes Rendus*, 1905, **140**, 1504–1510.
6. Fushchych W., Zhdanov R., Lahno V., On nonlinear representation of the conformal algebra $AC(2,2)$, *Dopovidi of the Ukrainian Academy of Sciences,* 1993, № 9, 44–47.
7. Rideau G., Winternitz P., Nonlinear equations invariant under the Poincaré, similitude and conformal groups in two-dimensional space-time, *J. Math. Phys.*, 1990, **31**, 1095–1105.
8. Yehorchenko I., Nonlinear representation of the Poincaré algebra and invariant equations, in Symmetry Analysis of Equations of Mathematical Physics, Kyiv, Inst. of Math., 1992, 62–66.

On the new approach to variable separation in the two-dimensional Schrödinger equation

R.Z. ZHDANOV, I.V. REVENKO, W.I. FUSHCHYCH

Для двовимірного рівняння Шредінгера з потенціалом, який не залежить від часової змінної, повністю розв'язано задачу класифікації потенціалів, при яких воно допускає розділення змінних. Для кожного з потенціалів описано всі системи координат, в яких розділюється відповідне рівняння Шредінгера.

There is a lot of papers devoted to separation of variables (SV) in the two-dimensional Schrödinger equation

$$iu_1 + u_{x_1x_2} + u_{x_2x_2} = V(x_1, x_2)u \tag{1}$$

with some specific $V(x_1, x_2)$ (see, e.g. [1–3] and references therein). Saying about the problem of SV in the Eq. (1), we imply two mutually connected problems. The first one is to describe all functions $V(x_1, x_2)$ such that the equation (1) admits separation of variables (classification problem). The second problem is to construct for each function $V(x_1, x_2)$ all coordinate systems making it possible to separate corresponding Schrödinger equation.

As far as we know, the first problem has been solved provided $V = 0$ [3] and $V = \alpha x_1^{-2} + \beta x_2^{-2}$ [1] and the second one has not been considered in the literature at all. We guess that a possible reason for this was absence of an adequate mathematical technique to handle the classification problem. In the paper [4] we suggested a new approach to SV in partial differential equations which enabled us to solve the problem of SV in two-dimensional wave equation with time independent potential [4]. In the present paper we give the complete solution of the problem of SV in the Schrödinger equation (1) obtained within the framework of the above said approach.

Solution with separated variables is looked for in the form of the ansatz [4]

$$u = Q(t, \vec{x})\varphi_0(t)\varphi_1(\omega_1(t, \vec{x}))\varphi_2(\omega_2(t, \vec{x})), \tag{2}$$

where $\varphi_0(t)$, $\varphi_1(\omega_1(t, \vec{x}))$, $\varphi_2(\omega_2(t, \vec{x}))$ are smooth functions satisfying ordinary differential equations (ODE)

$$\begin{aligned} &\frac{d\varphi_0}{dt} = U_0(t, \varphi_0, \lambda_1, \lambda_2), \\ &\frac{d^2\varphi_a}{d\omega_a^2} = U_a\left(\omega_a, \varphi_a, \frac{d\varphi_a}{d\omega_a}; \lambda_1, \lambda_2\right), \quad a = \overline{1, 2}, \end{aligned} \tag{3}$$

and Q, ω_1, ω_2 are functions to be determined from the requirement that ansatz (2) reduces Eq. (1) to ODE, $\lambda_1, \lambda_2 \subset \mathbb{R}^1$ are arbitrary parameters (separation constants). It is important to emphasize that functions Q, ω_1, ω_2 do not depend on the parameters λ_1, λ_2.

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Because of the lack of space we have no possibility to adduce all necessary computations. That is why, we shall restrict ourselves by pointing out main steps of realization of the approach to SV suggested in [4].

First of all, we note that the substitution

$$\omega_1 \to \omega_1' = \Omega_1(\omega_1), \quad \omega_2 \to \omega_2' = \Omega_2(\omega_2), \quad Q \to Q' = Q\psi_1(\omega_1)\psi_2(\omega_2) \tag{4}$$

does not alter the structure of relations (2), (3). That is why, we introduce the following equivalence relation: (ω_1, ω_2, Q) is equivalent to $(\omega_1', \omega_2', Q')$ provided (4) holds with some Ω_a, ψ_a.

Substituting (2) into (1) with account of equalities (3) and splitting obtained relation with respect to independent variables φ_0, φ_a, φ_{aa}, λ_a, $a = 1, 2$ we conclude that up to the equivalence relation (4) equations (3) take the form

$$\frac{d\varphi_0}{dt} = (\lambda_1 R_1(t) + \lambda_2 R_2(t) + R_0(t))\varphi_0,$$
$$\frac{d^2\varphi_a}{d\omega_a^2} = (\lambda_1 B_{1a}(\omega_a) + \lambda_2 B_{2a}(\omega_a) + B_{0a}(\omega_a))\varphi_a$$

and what is more, functions ω_1, ω_2, Q satisfy the over-determined system of nonlinear partial differential equations

$$\begin{aligned}
&1) \quad \sum_{b=1}^{2} \omega_{1x_b}\omega_{2x_b} = 0,\\
&2) \quad \sum_{b=1}^{2} [B_{a1}(\omega_1)\omega_{1x_b}\omega_{2x_b} + B_{a2}(\omega_2)\omega_{2x_b}\omega_{2x_b}] + R_a(t) = 0, \quad a = 1, 2,\\
&3) \quad 2\sum_{b=1}^{2} Q_{x_b}\omega_{ax_b} + Q\left(i\omega_{at} + \sum_{b=1}^{2}\omega_{ax_bx_b}\right) = 0, \quad a = 1, 2,\\
&4) \quad \sum_{b=1}^{2} [B_{01}(\omega_1)\omega_{1x_b}\omega_{2x_b} + B_{02}(\omega_2)\omega_{2x_b}\omega_{2x_b}]Q + iQ_i +\\
&\qquad + \sum_{b=1}^{2} Q_{x_bx_b} + R_0(t)Q - V(\vec{x})Q = 0.
\end{aligned} \tag{5}$$

Thus, to solve the problem of SV for the linear Schrödinger equation it is necessary to construct the general solution of the system of nonlinear equations (5). Roughly speaking, to solve a linear equation one has to solve a system of nonlinear equations! This is the reason why so far there is no complete description of all coordinate systems providing separability of the four-dimensional d'Alembert equation.

But in the case involved we have succeeded in integration of nonlinear system (5) for ω_1, ω_2, Q. First, we have established that the general solution of equations 1–3 from (5) determined up to the equivalence relation (4) splits into four inequivalent classes

$$1) \quad \omega_1 = A(t)x_1 + W_1(t), \quad \omega_2 = B(t)x_2 + W_2(t),$$
$$Q(t, \vec{x}) = \exp\left[-\frac{i}{4}\left(\frac{\dot{A}}{A}x_1^2 + \frac{\dot{B}}{B}x_2^2\right) - \frac{i}{2}\left(\frac{\dot{W}_1}{A}x_1 + \frac{\dot{W}_2}{B}x_2\right)\right],$$

$$
\begin{aligned}
&2)\quad \omega_1 = \frac{1}{2}\ln(x_1^2+x_2^2)+W(t), \quad \omega_2 = \operatorname{arctg}\frac{x_1}{x_2},\\
& Q(t,\vec{x}) = \exp\left[-\frac{I\dot{W}}{4}(x_1^2+x_2^2)\right],\\
&3)\quad x_1 = \frac{1}{2}W(t)(\omega_1^2-\omega_2^2)+W_1(t), \quad x_2 = W(t)\omega_1\omega_2+W_2(t),\\
& Q(t,\vec{x}) = \exp\left[\frac{i}{4}\frac{\dot{W}}{W}[(x_1-W_1)^2+(x_2-W_2)^2]+\frac{i}{2}(\dot{W}_1x_1+\dot{W}_2x_2)\right],\\
&4)\quad x_1 = W(t)\operatorname{ch}\omega_1\cos\omega_2+W_1(t), \quad x_2 = W(t)\operatorname{sh}\omega_1\sin\omega_2+W_2(t),\\
& Q(t,\vec{x}) = \exp\left[\frac{i}{4}\frac{\dot{W}}{W}[(x_1-W_1)^2+(x_2-W_2)^2]+\frac{i}{2}(\dot{W}_1x_1+\dot{W}_2x_2)\right].
\end{aligned}
\tag{6}
$$

Here A, B, W, W_1, W_2 are arbitrary smooth functions on t. Dot means differentiation with respect to t.

Substituting obtained expressions for Q, ω_1, ω_2 into the last equation from the system (5) and splitting with respect to the variables x_1, x_2 we get explicit forms of potentials $V(x_1,x_2)$ and systems of nonlinear ODE for functions $A(t)$, $B(t)$, $W(t)$, $W_1(t)$, $W_2(t)$. We have suceeded in integrating these and in constructing all coordinate systems providing SV in the initial equation (1). Complete list of these systems takes two dozens of pages, so we are to restrict ourselves to adducing explicit forms of potentials $V(x_1,x_2)$ such that the Schrödinger equation (1) admits SV.

$$
\begin{aligned}
&1)\quad V(\vec{x}) = V_1(x_1^2+x_2^2)+V_2\left(\frac{x_1}{x_2}\right)(x_1^2+x_2^2)^{-1};\\
&2)\quad V(\vec{x}) = V_2\left(\frac{x_1}{x_2}\right)(x_1^2+x_2^2)^{-1};\\
&3)\quad V(\vec{x}) = [V_1(\omega_1)+V_2(\omega_2)](\omega_1^2+\omega_2^2)^{-1},\\
& \text{where } x_1 = \frac{1}{2}(\omega_1^2-\omega_2^2), \quad x_2 = \omega_1\omega_2;\\
&4)\quad V(\vec{x}) = [V_1(\omega_1)+V_2(\omega_2)](\operatorname{sh}^2\omega_1+\sin^2\omega_2)^{-1},\\
& \text{where } x_1 = \operatorname{ch}\omega_1\cos\omega_2, \quad x_2 = \operatorname{sh}\omega_1\sin\omega_2;\\
&5)\quad V(\vec{x}) = V_1(x_1)+V_2(x_2);\\
&6)\quad V(\vec{x}) = kx_1^2+V_2(x_2);\\
&7)\quad V(\vec{x}) = k_1x_1^2+k_2x_1^{-2}+V_2(x_2), \quad k_2\neq 0;\\
&8)\quad V(\vec{x}) = kx_1^2, \quad k\neq 0;\\
&9)\quad V(\vec{x}) = k_1x_1^2+k_2x_2^2, \quad k_1k_2\neq 0;\\
&10)\quad V(\vec{x}) = k_1x_1^2+k_2x_1^{-2}, \quad k_1k_2\neq 0;\\
&11)\quad V(\vec{x}) = k_1x_1^2+k_2x_2^2+k_3x_2^{-2}, \quad k_1k_3\neq 0;\\
&12)\quad V(\vec{x}) = k_1x_1^2+k_2x_2^2+k_3x_1^{-2}+k_4x_2^{-2}, \quad k_3k_4\neq 0, \quad k_1^2+k_2^2\neq 0;\\
&13)\quad V(\vec{x}) = k_1x_1^{-2}+k_2x_2^{-2};\\
&14)\quad V(\vec{x}) = 0.
\end{aligned}
\tag{7}
$$

In the above formulae V_1, V_2 are arbitrary smooth functions, k, k_1, k_2, k_3, k_4 are arbitrary real constants.

Note 1. The Schrödinger equation with the potential

$$V(\vec{x}) = k(x_1^2 + x_2^2) + V_1\left(\frac{x_1}{x_2}\right)(x_1^2 + x_2^2)^{-1}, \quad k = \text{const}, \tag{8}$$

is reduced to the Schrödinger equation with the potential

$$V'(\vec{x}') = V_1'\left(\frac{x_1'}{x_2'}\right)(x_1'^2 + x'^2)^{-1} \tag{9}$$

by the change of variables

$$t' = \alpha(t), \quad \vec{x}' = \beta(t)\vec{x}, \quad u' = \exp(i\gamma(t)\vec{x}^2 + i\delta(t)).$$

(explicit form of the functions $\alpha(t)$, $\beta(t)$, $\gamma(t)$, $\delta(t)$ depends on the sign of the parameter k in (8)). Since the above change of variables does not alter the structure of ansatz (2), when classifying potentials $V(x_1, x_2)$ providing separability of Eq. (1) we consider potentials (7), (8) as equivalent.

Note 2. It is well-known (see, e.g. [5, 6]) that the general form of the invariance group admitted by Eq. (1) is as follows:

$$t' = f(t, \vec{\theta}), \quad x_a' = g_a(t, \vec{x}, \vec{\theta}), \quad a = 1, 2, \quad u' = h(t, \vec{x}, \vec{\theta})u, \tag{10}$$

where $\vec{\theta} = (\theta_1, \theta_2, \ldots, \theta_n)$ are group parameters.

Since transformations (10) do not alter the structure of the ansatz (2), systems of coordinates t', x_1', x_2' and t, x_1, x_2 are considered as equivalent.

Thus, there exist fourteen inequivalent types of the Schrödinger equations of the form (1) admitting SV. Consequently, the classification problem for Eq. (1) is solved.

Next, we shall obtain all coordinate systems providing separability of the Schrödinger equation having the potential $V = k_1x_1^2 + k_2x_2^2$ (the harmonic oscillator type equation). Explicit forms of the coordinate systems to be found depend essentially on the signs of the parameters k_1, k_2. Here we consider the case, when $k_1 < 0$, $k_2 > 0$ (the cases $k_1 > 0$, $k_2 > 0$ and $k_1 < 0$, $k_2 < 0$ will be considered in a separate publication). It means that Eq. (1) can be written in the form

$$iu_t + u_{x_1x_1} + u_{x_2x_2} + \frac{1}{4}(a^2x_1^2 - b^2x_2^2)u = 0, \tag{11}$$

where a, b, are arbitrary real constants (the factor 1/4 is introduced for further convenience).

We have proved above that to describe all coordinate systems t, ω_1, ω_2 providing separability of Eq. (11) one has to construct the general solution of system (5). The general solution of equations 1–3 from (5) splits into four inequivalent classes listed in (6).

Analysis shows that only solutions belonging to the first class can satisfy equation 4 from (5). Substituting corresponding formulae for ω_1, ω_2, Q into equation 4 from (5) with $V = \frac{1}{4}(a^2x_1^2 - b^2x_2^2)$ and splitting with respect to x_1, x_2 one gets

$$B_{01}(\omega_1) = \alpha_1\omega_1^2 + \alpha_2\omega_1, \quad B_{02}(\omega_2) = \beta_1\omega_2^2 + \beta_2\omega_2,$$

$$\left(\frac{\dot{A}}{A}\right)^{\cdot} - \left(\frac{\dot{A}}{A}\right)^2 - 4\alpha_1A^4 - a^2 = 0, \tag{12a}$$

$$\left(\frac{\dot B}{B}\right)^{\cdot} - \left(\frac{\dot B}{B}\right)^2 - 4\beta_1 B^4 + b^2 = 0, \tag{12b}$$

$$\ddot\theta_1 - 2\frac{\dot A}{A}\dot\theta_1 - 2(2\alpha_1\theta_1 + \alpha_2)A^4 = 0, \tag{12c}$$

$$\ddot\theta_2 - 2\frac{\dot A}{A}\dot\theta_2 - 2(2\beta_1\theta_2 + \beta_2)B^4 = 0, \tag{12d}$$

Here α_1, α_2, β_1, β_2 are arbitrary real constants.

Integration of the system of nonlinear ODE (12a–d) is carried out in the Appendix. Substitution of the formulae (A.4)–(A.9) into expressions 1 from (5) yields the complete list of coordinate systems providing separability of the Schrödinger equation (11). These systems can be reduced to the canonical form if we use the Note 2. The invariance group of Eq. (11) is generated by the following basis operators [6]:

$$\begin{gathered} P_0 = \partial_t, \quad I = u\partial_u, \quad M = iu\partial_u, \quad P_1 = \operatorname{ch} at\partial_{x_1} + \frac{ia}{2}(x_1 \operatorname{sh} at)u\partial_u, \\ P_2 = \cos bt\partial_{x_2} - \frac{ib}{2}(x_2 \sin bt)u\partial_u, \quad G_1 = \operatorname{sh} at\partial_{x_1} + \frac{ia}{2}(x_1 \operatorname{ch} at)u\partial_u, \\ G_2 = \sin bt\partial_{x_1} + \frac{ib}{2}(x_2 \cos bt)u\partial_u. \end{gathered} \tag{13}$$

Using the finite transformations generated by the infinitesimal operators (13) and the Note 2 we may choose in the formulae (A.4)–(A.9) $C_3 = C_4 = D_1 = 0$, $C_2 = D_2 = 1$, $D_3 = D_4 = 0$. As a result we come to the following assertion.

Theorem. *The Schrödinger equation (11) admits SV in 21 inequivalent coordinate systems of the form*

$$\omega_0 = t, \quad \omega_1 = \omega_1(t, \vec x), \quad \omega_2 = \omega_2(t, \vec x), \tag{14}$$

where ω_1 is given by one of the following formulae:

$$\begin{gathered} x_1(\operatorname{sh} a(t+C))^{-1} + a(\operatorname{sh} a(t+C))^{-2}, \quad x_1(\operatorname{ch} a(t+C))^{-1} + a(\operatorname{ch} a(t+C))^{-2}, \\ x_1 \exp(\pm a(t+C)) + a\exp(\pm 4a(t+C)), \quad x_1(a + \operatorname{sh} 2a(t+C))^{-1/2}, \\ x_1(a + \operatorname{ch} 2a(t+C))^{-1/2}, \quad x_1(a + \exp(\pm 2a(t+C)))^{-1/2}, \quad x_1 \end{gathered} \tag{15}$$

and ω_2 is given by one of the following formulae:

$$x_2(\sin bt)^{-1} + \beta(\sin bt)^{-2}, \quad x_2(\beta + \sin 2bt)^{-1/2}, \quad x_2. \tag{16}$$

In the above formulae C, α, β are arbitrary real parameters.

It is important to note that explicit form of the coordinate systems providing separability of Eq. (11) depends essentially on the parameters a, b contained in the potential $V(x_1, x_2)$. It means that in the free case ($V = 0$) the Schrödinger equation does not admit SV in such coordinate systems. Consequently, they are essentially new.

Appendix. Integration of nonlinear ODE (12a–d).

Evidently, equations (12a–d) can be rewritten in the following unified form:

$$\left(\frac{\dot y}{y}\right)^{\cdot} - \left(\frac{\dot y}{y}\right)^2 - \alpha y^4 = k, \quad \ddot z - \frac{\dot y}{y}\dot z - (\alpha z + \beta)y^4 = 0. \tag{A.1}$$

Provided $k = -a^2 < 0$, system (A.1) coincides with equations (12a,c) and under $k = b^2 > 0$ — with equations (12b,d).

First of all, we note that the function $z = z(t)$ is determined up to addition of an arbitrary constant. Really, the coordinate functions ω_a has the following structure:

$$\omega_a = yx_a + z, \quad a = 1, 2.$$

But the coordinate system t, ω_1, ω_2 is equivalent to the coordinate system t, $\omega_1 + C_1$, $\omega_2 + C_2$, $C_a \in \mathbb{R}^1$. Hence, it follows that the function $z(t)$ is equivalent to the function $z(t) + C$ with arbitrary real constant C. Consequently, provided $\alpha \neq 0$, we may choose in (A.1) $\beta = 0$.

The case 1. $a = 0$. On making the change of variables in (A.1)

$$w = \frac{\dot y}{y}, \quad v = \frac{z}{y} \tag{A.2}$$

we get

$$\dot w = w^2 + k, \quad \ddot v + kv = \beta y^3. \tag{A.3}$$

First, we consider the case $k = -a^2 < 0$. Then the general solutions of the first equation from (A.3) is given by the formulae $w = -a \operatorname{cth} a(t + C_1)$, $w = -a \operatorname{th} a(t + C_1)$, $w = \pm a$, $C_1 \in \mathbb{R}^1$, whence

$$\begin{gathered} y = C_2 \operatorname{sh}^{-1} a(t + C_1), \quad y = C_2 \operatorname{ch}^{-1} a(t + C_1), \\ y = \exp[\pm a(t + C_1)], \quad C_2 \in \mathbb{R}^1. \end{gathered} \tag{A.4}$$

The second equation of system (A.3) is linear inhomogeneous ODE. Its general solution after being substituted into (A.2) yields:

$$\begin{gathered} z = (C_3 \operatorname{ch} at + C_4 \operatorname{sh} at) \operatorname{sh}^{-1} a(t + C_1) + \frac{\beta C_2^4}{2a^2} \operatorname{sh}^{-2} a(t + C_1), \\ z = (C_2 \operatorname{ch} at + C_4 \operatorname{sh} at) \operatorname{ch}^{-1} a(t + C_1) + \frac{\beta C_2^4}{2a^2} \operatorname{ch}^{-2} a(t + C_1), \\ z = (C_3 \operatorname{ch} at + C_4 \operatorname{sh} at) \exp[\pm a(t + C_1)] + \frac{\beta}{8a^2} \exp[\pm 4a(t + C_1)], \\ C_3, C_4 \in \mathbb{R}^1. \end{gathered} \tag{A.5}$$

The case $k = b^2 > 0$ is treated in the analogous way, the general solution of (A.3) being given by the formulae

$$\begin{gathered} y = D_2 \sin^{-1} b(t + D_1), \\ z = (C_3 \cos bt + C_4 \sin bt) \sin^{-1} b(t + D_1) + \frac{\beta D_2^4}{2b^2} \sin^{-2} b(t + D_1), \end{gathered} \tag{A.6}$$

The case 2. $\alpha \neq 0$, $\beta = 0$. On making the change of variables in (A.1)

$$y = \exp w, \quad v = \frac{z}{y}$$

we get

$$\ddot{w} - \dot{w}^2 = k + \alpha \exp 4w, \quad \ddot{v} + kv = 0. \tag{A.1a}$$

The first ODE from (A.1a) is reduced to the first-order linear ODE

$$\frac{1}{2}p'(w) - p(w) = k + \alpha \exp 4w$$

to by the substitution $\dot{w} = (p(w))^{1/2}$, whence

$$p(w) = \alpha \exp 4w + \gamma \exp 2w - k, \quad \gamma \in \mathbb{R}^1.$$

Equation $\dot{w} = p(w)$ has a singular solution $w = C = \text{const}$ such that $p(C) = 0$. If $\dot{w} \neq 0$ then integrating equation $\dot{w} = p(w)$ and returning to the initial variable y, we get

$$\int^{y(t)} \frac{d\tau}{\tau(\alpha\tau^4 + \gamma\tau^2 - k)^{1/2}} = t + C_1.$$

Taking the integral in the left-hand side of the above equality we obtain the general solution of the first ODE from (A.1). It is given by the following formulae:

under $k = -a^2 < 0$

$$\begin{aligned} &y = C_2(\alpha + \operatorname{sh} 2a(t + C_1))^{-1/2}, \quad y = C_2(\alpha + \operatorname{ch} 2a(t + C_1))^{-1/2},\\ &y = C_2(\alpha + \exp[\pm 2a(t + C_1)])^{-1/2}, \end{aligned} \tag{A.7}$$

under $k = b^2 > 0$

$$y = D_2(\alpha + \sin 2b(t + D_1))^{-1/2}. \tag{A.8}$$

Here C_1, C_2, D_1, D_2 are arbitrary real constants.

Integrating the second ODE from (A.1a) and returning to the initial variable z we have

under $k = -a^2 < 0$

$$z = y(t)(C_3 \operatorname{sh} at + C_4 \operatorname{ch} at) \tag{A.9}$$

under $k = b^2 > 0$

$$z = y(t)(D_3 \cos bt + D_4 \sin bt)$$

where C_3, C_4, D_3, D_4 are arbitrary real constants.

Thus, we have constructed the general solution of the system of nonlinear ODE (A.1) which is given by formulae (A.5)–(A.9).

1. Boyer C., Lie theory and separation of variables for the equation $iU_t + U_{x_1x_1} + U_{x_2x_2} - (\alpha x_1^{-2} + \beta x_2^{-2})U = 0$, *SIAM J. Math. Anal.*, 1976, **7**, № 3, 230–263.
2. Miller W., Symmetry and separation of variables, Massachusetts, Addison-Wesley, 1977, 285 p.
3. Boyer C., Kalnins E., Miller W., Lie the theory and separation of variables, 6: The equation $iU_t + U_{x_1x_1} + U_{x_2x_2} = 0$, *J. Math. Phys.*, 1975, **16**, № 3, 499–511.
4. Fushchych W.I., Zhdanov R.Z., Revenko I.V., On the new approach to the variable separation in the wave equation with potential, *Докл. АН Украины*, 1993, № 1, 27–32.
5. Niederer U., The maximal kinematical group of the free Schödinger equation, *Helv. Phys. Acta*, 1972, **45**, № 5, 802–810.
6. Boyer C., The maximal kinematical invariance group for an arbitrary potential, *Helv. Phys. Acta*, 1974, **47**, № 5, 589–605.

Separation of variables in the two-dimensional wave equation with potential

R.Z. ZHDANOV, I.V. REVENKO, W.I. FUSHCHYCH

The paper is devoted to solution of a problem of separation of variables in the wave equation $u_{tt} - u_{xx} + V(x)u = 0$. We give a complete classification of potentials $V(x)$ for which this equation admits a nontrivial separation of variables. Furthermore, we obtain all coordinate systems that provide separability of the equation considered.

Дана стаття присвячена розв'язанню проблеми розділення змінних для хвильового рівняння $u_{tt} - u_{xx} + V(x)u = 0$. Вказані всі потенціали $V(x)$, для яких дане рівняння допускає нетривіальне розділення змінних. Крім того, одержані всі системи координат, в яких розділюється досліджуване рівняння.

1. Introduction. In this paper, we study the two-dimensional wave equation with potential

$$(\Box + V(x))u \equiv u_{tt} - u_{xx} + V(x)u = 0, \tag{1}$$

where $u = u(t,x) \in C^2(\mathbb{R}^2, \mathbb{R}^1)$ and $V(x) \in C(\mathbb{R}^1, \mathbb{R}^1)$, by using the method of separation of variables (SV). Equations belonging to class (1) are widely used in the modern quantum physics and can be related to other linear and nonlinear equations of mathematical physics (these relations will be discussed below, at the end of the article). In particular, class (1) contains the d'Alembert equation (with $V(x) = 0$) and the Klein– Gordon–Fock equation (with $V(x) = m \equiv \text{const}$).

The separation of variables in two- and three-dimensional Laplace, Helmholtz, d'Alembert, and Klein–Gordon–Fock equations had been carried out in the classical works by Bocher [1], Darboux [2], Eisenhart [3], Stepanov [4], Olevsky [5], and Kalnins and Miller (see [6] and references therein). Nevertheless, a complete solution of the problem of SV in equation (1) is not obtained yet.

When speaking about solution of equation (1) with separated variables ω_1, ω_2, we mean the ansatz

$$u(t,x) = A(t,x)\varphi_1(\omega_1(t,x))\varphi_2(\omega_2(t,x)) \tag{2}$$

reducing (1) to two ordinary differential equations for the functions $\varphi_i(\omega_i)$

$$\ddot{\varphi}_i = A_i(\omega_i, \lambda)\dot{\varphi}_i + B_i(\omega_i, \lambda)\varphi_i, \quad i = 1, 2, \tag{3}$$

In formulas (2) and (3), $A, \omega_1, \omega_2 \subset C^2(\mathbb{R}^2, \mathbb{R}^1)$, $A_i, B_i \subset C^2(\mathbb{R}^1 \times \Lambda, \mathbb{R}^1)$ are some unknown functions, $\lambda \in \Lambda \subset \mathbb{R}^1$ is a real parameter (separation constant).

Definition 1. *Equation (1) admits SV in the coordinates $\omega_1(t,x)$, $\omega_2(t,x)$ if the substitution of ansatz (2) into (1) with subsequent exclusion of the second derivatives $\ddot{\varphi}_1$, $\ddot{\varphi}_2$ according to (3) yields an identity with respect to the variables $\dot{\varphi}_i$, φ_i, λ (considered as independent ones).*

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On the basis of the above definition, one can formulate the procedure of SV in equation (1). At the first step; one has to substitute expression (2) into (1) and to express the second derivatives $\ddot{\varphi}_1$, $\ddot{\varphi}_2$ via the functions $\dot{\varphi}_i$, φ_i according to equations (3). At the second step, the obtained equality is splitted with respect to the independent variables $\dot{\varphi}_i$, φ_i. As a result, one gets an overdetermined system of partial differential equations for the functions A, ω_1, ω_2 with undefined coefficients. The general solution of this system gives rise to all systems of coordinates that provide separability of equation (1).

Let us emphasize that the above approach to SV in equation (1) has much in common with the non-Lie method of reduction of nonlinear differential equations suggested in [7–9]. It is also important to note that the idea of representing solutions of linear differential equations in the "separated" form (2) goes as far as to classical works of Euler and Fourier (for a modern exposition of the problem of SV, see Miller [6] and Koornwinder [10]).

The present paper is organized as follows: In the first section, we adduce principal assertions about SV in equation (1). In the second section, the detailed proof of these assertions is given. In the last section, we briefly discuss the obtained results.

2. List of principal results. It is evident that equation (1) admits SV in the Cartesian coordinates $\omega_1 = t$, $\omega_2 = x$ for an arbitrary $V = V(x)$.

Definition 2. *Equation (1) admits a nontrivial SV if there exists at least one coordinate system $\omega_1 = (t,x)$, $\omega_2(t,x)$, different from the Cartesian system, that provides its separability.*

Next, if, in equation (1), one makes the transformations

$$t \to C_1 t, \quad x \to C_1 x, \quad t \to t, \quad x \to x + C_2, \quad C_i \in \mathbb{R}^1,$$

then the class of equations (1) transforms into itself and, moreover,

$$\begin{aligned} &V(x) \to V'(x) = C_1^2 V(C_1 x), \\ &V(x) \to V'(x) = V(x + C_2). \end{aligned} \tag{4}$$

This is why the potentials $V(x)$ and $V'(x)$ connected by one of the above relations are regarded as equivalent ones.

Theorem 1. *Equation (1) admits a nontrivial SV iff the function $V(x)$ is given up to the equivalence relations (4) by one of the result formulas:*

$$\begin{aligned} &(1) \quad V = mx; \\ &(2) \quad V = mx^{-2}; \\ &(3) \quad V = m \sin^{-2} x; \\ &(4) \quad V = m \operatorname{sh}^{-2} x; \\ &(5) \quad V = m \operatorname{ch}^{-2} x; \\ &(6) \quad V = m \exp x; \\ &(7) \quad V = \cos^{-2} x (m_1 + m_2 \sin x); \\ &(8) \quad V = \operatorname{ch}^{-2} x (m_1 + m_2 \operatorname{sh} x); \\ &(9) \quad V = \operatorname{sh}^{-2} x (m_1 + m_2 \operatorname{ch} x); \end{aligned} \tag{5}$$

(10) $V = m_1 \exp x + m_2 \exp 2x$;
(11) $V = m_1 + m_2 x^{-2}$;
(12) $V = m$.

Here, m, m_1, m_2 are arbitrary real parameters, $m_2 \neq 0$.

Note 1. Equation (1) with the potential $V(x) = m \exp x$ is transformed by the change of variables [11]

$$x' = \exp\frac{x}{2}\operatorname{ch} t, \quad t' = \exp\frac{x}{2}\operatorname{sh} t$$

into equation (1) with $V(x) = m$ (i.e., into the Klein–Gordon–Fock equation).

Note 2. Equations (1) with potentials 3, 4, 5 from (5) are transformed into equation (1) with $V(x) = mx^{-2}$ by the changes of variables [11]

$$x' = \operatorname{tg}\xi + \operatorname{tg}\eta, \quad t' = \operatorname{tg}\xi - \operatorname{tg}\eta,$$
$$x' = \operatorname{th}\xi + \operatorname{th}\eta, \quad t' = \operatorname{th}\xi - \operatorname{th}\eta,$$
$$x' = \operatorname{cth}\xi + \operatorname{th}\eta, \quad t' = \operatorname{cth}\xi - \operatorname{th}\eta.$$

Here, $\xi = (x+t)/2$, $\eta = (x-t)/2$ are cone variables.

In virtue of the above remarks, Theorem 1 implies the following assertion:

Theorem 2. *Provided that equation (1) admits a nontrivial SV, it is locally equivalent to one of the following equations:*

$$\begin{aligned}
&(1)\quad \Box u + mxu = 0;\\
&(2)\quad \Box u + mx^{-2}u = 0;\\
&(3)\quad \Box u + \cos^{-2}x(m_1 + m_2\sin x)u = 0;\\
&(4)\quad \Box u + \operatorname{ch}^{-2}x(m_1 + m_2\operatorname{sh} x) = 0;\\
&(5)\quad \Box u + \operatorname{sh}^{-2}x(m_1 + m_2\operatorname{ch} x) = 0;\\
&(6)\quad \Box u + e^x(m_1 + m_2 e^x)u = 0;\\
&(7)\quad \Box u + (m_1 + m_2 x^{-2})u = 0;\\
&(8)\quad \Box u + mu = 0.
\end{aligned} \tag{6}$$

Thus, there exist eight inequivalent types of equations of the form (1) that admit a nontrivial SV.

It is well known that there are eleven coordinate systems that provide separability of the Klein–Gordon–Fock equation $(\Box + m)u = 0$ (see, e.g., [12]). This is why the case $V(x) = \text{const}$ is not considered here.

As is shown in Section 2, the general form of the solution of equations (6) with separated variables is as follows:

$$u(t,x) = \varphi_1(\omega_1(t,x))\varphi_2(\omega_2(t,x)); \tag{7}$$

here, $\varphi_1(\omega_1)$, $\varphi_2(\omega_2)$ are arbitrary solutions of the separated ordinary differential (6) here, equations

$$\ddot{\varphi}_i = (\lambda + g_i(\omega_i))\varphi_i, \quad i = 1, 2, \tag{8}$$

and the explicit form of the systems $\omega_i(t,x)$, $g_i(\omega_i)$ is given below.

Theorem 3. *The equation* $\Box u + mxu = 0$ *separated in two coordinate systems*

$$
\begin{aligned}
&(1)\quad \omega_1 = t, \quad \omega_2 = x, \quad g_1 = 0, \quad g_2 = m\omega_2;\\
&(2)\quad \omega_1 = (x+t)^{1/2} + (x-t)^{1/2}, \quad \omega_2 = (x+t)^{1/2} - (x-t)^{1/2},\\
&\qquad g_1 = -\frac{m}{4} m\omega_1^4, \quad g_2 = -\frac{m}{4}\omega_2^4.
\end{aligned}
\tag{9}
$$

Theorem 4. *The equation* $\Box u + \sin^{-2} x(m_1 + m_2\cos x)u = 0$ *is separated in four coordinate systems*

$$
\begin{aligned}
&(1)\quad \omega_1 = t, \quad \omega_2 = x; \quad g_1 = 0, \quad g_2 = \sin^{-2}\omega_2(m_1 + m_2\cos\omega_2);\\
&(2)\quad \left\{\begin{matrix} x \\ t \end{matrix}\right\} = \operatorname{arctg}\, \operatorname{sh}(\omega_1+\omega_2) \pm \operatorname{arctg}\operatorname{sh}(\omega_1-\omega_2),\\
&\qquad g_1 = (m_1+m_2)\operatorname{sh}^{-2}\omega_1, \quad g_2 = -(m_1-m_2)\operatorname{ch}^{-2}\omega_2;\\
&(3)\quad \left\{\begin{matrix} x \\ t \end{matrix}\right\} = \operatorname{arctg}\operatorname{tn}(\omega_1+\omega_2) \pm \operatorname{arctg}\ \operatorname{tn}\ (\omega_1-\omega_2)\\
&\qquad g_1 = m_1\ \operatorname{dn}^2\omega_1\operatorname{cn}^{-2}\omega_1\operatorname{sn}^{-2}\omega_1 + m_2[\operatorname{sn}^{-2}\omega_1 - \operatorname{dn}^2\omega_1\operatorname{cn}^{-2}\omega_1],\\
&\qquad g_2 = m_1k^4\ \operatorname{sn}^2\omega_2\operatorname{cn}^2\omega_2\operatorname{dn}^{-2}\omega_2 + m_2k^2[\operatorname{cn}^2\omega_2\operatorname{dn}^{-2}\omega_2 - \operatorname{sn}^2\omega_2];\\
&(4)\quad \left\{\begin{matrix} x \\ t \end{matrix}\right\} = \operatorname{arctg}\left(\left(\frac{k}{k'}\right)^{1/2}\operatorname{cn}(\omega_1+\omega_2)\right) \pm \operatorname{arctg}\left(\left(\frac{k}{k'}\right)^{1/2}\operatorname{cn}(\omega_1-\omega_2)\right),\\
&\qquad g_1 = m_1[\operatorname{dn}^2\omega_1\operatorname{cn}^{-2}\omega_1 + k^2\operatorname{sn}^2\omega_1] + m_2[(k')^2\operatorname{cn}^{-2}\omega_1 + k^2\operatorname{cn}^2\omega_1],\\
&\qquad g_2 = m_1[\operatorname{dn}^2\omega_2\operatorname{cn}^{-2}\omega_2 + k^2\operatorname{sn}^2\omega_2] + m_2[(k')^2\operatorname{cn}^{-2}\omega_2 + k^2\operatorname{cn}^2\omega_2].
\end{aligned}
\tag{10}
$$

In formulas (10), $k, k' = \sqrt{1-k^2}$ *are the moduli of the corresponding elliptic Jacobi functions and* k *is an arbitrary constant satisfying the inequality* $0 < k < 1$.

Theorem 5. *The equation* $\Box u + \operatorname{ch}^{-2} x(m_1 + m_2\operatorname{sh} x)u = 0$ *is separated in four coordinate systems*

$$
\begin{aligned}
&(1)\quad \omega_1 = t, \quad \omega_2 = x, \quad g_1 = 0, \quad g_2 = \operatorname{ch}^{-2}\omega_2(m_1 + m_2\operatorname{sh}\omega_2);\\
&(2)\quad \left\{\begin{matrix} x \\ t \end{matrix}\right\} = -\ln\left(\left(\frac{k'}{k}\right)^{1/2}\operatorname{cn}(\omega_1+\omega_2)\right) \mp \ln\left(\left(\frac{k'}{k}\right)^{1/2}\operatorname{cn}(\omega_1-\omega_2)\right),\\
&\qquad g_1 = m_1(k')^2\,(\operatorname{dn}2\omega_1)^{-2} + m_2\operatorname{cn}2\omega_1\,(\operatorname{dn}2\omega_1)^{-2},\\
&\qquad g_2 = m_1(k')^2\,(\operatorname{dn}2\omega_2)^{-2} + m_2\operatorname{cn}2\omega_2\,(\operatorname{dn}2\omega_2)^{-2};\\
&(3)\quad \left\{\begin{matrix} x \\ t \end{matrix}\right\} = -\ln\operatorname{sh}\frac{1}{2}(\omega_1+\omega_2) \pm \ln\operatorname{ch}\frac{1}{2}(\omega_1-\omega_2),\\
&\qquad g_1 = \operatorname{ch}^{-2}\omega_1(m_1 - m_2\operatorname{sh}\omega_1), \quad g_2 = \operatorname{ch}^{-2}\omega_2(m_1 - m_2\operatorname{sh}\omega_2);\\
&(4)\quad \left\{\begin{matrix} x \\ t \end{matrix}\right\} = \ln\operatorname{tn}\frac{1}{2}(\omega_1+\omega_2) \pm \ln\operatorname{dn}\frac{1}{2}(\omega_1+\omega_2).\\
&\qquad g_1 = -m_1k^2\operatorname{sn}^2\omega_1 + k^2m_2\operatorname{sn}\omega_1\operatorname{cn}\omega_1,\\
&\qquad g_2 = -m_1k^2\operatorname{sn}^2\omega_2 + k^2m_2\operatorname{sn}\omega_2\operatorname{cn}\omega_2.
\end{aligned}
\tag{11}
$$

Here, $k, k' = \sqrt{1-k^2}$ *are the moduli of the corresponding elliptic functions,* $0 < k < 1$.

Theorem 6. *The equation* $\Box u + \mathrm{sh}^{-2}x(m_1 + m_2\mathrm{ch}x)u = 0$ *is separated in eleven coordinate systems*

$$
\begin{aligned}
&(1) \quad \omega_1 = t, \quad \omega_2 = x, \quad g_1 = 0, \quad g_2 = \mathrm{sh}^{-2}\omega_2(m_1 + m_2\mathrm{ch}\omega_2);\\
&(2) \quad \left\{ \begin{matrix} x \\ t \end{matrix} \right\} = -\ln\frac{1}{2}(\omega_1+\omega_2) \pm \ln\frac{1}{2}(\omega_1-\omega_2),\\
&\qquad g_1 = (m_1 - m_2)\omega_1^{-2}, \quad g_2 = (m_1+m_2)\omega_2^{-2};\\
&(3) \quad \left\{ \begin{matrix} x \\ t \end{matrix} \right\} = -\ln\sin\frac{1}{2}(\omega_1+\omega_2) \pm \ln\sin\frac{1}{2}(\omega_1-\omega_2),\\
&\qquad g_1 = (m_1 - m_2)\sin^{-2}\omega_1, \quad g_2 = (m_1+m_2)\sin^{-2}\omega_2;\\
&(4) \quad \left\{ \begin{matrix} x \\ t \end{matrix} \right\} = -\ln\mathrm{sh}\frac{1}{2}(\omega_1+\omega_2) \mp \ln\mathrm{sh}\frac{1}{2}(\omega_1-\omega_2),\\
&\qquad g_1 = \mathrm{sh}^{-2}\omega_1(m_1 + m_2\mathrm{ch}\,\omega_1), \quad g_2 = \mathrm{sh}^{-2}\omega_2(m_1 - m_2\mathrm{ch}\,\omega_2);\\
&(5) \quad \left\{ \begin{matrix} x \\ t \end{matrix} \right\} = -\ln\mathrm{ch}\frac{1}{2}(\omega_1+\omega_2) \mp \ln\mathrm{ch}\,\frac{1}{2}(\omega_1-\omega_2),\\
&\qquad g_1 = \mathrm{sh}^{-2}\omega_1(m_1 - m_2\mathrm{ch}\,\omega_1), \quad g_2 = \mathrm{sh}^{-2}\omega_2(m_1 - m_2\mathrm{ch}\,\omega_2);\\
&(6) \quad \left\{ \begin{matrix} x \\ t \end{matrix} \right\} = \ln\mathrm{th}\frac{1}{2}(\omega_1+\omega_2) \pm \ln\mathrm{th}\frac{1}{2}(\omega_1-\omega_2),\\
&\qquad g_1 = \mathrm{ch}^{-2}\omega_1(m_2 - m_1), \quad g_2 = -\mathrm{ch}^{-2}\omega_2(m_2 + m_1);\\
&(7) \quad \left\{ \begin{matrix} x \\ t \end{matrix} \right\} = \ln\mathrm{tg}\frac{1}{2}(\omega_1+\omega_2) \pm \ln\mathrm{tg}\,\frac{1}{2}(\omega_1-\omega_2),\\
&\qquad g_1 = \cos^{-2}\omega_1(m_1 + m_2), \quad g_2 = \cos^{-2}\omega_2(m_1 - m_2);\\
&(8) \quad \left\{ \begin{matrix} x \\ t \end{matrix} \right\} = \mathrm{arth}\,\mathrm{cn}\,(\omega_1+\omega_2) \pm \mathrm{arth}\,\mathrm{cn}\,(\omega_1-\omega_2),\\
&\qquad g_1 = (m_1+m_2)\,\mathrm{dn}^2\omega_1\,\mathrm{cn}^{-2}\omega_1 + (m_1-m_2)k^2\,\mathrm{sn}^2\omega_1,\\
&\qquad g_2 = (m_1-m_2)\,\mathrm{dn}^2\omega_2\,\mathrm{cn}^{-2}\omega_2 + (m_1+m_2)k^2\,\mathrm{sn}^2\omega_2;\\
&(9) \quad \left\{ \begin{matrix} x \\ t \end{matrix} \right\} = \mathrm{arth}\,\mathrm{dn}\,(\omega_1+\omega_2) \pm \mathrm{arth}\,\mathrm{dn}\,(\omega_1-\omega_2),\\
&\qquad g_1 = (m_1+m_2)k^2\,\mathrm{cn}^2\omega_1\,\mathrm{dn}^{-2}\omega_1 + (m-m_2)k^2\,\mathrm{sn}^2\omega_1,\\
&\qquad g_2 = (m_1-m_2)k^2\,\mathrm{cn}^2\omega_2\,\mathrm{cn}^{-2}\omega_2 + (m_1+m_2)k^2\,\mathrm{sn}^2\omega_2;\\
&(10) \quad \left\{ \begin{matrix} x \\ t \end{matrix} \right\} = \mathrm{arth}\,\mathrm{sn}\,(\omega_1+\omega_2) \pm \mathrm{arth}\,\mathrm{sn}\,(\omega_1-\omega_2),\\
&\qquad g_1 = (m_1+m_2)\,\mathrm{sn}^{-2}\omega_1 + (m_1-m_2)k^2\,\mathrm{sn}^2\omega_1,\\
&\qquad g_2 = (m_1+m_2)k^2\,\mathrm{cn}^2\omega_2\,\mathrm{dn}^{-2}\omega_2 + (m_1-m_2)\,\mathrm{dn}^2\omega_2\,\mathrm{cn}^{-2}\omega_2;\\
&(11) \quad \left\{ \begin{matrix} x \\ t \end{matrix} \right\} = -\ln\,\mathrm{cn}\,(\omega_1+\omega_2) \pm \ln\,\mathrm{cn}\,(\omega_1-\omega_2),\\
&\qquad g_1 = -m_1\,\mathrm{sn}^{-2}\omega_1 - m_2\,\mathrm{cn}\,\omega_1\,\mathrm{sn}^{-2}\omega_1,\\
&\qquad g_2 = -m_1\,\mathrm{sn}^{-2}\omega_2 - m_2\,\mathrm{cn}\,\omega_2\,\mathrm{sn}^{-2}\omega_2.
\end{aligned}
\tag{12}
$$

Here, k *is the modulus of the corresponding elliptic functions,* $0 < k < 1$.

Theorem 7. *The equation* $\Box u + e^x(m_1 + m_2e^x)u = 0$ *is separated in six coordinate systems*

$$(1) \quad \omega_1 = t, \quad \omega_2 = x, \quad g_1 = 0, \quad g_2 = e^{\omega_2}(m_1 + m_2e^{\omega_2});$$

$$
\begin{aligned}
&(2)\quad \begin{Bmatrix} x \\ t \end{Bmatrix} = -\ln\cos(\omega_1+\omega_2) \mp \ln\cos(\omega_1-\omega_2),\\
&g_1 = -2m_1\cos 2\omega_1 - \frac{m_2}{2}\cos 4\omega_1,\\
&g_2 = -2m_1\cos 2\omega_2 - \frac{m_2}{2}\cos 4\omega_2;\\
&(3)\quad \begin{Bmatrix} x \\ t \end{Bmatrix} = \ln\mathrm{sh}\,(\omega_1+\omega_2) \pm \ln\mathrm{sh}\,(\omega_1-\omega_2),\\
&g_1 = -2m_1\mathrm{ch}2\omega_1 - \frac{m_2}{2}\mathrm{ch}\,4\omega_1,\\
&g_2 = -2m_1\mathrm{ch}\,2\omega_2 - \frac{m_2}{2}\mathrm{ch}\,4\omega_2;\\
&(4)\quad \begin{Bmatrix} x \\ t \end{Bmatrix} = \ln\mathrm{ch}(\omega_1+\omega_2) \pm \ln\mathrm{ch}(\omega_1-\omega_2),\\
&g_1 = -2m_1\mathrm{ch}\,2\omega_1 - \frac{m_2}{2}\mathrm{ch}\,4\omega_1,\\
&g_2 = -2m_1\mathrm{ch}2\omega_2 - \frac{m_2}{2}\mathrm{ch}\,4\omega_2;\\
&(5)\quad \begin{Bmatrix} x \\ t \end{Bmatrix} = \ln\mathrm{ch}(\omega_1+\omega_2) \pm \ln\mathrm{sh}(\omega_1-\omega_2),\\
&g_1 = -2m_1\mathrm{sh}\,2\omega_1 - \frac{m_2}{2}\mathrm{ch}\,4\omega_1,\\
&g_2 = -2m_1\mathrm{sh}\,2\omega_2 - \frac{m_2}{2}\mathrm{ch}\,4\omega_2;\\
&(6)\quad \begin{Bmatrix} x \\ t \end{Bmatrix} = \ln(\omega_1+\omega_2) \pm \ln(\omega_1-\omega_2),\\
&g_1 = 2m_1 + 2m_2\omega_1^2, \quad g_2 = 2m_1 + 2m_2\omega_2^2.
\end{aligned}
\tag{13}
$$

Theorem 8. *The equation $\Box u + (m_1 + m_2 x^{-2})u = 0$ separated in six coordinate systems*

$$
\begin{aligned}
&(1)\quad \omega_1 = t, \quad \omega_2 = x, \quad g_1 = 0, \quad g_2 = m_1 + m_2\omega_2^{-2};\\
&(2)\quad \begin{Bmatrix} x \\ t \end{Bmatrix} = \exp(\omega_1+\omega_2) \pm \exp(\omega_1-\omega_2),\\
&g_1 = 4m_1\exp 2\omega_1, \quad g_2 = m_2\mathrm{ch}^{-2}\omega_2;\\
&(3)\quad \begin{Bmatrix} x \\ t \end{Bmatrix} = \sin(\omega_1+\omega_2) \pm \sin(\omega_1-\omega_2),\\
&g_1 = 2m_1\cos 2\omega_1 + m_2\sin^{-2}\omega_1, \qquad g_2 = -2m_1\cos 2\omega_2 + m_2\cos^{-2}\omega_2;\\
&(4)\quad \begin{Bmatrix} x \\ t \end{Bmatrix} = \mathrm{sh}(\omega_1+\omega_2) \pm \mathrm{sh}(\omega_1-\omega_2),\\
&g_1 = 2m_1\mathrm{sh}2\omega_1 + m_2\mathrm{sh}^{-2}\omega_1,\\
&g_2 = -2m_1\mathrm{sh}2\omega_2 - m_2\mathrm{sh}^{-2}\omega_2;\\
&(5)\quad \begin{Bmatrix} x \\ t \end{Bmatrix} = \mathrm{ch}(\omega_1+\omega_2) \pm \mathrm{ch}(\omega_1-\omega_2),\\
&g_1 = 2m_1\mathrm{ch}2\omega_1 - m_2\mathrm{ch}^{-2}\omega_1, \quad g_2 = 2m_1\mathrm{ch}2\omega_2 - m_2\mathrm{ch}^{-2}\omega_2;\\
&(6)\quad \begin{Bmatrix} x \\ t \end{Bmatrix} = (\omega_1+\omega_2)^2 \pm (\omega_1-\omega_2)^2,\\
&g_1 = -16m_1\omega_1^2 + m_2\omega_1^{-2}, \quad g_2 = -16m_1\omega_2^2 + m_2\omega_2^{-2}.
\end{aligned}
\tag{14}
$$

It was established in [13] that the Euler–Poisson–Darboux equation

$$V_{tt} - V_{xx} - x^{-1}V_x + m^2x^{-2}V = 0$$

is separated in nine coordinate systems. Since the above equation is reduced to the equation $u_{tt}-u_{xx}+(m^2-1/4)x^{-2}u=0$ by the change of dependent variable $\nu(t,x) = x^{-1/2}u(t,x)$, equation (1) with $V(x) = \lambda x^{-2}$ is also separated in nine coordinate systems.

It has been understood not long ago [6, 14] that SV is intimately connected with the symmetry properties of the equation under the study. Therefore, it is important to investigate the symmetry of equation (1).

Clearly, equation (1) with an arbitrary $V(x)$ is invariant under the two-dimensional Lie algebra that has the basis elements $Q_1 = \partial_t$, $Q_2 = u\partial_u$. Below, we adduce without a proof the assertion which gives a complete description of the potentials $V(x)$ that provide an extension of the symmetry algebra admitted by equation (1).

Theorem 9. *Equation (1) admits additional symmetry operators (i.e., operators not belonging to the algebra $\langle\partial_t, u\partial_u\rangle$) iff the potential $V(x)$ is given by one of the following formulas:*

(1) $V(x) = m\exp x$;

(2) $V(x) = mx^{-2}$;

(3) $V(x) = m\sin^{-2} x$;

(4) $V(x) = m\,\mathrm{sh}^{-2} x$;

(5) $V(x) = m\,\mathrm{ch}^{-2} x$;

(6) $V(x) = m, \quad m\in\mathbb{R}^1$,

with the additional symmetry operators having the form

(1) $Q_3 = \exp\left\{\frac{1}{2}(t-x)\right\}(\partial_x - \partial_t), \quad Q_4 = \exp\left\{-\frac{1}{2}(x+t)\right\}(\partial_x + \partial_t)$;

(2) $Q_3 = x\partial_x + t\partial_t, \quad Q_4 = (x^2+t^2)\partial_t + 2tx\partial_x$;

(3) $Q_3 = \sin t\cos x\partial_t + \sin x\cos t\partial_x, \quad Q_4 = -\cos t\cos x\partial_t + \sin x\sin t\partial_x$;

(4) $Q_3 = \mathrm{sh}\, t\,\mathrm{ch}\, x\partial_t + \mathrm{sh}\, x\,\mathrm{ch}\, t\partial_x, \quad Q_4 = \mathrm{ch}\, t\,\mathrm{ch}\, x\partial_t + \mathrm{sh}\, t\,\mathrm{sh}\, x\partial_x$;

(5) $Q_3 = \mathrm{sh}\, x\,\mathrm{ch}\, t\partial_t + \mathrm{sh}\, t\,\mathrm{ch}\, x\partial_x, \quad Q_4 = \mathrm{sh}\, t\,\mathrm{sh}\, x\partial_t + \mathrm{ch}\, t\,\mathrm{ch}\, x\partial_x$;

(6) $Q_3 = \partial_x, \quad Q_4 = t\partial_x + x\partial_t$.

This theorem is proved by the standard Lie method (see, e.g., [15, 16]).

Corollary. *If equation (1) admits additional symmetry operators, then it is locally equivalent to one of the equations $\Box u + mu = 0$ or $\Box u + mx^{-2}u = 0$.*

Thus, separability of equations 1, 3–7 from (6) is not connected with their Lie symmetry. To explain this fact one has to take into account the second-order (non-Lie) symmetry operators of equation (1). This problem will be briefly discussed in the last section.

3. Proof of Theorems 1–8. To prove the assertions listed in the previous section one has to apply the above described procedure of SV to equation (1).

By substituting ansatz (2) into equation (1), expressing the functions $\ddot\varphi_i$ in terms of the functions $\dot\varphi_i$, φ_i, with the help of equalities (3), and splitting the obtained equation with respect to independent variables $\dot{\varphi}_i$, φ_i, we get the following system of nonlinear partial differential equations:

$$1)\quad A\Box\omega_1 + 2(A_t\omega_{1t} - A_x\omega_{1x}) + AA_1(\omega_1,\lambda)(\omega_{1t}^2 - \omega_{1x}^2) = 0, \tag{15}$$

$$2)\quad A\Box\omega_2 + 2(A_t\omega_{2t} - A_x\omega_{2x}) + AA_2(\omega_2,\lambda)(\omega_{2t}^2 - \omega_{2x}^2) = 0, \tag{16}$$

$$3)\quad \Box A + A[B_1(\omega_1,\lambda)(\omega_{1t}^2 - \omega_{1x}^2) + B_2(\omega^2,\lambda)(\omega_{2t}^2 - \omega_{2x}^2) + AV(x) = 0, \tag{17}$$

$$4)\quad \omega_{1t}\omega_{2t} - \omega_{1x}\omega_{2x} = 0. \tag{18}$$

Here, $\Box \equiv \partial_t^2 - \partial_x^2$.

Thus, to separate variables in the linear differential equation (1) one has to construct a general solution of system of nonlinear partial differential equations (15)–(18). The same assertion holds true for a general linear differential equation, i.e., the problem of SV is essentially nonlinear. This is the reason why, even for the classical d'Alembert equation $\Box_4 u \equiv u_{tt} - \Delta_3 u = 0$, there is no complete description of all coordinate systems that provide its separability [6].

It is not difficult to become convinced of that from (18). Since the functions ω_1, ω_2 are real, we have

$$(\omega_{1t}^2 - \omega_{1x}^2)(\omega_{2t}^2 - \omega_{2x}^2) \neq 0. \tag{19}$$

Differentiating equations (15), (16) with respect to λ and using (19), we get $A_{1\lambda} = A_{2\lambda} = 0$.

Consequently, the relation $B_{1\lambda}B_{2\lambda} \neq 0$ holds. Differentiating with respect to λ we have

$$B_{1\lambda}(\omega_{1t}^2 - \omega_{1x}^2) + B_{2\lambda}(\omega_{2t}^2 - \omega_{2x}^2) = 0$$

or $B_{1\lambda}/B_{2\lambda} = -(\omega_{2t}^2 - \omega_{2x}^2)/(\omega_{1t}^2 - \omega_{1x}^2)$. Differentiation of the above equality with respect to λ yields $B_{1\lambda\lambda}/B_{1\lambda} = B_{2\lambda\lambda}/B_{2\lambda}$. But the functions $B_1 = B_1(\omega_1)$, $B_2 = B_2(\omega_2)$ are independent, whence it follows that there exists a function such that $B_{i\lambda\lambda} = K(\lambda)B_{i\lambda}$, $i = 1,2$.

Integrating the above differential equation with respect to λ, we get

$$B_i(\omega_i) = \Lambda(\lambda)f_i(\omega_i) + g_i(\omega_i), \quad i = 1,2,$$

where f_i, g_i are arbitrary smooth functions.

On redefining the parameter $\lambda \to \Lambda(\lambda)$, we have

$$B_i(\omega_i) = \Lambda f_i(\omega_i) + g_i(\omega_i), \quad i = 1,2, \tag{20}$$

Substitution of (20) into (17) with a subsequent splitting with respect to λ yields the following equations:

$$\Box A + A[g_1(\omega_1)(\omega_{1t}^2 - \omega_{1x}^2) + g_2(\omega_2)(\omega_{2t}^2 - \omega_{2x}^2) + V(x)A = 0, \tag{21}$$

$$f_1(\omega_1)(\omega_{1t}^2 - \omega_{1x}^2) + f_2(\omega_2)(\omega_{2t}^2 - \omega_{2x}^2) = 0. \tag{22}$$

Thus, system (15)–(18) is equivalent to the system of equations (15), (16), (20)–(22). Before integrating it, we make a remark. It is evident that the structure of ansatz (2) is not changed by the transformation

$$\begin{aligned} &A \to A' = Ah_1(\omega_1)h_2(\omega_2), \\ &\omega_i \to \omega_i' = R_i(\omega_i), \quad i = 1, 2, \end{aligned} \tag{23}$$

where h_i, R_i are some smooth functions.

This is why solutions of the system under the study, connected by relations (23), are considered as equivalent ones.

By a proper choice of the functions h_i, we can put R_i, $f_1 = f_2 = 1$ and $A_1 = A_2 = 0$ in equations (15), (16), (22).

Consequently, the functions ω_1, ω_2 satisfy equations of the form

$$\omega_{1t}\omega_{2t} - \omega_{1x}\omega_{2x} = 0, \quad \omega_{1t}^2 - \omega_{1x}^2 + \omega_{2t}^2 - \omega_{2x}^2 = 0,$$

whence $(\omega_1 \pm \omega_2)_t^2 - (\omega_1 \pm \omega_2)_x^2 = 0$. Integrating the above equations, we get

$$\omega_1 = f(\xi) + g(\eta), \quad \omega_2 = f(\xi) - g(\eta), \tag{24}$$

where $f, g \subset C^2(\mathbb{R}^1, \mathbb{R}^1)$ are arbitrary functions, $\xi = (x + t)/2$, $\eta = (x - t)/2$.

Substitution of (24) into equations (15), (16) with $A_1 = A_2 = 0$ yields the following equations for a function $A = A(t, x)$: $(\ln A)_t = 0$, $(\ln A)_x = 0$, whence $A = 1$.

At last, substituting the obtained results into equation (21), we have

$$V(x) = [g_1(f + g) - g_2(f - g)]\frac{df}{d\xi}\frac{dg}{d\eta}. \tag{25}$$

Thus, the problem of integration of the overdetermined system of nonlinear differential equations (15)–(18) is reduced to the integration of the functional-differential equation (25).

Let us sum up the obtained results. The general form of the solution of equation (1) with separated variables is as follows:

$$u_1 = \varphi_1(f(\xi) + g(\eta))\varphi_2(f(\xi) - g(\eta)); \tag{26}$$

here, φ_i are arbitrary solutions of equations (8) and the functions $f(\xi)$, $g(\eta)$, $g_1(f+g)$, $g_2(f - g)$, $V(x)$ are determined by (25).

To integrate equation (25) we make the hodograph transformation

$$\xi = P(f), \quad \eta = R(g), \tag{27}$$

where $\dot{P} \not\equiv 0$, $\dot{R} \not\equiv 0$.

After making transformation (27), we get

$$g_1(f + g) - g_2(f - g) = \dot{P}(f)\dot{R}(g)V(P + R). \tag{28}$$

Evidently, equation (28) is equivalent to the equation

$$(\partial_f^2 - \partial_g^2)[\dot{P}(f)\dot{R}(g)V(P + R)] = 0$$

or

$$(\dddot{P}\dot{P}^{-1} - \dddot{R}\dot{R}^{-1})V + 3(\ddot{P} - \ddot{R})\dot{V} + (\dot{P}^2 - \dot{R}^2)\ddot{V} = 0. \tag{29}$$

Thus, to integrate equation (25) it suffices to construct all functions $P(f)$, $R(g)$, $V(P+R)$ satisfying (29) and substitute them into equation (28).

Let us prove the following assertion.

Lemma. *The general solution of equation (29), determined up to transformations (4), is given by the one of the following formulas:*

(1) $V = V(x)$ *is an arbitrary function,* $\dot{P} = \alpha$, $\dot{R} = \alpha$;

(2) $V = mx$, $\dot{P}^2 = \alpha P + \beta$, $\dot{R}^2 = \alpha R + \gamma$; (30)

(3) $V = mx^{-2}$, $P = F(f)$, $R = G(g)$,

$$\dot{F}^2 = \alpha F^4 + \beta F^3 + \gamma F^2 + \delta F + \rho, \tag{31}$$

$$\dot{G}^2 = \alpha G^4 - \beta G^3 + \gamma G^2 - \delta G + \rho;$$

(4) $V = m\sin^{-2} x$, $P = \operatorname{arctg} F(f)$, $R = \operatorname{arctg} G(g)$,
and F, G *are determined by (31)*;

(5) $V = m\,\mathrm{sh}^{-2} x$, $P = \operatorname{arth} F(f)$, $R = \operatorname{arth} G(g)$
and F, G *are determined by (31)*;

(6) $V = m\,\mathrm{ch}^{-2} x$, $P = \operatorname{arcth} F(f)$, $R = \operatorname{arcth} G(g)$
and F, G *are determined by (31)*;

(7) $V = m\exp x$,
$\dot{P}^2 = \alpha\exp 2P + \beta\exp P + \gamma$, $\dot{R}^2 = \alpha\exp 2R + \delta\exp R + \rho$;

(8) $V = \cos^{-2} x(m_1 + m_2\sin x)$,
$\dot{P}^2 = \alpha\sin 2P + \beta\cos 2P + \gamma$, $\dot{R}^2 = \alpha\sin 2R + \beta\cos 2R + \gamma$; (32)

(9) $V = \mathrm{ch}^{-2} x(m_1 + m_2\mathrm{sh}\, x)$,
$\dot{P}^2 = \alpha\,\mathrm{sh}\,2P + \beta\mathrm{ch}\,2P + \gamma$, $\dot{R}^2 = \alpha\,\mathrm{sh}\,2R - \beta\mathrm{ch}\,2R + \gamma$; (33)

(10) $V = \mathrm{sh}^{-2} x(m_1 + m_2\mathrm{ch} x)$,
$\dot{P}^2 = \alpha\,\mathrm{sh}\,2P + \beta\mathrm{ch}\,2P + \gamma$, $\dot{R}^2 = -\alpha\,\mathrm{sh}\,2R + \beta\mathrm{ch}\,2R + \gamma$; (34)

(11) $V = (m_1 + m_2\exp x)\exp x$,
$\ddot{P} = -\dot{P}^2 + \beta$, $\ddot{R} = -\dot{P}^2 + \beta$; (35)

(12) $V = m_1 + m_2 x^{-2}$,
$\dot{P}^2 = \alpha P^2 + \beta P + \gamma$, $\ddot{R}^2 = \alpha R^2 - \beta R + \gamma$, (36)

(13) $V = m$,
$\dot{P}^2 = \alpha P^2 + \beta P + \gamma$, $\dot{R}^2 = \alpha R^2 + \delta R + \rho$.

Here α, β, γ, δ, ρ, m_1, m_2, m *are arbitrary real parameters;* $x = \xi + \eta = P + R$.

Proof. Since the functions P, R in (29) are arbitrary, equation (29) is equivalent to the following system of equations:

$$(H_{ffff}H_f^{-1} - H_{ggg}H_g^{-1})V(H) + 3(H_{ff} - H_{gg})\dot{V}(H) + (H_f^2 - H_g^2)\ddot{V}(H) = 0, \tag{37}$$

$$H_{fg} = 0; \tag{38}$$

here, $H = P(f) + R(g)$.

Taking differential consequences of equation (37), we have

$$\begin{aligned} H_{ffff} &= H_{fff}H_{ff}H_f^{-1} + \dot{V}V^{-1}(H_{ggg}H_g^{-1}H_f^2 - 4H_{fff}H_f) + \\ &\quad + \ddot{V}V^{-1}(3H_{gg}H_f^2 - 5H_{ff}H_f^2) + \dddot{V}V^{-1}(H_g^2H_f^2 - H_f^4), \\ H_{gggg} &= H_{ggg}H_{gg}H_g^{-1} + \dot{V}V^{-1}(H_{fff}H_f^{-1}H_g^2 - 4H_{ggg}H_g) + \\ &\quad + \ddot{V}V^{-1}(3H_{ff}H_g^2 - 5H_{gg}H_g^2) + \dddot{V}V^{-1}(H_f^2H_g^2 - H_g^4). \end{aligned} \tag{39}$$

For system (39) to be compatible, it is necessary that relations $H_{ffffg} = H_{ggggf} = 0$ hold. Differentiating the first equation in (39) with respect to g and taking into account relations (39), we get

$$\begin{aligned} &(H_{fff}H_f^{-1} - H_{ggg}H_g^{-1})(5\dot{V}^2V^{-2} - 4\ddot{V}V^{-1}) + (H_{ff} - H_{gg}) \times \\ &\quad \times (8\ddot{V}\dot{V}V^{-2} - 5\dddot{V}V^{-1}) + (H_f^2 - H_g^2)(\dddot{V}\dot{V}V^{-2} - (\dddot{V}V^{-1})^{\cdot}) = 0. \end{aligned} \tag{40}$$

Since equation (40) is a necessary compatibility condition for a system (39), one has to supplement the system under study (equations (37), (38)) by equation (40). To investigate the system of equations (37), (38), (40) it is necessary to consider several inequivalent cases.

Case 1. Let $\ddot{V} = 0$, $\dot{V} \neq 0$. Then equalities $H_{ff} = H_{gg} = 2\alpha$, $\alpha = \text{const}$ hold. Hence, we have

$$\begin{aligned} &V = m(H + C) \equiv m(x + C), \\ &P(f) = \alpha f^2 + \beta, \quad R(g) = \alpha g^2 + \gamma, \quad \beta, \gamma \subset \mathbb{R}^1, \end{aligned}$$

i.e., we obtain the potential listed in the lemma under number 2.

Case 2. Let $\ddot{V} \neq 0$ and let equation (40) be a consequence of equation (37). In this case, the coefficients of V, $\dot{V}$, $\ddot{V}$ must be proportional

$$\begin{aligned} &(5\dot{V}^2V^{-2} - 4\ddot{V}V^{-1}) = (8\ddot{V}\dot{V}V^{-2} - 5\dddot{V}V^{-1})(3V)^{-1} = \\ &\quad = (2\dddot{V}\dot{V}V^{-2} - \ddddot{V}V^{-1})(\ddot{V})^{-1}. \end{aligned}$$

From the above equalities, we get a system of two ordinary differential equations for a function $V = V(H)$

$$\dddot{V} = 4\dot{V}V^{-1} - 3\dot{V}^3V^{-2}, \tag{41}$$

$$\ddddot{V} = 2\dddot{V}\dot{V}V^{-1} - 4\ddot{V}^2V^{-1} + 5\dot{V}^2\ddot{V}V^{-2}. \tag{42}$$

But equation (42) is the differential consequence of equation (41). The general solution of equation (41), determined up to equivalence relations (4), is given by one of the following formulas [17]:

$$\begin{aligned} &V_1 = mH^{-2}, \quad V_2 = m\sin^{-2}H, \\ &V_3 = m\,\mathrm{sh}^{-2}H, \quad V_4 = m\,\mathrm{ch}^{-2}H, \quad V_5 = m\exp H; \end{aligned} \tag{43}$$

i.e., we obtain potentials listed in the lemma under numbers 3–7.

By substituting $V = V_1 = mH^{-2}$ into (37) and replacing H by $P(f) + R(g)$, we get

$$(P+R)^2(\dddot{P}\dot{P}^{-1} - \dddot{R}\dot{R}^{-1}) - 6(P+R)(\ddot{P} - \ddot{R}) + 6(\dot{P}^2 - \dot{R}^2) = 0. \tag{44}$$

By differentiating (44) with respect to f and g, we obtain

$$(P+R)(\dot{h}_1\dot{P}^{-1} - \dot{h}_2\dot{R}^{-1}) = 2(h_1 - h_2),$$

where $h_1 = \dddot{P}\dot{P}^{-1}$ and $h_2 = \dddot{R}\dot{R}^{-1}$.

Differentiation of the above equation with respect to f and g yields the following relation:

$$(\dot{h}_1\dot{P}^{-1})^{\cdot}\dot{P}^{-1} = (\dot{h}_2\dot{R}^{-1})^{\cdot}\dot{R}^{-1}. \tag{45}$$

Since the functions $P(f)$, $R(g)$ are independent, it follows from (45) that the equalities

$$(\dot{h}_1\dot{P}^{-1})^{\cdot} = 12\alpha\dot{P}, \quad (\dot{h}_2\dot{R}^{-1})^{\cdot} = 12\alpha\dot{R}. \tag{46}$$

hold, where α is an arbitrary real parameter.

Integration of equations (46) yields

$$\dot{P} = \alpha P^4 + C_1P^3 + C_2P^2 + C_3P + C_4,$$
$$\dot{R}^2 = \alpha R^4 + D_1R^3 + D_2R^2 + D_3R + D_4,$$

where $C_1, \ldots, C_4$, $D_1, \ldots, D_4$ are arbitrary real constants. Substituting the above result into the initial equation (44), we get restrictions on the choice of arbitrary constants

$$C_1 = -D_1 = \beta, \quad C_2 = D_2 = \gamma, \quad C_3 = -D_3 = \delta, \quad C_4 = D_2 = \rho.$$

Thus, we have obtained the potential listed in the lemma under number 3.

It is straightforward to verify that the equations obtained by the substitution of functions $V = m\sin^{-2}H$, $V = m\,\mathrm{ch}^{-2}$ with $H = P(f) + R(g)$ into (37) are reduced to equation (44) by the following changes of variables:

$$P \to \operatorname{arctg} P, \quad R \to \operatorname{arctg} R,$$
$$P \to \operatorname{arth} P, \quad R \to \operatorname{arth} R,$$
$$P \to \operatorname{arcth} P, \quad R \to \operatorname{arcth} R;$$

i.e., the potentials listed in the lemma under numbers 4–6 are obtained.

Equation (1) with the potential $V = m\exp H$ is reduced to the Klein–Gordon–Fock equation (see case 4 and Note 2 below).

Case 3. Let $\ddot{V} \neq 0$ and assume, in addition, that equation (41) does not hold. In this case, we can exclude from equations (37), (40) the third derivatives of the function H

$$H_{ff} - H_{gg} + A(H)(H_f^2 - H_g^2) = 0, \tag{47a}$$

where

$$A(H) = (\ddddot{V} - 2\dddot{V}\dot{V}V^{-1} - 4\ddot{V}^2V + 5\ddot{V}\dot{V}^2V^{-2})(\dddot{V} - 4\ddot{V}\dot{V}V^{-1} + 3\dot{V}^3V^{-2})^{-1}.$$

It follows from (47a)

$$H_{fff} = \dot{A}H_f(H_g^2 - H_f^2) - 2H_{ff}H_fA, \quad H_{ggg} = \dot{A}H_g(H_f^2 - H_g^2) - 2H_{gg}H_gA,$$

(we have used equation (38)).

By taking the first differential consequence of the above equations with account of equation (38), we get

$$2\dot{A}(H_{ff} - H_{gg}) + \ddot{A}(H_f^2 - H_g^2) = 0. \tag{48}$$

Clearly, equations (47a) and (48) are consistent iff the function $A(H)$ satisfies the following ordinary differential equation:

$$\ddot{A} = 2\dot{A}A,$$

the general solution of which is given by one of the formulas (up to scaling $H \to CH$).

$$A = C, \quad A = \mathrm{tg}\,(H + C), \quad A = -\mathrm{th}\,(H + C),$$
$$A = -\mathrm{cth}\,(H + C), \quad A = -(H + C)^{-1}, \quad C \in \mathbb{R}^1.$$

Next, we consider the above cases separately.

Case 3.1. $A(H) = C$, $C \neq 0$. In this case, equation (47a) takes the form

$$P_{ff} - R_{gg} + C(P_f^2 - R_g^2) = 0 \tag{47b}$$

or

$$P_{ff} + CP_f^2 = R_{gg} + CR_g^2 = \beta, \quad \beta \in \mathbb{R}^1.$$

Finally, we get

$$P_{ff} = -CP_f^2 + \beta, \quad R_{gg} = -CR_g^2 + \beta. \tag{49}$$

Differentiating the first equation with respect to f, the second equation with respect to g, and subtracting, we get

$$P_{fff}P_f^{-1} - P_{ggg}P_g^{-1} = -2C(P_{ff} - R_{gg}). \tag{50}$$

Substituting (49), (50) into equation (37), we come to the equation for $V = V(H)$,

$$\ddot{V} - 3C\dot{V} + 2C^2V = 0$$

the general solution of which reads

$$V = m_1 \exp CH + m_2 \exp 2CH, \quad m_2, m_2 \subset \mathbb{R}^1. \tag{51}$$

It is not difficult to check that function (51) satisfies equation (47b) provided that $A(H) = C$. Consequently, if the potential is given by formula (51) (after rescaling $H \to CH$, we can choose $C = 1$), then the functions $P(f)$, $R(g)$ are determined by equations (35).

Case 3.2. $A = \mathrm{tg}(H + C)$. Multiplying equation (47) by $\mathrm{ctg}\,(H + C)$ and differentiating the obtained expression with respect to f and g, we arrive at the equation

$$(P_{fff}P_f^{-1} - P_{ggg}P_g^{-1}) - 2\mathrm{ctg}\,(H + C)(P_{ff} - R_{gg}) = 0. \tag{52}$$

After excluding the function $\operatorname{ctg}(H+C)$ from (47) and (52), we get an equation with separated variables

$$(P_{fff}P_f^{-1} - P_{ggg}P_g^{-1}) + 2(P_f^2 - R_g^2) = 0,$$

whence

$$P_{fff}P_f^{-1} + 2P_f^2 = \theta, \quad R_{ggg}R_g^{-1} + 2R_g^2 = \theta. \tag{53}$$

In (53), θ is an arbitrary real constant.

Substitution of formulas (52), (53) into equation (37) gives the equation for $V = V(H)$,

$$\ddot{V} - 3\operatorname{tg}(H+C)\dot{V} - 2V = 0$$

the general solution of which has the form [17]

$$V = \cos^{-2}(H+C)[m_1 + m_2\sin(H+C)]. \tag{54}$$

As a direct check shows, the function $V(H)$ (54) satisfies equation (47b) with $A = \operatorname{tg}(H+C)$.

Integrating equations (53), we get

$$P_f^2 = C_1\sin 2P + C_2\cos 2P + \gamma, \quad P_g^2 = D_1\sin 2R + D_2\cos 2R + \gamma, \tag{55}$$

where C_i, D_i, and γ are arbitrary real constants.

Substitution of (55) into (47) with $A = \operatorname{tg}(H+C)$ yields the following restrictions on the choice of the constants C_i, D_i: $C_1 = D_1 = \alpha$, $C_2 = D_2 = \beta$.

Thus, provided that the function $V(H)$ is given by (44), the functions $P(f)$, $R(g)$ are determined by equations (32).

Case 3.3. $A = -\operatorname{th}(H+C)$. In this case, one can obtain the following differential consequence of equation (47):

$$P_{fff}P_f^{-1} - P_{ggg}P_g^{-1} = 2\operatorname{cth}(P+R+C)(P_{ff} - R_{gg}). \tag{56}$$

Excluding the function $\operatorname{ctg}(H+C)$ from equations (47), (56), we get the equation

$$P_{fff}P_f^{-1} - P_{ggg}P_g^{-1} = 2(P_f^2 - R_g^2),$$

whence

$$P_{fff}P_f^{-1} - 2P_f^2 = \theta, \quad R_{ggg}R_g^{-1} - 2R_g^2 = \theta. \tag{57}$$

In (57), θ is an arbitrary real constant.

Integration of equations (57) gives

$$P_f^2 = C_1\operatorname{sh}2P + C_2\operatorname{ch}2P + \gamma, \quad R_g^2 = D_1\operatorname{sh}2R + D_2\operatorname{ch}2R + \gamma, \tag{58}$$

where C_i, D_i, and γ are arbitrary real constants.

Substituting expressions (56), (57) into (37), we obtain an equation for $V(H)$,

$$\ddot{V} + 3\operatorname{th}(H+C)\dot{V} + 2V = 0,$$

the general solution of which has the form [17]

$$V = \operatorname{ch}^{-2}(H+C)(m_1 + m_2\operatorname{sh}(H+C)), \quad m_i \in \mathbb{R}^1. \tag{59}$$

It is not difficult to become convinced of the fact that function (59) satisfies equation (47b) with $A = -\mathrm{th}\,(H+C)$.

At last, substituting (57) and (58) into (47), we get $C_1 = D_1 = \alpha$, $C_2 = -D_2 = \beta$. Consequently, if the potential $V(H)$ is given by formula (59), then functions $P(f)$ and $R(g)$ are determined by equations (33).

Case 3.4. $A = -\mathrm{cth}\,(H+C)$. In this case, one can obtain the following differential consequence of equation (47):

$$P_{fff}P_f^{-1} - R_{ggg}R_g^{-1} = 2\,\mathrm{th}\,(P+R+C)(P_{ff} - R_{gg}). \tag{60}$$

Using equations (37), (47), and (60), we get an equation for $V(H)$,

$$\ddot{V} + 3\,\mathrm{cth}\,(H+C)\dot{V} + 2V = 0,$$

the general solution of which has the form [17]

$$V = \mathrm{sh}^{-2}(H+C)(m_1 + m_2\,\mathrm{ch}(H+C)), \quad m_i \in \mathbb{R}^1. \tag{61}$$

By direct computation, one can check that function (61) satisfies equation (47b) with $A = -\mathrm{cth}\,(H+C)$.

Next, by eliminating the function $\mathrm{th}\,(H+C)$ from equations (47) and (60), we get an equation with separated variables

$$P_{fff}P_f^{-1} - P_{ggg}P_g^{-1} - 2P_f^2 + 2R_g^2 = 0,$$

whence

$$P_{fff}P_f^{-1} - 2P_f^3 = \theta, \quad R_{ggg}R_g^{-1} - 2R_g^2 = \theta.$$

Here, θ is an arbitrary real constant.

Integration of the above ordinary differential equations shows that the functions $P(f)$ and $R(g)$ are determined by equations (58), where C_i, D_i, and γ are arbitrary real constants. Substituting (58) into equation (47), we have the following restrictions on the choice of C_i, D_i:

$$C_1 = -D_1 = \alpha, \quad C_2 = D_2 = \beta.$$

Thus, if the function $V(H)$ is given by (61), then functions $P(f)$ and $R(g)$ are determined by equations (34).

Case 3.5. $A = -(H+C)^{-1}$. In this case, it follows from (47a) that the equality $P_{fff}P_f^{-1} = R_{ggg}R_g^{-1}$ holds. Hence, we get equations for $P(f)$, $R(g)$,

$$P_{fff} = \theta P_f, \quad R_{ggg} = \theta R_g \tag{62}$$

with arbitrary $\theta \in \mathbb{R}^1$. Moreover, the equation for $V(H)$ has the form $\ddot{V} + 3(H+C)\dot{V} = 0$, whence

$$V = m_1 + m_2(H+C)^{-2}, \quad m_i \in \mathbb{R}^1. \tag{63}$$

It is not difficult to check that function (63) satisfies (47b) with $A = -(H+C)^{-1}$.

Integration of equations (62) yields the following result:

$$P_f^2 = \alpha P^2 + C_1 P + C_2, \quad R_g^2 = \alpha R^2 + D_1 R + D_2, \tag{64}$$

here α, C_i, and D_i are arbitrary real constants.

Next, substituting (64) into (47), we get $C_1 = -D_1 = \beta$, $C_2 = D_2 = \gamma$.

Thus, if the potential V is given by (63), then the functions $P(f)$, $R(g)$ are determined by equations (36).

Case 4. $V(H) = m =$ const. In this case, equation (37) reads $P_{fff}P_f^{-1} = R_{ggg}R_g^{-1}$, whence

$$P_{fff} = \theta P_f, \quad R_{ggg} = \theta R_g, \tag{65}$$

where $\theta \in \mathbb{R}^1$ is an arbitrary constant.

Integrating (65), we get equations listed in the lemma under number 13.

Case 5. $V(H)$ is an arbitrary function. In this case, the coefficients of $\ddot{V}$, $\dot{V}$, V in (37) must vanish. Consequently, the relations

$$H_{fff}H_f^{-1} = H_{ggg}H_g^{-1}, \quad H_{gg} = H_{ff}, \quad H_f^2 = H_g^2$$

hold. Hence, we have $H_f = \alpha$, $H_g = \alpha$, $\alpha \in \mathbb{R}^1$. The lemma is proved.

Theorems 1, 2 are direct consequences of the above lemma. To prove Theorems 3–8, one has to integrate ordinary differential equations (30), (32)–(36) and substitute the obtained expressions into (27),

$$\frac{1}{2}(x+t) = P(f) \equiv P\left(\frac{\omega_1+\omega_2}{2}\right), \quad \frac{1}{2}(x-t) = R(g) \equiv R\left(\frac{\omega_1-\omega_2}{2}\right),$$

and (28).

Integration of equations (30), (32)–(36) is carried out in a standard way [17, 18], the obtained result depends essentially on relations between parameters α, β, γ, δ, ρ. This procedure demands very cumbersome computations; this is why we omit details.

With the above remarks, the proof of Theorems 1–8 is completed.

4. Discussion. Let us say a few words about intrinsic characterization of SV in equation (1). It is well known that the solution of a second-order partial differential equation with separated variables is a joint eigenfunction of mutually commuting second-order symmetry operators of the equation under study (for more details, see [6, 10, 14]). Below we construct, in an explicit form, a second-order symmetry operator of equation (1) such that the solution with separated variables is its eigenfunction and the parameter λ is an eigenvalue.

Making the change of variables (24) in equation (1), we get

$$u_{\omega_1\omega_1} - u_{\omega_2\omega_2} = V(\xi+\eta)(\dot{f}(\xi)\dot{g}(\eta))^{-1}u. \tag{66}$$

Provided that equation (1) admits SV, by virtue of equation (25), there exist functions $g_1(f+g)$ and $g_2(f-g)$ such that

$$V(\xi+\eta)(\dot{f}(\xi)\dot{g}(\eta))^{-1} = g_1(f+g) - g_2(f-g).$$

Since $f+g = \omega_1$ and $f-g = \omega_2$, equation (66) takes the form

$$u_{\omega_1\omega_1} - u_{\omega_2\omega_2} = (g_1(\omega_1) - g_2(\omega_2))u$$

or

$$Xu = 0, \quad X = \partial_{\omega_1}^2 - \partial_{\omega_2}^2 - g_1(\omega_1) + g_2(\omega_2).$$

It is evident that the operators $Q_i = \partial^2_{\omega_i} - g_i(\omega_i)$, $i = 1, 2$, commute with the operator X, i.e., they are symmetry operators of equation (1) and, moreover, the relations

$$Q_i u = Q_i \varphi_1(\omega_1)\varphi_2(\omega_2) = \lambda\varphi_1(\omega_1)\varphi_2(\omega_2) = \lambda u, \quad i = 1, 2$$

hold.

Thus, each solution of equation (1) with separated variables is an eigenfunction of some second-order symmetry operator admitted by equation (1).

Now, let us turn to partial differential equations related to equation (1). First, we consider the wave equation

$$\Box u + U(y_0^2 - y_1^2)u = 0. \tag{67}$$

It occurs [11] that equation (67) is reduced to the form (1) by the change of variables

$$t = \exp\left(\frac{1}{2}y_1\right)\operatorname{ch} y_0, \quad t = \exp\left(\frac{1}{2}y_1\right)\operatorname{sh} y_0$$

and, moreover, the potentials $V(\tau)$, $U(\tau)$ are connected by the relation

$$U(\tau) = \frac{1}{4\tau}V(\tau). \tag{68}$$

Consequently, to obtain all potentials $U(y_0^2 - y_1^2)$ such that equation (67) admits a nontrivial SV, one has to substitute potentials $V(x)$ listed in Theorem 2 into formula (68). The solution with separated variables has the form (7), where

$$y_1 + y_0 = \exp\{P((\omega_1 + \omega_2)/2)\}, \quad y_1 - y_0 = \exp\{R((\omega_1 - \omega_2)/2)\}.$$

The explicit form of the functions P and R is given in Theorems 3–8.

Another related equation is the following equation of hyperbolic type

$$v_{x_0x_0} - v_{x_1x_1}c^2(x_1) = 0, \tag{69}$$

which is widely used in various areas of mathematical physics (see, e.g. [19] and references therein).

Equation (69) is reduced to the form (1) by the change of variables

$$u(t, x) = [c(x_1)]^{-1/2}v(x_0, x_1) \quad t = x_0, \quad x = \int [c(x_1)]^{-1}dx_1,$$

and, moreover,

$$V(x) = -c^{3/2}(x_1)(c^{1/2}(x_1))^{\cdot\cdot}\Big|_{x=\int \frac{dx_1}{c(x_1)}}. \tag{70}$$

Thus, to describe all functions $c(x_1)$ that provide separability of equation (69), it suffices to integrate the ordinary differential equation (70). Let us show how to reduce the nonlinear equation (70) to a linear one.

On making in (70) the change of the variable

$$c(x_1) = (\dot{y}(x_1))^{-1},$$

we get

$$\dddot{y} = \frac{3}{2}\ddot{y}(\dot{y})^{-1} + 2V(y)\dot{y}^3.$$

The above equation with the change of the variable $\dot{y} = z^2(y)$ is reduced to the form

$$z_{yy} - V(y)z = 0. \tag{71}$$

So, the general solution of the nonlinear equation (70) is given by the formula

$$c(x_1) = z^{-2}(y(x_1)), \tag{72}$$

where $z(y)$ is a general solution of the linear differential equation (71) and the function $y(x_1)$ is determined by the quadrature

$$\int^{y(x_1)} z^{-2}(\tau)d\tau = x_1 + C, \quad C \in \mathbb{R}^1. \tag{73}$$

Consequently, the problem of description of all functions $c(x)$ such that equation (69) admits a nontrivial SV is reduced to the integration of the linear ordinary differential equation (71), where V is given by (6). Solutions with separated variables have the form

$$\nu = \sqrt{c(x_1)}\varphi_1(\omega_1(x_0, x_1))\varphi_2(\omega_2(x_0, x_1)),$$

where the functions ω_i, are determined by the equalities

$$\frac{1}{2}\left(x_0 + \int \frac{dx_1}{c(x_1)}\right) = P\left(\frac{\omega_1 + \omega_2}{2}\right), \quad \frac{1}{2}\left(x_0 - \int \frac{dx_1}{c(x_1)}\right) = R\left(\frac{\omega_1 - \omega_2}{2}\right),$$

and the explicit form of P and R is given in Theorems 3–8.

Let us also note that, by using the corollary of Theorem 9 and formulas (71)–(73), it is not difficult to obtain the results of Bluman and Kumei [19]. In that paper, they have pointed out all the functions $c(x_1)$ that provide the extension of the symmetry group admitted by equation (69).

The third related equation is the nonlinear wave equation

$$U_{tt} - [c^{-2}(U)U_x]_x = 0. \tag{74}$$

By substitution $U = V_x$, equation (74) is reduced to the form

$$V_{tt} - c^{-2}(V_x)V_{xx} = 0.$$

Applying the Legendre transformation

$$x_0 = V_t, \quad x_1 = V_x, \quad v_{x_0} = t, \quad v_{x_1} = x, \quad v + V = tV_t + xV_x,$$

we get equation (69). Consequently, the method of SV in the linear equation (1) makes it possible to construct exact solutions of the nonlinear wave equation (74).

In conclusion, we suggest a possible generalization of the definition of SV in order to take into consideration nonlinear partial differential equations,

$$U\left(x, u, \underset{1}{u}, \underset{2}{u}, \ldots, \underset{N}{u}\right) = 0, \tag{75}$$

where $x = (x_0, x_1, \ldots, x_{n-1})$ and the symbol $\underset{k}{u}$ denotes the collection of k-th order derivatives of the function $u(x)$.

When speaking about a solution of equation (75) with separated variables $\omega_i = \omega_i(x, u)$, $i = \overline{1, n}$, we mean the ansatz

$$F(x, u, \varphi_1(\omega_1), \ldots, \varphi_n(\omega_n)) = 0, \tag{76}$$

which reduces equation (75) to n ordinary differential equations

$$\varphi_i^{(N)} = f_i\big(\omega_i, \varphi_i, \dot{\varphi}_i, \ldots, \varphi_i^{(N-1)}, \vec{\lambda}\big). \tag{77}$$

In the above formulas, $\omega_i \in C^N(\mathbb{R}^{n+1}, \mathbb{R}^1)$, f_i are some sufficiently smooth functions, and $\vec{\lambda} = (\lambda_1, \ldots, \lambda_{n-1})$ are real parameters.

We say that equation (75) admits SV in the coordinates $\omega_i(x, u)$, $i = \overline{1, n}$, if the substitution of ansatz (76) into (75) with subsequent elimination of the N-th order derivatives $\varphi_i^{(N)}$, $i = \overline{1, n}$, yields an identity with respect to the variables φ_i, $\dot{\varphi}_i$, …, $\varphi_i^{(N-1)}$, $i = \overline{1, n}$, $\vec{\lambda}$ (considered as independent ones).

An application of the above approach to SV in nonlinear equations will be the topic of our future publications.

Here, we present without derivation some results on separation of variables in a two-dimensional nonlinear wave equation obtained with the use of the above described approach.

We have succeeded in separating variables in the following PDE:

1) $\Box_2 u = \lambda_1(\operatorname{ch} u + (\operatorname{sh} 2u) \operatorname{arctg} e^u) + \lambda_2 \operatorname{sh} 2u$;
2) $\Box_2 u = \lambda_1 e^u + \lambda_2 e^{-2u}$;
3) $\Box_2 u = \lambda_1(\operatorname{sh} u - (\operatorname{sh} 2u) \operatorname{arctg} e^u) + \lambda_2 \operatorname{sh} 2u$;
4) $\Box_2 u = \lambda_1 \left(2 \sin u + (\sin 2u) \ln \operatorname{tg} \dfrac{u}{2}\right) + \lambda_2 \sin 2u$;
5) $\Box_2 u = \lambda_1 u + \lambda_2 u \ln u$,

where λ_1 and λ_2 are arbitrary constants.

Below, we adduce ansatzes for $u(x)$ which provide a separation of equations 1–5 and corresponding reduced ordinary differential equations.

1) $u(x) = \ln \operatorname{tg}(\varphi_1(x_0) + \varphi_2(x_1))$,
$\dot{\varphi}_1^2 = C \cos 4\varphi_1 + A\varphi_1 + B_1, \quad \dot{\varphi}_2^2 = C \cos 4\varphi_2 - A\varphi_2 + B_2$,

where C, A, B_1 and B_2 are arbitrary constants satisfying the relations $A = \lambda_1/2$, $B_1 - B_2 = \lambda_2/2$;

2) $u(x) = \ln(\varphi_1(x_0) + \varphi_2(x_1))$,
$\dot{\varphi}_1^2 = 2A\varphi_1^3 + B\varphi_1^2 + C\dot{\varphi}_1 + D_1, \quad \dot{\varphi}_2^2 = -2A\varphi_2^3 + B\varphi_2^2 - C\varphi_2 + D_2$,

where A, B, C, D_1 and D_2 are arbitrary constants satisfying relations $A = \lambda_1$, $D_2 - D_1 = \lambda_2/2$;

3) $u(x) = \ln\, \operatorname{th}(\varphi_1(x_0) + \varphi_2(x_1))$,
$\dot{\varphi}_1^2 = C \operatorname{ch} 4\varphi_1 + A\varphi_1 + B_1, \quad \dot{\varphi}_2^2 = C \operatorname{ch} 4\varphi_2 - A\varphi_2 + B_2$,

where C, A, B_1 and B_2 are arbitrary constants satisfying the relations $A = \lambda_1/2$, $B_1 - B_2 = \lambda_2/2$;

$$4)\quad u(x) = 2\,\mathrm{arctg}\exp(\varphi_1(x_0) + \varphi_2(x_1)),$$
$$\dot\varphi_1^2 = C\,\mathrm{sh}\,2\varphi_1 + 2A\varphi_1 + 2B_1, \quad \dot\varphi_2^2 = C\,\mathrm{sh}\,2\varphi_2 - 2A\varphi_2 + 2B_2,$$

where C, A, B_1, and B_2 are arbitrary constants satisfying the relations $A = \lambda_1$, $B_1 - B_2 = \lambda_2$;

$$5)\quad u(x) = \exp(\varphi_1(x_0) + \varphi_2(x_1)),$$
$$\dot\varphi_1^2 = C_1 e^{-2\varphi_1} + A\varphi_1 + B_1, \quad \dot\varphi_2^2 = C_2 e^{-2\varphi_2} - A\varphi_2 + B_2,$$

where C_1, C_2, A, B_1, and B_2 are arbitrary constants satisfying the relations $A = \lambda_1$, $B_1 - B_2 = \lambda_2 - \lambda_1$.

1. Bocher M., Die Reihentwickelunger der Potentialtheorie, Leipzig, Teubner, 1894, 105 p.
2. Darboux G., Lecons sur les systemes orthogonaux et les coordonnees curvilignes, Paris, 1910, 270 p.
3. Eisenhart L.P., Separable systems of Stäkel, *Ann. Math.*, 1934, **35**, № 2, 284–305.
4. Степанов В.В., Об уравнении Лапласа и некоторых трижды ортогональных системах, *Мат. сб.*, 1942, **11**, 204–238.
5. Олевский М.Н., Триортогональные системы в пространствах постоянной кривизны, в которых уравнение $\Delta_2 u_+ \lambda u = 0$ допускает полное разделение переменных, *Мат. сб.*, 1950, **27**, 379–426.
6. Miller W., Symmetry and separation of variables, Massachusetts, Addison-Wesley, 1977, 340 p.
7. Fushchych W.I., Serov N.I., The symmetry and some exact solutions of many-dimensional nonlinear d'Alembert, Liouville and eikonal equations, *J. Phys. A: Math. Gen.*, 1983, **16**, № 15, 3645–3656.
8. Fushchych W.I., Zhdanov R.Z., On some new exact solutions of the nonlinear d'Alembert–Hamiltonian system, *Phys. Lett. A*, 1989, **141**, № 3–4, 113–115.
9. Fushchych W.I., Zhdanov R.Z., Yegorchenko I.A., On the reduction of the nonlinear multidimensional wave equations and compatibility of the d'Alembert–Hamiltonian system, *J. Math. Anal. and Appl.*, 1991, **161**, № 2, 352–360.
10. Koornwinder T.H., A precise definition of separation of variables, *Lect. Notes Math.*, 1980, **810**, 240–263.
11. Zhdanov R.Z., Revenko I.V., Fushchych W.I., On the new approach to the variable separation in the wave equation with potential, *Dokl. AN Ukrainy, Ser. A*, 1993, № 1, 9–11.
12. Kalnins E., Miller W., Lie theory and separation of variables, II: The EPD equation, *J. Math. Phys.*, 1976, **17**, № 3, 369–377.
13. Kalnins E., Miller W., Lie theory and separation of variables, III: The equation $f_{tt} - f_{ss} = \gamma^2 f$, *J. Math. Phys.*, 1974, **15**, № 9, 1025–1032.
14. Шаповалов В.Н., Разделение переменных в линейном дифференциальном уравнении второго порядка, *Диффереренц. уравнения*, 1980, **16**, № 10, 1864–1874.
15. Овсянников Л.В., Групповой анализ дифференциальных уравнений, М., Наука, 1978, 400 с.
16. Olver P., Applications of Lie groups to differential equations, New York, Springer, 1986, 497 p.
17. Камке Э., Справочник по обыкновенным дифференциальным уравнениям, М., Наука, 1976, 576 с.
18. Erdelyi A. et al, Higher transcendental functions, New York, McGraw and Hill, 1953, Vol. 1, 253 p., Vol. 2, 320 p.
19. Bluman G., Kumei S., On invariance properties of the wave equation, *J. Math. Phys.*, 1987, **28**, № 2, 307–318.

New solutions of the wave equation by reduction to the heat equation

P. BASARAB-HORWATH, L. BARANNYK, W.I. FUSHCHYCH

In this article we make a new connection between the linear wave equation and the linear heat equation. In this way we are able to construct new solutions of the linear wave equation, using symmetries and conditional symmetries of the heat equation.

1. Introduction

The linear wave equation in $(1+n)$-dimensional timespace $\mathbb{R}(1,n)$

$$\Box u = \frac{\partial^2 u}{\partial x_0^2} - \frac{\partial^2 u}{\partial x_1^2} - \cdots - \frac{\partial^2 u}{\partial x_n^2} = -m^2 u \tag{1}$$

is fundamental to mathematical physics: it describes spinless mesons when $n=3$, and is the paradigm of a hyperbolic equation. Its symmetry properties are also known [1, 2], and one has the following result concerning the Lie point symmetries of (1):

Proposition 1. *The maximal Lie point symmetry algebra of equation (1) has basis*

$$P_\mu = \partial_\mu, \quad I = u\partial_u, \quad J_{\mu\nu} = x_\mu\partial_\nu - x_\nu\partial_\mu \tag{2}$$

when $m \neq 0$ *and*

$$\begin{aligned} &P_\mu = \partial_\mu, \quad I = u\partial_u, \quad J_{\mu\nu} = x_\mu\partial_\nu - x_\nu\partial_\mu, \\ &D = x^\mu\partial_\mu, \quad K_\mu = 2x_\mu D - x^2\partial_\mu - 2x_\mu u\partial_u \end{aligned} \tag{3}$$

when $m=0$*, where*

$$\partial_u = \frac{\partial}{\partial u}, \quad \partial_\mu = \frac{\partial}{\partial x^\mu}, \quad x_\mu = g_{\mu\nu}x^\nu,$$
$$g_{\mu\nu} = \mathrm{diag}\,(1,-1,\ldots,-1), \quad \mu,\nu = 0,1,2,\ldots,n.$$

The symmetries can be used to build ansatzes for exact solutions of (1), which then reduce the equation to a partial differential equation with fewer independent variables or even to an ordinary differential equation [1, 2]. These ansatzes and reductions are based on a subalgebra analysis of parts of the symmetry algebra. The reduced equations do not always have nice symmetry properties, so that a full analysis of the resulting equations has not been carried out to this date. In this article we study a reduction which, as far as we know, has not been done before, and which links up solutions of the wave equation (1) in $\mathbb{R}(1,n)$ with those of the linear heat equation in $\mathbb{R}(1,n-1)$. We consider equation (1) with real u: the complex case with nonlinearities is studied in [3].

In [1, 2, 4], the reduction of the nonlinear wave equation

$$\Box u = F(u) \tag{1a}$$

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is considered and its reduction (to equations with a smaller number of independent variables) is studied with respect to the following algebras: $AP(1,n) = \langle P_\mu, J_{\mu\nu}\rangle$ when $F(u)$ is arbitrary; $A\tilde{P}(1,n) = \langle P_\mu, J_{\mu\nu}, D\rangle$ when $F(u) = \lambda u^p$ with p an arbitrary constant; $AC(1,3) = \langle P_\mu, J_{\mu\nu}, D, K_\mu\rangle$ when $F(u) = \lambda u^3$.

The linear equation (1), unlike the nonlinear one (1a), admits a new symmetry operator: $I = u\partial_u$ so that (1) is invariant under the algebras $\langle P_\mu, J_{\mu\nu}, I\rangle$ for $m \neq 0$ and $\langle P_\mu, J_{\mu\nu}, I, D, K_\mu\rangle$ for $m = 0$. However, until now, reductions of (1) have been based only on subalgebras of $\langle P_\mu, J_{\mu\nu}\rangle$ and $\langle P_\mu, J_{\mu\nu}, D, K_\mu\rangle$. In this paper we take the subalgebra $\langle P_\mu, I\rangle$ in both cases, it allows us to reduce the hyperbolic equation (1) to the parabolic heat equation and, in this way, we are then able to exploit the exact solutions of the heat equation to construct solutions of the wave equation. This is the central result of our paper. It may at first sight seem rather strange that a Poincaré-invariant equation is reducible (with an appropriate ansatz) to one that is Galilei-invariant. However, it is known (see [5]) that the Galilei algebra can be found within the Poincaré algebra, so that one may even expect the original equation to 'contain' a Galilei-invariant one.

2. Reduction to the heat equation

In this paper we limit ourselves to $(1+3)$-dimensional time-space $\mathbb{R}(1,3)$, but the generalization of our result to higher dimensions is obvious as the reduction remains the same.

We now turn to the construction of the ansatz which reduces (1) to the heat equation. Equation (1) is invariant under the operators P_μ, I and is therefore also invariant under any constant linear combination of them:

$$\tau^\mu \partial_\mu + ku\partial_u,$$

where k, τ^μ are constants. This latter operator then gives us the following invariant-surface condition

$$\tau^\mu u_\mu = ku$$

which gives the Lagrangian system

$$\frac{dx_\mu}{\tau_\mu} = \frac{du}{ku}$$

and it is not difficult to show that this, in turn, is equivalent to the Lagrangian system

$$\frac{d(cx)}{c\tau} = \frac{du}{ku} \tag{4}$$

for any constant four-vector c, with $cx = c^\mu x_\mu$, $c\tau = c^\mu \tau^\mu$. Choose now τ so that $\tau^2 = \tau^\mu \tau_\mu = 0$, namely τ is light-like, and choose four-vectors β, δ, ϵ so that

$$\beta^2 = \delta^2 = -1, \quad \epsilon^2 = -\frac{m^2}{k^2}, \quad \tau\beta = \tau\delta = \beta\delta = \beta\epsilon = \delta\epsilon = 0, \quad \tau\epsilon = 1. \tag{5}$$

On choosing c in (4) to be τ, β, δ, ϵ we obtain the system

$$\frac{d(\tau x)}{0} = \frac{d(\beta x)}{0} = \frac{d(\delta x)}{0} = \frac{d(\epsilon x)}{1} = \frac{du}{ku}. \tag{6}$$

The general integral of (6) is given by

$$u = e^{k(\epsilon x)} v(\tau x, \beta x, \delta x), \tag{7}$$

where v is a smooth function of its arguments (we assume that all our operations are smooth, at least locally). Treating (7) as an ansatz for equation (1), we find, on substituting (7) into (1), writing $t = \tau x$, $y_1 = \beta x$, $y_2 = \delta x$, performing some elementary computations and using (5), that v satisfies the linear heat equation (we have chosen $k = \frac{1}{2}$ for convenience)

$$\frac{\partial v}{\partial t} = \frac{\partial^2 v}{\partial y_1^2} + \frac{\partial^2 v}{\partial y_2^2}. \tag{8}$$

The Cauchy problem for equation (8) is well posed for $t > 0$, and (8) has solutions which are singular for $t = 0$. This then leads to a similar problem for the wave equation when $\tau x = 0$, which is a characteristic ($\tau^2 = 0$), so that the initial-value problem for (8) at $t = 0$ is related to the initial-value problem of (1) on a characteristic. This latter is known as Goursat's problem, and has been studied in [12], to which we refer the reader for more details.

The linear heat equation in $(1+1)$ spacetime dimensions has been studied extensively: its symmetry properties [2, 6, 7] and its conditional symmetries (also known as non-classical symmetries [6], Q-conditional symmetries in [2]) are known. The symmetry algebra of the linear heat equation in $1+2$ timespace can be found in [7] but for the sake of completeness, we give it in the following proposition.

Proposition 2. *The maximal Lie point symmetry algebra of equation (8) is the extended Galilei algebra $AG_3(1,2)$ with a basis given by the following vector fields*

$$\begin{aligned}
&T = \partial_t \qquad P_a = -\partial_{y_a}, \quad G_a = t\partial_{y_a} - \frac{1}{2} y_a v \partial_v, \quad M = -\frac{1}{2} v\partial_v, \\
&J_{12} = y_1 \partial_{y_2} - y_2 \partial_{y_1}, \quad D = 2t\partial_t + y_1\partial_{y_1} + y_2 \partial_{y_2} - v\partial_v, \\
&S = t^2 \partial_t + t y_1 \partial_{y_1} + t y_2 \partial_{y_2} - \left(t + \frac{1}{4}\left(y_1^2 + y_2^2\right)\right) v\partial_v.
\end{aligned} \tag{9}$$

Remark 1. We have not included the symmetry $v \to v + v_1$, where v_1 is an arbitrary solution of (8).

If we had considered equation (1) in $\mathbb{R}(1,4)$, then we would have obtained the linear heat equation in $1+3$ dimensions with our reduction. Note also that there is a Lie-algebraic reduction of (1) in $\mathbb{R}(1,4)$ to equation (1) in $\mathbb{R}(1,3)$, which amounts to omitting dependency on one of the spatial variables. In this way, we are able to use the wave equation in $\mathbb{R}(1,4)$ as a bridge in constructing solutions of the wave equation in $\mathbb{R}(1,3)$ from those of the heat equation in $1+3$ dimensions.

The invariance of equation (8) under the group $G_2(1,2)$ which the above algebra generates then allows us to obtain a nine-parameter family of exact solutions whenever one solution is given.

The commutation relations of the algebra (9) are

$$\begin{aligned}
&[P_a, G_b] = \delta_{ab} M, \quad [P_1, J_{12}] = P_2, \quad [P_2, J_{12}] = -P_1, \\
&[P_a, D] = P_a, \quad [P_a, S] = G_a, \quad [P_a, T] = 0, \quad [M, X] = 0 \text{ for all } X \in AG_3(2), \\
&[G_a, G_b] = 0, \quad [D, G_a] = G_a, \quad [T, G_a] = P_a, \quad [S, G_a] = 0, \\
&[J_{12}, T] = [J_{12}, D] = [J_{12}, S] = 0, \quad [T, D] = 2T, \quad [T, S] = D, \quad [D, S] = 2S.
\end{aligned}$$

Clearly, we see that the subalgebra $\langle P_a, G_a, M\rangle$, $a = 1, 2$ is an ideal (maximal and solvable, and therefore the radical of the algebra [8, 9]). Our algebra is seen to be the semi-direct sum $\langle J_{12}, S, T, D\rangle + \langle P_a, G_a, M\rangle$. In turn, we can verify that $\langle S, T, D\rangle$ is a semi-simple Lie algebra which we can take as being a realization of $ASL(2,\mathbb{R})$, the Lie algebra of $SL(2,\mathbb{R})$. To see this, we take $X_1 = \frac{1}{2}D$, $X_2 = \frac{1}{2}(T-S)$, $X_3 = \frac{1}{2}(T+S)$ as a new basis, and obtain the commutation relations of $SL(2,\mathbb{R})$:

$$[X_1, X_2] = -X_3, \quad [X_2, X_3] = X_1, \quad [X_3, X_1] = X_2.$$

Thus we obtain

$$\langle J_{12}, S, T, D\rangle = \langle J_{12}\rangle \oplus \langle S, T, D\rangle = \langle J_{12}\rangle \oplus ASL(2,\mathbb{R})$$

which is the Lie algebra of $O(2) \otimes SL(2,\mathbb{R})$.

The elements of the group $G_2(1,2)$ are considered as transformations of a space with local coordinates (t, y_1, y_2, v) and points with these coordinates are mapped to points (t', y_1', y_2', v'). The finite transformations defining this action are obtained by solving the corresponding Lie equations. For the subalgebra $\langle J_{12}, S, T, D\rangle = \langle J_{12}, X_1, X_2, X_3\rangle$ we solve the Lie equations as follows:

$$J_{12}: \quad \frac{dt'}{d\rho} = 0, \quad \frac{dy_1'}{d\rho} = -y_2', \quad \frac{dy_2'}{d\rho} = y_1', \quad \frac{dv'}{d\rho} = 0,$$
$$t'|_{\rho=0} = t, \quad y_a'|_{\rho=0} = y_a, \quad v'|_{\rho=0} = v$$

which gives the finite transformations

$$t' = t, \quad y_1' = y_1\cos\rho - y_2\sin\rho, \quad y_2' = y_1\sin\rho + y_2'\cos\rho, \quad v' = v.$$

Then we have the corresponding equations for X_1, X_2, X_3

$$X_1: \quad t' = e^{\nu_1}t = \frac{e^{\nu_1/2}t + 0}{0\cdot t + e^{-\nu_1/2}}, \quad y_a' = e^{\nu_1/2}y_a = \frac{y_a}{0\cdot t + e^{-\nu_1/2}},$$
$$v' = e^{-\nu_1/2}v,$$
$$X_2: \quad t' = \frac{t\cosh\nu_2 + \sinh\nu_2}{t\sinh\nu_2 + \cosh\nu_2}, \quad y_a' = \frac{y_a}{t\sinh\nu_2 + \cosh\nu_2},$$
$$v' = v(t\sinh\nu_2 + \cosh\nu_2)\exp\left(\frac{(y_1^2 + y_2^2)\sinh\nu_2}{4(t\sinh\nu_2 + \cosh\nu_2)}\right),$$
$$X_3: \quad t' = \frac{t\cos\nu_3 + \sin\nu_3}{\cos\nu_3 - t\sin\nu_3}, \quad y_a' = \frac{y_a}{\cos\nu_3 - t\sin\nu_3},$$
$$v' = v(\cos\nu_3 - t\sin\nu_3)\exp\left(-\tfrac{(y_1^2+y_2^2)\sin\nu_3}{4(\cos\nu_3 - t\sin\nu_3)}\right).$$

Thus, we see that the action of the group generated by $\langle J_{12}, S, T, D\rangle$ can be given in the form

$$t' = \frac{\zeta t + \eta}{\kappa t + \sigma}, \quad y_1' = \frac{y_1\varepsilon\cos\rho - y_2\varepsilon\sin\rho}{\kappa t + \sigma}, \quad y_2' = \frac{y_1\sin\rho + y_2\cos\rho}{\kappa t + \sigma},$$
$$v' = (\kappa t + \sigma)v\exp\left(\frac{\kappa(y_1^2 + y_2^2)}{4(\kappa t + \sigma)}\right)$$

with $\zeta\sigma - \eta\kappa = 1$, and $\varepsilon = \pm 1$ corresponds to the possibility of space reflections under which (8) is manifestly invariant (the group $O(2)$ has two components). The parameters ζ, η, κ, σ correspond to the action of $SL(2,\mathbb{R})$.

Solving the Lie equations defined by each of the other infinitesimal generators in (9), we obtain finite transformations such that $(t, y_1, y_2, v) \to (t', y'_1, y'_2, v')$ as follows:

$$
\begin{aligned}
G_i:\quad & t' = t, \quad y'_i = \mu_i t + y_i, \quad y'_j = y_j \quad \text{for } j \neq i,\\
& v' = v \exp\left(-\frac{1}{2}\left(\frac{\mu_i^2}{2}t + \mu_i y_i\right)\right),\\
P_i:\quad & t' = t, \quad y'_i = y_i - \lambda_i, \quad y'_j = y_j \quad \text{for } j \neq i, \quad v' = v,\\
M:\quad & t' = t, \quad y'_i = y_i, \quad v' = v \exp\left(-\frac{1}{2}\theta\right).
\end{aligned}
$$

3. Subalgebras and ansatzes

Having obtained and discussed the symmetry algebra of equation (8), we now pass to listing the subalgebras of $AG_2(1,2)$ which are inequivalent up to conjugation by $G_2(1,2)$, and giving the corresponding reduced equations. In those cases where it is possible, we integrate these equations. The method of obtaining subalgebras up to conjugation is described in [4, 10]; here we simply present our results. The reductions we have obtained have been verified with MAPLE.

3.1. Reduction to ordinary differential equations by two-dimensional subalgebras. Here we list the subalgebras, with restrictions on any parameters entering into the algebra, and then we give the corresponding ansatz and finally the differential equation which arises, with its solution. In all the cases, we can take the real and imaginary parts of the solutions, as the reduced equations are linear. This is understood when complex arguments appear.

3.1.1.

$$\langle P_2, T + \alpha M\rangle \ (\alpha = 0, \pm 1): \quad v = e^{-\alpha t/2}\varphi(\omega), \quad \omega = y_1, \quad \ddot{\varphi} + \frac{1}{2}\alpha\varphi = 0.$$

Integrating this reduced equation, we find the following cases

$$
\begin{aligned}
&\varphi = C_1\omega + C_2 \quad \text{for } \alpha = 0,\\
&\varphi = C_1 \exp\left(\frac{\omega}{\sqrt{2}}\right) + C_2 \exp\left(-\frac{\omega}{\sqrt{2}}\right) \quad \text{for } \alpha = -1,\\
&\varphi = C_1 \cos\left(\frac{\omega}{\sqrt{2}} + C_2\right) \quad \text{for } \alpha = 1.
\end{aligned}
$$

From these we obtain the following exact solutions of (8):

$$
\begin{aligned}
&v = C_1 y_2 + C_2 \quad \text{for } \alpha = 0,\\
&v = e^{t/2}\left(C_1 \exp\left(\frac{y_1}{\sqrt{2}}\right) + C_2 \exp\left(-\frac{y_2}{\sqrt{2}}\right)\right) \quad \text{for } \alpha = -1,\\
&v = e^{-t/2} C_1 \cos\left(\frac{y_1}{\sqrt{2}} + C_2\right) \quad \text{for } \alpha = 1
\end{aligned}
$$

with C_1, C_2 being arbitrary constants.

3.1.2.

$$\langle D + (2\alpha+1)M, T\rangle \ (\alpha \in \mathbb{R}): \quad v = y_1^{-(\alpha+3/2)}\varphi(\omega), \quad \omega = \frac{y_2}{y_1},$$

$$(\omega^2+1)\ddot{\varphi} + (5+2\alpha)\omega\dot{\varphi} + \left(\frac{3}{2}+\alpha\right)\left(\frac{5}{2}+\alpha\right)\varphi = 0.$$

For $\alpha = -\frac{5}{2}$ we have

$$\varphi = C_1\omega + C_2.$$

If $\alpha = -\frac{3}{2}$ then

$$\varphi = C_1 \arctan\omega + C_2.$$

For $\alpha \neq -\frac{3}{2}, -\frac{5}{2}$ then

$$\varphi = C_1(1+\omega^2)^{-(\alpha/2+3/4)} \cos\left(\left(\frac{3}{2}+\alpha\right)\arctan\omega + C_2\right).$$

The exact solutions being:

$$v = C_1 y_2 + C_2 y_2, \quad \alpha = -\frac{5}{2},$$
$$v = C_1 \arctan\frac{y_2}{y_1} + C_2, \quad \alpha = -\frac{3}{2},$$
$$v = C_1(y_1^2+y_2^2)^{-(\alpha/2+3/4)} \cos\left(\left(\frac{3}{2}+\alpha\right)\arctan\frac{y_2}{y_1} + C_2\right) \quad \alpha \neq -\frac{3}{2}, -\frac{5}{2}.$$

3.1.3.

$$\langle D + (4\alpha+1)M, P_2\rangle \ (\alpha \in \mathbb{R}): \quad v = t^{-(\alpha+3/4)}\varphi(\omega), \quad \omega = \frac{y_1^2}{t},$$
$$4\omega\ddot{\varphi} + (2+\omega)\dot{\varphi} + \left(\frac{3}{4}+\alpha\right)\varphi = 0.$$

If we make the transformation $\omega \to \xi = -\frac{\omega}{4}$ in this ODE, we obtain

$$\xi\varphi'' + \left(\frac{1}{2}-\xi\right)\varphi'' - \left(\alpha+\frac{3}{4}\right)\varphi = 0,$$

where φ' denotes differentiation with respect to ξ. The solutions of this equation are given in terms of the Pochhammer–Barnes confluent hypergeometric function (see for example Vol. 1, ch. 6 of [11])

$$\Phi(a;b;z) = \sum_{n=0}^{\infty}\frac{(a)_n z^n}{(b)_n n!}$$

with $b \neq 0$ and where $(a)_n = a(a+1)(a+2)\cdots(a+n-1), n \geq 1$. We find then [11]

$$\varphi = C_1\Phi\left(\alpha+\frac{3}{4};\frac{1}{2};-\frac{1}{4}\omega\right) + C_2\left(-\frac{1}{4}\omega\right)^{1/2}\Phi\left(\alpha+\frac{5}{4};\frac{3}{2};-\frac{1}{4}\omega\right).$$

Thus we find the exact solution

$$v = t^{-\left(\alpha+\frac{3}{4}\right)}\left[C_1\Phi\left(\alpha+\frac{3}{4};\frac{1}{2}-\frac{y_1^2}{4t}\right) + C_2\left(-\frac{y_1^2}{4t}\right)^{1/2}\Phi\left(\alpha+\frac{5}{4};\frac{3}{2};-\frac{y_1^2}{4t}\right)\right].$$

3.1.4.

$$\langle G_1, P_2\rangle: \quad v = \exp\left(-\frac{y_1^2}{4t}\right)\varphi(\omega), \quad \omega = t, \quad \dot{\varphi} + \frac{1}{2\omega}\varphi = 0$$

which integrates to give the exact solution

$$v = C|t|^{-1/2}\exp\left(-\frac{y_1^2}{4t}\right).$$

3.1.5.

$$\langle P_2, T+G_1\rangle: \quad v = \exp\left(\frac{t^3}{6} - \frac{y_1 t}{2}\right)\varphi(\omega), \quad \omega = t^2 - 2y_1,$$
$$16\ddot{\varphi} - \omega\varphi = 0.$$

To treat this ODE, first write $\varphi = \sqrt{\omega}\psi(z)$ with $z = \omega^{3/2}/6$. Then ψ satisfies

$$\psi'' + \frac{1}{z}\psi' - \left(1 + \frac{1}{9z^2}\right)\psi = 0$$

which is the equation for the Bessel function $J_{\pm 1/3}(iz)$ (these two are linearly independent solutions) (see Vol. 2, section 7.2.2 of [11]). Consequently, we have

$$v = \left(t^2 - 2y_1\right)^{1/2}\exp\left(\frac{t^3}{6} - \frac{ty_1}{2}\right)\times$$
$$\times\left[C_1 J_{1/3}\left(\frac{i\left(t^2-2y_1\right)^{3/2}}{6}\right) + C_2 J_{-1/3}\left(\frac{i\left(t^2-2y_1\right)^{3/2}}{6}\right)\right]$$

as an exact solution of the heat equation.

3.1.6.

$$\langle J_{12} + \alpha D - \alpha(4\beta+2)M, T\rangle \quad (\alpha > 0,\ \beta \in \mathbb{R}),$$
$$v = \left(y_1^2 + y_2^2\right)^{\beta}\varphi(\omega), \quad \omega = \alpha\arctan\left(\frac{y_1}{y_2}\right) + \frac{1}{2}\ln\left(y_1^2 + y_2^2\right),$$
$$(\alpha^2+1)\ddot{\varphi} + 4\beta\dot{\varphi} + 4\beta^2\varphi = 0.$$

Integrating this equation, we obtain

$$\varphi = C_1\omega + C_2 \quad \text{for} \quad \beta = 0$$

and

$$\varphi = C_1\exp\left(-\frac{2\beta\omega}{1+\alpha^2}\right)\cos\left(-\frac{2\alpha\beta\omega}{1+\alpha^2} + C_2\right) \quad \text{for} \quad \beta \neq 0.$$

These then give us the exact solutions

$$v = C_1\left[\alpha\arctan\left(\frac{y_1}{y_2}\right) + \frac{1}{2}\ln\left(y_1^2+y_2^2\right)\right] + C_2 \quad \text{for} \quad \beta = 0,$$
$$v = C_1\left(y_1^2+y_2^2\right)^{\beta}\exp\left(-\frac{2\beta\omega}{1+\alpha^2}\right)\cos\left(\frac{2\alpha\beta\omega}{1+\alpha^2} + C_2\right) \quad \text{for} \quad \beta \neq 0,$$

where

$$\omega = \alpha\arctan\left(\frac{y_1}{y_2}\right) + \frac{1}{2}\ln\left(y_1^2+y_2^2\right).$$

3.1.7.

$$\langle J_{12}+2\alpha M, D-(4\beta+2)M\rangle \quad (\alpha\ge 0,\ \beta\in\mathbb{R}),$$
$$v=t^{\beta}\exp\left(\alpha\arctan\frac{y_1}{y_2}\right)\varphi(\omega), \quad \omega=\frac{y_1^2+y_2^2}{t},$$
$$\omega^2\ddot\varphi+\left(\omega+\frac{\omega^2}{4}\right)\dot\varphi+\left(\frac{\alpha^2}{4}-\frac{\beta\omega}{4}\right)\varphi=0.$$

This equation gives

$$\ddot\varphi+\left(\frac14+\frac1\omega\right)\dot\varphi+\left(\frac{\alpha^2}{4\omega^2}-\frac{\beta}{4\omega}\right)\varphi=0.$$

Its solutions can be given in terms of Whittaker functions $W(k;m;z)$ (see Vol. 1, ch. 6, pp. 248–251 of [11]) and one obtains

$$\varphi=\frac{e^{-\omega/8}}{\sqrt{\omega}}W\left(-(\beta+1/2);\frac{i\alpha}{2};\frac{\omega}{4}\right)$$

and hence

$$v=\frac{t^{\beta+1/2}}{\sqrt{y_1^2+y_2^2}}\exp\left(-\frac{y_1^2+y_2^2}{8t}\right)\exp\left(\alpha\arctan\frac{y_1}{y_2}\right)W\left(-(\beta+1/2);\frac{i\alpha}{2};\frac{y_1^2+y_2^2}{4t}\right).$$

3.1.8.

$$\langle J_{12}+2\alpha M, T+\beta M\rangle \quad (\alpha\ge 0,\beta=0,\pm1),$$
$$v=\exp\left(\alpha\arctan\frac{y_1}{y_2}-\frac{\beta t}{2}\right)\varphi(\omega), \quad \omega=y_1^2+y_2^2,$$
$$\omega^2\ddot\varphi+\omega\dot\varphi+\left(\frac{\alpha^2}{4}+\frac{\beta\omega}{8}\right)\varphi=0.$$

We have the following cases:

$$\varphi=C_1+C_2\log\omega \quad \text{for} \quad \alpha=\beta=0,$$
$$\varphi=C_1\cos\left(-\frac{\alpha}{2}\log\omega+C_2\right) \quad \text{for} \quad \alpha\ne0,\ \beta=0,$$
$$\varphi=J_{i\alpha}\left(\sqrt{\frac{\beta\omega}{2}}\right) \quad \text{for} \quad \alpha\ge0,\ \beta\ne0.$$

Consequently, we have the following solutions of (8)

$$v=C_1+C_2\log\left(y_1^2+y_2^2\right) \quad \text{for} \quad \alpha=\beta=0,$$
$$v=\exp\left(\alpha\arctan\frac{y_1}{y_2}\right)C_1\cos\left(-\frac{\alpha}{2}\log\left(y_1^2+y_2^2\right)+C_2\right) \quad \text{for} \quad \alpha\ne0,\ \beta=0,$$
$$v=\exp\left(\alpha\arctan\frac{y_1}{y_2}-\frac{\beta t}{2}\right)J_{i\alpha}\left(\sqrt{\frac{\beta\left(y_1^2+y_2^2\right)}{2}}\right) \quad \text{for} \quad \alpha\ge0,\ \beta\ne0.$$

3.1.9.

$$\langle J_{12}+S+T+2\alpha M, G_1+P_2\rangle \quad (\alpha\in\mathbb{R}),$$

$$v=\left(t^2+1\right)^{-1/2}\exp\left[\left(\frac{1-t^2}{4t}\right)\left(\frac{y_1+ty_2}{t^2+1}\right)^2-\frac{y_1^2}{4t}-\alpha\arctan t\right]\varphi(\omega),$$

$$\omega=\frac{y_1+ty_2}{t^2+1},\quad \ddot{\varphi}+(\alpha+\omega^2)\varphi=0.$$

This equation is known as the Weber equation. Its solutions are the real and imaginary parts of the functions

$$D_{-\sqrt{\alpha}}(\pm(1+i)\omega),$$

where $D_\nu(z)$ are the Weber–Hermite (parabolic cylinder) functions (Vol. 2, ch. 8, section 8.2 of [11]). This gives the following exact solutions of (9):

$$v=\left(t^2+1\right)^{-1/2}\exp\left[\left(\frac{1-t^2}{4t}\right)\left(\frac{y_1+ty_2}{t^2+1}\right)^2-\frac{y_1^2}{4t}-\alpha\arctan t\right]\times$$
$$\times D_{\sqrt{\alpha}}\left(\pm(1+i)\frac{y_1+ty_2}{t^2+1}\right)$$

and the real and imaginary parts of this function give us exact solutions of the heat equation (9).

3.1.10.

$$\langle J_{12}+2\alpha M, S+T+2\beta M\rangle \quad (\alpha\geq 0,\ \beta\in\mathbb{R}),$$

$$v=\left(t^2+1\right)^{-1/2}\exp\left[-\beta\arctan t+\alpha\arctan\frac{y_1}{y_2}-\frac{t\left(y_1^2+y_2^2\right)}{4\left(t^2+1\right)}\right]\varphi(\omega),$$

$$\omega=\frac{y_1^2+y_2^2}{t^2+1},\quad \ddot{\varphi}+\frac{1}{\omega}\dot{\varphi}+\left(\frac{1}{16}+\frac{\beta}{4\omega}+\frac{\alpha^2}{4\omega^2}\right)\varphi=0.$$

The solutions of this equation can be given in terms of Whittaker functions [11], and we obtain the following exact solutions of the heat equation as a result:

$$v=\left(y_1^2+y_2^2\right)^{-1/2}\exp\left[-\beta\arctan t+\alpha\arctan\frac{y_1}{y_2}-\frac{t(y_1^2+y_2^2)}{4(t^2+1)}\right]\times$$
$$\times W\left(\frac{i\beta}{8};\frac{i\alpha}{2};\frac{i\left(y_1^2+y_2^2\right)}{2\left(t^2+1\right)}\right).$$

In the above cases we have been able to describe exact solutions of (8) in terms of elementary functions or confluent hypergeometric functions. Using the notation introduced in equations (8) and (7), we are thus able to construct strikingly new exact solutions of the linear wave equation (1).

3.2. Reduction to partial differential equations by one-dimensional subalgebras. Here we list the subalgebras, the relevant parameters, ansatzes and reduced equations, without constructing their exact solutions. We use φ_1 to denote the partial derivative with respect to ω_1, and φ_{22} means the second derivative with respect to ω_2, and so on.

3.2.1.

$$\langle P_2\rangle: \quad v=\varphi(\omega_1,\omega_2), \quad \omega_1=t, \quad \omega_2=y_1, \quad \varphi_1=\varphi_{22}.$$

This is the heat equation in $1+1$ spacetime dimensions. The symmetries and conditional symmetries of the heat equation are well known. A discussion of these can be found in [6] and in appendix 7 of [2].

3.2.2.

$$\langle G_1+P_2\rangle: \quad v=\exp\left(-\frac{y_1^2}{4t}\right)\varphi(\omega_1,\omega_2), \quad \omega_1=t, \quad \omega_2=y_1+ty_2,$$
$$(1+\omega_1^2)\varphi_{22}-\varphi_1-\frac{\omega_2}{\omega_1}\varphi_2-\frac{1}{2\omega_1}\varphi=0.$$

3.2.3.

$$\langle T+\alpha M\rangle \quad (\alpha=0,\pm1): \quad v=\exp\left(-\frac{\alpha t}{2}\right)\varphi(\omega_1,\omega_2), \quad \omega_1=y_1, \quad \omega_2=y_2,$$
$$\varphi_{11}+\varphi_{22}+\frac{1}{2}\alpha\varphi=0.$$

This equation is the Laplace equation for $\alpha=0$. Solutions can be obtained by using separation of variables.

3.2.4.

$$\langle T+G_1\rangle: \quad v=\exp\left(\frac{t^3}{6}-\frac{y_1 t}{2}\right)\varphi(\omega_1,\omega_2), \quad \omega_1=t^2-2y_1, \quad \omega_2=y_2,$$
$$4\varphi_{11}+\varphi_{22}-\frac{1}{4}\omega_1\varphi=0.$$

3.2.5.

$$\langle J_{12}+2\alpha M\rangle \quad (\alpha\ge 0),$$
$$v=\exp\left(\alpha\arctan\frac{y_1}{y_2}\right)\varphi(\omega_1,\omega_2), \quad \omega_1=y_1^2+y_2^2, \quad \omega_2=t,$$
$$4\omega_1^2\varphi_{11}+4\omega_1\varphi_1+\omega_1\varphi_2+\alpha^2\varphi=0.$$

3.2.6.

$$\langle J_{12}+T+2\alpha M\rangle \quad (\alpha\in\mathbb{R}),$$
$$v=\exp(-\alpha t)\varphi(\omega_1,\omega_2), \quad \omega_1=y_1^2+y_2^2, \quad \omega_2=t+\arctan\frac{y_1}{y_2},$$
$$4\omega_1^2\varphi_{11}+\varphi_{22}+4\omega_1\varphi_1-\omega_1\varphi_2+\alpha\omega_1\varphi=0.$$

3.2.7.

$$\langle J_{12}+\frac{\alpha}{2}D+\alpha(2\beta-1)M\rangle \quad (\alpha\ge 0,\ \beta\ge 1/2),$$
$$v=t^\beta\varphi(\omega_1,\omega_2), \quad \omega_1=\log t+\alpha\arctan\frac{y_1}{y_2}, \quad \omega_2=\frac{y_1^2+y_2^2}{t},$$
$$\alpha^2\varphi_{11}+4\omega_2^2\varphi_{22}-\omega_2\varphi_1+(4\omega_2+\omega_2^2)\varphi_2+\beta\omega_2\varphi=0.$$

3.2.8.

$$\langle D+(4\alpha-2)M\rangle \quad (\alpha\geq 1/2): \quad v=t^{-\alpha}\varphi(\omega_1,\omega_2), \quad \omega_1=\frac{y_1^2}{t}, \quad \omega_2=\frac{y_2^2}{t},$$
$$4\omega_1\varphi_{11}+4\varphi_2\varphi_{22}+(2+\omega_1)\varphi_1+(2+\omega_2)\varphi_2+\alpha\varphi=0.$$

3.2.9.

$$\langle S+T+\alpha J_{12}+2\beta M\rangle \quad (\alpha>0,\ \beta\in\mathbb{R}),$$
$$v=\left(t^2+1\right)^{-1/2}\exp\left[-\beta\arctan t-\frac{t(y_1^2+y_2^2)}{4\,(t^2+1)}\right]\varphi(\omega_1,\omega_2)$$
$$\omega_1=\frac{y_1^2+y_2^2}{t^2+1}, \quad \omega_2=\arctan\frac{y_1}{y_2}+\alpha\arctan t,$$
$$4\omega_1\varphi_{11}+\frac{1}{\omega_1}\varphi_{22}+4\varphi_1-\alpha\varphi_2+\left(\beta+\frac{\omega_1}{4}\right)\varphi=0.$$

3.2.10.

$$\langle S+T+2\alpha M\rangle \quad (\alpha\in\mathbb{R}),$$
$$v=\left(t^2+1\right)^{-1/2}\exp\left[-\alpha\arctan t-\frac{t(y_1^2+y_2^2)}{4(t^2+1)}\right]\varphi(\omega_1,\omega_2),$$
$$\omega_1=\frac{y_1^2}{t^2+1}, \quad \omega_2=\frac{y_2^2}{t^2+1},$$
$$4\omega_1\varphi_{11}+4\omega_2\varphi_{22}+2\varphi_1+2\varphi_2+\left(\alpha+\frac{\omega_1+\omega_2}{4}\right)\varphi=0.$$

3.2.11.

$$\langle S+T+J_{12}+\alpha(G_1+P_2)\rangle \quad (\alpha>0),$$
$$v=(t^2+1)^{-1/2}\exp\left[\frac{(1-t^2)(y_1+ty_2)^2}{4t(t^2+1)^2}-\frac{y_1^2}{4t}\right]\varphi(\omega_1,\omega_2),$$
$$\omega_1=\frac{y_1+ty_2}{t^2+1}, \quad \omega_2=\frac{ty_1-y_2}{t^2+1}=\alpha\arctan t,$$
$$\varphi_{11}+\varphi_{22}-(2\omega_1-\alpha)\varphi_2+\omega_1^2\varphi=0.$$

4. Some conditional symmetries of the 2 + 1 heat equation

In this section we give the conditional symmetries of equation (8). The defining equations are nonlinear coupled partial differential equations, which we do not solve, except in one case, leaving the others for consideration in a later publication. We have the following result.

Proposition 3. *Equation (8) is conditionally invariant under*

$$X=\xi^0\frac{\partial}{\partial t}+\xi^1\frac{\partial}{\partial y_1}+\xi^2\frac{\partial}{\partial y_2}+\eta\frac{\partial}{\partial v}$$

when the coefficients satisfy the following conditions:

$$(i) \quad \xi^0=1: \quad \xi^1_{y_1}=\xi^2_{y_2}, \quad \xi^1_{y_2}=-\xi^2_{y_1}, \quad \eta=Av+B,$$

where ξ^1, ξ^2, A, B are functions of t, y_1, y_2 and satisfy the system

$$\xi^1_t+2\xi^1\xi^1_{y_1}+2A_{y_1}=0, \quad \xi^2_t+2\xi^2\xi^2_{y_2}+2A_{y_2}=0,$$
$$A_t=A_{y_1y_1}+A_{y_2y_2}-2A\xi^2_{y_2}, \quad B_t=B_{y_1y_1}+B_{y_2y_2}-2B\xi^2_{y_2}.$$

$$(ii)\quad \xi^0 = 0, \quad \xi^1 = 1: \quad \xi^2_{y_2} = \xi^2 \xi^2_{y_1}, \quad \eta = Av + B,$$

where ξ^2, A, B *are functions of* t, y_1, y_2 *and satisfy the system*

$$\begin{aligned}
&\xi^2_t - \xi^2_{y_1y_1} - \xi^2_{y_2y_2} + 2\xi^2_{y_1}\xi^2_{y_2} - 2\xi^2 A_{y_1} - 2A\xi^2_{y_1} = 0,\\
&A_t = A_{y_1y_1} + A_{y_2y_2} + 2AA_{y_1} - 2A_{y_2}\xi^2_{y_1},\\
&B_t = B_{y_1y_1} + B_{y_2y_2} + 2BA_{y_1} - 2By_2\xi^2_{y_1}.
\end{aligned}$$

$$(iii)\quad \xi^0 = \xi^1 = 0, \quad \xi^2 = 1: \quad \eta = Av + B,$$

where A *is a function of* t, y_2 *only, and* B *is a function of* t, y_1, y_2 *and satisfy the equations*

$$A_t = A_{y_2y_2} + 2AA_{y_2}, \quad B_t = B_{y_1y_1} + B_{y_2y_2} + 2BA_{y_2}.$$

As is clear in the above three cases, the systems of equations involved are highly nonlinear, and cannot be solved in general. However, the equation for the function A in case (iii) is recognized to be the Burgers equation. This equation can be linearized by the Hopf–Cole transformation $A = w_{y_2}/w$, where w is a solution of the heat equation $w_t = w_{y_2y_2}$ (see for example [2]). The solutions obtained in this way can then be used to build ansatzes first for the $2+1$ heat equation (8) and then, in turn, the linear wave equation (1), using the ansatz (7).

Ansatzes can also be obtained from the symmetry algebra of the Burgers equation. Indeed, the symmetry algebra of the equation

$$A_t = A_{y_2y_2} + 2AA_{y_2} \tag{10}$$

is generated by the operators

$$\begin{aligned}
&\partial_t, \quad \partial_{y_2}, \quad 2t\partial_{y_2} - \partial_A, \quad 2t\partial_t + y_2\partial_{y_2} - A\partial_A,\\
&t^2\partial_t + ty_2\partial_{y_2} - \left(tA + \frac{y_2}{2}\right)\partial_A.
\end{aligned} \tag{11}$$

The operator (11) gives the ansatz

$$A = -\frac{y_2}{2t} + \frac{1}{t}\psi\left(\frac{y_2}{t}\right) \tag{12}$$

which gives, on substituting into (10), the equation

$$\ddot{\psi} + 2\psi\dot{\psi} = 0$$

for ψ, where the dot denotes differentiation with respect to the variable $\omega = y_2/t$. This equation readily integrates to

$$\dot{\psi} + \psi^2 = c,$$

where c is a constant. This gives us three cases:

$$c = 0: \quad \psi = t/(kt + y_2), \tag{13}$$

where k is a constant;

$$c = a^2, \quad a > 0: \quad \psi = a\left(t\exp\left(\frac{2ay_2}{t}\right) - 1\right)\Big/\left(t\exp\left(\frac{2ay_2}{t}\right) + 1\right) \tag{14}$$

with $l \neq 0$ a constant;

$$c = -a^2, \quad a > 0: \quad \psi = -a \tan\left(a^2 + \frac{ay_2}{t}\right). \tag{15}$$

Substituting these into (12), one obtains exact solutions of (10). We use these exact solutions for A together with theorem 3 (iii) (with $B = 0$) as follows. The equation (8) is conditionally invariant under

$$\partial_{y_2} + Av\partial_v \tag{16}$$

and this gives us an ansatz for v to be substituted into (8), and this, in turn, gives us an exact solution of (8) which, when we combine it with (7), gives an exact solution of (1). We list the results of these stages for each of the equations (13)–(15).

The ansatz for v from (13) is

$$v = (kt + y_2) \exp\left(-y_2^2/4t\right) \Phi(t, y_1),$$

where $\Phi(t, y_1)$ satisfies

$$\Phi_t = \Phi_{y_1 y_1} - \frac{3}{2}\Phi$$

and consequently we find that v is given by

$$v = (kt + y_2) \exp\left(-y_2^2/4t - 3t/2\right) \Phi(t, y_1),$$

where $\Psi(t, y_1)$ satisfies the $(1+1)$-dimensional heat equation.

The ansatz for v from (14) is

$$v = e^{a^2/t}\left[l \exp\left(-(y_2 - 2a)^2/4t\right) + \exp\left(-(y_2 + 2a)^2/4t\right)\right] \Phi(t, y_1),$$

where $\Phi(t, y_1)$ satisfies

$$\Phi_t + \left(\frac{1}{2t} - \frac{a^2}{t^2}\right)\Phi = \Phi_{y_1 y_1}$$

and using this we eventually find that v is given by

$$v = \frac{1}{\sqrt{t}}\left[l e^{ay_2/t} + e^{-ay_2/t}\right] \exp\left(-\left(4a^2 + y_2^2\right)/4t\right) \Psi(t, y_1), \tag{17}$$

where $\Psi(t, y_1)$ satisfies the $(1+1)$-dimensional heat equation.

The ansatz for v from (15) is

$$v = \cos\left(a^2 + \frac{ay_2}{t}\right) \exp\left(-y_2^2/4t\right) \Phi(t, y_1),$$

where $\Phi(t, y_1)$ satisfies

$$\Phi_t + \left(\frac{1}{2t} - \frac{a^2}{t^2}\right)\Phi = \Phi_{y_1 y_1},$$

so that we obtain

$$v = \frac{1}{\sqrt{t}} \cos\left(a^2 + \frac{ay_2}{t}\right) \exp\left(-\left(4a^2 + y_2^2\right)/4t\right) \Psi(t, y_1), \tag{18}$$

where $\Psi(t, y_1)$ satisfies the $(1+1)$-dimensional heat equation.

We can now combine equations (17)–(19) with equation (7) to obtain new solutions of (1):

$$u = [k(\tau x) + (\delta x)] \exp\left(\frac{(\epsilon x)}{2} - \frac{(\delta x)^2}{4(\tau x)} - \frac{3(\tau x)}{2}\right) \Psi((\tau x), (\beta x)),$$

$$u = \frac{1}{\sqrt{(\tau x)}}\left[le^{a(\delta x)/(\tau x)} + e^{-a(\delta x)/(\tau x)}\right]\exp\left(\frac{(\epsilon x)}{2} - \frac{(4a^2+(\delta x)^2)}{4(\tau x)}\right)\Psi((\tau x),(\beta x)),$$

$$u = \frac{1}{\sqrt{(\tau x)}} \cos\left(a^2 + \frac{a(\delta x)}{(\tau x)}\right) \exp\left(\frac{(\epsilon x)}{2} - \frac{(4a^2 + (\delta x)^2)}{4(\tau x)}\right) \Psi((\tau x), (\beta x)),$$

where $\Psi(t, x)$ is any solution of the $(1 + 1)$-dimensional heat equation.

One can, in principle, perform the same procedure for the other conditional symmetry operators defined in theorem 3; however, it is first necessary to obtain some exact solutions of the systems. These latter are quite nonlinear and require further treatment, and we leave this to a future publication.

5. Conclusion

We have been able to give a new reduction of the linear wave equation in $1 + 3$ timespace dimensions to a linear heat equation in $1 + 2$ timespace dimensions, that is, a reduction of a hyperbolic equation to a parabolic one. The further reductions of this heat equation by two-dimensional subalgebras (inequivalent under the action of $G_2(1, 2)$) to ordinary differential equations leads to exact solutions in terms of special functions. These are of interest in their own right. Conditional symmetries can also be used to obtain new exact solutions. Using these solutions of the heat equation, one can construct new solutions of the linear wave equation. In concluding, we remark that the complex nonlinear wave equation

$$\Box\Psi + F(|\Psi|, \partial_\mu|\Psi|\partial^\mu|\Psi|)\Psi = 0,$$

where F is an arbitrary smooth function of its arguments and Ψ is a complex function, can be reduced by the same ansatz as (7) (but with k imaginary) to a nonlinear Schrödinger equation with the same nonlinearity. Some of these equations admit soliton solutions. We report on these results in [3].

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1. Fushchych W.I., Serov N.I., Symmetry and exact solutions of the nonlinear multi-dimensional Liouville, d'Alembert and eikonal equations, *J. Phys. A: Math. Gen.*, 1983, **16**, 3645–3658.
2. Fushchych W.I., Shtelen W.M., Serov N.I., Symmetry analysis and exact solutions of equations of nonlinear mathematical physics, Dordrecht, Kluwer, 1993.
3. Basarab-Horwath P., Fushchych W.I., Barannyk L.F., Exact solutions of the nonlinear wave equation by reduction to the nonlinear Schrödinger equation, Preprint, Linköping University, 1994.

4. Fushchych W.I., Barannyk L.F., Barannyk A.F., Subgroup analysis of the Galilei and Poincaré groups and reduction of nonlinear equations, Kiev, Nauka Dumka, 1991 (in Russian).
5. Fushchych W.I., Nikitin A.G., Symmetries of equations of quantum mechanics, New York, Allerton Press Inc., 1994.
6. Bluman G.W., Cole J.D., The general similarity solution of the heat equation, *J. Math. Mech.*, 1969, **18**, 1025–1042.
7. Bluman G.W., Kumei S., Symmetries and differential equations, New York, Springer, 1989.
8. Jacobson N., Lie algebras, New York, Interscience, 1962.
9. Naimark M., Stem A., Theory of group representations, Berlin, Springer, 1982.
10. Patera J., Winternitz P., Zassenhaus H., Continuous subgroups of the fundamental groups of physics. I, *J. Math. Phys.*, 1975, **16**, 1597–1614.
11. Erdélyi A., Magnus W., Oberhettinger F., Tricomi F.G., Higher transcendental functions (Bateman manuscript project), New York, McGraw-Hill, 1953.
12. Borhardt A.A., Karpenko D.Ja., The characteristic problem for the wave equation with mass, *Differential Equations*, 1984, **20**, 239–245.

Amplitude-phase representation for solutions of nonlinear d'Alembert equations

P. BASARAB-HORWATH, N. EULER, M. EULER, W.I. FUSHCHYCH

We consider the nonlinear complex d'Alembert equation $\Box\Psi = F(|\Psi|)\Psi$ with Ψ represented in terms of amplitude and phase, in $(1+n)$-dimensional Minkowski space. We exploit a compatible d'Alembert–Hamilton system to construct new types of exact solutions for some nonlinearities.

1. Introduction

Let us consider the general nonlinear complex d'Alembert equation in $(1+n)$-dimensional Minkowski space

$$\Box\Psi = F(|\Psi|)\Psi, \tag{1}$$

where F is a smooth, real function of its argument, Ψ is a complex function of $1+n$ real variables, and

$$\Box = \frac{\partial^2}{\partial x_0^2} - \frac{\partial^2}{\partial x_1^2} - \cdots - \frac{\partial^2}{\partial x_n^2}.$$

Equation (1) plays a fundamental role in classical and quantum field theories, and in superfluidity and liquid crystal theory. Many exact solutions have been found using Lie symmetry methods [6, 11, 12, 13, 8, 7], as well as with conditional symmetries [7].

In this paper we use the representation $\Psi = ue^{iv}$, where u is the amplitude and v is the phase (both real functions). On substituting this in (1), we find the following system:

$$\Box u - u(v_\mu v_\mu) = uF(u), \tag{2}$$

$$u\Box v + 2u_\mu v_\mu = 0. \tag{3}$$

We use the notation

$$u_\mu v_\mu = \frac{\partial u}{\partial x_0}\frac{\partial v}{\partial x_0} - \frac{\partial u}{\partial x_1}\frac{\partial v}{\partial x_1} - \cdots - \frac{\partial u}{\partial x_n}\frac{\partial v}{\partial x_n}.$$

The system (2), (3) is obviously equivalent to the starting equation (1). However, equations (2), (3) has the advantage that it gives us the possibility of making functional and differential connections between the amplitude and phase, which substantially simplifies the problem of integrating equation (1). Moreover, in assuming the simplest possible relations between the amplitude and phase, we are able to construct exact solutions of (2), (3), and hence of (1).

We now seek solutions of (2), (3). We consider two cases: (i) the amplitude as a function of the phase, $u = g(v)$; (ii) the phase as a function of the amplitude, $v = g(u)$. This is reminiscent of the polar description of plane curves in geometry. The

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system (2), (3) then yields a pair of equations for the phase v in the first case and for the amplitude u in the second case. There then arises the question of the compatibility of the two equations obtained, and we solve it by exploiting the compatible system

$$w = \frac{\lambda N}{w}, \quad w_\mu w_\mu = \lambda, \tag{4}$$

where $\lambda = -1, 0, 1$ and $N = 0, 1, \ldots, n$. Exact solutions for the system (4) are given in table 1 in section 2.

The system (4) is a particular case of the d'Alembert–Hamilton system

$$\Box w = F_1(w), \quad w_\mu w_\mu = F_2(w), \tag{5}$$

The system (5) was studied by Smirnov and Sobolev in 1932, with $w = w(x_0, x_1, x_2)$ and $F_1 = F_2 = 0$. Collins [2–4] studied (5) with w a function of three complex variables, and obtained compatibility conditions for the functions $F_1(w)$, $F_2(w)$. For $(1+3)$ and higher dimensional Minkowski space, (5) was studied by Fushchych and co-workers [9, 10]: they obtained compatibility conditions for $F_1(w)$, $F_2(w)$ and some exact solutions.

Here, we exploit the results of Fushchych et. al [10], applying them to the system (4). Moreover, the compatibility of (4) dictates the type of nonlinearity $F(u)$ which can appear in (1). This is the novelty of our approach to finding some exact solutions of (1).

2. Solutions

2.1. $u = g(\nu)$. We now assume that the amplitude is a function of the phase: $u = g(v)$. Inserting this assumption in (2), (3), we obtain

$$\Box v = \frac{-2g\dot{g}F(g)}{g\ddot{g} - 2\dot{g}^2 - g^2} = F_1(v), \tag{6}$$

$$v_\mu v_\mu = \frac{g^2 F(g)}{g\ddot{g} - 2\dot{g}^2 - g^2} = F_2(v) \tag{7}$$

with $\dot{g} = dg/dv$.

We now deal with (6), (7) in two ways: (i) assume forms for F_1, F_2 so as to make equations (6), (7) compatible; (ii) transform equation (4) locally so as to agree with (6), (7).

First, let us make the assumption

$$F_1(v) = \frac{\lambda N}{v}, \quad F_2(v) = \lambda$$

with $N, \lambda \neq 0$. Then equations (6), (7) become a compatible system [10], and we also find that g and F must satisfy

$$g\ddot{g} - 2\dot{g}^2 - g^2 - \frac{g^2}{\lambda} F(g) = 0, \quad \frac{-2\dot{g}}{g} = N. \tag{8}$$

From (8) it now follows that

$$g(v) = \sigma v^{-N/2}, \quad F(v) = -\lambda + \lambda \frac{N}{2}\left(1 - \frac{N}{2}\right)\sigma^{-4/N} v^{4/N},$$

where $\sigma \neq 0$ is an arbitrary real constant.

With this, we have obtained the following:

Result 1. An exact solution of (1) with nonlinearity

$$F(|\Psi|) = -\lambda + \sigma^{-4/N}\lambda\frac{N}{2}\left(1-\frac{N}{2}\right)|\Psi|^{4/N}$$

is given by

$$\Psi(x) = \sigma v(x)^{-N/2}e^{iv(x)},$$

where $v(x)$ is a solution of the compatible system (4) for $N, \lambda \neq 0$.

Our next step is to perform a local transformation of (4). We do this by setting $w = f(v)$ in (4) (with $\lambda \neq 0$), with f a real, smooth function such that $\dot{f} \neq 0$. With this substitution, we obtain the system:

$$\Box v = \frac{\lambda N}{f(v)\dot{f}(v)} - \frac{\lambda\ddot{f}(v)}{\dot{f}^3(v)}, \tag{9}$$

$$v_\mu v_\mu = \frac{\lambda}{\dot{f}^2(v)}. \tag{10}$$

The system (9), (10) is obviously compatible since it is the local transformation of an already compatible system. However, it should be noted that this does not mean that the exact solutions we obtain by using (9), (10) are equivalent to those obtained from (8), since we have introduced some extra freedom via the function f.

We now equate the right-hand sides of (6), (7) with the right-hand sides of (9), (10), respectively. A little algebraic manipulation gives us

$$g(v) = \sigma\left(\frac{\dot{f}(v)}{f^N(v)}\right)^{1/2}, \tag{11}$$

where σ is an arbitrary non-zero constant. Thus we have a differential relation between f and g which we can integrate. For $N = 1$ we obtain

$$f(v) = C\exp\left(\frac{1}{\sigma^2}\int^v g^2(\xi)d\xi\right) \tag{12}$$

and g has to satisfy the integro-differential equation

$$g\ddot{g} - 2\dot{g}^2 - g^2 - \frac{C^2}{\lambda\sigma^4}g^6\exp\left(\frac{1}{\sigma^2}\int^v g^2(\xi)d\xi\right)F(g) = 0. \tag{13}$$

For $N \neq 1$ we find

$$f(v) = \left(\frac{1-N}{\sigma^2}\int^v g^2(\xi)d\xi + C\right)^{1/(1-N)}, \tag{14}$$

C being an arbitrary real constant, and with the following condition on g:

$$g\ddot{g} - 2\dot{g}^2 - g^2 - \frac{1}{\lambda\sigma^4}g^6\left(\frac{1-N}{\sigma^2}\int^v g^2(\xi)d\xi + C\right)^{2N/(1-N)}F(g) = 0. \tag{15}$$

Our result is summarized in the following:

Result 2. (i) The function

$$\Psi(x) = g(v(x)) \exp[iv(x)]$$

is a solution of (1) whenever g is a solution of (13) and $w(x) = f(v(x))$ is a solution of

$$\Box w = \frac{\lambda}{w}, \quad w_\mu w_\mu = \lambda$$

with f given by (12).

(ii) The function

$$\Psi(x) = g(v(x)) \exp[iv(x)]$$

is a solution of (1) whenever g is a solution of (15) and $w(x) = f(v(x))$ is a solution of

$$\Box w = \frac{\lambda N}{w}, \quad w_\mu w_\mu = \lambda$$

with f given by (14) for $N \neq 1$.

One may treat (13) and (15) in two ways: consider F as given, and then attempt to solve for g, or make an assumption about g and then find the corresponding F. We take this second approach, and in doing so, we determine the function f which appears in (12) and (14), which also relates (4) to the system (6), (7).

This is illustrated in the following example, where we take g as $g(v) = v^\beta$. Then we obtain after some elementary manipulation

$$w = f(v) = Cv^{1/\sigma^2}$$

when $N = 1$, $\beta = -\frac{1}{2}$ In this case we find the corresponding nonlinear version of (1) and an exact solution:

$$\Box\psi + \frac{\lambda\sigma^4}{C^2}\left(\frac{1}{4}|\psi|^{4/\sigma^2} - |\psi|^{4(1-\sigma^2)/\sigma^2}\right)\psi = 0,$$

$$\psi(x) = \left(\frac{1}{C}w(x)\right)^{-\sigma^2/2} \exp\left[i\left(\frac{1}{C}w(x)\right)^{\sigma^2}\right],$$

where w is a solution of

$$\Box w = \frac{\lambda}{w}, \quad w_\mu w_\mu = \lambda.$$

The solutions of this system are given in table 1. We can choose the nonlinearity in the above wave equation by choosing σ. For instance, for $\sigma^2 = \frac{2}{3}$ we obtain the equation

$$\Box\Psi - \left(\frac{2}{3}\right)^2 \frac{\lambda}{C^2}\left(|\Psi|^2 - \frac{1}{4}|\Psi|^6\right)\Psi = 0. \tag{16}$$

Equation (16) is of the type considered by Grundland and Tuczynski [12].

Table 1. Solutions for the system $\Box_n w = \lambda N/w$, $(\nabla_n w)^2 = \lambda$. Inner products are with respect to the Minkowski metric.

λ	N	Solutions w	Conditions on $a_\mu \in \mathbb{R}^n, b_\mu \in \mathbb{R}^n$
± 1	$N \in \{1, 2, \ldots, n-1\}$	$\left[(a_0 \cdot x)^2 \pm (a_1 \cdot x)^2 \pm \cdots \pm (a_N \cdot x)^2\right]^{1/2}$	$a_0 \cdot a_0 = 1,\ a_0 \cdot a_j = 1,\ a_j \cdot a_k = \pm\delta_{jk},$ $j, k = 1, 2, \ldots, N$
1	$N \in \{1, 2, \ldots, n-1\}$	$\left[(a_0 \cdot x)^2 \pm (a_1 \cdot x)^2 \pm \cdots \pm (a_N \cdot x)^2\right]^{1/2}$	$a_\mu \cdot a_\mu = -1,\ a_\mu \cdot a_\nu = 0,,$ $\nu, \mu = 0, 1, \ldots, N$
1	$N \in \{1, 2, \ldots, n-3\}$	$\left[\left(b_1 \cdot x + h_1\left(\sqrt{k}a_0 \cdot x + \sum\limits_{i=1}^{k} a_i \cdot x\right)\right)^2 + \right.$ $+ \left(b_2 \cdot x + h_2\left(\sqrt{k}a_0 \cdot x + \sum\limits_{i=1}^{k} a_i \cdot x\right)\right)^2 + \cdots$ $\left.\cdots + \left(b_{N+1} \cdot x + h_{N+1}\left(\sqrt{k}a_0 \cdot x + \sum\limits_{i=1}^{k} a_i \cdot x\right)\right)^2\right]^{1/2}$ $h_1, h_2 \ldots, h_{N+1}$ are arbitrary real functions	$a_0 \cdot a_0 = 1$ $a_i \cdot a_i = -1$ $b_j \cdot b_j = -1$ $a_0 \cdot a_i = 0,\ a_0 \cdot b_j = 0,\ a_i \cdot b_j = 0,\ b_j \cdot b_l = 0$ $i = 1, \ldots, k,\ (k \le n-1),\ j \ne l = 1, \ldots, N+1$
± 1	$N = 0$	$b_1 \cdot x \cos h_1\left(\sqrt{k}a_0 \cdot x + \sum\limits_{i=1}^{k} a_i \cdot x\right) +$ $+ b_2 \cdot x \sin h_1\left(\sqrt{k}a_0 \cdot x + \sum\limits_{i=1}^{k} a_i \cdot x\right) + h_2\left(\sqrt{k}a_0 \cdot x + \sum\limits_{i=1}^{k} a_i \cdot x\right)$ h_1 and h_2 are arbitrary real functions	$a_0 \cdot a_0 = \pm 1,\ a_i \cdot a_i = \mp 1,\ b_j \cdot b_j = \mp 1$ $a_0 \cdot a_i = 0,\ a_0 \cdot b_j = 0,\ a_i \cdot b_j = 0,\ b_j \cdot b_l = 0$ $i = 1, \ldots, k,\ (k \le n-1),\ j \ne l = 1, 2$
± 1	$N = 0$	$b_1 \cdot x \cos h_1\left(\sqrt{k}a_0 \cdot x + \sum\limits_{i=1}^{k} a_i \cdot x\right) +$ $+ b_2 \cdot x \sin h_1\left(\sqrt{k}a_0 \cdot x + \sum\limits_{i=1}^{k} a_i \cdot x\right) + h_2\left(\sqrt{k}a_0 \cdot x + \sum\limits_{i=1}^{k} a_i \cdot x\right)$ h_1 and h_2 are arbitrary real functions	$a_0 \cdot a_0 = 1,\ a_i \cdot a_i = -1$ $b_1 \cdot b_1 = \pm 1,\ b_2 \cdot b_2 = \mp 1$ $a_0 \cdot a_i = 0,\ a_0 \cdot b_j = 0,\ a_i \cdot b_j = 0,\ b_j \cdot b_l = 0$ $i = 1, \ldots, k,\ (k \le n-1),\ j \ne l = 1, 2$
0	$N = 0$	$h_1\left(\sqrt{k}a_0 \cdot x + \sum\limits_{i=1}^{k} a_i \cdot x\right)$ h_1 is an arbitrary real function	$a_0 \cdot a_0 = 1,\ a_i \cdot a_i = -1,\ a_0 \cdot a_i = 0$ $i = 1, \ldots, k\ (k \le n-1)$

For $N = 2$, $\beta = -1$ we obtain the following wave equation and exact solution:

$$\Box\Psi + \frac{1}{\sigma^8}\left(|\Psi|^2 + \frac{1}{\lambda\sigma^4}|\Psi|^6\right)\Psi = 0, \tag{17}$$

$$\Psi(x) = (C - w(x))\sigma^2 \exp\left[i\frac{1}{(C - w(x))\sigma^2}\right], \tag{18}$$

where w is a solution of the compatible system

$$\Box w = \frac{2\lambda}{w}, \quad w_\mu w_\mu = \lambda$$

and exact solutions of this system are given in table 1. Equation (17) is also of a type considered by Grundland and Tuczynski [12]. Our exact solutions are new.

2.2. $\nu = g(u)$. We now assume that the phase is a function of the amplitude: $v = g(u)$. On substituting this in equations (2), (3), we obtain

$$\Box u = \frac{(u^2\ddot{g} + 2u\dot{g})F(u)}{u\ddot{g} + 2\dot{g} + u^2\dot{g}^3} \equiv F_1(u), \tag{19}$$

$$u_\mu u_\mu = \frac{-u^2\dot{F}(u)}{u\ddot{g} + 2\dot{g} + u^2\dot{g}^3} \equiv F_2(u). \tag{20}$$

Here $\dot{g} = dg/du$.

We perform the same analysis as before. First, letting $F_1(u) = \lambda N/u$, $F_2(u) = \lambda$, $\lambda \neq 0$, we find (after some computation)

$$g(u) = -\frac{\sigma u^{N+1}}{N+1} + \sigma_1, \quad F(u) = \frac{\lambda N}{u^2} - \frac{\lambda\sigma_1}{u^{2(N+2)}}.$$

Having determined g and the nonlinearity of the wave equation (19), we have the following:

Result 3. An exact solution of (1) with nonlinearity

$$F(|\Psi|) = \lambda N|\Psi|^{-2} - \lambda\sigma_1|\Psi|^{-2(N+2)}$$

is given by

$$\Psi(x) = Cu(x)\exp\left(\frac{-i\sigma}{(N+1)u(x)^{(N+1)}}\right),$$

where $\lambda \neq 0$ and $C \neq 0$ is an arbitrary real constant, and where $u(x)$ is a solution of the system (4).

Another way of dealing with (19), (20) is to transform (4) locally using the transformation $w = f(u)$ with $\dot{f} \neq 0$, which gives us

$$\Box u = \frac{\lambda N}{f(u)\dot{f}(u)} - \frac{\lambda\ddot{f}(u)}{\dot{f}^3(u)}, \tag{21}$$

$$u_\mu u_\mu = \frac{\lambda}{\dot{f}^2(u)}. \tag{22}$$

Then, equating the right-hand sides of (21), (22) with the right-hand sides of (19), (20), we find (for $\lambda \neq 0$, as before) that

$$u^2\dot{g}(u) = \sigma\frac{\dot{f}(u)}{f^N(u)},$$

where $\sigma \neq 0$ is an arbitrary real constant. Again we see that there are two cases to consider: $N = 1$ and $N \neq 1$.

For $N = 1$ we obtain

$$f(u) = C\exp\left(\frac{1}{\sigma}\int^u \xi^2\frac{dg(\xi)}{d\xi}d\xi\right) \tag{23}$$

with C an arbitrary real constant. The condition on g is

$$u\ddot{g} + 2\dot{g} + u^2\dot{g}^3 + \frac{u^4C^2}{\lambda\sigma^2}\dot{g}^3\exp\left(\frac{1}{\sigma}\int^u \xi^2\frac{dg(\xi)}{d\xi}d\xi\right)F(u) = 0. \tag{24}$$

When $N \neq 1$, f is given by

$$f(u) = \left(C - \frac{N-1}{\sigma}\int^u \xi^2\frac{dg(\xi)}{d\xi}d\xi\right)^{1/(1-N)} \tag{25}$$

with C an arbitrary real constant and with the following condition on g:

$$u\ddot{g} + 2\dot{g} + u^2\dot{g}^3 + \frac{u^4}{\lambda\sigma^2}\left(C - \frac{N-1}{\sigma}\int^u \xi^2\frac{dg(\xi)}{d\xi}d\xi\right)^{2N/(1-N)}F(u) = 0.$$

This reasoning can be summarized in the following:

Result 4. (i) The function

$$\Psi(x) = u(x)\exp(ig(u(x)))$$

is a solution of (1) whenever g is a solution of (24) and $w(x) = f(u(x))$ is a solution of

$$\Box w = \frac{\lambda}{w}, \quad w_\mu w_\mu = \lambda$$

with f given by (23).

(ii) The function

$$\Psi(x) = u(x)\exp(ig(u(x)))$$

is a solution of (1) whenever g is a solution of (26) and $w(x) = f(u(x))$ is a solution of

$$\Box w = \frac{\lambda N}{w}, \quad w_\mu w_\mu = \lambda$$

with f given by (25) for $N \neq 1$.

We treat equations (24) and (26) relating g to the nonlinearity F as before: we assume a form for g and treat the equations as determining F. Taking $g(u) = u^\beta$, we have the following examples of the wave equation, exact solution and relation between u and w:

$N = 1$, $\beta \neq -2$.

$$\Box\Psi + \frac{\lambda\sigma^2}{C^2}|\Psi|^{-2}\left(1 + \frac{\beta+1}{\beta^2}|\Psi|^{-2\beta}\right)\exp\left[\frac{-\beta}{\sigma(\beta+2)}|\Psi|^{\beta+2}\right)\Psi = 0,$$
$$\Psi(x) = u(x)\exp[iu(x)^\beta],$$
$$u = \left(\frac{\sigma(\beta+2)}{\beta}\ln\left|\frac{w}{C}\right|\right)^{1/(\beta+2)},$$

where w is a solution (listed in table 1) of the compatible system

$$\Box w = \frac{\lambda}{w}, \quad w_\mu w_\mu = \lambda.$$

$N \neq 1$, $\beta \neq -2$.

$$\Box\Psi + \lambda\sigma^2|\Psi|^{-2}\left(1 + \frac{\beta+1}{\beta^2}|\Psi|^{-2\beta}\right)\left(C - \frac{(N-1)\beta}{\sigma(\beta+2)}|\Psi|^{\beta+2}\right)^{2N/(N-1)}\Psi = 0,$$
$$\Psi(x) = u(x)\exp[iu(x)^\beta],$$
$$u = \left(\frac{\sigma(\beta+2)}{(N-1)\beta}(C - w^{1-N})\right)^{1/(\beta+2)},$$

where w is a solution (listed in table 1) of the compatible system

$$\Box w = \frac{\lambda N}{w}, \quad w_\mu w_\mu = \lambda.$$

If we choose $\beta = -1$, $N = 2$, $C = 0$, then we find that the wave equation is

$$\Box\Psi + \lambda\sigma^{-2}|\Psi|^2\Psi = 0 \tag{26}$$

with the exact solution

$$\Psi(x) = u(x)\exp\left(\frac{i}{u(x)}\right)$$

and

$$u(x) = \sigma w(x),$$

where w solves

$$\Box w = \frac{2\lambda}{w}, \quad w_\mu w_\mu = \lambda.$$

Equation (27) is of some interest: of all the possible nonlinearities $F(|\Psi|)$, the nonlinearity $F(|\Psi|) = |\Psi|^2$ gives the widest possible symmetry group, admitting the conformal group. Equation (27) (and indeed equation (1)) can be reduced to the nonlinear Schrödinger equation in $(1+2)$-dimensional time-space (see [1]) with the same nonlinearity. This equation also admits the widest possible symmetry group for nonlinearities of the given type. It can be reduced to the $(1+1)$-dimensional nonlinear Schrödinger equation with the same nonlinearity, and this equation has soliton solutions (the well known Zakharov–Shabat soliton). Using this soliton, we can construct a new type of solution of the hyperbolic wave equation (27). Of course, this does not imply that (27) has soliton solutions located in three-dimensional space.

3. Conclusion

We have demonstrated an approach which can give new exact solutions of some nonlinear wave equations of the same type as (1). The novelty in our approach lies in the fact that we exploit the compatibility conditions for the d'Alembert–Hamilton system to dictate the type of nonlinearity and the exact solution(s). Moreover, some of the equations we obtain appear to be of interest in physics, but we are unable to make any statement about the physical nature of the exact solutions we obtain, as our approach has not used any physical criteria to single out any type of solution.

Of course, this is not the only approach possible; we could, for instance, reduce (1) to the Schrödinger equation (as in [1]) and then apply a similar method to this new equation in the amplitude-phase representation. Also, it is possible to consider a more general connection between the amplitude and phase, such as $u = G(v_\mu v_\mu)$ for some function G. This leads to a system involving the Born–Infeld equation, which has a very wide symmetry group, and we obtain new exact solutions of (1). This differential connection between amplitude and phase will of course be important when we allow nonlinearities dependent on derivatives, such as $F(|\Psi|, \Psi^*_\mu \Psi_\mu)$. We will report on this work in a forthcoming paper.

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1. Basarab-Horwath P., Barannyk L., Fushchych W., *J. Phys. A: Math. Gen.*, 1995, **28**, 1–14.
2. Collins C.B., *Math. Proc. Camb. Phil. Soc.*, 1976, **80**, 165–187.
3. Collins C.B., *Math. Proc. Camb. Phil. Soc.*, 1976, **80**, 349–355.
4. Collins C.B., *J. Math. Phys.*, 1980, **21**, 349–355.
5. Collins C.B., *J. Math. Phys.*, 1983, **24**, 22–28.
6. Fushchych W.I , Serov N.I., *J. Phys. A: Math. Gen.*, 1983, **16**, 3645–3658.
7. Fushchych W.I., Shtelen W., Serov M., Symmetry analysis and exact solutions of equations of nonlinear mathematical physics, Dordrecht, Kluwer, 1993.
8. Fushchych W.I., Yehorchenko I.A., *J. Phys. A: Math. Gen.*, 1989, **22**, 2643–2652.
9. Fushchych W.I., Zhdanov R.Z., On some exact solutions of nonlinear d'Alembert and Hamilton equations, Preprint Institute for Mathematics and Applications, University of Minnesota, 1988.
10. Fushchych W.I., Zhdanov R.Z., Yehorchenko I.A., *Mathematical Analysis and Applications*, 1991, **161**, 352–360.
11. Grundland M.A, Harnad J., Winternitz P., *J. Math. Phys.*, 1984, **25**, 791–805.
12. Grundland M.A., Tuszynski J.A., *J. Phys. A: Math. Gen.*, 1987, **20**, 6243–6258.
13. Grundland M.A., Tuszynski J.A., Winternitz P., *Phys. Lett. A*, 1987, **119**, 340–344.

Planck's constant is not constant in different quantum phenomena

O. BEDRIJ, W.I. FUSHCHYCH

Висунута ідея про те, що стала Планка h в мезодинаміці істотно відрізняється від сталої Планка в електродинаміці. Запропоновані рівняння руху для електрона, протона і нейтрона, в яких стала Планка має різні значення.

At present, it is a generally accepted axiom that the Planck's constant

$$h = 6.626 \cdot 10^{-34} J \cdot s \tag{1}$$

has the same meaning and value in electrodynamics, mesodynamics, quantum theory, theory of quarks, gravidynamics, etc. Planck's fundamental quantum hypothesis, put forward for the explanation of the energy spectrum of black body radiation, is ad hoc employed in all quantum physics.

In [1] we have suggested the following hypothesis: the fundamental value of Planck's constant h in mesodynamics is considerably different from (1). This assumption, for example, can be explained by the fact that in mesodynamics, not a photon but a meson is emitted, which mass does not equal zero. There are no fundamental grounds to assume that h in mesodynamics has to have the value of (1) [1–4].

In this short note we focus on the equation of motion for the fundamental particles (e — electron, p — proton, n — neutron) based on the aforementioned hypothesis. Schrödinger equations for electron, proton and neutron have the following form in our approach:

$$i\hbar_e \frac{\partial \Psi_e}{\partial t} = -\frac{\hbar_e^2}{2m_e} \Delta \Psi_e + V_c(x)\Psi_e, \tag{2}$$

$$i\hbar_p \frac{\partial \Psi_p}{\partial t} = -\frac{\hbar_p^2}{2m_p} \Delta \Psi_p + V_p\Psi_p, \tag{3}$$

$$i\hbar_n \frac{\partial \Psi_n}{\partial t} = -\frac{\hbar_n^2}{2m_n} \Delta \Psi_n + V_n\Psi_n, \tag{4}$$

$$\Delta \equiv \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2},$$

where

$$\hbar_e = \hbar, \quad \hbar_p \neq \hbar, \quad \hbar_n \neq \hbar,$$

$\hbar_e$ — Planck's constant for electron (electrodynamics); $\hbar_p$ — Planck's constant for proton (mesodynamics); $\hbar_n$ — Planck's constant for neutron (mesodynamics).

Доповіді НАН України, 1995, № 12, С. 26–27.

The V_e, V_p, V_n potentials are assumed to be depended on h_e, h_p and h_n, the wave functions Ψ_e, Ψ_p and Ψ_n as well as the coordinates of particles.

In addition, let us consider Poincaré invariant equations of motion for meson and for e, p, n. As is known, the energy of an elementary particle is defined by formulae:

$$E^2 = c^2 p_a^2 + m^2 c^4, \quad p_a^2 = p_1^2 + p_2^2 + p_3^2, \tag{5}$$

where m is the mass of a particle; c is the velocity of light in vacuum; p_a is momentum. Formulae (5) give us the following Poincaré-invariant equations for particles

$$-\hbar_\mu^2 \frac{\partial^2 u}{\partial t^2} = \left(- \hbar_\mu^2 c^2 \Delta + m_\mu^2 c^4 \right) + V_\mu u, \tag{6}$$

$$i\hbar_e \frac{\partial \Psi_e}{\partial t} = \left\{ -i\hbar_e \gamma_0 \gamma_k \frac{\partial}{\partial x_k} + m_e c^2 \gamma_0 \right\} \Psi_e + V_e \Psi_e, \tag{7}$$

$$i\hbar_p \frac{\partial \Psi_p}{\partial t} = \left\{ -i\hbar_p \gamma_0 \gamma_k \frac{\partial}{\partial x_k} + m_p c^2 \gamma_0 \right\} \Psi_p + V_p \Psi_p, \tag{8}$$

$$i\hbar_n \frac{\partial \Psi_n}{\partial t} = \left\{ -i\hbar_n \gamma_0 \gamma_k \frac{\partial}{\partial x_k} + m_n c^2 \gamma_0 \right\} \Psi_n + V_n \Psi_n, \tag{9}$$

where Ψ_e, Ψ_p, Ψ_n are four-component wave functions; u is a scalar wave function for meson with mass m_μ; γ_μ are 4×4 Dirac's matrices. Equations (7), (8) and (9) are Dirac's equations with different Planck's constants, $V_e = V_e(\Psi_e, \Psi_p, \Psi_n, x, t)$, $V_p = V_p(\Psi_e, \Psi_p, \Psi_n, x, t)$, $V_n = V_n(\Psi_e, \Psi_p, \Psi_n, x, t)$.

Consequently, to describe interactions between electron and proton, electron and neutron, etc., it is necessary to use different values for $\hbar_e$, $\hbar_p$ and $\hbar_n$ in equations (6)–(9).

A phenomenological approach, proposed in [2–4] for determining fundamental constants and based on a few known constants, gives us the following values [2, 3]:

$$h_e = 6.626 \cdot 10^{-34} J \cdot s, \quad h_p = 2.612 \cdot 10^{-30} J \cdot s, \quad h_n = 2.668 \cdot 10^{-30} J \cdot s.$$

Obviously, because h_e, h_p and h_n enter most of quantum relationships, we must review the standard theoretical schemes and possibly explore new physical experiments. This fundamental challenge will take time. Our main objective is to show a new possibility for description of interactions of particles which is related to a new value of h.

According to [5, 6], formulae (5) can be used for nonlinear generalization of equations of motion for elementary particles. Assume that in formulae (5) c is not constant but a function of field (or a functional with respect to fields)

$$c = c\left(\bar{\Psi}\Psi, \frac{\partial \bar{\Psi}\Psi}{\partial x^\mu} \frac{\partial \bar{\Psi}\Psi}{\partial x_\mu} \right). \tag{10}$$

Therefore, we can obtain from (10) a nonlinear equation of the type (6)–(8). This assumption means that velocity of a signal is a function of field [5, 6) and not a constant, as is presently accepted for the velocity of light in vacuum. The latter statement is a cornerstone of modern quantum physics. We should like to emphasize that here we have discussed a new glance on this fundamental point.

A more detailed development of these ideas will be published elsewhere.

1. Bedrij O., Fushchych W.I., Fundamental constants of nucleonmeson dynamics, Доп. АН України, 1993, № 5, 62–65.

2. Bedrij O., Fundamental constants in quantum electrodynamics, Доп. АН України, 1993, № 3, 40–45.

3. Bedrij O., Connection of π with the fine structure constants, Доп. АН України, 1994, № 10, 63–66.

4. Bedrij O., Scale invariance, unifying principle, order and sequence of physical quantities and fundamental constants, Доп. АН України, 1993, № 4, 67–74.

5. Fushchych W., New nonlinear equations for electromagnetic field having a velocity different from c, Доп. АН України, 1992, № 4, 24—27.

6. Fushchych W., Nikitin A., Symmetries of equations of quantum mechanics, New York, Allerton Press, 1994, 458 p.

Ansatz ’95

W.I. FUSHCHYCH

In this talk I am going to present a brief review of some key ideas and methods which were given start and were developed in Kyiv, at the Institute of Mathematics of National Academy of Sciences of Ukraine during recent years.

Plan of the talk

The simplest classification of equations.

What is ansatz? The problem of PDE reduction without symmetry.

Conditional symmetry. How can we expand symmetry of PDE?

Conditional symmmetry of Maxwell and Schrödinger systems.

Q-conditional symmetry of the nonlinear wave equation, which is not invariant with respect to the Lorentz group.

Conditional symmetry of the Poincaré–invariant d’Alembert equation.

Conditional symmetry of the nonlinear heat equation.

Reduction and Antireduction.

Antireduction of the nonlinear acoustics equation.

Antireduction of the equation for short waves in gas dynamics.

Antireduction of nonlinear heat equation.

Nonlocal symmetry, new relativity principles.

Non–Lie symmetry of the Schrödinger equation.

Time is absolute in relativistic physics.

New equations of motions.

High–order parabolic equation in Quantum Mechanics.

Nonlinear generalization of the Maxwell equations.

Equations for fields with the spin 1/2.

How to extend symmetry of on equation with arbitrary coefficients?

1 Classification of equations

Every field of science must begin from some classification. We have today a lot of classifications of differential equations: parabolic, hyperbolic, elliptic, ultrahyperbolic

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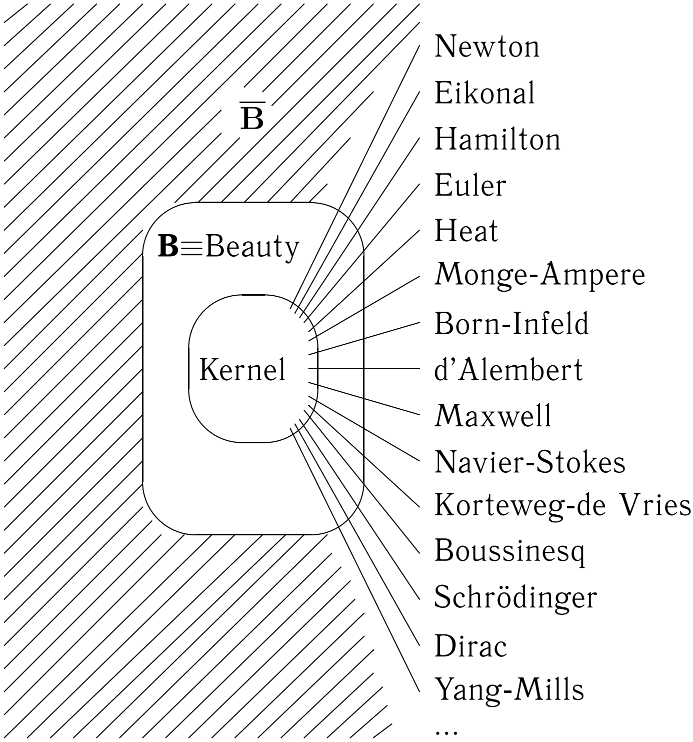
etc. I believe that it is most appropriate for our Conference to divide all equations of mathematics into two classes: B and $\bar{B}$



It is seen from the adduced picture that all fundamental equations of mathematical physics are united into one class B. From the point of view of existing now classifications they belong to essentially different classes. Equations from the class B have wide symmetry, and by this feature they are substantially different from other equations of mathematics.

It is important to point out that there are close relations among these different equations, which have not been investigated yet till now. For example, if we know solutions of the heat equation, we can construct solutions for the wave (d'Alembert) equation. By means of solutions of the Dirac equation, solutions of the Maxwell, heat, Yang–Mills, and other equations [18] can be obtained.

2 Ansatz reduction of PDE without using symmetry

Let us consider a PDE

$$L(x, u, u_{(1)}, u_{(2)}, \dots u_{(n)}) = 0,$$
$$u = u(x), \quad x = (x_0, x_1, \dots, x_n), \quad u_{(1)} = (u_0, u_1, u_2, \dots, u_n), \; u_\mu = \frac{\partial u}{\partial x_\mu}, \tag{2.1}$$
$$u_{(2)} = (u_{00}, u_{01}, \dots, u_{nn}), \; u_{\mu\nu} = \frac{\partial^2 u}{\partial x_\mu \partial x_\nu}.$$

Depending on the explicit form of L, equation (2.1) can belong to B or $\bar{B}$. In mathematical physics we often come across equations of the following type:

$$Lu \equiv \Box u - F(x, u, u_{(1)}) = 0. \tag{2.2}$$

What can we say today about solutions of equations (2.1), (2.2)? The answer is trivial: Nothing.

If equation (2.2) belongs to the class B and is invariant with respect to the Poincaré group $P(1,n)$, that is, a nonlinear function $F(x, u, u_{(1)})$ has the special form

$$F(x, u, u_{(1)}) = F\left(u, \frac{\partial u}{\partial x_\mu}\frac{\partial u}{\partial x^\mu}\right) \tag{2.3}$$

then for equation (2.2) we can construct some classes of exact solutions, study Painlevé properties, construct approximate solutions, study asymptotic properties, etc.

Definition 1. (W. Fushchych, 1981, 1983 [1, 2, 3]) We shall call a formula

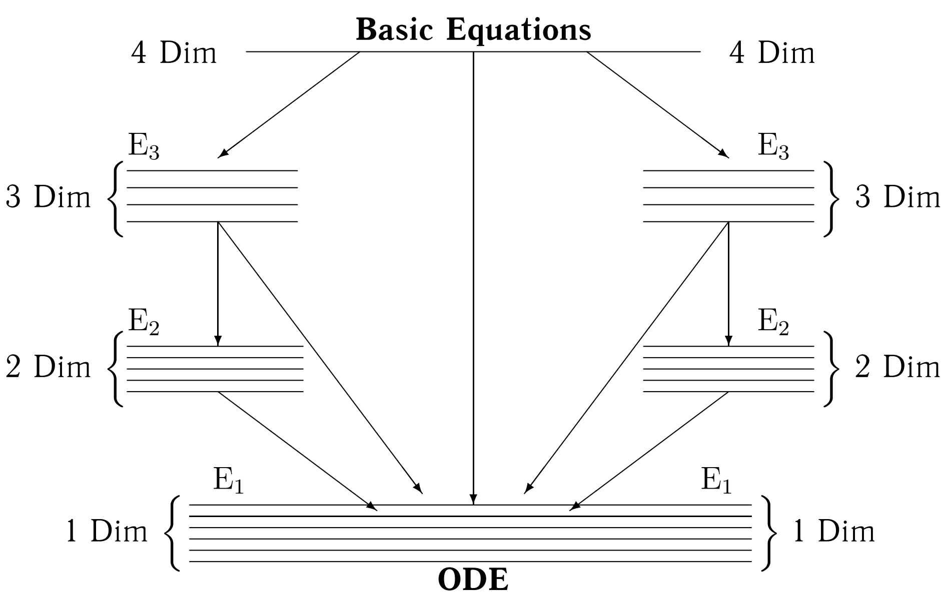
$$u = f(x)\varphi(\omega) + g(x), \tag{2.4}$$

an ansatz for equation (2.2) if after substitution of (2.4) we get an equation for the function $\varphi(\omega)$ which depends only on new variables $\omega = (\omega_1, \omega_2, \ldots, \omega_{n-1})$, where $f(x)$, $g(x)$ are given functions.

If (2.4) is an ansatz for (2.2), then the latter is reduced (the number of independent variables decreases by one) to an equation for the function $\varphi(\omega)$.

Thus the problem of reduction of an equation reduces to description of three functions $\langle f(x), g(x), \omega\rangle$ which leads to an equation for $\varphi(\omega)$ with less number of variables.

We can display schematically the process of reduction for an 4–dimensional equation in the following way:



E_3 is a set of three-dimensional equations, E_2 is a set of two-dimensional equations, E_1 is a set of one-dimensional equations with the following inclusion $E_3 \subset E_2 \subset E_1$.

That is, from one principal equation we obtain the whole set of ODE. Having solved the ODE, we find exact solutions of a multidimensional equation.

Description of ansatzes of the form (2.4) for the nonlinear wave equation is an extremely difficult nonlinear problem. In the simplest case, when we put $f(x) = 1$, $g(x) = 0$ for the nonlinear Poincaré–invariant d'Alembert equation

$$\Box u = F(u), \tag{2.5}$$

the problem of reduction of (2.5) to ODE reduces to construction of solutions for the following overdetermined system for ω (Fushchych W., Serov M. 1983 [3])

$$\begin{gathered}\Box\omega = F_2(\omega),\\ \frac{\partial\omega}{\partial x_\mu}\frac{\partial\omega}{\partial x^\mu} = \left(\frac{\partial\omega}{\partial x_0}\right)^2 - \left(\frac{\partial\omega}{\partial x_1}\right)^2 - \left(\frac{\partial\omega}{\partial x_2}\right)^2 - \cdots - \left(\frac{\partial\omega}{\partial x_n}\right)^2 = F_2(\omega).\end{gathered} \tag{2.6}$$

If ω is a solution of the system (2.6), then the multidimensional equation (2.5) reduces to ODE with variable coefficients

$$a_2(\omega)\ddot{\varphi}(\omega) + a_1(\omega)\dot{\varphi}(\omega) + a_0(\omega)\varphi(\omega)\ F(\varphi) = 0 \tag{2.7}$$

A solution of equation (2.5) has the form

$$u(x_0, \ldots x_n) = \varphi(\omega), \quad \omega = \omega(x_0, x_1, \ldots, x_n), \tag{2.8}$$

φ is a solution of equation (2.7).

Compatibility and general solutions of system (2.6) are described in detail in papers of Zhdanov, Revenko, Yehorchenko, Fushchych (1987–1993, [4–6]). As we see, without using explicitly the symmetry of equation (2.5), we can reduce a multidimensional wave equation to ODE. It is obvious that all ansatzes and solutions, which are constructed on the basis of the classical method by Sophus Lie, can be obtained within the framework of our approach. The subgroup analysis of the Poincaré group $P(1,n)$ (Patera J., Winternitz P., Zassenhaus H., 1975–1983, [7, 8] Fedorchuk V., Barannyk A., Barannyk L., Fushchych W., 1985–1991 [9–11]) gives only a part of possible ansatzes.

Note 1. P. Clarkson and M. Kruskal (1989 [12]) implemented the approach suggested by us in 1981–1983 [1, 2, 3] for the one-dimensional Boussinesq equation and constructed in explicit form ansatzes and solutions which cannot be obtained within the framework of the classical S. Lie method. In the literature, this approach is often called the "direct method of reduction". I believe that it would be more consistent and correct to call this method of construction of PDE solutions a method of ansatzes.

3 Conditional symmetry

The Lie symmetry, as known, is a local symmetry of the whole set of solutions. The Lie algorithm enables us to define the invariance algebra for an arbitrary given equation and to construct ansatzes.

The term and the concept “conditional symmetry” was introduced and developed in our papers (1983–1993, [2, 3, 13–18]). This extremely simple concept has appeared to be efficient and enabled us to discover a nature of many ansatzes which could not be obtained within the framework of the Lie method.

Conditional symmetry is the symmetry of subsets of equation’s solutions. Knowing conditional symmetry of an equation, we can construct non–Lie ansatzes and solutions. It is more difficult to study conditional symmetry of a given equation than to study its classical Lie symmetry. The difficulty is related to the fact that to find conditional symmetry of an equation, it is necessary to solve nonlinear determining equations.

During recent years, there are intensive studies in this promising direction, and today we can make following general conclusion:

Corollary 1. *Principal nonlinear equations of mathematical physics have conditional symmetry.*

Let us denote by the symbol

$$Q = \langle Q_1, Q_2, \dots, Q_r \rangle \tag{3.1}$$

some set of operators which does not belong to the invariance algebra (IA) of equation (2.1).

Definition 2. (Fushchych W., Nikitin A., Shtelen W. and Serov M., 1987 [13, 14, 18], Fushchych W. and Tsyfra I. (1987 [15])). Equation (2.1) is said to be conditionally invariant under the operators Q from (3.1), if there exists a supplementary condition on the solutions of (3.1) of the form

$$L_1(x, u, u_{(1)}, \dots, u_{(n)}) = 0 \tag{3.2}$$

such that (3.1) together with (3.2) is invariant under Q.

Thus, one has the following criterion of conditional invariance [13, 15, 18]

$$Q_s L = \lambda_0 L + \lambda_1 L_1, \tag{3.3}$$

$$Q_s L_1 = \lambda_2 L + \lambda_3 L_1, \tag{3.4}$$

where λ_0, λ_1, λ_2, λ_3 are some differential expressions, Q_s is the s-th prolongation by Lie.

Definition 3. We shall say that an equation is Q-conditionally invariant if the additional equation $L_1 = 0$ is a quasilinear equation of the first order

$$L_1(x, u, u_{(1)}) \equiv Qu = 0, \tag{3.5}$$

$$Q = \xi_\mu(x, u)\frac{\partial}{\partial x^\mu} + \eta(x, u)\frac{\partial}{\partial u}, \tag{3.6}$$

with η being a smooth function.

Thus, the problem of finding the conditional symmetry of a equation reduces to the solution of equations (3.3), (3.4). As a rule, the determining equations for calculating ξ_μ and η are nonlinear equations.

As is known, in the classical approach ξ_μ, η satisfy a linear system of differential equations which, because of being overdetermined, can be solved.

3.1 Conditional symmetry of the Maxwell equations

The first equation where we had noticed conditional symmetry was the Maxwell subsystem [13]

$$\frac{\partial \vec{E}}{\partial t} = \mathrm{rot}\,\vec{H}, \quad \frac{\partial \vec{H}}{\partial t} = -\mathrm{rot}\,\vec{E}. \tag{3.7}$$

It is possible to prove by means the standard Lie method that the maximal invariance algebra of system (3.7) is an 8-dimensional extended Euclid algebra $AE_1(4)$ with basis elements:

$$P_\mu = i\frac{\partial}{\partial x_\mu}, \quad J_{ab} = x_a p_b - x_b p_a + S_{ab}, \quad D = x_\mu P^\mu, \tag{3.8}$$

where S_{ab} are 6×6 matrices, which realize a reduced representation of the Lie algebra of the group $SU(2)$.

Thus, system (3.7) is not invariant with respect to the Lorentz transformations, which are generated by operators

$$J_{oa} = x_o P_a - x_a P_0 + S_{0a}, \tag{3.9}$$

$\langle S_{ab}, S_{0a}\rangle$ are matrices which realize a finite-dimensional representation of the Lie algebra of the Lorentz group $S(1,3)$.

Theorem 1. (Fushchych W. and Nikitin A. 1983 [13]). *System (3.7) is conditionally invariant under the Lorentz boosts (3.9) if and only if the solutions of (3.7) satisfy the conditions*

$$\mathrm{div}\,\vec{E} = 0, \quad \mathrm{div}\,\vec{H} = 0. \tag{3.10}$$

Thus, system (3.7) only together with equations (3.10) is invariant under the Lorentz group.

Note 2. 90 years ago H. Lorentz (1904, April 23), H. Poincaré (1905, June 5, July 23), A. Einstein (1905, June 30) discovered the theorem about invariance of the full Maxwell system (3.7), (3.10) with respect to rotations in the four-dimensional pseudo-Euclidean space-time. This theorem is a mathematical formulation of the fundamental Lorentz–Poincaré–Einstein principle of relativity.

3.2 Conditional symmetry of linear Schrödinger systems

Let us consider the multicomponent system of disconnected Schrödinger equations:

$$\begin{aligned} &S\Psi = \left(p_0 - \frac{p_a^2}{2m}\right)\Psi_r = 0, \quad r = 1,2,\ldots,n, \\ &p_0 = i\frac{\partial}{\partial x_0}, \quad p_a = -i\frac{\partial}{\partial x_a}, \quad a = 1,2,3, \\ &\Psi = (\Psi_1, \Psi_2, \ldots, \Psi_n), \quad \Psi = \Psi(x_0 = t, x_1, x_2, x_3). \end{aligned} \tag{3.11}$$

It is evident that every separate Schrödinger equation (3.11) is invariant with respect to a scalar representation of the group $G_2(1,3)$, a full Galilei group.

Let us consider a problem of existence of nontrivial vector, spinor, tensor representations of the full Galilei group, which are realized on the set of solutions of system (3.11).

We demand system (3.11) be invariant with respect to the following linear representations of the algebra $AG_2(1,3)$

$$\begin{aligned} &P_0 = i\frac{\partial}{\partial x_0}, \quad P_a = -i\frac{\partial}{\partial x_a}, \quad M = im, \\ &J_a = x_a p_b - x_b p_a + S_a, \quad S_a = \frac{1}{2}\varepsilon_{abc}S_{bc}, \\ &G_a = x_0 p_a - x_a p_0 + \lambda_a, \quad D = 2x_0 P_0 - x_k P_k + \lambda_0, \\ &A = x_0 D - x_0^2 P_0 + \frac{1}{2}m x_k^2 - \lambda_a x_a, \end{aligned} \tag{3.12}$$

where matrices S_a, λ_0, λ_a satisfy the commutation relations [29]

$$\begin{aligned} &[S_a, S_b] = i\varepsilon_{abc}S_c, \quad [\lambda_a, \lambda_b] = 0, \quad [\lambda_0, S_a] = 0, \\ &[\lambda_a S_b] = i\varepsilon_{abc}S_c, \quad [\lambda_0, \lambda_a] = i\lambda_a. \end{aligned} \tag{3.13}$$

Theorem 2 (Fushchych and Shtelen, 1983, [19]). *System of equations (3.11) is conditional invariant under representation* $AG_2(1,3)$ *(3.12) if*

$$\left(\lambda_0 - \frac{3}{2}i - \frac{1}{m}\lambda_k P_k\right)\Psi = 0, \tag{3.14}$$

$$(\lambda_1^2 + \lambda_2^2 + \lambda_3^2)\Psi = 0. \tag{3.15}$$

3.3 Q-conditional symmetry of Lorentz noninvariant nonlinear wave equation

Let us consider the following wave equation (Fushchych and Tsyfra 1987, [15])

$$Lu \equiv \Box u + F(x, \underset{1}{u}) = 0 \tag{3.16}$$

$$\begin{aligned} F(x, \underset{1}{u}) = &-\left(\frac{\lambda_0}{x_0}\right)^2\left(\frac{\partial u}{\partial x_0}\right)^2 + \left(\frac{\lambda_1}{x_1}\right)^2\left(\frac{\partial u}{\partial x_1}\right)^2 + \\ &+\left(\frac{\lambda_2}{x_2}\right)^2\left(\frac{\partial u}{\partial x_2}\right)^2 + \left(\frac{\lambda_3}{x_3}\right)^2\left(\frac{\partial u}{\partial x_3}\right)^2, \quad x_\mu \neq 0, \end{aligned} \tag{3.17}$$

λ_μ are arbitrary parameters.

Equation (3.16) is invariant only with respect to scale transformations and translations:

$$x_\mu \to x'_\mu = e^b x_\mu, \quad u \to u' = e^{2b}u, \quad u \to u' = u + c,$$

b is a real parameter.

Let us consider a Lorentz-invariant ansatz

$$u = \varphi(\omega), \quad \omega = x_\mu x^\mu = x_0^2 - x_1^2 - x_2^2 - x_3^2. \tag{3.18}$$

This ansatz, despite the fact that (3.16) is not invariant with respect to the Lorentz group, reduces equation (3.16) to ODE

$$\omega\frac{d^2\varphi}{d\omega^2}+2\frac{d\varphi}{d\omega}+\lambda^2\left(\frac{d\varphi}{d\omega}\right)^2=0 \tag{3.19}$$

whose solutions are given by the functions

$$\lambda=\lambda_0^2-\lambda_1^2-\lambda_2^2-\lambda_3^2,$$
$$\varphi(\omega)=2(-\lambda^2)^{1/2}\tan^{-1}\omega(-\lambda^2)^{-1/2}\quad \text{for} -\lambda^2>0,$$
$$\varphi(\omega)=-(\lambda^2)^{-1/2}ln\left\{\frac{(\lambda^2)^{1/2}+\omega}{(\lambda^2)^{1/2}-\omega}\right\}\quad \text{for} -\lambda^2<0.$$

What is the reason of such reduction? From the classical point of view, ansatz (3.18) must not reduce the Lorentz non–invariant equation (3.16) to ODE.

The reason of all this is the fact that equation (3.16) is conditionally invariant with respect to the Lorentz group.

Theorem 3 (Fushchych and Tsyfra, 1987 [15]). *Equation (3.16), (3.17) is conditionally invariant with respect to the Lorentz group if the following six conditions are added:*

$$J_{\mu\nu}u=0,\quad J_{\mu\nu}=x_\mu\frac{\partial}{\partial x_\nu}-x_\nu\frac{\partial}{\partial x_\mu},\quad \mu,\nu=0,1,2,3. \tag{3.20}$$

Thus, equation (3.16) together with the additional condition (3.20) is invariant with respect to the Lorentz group. The condition (3.20) picks out the subset from the whole set of solutions which is invariant with respect to the Lorentz group.

3.4 Conditionally conformal symmetry of the Poincaré-invariant d'Alembert equation

Let us consider the nonlinear d'Alembert equation with an additional condition

$$\Box u+F(u)=0, \tag{3.21}$$

$$\frac{\partial u}{\partial x_\mu}\frac{\partial u}{\partial x^\mu}=F_1(u). \tag{3.22}$$

Theorem 4 (Fushchych, Zhdanov, Serov 1989 [18]). *Equation (3.21) is conditionally invariant under the conformal group if*

$$F=3\lambda(u+c)^{-1}, \tag{3.23}$$

$$\frac{\partial u}{\partial x_\mu}\frac{\partial u}{\partial x^\mu}=\lambda, \tag{3.24}$$

where λ, c are arbitrary constants. The operators of conformal symmetry are

$$K_\mu=2x_\mu D-(x_\alpha x^\alpha-u^2)\frac{\partial}{\partial x^\mu},\quad \mu=0,1,2,3 \tag{3.25}$$

$$D = x^\mu \frac{\partial}{\partial x^\mu} + u\frac{\partial}{\partial u}. \tag{3.26}$$

Remark 3. Formulae (3.25), (3.26) give a nonlinear representation for the conformal algebra $AC(1,3)$.

An ansatz for the system

$$\Box u = u^{-1}, \quad \partial_\mu u \partial^\mu u = 1 \tag{3.27}$$

has the form (Fushchych and Zhdanov, 1989 [4])

$$u^2 = (a_\mu x^\mu + g_1)^2 - (b_\mu x^\mu + g_2)^2, \tag{3.28}$$

where $g_1 = g_1(\theta_\mu x^\mu)$, $g_2 = g_2(\theta_\mu x^\mu)$ are arbitrary smooth functions, a_μ, b_μ, θ_μ are arbitrary complex parameters satisfying the condition

$$a_\mu a^\mu = -b_\mu b^\mu = 1, \quad a_\mu b^\mu = a_\mu \theta^\mu = b_\mu \theta^\mu = \theta_\mu \theta^\mu = 0.$$

Remark 5. The problem of compatibility and construction of solutions of the d'Alembert–Hamilton system are considered in detail in [5, 6].

3.5 Conditional symmetry of the nonlinear heat equation

Let us consider the equation

$$u_0 + \vec\nabla[f(u)\vec\nabla u] = 0, \quad f(u) \neq \text{const}. \tag{3.29}$$

Ovsyannikov L. (1962, [20]) carried out the complete classification of the one-dimensional equation (3.29). Dorodnitsyn A., Knyaseva Z., Svirshchevskii S. (1983, [21]) carried out group classification of the three-dimensional equation (3.29) From the analysis of these results it follows.

Conclusion 1. (Fushchych 1983 [2]). *Among equations of the class (3.29), there are no nonlinear equations invariant with respect to Galilei transformations which are generated by the operators*

$$G_a = x_0\partial_a + M(u)x_a\frac{\partial}{\partial u}, \tag{3.30}$$

$M(u)$ is constant.

Theorem 5 (Fushchych, Serov, Chopyk 1988 [16]). *The equation (3.29) is conditional invariant under the Galilean operators (3.30) if*

$$u_0 + \frac{(\vec\nabla u)^2}{2M(u)} = 0, \tag{3.31}$$

$$M(u) = \frac{u}{2f(u)}. \tag{3.32}$$

Conclusion 2. *The nonlinear equation (3.29) with the additional condition (3.31) is compatible with the Galilei relativity principle.*

Conclusion 3. *If*

$$f(u) = \frac{1}{2m}u^k, \qquad M(u) = \frac{2m}{kn+2}u^{1-k}, \tag{3.33}$$

$$f(u) = e^u, \qquad M(u) = 1, \tag{3.34}$$

where m, k are arbitrary constants, $kn+2 \neq 0$, then equation (3.29) is conditionally invariant with respect to Galilei transformations.

Q-conditional symmetry of the one-dimensional equation

$$u_0 - u_{11} = F(u)$$

was studied in detail (Fushchych and Serov, 1990, [22, 23]). Recently these results were obtained by Clarkson P. and Mansfield E. (1994, [24]).

4 Reduction and antireduction

Under the term "reduction–antireduction", we understand a decreasing of dimension of an equation with respect to independent variables and increasing (antireduction) by the number of dependent variables. That is we have simultaneously the process of reduction (by the number of independent variables) and antireduction (increasing the number of reduced systems with respect to the original equation) [25].

In the classical Lie approach as a rule the number of components of dependent variables for reduced systems does not increase.

Example 1. Let us consider the nonlinear acoustics equation (Khokhlov–Zabolotskaja equation)

$$u_{01} - (u_1 u)_1 - u_{22} - u_{33} = 0, \tag{4.1}$$

$$u = u(x_1, x_2, x_3).$$

The ansatz (Fushchych and Myronyuk, 1991 [26])

$$u = \frac{1}{3}x_1\varphi^{(1)}(\omega_0,\omega_2,\omega_3) + \frac{1}{6}x_1^2\varphi^{(2)}(\omega_0,\omega_2,\omega_3) + \varphi^{(3)}(\omega_0,\omega_2,\omega_3), \tag{4.2}$$

$$\omega_0 = x_0, \quad \omega_2 = x_2, \quad \omega = x_3$$

antireduces four-dimensional equation (4.1) to the system of coupled three-dimensional equations for functions $\varphi^{(1)}$, $\varphi^{(2)}$, $\varphi^{(3)}$

$$\begin{aligned}
&\frac{\partial^2\varphi^{(1)}}{\partial\omega_2^2} + \frac{\partial^2\varphi^{(2)}}{\partial\omega_3^2} = (\varphi^{(2)})^2,\\
&\frac{\partial^2\varphi^{(1)}}{\partial\omega_2^2} + \frac{\partial^2\varphi^{(1)}}{\partial\omega_3^2} + \frac{\partial\varphi^{(1)}}{\partial\omega_0} - \varphi^{(1)}\varphi^{(2)} = 0,\\
&\frac{\partial^2\varphi^{(3)}}{\partial\omega_2^2} + \frac{\partial^2\varphi^{(3)}}{\partial\omega_3^2} - \frac{1}{3}\varphi^{(2)}\varphi^{(3)} - \frac{1}{3}\frac{\partial\varphi^{(1)}}{\partial\omega_0} + \frac{1}{9}(\varphi^{(1)})^2 = 0.
\end{aligned} \tag{4.3}$$

The formula (4.2) gives a non-Lie ansatz for equation (4.1).

Example 2. Let us consider the equation for short waves in gas dynamics

$$\begin{aligned} &2u_{01} - 2(2x_1 + u_1)u_{11} + u_{22} + 2\lambda u_1 = 0, \\ &u = u(x_0 = t, x_1, x_2). \end{aligned} \tag{4.4}$$

The ansatz (Fushchych and Repeta 1991, [27])

$$\begin{aligned} &u = x_1\varphi^{(1)}(\omega_0, \omega_2) + x_1^2\varphi^{(2)}(\omega_0, \omega_2) + x_1^{3/2}\varphi^{(3)} + \varphi^{(4)}, \\ &\omega_0 = x_0, \qquad \omega_2 = x_2 \end{aligned} \tag{4.5}$$

antireduces one three-dimensional scalar equation (4.4) to a system of two-dimensional equations for four functions

$$\begin{aligned} &\varphi^{(3)} = 0, \quad \frac{\partial^2\varphi^{(1)}}{\partial\omega_2^2} = 0, \quad \frac{\partial^2\varphi^{(2)}}{\partial\omega_2^2} = 0, \\ &\frac{\partial^2\varphi^{(4)}}{\partial\omega_2^2} = \frac{9}{4}\left(\varphi^{(1)}\right)^2, \quad \frac{\partial\varphi^{(1)}}{\partial\omega_0} = \varphi^{(1)}\left(3\varphi^{(2)} + \frac{1}{2} - \lambda\right). \end{aligned} \tag{4.6}$$

4.1 Antireduction and ansatzes for the nonlinear heat equation

Let us consider the nonlinear one-dimensional heat equation

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left\{a(u)\frac{\partial u}{\partial x}\right\} + F(u), \tag{4.7}$$

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + F(u). \tag{4.8}$$

We consider an implicit ansatz

$$h\big(t, x, u, \varphi^{(1)}(\omega), \varphi^{(2)}(\omega) \ldots, \varphi^{(N)}(\omega)\big) = 0, \tag{4.9}$$

which reduces the two–dimensional equation (4.7) to the system of ODE for functions $\varphi^{(1)}, \ldots, \varphi^{(N)}$. We have constructed a quite long list of ansatzes which reduce equation (4.7) to the system of ODE (Zhdanov R. and Fushchych W. 1994, [33]).

Example 3. If in (4.7)

$$a(u) = \lambda u^{-3/2}, \quad F(u) = \lambda_1 u + \lambda_2 u^{5/2}, \tag{4.10}$$

then the ansatz [33] is as follows

$$u^{-3/2} = \varphi^{(1)}(t) + \varphi^{(2)}(t)x + \varphi^{(3)}(t)x^2 + \varphi^{(4)}(t)x^3, \tag{4.11}$$

$$\begin{aligned} &\dot\varphi^{(1)} = 2\lambda\varphi^{(1)}\varphi^{(3)} - \frac{2}{3}\lambda(\varphi^{(2)})^2 - \frac{3}{2}\lambda_1\varphi^{(1)} - \frac{3}{2}\lambda_2, \\ &\dot\varphi^{(2)} = -\frac{2}{3}\lambda\varphi^{(2)}\varphi^{(3)} + 6\lambda\varphi^{(1)}\varphi^{(4)} - \frac{3}{2}\lambda_1\varphi^{(2)}, \\ &\dot\varphi^{(3)} = -\frac{2}{3}\lambda(\varphi^{(3)})^2 + 2\lambda\varphi^{(2)}\dot\varphi^{(4)} - \frac{3}{2}\lambda_1\varphi^{(3)}, \\ &\dot\varphi^{(4)} = -\frac{3}{2}\lambda_1\varphi^{(4)}. \end{aligned} \tag{4.12}$$

Having solved the system of ODE (4.12), by formula (4.11) we construct exact solutions of the equation (4.7).

Example 4. If in (4.8)

$$F(u) = \left\{\alpha + \beta \ln u - \gamma^2 (\ln u)^2\right\} u, \tag{4.13}$$

then the ansatz

$$\ln u = \varphi^{(1)}(t) + e^{\gamma x}\varphi^{(2)}(t) \tag{4.14}$$

reduces (4.8) to the system of ODE

$$\begin{aligned} \dot\varphi^{(1)} &= 2 + \beta\varphi^{(1)} - \gamma^2(\varphi^{(1)})^2, \\ \dot\varphi^{(2)} &= \left\{\beta + \gamma^2 - 2\gamma^2\varphi^{(1)}\right\}\varphi^{(2)}. \end{aligned} \tag{4.15}$$

It is possible to construct solutions of system (4.15) in the explicit form. Depending on the sign of the quantity $d = \beta^2 + 4\alpha\gamma^2$ we get the following solutions of the nonlinear equation (4.8), (4.13).

Case 4.1 $d > 0$

$$u = c\left(\cos\frac{d^{1/2}t}{2}\right)^{-2}\exp\left(\gamma x + \gamma^2 t\right) + \frac{1}{2\gamma^2}\left(\beta - d^{1/2}\mathrm{tg}\,\frac{d^{1/2}t}{2}\right). \tag{4.16}$$

Case 4.2 $d < 0$

$$u = c\left(\mathrm{ch}\,\frac{|d|^{1/2}t}{2}\right)^{-2}\exp\left(\gamma x + \gamma^2 t\right) + \frac{1}{2\gamma^2}\left(\beta + |d|^{1/2}\mathrm{th}\,\frac{|d|^{1/2}t}{2}\right). \tag{4.17}$$

Case 4.3 $d = 0$

$$u = ct^{-2}\exp\left(\gamma x + \gamma^2 t\right) + \frac{1}{2\gamma^2 t}(\beta t + 2). \tag{4.18}$$

Example 5. If in (4.7)

$$a(u) = \lambda u^k, \quad F(u) = \lambda_1 u + \lambda_2 u^{1-k}, \tag{4.19}$$

then the ansatz

$$u^k = \varphi^{(1)}(t) + \varphi^{(2)}(t)x + \varphi^{(3)}(t)x^2 \tag{4.20}$$

antireduces (4.7) to the system of ODE

$$\begin{aligned} \dot\varphi^{(1)} &= 2\lambda\varphi^{(1)}\varphi^{(3)} + \lambda k^{-1}(\varphi^{(2)})^2 + k\lambda_2, \\ \dot\varphi^{(2)} &= 2\lambda(1 + 2k^{-1})\varphi^{(2)}\varphi^{(3)} + k\lambda_1\varphi^{(2)}, \\ \dot\varphi^{(3)} &= 2\lambda(1 + 2k^{-1})(\varphi^{(3)})^2 + k\lambda_1\varphi^{(3)}. \end{aligned} \tag{4.21}$$

5 Non-Lie symmetry, new relativity principles

5.1 Non-Lie symmetry Schrödinger equation

Let us consider the Schrödinger equation

$$\left(i\frac{\partial}{\partial x_0} - \frac{p_a^2}{2n}\right) u(x_0, \vec{x}) = 0. \tag{5.1}$$

It is well known that the maximal (in the Lie sense) invariance algebra (5.1) is the full Galilei algebra $AG_2(1,3) = \langle P_0, P_a, J_{ab}, G_a, D, A\rangle$

$$\begin{aligned} &P_0 = i\frac{\partial}{\partial x_0}, \quad P_a = -i\frac{\partial}{\partial x_0}, \quad a = 1,2,3, \\ &J_{ab} = x_a p_b - x_b p_a, \quad G_a = x_0 p_a - m x_a, \\ &D = 2x_0 P_0 - x_k P_k, \quad A = x_0 D - x_0^2 P_0 + \frac{1}{2} m x_a^2. \end{aligned} \tag{5.2}$$

Operators G_a generate the standard Galilei transformations:

$$t \to t' = \exp\{iG_a v_a\}\, t \exp\{-iG_a v_a\} = t, \tag{5.3}$$

$$x_a \to x_a' = \exp\{iG_b v_b\}\, x_a \exp\{-iG_c v_c\} = x_a + v_a t. \tag{5.4}$$

Let us put the following question: do symmetries which are not reduced for the algebra (5.2) exhaust for equation (5.1)?

Answer: The Schrödinger equation (5.1) has additional symmetries (supersymmetries, non-Lie, nonlocal) which are not reduced to the Galilei algebra $AG_2(1,3)$ [29].

One of results in this direction is the following:

Theorem 6. (Fushchych and Seheda 1977 [28]). *The Schrödinger equation (5.1) is invariant with respect to the Lorentz algebra* $AL(1,3)$

$$J_{ab} = x_a p_b - x_b p_a, \tag{5.5}$$

$$J_{0a} = \frac{1}{2m}(pG_a + G_a p), \quad p = (p_1^2 + p_2^2 + p_3^2)^{1/2} = (-\Delta)^{1/2}. \tag{5.6}$$

It is not difficult to check that the operators $\langle J_{ab}, J_{0c}\rangle \equiv AL(1,3)$ satisfy the commutation relations

$$[J_{ab}, J_{0c}] = i(g_{ac}J_{b0} - g_{bc}J_{a0}), \quad [J_{0a}, J_{0b}] = -iJ_{ab}.$$

It is important to point out that J_{0a} are integral-differential symmetry operators and generate nonlocal transformations

$$x_a \to x_a' = \exp\{iJ_{ob}V_b\}\, x_a \exp\{-iJ_{0c}V_c\} \neq \text{ Galilei transform. (5.4)}, \tag{5.7}$$

$$t \to t' = \exp\{iJ_{0a}V_a\}\, t \exp\{-iJ_{0b}V_b\} = t. \tag{5.8}$$

Hence the operators J_{0a} (5.6) generate new transformations which do not coincide with the known Galilei and Lorentz transformation. Thus we have new relativity principle. It is defined by formulae (5.7), (5.8).

5.2 Time is absolute in relativistic physics

The four-component Dirac equation lies in the foundation of the modern quantum mechanics

$$\gamma_\mu p^\mu \Psi = m\Psi(x_0, x_1, x_2, x_3). \tag{5.9}$$

Here γ_μ are 4×4 Dirac matrices.

Since the time of discovery of this equation it is known that (5.9) is invariant with respect to the Poincaré algebra $AP(1,3) = \langle P_\mu, J_{\mu\nu}\rangle$ with the basis elements

$$P_\mu = i\frac{\partial}{\partial x^\mu}, \quad J^{(1)}_{\mu\nu} = x_\mu p_\nu - x_\nu p_\mu + S_{\mu\nu}, \quad S_{\mu\nu} = \frac{i}{4}(\gamma_\mu\gamma_\nu - \gamma_\nu\gamma_\mu). \tag{5.10}$$

Operators $J^{(1)}_{\mu\nu}$ generate the standard Lorentz transformations

$$t \to t' = \exp\left\{iJ^{(1)}_{0a}v_a\right\} t \exp\left\{-iJ_{0b}v_b\right\}, \tag{5.11}$$

$$x_a \to x'_a = \exp\left\{iJ^{(1)}_{0b}v_b\right\} x_a \exp\left\{-iJ_{0c}v_c\right\}. \tag{5.12}$$

Hence, the fundamental statement follows that time $t \in T(1)$ and space $\vec{x} \in R(3)$ are the single pseudo-Euclidean space-time with the metric

$$s^2 = x_0^2 - x_1^2 - x_2^2 - x_3^2. \tag{5.13}$$

Let us put another question: Do there exist symmetries in equation (5.10) which cannot be reduced to the algebra AP(1,3) (5.11)?

Answer: The Dirac equation (5.9) has a wide additional symmetry (supersymmetry, non-Lie symmetry) which cannot be reduced to the algebra $AP(1,3)$ (5.10) [13, 29].

I shall say here briefly about one of such symmetries.

Theorem 7. (Fushchych 1971, 1974 [30, 31]. *The Dirac equation (5.9) is invariant with respect to the following representation of the Poincaré algebra*

$$P^{(2)}_0 = H = \gamma_0\gamma_a p_a + \gamma_0 m, \quad P^{(2)}_a = -i\frac{\partial}{\partial x_a}, \quad a = 1, 2, 3, \tag{5.14}$$

$$J^{(2)}_{ab} = x_a p_b - x_b p_a + S_{ab}, \qquad S_{ab} = \frac{i}{4}(\gamma_a\gamma_b - \gamma_b\gamma_a), \tag{5.15}$$

$$J^{(2)}_{0a} = x_0 p_a - \frac{1}{2}(x_a H + H x_a). \tag{5.16}$$

Thus we have two different representations of the Poincaré algebra $AP(1,3)$ (5.10) and (5.14)–(5.16).

The representation (5.15) and (5.16) generates nonlocal transformations

$$x_a \to x'_a = \exp\{iJ^{(2)}_{ab}v_b\}x_a \exp\{iJ^{(2)}_{0c}v_c\} \neq \text{Lorentz transform}, \tag{5.17}$$

$$t \to t' = \exp\{iJ^{(2)}_{0b}v_b\}t \exp\{-iJ^{(2)}_{0c}v_c\} = t. \tag{5.18}$$

Thus, time does not change in relativistic physics. Time is absolute in relativistic physics.

There are two nonequivalent possibilities (duality) for transformations of coordinates and time: Lorentz transformation (5.11), (5.12) and non-Lorentz transformation (5.17), (5.18).

The Maxwell and Klein–Gordon–Fock equations are also invariant under nonlocal transformations (5.17), (5.18) when time does not change. However energy and momentum are transformed by the Lorentz law [31,32]. We have new relativity principle (5.17), (5.18).

What is the reason of such a paradoxical statement? The reason is that the operators $J^{(2)}_{0a}$ are non-Lie symmetry operators and the standard relation (S. Lie's theorems) between Lie groups and Lie algebras is broken.

So, physics is not equivalent to geometry and geometry is not physics. Physics is Nature. Theoretical Physics is only a Model of Nature!

6 On some new motion equations

Some new motion equations are adduced in this section. These equations are generalizations of known classical equations. Symmetry of these equations has not been investigated.

6.1 High order parabolic equation in quantum mechanics

The Schrödinger equation (5.1) is not the only equation compatible with the Galilei relativity principle. A more general equation was suggested in [1, 2]

$$
\begin{aligned}
&(\lambda_1 S + \lambda_2 S^2 + \cdots + \lambda_n S^n)u = \lambda u,\\
&S \equiv p_0 - \frac{p_a^2}{2m}, \quad S^2 = S \cdot S, \quad S^n = S^{n-1}S,
\end{aligned}
\tag{6.1}
$$

$\lambda, \lambda_1, \lambda_2, \ldots, \lambda_n$ are arbitrary parameters. Equation (6.1) as well as the classical equation (5.1) is invariant with respect to the Galilei transformations but it is not invariant with respect to scale and projective transformations.

A new equation for two particles (waves):

$$
\begin{aligned}
&p_0 u_1 = \frac{1}{2m_1} p_a^2 u_1 + V_1\big(t, x_1, x_2, \ldots, x_6, u_1, u_2\big),\\
&p_0 u_2 = \frac{1}{2m_2} p_{a+3}^2 u_2 + V_2\big(t, x_1, x_2, \ldots, x_6, u_1, u_2\big),\\
&u_1 = u_1(t, x_1, x_2, x_3), \quad u_2 = u_2(t, x_4, x_5, x_6), \quad V_1 \text{ and } V_2 \text{ are potentials.}
\end{aligned}
$$

6.2 Nonlinear generalization of Maxwell equations

If we assume that the light velocity is not constant [34], we can suggest some generalizations of the Maxwell equations

$$
\begin{aligned}
&\tfrac{\partial \vec{E}}{\partial t} = \mathrm{rot}\,\{c(\vec{E}^2, \vec{H}^2, \vec{E}\vec{H})\vec{H}\}, \qquad \frac{\partial \vec{H}}{\partial t} = -\mathrm{rot}\,\{c(\vec{E}^2, \vec{H}^2, \vec{E}\vec{H})\vec{E}\},\\
&\mathrm{div}\,\{a(\vec{E}^2, \vec{H}^2, \vec{E}\vec{H})\vec{E}\} = 0, \qquad \mathrm{div}\,\{b(\vec{E}^2, \vec{H}^2, \vec{E}\vec{H})\vec{H}\} = 0,
\end{aligned}
\tag{6.2}
$$

where a, b and c are some functions of electromagnetic field;

$$\begin{aligned} \frac{\partial \vec{E}}{\partial t} &= \text{rot}\,\{c(\vec{B}^2, \vec{D}^2, \vec{B}\vec{D})\vec{B}\} + \vec{j}, \\ \frac{\partial \vec{H}}{\partial t} &= -\text{rot}\,\{c(\vec{B}^2, \vec{D}^2, \vec{B}\vec{D})\vec{D}\}, \end{aligned} \quad \text{or} \quad \begin{aligned} \frac{\partial \vec{D}}{\partial t} &= \text{rot}\,\{c(\vec{E}^2, \vec{H}^2, \vec{E}\vec{H})\vec{E}\} + \vec{j}, \\ \frac{\partial \vec{B}}{\partial t} &= -\text{rot}\,\{c(\vec{E}^2, \vec{H}^2, \vec{E}\vec{H})\vec{E}\}, \end{aligned} \tag{6.3}$$

$$\begin{aligned} &\lambda_1 \vec{D} + \lambda_2 \Box \vec{D} = F_1(\vec{E}^2, \vec{H}^2, \vec{E}\vec{H})\vec{E} + F_2(\vec{E}^2, \vec{H}^2, \vec{E}\vec{H})\vec{H}, \\ &\lambda_3 \vec{B} + \lambda_4 \Box \vec{B} = R_1(\vec{E}^2, \vec{H}^2, \vec{E}\vec{H})\vec{E} + R_2(\vec{E}^2, \vec{H}^2, \vec{E}\vec{H})\vec{H}, \end{aligned} \tag{6.4}$$

$$\text{div}\,\vec{D} = \rho, \quad \text{div}\,\vec{B} = 0, \tag{6.5}$$

where F_1, F_2, R_1, R_2 are functions of fields $\vec{E}$ and $\vec{H}$, c in equations (6.2), (6.3) can be a function of $(t, \vec{x})$, $c = c(t, \vec{x})$, or depend on the gravity potential V, $c = C(V)$. Nonlinear wave equations for $\vec{E}$ and $\vec{H}$ have form

$$\frac{\partial^2 \vec{E}}{\partial t^2} - c^2 \Delta \vec{E} = 0, \quad \frac{\partial^2 \vec{H}}{\partial t^2} - c^2 \Delta \vec{H} = 0, \tag{6.6}$$

or

$$\frac{\partial^2 \vec{E}}{\partial t^2} - \Delta(c^2 \vec{E}) = 0, \quad \frac{\partial^2 \vec{H}}{\partial t^2} - \Delta(c^2 \vec{H}) = 0; \tag{6.7}$$

or

$$\frac{\partial^2}{\partial t^2}\left(\frac{1}{c^2}\vec{E}\right) - \Delta \vec{E} = 0, \quad \frac{\partial^2}{\partial t^2}\left(\frac{1}{c^2}\vec{H}\right) - \Delta H = 0, \tag{6.8}$$

with one of the conditions

$$c^2 = \frac{1}{2}\frac{\left(\frac{\partial \vec{E}}{\partial t}\right) + \left(\frac{\partial \vec{H}}{\partial t}\right)}{(\text{rot}\,\vec{H})^2 + (\text{rot}\,\vec{E})^2} \tag{6.9}$$

or

$$\frac{\partial c^2}{\partial x^\mu}\frac{\partial c^2}{\partial x^\mu} = 0. \tag{6.10}$$

or

$$c_\mu \frac{\partial c_2}{\partial x_\mu} = \lambda(E^2 \vec{H}^2, \vec{E}\vec{H}) F_{\alpha\beta} c^\beta, \tag{6.11}$$

c_α is the four-velocity of the light (electromagnetic field), $c^2 = c_\alpha c^\alpha$.

Equations of hydrodynamical type for electromagnetic field have form

$$\begin{aligned} \frac{\partial \vec{E}}{\partial t} &= a_1 \left\{\vec{\nabla} \times (\vec{c} \times \vec{H})\right\} + a_2 \left\{\vec{\nabla} \times (\vec{c} \times \vec{E})\right\}, \\ \frac{\partial \vec{H}}{\partial t} &= b_1 \left\{\vec{\nabla} \times (\vec{c} \times \vec{E})\right\} + b_2 \left\{\vec{\nabla} \times (\vec{c} \times \vec{H})\right\}, \\ \frac{\partial \vec{c}}{\partial t} &+ (\vec{c}\vec{\nabla})\vec{c} = R_1 \vec{E} + R_2 \vec{H}, \end{aligned} \tag{6.12}$$

$\vec{c}$ is the three-velocity of the light, where a_1, a_2, b_1, b_2, R_1, R_2 are functions of $\vec{E}^2$, $\vec{H}^2$, $\vec{E}\vec{H}$.

Maxwell's equations in a moving frame with the velocity can be generalized in such forms

$$\frac{\partial \vec{E}}{\partial t} + \lambda_1 v_k \frac{\partial \vec{E}}{\partial x_k} + \lambda_2 \operatorname{rot} \vec{H} = 0, \quad \frac{\partial \vec{H}}{\partial t} + \lambda_3 v_k \frac{\partial \vec{H}}{\partial x_k} + \lambda_4 \operatorname{rot} \vec{E} = 0,$$

or

$$\frac{\partial \vec{E}}{\partial t} + \lambda_1 v_k \frac{\partial \vec{H}}{\partial x_k} + \lambda_2 \operatorname{rot} \vec{H} = 0, \quad \frac{\partial \vec{H}}{\partial t} + \lambda_3 v_k \frac{\partial \vec{E}}{\partial x_k} + \lambda_4 \operatorname{rot} \vec{E} = 0,$$

with the conditions $\frac{\partial v_k}{\partial t} + v_l \frac{\partial v_k}{\partial x_l} = 0$.

6.3 Equations for fields with the spin 1/2

Fields with the spin 1/2 are described, as a rule, by first-order equations, by the Dirac equation. However, such fields can be also described by second-order equations. Some of such equations are adduced below:

$$p_\mu p^\mu \Psi = F_1(\bar{\psi}\psi)\Psi, \quad \bar{\psi}\gamma_\mu p^\mu \Psi = F_2(\bar{\psi}\Psi); \tag{6.13}$$

$$p_\mu p^\mu \Psi = R_1(\bar{\psi}\psi)\Psi, \quad (\bar{\psi}\gamma_\mu \Psi) p^\mu \Psi = F_2(\bar{\psi}\psi)\Psi; \tag{6.14}$$

$$p_\mu p^\mu \Psi = F_1(\bar{\psi}\psi)\Psi, \quad (\bar{\psi}\gamma_\mu \Psi)(\bar{\psi} p^\mu \Psi) = F_3(\bar{\psi}\psi); \tag{6.15}$$

$$p_\mu p^\mu \Psi + \lambda \gamma_\mu p^\mu \Psi = F(\bar{\psi}\psi)\Psi; \tag{6.16}$$

$$p_\mu p^\mu \Psi = F_1(\bar{\psi}\psi)\Psi, \quad p_0 \Psi = \{(\bar{\psi}\gamma_0 \Psi)(\bar{\psi}\gamma_k \psi) p_k + m\bar{\Psi}\gamma_0 \Psi\}\Psi.$$

6.4 How to extend symmetry of an equation with arbitrary coefficients?

Let us consider the a second-order equation

$$a_{\mu\nu}(x) \frac{\partial^2 u}{\partial x^\mu \partial x^\nu} + b_\mu(x) \frac{\partial u}{\partial x^\mu} + F(u) = 0. \tag{6.17}$$

Equation (6.17) with arbitrary fixed coefficients has only a trivial symmetry ($x \to x' = x$, $u \to u' = u$). However, if we do not fix coefficient functions $a_{\mu\nu}(x), b_\mu(x)$, such an equation can have wide symmetry. E.g., if $a_{\mu\nu}$, b_μ satisfy the equations

$$\Box a_{\mu\nu} = \frac{\partial u}{\partial x_\mu} \frac{\partial u}{\partial x_\nu} F_1(u) \tag{6.18}$$

or

$$\Box b_\mu = F_2(u) \frac{\partial u}{\partial x_\mu}, \quad \Box a_{\mu\nu} = \frac{\partial^2 u}{\partial x_\mu \partial x_\nu} F_3(u), \tag{6.19}$$

then the nonlinear system (6.17), (6.18), (6.19) is invariant with respect to the Poincaré group P(1,3). Let us emphasize that here even if we put $F_1 = 0$, $F_2 = 0$, equations (6.17), (6.18), (6.19) are a nonlinear system of equations. With some particular functions F_1 and F_2, it is possible to construct ansatzes which reduce system (6.17), (6.18), (6.19) to the system of ordinary differential equations.

So, considering (6.17) as a nonlinear equation with additional conditions for $a_{\mu\nu}$, b_ν, we can construct the exact solution for eqation (6.17). The adduced idea about extension of the symmetry of (6.17) can be used for construction of exact solutions for motion equations in gravity theory.

The second example of equations which have wide symmetry is

$$v_\mu v_\nu \frac{\partial^2 F_{\alpha\beta}}{\partial x^\mu \partial x^\nu} = 0, \qquad (6.20)$$

$$v_\mu \frac{\partial v_\nu}{\partial x^\mu} = 0. \qquad (6.21)$$

If in (6.20) v_μ are fixed functions the equation, as a rule, has trivial symmetry.

1. Fushchych W., Symmetry in problems of mathematical physics, in Algebraic-Theortic Studies in Mathematical Physics, Kiev, Inst. of Math. Ukrainian Acad. Sci., 1981, 6–44.
2. Fushchych W., On symmetry and exact solutions of some multidimensional equations of mathematical physics problems, in Algebraic-Theoretical Methods in Mathematical Physics, Kiev, Inst. of Math. Ukrainian Acad. Sci., 1983, 4–23.
3. Fushchych W.,Serov M., The symmetry and exact solutions of the nonlinear multi-dimensional Liouville, d'Alembert and eikonal equations, *J. Phys. A: Math. Gen.*, 1983, **16**, № 15, 3645–3658.
4. Fushchych W., Zhdanov R., On some new exact solutions of the nonlinear d'Alembert–Hamilton system, *Phys. Lett. A*, 1989, **141**, № 3–4, 113–115.
5. Fushchych W., Zhdanov R., Revenko I., General solutions of the nonlinear wave and eikonal equations, *Ukrainian Math. J.*, 1991, **43**, № 11, 1471–1486.
6. Fushchych W., Zhdanov R., Yehorchenko I., On the reduction of the nonlinear multi-dimensional wave eqautions and compatibility of the d'Alembert–Hamilton system, *J. Math. Anal. Appl.*, 1991, **161**, № 2, 352–360.
7. Patera J., Winternitz P., Zassenhaus H., Continuous subgroups of the fundamental groups of physics, *J. Math. Phys.*, 1975, **16**, № 8, 1597–1624.
8. Patera J., Winternitz P., Zassenhaus H., Maximal Abelian subalgebras of real and complex symplectic Lie algebras, *J. Math. Phys.*, 1983, **24**, № 8, 1973–1985.
9. Fushchych W., Barannyk A., Barannyk L., Fedorchuk V., Continuous subgroups of the Poincare group $P(1,4)$, *J. Phys. A: Math. Gen.*, 1985, **18**, № 5, 2893–2899.
10. Barannyk L., Fushchych W., On continuous subgroups of the generalized Schrödinger groups, *J. Math. Phys.*, 1989, **30**, № 2, 280–290.
11. Fushchych W., Barannyk L., Barannyk A., Subgroup analysis of the Galilei and Poincaré groups and reductions of nonlinear equations, Kiev, Naukova Dumka,1991, 300 p.
12. Clarkson P., Kruskal M., New similarity reductions of the Boussinesq equation, *J. Math. Phys.*, 1989, **30**, № 10, 2201–2213.
13. Fushchych W., Nikitin A., Symmetries of Maxwell's equation, Reidel Dordrecht, 1987, 230 p. (Russian version 1983, Naukova Dumka 1983).
14. Fushchych W., How to extend symmetry of differential equations?, in Symmetry and Solutions of Nonlinear Mathematical Physics, Kiev, Inst. of Math. Ukrainian Acad. Sci., 1987, 4–16.
15. Fushchych W., Tsyfra I., On reduction and solutions of nonlinear wave equations with broken symmetry, *J.Phys. A: Math. Gen.*, 1987, **20**, № 2, 45–47.
16. Fushchych W., Serov M., Chopyk V., Conditional invariance and nonlinear heat equations, *Dopovidi Ukrainian Acad. Sci.*, 1988, № 9, 17–20.

17. Fushchych W., Serov M., Conditional invariance and exact solutions of nonlinear acoustic equation, *Dopovidi Ukrainian Acad. Sci.*, 1988, № 10, 27–31.
18. Fushchych W., Shtelen W., Serov M., Symmetry analysis and exact solutions of equations of nonlinear mathematical physics, Kluwer Academic Publishers, 1993, 430 p. (Russian version, 1989, Naukova Dumka).
19. Fushchych W., Shtelen W., On linear and nonlinear systems of differential equations invariant under the Schrödinger group, *Theor. and Math. Fisika*, 1983, **56**, № 3, 387–394.
20. Ovsyannikov L., Group analysis of differential equations, Moscow, Nauka, 1978.
21. Dorodnyzyn W., Knyazeva I., Svirshchevski S., Group properties of the heat equation with a source in two and three dimensions, *Defferentsial'nye Uravneniya*, 1983, **19**, № 7, 1215–1224.
22. Fushchych W., Serov M., Conditional symmetry and reduction of nonlinear heat equation, *Dopovidi Ukrainian Acad. Sci.*, 1990, № 7, 24–28.
23. Serov M., Conditional invariance and exact solutions of nonlinear heat equation, *Ukr. Math. J.,* 1990, **42**, № 10, 1370–1376.
24. Clarkson P., Mansfield E., Algorithms for the nonclassical method of symmetry reductions, *SIAM J. Appl. Math.*, 1994, **54**, № 6, 1693–1719.
25. Fushchych W., Conditional symmetry of the equations of mathematical physics, *Ukr. Math. J.*, 1991, **43**, № 1, 1456–1470.
26. Fushchych W., Myronyuk P., Conditional symmetry and exact solutions of nonlinear acoustics equation, *Dopovidi of the Ukrainian Acad. Sciences*, 1991, № 8, 23–26.
27. Fushchych W., Repeta V., Exact solutions of equations of gas dynamics and nonlinear acoustics, *Dopovidi of the Ukrainian Acad. Sciences*, 1991, № 8, 35–38.
28. Fushchych W., Seheda Yu., On a new invariance algebra for the free Schrödinger equation, *Doklady Acad. sci. USSR*, 1977, **232**, № 4, 800–801.
29. Fushchych W., Nikitin A., Symmetries of equations of quantum mechanics, Allerton Press, 1994, 500 p. (in Russian 1990, Moscow, Nauka).
30. Fushchych W., On additional invariance of relativistic equations of motion, *Teor. and Matem. Fizika*, 1971, **7**, № 1, 3–12.
31. Fushchych W., On additional invariance of the Dirac and Maxwell equations, *Lettere Nuovo Cimento*, 1974, **11**, № 10, 508–512.
32. Fushchych W., On additional invariance of the Klein–Gordon–Fock equation, *Dokl. Acad. Sci. USSR*, 1976, **230**, № 3, 570–573.
33. Fushchych W., Zhdanov R., Antireduction and exact solutions of nonlinear heat equation, *J. Nonlinear Math. Phys.*, 1994, **1**, № 1, 60–64.
34. Fushchych W., New nonlinear equations for electromagnetic field having the velocity different from c, Dopovidi of the Ukrainian Academy of Sciences, 1992, № 4, 24–28.

Galilei invariant nonlinear Schrödinger type equations and their exact solutions

W.I. FUSHCHYCH

In this paper we describe wide classes of nonlinear Schrödinger-type PDEs which are invariant under the Galilei group and its generalizations. We construct sets of ansatzes for Galilei invariant equations, and exact classes solutions are found for some nonlinear Schrödinger equations.

1. Introduction

Let us consider the following nonlinear equations

$$L_1(\psi,\psi^*) \equiv S\Psi - F_1(x,\psi,\psi^*),$$
$$S = p_0 - \frac{p_a^2}{2m}, \quad p_0 = i\frac{\partial}{\partial x_0}, \quad p_a = -i\frac{\partial}{\partial x_a}, \quad a = 1,2,3, \tag{1}$$

$$L_2 \equiv p_0\psi + g_{ab}(x_0,\vec{x},\psi,\psi^*)\frac{\partial^2\psi}{\partial x_a \partial x_b} + F_2(x_0,\vec{x},\psi,\psi^*,\underset{1}{\psi},\underset{1}{\psi}^*) = 0,$$
$$\psi \equiv \psi(x_0,x_1,x_2,x_3), \quad x_0 \equiv t, \quad \underset{1}{\psi}(x) = \left\{\frac{\partial\psi}{\partial x_a}\right\}, \quad \underset{1}{\psi}^* = \left\{\frac{\partial\psi^*}{\partial x_a}\right\}, \tag{2}$$

where F_1, F_2, g_{ab} are some smooth functions,

$$L_3\psi \equiv S\psi - F_3(\psi,\psi^*,\underset{1}{\psi},\underset{1}{\psi}^*,\underset{2}{\psi},\underset{2}{\psi}^*),$$
$$\underset{2}{\psi}(x) \equiv \left\{\frac{\partial^2\psi}{\partial x_a \partial x_b}\right\}, \quad \underset{2}{\psi}^*(x) \equiv \left\{\frac{\partial^2\psi^*}{\partial x_a \partial x_b}\right\}. \tag{1'}$$

In the present paper we consider the following problems.

Problem 1. Describe all nonlinear equations (1), (2) which are invariant with respect to the Galilei group and its various generalizations.

Problem 2. Study the conditional symmetry of equation (1).

Problem 3. Construct classes of exact solutions for Galilei invariant equations.

The results of this talk have been obtained in collaboration with R. Cherniha, V. Chopyk and M. Serov.

2. Galilei invariant quazilinear equations

Theorem 1 [1]. *There are only three types of equations of the form (1)*

$$S\psi = \lambda F(|\psi|)\psi, \tag{3}$$

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$$S\psi = \lambda|\psi|^k\psi, \quad k \in \mathbb{R}, \tag{4}$$

$$S\psi = \lambda|\psi|^{4/n}\psi, \quad n = 1, 2, 3\ldots, \tag{5}$$

which are invariant, correspondingly, with respect to the following algebras:

$$\begin{gathered} AG(1,n) = \langle P_0, P_a, J_{ab}, G_a, Q\rangle, \quad a = 1, 2, \ldots, n, \\ P_0 = i\frac{\partial}{\partial x_0} \equiv p_0, \quad P_a = -i\frac{\partial}{\partial x_a} \equiv p_a, \\ J_{ab} = x_a p_b - x_b p_a, \quad G_a = x_0 p_a - m x_a Q, \quad Q = i\left(\psi\frac{\partial}{\partial\psi} - \psi^*\frac{\partial}{\partial\psi^*}\right); \end{gathered} \tag{6}$$

$$\begin{gathered} AG(1,n) = \langle AG(1,n), D\rangle, \\ D = 2x_0 p_0 - x_a p_a - kI, \quad k \in \mathbb{R}, \quad I = \psi\frac{\partial}{\partial\psi} + \psi^*\frac{\partial}{\partial\psi^*}; \end{gathered} \tag{7}$$

$$\begin{gathered} AG(1,n) = \langle AG_1(1,n), \Pi\rangle, \\ \Pi = x_0^2 p_0 + x_0 x_a p_a + \frac{m}{2}x^2 Q + \frac{n}{2}x_0 I, \end{gathered} \tag{8}$$

λ is arbitrary parameter, n is the number of space variables.

Note 1. If we put $F = 0$ in (1) we obtain the standard linear Schrödinger equation and its maximal invariance algebra is $AG_2(1,n)$.

Corollary 1. *There is only one nonlinear equation in the class of Schrödinger equations (1)*

$$\left(p_0 - \frac{p_a^2}{2m}\right)\psi = \lambda|\psi|^{4/n}\psi \tag{9}$$

which has the same symmetry as the linear Schrödinger equation.

Let us answer the following question: whether there exist other equations in the class (1) invariant under the Galilei algebra $AG(1,n)$ but not invariant under operators D and Π (7), (8).

The following theorem answers this question.

Theorem 2 [2]. *There is only one equation of the form (1)*

$$\left(p_0 - \frac{p_a^2}{2m}\right)\psi = \lambda\ln(\psi\psi^*)\psi, \quad \lambda = \lambda_1 + i\lambda_2 \tag{10}$$

which is invariant with respect to the following algebras:

$$\begin{gathered} AG_3(1,n) \equiv \langle AG(1,n), B_1\rangle, \quad \lambda_1 \neq 0, \quad \lambda_2 = 0, \quad B_1 = I - 2\lambda_1 x_0 Q; \\ AG_4(2,n) = \langle AG(1,n), B_2\rangle, \quad \lambda_3 \neq 0, \quad B_2 = \exp(2\lambda_2 x_0)(I + i\lambda_1\lambda_2^{-1}Q). \end{gathered}$$

Note 2. The maximal invariance algebra of equation (10) with logarithmic nonlinearity contains operators not admitted by the linear equation (1).

Corollary 2. *Operators B_1, B_2 generate the following transformations for ψ:*

$$\begin{gathered} \psi \to \psi' = \exp\{(1 - 2i\lambda_1 x_0)\theta_1\}\psi \quad \textit{for } \lambda_2 = 0, \\ \psi \to \psi' = \exp\{\theta_2[2x_0\lambda_2(1 - i\lambda_1\lambda_2^{-1})\}\psi, \quad \lambda_2 \neq 0, \\ \psi \to \psi' = \exp\{\theta_3\exp(2\lambda_1 x_0)\}\psi, \quad \lambda_1 \neq 0, \quad \lambda_2 \neq 0, \end{gathered}$$

where θ_1, θ_2, θ_3 are group parameters.

The following equation is widely used for description of dissipative systems

$$i\frac{\partial\psi}{\partial t} = \frac{1}{2m}\Delta\psi - i\beta\ln(\psi(\psi^*)^{-1})\psi + F_2(\psi\psi^*)\psi. \tag{11}$$

Equation (11) is usually called the phase Schrödinger equation or the Schrödinger–Langevin equation [4].

The main difference of equation (11) from equations (3), (4), (5), (10), (11) is that it is not invariant with respect to the Galilei transformations. This equation does not the standard Galilei relativity principle. However equation (11) has interesting symmetry properties.

Theorem 3 [2]. *The maximal invariance algebra of equation (11) is a 11-dimensional Lie algebra*

$$A = \langle P_0, P_a, J_{ab}, G_a^{(1)}, Q\rangle,$$
$$G_a^{(1)} = \exp\{2\beta x_0\}\left(p_a + \frac{\beta m}{2}x_a\right)Q, \quad Q_1 = \exp\{2\beta x_0\}Q.$$

Corollary 3. *Operators $G_a^{(1)}$ generate the transformations*

$$x_0 \to x_0' = x_0, \quad x_a \to x_a' = x_a + \exp\{2\beta x_0\}\theta_a, \tag{12}$$

$$\psi \to \psi' = \psi\exp\{i[\beta m\exp(4\beta x_0)\theta^2 + \exp(2\beta x_0)x_a\theta_a]\}, \tag{13}$$

where $\theta^2 = \theta_a\theta_a$, θ_a are group parameters.

So operators $G_a^{(1)}$ as distinguished from the Galilei operators, generate nonlinear transformations (12). In the first approximation by β (12) coincides with the Galilei transformations. It is known that the Galilei transformations are of the form

$$\begin{aligned} &x_0 \to x_0' = x_0, \quad x_a \to x_a' = x_a + x_0\theta_a,\\ &\psi \to \psi' = \exp\left\{im\left(\vec{\theta}\vec{x} + \frac{1}{2}(\vec{\theta})^2x_0\right)\right\}\psi(x'). \end{aligned} \tag{14}$$

3. Galilei invariant nonlinear equations with first order derivatives

Let us consider equations

$$S\psi = F(x, \psi, \psi^*, \underset{1}{\psi}, \underset{1}{\psi}^*). \tag{15}$$

Theorem 4 [5]. *There exist four classes of equations of the form (15) which are invariant with respect to Galilei algebras:*

$$AG_1(1,n): \quad S\psi = F_1(|\psi|, (\vec{\nabla}|\psi|)^2)\psi; \tag{16}$$

$$AG_1(1,n): \quad S\psi = |\psi|^{-2/k}F_2(|\psi|^{-2+2/k}(\vec{\nabla}|\psi|)^2)\psi, \tag{17}$$

$$S\psi = (\vec{\nabla}|\psi|)^2F_3(|\psi|)\psi; \tag{18}$$

$$AG_2(1,n): \quad S\psi = |\psi|^{4/n}F_4(|\psi|^{-2-4/n}(\vec{\nabla}|\psi|)^2). \tag{19}$$

Let us adduce some simplest $G_2(1,3)$ invariant equations:

$$S\psi = \lambda|\psi|^{4/3}\psi, \tag{20}$$

$$S\psi = \lambda|\psi|^2 \frac{\partial|\psi|\partial|\psi|}{\partial x_a \partial x_a}\psi. \tag{21}$$

4. Conditional symmetry of the nonlinear Schrödinger equation

Let us consider some nonlinear differential equation of s-th order:

$$L(x, \psi, \underset{1}{\psi}, \underset{2}{\psi}, \ldots, \underset{s}{\psi}) = 0, \tag{22}$$

$\underset{s}{\psi}$ designates the set of all s-th order derivatives.

Let us assume that equation (22) is invariant with respect to a certain Lie algebra $A = \langle X_1, X_2, \ldots, X_n\rangle$, where X_k are basis vectors of the algebra A.

This means that the following conditions must be satisfied:

$$\underset{s}{X_k} L = RL, \tag{23}$$

where $\underset{s}{X_k}$ is the s-th prolongation of the operator $X_k \in A$, $R = R(x, \psi, \underset{1}{\psi}, \ldots)$ is some differential expression.

Let us consider a set of operators which do not belong to the invariance algebra of equation (22)

$$Y = \langle Y_1, Y_2, \ldots, Y_r\rangle, \quad Y_k \in A.$$

Definition 1 [6, 7]. We shall say that equation (22) is conditionally invariant with respect to the operators Y if there exists an additional condition

$$\tilde{L}_1(x, \psi, \underset{1}{\psi}, \ldots,) = 0 \tag{24}$$

on solutions of equation (22), such that equation (22) together with (24) is invariant with respect to the set of operators Y. This means that the following conditions are satisfied:

$$\underset{s}{Y_k} L = R_0 L + R_1 \tilde{L}_1, \quad \underset{s}{Y_k} \tilde{L} = R_2 L + R_3 \tilde{L}_1,$$

where R_0, R_1, R_2, R_3 are some smooth functions, $\underset{s}{Y_k}$ is the s-th Lie prolongation of the operator $Y_k \in Y$.

It is evident that Definition 1 makes sense only if system (23), (24) is compatible.

The notion of conditional symmetry has turned out extremely efficient, and during recent years it was established that d'Alembert, Schrödinger, Maxwell, heat, Boussinesq equations possess nontrivial conditional symmetry. The problem of detailed description of conditional symmetry for principal equations of mathematical physics remains open [6, 7].

Theorem 5 [2, 8]. *Equation*

$$\left(p_0 - \frac{p_a^2}{2m}\right)\psi = F(|\psi|)\psi \tag{25}$$

is conditionally invariant with respect to the operator

$$Y = x_a p_a + r\left(\psi\frac{\partial}{\partial\psi} + \psi^*\frac{\partial}{\partial\psi^*}\right) - i\ln(\psi(\psi^*)^{-1})Q, \tag{26}$$

if

$$F = a_1|\psi|^{2r^{-1}} + a_2|\psi|^{-2r^{-1}}, \quad r \neq 0 \tag{27}$$

$$\tilde{L}_1(u) = \Delta|\psi| - a_3|\psi|^{(r-2)/r} = 0, \quad a_3 = \frac{1}{2}a_2 m, \quad r, a_1, a_2 \in \mathbb{R}. \tag{28}$$

Corollary 4. *Operator (26) generates the following finite transformations*

$$x_0 \to x_0' = x_0, \quad x_a \to x_a' = \exp\theta \cdot x_a, \tag{29}$$

$$\psi \to \psi' = \exp(r\theta)\exp\{\exp(2\theta)\}(\psi(\psi^*)^{-1})^{1/2}|\psi|, \tag{30}$$

θ is the group parameter.

Formula (30) gives nonlinear transformations for the function ψ.

So equation (25), (27) together with (28) admits an additional operator Y (26). Equation (25) with the nonlinearity (27) without the additional condition (28) is not invariant with respect to the operator (26).

Having the additional symmetry operator (26) we can construct new ansatzes.

5. Reduction and exact solutions of nonlinear equations

Let us consider the simplest equations (1), (2) which are invariant with respect to algebra $AG_2(1,3)$:

$$i\frac{\partial\psi}{\partial t} = -\Delta\psi + \lambda|\psi|^{4/3}\psi, \tag{31}$$

$$i\frac{\partial\psi}{\partial t} = -\Delta\psi + \lambda|\psi|^{-2}\frac{\partial|\psi|}{\partial x_k}\frac{\partial|\psi|}{\partial x_k}\psi. \tag{32}$$

We shall search for the solutions in the form [7]

$$\psi = f(t,\vec{x})\varphi(w), \quad w \equiv (w_1, w_2, w_3), \quad w_k = w_k(t,\vec{x}). \tag{33}$$

Definition 2. We shall say that the formula (33) is an ansatz for equations (31), (32) if functions $f(x)$, w_1, w_2, w_3 have such structure that four-dimensional equations are reduced to three-dimensional ones for the function $\varphi(w)$. Equations obtained for $\varphi(w)$ depend only on w.

The problem of reduction in the general formulation is an extremely difficult problem and it requires explicit description of functions $f(x)$, w_1, w_2, w_3 which satisfy a nonlinear system of equations. We do not think that it is possible now to construct the general solution of these equations. But in case of an equation having rich symmetry properties the problem of reduction and description of $f(x)$ and w can be partially reduced to an algebraic problem of description of inequivalent subalgebras of this equation [7].

By means of subalgebraic structure of the algebra $AG_2(1,3)$ we have constructed quite a large list of ansatzes which reduce four-dimensional equations (31), (32) to three-dimensional ones. I adduce some of them.

Ansatzes for equations (31), (32).

$$1.\quad \psi(x)=\exp\left(i\frac{x_3^2}{4t}\right)\varphi(w),\qquad w_1=t,\quad w_2=x_1^2+x_2^2,\quad w_3=x_3-t\arctan\frac{x_2}{x_1}. \tag{34}$$

The reduced equation

$$i\left(\frac{\partial\varphi}{\partial w_1}+\frac{\varphi}{2w_1}+\frac{w_3}{w_1}\frac{\partial\varphi}{\partial w_3}\right)=-4w_2\frac{\partial^2\varphi}{\partial w_2^2}-\left(1+\frac{w_1^2}{w_2^2}\right)\frac{\partial^2\varphi}{\partial w_3^2}+\lambda|\varphi|^{4/3}\varphi. \tag{35}$$

$$2.\quad \psi=(t^2+1)^{-3/4}\exp\left\{\frac{i}{4}\left(\frac{|\vec{x}|^2t}{1+t^2}+2\alpha\arctan t\right)\right\}\varphi(w),\qquad w_1=\frac{x_1}{\sqrt{1+t^2}},\quad w_2=\frac{x_2}{\sqrt{1+t^2}},\quad w_3=\frac{x_3}{\sqrt{1+t^2}}. \tag{36}$$

The reduced equation

$$-\frac{\partial^2\varphi}{\partial w_1^2}-\frac{\partial^2\varphi}{\partial w_2^2}-\frac{\partial^2\varphi}{\partial w_3^2}-\frac{(2\alpha-\vec{w}\vec{w})}{4}\varphi+\lambda|\varphi|^{4/3}\varphi=0, \tag{37}$$

where α is an arbitrary real parameter.

$$3.\quad \psi=(t^2+1)^{-3/4}\exp\left\{\frac{i}{4}\left(\frac{|\vec{x}|^2t}{1+t^2}+2\beta\frac{tx_2-x_1}{t^2+1}\arctan t\right)\right\}\varphi(w),\qquad w_1=\frac{tx_1+x_2}{t^2+1}+\beta\arctan t,\quad w_2=\frac{tx_2+x_1}{t^2+1},\quad w_3=\frac{x_3}{\sqrt{t^2+1}}. \tag{38}$$

The reduced equation

$$i\left(\beta\frac{\partial\varphi}{\partial w_1}+w_1\frac{\partial\varphi}{\partial w_2}-w_2\frac{\partial\varphi}{\partial w_1}\right)=\Delta\varphi+\frac{1}{4}(2\beta w_2+\vec{w}\vec{w})\varphi+\lambda|\varphi|^{4/3}\varphi. \tag{39}$$

Having investigated symmetry of reduced equations which depend on three variables and then of ones depending on two variables we come finally to ordinary differential equations of the form

$$A(w)\frac{d^2\varphi}{dw^2}+B(w)\frac{d\varphi}{dw}+C(w)\varphi+\lambda|\varphi|^{4/3}\varphi=0, \tag{40}$$

where $A(w)$, $B(w)$, $C(w)$ are second degree polynomials.

Having solved equations (40) we construct exact solutions of the four-dimensional nonlinear Schrödinger equations (31) by means of the formulae (34), (36), (38).

Solutions of equation (32) constructed by means of ansatzes (34), (36), (38).

$$\psi(t,\vec{x})=\frac{\exp(ia_0t)}{\{\sqrt{-\gamma}\cos(\vec{a}\vec{x})\}^{3/2}},\qquad \lambda>0,\quad a_0<0;$$

$$\psi(t,\vec{x})=\frac{\exp(ia_0t)}{\{\sqrt{-\gamma}\,\mathrm{sh}(\vec{a}\vec{x})\}^{3/2}},\qquad \lambda>0,\quad a_0>0;$$

$$\psi(t,\vec{x})=\frac{\exp(ia_0t)}{\{\sqrt{-\gamma}\,\mathrm{ch}(\vec{a}\vec{x})\}^{3/2}},\qquad \lambda<0,\quad a_0>0;$$

a_k are arbitrary real parameters and what is more $\vec{a}\vec{a} = a^2 = \frac{4}{9}|a_0|$, $\gamma = 3\lambda/5a_0$.

One can see that all obtained solutions depend non-analytically on the parameter λ (constant of interaction).

The obtained three-dimensional partial solutions can be used for construction of multi-parameter families of exact solutions. Really, as equation (31) is invariant with respect to 13-parameter group $G(1,3)$, that means the following.

If $\psi_1(t,\vec{x})$ is a solution of equation (31), then functions

$$\begin{aligned} \psi_2(t,\vec{x}) &= \exp\left\{\frac{i}{2}\left(\vec{v}\vec{x} + \frac{\vec{v}^{\,2}t}{2}\right)\right\}\psi_1(t,\vec{x}+\vec{v}t), \\ \psi_3(t,\vec{x}) &= \exp\left\{-\frac{i}{4}\frac{\theta\vec{x}^{\,2} + 2\vec{v}\vec{x} + \vec{v}^{\,2}t}{1-\theta t}\right\}(1-\theta t)^{-3/2}\psi_1\left(\frac{t}{1-\theta t}, \frac{\vec{x}-\vec{v}t}{1-\theta t}\right) \end{aligned} \qquad (41)$$

are also solutions of the same equation. $\vec{v}$, θ are real parameters.

6. Galilei invariant nonlinear equations with second order derivatives

Now we formulate one result about the equations ($1'$) which are invariant under $AG_2(1,n)$ (for more details, see [9]).

Theorem 6 [9]. *The equations*

$$\begin{aligned} S\psi = A_0 &\left\{\Delta\psi - \psi^{-1}\frac{\partial\psi}{\partial x_a}\frac{\partial\psi}{\partial x_a} + (\psi^*)^{-1}\psi\left[\Delta\psi^* - (\psi^*)^{-1}\frac{\partial\psi^*}{\partial x_a}\frac{\partial\psi^*}{\partial x_a}\right]\right\} + \\ &+ A_1|\psi|^{4/n}\psi + A_2|\psi|^{-\frac{2n+4}{n}}\frac{\partial|\psi|}{\partial x_a}\frac{\partial|\psi|}{\partial x_b}\times \\ &\times\left\{\frac{\partial^2\psi}{\partial x_a\partial x_b} - \psi^{-1}\frac{\partial\psi}{\partial x_a}\frac{\partial\psi}{\partial x_b} + (\psi^*)^{-1}\psi\left[\frac{\partial^2\psi^*}{\partial x_a\partial x_b} - (\psi^*)^{-1}\frac{\partial\psi^*}{\partial x_a}\frac{\partial\psi^*}{\partial x_b}\right]\right\}, \\ A_0 \equiv A_0(w), &\quad A_1 \equiv A_1(w), \quad A_2 \equiv A_2(w) \quad w = \frac{\partial|\psi|}{\partial x_a}\frac{\partial|\psi|}{\partial x_a}|\psi|^{-\frac{2n+4}{n}} \end{aligned}$$

are invariant under $AG_2(1,n)$ algebra. A_0, A_1, A_2 are arbitrary smooth functions.

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1. Fushchych W., Serov M., On some exact solutions of the three-dimensional nonlinear Schrödinger equation, *J. Phys. A*, 1987, **20**, № 16, L29–L33.
2. Fushchych W., Chopyk V., Symmetry and non-Lie reduction of the nonlinear Schrödinger equation, *Ukr. Math. J.*, 1993, **45**, № 4, 581–597.
3. Schuh D., Chung K.-M., Hartman H.N., Nonlinear Schrödinger-type field equation for the description of dissipative systems, *J. Math. Phys.*, 1983, **24**, № 6, 1652–1660.
4. Briihl L., Lange H., The Schrödinger–Langevin equation: special solutions and nonexistence of solitary waves, *J. Math. Phys.*, 1984, **25**, № 4, 786–790.
5. Fushchych W., Cherniha R., Galilei invariant nonlinear equation of Schrödinger type and their exact solutions: I, II, *Ukr. Math. J.*, 1989, **41**, № 10, 1349–1357; № 12, 1687–1694.
6. Fushchych W., Nikitin A., Symmetries of Maxwell’s equations, D. Reidel, 1987.

7. Fushchych W., Shtelen W., Serov M., Symmetry analysis and exact solutions of equations of nonlinear mathematical physics, Kluwer Academic Publishers, 1993.

8. Fushchych W., Chopyk V., Conditional invariance of the nonlinear Schrödinger equation, *Proc. Ukrainian Acad. Sci.*, 1990, № 4, 30–33.

9. Fushchych W., Cherniha R., Systems of nonlinear evolution equations of the second order invariant under the Galilean algebra and its extensions, *Proc. Ukrainian Acad. Sci.*, 1993, № 8, 44–51.

On new exact solutions of the multidimensional nonlinear d'Alembert equation

W.I. FUSHCHYCH, A.F. BARANNYK, YU.D. MOSKALENKO

On the present paper new classes of exact solutions of the nonlinear d'Alembert equation in the space $R_{1,n}$, $n \geq 2$,

$$\Box u + \lambda u^k = 0 \tag{1}$$

are built. Here $\Box u = u_{00} - u_{11} - \cdots - u_{nn}$, $u_{\mu\nu} = \frac{\partial^2 u}{\partial x_\mu \partial x_\nu}$, $u = u(x)$, $x = (x_0, x_1, \ldots, x_n)$; $\mu, \nu = 0, 1, \ldots, n$. Symmetry properties of equation (1) have been studied in papers [1, 2] in which it was established that equation (1) is invariant under the extended Poincaré algebra $A\tilde{P}(1,n)$:

$$J_{0a} = x_0\partial_a + x_a\partial_0, \quad J_{ab} = x_b\partial_a - x_a\partial_b, \quad P_\mu = \partial_\mu,$$
$$S = -x^\mu\partial_\mu + \frac{2u}{k-1}\partial_u \quad (a, b = 1, \ldots, n;\ \mu = 0, 1, \ldots, n).$$

Using the subgroup structure of the group $\tilde{P}(1,2)$ in papers [1, 2] some classes of exact solutions of equation (1) in the space $R_{1,2}$ were built. The analogous results in the space $R_{1,3}$ were obtained in [3, 4]. The generalization of results for the n-dimensional case was considered in [5, 6]. In order to find exact solutions, symmetry ansatzes reducing equation (1) to ordinary differential equations were applied in above mentioned papers.

In the present paper in order to build exact solutions of equation (1), symmetry ansatzes reducing equation (1) to equations of two invariant variables are used. We are interested in these ansatzes because a reduced equation often has additional symmetries. This fact permits to apply these ansatzes for finding new solutions of the present equation. Let us cite as an example the ansatz $u = u(x_0 - x_n, x_1, \ldots, x_{n-1})$ which was considered in [6]. The corresponding reduced equation has the infinite group of invariance. Note that this ansatz is built by one-dimensional subalgebra $\langle P_0 + P_n \rangle$.

In the present paper the series of ansatzes of such a kind as $u = u(\omega_1, \omega_2)$, where $\omega_1 = x_0 - x_m$, $\omega_2 = x_0^2 - x_1^2 - \cdots - x_m^2$, $2 \leq m \leq n$, is considered. These ansatzes are built by the subalgebras $AE_1[1, m-1] \oplus AE[m+1, n]$, where $AE_1[1, m-1] = \langle G_1, \ldots, G_{m-1}, J_{12}, \ldots, J_{m-2,m-1} \rangle$, $AE[m+1, n] = \langle P_{m+1}, \ldots, P_n, J_{m+1,m+2}, \ldots, J_{n-1,n} \rangle$, $G_a = J_{0a} - J_{am}$, $a = 1, \ldots, m-1$, and if $m = n$ we think $AE[m+1, n] = 0$. The ansatz $u = u(\omega_1, \omega_2)$ reduces equation (1) to the equation

$$4\omega_1 u_{12} + 4\omega_2 u_{22} + 2(m+1)u_2 + \lambda u^k = 0. \tag{2}$$

Let us investigate symmetry of the equation (2).

Доповіді НАН України, 1995, № 2, P. 33–37.

Theorem 1. *The maximal algebra of invariance of equation (2) in the case of $k \neq 0$, $\frac{m+1}{m-1}$ and $m > 1$ in the Lie sense is the 4-dimensional Lie algebra $A(4)$ which is generated by such operators:*

$$X_1 = \omega_1 \frac{\partial}{\partial w_1} + \omega_2 \frac{\partial}{\partial \omega_2} - \frac{1}{k-1} u \frac{\partial}{\partial u}, \quad X_2 = \omega_2 \frac{\partial}{\partial w_2} - \frac{1}{k-1} u \frac{\partial}{\partial u},$$
$$M = \omega_1^l \left(\omega_1 \frac{\partial}{\partial w_1} + \omega_2 \frac{\partial}{\partial \omega_2} - \frac{m-1}{2} u \frac{\partial}{\partial u} \right), \quad l = \frac{(m-1)(k-1)}{2} - 1.$$

Theorem 2. *The maximal algebra of invariance of equation (2) in the case of $k = \frac{m+1}{m-1}$ and $m > 1$ in the sense of Lie is the 4-dimensional Lie algebra $B(4)$ which is generated by such operators:*

$$S = \omega_1 \ln \omega_1 \frac{\partial}{\partial \omega_1} + \omega_2 \ln \omega_1 \frac{\partial}{\partial \omega_2} - \frac{m-1}{2} (\ln \omega_1 + 1) u \frac{\partial}{\partial u},$$
$$Z_1 = \omega_1 \frac{\partial}{\partial \omega_1} + \omega_2 \frac{\partial}{\partial \omega_2} - \frac{m-1}{2} u \frac{\partial}{\partial u},$$
$$Z_2 = \omega_2 \frac{\partial}{\partial \omega_2} - \frac{m-1}{2} u \frac{\partial}{\partial u}, \quad Z_3 = \omega_1 \frac{\partial}{\partial \omega_2}.$$

Let us consider two cases.

1. The case $k \neq \frac{m+1}{m-1}$. Classify one-dimensional subalgebras of the algebra $A(4)$ with respect to G-conjugation, where $G = \exp A(4)$. Ansatzes, built by these subalgebras, reduce the equation (2) to ordinary differential equations. Note that the operators of the algebra $A(4)$ satisfy the following commutation relations: $[X_1, X_2] = 0$, $[X_1, X_3] = 0$, $[X_1, M] = lM$, $[X_2, X_3] = -X_3$, $[X_2, M] = 0$, $[X_3, M] = 0$.

Theorem 3. *Let K be one-dimensional subalgebra of the algebra $A(4)$. Then K is conjugated with one of the following algebras:* 1) $K_1 = \langle X_1 + \alpha X_2 \rangle$; 2) $K_2 = \langle X_2 \rangle$; 3) $K_3 = \langle X_1 + \alpha X_3 \rangle$ $(\alpha = \pm 1)$; 4) $K_4 = \langle X_3 \rangle$; 5) $K_5 = \langle M + \alpha X_2 \rangle$ $(\alpha = 0, \pm 1)$; 6) $K_6 = \langle M + \alpha X_3 \rangle$ $(\alpha = \pm 1)$.

The following ansatzes correspond to the subalgebras K_1–K_6 of the theorem 3:

$$K_1: \quad u = \omega^{\frac{\alpha+1}{1-k}} \varphi(\omega), \quad \omega = \omega_2 \omega_1^{-\alpha-1};$$
$$K_2: \quad u = \omega_2^{\frac{1}{1-k}} \varphi(\omega), \quad \omega = \omega_1;$$
$$K_3: \quad u = \omega_1^{\frac{1}{1-k}} \varphi(\omega), \quad \omega = \frac{\omega_2}{\omega_1} - \alpha \ln \omega_1;$$
$$K_4: \quad u = \varphi(\omega), \quad \omega = \omega_1;$$
$$K_5: \quad u = (\omega_1^1 \omega_2)^{\frac{1}{1-k}} \varphi(\omega), \quad \omega = \frac{\alpha}{l} \omega_1^{-l} + \ln \frac{\omega_2}{\omega_1};$$
$$K_6: \quad u = \omega_1^{\frac{1-m}{2}} \varphi(\omega), \quad \omega = \frac{\omega_2}{\omega_1} + \frac{\alpha}{l} \omega_1^{-l}.$$

These ansatzes reduce equation (2) to ordinary differential equations with an unknown function $\varphi(\omega)$:

$$K_1: \quad -4\alpha\omega\ddot{\varphi} + \frac{4(l - \alpha k)}{k-1} \dot{\varphi} + \lambda \varphi^k = 0;$$
$$K_2: \quad -\frac{4\omega}{k-1} \dot{\varphi} - \frac{4l}{(k-1)^2} \varphi + \lambda \varphi^k = 0;$$
$$K_3: \quad -4\alpha\ddot{\varphi} + \frac{4l}{k-1} \dot{\varphi} + \lambda \varphi^k = 0;$$

$$K_4: \quad \lambda\varphi^k = 0;$$
$$K_5: \quad -4\alpha\ddot{\varphi} + \frac{4\alpha}{k-1}\dot{\varphi} + \lambda\varphi^k = 0;$$
$$K_6: \quad -4\alpha\ddot{\varphi} + \lambda\varphi^k = 0.$$

The equation corresponding to the subalgebra K_1, in case $\alpha = 0$ has the solution

$$\varphi^{1-k} = \frac{\lambda(k-1)^2}{4l}(\omega + C).$$

In consequence we obtain the following solution of equation (1)

$$u^{1-k} = \frac{\lambda(k-1)^2}{4l}(\omega_2 + C\omega_1). \tag{3}$$

If $\alpha = \frac{2l}{k+1}$, then the equation corresponding to the subalgebra K_1 assumes

$$-\frac{8l\omega}{k+1}\ddot{\varphi} - \frac{4l}{k+1}\dot{\varphi} + \lambda\varphi^k = 0.$$

The particular solution of this equation is

$$\varphi^{1-k} = \frac{\lambda(k-1)^2}{4l}(\omega^{\frac{1}{2}} + C)^2.$$

Therefore, equation (1) has the following solution:

$$u^{1-k} = \frac{\lambda(k-1)^2}{4l}\left(\omega_2^{\frac{1}{2}} + C\omega_1^{\frac{k-1}{2(k+1)}}\right)^2. \tag{4}$$

If $\alpha = \frac{l(k+1)}{2}$, then the equation corresponding to the subalgebra K_1 assumes

$$-2l(k+1)\varphi\ddot{\varphi} - 2(k+2)\dot{\varphi} + \lambda\varphi^k = 0.$$

This equation has the solution

$$\varphi^{1-k} = \frac{\lambda(k-1)^2}{4l}\left(\omega^{\frac{1}{2}} + Cw^{\frac{k-1}{2(k+1)}}\right)^2.$$

Therefore, equation (1) has the following solution

$$u^{1-k} = \frac{\lambda(k-1)^2}{4l}\left\{\omega_2^{\frac{1}{2}} + C\omega_1^{\frac{l(k+1)+2}{2(k+1)}}\omega_2^{\frac{k-1}{2(k+1)}}\right\}^2. \tag{5}$$

The equations corresponding to the subalgebras K_5 and K_6 have such solutions:

$$\varphi^{1-k} = \frac{\lambda(k-1)^2}{4l}(1 + C\omega^l), \quad \varphi^{1-k} = \frac{\lambda(k-1)^2}{8\alpha(k+1)}(\omega + C)^2.$$

Therefore, equation (1) has the following solutions:

$$u^{1-k} = \frac{\lambda(k-1)^2}{4l}\omega_2(1 + C\omega_1^l), \tag{6}$$

$$u^{1-k} = \frac{\lambda(k-1)^2}{8\alpha(k+1)}\omega_1^{l-1}\left(\omega_2 + \frac{\alpha}{l}\omega_1^{1-l} + C\omega_1\right)^2. \tag{7}$$

2. The case $k = \frac{m+1}{m-1}$. The basis elements of the algebra $B(4)$ satisfy the following commutation relations: $[S, Z_1] = -Z_1$, $[S, Z_2] = 0$, $[S, Z_3] = 0$, $[Z_1, Z_2] = 0$, $[Z_1, Z_3] = 0$, $[Z_2, Z_3] = -Z_3$.

Theorem 4. *Let L be one-dimensional subalgebra of the algebra $B(4)$. Then L is conjugated with one of the following algebras:* 1) $L_1 = \langle Z_1 + \alpha Z_2\rangle$ $(\alpha = 0, \pm 1)$; 2) $L_2 = \langle Z_2\rangle$; 3) $L_3 = \langle Z_1 + \alpha Z_3\rangle$ $(\alpha = \pm 1)$; 4) $L_4 = \langle Z_3\rangle$; 5) $L_5 = \langle S + \alpha Z_2\rangle$; 6) $L_6 = \langle S + \alpha Z_3\rangle$ $(\alpha = \pm 1)$.

The following ansatzes correspond to the subalgebras L_1–L_6 of theorem 4:

$$
\begin{aligned}
&L_1: \quad u = \omega_2^{\frac{1-m}{2}}\varphi(\omega), \quad \omega = \omega_2\omega_1^{-d-1};\\
&L_2: \quad u = \omega_2^{\frac{1-m}{2}}\varphi(\omega), \quad \omega = \omega_1;\\
&L_3: \quad u = \omega_1^{\frac{1-m}{2}}\varphi(\omega), \quad \omega = \frac{\omega_2}{\omega_1} - \alpha\ln\omega_1;\\
&L_4: \quad u = \varphi(\omega), \quad \omega = \omega_1;\\
&L_5: \quad u = (\omega_1\ln^{\alpha+1}\omega_1)^{\frac{1-m}{2}}\varphi(\omega), \quad \omega = \frac{\omega_1\ln^{\alpha}\omega_1}{\omega_2};\\
&L_6: \quad u = (\omega_1\ln\omega_1)^{\frac{1-m}{2}}\varphi(\omega), \quad \omega = \frac{\omega_2}{\omega_1} - \alpha\ln(\ln\omega_1).
\end{aligned}
$$

These ansatzes reduce equation (2) to ordinary differential equations with an unknown function $\varphi(\omega)$:

$$
\begin{aligned}
&L_1: \quad -4\alpha\omega^2\ddot\varphi + 2\alpha(m-3)\omega\dot\varphi + \lambda\varphi^{\frac{m+1}{m-1}} = 0;\\
&L_2: \quad -2(m-1)\omega\dot\varphi + \lambda\varphi^{\frac{m+1}{m-1}} = 0;\\
&L_3: \quad -4\alpha\ddot\varphi + \lambda\varphi^{\frac{m+1}{m-1}} = 0;\\
&L_4: \quad \lambda\varphi^{\frac{m+1}{m-1}} = 0;\\
&L_5: \quad -4\alpha\omega^3\ddot\varphi + 2((m-3)\alpha + m - 1)\omega^2\dot\varphi + \lambda\varphi^{\frac{m+1}{m-1}} = 0;\\
&L_6: \quad -4\alpha\ddot\varphi - 2(m-1)\dot\varphi + \lambda\varphi^{\frac{m+1}{m-1}} = 0.
\end{aligned}
$$

The equation corresponding to the subalgebra L_2 has the solution

$$\varphi^{\frac{2}{1-m}} = -\frac{\lambda}{(m-1)^2}(\ln\omega + C).$$

Therefore, the equation (2) has the following solution

$$u^{\frac{2}{1-m}} = -\frac{\lambda}{(m-1)^2}(\omega_2\ln\omega_1 + C\omega_2). \tag{8}$$

The equation corresponding to the subalgebra L_3 has the particular solution

$$\varphi^{\frac{2}{1-m}} = -\frac{\lambda}{4\alpha m(m-1)}(\omega + C)^2.$$

Therefore, equation (2) has the following solution

$$u^{\frac{2}{1-m}} = -\frac{\lambda}{4\alpha m(m-1)\omega_1}(\omega_2 - \alpha\omega_1\ln\omega_1 + C\omega_1)^2. \tag{9}$$

In the case of an equation corresponding to the subalgebra L_5 $\alpha = 0$ or $\alpha = \frac{1-m}{m-3}$ $(m \neq 3)$ we obtain the equations:

$$2(m-1)\omega^2\dot{\varphi} + \lambda\varphi^{\frac{m+1}{m-1}} = 0, \quad -\frac{4(1-m)}{m-3}\omega^3\ddot{\varphi} + \lambda\varphi^{\frac{m+1}{m-1}} = 0.$$

The solutions of these equation are:

$$\varphi^{\frac{2}{1-m}} = -\frac{\lambda}{(m-1)^2}(\omega^{-1} + C), \quad \varphi^{\frac{2}{1-m}} = -\frac{\lambda}{(m-1)^2}\omega^{-1}.$$

Hence equation (2) has the following solutions:

$$u^{\frac{2}{1-m}} = -\frac{\lambda}{(m-1)^2}\ln\omega_1(\omega_2 + C\omega_1), \tag{10}$$

$$u^{\frac{2}{1-m}} = -\frac{\lambda}{(m-1)^2}\omega_2\ln\omega_1. \tag{11}$$

Using the groups of invariance of equations (1) and (2) we can duplicate the solutions (3)–(11). In consequence we obtain multiparametric exact solutions of equation (1). Write out these solutions for equation (1) in the space $R_{1,3}$ using the following notations: $a = (a_0, a_1, a_2, a_3)$, $b = (b_0, b_1, b_2, b_3)$, $c = (c_0, c_1, c_2, c_3)$, $y_\mu = x_\mu + \alpha_\mu$ $(\mu = 0, 1, 2, 3)$, $a \cdot b = a_0b_0 - a_1b_1 - a_2b_2 - a_3b_3$, $\varepsilon = \pm 1$.

1) $u^{1-k} = \sigma(y \cdot y)(1 + \varepsilon(b \cdot y)^{k-2}), \quad \sigma = \dfrac{\lambda(k-1)^2}{4(k-2)}, \quad b \cdot b = 0;$

2) $u^{1-k} = \sigma\Big\{[(y \cdot y)(1 + \varepsilon(b \cdot y)^{k-2})]^{\frac{1}{2}} + \alpha(b \cdot y)^{\frac{3(k-1)}{2(k+1)}}[1 + \varepsilon(b \cdot y)^{k-2}]^{\frac{k-1}{2(k+1)}}\Big\}^2,$

$\sigma = \dfrac{\lambda(k-1)^2}{4(k-2)}, \quad b \cdot b = 0, \quad \alpha \in R;$

3) $u^{1-k} = \sigma\left\{[(y \cdot y)(1 + \varepsilon(b \cdot y)^{k-2})]^{\frac{1}{2}} + \alpha(b \cdot y)^{\frac{k(k-1)}{2(k+1)}}(y \cdot y)^{\frac{k-1}{2(k+1)}}\right\}^2,$

$\sigma = \dfrac{\lambda(k-1)^2}{4(k-2)}, \quad b \cdot b = 0, \quad \alpha \in R;$

4) $u^{1-k} = \sigma\varepsilon(b \cdot y)^{k-3}[(y \cdot y) + \varepsilon(b \cdot y)^{3-k}]^2, \quad \sigma = \dfrac{\lambda(k-1)^2}{8(k-2)(k+1)}, \; b \cdot b = 0;$

5) $u^{1-k} = \sigma[(y \cdot y) + (a \cdot y)^2][1 + \varepsilon(b \cdot y)^{\frac{k-3}{2}}], \quad \sigma = \dfrac{\lambda(k-1)^2}{2(k-3)},$

$a \cdot a = -1, \quad a \cdot b = 0, \quad b \cdot b = 0;$

6) $u^{1-k} = \sigma\left\{[((y \cdot y) + (a \cdot y)^2)(1 + \varepsilon(b \cdot y)^{\frac{k-3}{2}})]^{\frac{1}{2}} + \right.$

$\left. + \alpha(b \cdot y)^{\frac{k-1}{k+1}}(1 + \varepsilon(b \cdot y)^{\frac{k-3}{2}})^{\frac{k-1}{2(k+1)}}\right\}^2,$

$\sigma = \dfrac{\lambda(k-1)^2}{2(k-3)}, \quad a \cdot a = -1, \quad a \cdot b = 0, \quad b \cdot b = 0, \quad \alpha \in R;$

7) $u^{1-k} = \sigma\left\{[((y \cdot y) + (a \cdot y)^2)(1 + \varepsilon(b \cdot y)^{\frac{k-3}{2}})]^{\frac{1}{2}} + \right.$

$\left. + \alpha(b \cdot y)^{\frac{(k-1)^2}{4(k+1)}}((y \cdot y) + (b \cdot y)^2)^{\frac{k-1}{2(k+1)}}\right\}^2,$

$$\sigma = \frac{\lambda(k-1)^2}{2(k-3)}, \quad a\cdot a = -1, \quad a\cdot b = 0, \quad b\cdot b = 0, \quad \alpha\in R;$$

8) $$u^{1-k} = \sigma\varepsilon(b\cdot y)^{\frac{k-5}{2}}[(y\cdot y) + (a\cdot y)^2 + \varepsilon(b\cdot y)^{\frac{5-k}{2}}]^2;$$
$$\sigma = \frac{\lambda(k-1)^2}{4(k-3)(k+1)}, \quad a\cdot a = -1, \quad a\cdot b = 0, \quad b\cdot b = 0;$$

9) $$u^{-1} = -\frac{\lambda}{4}(y\cdot y)\ln(b\cdot y), \quad b\cdot b = 0, \quad k = 2;$$

10) $$u^{-1} = -\frac{\lambda\varepsilon}{24}(b\cdot y)^{-1}[(y\cdot y) - \varepsilon(b\cdot y)\ln(b\cdot y)]^2, \quad b\cdot b = 0, \quad k = 2;$$

11) $$u^{-2} = -\lambda\ln(b\cdot y)[(y\cdot y) + (a\cdot y)^2],$$
$$a\cdot a = -1, \quad a\cdot b = 0, \quad b\cdot b = 0, \quad k = 3;$$

12) $$u^{-2} = \frac{\lambda\varepsilon}{8}(b\cdot y)^{-1}[(y\cdot y) + (a\cdot y)^2 - \varepsilon(b\cdot y)]^2,$$
$$a\cdot a = -1, \quad a\cdot b = 0, \quad b\cdot b = 0, \quad k = 3.$$

1. Фущич В.И., Штелень В.М., Серов Н.И., Симметрийный анализ и точные решения нелинейных уравнений математической физики, Киев, Наук. думка, 1989, 336 с.

2. Fushchych W.I., Serov N.I., The symmetry and some exact solutions of the nonlinear many-dimensional Liouville, d'Alembert and eikonal equations, *J. Phys. A*, 1983, **16**, 3645–3658.

3. Grundland A.M., Harnad J., Winternitz P., Symmetry reduction for nonlinear relativistically invariant equations, *J. Math. Phys.*, 1984, **25**, № 4, 791–806.

4. Фущич В.И., Баранник Л.Ф., Баранник А.Ф., Подгрупповой анализ групп Галилея, Пуанкаре и редукция нелинейных уравнений, Киев, Наук. думка, 1991, 304 с.

5. Фущич В.И., Баранник А.Ф., Максимальные подалгебры ранга $n-1$ алгебры $AP(1,n)$ и редукция нелинейных волновых уравнений. I, *Укр. мат. журн.*, 1990, **42**, № 11, 1250–1256.

6. Фущич В.И., Баранник А.Ф., Максимальные подалгебры ранга $n-1$ алгебры $AP(1,n)$ и редукция нелинейных волновых уравнений, II, *Укр. мат. журн.*, 1990, **42**, № 12, 1693–1700.

Symmetry classification of the one-dimensional second order equation of hydrodynamical type

W.I. FUSHCHYCH, V.M. BOYKO

The paper contains a symmetry classification of the one-dimensional second order equation of hydrodynamical type $L(Lu) + \lambda Lu = F(u)$, where $L \equiv \partial_t + u\partial_x$. Some classes of exact solutions of this equation are pointed out.

In [1, 2] the following generalized Navier–Stokes equation

$$\lambda_1 L\vec{v} + \lambda_2 L(L\vec{v}) = F\left(\vec{v}^{\,2}\right)\vec{v} + \lambda_4 \nabla p, \tag{1}$$

was proposed, where

$$L \equiv \frac{\partial}{\partial t} + v^l \frac{\partial}{\partial x_l} + \lambda_3 \triangle, \quad l = 1, 2, 3,$$

$\vec{v} = \left(v^1, v^2, v^3\right)$, $v^l = v^l(t, \vec{x})$, $p = p(t, \vec{x})$, ∇ is the gradient, $\triangle$ is the Laplace operator, λ_1, λ_2, λ_3, λ_4 are arbitrary real parameters, $F\left(\vec{v}^{\,2}\right)$ is an arbitrary differentiable function.

In the one-dimensional scalar case, when $\lambda_3 = 0$, $\lambda_4 = 0$, equation (1) has the form

$$\lambda_1 Lu + \lambda_2 L(Lu) = F(u), \tag{2}$$

where $u = u(t, x)$, $L \equiv \partial_t + u\partial_x$.

In the case when $\lambda_2 = 0$ and $F(u) = 0$, equation (2) is known to describe the simple wave

$$u = \varphi(x - tu), \tag{3}$$

where φ is an arbitrary function. Formula (3) gives the general solution of the equation

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0.$$

If $\lambda_2 \neq 0$, then equation (2) can be rewritten in the form

$$L(Lu) + \lambda Lu = F(u), \quad \lambda = \text{const}. \tag{4}$$

Equation (4), in expanded form, is written as follows

$$\frac{\partial^2 u}{\partial t^2} + 2u\frac{\partial^2 u}{\partial t \partial x} + \frac{\partial u}{\partial t}\frac{\partial u}{\partial x} + u\left(\frac{\partial u}{\partial x}\right)^2 + u^2\frac{\partial^2 u}{\partial x^2} + \lambda\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x}\right) = F(u).$$

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This equation with arbitrary $F(u)$ is evidently invariant under the two-dimensional algebra of translations that is determined by the operators

$$P_0 = \partial_t, \quad P_1 = \partial_x. \tag{5}$$

In the present paper we carry out a symmetry classification of the equation (4), i.e., we describe functions $F(u)$, with which the equation (4) admits more extensive Lie algebras than the two-dimensional algebra of translations (5).

Symmetry classification

Symmetry classification of (4) is performed on the base of the Lie algorithm [3, 4, 5] in the class of first-order differential operators

$$X = \xi^0(t,x,u)\partial_t + \xi^1(t,x,u)\partial_x + \eta(t,x,u)\partial_u. \tag{6}$$

Remark. In cases **1.4, 2.3, 2.4** we assume that

$$\frac{\partial \xi^0}{\partial u} = 0, \quad \frac{\partial \xi^1}{\partial u} = 0.$$

It is obvious, that the cases $\lambda = 0$ and $\lambda \neq 0$ will be essentially different for the investigation of symmetries of the equation (4). If $\lambda \neq 0$, then one can always set $\lambda \equiv 1$ (there exists a change of variables). For this reason we consider the cases $\lambda = 0$ and $\lambda = 1$ separately.

I. Let us consider equation (4), when $\lambda = 0$, i.e., the equation

$$L(Lu) = F(u). \tag{7}$$

Symmetry classification of (7) leads to five distinct cases.

Case 1.1. $F(u)$ is an arbitrary continuously differentiable function. The maximal invariance algebra in this case is the two-dimensional algebra (5).

Case 1.2. $F(u) = a\exp(bu)$, $a, b = \mathrm{const}$, $a \neq 0$, $b \neq 0$. Without loss of generality we can put $b \equiv 1$ (there exists a change of variables). The maximal invariance algebra of the equation

$$L(Lu) = a\exp(u) \tag{8}$$

is a three-dimensional algebra, whose basis elements are given by the operators

$$P_0 = \partial_t, \quad P_1 = \partial_x, \quad Y = t\partial_t + (x - 2t)\partial_x - 2\partial_u. \tag{9}$$

The finite transformations which are generated by the operator Y in (9) have the form:

$$\begin{aligned} &t \to \tilde{t} = t\exp(\theta), \\ &x \to \tilde{x} = (x - 2\theta t)\exp(\theta), \\ &u \to \tilde{u} = u - 2\theta. \end{aligned}$$

Hereafter θ is a real group parameter of the corresponding Lie group.

We note that Y in (9) can be represented as the linear combination of the dilatation and Galilei operators

$$Y = (t\partial_t + x\partial_x) - 2(t\partial_x + \partial_u) = D - 2G.$$

The operators D and G commute, thus the transformations corresponding to Y can be interpreted as a composition of dilatation and Galilei transformations, i.e., as a composition of dilatation on t and x with a change of inertial system. On the other hand, the operators (9) form a subalgebra of extended Galilei algebra, although the extended Galilei algebra is not the invariance algebra of the equation (8). The same results are valid for other cases of equation (4).

Case 1.3. $F(u) = a(u+b)^p$, $a, b, p = \text{const}$, $a \neq 0$, $p \neq 0$, $p \neq 1$. The maximal invariance algebra of the equation

$$L(Lu) = a(u+b)^p \tag{10}$$

is a three-dimensional algebra, whose basis elements are given by the operators

$$\begin{aligned} &P_0 = \partial_t, \quad P_1 = \partial_x, \\ &R = t\partial_t + \left(\frac{p-3}{p-1}x - \frac{2b}{p-1}t\right)\partial_x - \frac{2}{p-1}(u+b)\partial_u. \end{aligned} \tag{11}$$

The operator R generates the following finite transformations:

$$\begin{aligned} &t \to \tilde{t} = t\exp(\theta), \\ &x \to \tilde{x} = x\exp\left(\frac{p-3}{p-1}\theta\right) - bt\exp(\theta), \\ &u \to \tilde{u} = (u+b)\exp\left(-\tfrac{2}{p-1}\theta\right) - b. \end{aligned}$$

If $b \neq 0$, then R can be again represented as a linear combination of dilatation and Galilei operators.

Case 1.4. $F(u) = au + b$, $a, b = \text{const}$, $a \neq 0$. In consequence of a change of variables one can always set $a \equiv 1$ or $a \equiv -1$. Let us consider these cases.

a) The invariance algebra of the equation

$$L(Lu) = u + b \tag{12}$$

is a seven-dimensional algebra, whose basis elements are given by the operators

$$\begin{aligned} &P_0 = \partial_t, \quad P_1 = \partial_x, \\ &Y_1 = (x+bt)\partial_x + (u+b)\partial_u, \\ &Y_2 = \cosh t\partial_x + \sinh t\partial_u, \\ &Y_3 = \sinh t\partial_x + \cosh t\partial_u, \\ &Y_4 = \cosh t\partial_t + (x+bt)\sinh t\partial_x + ((x+bt)\cosh t + b\sinh t)\partial_u, \\ &Y_5 = \sinh t\partial_t + (x+bt)\cosh t\partial_x + ((x+bt)\sinh t + b\cosh t)\partial_u. \end{aligned} \tag{13}$$

The operators Y_1–Y_3 generate the following finite transformations (because the transformations for Y_4 and Y_5 are cumbersome we omit their explicit form):

$$\begin{aligned} Y_1: \quad &t \to \tilde{t} = t, \\ &x \to \tilde{x} = (x+bt)\exp(\theta) - bt, \\ &u \to \tilde{u} = (u+b)\exp(\theta) - b. \end{aligned}$$

$$\begin{aligned} Y_2: \quad &t \to \tilde{t} = t, \\ &x \to \tilde{x} = x + \theta\cosh t, \\ &u \to \tilde{u} = u + \theta\sinh t. \end{aligned}$$

$$\begin{aligned} Y_3: \quad & t \to \tilde{t} = t, \\ & x \to \tilde{x} = x + \theta \sinh t, \\ & u \to \tilde{u} = u + \theta \cosh t. \end{aligned}$$

The operator Y_1 in (13) can be again represented as a linear combination of the dilatation and Galilei operators.

b) The invariance algebra of the equation

$$L(Lu) = -u + b \tag{14}$$

is a seven-dimensional algebra, whose basis elements are given by the operators

$$\begin{aligned} & P_0 = \partial_t, \quad P_1 = \partial_x, \\ & R_1 = (x - bt)\partial_x + (u - b)\partial_u, \\ & R_2 = \cos t\partial_x - \sin t\partial_u, \\ & R_3 = \sin t\partial_x + \cos t\partial_u, \\ & R_4 = -\cos t\partial_t + (x - bt)\sin t\partial_x + ((x - bt)\cos t - b\sin t)\partial_u, \\ & R_5 = \sin t\partial_t + (x - bt)\cos t\partial_x - ((x - bt)\sin t + b\cos t)\partial_u. \end{aligned} \tag{15}$$

The operators R_1–R_3 generate the following finite transformations (because the transformations for R_4 and R_5 are cumbersome we omit their explicit form):

$$\begin{aligned} R_1: \quad & t \to \tilde{t} = t, \\ & x \to \tilde{x} = (x - bt)\exp(\theta) + bt, \\ & u \to \tilde{u} = (u - b)\exp(\theta) + b. \end{aligned}$$

$$\begin{aligned} R_2: \quad & t \to \tilde{t} = t, \\ & x \to \tilde{x} = x + \theta \cos t, \\ & u \to \tilde{u} = u - \theta \sin t. \end{aligned}$$

$$\begin{aligned} R_3: \quad & t \to \tilde{t} = t, \\ & x \to \tilde{x} = x + \theta \sin t, \\ & u \to \tilde{u} = u + \theta \cos t. \end{aligned}$$

The operator R_1 in (15) can be again represented as a linear combination of dilatation and Galilei operators.

Case 1.5. $F(u) = a$, $a = \text{const}$. In the case $a \neq 0$ (there exists a change of variables) without loss of generality we can admit that $a \equiv 1$. Thus we consider the cases $a = 0$ and $a = 1$ separately.

a) The maximal invariance algebra of the equation

$$L(Lu) = 0 \tag{16}$$

is a ten-dimensional algebra, whose basis elements are given by the operators

$$\begin{aligned} & P_0 = \partial_t, \quad P_1 = \partial_x, \\ & G = t\partial_x + \partial_u, \quad D = t\partial_t + x\partial_x, \quad D_1 = x\partial_x + u\partial_u, \\ & A_1 = \frac{1}{2}t^2\partial_t + tx\partial_x + x\partial_u, \quad A_2 = \frac{1}{2}t^2\partial_x + t\partial_u, \quad A_3 = u\partial_t + \frac{1}{2}u^2\partial_x, \\ & A_4 = (tu - x)\partial_t + \frac{1}{2}tu^2\partial_x + \frac{1}{2}u^2\partial_u, \\ & A_5 = \left(t^2u - 2tx\right)\partial_t + \left(\frac{1}{2}t^2u^2 - 2x^2\right)\partial_x + \left(tu^2 - 2xu\right)\partial_u. \end{aligned} \tag{17}$$

We note, that subalgebras $\langle P_0, P_1, G\rangle$ and $\langle A_1, -A_2, G\rangle$ in the representation (17) define two different representations of the Galilei algebra $AG(1,1)$ [3].

The finite transformations which are generated by the operators (17) have the form (because the transformations for A_4 and A_5 are cumbersome we omit their explicit form):

$$
\begin{aligned}
G:\quad & t \to \tilde{t} = t,\\
& x \to \tilde{x} = x + \theta t,\\
& u \to \tilde{u} = u + \theta.
\end{aligned}
$$

$$
\begin{aligned}
D:\quad & t \to \tilde{t} = t\exp(\theta),\\
& x \to \tilde{x} = x\exp(\theta),\\
& u \to \tilde{u} = u.
\end{aligned}
$$

$$
\begin{aligned}
D_1:\quad & t \to \tilde{t} = t,\\
& x \to \tilde{x} = x\exp(\theta),\\
& u \to \tilde{u} = u\exp(\theta).
\end{aligned}
$$

$$
\begin{aligned}
A_1:\quad & t \to \tilde{t} = \frac{2t}{2-\theta t},\\
& x \to \tilde{x} = \frac{4x}{(2-\theta t)^2},\\
& u \to \tilde{u} = u + \frac{2x\theta}{2-\theta t}.
\end{aligned}
$$

$$
\begin{aligned}
A_2:\quad & t \to \tilde{t} = t,\\
& x \to \tilde{x} = x + \frac{1}{2}\theta t^2,\\
& u \to \tilde{u} = u + \theta t.
\end{aligned}
$$

$$
\begin{aligned}
A_3:\quad & t \to \tilde{t} = t + \theta u,\\
& x \to \tilde{x} = x + \frac{1}{2}\theta u^2,\\
& u \to \tilde{u} = u.
\end{aligned}
$$

b) The maximal invariance algebra of the equation

$$
L(Lu) = 1 \tag{18}
$$

is a ten-dimensional algebra, whose basis elements are given by the operators

$$
\begin{aligned}
& P_0 = \partial_t, \quad P_1 = \partial_x, \quad G = t\partial_x + \partial_u,\\
& B_1 = t\partial_t + 3x\partial_x + 2u\partial_u, \quad B_2 = \left(x - \frac{1}{6}t^3\right)\partial_x + \left(u - \frac{1}{2}t^2\right)\partial_u,\\
& B_3 = \frac{1}{2}t^2\partial_t + \left(tx + \frac{1}{12}t^4\right)\partial_x + \left(x + \frac{1}{3}t^3\right)\partial_u, \quad A_2 = \frac{1}{2}t^2\partial_x + t\partial_u,\\
& B_4 = \left(u - \frac{1}{2}t^2\right)\partial_t + \left(\frac{1}{2}u^2 - \frac{1}{8}t^4\right)\partial_x + \left(tu - \frac{1}{2}t^3\right)\partial_u,
\end{aligned}
\tag{19}
$$

$$B_5 = \left(tu - x - \frac{1}{3}t^3\right)\partial_t + \left(\frac{1}{2}tu^2 - \frac{1}{2}t^2x - \frac{1}{24}t^5\right)\partial_x + \\ + \left(\frac{1}{2}u^2 + \frac{1}{2}t^2u - tx - \frac{5}{24}t^4\right)\partial_u,$$

$$B_6 = \left(t^2u - 2tx - \frac{1}{6}t^4\right)\partial_t + \left(\frac{1}{2}t^2u^2 - 2x^2 - \frac{1}{3}t^3x - \frac{1}{72}t^6\right)\partial_x + \\ + \left(tu^2 - 2xu + \frac{1}{3}t^3u - t^2x - \frac{1}{12}t^5\right)\partial_u.$$

The algebra, generated by the operators (19), includes again two different Galilei algebras $\langle P_0, P_1, G\rangle$ and $\langle B_3, -A_2, G\rangle$ as subalgebras.

The finite transformations which are generated by the operators (19) have the form (because the transformations for B_4, B_5 and B_6 are cumbersome we omit their explicit form):

$$B_1: \quad t \to \tilde{t} = t\exp(\theta), \quad x \to \tilde{x} = x\exp(3\theta), \quad u \to \tilde{u} = u\exp(2\theta).$$

$$B_2: \quad t \to \tilde{t} = t, \quad x \to \tilde{x} = \left(x - \frac{1}{6}t^3\right)\exp(\theta) + \frac{1}{6}t^3, \quad u \to \tilde{u} = \left(u - \frac{1}{2}t^2\right)\exp(\theta) + \frac{1}{2}t^2.$$

$$B_3: \quad t \to \tilde{t} = \frac{2t}{2 - \theta t}, \quad x \to \tilde{x} = \frac{12x - 2t^3}{3(2-\theta t)^2} + \frac{4t^3}{3(2-\theta t)^3}, \quad u \to \tilde{u} = u + \frac{2t^2}{(2-\theta t)^2} + \frac{12x - 2t^3}{3t(2-\theta t)} - \frac{12x + t^3}{6t}.$$

II. Let us consider equation (4) for $\lambda \neq 0$. As it was noticed above, we can set $\lambda \equiv 1$. Symmetry classification gives in this case four principally distinct cases.

Case 2.1. $F(u)$ is an arbitrary continuously differentiable function. The maximal invariance algebra of the equation

$$L(Lu) + Lu = F(u), \tag{20}$$

is the two-dimensional algebra (5).

Case 2.2. $F(u) = au^3 - \frac{2}{9}u$, $a = \text{const}$, $a \neq 0$. The maximal invariance algebra of the equation

$$L(Lu) + Lu = au^3 - \frac{2}{9}u \tag{21}$$

is a three-dimensional algebra, whose basis elements are given by the operators

$$P_0 = \partial_t, \quad P_1 = \partial_x, \quad Z = \exp\left(\frac{1}{3}t\right)\left(\partial_t - \frac{1}{3}u\partial_u\right). \tag{22}$$

The operator Z generates the following finite transformations:

$$t \to \tilde{t} = -3\ln\left(\exp\left(-\frac{1}{3}t\right) - \frac{\theta}{3}\right),$$
$$x \to \tilde{x} = x,$$
$$u \to \tilde{u} = u\left(1 - \frac{1}{3}\theta\exp\left(\frac{1}{3}t\right)\right).$$

Case 2.3. $F(u) = au + b$, $a, b = \text{const}$, $a \neq 0$. The invariance algebra of the equation

$$L(Lu) + Lu = au + b \tag{23}$$

is a five-dimensional algebra, whose basis elements are given by the operators

$$P_0 = \partial_t, \quad P_1 = \partial_x, \quad Z_1 = \left(x + \frac{b}{a}t\right)\partial_x + \left(u + \frac{b}{a}\right)\partial_u,$$

and two other operators depending on constant a have the form

a) $a = -\frac{1}{4}$

$$Z_2 = \exp\left(-\frac{1}{2}t\right)\left(\partial_x - \frac{1}{2}\partial_u\right), \quad Z_3 = \exp\left(-\frac{1}{2}t\right)\left(t\partial_x + \left(1 - \frac{1}{2}t\right)\partial_u\right),$$

b) $a > -\frac{1}{4}$, $a \neq 0$

$$Z_4 = \exp(\alpha t)(\partial_x + \alpha\partial_u), \quad Z_5 = \exp(\beta t)(\partial_x + \beta\partial_u),$$

where

$$\alpha = \frac{-1 - \sqrt{4a+1}}{2}, \quad \beta = \frac{-1 + \sqrt{4a+1}}{2},$$

c) $a < -\frac{1}{4}$

$$Z_6 = \exp(\gamma t)(\sin\delta t\partial_x + (\gamma\sin\delta t + \delta\cos\delta t)\partial_u),$$
$$Z_7 = \exp(\gamma t)(\cos\delta t\partial_x + (\gamma\cos\delta t - \delta\sin\delta t)\partial_u),$$

where

$$\gamma = -\frac{1}{2}, \quad \delta = \frac{\sqrt{-(4a+1)}}{2}.$$

The corresponding finite transformations have the form:

$$Z_1: \quad t \to \tilde{t} = t,$$
$$x \to \tilde{x} = \left(x + \frac{b}{a}t\right)\exp(\theta) - \frac{b}{a}t,$$
$$u \to \tilde{u} = \left(u + \frac{b}{a}\right)\exp(\theta) - \frac{b}{a}.$$

$$Z_4: \quad t \to \tilde{t} = t,$$
$$x \to \tilde{x} = x + \theta\exp(\alpha t),$$
$$u \to \tilde{u} = u + \alpha\theta\exp(\alpha t).$$

$$Z_3: \quad t \to \tilde{t} = t,$$
$$x \to \tilde{x} = x + \theta t \exp\left(-\frac{1}{2}t\right),$$
$$u \to \tilde{u} = u + \theta\left(1 - \frac{1}{2}t\right)\exp\left(-\frac{1}{2}t\right).$$

$$Z_6: \quad t \to \tilde{t} = t,$$
$$x \to \tilde{x} = x + \theta \sin\delta t \exp(\gamma t),$$
$$u \to \tilde{u} = u + \theta(\gamma\sin\delta t + \delta\cos\delta t)\exp(\gamma t).$$

$$Z_7: \quad t \to \tilde{t} = t,$$
$$x \to \tilde{x} = x + \theta \cos\delta t \exp(\gamma t),$$
$$u \to \tilde{u} = u + \theta(\gamma\cos\delta t - \delta\sin\delta t)\exp(\gamma t).$$

Case 2.4. $F(u) = a$, $a = \text{const}$. The invariance algebra of the equation

$$L(Lu) + Lu = a \tag{24}$$

is a five-dimensional algebra, whose basis elements are given by the operators

$$P_0 = \partial_t, \quad P_1 = \partial_x, \quad G = t\partial_x + \partial_u,$$
$$Q_1 = \left(x - \frac{a}{2}t^2\right)\partial_x + (u - at)\partial_u, \quad Q_2 = \exp(-t)(\partial_x - \partial_u). \tag{25}$$

The finite transformations for Q_1, Q_2 have the form:

$$Q_1: \quad t \to \tilde{t} = t,$$
$$x \to \tilde{x} = \left(x - \frac{a}{2}t^2\right)\exp(\theta) + \frac{a}{2}t^2,$$
$$u \to \tilde{u} = (u - at)\exp(\theta) + at.$$

$$Q_2: \quad t \to \tilde{t} = t,$$
$$x \to \tilde{x} = x + \theta\exp(-t),$$
$$u \to \tilde{u} = u - \theta\exp(-t).$$

Construction of solutions

In the case when the equation (4) has the form

$$L(Lu) + \lambda Lu = a, \quad a, \lambda = \text{const} \tag{26}$$

the change of variables

$$t = \tau, \quad x = \omega + u\tau, \quad u = u \tag{27}$$

enable us to construct the general solution of (26). In consequence of the change of variables (27) we obtain:

$$L = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} \quad \to \quad \partial_\tau,$$
$$Lu = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} \quad \to \quad \frac{u_\tau}{1 + \tau u_\omega}.$$

After the change of variables the equation (26) has the form

$$\partial_\tau \left(\frac{u_\tau}{1+\tau u_\omega}\right) + \lambda \left(\frac{u_\tau}{1+\tau u_\omega}\right) = a. \tag{28}$$

Integrating (28) one time, we get the linear nonhomogeneous partial differential equation. Finding first integrals of the corresponding system of characteristic equations and doing the inverse change of variables we find the solutions of (26).

Remark. We notice that the solution of equation $1+\tau u_\omega = 0$ in variables (t, x, u) is $x = f(t)$, where $f(t)$ is an arbitrary function. Thus (26) is equivalent to an ordinary differential equation in this singular case.

Let us illustrate it on the example of equations (16). After the change of variables (27), equation (16) is rewritten in the form:

$$\partial_\tau \left(\frac{u_\tau}{1+\tau u_\omega}\right) = 0. \tag{29}$$

Integrating (29) we obtain

$$\frac{u_\tau}{1+\tau u_\omega} = g(\omega), \tag{30}$$

where $g(\omega)$ is an arbitrary function.

If $g(\omega) \equiv 0$, then $u_\tau = 0$ and we get the solution of type (3) (because, it is obvious that the solution of equation $Lu = 0$ is a solution of (16)). When $g(\omega) \neq 0$, in accordance with arbitrary choice of $g(\omega)$ we can set $g(\omega) = -2(dh(\omega)/d\omega)^{-1}$. Therefore (30) has the form

$$u_\tau + \frac{2\tau}{h'(\omega)} u_\omega = -\frac{2}{h'(\omega)}. \tag{31}$$

The system of characteristic equation for (31) is

$$\frac{d\tau}{1} = \frac{h'(\omega)d\omega}{2\tau} = \frac{h'(\omega)du}{-2}. \tag{32}$$

Hence, we obtain two first integrals:

$$\tau^2 - h(\omega) = C_1, \quad u \pm \int \frac{d\omega}{\sqrt{h(\omega)+C_1}} = C_2. \tag{33}$$

Integrating (33) and expressing C_1 and C_2 by (τ, ω, u) we find a solution of (30) in the form

$$\Phi\left(C_1, C_2\right) = 0, \tag{34}$$

where Φ is an arbitrary function. Performing in (34) the inverse change of variables we get a solution of (16). For instance, we set $h(\omega) = \omega$. Then the expression

$$x - ut - t^2 = \varphi(u + 2t), \tag{35}$$

defines the class of implicit solutions of equation (16), where φ is an arbitrary function.

The same results we can obtain for other cases of (26). If $F(u) \neq \mathrm{const}$ in (4) then this method does not lead to solutions. Below we give some classes of solutions of equations (26):

1. $L(Lu) = 0$

1.1. $x - ut + \dfrac{C}{2}t^2 = \varphi(u - Ct);$

1.2. $u \pm \ln(x - ut \mp t) = \varphi\left(t^2 - (x - ut)^2\right);$

1.3. $u + \dfrac{t(x-ut)^3}{t^2(x-ut)^2 - 1} = \varphi\left(t^2 - \dfrac{1}{(x-ut)^2}\right);$

1.4. $u = \varphi\left(\dfrac{x - ut}{\exp\left(t^2\right)}\right) - \dfrac{x - ut}{\exp\left(t^2\right)}\displaystyle\int \exp\left(t^2\right) dt;$

2. $L(Lu) = a$

$$x - ut + \frac{a}{3}t^3 + \frac{C}{2}t^2 = \varphi\left(u - \frac{a}{2}t^2 - Ct\right);$$

3. $L(Lu) + Lu = a$

$$x - ut - C(t+1)\exp(-t) + \frac{a}{2}t^2 = \varphi\left(u + C\exp(-t) - at\right),$$

$C = \mathrm{const}$, φ is an arbitrary function.

Thus, we have investigated the symmetry classifications of (4) and pointed out all functions $F(u)$ under which the invariance algebra of (4) admits the extension. The new representations which may have an interesting physical interpretation are obtained. In the case $F(u) = \mathrm{const}$ we described the algorithm of construction of the general solution of (4) and pointed out some solutions. The symmetry properties of (4) can be used for a symmetry reduction and construction of the solutions and for their generation by finite group transformations [3, 4, 5].

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1. Fushchych W.I., New nonlinear equation for electromagnetic field having the velocity different from c, *Dopovidi Akademii Nauk Ukrainy (Proceedings of the Academy of Sciences of Ukrainy)*, 1992, № 1, 24–27.
2. Fushchych W., Symmetry analysis. Preface, in "Symmetry Analysis of Equations of Mathematical Physics", Kiev, Inst. of Math., 1992, 5–6.
3. Fushchych W., Shtelen W. and Serov N., Symmetry analysis and exact solutions of equations of nonlinear mathematical physics, Dordrecht, Kluwer Academic Publishers, 1993, 436 p.
4. Olver P., Application of Lie groups to differential equations, New York, Springer, 1986, 497 p.
5. Ovsyannikov L.V., Group analysis of differential equations, New York, Academic Press, 1982, 400 p.

Galilei-invariant nonlinear systems of evolution equations

W.I. FUSHCHYCH, R.M. CHERNIHA

All systems of $(n+1)$-dimensional quasilinear evolution second-order equations invariant under chain of algebras $AG(1,n) \subset AG_1(1,n) \subset AG_2(1,n)$ are described. The results obtained are illustrated by the examples of the nonlinear Schrödinger equations, Hamilton–Jacobi-type systems and of reaction-diffusion equations.

1. Introduction

The $(n+1)$-dimensional diffusion (heat) system of equations

$$\lambda_1 U_t = \Delta U,$$
$$\lambda_2 V_t = \Delta V,$$

where $U = U(t,x)$, $V = V(t,x)$ are unknown differentiable real functions, $U_t = \partial U/\partial t$, $V_t = \partial V/\partial t$, $x = (x_1, \dots x_n)$, $\lambda_1, \lambda_2 \in \mathbb{R}$, is known to be invariant under the generalized Galilei algebra $AG_2(1,n)$ [1, 2]

$$P_t = \partial_t, \quad P_a = \partial_a, \tag{2a}$$

$$Q_\lambda = \lambda_1 U\partial_U + \lambda_2 V\partial_V, \quad G_a = tP_a - \frac{x_a}{2}Q_\lambda, \quad J_{ab} = x_a P_b - x_b P_a, \tag{2b}$$

$$D = 2tP_t + x_a P_a + I_\alpha, \tag{2c}$$

$$\Pi = t^2 P_t + t x_a P_a - \frac{1}{4}|x|^2 Q_\lambda + tI_\alpha, \quad \alpha_k = -\frac{1}{2}n. \tag{2d}$$

In relations (2) and elsewhere hereinafter $I_\alpha = \alpha_1 U\partial_U + \alpha_2 V\partial_V$, $\partial_U \equiv \partial/\partial U$, $\partial_V \equiv \partial/\partial V$, $\partial_t \equiv \partial/\partial t$, $\partial_a \equiv \partial/\partial x_a$, $\alpha_k \in \mathbb{R}$, $k = 1,2$ and a summation is assumed from 1 to n over repeated indices.

The algebra produced by the operators (2a), (2b) is called the Galilei algebra $AG(1,n)$, and its extension by using the operator (2c) will be refereed to as $AG_1(1,n)$ [1, 2].

Clearly, the unit operators I_α and Q_λ are linearly dependent only in the case when the determinant

$$\delta = \begin{vmatrix} \alpha_1 & \alpha_2 \\ \lambda_1 & \lambda_2 \end{vmatrix} = 0.$$

As a result we obtain two essentially different representations of algebras $AG_1(1,n)$ and $AG_2(1,n)$ for $\delta = 0$ and $\delta \neq 0$, in contrast to the case of a single diffusion equation (the nonlinear diffusion equation invariant with respect of a set of $AG_2(1,n)$ subalgebras was studied in [2, 3]).

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Note that in the case when the system (1) is a pair of complex conjugate Schrödinger equations, i.e. $U = \overset{*}{V}$, $\lambda_1 = \overset{*}{\lambda}_1 = i$, the operators I_α and Q_λ are linearly independent. This results in the fact that nonlinear generalizations of Schrödinger equations, preserving its symmetry [1], differ essentiallyfrom nonlinear generalizations of the diffusion system (1) at $\delta = 0$.

Now consider a system of quasilinear generalizations of diffusion equations (1) of the form

$$\begin{aligned} \lambda_1 U_t &= A_{ab} U_{ab} + C_{ab} V_{ab} + B_1, \\ \lambda_1 V_t &= D_{ab} U_{ab} + E_{ab} V_{ab} + B_2, \end{aligned} \tag{3}$$

A_{ab}, C_{ab}, D_{ab}, E_{ab}, B_1, B_2 being arbitrary real or complex differentiable functions of $2n+2$ variables $U, V, U_1, \ldots, U_n, V_1, \ldots, V_n$. The indices $a = 1, \ldots, n$ and $b = 1, \ldots, n$ of functions U and V denote differentiating with respect to x_a and x_b.

The system (3) generalizes practically all the known nonlinear systems of first- and second-order evolution equations, describing various processes in physics, chemistry and biology (heat and mass transfer, filtration of two-phase liquid, diffusion in chemical reactions etc.) [4–7].

In the case of complex $U = \overset{*}{V}$, $A_{ab} = \overset{*}{E}_{ab}$, $C_{ab} = \overset{*}{D}_{ab}$, $B_1 = \overset{*}{B}_2 = B$, $\lambda_1 = \overset{*}{\lambda}_2 = i$ the system (3) is transformed into a pair of complex conjugate equations. We treat them as a class of nonlinear generalizations of Schrödinger equations, namely:

$$iU_t = A_{ab} U_{ab} + \overset{*}{D}_{ab} \overset{*}{U}_{ab} + B, \tag{4a}$$

$$-i \overset{*}{U}_t = \overset{*}{A}_{ab} \overset{*}{U}_{ab} + D_{ab} U_{ab} + \overset{*}{B} \tag{4b}$$

(hereinafter complex conjugate equations (4b) are omitted).

For $A_{ab} = D_{ab} = D_{aa} = 0$, $a \neq b$, $A_{aa} = -h$ equation (4a) is obviously transformed into a Schrödinger equation with nonlinear potential B:

$$iU_t + h\Delta U = B. \tag{4$'$}$$

By choice of the corresponding potential $B = B(U, \overset{*}{U}, U_1, \ldots, U_n, \overset{*}{U}_1, \ldots, \overset{*}{U}_n)$ a great variety of Schrödinger equation generalizations, known from the literature (see e.g. [1, 2, 8, 9, 10]) can be obtained.

In case of zero potential B a classical Schrödinger equation is obtained

$$iU_t + h\Delta U = 0 \tag{5}$$

invariant under $AG_2(1,n)$ algebra with the basic operators (2) [11], where

$$Q_\lambda = -\frac{i}{h}(U\partial_U - \overset{*}{U} \partial_{\overset{*}{U}}), \quad I_\alpha = \alpha(U\partial_U + \overset{*}{U} \partial_{\overset{*}{U}}). \tag{6}$$

Note that the algebra $AG_2(1,n)$ in the case of the Schrödinger equations is called the Schrödinger algebra [11].

In the present paper all the systems of evolution equations of the form (3), invariant under the chain of algebras $AG(1,n) \subset AG_1(1,n) \subset AG_2(1,n)$, are described. The results obtained are illustrated by the examples of the nonlinear Schrödinger equations, reaction-diffusion equations and Hamilton–Jacobi type systems.

2. Description of systems (3) with Galilean symmetry

The algebra of symmetries for the system of equations (1) contains the Galilei operators G_a, $a = 1, \dots, n$, being a mathematical expression of the Galilei relativistic principle for equations (1). The Galilei operators are also known [3] to be closely related with the fundamental solution of the diffusion equation. We recall that if some system of PDEs is invariant with respect to the Galilei algebra or its extention, then it gives a wide range of possibilities for the construction of multiparametric families of exact solutions [1, 12, 22]. Moveover the Galilei operators and the projective operator (2d) generate non-trivial formulae of multiplication of solutions. These formulae can be used to convert stationary (time-independent) into non-stationary ones with a different structure.

In view of this it seems reasonable to search for Galilei-invariant nonlinear generalizations of system (1) in the class of system (3).

Theorem 1. *The system of nonlinear equations (3) is invariant under the Galilei algebra in the represention (2a), (2b) if and only if it has the form:*

$$\begin{aligned} \lambda_1 U_t &= \Delta U + U[A_1 \Delta \ln U + C_1 \Delta \ln V + B_1] + \\ &\quad + U[A_2 \omega_a \omega_b (\ln U)_{ab} + C_2 \omega_a \omega_b (\ln V)_{ab}], \\ \lambda_2 V_t &= \Delta V + V[D_1 \Delta \ln U + E_1 \Delta \ln V + B_2] + \\ &\quad + V[D_2 \omega_a \omega_b (\ln U)_{ab} + E_2 \omega_a \omega_b (\ln V)_{ab}], \end{aligned} \tag{7}$$

where $(\ln U)_{ab} \equiv \partial^2 \ln U / \partial x_a \partial x_b$, $(\ln V)_{ab} \equiv \partial^2 \ln V / \partial x_a \partial x_b$, $\Delta \ln U \equiv (\ln U)_{11} + \cdots + (\ln U)_{nn}$, $\Delta \ln V \equiv (\ln V)_{11} + \cdots + (\ln V)_{nn}$, $\omega = U^{\lambda_2} V^{-\lambda_1}$, $\omega_a = \partial \omega / \partial x_a \equiv (\lambda_2 U_a / U - \lambda_1 V_a / V)\omega$ *and* A_k, B_k, C_k, D_k, E_k, $k = 1, 2$ *are arbitrary functions of absolute invariants of the* $AG(1, n)$ *algebra* ω *and* $\theta = \omega_a \omega_a$.

The proof of this and the following theorems is based on the classical Lie scheme, which is realized in [3, 12] for obtaining the Galilei invariant equations. The detailed cumbersome calculations are omitted.

Note that in case where $\lambda_1 = 0$, i.e. the first equation of system (3) being elliptical, the absolute invariants of the Galilei algebra simplify consideretly: $\omega = U$, $\theta = U_a U_a$.

In case of systems of the form (3) being $AG_1(1, n)$- and $AG_2(1, n)$-invariant the structure of such systems essentially depends on the determinant δ.

Theorem 2. *The nonlinear system (3) is invariant with respect to algebra* $AG_1(1, n)$ *with basic operators (2a)–(2c) if and only if it has the form:*

(i) In case when $\delta \neq 0$

$$\begin{aligned} \lambda_1 U_t &= \Delta U + U[A_1(\hat\theta) \Delta \ln U + A_2(\hat\theta) \Delta \ln V + \omega^{-2/\delta} B_1(\hat\theta)] + \\ &\quad + U \omega^{2/\delta - 2}[C_1(\hat\theta) \omega_a \omega_b (\ln U)_{ab} + C_2(\hat\theta) \omega_a \omega_b (\ln V)_{ab}], \\ \lambda_2 V_t &= \Delta V + V[D_1(\hat\theta) \Delta \ln U + D_2(\hat\theta) \Delta \ln V + \omega^{-2/\delta} B_2(\hat\theta)] + \\ &\quad + V \omega^{2/\delta - 2}[E_1(\hat\theta) \omega_a \omega_b (\ln U)_{ab} + E_2(\hat\theta) \omega_a \omega_b (\ln V)_{ab}]. \end{aligned} \tag{8}$$

(ii) In case when $\delta = 0$

$$\begin{aligned} \lambda_1 U_t &= \Delta U + U[A_1(\omega) \Delta \ln V + A_2(\omega) \Delta \ln V + \omega_a \omega_a B_1(\omega)] + \\ &\quad + U(\omega_{a_1} \omega_{a_1})^{-1} \omega_a \omega_b [C_1(\omega)(\ln U)_{ab} + C_2(\omega)(\ln V)_{ab}], \\ \lambda_2 V_t &= \Delta V + V[D_1(\omega) \Delta \ln U + D_2(\omega) \Delta \ln V + \omega_a \omega_a B_2(\omega)] + \\ &\quad + V(\omega_{a_1} \omega_{a_1})^{-1} \omega_\alpha \omega_b [E_1(\omega)(\ln U)_{ab} + E_2(\omega)(\ln U)_{ab}], \end{aligned} \tag{9}$$

where A_k, B_k, C_k, D_k, E_k, $k = 1, 2$ being arbitrary functions, $\hat{\theta} = \omega_a\omega_a\omega^{2/\delta-2}$ and ω are the absolute first-order invariants of the algebra $AG_1(1,n)$, $a_1 = 1,\ldots,n$ (ω_a, ω see theorem 1).

In the case when the first equation of system (3) degenerates into an elliptical ($\lambda_1 = 0$) equation, the absolute invariants in systems (8) and (9) simplify and $\hat{\theta} = U_aU_aU^{2/\alpha_1-2}$ for $\delta \neq 0$, $\omega = U$ for $\delta = 0$.

Theorem 3. *The nonlinear system of equations (3) is invariant with respect to algebra $AG_2(1,n)$ with basic operators (2) (α_1, α_2 are arbitrary constants) iff it has the form:*

(i) In case when $\delta \neq 0$

$$\begin{aligned}\lambda_1U_t &= \hat{\alpha}_1\Delta U + UA(\hat{\theta})(\lambda_2\Delta\ln U - \lambda_1\Delta\ln V) + U\omega^{-2/\delta}B_1(\hat{\theta}) + \\ &\quad + (1-\hat{\alpha}_1)U_aU_a/U + U\omega^{2/\delta-2}\omega_a\omega_b[\lambda_2(\ln U)_{ab} - \lambda_1(\ln V)_{ab}]C(\hat{\theta}), \\ \lambda_2V_t &= \hat{\alpha}_2\Delta V + VD(\hat{\theta})(\lambda_2\Delta\ln U - \lambda_1\Delta\ln V) + V\omega^{-2/\delta}B_2(\hat{\theta}) + \\ &\quad + (1-\hat{\alpha}_2)V_aV_a/V + V\omega^{2/\delta-2}\omega_a\omega_b[\lambda_2(\ln U)_{ab} - \lambda_1(\ln V)_{ab}]E(\hat{\theta}).\end{aligned} \tag{10}$$

(ii) In case when $\delta = 0$

$$\begin{aligned}\lambda_1U_t &= \hat{\alpha}_1\Delta U + UA(\omega)(\lambda_2\Delta\ln U - \lambda_1\Delta\ln V) + U\omega_a\omega_aB_1(\omega) + \\ &\quad + (1-\hat{\alpha}_1)U_aU_a/U + U(\omega_{a_1}\omega_{a_1})^{-1}\omega_a\omega_b[\lambda_2(\ln U)_{ab} - \lambda_1(\ln V)_{ab}]C(\omega), \\ \lambda_2V_t &= \hat{\alpha}_2\Delta V + VD(\omega)(\lambda_2\Delta\ln U - \lambda_1\Delta\ln V) + V\omega_a\omega_aB_2(\omega) + \\ &\quad + (1-\hat{\alpha}_2)V_aV_a/V + V(\omega_{a_1}\omega_{a_1})^{-1}\omega_a\omega_b[\lambda_2(\ln U)_{ab} - \lambda_1(\ln V)_{ab}]E(\omega),\end{aligned} \tag{11}$$

where A, B_1, B_2, C, D, E being arbitrary functions, $\hat{\alpha}_k = -2\alpha_k/n$, $k = 1, 2$ (α_k see operator I_α).

It can be noticed that in case where $\alpha_1\alpha_2 \neq 0$ systems (10) and (11) can be reduced by the local substitution $U \to U^{\hat{\alpha}_1}$, $V \to V^{\hat{\alpha}_2}$ to the systems of the same structure, but with $\hat{\alpha}_k = 1$, i.e. $\alpha_k = -n/2$. The specific case of $\alpha_1 = \alpha_2 = 0$ will be considered in what following.

The classes of $AG_2(1,n)$-invariant systems (10) and (11) thus obtained contain, in particular, such genaralizations of equations (1) as ($\delta \neq 0$)

$$\begin{aligned}\lambda_1U_t &= \Delta U + e_1U(\lambda_2\Delta\ln U - \lambda_1\Delta\ln V), \\ \lambda_2U_t &= \Delta V + e_2V(\lambda_2\Delta\ln U - \lambda_1\Delta\ln V)\end{aligned}$$

and ($\delta = 0$)

$$\begin{aligned}U_t &= \Delta U + e_1U\frac{\partial(UV^{-1})}{\partial x_a}\frac{\partial(UV^{-1})}{\partial x_a}, \\ V_t &= \Delta V + e_2V\frac{\partial(UV^{-1})}{\partial x_a}\frac{\partial(UV^{-1})}{\partial x_a},\end{aligned}$$

where $e_1, e_2 \in \mathbb{R}$.

In the case where the first of equations (3) degenerates into an elliptical one ($\lambda_1 = 0$), the $AG_2(1,n)$-invariant systems of equations are simply

$$\begin{aligned}
&0 = A_1(\hat\theta)\Delta U + A_2(\hat\theta)(U_{a_1}U_{a_1})^{-1}U_aU_bU_{ab} + U^{1-2/\alpha_1}B_1(\hat\theta) + \\
&\quad + UC(\hat\theta)[\Delta\ln V - (U_{a_1}U_{a_1})^{-1}U_aU_b(\ln V)_{ab}], \\
&\lambda_2V_t = \hat\alpha_2\Delta V + \frac{V}{U}D_1(\hat\theta)\Delta U + \frac{V}{U}D_2(\hat\theta)(U_{a_1}U_{a_1})^{-1}U_aU_bU_{ab} + \\
&\qquad + (1-\hat\alpha_2)V_aV_a/V + VU^{-2/\alpha_1}B_2(\hat\theta) + \\
&\qquad + VE(\hat\theta)[\Delta\ln V - (U_{a_1}U_{a_1})^{-1}U_aU_b(\ln V)_{ab}]
\end{aligned} \tag{12}$$

if $\delta \neq 0$, and

$$\begin{aligned}
&0 = A_1(U)\Delta U + A_2(U)(U_{a_1}U_{a_1})^{-1}U_aU_bU_{ab} + U_aU_aB_1(U) + \\
&\quad + C(U)[\Delta\ln V - (U_{a_1}U_{a_1})^{-1}U_aU_b(\ln V)_{ab}], \\
&\lambda_2V_t = \hat\alpha_2\Delta V + VD_1(U)\Delta U + VD_2(U)(U_{a_1}U_{a_1})^{-1}U_aU_bU_{ab} + VU_aU_aB_2(U) + \\
&\qquad + (1-\hat\alpha_2)V_aV_a/V + VE(U)[\Delta\ln V - (U_{a_1}U_{a_1})^{-1}U_aU_b(\ln V)_{ab}],
\end{aligned} \tag{13}$$

if $\delta = 0$. In equations (12), (13) A_k, B_k, D_k, E, C are arbitrary functions, $\hat\theta = U_aU_aU^{2/\alpha_1-2}$, $\hat\alpha_2 = -2\alpha_2/n$. In [13] integration of two-dimensional systems of equations (12), (13) form was reduced to the integration of linear heat equation with a source.

3. Galilei-invariant nonlinear generalizations of the Schrödinger equation

As noted above, a class of nonlinear generalization of Schrödinger equation (4) is a specific case of evolution equations (3). On the basis of theorems 1, 2 and 3 this enables one to describe all quasilinear generalizations of Schrödinger equation (5), which are invariant with respect to a chain of algebras $AG(1,n) \subset AG_1(1,n) \subset AG_2(1,n)$.

Corollary 1. *In the class of nonlinear equations of the form (4) algebra AG(1,n) (2a),(2b) with* $Q_\lambda = -\frac{i}{h}(U\partial_U - \overset{*}{U}\partial_{\overset{*}{U}})$ *is admitted only for equations given by*

$$\begin{aligned}
iU_t + h\Delta U &= U[A_1\Delta\ln U + A_2\Delta\ln\overset{*}{U} + B] + \\
&\quad + U[A_3|U|_a|U|_b(\ln U)_{ab} + A_4|U|_a|U|_b(\ln\overset{*}{U})_{ab}],
\end{aligned} \tag{14}$$

where $A_j = 0$, $j = 1,2,3,4$ *and* B *are arbitrary complex functions of two arguments* $|U|$ *and* $|U|_a|U|_a$; $|U|^2 = U\overset{*}{U}$, $|U|_a = \partial|U|/\partial x_a$.

In case $A_j = 0$ the class of equations (14) is reduced to an equation

$$iU_t + h\Delta U = UB(|U|, |U|_a|U|_a) \tag{15}$$

obtained in [1, 12], whose specific case is a Schrödinger equation with power nonlinearity $U|U|^\beta$, $\beta = \mathrm{const}$.

By using the identities

$$\begin{aligned}
&\Delta\ln|U|^2 = (\Delta|U|^2 - 4|U|_a|U|_a)/|U|^2, \\
&\mathrm{Re}\,(\Delta U/U) + |\nabla U|^2/|U|^2 = \Delta\ln|U| + |U|_a|U|_a/|U|^2, \\
&\mathrm{Im}\,(\Delta U/U - U_aU_a/U^2) = (\Delta\ln U - \Delta\ln\overset{*}{U})/2i
\end{aligned}$$

it is easily to show that the class of the Galilei-invariant equations (14) contains the equation

$$iU_t + \Delta U = \frac{id}{2} U\Delta|U|^2/|U|^2 + U[d_1(\mathrm{Re}\,(\Delta U/U) + |\nabla U|^2/|U|^2) + \\ + d_2 \mathrm{Im}\,(\Delta U/U - (\nabla U/U)^2) + d_3(\mathrm{Re}\,(\nabla U/U)^2) + |\nabla U|^2/|U|^2)],$$

where $\nabla U = (\partial U/\partial x_1, \dots, \partial U/\partial x_n)$, $d_1, d_2, d_3 \in \mathbb{R}$, proposed in [9] from certain physical considerations. By the way, a nonlinear generalization of the Schrödinger equation [8]

$$iU_t = (id_1 - h)\Delta U + id_1 U|\nabla U|^2/|U|^2 + UB(|U|),$$

does not preserve Galilean symmetry of the linear Schrödinger equation. Instead it would be appropriate to propose Galilei-invariant nonlinear equations of the class (14)

$$iU_t = c\Delta U + (h - c) \overset{*}{U} (\nabla U)^2/|U|^2 + UB(|U|),$$

and [13]

$$iU_t = -h\Delta U + cU\Delta|U|^2/|U|^2 + UB(|U|),$$

where c is arbitrary complex constant and B is an arbitrary complex function.

Corollary 2. *In the class of nonlinear equations of the form (4) algebra $AG_1(1,n)$ (2a), (2b), (2c), (6) is admitted only for equations given by*

(i) In the case $\alpha \neq 0$

$$iU_t + h\Delta U = U[D_1\Delta \ln U + D_2 \Delta \ln \overset{*}{U} + |U|^{-2/\alpha} B] + \\ + U|U|^{2/\alpha - 2}[D_3 |U|_a |U|_b (\ln U)_{ab} + D_4 |U|_a |U|_b (\ln \overset{*}{U})_{ab}], \tag{16}$$

where D_j, $j = 1,2,3,4$ and B are arbitrary complex functions of the argument $|U|^{2/\alpha-2}|U|_a|U|_a$;

(ii) In the case $\alpha = 0$

$$iU_t + h\Delta U = U[D_1\Delta \ln U + D_2 \Delta \ln \overset{*}{U} + |U|_a|U|_a B] + \\ + U(|U|_{a_1}|U|_{a_1})^{-1}[D_3 |U|_a |U|_b (\ln U)_{ab} + D_4 |U|_a |U|_b (\ln \overset{*}{U})_{ab}], \tag{17}$$

where $D_j = D_j(|U|)$, $j = 1,2,3,4$ and $B = B(|U|)$ are arbitrary complex functions.

It is easily seen that the class of the $AG_1(1,n)$-invariant equations (14) contains the well-known nonlinear Schrödinger equation

$$iU_t + h\Delta U + cU|U|^2 = 0 \tag{18}$$

which in the case $n = 1$ is integrated by inverse scattering method [14]. Note that in the case $n = 2$ equation (17) is invariant under the $AG_2(1,2)$ algebra [12, 15].

Corollary 3. *Within the class of nonlinear equations of the form (4) algebra $AG_2(1,n)$ (2), (6) for $\alpha = -n/2$ of the linear Schrödinger equation (5) is conserved only for equations given by*

$$iU_t + h\Delta U = UE_1\Delta \ln|U| + U|U|^{4/n}B + U|U|^{-4/n-2}E_2|U|_a|U|_b(\ln|U|)_{ab}. \tag{19}$$

In equation (19) E_1, E_2 and B are arbitrary complex functions of the argument $|U|^{-4/n-2}|U|_a|U|_a$, which is an absolute invariant of the generalized Galilei algebra $AG_2(1,n)$.

If we consider a representation of $AG_2(1,n)$ algebra with basic operators (2), (6) for $\alpha = 0$, a principally different class of quasilinear second-order equations, invariant with respect to this algebra, namely

$$iU_t + hU_aU_a/U = UE_1(|U|)\Delta \ln|U| + U|U|_a|U|_aB(|U|) + \\ + UE_2(|U|)(|U|_{a_1}|U|_{a_1})^{-1}|U|_a|U|_b(\ln|U|)_{ab}. \tag{20}$$

is obtained.

It is easily seen that within the class of equations (20) there is not a single linear equation, the simplest one among them being Hamilton–Jacobi equation for a complex function

$$iU_t + hU_aU_a/U = 0$$

which is reduced to a standard form

$$iW_t + hW_aW_a = 0, \quad W_a = \frac{\partial W}{\partial x_a}, \quad W_t = \frac{\partial W}{\partial t}$$

by a local substitution $U = \exp W$, $W = W(t, x_1, \ldots, x_n)$.

In case $E_1 = E_2 = 0$ equation

$$iU_t + h\Delta U = U|U|^{4/n}B \tag{21}$$

is obtained from the class of equations (19) which had been obtained in [1, 12]. Note that at $B = c = \text{const}$ equation (21) is transformed into an equation with fixed power nonlinearity, studied in a series of papers (for $n = 1$ [16, 17], $n = 2$ [18] and $n = 3$ [1, 2, 12, 19]). In [1, 12] multiparametric families of invariant solutions of equation (21) of the form

$$iU_t + h\Delta U = cU\frac{|U|_a|U|_a}{|U|^2}$$

are also constructed and systematized.

Being written in the case of one spatial variable ($n = 1$), after simple transformations the class of equations (19) is given by

$$iU_t + hU_{xx} = UE_1(\ln|U|)_{xx} + U|U|^4B, \quad U = U(t,x), \quad x = x_1, \tag{22}$$

E_1 and B being arbitrary complex functions of the argument $|U|^{-3}|U|_x$.

Obviously, a specific case of equation (22) is given by

$$iU_t + hU_{xx} + c_1U|U|^4 + c_2U|U||U|_x = 0 \tag{23}$$

which at $h = 1$, $c_1 = 1$, $c_2 = 4$ is known as Eckhaus equation [20, 21]. Equation (23) has been studied in detail for arbitrary constant values of c_1 and c_2 in [22]. A multidimensional generalization of equation (23), posessing $AG_2(1,n)$ symmetry, can be proposed

$$iU_t + h\Delta U + c_1U|U|^{4/n} + c_2U|U|^{-1+2/n}(|U|_a|U|_a)^{1/2} = 0. \tag{24}$$

4. Galilei-invariant systems of Hamilton–Jacobi-type

It should be noted that the local substitution $U = M(\hat{U})$, $V = N(\hat{V})$, where M, N are arbitrary differentiable functions, reduces any equation system with the symmetry $AG(1,n)$, $AG_1(1,n)$ or $AG_2(1,n)$ to a locally equivalent system with the same symmetry, but with different representation of operators Q_λ and I_α, namely

$$\hat{Q}_\lambda = \lambda_1 M \left(\frac{dM}{d\hat{U}}\right)^{-1} \partial_{\hat{U}} + \lambda_2 N \left(\frac{dN}{d\hat{V}}\right)^{-1} \partial_{\hat{V}},$$
$$I_\alpha = \alpha_1 M \left(\frac{dM}{d\hat{U}}\right)^{-1} \partial_{\hat{U}} + \alpha_2 N \left(\frac{dN}{d\hat{V}}\right)^{-1} \partial_{\hat{V}}.$$

In the particular case where $M = \exp(\hat{U})$, $N = \exp(\hat{V})$, we obtain

$$\hat{Q}_\lambda = \lambda_1 \partial_{\hat{U}} + \lambda_2 \partial_{\hat{V}}, \quad I_\alpha = \alpha_1 \partial_{\hat{U}} + \alpha_2 \partial_{\hat{V}}. \tag{25}$$

In this case the class of equation systems, invariant with respect to $AG_2(1,n)$ algebra in the representation (2), (25), at $\delta = 0$ is given by

$$\begin{aligned} \lambda_1 \hat{U}_t &= \hat{\alpha}_1 \Delta \hat{U} + A(\hat{\omega})(\lambda_2 \Delta \hat{U} - \lambda_1 \Delta \hat{V}) + \hat{\omega}_a \hat{\omega}_a B_1(\hat{\omega}) + \\ &\quad + \hat{U}_a \hat{U}_a + C(\hat{\omega})(\hat{\omega}_{a_1} \hat{\omega}_{a_1})^{-1} \hat{\omega}_a \hat{\omega}_b [\lambda_2 \hat{U}_{ab} - \lambda_1 \hat{V}_{ab}], \\ \lambda_2 \hat{V}_t &= \hat{\alpha}_2 \Delta \hat{V} + D(\hat{\omega})(\lambda_2 \Delta \hat{U} - \lambda_1 \Delta \hat{V}) + \hat{\omega}_a \hat{\omega}_a B_2(\hat{\omega}) + \\ &\quad + \hat{V}_a \hat{V}_a + E(\hat{\omega})(\hat{\omega}_{a_1} \hat{\omega}_{a_1})^{-1} \hat{\omega}_a \hat{\omega}_b [\lambda_2 \hat{U}_{ab} - \lambda_1 \hat{V}_{ab}], \end{aligned} \tag{26}$$

where $\hat{\omega} = \lambda_2 \hat{U} - \lambda_1 \hat{V}$, $\hat{\omega}_a = \lambda_2 \hat{U}_a - \lambda_1 \hat{V}_a$ and A, B_1, B_2, C, D, E are arbitrary differentiable functions.

In case where $\hat{\alpha}_1 = \hat{\alpha}_2 = 0$, $A = C = D = E = 0$ the system of equations (26) is reduced to the systems of the form (the symbols $\hat{}$ being omitted below)

$$\begin{aligned} \lambda_1 U_t &= U_a U_a + \omega_a \omega_a B_1(\omega), \\ \lambda_1 V_t &= V_a V_a + \omega_a \omega_a B_2(\omega), \quad \lambda_1 \lambda_2 \neq 0 \end{aligned} \tag{27}$$

It is natural to call system (27) a generalization of the noncoupled system of the Hamilton–Jacobi (HJ) equations

$$\lambda_1 U_t = U_a U_a, \quad \lambda_1 V_t = V_a V_a. \tag{28}$$

In contrast to the symmetry of a single HJ equation [2, 23], the local symmetry of the system (28) is exhausted by $AG_2(1,n)$ algebra (2), (25) at $\alpha_1 = \alpha_2 = 0$ with additional operators

$$P_V = \partial_V, \quad D_1 = -t\partial_t + U\partial_U + V\partial_V. \tag{29}$$

Thus, all the nonlinear generalizations of the form

$$\begin{aligned} \lambda_1 U_t &= U_a U_a + B_1(U, V, U_1, \ldots, U_n, V_1, \ldots, V_n), \\ \lambda_1 V_t &= V_a V_a + B_2(U, V, U_1, \ldots, U_n, V_1, \ldots, V_n) \end{aligned} \tag{30}$$

of HJ system, preserving its symmetry $AG_2(1,n)$, are exhaused by system (27).

Among the non-linear generalizations of HJ system (27), a system of equations with unique symmetry properties exists, namely for $B_1 = 0$, $B_2 = -1/(\lambda_2)^2$ (in the following $\lambda_1 = 1$, $\lambda_2 = \lambda$).

Theorem 4. *The maximal (in the sense of Lie) algebra of the invariance for the system of equations*

$$\begin{aligned} &U_t = U_a U_a, \\ &V_t = -\lambda U_a U_a + 2U_a V_a \end{aligned} \tag{31}$$

is generated by the basic operators

$$\begin{aligned} &P_t, \quad P_a, \quad J_{ab}, \quad Q_\lambda = \lambda\partial_U - \partial_V, \quad X = W\partial_V, \quad G_a = tP_a - \frac{x_a}{2}Q_\lambda, \\ &D = 2tP_t + x_a P_a, \quad \Pi = t^2 P_t + t x_a P_a - \frac{1}{4}|x|^2 Q_\lambda, \\ &G_a^1 = UP_a - \frac{x_a}{2}P_t, \quad D_1 = 2UP_U + x_a P_a, \\ &\Pi_1 = U^2 P_U + U x_a P_a - \frac{1}{4}|x|^2 P_t, \\ &K_a = x_a t P_t - \left(2tU + \frac{1}{2}|x|^2\right) P_a + x_a x_b P_b + x_a U Q_\lambda, \end{aligned} \tag{32}$$

where W are an arbitrary differentiable function of $\lambda U - V$.

Note that the presence of the operator$X$ including an arbitrary function W in the invariance algebra for the system (31) is natural, since the second equation of the system is linear with respect to the function V. Much more interesting is the fact the system (31) can be considered as a generalization of classical HJ equation to the case of two unknown functions, since for $W = 1$ the operators (32) generate the same algebra as the HJ equation. We consider this fact to be very important, since a *trivial* generalization of the above-mentioned equation to the system of (28) does not preserve the symmetry of the HJ equation.

5. Galilei-invariant reaction-diffusion systems

Now consider a nonlinear system of evolution equations, given by

$$\begin{aligned} &\lambda_1 U_t = \Delta U + f(U,V), \\ &\lambda_2 V_t = \Delta V + g(U,V), \end{aligned} \tag{33}$$

where f, g are arbitrary differentiable functions. The systems of reaction-diffusion equations (33) has been studied intensively of late (see, e.g., [4, 6, 7]). As follows from theorems 1, 2 and 3, the class of systems (33) contains systems with broad symmetry. In particular, all the systems of equations of the form

$$\begin{aligned} &\lambda_1 U_t = \Delta U + Uf(\omega), \quad \omega = U^{\lambda_2} V^{-\lambda_1}, \\ &\lambda_2 V_t = \Delta V + Vg(\omega) \end{aligned} \tag{34}$$

will be invariant under the Galilei algebra $AG(1,n)$.

Note 1. In the case, where $\lambda_2 = \lambda_1 = \lambda$, $f = d_1((U+V)/V)^{d_0} - 1$, $g = d_2((U+V)/V)^{d_0} - d_3$ and $d_0, d_1, d_2, d_3 \in \mathbb{R}$ the system (34) is the particular case of the conservation equations for normal and mutant cells [7, 24].

In case where $f = \beta_1\omega^{-2/\delta}$, $g = \beta_2\omega^{-2/\delta}$, $\delta \neq 0$ (δ is defined in the introduction) there will be invariance under the algebra $AG_1(1,n)$. Finally, for $\delta = -n(\lambda_2 - \lambda_1)/2$, i.e. $\alpha_1 = \alpha_2 = -n/2$, the system of equations

$$\begin{aligned} &\lambda_1 U_t = \Delta U + \beta_1 U^{1+\lambda_2\gamma} V^{-\lambda_1\gamma}, \\ &\lambda_2 V_t = \Delta V + \beta_2 V^{1-\lambda_1\gamma} U^{\lambda_2\gamma} \end{aligned} \tag{35}$$

is obtained (where $\gamma = 4/(n(\lambda_2 - \lambda_1))$, $\lambda_2 \neq \lambda_1$, $\beta_k \in \mathbb{R}$), preserving the $AG_2(1,n)$-symmetry of the linear system (1).

Note 2. For $\lambda_2 = -\lambda_1 = \lambda$ the diffusion system (33) is reduced by substitution

$$U = Y + Z, \quad V = Y - Z, \quad Y = Y(t,x), \quad Z = Z(t,x) \tag{36}$$

to the system of equations

$$-\lambda Y_t = \Delta Z + f_1(Y,Z),$$
$$\lambda Z_t = \Delta Y + g_1(Y,Z),$$

whose invariance under the chain of algebras $AG(1,n) \subset AG_1(1,n) \subset AG_2(1,n)$ with the unit operator $Q_\lambda = \lambda Y \partial_Z + \lambda Z \partial_Y$ is described by the substitution (36) being applied to the system of equations of the form (33) with the corresponding symmetry.

It is interesting to consider system (33) in case where one of the equations degenerates into an elliptical one. Without reducing generality we consider $\lambda_2 = 0$, $\lambda_1 = 1$. Then according to the theorem 1, all systems of the form (33) for $\lambda_2 = 0$, $\lambda_1 = 1$ and posessing $AG(1,n)$ symmetry are given by

$$U_t = \Delta U + U f(V), \tag{37a}$$

$$0 = \Delta V + g(V), \tag{37b}$$

where f and g being arbitrary functions.

For the system (37) a clear physical treatment can be suggested. Namely, eqution (37a) is the heat equation with spatial source of energy absorption (extraction) $q = Uf(V)$, proportional to the temperature U, with an additional condition of elliptical equation (37b) being imposed on proportionality coeficient $f(V)$ (in particular we can consider $f(V) = V$). Thus we have obtained a class of nonlinear heat equations with an additional condition for the source that preserve Galilean symmetry of the linear heat equation. This result is quite non-trivial, since it is well-known fact that among nonlinear heat equations with a source

$$U_t = \Delta U + q(U)$$

not a single one is invariant with respect to Galilei algebra $AG(1,n)$ [3]. As it is seen, this "symmetry contradiction" between the linear and nonlinear heat equations can be solved in two ways: either the source is supposed to depend explicitly on temperature and independent variables $t, x_1, \ldots, x_n$ [3], or an additional condition equation (37b) upon the source is imposed as above.

It should be noted that in case $f = \beta_1 V^{2/\alpha_2}$, $g = \beta_2 V^{1+2/\alpha_2}$, $0 \neq \alpha_2$, $\beta_k \in \mathbb{R}$ system (37) is invariant under $AG_1(1,n)$ algebra (2a)–(2c). If the system (37) has the form

$$U_t = \Delta U + \beta_1 U V^{4/n}, \tag{38a}$$

$$0 = \Delta V + \beta_2 V^{1+4/n}, \tag{38b}$$

it is invariant under $AG_2(1,n)$ algebra with basic operators (2) for $\lambda_2 = 0$, $\lambda_1 = 1$, i.e. heat equation (38a) with nonlinear condition (38b) for the source conserves all the non-trivial Lie symmetry of the linear heat equation

$$U_t = \Delta U.$$

Note 3. If V is a fixed given function on independent variables $t, x_1, \dots, x_n$, equation (38a) can lose any symmetry.

In conclusion, the interesting system of the form (33) should be considered, namely

$$\begin{aligned} &\lambda U_t = \Delta U + \beta_1 U^2 V^{-1}, \\ &\lambda V_t = \Delta V + \beta_2 U, \quad \beta_1 \neq \beta_2. \end{aligned} \tag{39}$$

Theorem 5. *The maximal algebra of invariance for the system (39) is the generalized Galilei algebra with the basic operators (2a), (2b) and*

$$\begin{aligned} &D = 2tP_t + x_a P_a - 2U\partial_U - \left(\frac{n}{2} + \frac{\beta_2}{\beta_1 - \beta_2}\right) Q_\lambda, \\ &\Pi = -t^2 P_t + tD - \frac{1}{4}|x|^2 Q_\lambda - \frac{\lambda}{\beta_1 - \beta_2} V\partial_U. \end{aligned}$$

By the way, among the systems of the form (33) in case where $\lambda_2 = \lambda_1 = \lambda$ there is not an $AG_2(1,n)$-invariant system in the standard representation (2). Note that the system (39) can be considered as a particular case of the conservation equations for normal and mutant cells [7, 24].

Some classes of exact solutions for the system (39) are obtained in [25].

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1. Fushchych W.I., Cherniha R.M., *Ukr. Math. J.*, 1989, **41**, № 10, 1161–1167; № 12, 1456–1463.
2. Fushchych W., Shtelen W., Serov N., Symmetry analysis and exact solutions of equations of nonlinear mathematical physics, Dordrecht, Kluwer Academic Publishers, 1993.
3. Fushchych W.I., Cherniha R.M., *J. Phys. A*, 1985, **18**, 3491–3503.
4. Aris R., The theory of reaction and diffusion in permeable catalyst, Oxford, 1975.
5. Wilhelmsson H., *Ukr. Phys. J*, 1993, **38**, № 1, 44–53; *Physica Scripta*, 1992, **46**, 177–181.
6. Fife P.C., Mathematical aspects of reacting and diffusing systems, Berlin, Springer, 1979.
7. Murray J.D., Mathematical biology, Berlin, Springer, 1989.
8. Doebner H.-D., Goldin G.A., *Phys. Lett. A*, 1992, **162**, 397–407.
9. Goldin G.A., Svetlichny G., *J. Math. Phys.*, 1994, **35**, 3322–3332.
10. Malomed B.A., Steflo L., *J. Phys. A*, 1991, **24**, L1149–L1153.
11. Niederer U., *Helv. Phys. Acta*, 1972, **45**, 808–816.
12. Fushchych W.I., Cherniha R.M., On exact solutions of two multidimensional nonlinear Schrödinger-type equations, Preprint № 86.85, Kiev, Institute of Mathematics of Ukrainian Acad. Sci., 1986.
13. Fushchych W.I, Cherniha R.M., *Proc. Ukr. Acad. Sci.*, 1993, № 8, 44–51.
14. Zakharov V., Shabat B., *Zhurn. Eksper. Teor. Fiz. (JETP)*, 1971, **61**, № 1, 118–134.
15. Fushchych W.I., Moskaliuk S.S., *Lett. Nuovo Cimento*, 1981, **31**, № 16, 571–576.

16. Clarkson P.A., Cosgrove C.M., *J. Phys. A*, 1987, **20**, 2003–2024.
17. Gagnon L., Winternitz P., *J. Phys. A*, 1988, **21**, 1493–1511.
18. Gagnon L., *J. Phys. A*, 1992, **25**, 2649–2667.
19. Fushchych W.I., Serov N.I., *J. Phys. A*, 1987, **20**, L929–L933.
20. Kundu A., *J. Math. Phys.*, 1984, **25**, 3433–3438.
21. Calogero F., De Lillo S., *Inverse Problems*, 1988, **4**, L33–L37.
22. Cherniha R.M., *Ukr. Phys. J.*, 1995, **40**, № 4, 376–384.
23. Boyer C.P., Penafiel M.N., *Nuovo Cimento B*, 1976, **31**, № 2, 195–210.
24. Sherratt J.A., *Physica D*, 1994, **70**, 370–382.
25. Cherniha R.M., On exact solutions of a nonlinear diffusion-type system, in Symmetry Analysis and Exact Solutions of Equations of Mathematical Physics, Kyiv, Institute of Mathematics of Ukrainian Acad. Sci., 1988, 49–53.

Symmetries and reductions of nonlinear Schrödinger equations of Doebner–Goldin type

W.I. FUSHCHYCH, V. CHOPYK, P. NATTERMANN, W. SCHERER

We compute symmetry algebras for nonlinear Schrödinger equations which contain an imaginary nonlinearity as derived by Doebner and Goldin and certain real nonlinearities not depending on the derivatives. In the three-dimensional case we find the maximal symmetry algebras for equations of this type. Admitting other imaginary nonlinearities does lead to similar symmetry algebras. These symmetries are used to obtain explicit solutions of these equations by means of reduction.

1. Introduction

Recently, a new nonlinear Schrödinger equation as the evolution equation of a quantum mechanical system on $\mathbb{R}^n$ has been derived from general principles by Doebner and Goldin [1–4]. Their derivation is based on the representation theory of the semidirect product of the group of diffeomorphisms with the smooth functions on $\mathbb{R}^n$ and results in the replacement of the usual continuity equation $\dot\rho = -\vec\nabla\vec j$ (where $\rho = \bar\psi\psi$ and $\vec j = \frac{\hbar}{2mi}(\bar\psi\vec\nabla\psi - \vec\nabla\bar\psi\psi)$) associated with the linear Schrödinger equation by the Fokker–Planck equation $\dot\rho = -\vec\nabla\vec j + d\Delta\rho$ describing diffusion of the probability density ρ. This Fokker–Planck equation for the probability density can be derived from a nonlinear Schrödinger equation which has to be of the form

$$i\hbar\dot\psi = \left(-\frac{\hbar}{2m}\Delta + V + i\frac{\hbar d}{2}\frac{\Delta\rho}{\rho}\psi + F[\psi,\bar\psi]\right)\psi, \tag{1}$$

where F is assumed to be an arbitrary real functional. Doebner and Goldin proceeded with the requirement that $F[\psi,\bar\psi]$ should have similar properties as the imaginary nonlinear functional, and were thus led to a five parameter functional including derivative terms [4]. Galilei-invariant nonlinear Schrödinger equations of type (1), where $d = 0$ and F depends on the wave function and its first order derivatives, were described by Fushchych and Cherniha [5].

On the other hand, equations similar to (1) have been considered in plasma physics [6] and for $d = 0$ and $F[\psi,\bar\psi] = a\rho$ it reduces to the usual nonlinear Schrödinger equation which appears in many subfields of physics. It seems therefore worthwhile to investigate the Lie symmetries for equations of this type and to use them to construct solutions. This is what we shall do in this paper.

Obviously, we shall have to restrict the functional F suitably since otherwise it would be impossible to say anything at all about the symmetries of this equation. Whereas the maximal Lie symmetry of the Doebner–Goldin equation has already been calculated [7], we shall restrict our considerations in this paper to another class of functional F given by (sufficiently smooth) functions f of a single real variable:

$$F[\psi,\bar\psi] := \hbar f(\rho), \tag{2}$$

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which includes many physically interesting models [8, 9]. Although we leave the framework set by Doebner and Goldin if f is not real, we will consider a slightly morel general case of *complex* valued functions f since calculations are similar. For $d = 0$ the Lie symmetry of this nonlinear Schrödinger equation has been discussed in [10, 11, 12].

In Section 2 we will determine the maximal Lie symmetries of the nonlinear Schrödinger equations (1) with functional of type (2). It turns out that the most prominent cases, i.e. $f(p) \equiv \rho^k$ and $f(\rho) \equiv \ln\rho$, admit the largest symmetry algebras.) Subalgebras of the maximal symmetry algebras will be used in Section 3 to reduce equation (1) and find exact solutions. We close this paper with some further remarks) on the equations and the solutions obtained.

2. Lie symmetry algebra

2.1. $n \le 3$. First, we shall treat the physically most interesting case of three space dimensions ($n = 3$) for which we will determine the *maximal* Lie symmetry algebra of equation (1) with the *complex* valued functional (2). In order to do so, we write ψ in terms of an amplitude function R and a phase function S:

$$\psi(\vec{x}, t) = R(\vec{x}, t)e^{iS(\vec{x},t)}.$$

With the decomposition of f into the real and imaginary parts, $f = u+iv$, equation (1) is thus equivalent to two real evolution equations:

$$\partial_t R + \frac{\hbar}{2m}\left(R\Delta S + 2\vec{\nabla}R\cdot\vec{\nabla}S\right) - d\left(\Delta R + \frac{(\vec{\nabla}R)^2}{R}\right) - Rv(R^2) = 0, \tag{3}$$

$$\partial_t S + \frac{\hbar}{2m}\left((\vec{\nabla}S)^2 - \frac{\Delta R}{R}\right) + u(R^2) = 0. \tag{4}$$

Vector fields acting on the space of independent (x_1, x_2, x_3, t) and dependent (R, S) variables

$$X = \xi_j\partial_{x_j} + \tau\partial_t + \phi\partial_R + \sigma\partial_S,$$

are generators of a Lie symmetry of the equations (3) and (4), if the coefficients ξ_j, τ, ϕ, σ satisfy the so-called determining equations. A detailed description of the theory can be found in the monographs [10, 13, 14]. Since the procedure is purely algorithmic, we use a **Mathematica** program [15] to obtain these equations. This leads to 62 determining equations among which only two contain the real and imaginary part of f. These two equations determine the functional F of equation (1). The integration of the 60 remaining equations yields the following coefficients of the vector field X:

$$\begin{aligned}
&\xi_j = (2c_1t + c_2)x_j + w_{jl}x_l + v_jt + a_j,\\
&\tau = 2c_1t^2 + 2c_2t + 2c_3,\\
&\phi = \alpha(t)R,\\
&\sigma = \frac{m}{\hbar}(c_1\vec{x}^2 + v_kx_k) + \beta(t),
\end{aligned} \tag{5}$$

where c_i, v_j and a_j are real constants, w_{jl} is an antisymmetric matrix with real constant coefficients, and α and β are real functions of time. The two remaining determining equations which contain the functions u and v thus read

$$\alpha(t)R^2u'(R^2)+(2c_1t+c_2)u(R^2)+\frac{1}{2}\beta'(t)=0, \tag{6}$$

$$\alpha(t)R^2v'(R^2)+(2c_1t+c_2)v(R^2)-\frac{1}{2}(\alpha'(t)+2nc_1)=0. \tag{7}$$

For the cases $n=1,2$ the resulting equations are exactly the same, with the understanding that in equation (7) the dimension n has to be inserted. In order to calculate the maximal symmetry, we solve the ordinary differential equation (7) for α and then (6) for β, requiring that the resulting functions do not depend on R. Neglecting the case of constant functions $u=C$ — which can be transformed to zero by the map $\psi\longmapsto e^{iCt}\psi$ — this leads to the following six possible cases.

1. For arbitrary functions u and v one has to require that their coefficients and the inhomogeneous terms in equations (6) and (7) vanish, which leaves only the centrally extended Galilei algebra $\mathfrak{g}(n=3)=\langle H,P_j,J_{jk},G_j,Q\rangle$ with ten generators:

$$\begin{gathered}H=\partial_t,\quad P_j=\partial_{x_j},\quad J_{jk}=x_j\partial_{x_k}-x_k\partial_{x_j},\\ G=t\partial_{x_j}+\frac{m}{\hbar}x_j\partial_S,\quad Q=\partial_S.\end{gathered} \tag{8}$$

2. A larger algebra is obtained if u and v are of the from

$$u(R^2)=\lambda_1R^{2k},\quad v(R^2)=\lambda_2R^{2k},$$

in which case equations (6) and (7) reduce to linear inhomogeneous equations in u and v, respectively. Requiring the coefficients and the inhomogeneous term to vanish allows the maximal Lie symmetry to contain an additional generator

$$D=2t\partial_t+x_k\partial_{x_k}-\frac{1}{k}R\partial_R, \tag{9}$$

and this algebra $\langle H,P_j,J_{jk},G_j,Q,D\rangle$ has been named the Galilei similitude algebra [16]. D generates the dilations.

3. Calculations of the previous case show that the Lie symmetry has an extra generator if $k=\frac{1}{n}=\frac{1}{3}$:

$$C=t^2\partial_t+tx_k\partial_{x_k}+\frac{m}{2\hbar}\vec{x}^2\partial_S-ntR\partial_R, \tag{10}$$

yielding the maximal Lie symmetry algebra of the free linear Schrödinger equation [17] $\langle H,P_j,J_{jk},G_j,Q,D,C\rangle$ (Schrödinger algebra). The transformations generated by C are called projective or conformal transformations.

4. If $u(R^2)=\lambda_1\ln(R^2)$ and $v=\lambda_3$ is a constant, we obtain the maximal Lie symmetry algebra $\langle H,P_j,J_{jk},G_j,Q,D,B\rangle$, where

$$B=R\partial_R-2\lambda_1t\partial_S. \tag{11}$$

Note that for nonvanishing λ_1 the constant λ_3 can be transformed to zero by the map $\psi\longmapsto e^{-\lambda_3t+i\lambda_1\lambda_3t^2}\psi$.

5. If $u(R^2) = \lambda_1 \ln(R^2)$ and $v(R^2) = \lambda_2 \ln(R^2) + \lambda_3$ with $\lambda_2 \neq 0$, equation (7) leads to a simple differential equation for $\alpha(t)$ and equation (6) determines $\beta(t)$ up to a constant. Hence, the maximal symmetry algebra is $\langle H, P_j, J_{jk}, G_j, Q, D, A\rangle$, where

$$A = e^{2\lambda_2 t}\left(R\partial_R - \frac{\lambda_1}{\lambda_2}\partial_S\right). \tag{12}$$

6. Finally, if u and v vanish identically, the maximal Lie symmetry algebra is $\langle H, P_j, J_{jk}, G_j, Q, D', I\rangle$, the direct sum of the Schrödinger algebra (though with a different representative D' of the generator of dilations) with a one-dimensional algebra generated by I, where

$$D' = 2t\partial_t + x_k\partial_{x_k}, \tag{13}$$

$$I = R\partial_R. \tag{14}$$

The invariance under I reflects real homogeneity of the equation (1); together with Q it generates complex rescalings of ψ.

2.2. $n > 3$. In all cases the algebras remain symmetry algebras for arbitrary dimension n. We believe that they are still maximal, but we have no proof of maximality for arbitrary n. The algebras of the cases 1–3 and 6 have been studied in [18], and the finite transformations they generate are well known. The structure of the algebra of case 4 was investigated in [12, 19, 20].

As for the generators B and A, they generate the following finite transformations:

$$\psi \mapsto g_\epsilon^B\psi, \qquad g_\epsilon^B\psi(\vec{x},t) = \exp(\epsilon(1 - i2\lambda_1 t))\psi(\vec{x},t),$$
$$\psi \mapsto g_\epsilon^A\psi, \qquad g_\epsilon^A\psi(\vec{x},t) = \exp\left(\epsilon\left(1 - i\frac{\lambda_1}{\lambda_2}\right)\right)e^{2\lambda_2 t}\psi(\vec{x},t),$$

3. Reduction and exact solutions

Using the operators of symmetry we will construct ansätze reducing equation (1) to a system of ordinary differential equations (ODEs). The algebras of the cases 1–3 and 6 are subalgebras of the maximal symmetry algebra of the linear Schrödinger equation; their structure was studied in detail and corresponding ansätze are well known. Thus we concentrate on the cases 4 and 5, and particularly on the reduction by those subalgebras containing the "new" generators A and B. The solutions obtained in this way might reflect the nonlinear structure of equation (1) with $f(\rho) := (\lambda_1 + i\lambda_2)\ln\rho + i\lambda_3$. We consider mainly the case of three spatial variables, $n = 3$.

3.1. Case 4: $f(\rho) := \lambda_1 \ln\rho + i\lambda_3$; or $u(R^2) = \lambda_1\ln(R^2)$, $v(R^2) = \lambda_3$

1. $\langle B + G_1, G_2, G_3\rangle$. The ansatz

$$\psi(\vec{x},t) = \exp\left\{\frac{x_1}{t} + g(t) + i\left[-2\lambda_1 x_1 + \frac{m}{2\hbar}\frac{\vec{x}^2}{t} + h(t)\right]\right\} \tag{15}$$

reduces equation (1) to the system

$$\frac{dg}{dt} = \left(\frac{2\hbar\lambda_1}{m} - \frac{3}{2}\right)\frac{1}{t} + 2d\frac{1}{t^2} + \lambda_3,$$
$$\frac{dh}{dt} = -\frac{2\hbar\lambda_1^2}{m} + \frac{\hbar}{2m}\frac{1}{t^2} - 2\lambda_1 g(t).$$

Having solved this system we find the solution

$$\psi(\vec{x},t) = t^k \exp\left\{\lambda_3 t + (x_1 - 2d)\frac{1}{t} + c_1 + i\left[-2\lambda_1 x_1 + \frac{m}{2\hbar}\frac{\vec{x}^2}{t} - \lambda_1\lambda_3 t^2 - \right.\right.$$
$$\left.\left. - 2\lambda_1 k t\ln t + 2\lambda_1\left(k - c_1 - \frac{\hbar\lambda_1}{m}\right)t + 4d\lambda_1 \ln t - \frac{\hbar}{2m}\frac{1}{t} + c_2\right]\right\},$$

where $k := \frac{2\hbar\lambda_1}{m} - \frac{3}{2}$ and c_1, c_2 are real constants.

2. $\langle B + \alpha H, J_{12} + \beta P_3\rangle$, $\alpha \in \mathbb{R}_{\neq 0}$, $\beta \in \mathbb{R}$. For $\lambda_3 = 0$, the ansatz

$$\psi(\vec{x},t) = \exp\left\{\frac{t}{\alpha} + g(\omega_1,\omega_2) + i\left[-\frac{\lambda_1}{\alpha}t^2 + h(\omega_1\omega_2)\right]\right\}, \tag{16}$$

with $\omega_1 := (x_1^2 + x_2^2)^{\frac{1}{2}}$ and $\omega_2 := \arctan\left(\frac{x_2}{x_1}\right) - \beta x_3$, reduces equation (1) to the system

$$h_{11} + h_{22}\left(1 + \frac{\beta^2}{\omega_1^2}\right) + \frac{h_1}{\omega_1} + 2g_1h_1 + 2g_2h_2\left(1 + \frac{\beta^2}{\omega_1^2}\right) -$$
$$- \frac{2md}{\hbar}\left(g_{11} + g_{22}\left(1 + \frac{\beta^2}{\omega_1^2}\right) + \frac{g_1}{\omega_1} + 2g_1^2 + 2g_1^2 + 2g_2^2\left(1 + \frac{\beta^2}{\omega_1^2}\right)\right) =$$
$$= \frac{2m}{\hbar}\left(\lambda_3 - \frac{1}{\alpha}\right),$$
$$g_{11} + g_{22}\left(1 + \frac{\beta^2}{\omega_1^2}\right) + \frac{g_1}{\omega_1} + g_1^2 + g_2^2\left(1 + \frac{\beta^2}{\omega_1^2}\right) -$$
$$- \frac{4m\lambda_1}{\hbar}g - h_1^2 - h_2^2\left(1 + \frac{\beta^2}{\omega_1^2}\right) = 0,$$

where subscripts denote derivatives, i.e. $g_1 := \partial g/\partial\omega_1$, etc.

3. $\langle B + \alpha H + \beta G_1, J_{23}\rangle$, $\alpha \in \mathbb{R}_{\neq 0}$, $\beta \in \mathbb{R}$. The ansatz

$$\psi(\vec{x},t) = \exp\left\{\frac{t}{\alpha} + g(\omega_1,\omega_2) + i\left[\frac{m\beta}{\hbar\alpha}x_1 t - \frac{\lambda_1}{\alpha}t^2 - \frac{m\beta^2}{3\hbar\alpha^2}t^3 + h(\omega_1\omega_2)\right]\right\}, \tag{17}$$

with $\omega_1 := \frac{\beta t^2}{2\alpha} - x_1$ and $\omega_2 := (x_2^2 + x_3^2)^{\frac{1}{2}}$ reduces equation (1) to the system

$$2g_1h_1 + 2g_2h_2 + h_{11} + h_{22} + \frac{h_2}{\omega_2} - \frac{2md}{\hbar}\left(g_{11} + g_{22} + \frac{g_2}{\omega_2} - 2g_1^2 - 2g_2^2\right) =$$
$$= \frac{2m}{\hbar}\left(\lambda_3 - \frac{1}{\alpha}\right),$$
$$h_1^2 + h_2^2 - \frac{2\beta m^2\omega_1}{\hbar^2\alpha} - g_{11} - g_{22} - \frac{g_2}{\omega_2} - g_1^2 - g_2^2 + \frac{4\lambda_1 m}{\hbar}g = 0.$$

For $\alpha = 1/\lambda_3$, $\lambda_3 \neq 0$ and $d \neq \hbar/2m$ we have found the following partial solution of this system:

$$g(\omega_1,\omega_2) = \frac{m\beta}{2\hbar\alpha\lambda_1}\omega_1 + \frac{m\hbar\lambda_1}{\hbar^2 - 4m^2d^2}\omega_2^2 + \frac{\hbar^2}{\hbar^2 - 4m^2d^2} + \frac{(\hbar^2 - 4m^2d^2)m\beta^2}{16\hbar^3\alpha^2\lambda_1^3},$$
$$h(\omega_1,\omega_2) = \frac{2md}{\hbar}f(\omega_1,\omega_2).$$

The corresponding solution of equation (1) has then the form

$$\psi(\vec{x},t) = \exp\left\{\frac{t}{\alpha} + \frac{m\beta^2}{4\hbar\alpha^2\lambda_1}t^2 - \frac{m\beta}{2\hbar\alpha\lambda_1}x_1 + \frac{m\hbar\lambda_1}{\hbar^2 - 4m^2d^2}(x_1^2 + x_2^2) + \right.$$
$$+ \frac{\hbar^2}{\hbar^2 - 4m^2d^2} + \frac{(\hbar^2 - 4m^2d^2)m\beta^2}{16\hbar^3\alpha^2\lambda_1^3} +$$
$$+ i\left[\frac{m\beta}{\hbar\alpha}tx_1 + \left(\frac{m^2\beta^2}{2\hbar^2\alpha^2\lambda_1} - \frac{\lambda_1}{\alpha}\right)t^2 - \frac{m\beta^2}{3\hbar\alpha^2}t^3 - \frac{m^2d\beta}{\hbar^2\alpha\lambda - 1}x_1 + \right.$$
$$\left.\left. + \frac{2m^2d\lambda_1}{\hbar^2 - 4m^2d^2}(x_2^2 + x_3^2) + c\right]\right\}.$$

4. $\langle B + \alpha H, J_{jk}\rangle$, $\alpha \in \mathbb{R}_{\neq 0}$. The ansatz

$$\psi(\vec{x},t) = \exp\left\{\frac{t}{\alpha} + g(\omega) + i\left[-\frac{\lambda_1}{\alpha}t^2 + h(\omega)\right]\right\}, \tag{18}$$

where $\omega := \sqrt{x_1^2 + x_2^2 + x_3^2}$, reduces equation (1) to the system

$$\frac{d^2h}{d\omega^2} + \frac{2}{\omega}\frac{dh}{d\omega} + 2\frac{dg}{d\omega}\frac{dh}{d\omega} - \frac{2md}{\hbar}\left(\frac{d^2g}{d\omega^2} + \frac{2}{\omega}\frac{dg}{d\omega} + 2\left(\frac{dg}{d\omega}\right)^2\right) = \frac{2m}{\hbar}\left(\lambda_3 - \frac{1}{\alpha}\right),$$
$$\frac{d^2g}{d\omega^2} + \frac{2}{\omega}\frac{dg}{d\omega} + \left(\frac{dg}{d\omega}\right)^2 - \left(\frac{dh}{d\omega}\right)^2 - \frac{4m\lambda_1}{\hbar}g = 0.$$

Its partial solution for the case $\alpha = 1/\lambda_3$ and $d \neq \hbar/2m$ is

$$g(\omega) = \frac{\hbar}{\hbar^2 - 4m^2d^2}\left(m\lambda_1\vec{x}^2 + \frac{3}{2}\hbar\right),$$
$$h(\omega) = \frac{2m^2d\lambda_1}{\hbar^2 - 4m^2d^2}\vec{x}^2 + c,$$

where c is an arbitrary real constant. The corresponding solution of equation (1) has then the form

$$\psi(\vec{x},t) = \exp\left\{\frac{t}{\alpha} + \frac{\hbar}{\hbar^2 - 4m^2d^2}\left(m\lambda_1\vec{x}^2 + \frac{3}{2}\hbar\right) + \right.$$
$$\left. + i\left[\frac{2m^2d\lambda_1}{\hbar^2 - 4m^2d^2}\vec{x}^2 - \frac{\lambda_1}{\alpha}t^2 + c\right]\right\}.$$

3.2. Case 5: $f(\rho) := (\lambda_1 + i\lambda_2)\ln\rho$; or $u(R^2) = \lambda_1\ln(R^2)$, $v(R^2) = \lambda_2\ln(R^2)$; $\lambda_2 \neq 0$.

1. $\langle A + \alpha P_1, G_2, G_3\rangle$, $\alpha \in \mathbb{R}_{\neq 0}$. The ansatz

$$\psi(\vec{x},t) = \exp\left\{\frac{1}{\alpha}e^{2\lambda_2 t}x_1 + g(t) + i\left[-\frac{\lambda_1}{\alpha\lambda_2}e^{2\lambda_2 t}x_1 + \frac{m}{2\hbar}\frac{x_2^2 + x_3^2}{t} + h(t)\right]\right\}$$

reduces equation (1) to the system of ODEs

$$\frac{dg}{dt} - 2\lambda_2 g = -\frac{1}{t} + \frac{1}{\alpha^2}\left(\frac{\hbar\lambda_1}{m\lambda_2} + 2d\right)e^{4\lambda_2 t},$$
$$\frac{dh}{dt} = -2\lambda_1 g + \frac{\hbar}{2m\alpha^2}\left(1 - \frac{\lambda_1^2}{\lambda_2^2}\right)e^{4\lambda_2 t}.$$

Having solved this system we obtain the following exact solution of equation (1):

$$\psi(\vec{x},t) = \exp\left\{\frac{1}{\alpha}e^{2\lambda_2 t}x_1 + ce^{2\lambda_2 t} + \frac{1}{2\lambda_2\alpha^2}\left(\frac{\hbar\lambda_1}{m\lambda_2} + 2d\right)e^{4\lambda_2 t} - \right.$$
$$-Ei(-2\lambda_2 t)e^{2\lambda_2 t} + i\left[-\frac{\lambda_1}{\alpha\lambda_2}e^{2\lambda_2 t}x_1 + \frac{m}{2\hbar}\frac{x_2^2+x_3^2}{t} - \frac{\lambda_1 c}{\lambda_2}e^{2\lambda_2 t} - \frac{\lambda_1}{\lambda_2}\ln(2\lambda_2 t) + \right.$$
$$\left.\left. +\frac{\hbar}{8m\alpha^2\lambda_2}\left(1 - 3\frac{\lambda_1^2}{\lambda_2^2} - \frac{4md}{\hbar}\right)e^{4\lambda_2 t} + \frac{\lambda_1}{\lambda_2}Ei(-2\lambda_2 t)e^{2\lambda_2 t}\right]\right\},$$

where c is a real constant and $Ei(ax) = \int \frac{\exp(ax)}{x}dx = \ln x + \sum\limits_{k=1} \frac{a^k x^k}{k!k}$. This solution is non-analytical in λ_2, and for $n = 1$ can be written in explicit form.

2. $\langle A + \alpha J_{12}, G_3\rangle$, $\alpha \in \mathbb{R}_{\neq 0}$. The ansatz

$$\psi(\vec{x},t) = \exp\left\{\frac{1}{\alpha}e^{2\lambda_2 t}\arctan\left(\frac{x_2}{x_1}\right) + g(t,\omega) + \right.$$
$$\left. + i\left[-\frac{\lambda_2}{\alpha\lambda_1}e^{2\lambda_2 t}\arctan\left(\frac{x_2}{x_1}\right) + \frac{mx_3^2}{2\hbar t} + g(t,\omega)\right]\right\},$$

where $\omega = \sqrt{x_1^2 + x_2^2}$, reduces equations (1) to the system

$$g_1 + \frac{\hbar}{2m}\left(h_{22} + \frac{h_2}{\omega} + 2g_2 h_2 - \frac{2\lambda_1}{\alpha^2\lambda_2}e^{4\lambda_2 t}\frac{1}{r^2}\right) -$$
$$- d\left(g_{22} + \frac{g_2}{\omega} + 2g_2^2 + \frac{2}{\alpha^2}e^{4\lambda_2 t}\frac{1}{r^2}\right) - 2\lambda_2 g = 0,$$
$$h_1 + \frac{\hbar}{2m}\left(h_2^2 - g_{22} - \frac{g_2}{\omega} - g_2^2 - \frac{1}{\alpha^2}\left(1 - \frac{\lambda_1^2}{\lambda_2^2}\right)e^{4\lambda_2 t}\frac{1}{r^2}\right) + 2\lambda_1 g = 0,$$

3. $\langle A + \alpha H, J_{12} + \beta P_3\rangle$, $\alpha \in \mathbb{R}_{\neq 0}$, $\beta \in \mathbb{R}$. The ansatz

$$\psi(\vec{x},t) = \exp\left\{\frac{1}{2\alpha\lambda_2}e^{2\lambda_2 t} + g(\omega_1,\omega_2) + i\left[-\frac{\lambda_1}{2\alpha\lambda_2^2}e^{2\lambda_2 t} + h(\omega_1,\omega_2)\right]\right\},$$

where $\omega_1 = \sqrt{x_1^2 + x_2^2}$, $\omega_2 = \beta\arctan\left(\frac{x_1}{x_2}\right) - x_3$, reduces equations (1) to the system

$$h_{11} + \frac{h_1}{\omega_1} + 2g_1 h_1 + h_{22}\left(1 + \frac{\beta^2}{\omega_1^2}\right) + h_2^2\left(1 + \frac{\beta^2}{\omega_1^2}\right) -$$
$$- \frac{2md}{\hbar}\left(g_{11} + \frac{g_1}{\omega_1} + 2g_1^2 + g_{22}\left(1 + \frac{\beta^2}{\omega_1^2}\right) + g_2^2\left(1 + \frac{\beta^2}{\omega_1^2}\right)\right) - \frac{4m\lambda_2}{m}g = 0,$$
$$g_{11} + \frac{g_1}{\omega_1} + g_1^2 - h_1^2 + g_{22}\left(1 + \frac{\beta^2}{\omega_1^2}\right) + g_2^2\left(1 + \frac{\beta^2}{\omega_1^2}\right) -$$
$$- h_2^2\left(1 + \frac{\beta^2}{\omega_1^2}\right) - \frac{4m\lambda_1}{m}g = 0.$$

4. $\langle A + \alpha P_3, J_{12}\rangle$, $\alpha \in \mathbb{R}_{\neq 0}$. The ansatz

$$\psi(\vec{x},t) = \exp\left\{\frac{1}{\alpha}e^{2\lambda_2 t}x_3 + g(t,\omega) + i\left[-\frac{\lambda_1}{\alpha\lambda_2}e^{2\lambda_2 t}x_3 + h(t,\omega)\right]\right\},$$

where $\omega = \sqrt{x_1^2 + x_2^2}$, reduces equations (1) to the system

$$g_1 + \frac{\hbar}{2m}\left(h_{22} + \frac{h_2}{\omega} + 2g_2h_2 - \frac{\lambda_1}{\alpha^2\lambda_2}e^{4\lambda_2 t}\right) - \\ - d\left(g_{22} + \frac{g_2}{\omega} + 2g_2^2 + \frac{2}{\alpha^2}e^{4\lambda_2 t}\right) - 2\lambda_2 g = 0,$$

$$h_1 + \frac{\hbar}{2m}\left(h_2^2 - g_{22} - \frac{g_2}{\omega} - g_2^2 - \frac{1}{\alpha^2}\left(1 - \frac{\lambda_1^2}{\lambda_2^2}\right)e^{4\lambda_2 t}\right) + 2\lambda_1 g = 0.$$

4. Conclusions

We have determined the maximal Lie symmetries of equation (1) with an F of the form $F[\psi, \bar{\psi}] := \hbar f(\rho)$, and have found six different algebras containing among others the centrally extended Galilei algebra, the Galilei similitude algebra, and the Schrödinger algebra. Reduction and ansätze for these algebras have been studied previously.

New maximal symmetry algebras, due to the nonlinear character of the equation, appear in the case $f(\rho) = (\lambda_1 + i\lambda_2)\ln(\rho)$ (see cases 5 and 6 in Section 2.1). For these cases we have obtained reduced equations for various subalgebras. The ansätze resulting from these reductions lead to differential equations which we have solved explicitly in some cases and thus we have obtained explicit solutions of (1). Those reduced equations, which we have not been able to solve explicitly, are still much more suitable to numerical treatments than the original equation (1). The list of subalgebras which we have used for reduction in the case of the new algebras is by no means complete. In view of the successes of the reduction technique it seems warranted to obtain a classification of their subalgebras. The non-Lie anzätze for the nonlinear Schrödinger equation were constructed by Fushchych and Chopyk [21].

1. Doebner H.D., Goldin G.A., *Phys. Lett. A*, 1992, **162**, 397.
2. Doebner H.D., Goldin G.A., *Annales de Fisica*, Monografias, Vol. II, CIEMAT, 1993, 442–445.
3. Doebner H.D., Goldin G.A., Manifolds, general symmetries, quantization and nonlinear quantum mechanics, in Proceedings of the First German-Polish Symposium on Paricles and Fields (Rydzyna Castle, 1992), Singapore, World Scientific, 1993, 115.
4. Doebner H.D. and Goldin G.A., *J. Phys. A*, 1994, **27**, 1771.
5. Fushchych W., Cherniha R., *Ukr. Math. J.*, 1989, **41**, № 10, 1349; № 12, 1487.
6. Malomed B.A., Stenflo L., *J. Phys. A*, 1991, **24**, L1149.
7. Nattermann P., Maximal Lie symmetry of the free general Doebner–Goldin equation in 1 + 1 dimensions, Clausthal-preprint ASI-TPA/9/94.
8. Biatynicki-Birula I., Mycielski J., *Ann. Phys.*, 1976, **100**, 62.
9. Malomed B.A. and Kivshar Y.S., *Rev. Mod. Phys.*, **61**, 1989, 763.
10. Fushchych W.I., Shtelen V.M., Serov N.I., Symmetry analysis and exact solutions of equations of nonlinear mathematical physics, Dordrecht, Kluwer, 1993.
11. Fushchych W., in Algebraic-Theoretical Studies in Mathematical Physics, Kiev, Inst. Math. Ukrainian Acad. Sci., 1981, 6–44 (in Russian).
12. Fushchych W., Chopyk V., Symmetry analysis and ansatzes for Schrödinger equation with the logarithmic nonlinearity, Preprint, Linköping University, 1993.
13. Ovsiannikov L.V., Group analysis of differential equations, Academic Press, 1982.
14. Olver J., Appplications of Lie groups to differential equations, New York, Springer, 1986.

15. Baumann G., Lie symmetries of differential equations: A Mathematica program to determine Lie symmetries, Wolfram Research Inc., Champaign, Ilinois, MathSource 0202-622, 1992.
16. Rideau G., Winternitz P., *J. Math. Phys.*, 1993, **34**, 558.
17. Niederer U., *Helvet. Phys. Acta*, 1972, **45**, 802.
18. Fushchych W., Barannik L., Barannik A., Subgroup analysis of the Galilei and Poincaré Groups, and reduction of nonlinear equations, Kiev, Naukova Dumka, 1991 (in Russian).
19. Fushchych W., Chopyk V., Cherkasenko V., *Dokl. Ukr. Acad. Sci.*, 1993, № 2, 32.
20. Chopyk V.I., Symmetry and reduction of multi-dimensional Schrödinger equation with the logarithmic nonlinearity, in Symmetry Analysis of Equations of Mathematical Physics, Kiev, Inst. Math. Ukrainian Acad. Sci., 1992.
21. Fuschchych W., Chopyk V., *Ukr. Math. J.*, 1993, **45**, 581.

Symmetry reduction and exact solutions of nonlinear biwave equations

W.I. FUSHCHYCH, O.V. ROMAN, R.Z. ZHDANOV

Symmetry analysis of a class of the biwave equations $\Box^2 u = F(u)$ and of a system of wave equations which is equivalent to it is performed. Reduction of the nonlinear biwave equations by means of the Ansätze invariant under non-conjugated subalgebras of the extended Poincaré algebra $A\widetilde{P}(1,1)$ and the conformal algebra $AC(1,1)$ is carried out. Some exact solutions of these equations are obtained.

1 Introduction

It was customary for the classical mathematical physics to use as the mathematical models for describing real physical processes linear partial differential equations (PDE) of the order not higher than two. All fundamental equations of mathematical physics such as the Laplace, heat, Klein–Gordon–Fock, Maxwell, Dirac, Schrödinger equations are the first- or the second-order linear partial differential equations. But now there are strong evidences that linear description is not satisfactory (especially it is the case in the quantum field theory [1]). That is why, it was attempted to generalize the classical equations in a non-linear way in order to get more satisfactory models. There exist different principles of the choice of such generalizations but up to our mind the most natural and systematic is the *symmetry selection principle*. A classical illustration is a group classification of nonlinear wave equations

$$\Box u = F(u). \tag{1}$$

Here and further $\Box = \partial^2/\partial x_0 - \partial^2/\partial x_1 - \cdots - \partial^2/\partial x_n$ is the d'Alembertian in the $(n+1)$-dimensional pseudo-Euclidean space $\mathbb{R}(1,n)$ with the metric tensor $g_{\mu\nu} = \mathrm{diag}\,(1,-1,\ldots,-1)$, $\mu,\nu = \overline{0,n}$; $x_\mu = x^\nu g_{\mu\nu}$; $F(u)$ is an arbitrary smooth function; $u = u(x)$ is a real function; the summation over the repeated indices from 0 to n is understood.

With an arbitrary $F(u)$ equation (1) is invariant under the $\frac{(n+1)(n+2)}{2}$-parameter Poincaré group $P(1,n)$ having the following generators:

$$P_\mu = \frac{\partial}{\partial x_\mu}, \qquad J_{\mu\nu} = x^\mu \frac{\partial}{\partial x_\nu} - x^\nu \frac{\partial}{\partial x_\mu}, \qquad \mu,\nu = \overline{0,n}. \tag{2}$$

But equation (1) taken with an arbitrary nonlinearity $F(u)$ is too "general" to be a reasonable mathematical model for describing a specific physical phenomena. To specify a form of $F(u)$ symmetry properties of the *linear* wave equation are utilized. It is well-known that PDE (1) with $F(u) = 0$ in addition to the Poincaré group admits a one-parameter scale transformation group and a $(n+1)$-parameter group of special conformal transformations (see, e.g. [2]). Therefore, it is not but natural to

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postulate that those nonlinearities are admissible which preserve a symmetry of the linear equation. It has been proved in [3] that there are only two functions $F(u)$, namely

$$F(u) = \lambda(u + C)^k, \quad F(u) = \lambda \exp Cu, \tag{3}$$

where C, $k \neq 0$ are arbitrary constants, such that Poincaré-invariant equation (1) admits a one-parameter scale transformation group. Furthermore, it was known long ago that the only equation of the form (1) admitting the conformal group $C(1, n)$ is the one with $F(u) = \lambda(u + C)^{\frac{n+3}{n-1}}$. Consequently, choosing from the whole set of PDE (1) equations having the highest symmetry we get the ones with very specific nonlinearities.

A procedure described above is called group or symmetry classification of PDE (1). A method used is the classical infinitesimal Lie's method. Given a representation of a Lie transformation group (which is fixed by a requirement that this group should be admitted by the linear wave equation), the problem of symmetry classification of equations (1) is reduced to solving some linear over-determined system of PDE. This system is called determining equations (for more detail, see [2, 5]).

But what is most important, the Lie's method can be applied not only to classify invariant equations but also to construct their explicit solutions by means of *symmetry reduction procedure*. And one more important remark is that equations having broad Lie symmetry often admit non-trivial *conditional symmetry*, which can be also used to obtain their particular solutions [2].

In [6] the description has been suggested of different physical processes with the help of nonlinear partial differential equations of high order, namely

$$\Box^l u = F\Big(u, \frac{\partial u}{\partial x_\mu}\frac{\partial u}{\partial x^\mu}\Big). \tag{4}$$

where $\Box^l = \Box(\Box^{l-1})$, $l \in \mathrm{N}$; $F(\cdot\,,\cdot)$ is an arbitrary smooth function.

The equations (4) were considered from different points of view in [2, 7, 8], where the pseudo-differential equations of type (4) were also studied (in this case l is fractional or negative).

Assuming $l = 1$ and $F = F(u)$ in (4) we obtain the standard nonlinear wave equation (1), which describes a scalar spin-less uncharged particle in the quantum field theory. Symmetry properties of the equation (1) were studied in [2, 3, 4] and wide classes of its exact solutions with certain concrete values of the function $F(u)$ were obtained in [2, 3, 9, 10, 11].

In this paper we restrict ourselves to symmetry analysis of the biwave equation

$$\Box^2 u = F(u), \tag{5}$$

which is one of the simplest equations of type (4) of the order higher than two ($l = 2$, $F = F(u)$).

2 Symmetry classification of biwave equations

In order to carry out a symmetry classification of the equation (5) we shall establish at first the maximal transformation group admitted by the equation (5), provided $F(u)$

is an arbitrary function. Next, we shall determine all the functions $F(u)$ such that the equation (5) admits a more extended symmetry.

Results of symmetry classification of the equation (5) are presented below.

Lemma 1 *The maximal invariance group of the equation (5) with an arbitrary function $F(u)$ is the Poincaré group $P(1,n)$ generated by the operators (2).*

Theorem 1 *Any equation of type (5) admitting a more extended invariance algebra than the Poincaré algebra $AP(1,n)$ is equivalent to one of the following PDE:*

$$1.\ \Box^2 u = \lambda_1 u^k, \qquad \lambda_1 \neq 0,\ k \neq 0, 1; \tag{6}$$

$$2.\ \Box^2 u = \lambda_2 e^u, \qquad \lambda_2 \neq 0; \tag{7}$$

$$3.\ \Box^2 u = \lambda_3 u, \qquad \lambda_3 \neq 0; \tag{8}$$

$$4.\ \Box^2 u = 0. \tag{9}$$

Here λ_1, λ_2, λ_3, k are arbitrary constants.

Theorem 2 *The symmetry of the equations (6)–(9) is described as follows:*

1. (a) The maximal invariance group of the equation (6) when $k \neq (n+5)/(n-3)$, $k \neq 0, 1$ is the extended Poincaré group $\widetilde{P}(1,n)$ generated by the operators (2) and

$$D = x_\mu \frac{\partial}{\partial x_\mu} + \frac{4}{1-k} u \frac{\partial}{\partial u}. \tag{10}$$

(b) The maximal invariance group of the equation (6) when $k = (n+5)/(n-3)$, $n \neq 3$ is the conformal group $C(1,n)$ generated by the operators (2) and operators

$$\begin{aligned} D^{(1)} &= x_\mu \frac{\partial}{\partial x_\mu} + \frac{3-n}{2} u \frac{\partial}{\partial u}, \\ K_\mu^{(1)} &= 2x^\mu D^{(1)} - (x_\nu x^\nu) \frac{\partial}{\partial x_\mu}. \end{aligned} \tag{11}$$

2. (a) The maximal invariance group of the equation (7) when $n \neq 3$ is the extended Poincaré group $\widetilde{P}(1,n)$ generated by the operators (2) and

$$D^{(2)} = x_\mu \frac{\partial}{\partial x_\mu} - 4 \frac{\partial}{\partial u}. \tag{12}$$

(b) The maximal invariance group of the equation (7) when $n = 3$ is the conformal group $C(1,n)$ generated by the operators (2) and operators

$$K_\mu^{(2)} = 2x^\mu D^{(2)} - (x_\nu x^\nu) \frac{\partial}{\partial x_\mu}. \tag{13}$$

3. The maximal invariance group of the equation (8) is generated by the operators (2) and

$$Q = h(x) \frac{\partial}{\partial u}, \quad I = u \frac{\partial}{\partial u},$$

where $h(x)$ is an arbitrary solution of the equation (8).

4. The maximal invariance group of the equation (9) is generated by the operators (2), (11) and

$$Q = q(x)\frac{\partial}{\partial u}, \qquad I = u\frac{\partial}{\partial u},$$

where $q(x)$ is an arbitrary solution of the equation (9).

The proof of the Lemma 1 and the Theorems 1, 2 is carried out by means of the infinitesimal algorithm of S. Lie [2, 5]. Since it requires very cumbersome computations, we adduce a general scheme of the proof only.

Within the framework of the Lie's approach an infinitesimal operator of the equation (5) invariance group is looked for in the form

$$X = \xi^\mu(x,u)\frac{\partial}{\partial x_\mu} + \eta(x,u)\frac{\partial}{\partial u}. \tag{14}$$

The criterion of invariance of the equation (5) with respect to a group generated by the operator (14) reads

$$\underset{4}{X}(\Box^2 u - F(u))\Big|_{\Box^2 u = F(u)} = 0, \tag{15}$$

where $\underset{4}{X}$ is the 4-th prolongation of the operator X.

Splitting the equation (15) with respect to the independent variables, we come to the system of partial differential equations for functions $\xi^\mu(x,u)$ and $\eta(x,u)$:

$$\begin{aligned}
&\xi^\mu_u = 0, \quad \eta_{uu} = 0, \quad \mu = \overline{0,n},\\
&\xi^i_0 = \xi^0_i, \quad \xi^i_j = -\xi^j_i, \quad i \neq j, \ i,j = \overline{1,n},\\
&\xi^0_0 = \xi^1_1 = \cdots = \xi^n_n,\\
&2\eta_{\nu u} = (3-n)\xi^\nu_{00}, \quad \nu = \overline{0,n},
\end{aligned} \tag{16}$$

$$\Box^2\eta - \eta F'(u) + F(u)(\eta_u - 4\xi^0_0) = 0. \tag{17}$$

Besides, when $n = 1$, there are additional equations

$$\eta_{00u} = 0, \quad \eta_{01u} = 0, \tag{18}$$

that do not follow from the equations (16) and (17).

In the above formulae we use the notations $\xi^\mu_\nu = \partial\xi^\mu/\partial x_\nu$, $\eta_\mu = \partial\eta/\partial x_\mu$ and so on.

System (16) is one of the Killing equations in the Minkowski space-time. Its general solution is well-known and can be represented in the following form:

$$\begin{aligned}
&\xi^\nu = 2x^\nu x_\mu c^\mu - x_\mu x^\mu c^\nu + b_{\nu\mu}x^\mu + dx_\nu + a_\nu,\\
&\eta = ((3-n)c^\mu x_\mu + p)u + \varkappa(x),
\end{aligned} \tag{19}$$

where c_μ, $b_{\nu\mu} = -b_{\mu\nu}$, d, a_ν, p are arbitrary constants, $\varkappa(x)$ is an arbitrary smooth function.

Substituting the expression (19) into the classifying equation (17) and splitting it with respect to u we arrive at the statements of the Lemma 1 and the Theorems 1, 2 according to the form of $F(u)$.

It follows from the assertions proved that the equation of type (4) is invariant under the extended Poincaré group $\widetilde{P}(1,n)$ if and only if it is equivalent to one of the equations (6), (7) or (9). Let us note that an analogous result was obtained for the wave equation (1) in [3].

The following statement is also a consequence of the Theorems 1, 2 but because of its importance we adduce it as a theorem.

Theorem 3 *Equation (5) admits the conformal group $C(1,n)$ if and only if it is equivalent to the following:*

$$1.\ \Box^2 u = \lambda_1 u^{(n+5)/(n-3)}, \quad n \neq 3; \tag{20}$$

$$2. \Box^2 u = \lambda_2 e^u, \quad n = 3. \tag{21}$$

Let us note that conformal invariance of the equation (20) has been first ascertained in [12] and that of equation (21) – in [2] by means of the Baker–Campbell–Hausdorff formula. It is also worth noting that conformal invariance of the nonlinear polyharmonic equations has been studied in [13], which enables constructing some their exact solutions.

In conclusion of the Section let us emphasize an important property of the linear biwave equation (9) with $n = 3$, which is a consequence of the Theorems 2, 3.

Corollary. *There exist two inequivalent representations of the Lie algebra of the conformal group $C(1,3)$ on the solution set of the equation (9) [2, 6, 8]:*

$$\begin{aligned}
1.\quad & P_\mu^{(1)} = P_\mu, \quad J_{\mu\nu}^{(1)} = J_{\mu\nu}, \\
& D^{(1)} = x_\mu \frac{\partial}{\partial x_\mu}, \quad K_\mu^{(1)} = 2x^\mu D^{(1)} - (x_\nu x^\nu)\frac{\partial}{\partial x_\mu}; \\
2.\quad & P_\mu^{(2)} = P_\mu, \quad J_{\mu\nu}^{(2)} = J_{\mu\nu}, \\
& D^{(2)} = x_\mu \frac{\partial}{\partial x_\mu} + \frac{\partial}{\partial u}, \quad K_\mu^{(2)} = 2x^\mu D^{(2)} - (x_\nu x^\nu)\frac{\partial}{\partial x_\mu},
\end{aligned}$$

where the operators P_μ, $J_{\mu\nu}$ are determined in (2).

3 Symmetry classification of system of wave equations

Introducing a new variable $v = \Box u$ in (5) we get a system of partial differential equations

$$\begin{aligned}
&\Box u = v, \\
&\Box v = F(u),
\end{aligned} \tag{22}$$

which is equivalent to the biwave equation (5).

Symmetry properties of the system (22) are investigated by analogy with the previous Section. That is why, we restrict ourselves to formulating the corresponding assertions omitting their proofs.

Lemma 2 *The maximal invariance group of the system (22) with an arbitrary function $F(u)$ is the Poincaré group $P(1,n)$ generated by the operators (2).*

Theorem 4 *Any system of type (22) admitting a more extended invariance algebra than the Poincaré algebra $AP(1,n)$ is equivalent to one of the following:*

$$1. \quad \begin{aligned} &\Box u = v, \\ &\Box v = \lambda_1 u^k, \quad \lambda_1 \neq 0,\ k \neq 0,1; \end{aligned} \tag{23}$$

$$2. \quad \begin{aligned} &\Box u = v, \\ &\Box v = \lambda_2 u, \quad \lambda_2 \neq 0; \end{aligned} \tag{24}$$

$$3. \quad \begin{aligned} &\Box u = v, \\ &\Box v = 0. \end{aligned} \tag{25}$$

Theorem 5 *The symmetry of the systems (23)–(25) is described in the following way:*

1. The maximal invariance group of the system (23) is the extended Poincaré group $\widetilde{P}(1,n)$ generated by the operators (2) and

$$D = x_\mu \frac{\partial}{\partial x_\mu} + \frac{4}{1-k} u \frac{\partial}{\partial u} + \frac{2(1+k)}{1-k} v \frac{\partial}{\partial v}.$$

2. The maximal invariance group of the system (24) is generated by the operators (2) and

$$Q_1 = u\frac{\partial}{\partial u} + v\frac{\partial}{\partial v}, \qquad Q_2 = v\frac{\partial}{\partial u} + \lambda_2 u \frac{\partial}{\partial v},$$
$$Q_3 = h_1(x)\frac{\partial}{\partial u} + h_2(x)\frac{\partial}{\partial v},$$

where $(h_1(x), h_2(x))$ is an arbitrary solution of the system (24).

3. The maximal invariance group of the system (25) is generated by the operators (2) and

$$D = x_\mu \frac{\partial}{\partial x_\mu} + 2u\frac{\partial}{\partial u}, \qquad Q_1 = u\frac{\partial}{\partial u} + v\frac{\partial}{\partial v},$$
$$Q_2 = v\frac{\partial}{\partial u}, \quad Q_3 = q_1(x)\frac{\partial}{\partial u} + q_2(x)\frac{\partial}{\partial v},$$

where $(q_1(x), q_2(x))$ is an arbitrary solution of the system (25).

It follows from the statements above that, unlike the biwave equations, the extended Poincaré group $\widetilde{P}(1,n)$ is the invariance group of the system (22) only in two cases, namely, when the system (22) is equivalent to (23) or (25). Moreover, there are no systems of the form (22) which are invariant under the conformal group. Therefore, in the class of Lie operators, the invariance algebras of the biwave equations and the corresponding systems of the wave equations are essentially different.

4 Reduction and exact solutions of the equation $\Box^2 u = \lambda e^u$

As follows from the Theorem 2 the maximal invariance group of the equation (7) with $n = 1$ is the extended Poincaré group $\widetilde{P}(1,1)$ with the generators

$$P_0 = \frac{\partial}{\partial x_0}, \quad P_1 = \frac{\partial}{\partial x_1}, \quad J_{01} = x_0 \frac{\partial}{\partial x_1} + x_1 \frac{\partial}{\partial x_0}, \tag{26}$$

$$D^{(2)} = x_0 \frac{\partial}{\partial x_0} + x_1 \frac{\partial}{\partial x_1} - 4\frac{\partial}{\partial u}. \tag{27}$$

To construct exact solutions of the above equation we shall make use of the symmetry reduction procedure. A principal idea of the said procedure is a special choice of a solution to be found. This choice is motivated by a representation of symmetry group admitted. It is known that if an equation admits a Lie transformation group having a symmetry operator

$$X = \xi^\mu(x) \frac{\partial}{\partial x_\mu} + \eta(x) \frac{\partial}{\partial u}, \tag{28}$$

then its solutions can be looked for in the form [2]:

$$u(x) = \varphi(\omega) + g(x), \tag{29}$$

where $\varphi(\omega)$ is an arbitrary smooth function, and what is more, functions $\omega(x)$ and $g(x)$ are to satisfy the following conditions:

$$\xi^\mu(x) \frac{\partial \omega}{\partial x_\mu} = 0, \qquad \xi^\mu(x) \frac{\partial g(x)}{\partial x_\mu} = \eta(x).$$

To obtain all the $\widetilde{P}(1,1)$ non-conjugated Ansätze (29) we have to describe all the inequivalent one-dimensional subalgebras of the Lie algebra $\widetilde{AP}(1,1)$ spanned by the operators (26) and (27) (see [2, 11]). In the paper we make use of a classification adduced in [11]. Omitting cumbersome intermediate computations we give $\widetilde{P}(1,1)$ non-conjugated Ansätze in the Table 1.

Table 1

N	Algebra	Invariant variable ω	Ansatz
1°	$D - J_{01}$	$x_0 + x_1$	$u = \varphi(\omega) - 2\ln(x_0 - x_1)$
2°	$D + \alpha J_{01},\ \alpha \neq -1$	$(1+\alpha)\ln(x_1 - x_0) - (1-\alpha)\ln(x_0 + x_1)$	$u = \varphi(\omega) - \dfrac{4}{\alpha+1}\ln(x_0 + x_1)$
3°	$D - J_{01} + P_0$	$\ln(x_0 - x_1 + 1/2) - 2(x_0 + x_1)$	$u = \varphi(\omega) - 2\ln(x_0 - x_1 + 1/2)$
4°	J_{01}	$x_0^2 - x_1^2$	$u = \varphi(\omega)$
5°	P_0	x_1	$u = \varphi(\omega)$
6°	$P_0 + P_1$	$x_0 - x_1$	$u = \varphi(\omega)$

Remark. Inequivalent subalgebras adduced in the Table 1 are constructed by taking into account an obvious fact that equation (7) is invariant under transformations of the form:

$$\begin{aligned} &x_0' \to x_0, \\ &x_1' \to -x_1; \end{aligned} \qquad \text{and} \qquad \begin{aligned} &x_0' \to x_1, \\ &x_1' \to x_0. \end{aligned} \tag{30}$$

Substituting the Ansätze obtained into the equation (7) we get the following ordinary differential equations (ODE) for a function $\varphi(\omega)$:

$1^\circ \quad 0 = \lambda e^{\varphi}$,

$2^\circ \quad \varphi^{(4)}(\alpha^2-1)^2 + 2\varphi^{(3)}\alpha(1-\alpha^2) - \varphi^{(2)}(1-\alpha^2) = \dfrac{\lambda}{16}\exp\left(\varphi + \dfrac{2\omega}{\alpha+1}\right)$,

$3^\circ \quad \varphi^{(4)} - \varphi^{(3)} = \dfrac{\lambda}{64}e^{\varphi}$,

$4^\circ \quad \varphi^{(4)}\omega^2 + 4\varphi^{(3)}\omega + 2\varphi^{(2)} = \dfrac{\lambda}{16}e^{\varphi}$,

$5^\circ \quad \varphi^{(4)} = \lambda e^{\varphi}$,

$6^\circ \quad 0 = \lambda e^{\varphi}$.

Equation 5° has a particular solution

$$\varphi = \ln\left(\frac{24}{\lambda}(\omega + c)^{-4}\right), \quad \lambda > 0,$$

that leads to the following exact solutions of the equation (7):

$$\begin{aligned} &u = \ln\left(\frac{24}{\lambda}(x_0 + c_1)^{-4}\right), \quad \lambda > 0, \\ &u = \ln\left(\frac{24}{\lambda}(x_1 + c_2)^{-4}\right), \quad \lambda > 0. \end{aligned} \tag{31}$$

Here c, c_1, c_2 are arbitrary constants. This solutions are invariant under the operators P_0 and P_1 accordingly.

In conclusion of the section let us note that the solutions (31) can be also obtained by making use of the Ansatz in a Liouville form [2]:

$$u = \ln\left\{\frac{24}{\lambda}\frac{\left(\dot\varphi_1(\omega_1)\dot\varphi_2(\omega_2)\right)^2}{(\varphi_1(\omega_1)+\varphi_2(\omega_2))^4}\right\}, \quad \omega_1 = x_0 + x_1, \quad \omega_2 = x_0 - x_1,$$

that reduces the equation (7) to one of the following systems:

1. $\ddot\varphi_1 = 0, \quad \ddot\varphi_2 = 0;$
2. $\ddot\varphi_1 = \dfrac{2\dot\varphi_1^2}{\varphi_1}, \quad \ddot\varphi_2 = \dfrac{2\dot\varphi_2^2}{\varphi_2}.$

Here $\dot\varphi$ and $\ddot\varphi$ stand for the first and the second derivatives with respect to a corresponding argument.

Integrating the above systems we get the following exact solutions of the equation (7):

$$u = \ln\left(\frac{24}{\lambda}\frac{(a^2-b^2)^2}{(ax_0+bx_1+c)^4}\right), \tag{32}$$

where a, b, c are arbitrary constants.

The solution (32) can be obtained from (31) by means of the final transformations of the extended Poincaré group with generators (26) and (27).

5 Reduction and exact solutions of the equation $\Box^2 u = \lambda u^k$

It follows from the Theorem 2 that the equation (6) with $n=1$ is invariant under the extended Poincaré group $\widetilde{P}(1,1)$ with generators (26) and

$$D = x_0\frac{\partial}{\partial x_0} + x_1\frac{\partial}{\partial x_1} + \frac{4}{1-k}u\frac{\partial}{\partial u}. \tag{33}$$

If some equation admits a symmetry operator

$$X = \xi^\mu(x)\frac{\partial}{\partial x_\mu} + \eta(x)u\frac{\partial}{\partial u}, \tag{34}$$

then its solutions can be looked for in the form [2]:

$$u(x) = f(x)\varphi(\omega), \tag{35}$$

provided functions $\omega(x)$ and $f(x)$ satisfy the following system:

$$\xi^\mu(x)\frac{\partial\omega}{\partial x_\mu} = 0, \quad \xi^\mu(x)\frac{\partial f(x)}{\partial x_\mu} = \eta(x)f(x). \tag{36}$$

A complete list of $\widetilde{P}(1,1)$ non-conjugated Ansätze invariant under the inequivalent one-dimensional subalgebras of the algebra $\widetilde{P}(1,1)$ is given in the Table 2.

Table 2

N	Algebra	Invariant variable ω	Ansatz
1°	$D - J_{01}$	$x_0 + x_1$	$u = (x_0 - x_1)^{\frac{2}{1-k}}\varphi(\omega)$
2°	$D + \alpha J_{01}, \quad \alpha \neq -1$	$(x_0 - x_1)(x_0 + x_1)^{\frac{\alpha-1}{\alpha+1}}$	$u = (x_0 + x_1)^{\frac{4}{(1-k)(\alpha+1)}}\varphi(\omega)$
3°	$D + J_{01} + P_0$	$(x_0 + x_1 + \frac{1}{2})\times$ $\times\exp\left(2(x_1 - x_0)\right)$	$u = \exp\left(\frac{4}{k-1}(x_1 - x_0)\right)\varphi(\omega)$
4°	J_{01}	$x_0^2 - x_1^2$	$u = \varphi(\omega)$
5°	P_0	x_1	$u = \varphi(\omega)$
6°	$P_0 + P_1$	$x_0 + x_1$	$u = \varphi(\omega)$

Let us note that similar Ansätze for the nonlinear wave equation

$$\Box u = \lambda u^k, \tag{37}$$

were obtained in [3].

Substituting the Ansätze obtained into the equation (6) we get the following ODE for a function $\varphi(\omega)$:

$$1^\circ \quad \frac{1+k}{(1-k)^2}\varphi^{(2)} = \frac{\lambda}{32}\varphi^k,$$

$$2^\circ \quad (\alpha-1)^2\varphi^{(4)}\omega^2 + 2(\alpha-1)(\alpha+1)^2\left(\frac{3k+1}{1-k}+2\alpha\right)\omega\varphi^{(3)} + \\ + 2\left(\alpha^2-4\alpha+3+\frac{6\alpha-10}{1-k}+\frac{8}{(1-k)^2}\right)\varphi^{(2)} = \frac{\lambda}{16}(\alpha+1)^2\varphi^k,$$

$$3^\circ \quad \varphi^{(4)}\omega^2 + \frac{5k-1}{k-1}\varphi^{(3)}\omega + \frac{4k^2}{(1-k)^2}\varphi^{(2)} = \frac{\lambda}{64}\varphi^k,$$

$$4^\circ \quad \varphi^{(4)}\omega^2 + 4\varphi^{(3)}\omega + 2\varphi^{(2)} = \frac{\lambda}{16}\varphi^k,$$

$$5^\circ \quad \varphi^{(4)} = \lambda\varphi^k,$$

$$6^\circ \quad \lambda\varphi^k = 0.$$

Equations 1°, 2°, 4° have particular solutions of the form:

$$\varphi = \left(\frac{64}{\lambda}\frac{(k+1)^2}{(k-1)^4}\right)^{\frac{1}{k-1}}\omega^{-\frac{2}{k-1}}, \quad k \neq -1$$

and equation 5° has a particular solution of the form

$$\varphi = \left(\frac{8}{\lambda}\frac{(k+1)(k+3)(3k+1)}{(k-1)^4}\right)^{\frac{1}{k-1}}\omega^{-\frac{4}{k-1}}, \quad k \neq -1, -3, -\frac{1}{3},$$

which lead to the following solutions of the equation (6):

$$u = \left(\frac{64}{\lambda}\frac{(k+1)^2}{(k-1)^4}\right)^{\frac{1}{k-1}}\Big((x_0+x_1+c_1)(x_0-x_1+c_2)\Big)^{-\frac{2}{k-1}}, \quad k \neq -1,$$

$$u = \left(\frac{8}{\lambda}\frac{(k+1)(k+3)(3k+1)}{(k-1)^4}\right)^{\frac{1}{k-1}}(x_0+c_3)^{\frac{4}{1-k}}, \quad k \neq -1, -3, -\frac{1}{3},$$

$$u = \left(\frac{8}{\lambda}\frac{(k+1)(k+3)(3k+1)}{(k-1)^4}\right)^{\frac{1}{k-1}}(x_1+c_4)^{\frac{4}{1-k}}, \quad k \neq -1, -3, -\frac{1}{3},$$

where c_1, c_2, c_3, c_4 are arbitrary constants.

Note that equation (37) has analogous solutions (see e.g. [2]).

6 Reduction and exact solutions of the equation $\Box^2 u = \lambda u^{-3}$

It follows from the Theorems 2, 3 that the equation

$$\Box^2 u = \lambda u^{-3} \tag{38}$$

with $n = 1$ is invariant under the conformal group $C(1,1)$ with generators (26) and

$$D^{(1)} = x_0\frac{\partial}{\partial x_0} + x_1\frac{\partial}{\partial x_1} + u\frac{\partial}{\partial u},$$
$$K^{(1)}_\mu = 2x^\mu D^{(1)} - (x_\nu x^\nu)\frac{\partial}{\partial x_\mu}, \quad \mu,\nu = 0,1. \tag{39}$$

By analogy with the preceding Section solutions of the equation (38) are looked for in the form (35), where functions $\omega(x)$ and $f(x)$ are the solutions of the system (36), and what is more, the operator (34) belongs to the invariance algebra of the equation (38).

To obtain all the $C(1,1)$ non-conjugated Ansätze we use the one-dimensional inequivalent subalgebras of the conformal algebra $AC(1,1)$ adduced in [11].

Solving for each subalgebra equations (36) we arrive at the collection of $C(1,1)$-invariant Ansätze which are presented in the Table 3.

Table 3.

N	Algebra	Invariant variable ω	Ansatz
1°	$P_0 + K_0^{(1)}$	$\mathrm{arctg}(x_1 - x_0) + \mathrm{arctg}(x_1 + x_0)$	$u = \left((x_0 - x_1)^2 + 1\right)^{1/2} \times \left((x_0 + x_1)^2 + 1\right)^{1/2} \varphi(\omega)$
2°	$P_0 + K_0^{(1)} + \alpha(K_1^{(1)} - P_1)$, $0 < \alpha < 1$	$(\alpha - 1)\mathrm{arctg}(x_0 - x_1) + (\alpha + 1)\mathrm{arctg}(x_1 + x_0)$	$u = \left((x_0 - x_1)^2 + 1\right)^{1/2} \times \left((x_0 + x_1)^2 + 1\right)^{1/2} \varphi(\omega)$
3°	$P_0 + K_0^{(1)} + K_1^{(1)} - P_1$	$x_0 + x_1$	$u = \left((x_0 - x_1)^2 + 1\right)^{1/2} \times \varphi(\omega)$
4°	$2P_1 + K_0^{(1)} + K_1^{(1)}$	$x_0 + x_1 + \frac{1}{2}\ln\frac{1 + x_0 - x_1}{1 - x_0 + x_1}$	$u = \left((x_0 - x_1)^2 + 1\right)^{1/2} \times \varphi(\omega)$
5°	$2P_1 - K_0^{(1)} - K_1^{(1)}$	$x_0 + x_1 + \mathrm{arctg}(x_0 - x_1)$	$u = \left((x_0 - x_1)^2 + 1\right)^{1/2} \times \varphi(\omega)$
6°	$P_0 + K_0^{(1)} + K_1^{(1)} - P_1 - \beta(J_{01} + D^{(1)})$, $\beta > 0$	$\ln(x_0 + x_1) - \beta\,\mathrm{arctg}(x_1 - x_0)$	$u = \left((x_0 - x_1)^2 + 1\right)^{1/2} \times \left(x_0 + x_1\right)^{1/2} \varphi(\omega)$

We omit subalgebras not containing the conformal operator (39) since they were considered in the preceding Section.

Substituting Ansätze obtained in the equation (38) we get the following reduced ODE for a function $\varphi(\omega)$:

1° $\quad \varphi^{(4)} + 2\varphi^{(2)} + \varphi = \dfrac{\lambda}{16}\varphi^{-3};$

2° $\quad (\alpha^2 - 1)^2\varphi^{(4)} + 2(\alpha^2 + 1)\varphi^{(2)} + \varphi = \dfrac{\lambda}{16}\varphi^{-3};$

3° $\quad \varphi^{(2)} = \dfrac{\lambda}{16}\varphi^{-3};$

4° $\quad \varphi^{(4)} - \varphi^{(2)} = \dfrac{\lambda}{16}\varphi^{-3};$

$$5^\circ \quad \varphi^{(4)} + \varphi^{(2)} = \frac{\lambda}{16}\varphi^{-3};$$

$$6^\circ \quad 4\beta^2\varphi^{(4)} + (4-\beta^2)\varphi^{(2)} - \varphi = \frac{\lambda}{4}\varphi^{-3}.$$

The general solution of the equation 3° is of the form

$$\varphi = \pm\sqrt{\frac{(c_1\omega + c_2)^2}{c_1} + \frac{\lambda}{16c_1}}, \quad \varphi = \pm\sqrt{\frac{1}{2}\sqrt{-\lambda}\omega + c},$$

where c, c_1, c_2 are arbitrary constants, $c_1 \neq 0$.

Hence we obtain the following exact solutions of the equation (38):

$$1. \quad u = \pm\frac{1}{\sqrt{2}}\left(\frac{\lambda}{a_1}\right)^{1/4}\left|(x_0 + x_1 + a_2)^2 - a_1\right|^{1/2}\left|x_0 - x_1 + a_3\right|^{1/2},$$

$$2. \quad u = \pm\frac{1}{\sqrt{2}}\left(\frac{\lambda}{b_1}\right)^{1/4}\left|(x_0 - x_1 + b_2)^2 - b_1\right|^{1/2}\left|x_0 + x_1 + b_3\right|^{1/2},$$

$$3. \quad u = \pm\frac{1}{2}\left(\frac{\lambda}{c_1c_2}\right)^{1/4}\left|(x_0 - x_1 + c_3)^2 + c_1\right|^{1/2}\left|(x_0 + x_1 + c_4)^2 + c_2\right|^{1/2},$$

where a_i, b_i, c_j, $i = \overline{1,3}$, $j = \overline{1,4}$ are arbitrary constants.

Besides, the expression

$$u = \pm\lambda^{1/4}\left|(x_0 - x_1 + c_1)(x_0 + x_1 + c_2)\right|^{1/2}$$

(c_1, c_2 are arbitrary constants) was proved in the Section 4 to be the exact solution of the equation (38).

Conclusion

Thus, we have shown that the symmetry selection principle is a natural way of classification of physically admissible nonlinear biwave equations. Requiring an invariance with respect to the extended Poincaré group picks out very specific nonlinearities (3). And the demand of a conformal invariance yields, in fact, a unique nonlinear PDE (20), (21).

As equations obtained in this way admit broad Lie symmetry, one can apply the symmetry reduction procedure to find their exact solutions. An important part of the said procedure is a construction of special substitutions which reduce the equation under study to PDE with less number of independent variables. Given a subgroup classification of the equation under study, a procedure of construction of such substitutions is entirely algorithmic. Of course, there is no guarantee that the reduced equations can be solved explicitly. But our experience as well as a rich experience of other groups engaged in the field of group-theoretical, symmetry analysis of nonlinear partial differential equations evidence that it is almost always possible [2, 5, 14, 16]. The reason is that PDE obtained by means of reduction of some initial PDE admitting broad Lie symmetry also possess a hereditary symmetry. Moreover, in some exceptional cases this symmetry can be much more extensive than the one of the initial equation. An example is given in [15], where it is established that some equations obtained by means of reduction of the nonlinear Poincaré-invariant Dirac equation admit infinite-parameter symmetry groups. Since a maximal symmetry group of the initial equation is the ten-parameter Poincaré group, this symmetry is essentially new. The source of it is the *conditional symmetry* of the nonlinear Dirac equation [2, 15].

In the present paper we have applied the symmetry reduction procedure to reduce to ODE the fourth-order nonlinear biwave equations of the form (5) having two independent variables x_0, x_1 and to construct its explicit solutions. A problem of symmetry reduction of these equations has been completely solved in a sense that any solution of PDE (5) invariant under a subgroup of the conformal group $C(1,1)$ (which is a most extensive group that can be admitted by equation of the form (5)) is equivalent to one of the Ansätze given in the Tables 1–3. And what is more, these Ansätze can be applied to reduce any two-dimensional PDE, provided it is invariant under the Poincaré, extended Poincaré and conformal groups having the generators (2), (10)–(13). But it does not mean that *all* possibilities to reduce PDE (5) to ODE are exhausted. New reductions can be obtained by utilizing conditional symmetry of the biwave equation in the way as it has been done for a number of nonlinear mathematical physics equations in [2]. This problem is under investigation now.

An another interesting problem is to carry out symmetry reduction of the biwave equation in the four-dimensional Minkowski space-time. This work is now in progress and will be reported elsewhere.

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1. Heisenberg W., Einführung in die Einheitliche Feldtheorie der Elementarteilchen, Stuttgart, S. Hirzel Verlag, 1967.
2. Fushchych W.I., Shtelen W.M., Serov N.I., Symmetry analysis and exact solutions of equations of nonlinear mathematical physics, Dordrecht, Kluwer Academic Publishers, 1993.
3. Fushchych W.I., Serov N.I., *J. Phys. A: Math. Gen.*, 1983, **16**, 3645–3658.
4. Ibragimov N.H., Lie groups in some questions of mathematical physics, University Press, Novosibirsk, 1972.
5. Ovsiannikov L.V., Group analysis of differential equations, Moscow, Nauka, 1978.
6. Fushchych W.I., in Algebraic-Theoretical Studies in Mathematical Physics, Kiev, Institute of Mathematics, 1981, 6–11.
7. Bollini C.G., Giambia J.J., *J. Math. Phys.*, 1993, **34**, 610–621.
8. Fushchych W.I., Selehman M.A., *Dopovidi Acad. Sci. Ukrainy*, 1983, № 5, 21–24.
9. Winternitz P., Grundland A.M., Tuszyński J.A., *J. Math. Phys.*, 1987, **28**, 2194–2212.
10. Fedorchuk V.M., in Symmetry and Solutions of Nonlinear Equations of Mathematical Physics, Kiev, Institute of Mathematics, 1987, 73–76.
11. Fushchych W.I., Barannik L.F., Barannik A.F., Subgroup analysis of Galilei and Poincaré groups and reduction of nonlinear equations, Kiev, Naukova Dumka, 1991.
12. Serov N.I., in Algebraic-Theoretical Studies in Mathematical Physics, Kiev, Institute of Mathematics, 1981.
13. Svirshchevskii S.R., *Differential Equations*, 1993, **29**, 1538–1547.
14. Olver P., Applications of Lie groups to differential equations, New York, Springer, 1986.
15. Fushchych W.I.,Zhdanov R.Z., Nonlinear spinor equations: symmetry and exact solutions, Kiev, Naukova Dumka, 1992.
16. Winternitz P., in Partially Integrable Evolution Equations in Physics, Dordrecht, Kluwer, 1990, 515–567.

Symmetry and some exact solutions of non-linear polywave equations

W.I. FUSHCHYCH, O.V. ROMAN, R.Z. ZHDANOV

We have studied the maximal symmetry group admitted by the non-linear polywave equation $\Box^l u = F(u)$. In particular, we establish that equation in question admits the conformal group $C(1,n)$ if and only if $F(u) = \lambda e^u$, $n+1 = 2l$ or $F(u) = \lambda u^{(n+1+2l)/(n+1-2l)}$, $n+1 \neq 2l$. Symmetry reduction for the biwave equation $\Box^2 u = \lambda u^{-3}$ is carried out and some exact solutions are obtained.

Recently a number of works (see, e.g., [1, 2, 3]) have appeared pointing out the possibility to choose linear and non-linear polywave equations

$$\Box^l u = F(u) \qquad (1)$$

as possible mathematical models describing an uncharged scalar particle in quantum field theory.

Here $\Box^l = \Box(\Box^{l-1})$, $l \in \mathbb{N}$; $\Box = \partial^2_{x_0} - \partial^2_{x_1} - \cdots - \partial^2_{x_n}$ is d'Alembertian in $(n+1)$-dimensional pseudo-Euclidean space $\mathbb{R}(1,n)$ with metric tensor $g_{\mu\nu} = \mathrm{diag}(1,-1,\ldots,-1)$, $\mu,\nu = \overline{0,n}$; $F(u)$ is an arbitrary smooth function and $u = u(x)$ is a real function (the case $l = 1$, $n = 1$ has been studied earlier [4], that is why we put $l + n > 2$). In the following, a summation over the repeated indices from 0 to n is understood, rising and lowering of the vector indices is performed by means of the tensor $g_{\mu\nu}$, i.e. $x^\mu = g_{\mu\nu} x_\nu$.

But the fact that the non-linear partial differential equation (PDE) in question is of high order makes the prospects of studying such a model rather obscure. Using group properties of equation (1) seems to be the only way to get some non-trivial information about the said equation and its solutions. It occurs that PDE (1) admits wide symmetry group which, in fact, is the same as the one of the standard wave equation

$$\Box u = F(u). \qquad (2)$$

The main tool used is the infinitesimal Lie method (see, e.g., [5]). But an application of it to study of symmetry properties of equation (1) is by itself a non-trivial problem in the case $l > 1$. It should be emphasized that because of arbitrariness of the order (l) and of the number of independent variables (n) one can not apply symbolic manipulation programs [6, 7]. We have succeeded in constructing the maximal symmetry group admitted by equation (1) using the remarkable combinatorial properties of the prolongation formulae.

Theorem 1. *The maximal invariance group of PDE (1) with arbitrary smooth function $F(u)$ is the Poincaré group $P(1,n)$ generated by the operators*

$$P_\mu = \partial_{x_\mu}, \quad J_{\mu\nu} = x^\mu \partial_{x_\nu} - x^\nu \partial_{x_\mu}, \quad \mu,\nu = \overline{0,n}. \qquad (3)$$

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It is established below that the equation of the type (1) admitting the group, which is more extensive than the Poincaré group, is equivalent up to the change of variables to one of the following equations:

$$1.\ \Box^l u = \lambda_1 u^k, \qquad \lambda_1 \neq 0,\ k \neq 0, 1; \tag{4}$$

$$2.\ \Box^l u = \lambda_2 e^u, \qquad \lambda_2 \neq 0; \tag{5}$$

$$3. \Box^l u = \lambda_3 u, \qquad \lambda_3 \neq 0; \tag{6}$$

$$4.\ \Box^l u = 0. \tag{7}$$

Here λ_1, λ_2, λ_3, k are arbitrary constants.

Maximal invariance groups of the equations (4)–(7) are described by the following statements.

Theorem 2. *Equation (4) has the following symmetry:*

Case 1. $k \neq (n+1+2l)/(n+1-2l)$, $k \neq 0, 1$. *The maximal invariance group of (4) is the extended Poincaré group* $\widetilde{P}(1,n)$ *generated by the operators (3) and*

$$D = x_\mu \partial_{x_\mu} + \frac{2l}{1-k} u\partial_u.$$

Case 2. $k = (n+1+2l)/(n+1-2l)$, $n+1 \neq 2l$. *The maximal invariance group of (4) is the conformal group* $C(1,n)$ *generated by the operators (3) and operators*

$$\begin{aligned} D^{(1)} &= x_\mu \partial_{x_\mu} + \frac{(2l-n-1)}{2} u\partial_u, \\ K_\mu^{(1)} &= 2x^\mu D^{(1)} - (x_\nu x^\nu)\partial_{x_\mu}, \quad \mu, \nu = \overline{0, n}. \end{aligned} \tag{8}$$

Theorem 3. *Equation (5) has the following symmetry:*

Case 1. $n \neq 2l-1$. *The maximal invariance group of (5) is the extended Poincaré group* $\widetilde{P}(1,n)$ *generated by the operators (3) and*

$$D^{(2)} = x_\mu \partial_{x_\mu} - 2l\partial_u. \tag{9}$$

Case 2. $n = 2l-1$. *The maximal invariance group of (5) is the conformal group* $C(1,n)$ *generated by the operators (3) and operators*

$$K_\mu^{(2)} = 2x^\mu D^{(2)} - (x_\nu x^\nu)\partial_{x_\mu}, \quad \mu, \nu = \overline{0, n}. \tag{10}$$

Theorem 4. *The maximal invariance group of the equation (6) is generated by the operators (3) and*

$$Q_\infty = f(x)\partial_u, \quad I = u\partial_u,$$

where $f(x)$ *is an arbitrary solution of PDE (6).*

Theorem 5. *The maximal invariance group of the equation (7) is generated by the operators (3), (8) and*

$$Q_\infty = q(x)\partial_u, \quad I = u\partial_u,$$

where $q(x)$ *is an arbitrary solution of PDE (7).*

The proof of the Theorems 1–5 carried out by means of the infinitesimal algorithm of S. Lie [5] requires very cumbersome computations. That is why, we omit it.

An important consequence of the Theorems 1–5 is the following statement.

Theorem 6. *The non-linear PDE (1) is invariant under the conformal group $C(1,n)$ iff it is equivalent to the following*

$$1. \Box^l u = \lambda_1 u^{\frac{n+1+2l}{n+1-2l}}, \quad n+1 \neq 2l; \tag{11}$$

$$2. \ \Box^l u = \lambda_2 e^u, \quad n+1 = 2l. \tag{12}$$

Remark 1. Conformal invariance of the equation (11) was first ascertained in [8] and that of equation (12) was done in [3] by means of Baker–Campbell–Hausdorff formulae.

Assuming $l=1$ in (11) we obtain the well-known result [3]; that non-linear wave equation (2) admits the conformal group if it is equivalent to the PDE

$$\Box u = \lambda u^{\frac{n+3}{n-1}} \quad \text{when} \quad n \neq 1.$$

Remark 2. When $l=2$ it follows from the Theorem 6 that in the four-dimensional space $\mathbb{R}(1,3)$ there is only one $C(1,3)$-invariant equation

$$\Box^2 u = \lambda e^u.$$

One of the important applications of the Lie groups in mathematical physics is the finding exact solutions of non-linear PDE. To this end one has to construct so called invariant solutions [2, 3, 5] which reduce PDE under study to equations with less number of independent variables (in particular, to ordinary differential equations). Integrating these one gets exact solutions of the initial PDE. A procedure described is called symmetry (or group-theoretical) reduction of differential equations. Here we perform symmetry reduction of the conformally-invariant biwave equation in the two-dimensional space $\mathbb{R}(1,1)$:

$$\Box^2 u = \lambda u^{-3}. \tag{13}$$

Making use of inequivalent one-dimensional subalgebras of the conformal algebra $AC(1,1)$ [9] one can obtain the following $C(1,1)$-inequivalent Ansätze which reduce the equation (13) to ordinary differential equations. For each case the reduced equations are given:

1. $u = \varphi(\omega), \quad \omega = x_0 \ \text{or} \ \omega = x_1,$
   $$\varphi^{(4)} = \lambda \varphi^{-3};$$
2. $u = \varphi(\omega), \quad \omega = x_0^2 - x_1^2,$
   $$\varphi^{(4)}\omega^2 + 4\varphi^{(3)}\omega + 2\varphi^{(2)} = \frac{\lambda}{16}\varphi^{-3};$$
3. $u = (x_0 + x_1)^{1/2}\varphi(\omega), \quad \omega = x_0 - x_1,$
   $$\varphi^{(2)} = -\frac{\lambda}{4}\varphi^{-3};$$
4. $u = (x_0 + x_1)^{1/(\alpha+1)}\varphi(\omega), \quad \omega = (x_0 - x_1)(x_0 + x_1)^{(\alpha-1)/(\alpha+1)};$
   $$\varphi^{(4)}\omega^2 + 4\varphi^{(3)}\omega + \frac{(\alpha-2)(2\alpha-1)}{(\alpha-1)^2}\varphi^{(2)} = \frac{\lambda}{16}\frac{(\alpha+1)^2}{(\alpha-1)^2}\varphi^{-3}, \ \alpha > 1;$$

5. $u=\exp(x_0-x_1)\varphi(\omega), \quad \omega=\left(x_0+x_1+\frac{1}{2}\right)\exp\bigl(-2(x_0-x_1)\bigr),$

$\varphi^{(4)}\omega^2+4\varphi^{(3)}\omega+\frac{9}{4}\varphi^{(2)}=\frac{\lambda}{64}\varphi^{-3};$

6. $u=\bigl((x_0-x_1)^2+1\bigr)^{1/2}\varphi(\omega), \quad \omega=x_0+x_1,$

$\varphi^{(2)}=\frac{\lambda}{16}\varphi^{-3};$

7. $u=\bigl((x_0-x_1)^2+1\bigr)^{1/2}\varphi(\omega), \quad \omega=x_0+x_1+\arctan(x_0-x_1),$

$\varphi^{(4)}+\varphi^{(2)}=\frac{\lambda}{16}\varphi^{-3};$

8. $u=\bigl((x_0-x_1)^2+1\bigr)^{1/2}\varphi(\omega), \quad \omega=x_0+x_1+\frac{1}{2}\ln\frac{1+x_0-x_1}{1-x_0+x_1},$

$\varphi^{(4)}-\varphi^{(2)}=\frac{\lambda}{16}\varphi^{-3};$

9. $u=\bigl((x_0-x_1)^2+1\bigr)^{1/2}\bigl(x_0+x_1\bigr)^{1/2}\varphi(\omega),$

$\omega=\ln(x_0+x_1)-\beta\arctan(x_1-x_0),$

$4\beta^2\varphi^{(4)}+(4-\beta^2)\varphi^{(2)}-\varphi=\frac{\lambda}{4}\varphi^{-3}, \quad \beta>0;$

10. $u=\bigl((x_0-x_1)^2+1\bigr)^{1/2}\bigl((x_0+x_1)^2+1\bigr)^{1/2}\varphi(\omega),$

$\omega=(\gamma-1)\arctan(x_0-x_1)+(\gamma+1)\arctan(x_0+x_1),$

$(\gamma^2-1)^2\varphi^{(4)}+2(\gamma^2+1)\varphi^{(2)}+\varphi=\frac{\lambda}{16}\varphi^{-3}, \ 0\le\gamma<1.$

Integration of the reduced equations gives rise to exact solutions of the non-linear biwave equation (13). Here we present some exact solutions of this equation obtained with the use of Ansätze 3 and 6:

$$
\begin{aligned}
u&=\pm\lambda^{1/4}\bigl(x_0^2-x_1^2\bigr)^{1/2},\\
u&=\pm\frac{1}{\sqrt{2}}\left(\frac{\lambda}{c_1}\right)^{1/4}\bigl|(x_0-x_1)^2-c_1\bigr|^{1/2}(x_0+x_1)^{1/2},\\
u&=\pm\frac{1}{2}\left(\frac{\lambda}{c_2}\right)^{1/4}\bigl((x_0-x_1)^2+1\bigr)^{1/2}\bigl|(x_0+x_1)^2+c_2\bigr|^{1/2},
\end{aligned}
\tag{14}
$$

where c_1, c_2 are arbitrary constants.

Since the conformal group $C(1,1)$ is a maximal symmetry group of equation (13), formulae 1–10 give “maximal” information about its solutions which can be obtained within the framework of the Lie approach. It means that any solution invariant under a subgroup of the symmetry group of PDE (13) can be reduced by a transformation from the group $C(1,1)$ to one of the Ansätze 1–10.

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1. Bollini C.G., Giambia J.J., Arbitrary powers of d'Alembertians and the Huygens' principle, *J. Math. Phys.*, 1993, **34**, 610.
2. Fushchych W.I., Symmetry in problems of mathematical physics, in Algebraic-Theoretical Studies in Mathematical Physics, Kiev, Institute of Mathematics, 1981, 6.
3. Fushchych W.I., Shtelen W.M., Serov N.I., Symmetry analysis and exact solutions of equations of nonlinear mathematical physics, Dordrecht, Kluwer, 1993.
4. Fushchych W.I., Serov N.I., The symmetry and some exact solutions of the nonlinear many-dimensional d'Alembert, Liouville and eikonal equations, *J. Phys. A*, 1983, **16**, 3645.
5. Olver P., Applications of Lie groups to differential equations, New York, Springer, 1986.
6. Hereman W., Review of symbolic software for the computation of Lie symmetries of differential equations, *Euromath. Bull.*, 1994, **1**, 45.
7. Mansfield E.L., Clarkson P.A., Applications of the differential algebra package `diffgrob2` to reductions of PDE, in Proceedings of the Fourteenth IMACS World Congress on Computation and Applied Mathematics, Editor W.F. Ames (Georgia Inst. Tech.), 1994, **1**, 336.
8. Serov N.I., Conformal symmetry of nonlinear wave equations, in Algebraic-Theoretical Studies in Mathematical Physics, Kiev, Institute of Mathematics, 1981, 59.
9. Fushchych W.I., Barannik L.F., Barannik A.F., Subgroup analysis of the groups of Galilei and Poincaré and reduction of nonlinear equations, Kiev, Naukova Dumka, 1991.

W.I. Fushchych, Scientific Works 2003, Vol. 5, 420–430.

Symmetry properties, reduction and exact solutions of biwave equations

W.I. FUSHCHYCH, R.Z. ZHDANOV, O.V. ROMAN

We have studied symmetry properties of the biwave equations $\Box^2 u = F(u)$ and the systems of wave equations which are equivalent to them. Reduction of the nonlinear biwave equations with the use of subalgebras of the extended Poincaré algebra $\widetilde{AP}(1,1)$ and the conformal algebra $C(1,1)$ was carried out. Some exact solutions of these equations were obtained.

It was suggested in [1] to describe different physical processes with the help of nonlinear partial equations of high order, namely

$$\Box^l u = F\Big(u, \frac{\partial u}{\partial x_\mu}\frac{\partial u}{\partial x^\mu}\Big). \tag{1}$$

Here and further $\Box = \partial^2/\partial x_0 - \partial^2/\partial x_1 - \cdots - \partial^2/\partial x_n$ is d'Alembertian in $(n+1)$-dimensional pseudo-Euclidean space $\mathbb{R}(1,n)$ with metric tensor $g_{\mu\nu} = \mathrm{diag}(1,-1,\ldots,-1)$, $\mu,\nu = \overline{0,n}$; $\Box^l = \Box(\Box^{l-1})$, $l \in \mathbb{N}$; $x_\mu = x^\nu g_{\mu\nu}$; $F(\cdot,\cdot)$ is an arbitrary smooth function; $u = u(x)$ is a real function; the summation over the repeated indices from 0 to n is understood.

Equations (1) were considered from different points of view in [2, 3, 4], where the pseudodifferential equations of type (1) were also studied (in this case l is fractional or negative).

Assuming $l = 1$ and $F = F(u)$ in (1) we obtain the standard wave equation

$$\Box u = F(u) \tag{2}$$

which describes a scalar spinless uncharged particle in quantum field theory. Symmetry properties of equation (2) were studied in [4, 5, 6] and wide classes of its exact solutions with certain concrete values of the function $F(u)$ were obtained in [4, 5, 7, 8, 9].

In this paper we restrict ourselves by considering the biwave equation

$$\Box^2 u = F(u) \tag{3}$$

which is one of the simplest equations of type (1) of high order ($l = 2$, $F = F(u)$).

1 Symmetry classification of the biwave equation

In order to carry out a symmetry classification of equation (3) we shall establish at first the maximal transformation group admitted by equation (3) provided $F(u)$ is an arbitrary function. After that we shall determine all the functions $F(u)$ when equation (3) admits more extended symmetry.

Results of symmetry classification of equation (3) are cited in the following statements.

Preprint LiTH-MAT-R-95-20, Department of Mathematics, Linköping University, Sweden, 16 p.

Lemma 1 *The maximal invariance group of equation (3) with an arbitrary function $F(u)$ is the Poincaré group $P(1,n)$ generated by the operators*

$$P_\mu = \frac{\partial}{\partial x_\mu}, \quad J_{\mu\nu} = x^\mu \frac{\partial}{\partial x_\nu} - x^\nu \frac{\partial}{\partial x_\mu}, \quad \mu,\nu = \overline{0,n}. \tag{4}$$

Theorem 1 *All the equations of type (3) admitting more extended invariance algebra than the Poincaré algebra $AP(1,n)$ are equivalent one of the following:*

$$1.\ \Box^2 u = \lambda_1 u^k, \quad \lambda_1 \neq 0,\ k \neq 0,1; \tag{5}$$

$$2.\ \Box^2 u = \lambda_2 e^u, \quad \lambda_2 \neq 0; \tag{6}$$

$$3.\ \Box^2 u = \lambda_3 u, \quad \lambda_3 \neq 0; \tag{7}$$

$$4.\ \Box^2 u = 0. \tag{8}$$

Here λ_1, λ_2, λ_3 are arbitrary constants.

Theorem 2 *The symmetry of the equations (5)–(8) is described in the following way:*

1. (a) The maximal invariance group of equation (5) when $k \neq (n+5)/(n-3)$, $k \neq 0,1$ is the extended Poincaré group $\widetilde{P}(1,n)$ generated by the operators (4) and

$$D = x_\mu \frac{\partial}{\partial x_\mu} + \frac{4}{1-k} u \frac{\partial}{\partial u}.$$

(b) The maximal invariance group of equation (5) when $k = (n+5)/(n-3)$, $n \neq 3$ is the conformal group $C(1,n)$ generated by the operators (4) and

$$\begin{aligned} D^{(1)} &= x_\mu \frac{\partial}{\partial x_\mu} + \frac{3-n}{2} u \frac{\partial}{\partial u}, \\ K_\mu^{(1)} &= 2x^\mu D^{(1)} - (x_\nu x^\nu) \frac{\partial}{\partial x_\mu}. \end{aligned} \tag{9}$$

2. (a) The maximal invariance group of equation (6) when $n \neq 3$ is the extended Poincaré group $\widetilde{P}(1,n)$ generated by the operators (4) and

$$D^{(2)} = x_\mu \frac{\partial}{\partial x_\mu} - 4\frac{\partial}{\partial u}. \tag{10}$$

(b) The maximal invariance group of equation (6) when $n = 3$ is the conformal group $C(1,n)$ generated by the operators (4) and

$$K_\mu^{(2)} = 2x^\mu D^{(2)} - (x_\nu x^\nu) \frac{\partial}{\partial x_\mu}. \tag{11}$$

3. The maximal invariance group of equation (7) is generated by the operators (4) and

$$Q = h(x) \frac{\partial}{\partial u}, \quad I = u \frac{\partial}{\partial u},$$

where $h(x)$ is an arbitrary solution of equation (7).

4. The maximal invariance group of equation (8) is generated by the operators (4), (9) and

$$Q = q(x)\frac{\partial}{\partial u}, \quad I = u\frac{\partial}{\partial u},$$

where $q(x)$ is an arbitrary solution of equation (8).

The proof of Lemma 1 and Theorems 1, 2 is carried out by means of the infinitesimal algorithm of S. Lie [4, 10]. Since it requires very cumbersome computations we only give a general scheme of the proof.

In the Lie approach the infinitesimal operator of equation (3) invariance group is of the form

$$X = \xi^\mu(x,u)\frac{\partial}{\partial x_\mu} + \eta(x,u)\frac{\partial}{\partial u}. \tag{12}$$

The invariance criterion of equation (3) under group generated by the operators (12) is

$$\underset{4}{X}(\Box^2 u - F(u))\Big|_{\Box^2 u = F(u)} = 0, \tag{13}$$

where $\underset{4}{X}$ is the 4-th prolongation of the operator X.

Splitting equation (13) with respect to the independent variables, we come to the system of partial differential equations for functions $\xi^\mu(x,u)$ and $\eta(x,u)$:

$$\begin{aligned} &\xi^\mu_u = 0, \quad \eta_{uu} = 0, \quad \mu = \overline{0,n}, \\ &\xi^i_0 = \xi^0_i, \quad \xi^i_j = -\xi^j_i, \quad i \neq j, \ i,j = \overline{1,n}, \\ &\xi^0_0 = \xi^1_1 = \cdots = \xi^n_n, \\ &2\eta_{\nu u} = (3-n)\xi^\nu_{00}, \quad \nu = \overline{0,n}, \end{aligned} \tag{14}$$

$$\Box^2\eta - \eta F'(u) + F(u)(\eta_u - 4\xi^0_0) = 0. \tag{15}$$

Besides, when $n = 1$, there are additional equations:

$$\eta_{00u} = 0, \quad \eta_{01u} = 0, \tag{16}$$

that do not follow from equations (14) and (15).

In the above formulae we use the notations $\xi^\mu_\nu = \partial\xi^\mu/\partial x_\nu$, $\eta_\mu = \partial\eta/\partial x_\mu$ and so on.

System (14) is a system of Killing equations. The general solution of equations (14), (16) is of the form:

$$\begin{aligned} &\xi^\nu = 2x^\nu x_\mu c^\mu - x_\mu x^\mu c^\nu + b_{\nu\mu}x^\mu + dx_\nu + a_\nu, \\ &\eta = ((3-n)c^\mu x_\mu + p)u + \varkappa(x), \end{aligned} \tag{17}$$

where c^μ, $b_{\nu\mu} = -b_{\mu\nu}$, d, a_ν, p are arbitrary constants, $\varkappa(x)$ is an arbitrary smooth function.

Substituting (17) into the classifying equation (15) and splitting it with respect to u we obtain statements of Lemma 1 and Theorems 1, 2 according to the form of $F(u)$.

It follows from the statements proved that the equation of type (1) is invariant under the extended Poincaré group $\widetilde{P}(1,n)$ iff it is equivalent one of equations (5), (6) or (8). Let us note that the analogous result was obtained for the wave equations (2) in [5].

The following statement also is the consequence of the Theorems but since it is important we adduce it as a Theorem.

Theorem 3 *Equation (3) is invariant under the conformal group $C(1,n)$ iff it is equivalent to the following:*

$$1.\ \Box^2 u = \lambda_1 u^{(n+5)/(n-3)}, \quad n \neq 3; \tag{18}$$

$$2.\ \Box^2 u = \lambda_2 e^u, \quad n = 3. \tag{19}$$

Let us note that conformal invariance of equation (18) was first ascertained in [11] and that of equation (19) was done in [4] by means of Baker–Campbell–Hausdorff formulae.

In conclusion of the Section let us emphasize an important property of the linear biwave equation (8), when $n = 3$, which is the consequence of Theorems 2 and 3.

Corollary *There exist two nonequivalent representations of the Lie algebra of the conformal group $C(1,n)$ on the set of solutions of equation (8) [1, 3, 4]:*

$$\begin{aligned} 1.\quad & P^{(1)}_\mu = P_\mu, \quad J^{(1)}_{\mu\nu} = J_{\mu\nu}, \\ & D^{(1)} = x_\mu \frac{\partial}{\partial x_\mu}, \quad K^{(1)}_\mu = 2x^\mu D^{(1)} - (x_\nu x^\nu)\frac{\partial}{\partial x_\mu}; \\ 2.\quad & P^{(2)}_\mu = P_\mu, \quad J^{(2)}_{\mu\nu} = J_{\mu\nu}, \\ & D^{(2)} = x_\mu \frac{\partial}{\partial x_\mu} + \frac{\partial}{\partial u}, \quad K^{(2)}_\mu = 2x^\mu D^{(2)} - (x_\nu x^\nu)\frac{\partial}{\partial x_\mu}, \end{aligned}$$

where the operators P_μ, $J_{\mu\nu}$ are determined in (4).

2 Symmetry classification of system of wave equations

Introducing a new variable $v = \Box u$ in (3) we get the system of partial differential equations

$$\begin{aligned} &\Box u = v, \\ &\Box v = F(u), \end{aligned} \tag{20}$$

which is equivalent to the biwave equation (3).

Symmetry properties of the system (20) are investigated by analogy with the previous Section. So we only formulate statements analogous to the preceding ones without proving them.

Lemma 2 *The maximal invariance group of the system (20) with an arbitrary function $F(u)$ is the Poincaré group $P(1,n)$ generated by the operators (4).*

Theorem 4 *All the systems of type (20) admitting more extended invariance algebra than the Poincaré algebra $AP(1,n)$ are equivalent one of the following:*

$$1.\quad \begin{aligned} &\Box u = v,\\ &\Box v = \lambda_1 u^k, \quad \lambda_1 \neq 0,\ k \neq 0,1; \end{aligned} \tag{21}$$

$$2.\quad \begin{aligned} &\Box u = v,\\ &\Box v = \lambda_2 u, \quad \lambda_2 \neq 0; \end{aligned} \tag{22}$$

$$3.\quad \begin{aligned} &\Box u = v,\\ &\Box v = 0. \end{aligned} \tag{23}$$

Theorem 5 *The symmetries of the systems (21)–(23) is described in the following way:*

1. The maximal invariance group of the system (21) is the extended Poincaré group $\widetilde{P}(1,n)$ generated by the operators (4) and

$$D = x_\mu \frac{\partial}{\partial x_\mu} + \frac{4}{1-k} u \frac{\partial}{\partial u} + \frac{2(1+k)}{1-k} v \frac{\partial}{\partial v}.$$

2. The maximal invariance group of the system (22) is generated by the operators (4) and

$$Q_1 = u\frac{\partial}{\partial u} + v\frac{\partial}{\partial v}, \quad Q_2 = v\frac{\partial}{\partial u} + \lambda_2 u \frac{\partial}{\partial v}, \quad Q_3 = h_1(x)\frac{\partial}{\partial u} + h_2(x)\frac{\partial}{\partial v},$$

where $(h_1(x), h_2(x))$ is an arbitrary solution of the system (22).

3. The maximal invariance group of the system (23) is generated by the operators (4) and

$$D = x_\mu \frac{\partial}{\partial x_\mu} + 2u\frac{\partial}{\partial u}, \quad Q_1 = u\frac{\partial}{\partial u} + v\frac{\partial}{\partial v},$$
$$Q_2 = v\frac{\partial}{\partial u}, \quad Q_3 = q_1(x)\frac{\partial}{\partial u} + q_2(x)\frac{\partial}{\partial v},$$

where $(q_1(x), q_2(x))$ is an arbitrary solution of the system (23).

It follows from the foregoing statements that unlike the biwave equations, the extended Poincaré group $\widetilde{P}(1,n)$ is the invariance group of the system (20) only in two cases, namely, when (20) is equivalent to (21) or (23). Moreover, the system (20) is not invariant under the conformal group for any functions $F(u)$. Therefore, in the class of Lie operators, the invariance algebras of the biwave equations and the corresponding systems of wave equations are essentially different.

3 Reduction and exact solutions of the equation $\Box^2 u = \lambda e^u$

As follows from Theorem 2 the maximal invariance group of the equation (6), when $n = 1$ is the extended Poincaré group $\widetilde{P}(1,1)$ with generators

$$P_0 = \frac{\partial}{\partial x_0}, \quad P_1 = \frac{\partial}{\partial x_1}, \quad J_{01} = x^0\frac{\partial}{\partial x_1} - x^1\frac{\partial}{\partial x_0}, \tag{24}$$

$$D^{(2)} = x_0\frac{\partial}{\partial x_0} + x_1\frac{\partial}{\partial x_1} - 4\frac{\partial}{\partial u}. \tag{25}$$

It is known that if an equation admits the symmetry operator

$$X = \xi^\mu(x)\frac{\partial}{\partial x_\mu} + \eta(x)\frac{\partial}{\partial u} \tag{26}$$

then its solutions can be found in the form [4]:

$$u(x) = \varphi(\omega) + g(x). \tag{27}$$

For the substitution (27) to be an ansatz for the equation with the symmetry operator (26), the functions $\omega(x)$ and $g(x)$ are to satisfy the following conditions:

$$\xi^\mu(x)\frac{\partial\omega}{\partial x_\mu} = 0, \quad \xi^\mu(x)\frac{\partial g(x)}{\partial x_\mu} = \eta(x).$$

To obtain all the $\widetilde{P}(1,1)$-nonequivalent ansatzes (27) we have to describe all the nonequivalent one-dimensional subalgebras of the Lie algebra $\widetilde{AP}(1,1)$ spanned by the operators (24) and (25) (see [4, 9]). In the paper we make use of classification given in [9] and omitting rather cumbersome computations we write $\widetilde{P}(1,1)$-nonequivalent ansatzes in Table 1.

Table 1.

N	Algebra	Invariant variables ω	Ansatz
1°	$D - J_{01}$	$x_0 + x_1$	$u = \varphi(\omega) - 2\ln(x_0 - x_1)$
2°	$D + \alpha J_{01},\ \alpha \neq -1$	$(1+\alpha)\ln(x_1 - x_0) - (1-\alpha)\ln(x_0 + x_1)$	$u = \varphi(\omega) - \dfrac{4}{\alpha+1}\ln(x_0 + x_1)$
3°	$D - J_{01} + P_0$	$\ln(x_0 - x_1 + 1/2) - 2(x_0 + x_1)$	$u = \varphi(\omega) - 2\ln(x_0 - x_1 + 1/2)$
4°	J_{01}	$x_0^2 - x_1^2$	$u = \varphi(\omega)$
5°	P_0	x_1	$u = \varphi(\omega)$
6°	$P_0 + P_1$	$x_0 - x_1$	$u = \varphi(\omega)$

Remark. Inequivalent subalgebras listed in Table 1 are built by taking account of the obvious fact that equation (6) is invariant under the transformations of the form:

$$\begin{aligned} x_0' &\to x_0, \\ x_1' &\to -x_1; \end{aligned} \quad \text{and} \quad \begin{aligned} x_0' &\to x_1, \\ x_1' &\to x_0. \end{aligned} \tag{28}$$

Substituting ansatzes obtained in (6) we get the following equations for the function $\varphi(\omega)$:

1° $\quad 0 = \lambda e^\varphi,$

2° $\quad \varphi^{(4)}(\alpha^2-1)^2 + 2\varphi^{(3)}\alpha(1-\alpha^2) - \varphi^{(2)}(1-\alpha^2) = \dfrac{\lambda}{16}\exp\left(\varphi + \dfrac{2\omega}{\alpha+1}\right),$

3° $\quad \varphi^{(4)} - \varphi^{(3)} = \dfrac{\lambda}{64}e^\varphi,$

4° $\quad \varphi^{(4)}\omega^2 + 4\varphi^{(3)}\omega + 2\varphi^{(2)} = \dfrac{\lambda}{16}e^\varphi,$

5° $\quad \varphi^{(4)} = \lambda e^\varphi,$

6° $\quad 0 = \lambda e^\varphi.$

Equation 5° has the partial solution

$$\varphi = \ln\left(\frac{24}{\lambda}(\omega + c)^{-4}\right), \quad \lambda > 0,$$

that leads us to the following exact solutions of equation (6):

$$\begin{aligned} u &= \ln\left(\frac{24}{\lambda}(x_0 + c_1)^{-4}\right), \quad \lambda > 0, \\ u &= \ln\left(\frac{24}{\lambda}(x_1 + c_2)^{-4}\right), \quad \lambda > 0. \end{aligned} \tag{29}$$

Here c, c_1, c_2 are arbitrary constants. This solutions are invariant under the operators P_0 and P_1 accordingly.

To finish the Section let us note that the solutions (29) can be obtained by making use of the ansatz in Liouville form [4]:

$$u = \ln\left\{\frac{24}{\lambda}\frac{\Big(\dot\varphi_1(\omega_1)\dot\varphi_2(\omega_2)\Big)^2}{(\varphi_1(\omega_1) + \varphi_2(\omega_2))^4}\right\}, \quad \omega_1 = x_0 + x_1, \quad \omega_2 = x_0 - x_1,$$

which reduces equation (6) to one of the following systems:

$$1. \quad \ddot\varphi_1 = 0, \quad \ddot\varphi_2 = 0;$$

$$2. \quad \ddot\varphi_1 = \frac{2\dot\varphi_1^2}{\varphi_1}, \quad \ddot\varphi_2 = \frac{2\dot\varphi_2^2}{\varphi_2}.$$

Here $\dot\varphi$ and $\ddot\varphi$ mean the first derivative and the second one of the corresponding argument.

Finding the general solution of the systems we get the following exact solutions of equation (6):

$$u = \ln\left(\frac{24}{\lambda}\frac{(a^2 - b^2)^2}{(ax_0 + bx_1 + c)^4}\right), \tag{30}$$

where a, b, c are arbitrary constants.

Solution (30) can be obtained from (29) by the transformations of the extended Poincaré group with the generators (24) and (25).

4 Reduction and exact solutions of the equation $\Box^2 u = \lambda u^k$

It follows from Theorem 2 that when $n = 1$ the equation (5) is invariant under the extended Poincaré group $\widetilde{P}(1,1)$ with the generators (24) and

$$D = x_0\frac{\partial}{\partial x_0} + x_1\frac{\partial}{\partial x_1} + \frac{4}{1-k}u\frac{\partial}{\partial u}. \tag{31}$$

If an equation admits the symmetry operator

$$X = \xi^\mu(x)\frac{\partial}{\partial x_\mu} + \eta(x)u\frac{\partial}{\partial u} \tag{32}$$

then its solutions can be found in the form [4]:

$$u(x) = f(x)\varphi(\omega) \tag{33}$$

provided functions $\omega(x)$ and $f(x)$ satisfy the following system:

$$\xi^\mu(x)\frac{\partial\omega}{\partial x_\mu} = 0, \quad \xi^\mu(x)\frac{\partial f(x)}{\partial x_\mu} = \eta(x)f(x). \tag{34}$$

With an allowance for invariance of equation (5) under the changes of variables (28) we write $\widetilde{P}(1,1)$-nonequivalent ansatzes of the form (33) in Table 2.

Table 2.

N	Algebra	Invariant variables ω	Ansatz
1°	$D - J_{01}$	$x_0 + x_1$	$u = (x_0 - x_1)^{\frac{2}{1-k}}\varphi(\omega)$
2°	$D + \alpha J_{01},\ \alpha \neq -1$	$(x_0 - x_1)(x_0 + x_1)^{\frac{\alpha-1}{\alpha+1}}$	$u = (x_0 + x_1)^{\frac{4}{(1-k)(\alpha+1)}}\varphi(\omega)$
3°	$D + J_{01} + P_0$	$(x_0 + x_1 + \frac{1}{2})\times$ $\times\exp\Big(2(x_1 - x_0)\Big)$	$u = \exp\Big(\frac{4}{k-1}(x_1 - x_0)\Big)\varphi(\omega)$
4°	J_{01}	$x_0^2 - x_1^2$	$u = \varphi(\omega)$
5°	P_0	x_1	$u = \varphi(\omega)$
6°	$P_0 + P_1$	$x_0 + x_1$	$u = \varphi(\omega)$

Let us note that analogous ansatzes were obtained in [4] for the nonlinear wave equation

$$\Box u = \lambda u^k. \tag{35}$$

Substituting the ansatzes obtained to equation (5) we get the following equations for the function $\varphi(\omega)$:

$$1^\circ \quad \frac{1+k}{(1-k)^2}\varphi^{(2)} = \frac{\lambda}{32}\varphi^k,$$

$$2^\circ \quad (\alpha-1)^2\varphi^{(4)}\omega^2 + 2(\alpha-1)(\alpha+1)^2\left(\frac{3k+1}{1-k} + 2\alpha\right)\omega\varphi^{(3)} + $$
$$+ 2\left(\alpha^2 - 4\alpha + 3 + \frac{6\alpha-10}{1-k} + \frac{8}{(1-k)^2}\right)\varphi^{(2)} = \frac{\lambda}{16}(\alpha+1)^2\varphi^k,$$

$$3^\circ \quad \varphi^{(4)}\omega^2 + \frac{5k-1}{k-1}\varphi^{(3)}\omega + \frac{4k^2}{(1-k)^2}\varphi^{(2)} = \frac{\lambda}{64}\varphi^k,$$

$$4^\circ \quad \varphi^{(4)}\omega^2 + 4\varphi^{(3)}\omega + 2\varphi^{(2)} = \frac{\lambda}{16}\varphi^k,$$

$$5^\circ \quad \varphi^{(4)} = \lambda\varphi^k,$$

$$6^\circ \quad \lambda\varphi^k = 0.$$

Equations 1°, 2°, 4° have the partial solutions of the form:

$$\varphi = \left(\frac{64}{\lambda}\frac{(k+1)^2}{(k-1)^4}\right)^{\frac{1}{k-1}}\omega^{-\frac{2}{k-1}}, \quad k \neq -1,$$

and equation 5° has the partial solution of the form

$$\varphi = \left(\frac{8}{\lambda}\frac{(k+1)(k+3)(3k+1)}{(k-1)^4}\right)^{\frac{1}{k-1}} \omega^{-\frac{4}{k-1}}, \quad k \neq -1, -3, -\frac{1}{3}$$

which lead us to the following solutions of equation (5):

$$u = \left(\frac{64}{\lambda}\frac{(k+1)^2}{(k-1)^4}\right)^{\frac{1}{k-1}} \Big((x_0+x_1+c_1)(x_0-x_1+c_2)\Big)^{-\frac{2}{k-1}}, \quad k \neq -1,$$
$$u = \left(\frac{8}{\lambda}\frac{(k+1)(k+3)(3k+1)}{(k-1)^4}\right)^{\frac{1}{k-1}} (x_0+c_3)^{\frac{4}{1-k}}, \quad k \neq -1, -3, -\frac{1}{3},$$
$$u = \left(\frac{8}{\lambda}\frac{(k+1)(k+3)(3k+1)}{(k-1)^4}\right)^{\frac{1}{k-1}} (x_1+c_4)^{\frac{4}{1-k}}, \quad k \neq -1, -3, -\frac{1}{3},$$

where c_1, c_2, c_3, c_4 are arbitrary constants.

Note that equation (35) has analogous solutions (see [4]).

5 Reduction and exact solutions of the equation $\Box^2 u = \lambda u^{-3}$

It follows from Theorems 2 and 3 that when $n = 1$ the equation

$$\Box^2 u = \lambda u^{-3} \tag{36}$$

is invariant under the conformal group $C(1,1)$ with the generators (24) and

$$D^{(1)} = x_0\frac{\partial}{\partial x_0} + x_1\frac{\partial}{\partial x_1} + u\frac{\partial}{\partial u},$$
$$K_\mu^{(1)} = 2x^\mu D^{(1)} - (x_\nu x^\nu)\frac{\partial}{\partial x_\mu}, \quad \mu, \nu = 0, 1. \tag{37}$$

By analogy with the previous Section solutions of equation (36) can be found in the form (33) where functions $\omega(x)$ and $f(x)$ are the solutions of the system (34) provided the operator (32) belongs to the invariance algebra of equation (36).

To obtain all the $C(1,1)$–nonequivalent ansatzes we use the one-dimensional nonequivalent subalgebras of the conformal algebra $AC(1,1)$ adduced in [9].

Omitting rather cumbersome computations and taking account of equation (36) being invariant under the changes of variables (28) we write nonequivalent ansatzes in Table 3.

We omit subalgebras not containing conformal the operator (37) since they were considered in the previous Section.

Substituting ansatzes obtained in (36) we get the following equations for the function $\varphi(\omega)$:

$1^\circ \quad \varphi^{(4)} + 2\varphi^{(2)} + \varphi = \dfrac{\lambda}{16}\varphi^{-3};$

$2^\circ \quad (\alpha^2-1)^2\varphi^{(4)} + 2(\alpha^2+1)\varphi^{(2)} + \varphi = \dfrac{\lambda}{16}\varphi^{-3};$

Table 3.

N	Algebra	Invariant variables ω	Ansatz
1°	$P_0 + K_0^{(1)}$	$\mathrm{arctg}(x_1 - x_0) + \mathrm{arctg}(x_1 + x_0)$	$u = \left((x_0 - x_1)^2 + 1\right)^{1/2} \times \left((x_0 + x_1)^2 + 1\right)^{1/2} \varphi(\omega)$
2°	$P_0 + K_0^{(1)} + \alpha(K_1^{(1)} - P_1)$ $0 < \alpha < 1$	$(\alpha - 1)\mathrm{arctg}(x_0 - x_1) + (\alpha + 1)\mathrm{arctg}(x_1 + x_0)$	$u = \left((x_0 - x_1)^2 + 1\right)^{1/2} \times \left((x_0 + x_1)^2 + 1\right)^{1/2} \varphi(\omega)$
3°	$P_0 + K_0^{(1)} + K_1^{(1)} - P_1$	$x_0 + x_1$	$u = \left((x_0 - x_1)^2 + 1\right)^{1/2} \times \varphi(\omega)$
4°	$2P_1 + K_0^{(1)} + K_1^{(1)}$	$x_0 + x_1 + \frac{1}{2} \ln \frac{1 + x_0 - x_1}{1 - x_0 + x_1}$	$u = \left((x_0 - x_1)^2 + 1\right)^{1/2} \times \varphi(\omega)$
5°	$2P_1 - K_0^{(1)} - K_1^{(1)}$	$x_0 + x_1 + \mathrm{arctg}(x_0 - x_1)$	$u = \left((x_0 - x_1)^2 + 1\right)^{1/2} \times \varphi(\omega)$
6°	$P_0 + K_0^{(1)} + K_1^{(1)} - P_1 - \beta(J_{01} + D^{(1)}),\ \beta > 0$	$\ln(x_0 + x_1) - \beta\mathrm{arctg}(x_1 - x_0)$	$u = \left((x_0 - x_1)^2 + 1\right)^{1/2} \times \left(x_0 + x_1\right)^{1/2} \varphi(\omega)$

$$3^\circ \quad \varphi^{(2)} = \frac{\lambda}{16}\varphi^{-3};$$
$$4^\circ \quad \varphi^{(4)} - \varphi^{(2)} = \frac{\lambda}{16}\varphi^{-3};$$
$$5^\circ \quad \varphi^{(4)} + \varphi^{(2)} = \frac{\lambda}{16}\varphi^{-3};$$
$$6^\circ \quad 4\beta^2\varphi^{(4)} + (4 - \beta^2)\varphi^{(2)} - \varphi = \tfrac{\lambda}{4}\varphi^{-3}.$$

The general solution of equation 3° is of the form

$$\varphi = \pm\sqrt{\frac{(c_1\omega + c_2)^2}{c_1} + \frac{\lambda}{16c_1}}, \quad c_1 \neq 0;$$
$$\varphi = \pm\sqrt{\frac{1}{2}\sqrt{-\lambda}\omega + c},$$

where c, c_1, c_2 are arbitrary constants.

Hence we obtain the following exact solutions of equation (36):

$$1. \quad u = \pm\frac{1}{\sqrt{2}}\left(\frac{\lambda}{a_1}\right)^{1/4} \left|(x_0 + x_1 + a_2)^2 - a_1\right|^{1/2} \left|x_0 - x_1 + a_3\right|^{1/2},$$
$$2. \quad u = \pm\frac{1}{\sqrt{2}}\left(\frac{\lambda}{b_1}\right)^{1/4} \left|(x_0 - x_1 + b_2)^2 - b_1\right|^{1/2} \left|x_0 + x_1 + b_3\right|^{1/2},$$
$$3. \quad u = \pm\frac{1}{2}\left(\frac{\lambda}{c_1 c_2}\right)^{1/4} \left|(x_0 - x_1 + c_3)^2 + c_1\right|^{1/2} \left|(x_0 + x_1 + c_4)^2 + c_2\right|^{1/2},$$

where a_i, b_i, c_j, $i = \overline{1,3}$, $j = \overline{1,4}$ are arbitrary constants.

Besides, the expression

$$u = \pm\lambda^{1/4}\big|(x_0 - x_1 + c_1)(x_0 + x_1 + c_2)\big|^{1/2}$$

(c_1, c_2 are arbitrary constants) was proved in Section 4 to be the exact solution of equation (36).

In conclusion let us note that we can obtain the same solutions using the following ansatz

$$u = \varphi_1(\omega_1)\varphi_2(\omega_2), \quad \omega_1 = x_0 + x_1, \quad \omega_2 = x_0 - x_1,$$

which reduces equation (36) to the system of ordinary differential equations for the unknown functions $\varphi_1(\omega_1)$ and $\varphi_2(\omega_2)$, namely

$$\ddot{\varphi}_1 = \frac{c}{4}\varphi_1^{-3}, \qquad \ddot{\varphi}_2 = \frac{\lambda}{4c}\varphi_2^{-3}, \tag{38}$$

where c is an arbitrary constant.

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1. Fushchych W.I., Symmetry in problems of mathematical physics, in Algebraic-Theoretical Studies in Mathematical Physics, Kiev, Inst. of Math., 1981, 6.
2. Bollini C.G., Giambia J.J., Arbitrary powers of d'Alembertians and the Huygens' principle, *J. Math. Phys.*, 1993, **34**, № 2, 610–621.
3. Fushchych W.I., Selehman M.A., Integro-differential equations invariant under group of Galilei, Poincare, Schrödinger and conformal group, *Proceedings of the Academy of Sciences of Ukrainy*, 1983, № 5, 21–24.
4. Fushchych W., Shtelen W., Serov N., Symmetry analysis and exact solutions of equations of nonlinear mathematical physics, Dordrecht, Kluwer Academic Publishers, 1993, 436 p.
5. Fushchych W.I., Serov N.I., The symmetry and some exact solutions of the nonlinear many-dimensional Liouville, d'Alembert and eikonal equations, *J. Phys. A: Math. Gen.*, 1983, № 16, 3645–3658.
6. Ibragimov N.H., Lie groups in some questions of mathematical physics, Novosibirsk, Novosibirsk University, 1972, 200 p.
7. Winternitz P., Grundland A.M., Tuszynski J.A., Exact solutions of the multidimensional classical Φ^6 field equations obtained by symmetry reduction, *J. Math. Phys.*, 1987, **28**, № 9, 2194–2212.
8. Fedorchuk V.M., On reduction and some exact solutions of nonlinear wave equation, in Symmetry and Solutions of Nonlinear Equations of Mathematical Physics, Kiev, Inst. of Math., 1987, 73–76.
9. Fushchych W.I., Barannik L.F., Barannik A.F., Subgroup analysis of Galilei and Poincaré groups and reduction of nonlinear equations, Kiev, Naukova Dumka, 1991, 304 p.
10. Ovsiannikov L.V., Group analysis of differential equations, Moscow, Nauka, 1978, 400p.
11. Serov N.I., Conformal symmetry of nonlinear wave equations, in Algebraic-Theoretical Studies in Mathematical Physics, Kiev, Inst. of Math., 1981, 59.

Symmetry reduction and exact solutions of the Yang–Mills equations

V.I. LAHNO, R.Z. ZHDANOV, W.I. FUSHCHYCH

We present a detailed account of symmetry properties of $SU(2)$ Yang–Mills equations. Using a subgroup structure of the Poincaré group $P(1,3)$ we have constructed all $P(1,3)$-inequivalent ansatzes for the Yang-Mills field which are invariant under the three-dimensional subgroups of the Poincaré group. With the aid of these ansatzes reduction of Yang-Mills equations to systems of ordinary differential equations is carried out and wide families of their exact solutions are constructed.

1 Introduction

Since Newton's and Euler's works, exact solutions of differential equations describing physical processes were highly estimated. Green, Lame, Liouville, Cayley, Donkin, Stokes, Kirchhoff, Poincaré, Stieltjes, Forsyth, Volterra, Appel, Macdonald, Weber, Bateman, Whittaker, Sommerfeld and many other famous researchers constructed different classes of exact solutions of linear Laplace, d'Alembert, heat, and Maxwell equations.

Nowadays, this constructive branch of mathematical physics is not so popular as earlier. But if one wants to have some nontrivial information on solutions of basic motion equations in quantum mechanics, field theory, gravitation theory, acoustics, and hydrodynamics, then the more intensive research work should be carried out in order to develop analytical methods of solution of partial differential equations (PDE). And what is more, unlike the mathematical physics of the 19th century, modern mathematical physics is essentially nonlinear. It means that all principal equations of modern physics, biology and chemistry are nonlinear. This fact complicates very much the problem of constructing their exact solutions (see, e.g. [1] and references therein).

Up to now, we have comparatively few papers devoted to construction of exact solutions of nonlinear multi-dimensional d'Alembert, Maxwell, Schrödinger, Dirac, Maxwell–Dirac, Yang–Mills equations. Whereas, a huge amount of papers and monographs are devoted to construction of exact solutions of equations for gravitational field. It is difficult even to estimate the number of papers and monographs, where the soliton solutions of the one-dimensional nonlinear KdV, Schrödinger and Sine-Gordon equations are studied. We are sure that the above mentioned equations should deserve much more attention of researchers in mathematical physics.

With the present paper we start a series of papers devoted to construction of new classes of exact solutions of the classical Yang–Mills equations (YME) with the use of their Lie and non-Lie symmetry. Here we study in detail symmetry reduction of YME by Poincaré-invariant ansatzes and obtain wide families of its exact Poincaré-invariant solutions.

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By the classical YME, we mean the following nonlinear system of twelve second-order PDE:

$$\begin{aligned} &\partial_\nu \partial^\nu \vec{A}_\mu - \partial^\mu \partial_\nu \vec{A}_\nu + e[(\partial_\nu \vec{A}_\nu) \times \vec{A}_\mu - 2(\partial_\nu \vec{A}_\mu) \times \vec{A}_\nu + (\partial^\mu \vec{A}_\nu) \times \vec{A}^\nu] + \\ &\qquad + e^2 \vec{A}_\nu \times (\vec{A}^\nu \times \vec{A}_\mu) = \vec{0}. \end{aligned} \tag{1.1}$$

Here $\partial_\nu = \frac{\partial}{\partial x_\nu}$, $\mu, \nu = \overline{0,3}$, $e = \mathrm{const}$, $\vec{A}_\mu = \vec{A}_\mu(x_0, x_1, x_2, x_3)$ is the three-component vector-potential of the Yang–Mills field (called, for bravity, the Yang–Mills field). Hereafter, the summation over the repeated indices μ, ν from 0 to 3 is understood. Raising and lowering the vector indices is performed with the aid of the metric tensor

$$g_{\mu\nu} = \begin{cases} 1, & \mu = \nu = 0, \\ -1, & \mu = \nu = 1, 2, 3, \\ 0, & \mu \neq \nu \end{cases}$$

(i.e. $\partial^\mu = g_{\mu\nu} \partial_\nu$).

It should be said that there were several reviews devoted to classical solutions of YME (see [2] and the literature cited there). But, in fact, symmetry properties of YME were not used. The solutions were obtained with the help of ad hoc substitutions suggested by Wu and Yang, Rosen, 't Hooft, Corrigan and Fairlie, Wilczek, Witten (for more detail, see [2]).

The structure of our paper is as follows. In the second Section we give all necessary information about symmetry properties of YME and about a solution generation procedure by virtue of the finite transformations of the symmetry group admitted by YME. In Section 3 we construct $P(1,3)$-inequivalent ansatzes for the Yang–Mills field invariant under the three-parameter subgroups of the Poincaré group. Section 4 is devoted to reduction of YME to systems of ordinary differential equations (ODE). Integrating these in Section 5 we construct multi-parameter families of exact solutions of YME. In Section 6 we consider some generalizations of the solutions obtained and, in particular, construct the generalization of Coleman's solution.

2 Symmetry and solution generation for the Yang–Mills equations

It was known long ago that YME are invariant with respect to the group $C(1,3) \otimes SU(2)$, where $C(1,3)$ is the 15-parameter conformal group having the following generators:

$$\begin{aligned} &P_\mu = \partial_\mu, \\ &J_{\alpha\beta} = x^\alpha \partial_\beta - x^\beta \partial_\alpha + A^{a\alpha} \partial_{A^a_\beta} - A^{a\beta} \partial_{A^a_\alpha}, \\ &D = x_\mu \partial_\mu - A^a_\mu \partial_{A^a_\mu}, \\ &K_\mu = 2x^\mu D - (x_\nu x^\nu) \partial_\mu + 2A^{a\mu} x_\nu \partial_{A^a_\nu} - 2A^a_\nu x^\nu \partial_{A^a_\mu}, \end{aligned} \tag{2.1}$$

and $SU(2)$ is the infinite-parameter special unitary group with the following basis generator:

$$Q = (\varepsilon_{abc} A^b_\mu w^c(x) + e^{-1} \partial_\mu w^a(x)) \partial_{A^a_\mu}. \tag{2.2}$$

In (2.1), (2.2) $\partial_{A^a_\mu} = \frac{\partial}{\partial A^a_\mu}$, $w^c(x)$ are arbitrary smooth functions, ε_{abc} is the third-order anti-symmetrical tensor with $\varepsilon_{123} = 1$. Hereafter, summation over the repeated indices a, b, c from 1 to 3 is understood.

But the fact that the group with generators (2.1), (2.2) is a maximal (in Lie's sense) invariance group admitted by YME was established only recently [3] with the use of a symbolic computation technique. The only explanation for this situation is a very cumbersome structure of the system of PDE (1.1). As a consequence, realization of the Lie algorithm of finding the maximal invariance group admitted by YME demands a huge amount of computations. This difficulty had been overcome with the aid of computer facilities.

One of the remarkable possibilities provided by the fact that the considered equation admits a nontrivial symmetry group gives the possibility of getting new solutions from the known ones by the solution generation technique [1, 4]. This technique is based on the following assertion.

Lemma. *Let*

$$x'_\mu = f_\mu(x, u, \tau), \quad \mu = \overline{0, n-1},$$
$$u'_a = g_a(x, u, \tau), \quad a = \overline{1, N},$$

where $\tau = (\tau_1, \tau_2, \ldots, \tau_r)$ *be the* r*-parameter invariance group of some system of PDE and* $U_a(x)$, $a = \overline{1, N}$ *be its particular solution. Then the* N*-component function* $u_a(x)$ *determined by implicit formulae*

$$U_a(f(x, u, \tau)) = g_a(x, u, \tau), \quad a = \overline{1, N} \tag{2.3}$$

is also a solution of the same system of PDE.

To make use of the above assertion we need formulae for finite transformations generated by infinitesimal operators (2.1), (2.2). We adduce these formulae following [1, 2].

1. The group of translations (generator $X = \tau_\mu P_\mu$)

$$x'_\mu = x_\mu + \tau_\mu, A^{d\prime}_\mu = A^d_\mu.$$

2. The Lorentz group $O(1,3)$

a) the group of rotations (generator $X = \tau J_{ab}$)

$$x'_0 = 0, \quad x'_c = x_c, \quad c \neq a, \quad c \neq b,$$
$$x'_a = x_a \cos\tau + x_b \sin\tau,$$
$$x'_b = x_b \cos\tau - x_a \sin\tau,$$
$$A^{d\prime}_0 = A^d_0, \quad A^{d\prime}_c = A^d_c, \quad c \neq a, \quad c \neq b,$$
$$A^{d\prime}_a = A^d_a \cos\tau + A^d_b \sin\tau,$$
$$A^{d\prime}_b = A^d_b \cos\tau - A^d_a \sin\tau;$$

b) the group of Lorentz transformations (generator $X = \tau J_{0a}$)

$$x'_0 = x_0 \cosh\tau + x_a \sinh\tau,$$
$$x'_a = x_a \cosh\tau + x_0 \sinh\tau, \quad x'_b = x_b, \quad b \neq a,$$
$$A^{d\prime}_0 = A^d_0 \cosh\tau + A^d_a \sinh\tau,$$
$$A^{d\prime}_a = A^d_a \cosh\tau + A^d_0 \sinh\tau, \quad A^{d\prime}_b = A^d_b, \quad b \neq a.$$

3. The group of scale transformations (generator $X = \tau D$)

$$x'_\mu = x_\mu e^\tau, \quad A^{d\prime}_\mu = A^d_\mu e^{-\tau}.$$

4. The group of conformal transformations (generator $X = \tau_\mu K^\mu$)

$$x'_\mu = (x_\mu - \tau_\mu x_\nu x^\nu)\sigma^{-1}(x),$$
$$A^{d\prime}_\mu = [g_{\mu\nu}\sigma(x) + 2(x_\mu\tau_\nu - x_\nu\tau_\mu + 2\tau_\alpha x^\alpha \tau_\mu x_\nu - x_\alpha x^\alpha \tau_\mu\tau_\nu - \tau_\alpha\tau^\alpha x_\mu x_\nu]A^{d\nu}.$$

5. The group of gauge transformations (generator $X = Q$)

$$x'_\mu = x_\mu,$$
$$A^{d\prime}_\mu = A^d_\mu \cos w + \varepsilon_{dbc}A^b_\mu n^c \sin w + 2n^d n^b A^b_\mu \sin^2\frac{w}{2} + \\ + e^{-1}\left[\frac{1}{2}n^d\partial_\mu w + \frac{1}{2}(\partial_\mu n^d)\sin w + \varepsilon_{dbc}(\partial_\mu n^b)n^c\right].$$

In the above formulae $\sigma(x) = 1 - \tau_\alpha x^\alpha + (\tau_\alpha\tau^\alpha)(x_\beta x^\beta)$, $n^a = n^a(x)$ is a unit vector determined by the equality $w^a(x) = w(x)n^a(x)$, $a = \overline{1,3}$.

Using the Lemma it is not difficult to obtain formulae for generating solutions of YME by the above transformation groups. We adduce them omitting derivation (see also [3]).

1. The group of translations

$$A^a_\mu(x) = u^a_\mu(x+\tau).$$

2. The Lorentz group

$$A^d_\mu(x) = a_\mu u^d_0(ax, bx, cx, dx) + b_\mu u^d_1(ax, bx, cx, dx) + \\ + c_\mu u^d_2(ax, bx, cx, dx) + d_\mu u^d_3(ax, bx, cx, dx).$$

3. The group of scale transformations

$$A^d_\mu(x) = e^\tau u^d_\mu(xe^\tau).$$

4. The group of conformal transformations

$$A^d_\mu(x) = [g_{\mu\nu}\sigma^{-1}(x) + 2\sigma^{-2}(x)(x_\mu\tau_\nu - x_\nu\tau_\mu + 2\tau_\alpha x^\alpha\tau_\mu x_\nu - \\ - x_\alpha x^\alpha\tau_\mu\tau_\nu - \tau_\alpha\tau^\alpha x_\mu x_\nu)]u^{d\nu}((x - \tau(x_\alpha x^\alpha))\sigma^{-1}(x)).$$

5. The group of gauge transformations

$$A^d_\mu(x) = u^d_\mu \cos w + \varepsilon_{dbc}u^b_\mu n^c \sin w + 2n^d n^b u^b_\mu \sin^2\frac{w}{2} + \\ + e^{-1}\left[\frac{1}{2}n^d\partial_\mu w + \frac{1}{2}(\partial_\mu n^d)\sin w + \varepsilon_{dbc}(\partial_\mu n^b)n^c\right].$$

Here $u^d_\mu(x)$ is an arbitrary given solution of YME; $A^d_\mu(x)$ is a new solution of YME; τ, τ_μ are arbitrary parameters; a_μ, b_μ, c_μ, d_μ are arbitrary parameters satisfying the equalities

$$a_\mu a^\mu = -b_\mu b^\mu = -c_\mu c^\mu = -d_\mu d^\mu = 1,$$
$$a_\mu b^\mu = a_\mu c^\mu = a_\mu d^\mu = b_\mu c^\mu = b_\mu d^\mu = c_\mu d^\mu = 0.$$

Besides that, we use the following designations: $x + \tau = \{x_\mu + \tau_\mu, \mu = \overline{0,3}\}$, $ax = a_\mu x^\mu$.

Thus, each particular solution of YME gives rise to a multi-parameter family of exact solutions by virtue of the above solution generation formulae.

3 Ansatzes for the Yang–Mills field

A key idea of the symmetry approach to the problem of reduction of PDE is a special choice of the form of a solution. This choice is dictated by a structure of the symmetry group admitted by the equation under study.

In the case involved, to reduce YME by N variables one has to construct ansatzes for the Yang–Mills field $A^a_\mu(x)$ invariant under N-dimensional subalgebras of the algebra with the basis elements (2.1), (2.2) [1, 5]. Since we are looking for Poincaré-invariant ansatzes reducing YME to systems of ODE, N is equal to 3. Due to invariance of YME under the Poincaré group $P(1,3)$, it is enough to consider only subalgebras which can not be transformed one into another by group transformation, i.e. $P(1,3)$-inequivalent subalgebras. Complete description of $P(1,3)$-inequivalent subalgebras of the Poincaré algebra was obtained in [6] (see also [7]).

According to the classical symmetry approach, to construct the ansatz invariant under the invariance algebra having the basis elements

$$X_a = \xi_{a\mu}(x,A)\partial_\mu + \eta^b_{a\mu}(x,A)\partial_{A^b_\mu}, \quad a=\overline{1,3}, \tag{3.1}$$

where $A=\{A^a_\mu, a=\overline{1,3}, \mu=\overline{0,3}\}$, one has

1) to construct a complete system of functionally-independent invariants of the operators (3.1) $\Omega=\{w_i(x,A), i=\overline{1,13}\}$;

2) to resolve relations

$$F_j(w_1(x,A),\dots,w_{13}(x,A))=0, \quad j=\overline{1,13} \tag{3.2}$$

with respect to the function A^a_μ.

As a result, one gets the ansatz for the field $A^a_\mu(x)$ which reduces YME to the system of twelve nonlinear ODE.

Note. Equalities (3.2) can be resolved with respect to A^a_μ, $a=\overline{1,3}$, $\mu=\overline{0,3}$ if the condition

$$\operatorname{rank}\|\xi_{a\mu}(x,A)\|^{3}_{a=1}\,{}^{3}_{\mu=0}=3 \tag{3.3}$$

holds. If (3.3) does not hold, the above procedure leads to partially-invariant solutions [5], which are not considered in the present paper.

In [1, 4] we established that the procedure of construction of invariant ansatzes could be essentially simplified if coefficients of operators X_a have the following structure:

$$\xi_{a\mu}=\xi_{a\mu}(x), \quad \eta^b_{a\mu}=\rho^{bc}_{a\mu\nu}(x)A^c_\nu \tag{3.4}$$

(i.e. basis elements of the invariance algebra realize the linear representation). In this case, the invariant ansatz for the field $A^a_\mu(x)$ is searched for in the form

$$A^a_\mu(x)=Q^{ab}_{\mu\nu}(x)B^b_\nu(w(x)). \tag{3.5}$$

Here $B^b_\nu(w)$ are arbitrary smooth functions and $w(x)$, $Q^{ab}_{\mu\nu}(x)$ are particular solutions of the system of PDE

$$\begin{gathered}\xi_{a\mu}w_{x_\mu}=0, \quad a=\overline{1,3},\\ (\xi_{a\nu}\partial_\nu-\rho^{bc}_{a\mu\alpha})Q^{cd}_{\alpha\beta}=0, \quad \mu=\overline{0,3}, \quad a,b,d=\overline{1,3}.\end{gathered} \tag{3.6}$$

Basis elements of the Poincaré algebra P_μ, $J_{\alpha\beta}$ from (2.1) evidently satisfy the conditions (3.4) and besides the equalities

$$\eta^b_{a\mu} = \rho_{a\mu\nu}(x)A^b_\nu, \quad a, b = \overline{1,3}, \quad \mu = \overline{0,3} \tag{3.7}$$

hold.

This fact permits further simplification of formulae (3.5), (3.6). Namely, the ansatz for the Yang–Mills field invariant under the 3-dimensional subalgebra of the Poincaré algebra with basis elements of the form (3.1), (3.7) should be looked for in the form

$$A^a_\mu = Q_{\mu\nu}(x)B^a_\nu(w(x)), \tag{3.8}$$

where $B^a_\nu(w)$ are arbitrary smooth functions and $w(x)$, $Q_{\mu\nu}(x)$ are particular solutions of the following system of PDE:

$$\xi_{a\mu}w_{x_\mu} = 0, \qquad a = \overline{1,3}, \tag{3.9}$$

$$\xi_{a\alpha}\partial_\alpha Q_{\mu\nu} - \rho_{a\mu\alpha}Q_{\alpha\nu} = 0, \quad a = \overline{1,3}, \quad \mu, \nu = \overline{0,3}. \tag{3.10}$$

Thus, to obtain the complete description of $P(1,3)$-inequivalent ansatzes for the field $A^a_\mu(x)$ invariant under 3-dimensional subalgebras of the Poincaré algebra, one has to integrate the over-determined system of PDE (3.9), (3.10) for each $P(1,3)$-inequivalent subalgebra. Let us note that compatibility of (3.9), (3.10) is guaranteed by the fact that operators X_1, X_2, X_3 form a Lie algebra.

Consider, as an example, the procedure of constructing ansatz (3.8) invariant under the subalgebra $\langle P_1, P_2, J_{03}\rangle$. In this case system (3.9) reads

$$w_{x_1} = 0, \quad w_{x_2} = 0, \quad x_0 w_{x_3} + x_3 w_{x_0} = 0,$$

whence $w = x_0^2 - x_3^2$.

Next, we note that coefficients $\rho_{1\mu\nu}$, $\rho_{2\mu\nu}$ of the operators P_1, P_2 are equal to zero, while coefficients $\rho_{3\mu\nu}$ form the following (4×4) matrix

$$\|\rho_{3\mu\nu}\|^{\,3}_{\,\mu,\nu=0} = \left\|\begin{matrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{matrix}\right\|$$

(we designate this constant matrix by the symbol S).

With account of the above fact, equations (3.10) take the form

$$Q_{x_1} = 0, \quad Q_{x_2} = 0, \quad x_0 Q_{x_3} + x_3 Q_{x_0} - SQ = 0, \tag{3.11}$$

where $Q = \|Q_{\mu\nu}(x)\|^3_{\mu,\nu=0}$ is a (4×4)-matrix.

From the first two equations of system (3.11) it follows that $Q = Q(x_0, x_3)$. Since S is a constant matrix, a solution of the third equation can be looked for in the form (see, for example, [4])

$$Q = \exp\{f(x_0, x_3)S\}.$$

Substituting this expression into (3.11) we get

$$(x_0 f_{x_3} + x_3 f_{x_0} - 1)\exp\{fS\} = 0$$

or, equivalently,

$$x_0 f_{x_3} + x_3 f_{x_0} = 1,$$

whence $f = \ln(x_0 + x_3)$.

Consequently, a particular solution of equations (3.11) reads

$$Q = \exp\{\ln(x_0 + x_3)S\}.$$

Using an evident identity $S = S^3$ we get the following equalities:

$$\begin{aligned}Q &= \sum_{n=0}^{\infty}(n!)^{-1}(\ln(x_0+x_3))^n S^n = \\ &= I + S[\ln(x_0+x_3) + (3!)^{-1}(\ln(x_0+x_3))^3 + \cdots] + \\ &\quad + S^2[(2!)^{-1}(\ln(x_0+x_3))^2 + (4!)^{-1}(\ln(x_0+x_3))^4 + \cdots] = \\ &= I + S\sinh(\ln(x_0+x_3)) + S^2(\cosh(\ln(x_0+x_3)) - 1),\end{aligned}$$

where I is a unit (4×4)-matrix.

Substitution of the obtained expressions for functions $w(x)$, $Q_{\mu\nu}(x)$ into (3.8) yields the ansatz for the Yang–Mills field $A^a_\mu(x)$ invariant under the algebra $\langle P_1, P_2, J_{03}\rangle$

$$\begin{aligned}&A^a_0 = B^a_0(x_0^2 - x_3^2)\cosh\ln(x_0+x_3) + B^a_3(x_0^2 - x_3^2)\sinh\ln(x_0+x_3),\\ &A^a_1 = B^a_1(x_0^2 - x_3^2), \quad A^a_2 = B^a_2(x_0^2 - x_3^2),\\ &A^a_3 = B^a_3(x_0^2 - x_3^2)\cosh\ln(x_0+x_3) + B^a_0(x_0^2 - x_3^2)\sinh\ln(x_0+x_3).\end{aligned} \tag{3.12}$$

Substituting (3.12) into YME we get a system of ODE for functions B^a_μ. If we will succeed in constructing its general or particular solutions, then substituting it into formulae (3.12) we get an exact solution of YME. But such a solution will have an unpleasant feature: independent variables x_μ will be included into it in asymmetrical way. At the same time, in the initial equation (1.1) all independent variables are on equal rights. To remove this defect one has to apply solution generation procedure by transformations from the Lorentz group. As a result, we will obtain an ansatz for the Yang–Mills field in the manifestly-covariant form with symmetrical dependence on x_μ.

In the same way, we construct the rest of ansatzes invariant under three-dimensional subalgebras of the Poincaré algebra. They are represented in the unified form

$$\begin{aligned}A^a_\mu(x) = \{&(a_\mu a_\nu - d_\mu d_\nu)\cosh\theta_0 + (d_\mu a_\nu - d_\nu a_\mu)\sinh\theta_0 + \\ &+ 2(a_\mu + d_\mu)[(\theta_1\cos\theta_3 + \theta_2\sin\theta_3)b_\nu + (\theta_2\cos\theta_3 - \theta_1\sin\theta_3)c_\nu + \\ &+ (\theta_1^2 + \theta_2^2)e^{-\theta_0}(a_\nu + d_\nu)] + (b_\mu c_\nu - b_\nu c_\mu)\sin\theta_3 - \\ &- (c_\mu c_\nu + b_\mu b_\nu)\cos\theta_3 - 2e^{-\theta_0}(\theta_1 b_\mu + \theta_2 c_\mu)(a_\nu + d_\nu)\}B^{a\nu}(w).\end{aligned} \tag{3.13}$$

Here θ_μ, $\mu = \overline{0,3}$, w are some functions whose explicit form is determined by the choice of a subalgebra of the Poincaré algebra $AP(1,3)$.

Below, we adduce a complete list of 3-dimensional $P(1,3)$-inequivalent subalgebras of the Poincaré algebra following [7]

$$
\begin{array}{ll}
L_1 = \langle P_0, P_1, P_2\rangle; & L_2 = \langle P_1, P_2, P_3\rangle;\\
L_3 = \langle P_0 + P_3, P_1, P_2\rangle; & L_4 = \langle J_{03} + \alpha J_{12}, P_1, P_2\rangle;\\
L_5 = \langle J_{03}, P_0 + P_3, P_1\rangle; & L_6 = \langle J_{03} + P_1, P_0, P_3\rangle;\\
L_7 = \langle J_{03} + P_1, P_0 + P_3, P_2\rangle; & L_8 = \langle J_{12} + \alpha J_{03}, P_0, P_3\rangle;\\
L_9 = \langle J_{12} + P_0, P_1, P_2\rangle; & L_{10} = \langle J_{12} + P_3, P_1, P_2\rangle;\\
L_{11} = \langle J_{12} + P_0 - P_3, P_1, P_2\rangle; & L_{12} = \langle G_1, P_0 + P_3, P_2 + \alpha P_1\rangle;\\
L_{13} = \langle G_1 + P_2, P_0 + P_3, P_1\rangle; & L_{14} = \langle G_1 + P_0 - P_3, P_0 + P_3, P_2\rangle;\\
L_{15} = \langle G_1 + P_0 - P_3, P_0 + P_3, P_1 + \alpha P_2\rangle; & L_{16} = \langle J_{12}, J_{03}, P_0 + P_3\rangle;\\
L_{17} = \langle G_1 + P_2, G_2 - P_1 + \alpha P_2, P_0 + P_3\rangle; & L_{18} = \langle J_{03}, G_1, P_2\rangle;\\
L_{19} = \langle G_1, J_{03}, P_0 + P_3\rangle; & L_{20} = \langle G_1, J_{03} + P_2, P_0 + P_3\rangle;\\
L_{21} = \langle G_1, J_{03} + P_1 + \alpha P_2, P_0 + P_3\rangle; & L_{22} = \langle G_1, G_2, J_{03} + \alpha J_{12}\rangle;\\
L_{23} = \langle G_1, P_0 + P_3, P_1\rangle; & L_{24} = \langle J_{12}, P_1, P_2\rangle;\\
L_{25} = \langle J_{03}, P_0, P_3\rangle; & L_{26} = \langle J_{12}, J_{13}, J_{23}\rangle;\\
L_{27} = \langle J_{01}, J_{02}, J_{12}\rangle. &
\end{array}
\qquad (3.14)
$$

Here $G_i = J_{0i} - J_{i3}$ $(i = 1, 2)$, $\alpha \in \mathbb{R}$.

Ansatzes for the Yang–Mills field $A^a_\mu(x)$ are of the form (3.13), functions $\theta_\mu(x)$, $\mu = \overline{0,3}$, $w(x)$ being determined by one of the following formulae:

$$
\begin{array}{ll}
L_1: & \theta_\mu = 0, \quad w = dx; \quad L_2: \quad \theta_\mu = 0, \quad w = ax; \quad L_3: \quad \theta_\mu = 0, w = kx;\\
L_4: & \theta_0 = -\ln|kx|, \quad \theta_1 = \theta_2 = 0, \quad \theta_3 = \alpha \ln|kx|, \quad w = (ax)^2 - (dx)^2;\\
L_5: & \theta_0 = -\ln|kx|, \quad \theta_1 = \theta_2 = \theta_3 = 0, \quad w = cx;\\
L_6: & \theta_0 = -bx, \quad \theta_1 = \theta_2 = \theta_3 = 0, \quad w = cx;\\
L_7: & \theta_0 = -bx, \quad \theta_1 = \theta_2 = \theta_3 = 0, \quad w = bx - \ln|kx|;\\
L_8: & \theta_0 = \alpha \arctan(bx(cx)^{-1}), \quad \theta_1 = \theta_2 = 0,\\
 & \theta_3 = -\arctan(bx(cx)^{-1}), \quad w = (bx)^2 + (cx)^2;\\
L_9: & \theta_0 = \theta_1 = \theta_2 = 0, \quad \theta_3 = -ax, \quad w = dx;\\
L_{10}: & \theta_0 = \theta_1 = \theta_2 = 0, \quad \theta_3 = dx, \quad w = ax;\\
L_{11}: & \theta_0 = \theta_1 = \theta_2 = 0, \quad \theta_3 = -\dfrac{1}{2}kx, \quad w = ax - dx;\\
L_{12}: & \theta_0 = 0, \quad \theta_1 = \dfrac{1}{2}(bx - \alpha cx)(kx)^{-1}, \quad \theta_2 = \theta_3 = 0, \quad w = kx;\\
L_{13}: & \theta_0 = \theta_2 = \theta_3 = 0, \quad \theta_1 = \frac{1}{2}cx, \quad w = kx;\\
L_{14}: & \theta_0 = \theta_2 = \theta_3 = 0, \quad \theta_1 = -\dfrac{1}{4}kx, \quad w = 4bx + (kx)^2;\\
L_{15}: & \theta_0 = \theta_2 = \theta_3 = 0, \quad \theta_1 = -\dfrac{1}{4}kx, \quad w = 4(\alpha bx - cx) + \alpha(kx)^2;\\
L_{16}: & \theta_0 = -\ln|kx|, \quad \theta_1 = \theta_2 = 0, \quad \theta_3 = -\arctan(bx(cx)^{-1}),\\
 & w = (bx)^2 + (cx)^2;\\
L_{17}: & \theta_0 = \theta_3 = 0, \quad \theta_1 = \dfrac{1}{2}(cx + (\alpha + kx)bx)(1 + kx(\alpha + kx))^{-1},
\end{array}
\qquad (3.15)
$$

$$\theta_2 = -\frac{1}{2}(bx - cxkx)(1 + kx(\alpha + kx))^{-1}, \quad w = kx;$$

$$L_{18}: \quad \theta_0 = -\ln|kx|, \quad \theta_1 = \frac{1}{2}bx(kx)^{-1}, \quad \theta_2 = \theta_3 = 0,$$
$$w = (ax)^2 - (bx)^2 - (dx)^2;$$

$$L_{19}: \quad \theta_0 = -\ln|kx|, \quad \theta_1 = \frac{1}{2}bx(kx)^{-1}, \quad \theta_2 = \theta_3 = 0, w = cx;$$

$$L_{20}: \quad \theta_0 = -\ln|kx|, \quad \theta_1 = \frac{1}{2}bx(kx)^{-1}, \quad \theta_2 = \theta_3 = 0, \quad w = \ln|kx| - cx;$$

$$L_{21}: \quad \theta_0 = -\ln|kx|, \quad \theta_1 = \frac{1}{2}(bx - \ln|kx|)(kx)^{-1}, \quad \theta_2 = \theta_3 = 0,$$
$$w = \alpha\ln|kx| - cx;$$

$$L_{22}: \quad \theta_0 = -\ln|kx|, \quad \theta_1 = \frac{1}{2}bx(kx)^{-1}, \quad \theta_2 = \frac{1}{2}cx(kx)^{-1},$$
$$\theta_3 = \alpha\ln|kx|, \quad w = (ax)^2 - (bx)^2 - (cx)^2 - (dx)^2.$$

Here $ax = a_\mu x^\mu$, $bx = b_\mu x^\mu$, $cx = c_\mu x^\mu$, $dx = d_\mu x^\mu$, $\mu = \overline{0,3}$, $kx = ax + dx$.

Note. Basis elements of subalgebras L_{23}, L_{24}, L_{25}, L_{26}, L_{27} do not satisfy (3.3). That is why, ansatzes invariant under these subalgebras are partially-invariant solutions and are not considered here.

4 Reduction of the Yang–Mills equations

In order to reduce YME to ODE it is necessary to substitute ansatz (3.13) into (1.1) and convolute the expression obtained with $Q^\mu_\alpha(x)$. As a result, we get a system of twelve nonlinear ODE for functions $B^\alpha_\nu(w)$ of the form

$$k_{\mu\gamma}\ddot{\vec{B}}^\gamma + l_{\mu\gamma}\dot{\vec{B}}^\gamma + m_{\mu\gamma}\vec{B}^\gamma + eg_{\mu\nu\gamma}\dot{\vec{B}}^\nu \times \vec{B}^\gamma + eh_{\mu\nu\gamma}\vec{B}^\nu \times \vec{B}^\gamma + \\ + e^2\vec{B}_\gamma \times (\vec{B}^\gamma \times \vec{B}_\mu) = \vec{0}. \tag{4.1}$$

Coefficients of the reduced ODE are given by the following formulae:

$$k_{\mu\gamma} = g_{\mu\gamma}F_1 - G_\mu G_\gamma, \quad l_{\mu\gamma} = g_{\mu\gamma}F_2 + 2S_{\mu\gamma} - G_\mu H_\gamma - G_\mu\dot{G}_\gamma, \\ m_{\mu\gamma} = R_{\mu\gamma} - G_\mu\dot{H}_\gamma, \quad g_{\mu\nu\gamma} = g_{\mu\gamma}G_\nu + g_{\nu\gamma}G_\mu - 2g_{\mu\nu}G_\gamma, \\ h_{\mu\nu\gamma} = (1/2)(g_{\mu\gamma}H_\nu - g_{\mu\nu}H_\gamma) - T_{\mu\nu\gamma}, \tag{4.2}$$

where $g_{\mu\nu}$ is a metric tensor of the Minkowski space $\mathbb{R}(1,3)$ and $F_1, F_2, G_\mu, \ldots, T_{\mu\nu\gamma}$ are functions on w determined by the relations

$$F_1 = w_{x_\mu}w_{x^\mu}, \quad F_2 = \Box w, \quad G_\mu = Q_{\alpha\mu}w_{x_\alpha}, \quad H_\mu = Q_{\alpha\mu x_\alpha}, \\ S_{\mu\nu} = Q^\alpha_\mu Q_{\alpha\nu x_\beta}w_{x^\beta}, \quad R_{\mu\nu} = Q^\alpha_\mu \Box Q_{\alpha\nu}, \\ T_{\mu\nu\gamma} = Q^\alpha_\mu Q_{\alpha\nu x_\beta}Q_{\beta\gamma} + Q^\alpha_\nu Q_{\alpha\gamma x_\beta}Q_{\beta\mu} + Q^\alpha_\gamma Q_{\alpha\mu x_\beta}Q_{\beta\nu}. \tag{4.3}$$

Substituting functions $Q_{\mu\nu}(x)$ from (3.13), where $\theta_\mu(x)$, $w(x)$ are determined by one of the formulae (3.15) into (4.2), (4.3) we obtain coefficients of the corresponding systems of ODE (4.1)

$$L_1: \quad k_{\mu\gamma} = -g_{\mu\gamma} - d_\mu d_\gamma, \quad l_{\mu\gamma} = m_{\mu\gamma} = 0,$$
$$g_{\mu\nu\gamma} = g_{\mu\gamma}d_\nu + g_{\nu\gamma}d_\mu - 2g_{\mu\nu}d_\gamma, \quad h_{\mu\nu\gamma} = 0;$$

$$
\begin{aligned}
L_2:\quad & k_{\mu\gamma} = g_{\mu\gamma} - a_\mu a_\gamma, \quad l_{\mu\gamma} = m_{\mu\gamma} = 0,\\
& g_{\mu\nu\gamma} = g_{\mu\gamma}a_\nu + g_{\nu\gamma}a_\mu - 2g_{\mu\nu}a_\gamma, \quad h_{\mu\nu\gamma} = 0;\\
L_3:\quad & k_{\mu\gamma} = -k_\mu k_\gamma, \quad l_{\mu\gamma} = m_{\mu\gamma} = 0, \quad g_{\mu\nu\gamma} = g_{\mu\gamma}k_\nu + g_{\nu\gamma}k_\mu - 2g_{\mu\nu}k_\gamma,\\
& h_{\mu\nu\gamma} = 0;\\
L_4:\quad & k_{\mu\gamma} = 4g_{\mu\gamma}w - a_\mu a_\gamma(w+1)^2 - d_\mu d_\gamma(w-1)^2 - (a_\mu d_\gamma + a_\gamma d_\mu)(w^2-1),\\
& l_{\mu\gamma} = 4(g_{\mu\gamma} + \alpha(b_\mu c_\gamma - c_\mu b_\gamma)) - 2k_\mu(a_\gamma - d_\gamma + k_\gamma w), \quad m_{\mu\gamma} = 0,\\
& g_{\mu\nu\gamma} = \epsilon(g_{\mu\gamma}(a_\nu - d_\nu + k_\nu w) + g_{\nu\gamma}(a_\mu - d_\mu + k_\mu w) -\\
& \qquad - 2g_{\mu\nu}(a_\gamma - d_\gamma + k_\gamma w)),\\
& h_{\mu\nu\gamma} = \frac{\epsilon}{2}[g_{\mu\gamma}k_\nu - g_{\mu\nu}k_\gamma] + \alpha\epsilon[(b_\mu c_\nu - c_\mu b_\nu)k_\gamma + (b_\nu c_\gamma - c_\nu b_\gamma)k_\mu +\\
& \qquad + (b_\gamma c_\mu - c_\gamma b_\mu)k_\nu];\\
L_5:\quad & k_{\mu\gamma} = -g_{\mu\gamma} - c_\mu c_\gamma, \quad l_{\mu\gamma} = -\epsilon c_\mu k_\gamma, \quad m_{\mu\gamma} = 0,\\
& g_{\mu\nu\gamma} = g_{\mu\gamma}c_\nu + g_{\nu\gamma}c_\mu - 2g_{\mu\nu}c_\gamma, \quad h_{\mu\nu\gamma} = \frac{\epsilon}{2}(g_{\mu\gamma}k_\nu - g_{\mu\nu}k_\gamma);\\
L_6:\quad & k_{\mu\gamma} = -g_{\mu\gamma} - c_\mu c_\gamma, \quad l_{\mu\gamma} = 0,\\
& m_{\mu\gamma} = -(a_\mu a_\gamma - d_\mu d_\gamma), \quad g_{\mu\nu\gamma} = g_{\mu\gamma}c_\nu + g_{\nu\gamma}c_\mu - 2g_{\mu\nu}c_\gamma,\\
& h_{\mu\nu\gamma} = -[(a_\mu d_\nu - a_\nu d_\mu)b_\gamma + (a_\nu d_\gamma - a_\gamma d_\nu)b_\mu + (a_\gamma d_\mu - a_\mu d_\gamma)b_\nu];\\
L_7:\quad & k_{\mu\gamma} = -g_{\mu\gamma} - (b_\mu - \epsilon k_\mu)(b_\gamma - \epsilon k_\gamma), \quad l_{\mu\gamma} = -2(a_\mu d_\gamma - a_\gamma d_\mu),\\
& m_{\mu\gamma} = -(a_\mu a_\gamma - d_\mu d_\gamma),\\
& g_{\mu\nu\gamma} = g_{\mu\gamma}(b_\nu - \epsilon k_\nu) + g_{\nu\gamma}(b_\mu - \epsilon k_\mu) - 2g_{\mu\nu}(b_\gamma - \epsilon k_\gamma),\\
& h_{\mu\nu\gamma} = -[(a_\mu d_\nu - a_\nu d_\mu)b_\gamma + (a_\nu d_\gamma - a_\gamma d_\nu)b_\mu + (a_\gamma d_\mu - a_\mu d_\gamma)b_\nu]; \qquad (4.4)\\
L_8:\quad & k_{\mu\gamma} = -4w(g_{\mu\gamma} + c_\mu c_\gamma), \quad l_{\mu\gamma} = -4(g_{\mu\gamma} + c_\mu c_\gamma),\\
& m_{\mu\gamma} = -\frac{1}{w}(\alpha^2(a_\mu a_\gamma - d_\mu d_\gamma) + b_\mu b_\gamma),\\
& g_{\mu\nu\gamma} = 2\sqrt{w}(g_{\mu\gamma}c_\nu + g_{\nu\gamma}c_\mu - 2g_{\mu\nu}c_\gamma),\\
& h_{\mu\nu\gamma} = \frac{1}{2\sqrt{w}}(g_{\mu\gamma}c_\nu - g_{\mu\nu}c_\gamma) + \frac{\alpha}{\sqrt{w}}((a_\mu d_\nu - a_\nu d_\mu)b_\gamma +\\
& \qquad + (a_\nu d_\gamma - d_\nu a_\gamma)b_\mu + (a_\gamma d_\mu - a_\mu d_\gamma)b_\nu);\\
L_9:\quad & k_{\mu\gamma} = -g_{\mu\gamma} - d_\mu d_\gamma, \quad l_{\mu\gamma} = 0,\\
& m_{\mu\gamma} = b_\mu b_\gamma + c_\mu c_\gamma, \quad g_{\mu\nu\gamma} = g_{\mu\gamma}d_\nu + g_{\nu\gamma}d_\mu - 2g_{\mu\nu}d_\gamma,\\
& h_{\mu\nu\gamma} = a_\gamma(b_\mu c_\nu - c_\mu b_\nu) + a_\mu(b_\nu c_\gamma - c_\nu b_\gamma) + a_\nu(b_\gamma c_\mu - c_\gamma b_\mu);\\
L_{10}:\quad & k_{\mu\gamma} = g_{\mu\gamma} - a_\mu a_\gamma, \quad l_{\mu\gamma} = 0,\\
& m_{\mu\gamma} = -(b_\mu b_\gamma + c_\mu c_\gamma), \quad g_{\mu\nu\gamma} = g_{\mu\gamma}a_\nu + g_{\nu\gamma}a_\mu - 2g_{\mu\nu}a_\gamma,\\
& h_{\mu\nu\gamma} = -[d_\gamma(b_\mu c_\nu - c_\mu b_\nu) + d_\mu(b_\nu c_\gamma - c_\nu b_\gamma) + d_\nu(b_\gamma c_\mu - c_\gamma b_\mu)];\\
L_{11}:\quad & k_{\mu\gamma} = -(a_\mu - d_\mu)(a_\gamma - d_\gamma), \quad l_{\mu\gamma} = -2(b_\mu c_\gamma - c_\mu b_\gamma), \quad m_{\mu\gamma} = 0,\\
& g_{\mu\nu\gamma} = g_{\mu\gamma}(a_\nu - d_\nu) + g_{\nu\gamma}(a_\mu - d_\mu) - 2g_{\mu\nu}(a_\gamma - d_\gamma),\\
& h_{\mu\nu\gamma} = \tfrac{1}{2}[k_\gamma(b_\mu c_\nu - c_\mu b_\nu) + k_\mu(b_\nu c_\gamma - c_\nu b_\gamma) + k_\nu(b_\gamma c_\mu - c_\gamma b_\mu)];\\
L_{12}:\quad & k_{\mu\gamma} = -k_\mu k_\gamma, \quad l_{\mu\gamma} = -\frac{1}{w}k_\mu k_\gamma, \quad m_{\mu\gamma} = -\frac{\alpha^2}{w^2}k_\mu k_\gamma,\\
& g_{\mu\nu\gamma} = g_{\mu\gamma}k_\nu + g_{\nu\gamma}k_\mu - 2g_{\mu\nu}k_\gamma,\\
& h_{\mu\nu\gamma} = \frac{1}{2w}(g_{\mu\gamma}k_\nu - g_{\mu\nu}k_\gamma) +
\end{aligned}
$$

$$+\frac{\alpha}{w}((k_\mu b_\nu - k_\nu b_\mu)c_\gamma + (k_\nu b_\gamma - k_\gamma b_\nu)c_\mu + (k_\gamma b_\mu - k_\mu b_\gamma)c_\nu);$$

$$L_{13}: \quad k_{\mu\gamma} = -k_\mu k_\gamma, \quad l_{\mu\gamma} = 0, \quad m_{\mu\gamma} = -k_\mu k_\gamma,$$
$$g_{\mu\nu\gamma} = g_{\mu\gamma}k_\nu + g_{\nu\gamma}k_\mu - 2g_{\mu\nu}k_\gamma, \quad h_{\mu\nu\gamma} = -((k_\mu b_\nu - k_\nu b_\mu)c_\gamma +$$
$$+ (k_\nu b_\gamma - k_\gamma b_\nu)c_\mu + (k_\gamma b_\mu - k_\mu b_\gamma)c_\nu);$$

$$L_{14}: \quad k_{\mu\gamma} = -16(g_{\mu\gamma} + b_\mu b_\gamma), \quad l_{\mu\gamma} = m_{\mu\gamma} = h_{\mu\nu\gamma} = 0,$$
$$g_{\mu\nu\gamma} = 4(g_{\mu\gamma}b_\nu + g_{\nu\gamma}b_\mu - 2g_{\mu\nu}b_\gamma);$$

$$L_{15}: \quad k_{\mu\gamma} = -16[(1+\alpha^2)g_{\mu\gamma} + (c_\mu - \alpha b_\mu)(c_\gamma - \alpha b_\gamma)],$$
$$l_{\mu\gamma} = m_{\mu\gamma} = h_{\mu\nu\gamma} = 0,$$
$$g_{\mu\nu\gamma} = -4[g_{\mu\gamma}(c_\nu - \alpha b_\nu) + g_{\nu\gamma}(c_\mu - \alpha b_\mu) - 2g_{\mu\nu}(c_\gamma - \alpha b_\gamma)];$$

$$L_{16}: \quad k_{\mu\gamma} = -4w(g_{\mu\gamma} + c_\mu c_\gamma), \quad l_{\mu\gamma} = -4(g_{\mu\gamma} + c_\mu c_\gamma) - 2\epsilon k_\gamma c_\mu \sqrt{w},$$
$$m_{\mu\gamma} = -\frac{1}{w}b_\mu b_\gamma, \quad g_{\mu\nu\gamma} = 2\sqrt{w}(g_{\mu\gamma}c_\nu + g_{\nu\gamma}c_\mu - 2g_{\mu\nu}c_\gamma),$$
$$h_{\mu\nu\gamma} = \frac{1}{2}[\epsilon(g_{\mu\gamma}k_\nu - g_{\mu\nu}k_\gamma) + \frac{1}{\sqrt{w}}(g_{\mu\gamma}c_\nu - g_{\mu\nu}c_\gamma)];$$

$$L_{17}: \quad k_{\mu\gamma} = -k_\mu k_\gamma, \quad l_{\mu\gamma} = -\frac{2w+\alpha}{w(w+\alpha)+1}k_\mu k_\gamma,$$
$$m_{\mu\gamma} = -4k_\mu k_\gamma(1 + w(\alpha + w))^{-2}, \quad g_{\mu\nu\gamma} = g_{\mu\gamma}k_\nu + g_{\nu\gamma}k_\mu - 2g_{\mu\nu}k_\gamma,$$
$$h_{\mu\nu\gamma} = \tfrac{1}{2}(\alpha + 2w)(g_{\mu\gamma}k_\nu - g_{\mu\nu}k_\gamma)(1 + w(\alpha + w))^{-1} -$$
$$2(1 + w(w+\alpha))^{-1}((k_\mu b_\nu - k_\nu b_\mu)c_\gamma + (k_\nu b_\gamma - k_\gamma b_\nu)c_\mu +$$
$$+ (k_\gamma b_\mu - k_\mu b_\gamma)c_\nu);$$

$$L_{18}: \quad k_{\mu\gamma} = 4wg_{\mu\gamma} - (k_\mu w + a_\mu - d_\mu)(k_\gamma w + a_\gamma - d_\gamma),$$
$$l_{\mu\gamma} = 6g_{\mu\gamma} + 4(a_\mu d_\gamma - a_\gamma d_\mu) - 3k_\gamma(k_\mu w + a_\mu - d_\mu), \quad m_{\mu\gamma} = -k_\mu k_\gamma,$$
$$g_{\mu\nu\gamma} = \epsilon(g_{\mu\gamma}(k_\nu w + a_\nu - d_\nu) + g_{\nu\gamma}(k_\mu w + a_\mu - d_\mu) -$$
$$- 2g_{\mu\nu}(k_\gamma w + a_\gamma - d_\gamma)),$$
$$h_{\mu\nu\gamma} = \epsilon(g_{\mu\gamma}k_\nu - g_{\mu\nu}k_\gamma);$$

$$L_{19}: \quad k_{\mu\gamma} = -g_{\mu\gamma} - c_\mu c_\gamma, \quad l_{\mu\gamma} = 2\epsilon k_\gamma c_\mu, \quad m_{\mu\gamma} = -k_\mu k_\gamma,$$
$$g_{\mu\nu\gamma} = g_{\mu\gamma}c_\nu + g_{\nu\gamma}c_\mu - 2g_{\mu\nu}c_\gamma, \quad h_{\mu\nu\gamma} = \epsilon(g_{\mu\gamma}k_\nu - g_{\mu\nu}k_\gamma);$$

$$L_{20}: \quad k_{\mu\gamma} = -g_{\mu\gamma} - (c_\mu - \epsilon k_\mu)(c_\gamma - \epsilon k_\gamma), \quad l_{\mu\gamma} = 2\epsilon k_\gamma c_\mu - 2k_\mu k_\gamma,$$
$$m_{\mu\gamma} = -k_\mu k_\gamma,$$
$$g_{\mu\nu\gamma} = g_{\mu\gamma}(\epsilon k_\nu - c_\nu) + g_{\nu\gamma}(\epsilon k_\mu - c_\mu) - 2g_{\mu\nu}(\epsilon k_\gamma - c_\gamma),$$
$$h_{\mu\nu\gamma} = \epsilon(g_{\mu\gamma}k_\nu - g_{\mu\nu}k_\gamma);$$

$$L_{21}: \quad k_{\mu\gamma} = -g_{\mu\gamma} - (c_\mu - \alpha\epsilon k_\mu)(c_\gamma - \alpha\epsilon k_\gamma), \quad l_{\mu\gamma} = 2(\epsilon k_\gamma c_\mu - \alpha k_\mu k_\gamma),$$
$$m_{\mu\gamma} = -k_\mu k_\gamma,$$
$$g_{\mu\nu\gamma} = -g_{\mu\gamma}(c_\nu - \alpha\epsilon k_\nu) - g_{\nu\gamma}(c_\mu - \alpha\epsilon k_\mu) + 2g_{\mu\nu}(c_\gamma - \alpha\epsilon k_\gamma),$$
$$h_{\mu\nu\gamma} = \epsilon(g_{\mu\gamma}k_\nu - g_{\mu\nu}k_\gamma);$$

$$L_{22}: \quad k_{\mu\gamma} = 4wg_{\mu\gamma} - (a_\mu - d_\mu + k_\mu w)(a_\gamma - d_\gamma + k_\gamma w),$$
$$l_{\mu\gamma} = 4[2g_{\mu\gamma} + \alpha(b_\mu c_\gamma - c_\mu b_\gamma) - a_\mu a_\gamma + d_\mu d_\gamma - wk_\mu k_\gamma],$$
$$m_{\mu\gamma} = -2k_\mu k_\gamma,$$
$$g_{\mu\nu\gamma} = \epsilon(g_{\mu\gamma}(a_\nu - d_\nu + k_\nu w) + g_{\nu\gamma}(a_\mu - d_\mu + k_\mu w) -$$
$$- 2g_{\mu\nu}(a_\gamma - d_\gamma + k_\gamma w),$$

$$h_{\mu\nu\gamma} = \frac{3\epsilon}{2}(g_{\mu\gamma}k_\nu - g_{\mu\nu}k_\gamma) - \epsilon\alpha[k_\gamma(b_\mu c_\nu - c_\mu b_\nu) + \\ + k_\mu(b_\nu c_\gamma - c_\nu b_\gamma) + k_\nu(b_\gamma c_\mu - c_\gamma b_\mu)];$$

where $k_\mu = a_\mu + d_\mu$, $\epsilon = 1$ for $ax + dx > 0$ and $\epsilon = -1$ for $ax + dx < 0$.

5 Exact solutions of the Yang–Mills equations

When applying the symmetry reduction procedure to the nonlinear Dirac equation, we succeeded in constructing general solutions of a large part of reduced systems of ODE. In the case involved we are not so lucky. Nevertheless, we obtain some particular solutions of equations (4.2), (4.4).

The principal idea of our approach to integration of systems of ODE (4.2), (4.4) is rather simple and quite natural. It is a reduction of these systems by the number of components with the aid of ad hoc substitutions. Using this trick we construct particular solutions of equations 1, 2, 5, 8, 14, 15, 16, 18, 19, 20, 21, 22 ($\alpha = 0$). Below we adduce substitutions for $\vec{B}_\mu(w)$ and corresponding equations.

$$\begin{array}{ll}
1. & \vec{B}_\mu = a_\mu\vec{e}_1 f(w) + b_\mu\vec{e}_2 g(w) + c_\mu\vec{e}_3 h(w), \\
& \ddot{f} - e^2(g^2 + h^2)f = 0, \quad \ddot{g} + e^2(f^2 - h^2)g = 0, \quad \ddot{h} + e^2(f^2 - g^2)h = 0. \\
2. & \vec{B}_\mu = b_\mu\vec{e}_1 f(w) + c_\mu\vec{e}_2 g(w) + d_\mu\vec{e}_3 h(w), \\
& \ddot{f} + e^2(g^2 + h^2)f = 0, \quad \ddot{g} + e^2(f^2 + h^2)g = 0, \quad \ddot{h} + e^2(f^2 + g^2)h = 0. \\
5. & \vec{B}_\mu = k_\mu\vec{e}_1 f(w) + b_\mu\vec{e}_2 g(w), \quad \ddot{f} - e^2 g^2 f = 0, \ddot{g} = 0. \\
8.1. & (\alpha = 0)\ \vec{B}_\mu = k_\mu\vec{e}_1 f(w) + b_\mu\vec{e}_2 g(w), \\
& 4w\ddot{f} + 4\dot{f} - e^2 g^2 f = 0, \quad 4w\ddot{g} + 4\dot{g} - w^{-1}g = 0. \\
8.2. & \vec{B}_\mu = a_\mu\vec{e}_1 f(w) + d_\mu\vec{e}_2 g(w) + b_\mu\vec{e}_3 h(w), \\
& 4w\ddot{f} + 4\dot{f} - \dfrac{\alpha^2}{w}f - \dfrac{2\alpha e}{\sqrt{w}}gh + e^2(h^2 + g^2)f = 0, \\
& 4w\ddot{g} + 4\dot{g} + \dfrac{\alpha^2}{w}g + \dfrac{2\alpha e}{\sqrt{w}}fh + e^2(f^2 - h^2)g = 0, \\
& 4w\ddot{h} + 4\dot{h} - w^{-1}h + \dfrac{2\alpha e}{\sqrt{w}}fg + e^2(f^2 - g^2)h = 0. \\
14.1. & \vec{B}_\mu = a_\mu\vec{e}_1 f(w) + d_\mu\vec{e}_2 g(w) + c_\mu\vec{e}_3 h(w), \\
& 16\ddot{f} - e^2(h^2 + g^2)f = 0, \quad 16\ddot{g} + e^2(f^2 - h^2)g = 0, \\
& 16\ddot{h} + e^2(f^2 - g^2)h = 0. \\
14.2. & \vec{B}_\mu = k_\mu\vec{e}_1 f(w) + c_\mu\vec{e}_2 g(w), \quad 16\ddot{f} - e^2 g^2 f = 0, \ddot{g} = 0. \\
15.1. & \vec{B}_\mu = a_\mu\vec{e}_1 f(w) + d_\mu\vec{e}_2 g(w) + (1 + \alpha^2)^{-\frac{1}{2}}(\alpha c_\mu + b_\mu)\vec{e}_3 h(w), \\
& 16(1 + \alpha^2)\ddot{f} - e^2(h^2 + g^2)f = 0, \quad 16(1 + \alpha^2)\ddot{g} + e^2(f^2 - h^2)g = 0, \\
& 16(1 + \alpha^2)\ddot{h} + e^2(f^2 - g^2)h = 0. \\
15.2. & \vec{B}_\mu = k_\mu\vec{e}_1 f(w) + (1 + \alpha^2)^{-\frac{1}{2}}(\alpha c_\mu + b_\mu)\vec{e}_2 g(w), \\
& 16(1 + \alpha^2)\ddot{f} - e^2 f g^2 = 0, \quad \ddot{g} = 0.
\end{array} \tag{5.1}$$

16. $\vec{B}_\mu = k_\mu \vec{e}_1 f(w) + b_\mu \vec{e}_2 g(w)$,
$4w\ddot{f} + 4\dot{f} - e^2 g^2 f = 0, \quad 4w\ddot{g} + 4\dot{g} - w^{-1} g = 0.$
18. $\vec{B}_\mu = b_\mu \vec{e}_1 f(w) + c_\mu \vec{e}_2 g(w)$,
$4w\ddot{f} + 6\dot{f} + e^2 g^2 f = 0, \quad 4w\ddot{g} + 6\dot{g} + e^2 f^2 g = 0.$
19. $\vec{B}_\mu = k_\mu \vec{e}_1 f(w) + b_\mu \vec{e}_2 g(w)$,
$\ddot{f} - e^2 g^2 f = 0, \quad \ddot{g} = 0.$
20. $\vec{B}_\mu = k_\mu \vec{e}_1 f(w) + b_\mu \vec{e}_2 g(w)$,
$\ddot{f} - e^2 g^2 f = 0, \quad \ddot{g} = 0.$
21. $\vec{B}_\mu = k_\mu \vec{e}_1 f(w) + b_\mu \vec{e}_2 g(w)$,
$\ddot{f} - e^2 g^2 f = 0, \quad \ddot{g} = 0.$
22. $(\alpha = 0)$ $\vec{B}_\mu = b_\mu \vec{e}_1 f(w) + c_\mu \vec{e}_2 g(w)$,
$4w\ddot{f} + 8\dot{f} + e^2 g^2 f = 0, \quad 4w\ddot{g} + 8\dot{g} + e^2 f^2 g = 0.$

In the above formulae we use designations $\vec{e}_1 = (1, 0, 0)$, $\vec{e}_2 = (0, 1, 0)$, $\vec{e}_3 = (0, 0, 1)$.

Thus, combining symmetry reduction by the number of independent variables and reduction by the number of dependent variables we reduce YME to rather simple ODE. It is worth reminding that effectiveness of the widely used ansatz for the Yang–Mills field suggested by t'Hooft et al [2] is closely connected with the fact that it reduces the system of twelve PDE to one nonlinear wave equation.

Next, we will briefly consider a procedure of integration of equations (5.1).

Substitution $f = 0$, $g = h = u(w)$ reduces the system of ODE 1 from (5.1) to the equation

$$\ddot{u} = e^2 u^3, \tag{5.2}$$

which is integrated in elliptic functions [8]. Besides that, ODE (5.2) has a solution which is expressed in terms of elementary functions $u = \sqrt{2}(ew - C)^{-1}$, $C \in \mathbb{R}^1$.

ODE 2 with $f = g = h = u(w)$ reduces to the form $\ddot{u} + 2e^2 u^3 = 0$.

This equation is also integrated in elliptic functions [8].

Integrating the second equation of system of ODE 5 we get $g = C_1 w + C_2$, $C_i \in \mathbb{R}^1$. If $C_1 \neq 0$, then the constant C_2 can be neglected, and we may put $C_2 = 0$. Provided $C_1 \neq 0$, the first equation from system 5 reads

$$\ddot{f} - e^2 C_1^2 w^2 f = 0. \tag{5.3}$$

A general solution of ODE (5.3) is given by formula $f = w^{1/2} Z_{\frac{1}{4}}(\frac{ie}{2} C_1 w^2)$.

Hereafter, we use the designation $Z_\nu(w) = C_3 J_\nu(w) + C_4 Y_\nu(w)$, where J_ν, Y_ν are Bessel functions, C_3, C_4 are arbitrary constants.

In the case $C_1 = 0$, $C_2 \neq 0$ a general solution of the first equation from system 5 reads $f = C_3 \cosh C_2 ew + C_4 \sinh C_2 ew$, where C_3, C_4 are arbitrary constants.

At last, provided $C_1 = C_2 = 0$, a general solution of the first equation from system 5 has the form $f = C_3 w + C_4$, $C_3, C_4 \in \mathbb{R}^1$.

A general solution of the second ODE from system 8.1 is of the form $g = C_1\sqrt{w} + C_2(\sqrt{w})^{-1}$, where C_1, C_2 are arbitrary constants.

Substituting the expression obtained into the first equation we get

$$4w^2\ddot{f} + 4w\dot{f} - e^2(C_1w + C_2)^2 f = 0. \tag{5.4}$$

Under $C_1, C_2 \neq 0$ a solution of ODE (5.4) is not known. In the remaining cases its general solution reads

$$\begin{array}{llll} a) & C_1 \neq 0, & C_2 = 0 & f = Z_0\left[\dfrac{ie}{2}C_1 w\right], \\ b) & C_1 = 0, & C_2 \neq 0 & f = C_3 w^{\frac{eC_2}{2}} + C_4 w^{-\frac{eC_2}{2}}, \\ c) & C_1 = 0, & C_2 = 0 & f = C_3 \ln w + C_4. \end{array}$$

Here C_3, C_4 are arbitrary constants.

We do not succeed in obtaining particular solutions of system 8.2. Equations 14.1 coincide with equations 1, if one changes e by $\frac{e}{4}$. Similarly, equations 14.2 coincide with equations 5, if one changes e by $\frac{e}{4}$. Next, equations 15.1 coincide with equations 1 and equations 15.2 – with equations 5, if one replaces e by $\frac{e}{4}(1+\alpha^2)^{-\frac{1}{2}}$.

System of ODE 16 coincides with system 8.1 and systems 19, 20, 21 – with system 5. We did not succeed in integrating equations 18.

At last, system 22 ($\alpha = 0$) with the substitution $f = g = u(w)$ reduces to the form

$$w\ddot{u} + 2\dot{u} + \frac{e^2}{4}u^3 = 0. \tag{5.5}$$

ODE (5.5) is Emden–Fowler equation and the function $u = e^{-1}w^{-\frac{1}{2}}$, is its particular solution.

Substituting the results obtained into corresponding formulae from (5.1) and then into the ansatz (3.13), we get exact solutions of the nonlinear YME (1.1). Let us note that solutions of systems of ODE 5, 8.1, 14.2, 15.2, 16, 19, 20, 21 satisfying the condition $g = 0$ give rise to Abelian solutions of YME. We do not adduce them and present only non-Abelian solutions of YME.

1. $\vec{A}_\mu = (\vec{e}_2 b_\mu + \vec{e}_3 c_\mu)\sqrt{2}(edx - \lambda)^{-1};$
2. $\vec{A}_\mu = (\vec{e}_2 b_\mu + \vec{e}_3 c_\mu)\left[\lambda\,\mathrm{sn}\left(\frac{\sqrt{2}}{2}e\lambda dx\right)\mathrm{dn}\left(\frac{\sqrt{2}}{2}e\lambda dx\right)\right]\left[\mathrm{cn}\left(\frac{\sqrt{2}}{2}e\lambda dx\right)\right]^{-1};$
3. $\vec{A}_\mu = (\vec{e}_2 b_\mu + \vec{e}_3 c_\mu)\lambda[\mathrm{cn}\,(e\lambda dx)]^{-1};$
4. $\vec{A}_\mu = (\vec{e}_1 b_\mu + \vec{e}_2 c_\mu + \vec{e}_3 d_\mu)\lambda\,\mathrm{cn}\,(e\lambda ax);$
5. $\vec{A}_\mu = \vec{e}_1 k_\mu |kx|^{-1}\sqrt{cx}\,Z_{\frac{1}{4}}\left[\frac{i}{2}e\lambda(cx)^2\right] + \vec{e}_2 b_\mu \lambda cx;$
6. $\vec{A}_\mu = \vec{e}_1 k_\mu |kx|^{-1}[\lambda_1\cosh(e\lambda cx) + \lambda_2\sinh(e\lambda cx)] + \vec{e}_2 b_\mu \lambda;$
7. $\vec{A}_\mu = \vec{e}_1 k_\mu Z_0\left[\frac{i}{2}e\lambda((bx)^2 + (cx)^2)\right] + \vec{e}_2(b_\mu cx - c_\mu bx)\lambda;$
8. $\vec{A}_\mu = \vec{e}_1 k_\mu[\lambda_1((bx)^2 + (cx)^2)^{\frac{e\lambda}{2}} + \lambda_2((bx)^2 + (cx)^2)^{-\frac{e\lambda}{2}}] +$
$+ \vec{e}_2(b_\mu cx - c_\mu bx)\lambda((bx)^2 + (cx)^2)^{-1};$

$$\begin{aligned}
9.\quad \vec{A}_\mu &= \left[\vec{e}_2\left(\frac{1}{8}(d_\mu - k_\mu(kx)^2) + \frac{1}{2}b_\mu kx\right) + \vec{e}_3 c_\mu\right]\lambda\,\mathrm{sn}\left(\frac{e\sqrt{2}}{8}\lambda(4bx+(kx)^2)\right)\times\\
&\times \mathrm{dn}\left(\frac{e\sqrt{2}}{8}\lambda(4bx+(kx)^2)\right)\left(\mathrm{cn}\left(\frac{e\sqrt{2}}{8}\lambda(4bx+(kx)^2)\right)\right)^{-1};\\
10.\quad \vec{A}_\mu &= \left[\vec{e}_2\left(\frac{1}{8}(d_\mu - k_\mu(kx)^2) + \frac{1}{2}b_\mu kx\right) + \vec{e}_3 c_\mu\right]\times\\
&\times \lambda\left[\mathrm{cn}\left(\frac{e\sqrt{2}\lambda}{8}(4bx+(kx)^2)\right)\right]^{-1};\\
11.\quad \vec{A}_\mu &= \left[\vec{e}_2\left(\frac{1}{8}(d_\mu - k_\mu(kx)^2) + \frac{1}{2}b_\mu kx\right) + \vec{e}_3 c_\mu\right]\times\\
&\times 4\sqrt{2}(e(4bx+(kx)^2)-\lambda)^{-1};\\
12.\quad \vec{A}_\mu &= \vec{e}_1 k_\mu\sqrt{4bx+(kx)^2}\,Z_{\frac{1}{4}}\left(\frac{ie\lambda}{8}(4bx+(kx)^2)^2\right) + \vec{e}_2 c_\mu\lambda(4bx+(kx)^2);\\
13.\quad \vec{A}_\mu &= \vec{e}_1 k_\mu\left(\lambda_1\cosh\left(\frac{e\lambda}{4}(4bx+(kx)^2)\right)+\right.\\
&\left.+\lambda_2\sinh\left(\frac{e\lambda}{4}(4bx+(kx)^2)\right)\right) + \vec{e}_2 c_\mu\lambda;\\
14.\quad \vec{A}_\mu &= \left\{\vec{e}_2\left(d_\mu - \frac{1}{8}k_\mu(kx)^2 - \frac{1}{2}b_\mu kx\right)+\right.\\
&\left.+\vec{e}_3\left(\alpha c_\mu + b_\mu + \frac{1}{2}k_\mu kx\right)(1+\alpha^2)^{-\frac{1}{2}}\right\}\times \qquad (5.6)\\
&\times\lambda\,\mathrm{sn}\left[\frac{e\lambda\sqrt{2}}{8}(4(\alpha bx - cx)+\alpha(kx)^2)(1+\alpha^2)^{-\frac{1}{2}}\right]\times\\
&\times\mathrm{dn}\left[\frac{e\lambda\sqrt{2}}{8}(4(\alpha bx - cx)+\alpha(kx)^2)(1+\alpha^2)^{-\frac{1}{2}}\right]\times\\
&\times\left\{\mathrm{cn}\left[\frac{e\lambda\sqrt{2}}{8}(4(\alpha bx - cx)+\alpha(kx)^2)(1+\alpha^2)^{-\frac{1}{2}}\right]\right\}^{-1};\\
15.\quad \vec{A}_\mu &= \left\{\vec{e}_2\left(d_\mu - \frac{1}{8}k_\mu(kx)^2 - \frac{1}{2}b_\mu kx\right)+\right.\\
&\left.+\vec{e}_3\left(\alpha c_\mu + b_\mu + \frac{1}{2}k_\mu kx\right)(1+\alpha^2)^{-\frac{1}{2}}\right\}\times\\
&\times\left\{\mathrm{cn}\left[\frac{e\lambda}{4}(4(\alpha bx - cx)+\alpha(kx)^2)(1+\alpha^2)^{-\frac{1}{2}}\right]\right\}^{-1};\\
16.\quad \vec{A}_\mu &= \left\{\vec{e}_2\left(d_\mu - \frac{1}{8}k_\mu(kx)^2 - \frac{1}{2}b_\mu kx\right)+\right.\\
&\left.+\vec{e}_3\left(\alpha c_\mu + b_\mu + \frac{1}{2}k_\mu kx\right)(1+\alpha^2)^{-\frac{1}{2}}\right\}\times\\
&\times 4\sqrt{2}(1+\alpha^2)^{\frac{1}{2}}[e(4(\alpha bx - cx)+\alpha(kx)^2)]^{-1};\\
17.\quad \vec{A}_\mu &= \vec{e}_1 k_\mu\left\{\sqrt{4(\alpha bx - cx)+\alpha(kx)^2}\times\right.
\end{aligned}$$

$$\times Z_{\frac{1}{4}}\left(\frac{ie\lambda}{8}(4(\alpha bx-cx)+\alpha(kx)^2)^2(1+\alpha^2)^{-\frac{1}{2}}\right)\Big\}+$$
$$+\vec{e}_2\left(\alpha c_\mu+b_\mu+\frac{1}{2}k_\mu kx\right)\lambda(4(\alpha bx-cx)+\alpha(kx)^2)(1+\alpha^2)^{-\frac{1}{2}};$$

18. $$\vec{A}_\mu=\vec{e}_1k_\mu\Big\{\lambda_1\cosh\left[\frac{e\lambda}{4}(1+\alpha^2)^{-\frac{1}{2}}(4(\alpha bx-cx)+\alpha(kx)^2)\right]+$$
$$+\lambda_2\sinh\left[\frac{e\lambda}{4}(1+\alpha^2)^{-\frac{1}{2}}(4(\alpha bx-cx)+\alpha(kx)^2\right]\Big\}+$$
$$+\vec{e}_2\left(\alpha c_\mu+b_\mu+\frac{1}{2}k_\mu kx\right)\lambda(1+\alpha^2)^{-\frac{1}{2}};$$

19. $$\vec{A}_\mu=\vec{e}_1k_\mu|kx|^{-1}Z_0\left[\frac{ie\lambda}{2}((bc)^2+(cx)^2)\right]+\vec{e}_2(b_\mu cx-c_\mu bx)\lambda;$$

20. $$\vec{A}_\mu=\vec{e}_1k_\mu|kx|^{-1}[\lambda_1((bx)^2+(cx)^2)^{\frac{e\lambda}{2}}+\lambda_2((bx)^2+(cx)^2)^{-\frac{e\lambda}{2}}]+$$
$$+\vec{e}_2(b_\mu cx-c_\mu bx)\lambda((bx)^2+(cx)^2)^{-1};$$

21. $$\vec{A}_\mu=\vec{e}_1k_\mu|kx|^{-1}\sqrt{cx}Z_{\frac{1}{4}}\left(\frac{ie\lambda}{2}(cx)^2\right)+\vec{e}_2(b_\mu-k_\mu bx(kx)^{-1})\lambda cx;$$

22. $$\vec{A}_\mu=\vec{e}_1k_\mu|kx|^{-1}[\lambda_1\cosh(\lambda ecx)+\lambda_2\sinh(\lambda ecx)]+\vec{e}_2(b_\mu-k_\mu bx(kx)^{-1})\lambda;$$

23. $$\vec{A}_\mu=\vec{e}_1k_\mu|kx|^{-1}\sqrt{\ln|kx|-cx}Z_{\frac{1}{4}}\left(\frac{ie\lambda}{2}(\ln|kx|-cx)^2\right)+$$
$$+\vec{e}_2(b_\mu-k_\mu bx(kx)^{-1})\lambda(\ln|kx|-cx);$$

24. $$\vec{A}_\mu=\vec{e}_1k_\mu|kx|^{-1}[\lambda_1\cosh(\lambda e(\ln|kx|-cx))+\lambda_2\sinh(\lambda e(\ln|kx|-cx))]+$$
$$+\vec{e}_2(b_\mu-k_\mu bx(kx)^{-1})\lambda;$$

25. $$\vec{A}_\mu=\vec{e}_1k_\mu|kx|^{-1}\sqrt{\alpha\ln|kx|-cx}Z_{\frac{1}{4}}\left(\frac{ie\lambda}{2}(\alpha\ln|kx|-cx)^2\right)+$$
$$+\vec{e}_2(b_\mu-k_\mu(bx-\ln|kx|)(kx)^{-1})\lambda(\alpha\ln|kx|-cx);$$

26. $$\vec{A}_\mu=\vec{e}_1k_\mu|kx|^{-1}[\lambda_1\cosh(\lambda e(\alpha\ln|kx|-cx))+$$
$$+\lambda_2\sinh(\lambda e(\alpha\ln|kx|-cx))]+\vec{e}_2(b_\mu-k_\mu(bx-\ln|kx|)(kx)^{-1})\lambda;$$

27. $$\vec{A}_\mu=\{\vec{e}_1(b_\mu-k_\mu bx(kx)^{-1})+\vec{e}_2(c_\mu-k_\mu cx(kx)^{-1})\}e^{-1}(x_\mu x^\mu)^{-\frac{1}{2}};$$

28. $$\vec{A}_\mu=\{\vec{e}_1(b_\mu-k_\mu bx(kx)^{-1})+\vec{e}_2(c_\mu-k_\mu cx(kx)^{-1})\}f(x_\mu x^\mu),$$
$$w\ddot{f}+2\dot{f}+(e^2f^3/4)=0,\quad w=x_\mu x^\mu=(ax)^2-(bx)^2-(cx)^2-(dx)^2.$$

In the above formulae $Z_\alpha(w)$ is the Bessel function; sn, dn, cn are Jacobi elliptic functions having the modulus $\frac{\sqrt{2}}{2}$; $\lambda,\lambda_1,\lambda_2=\text{const}$.

In the present paper we do not analyze in detail the obtained solution. We only note that the solutions numbered by 27 is nothing more but the meron solution of YME [2]. In the Euclidean space meron and instanton solutions were obtained by Alfaro, Fubini, Furlan [9] and Belavin, Polyakov, Schwartz, Tyupkin [10] with the use of the ansatz suggested by 't Hooft [11], Corrigan and Fairlie [12] and Wilczek [13].

Another important point is that we can obtain new exact solutions of YME by applying to solutions (5.6) the solution generation technique. We do not adduce corresponding formulae because of their cumbersomity.

6 Some generalizations

It was noticed in [14] that group-invariant solutions of nonlinear PDE could provide us with rather general information about the structure of solutions of the equation under study. Using this fact, we constructed in [4, 14] a number of new exact solutions of the nonlinear Dirac equation which could not be obtained by symmetry reduction procedure. We will demonstrate that the same idea will be effective for constructing new solutions of YME.

Solutions of YME numbered by 7, 8, 19, 20 can be presented in the following unified form:

$$\vec{A}_\mu = k_\mu \vec{B}(kx, cx) + b_\mu \vec{C}(kx, cx), \tag{6.1}$$

where $kx = k_\mu x^\mu$, $cx = c_\mu x^\mu$, $k_\mu = a_\mu + d_\mu$.

Substituting the ansatz (6.1) into YME and splitting the equality obtained with respect to linearly-independent four-vectors with components k_μ, b_μ, c_μ, we get

$$\begin{aligned} &1. \quad \vec{C}_{w_1 w_1} = \vec{0}, \\ &2. \quad \vec{C} \times \vec{C}_{w_1} = \vec{0}, \\ &3. \quad \vec{B}_{w_1 w_1} + e\vec{C}_{w_0} \times \vec{C} + e^2 \vec{C} \times (\vec{C} \times \vec{B}) = \vec{0}. \end{aligned} \tag{6.2}$$

Here we use designations $w_0 = kx$, $w_1 = cx$.

A general solution of the first two equations from (6.2) is given by one of the formulae

$$\begin{aligned} &\text{I.} \quad \vec{C} = \vec{f}(w_0), \\ &\text{II.} \quad \vec{C} = (w_1 + v_0(w_0))\vec{f}(w_0), \end{aligned}$$

where v_0, $\vec{f}$ are arbitrary smooth functions.

Consider the case $\vec{C} = \vec{f}(w_0)$. Substituting this expression into the third equation from (6.2) we have

$$\vec{B}_{w_1 w_1} + e\vec{f}_{w_0} \times \vec{f} + e^2 \vec{f}(\vec{f}\vec{B}) - e^2 \vec{f}^2 \vec{B} = \vec{0}. \tag{6.3}$$

Since equations (6.3) do not contain derivatives of $\vec{B}$ with respect to w_0, they can be considered as a system of ODE with respect to the variable w_1. Multiplying (6.3) by $\vec{f}$ we arrive at the relation $(\vec{B}\vec{f})_{w_1 w_1} = 0$, whence

$$\vec{B}\vec{f} = v_1(w_0) w_1 + v_2(w_0). \tag{6.4}$$

In (6.4) v_1, v_2 are arbitrary smooth enough functions.

With account of (6.4) system (6.3) reads

$$\vec{B}_{w_1 w_1} - e^2 \vec{f}^2 \vec{B} = e\vec{f} \times \vec{f}_{w_0} - e^2 (v_1 w_1 + v_2)\vec{f}.$$

The above linear system of ODE is easily integrated. Its general solution is given by the formula

$$\begin{aligned} \vec{B} = {} & \vec{g}(w_0) \cosh e|\vec{f}| w_1 + \vec{h}(w_0) \sinh e|\vec{f}| w_1 + \\ & + e^{-1} |\vec{f}|^{-2} \vec{f_{w_0}} \times \vec{f} + |\vec{f}|^{-2} (v_1 w_1 + v_2) \vec{f}, \end{aligned} \tag{6.5}$$

where $\vec{g}$, $\vec{h}$ are arbitrary smooth functions.

Substituting (6.5) into (6.4) we get the following restrictions on the choice of the functions $\vec{g}$, $\vec{h}$:

$$\vec{f}\vec{g} = 0, \quad \vec{f}\vec{h} = 0. \tag{6.6}$$

Thus, provided $\vec{C}_{w_1} = 0$, a general solution of the system of ODE (6.3) is given by the formulae (6.5), (6.6). Substituting (6.5) into the initial ansatz (6.1) we obtain the following family of exact solutions of YME:

$$\vec{A}_\mu = k_\mu\{\vec{g}(kx)\cosh e|\vec{f}|cx + \vec{h}(kx)\sinh e|\vec{f}|cx + \\ + e^{-1}|\vec{f}|^{-2}\dot{\vec{f}} \times \vec{f} + (v_1(kx)cx + v_2(kx))\vec{f}\} + b_\mu\vec{f}$$

where $\vec{f}(kx)$, $\vec{g}(kx)$, $\vec{h}(kx)$, $v_1(kx)$, $v_2(kx)$ are arbitrary smooth functions satisfying (6.6), $\dot{\vec{f}} = \frac{d\vec{f}}{d\omega_0}$.

The case $\vec{C} = (w_1 + v_0(w_0))\vec{f}(w_0)$ is treated in analogous way. As a result, we obtain the following family of exact solutions of YME:

$$\vec{A}_\mu = k_\mu\Bigg\{(cx + v_0(kx))^{\frac{1}{2}}\left[\vec{g}(kx)J_{\frac{1}{4}}\left(\frac{ie}{2}|\vec{f}|(\vec{c}\vec{x} + v_0(kx))^2\right) + \right. \\ \left. + \vec{h}(kx)Y_{\frac{1}{4}}\left(\frac{ie}{2}|\vec{f}|(cx + v_0(kx))^2\right)\right] + \\ + (v_1(kx)cx + v_2(kx))\vec{f} + e^{-1}|\vec{f}|^{-2}\dot{\vec{f}} \times \vec{f}\Bigg\} + b_\mu(cx + v_0(kx))\vec{f},$$

where $\vec{f}(kx)$, $\vec{g}(kx)$, $\vec{h}(kx)$, $v_0(kx)$, $v_1(kx)$, $v_2(kx)$ are arbitrary smooth functions satisfying (6.6), $J_{\frac{1}{4}}(w)$, $Y_{\frac{1}{4}}(w)$ are the Bessel functions.

Another effective ansatz for the Yang–Mills field is obtained if one replaces in (6.1) cx by bx

$$\vec{A}_\mu = k_\mu\vec{B}(kx, bx) + b_\mu\vec{C}(kx, bx). \tag{6.7}$$

Substitution of (6.7) into YME yields the following system of PDE for $\vec{B}$, $\vec{C}$:

$$\vec{B}_{w_1w_1} - \vec{C}_{w_0w_1} - e(\vec{B} \times \vec{C}_{w_1} + 2\vec{B}_{w_1} \times \vec{C} + \vec{C} \times \vec{C}_{w_0}) + e^2\vec{C} \times (\vec{C} \times \vec{B}) = \vec{0}. \tag{6.8}$$

We succeeded in integrating system (6.8), provided $\vec{C} = \vec{f}(w_0)$. Substituting the result obtained into (6.7), we come to the following family of exact solutions of YME:

$$\vec{A}_\mu = k_\mu\{(\vec{g} + |\vec{f}|^{-1}\vec{g} \times \vec{f}bx)\cos(e|\vec{f}|bx) + (\vec{h} + |\vec{f}|^{-1}\vec{h} \times \vec{f}bx)\sin(e|\vec{f}|bx) + \\ + e^{-1}|\vec{f}|^{-2}\dot{\vec{f}} \times \vec{f} + (v_1(kx)bx + v_2(kx))\vec{f}\} + b_\mu\vec{f},$$

where $\vec{f}(kx)$, $\vec{g}(kx)$, $\vec{h}(kx)$, $v_1(kx)$, $v_2(kx)$ are arbitrary smooth functions.

Besides that, we obtained the following class of exact solutions of YME:

$$\vec{A}_\mu = k_\mu\vec{e}_1v_0(kx)u^2(bx) + b_\mu\vec{e}_2u(bx),$$

where $\vec{e}_1 = (1, 0, 0)$, $\vec{e}_2 = (0, 1, 0)$; $v_0(kx)$ is an arbitrary smooth function; $u(bx)$ is a solution of the nonlinear ODE $\ddot{u} = e^2u^5$, which is integrated in elliptic functions.

In conclusion of this Section we will obtain a generalization of the plane-wave Coleman solution [15]]

$$\vec{A}_\mu = k_\mu(\vec{f}(kx)bx + \vec{g}(kx)cx). \tag{6.9}$$

It is not difficult to verify that (6.9) satisfy YME with arbitrary $\vec{f}$, $\vec{g}$.

Evidently, solution (6.9) is a particular case of the ansatz

$$\vec{A}_\mu = k_\mu \vec{B}(kx, bx, cx). \tag{6.10}$$

Substituting (6.10) into YME we get

$$\vec{B}_{w_1 w_1} + \vec{B}_{w_2 w_2} = \vec{0}, \tag{6.11}$$

where $w_1 = bx$, $w_2 = cx$.

Integrating the Laplace equations (6.11) and substituting the result obtained into (6.10) we have

$$\vec{A}_\mu = k_\mu(\vec{U}(kx, bx + icx) + \vec{U}(kx, bx - icx)).$$

Here $\vec{U}(kx, z)$ is an arbitrary analytical with respect to z function. Choosing $\vec{U} = \frac{1}{2}(\vec{f}(kx) - i\vec{g}(kx))z$ we get Coleman solution (6.9).

7 Conclusion

Thus, starting from the invariance of YME under the Poincaré group we have obtained wide families of its exact solutions including arbitrary functions. In our future papers we intend to describe exact solutions of YME invariant under the extended Poincaré group and conformal group.

Besides that, we will study exact solutions which correspond to the conditional and non-local symmetries of the Yang–Mills equations (1.1)

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1. Fushchych W.I., Shtelen W.M., Serov N.I., Symmetry analysis and exact solutions of nonlinear equations of mathematical physics, Kiev, Naukova Dumka, 1989.
2. Actor A., Classical solutions of $SU(2)$ Yang–Mills theories, *Rev. Mod. Phys.*, 1979, **51**, № 3, 461–525.
3. Schwarz F., Symmetry of $SU(2)$-invariant Yang–Mills theories, *Lett. Math. Phys.*, 1982, **6**, № 5, 355–359.
4. Fushchych W.I., Zhdanov R.Z. Symmetry and exact solutions of nonlinear spinor equations, *Phys. Repts.*, 1989, **46**, № 2, 325–365.
5. Ovsiannikov L.V., Group analysis of differential equations, Moscow, Nauka, 1978.
6. Patera J., Winternitz P., Zassenhaus H., Continuous subgroups of the fundamental groups of physics. 1. General method and the Poincaré group, *J. Math. Phys.*, 1975, **16**, 1597–1624.
7. Fushchych W.I., Barannik L.F., Barannik A.F., Subgroup analysis of the Galilei and Poincaré groups and reduction of nonlinear equations, Kiev, Naukova Dumka, 1991.
8. Kamke E., Handbook on Ordinary Differential Equations, Nauka, Moscow, 1976.
9. De Alfaro V., Fubini S., Furlan G., A new classical solutions of the Yang–Mills field equations, *Phys. Lett. B*, 1977, **65**, № 2, 163–166.

10. Belavin A.A., Polyakov A.M., Tyupkin Yu.S., Schwartz A.S., Pseudoparticle solutions of the Yang–Mills equations, *Phys. Lett. B*, 1975, **59**, № 1, 85–87.

11. ’t Hooft G., Computation of the quantum effects due to a four-dimensional pseudoparticle, *Phys. Rev. D*, 1976, **14**, № 12, 3432–3450.

12. Corrigan E., Fairlie D.B., Scalar field theory and exact solutions to a classical $SU(2)$ gange theory, *Phys. Lett. B*, 1977, **67**, 69–74.

13. Wilczek F., Geometry and interaction of instantons, in Quark Confinement and Field Theory, New York, Wiley, 1977, 211–219.

14. Fushchych W.I. and Zhdanov R.Z., Nonlinear spinor equations: symmetry and exact solutions, Kiev, Naukova Dumka, 1992.

15. Coleman S., Classical lumps and their quantum descendants, in New Phenomena in Subnuclear Physics (Erice, 1975), Editor A. Zichichi, New York, 1977.

Conditional symmetry and new classical solutions of the Yang–Mills equations

R.Z. ZHDANOV, W.I. FUSHCHYCH

We suggest an effective method for reducing the Yang–Mills equations to systems of ordinary differential equations. With the use of this method we construct the extensive families of new exact solutions of the Yang–Mills equations. Analysis of the solutions thus obtained shows that they correspond to the conditional (non-classical) symmetry of the equations under study.

1 Introduction

A majority of papers devoted to construction of explicit form of the exact solutions of $SU(2)$ Yang–Mills equations (YMEs)

$$\begin{gathered}\partial_\nu\partial^\nu \boldsymbol{A}_\mu - \partial^\mu\partial_\nu \boldsymbol{A}_\nu + e\big((\partial_\nu \boldsymbol{A}_\nu)\times\boldsymbol{A}_\mu - 2(\partial_\nu\boldsymbol{A}_\mu)\times\boldsymbol{A}_\nu + \\ +(\partial^\mu\boldsymbol{A}_\nu)\times\boldsymbol{A}^\nu\big) + e^2\boldsymbol{A}_\nu\times(\boldsymbol{A}^\nu\times\boldsymbol{A}_\mu) = \boldsymbol{0}\end{gathered} \tag{1}$$

are based on the ansätze for the Yang–Mills field $\boldsymbol{A}_\mu(x)$ suggested by Wu and Yang, Rosen, 't Hooft, Corrigan and Fairlie, Wilczek, Witten (see [1] and references therein). There were further developments for the self-dual YMEs (which form the first-order system of nonlinear partial differential equations such that system (1) is its differential consequence). Let us mention the Atiyah–Hitchin–Drinfeld–Manin method for obtaining instanton solutions [2] and its generalization due to Nahm. However, the solution set of the self-dual YMEs is only a subset of solutions of YMEs (1) and the problem of construction of new non self-dual solutions of system (1) is, in fact, completely open (see, also [1]). As the development of new approaches to the construction of exact solutions of YMEs is a very interesting mathematical problem, it may also be of importance for physics. The reason is that all famous mathematical models of elementary particles such as solitons, instantons, merons are quite simply particular solutions of some nonlinear partial differential equations.

A natural approach to construction of particular solutions of YMEs (1) is to utilize their symmetry properties in the way as it is done in [9, 10, 16] (see, also [15], where the reduction of the Euclidean self-dual YMEs is considered). The apparatus of the theory of Lie transformation groups makes it possible to reduce system of partial differential equations (PDEs) (1) to systems of nonlinear ordinary differential equations (ODEs) by using special ansätze (invariant solutions) [10, 18, 20]. If one succeeds in constructing general or particular solutions of the said ODEs (which is an extremely difficult problem), then on substituting the results in the corresponding ansätze one gets exact solutions of the initial system of PDEs (1).

Another possibility of construction of exact solutions of YMEs is to use their conditional (non-Lie) symmetry (for more details about conditional symmetry of equations of mathematical physics, see [6, 8] and also [10, 12]) which has much in common with

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a “non-classical symmetry” of PDEs by Bluman and Cole [3] (see also [17, 19]) and “direct method of reduction of PDEs” by Clarkson and Kruskal [4]. But the prospects of a systematic and exhaustive study of conditional symmetry of system of twelve second-order nonlinear PDEs (1) seem to be rather remote. It should be said that so far there is no complete description of conditional symmetry of the nonlinear wave equation even in the case of one space variable.

A principal idea of the method of ansätze, as well as of the direct method of reduction of PDEs, is a special choice of the class of functions to which a possible solution should belong. Within the framework of the above methods, a solution of system (1) is sought in the form

$$\boldsymbol{A}_\mu = H_\mu\big(x, \boldsymbol{B}_\nu(\omega(x))\big), \quad \mu = \overline{0,3},$$

where H_μ are smooth functions chosen in such a way that substitution of the above expressions into the Yang–Mills equations results in a system of ODEs for “new” unknown vector-functions $\boldsymbol{B}_\nu$ of one variable ω. However, the problem of reduction of YMEs posed in this way seemed to be hopeless. Really, if we restrict ourselves to the case of a linear dependence of the above ansatz on $\boldsymbol{B}_\nu$

$$\boldsymbol{A}_\mu(x) = R_{\mu\nu}(x)\boldsymbol{B}^\nu(\omega), \tag{2}$$

where $\boldsymbol{B}_\nu(\omega)$ are new unknown vector-functions, $\omega = \omega(x)$ is a new independent variable, then a requirement of reduction of (1) to a system of ODEs by virtue of (2) gives rise to a system of nonlinear PDEs for 17 unknown functions $R_{\mu\nu}$, ω. What is more, the system obtained is no way simpler than the initial Yang–Mills equations (1). It means that some additional information about the structure of the matrix function $R_{\mu\nu}$ should be input into the ansatz (2). This can be done in various ways. But the most natural one is to use the information about the structure of solutions provided by the Lie symmetry of the equation under study.

In [11] we suggest an effective approach to the study of conditional symmetry of the nonlinear Dirac equation based on its Lie symmetry. We have observed that all Poincaré-invariant ansätze for the Dirac field $\psi(x)$ can be represented in the unified form by introducing several arbitrary elements (functions) $u_1(x), u_2(x), \ldots, u_N(x)$. As a result, we get an ansatz for the field $\psi(x)$ which reduces the nonlinear Dirac equation to system of ODEs provided functions $u_i(x)$ satisfy some compatible over-determined system of nonlinear PDEs. After integrating it, we have obtained a number of new ansätze that cannot in principle be obtained within the framework of the classical Lie approach.

In the present paper we will demonstrate that the same idea proves to be fruitful for obtaining new (non-Lie) reductions of YMEs and for constructing new exact solutions of system (1).

2 Reduction of YMEs

In the paper [16] we have obtained a complete list of $P(1,3)$-inequivalent ansätze for the Yang–Mills field which are invariant under the three-parameter subgroups of the Poincaré group $P(1,3)$. Analyzing these ansätze we come to conclusion that they can be represented in the unified form (2), where $\boldsymbol{B}_\nu(\omega)$ are new unknown vector functions, $\omega = \omega(x)$ is a new independent variable and functions $R_{\mu\nu}(x)$ are given by the formulae

$$\begin{aligned} R_{\mu\nu}(x) = {} & (a_\mu a_\nu - d_\mu d_\nu)\cosh\theta_0 + (a_\mu d_\nu - d_\mu a_\nu)\sinh\theta_0 + 2(a_\mu + d_\mu)\times \\ & \times[(\theta_1\cos\theta_3 + \theta_2\sin\theta_3)b_\nu + (\theta_2\cos\theta_3 - \theta_1\sin\theta_3)c_\nu + \\ & + (\theta_1^2+\theta_2^2)e^{-\theta_0}(a_\nu + d_\nu)] - (c_\mu c_\nu + b_\mu b_\nu)\cos\theta_3 - \\ & - (c_\mu b_\nu - b_\mu c_\nu)\sin\theta_3 - 2e^{-\theta_0}(\theta_1 b_\mu + \theta_2 c_\mu)(a_\nu + d_\nu). \end{aligned} \tag{3}$$

In (3) $\theta_\mu(x)$ are some smooth functions and what is more $\theta_a = \theta_a(\xi, b_\mu x^\mu, c_\mu x^\mu)$, $a = 1, 2$; $\xi = \frac{1}{2}k_\mu x^\mu = \frac{1}{2}(a_\mu x^\mu + d_\mu x^\mu)$; a_μ, b_μ, c_μ, d_μ are arbitrary constants satisfying the following relations:

$$\begin{aligned} & a_\mu a^\mu = -b_\mu b^\mu = -c_\mu c^\mu = -d_\mu d^\mu = 1, \\ & a_\mu b^\mu = a_\mu c^\mu = a_\mu d^\mu = b_\mu c^\mu = b_\mu d^\mu = c_\mu d^\mu = 0. \end{aligned}$$

Hereafter, summation over the repeated indices from 0 to 3 is understood. Raising and lowering of the indices is performed with the help of the tensor $g_{\mu\nu} = \mathrm{diag}(1,-1,-1,-1)$, e.g. $R^\alpha_\mu = g_{\alpha\beta}R_{\beta\mu}$.

A choice of the functions $\omega(x)$, $\theta_\mu(x)$ is determined by the requirement that substitution of the ansatz (2) in the YMEs yields a system of ODEs for the vector function $\boldsymbol{B}_\mu(\omega)$.

By the direct check one can convince one self that the following assertion holds true.

Lemma. *Ansatz (2), (3) reduces YMEs (1) to system of ODEs iff the functions $\omega(x)$, $\theta_\mu(x)$ satisfy the following system of PDEs:*

$$\omega_{x_\mu}\omega_{x^\mu} = F_1(\omega), \tag{4a}$$

$$\Box\omega = F_2(\omega), \tag{4b}$$

$$R_{\alpha\mu}\omega_{x_\alpha} = G_\mu(\omega), \tag{4c}$$

$$R_{\alpha\mu x_\alpha} = H_\mu(\omega), \tag{4d}$$

$$R^\alpha_\mu R_{\alpha\nu x_\beta}\omega_{x^\beta} = Q_{\mu\nu}(\omega), \tag{4e}$$

$$R^\alpha_\mu \Box R_{\alpha\nu} = S_{\mu\nu}(\omega), \tag{4f}$$

$$R^\alpha_\mu R_{\alpha\nu x_\beta}R_{\beta\gamma} + R^\alpha_\nu R_{\alpha\gamma x_\beta}R_{\beta\mu} + R^\alpha_\gamma R_{\alpha\mu x_\beta}R_{\beta\nu} = T_{\mu\nu\gamma}(\omega), \tag{4g}$$

where F_1, F_2, G_μ, …, $T_{\mu\nu\gamma}$ are some smooth functions, $\mu, \nu, \gamma = \overline{0,3}$. And what is more, a reduced equation has the form

$$\begin{aligned} & k_{\mu\gamma}\ddot{\boldsymbol{B}}^\gamma + l_{\mu\gamma}\dot{\boldsymbol{B}}^\gamma + m_{\mu\gamma}\boldsymbol{B}^\gamma + eq_{\mu\nu\gamma}\dot{\boldsymbol{B}}^\nu \times \boldsymbol{B}^\gamma + eh_{\mu\nu\gamma}\boldsymbol{B}^\nu \times \boldsymbol{B}^\gamma + \\ & \quad + e^2\boldsymbol{B}_\gamma \times (\boldsymbol{B}^\gamma \times \boldsymbol{B}_\mu) = \boldsymbol{0}, \end{aligned} \tag{5}$$

where

$$\begin{aligned} & k_{\mu\gamma} = g_{\mu\gamma}F_1 - G_\mu G_\gamma, \\ & l_{\mu\gamma} = g_{\mu\gamma}F_2 + 2Q_{\mu\gamma} - G_\mu H_\gamma - G_\mu \dot{G}_\gamma, \\ & m_{\mu\gamma} = S_{\mu\gamma} - G_\mu \dot{H}_\gamma, \\ & q_{\mu\nu\gamma} = g_{\mu\gamma}G_\nu + g_{\nu\gamma}G_\mu - 2g_{\mu\nu}G_\gamma, \\ & h_{\mu\nu\gamma} = \frac{1}{2}(g_{\mu\gamma}H_\nu - g_{\mu\nu}H_\gamma) - T_{\mu\nu\gamma}. \end{aligned} \tag{6}$$

Thus, to describe all ansätze of the form (2), (3) reducing the YMEs to a system of ODEs one has to construct the general solution of the over-determined system of PDEs (3), (4). Let us emphasize that system (3), (4) is compatible, since the ansätze for the Yang–Mills field $\boldsymbol{A}_\mu(x)$ invariant under the three-parameter subgroups of the Poincaré group satisfy equations (3), (4) with some specific choice of the functions $F_1, F_2, \ldots, T_{\mu\nu\gamma}$ [16].

Integration of system of nonlinear PDEs (3), (4) demands a huge amount of computations. That is why we present here only the principal idea of our approach to solving the system (3), (4). When integrating it we use essentially the fact that the general solution of system of equations (4a), (4b) is known [13]. With $\omega(x)$ already known we proceed to integration of linear PDEs (4c), (4d). Next, we substitute the results obtained in the remaining equations (4) and get the final form of the functions $\omega(x)$, $\theta_\mu(x)$.

Before presenting the results of integration of system of PDEs (3), (4) we make a remark. As the direct check shows, the structure of the ansatz (2), (3) is not altered by the change of variables

$$\begin{aligned}
&\omega \to \omega' = T(\omega), \quad \theta_0 \to \theta_0' = \theta_0 + T_0(\omega),\\
&\theta_1 \to \theta_1' = \theta_1 + e^{\theta_0}\big(T_1(\omega)\cos\theta_3 + T_2(\omega)\sin\theta_3\big),\\
&\theta_2 \to \theta_2' = \theta_2 + e^{\theta_0}\big(T_2(\omega)\cos\theta_3 - T_1(\omega)\sin\theta_3\big),\\
&\theta_3 \to \theta_3' = \theta_3 + T_3(\omega),
\end{aligned} \tag{7}$$

where $T(\omega)$, $T_\mu(\omega)$ are arbitrary smooth functions. That is why, solutions of system (3), (4) connected by the relations (7) are considered as equivalent.

Integrating the system of PDEs within the above equivalence relations we obtain the set of ansätze containing the ones equivalent to the Poincaré-invariant ansätze. We list below the corresponding expressions for the functions θ_μ, ω:

$$\theta_\mu = 0, \quad \omega = d\cdot x; \tag{8a}$$

$$\theta_\mu = 0, \quad \omega = a\cdot x; \tag{8b}$$

$$\theta_\mu = 0, \quad \omega = k\cdot x; \tag{8c}$$

$$\begin{aligned}
&\theta_0 = -\ln|k\cdot x|, \quad \theta_1 = \theta_2 = 0, \quad \theta_3 = \alpha\ln|k\cdot x|,\\
&\omega = (a\cdot x)^2 - (d\cdot x)^2;
\end{aligned} \tag{8d}$$

$$\theta_0 = -\ln|k\cdot x|, \quad \theta_1 = \theta_2 = \theta_3 = 0, \quad \omega = c\cdot x; \tag{8e}$$

$$\theta_0 = -b\cdot x, \quad \theta_1 = \theta_2 = \theta_3 = 0, \quad \omega = c\cdot x; \tag{8f}$$

$$\theta_0 = -b\cdot x, \quad \theta_1 = \theta_2 = \theta_3 = 0, \quad \omega = b\cdot x - \ln|k\cdot x|; \tag{8g}$$

$$\begin{aligned}
&\theta_0 = \alpha\arctan(b\cdot x/c\cdot x), \quad \theta_1 = \theta_2 = 0,\\
&\theta_3 = -\arctan(b\cdot x/c\cdot x), \quad \omega = (b\cdot x)^2 + (c\cdot x)^2;
\end{aligned} \tag{8h}$$

$$\theta_0 = \theta_1 = \theta_2 = 0, \quad \theta_3 = -a\cdot x, \quad \omega = d\cdot x; \tag{8i}$$

$$\theta_0 = \theta_1 = \theta_2 = 0, \quad \theta_3 = d\cdot x, \quad \omega = a\cdot x; \tag{8j}$$

$$\theta_0 = \theta_1 = \theta_2 = 0, \quad \theta_3 = -\frac{1}{2}k\cdot x, \quad \omega = a\cdot x - d\cdot x; \tag{8k}$$

$$\theta_0 = 0, \quad \theta_1 = \frac{1}{2}(b\cdot x - \alpha c\cdot x)(k\cdot x)^{-1}, \quad \theta_2 = \theta_3 = 0, \quad \omega = k\cdot x; \tag{8l}$$

$$\theta_0 = \theta_2 = \theta_3 = 0, \quad \theta_1 = \frac{1}{2}c\cdot x, \quad \omega = k\cdot x; \tag{8m}$$

$$\theta_0 = \theta_2 = \theta_3 = 0, \quad \theta_1 = -\frac{1}{4}k\cdot x, \quad \omega = 4b\cdot x + (k\cdot x)^2; \tag{8n}$$

$$\theta_0 = \theta_2 = \theta_3 = 0, \quad \theta_1 = -\frac{1}{4}k\cdot x, \quad \omega = 4(\alpha b\cdot x - c\cdot x) + \alpha(k\cdot x)^2; \tag{8o}$$

$$\begin{gathered}\theta_0 = -\ln|k\cdot x|, \quad \theta_1 = \theta_2 = 0, \\ \theta_3 = -\arctan(b\cdot x/c\cdot x), \quad \omega = (b\cdot x)^2 + (c\cdot x)^2;\end{gathered} \tag{8p}$$

$$\begin{gathered}\theta_0 = \theta_3 = 0, \quad \theta_1 = \frac{1}{2}(c\cdot x + (\alpha + k\cdot x)b\cdot x)(1 + k\cdot x(\alpha + k\cdot x))^{-1}, \\ \theta_2 = -\frac{1}{2}(b\cdot x - c\cdot xk\cdot x)(1 + k\cdot x(\alpha + k\cdot x))^{-1}, \quad \omega = k\cdot x;\end{gathered} \tag{8q}$$

$$\begin{gathered}\theta_0 = -\ln|k\cdot x|, \quad \theta_1 = \frac{1}{2}b\cdot x(k\cdot x)^{-1}, \quad \theta_2 = \theta_3 = 0, \\ \omega = (a\cdot x)^2 - (b\cdot x)^2 - (d\cdot x)^2;\end{gathered} \tag{8r}$$

$$\theta_0 = -\ln|k\cdot x|, \quad \theta_1 = \frac{1}{2}b\cdot x(k\cdot x)^{-1}, \quad \theta_2 = \theta_3 = 0, \quad \omega = c\cdot x; \tag{8s}$$

$$\begin{gathered}\theta_0 = -\ln|k\cdot x|, \quad \theta_1 = \frac{1}{2}b\cdot x(k\cdot x)^{-1}, \quad \theta_2 = \theta_3 = 0, \\ \omega = \ln|k\cdot x| - c\cdot x;\end{gathered} \tag{8t}$$

$$\begin{gathered}\theta_0 = -\ln|k\cdot x|, \quad \theta_1 = \frac{1}{2}(b\cdot x - \ln|k\cdot x|)(k\cdot x)^{-1}, \\ \theta_2 = \theta_3 = 0, \quad \omega = \alpha\ln|k\cdot x| - c\cdot x;\end{gathered} \tag{8u}$$

$$\begin{gathered}\theta_0 = -\ln|k\cdot x|, \quad \theta_1 = \frac{1}{2}b\cdot x(k\cdot x)^{-1}, \quad \theta_2 = \frac{1}{2}c\cdot x(k\cdot x)^{-1}, \\ \theta_3 = \alpha\ln|k\cdot x|, \quad \omega = (a\cdot x)^2 - (b\cdot x)^2 - (c\cdot x)^2 - (d\cdot x)^2,\end{gathered} \tag{8v}$$

where $a\cdot x$ stands for $a_\mu x^\mu$ and α is an arbitrary real constant.

We do not consider reduction of YMEs with the help of the above ansätze, because it is studied in a great detail in [16].

We concentrate on the cases when the new (non-Lie) ansätze are obtained. It occurs that the procedure described gives rise to non-Lie ansätze provided the functions $\omega(x)$, $\theta_\mu(x)$ within the equivalence relations (7) have the form

$$\theta_\mu = \theta_\mu(\xi, b_\nu x^\nu, c_\nu x^\nu), \quad \omega = \omega(\xi, b_\nu x^\nu, c_\nu x^\nu). \tag{9}$$

The list of inequivalent solutions of system of PDEs (3), (4) satisfying (9) is exhausted by the following solutions:

$$\theta_0 = \theta_3 = 0, \quad \omega = \frac{1}{2}k\cdot x, \quad \theta_1 = w_0(\xi)b\cdot x + w_1(\xi)c\cdot x, \\ \theta_2 = w_2(\xi)b\cdot x + w_3(\xi)c\cdot x; \tag{10a}$$

$$\omega = b\cdot x + w_1(\xi), \quad \theta_0 = \alpha\big(c\cdot x + w_2(\xi)\big), \\ \theta_a = -\frac{1}{4}\dot w_a(\xi), \quad a = 1,2, \quad \theta_3 = 0, \tag{10b}$$

$$\theta_0 = T(\xi), \quad \theta_3 = w_1(\xi), \quad \omega = b\cdot x\cos w_1 + c\cdot x\sin w_1 + w_2(\xi), \\ \theta_1 = \left(\frac{1}{4}(\varepsilon e^T + \dot T)(b\cdot x\sin w_1 - c\cdot x\cos w_1) + w_3(\xi)\right)\sin w_1 + \\ + +\frac{1}{4}\big(\dot w_1(b\cdot x\sin w_1 - c\cdot x\cos w_1) - \dot w_2\big)\cos w_1, \\ \theta_2 = -\left(\frac{1}{4}(\varepsilon e^T + \dot T)(b\cdot x\sin w_1 - c\cdot x\cos w_1) + w_3(\xi)\right)\cos w_1 + \\ + \frac{1}{4}\big(\dot w_1(b\cdot x\sin w_1 - c\cdot x\cos w_1) - \dot w_2\big)\sin w_1; \tag{10c}$$

$$\theta_0 = 0, \quad \theta_3 = \arctan\big([c\cdot x + w_2(\xi)][b\cdot x + w_1(\xi)]^{-1}\big), \\ \theta_a = -\frac{1}{4}\dot w_a(\xi), \quad a = 1,2, \quad \omega = \big([b\cdot x + w_1(\xi)]^2 + [c\cdot x + w_2(\xi)]^2\big)^{1/2}. \tag{10d}$$

Here $\alpha \neq 0$ is an arbitrary constant, $\varepsilon = \pm 1$, w_0, w_1, w_2, w_3 are arbitrary smooth functions on $\xi = \frac{1}{2}k\cdot x$, $T = T(\xi)$ is a solution of the nonlinear ODE

$$(\dot T + \varepsilon e^T)^2 + \dot w_1^2 = \varkappa e^{2T}, \quad \varkappa \in \mathbb{R}^1, \tag{11}$$

where a dot over the symbol denotes differentiation with respect to ξ.

Substitution of the ansatz (2), where $R_{\mu\nu}(x)$ are given by formulae (3), (10), in the YMEs yields systems of nonlinear ODEs of the form (5), where

$$k_{\mu\gamma} = -\frac{1}{4}k_\mu k_\gamma, \quad l_{\mu\gamma} = -(w_0 + w_3)k_\mu k_\gamma, \\ m_{\mu\gamma} = -4\ (w_0^2 + w_1^2 + w_2^2 + w_3^2)k_\mu k_\gamma - (\dot w_0 + \dot w_3)k_\mu k_\gamma, \\ q_{\mu\nu\gamma} = \frac{1}{2}(g_{\mu\gamma}k_\nu + g_{\nu\gamma}k_\mu - 2g_{\mu\nu}k_\gamma), \\ h_{\mu\nu\gamma} = (w_0 + w_3)(g_{\mu\gamma}k_\nu - g_{\mu\nu}k_\gamma) + \\ + 2(w_1 - w_2)\big((k_\mu b_\nu - k_\nu b_\mu)\ c_\gamma + (b_\mu c_\nu - b_\nu c_\mu)k_\gamma + (c_\mu k_\nu - c_\nu k_\mu)b_\gamma\big); \tag{12a}$$

$$k_{\mu\gamma} = -g_{\mu\gamma} - b_\mu b_\gamma, \quad l_{\mu\gamma} = 0, \quad m_{\mu\gamma} = -\alpha^2(a_\mu a_\gamma - d_\mu d_\gamma), \\ q_{\mu\nu\gamma} = g_{\mu\gamma}b_\nu + g_{\nu\gamma}b_\mu - 2g_{\mu\nu}b_\gamma, \\ h_{\mu\nu\gamma} = \alpha\big((a_\mu d_\nu - a_\nu d_\mu)c_\gamma + (d_\mu c_\nu - d_\nu c_\mu)a_\gamma + (c_\mu a_\nu - c_\nu a_\mu)d_\gamma\big); \tag{12b}$$

$$k_{\mu\gamma} = -g_{\mu\gamma} - b_\mu b_\gamma, \quad l_{\mu\gamma} = -\frac{\varepsilon}{2}b_\mu k_\gamma, \quad m_{\mu\gamma} = -\frac{\varkappa}{4}k_\mu k_\gamma, \\ q_{\mu\nu\gamma} = g_{\mu\gamma}b_\nu + g_{\nu\gamma}b_\mu - 2g_{\mu\nu}b_\gamma, \quad h_{\mu\nu\gamma} = \frac{\varepsilon}{4}(g_{\mu\gamma}k_\nu - g_{\mu\nu}k_\gamma); \tag{12c}$$

$$k_{\mu\gamma} = -g_{\mu\gamma} - b_\mu b_\gamma, \quad l_{\mu\gamma} = -\omega^{-1}(g_{\mu\gamma} + b_\mu b_\gamma), \quad m_{\mu\gamma} = -\omega^{-2} c_\mu c_\gamma,$$
$$q_{\mu\nu\gamma} = g_{\mu\gamma} b_\nu + g_{\nu\gamma} b_\mu - 2 g_{\mu\nu} b_\gamma, \quad h_{\mu\nu\gamma} = \frac{1}{2}\omega^{-1}(g_{\mu\gamma} b_\nu - g_{\mu\nu} b_\gamma). \tag{12d}$$

3 Exact solutions of the nonlinear Yang–Mills equations

The systems (5), (12) are systems of twelve nonlinear second-order ODEs with variable coefficients. That is why there is a little hope to construct their general solutions. But it is possible to obtain particular solutions of system (5) whose coefficients are given by expressions (12b)–(12d).

Consider, as an example, system of ODEs (5) with coefficients given by the expressions (12b). We seek its solutions in the form

$$\boldsymbol{B}_\mu = k_\mu \boldsymbol{e}_1 f(\omega) + b_\mu \boldsymbol{e}_2 g(\omega), \quad fg \neq 0, \tag{13}$$

where $\boldsymbol{e}_1 = (1,0,0)$, $\boldsymbol{e}_2 = (0,1,0)$.

On substituting the expression (13) into the above mentioned system we get

$$\ddot{f} + (\alpha^2 - e^2 g^2) f = 0, \quad f\dot{g} + 2\dot{f} g = 0. \tag{14}$$

The second ODE from (14) is easily integrated to give

$$g = \lambda f^{-2}, \quad \lambda \in \mathbb{R}^1, \quad \lambda \neq 0. \tag{15}$$

Substitution of the result obtained in the first ODE from (14) yields the Ermakov-type equation for $f(\omega)$

$$\ddot{f} + \alpha^2 f - e^2 \lambda^2 f^{-3} = 0,$$

which is integrated in elementary functions [14]

$$f = \big(\alpha^{-2} C^2 + \alpha^{-2}(C^4 - \alpha^2 e^2 \lambda^2)^{1/2} \sin 2|\alpha|\omega\big)^{1/2}. \tag{16}$$

Here $C \neq 0$ is an arbitrary constant.

Substituting (13), (15), (16) into the corresponding ansatz for $\boldsymbol{A}_\mu(x)$ we get the following class of exact solutions of YMEs (1):

$$\begin{aligned}\boldsymbol{A}_\mu = {}& \boldsymbol{e}_1 k_\mu \exp\,(-\alpha c\cdot x - \alpha w_2)\big(\alpha^{-2} C^2 + \alpha^{-2}(C^4 - \alpha^2 e^2\lambda^2)^{1/2} \times \\ & \times \sin 2|\alpha|(b\cdot x + w_1)\big)^{1/2} + \boldsymbol{e}_2 \lambda\big(\alpha^{-2}C^2 + \alpha^{-2}(C^4 - \alpha^2 e^2 \lambda^2)^{1/2} \times \\ & \times \sin 2|\alpha|(b\cdot x + w_1)\big)^{-1}\left(b_\mu + \frac{1}{2} k_\mu \dot{w}_1\right).\end{aligned}$$

In a similar way we have obtained five other classes of exact solutions of the Yang–Mills equations

$$\begin{aligned}\boldsymbol{A}_\mu = {}& \boldsymbol{e}_1 k_\mu e^{-T}(b\cdot x\cos w_1 + c\cdot x\sin w_1 + w_2)^{1/2} Z_{1/4}\big((ie\lambda/2)(b\cdot x\cos w_1 + \\ & + c\cdot x \sin w_1 + w_2)^2\big) + \boldsymbol{e}_2 \lambda\ (b\cdot x\cos w_1 + c\cdot x\sin w_1 + w_2)\times \\ & \times \big(c_\mu \cos w_1 - b_\mu \sin w_1 + 2k_\mu[(1/4)(\varepsilon e^T + \dot{T})(b\cdot x\sin w_1 - \\ & - c\cdot x\cos w_1) + w_3]\big);\end{aligned}$$

$$\begin{aligned}
\boldsymbol{A}_\mu &= \boldsymbol{e}_1 k_\mu e^{-T}\big(C_1\cosh[e\lambda(b\cdot x\cos w_1 + c\cdot x\sin w_1 + w_2)] + C_2\sinh[e\lambda\times\\
&\quad\times(b\cdot x\cos w_1 + c\cdot x\sin w_1 + w_2)]\big) + \boldsymbol{e}_2\lambda\big(c_\mu\cos w_1 - b_\mu\sin w_1 +\\
&\quad + 2k_\mu[(1/4)(\varepsilon e^T + \dot T)(b\cdot x\sin w_1 - c\cdot x\cos w_1) + w_3]\big);\\
\boldsymbol{A}_\mu &= \boldsymbol{e}_1 k_\mu e^{-T}\big(C^2(b\cdot x\cos w_1 + c\cdot x\sin w_1 + w_2)^2 + \lambda^2 e^2 C^{-2}\big)^{1/2} +\\
&\quad + \boldsymbol{e}_2\lambda\big(C^2(b\cdot x\cos w_1 + c\cdot x\sin w_1 + w_2)^2 + \lambda^2 e^2 C^{-2}\big)^{-1}\times\\
&\quad\times\big(b_\mu\cos w_1 + c_\mu\sin w_1 - (1/2)k_\mu[\dot w_1(b\cdot x\sin w_1 - c\cdot x\cos w_1) - \dot w_2]\big);\\
\boldsymbol{A}_\mu &= \boldsymbol{e}_1 k_\mu Z_0\big((ie\lambda/2)[(b\cdot x + w_1)^2 + (c\cdot x + w_2)^2]\big) + \boldsymbol{e}_2\lambda\big(c_\mu(b\cdot x + w_1) -\\
&\quad - b_\mu(c\cdot x + w_2) - (1/2)k_\mu[\dot w_1(c\cdot x + w_2) - \dot w_2(b\cdot x + w_1)]\big);\\
\boldsymbol{A}_\mu &= \boldsymbol{e}_1 k_\mu\big(C_1[(b\cdot x + w_1)^2 + (c\cdot x + w_2)^2]^{e\lambda/2} + C_2[(b\cdot x + w_1)^2 +\\
&\quad + (c\cdot x + w_2)^2]^{-e\lambda/2}\big) + \boldsymbol{e}_2\lambda[(b\cdot x + w_1)^2 + (c\cdot x + w_2)^2]^{-1}\times\\
&\quad\times\big(c_\mu(b\cdot x + w_1) - b_\mu(c\cdot x + w_2) - (1/2)k_\mu[\dot w_1(c\cdot x + w_2) -\\
&\quad - \dot w_2(b\cdot x + w_1)]\big).
\end{aligned}$$

Here $C_1, C_2, C \neq 0$, λ are arbitrary parameters; w_1, w_2, w_3 are arbitrary smooth functions on $\xi = \frac{1}{2}k\cdot x$; $T = T(\xi)$ is a solution of ODE (11). In addition, we use the following notations:

$$k\cdot x = k_\mu x^\mu, \quad b\cdot x = b_\mu x^\mu, \quad c\cdot x = c_\mu x^\mu,$$
$$Z_s(\omega) = C_1 J_s(\omega) + C_2 Y_s(\omega), \quad \boldsymbol{e}_1 = (1,0,0), \quad \boldsymbol{e}_2 = (0,1,0),$$

where J_s, Y_s are the Bessel functions.

Thus, we have obtained broad families of exact non-Abelian solutions of YMEs (1). It can be verified by direct and rather involved computation that the solutions obtained are not self-dual, i.e. that they do not satisfy self-dual YMEs.

4 Conclusion

Let us say a few words about symmetry interpretation of the ansätze (2), (3), (10). Consider as an example, the ansatz determined by expressions (10a). As a direct computation shows, generators of a three-parameter Lie group G leaving it invariant are of the form

$$\begin{aligned}
&Q_1 = k_\alpha\partial_\alpha, \quad Q_2 = b_\alpha\partial_\alpha - 2[w_0(k_\mu b_\nu - k_\nu b_\mu) + w_2(k_\mu c_\nu - k_\nu c_\mu)]\sum_{a=1}^{3} A^{a\nu}\partial_{A^{a\mu}},\\
&Q_3 = c_\alpha\partial_\alpha - 2[w_1(k_\mu b_\nu - k_\nu b_\mu) + w_3(k_\mu c_\nu - k_\nu c_\mu)]\sum_{a=1}^{3} A^{a\nu}\partial_{A^{a\mu}}.
\end{aligned} \tag{17}$$

Evidently, the system of PDEs (1) is invariant under the one-parameter group G_1 having the generator Q_1. But it is not invariant under the groups having the generators Q_2, Q_3. Consider, as an example, the generator Q_2. Acting by the second prolongation of the operator Q_2 (which is constructed in a standard way, see e.g. [18, 20]) on the system of PDEs (1), after some tedious algebra we obtain the following equality:

$$\begin{aligned}
\underset{2}{Q_2}\boldsymbol{L}_\mu &= 2\big(w_0(k_\mu b_\nu - k_\nu b_\mu) + w_2(k_\mu c_\nu - k_\nu c_\mu)\big)\boldsymbol{L}^\nu +\\
&\quad + 2\big(\dot w_0(k_\mu b_\nu - k_\nu b_\mu) + \dot w_2(k_\mu c_\nu - k_\nu c_\mu)\big)\underline{Q_1\boldsymbol{A}^\nu} -
\end{aligned} \tag{18}$$

$$
\begin{aligned}
&- \partial^\mu\big((w_0 b_\nu + w_2 c_\nu)\underline{Q_1\boldsymbol{A}^\nu} - k_\nu(w_0\underline{Q_2\boldsymbol{A}^\nu} + w_2\underline{Q_3\boldsymbol{A}^\nu})\big) - \\
&- (w_0 b_\mu + w_2 c_\mu)\partial_\nu \underline{Q_1\boldsymbol{A}_\nu} - k_\mu\big(w_0(w_0 b_\nu + w_2 c_\nu) + \\
&+ w_2(w_1 b_\nu + w_3 c_\nu)\big)\underline{Q_1\boldsymbol{A}^\nu} + e\big((w_0 b_\nu + w_2 c_\nu)\underline{Q_1\boldsymbol{A}^\nu} - \\
&- k_\nu(w_0\underline{Q_2\boldsymbol{A}^\nu} + w_2\underline{Q_3\boldsymbol{A}^\nu})\big) \times \boldsymbol{A}_\mu + 2e(w_0 b_\nu \boldsymbol{A}^\nu + w_2 c_\nu \boldsymbol{A}^\nu)\times\underline{Q_1\boldsymbol{A}_\mu} - \\
&- 2ek_\nu\boldsymbol{A}^\nu \times (w_0\underline{Q_2\boldsymbol{A}_\mu} + w_2\underline{Q_3\boldsymbol{A}_\mu}) + e\boldsymbol{A}_\nu \times (w_0 b_\mu + w_2 c_\mu)\underline{Q_1\boldsymbol{A}^\nu} - \\
&- ek_\mu\boldsymbol{A}_\nu \times (w_0\underline{Q_2\boldsymbol{A}^\nu} + w_2\underline{Q_3\boldsymbol{A}^\nu}).
\end{aligned}
$$

In the above expressions we use the designations

$$
\begin{aligned}
&\boldsymbol{L}_\mu \equiv \partial_\nu\partial^\nu\boldsymbol{A}_\mu - \partial^\mu\partial_\nu\boldsymbol{A}_\nu + e\big((\partial_\nu\boldsymbol{A}_\nu)\times\boldsymbol{A}_\mu - 2(\partial_\nu\boldsymbol{A}_\mu)\times\boldsymbol{A}_\nu + \\
&\qquad + (\partial^\mu\boldsymbol{A}_\nu)\times\boldsymbol{A}^\nu\big) + e^2\boldsymbol{A}_\nu\times(\boldsymbol{A}^\nu\times\boldsymbol{A}_\mu), \\
&Q_1\boldsymbol{A}_\mu \equiv k_\alpha\partial_\alpha\boldsymbol{A}_\mu, \\
&Q_2\boldsymbol{A}_\mu \equiv b_\alpha\partial_\alpha\boldsymbol{A}_\mu + 2\big(w_0(k_\mu b_\nu - k_\nu b_\mu) + w_2(k_\mu c_\nu - k_\nu c_\mu)\big)\boldsymbol{A}^\nu, \\
&Q_3\boldsymbol{A}_\mu \equiv c_\alpha\partial_\alpha\boldsymbol{A}_\mu + 2\big(w_1(k_\mu b_\nu - k_\nu b_\mu) + w_3(k_\mu c_\nu - k_\nu c_\mu)\big)\boldsymbol{A}^\nu
\end{aligned}
$$

and by the symbol $\underset{2}{Q_2}$ we denote the second prolongation of the operator Q_2.

As underlined terms in (18) do not vanish on the set of solutions of YMEs, system of PDEs (1) is not invariant under the Lie transformation group G_2 having the generator Q_2. On the other hand, system

$$
\boldsymbol{L}_\mu = \boldsymbol{0}, \quad Q_a\boldsymbol{A}_\mu = \boldsymbol{0}, \quad a = 1,2,3
$$

is evidently invariant under the group G_2. The same assertion holds for the Lie transformation group G_3 having the generator Q_3. Consequently, the YMEs are conditionally-invariant with respect to the three-parameter Lie transformation group $G = G_1 \otimes G_2 \otimes G_3$. This means that solutions of the YMEs obtained with the help of the ansatz invariant under the group with generators (17) can not be found by means of the classical symmetry reduction procedure.

As rather tedious computations show, the ansätze determined by the expressions (10b)–(10d) also correspond to conditional symmetry of YMEs. Hence it follows, in particular, that the YMEs should be included into the long list of mathematical and theoretical physics equations possessing non-trivial conditional symmetry [7].

Another interesting observation is that specifying the arbitrary functions contained in non-Lie ansätze in an appropriate way, one can obtain some Lie ansätze. Really, expressions (8c), (8l), (8m), (8q) are particular cases of expressions (10a), expressions (8a), (8e), (8f), (8g), (8n), (8o), (8s), (8t), (8u) are particular cases of expressions (10b), (10c) and expressions (8h), (8p) are particular cases of the expressions (10d). So if we denote the invariant solutions of the Yang–Mills equations symbolically by the dots in some space of solutions of system of PDEs (1), then some of them can be connected by curves which are *conditionally-invariant* solutions! Thus, at the first the distinct glance solutions are the particular cases of more general solutions. A similar assertion holds for the nonlinear wave [13] and the Dirac [11] equations. On the other hand, some invariant solutions (namely those determined by expressions (8b), (8d), (8i), (8j), (8k), (8r), (8v)) can not be connected with other solutions by the curve which is a conditionally-invariant solution of the form (10). A possible explanation of this fact is that there exist more general conditionally-invariant solutions of YMEs.

The above picture admits an analogy with a case when equation under study has general solution. In that case, each two solutions can be connected by a curve which is a solution of the equation. The only exceptions are the singular solutions which are obtained by some asymptotic procedure. So one can guess that there exists such collection of conditionally-invariant solutions of YMEs that the majority of invariant solutions are their particular cases and the remaining ones are obtained from these by an asymptotic procedure. However, this problem so far is completely open and needs further investigation.

One last remark is that the procedure suggested yields also some well-known exact solutions of YMEs. For example, the ansatz for the Yang–Mills field determined by expressions (2), (3) and (8v) gives rise to the meron and instanton solutions of the system (1), originally obtained with the help of the Ansatz suggested by 't Hooft [21], Corrigan and Fairlie [5] and Wilczek [22] (for more details, see [16]).

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1. Actor A., *Rev. Mod. Phys.*, 1979, **51**, 461.
2. Atiyah M.F., Hitchin N.J., Drinfeld V.G., Manin Yu.A., *Phys. Lett. A*, 1978, **65**, 185.
3. Bluman G., Cole J., *J. Math. Phys.*, 1969, **18**, 1025.
4. Clarkson P.A., Kruskal M.D., *J. Math. Phys.*, 1989, **30**, 2201.
5. Corrigan E., Fairlie D.B., *Phys. Lett. B*, 1977, **67**, 69.
6. Fushchych W.I., *Ukr. Math. J.*, 1991, **43**, 1456.
7. Fushchych W.I., in Proceedings of the International Workshop "Modern Group Analysis", Editors N. Ibragimov, M. Torrisi and A. Valenti, Dordrecht, Kluwer Academic Publishers, 1993, 231.
8. Fushchych W.I., Serov N.I., Chopyk V.I., *Dopovidi Akad. Nauk Ukrainy, Ser. A*, 1988, № 9, 17.
9. Fushchych W.I., Shtelen W.M., *Lett. Nuovo Cim.*, 1983, **38**, № 2, 37.
10. Fushchych W.I., Shtelen W.M., Serov N.I., Symmetry analysis and exact solutions of nonlinear equations of mathematical physics, Kiev, Naukova Dumka, 1989.
11. Fushchych W.I., Zhdanov R.Z., Nonlinear spinor equations: symmetry and exact solutions, Kiev, Naukova Dumka, 1992.
12. Fushchych W.I., Zhdanov R.Z., *Ukr. Math. J.*, 1992, **44**, 970.
13. Fushchych W.I., Zhdanov R.Z., Revenko I.V., *Ukr. Math. J.*, 1991, **43**, 1471.
14. Kamke E., Differentialgleichungen. Lösungmethoden und Lösungen, Leipzig: Akademische Verlagsgesellschaft, 1961.
15. Kovalyov M., Légaré M., Gagnon L., *J. Math. Phys.*, 1993, **34**, 3245.
16. Lahno V.I., Zhdanov R.Z., Fushchych W.I., *J. Nonlinear Math. Phys.*, 1995, **2**, 51.
17. Levi D., Winternitz P., *J. Phys. A: Math. Gen.*, 1989, **22**, 2922.
18. Olver P., Applications of Lie groups to differential equations, New York, Springer, 1993.
19. Olver P., Rosenau Ph., *Phys.Lett. A*, 1986, **112**, 107.
20. Ovsiannikov L.V., Group analysis of differential equations, Moscow, Nauka, 1978.
21. 't Hooft G., *Phys. Rev. D*, 1976, **14**, 3432.
22. Wilczek F., in Quark Confinement and Field Theory, New York, Wiley, 1977, 211.

On non-Lie ansatzes and new exact solutions of the classical Yang–Mills equations

R.Z. ZHDANOV, W.I. FUSHCHYCH

We suggest an effective method for reducing Yang–Mills equations to systems of ordinary differential equations. With the use of this method, we construct wide families of new exact solutions of the Yang–Mills equations. Analysis of the solutions obtained shows that they correspond to conditional symmetry of the equations under study.

1 Introduction

The majority of papers devoted to construction of the explicit form of exact solutions of the $SU(2)$ Yang–Mills equations (YME)

$$\partial_\nu\partial^\nu\vec{A}_\mu - \partial^\mu\partial_\nu\vec{A}_\nu + e[(\partial_\nu\vec{A}_\nu)\times\vec{A}_\mu - 2(\partial_\nu\vec{A}_\mu)\times\vec{A}_\nu + (\partial^\mu\vec{A}_\nu)\times\vec{A}^\nu] + \\ + e^2\vec{A}_\nu\times(\vec{A}^\nu\times\vec{A}_\mu) = \vec{0} \tag{1}$$

is based on the ansatzes for the Yang–Mills field $\vec{A}_\mu(x)$ suggested by Wu and Yang, Rosen, 't Hooft, Corrigan and Fairlie, Wilczek, Witten (see [1] and references therein). And what is more, the above ansatzes were obtained in a non-algorithmic way, i.e., there was no regular and systematic method for constructing such ansatzes.

Since one has only a few distinct exact solutions of YME, it is difficult to give their reliable and self-consistent physical interpretation. That is why, the problem of prime importance is the development of an effective regular approach for constructing new exact solutions of the system of nonlinear partial differential equations (PDE) (1) (see also [1]).

A natural approach to construction of particular solutions of YME (1) is to utilize their symmetry properties in the way as it is done in [2–4, 13]. Apparatus of the theory of Lie transformation groups makes it possible to reduce the system of PDE (1) to systems of nonlinear ordinary differential equations (ODE) by using special ansatzes (invariant solutions) [5, 6]. If one succeeds in constructing general or particular solutions of the said ODE (which is extremely difficult problem), then substituting results into the corresponding ansatzes, one gets exact solutions of the initial system of PDE (1).

Another possibility of construction of exact solutions of YME is to use their conditional (non-Lie) symmetry (for more details about conditional symmetry of equations of mathematical physics, see [7, 8] and also [9]). But the prospects of a systematic and exhaustive study of conditional symmetry of the system of twelve second-order nonlinear PDE (1) seem to be rather obscure. It should be said that so far we have no complete description of conditional symmetry of a nonlinear wave equation even in the case of one space variable.

In [9] we suggested an effective approach to study of conditional symmetry of the nonlinear Dirac equation based on its Lie symmetry. We have observed that all

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Poincaré-invariant ansatzes can be represented in the unified form by introducing several arbitrary elements (functions) $u_1(x), u_2(x), \ldots, u_N(x)$. As a result, we get an ansatz for the Dirac field which reduces the nonlinear Dirac equation to a system of ODE provided functions $u_i(x)$ satisfy some compatible over-determined system of nonlinear PDE. After integrating it, we have obtained a number of new ansatzes that cannot in principle be obtained within the framework of the classical Lie approach.

In the present paper we construct a number of new exact solutions of YME (1) with the aid of the above described approach.

2 Reduction of YME

In the papers [2, 13] we adduce a complete list of $P(1,3)$-inequivalent ansatzes for the Yang–Mills field which are invariant under three-parameter subgroups of the Poincaré group $P(1,3)$. Analyzing these ansatzes, we come to the conclusion that they can be represented in the following unified form:

$$\vec{A}_\mu(x) = R_{\mu\nu}(x)\vec{B}^\nu(\omega), \tag{2}$$

where $\vec{B}_\nu(\omega)$ are new unknown vector-functions, $\omega = \omega(x)$ is a new independent variable, functions $R_{\mu\nu}(x)$ are given by

$$\begin{aligned} R_{\mu\nu}(x) = {}& (a_\mu a_\nu - d_\mu d_\nu)\operatorname{ch}\theta_0 + (a_\mu d_\nu - d_\mu a_\nu)\operatorname{ch}\theta_0 + \\ & + 2(a_\mu d_\mu)[(\Theta_1\cos\theta_3 + \theta_2\sin\Theta_3)b_\nu + (\theta_2\cos\theta_3 - \theta_1\sin\theta_3)c_\nu + \\ & + (\theta_1^2 + \theta_2^2)e^{-\theta_0}(a_\nu + d_\nu)] - (c_\mu c_\nu + b_\mu b_\nu)\cos\theta_3 - \\ & - (c_\mu b_\nu - b_\mu c_\nu)\sin\theta_3 - 2e^{-\theta_0}(\theta_1 b_\mu + \theta_2 c_\mu)(a_\nu + d_\nu). \end{aligned} \tag{3}$$

In (3) $\theta_\mu(x)$ are some smooth functions and what is more, $\theta_a = \theta_a(\xi, b_\mu x^\mu, c_\mu x^\mu)$, $a = 1, 2$, $\xi = \frac{1}{2}k_\mu x^\mu = \frac{1}{2}(a_\mu x^\mu + d_\mu x^\mu)$; a_μ, b_μ, c_μ, d_μ are arbitrary constants satisfying the following relations:

$$\begin{aligned} & a_\mu a^\mu = -b_\mu b^\mu = -c_\mu c^\mu = -d_\mu d^\mu = 1, \\ & a_\mu b^\mu = a_\mu c^\mu = a_\mu d^\mu = b_\mu c^\mu = b_\mu d^\mu = c_\mu d^\mu = 0. \end{aligned}$$

Hereafter, summation over the repeated indices from 0 to 3 is understood. Rising and lowering of the indices is performed with the help of the tensor $g_{\mu\nu} = \operatorname{diag}(1, -1, -1, -1)$, i.e. $R^\alpha_\mu = g_{\alpha\beta}R_{\beta\mu}$.

The choice of the functions $\omega(x)$, $\theta_\mu(x)$ is determined by the requirement that substitution of the ansatz (2) into YME yields a system of ordinary differential equations for the vector function $\vec{B}_\mu(\omega)$.

By a direct check, one can become convinced of that the following assertion holds true.

Lemma. *Ansatz (2), (3) reduces YME (1) to a system of ODE if the functions $\omega(x)$, $\theta_\mu(x)$ satisfy the system of PDE*

$$\begin{aligned} & 1.\quad \omega_{x_\mu}\omega_{x^\mu} = F_1(\omega), \\ & 2.\quad \Box\omega = F_2(\omega), \\ & 3.\quad R_{\alpha\mu}\omega_{x_\alpha} = G_\mu(\omega), \\ & 4.\quad R_{\alpha\mu x_\alpha} = H_\mu(\omega), \end{aligned} \tag{4}$$

5. $R^\alpha_\mu R_{\alpha\nu x_\beta}\omega_{x\beta} = Q_{\mu\nu}(\omega)$,
6. $R^\alpha_\mu \Box R_{\alpha\nu} = S_{\mu\nu}(\omega)$,
7. $R^\alpha_\mu R_{\alpha\nu x_\beta} R_{\beta\gamma} + R^\alpha_\nu R_{\alpha\gamma x_\beta} R_{\beta\mu} + R^\alpha_\gamma R_{\alpha\mu x_\beta} R_{\beta\nu} = T_{\mu\nu\gamma}(\omega)$,

where $F_1, F_2, G_\mu, \ldots, T_{\mu\nu\gamma}$ are some smooth functions, $\mu,\nu,\gamma = \overline{0,3}$. And what is more, the reduced equation has the form

$$k_{\mu\gamma}\ddot{\vec{B}}^\gamma + l_{\mu\gamma}\dot{\vec{B}}^\gamma + m_{\mu\gamma}\vec{B}^\gamma + eq_{\mu\nu\gamma}\dot{\vec{B}}^\nu \times \vec{B}^\gamma + \\ + eh_{\mu\nu\gamma}\vec{B}^\nu \times \vec{B}^\gamma + e^2\vec{B}_\gamma \times (\vec{B}^\gamma \times \vec{B}_\mu) = \vec{0}, \tag{5}$$

where

$$\begin{aligned} &k_{\mu\gamma} = g_{\mu\gamma}F_1 - G_\mu g_\gamma, \\ &l_{\mu\gamma} = g_{\mu\gamma}F_2 + 2Q_{\mu\gamma} - G_\mu H_\gamma - G_\mu \dot{G}_\gamma, \\ &m_{\mu\gamma} = S_{\mu\gamma} - G_\mu \dot{H}_\gamma, \\ &q_{\mu\nu\gamma} = g_{\mu\gamma}G_\nu + g_{\nu\gamma}G_\mu - 2g\mu\nu G_\gamma, \\ &h_{\mu\nu\gamma} = \frac{1}{2}(g_{\mu\gamma}H_\nu - g_{\mu\nu}H_\gamma) - T_{\mu\nu\gamma}. \end{aligned} \tag{6}$$

Thus, to describe all ansatzes of the form (2), (3) reducing YME to a system of ODE, one has to construct the general solution of the over-determined system of PDE (3), (4). Let us emphasize that system (3), (4) is compatible, since ansatzes invariant under the Poincaré group satisfy equations (3), (4) with some specific choice of the functions $F_1, F_2, \ldots, T_{\mu\nu\gamma}$.

Integration of the system of nonlinear PDE (3), (4) demands a huge amount of computations. That is why, we present here only the principal idea of our approach to solving system (3), (4). When integrating it, we use essentially the fact that the general solution of the system of equations 1, 2 from (4) is known [10]. With already known $\omega(x)$, we proceed to integration of the linear PDE 3, 4 from (4). Next, we substitute the results obtained into the remaining equations and get the final form of the functions $\omega(x)$, $\theta_\mu(x)$.

Before presenting the results of integration of the system of PDE (3), (4), we make a remark. As a direct check shows, the structure of the ansatzes (2), (7) is not altered by the change of variables

$$\begin{aligned} &\omega \to \omega' = T(\omega), \quad \theta_0 \to \theta_0' = \theta_0 + T_0(\omega), \\ &\theta_1 \to \theta_1' = \theta_1 + e^{\theta_0}(T_1(\omega)\cos\theta_3 + T_2(\omega)\sin\theta_3), \\ &\theta_2 \to \theta_2' = \theta_2 + e^{\theta_0}(T_2(\omega)\cos\theta_3 - T_1(\omega)\sin\theta_3), \\ &\theta_3 \to \theta_3' = \theta_3 + T_3(\omega), \end{aligned} \tag{7}$$

where $T(\omega)$, $T_\mu(\omega)$ are arbitrary smooth functions. That is why, solutions of system (3), (4) connected by the relations (7) are considered as equivalent.

It occurs that new (non-Lie) ansatzes are obtained, if functions $\omega(x)$, $\theta_\mu(x)$ up to the equivalence relations (7) have the form

$$\begin{aligned} &\theta_\mu = \theta_\mu(\xi, b_\nu x^\nu, c_\nu x^\nu), \quad \mu = \overline{0,3}, \\ &\omega = \omega(\xi, b_\nu x^\nu, c_\nu x^\nu), \end{aligned} \tag{8}$$

where $\xi = \frac{1}{2}k_\nu x^\nu$, $k_\nu = a_\nu + d_\nu$.

The list of inequivalent solutions of the system of PDE (3), (4) satisfying (8) is exhausted by the following solutions:

$$
\begin{aligned}
1.\quad & \theta_0 = \theta_3 = 0, \quad \omega = \frac{1}{2}k_\nu x^\nu, \\
& \theta_1 = w_0(\xi)b_\mu x^\mu + w_1(\xi)c_\mu x^\mu, \quad \theta_2 = w_2(\xi)b_\mu x^\mu + w_3(\xi)c_\mu x^\mu; \\
2.\quad & \omega = b_\mu x^\mu + w_1(\xi), \quad \theta_0 = \alpha\big(c_\mu x^\mu + w_2(\xi)\big), \\
& \theta_a = -\frac{1}{4}\dot{w}_a(\xi), \quad a = \overline{1,2}, \quad \theta_3 = 0; \\
3.\quad & \theta_0 = T(\xi), \quad \theta_3 = w_1(\xi), \\
& \omega = b_\mu x^\mu \cos w_1 + c_\mu x^\mu \sin w_1 + w_2(\xi), \\
& \theta_1 = \left[\frac{1}{4}(\varepsilon e^T + \dot{T})(b_\mu x^\mu \sin w_1 - c_\mu x^\mu \cos w_1) + w_3(\xi)\right]\sin w_1 + \\
& \qquad + \frac{1}{4}\big[\dot{w}_1(b_\mu x^\mu \sin w_1 - c_\mu x^\mu \cos w_1) - \dot{w}_2\big]\cos w_1, \\
& \theta_2 = -\left[\frac{1}{4}(\varepsilon e^T + \dot{T})(b_\mu x^\mu \sin w_1 - c_\mu x^\mu \cos w_1) + w_3(\xi)\right]\cos w_1 + \\
& \qquad + \frac{1}{4}\big[\dot{w}_1(b_\mu x^\mu \sin w_1 - c_\mu x^\mu \cos w_1) - \dot{w}_2\big]\sin w_1; \\
4.\quad & \theta_0 = 0, \quad \theta_3 = \operatorname{arctg}\Big[\big(c_\mu x^\mu + w_2(\xi)\big)\big(b_\mu x^\mu + w_1(\xi)\big)^{-1}\Big], \\
& \theta_a = -\frac{1}{4}\dot{w}_a(\xi), \quad a = 1,2, \quad \omega = \Big[\big(b_\mu x^\mu + w_1(\xi)\big)^2 + \big(c_\mu x^\mu + w_2(\xi)\big)^2\Big]^{1/2}.
\end{aligned}
\tag{9}
$$

Here $\alpha \neq 0$, ε are arbitrary constants, w_0, w_1, w_2, w_3 are arbitrary smooth functions of $\xi = \frac{1}{2}k_\mu x^\mu$, $T = T(\xi)$ is a solution of the nonlinear ODE

$$
\big(\dot{T} + \varepsilon e^T\big)^2 + \dot{w}_1^2 = \varkappa e^{2T}, \quad \varkappa \in \mathbb{R}^1. \tag{10}
$$

Substitution of the ansatz (2), where $R_{\mu\nu}(x)$ are given by formulae (3), (9), into YME yields systems of nonlinear ODE of the form (5), where

$$
\begin{aligned}
1.\quad & k_{\mu\gamma} = -\frac{1}{4}k_\mu k_\gamma, \quad l_{\mu\gamma} = -(w_0 + w_3)k_\mu k_\gamma, \\
& m_{\mu\gamma} = -4(w_0^2 + w_1^2 + w_2^2 + w_3^2)k_\mu k_\gamma - (\dot{w}_0 + \dot{w}_3)k_\mu k_\gamma, \\
& q_{\mu\nu\gamma} = \frac{1}{2}(g_{\mu\gamma}k_\nu + g_{\nu\gamma}k_\mu - 2g_{\mu\nu}k_\gamma), \\
& h_{\mu\nu\gamma} = (w_0 + w_3)(g_{\mu\gamma}k_\nu - g_{\mu\nu}k_\gamma) + 2\ (w_1 - w_2)\big[(k_\mu b_\nu - k_\nu b_\mu)c_\gamma + \\
& \qquad + (b_\mu c_\nu - b_\nu c_\mu)k_\gamma + (c_\mu k_\nu - c_\nu k_\mu)b_\gamma\big]; \\
2.\quad & k_{\mu\gamma} = -g_{\mu\gamma} - b_\mu b_\gamma, \quad l_{\mu\gamma} = 0, \quad m_{\mu\gamma} = -\alpha^2(a_\mu a_\gamma - d_\mu d_\gamma), \\
& q_{\mu\nu\gamma} = g_{\mu\gamma}b_\nu + g_{\nu\gamma}b_\mu - 2g_{\mu\nu}b_\gamma, \\
& h_{\mu\nu\gamma} = \alpha\big[(a_\mu d_\nu - a_\nu d_\mu)c_\gamma + (d_\mu c_\nu - d_\nu c_\mu)a_\gamma + (c_\mu a_\nu c_\nu a_\mu)d_\gamma\big]; \\
3.\quad & k_{\mu\gamma} = -g_{\mu\gamma} - b_\mu b_\gamma, \quad l_{\mu\gamma} = -\frac{\varepsilon}{2}b_\mu k_\gamma, \quad m_{\mu\gamma} = -\frac{\varkappa}{4}k_\mu k_\gamma, \\
& q_{\mu\nu\gamma} = g_{\mu\gamma}b_\nu + g_{\nu\gamma}b_\mu - 2g_{\mu\nu}b_\gamma, \quad h_{\mu\nu\gamma} = \frac{\varepsilon}{4}(g_{\mu\gamma}k_\nu - g_{\mu\nu}k_\gamma); \\
4.\quad & k_{\mu\gamma} = -g_{\mu\gamma} - b_\mu b_\gamma, \quad l_{\mu\gamma} = -\omega^{-1}(g_{\mu\gamma} + b_\mu b_\gamma), \quad m_{\mu\gamma} = -\omega^{-2}c_\mu c_\gamma, \\
& q_{\mu\nu\gamma} = g_{\mu\gamma}b_\nu + g_{\nu\gamma}b_\mu - 2g_{\mu\nu}b_\gamma, \quad h_{\mu\nu\gamma} = \frac{1}{2}\omega^{-1}(g_{\mu\gamma}b_\nu - g_{\mu\nu}b_\gamma).
\end{aligned}
\tag{11}
$$

3 Exact solutions of the nonlinear Yang–Mills equations

Systems (5), (11) are systems of twelve nonlinear second-order ODE with variable coefficients. That is why, there is a little hope to construct their general solutions. But it is possible to obtain particular solutions of system (5), which coefficients are given by the formulae 2–4 from (11).

Consider, as an example, the system of ODE (5) with coefficients given by the formulae 2 from (11). We look for its solutions in the form

$$\vec{B}_\mu = k_\mu \vec{e}_1 f(\omega) + b_\mu \vec{e}_2 g(\omega), \quad fg \neq 0, \tag{12}$$

where $\vec{e}_1 = (1,0,0)$, $\vec{e}_2 = (0,1,0)$.

Substituting the expression (12) into the above mentioned system, we get

$$\ddot{f} + (\alpha^2 - e^2 g^2) f = 0, \quad f\dot{g} + 2\dot{f}g = 0. \tag{13}$$

The second ODE from (13) is easily integrated

$$g = \lambda\, f^{-2}, \quad \lambda \in \mathbb{R}^1, \quad \lambda \neq 0. \tag{14}$$

Substitution of the result obtained into the first ODE from (13) yields the Ermakov-type equation for $f(\omega)$

$$\ddot{f} + \alpha^2 f - e^2\lambda^2\, f^{-3} = 0,$$

which is integrated in elementary functions [11]

$$f = \left[\alpha^{-2}C^2 + \alpha^{-2}\left(C^4 - \alpha^2 e^2 \lambda^2\right)^{1/2} \sin 2|\alpha|\omega\right]^{1/2}. \tag{15}$$

Here $C \neq 0$ is an arbitrary constant.

Substituting (12), (14), (15) into the corresponding ansatz for $\vec{A}_\mu(x)$, we get the following class of exact solutions of YME (1):

$$\begin{aligned}\vec{A}_\mu = {} & \vec{e}_1 k_\mu \exp\left(-\alpha cx - \alpha w_2\right)\left[\alpha^{-2}C^2 + \alpha^{-2}\left(C^4 - \alpha^2 e^2\lambda^2\right)^{1/2} \times \right.\\ & \left.\times \sin 2|\alpha|(bx + w_1)\right]^{1/2} + \vec{e}_2 \lambda\left[\alpha^{-2}C^2 + \alpha^{-2}\left(C^4 - \alpha^2 e^2 \lambda^2\right) \times \right.\\ & \left.\times \sin 2|\alpha|(bx + w_1)\right]^{-1}\left(b_\mu + \frac{1}{2}\, k_\mu \dot{w}_1\right).\end{aligned}$$

In a similar way, we have obtained five other classes of exact solutions of the Yang–Mills equations

$$\begin{aligned}\vec{A}_\mu = {} & \vec{e}_1 k_\mu e^{-T}\left(bx\cos w_1 + cx \sin w_1 + w_2\right)^{1/2} Z_{1/4}\left(\frac{ie\lambda}{2}\left(bx \cos w_1 + \right.\right.\\ & \left.\left. + cx \sin w_1 + w_2\right)^2\right) + \vec{e}_2\lambda(bx\cos w_1 + cx \sin w_1 + w_2) \times \\ & \times \left[c_\mu \cos w_1 - b_\mu \sin w_1 + 2k_\mu\left(\frac{1}{4}\left(\varepsilon e^T + \dot{T}\right)\left(bx \sin w_1 - cx\cos w_1\right) + w_3\right)\right];\\ \vec{A}_\mu = {} & \vec{e}_1 k_\mu e^{-T}\left[C_1 \operatorname{ch} e\lambda\left(bx\cos w_1 + cx\sin w_1 + w_2\right) + C_2 \operatorname{sh} e\lambda\left(bx \cos w_1 + \right.\right.\\ & \left.\left. + cx \sin w_1 + w_2\right)\right] + \vec{e}_2\lambda\left[C_\mu \cos w_1 - b_\mu \sin w_1 + \right.\\ & \left. + 2k_\mu\left(\frac{1}{4}\left(\varepsilon e^T + \dot{T}\right)(bx\sin w_1 - cx\cos w_1) + w_3\right)\right];\end{aligned}$$

$$\vec{A}_\mu = \vec{e}_1 k_\mu e^{-T}\left[C^2\left(bx\cos w_1 + cx\sin w_1 + w_2\right)^2 + \lambda^2 e^2 C^{-2}\right]^{1/2} +$$
$$+ \vec{e}_2\lambda\left[C^2(bx\cos w_1 + cx\sin w_1 + w_2)^2 + \lambda^2 e^2 C^{-2}\right]^{-1} \times$$
$$\times \left\{b_\mu\cos w_1 + C_\mu\sin w_1 - \frac{1}{2}k_\mu\left[\dot{w}_1(bx\sin w_1 - cx\cos w_1) - \dot{w}_2\right]\right\};$$
$$\vec{A}_\mu = \vec{e}_1 k_\mu Z_0\left(\frac{ie\lambda}{2}\left[(bx+w_1)^2 + (cx+w_2)^2\right]\right) + \vec{e}_2\lambda \times$$
$$\times \left\{c_\mu(bx+w_1) - b_\mu(cx+w_2) - \frac{1}{2}k_\mu\left[\dot{w}_1(cx+w_2) - \dot{w}_2(bx+w_1)\right]\right\};$$
$$\vec{A}_\mu = \vec{e}_1 k_\mu\left[C_1\left((bx+w_1)^2 + (cx+w_2)^2\right)^{e\lambda/2} +\right.$$
$$\left. + C_2\left((bx+w_1)^2 + (cx+w_2)^2\right)^{-e\lambda/2}\right. + \vec{e}_2\lambda\left[(bx+w_1)^2 + (cx+w_2)^2\right]^{-1} \times$$
$$\times \left\{c_\mu(bx+w_1) - b_\mu(cx+w_2) - \frac{1}{2}\, k_\mu\left[\dot{w}_1(cx+w_2) - \dot{w}_2(bx+w_1)\right]\right\}.$$

Here $C_1, C_2, C \neq 0$, ε, λ are arbitrary parameters; w_1, w_2, w_3 are arbitrary smooth functions of $\xi = \frac{1}{2}kx$, $T = T(\xi)$ is a solution of ODE (10).

Besides that, we use the following notations:

$$kx = k_\mu x^\mu, \quad bx = b_\mu x^\mu, \quad cx = c_\mu x^\mu,$$
$$Z_s(\omega) = C_1 J_s(\omega) + C_2 Y_s(\omega), \quad \vec{e}_1 = (1,0,0), \quad \vec{e}_2 = (0,1,0),$$

where J_s, Y_s are Bessel functions. Thus, we have obtained the wide families of exact non-Abelian solutions of YME (1).

In conclusion we say a few words about a symmetry interpretation of the ansatzes (2), (7), (10). Let us consider, as an example, the ansatz determined by the formulae 1 from (9). As a direct computation shows, generators of the three-parameter Lie group leaving it invariant are of the form

$$Q_1 = k_\alpha\partial_\alpha,$$
$$Q_2 = b_\alpha\partial_\alpha - \left\{\left[w_0(k_\mu b_\nu - k_\nu b_\mu) + w_2(k_\mu c_\nu - k_\nu c_\mu)\right]\sum_{a=1}^{3} A^{a\nu}\right\}\partial_{A^{a\mu}}, \tag{16}$$
$$Q_3 = c_\alpha\partial_\alpha - 2\left\{\left[w_1(k_\mu b_\nu - k_\nu b_\mu) + w_3(k_\mu c_\nu - k_\nu c_\mu)\right]\sum_{a=1}^{3} A^{a\nu}\right\}\partial_{A^{a\mu}}.$$

Evidently, the system of PDE (1) is invariant under the one-parameter group having the generator Q_1. But it is not invariant under the groups having the generators Q_2, Q_3. At the same time, the system of PDE

$$\partial_\nu\partial^\nu\vec{A}_\mu - \partial^\mu\partial_\nu\vec{A}_\nu + e\left[(\partial_\nu\vec{A}_\nu)\times\vec{A}_\mu - 2(\partial_\nu\vec{A}_\mu)\times\vec{A}_\nu + (\partial^\mu\vec{A}_\nu)\times\vec{A}^\nu\right] +$$
$$+ e^2\vec{A}_\nu\times(\vec{A}^\nu\times\vec{A}_\mu) = \vec{0},$$
$$Q_0\vec{A}_\mu \equiv k_\alpha\partial_\alpha\vec{A}_\mu = \vec{0},$$
$$Q_1\vec{A}_\mu \equiv b_\alpha\partial_\alpha\vec{A}_\mu + 2\left[w_0(k_\mu b_\nu - k_\nu b_\mu) + w_2(k_\mu c_\nu - k_\nu c_\mu)\right]\vec{A}^\nu = \vec{0},$$
$$Q_2\vec{A}_\mu \equiv c_\alpha\partial_\alpha\vec{A}_\mu + 2\left[w_1(k_\mu b_\nu - k_\nu b_\mu) + w_3(k_\mu c_\nu - k_\nu c_\mu)\right]\vec{A}^\nu = \vec{0}$$

is invariant under the said group. Consequently, YME (1) are conditionally-invariant under the Lie algebra (16). It means that the solutions of YME obtained with the help of the ansatz invariant under the group with the generators (16) can not be found by the classical symmetry reduction procedure.

As rather tedious computations show, the ansatzes determined by the formulae 2–4 from (9) also correspond to conditional symmetry of YME. Hence it follows, in particular, that YME should be included into the long list of mathematical and theoretical physics equations possessing a nontrivial conditional symmetry [12].

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1. Actor A., Classical solutions of $SU(2)$ Yang–Mills theories, *Rev. Mod. Phys.,* 1979, **51**, № 3, 461–525.
2. Zhdanov R.Z., Lahno V.I., On the new exact solutions of the Yang–Mills equations, *Dopovidi Akad. Nauk Ukrainy*, 1994 (to appear).
3. Fushchych W.I., Shtelen W.M., Conformal symmetry and new exact solutions of $SU(2)$ Yang–Mills equations, *Lett. Nuovo Cim.*, 1983, **38**, № 2, 37–40.
4. Fushchych W.I., Shtelen W.M., Serov N.I., Symmetry analysis and exact solutions of nonlinear equations of mathematical physics, Kiev, Naukova Dumka, 1989.
5. Ovsiannikov L.V., Group analysis of differential equations, Moscow, Nauka, 1978.
6. Ibragimov N.Kh., Transformation groups in mathematical physics, Moscow, Nauka, 1983.
7. Fushchych W.I., Serov N.I., Chopyk V.I., Conditional invariance and nonlinear heat equations, *Dopovidi Akad. Nauk Ukrainy, Ser. A*, 1988, № 9, 17–21.
8. Fushchych W.I., Conditional symmetry of equations of nonlinear mathematical physics, *Ukr. Math. J.*, 1991, **43**, № 11, 1456–1470.
9. Fushchych W.I., Zhdanov R.Z., Nonlinear spinor equations: symmetry and exact solutions, Kiev, Naukova dumka, 1992.
10. Fushchych W.I., Zhdanov R.Z., Revenko I.V., General solutions of the nonlinear wave equation and eikonal equation, *Ukr. Math. J.*, 1991, **43**, № 11, 1471–1476.
11. Kamke E., Handbook on ordinary differential equations, Moscow, Nauka, 1976.
12. Fushchych W.I., Conditional symmetry of equations of mathematical physics, in Proceedings of the International Workshop “Modern Group Analysis”, Editors N. Ibragimov, M. Torrisi and A. Valenti, Kluwer Academic Publishers, 1993, 231–240.
13. Lahno V., Zhdanov R., Fushchych W., Symmetry reduction and exact solution of the Yang–Mills equations, *J. Nonlinear Math. Phys.*, 1995, **2**, № 1, 51–72.

Symmetry and reduction of nonlinear Dirac equations

R.Z. ZHDANOV, W.I. FUSHCHYCH

We present results of symmetry classification of the nonlinear Dirac equations with respect to the conformal group $C(1,3)$ and its principal subgroups. Next we briefly consider the problem of classical and non-classical symmetry reduction and construction of exact solutions for the nonlinear Poincaré-invariant Dirac equations. In particular, a class of exact solutions is constructed which can not be in principle obtained within the framework of the classical Lie approach.

The Dirac equation is a system of four complex partial differential equations of the form

$$(i\gamma_\mu\partial_{x_\mu} - m)\psi(x) = 0, \tag{1}$$

where $\psi = \psi(x_0, \vec{x})$ is a four-component function-column, γ_μ are 4×4 Dirac matrices

$$\gamma_0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \quad \gamma_a = \begin{pmatrix} 0 & \sigma_a \\ -\sigma_a & 0 \end{pmatrix}$$

and σ_a are usual 2×2 Pauli matrices.

In fact in the following we do not use an explicit representation of the Dirac matrices, we use the commutational relations

$$\gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu = 2g_{\mu\nu} = 2\begin{cases} 1, & \mu=\nu=0, \\ -1, & \mu=\nu=1,2,3, \\ 0, & \mu\neq\nu \end{cases}$$

only.

Nonlinear generalizations of the Dirac equation were suggested by Ivanenko [1]

$$[i\gamma_\mu\partial_{x_\mu} - m + \lambda(\bar\psi\psi)]\psi = 0 \tag{2}$$

and by Heisenberg [2]

$$[i\gamma_\mu\partial_{x_\mu} + \lambda(\bar\psi\gamma_\mu\gamma_4\psi)\gamma^\mu\gamma_4]\psi = 0. \tag{3}$$

Here $\bar\psi = (\psi_0^*, \psi_1^*, -\psi_2^*, -\psi_3^*)$ is a four-component function-row, $\gamma_4 = \gamma_0\gamma_1\gamma_2\gamma_3$, $\lambda = \text{const}$.

The above equations can be obtained in a unified way within the framework of symmetry approach. For the equation of the form

$$[i\gamma_\mu\partial_{x_\mu} + F(\bar\psi, \psi)]\psi = 0 \tag{4}$$

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to be physically acceptable generalization of the linear Dirac equation (1) it must obey the Einstein relativity principle. From mathematical point of view, it means that on the set of solutions of Eq. (4) some representation of the Poincaré group $P(1,3)$ is to be realized. Consequently, one has to describe all the matrices $F(\bar{\psi},\psi)$ such that Eq. (4) is invariant under the Poincaré group. Furthermore, it is known that the massless Dirac equation is invariant under the 15-parameter conformal group $C(1,3)$. Therefore it is of interest to describe nonlinear equations (4) admitting conformal group. Such procedure is usually called symmetry or group-theoretical classification of nonlinear equations (4).

First, we give the results of symmetry classification and then turn to the problem of constructing exact solutions of the nonlinear Dirac equations (4).

Theorem 1. *System of partial differential equations (4) is Poincaré invariant iff*

$$F(\bar{\psi},\psi) = F_1(\bar{\psi}\psi, \bar{\psi}\gamma_4\psi) + F_2(\bar{\psi}\psi, \bar{\psi}\gamma_4\psi)\gamma_4, \tag{5}$$

where F_1, F_2 are arbitrary complex functions.

Theorem 2. *System of PDE (4) is invariant under the extended Poincaré group $\tilde{P}(1,3) = P(1,3) \otimes D(1)$, where $D(1)$ is a one-parameter group of scale transformations generated by the following infinitesimal operator:*

$$D = x_\mu \partial_\mu + k, \quad k \in \mathbb{R}^1, \tag{6}$$

iff the matrix-function $F(\bar{\psi}\psi)$ is of the form (5), F_i being determined by the formulae

$$F_i = (\bar{\psi}\psi)^{1/2k} \tilde{F}_i(\bar{\psi}\psi/\bar{\psi}\gamma_4\psi), \quad i = 1, 2, \tag{7}$$

with arbitrary complex functions $\tilde{F}_i$.

Theorem 3. *System of PDE (4) is invariant under the 15-parameter conformal group $C(1,3) = \tilde{P}(1,3) \otimes K(4)$, where $K(4)$ is a 4-parameter group of special conformal transformations which is generated by the following infinitesimal operators:*

$$K_\mu = 2x_\mu D - x_\nu x^\nu \partial^\mu + \frac{1}{2}(\gamma_\mu\gamma_\nu - \gamma_\nu\gamma_\mu)x^\nu, \quad \mu = 0, \ldots, 3, \tag{8}$$

iff $F(\bar{\psi},\psi)$ is of the form (5), (7) with $k = 3/2$. In formula (8) D is the operator (6) with $k = 3/2$, $x^\mu = g_{\mu\nu}x_\nu$, $\partial^\mu = g_{\mu\nu}\partial_\nu$, $\mu,\nu = 0, \ldots, 3$.

Proof of the Theorems 1–3 is carried out with the help of the infinitesimal Lie algorithm [3, 4].

Thus, there exists rather narrow class of Poincaré invariant equations of the form (4)

$$[i\gamma_\mu \partial_{x_\mu} + F_1(\bar{\psi}\psi, \bar{\psi}\gamma_4\psi) + F_2(\bar{\psi}\psi, \bar{\psi}\gamma_4\psi)\gamma_4]\psi = 0. \tag{9}$$

To construct exact solutions of the nonlinear Dirac equation (9) we apply the symmetry reduction technique.

The general idea of symmetry reduction of PDEs can be formulated in a very simple and natural way. Since coefficients of Eq. (9) do not depend explicitly on the variable x_0, we can look for a particular solution which is also independent of x_0

$$\psi = \varphi(x_1, x_2, x_3). \tag{10}$$

After substituting (10) into Eq. (9) we get system of PDEs with three independent variables

$$[i\gamma_1\partial_{x_1} + i\gamma_2\partial_{x_2} + i\gamma_3\partial_{x_3} + F_1(\bar\varphi\varphi, \bar\varphi\gamma_4\varphi) + F_2(\bar\varphi\varphi, \bar\varphi\gamma_4\varphi)\gamma_4]\varphi = 0. \tag{11}$$

But from the group-theoretical point of view independence of Eq. (9) of the variable x_0 means that this equation is invariant under the one-parameter group of displacements with respect to x_0

$$x_0' = x_0 + \theta, \quad \vec{x}\,' = \vec{x}, \quad \psi' = \psi. \tag{12}$$

Similarly, (10) is a manifold in the space of variables x, ψ invariant under the group of displacements with respect to x_0.

Thus, imposing on the solution to be found requirement of invariance with respect to the one-parameter group (12) which is a subgroup of the invariance group of Eq. (9) we reduce it by one independent variable.

Now we turn to the general case. Let Eq. (9) be invariant under the one-parameter transformation group

$$x_\mu' = f_\mu(x,\theta), \quad \psi' = F(x,\theta)\psi, \tag{13}$$

where f_μ are some real functions and F is a variable 4×4 matrix.

It is known, that there exists such change of variables

$$\omega_\mu = \omega_\mu(x), \quad \varphi = B(x)\psi, \tag{14}$$

where $B(x)$ is some invertible 4×4 matrix, that the group (13) in the space of variables ω_μ, φ takes the form

$$\omega_0' = \omega_0 + \theta, \quad \vec{\omega}\,' = \vec{\omega}, \quad \varphi' = \varphi. \tag{15}$$

Consequently, if we make in the initial equation (9) the change of variables (14), then the equation obtained will be invariant under the one-parameter group of displacements (15). Therefore, a substitution $\varphi = \varphi(\omega_1, \omega_2, \omega_3)$ reduce it to a system of PDEs with three independent variables ω_1, ω_2, ω_3.

In the initial variables the above said substitution reads

$$\psi(x) = A(x)\varphi(\omega_1(x), \omega_2(x), \omega_3(x)), \tag{16}$$

where $A(x) = B^{-1}(x)$.

And what is more, substitution of the expression (16) into Eq. (9) reduce it to a system of PDEs with three independent variables ω_1, ω_2, ω_3.

In fact, we gave a sketch of the proof of the reduction theorem, which is of utmost importance for applications of Lie transformation groups in mathematical physics. Namely, solution invariant under the one-parameter subgroup of the invariance group of the nonlinear Dirac equation reduce it to a system of PDEs with three independent variables. Obviously, a solution invariant under a three-parameter subgroup of invariance group reduce the nonlinear Dirac equation to a system of ordinary differential equations (ODEs).

So each three-parameter subgroup of the Poincaré group $P(1,3)$ gives rise to an Ansatz

$$\psi(x) = A(x)\varphi(\omega(x)), \tag{17}$$

which reduces the nonlinear Dirac equation (9) to a system of ODEs for a function $\varphi(\omega)$.

In practice it is more convenient to work with Lie algebras. Let the operators

$$Q_a = \xi_{a\mu}(x)\partial_{x_\mu} + \eta_a(x), \quad a = 1, 2, 3 \tag{18}$$

form a three-dimensional Lie algebra corresponding to a given three-parameter subgroup G_3 of the group $P(1,3)$. Then a solution invariant with respect to the group G_3 has the form (18), where function $\omega(x)$ and matrix function $A(x)$ are determined by the following equations:

$$\begin{array}{ll} 1. & \xi_{a\mu}(x)\partial_{x_\mu}\omega(x) = 0, \quad a = 1, 2, 3, \\ 2. & (\xi_{a\mu}(x)\partial_{x_\mu} + \eta_a(x))A(x) = 0, \quad a = 1, 2, 3. \end{array} \tag{19}$$

Classification of $P(1,3)$

The principal idea is based on the following observation: all Ansätze invariant under three-parameter subgroups of the group $P(1,3)$ can be represented in the following unified form [6]:

$$\psi(x) = \exp\left((\gamma_1\theta_1(x) + \gamma_2\theta_2(x))(\gamma_0 + \gamma_3)\right) \times \\ \times \exp\left(\theta_0(x)\gamma_0\gamma_3 + \theta_3(x)\gamma_1\gamma_2\right)\varphi(\omega(x)). \tag{21}$$

Specifying the functions $\theta_\mu(x)$, $\omega(x)$ we get from (21) all Poincaré invariant Ansätze mentioned above.

The idea is not to impose ad hoc conditions on the functions θ_μ, ω. The only condition is a requirement that substitution of expression (21) into Eq. (9) yields a system of ODEs for a four-component function $\varphi(\omega)$.

As a result, one gets a system of twelve nonlinear PDEs for five functions. From the first sight it looks even more complicated than the initial Eq. (9). But the fact that said system is strongly over-determined enabled us to construct its general solution.

In this way we have obtained not only all Poincaré invariant Ansätze (which is quite predictable) but also six principally new classes of Ansätze which correspond to conditional symmetry of the equation under study.

We adduce, as an example, the following Ansatz

$$\psi(x) = \exp\Big(\frac{1}{2}(w_1'\gamma_1 + w_2'\gamma_2)(\gamma_0 + \gamma_3) + C(y_1^2 + y_2^2)^{-1/2}(y_2\gamma_1 - y_1\gamma_2) \times \\ \times (\gamma_0 + \gamma_3)\Big)\exp\Big(-\frac{1}{2}\gamma_1\gamma_2\arctan\frac{y_1}{y_2}\Big)\varphi(y_1^2 + y_2^2), \tag{22}$$

where $y_a = x_a + w_a$, $a = 1, 2$, $w_a = w_a(x_0 + x_3)$ are arbitrary functions, C is an arbitrary constant.

It is readily seen that provided $C = 0, w_1 = w_2 = 0$ formula (22) gives the Ansatz (20) which has been obtained with the use of the invariance group of Eq. (9). This example demonstrates that invariant solutions are very special cases of conditionally invariant solutions.

It is important to emphasize a principal difference between invariant and conditionally-invariant Ansätze. Ansatz (20) invariant under the three-parameter subgroup of the Poincaré group can be used to reduce any Poincaré invariant system of PDE. But conditionally-invariant Ansatz (22) can be used for Eq. (9) only. It means that the last Ansatz contains more precise information about structure of solutions of the equation under study.

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1. Ivanenko D.D., *Zhurn. Eksperim. Teoret. Fiziki*, 1938, **8**, № 3, 1938, 260–266.
2. Heisenberg W., *Zs. Naturforsch. A*, 1954, **9**, 1954, 292–303.
3. Olver P., Applications of Lie groups to differential equations, New York, Springer, 1986.
4. Fushchych W.I., Zhdanov R.Z., *Phys. Rep.*, 1989, **172**, № 4, 1989, 123–174.
5. Patera J., Sharp R.T., Winternitz P., Zassenhaus H., *J. Math. Phys.*, 1975, **16**, № 8, 1613–1624.
6. Fushchych W.I., Zhdanov R.Z., Nonlinear spinor equations: symmetry and exact solutions, Kiev, Naukova Dumka, 1992.

Reduction of the self-dual Yang–Mills equations. I. The Poincaré group

R.Z. ZHDANOV, V.I. LAHNO, W.I. FUSHCHYCH

We have obtained a complete description of ansatzes for the vector-potential of the Yang–Mills field invariant under 3-parameter $P(1,3)$-inequivalent subgroups of the Poincaré group. Using these, we carry out a reduction of the self-dual Yang–Mills equations to system of ordinary differential equations.

Для вектор-потенціалу поля Янга–Міллса побудовано повний набір інваріантних відносно $P(1,3)$-нееквівалентних підгруп групи Пуанкаре анзаців, з використанням яких проведено редукцію самодуальних рівнянь Янга–Мілса до систем звичайних диференціальних рівнянь.

Classical $SU(2)$ Yang–Mills equations form a system of twelve nonlinear second-order partial differential equations (PDE) in the Minkowski space $\mathbb{R}(1,3)$. But one can obtain an important subclass of solutions by considering the following first-order system of PDE:

$$\vec{F}_{\mu\nu} = \frac{i}{2}\varepsilon_{\mu\nu\alpha\beta}\vec{F}^{\alpha\beta}, \tag{1}$$

where $\vec{F}_{\mu\nu} = \partial^\mu \vec{A}_\nu - \partial^\nu \vec{A}_\mu + e\vec{A}_\mu \times \vec{A}_\nu$ is a tensor of the Yang–Mills field; $\partial_\mu = \partial/\partial x_\mu$, $\varepsilon_{\nu\alpha\beta}$ is the antisymmetric fourth-order tensor; $\mu,\nu,\alpha,\beta = \overline{0,3}$. Hereafter, the summation over the repeated indices from 0 to 3 is understood, rising and lowering of the tensor indices is carried out with the help of the metric tensor $g_{\mu\nu} = \mathrm{diag}\,(1,-1,-1,-1)$ of the Minkowski space.

Equations (1) are called self-dual Yang–Mills equations. They are very interesting because of the fact that any solution of equations (1) automatically satisfies Yang–Mills equations (see, e.g. [1]). Moreover, symmetry groups of the Yang–Mills and of the self-dual Yang–Mills equations are the same. Maximal symmetry group admitted by equations (1) is the conformal group $C(1,3)$ supplemented by the gauge group $SU(2)$ [2].

In the present paper, we carry out a symmetry reduction of the self-dual Yang–Mills equations (1) by using ansatzes for the vector-potential of the Yang–Mills $\vec{A}_\mu(x)$ invariant under the three-parameter subgroups of the Poincaré group $P(1,3) \subset C(1,3)$.

It is known that the problem of classification of inequivalent subgroups of a Lie transformation group is equivalent to the one of classification of inequivalent subalgebras of the Lie algebra (see, e.g. [3, 4]). Complete description of $P(1,3)$-inequivalent three-dimensional subalgebras of the Poincaré algebra $AP(1,3)$ had been obtained in [3].

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To establish correspondence between the three-dimensional subalgebra of the symmetry algebra of equations (1) having the basis elements

$$X_a = \xi_{a\mu}(x,A)\partial_\mu + \sum_{b=1}^{3} \eta^b_{a\mu}(x,A)\frac{\partial}{\partial A^b_\mu}, \quad a=\overline{1,3}, \tag{2}$$

where $\{A^a_\mu, a=\overline{1,3}, \mu=\overline{0,3}\}$, and the ansatz for $\vec{A}_\mu(x)$ reducing equations (1) to a system of ordinary differential equations, one has:

(1) to construct a complete system of functionally-different invariants of the operators (2) $\omega=\{\omega_i(x,A), i=\overline{1,13}\}$;

(2) to resolve the relations

$$F_j(\omega_1(x,A),\ldots,\omega_{13}(x,A))=0, \quad j=\overline{1,13} \tag{3}$$

with respect to the functions A^a_μ.

As proved in [5], the above procedure can be significantly simplified if coefficients of operators (2) have the following structure:

$$\xi_{a\mu}=\xi_{a\mu}(x), \quad \eta^b_{a\mu}=\sum_{c=1}^{3} R^{bc}_{a\mu\nu}(x)A^c_\nu. \tag{4}$$

The ansatz for $\vec{A}_\mu$ can be searched for in the form

$$A^a_\mu(x)=\sum_{c=1}^{3} Q^{ab}_{\mu\nu}(x)B^b_\nu(\omega(x)), \tag{5}$$

where $B^b_\nu(\omega)$ are arbitrary smooth and the functions $\omega(x)$, $Q^{ab}_{\mu\nu}(x)$ satisfy the system of PDE

$$\begin{gathered}\xi_{a\mu}(x)\omega_{x_\mu}=0,\\ \sum_{c=1}^{3}(\xi_{a\mu}\delta^{bc}\partial_\mu - R^{bc}_{a\mu\nu})Q^{cd}_{\nu\alpha}=0.\end{gathered} \tag{6}$$

Here, δ^{bc} is the Kronecker symbol, $a,b,d=\overline{1,3}$, $\alpha=\overline{0,3}$.

On the set of solutions of equations (1), the following representation of the Poincaré algebra is realized:

$$P_\mu=\partial^\mu, \quad J_{\mu\nu}=x_\mu P_\nu - x_\nu P_\mu + \sum_{a=1}^{3}\left(A^a_\mu\frac{\partial}{\partial A^{a\nu}} - A^a_\nu\frac{\partial}{\partial A^{a\mu}}\right), \quad \mu,\nu=\overline{0,3}. \tag{7}$$

Consequently, relations (4) hold true. Moreover, expression for $\eta^b_{a\mu}$ has the form

$$\eta^b_{a\mu}=R_{a\mu\nu}(x)A^b_\nu, \quad a,b=\overline{1,3}, \quad \mu=\overline{0,3}. \tag{8}$$

That is why formulae (5), (6) can be rewritten in a simpler way. Namely, an ansatz for the vector-potential of the Yang–Mills field $\vec{A}(x)$ invariant under a subalgebra of the algebra $AP(1,3)$ with basis operators (7) should be searched for in the form

$$A^a_\mu(x)=Q_{\mu\nu}(x)B^a_\nu(\omega(x)), \tag{9}$$

where $B^a_\nu(\omega)$ are arbitrary smooth functions and functions $\omega(x)$, $Q_{\mu\nu}(x)$ satisfy the system of PDE

$$\begin{aligned}&\xi_{a\mu}(x)\omega_{x_\mu}=0,\\&\xi_{a\alpha}(x)\partial_\alpha Q_{\mu\nu}-R_{a\mu\alpha}(x)Q_{\alpha\nu}=0,\end{aligned}\tag{10}$$

where $a=\overline{1,3}$, $\mu,\nu=\overline{0,3}$.

Thus, to get a complete description of $P(1,3)$-inequivalent ansatzes invariant under three-dimensional subalgebras of the Poincaré algebra, one has to integrate over-determined system of PDE (10) for each subalgebra. Let us note that compatibility of equations (10) is guaranteed by the fact that the operators X_1, X_2, X_3 form a Lie algebra.

Bellow, we adduce a complete list of $C(1,3)$-inequivalent three-dimensional subalgebras of the Poincaré algebra $AP(1,3)$ following [4]:

$$\begin{aligned}
&L_1=\langle P_0,P_1,P_2\rangle, && L_2=\langle P_1,P_2,P_3\rangle,\\
&L_3=\langle P_0+P_3,P_1,P_2\rangle, && L_4=\langle J_{03}+\alpha J_{12},P_1,P_2\rangle,\\
&L_5=\langle J_{03},P_0+P_3,P_1\rangle, && L_6=\langle J_{03}+P_1,P_0,P_3\rangle,\\
&L_7=\langle J_{03}+P_1,P_0+P_3,P_1\rangle, && L_8=\langle J_{12}+\alpha J_{03},P_0,P_3\rangle,\\
&L_9=\langle J_{12}+P_0,P_1,P_2\rangle, && L_{10}=\langle J_{12}+P_3,P_1,P_2\rangle,\\
&L_{11}=\langle J_{12}+P_0-P_3,P_1,P_2\rangle, && L_{12}=\langle G_1,P_0+P_3,P_2+\alpha P_1\rangle,\\
&L_{13}=\langle G_1+P_2,P_0+P_3,P_1\rangle, && L_{14}=\langle G_1+P_0-P_3,P_0+P_3,P_2\rangle,\\
&L_{15}=\langle G_1+P_0-P_3,P_1+\alpha P_2,P_0+P_3\rangle, && L_{16}=\langle J_{12},J_{03},P_0+P_3\rangle,\\
&L_{17}=\langle G_1+P_2,G_2-P_1+\alpha P_2,P_0+P_3\rangle, && L_{18}=\langle G_1,J_{03},P_2\rangle,\\
&L_{19}=\langle J_{03},G_1,P_0+P_3\rangle, && L_{20}=\langle J_{03}+P_2,G_1,P_0+P_3\rangle,\\
&L_{21}=\langle G_1,J_{03}+P_1+\alpha P_2,P_0+P_3\rangle, && L_{22}=\langle G_1,G_2,J_{03}+\alpha J_{12}\rangle,\\
&L_{23}=\langle G_1,P_0+P_3,P_1\rangle, && L_{24}=\langle J_{12},P_1,P_2\rangle,\\
&L_{25}=\langle J_{03},P_0,P_3\rangle, && L_{26}=\langle J_{01},J_{02},J_{12}\rangle,\\
&L_{27}=\langle J_{12},J_{23},J_{13}\rangle,
\end{aligned}$$

Here, $G_i=J_{0i}-J_{i3}$ $(i=1,2)$, $\alpha\in\mathbb{R}$.

Let us consider, as an example, the procedure of construction of ansatz (9) invariant under subalgebra L_4 $(\alpha=0)$. In this case, system (10) reads

$$\omega_{x_1}=\omega_{x_2}=0,\quad x_0\omega_{x_3}+x_3\omega_{x_0}=0,\tag{11a}$$

$$Q_{x_1}=Q_{x_2}=0,\quad x_0Q_{x_3}+x_3Q_{x_0}-SQ=0,\tag{11b}$$

where $Q=\|Q_{\mu\nu}(x)\|^3_{\mu,\nu}=0$,

$$S=\left\|\begin{matrix}0&0&0&1\\0&0&0&0\\0&0&0&0\\1&0&0&0\end{matrix}\right\|.$$

The first integral of system (11a) has the form $\omega=x_0^2-x_3^2$. Next, from first two equations of system (11b), it follows that $Q=Q(x_0,x_3)$. Since S is a constant matrix, solutions of the third equation from (11b) can be looked for in the form (see, e.g. [6])

$$Q=\exp\{f(x_0,x_3)S\}.$$

By substituting this expression into (11b), we get

$$(x_0 f_{x_3}, x_3 f_{x_0} - 1)\exp\{fS\} = 0,$$

where $f = \ln(x_0 + x_3)$.

Consequently, a particular solution of equations (lib) can be chosen in the following way:

$$Q = \exp\{\ln(x_0 + x_3)S\}.$$

By using evident identity $S = S^3$, we obtain the equality

$$Q = I + S\,\mathrm{sh}(\ln(x_0 + x_3)) + S^2(\mathrm{ch}(\ln(x_0 + x_3)) - 1), \qquad (12)$$

where I is a unit (4×4)-matrix.

By substituting the obtained expressions into formula (9), we get an ansatz for $\vec{A}_\mu(x)$ which is invariant under the algebra L_4

$$\begin{aligned}
&A_0^a = B_0^a(x_0^2 - x_3^2)\,\mathrm{ch}(\ln(x_0 + x_3)) + B_3^a(x_0^2 - x_3^2)\,\mathrm{sh}(\ln(x_0 + x_3)),\\
&A_1^a = B_1^a(x_0^2 - x_3^2), \quad A_2^a = B_2^a(x_0^2 - x_3^2),\\
&A_3^a = B_3^a(x_0^2 - x_3^2)\,\mathrm{ch}(\ln(x_0^2 - x_3^2)) + B_0^a(x_0^2 - x_3^2)\,\mathrm{sh}(\ln(x_0^2 - x_3^2)), \quad a = \overline{1,3}.
\end{aligned} \qquad (13)$$

The above ansatz has such an unpleasant feature as an asymmetric dependence on independent variables x_μ. To remove this asymmetry, one has to use a solution generation procedure [7]. As a result, we arrive at the following representation of the Poincaré invariant ansatz for the vector-potential of the Yang–Mills field:

$$\begin{aligned}
\vec{A}_\mu(x) = Q_{\mu\nu}(x)\vec{B}^\nu(\omega) = \{&(a_\mu a_\nu - d_\mu d_\nu)\,\mathrm{ch}\,\theta_0 + (d_\mu a_\nu - d_\nu a_\mu)\,\mathrm{sh}\,\theta_0 + \\
&+ 2k_\mu[(\theta_1\cos\theta_3 + \theta_2\sin\theta_3)b_\nu + (\theta_2\cos\theta_3 - \theta_1\sin\theta_3)c_\nu + \\
&+ (\theta_1^2 + \theta_2^2)k_\nu\exp(-\theta_0)] + (b_\mu c_\nu - b_\nu c_\mu)\sin\theta_3 - (c_\mu c_\nu - b_\mu b_\nu)\cos\theta_3 - \\
&- 2(\theta_1 b_\mu + \theta_2 c_\mu)k_\nu\exp(-\theta_0)]\}\vec{B}^\nu(\omega).
\end{aligned}$$

Here, a_μ, b_μ, c_μ, d_μ are arbitrary constants satisfying the following equalities:

$$\begin{aligned}
&a_\mu a^\mu = -b_\mu b^\mu = -c_\mu c^\mu = -d_\mu d^\mu = 1,\\
&a_\mu b^\mu = a_\mu c^\mu = a_\mu d^\mu = b_\mu c^\mu = b_\mu d^\mu = c_\mu d^\mu = 0,
\end{aligned}$$

$k_\mu = a_\mu + d_\mu$, Q_μ, ω are some functionals of x whose explicit form depends on the choice of the algebra $AP(1,3)$, $\mu = \overline{0,3}$. Below, we adduce a complete list of functions Q_μ, $\mu = \overline{0,3}$, ω co corresponding to three-dimensional subalgebras of the Poincaré algebra (7).

$$\begin{aligned}
&L_1: \quad \theta_\mu = 0, \quad \omega = dx;\\
&L_2: \quad \theta_\mu = 0, \quad \omega = ax;\\
&L_3: \quad \theta_\mu = 0, \quad \omega = ax + dx;\\
&L_4: \quad \theta_0 = -\ln|ax + dx|, \quad \theta_1 = \theta_2 = 0, \quad \theta_3 = a\ln|ax + dx|,\\
&\qquad\ \ \omega = (ax)^2 - (dx)^2;\\
&L_5: \quad \theta_0 = -\ln|ax + dx|, \quad \theta_1 = \theta_2 = \theta_3 = 0, \quad \omega = cx;\\
&L_6: \quad \theta_0 = bx, \quad \theta_1 = \theta_2 = \theta_3 = 0, \quad \omega = cx;\\
&L_7: \quad \theta_0 = bx, \quad \theta_1 = \theta_2 = \theta_3 = 0, \quad \omega = -bx + \ln|ax + dx|;
\end{aligned}$$

$$L_8: \quad \theta_0 = \alpha \operatorname{arctg}[bx(cx)^{-1}], \quad \theta_1 = \theta_2 = 0, \quad \theta_3 = -\operatorname{arctg}[bx(cx)^{-1}], \quad \omega = (bx)^2 + (cx)^2;$$

$$L_9: \quad \theta_0 = \theta_1 = \theta_2 = 0, \quad \theta_3 = -ax, \quad \omega = dx;$$

$$L_{10}: \quad \theta_0 = \theta_1 = \theta_2 = 0, \quad \theta_3 = dx, \quad \omega = ax;$$

$$L_{11}: \quad \theta_0 = \theta_1 = \theta_2 = 0, \quad \theta_3 = -\frac{1}{2}(dx + ax), \quad \omega = ax + dx;$$

$$L_{12}: \quad \theta_0 = \theta_2 = \theta_3 = 0, \quad \theta_1 = \frac{1}{2}(bx - \alpha cx)(ax + dx)^{-1}, \quad \omega = ax + dx;$$

$$L_{13}: \quad \theta_0 = \theta_2 = \theta_3 = 0, \quad \theta_1 = \frac{1}{2}cx, \quad \omega = ax + dx;$$

$$L_{14}: \quad \theta_0 = \theta_2 = \theta_3 = 0, \quad \theta_1 = -\frac{1}{4}(ax + dx), \quad \omega = 4bx - (ax + dx)^2;$$

$$L_{15}: \quad \theta_0 = \theta_2 = \theta_3 = 0, \quad \theta_1 = -\frac{1}{4}(ax + dx), \quad \omega = 4(\alpha bx - cx) - \alpha(ax + dx)^2;$$

$$L_{16}: \quad \theta_0 = -\ln|ax + dx|, \quad \theta_1 = \theta_2 = 0, \quad \theta_3 = -\operatorname{arctg}[bx(cx)^{-1}], \quad \omega = (bx)^2 + (cx)^2;$$

$$L_{17}: \quad \theta_0 = 0, \quad \theta_1 = \frac{1}{2}\frac{cx + (\alpha + ax + dx)bx}{1 + (ax + dx)(\alpha + ax + dx)}, \quad \theta_2 = -\frac{1}{2}\frac{bx - cx(ax + dx)}{1 + (ax + dx)(\alpha + ax + dx)}, \quad \theta_3 = 0, \quad \omega = ax + dx;$$

$$L_{18}: \quad \theta_0 = -\ln|ax + dx|, \quad \theta_1 = \frac{1}{2}\frac{bx}{ax + dx}, \quad \theta_2 = \theta_3 = 0, \quad \omega = (ax)^2 - (bx)^2 - (dx)^2;$$

$$L_{19}: \quad \theta_0 = -\ln|ax + dx|, \quad \theta_1 = \frac{1}{2}\frac{bx}{ax + dx}, \quad \theta_2 = \theta_3 = 0, \quad \omega = cx;$$

$$L_{20}: \quad \theta_0 = -\ln|ax + dx|, \quad \theta_1 = \frac{1}{2}\frac{bx}{ax + dx}, \quad \theta_2 = \theta_3 = 0, \quad \omega = cx + \ln|ax + dx|;$$

$$L_{21}: \quad \theta_0 = -\ln|ax + dx|, \quad \theta_1 = -\frac{1}{2}\frac{-bx + \ln|ax - dx|}{ax + dx}, \quad \theta_2 = \theta_3 = 0, \quad \omega = cx + \alpha\ln|ax + dx|;$$

$$L_{22}: \quad \theta_0 = -\ln|ax + dx|, \quad \theta_1 = \frac{1}{2}\frac{bx}{ax - dx}, \quad \theta_2 = \frac{1}{2}\frac{cx}{ax - dx}, \quad \theta_3 = \alpha\ln|ax + dx|, \quad \omega = (ax)^2 - (bx)^2 - (cx)^2 - (dx)^2.$$

Here, $ax = a_\mu x^\mu$, $bx = b_\mu x^\mu$, $cx = c_\mu x^\mu$, $dx = d_\mu x^\mu$, $\mu = \overline{0,3}$.

Note. Ansatzes invariant under subalgebras L_{23}, L_{24}, L_{25}, L_{26}, L_{27} yield so-called partially-invariant solutions (the term was introduced by L.V. Ovsyannikov [8]) which cannot be represented in the form (13) and are not considered here.

Substitution of ansatzes (13), (14) into system of PDE (1) demands very cumbersome computations. This is why we omit these and adduce only the final result-system of ordinary differential equations for $\vec{B}_\mu(\omega)$.

General form of the reduced system is the following:

$$\vec{T}_{\mu\nu} = \frac{1}{2}\varepsilon_{\mu\alpha\beta}\vec{T}^{\alpha\beta}, \quad \mu,\nu = \overline{0,3}, \tag{14}$$

where

$$\vec{T}_{\mu\nu} = G_\mu(\omega)\vec{B}_\mu - G_\nu\vec{B}_\mu - H_{\mu\nu\gamma}(\omega)\vec{B}^\gamma + e\vec{B}_\mu \times \vec{B}_\nu$$

and functions $G_\mu(\omega), H_{\mu\gamma}(\omega)$ are calculated according to the following formulae:

$$G_\mu(\omega) = Q_{\mu\nu}\omega_{x_\nu},$$
$$H_{\mu\nu\gamma}(\omega) = Q^\alpha_\mu Q_{\alpha\gamma x_\beta}Q_{\beta\nu} - Q^\alpha_\nu - Q_{\alpha\gamma x_\beta}Q_{\beta\mu}.$$

In the above formulae, overdot means differentiation with respect to ω.

Thus, the form of the reduced equations for functions $\vec{B}_\mu(\omega)$ depends on the explicit forms of functions $G_\mu(\omega)$, $H_{\mu\nu\gamma}(\omega)$. Below, we adduce a list of these functions corresponding to ansatzes (13), (14).

$L_1: \quad G_\mu = -d_\mu, \quad H_{\mu\nu\gamma} = 0;$

$L_2: \quad G_\mu = a_\mu, \quad H_{\mu\nu\gamma} = 0;$

$L_3: \quad G_\mu = k_\mu, \quad H_{\mu\nu\gamma} = 0;$

$L_4: \quad G_\mu = \varepsilon[a_\mu - d_\mu + k_\mu\omega],$
$H_{\mu\nu\gamma} = -\varepsilon[(a_\mu d_\nu - d_\mu a_\nu)k_\gamma + \alpha(k_\nu(b_\gamma c_\mu - c_\gamma b_\mu) - k_\mu(b_\gamma c_\mu - c_\gamma b_\nu))];$

$L_5: \quad G_\mu = c_\mu, \quad H_{\mu\nu\gamma} = -\varepsilon(a_\mu d_\nu - d_\mu a_\nu)k_\gamma;$

$L_6: \quad G_\mu = c_\mu, \quad H_{\mu\nu\gamma} = (a_\mu d_\gamma - a_\gamma d_\mu)b_\nu + (a_\nu d_\gamma - a_\gamma d_\nu)b_\mu;$

$L_7: \quad G_\mu = -b_\mu + \varepsilon k_\mu, \quad H_{\mu\nu\gamma} = -(a_\mu d_\gamma - a_\gamma d_\mu)b_\nu + (a_\nu d_\gamma - a_\gamma d_\nu)b_\mu;$

$L_8: \quad G_\mu = 2c_\mu\sqrt{\omega},$
$H_{\mu\nu\gamma} = \dfrac{1}{\sqrt{\omega}}\{(c_\mu b_\nu - c_\nu b_\mu)b_\gamma + \alpha[(d_\mu a_\gamma - a_\mu d_\gamma)b_\nu - (d_\nu a_\gamma - a_\nu d_\gamma)b_\mu]\};$

$L_9: \quad G_\mu = -d_\mu, \quad H_{\mu\nu\gamma} = -a_\mu(b_\nu c_\gamma - c_\nu b_\gamma) + a_\nu(b_\mu c_\gamma - c_\mu b_\gamma);$

$L_{10}: \quad G_\mu = a_\mu, \quad H_{\mu\nu\gamma} = (b_\mu c_\gamma - c_\mu b_\gamma)d_\nu - (b_\nu c_\gamma - c_\nu b_\gamma)d_\mu;$

$L_{11}: \quad G_\mu = a_\mu - d_\mu, \quad H_{\mu\nu\gamma} = \dfrac{1}{2}[(b_\nu c_\gamma - c_\nu b_\gamma)b_\mu - (b_\mu c_\gamma - c_\mu b_\gamma)b_\nu];$

$L_{12}: \quad G_\mu = k_\mu,$
$H_{\mu\nu\gamma} = \dfrac{1}{\omega}\{(k_\mu b_\nu - k_\nu b_\mu)b_\gamma - \alpha[(k_\mu b_\gamma - k_\gamma b_\mu)c_\nu - (k_\nu b_\gamma - k_\gamma b_\nu)c_\mu]\};$

$L_{13}: \quad G_\mu = k_\mu, \quad H_{\mu\nu\gamma} = (k_\mu b_\gamma - k_\gamma b_\mu)c_\nu - (k_\nu c_\gamma - k_\gamma b_\nu)c_\mu;$

$L_{14}: \quad G_\mu = 4b_\mu, \quad H_{\mu\nu\gamma} = \dfrac{1}{2}(b_\mu k_\nu - b_\nu k_\mu)k_\gamma;$

$L_{15}: \quad G_\mu = 4(c_\mu - \alpha b_\mu), \quad H_{\mu\nu\gamma} = \dfrac{1}{2}(b_\mu k_\nu - b_\nu k_\mu)k_\gamma;$

$L_{16}: \quad G_\mu = 2c_\mu\sqrt{\omega},$
$H_{\mu\nu\gamma} = \varepsilon(a_\mu d_\nu - a_\nu d_\mu)k_\gamma - \dfrac{1}{\sqrt{\omega}}k_\gamma - \dfrac{1}{\sqrt{\omega}}(b_\mu c_\nu - c_\mu b_\nu)b_\gamma;$

$L_{17}: \quad G_\mu = k_\mu,$
$H_{\mu\nu\gamma} = \dfrac{1}{1+\omega(\omega+\alpha)}\{2(b_\nu c_\mu - b_\mu c_\nu)k_\gamma + (k_\mu c_\nu - k_\nu c_\mu)b_\gamma +$

$$
\begin{aligned}
&\qquad\qquad + (k_\nu b_\mu - k_\mu b_\nu)c_\gamma + (\alpha+\omega)(k_\mu b_\nu - k_\nu b_\mu)b_\gamma + \\
&\qquad\qquad + \omega(k_\mu c_\nu - k_\nu c_\mu)c_\gamma\};\\
L_{18}:\quad & G_\mu = \varepsilon(k_\mu\omega + a_\mu - d_\mu),\\
& H_{\mu\nu\gamma} = \varepsilon[(k_\mu b_\nu - k_\nu b_\mu)b_\gamma + (a_\mu d_\nu - k_\nu b_\mu)k_\gamma];\\
L_{19}:\quad & G_\mu = c_\mu, \quad H_{\mu\nu\gamma} = \varepsilon[(k_\mu b_\nu - k_\nu b_\mu)b_\gamma + (a_\mu d_\nu - a_\nu d_\mu)k_\gamma];\\
L_{20}:\quad & G_\mu = c_\mu + \varepsilon k_\mu, \quad H_{\mu\nu\gamma} = \varepsilon[(a_\mu d_\nu - a_\nu d_\mu)k_\gamma + (k_\mu b_\nu - k_\nu b_\mu)b_\gamma];\\
L_{21}:\quad & G_\mu = c_\mu + \varepsilon\alpha k_\mu,\\
& H_{\mu\nu\gamma} = \varepsilon[(a_\mu d_\nu - a_\nu d_\mu)k_\gamma + (k_\mu b_\nu - k_\nu b_\mu)b_\gamma - (k_\mu b_\nu - k_\nu b_\mu)k_\gamma];\\
L_{22}:\quad & G_\mu = \varepsilon(k_\mu\omega + a_\mu - d_\mu),\\
& H_{\mu\nu\gamma} = \varepsilon\{(k_\mu b_\nu - k_\nu b_\mu)b_\gamma + (k_\mu c_\nu - k_\nu c_\mu)c_\gamma + \\
&\qquad\qquad + \alpha[(b_\mu c_\gamma - c_\mu b_\mu)k_\nu - (b_\nu c_\gamma - c_\nu b_\gamma] + (a_\mu d_\nu - a_\nu d_\mu)k_\gamma\}.
\end{aligned}
$$

Here, $k_\mu = a_\mu + d_\mu$, $\varepsilon = 1$ for $ax + dx > 0$ and $\varepsilon = -1$ for $ax + dx < 0$.

Thus, using symmetry properties of the self-dual Yang–Mills equations and subalgebraic structure of the Poincaré algebra, we reduced system of PDE (1) to the system of ordinary differential equations (15). Let us emphasize that system (15) contains nine equations for twelve functions, which means that it is underdetermined. This fact simplifies essentially finding its particular solutions.

If one constructs a solution of one of equations (15) (general or particular), then substitution of the obtained result into the corresponding ansatz from (13). (14) yields an exact solution of the nonlinear self-dual Yang–Mills equations (1). We intend to study in detail the reduced system of ordinary differential equations (15) and construct new classes of exact solutions of equations (1) but this will be a topic of our future publication.

1. Actor A., Classical solutions of SU(2) Yang–Mills theories, *Rev. Mod. Phys.*, 1979, **51**, № 3, 461–526.
2. Schwarz F., Symmetry of $SU(2)$ invariant Yang–Mills theories, *Lett. Math. Phys.*, 1982, **6**, № 5, 355–359.
3. Patera J., Winternitz P., Zassenhaus H., Continuous subgroups of the fundament al groups of physics. I. General method and the Poincaré group, *J. Math. Phys.*, 1975, **16**, № 8, 1597–1624.
4. Фущич В.I., Баранник Л.Ф., Баранник А.Ф., Подгрупповой анализ групп Галилея и Пуанкаре и редукция нелинейных уравнений, Киев, Наук. думка, 1991, 304 с.
5. Fushchych W.I., Zhdanov R.Z., Conditional symmetry and reduction of partial differential equations, *Ukr. Math. J.*, 1992, **44**, № 7, 970–982.
6. Fushchych W.I., Zhdanov R.Z., Symmetry and exact solutions of nonlinear spinor equations, *Phys. Repts.*, 1982, **172**, № 4, 123–174.
7. Фущич В.И., Жданов Р.З., Нелинейные спинорные уравнения: симметрия и точные решения, Киев, Наук, думка, 1992, 388 с.
8. Овсянников Л.В., Групповой анализ дифференциальных уравнений, М: Наука, 1978, 400 с.

On the new approach to variable separation in the time-dependent Schrödinger equation with two space dimensions

R.Z. ZHDANOV, I.V. REVENKO, W.I. FUSHCHYCH

We suggest an effective approach to separation of variables in the Schrödinger equation with two space variables. Using it we classify inequivalent potentials $V(x_1, x_2)$ such that the corresponding Schrödinger equations admit separation of variables. Besides that, we carry out separation of variables in the Schrödinger equation with the anisotropic harmonic oscillator potential $V = k_1x_1^2 + k_2x_2^2$ and obtain a complete list of coordinate systems providing its separability. Most of these coordinate systems depend essentially on the form of the potential and do not provide separation of variables in the free Schrödinger equation ($V = 0$).

1 Introduction

The problem of separation of variables (SV) in the two-dimensional Schrödinger equation

$$iu_t + u_{x_1x_1} + u_{x_2x_2} = V(x_1, x_2)u \tag{1}$$

as well as the most of classical problems of mathematical physics can be formulated in a very simple way (but this simplicity does not, of course, imply an existence of easy way to its solution). To separate variables in Eq. (1) one has to construct such functions $R(t, \boldsymbol{x})$, $\omega_1(t, \boldsymbol{x})$, $\omega_2(t, \boldsymbol{x})$ that the Schrödinger equation (1) after being rewritten in the new variables

$$\begin{aligned} &z_0 = t, \quad z_1 = \omega_1(t, \boldsymbol{x}), \quad z_2 = \omega_2(t, \boldsymbol{x}), \\ &v(z_0, \boldsymbol{z}) = R(t, \boldsymbol{x})u(t, \boldsymbol{x}) \end{aligned} \tag{2}$$

separates into three ordinary differential equations (ODEs). From this point of view the problem of SV in Eq. (1) is studied in [1–4].

But no less of an important problem is the one of description of potentials $V(x_1, x_2)$ such that the Schrödinger equation admits variable separation. That is why saying about SV in Eq. (1) we imply two mutually connected problems. The first one is to describe all such functions $V(x_1, x_2)$ that the corresponding Schrödinger equation (1) can be separated into three ODEs in some coordinate system of the form (2) (classification problem). The second problem is to construct for each function $V(x_1, x_2)$ obtained in this way all coordinate systems (2) enabling us to carry out SV in Eq. (1).

Up to our knowledge, the second problem has been solved provided $V = 0$ [2, 3] and $V = \alpha x_1^{-2} + \beta x_2^{-2}$ [1]. The first one was considered in a restricted sense in [4]. Authors using symmetry approach to classification problem obtained some potentials providing separability of Eq. (1) and carried out SV in the corresponding

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Schrödinger equation. But their results are far from being complete and systematic. The necessary and sufficient conditions imposed on the potential $V(x_1, x_2)$ by the requirement that the Schrödinger equation admits symmetry operators of an arbitrary order are obtained in [5]. But so far there is no systematic and exhaustive description of potentials $V(x_1, x_2)$ providing SV in Eq. (1).

To be able to discuss the description of *all* potentials and *all* coordinate systems making it possible to separate the Schrödinger equation one has to give a definition of SV. One of the possible definitions of SV in partial differential equations (PDEs) is proposed in our article [6]. It is based on the concept of Ansatz suggested by Fushchych [7] and on ideas contained in the article by Koornwinder [8]. The said definition is quite algorithmic in the sense that it contains a regular algorithm of variable separation in partial differential equations which can be easily adapted to handle both linear [6, 9] and nonlinear [10] PDEs. In the present article we apply the said algorithm to solve the problem of SV in Eq. (1).

Consider the following system of ODEs:

$$\begin{gathered} i\frac{d\varphi_0}{dt} = U_0(t, \varphi_0; \lambda_1, \lambda_2), \\ \frac{d^2\varphi_1}{d\omega_1^2} = U_1\left(\omega_1, \varphi_1, \frac{d\varphi_1}{d\omega_1}; \lambda_1, \lambda_2\right), \quad \frac{d^2\varphi_2}{d\omega_2^2} = U_2\left(\omega_2, \varphi_2, \frac{d\varphi_2}{d\omega_2}; \lambda_1, \lambda_2\right), \end{gathered} \tag{3}$$

where U_0, U_1, U_2 are some smooth functions of the corresponding arguments, $\lambda_1, \lambda_2 \subset \mathbb{R}^1$ are arbitrary parameters (separation constants) and what is more

$$\operatorname{rank} \left\| \frac{\partial U_\mu}{\partial \lambda_a} \right\|_{\mu=0}^{2} \ {}_{a=1}^{2} = 2 \tag{4}$$

(the last condition ensures essential dependence of the corresponding solution with separated variables on λ_1, λ_2, see [8]).

Definition 1. *We say that Eq. (1) admits SV in the system of coordinates t, $\omega_1(t, \boldsymbol{x})$, $\omega_2(t, \boldsymbol{x})$ if substitution of the Ansatz*

$$u = Q(t, \boldsymbol{x})\varphi_0(t)\varphi_1\big(\omega_1(t, \boldsymbol{x})\big)\varphi_2\big(\omega_2(t, \boldsymbol{x})\big) \tag{5}$$

into Eq. (1) with subsequent exclusion of the derivatives $d\varphi_0/dt$, $d^2\varphi_1/d\omega_1^2$, $d^2\varphi_2/d\omega_2^2$ according to Eqs. (3) yields an identity with respect to φ_0, φ_1, φ_2, $d\varphi_1/d\omega_1$, $d\varphi_2/d\omega_2$, λ_1, λ_2.

Thus, according to the above definition to separate variables in Eq. (1) one has

(i) to substitute the expression (5) into (1),

(ii) to exclude derivatives $d\varphi_0/dt$, $\frac{d^2\varphi_1}{d\omega_1^2}$, $d^2\varphi_2/d\omega_2^2$ with the help of Eqs. (3),

(iii) to split the obtained equality with respect to the variables φ_0, φ_1, φ_2, $d\varphi_1/d\omega_1$, $d\varphi_2/d\omega_2$, λ_1, λ_2 considered as independent.

As a result one gets some over-determined system of PDEs for the functions $Q(t, \boldsymbol{x})$, $\omega_1(t, \boldsymbol{x})$, $\omega_2(t, \boldsymbol{x})$. On solving it one obtains a complete description of all coordinate systems and potentials providing SV in the Schrödinger equation. Naturally, an expression *complete description* makes sense only within the framework of our definition. So if one uses a more general definition it may be possible to construct new coordinate systems and potentials providing separability of Eq. (1). But all solutions of the Schrödinger equation with separated variables known to us fit into the scheme suggested by us and can be obtained in the above described way.

2 Classification of potentials $V(x_1, x_2)$

We do not adduce in full detail computations needed because they are very cumbersome. We shall restrict ourselves to pointing out main steps of the realization of the above suggested algorithm.

First of all we make a remark, which makes life a little bit easier. It is readily seen that a substitution of the form

$$\begin{aligned}&Q \to Q' = Q\Psi_1(\omega_1)\Psi_2(\omega_2),\\ &\omega_a \to \omega_a' = \Omega_a(\omega_a), \quad a = 1,2, \quad \lambda_a \to \lambda_a' = \Lambda_a(\lambda_1, \lambda_2), \quad a = 1,2,\end{aligned} \tag{6}$$

does not alter the structure of relations (3), (4), and (5). That is why, we can introduce the following equivalence relation:

$$(\omega_1, \omega_2, Q) \ \sim \ (\omega_1', \omega_2', Q')$$

provided Eq. (6) holds with some Ψ_a, Ω_a, Λ_a.

Substituting Eq. (5) into Eq. (1) and excluding the derivatives $d\varphi_0/dt$, $d^2\varphi_1/d\omega_1^2$, $d^2\varphi_2/d\omega_2^2$ with the use of equations (3) we get

$$\begin{aligned}&i(Q_t\varphi_0\varphi_1\varphi_2 + QU_0\varphi_1\varphi_2 + Q\omega_{1t}\varphi_0\dot\varphi_1\varphi_2 + Q\omega_{2t}\varphi_0\varphi_1\dot\varphi_2) + (\triangle Q)\varphi_0\varphi_1\varphi_2 +\\ &\quad + 2Q_{x_a}\omega_{1x_a}\varphi_0\dot\varphi_1\varphi_2 + 2Q_{x_a}\omega_{2x_a}\varphi_0\varphi_1\dot\varphi_2 + Q\big((\triangle\omega_1)\varphi_0\dot\varphi_1\varphi_2 +\\ &\quad + (\triangle\omega_2)\varphi_0\varphi_1\dot\varphi_2 + \omega_{1x_a}\omega_{1x_a}\varphi_0 U_1\varphi_2 + \omega_{2x_a}\omega_{2x_a}\varphi_0\varphi_1 U_2 +\\ &\quad + 2\omega_{1x_a}\omega_{2x_a}\varphi_0\dot\varphi_1\dot\varphi_2\big) = VQ\varphi_0\varphi_1\varphi_2,\end{aligned}$$

where the summation over the repeated index a from 1 to 2 is understood. Hereafter an overdot means differentiation with respect to a corresponding argument and $\triangle = \partial^2_{x_1} + \partial^2_{x_2}$.

Splitting the equality obtained with respect to independent variables φ_1, φ_2, $d\varphi_1/d\omega_1$, $d\varphi_2/d\omega_2$, λ_1, λ_2 we conclude that ODEs (3) are linear and up to the equivalence relation (6) can be written in the form

$$\begin{aligned}&i\frac{d\varphi_0}{dt} = \big(\lambda_1 R_1(t) + \lambda_2 R_2(t) + R_0(t)\big)\varphi_0,\\ &\frac{d^2\varphi_1}{d\omega_1^2} = \big(\lambda_1 B_{11}(\omega_1) + \lambda_2 B_{12}(\omega_1) + B_{01}(\omega_1)\big)\varphi_1,\\ &\frac{d^2\varphi_2}{d\omega_2^2} = \big(\lambda_1 B_{21}(\omega_2) + \lambda_2 B_{22}(\omega_2) + B_{02}(\omega_2)\big)\varphi_2\end{aligned}$$

and what is more, functions ω_1, ω_2, Q satisfy an over-determined system of nonlinear

PDEs

$$
\begin{aligned}
&(1)\quad \omega_{1x_b}\omega_{2x_b}=0,\\
&(2)\quad B_{1a}(\omega_1)\omega_{1x_b}\omega_{1x_b}+B_{2a}(\omega_2)\omega_{2x_b}\omega_{2x_b}+R_a(t)=0,\quad a=1,2,\\
&(3)\quad 2\omega_{ax_b}Q_{x_b}+Q(i\omega_{at}+\triangle\omega_a),\quad a=1,2,\\
&(4)\quad \big(B_{01}(\omega_1)\omega_{1x_b}\omega_{1x_b}+B_{02}(\omega_1)\omega_{2x_b}\omega_{2x_b}\big)Q+iQ_t+\triangle Q+R_0(t)Q-\\
&\qquad -V(x_1,x_2)Q=0.
\end{aligned}
\tag{7}
$$

Thus, to solve the problem of SV for the linear Schrödinger equation it is necessary to construct general solution of system of nonlinear PDEs (7). Roughly speaking, to solve a linear equation one has to solve a system of *nonlinear equations*! This is the reason why so far there is no complete description of all coordinate systems providing separability of the four-dimensional wave equation [3].

But in the case involved we have succeeded in integrating system of nonlinear PDEs (7). Our approach to integration of it is based on the following change of variables (hodograph transformation)

$$
z_0=t,\quad z_1=Z_1(t,\omega_1,\omega_2),\quad z_2=Z_2(t,\omega_1,\omega_2),\quad v_1=x_1,\quad v_2=x_2,
$$

where z_0, z_1, z_2 are new independent and v_1, v_2 are new dependent variables correspondingly.

Using the hodograph transformation determined above we have constructed the general solution of Eqs. (1)–(3) from Eq. (7). It is given up to the equivalence relation (6) by one of the following formulas:

$$
\begin{aligned}
&(1)\quad \omega_1=A(t)x_1+W_1(t),\quad \omega_2=B(t)x_2+W_2(t),\\
&\qquad Q(t,\boldsymbol{x})=\exp\left\{-\frac{i}{4}\left(\frac{\dot A}{A}x_1^2+\frac{\dot B}{B}x_2^2\right)-\frac{i}{2}\left(\frac{\dot W_1}{A}x_1+\frac{\dot W_2}{B}x_2\right)\right\};\\
&(2)\quad \omega_1=\frac{1}{2}\ln(x_1^2+x_2^2)+W(t),\quad \omega_2=\arctan\frac{x_1}{x_2},\\
&\qquad Q(t,\boldsymbol{x})=\exp\left\{-\frac{i\dot W}{4}(x_1^2+x_2^2)\right\};\\
&(3)\quad x_1=\frac{1}{2}W(t)(\omega_1^2-\omega_2^2)+W_1(t),\quad x_2=W(t)\omega_1\omega_2+W_2(t),\\
&\qquad Q(t,\boldsymbol{x})=\exp\left\{\frac{i\dot W}{4W}\left((x_1-W_1)^2+(x_2-W_2)^2\right)+\frac{i}{2}(\dot W_1x_1+\dot W_2x_2)\right\};\\
&(4)\quad x_1=W(t)\cosh\omega_1\cos\omega_2+W_1(t),\quad x_2=W(t)\sinh\omega_1\sin\omega_2+W_2(t),\\
&\qquad Q(t,\boldsymbol{x})=\exp\left\{\frac{i\dot W}{4W}\left((x_1-W_1)^2+(x_2-W_2)^2\right)+\frac{i}{2}(\dot W_1x_1+\dot W_2x_2)\right\};
\end{aligned}
\tag{8}
$$

Here A, B, W, W_1, W_2 are arbitrary smooth functions on t.

Substituting the obtained expressions for the functions Q, ω_1, ω_2 into the last equation from the system (7) and splitting with respect to variables x_1, x_2 we get explicit forms of potentials $V(x_1,x_2)$ and systems of nonlinear ODEs for unknown functions $A(t)$, $B(t)$, $W(t)$, $W_1(t)$, $W_2(t)$. We have succeeded in integrating these and in constructing all coordinate systems providing SV in the initial equation (1).

Here we consider in detail integration of the fourth equation of system (7) for the case 2 from Eq. (8), since computations needed are not so lengthy as for other cases.

First, we make several important remarks which introduce an equivalence relation on the set of potentials $V(x_1, x_2)$.

Remark 1. The Schrödinger equation with the potential

$$V(x_1, x_2) = k_1 x_1 + k_2 x_2 + k_3 + V_1(k_2 x_1 - k_1 x_2), \tag{9}$$

where k_1, k_2, k_3 are constants, is transformed to the Schrödinger equation with the potential

$$V'(x'_1, x'_2) = V_1(k_2 x'_1 - k_1 x'_2) \tag{10}$$

by the following change of variables:

$$\begin{aligned} &t' = t, \quad \boldsymbol{x}' = \boldsymbol{x} + t^2 \boldsymbol{k}, \\ &u' = u \exp\left\{ \frac{i}{3}(k_1^2 + k_2^2)t^3 + it(k_1 x_1 + k_2 x_2) + ik_3 t \right\}. \end{aligned} \tag{11}$$

It is readily seen that the class of Ansätze (5) is transformed into itself by the above change of variables. That is why, potentials (9) and (10) are considered as equivalent.

Remark 2. The Schrödinger equation with the potential

$$V(x_1, x_2) = k(x_1^2 + x_2^2) + V_1\left(\frac{x_1}{x_2}\right)(x_1^2 + x_2^2)^{-1} \tag{12}$$

with $k = \mathrm{const}$ is reduced to the Schrödinger equation with the potential

$$V'(x_1, x_2) = V_1\left(\frac{x'_1}{x'_2}\right)(x_1'^2 + x_1'^2)^{-1} \tag{13}$$

by the change of variables

$$t' = \alpha(t), \quad \boldsymbol{x}' = \beta(t)\boldsymbol{x}, \quad u' = u \exp\{i\gamma(t)(x_1^2 + x_2^2) + \delta(t)\},$$

where $\big(\alpha(t), \beta(t), \gamma(t), \delta(t)\big)$ is an arbitrary solution of the system of ODEs

$$\dot\gamma - 4\gamma^2 = k, \quad \dot\beta - 4\gamma\beta = 0, \quad \dot\alpha - \beta^2 = 0, \quad \dot\delta + 4\gamma = 0$$

such that $\beta \neq 0$.

Since the above change of variables does not alter the structure of the Ansatz (5), when classifying potentials $V(x_1, x_2)$ providing separability of the corresponding Schrödinger equation, we consider potentials (12), (13) as equivalent.

Remark 3. It is well-known (see e.g. [11, 12]) that the general form of the invariance group admitted by Eq. (1) is as follows

$$t' = F(t, \boldsymbol{\theta}), \quad x'_a = g_a(t, \boldsymbol{x}, \boldsymbol{\theta}), \quad a = 1, 2, \quad u' = h(t, \boldsymbol{x}, \boldsymbol{\theta})u + U(t, \boldsymbol{x}),$$

where $\boldsymbol{\theta} = (\theta_1, \theta_2, \dots, \theta_n)$ are group parameters and $U(t, \boldsymbol{x})$ is an arbitrary solution of Eq. (1).

The above transformations also do not alter the structure of the Ansatz (5). That is why, systems of coordinates t', x_1', x_2' and t, x_1, x_2 are considered as equivalent.

Now we turn to the integration of the fourth equation of system (7). Substituting into it the expressions for the functions ω_1, ω_2, Q given by formulas (2) from Eq. (8) we get

$$V(x_1,x_2) = \big(B_{01}(\omega_1) + B_{02}(\omega_2)\big)\exp\{-2(\omega_1 - W)\} + \frac{1}{4}(\ddot W - \dot W^2)\times \\ \times\exp\{2(\omega_1 - W)\} + R_0(t) - i\dot W. \tag{14}$$

In the above equality B_{01}, B_{02}, $R_0(t)$, $W(t)$ are unknown functions to be determined from the requirement that the right-hand side of (14) does not depend on t.

Differentiating Eq. (14) with respect to t and taking into account the equalities

$$\omega_{1t} = \dot W, \quad \omega_{2t} = 0$$

we have

$$\dot W \exp\{-2(\omega_1 - W)\}\dot B_{01} + \dot\alpha(t)\exp\{2(\omega_1 - W)\} + \dot\beta(t) = 0, \tag{15}$$

where $\alpha(t) = \frac{1}{4}(\ddot W - \dot W^2)$, $\beta(t) = R_0 - i\dot W$.

Cases $\dot W = 0$ and $\dot W \neq 0$ have to be considered separately.

Case 1. $\dot W = 0$. In this case $W = C = \mathrm{const}$, $R_0 = 0$. Since coordinate systems ω_1, ω_2 and $\omega_1 + C_1$, $\omega_2 + C_2$ are equivalent with arbitrary constants C_1, C_2, choosing $C_1 = -C$, $C_2 = 0$ we can put $C = 0$. Hence it immediately follows that

$$V(x_1,x_2) = \left[B_{01}\left(\frac{1}{2}\ln(x_1^2 + x_2^2)\right) + B_{02}\left(\arctan\frac{x_1}{x_2}\right)\right](x_1^2 + x_2^2)^{-1},$$

where B_{01}, B_{02} are arbitrary functions. And what is more, the Schrödinger equation (1) with such potential separates only in one coordinate system

$$\omega_1 = \frac{1}{2}\ln(x_1^2 + x_2^2), \quad \omega_2 = \arctan\frac{x_1}{x_2}. \tag{16}$$

Case 2. $\dot W \neq 0$. Dividing Eq. (14) into $\dot W\exp\{-2(\omega_1 - W)\}$ and differentiating the equality obtained with respect to t we get

$$\exp\{4\omega_1\}\frac{d}{dt}\big(\dot\alpha(\dot W)^{-1}\exp\{-4W\}\big) + \exp\{2\omega_1\}\frac{d}{dt}\big(\dot\beta(\dot W)^{-1}\exp\{-2W\}\big) = 0,$$

whence

$$\frac{d}{dt}\big(\dot\alpha(\dot W)^{-1}\exp\{-4W\}\big) = 0, \quad \frac{d}{dt}\big(\dot\beta(\dot W)^{-1}\exp\{-2W\}\big) = 0.$$

Integration of the above ODEs yields the following result:

$$\alpha = C_1\exp\{4W\} + C_2, \quad \beta = C_3\exp\{2W\} + C_4,$$

where C_j, $j = \overline{1,4}$ are arbitrary real constants.

Inserting the result obtained into Eq. (15) we get an equation for B_{01}

$$\dot B_{01} = -4C_1\exp\{4\omega_1\} - 2C_3\exp\{2\omega_1\},$$

which general solution reads

$$B_{01} = -C_1 \exp\{4\omega_1\} - C_3 \exp\{2\omega_1\} + C_5.$$

In the above equality C_5 is an arbitrary real constant.

Substituting the expressions for α, β, B_{01} into Eq. (14) we have the explicit form of the potential $V(x_1, x_2)$

$$V(x_1, x_2) = \left[B_{02}\left(\arctan\frac{x_1}{x_2}\right) + C_5\right](x_1^2 + x_2^2)^{-1} + C_2(x_1^2 + x_2^2) + C_4,$$

where B_{02} is an arbitrary function.

By force of the Remarks 1, 2 we can choose $C_2 = C_4 = 0$. Furthermore, due to arbitrariness of the function B_{02} we can put $C_5 = 0$.

Thus, the case $\dot{W} \neq 0$ leads to the following potential:

$$V(x_1, x_2) = B_{02}\left(\arctan\frac{x_1}{x_2}\right)(x_1^2 + x_2^2)^{-1}. \qquad (17)$$

Substitution of the above expression into Eq. (14) yields second-order nonlinear ODE for the function $W = W(t)$

$$\ddot{W} - \dot{W}^2 = 4C_1 \exp\{4W\}, \qquad (18)$$

while the function R_0 is given by the formula

$$R_0 = i\dot{W} + C_3 \exp\{2W\}.$$

Integration of ODE (18) is considered in detail in the Appendix A. Its general solution has the form
under $C_1 \neq 0$

$$W = -\frac{1}{2}\ln\big((at - b)^2 - 4C_1\big) + \frac{1}{2}\ln a,$$

under $C_1 = 0$

$$W = a - \ln(t + b).$$

Substituting obtained expressions for W into formulas (2) from (8) and taking into account the Remark 3 we arrive at the conclusion that the Schrödinger equation (1) with the potential (17) admits SV in two coordinate systems. One of them is the polar coordinate system (16) and another one is the following:

$$\omega_1 = \frac{1}{2}\ln(x_1^2 + x_2^2) - \frac{1}{2}\ln(t^2 \pm 1), \quad \omega_2 = \arctan\frac{x_1}{x_2}. \qquad (19)$$

Consequently, the case 2 from Eq. (8) gives rise to two classes of the separable Schrödinger equations (1).

Cases 1, 3, 4 from Eq. (8) are considered in an analogous way but computations involved are much more cumbersome. As a result, we obtain the following list of inequivalent potentials $V(x_1, x_2)$ providing separability of the Schrödinger equation.

(1) $V(x_1,x_2) = V_1(x_1) + V_2(x_2)$;

(a) $V(x_1,x_2) = k_1x_1^2 + k_2x_1^{-2} + V_2(x_2)$, $k_2 \neq 0$;

(i) $V(x_1,x_2) = k_1x_1^2 + k_2x_2^2 + k_3x_1^{-2} + k_4x_2^{-2}, \quad k_3k_4 \neq 0,$
$k_1^2 + k_2^2 \neq 0,\ k_1 \neq k_2$;

(ii) $V(x_1,x_2) = k_1x_1^2 + k_2x_1^{-2}, \quad k_1k_2 \neq 0$;

(iii) $V(x_1,x_2) = k_1x_1^{-2} + k_2x_2^{-2}$;

(b) $V(x_1,x_2) = k_1x_1^2 + V_2(x_2)$;

(i) $V(x_1,x_2) = k_1x_1^2 + k_2x_2^2 + k_3x_2^{-2}, \quad k_1k_3 \neq 0,\ k_1 \neq k_2$;

(ii) $V(x_1,x_2) = k_1x_1^2 + k_2x_2^2, \quad k_1k_2 \neq 0, \quad k_1 \neq k_2$;

(iii) $V(x_1,x_2) = k_1x_1^2 + k_2x_2^{-2}, \quad k_1 \neq 0$;

(2) $V(x_1,x_2) = V_1(x_1^2+x_2^2) + V_2(x_1/x_2)(x_1^2+x_2^2)^{-1}$;

(a) $V(x_1,x_2) = V_2(x_1/x_2)(x_1^2+x_2^2)^{-1}$;

(b) $V(x_1,x_2) = k_1(x_1^2+x_2^2)^{-1/2}$, $k_1 \neq 0$;

(3) $V(x_1,x_2) = (V_1(\omega_1) + V_2(\omega_2))(\omega_1^2+\omega_2^2)^{-1}$, where $\omega_1^2 - \omega_2^2 = 2x_1$, $\omega_1\omega_2 = x_2$;

(4) $V(x_1,x_2) = (V_1(\omega_1) + V_2(\omega_2))(\sinh^2\omega_1 + \sin^2\omega_2)^{-1}$, where $\cosh\omega_1\cos\omega_2 = x_1$, $\sinh\omega_1\sin\omega_2 = x_2$;

(5) $V(x_1,x_2) = 0$.

In the above formulas V_1, V_2 are arbitrary smooth functions, k_1, k_2, k_3, k_4 are arbitrary constants.

It should be emphasized that the above potentials are not inequivalent in a usual sense. These potentials differ from each other by the fact that the coordinate systems providing separability of the corresponding Schrödinger equations are different. As an illustration, we give the Fig. 1, where $r = (x_1^2+x_2^2)^{1/2}$ and by the symbol $V^{(\jmath)}$, $\jmath = 1,4$ we denote the potential given in the above list under the number $\jmath$. Down arrows in the Fig. 1 indicate specifications of the potential $V(x_1,x_2)$ providing new possibilities to separate the corresponding Schrödinger equation (1).

The Schrödinger equation (1) with arbitrary function $V(x_1,x_2)$ (level 1 of the Fig. 1) admits no separation of variables. Next, Eq. (1) with the "root" potentials $V^{(\jmath)}$ (level 2), V_1, V_2 being arbitrary smooth functions, separates in the Cartesian ($\jmath = 1$), polar ($\jmath = 2$), parabolic ($\jmath = 3$) and elliptic ($\jmath = 4$) coordinate systems, correspondingly. Specifying the functions V_1, V_2 (i.e. going down to the lower levels) new possibilities to separate variables in the Schrödinger equation (1) arise. For example, Eq. (1) with the potential $V_2(x_1/x_2)r^{-2}$, which is a particular case of the potential $V^{(2)}$, separates not only in the polar coordinate system (16) but also in the coordinate systems (19). The Schrödinger equation with the Coulomb potential k_1r^{-1}, which is a particular case of the potentials $V^{(2)}$, $V^{(3)}$, separates in two coordinate systems (namely, in the polar and parabolic coordinate systems, see below the Theorem 4). An another characteristic example is a transition from the potential $V^{(1)}$ to the potential $k_1x_1^2 + V_2(x_2)$. The Schrödinger equation with the potential $V^{(1)}$ admits SV in the Cartesian coordinate system $\omega_0 = t$, $\omega_1 = x_1$, $\omega_2 = x_2$ only, while the one with the potential $k_1x_1^2 + V_2(x_2)$ separates in seven ($k_1 < 0$) or in three ($k_1 > 0$) coordinate systems.

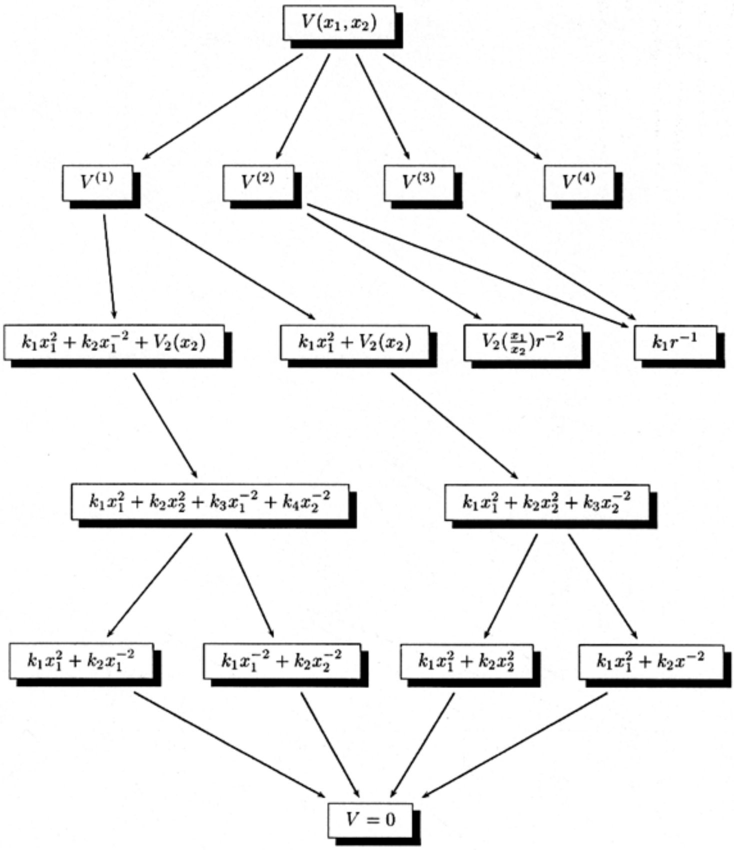

Figure 1.

A complete list of coordinate systems providing SV in the Schrödinger equations with the above given potentials takes two dozen pages. Therefore, we restrict ourself to considering the Schrödinger equation with anisotropic harmonic oscillator potential $V(x_1, x_2) = k_1x_1^2 + k_2x_2^2$, $k_1 \neq k_2$ and Coulomb potential $V(x_1, x_2) = k_1(x_1^2 + x_2^2)^{-1/2}$.

3 Separation of variables in the Schrödinger equation with the anisotropic harmonic oscillator and the Coulomb potentials

Here we will obtain all coordinate systems providing separability of the Schrödinger equation with the potential $V(x_1, x_2) = k_1x_1^2 + k_2x_2^2$

$$iu_t + u_{x_1x_1} + u_{x_2x_2} = (k_1x_1^2 + k_2x_2^2)u. \tag{20}$$

In the following, we consider the case $k_1 \neq k_2$, because otherwise Eq. (1) is reduced to the free Schrödinger equation (see the Remark 2) which has been studied in detail in [1–3].

Explicit forms of the coordinate systems to be found depend essentially on the signs of the parameters k_1, k_2. We consider in detail the case, when $k_1 < 0$, $k_2 > 0$ (the cases $k_1 > 0$, $k_2 > 0$ and $k_1 < 0$, $k_2 < 0$ are handled in an analogous way). It means that Eq. (20) can be written in the form

$$iu_t + u_{x_1x_1} + u_{x_2x_2} + \frac{1}{4}(a^2x_1^2 - b^2x_2^2)u = 0, \tag{21}$$

where a, b are arbitrary non-null real constants (the factor $\frac{1}{4}$ is introduced for further convenience).

As stated above to describe all coordinate systems t, $\omega_1(t,\boldsymbol{x})$, $\omega_2(t,\boldsymbol{x})$ providing separability of Eq. (20) one has to construct the general solution of system (8) with $V(x_1,x_2) = -\frac{1}{4}(a^2x_1^2 - b^2x_2^2)$. The general solution of Eqs. (1)–(3) from Eq. (7) splits into four inequivalent classes listed in Eq. (8). Analysis shows that only solutions belonging to the first class can satisfy the fourth equation of (7).

Substituting the expressions for ω_1, ω_2, Q given by the formulas (1) from (8) into the equation 4 from (7) with $V(x_1,x_2) = -\frac{1}{4}(a^2x_1^2 - b^2x_2^2)$ and splitting with respect to x_1, x_2 one gets

$$B_{01}(\omega_1) = \alpha_1\omega_1^2 + \alpha_2\omega_1, \quad B_{02}(\omega_2) = \beta_1\omega_2^2 + \beta_2\omega_2,$$

$$\left(\frac{\dot{A}}{A}\right)^{\cdot} - \left(\frac{\dot{A}}{A}\right)^2 - 4\alpha_1A^4 + a^2 = 0, \tag{22}$$

$$\left(\frac{\dot{B}}{B}\right)^{\cdot} - \left(\frac{\dot{B}}{B}\right)^2 - 4\beta_1B^4 - b^2 = 0, \tag{23}$$

$$\ddot{\theta}_1 - 2\dot{\theta}_1\frac{\dot{A}}{A} - 2(2\alpha_1\theta_1 + \alpha_2)A^4 = 0, \tag{24}$$

$$\ddot{\theta}_2 - 2\dot{\theta}_2\frac{\dot{B}}{B} - 2(2\beta_1\theta_2 + \beta_2)B^4 = 0. \tag{25}$$

Here α_1, α_2, β_1, β_2 are arbitrary real constants.

Integration of the system of nonlinear ODEs (22)–(25) is carried out in the Appendix A. Substitution of the formulas (A.2), (A.4)–(A.6), (A.8)–(A.11) into the corresponding expressions 1 from (8) yields a complete list of coordinate systems providing separability of the Schrödinger equation (21). These systems can be transformed to canonical form if we use the Remark 3.

The invariance group of Eq. (21) is generated by the following basis operators [11]:

$$\begin{aligned}
&P_0 = \partial_t, \quad I = u\partial_u, \quad M = iu\partial_u, \quad Q_\infty = U(t,\boldsymbol{x})\partial_u,\\
&P_1 = \cosh at\partial_{x_1} + \frac{ia}{2}(x_1\sinh at)u\partial_u,\\
&P_2 = \cos bt\partial_{x_2} - \frac{ib}{2}(x_2\sin bt)u\partial_u,\\
&G_1 = \sinh at\partial_{x_1} + \frac{ia}{2}(x_1\cosh at)u\partial_u,\\
&G_2 = \sin bt\partial_{x_2} + \frac{ib}{2}(x_2\cos bt)u\partial_u,
\end{aligned} \tag{26}$$

where $U(t,\boldsymbol{x})$ is an arbitrary solution of Eq. (21).

Using the finite transformations generated by the infinitesimal operators (26) and the Remark 3 we can choose in the formulas (A.4)–(A.6), (A.8), (A.10), (A.11) $C_3 = C_4 = D_1 = 0$, $D_3 = D_4 = 0$, $C_2 = D_2 = 1$. As a result, we come to the following assertion.

Theorem 1. *The Schrödinger equation (21) admits SV in 21 inequivalent coordinate systems of the form*

$$\omega_0 = t, \quad \omega_1 = \omega_1(t, \boldsymbol{x}), \quad \omega_2 = \omega_2(t, \boldsymbol{x}), \tag{27}$$

where ω_1 is given by one of the formulas from the first and ω_2 by one of the formulas from the second column of the Table 1.

Table 1. Coordinate systems proving SV in Eq. (21).

$\omega_1(t, \boldsymbol{x})$	$\omega_2(t, \boldsymbol{x})$
$x_1\big(\sinh a(t+C)\big)^{-1} + \alpha\big(\sinh a(t+C)\big)^{-2}$	$x_2(\sin bt)^{-1} + \beta(\sin bt)^{-2}$
$x_1\big(\cosh a(t+C)\big)^{-1} + \alpha\big(\cosh a(t+C)\big)^{-2}$	$x_2(\beta + \sin 2bt)^{-1/2}$
$x_1 \exp(\pm at) + \alpha \exp(\pm 4at)$	x_2
$x_1\big(\alpha + \sinh 2a(t+C)\big)^{-1/2}$	
$x_1\big(\alpha + \cosh 2a(t+C)\big)^{-1/2}$	
$x_1\big(\alpha + \exp(\pm 2at)\big)^{-1/2}$	
x_1	

Here C, α, β are arbitrary real constants.

There is no necessity to consider specially the case when in Eq. (20) $k_1 > 0$, $k_2 < 0$, since such an equation by the change of independent variables $u(t, x_1, x_2) \to u(t, x_2, x_1)$ is reduced to Eq. (21).

Below we adduce without proof the assertions describing coordinate systems providing SV in Eq. (20) with $k_1 < 0$, $k_2 < 0$ and $k_1 > 0$, $k_2 > 0$.

Theorem 2. *The Schrödinger equation*

$$iu_t + u_{x_1x_1} + u_{x_2x_2} + \frac{1}{4}(a^2x_1^2 + b^2x_2^2)u = 0 \tag{28}$$

with $a^2 \neq 4b^2$ admits SV in 49 inequivalent coordinate systems of the form (27), where ω_1 is given by one of the formulas from the first and ω_2 by one of the formulas from the second column of the Table 2. Provided $a^2 = 4b^2$ one more coordinate system should be included into the above list, namely

$$\omega_0 = t, \quad \omega_1^2 - \omega_2^2 = 2x_1, \quad \omega_1\omega_2 = x_2. \tag{29}$$

Table 2. Coordinate systems proving SV in Eq. (28).

$\omega_1(t,\boldsymbol{x})$	$\omega_2(t,\boldsymbol{x})$
$x_1\big(\sinh a(t+C)\big)^{-1}+\alpha\big(\sinh a(t+C)\big)^{-2}$	$x_2(\sinh bt)^{-1}+\beta(\sinh bt)^{-2}$
$x_1\big(\cosh a(t+C)\big)^{-1}+\alpha\big(\cosh a(t+C)\big)^{-2}$	$x_2(\cosh bt)^{-1}+\beta(\cosh bt)^{-2}$
$x_1\exp(\pm at)+\alpha\exp(\pm 4at)$	$x_2\exp(\pm bt)+\beta\exp(\pm 4bt)$
$x_1\big(\alpha+\sinh 2a(t+C)\big)^{-1/2}$	$x_2(\beta+\sinh 2bt)^{-1/2}$
$x_1\big(\alpha+\cosh 2a(t+C)\big)^{-1/2}$	$x_2(\beta+\cosh 2bt)^{-1/2}$
$x_1\big(\alpha+\exp(\pm 2at)\big)^{-1/2}$	$x_2\big(\beta+\exp(\pm 2bt)\big)^{-1/2}$
x_1	x_2

Here C, α, β are arbitrary constants.

Table 3. Coordinate systems proving SV in Eq. (30).

$\omega_1(t,\boldsymbol{x})$	$\omega_2(t,\boldsymbol{x})$
$x_1\big(\sin a(t+C)\big)^{-1}+\alpha\big(\sin a(t+C)\big)^{-2}$	$x_2(\sin bt)^{-1}+\beta(\sin bt)^{-2}$
$x_1\big(\beta+\sin 2a(t+C)\big)^{-1/2}$	$x_2(\beta+\sin 2bt)^{-1/2}$
x_1	x_2

Here C, α, β are arbitrary constants.

Theorem 3. *The Schrödinger equation*

$$iu_t+u_{x_1x_1}+u_{x_2x_2}-\frac{1}{4}(a^2x_1^2+b^2x_2^2)u=0 \tag{30}$$

with $a^2\neq 4b^2$ admits SV in 9 inequivalent coordinate systems of the form (27), where ω_1 is given by one of the formulas from the first and ω_2 by one of the formulas from the second column of the Table 3. Provided $a^2=4b^2$, the above list should be supplemented by the coordinate system (29).

Remark 4. If we consider Eq. (1) as an equation for a complex-valued function u of three complex variables t, x_1, x_2, then the cases considered in the Theorems 1–3 are equivalent. Really, replacing, when necessary, a with ia and b by ib we can always reduce Eqs. (21), (28) to the form (30). It means that coordinate systems presented in the Tables 1, 2 are complex equivalent to those listed in the Table 3. But if u is a complex-valued function of real variables t, x_1, x_2 it is not the case.

Theorem 4. *The Schrödinger equation with the Coulomb potential*

$$iu_t+u_{x_1x_1}+u_{x_2x_2}-k_1(x_1^2+x_2^2)^{-1/2}u=0$$

admits SV in two coordinate systems (16), (29).

It is important to note that explicit forms of coordinate systems providing separability of Eqs. (21), (28), (30) depend essentially on the parameters a, b contained in the potential $V(x_1,x_2)$. It means that the free Schrödinger equation ($V=0$) does not admit SV in such coordinate systems. Consequently, they are essentially new.

4 Conclusion

In the present paper we have studied the case when the Schrödinger equation (1) separates into one first-order and two second-order ODEs. It is not difficult to prove that there are no functions $Q(t,\boldsymbol{x})$, $\omega_\mu(t,\boldsymbol{x})$, $\mu=0,1,2$ such that the Ansatz

$$u = Q(t,\boldsymbol{x})\varphi_0(\omega_0(t,\boldsymbol{x}))\varphi_1\big(\omega_1(t,\boldsymbol{x})\big)\varphi_2\big(\omega_2(t,\boldsymbol{x})\big)$$

separates Eq. (1) into three second-order *ODEs* (see Appendix B). Nevertheless, there exists a possibility for Eq. (1) to be separated into two first-order and one second-order ODEs or into three first-order ODEs. This is a probable source of new potentials and new coordinate systems providing separability of the Schrödinger equation. It should be said that separation of the two-dimensional wave equation

$$u_{tt} - u_{xx} = V(x)u$$

into one first-order and one second-order ODEs gives no new potentials as compared with separation of it into two second-order ODEs. But for some already known potentials new coordinate system providing separability of the above equation are obtained [9].

Let us briefly analyze a connection between separability of Eq. (1) and its symmetry properties. It is well-known that each solution of the free Schrödinger equation with separated variables is a common eigenfunction of two mutually commuting second-order symmetry operators of the said equation [2, 3]. And what is more, separation constants λ_1, λ_2 are eigenvalues of these symmetry operators.

We will establish that the same assertion holds for the Schrödinger equation (1). Let us make in Eq. (1) the following change of variables:

$$u = Q(t,\boldsymbol{x})U\big(t,\omega_1(t,\boldsymbol{x}),\omega_2(t,\boldsymbol{x})\big), \tag{31}$$

where (Q,ω_1,ω_2) is an arbitrary solution of the system of PDEs (7).

Substituting the expression (31) into (1) and taking into account equations (7) we get

$$Q\big(iU_t + \big(U_{\omega_1\omega_1} - B_{01}(\omega_1)U\big)\omega_{1x_a}\omega_{1x_a} + \big(U_{\omega_2\omega_2} - B_{02}(\omega_2)U\big)\omega_{2x_a}\omega_{2x_a}\big) = 0. \tag{32}$$

Resolving Eqs. (2) from the system (7) with respect to $\omega_{1x_a}\omega_{1x_a}$ and $\omega_{2x_a}\omega_{2x_a}$ we have

$$\omega_{1x_a}\omega_{1x_a} = \frac{1}{\delta}\big(R_2(t)B_{21}(\omega_2) - R_1(t)B_{22}(\omega_2)\big),$$
$$\omega_{2x_a}\omega_{2x_a} = \frac{1}{\delta}\big(R_1(t)B_{12}(\omega_1) - R_2(t)B_{11}(\omega_1)\big),$$

where $\delta = B_{11}(\omega_1)B_{22}(\omega_2) - B_{12}(\omega_1)B_{21}(\omega_2)$ ($\delta\neq 0$ by force of the condition (4)).

Substitution of the above equalities into Eq. (32) with subsequent division by $Q\neq 0$ yields the following PDE:

$$\begin{aligned} iU_t &+ \frac{R_1(t)}{\delta}\Big(B_{12}(\omega_1)\big(U_{\omega_2\omega_2} - B_{02}(\omega_2)U\big) - B_{22}(\omega_2)\big(U_{\omega_1\omega_1} - B_{01}(\omega_1)U\big)\Big) + \\ &+ \frac{R_2(t)}{\delta}\Big(B_{21}(\omega_2)\big(U_{\omega_1\omega_1} - B_{01}(\omega_1)U\big) - B_{11}(\omega_1)\big(U_{\omega_2\omega_2} - B_{02}(\omega_2)U\big)\Big) = 0. \end{aligned} \tag{33}$$

Thus, in the new coordinates t, ω_1, ω_2, $U(t,\omega_1,\omega_2)$ Eq. (1) takes the form (33).

By direct (and very cumbersome) computation one can check that the following second-order differential operators:

$$X_1 = \frac{B_{22}(\omega_2)}{\delta}\big(\partial^2_{\omega_1} - B_{01}(\omega_1)\big) - \frac{B_{12}(\omega_1)}{\delta}\big(\partial^2_{\omega_2} - B_{02}(\omega_2)\big),$$
$$X_2 = -\frac{B_{21}(\omega_2)}{\delta}\big(\partial^2_{\omega_1} - B_{01}(\omega_1)\big) + \frac{B_{11}(\omega_1)}{\delta}\big(\partial^2_{\omega_2} - B_{02}(\omega_2)\big),$$

commute under arbitrary B_{0a}, B_{ab}, $a,b=1,2$, i.e.

$$[X_1,X_2] \equiv X_1X_2 - X_2X_1 = 0. \tag{34}$$

After being rewritten in terms of the operators X_1, X_2 Eq. (33) reads

$$\big(i\partial_t - R_1(t)X_1 - R_2(t)X_2\big)U = 0.$$

Since the relations

$$\big[i\partial_t - R_1(t)X_1 - R_2(t)X_2, X_a\big] = 0, \quad a=1,2 \tag{35}$$

hold, operators X_1, X_2 are mutually commuting symmetry operators of Eq. (33). Furthermore, solution of Eq. (33) with separated variables $U = \varphi_0(t)\varphi_1(\omega_1)\varphi_2(\omega_2)$ satisfies the identities

$$X_aU = \lambda_aU, \quad a=1,2. \tag{36}$$

Consequently, if we designate by X_1', X_2' the operators X_1, X_2 written in the initial variables t, $\boldsymbol{x}$, u, then we get from (34)–(36) the following equalities:

$$\big[i\partial_t + \triangle - V(x_1,x_2), X_a'\big] = 0, \quad a=1,2,$$
$$\big[X_1', X_2'\big] = 0, \quad X_a'u = \lambda_au, \quad a=1,2.$$

where $u = Q(t,\boldsymbol{x})\varphi_0(t)\varphi_1(\omega_1)\varphi_2(\omega_2)$.

It means that each solution with separated variables is a common eigenfunction of two mutually commuting symmetry operators X_1', X_2' of the Schrödinger equation (1), separation constants λ_1, λ_2 being their eigenvalues.

Detailed study of the said operators as well as analysis of separated ODEs for functions φ_μ, $\mu=\overline{0,2}$ (in the way as it is done for the free Schrödinger equation in [2, 3]) is in progress and will be a topic of our future publications.

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Appendix A. Integration of nonlinear ODEs (22)–(25)

Evidently, equations (22)–(25) can be rewritten in the following unified form:

$$\left(\frac{\dot y}{y}\right)^{\cdot} - \left(\frac{\dot y}{y}\right)^2 - 4\alpha y^4 = k, \quad \ddot z - 2\dot z\frac{\dot y}{y} - 2(2\alpha z + \beta)y^4 = 0. \tag{A1}$$

Provided $k = -a^2 < 0$, system (A.1) coincides with Eqs. (22), (24) and under $k = b^2 > 0$ – with Eqs. (23), (25).

First of all, we note that the function $z = z(t)$ is determined up to addition of an arbitrary constant. Really, the coordinate functions ω_a have the following structure:

$$\omega_a = yx_a + z, \quad a = 1, 2.$$

But the coordinate system t, ω_1, ω_2 is equivalent to the coordinate system t, $\omega_1 + C_1$, $\omega_2 + C_2$, $C_a \in \mathbb{R}^1$. Hence it follows that the function $z(t)$ is equivalent to the function $z(t) + C$ with arbitrary real constant C. Consequently, provided $\alpha \neq 0$, we can choose in (A.1) $\beta = 0$.

The case 1. $\alpha = 0$. On making in (A.1) the change of variables

$$w = \dot{y}/y, \quad v = z/y \tag{A2}$$

we get

$$\dot{w} = w^2 + k, \quad \ddot{v} + kv = 2\beta y^3. \tag{A3}$$

First, we consider the case $k = -a^2 < 0$. Then the general solution of the first equation from (A.3) is given by one of the formulas

$$w = -a \coth a(t + C_1), \quad w = -a \tanh a(t + C_1), \quad w = \pm a, \quad C_1 \in \mathbb{R}^1,$$

whence

$$\begin{gathered} y = C_2 \sinh^{-1} a(t + C_1), \quad y = C_2 \cosh^{-1} a(t + C_1), \\ y = C_2 \exp(\pm at), \quad C_2 \in \mathbb{R}^1. \end{gathered} \tag{A4}$$

The second equation of system (A.3) is a linear inhomogeneous ODE. Its general solution after being substituted into (A.2) yields the following expression for $z(t)$:

$$\begin{gathered} (C_3 \cosh at + C_4 \sinh at) \sinh^{-1} a(t + C_1) + \frac{\beta C_2^4}{a^2} \sinh^{-2} a(t + C_1), \\ (C_3 \cosh at + C_4 \sinh at) \cosh^{-1} a(t + C_1) + \frac{\beta C_2^4}{a^2} \cosh^{-2} a(t + C_1), \\ (C_3 \cosh at + C_4 \sinh at) \exp(\pm at) + \frac{\beta C_2^4}{4a^2} \exp(\pm 4at), \quad C_3, C_4 \subset \mathbb{R}^1. \end{gathered} \tag{A5}$$

The case $k = b^2 > 0$ is treated in an analogous way, the general solution of (A.1) being given by the formulas

$$\begin{gathered} y = D_2 \sin^{-1} b(t + D_1), \\ z = (D_3 \cos bt + D_4 \sin bt) \sin^{-1} b(t + D_1) + \frac{\beta D_2^4}{b^2} \sin^{-2} b(t + D_1), \end{gathered} \tag{A6}$$

where D_1, D_2, D_3, D_4 are arbitrary real constants.

The case 2. $\alpha \neq 0$, $\beta = 0$. On making in Eq. (A.1) the change of variables

$$y = \exp w, \quad v = z/y$$

we have

$$\ddot{w} - \dot{w}^2 = k + \alpha \exp 4w, \quad \ddot{v} + kv = 0. \tag{A7}$$

The first ODE from Eq. (A.7) is reduced to the first-order linear ODE

$$\frac{1}{2}\frac{dp(w)}{dw} - p(w) = k + \alpha \exp 4w$$

by the substitution $\dot{w} = (p(w))^{1/2}$, whence

$$p(w) = \alpha \exp 4w + \gamma \exp 2w - k, \quad \gamma \in \mathbb{R}^1.$$

Equation $\dot{w} = (p(w))^{1/2}$ has a singular solution $w = C = \text{const}$ such that $p(C) = 0$. If $\dot{w} \neq 0$, then integrating the equation $\dot{w} = p(w)$ and returning to the initial variable y we get

$$\int^{y(t)} \frac{d\tau}{\tau(\alpha\tau^4 + \gamma\tau^2 - k)^{1/2}} = t + C_1.$$

Taking the integral in the left-hand side of the above equality we obtain the general solution of the first ODE from Eq. (A.1). It is given by the following formulas:
under $k = -a^2 < 0$

$$\begin{aligned} y &= C_2\big(\alpha + \sinh 2a(t + C_1)\big)^{-1/2}, \\ y &= C_2\big(\alpha + \cosh 2a(t + C_1)\big)^{-1/2}, \\ y &= C_2\big(\alpha + \exp(\pm 2at)\big)^{-1/2}, \end{aligned} \tag{A8}$$

under $k = b^2 > 0$

$$y = D_2\big(\alpha + \sin 2b(t + D_1)\big)^{-1/2}. \tag{A9}$$

Here C_1, C_2, D_1, D_2 are arbitrary real constants.

Integrating the second ODE from Eq. (A.7) and returning to the initial variable z we have
under $k = -a^2 < 0$

$$z = y(t)(C_3 \cosh at + C_4 \sinh at) \tag{A10}$$

under $k = b^2 > 0$

$$z = y(t)(D_3 \cos bt + D_4 \sin bt), \tag{A11}$$

where C_3, C_4, D_3, D_4 are arbitrary real constants.

Thus, we have constructed the general solution of the system of nonlinear ODEs (A.1) which is given by the formulas (A.5)–(A.11).

Appendix B. Separation of Eq. (1) into three second-order ODEs

Suppose that there exists an Ansatz

$$u = Q(t, \boldsymbol{x})\varphi_0(\omega_0(t, \boldsymbol{x}))\varphi_1\big(\omega_1(t, \boldsymbol{x})\big)\varphi_2\big(\omega_2(t, \boldsymbol{x})\big) \tag{A12}$$

which separates the Schrödinger equation into three second-order ODEs

$$\begin{aligned}&\frac{d^2\varphi_0}{d\omega_0^2} = U_0\left(\omega_0, \varphi_0, \frac{d\varphi_0}{d\omega_0}; \lambda_1, \lambda_2\right), \quad \frac{d^2\varphi_1}{d\omega_1^2} = U_1\left(\omega_1, \varphi_1, \frac{d\varphi_1}{d\omega_1}; \lambda_1, \lambda_2\right),\\ &\frac{d^2\varphi_2}{d\omega_2^2} = U_2\left(\omega_2, \varphi_2, \frac{d\varphi_2}{d\omega_2}; \lambda_1, \lambda_2\right)\end{aligned} \tag{A13}$$

according to the Definition 1.

Substituting the Ansatz (A.12) into Eq. (1) and excluding the second derivatives $d^2\varphi_\mu/d\omega_\mu^2$, $\mu = \overline{0,2}$ according to Eqs. (A.13) we get

$$\begin{aligned}&i(Q_t\varphi_0\varphi_1\varphi_2 + Q\omega_{0t}\dot\varphi_0\varphi_1\varphi_2 + Q\omega_{1t}\varphi_0\dot\varphi_1\varphi_2 + Q\omega_{2t}\varphi_0\varphi_1\dot\varphi_2) + (\triangle Q)\varphi_0\varphi_1\varphi_2 + \\ &\quad + 2Q_{x_a}\omega_{0x_a}\dot\varphi_0\varphi_1\varphi_2 + 2Q_{x_a}\omega_{1x_a}\varphi_0\dot\varphi_1\varphi_2 + 2Q_{x_a}\omega_{2x_a}\varphi_0\varphi_1\dot\varphi_2 + \\ &\quad + Q\big((\triangle\omega_0)\dot\varphi_0\varphi_1\varphi_2 + (\triangle\omega_1)\varphi_0\dot\varphi_1\varphi_2 + (\triangle\omega_2)\varphi_0\varphi_1\dot\varphi_2 + \omega_{0x_a}\omega_{0x_a}U_0\varphi_1\varphi_2 + \\ &\quad + \omega_{1x_a}\omega_{1x_a}\varphi_0U_1\varphi_2 + \omega_{2x_a}\omega_{2x_a}\varphi_0\varphi_1U_2 + 2\omega_{0x_a}\omega_{1x_a}\dot\varphi_0\dot\varphi_1\varphi_2 + \\ &\quad + 2\omega_{0x_a}\omega_{2x_a}\dot\varphi_0\varphi_1\dot\varphi_2 + 2\omega_{1x_a}\omega_{2x_a}\varphi_0\dot\varphi_1\dot\varphi_2\big) = VQ\varphi_0\varphi_1\varphi_2.\end{aligned}$$

Splitting the above equality with respect to $\dot\varphi_0\dot\varphi_1$, $\dot\varphi_0\dot\varphi_2$, $\dot\varphi_1\dot\varphi_2$ we obtain the equalities:

$$\omega_{0x_a}\omega_{1x_a} = 0, \quad \omega_{0x_a}\omega_{2x_a} = 0, \quad \omega_{1x_a}\omega_{2x_a} = 0. \tag{A14}$$

Since the functions ω_μ, $\mu = \overline{0,2}$ are real-valued, equalities (A.14) mean that there are three real two-component vectors which are mutually orthogonal. This is possible only if one of them is a null-vector. Without loss of generality we may suppose that $(\omega_{0x_1}, \omega_{0x_2}) = (0,0)$, whence $\omega_0 = f(t) \sim t$.

Consequently, Ansatz (A.12) necessarily takes the form (5). But Ansatz (5) can not separate Eq. (1) into three second-order ODEs, since it contains no second-order derivative with respect to t.

Thus, we have proved that the Schrödinger equation (1) is not separable into three second-order ODEs.

1. Boyer C., *SIAM J. Math. Anal.*, 1976, **7**, 230.
2. Boyer C., Kalnins E., Miller W., *J. Math. Phys.*, 1975, **16**, 499.
3. Miller W., Symmetry and separation of variables, Massachusetts, Addison-Wesley, 1977.
4. Shapovalov V.N., Sukhomlin N.B., *Izvestiya Vuzov, Fizika*, 1974, № 12, 268.
5. Nikitin A.G., Onufriychuk S.P., Fushchych W.I., *Teoret. Mat. Fiz.*, 1992, **12**, 268.
6. Fushchych W.I., Zhdanov R.Z., Revenko I.V., *Proc. Ukrain. Acad. Sci.*, 1993, № 1, 27.
7. Fushchych W.I., in Symmetry in Mathematical Physics Problems, Kiev, Inst. of Math., 1981, 6.
8. Koornwinder T.H., *Lect. Notes in Math.*, 1980, **810**, 240.
9. Zhdanov R.Z., Revenko I.V., Fushchych W.I., *J. Phys. A: Math. Gen.*, 1993, **26**, 5959.
10. Zhdanov R.Z., *J. Phys. A: Math. Gen.*, 1994, **27**, L291.
11. Boyer C., *Helv. Phys. Acta*, 1974, **47**, 589.
12. Niederer U., *Helv. Phys. Acta*, 1972, **45**, 802.

On the general solution of the d'Alembert equation with a nonlinear eikonal constraint and its applications

R.Z. ZHDANOV, I.V. REVENKO, W.I. FUSHCHYCH

We construct the general solutions of the system of nonlinear differential equations $\Box_n u = 0$, $u_\mu u^\mu = 0$ in the four- and five-dimensional complex pseudo-Euclidean spaces. The results obtained are used to reduce the multi-dimensional nonlinear d'Alembert equation $\Box_4 u = F(u)$ to ordinary differential equations and to construct its new exact solutions.

1 Introduction

Kaluza [1] was the first who put forward an idea of extension of the four-dimensional Minkowski space in order to use it as a geometric basis for unification of the electromagnetic and gravitational fields. Nowadays, Kaluza's idea is well-known and there are a lot of papers where further development and various generalizations of this idea are obtained [2].

In [3–5] it was proposed to apply five-dimensional wave equations to describe particles (fields) having variable spins and masses. Such physical interpretation of the five-dimensional equations is based on the fact that the generalized Poincaré group $P(1,4)$ acting in the five-dimensional de Sitter space contains the Poincaré group $P(1,3)$ as a subgroup. It means that the mass and spin Casimir operators have continuous and discrete spectrum, respectively, in the space of irreducible representations of the group $P(1,4)$ [3–6].

The simplest $P(1,4)$-invariant scalar linear equation has the form

$$\Box_5 u + \chi^2 u = 0, \quad \chi = \text{const}, \tag{1}$$

where $\Box_5$ is the d'Alembert operator in the five-dimensional Minkowski space with the signature $(+,-,-,-,-)$.

The problem of construction of exact solutions of the above equation is, in fact, completely open. One can obtain some its particular solutions applying the symmetry reduction procedure or the method of separation of variables (both approaches use essentially symmetry properties of the whole set of solutions of Eq. (1)). In the present paper we suggest a method for construction of solutions of partial differential equation (1) which utilizes implicitly the symmetry of a *subset* of the set of its solutions. Namely, a special subset of its exact solutions obtained by imposing an additional constraint

$$u_{x_0}^2 - u_{x_1}^2 - u_{x_2}^2 - u_{x_3}^2 - u_{x_4}^2 = 0,$$

which is the eikonal equation in the five-dimensional space, will be investigated. As shown in [7, 8], the system obtained is compatible if and only if $\chi = 0$. We

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will construct general solutions of multi-dimensional systems of partial differential equations (PDE)

$$\Box_n u = 0, \quad u_\mu u^\mu = 0 \tag{2}$$

in the four- and five-dimensional complex pseudo-Euclidean spaces.

In (2) $u = u(x_0, x_1, \ldots, x_{n-1}) \in C^2(\mathbb{C}^n, \mathbb{C}^1)$. Hereafter, the summation over the repeated indices in the pseudo-Euclidean space $M(1, n-1)$ with the metric tensor $g_{\mu\nu} = \mathrm{diag}\,(1, \underbrace{-1, \ldots, -1}_{n-1})$ is understood, e.g. $\Box_n \equiv \partial_\mu \partial^\mu = \partial_0^2 - \partial_1^2 - \cdots - \partial_{n-1}^2$, $\partial_\mu = \partial/\partial x_\mu$.

It occurs that solutions of system of PDE (2), being very interesting by itself, can be used to reduce the *nonlinear* d'Alembert equation

$$\Box_4 u = F(u), \quad F(u) \in C(\mathbb{R}^1, \mathbb{R}^1), \tag{3}$$

to ordinary differential equations, thus giving rise to families of principally new exact solutions of (3). More precisely, we will establish that there exists a nonlinear map from the set solutions of the system of PDE (2) into the set of solutions of the nonlinear d'Alembert equation, such that each solution of (2) corresponds to a family of exact solutions of Eq. (3) containing two arbitrary functions of one argument. It will be shown that solutions of the nonlinear d'Alembert equation obtained in this way can be related to its *conditional* symmetry.

The paper is organized as follows. In Section 2 we give assertions describing the general solution of system of PDE (2) in the n-dimensional real and in the four- and five-dimensional complex pseudo-Euclidean spaces. In Section 3 we prove these assertions. Section 4 is devoted to discussion of the connection between exact solutions of system (2) and the problem of reduction of the nonlinear d'Alembert equation (3). In Section 5 we construct principally new exact solutions of Eq. (3).

2 Integration of the system (2): the list of principal results

Below we adduce assertions giving general solutions of the system of PDE (2) with arbitrary $n \in \mathbb{N}$ provided $u(x) \in C^2(\mathbb{R}^n, \mathbb{R}^1)$, and with $n = 4, 5$, provided $u(x) \in C^2(\mathbb{C}^n, \mathbb{C}^1)$.

Theorem 1. *Let $u(x)$ be a sufficiently smooth real function of n real variables $x_0, \ldots, x_{n-1}$. Then, the general solution of the system of nonlinear PDE (2) is given by the following formula:*

$$A_\mu(u) x^\mu + B(u) = 0, \tag{4}$$

where $A_\mu(u)$, $B(u)$ are arbitrary real functions which satisfy the condition

$$A_\mu(u) A^\mu(u) = 0. \tag{5}$$

Note 1. As far as we know, Jacobi, Smirnov and Sobolev were the first who obtained the formulas (4), (5) with $n = 3$ [9, 10]. That is why, it is natural to call (4), (5) the Jacoby–Smirnov–Sobolev formulas (JSSF). Later on, in 1944 Yerugin generalized

JSSF for the case $n = 4$ [11]. Recently, Collins [12] has proved that JSSF give the general solution of system (2) for an arbitrary $n \in \mathbb{N}$. He applied rather complicated differential geometry technique. Below we show that to integrate Eqs. (2) it is quite enough to make use of the classical methods of mathematical physics only.

Theorem 2. *The general solution of the system of nonlinear PDE (2) in the class of functions $u = u(x_0, x_1, x_2, x_3, x_4) \in C^2(\mathbb{C}^5, \mathbb{C}^1)$ is given by one of the following formulas:*

$$(1)\ A_\mu(\tau, u)x^\mu + C_1(\tau, u) = 0, \tag{6}$$

where $\tau = \tau(u, x)$ is a complex function determined by the equation

$$B_\mu(\tau, u)x^\mu + C_2(\tau, u) = 0, \tag{7}$$

and $A_\mu, B_\mu, C_1, C_2 \in C^2(\mathbb{C}^2, \mathbb{C}^1)$ are arbitrary functions satisfying the conditions

$$A_\mu A^\mu = A_\mu B^\mu = B_\mu B^\mu = 0, \quad B^\mu \frac{\partial A_\mu}{\partial \tau} = 0, \tag{8}$$

and what is more,

$$\Delta = \det \left\| \begin{matrix} x^\mu \dfrac{\partial A_\mu}{\partial \tau} + \dfrac{\partial C_1}{\partial \tau} & x^\mu \dfrac{\partial A_\mu}{\partial u} + \dfrac{\partial C_1}{\partial u} \\ x^\mu \dfrac{\partial B_\mu}{\partial \tau} + \dfrac{\partial C_2}{\partial \tau} & x^\mu \dfrac{\partial B_\mu}{\partial u} + \dfrac{\partial C_2}{\partial u} \end{matrix} \right\| \neq 0; \tag{9}$$

$$(2)\ A_\mu(u)x^\mu + C_1(u) = 0, \tag{10}$$

where $A_\mu(u)$, $C_1(u)$ are arbitrary smooth functions satisfying the relations

$$A_\mu A^\mu = 0 \tag{11}$$

(in the formulas (6)–(11) the index μ takes the values $0, 1, 2, 3, 4$).

Theorem 3. *The general solution of the system of nonlinear PDE (2) in the class of functions $u = u(x_0, x_1, x_2, x_3) \in C^2(\mathbb{C}^4, \mathbb{C}^1)$ is given by the formulas (6)–(11), where the index μ is supposed to take the values $0, 1, 2, 3$.*

Note 2. Investigating particular solutions of the Maxwell equations, Bateman [13] arrived at the problem of integrating the d'Alembert equation $\Box_4 u = 0$ with an additional nonlinear condition (the eikonal equation) $u_{x_\mu} u_{x^\mu} = 0$. He has obtained the following class of exact solutions of the said system of PDE:

$$u(x) = c_\mu(\tau)x^\mu + c_4(\tau), \tag{12}$$

where $\tau = \tau(x)$ is a complex-valued function determined in implicit way

$$\dot{c}_\mu(\tau)x^\mu + \dot{c}_4(\tau) = 0, \tag{13}$$

and $c_\mu(\tau)$, $c_4(\tau)$ are arbitrary smooth functions satisfying conditions

$$c_\mu c^\mu = \dot{c}_\mu \dot{c}^\mu = 0. \tag{14}$$

(hereafter, a dot over a symbol means differentiation with respect to a corresponding argument).

It is not difficult to check that solutions (12)–(14) are complex (see the Lemma 1 below). An another class of complex solutions of the system (2) with $n = 4$ was constructed by Yerugin [11]. But neither the Bateman's formulas (12)–(14) nor the Yerugin's results give the general solution of the system (2) with $n = 4$.

3 Proofs of Theorems 1–3

It is well-known that the maximal symmetry group admitted by equation (1) is finite-dimensional (we neglect a trivial invariance with respect to an infinite-parameter group $u(x) \to u(x) + U(x)$, where $U(x)$ is an arbitrary solution of Eq. (1), which is due to its linearity). But being restricted to a set of solutions of the eikonal equation the set solutions of PDE (1) admits an infinite-dimensional symmetry group [14]! It is this very fact that enables us to construct the general solution of (2).

Proof of the Theorem 1. Let us make in (2) the hodograph transformation

$$z_0 = u(x), \quad z_a = x_a, \quad a = \overline{1, n-1}, \quad w(z) = x_0. \tag{15}$$

Evidently, the transformation (15) is defined for all functions $u(x)$, such that $u_{x_0} \not\equiv 0$. But the system (2) with $u_{x_0} = 0$ takes the form

$$\sum_{a=1}^{n-1} u_{x_a x_a} = 0, \quad \sum_{a=1}^{n-1} u_{x_a}^2 = 0,$$

whence $u_{x_a} \equiv 0$, $a = \overline{1, n-1}$ or $u(x) = \mathrm{const}$.

Consequently, the change of variables (9) is defined on the whole set of solutions of the system (2) with the only exception $u(x) = \mathrm{const}$.

Being rewritten in the new variables z, $w(z)$ the system (2) takes the form

$$\sum_{a=1}^{n-1} w_{z_a z_a} = 0, \quad \sum_{a=1}^{n-1} w_{z_a}^2 = 1. \tag{16}$$

Differentiating the second equation with respect to z_b, z_c we get

$$\sum_{a=1}^{n-1} (w_{z_a z_b z_c} w_{z_a} + w_{z_a z_b} w_{z_a z_c}) = 0.$$

Choosing in the above equality $c = b$ and summing up we have

$$\sum_{a,b=1}^{n-1} (w_{z_a z_b z_b} w_{z_a} + w_{z_a z_b} w_{z_a z_b}) = 0,$$

whence, by force of (16),

$$\sum_{a,b=1}^{n-1} w_{z_a z_b}^2 = 0. \tag{17}$$

Since $u(z)$ is a real-valued function, it follows from (17) that an equality $w_{z_a z_b} = 0$ holds for all $a, b = \overline{1, n-1}$, whence

$$w(z) = \sum_{a=1}^{n-1} \alpha_a(z_0) z_a + \alpha(z_0). \tag{18}$$

In (18) $\alpha_a, \alpha \in C^2(\mathbb{R}^1, \mathbb{R}^1)$ are arbitrary functions.

Substituting (18) into the second equation of system (16), we have

$$\sum_{a,b=1}^{n-1} \alpha_a^2(z_0) = 1. \tag{19}$$

Thus, the formulas (18), (19) give the general solution of the system of nonlinear PDE (16). Rewriting (18), (19) in the initial variables x, $u(x)$, we get

$$x_0 = \sum_{a=1}^{n-1} \alpha_a(u)x_a + \alpha(u), \quad \sum_{a=1}^{n-1} \alpha_a^2(u) = 1. \tag{20}$$

To represent the formulas (20) in a manifestly covariant form (4), (5) we redefine the functions $\alpha_a(u)$ in the following way:

$$\alpha_a(u) = \frac{A_a(u)}{A_0(u)}, \quad \alpha(u) = -\frac{B(u)}{A_0(u)}, \quad a = \overline{1, n-1}.$$

Substituting the above expressions into (20) we arrive at the formulas (4), (5).

Next, as $u = \text{const}$ is contained in the class of functions $u(x)$ determined by the formulas (4), (5) under $A_\mu \equiv 0$, $\mu = \overline{0, n-1}$, $B(u) = u + \text{const}$, JSSF (4), (5) give the general solution of the system of the PDE (2) with an arbitrary $n \in \mathbb{N}$. The theorem is proved.

Let us emphasize that the reasonings used above can be applied to the case of a real-valued function $u(x)$ only. If a solution of the system (2) is looked for in a class of complex-valued functions $u(x)$, then JSSF (4), (5) do not give its general solution with $n > 3$. Each case $n = 4, 5\ldots$ requires a special consideration.

Proof of the Theorem 2. *Case 1:* $u_{x_0} \neq 0$. In this case the hodograph transformation (15) reducing the system (2) with $n = 5$ to the form

$$\sum_{a=1}^{4} w_{z_a z_a} = 0, \quad \sum_{a=1}^{4} w_{z_a}^2 = 1, \quad w_{z_0} \not\equiv 0 \tag{21}$$

is defined.

The general solution of nonlinear complex Eqs. (21) was constructed in [15]. It is given by one of the following formulas:

$$(1)\ w(z) = \sum_{a=1}^{4} \alpha_a(\tau, z_0)z_a + \gamma_1(\tau, z_0), \tag{22}$$

where $\tau = \tau(z_0, \ldots, z_4)$ is a function determined in implicit way

$$\sum_{a=1}^{4} \beta_a(\tau, z_0)z_a + \gamma_2(\tau, z_0) = 0 \tag{23}$$

and $\alpha_a, \beta_a, \gamma_1, \gamma_2 \in C^2(\mathbb{C}^2, \mathbb{C}^1)$ are arbitrary smooth functions satisfying the relations

$$\sum_{a=1}^{4} \alpha_a^2 = 1, \quad \sum_{a=1}^{4} \alpha_a\beta_a = \sum_{a=1}^{4} \beta_a^2 = 0, \quad \sum_{a=1}^{4} \alpha_a \frac{\partial \beta_a}{\partial \tau} = 0; \tag{24}$$

$$(2)\ w(z) = \sum_{a=1}^{4} \alpha_a(z_0) z_a + \gamma_1(z_0), \tag{25}$$

where $\alpha_a, \gamma_1 \in C^2(\mathbb{C}^1, \mathbb{C}^1)$ are arbitrary functions satisfying the relation

$$\sum_{a=1}^{4} \alpha_a^2 = 1. \tag{26}$$

Rewriting the formulas (23), (24) in the initial variables x, $u(x)$, we have

$$x_0 = \sum_{a=1}^{4} \alpha_a(\tau, u) x_a + \gamma_1(\tau, u), \tag{27}$$

where $\tau = \tau(u, x)$ is a function determined in implicit way

$$\sum_{a=1}^{4} \beta_a(\tau, u) x_a + \gamma_2(\tau, u) = 0, \tag{28}$$

and the relations (24) hold.

Evidently, the formulas (27), (28) are obtained from (6)–(8) with a particular choice of functions A_μ, B_μ, C_1, C_2

$$\begin{aligned} &A_0 = 1, \quad A_a = \alpha_a, \quad C_1 = -\gamma_1, \\ &B_0 = 0, \quad B_a = \beta_a, \quad C_2 = -\gamma_2, \end{aligned} \tag{29}$$

where $a = \overline{1,4}$.

Next, by force of inequality $w_{z_0} \not\equiv 0$ we get from (22)

$$\sum_{a=1}^{4} (\alpha_{az_0} + \alpha_{a\tau} \tau_{z_0}) x_a + \gamma_{1z_0} + \gamma_{1\tau} \tau_{z_0} \neq 0. \tag{30}$$

Differentiation of (23) with respect to z_0 yields the following expression for τ_{z_0}:

$$\tau_{z_0} = -\left(\sum_{a=1}^{4} \beta_{az_0} x_a + \gamma_{2z_0} \right) \left(\sum_{a=1}^{4} \beta_{a\tau} x_a + \gamma_{2\tau} \right)^{-1}$$

Substitution of the above result into (30) yields the relation

$$\left(\sum_{a=1}^{4} \beta_{a\tau} x_a + \gamma_{2\tau} \right)^{-1} \begin{vmatrix} \sum\limits_{a=1}^{4} \alpha_{az_0} x_a + \gamma_{1z_0} & \sum\limits_{a=1}^{4} \alpha_{a\tau} x_a + \gamma_{1\tau} \\ \sum\limits_{a=1}^{4} \beta_{az_0} x_a + \gamma_{2z_0} & \sum\limits_{a=1}^{4} \beta_{a\tau} x_a + \gamma_{2\tau} \end{vmatrix} \neq 0.$$

As the direct check shows, the above inequality is equivalent to (9) provided the conditions (29) hold.

Now we turn to solutions of the system (21) of the form (25). Rewriting the formulas (25), (26) in the initial variables x, $u(x)$ we get

$$x_0 = \sum_{a=1}^{4} \alpha_a(u) x_a + \gamma_1(u), \qquad \sum_{a=1}^{4} \alpha_a^2(u) = 1.$$

Making in the equalities obtained the change $\alpha_a = A_a A_0^{-1}$, $a = \overline{1,4}$, $\gamma_1 = -C_1 A_0^{-1}$, we arrive at the formulas (10), (11).

Thus, under $u_{x_0} \not\equiv 0$ the general solution of the system (2) is contained in the class of functions $u(x)$ given by the formulas (6)–(9) or (10), (11).

Case 2: $u_{x_0} \equiv 0$, $u \neq \text{const}$. It is a common knowledge that the system of PDE (2) is invariant under the generalized Poincaré group $P(1,n-1)$ (see, e.g. [16])

$$x'_\mu = \Lambda_{\mu\nu} x^\nu + \Lambda_\mu, \quad u'(x') = u(x),$$

where $\Lambda_{\mu\nu}$, Λ_μ are arbitrary complex parameters satisfying the relations $\Lambda_{\alpha\mu}\Lambda^\alpha{}_\nu = g_{\mu\nu}$, $\mu,\nu = \overline{0,n-1}$. Hence, it follows that the transformation

$$u(x) \to u(x') = u(\Lambda_{\mu\nu} x^\nu) \tag{31}$$

leaves the set of solutions of the system (2) invariant. Consequently, provided $u(x) \neq \text{const}$ we can always transform u to such a form that $u_{x_0} \neq 0$. Thus, in the case 2 the general solution is also given by the formulas (6)–(11) within the transformation (31).

Case 3: $u = \text{const}$. Choosing in (10), (11) $A_\mu = 0$, $\mu = \overline{0,4}$, $C_1 = u = \text{const}$ we come to the conclusion that this solution is described by the formulas (6)–(11).

Thus, we have proved that, within a transformation from the group $P(1,4)$ (31), the general solution of the system of PDE (2) with $n = 5$ is given by the formulas (6)–(11). But these formulas are represented in a manifestly covariant form and are not altered with the transformation (31). Consequently, to complete the proof of the theorem it is enough to demonstrate that each function $u = u(x)$ determined by the equalities (6)–(11) is a solution of the system of equations (2).

Differentiating the relations (6), (7) with respect to x_μ, we have

$$\begin{aligned} &A^\mu + \tau_{x_\mu}(A_{\nu\tau} x^\nu + C_{1\tau}) + u_{x_\mu}(A_{\nu u} x^\nu + C_{1u}) = 0, \\ &B^\mu + \tau_{x_\mu}(B_{\nu\tau} x^\nu + C_{2\tau}) + u_{x_\mu}(B_{\nu u} x^\nu + C_{2u}) = 0. \end{aligned}$$

Resolving the above system of linear algebraic equations with respect to u_{x_μ}, τ_{x_μ}, we get

$$\begin{aligned} u_{x_\mu} &= \frac{1}{\Delta}\big(B_\mu(A_{\nu\tau} x^\nu + C_{1\tau}) - A_\mu(B_{\nu\tau} x^\nu + C_{2\tau})\big), \\ \tau_{x_\mu} &= \frac{1}{\Delta}\big(A_\mu(B_{\nu u} x^\nu + C_{1u}) - B_\mu(A_{\nu u} x^\nu + C_{2u})\big), \end{aligned} \tag{32}$$

where $\Delta \neq 0$ by force of (9). Consequently,

$$\begin{aligned} u_{x_\mu} u_{x^\mu} = \Delta^{-2}\big(&B_\mu B^\mu (A_{\nu\tau} x^\nu + C_{1\tau})^2 - 2A_\mu B^\mu (A_{\nu\tau} x^\nu + C_{1\tau})(B_{\nu\tau} x^\nu + C_{2\tau}) + \\ &+ A_\mu A^\mu (B_{\nu\tau} x^\nu + C_{2\tau})^2\big) = 0. \end{aligned}$$

Analogously, differentiating (32) with respect to x_ν and convoluting the expression obtained with the metric tensor $g_{\mu\nu}$, we get $g^{\mu\nu} u_{x_\mu x_\nu} \equiv \Box_5 u = 0$.

Next, differentiating (10) with respect to x_μ we have

$$u_{x_\mu} = -A_\mu(\dot{A}_\nu x^\nu + \dot{C}_1)^{-1}, \quad \mu = \overline{0,4},$$

whence

$$u_{x_\mu x_\nu} = -(\dot{A}^\mu A^\nu + \dot{A}^\nu A^\mu)(\dot{A}_\alpha x^\alpha + \dot{C}_1)^{-2} + A^\mu A^\nu (\ddot{A}_\alpha x^\alpha + \ddot{C}_1)(\dot{A}_\alpha x^\alpha + \dot{C}_1)^{-2}.$$

Consequently,

$$u_{x_\mu}u_{x^\mu} = A_\mu A^\mu(\dot{A}_\nu x^\mu + \dot{C}_1)^{-2} = 0,$$
$$\Box_5 u \equiv u_{x_\mu x^\mu} = -2(A_\mu \dot{A}^\mu)(\dot{A}_\nu x^\nu + \dot{C}_1)^{-2} + \\ + A_\mu A^\mu(\ddot{A}_\nu x^\nu + \ddot{C}_1)(\dot{A}_\nu x^\nu + \dot{C}_1)^{-2} = 0.$$

The Theorem 2 is proved.

The Theorem 3 is a direct consequence of the Theorem 2. Really, solutions of the system of PDE (2) with $n = 4$ are obtained from solutions of the system of PDE (2) with $n = 5$ provided $u_{x_4} \equiv 0$. Imposing on functions $u(x)$ determined by the formulas (6)–(11) a condition $u_{x_4} \equiv 0$ we arrive at the following restrictions on the functions A_μ, B_μ, C_1, C_2:

$$A_4 = 0, \quad B_4 = 0$$

the same as what was to be proved.

4 Applications: reduction of the nonlinear d'Alembert equation

Following [8, 15, 16], we look for a solution of the nonlinear d'Alembert equation

$$\Box_4 w = F(w), \quad F \in C^1(\mathbb{R}^1, \mathbb{R}^1) \tag{33}$$

in the form

$$w = \varphi(\omega_1, \omega_2), \tag{34}$$

where $\omega_i = \omega_i(x) \in C^2(\mathbb{R}^4, \mathbb{R}^1)$ are supposed to be functionally-independent. The functions $\omega_1(x)$, $\omega_2(x)$ are determined by the requirement that the substitution of (34) into (33) yields two-dimensional PDE for a function $\varphi = \varphi(\omega_1, \omega_2)$. As a result, we obtain an over-determined system of PDE [16]

$$\begin{gathered} \Box_4\omega_1 = f_1(\omega_1, \omega_2), \quad \Box_4\omega_2 = f_2(\omega_1, \omega_2), \\ \omega_{1x_\mu}\omega_{1x^\mu} = g_1(\omega_1, \omega_2), \quad \omega_{2x_\mu}\omega_{2x^\mu} = g_2(\omega_1, \omega_2), \\ \omega_{1x_\mu}\omega_{2x^\mu} = g_3(\omega_1, \omega_2), \quad \operatorname{rank} \left\| \frac{\partial \omega_i}{\partial x_\mu} \right\|_{i=1}^{2}\,{}_{\mu=0}^{3} = 2, \end{gathered} \tag{35}$$

and besides, the function $\varphi(\omega_1, \omega_2)$ satisfies a two-dimensional PDE,

$$g_1\varphi_{\omega_1\omega_1} + g_2\varphi_{\omega_2\omega_2} + 2g_3\varphi_{\omega_1\omega_2} + f_1\varphi_{\omega_1} + f_2\varphi_{\omega_2} = F(\varphi). \tag{36}$$

Consider the following problem: to describe all smooth real functions $\omega_1(x)$, $\omega_2(x)$ such that the Ansatz (34) reduces Eq. (33) to an ordinary differential equation (ODE) with respect to the variable ω_1. It means that one has to put coefficients g_2, g_3, f_2 in (36) equal to zero. In other words, it is necessary to construct a general solution of the system of nonlinear PDE

$$\begin{gathered} \Box_4\omega_1 = f_1(\omega_1, \omega_2), \quad \omega_{1x_\mu}\omega_{1x^\mu} = g_1(\omega_1, \omega_2), \\ \omega_{1x_\mu}\omega_{2x^\mu} = 0, \quad \omega_{2x^\mu}\omega_{2x_\mu} = 0, \quad \Box_4\omega_2 = 0. \end{gathered} \tag{37}$$

The above system includes Eqs. (2) as a subsystem. So, the d'Alembert-eikonal system (2) arises in a natural way when solving the problem of reduction of Eq. (33) to PDE having a smaller dimension (see, also [15, 17]).

With an appropriate choice of a function $G(\omega_1, \omega_2)$ the change of variables

$$v = G(\omega_1, \omega_2), \quad u = \omega_2$$

reduces the system (37) to the form

$$\Box_4 v = f(u, v), \quad v_{x_\mu} v_{x^\mu} = \lambda, \tag{38}$$

$$u_{x_\mu} v_{x^\mu} = 0, \quad u_{x_\mu} u_{x^\mu} = 0, \quad \Box_4 u = 0, \tag{39}$$

$$\operatorname{rank} \left\| \begin{matrix} v_{x_0} v_{x_1} v_{x_2} v_{x_3} \\ u_{x_0} u_{x_1} u_{x_2} u_{x_3} \end{matrix} \right\| = 2, \tag{40}$$

where λ is a real parameter taking the values -1, 0, 1.

Before formulating the principal assertion, we will prove an auxiliary lemma.

Lemma 1. *Let $a = (a_0, a_1, a_2, a_3)$, $b = (b_0, b_1, b_2, b_3)$ be four-vectors defined in the real Minkowski space $M(1,3)$. Suppose they satisfy the relations*

$$a_\mu b^\mu = b_\mu b^\mu = 0, \quad \sum_{\mu=0}^{3} b_\mu^2 \neq 0. \tag{41}$$

Then, an inequality $a_\mu a^\mu \leq 0$ holds.

Proof. It is known that any isotropic non-null vector b in the space $M(1,3)$ can be reduced to the form $b' = (\alpha, \alpha, 0, 0)$, $\alpha \neq 0$ by means of a transformation from the group $P(1,3)$. Substituting $b' = (\alpha, \alpha, 0, 0)$ into the first equality from (41), we get

$$\alpha(a'_0 - a'_2) = 0 \ \Leftrightarrow \ a'_0 = a'_3.$$

Consequently, the vector a' has the following components: a'_0, a'_1, a'_2, a'_0. That is why, $a'_\mu a'^\mu = a'^2_0 - a'^2_1 - a'^2_2 - a'^2_0 = -(a'^2_1 + a'^2_2) \leq 0$. As the quadratic form $a_\mu a^\mu$ is invariant with respect to the group $P(1,3)$, hence it follows that $a_\mu a^\mu \leq 0$.

Let us note that $a_\mu a^\mu = 0$ if and only if $a_2 = a_3$, i.e. $a_\mu a^\mu = 0$ if and only if the vectors a and b are parallel.

Theorem 4. *Eqs. (38)–(40) are compatible if and only if*

$$\lambda = -1, \quad f = -N\big(v + h(u)\big)^{-1}, \tag{42}$$

where $h \in C^1(\mathbb{R}^1, \mathbb{R}^1)$ is an arbitrary function, $N = 0, 1, 2, 3$.

Theorem 4. *The general solution of the system of Eqs. (38)–(40) being determined within a transformation from the group $P(1,3)$ is given by the following formulas:*

a) under $f = -3\big(v + h(u)\big)^{-1}$, $\lambda = -1$

$$\begin{aligned} &\big(v + h(u)\big)^2 = (-\dot{A}_\nu \dot{A}^\nu)^{-1} (\dot{A}_\mu x^\mu + \dot{B})^2 + \\ &\qquad\qquad + (-\dot{A}_\nu \dot{A}^\nu)^{-3} (\varepsilon^{\mu\nu\alpha\beta} A_\mu \dot{A}_\nu \ddot{A}_\alpha x_\beta + C)^2, \\ &A_\mu x^\mu + B = 0; \end{aligned} \tag{43}$$

b) under $f = -2\big(v + h(u)\big)^{-1}$, $\lambda = -1$

$$\big(v + h(u)\big)^2 = (-\dot{A}_\nu \dot{A}^\nu)^{-1}(\dot{A}_\mu x^\mu + \dot{B})^2, \quad A_\mu x^\mu + B = 0, \tag{44}$$

where $A_\mu = A_\mu(u)$, $B = B(u)$, $C = C(u)$ are arbitrary smooth functions satisfying the relations

$$A_\mu A^\mu = 0, \quad \dot{A}_\mu \dot{A}^\mu \neq 0, \tag{45}$$

c) under $f = -\big(v + h(u)\big)^{-1}$, $\lambda = -1$

$$\begin{aligned} &\big(v + h(x_0 - x_3)\big)^2 = \big(x_1 + C_1(x_0 - x_3)\big)^2 + \big(x_2 + C_2(x_0 - x_3)\big)^2, \\ &u = C_0(x_0 - x_3), \end{aligned} \tag{46}$$

where C_0, C_1, C_2 are arbitrary smooth functions;
d) under $f = 0$, $\lambda = -1$

$$(1) \quad v = (-\dot{A}_\nu \dot{A}^\nu)^{-3/2} \varepsilon^{\mu\nu\alpha\beta} A_\mu \dot{A}_\nu \ddot{A}_\alpha x_\beta + C, \quad A_\mu x^\mu + B = 0, \tag{47}$$

where $A_\mu = A_\mu(u)$, $B = B(u)$, $C = C(u)$ are arbitrary smooth functions satisfying the relations (45);

$$\begin{aligned} (2) \quad &v = x_1 \cos\big(C_1(x_0 - x_3)\big) + x_2 \sin\big(C_1(x_0 - x_3)\big) + C_2(x_0 - x_3), \\ &u = C_0(x_0 - x_3), \end{aligned} \tag{48}$$

where C_0, C_1, C_2 are arbitrary smooth functions.

In the above formulas (43), (47) we denote by $\varepsilon_{\mu\nu\alpha\beta}$ the completely anti-symmetric fourth-order tensor (the Levi-Civita tensor), i.e.

$$\varepsilon_{\mu\nu\alpha\beta} = \begin{cases} 1, & (\mu,\nu,\alpha,\beta) = \text{cycle}\,(0,1,2,3), \\ -1, & (\mu,\nu,\alpha,\beta) = \text{cycle}\,(1,0,2,3), \\ 0, & \text{in the remaining cases.} \end{cases}$$

Proof of the Theorems 4, 5. By force of (40) $u \neq \text{const}$. Consequently, within a transformation from the group $P(1,3)$ $u_{x_0} \neq 0$. That is why, one can apply to Eqs. (38)–(40) the hodograph transformation

$$z_0 = u(x), \quad z_a = x_a, \quad a = \overline{1,3}, \quad w(z) = x_0, \quad v = v(z_0, z_a).$$

As a result, the system (38), (39) reads

$$\sum_{a=1}^{3} w_{z_a}^2 = 1, \quad \sum_{a=1}^{3} w_{z_a z_a} = 0, \tag{49}$$

$$\sum_{a=1}^{3} v_{z_a} w_{z_a} = 0, \tag{50}$$

$$\sum_{a=1}^{3} v_{z_a}^2 = -\lambda, \quad \sum_{a=1}^{3} (v_{z_a z_a} + 2w_{z_0}^{-1} v_{z_a} w_{z_a z_0}) = -f(v, z_0). \tag{51}$$

As $v(z)$ is a real-valued function, $\lambda \leq 0$. Scaling, if necessary, the function v we can put $\lambda = -1$ or $\lambda = 0$.

Case 1: $\lambda = -1$. As it is shown in the Section 2, the general solution of the system (49) in the class of real-valued functions $w(z)$ is given by the formulas (18), (19) with $n = 4$. Substituting (18) into (50), we obtain a first-order linear PDE

$$\sum_{a=1}^{3} \alpha_a(z_0) v_{z_a} = 0, \tag{52}$$

whose general solution is represented in the form

$$v = v(z_0, \rho_1, \rho_2). \tag{53}$$

In (53),

$$z_0, \quad \rho_1 = \left(\sum_{a=1}^{3} \dot\alpha_a^2\right)^{-1/2} \left(\sum_{a=1}^{3} \dot\alpha_a z_a + \dot\alpha\right),$$

$$\rho_2 = \left(\sum_{a=1}^{3} \dot\alpha_a^2\right)^{-3/2} \sum_{a,b,c=1}^{3} \varepsilon_{abc} z_a \alpha_b \dot\alpha_c$$

are the first integrals of Eq. (52) and what is more, $\sum\limits_{a=1}^{3} \dot\alpha_a^2 \neq 0$ (the case $\alpha_a = \text{const}$, $a = \overline{1,3}$ will be treated separately), ε_{abc} is the third-order anti-symmetric tensor with $\varepsilon_{123} = 1$.

Substitution the expression (53) into (51) yields the system of two PDE for a function $v = v(z_0, \rho_1, \rho_2)$

$$v_{\rho_1\rho_1} + v_{\rho_2\rho_2} + 2\rho_1^{-1} v_{\rho_1} = -f(v, z_0), \tag{54}$$

$$v_{\rho_1}^2 + v_{\rho_2}^2 = 1. \tag{55}$$

To get rid of an arbitrary element (function) $f(v, z_0)$ from (54) we consider instead of system (54), (55) its differential consequence

$$v_{\rho_2}(v_{\rho_1\rho_1} + v_{\rho_2\rho_2} + 2\rho_1^{-1} v_{\rho_1})_{\rho_1} - v_{\rho_1}(v_{\rho_1\rho_1} + v_{\rho_2\rho_2} + 2\rho_1^{-1} v_{\rho_1})_{\rho_1} = 0, \tag{56}$$

$$v_{\rho_1}^2 + v_{\rho_2}^2 = 1, \tag{57}$$

that is obtained by differentiating the first equation with respect to ρ_1, ρ_2, multiplying the expressions obtained by v_{ρ_2} and $-v_{\rho_1}$, respectively, and summing.

Further, we will consider the subcases $v_{\rho_2\rho_2} = 0$ and $v_{\rho_2\rho_2} \neq 0$ separately.

Subcase 1.A: $v_{\rho_2\rho_2} = 0$. Then,

$$v = g_1(z_0, \rho_1)\rho_2 + g_2(z_0, \rho_1), \tag{58}$$

where $g_1, g_2 \in C^2(\mathbb{R}^1, \mathbb{R}^1)$ are arbitrary functions.

Substituting (58) into (57) and splitting an equality obtained by the powers of ρ_2, we have

$$g_{1\rho_1} = 0, \quad g_1^2 + (g_{2\rho_2})^2 = 1,$$

whence

$$v = \alpha\rho_1 \pm \sqrt{1-\alpha^2}\rho_2 - h(z_0). \tag{59}$$

Here $\alpha \in \mathbb{R}^1$, h is an arbitrary smooth function.

Inserting (59) into (56) we get an algebraic equation $\alpha\sqrt{1-\alpha^2} = 0$, whence $\alpha = 0, \pm 1$.

Finally, substitution of (59) into (54) yields the equation for $f(v, z_0)$

$$2\alpha\rho_1^{-1} = -f\left(\alpha\rho_1 \pm \sqrt{1-\alpha^2}\rho_2 - h(z_0), z_0\right). \tag{60}$$

From Eq. (60) it follows that, under $\alpha = 0$,

$$f = 0, \quad v = \pm\rho_2 - h(z_0) \tag{61}$$

and under $\alpha = \pm 1$,

$$f = -2\big(v + h(z_0)\big)^{-1}, \quad v = \pm\rho_1 - h(z_0). \tag{62}$$

Subcase 1.B: $v_{\rho_2\rho_2} \neq 0$. In this case one can apply to Eqs. (56), (57) the Euler–Ampére transformation

$$\begin{aligned} &z_0 = y_0, \quad \rho_1 = y_1, \quad \rho_2 = G_{y_2}, \quad v + G = \rho_2 y_2, \quad v_{\rho_1} = -G_{y_1}, \quad v_{\rho_2} = y_2, \\ &v_{\rho_2\rho_2} = (G_{y_2y_2})^{-1}, \quad v_{\rho_1\rho_2} = -G_{y_1y_2}(G_{y_2y_2})^{-1}, \\ &v_{\rho_1\rho_1} = (G^2_{y_1y_2} - G_{y_1y_1}G_{y_2y_2})(G_{y_2y_2})^{-1}. \end{aligned} \tag{63}$$

Here y_0, y_1, y_2 are new independent variables, $G = G(y_0, y_1, y_2)$ is a new function.

Being rewritten in the new variables y, $G(y)$ the Eq. (57), becomes linear

$$G_{y_1} = \pm\sqrt{1 - y_2^2},$$

whence

$$G = \pm y_1\sqrt{1-y_2^2} + H(y_0, y_2), \quad H \in C^2(\mathbb{R}^2, \mathbb{R}^1). \tag{64}$$

Making in the Eq. (56) the change of variables (63) and inserting the expression (64), we transform it as follows

$$\big(y_2 - (1-y_2^2)^{3/2}H_{y_2y_2}\big)^{-2}\big(3y_2H_{y_2y_2} + (y_2^2-1)H_{y_2y_2y_2}\big) + 2y_1^{-2}y_2H_{y_2y_2} = 0. \tag{65}$$

Splitting (65) by the powers of y_1 and integrating the equations obtained, we get

$$H = h_1(y_0)y_2 + h_2(y_0).$$

Substituting the above result into (64) and returning to the initial variables z_0, ρ_1, ρ_2, $v(z_0, \rho_1, \rho_2)$ we obtain the general solution of the system of PDE (56), (57)

$$v + h_2(z_0) = \pm\big([\rho_2 - h_1(z_0)]^2 + \rho_1^2\big)^{1/2}. \tag{66}$$

At last, inserting (66) into the equation (54), we arrive at the conclusion that the function f is determined by the formula (42) with $N = 3$.

If $\alpha_a = \text{const}$, $a = \overline{1,3}$, then the equality $\alpha_1^2 + \alpha_2^2 + \alpha_3^2 = 1$ holds. Applying, if necessary, a transformation from the group $P(1,3)$ one can put $\alpha_1 = \alpha_2 = 0$, $\alpha_3 = 1$, i.e. $u = C_0(x_0 - x_3)$, $C_0 \in C^2(\mathbb{R}^1, \mathbb{R}^2)$.

As a consequence of Eqs. (39) we get $v = v(\xi, x_1, x_2)$, where $\xi = x_0 - x_3$, and what is more, Eqs. (38) take the form

$$v_{x_1}^2 + v_{x_2}^2 = 1, \quad v_{x_1x_1} + v_{x_2x_2} = -f\big(v, C_0(\xi)\big). \tag{67}$$

It is known [15, 18] that Eqs. (67) are compatible if and only if $f = 0$ or $f = -(v + h(u))^{-1}$, $h \in C^1(\mathbb{R}^1, \mathbb{R}^1)$. And besides, the general solution of (67) is given by the formulas (48) and (46), respectively.

Thus, we have completely investigated the case $\lambda = -1$.

Case 2: $\lambda = 0$. By force of the fact that the function v is a real one, it follows from (51) that $v = v(z_0)$. Consequently, an equality $v = v(u)$ holds that breaks the condition (40) which means that under $\lambda = 0$ the system (38)–(40) is incompatible.

Thus, we have proved that the system of nonlinear PDE (38)–(40) is compatible if and only if the relations (42) hold and that its general solution is given by one of the formulas (46), (48), (61), (62), and (66). To complete the proof, one has to rewrite the expressions (61), (62), (66) in the manifestly covariant from (43), (44), (47).

Consider, as an example, the formula (62)

$$v = \pm\rho_1 - h(z_0) \equiv \pm\left(\sum_{a=1}^{3}\dot\alpha_a^2(u)\right)^{-1/2}\left(\sum_{a=1}^{3}x_a\dot\alpha_a(u) + \dot\alpha(u)\right) - h(u), \tag{68}$$

the function $u(x)$ being determined by the formula (20),

$$\sum_{a=1}^{3}\alpha_a(u)x_a + \alpha(u) = x_0, \quad \sum_{a=1}^{3}\alpha_a^2(u) = 1. \tag{69}$$

Let us make in (68), (69) a substitution $\alpha_a = A_aA_0^{-1}$, $\alpha = -BA_0^{-1}$, whence

$$\begin{aligned}
&A_\mu(u)x^\mu + B(u) = 0, \quad A_\mu A^\mu = 0,\\
&v = \pm\left(\sum_{a=1}^{3}(\dot A_aA_0^{-1} - A_a\dot A_0A_0^{-2})^2\right)^{-1/2}\times\\
&\quad\times\left(\sum_{a=1}^{3}x_a(\dot A_aA_0^{-1} - A_a\dot A_0A_0^{-2}) + B\dot A_0A_0^{-2} - \dot BA_0^{-1}\right) - h(u) =\\
&= \pm\left(\sum_{a=1}^{3}(\dot A_a^2A_0^{-2} + A_a^2\dot A_0^2A_0^{-4} - 2\dot A_aA_a\dot A_0A_0^{-3})^{-1/2}\right)\times\\
&\quad\times\left(\sum_{a=1}^{3}x_a(\dot A_aA_0^{-1} - A_a\dot A_0A_0^{-2}) + B\dot A_0A_0^{-2} - \dot BA_0\right) - h(u) =\\
&= \pm\left(-\dot A_\mu\dot A^\mu A_0^{-2} - A_\mu A^\mu\dot A_0^2A_0^{-4} + 2\dot A_\mu A^\mu\dot A_0A_0^{-3}\right)^{-1/2}\times\\
&\quad\times\left(-A_0^{-1}(x_\mu\dot A^\mu + \dot B) + A_0^{-2}\dot A_0(x_\mu A^\mu + B)\right) - h(u) =\\
&= \mp(-\dot A_\mu\dot A^\mu)^{-1/2}(x_\mu\dot A^\mu + \dot B) - h(u).
\end{aligned}$$

The only thing left is to prove that $\dot{A}_\mu \dot{A}^\mu < 0$. Since $A_\mu A^\mu = 0$, the equality $\dot{A}_\mu A^\mu = 0$ holds. Consequently, by force of the Lemma $-\dot{A}_\mu \dot{A}^\mu \geq 0$, and what is more, the equality $\dot{A}_\mu \dot{A}^\mu = 0$ holds if and only if $\dot{A}_\mu = k(u)A_\mu$. General solution of the above system of ordinary differential equations reads $A_\mu = l(u)\theta_\mu$, where $l(u)$ is an arbitrary function, θ_μ are arbitrary real parameters obeying the equality $\theta_\mu\theta^\mu = 0$. Hence it follows that $\alpha_a = A_a A_0^{-1} = \theta_a \theta_0^{-1} = \text{const}$, and the condition $\sum\limits_{a=1}^{3} \dot{\alpha}_a^2 \neq 0$ does not hold. We come to the contradiction, whence it follows that $\dot{A}_\mu \dot{A}^\mu < 0$.

Thus, we have obtained the formula (44). Derivation of the remaining formulas from (43), (47) is carried out in the same way. The theorems are proved.

Substitution of the results obtained above into the formula (34) yields the following collection of Ansätze for the nonlinear d'Alembert equation (33):

$$
\begin{aligned}
&(1)\ w(x) = \varphi\Big(\big[\big(-\dot{A}_\nu(u)\dot{A}^\nu(u)\big)^{-1}\big(\dot{A}_\mu(u)x^\mu + \dot{B}(u)\big)^2 + \\
&\qquad + \big(-\dot{A}_\nu(u)\dot{A}^\nu(u)\big)^{-3}\big(\varepsilon^{\mu\nu\alpha\beta}A_\mu(u)\dot{A}_\nu(u)\ddot{A}_\alpha(u)x_\beta + C(u)\big)^2\big]^{1/2}, u\Big);\\
&(2)\ w(x) = \varphi\left(\big(-\dot{A}_\nu(u)\dot{A}^\nu(u)\big)^{1/2}\big(\dot{A}_\mu(u)x^\mu + \dot{B}(u)\big), u\right);\\
&(3)\ w(x) = \varphi\left(\big[\big(x_1 + C_1(x_0 - x_3)\big)^2 + \big(x_2 + C_2(x_0 - x_3)\big)^2\big]^{1/2}, x_0 - x_3\right);\\
&(4)\ w(x) = \varphi\left(\big(-\dot{A}_\nu(u)\dot{A}^\nu(u)\big)^{-3/2}\big(\varepsilon^{\mu\nu\alpha\beta}A_\mu(u)\dot{A}_\nu(u)\ddot{A}_\alpha(u)x_\beta + C(u)\big), u\right);\\
&(5)\ w(x) = \varphi\big(x_1\cos C_1(x_0 - x_3) + x_2 \sin C_1(x_0 - x_3) + C_2(x_0 - x_3), x_0 - x_3\big).
\end{aligned}
\tag{70}
$$

Here B, C, C_1, C_2 are arbitrary smooth functions of the corresponding arguments, $A_\mu(u)$ are arbitrary smooth functions satisfying the condition $A_\mu A^\mu = 0$ and the function $u = u(x)$ is determined by JSSF (10) with $C_1(u) = B(u)$, $n = 4$. Note that arbitrary functions h contained in the functions $v(x)$ (see above the formulas (43), (44), (46)) are absorbed by the function $\varphi(v, u)$ at the expense of the second argument.

Substitution of the expressions (70) into (33) gives the following equations for $\varphi = \varphi(u, v)$:

$$(1)\ \varphi_{vv} + \frac{3}{v}\varphi_v = -F(\varphi), \tag{71}$$

$$(2)\ \varphi_{vv} + \frac{2}{v}\varphi_v = -F(\varphi), \tag{72}$$

$$(3)\ \varphi_{vv} + \frac{1}{v}\varphi_v = -F(\varphi), \tag{73}$$

$$(4)\ \varphi_{vv} = -F(\varphi), \tag{74}$$

$$(5)\ \varphi_{vv} = -F(\varphi), \tag{75}$$

Equations (4), (5) from (71)–(75) are known to be integrable in quadratures. Therefore, any solution of the d'Alembert-eikonal system (2) corresponds to some class of exact solutions of the nonlinear wave equation (33) that contains arbitrary functions. Saying it in another way, the formulas (70) make it possible to construct wide families of exact solutions of the nonlinear PDE (33) using exact solutions of the linear d'Alembert equation $\Box_4 u = 0$ satisfying an additional constraint $u_{x_\mu} u_{x^\mu} = 0$.

It is interesting to compare our approach to the problem of reduction of Eq. (33) with the classical Lie approach. Within the framework of the Lie approach functions $\omega_1(x)$, $\omega_2(x)$ from (34) are looked for as invariants of the symmetry group of the equation under study (in the case involved it is the Poincaré group $P(1,3)$). Since the group $P(1,3)$ is a finite-parameter group, its invariants cannot contain an arbitrary function (a complete description of invariants of the group $P(1,3)$ had been carried out in [19]). Therefore, the Ansätze (70) cannot, in principle, be obtained by means of the Lie symmetry of the PDE (33).

All Ansätze listed in (70) correspond to a *conditional invariance* of the nonlinear d'Alembert equation (33). It means that for each Ansatz from (70) there exist two differential operators $Q_a = \xi_{a\mu}(x)\partial_{x_\mu}$, $a = 1,2$ such that

$$Q_a w(x) \equiv Q_a \varphi(\omega_1, \omega_2) = 0, \quad a = \overline{1,2}$$

and besides, the system of PDE

$$\Box_4 w - F(w) = 0, \quad Q_a w = 0, \quad a = 1,2$$

is invariant in Lie's sense under the one-parameter groups with the generators Q_1, Q_2. For example, the fourth Ansatz from (16) is invariant with respect to the operators: $Q_1 = A_\mu(u)\partial_\mu$, $Q_2 = \dot{A}_\mu(u)\partial_\mu$. A direct computation shows that the following relations hold:

$$\underset{2}{Q_i}(\Box_4 \omega) = -(\dot{A}^\alpha x_\alpha + \dot{B})^{-1} A^\mu \partial_\mu Q_i w, \quad i = 1,2,$$
$$[Q_1, Q_2] = 0,$$

where $\underset{2}{Q_i}$ stands for the second prolongation of the operator Q_i. Hence it follows that the nonlinear d'Alembert equation (33) is conditionally-invariant under the two-dimensional commutative Lie algebra having the basis elements Q_1, Q_2 (for more details about conditional symmetry of PDE see [20, 21]). It should be said that the notion of conditional symmetry of PDE is closely connected with the "non-classical reduction" [22–24] and "direct reduction" [25] methods.

5 On the new exact solutions of the nonlinear d'Alembert equation

According to [26], general solutions of Eqs. (74), (75) are given by the following quadrature:

$$v + D(u) = \int_0^{\varphi(u,v)} \left(-2 \int_0^\tau F(z)dz + C(u) \right)^{-1/2} d\tau, \tag{76}$$

where $D(u), C(u) \in C^2(\mathbb{R}^1, \mathbb{R}^1)$ are arbitrary functions.

Substituting the expressions for $u(x)$, $v(x)$ given by the formulas (4), (5) from (70) into (76) we obtain two classes of exact solutions of the nonlinear d'Alembert equation (33) that contain several arbitrary functions of one variable.

Equations (71) and (72) with $F(\varphi) = \lambda\varphi^k$ are Emden–Fowler type equations. They were investigated by many authors (see, e.g. [26]). In particular, it is known that the equations

$$\varphi_{vv} + 2v^{-1}\varphi_v = -\lambda\varphi^5, \tag{77}$$

$$\varphi_{vv} + 3v^{-1}\varphi_v = -\lambda\varphi^3 \tag{78}$$

are integrated in quadratures. In the paper [27] it has been established that Eqs. (77), (78) possess a Painlevé property. This fact makes it possible to integrate these by applying rather complicated technique. In [28] we have integrated Eqs. (77), (78) using a standard technique based on their Lie symmetry. Substituting the results obtained into the corresponding Ansätze from (70) we get exact solutions of the nonlinear PDE (33) with $F(w) = \lambda w^3$, λw^5, which include an arbitrary solution of the system (2) with $n = 4$. Consequently, we have constructed principally new exact solutions of the nonlinear d'Alembert equation (33) depending on several arbitrary functions. Let us stress that following the classical Lie symmetry reduction procedure one can not in principle obtain solutions with arbitrary functions since the maximal symmetry group of Eq. (33) is finite-dimensional (see, e.g. [16]).

Below we give new exact solutions of the nonlinear d'Alembert equation (33) obtained with the use of the technique described above. We adduce only those ones that can be written down explicitly

1. $F(w) = \lambda w^3$

(1) $$w(x) = \frac{1}{a\sqrt{2}}(x_1^2 + x_2^2 + x_3^2 - x_0^2)^{-1/2}\times \tan\left\{-\frac{\sqrt{2}}{4}\ln\bigl(C(u)(x_1^2 + x_2^2 + x_3^2 - x_0^2)\bigr)\right\},$$

where $\lambda = -2a^2 < 0$,

(2) $$w(x) = \frac{2\sqrt{2}}{a}C(u)\bigl(1 \pm C^2(u)(x_1^2 + x_2^2 + x_3^2 - x_0^2)\bigr)^{-1},$$

where $\lambda = \pm a^2$;

2. $F(w) = \lambda w^5$

(1) $$w(x) = a^{-1}(x_1^2 + x_2^2 - x_0^2)^{-1/4}\left\{\sin\ln\bigl(C(u)(x_1^2 + x_2^2 - x_0^2)^{-1/2}\bigr) + 1\right\}^{1/2}\times \left\{2\sin\ln\bigl(C(u)(x_1^2 + x_2^2 - x_0^2)^{-1/2}\bigr) - 4\right\}^{-1/2},$$

where $\lambda = a^4 > 0$,

(2) $$w(x) = \frac{3^{1/4}}{\sqrt{a}}C(u)\bigl(1 \pm C^4(u)(x_1^2 + x_2^2 - x_0^2)\bigr)^{-1/2},$$

where $\lambda = \pm a^2$.

In the above formulas $C(u)$ is an arbitrary twice continuously differentiable function on

$$u(x) = \frac{x_0x_1 \pm x_2\sqrt{x_1^2 + x_2^2 - x_0^2}}{x_1^2 + x_2^2},$$

$a \neq 0$ is an arbitrary real parameter.

6 Conclusion

The present paper demonstrates once more that possibilities to construct in explicit form new exact solutions of the nonlinear d’Alembert equation (33) (as compared with those obtainable by the standard symmetry reduction technique [16, 19, 27]) are far from being exhausted. A source of new (non-Lie) reductions is the conditional symmetry of Eq. (33).

Roughly speaking, a principal idea of the method of conditional symmetries is the following: to be able to reduce given PDE it is enough to require an invariance of a *subset* of its solutions with respect to some Lie transformation group. And what is more, this subset is not obliged to coincide with the whole set. This specific subsets can be chosen in different ways: one can fix in some way an Ansatz for a solution to be found (the method of Ansätze [16, 17] or the direct reduction method [25]) or one can impose an additional differential constraint (the method of non-classical [22–24] or conditional symmetries [20, 21]). But all the above approaches have a common feature: to find new (non-Lie) reduction of a given PDE one has to solve some nonlinear over-determined system of differential equations. For example, to describe Ansätze of the form (34) reducing Eq. (33) to ODE one has to integrate system of five nonlinear PDE (37). This is a “price” to be paid for the new possibilities to reduce a given nonlinear PDE to equations with less number of independent variables and to construct its explicit solutions.

As mentioned in the Introduction, the Ansatz (34) can also be interpreted as a map (more exactly, a family of maps) from the set of solutions of the linear d’Alembert equation,

$$\Box_4 u = 0 \tag{79}$$

into the set of solutions of the nonlinear d’Alembert equation (33).

Really, we started with a subset of solutions of Eq. (79) which was chosen by an additional eikonal constraint $u_{x_\mu} u_{x^\mu} = 0$. Then, we constructed the functions $v(x)$ and $\varphi(v, u)$ in such a way that the function $w(x)$ determined by the equality $w = \varphi(v(x), u(x))$ satisfied the nonlinear d’Alembert equation (33) (see below the Fig. 1).

There is an analogy between the map described above and Bäcklund transformations of partial differential equations. System of PDE (38)–(40) and the Ansatz (34) (level 2 of the Fig. 1) can be interpreted as a Bäcklund transformation of a set of solutions of linear PDE (level 1 of the Fig. 1) into a set of solutions of nonlinear PDE (level 3). A principal difference is that a classical Bäcklund transformation acts on the whole spaces of solutions of equations under study and the above map acts on subsets of solutions of the linear and nonlinear d’Alembert equations. It is known that technique of linearization of PDE with the use of Bäcklund transformations can be effectively applied to two-dimensional equations only. The results obtained in the present paper imply the following way of extension of applicability of Bäcklund transformations: one should consider Bäcklund transformations connecting subsets of solutions of linear and nonlinear equations. And these subsets may not coincide with the whole sets of solutions.

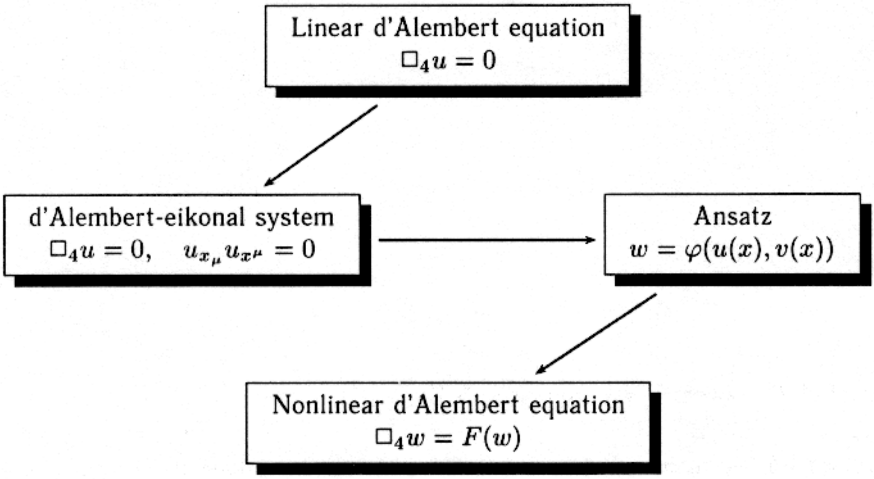

Figure 1.

As an illustration we consider the case when in (33) $F(w) = 0$, i.e. the case when the map constructed above transforms a subset of solutions of the linear d'Alembert equation into another subset of solutions of the same equation. Integrating ODE (71)–(75) we obtain explicit forms of functions $\varphi(v, u)$

$$\begin{aligned}
&(1) \quad \varphi(v,u) = f_1(u)v^{-2} + f_2(u),\\
&(2) \quad \varphi(v,u) = f_1(u)v^{-1} + f_2(u),\\
&(3) \quad \varphi(v,u) = f_1(u)\ln v + f_2(u),\\
&(4) \quad \varphi(v,u) = f_1(u)v + f_2(u),\\
&(5) \quad \varphi(v,u) = f_1(u)v + f_2(u),
\end{aligned}$$

where f_1, f_2 are arbitrary smooth enough functions. Consequently, we have the following maps transforming subsets of solutions of the linear d'Alembert equation (79) into another subsets of its solutions:

$$\begin{aligned}
(1) \quad & u \to f_1(u)\big[\big(-\dot{A}_\nu(u)\dot{A}^\nu(u)\big)^{-1}\big(\dot{A}_\mu(u)x^\mu + \dot{B}(u)\big)^2 + \\
& \quad + \big(-\dot{A}_\nu(u)\dot{A}^\nu(u)\big)^{-3}\big(\varepsilon^{\mu\nu\alpha\beta}A_\mu(u)\dot{A}_\nu(u)\ddot{A}_\alpha(u)x_\beta + C(u)\big)^2\big]^{-1} + f_2(u),\\
(2) \quad & u \to f_1(u)\big[\big(-\dot{A}_\nu(u)\dot{A}^\nu(u)\big)^{1/2}\big(\dot{A}_\mu(u)x^\mu + \dot{B}(u)\big)\big]^{-1} + f_2(u),\\
(3) \quad & x_0 - x_3 u \to f_1(x_0 - x_3)\ln\big[\big(x_1 + C_1(x_0 - x_3)\big)^2 + \\
& \quad + \big(x_2 + C_2(x_0 - x_3)\big)^2\big]^{-1/2} + f_2(x_0 - x_3),\\
(4) \quad & u \to \big(-\dot{A}_\nu(u)\dot{A}^\nu(u)\big)^{-3/2}\big(\varepsilon^{\mu\nu\alpha\beta}A_\mu(u)\dot{A}_\nu(u)\ddot{A}_\alpha(u)x_\beta + C(u)\big),\\
(5) \quad & x_0 - x_3 \to f_1(x_0 - x_3)(x_1\cos C_1(x_0 - x_3) + \\
& \quad + x_2 \sin C_1(x_0 - x_3) + C_2(x_0 - x_3).
\end{aligned}$$

Note that in the cases 4, 5 function f_2 is absorbed by arbitrary functions C, C_2.

And one more remark seems to be noteworthy. If one takes as a particular solution of the system (2) the function $u(x) = (x_0 x_1 \pm x_2\sqrt{x_1^2 + x_2^2 - x_0^2})/(x_1^2 + x_2^2)$ and substitutes it into the first, second and fourth Ansätze from (70), then the following Ansätze are obtained:

$$(1) \quad w(x) = \varphi\left(x_1^2 + x_2^2 + x_3^2 - x_0^2, \frac{x_0x_1 \pm x_2\sqrt{x_1^2 + x_2^2 - x_0^2}}{x_1^2 + x_2^2}\right),$$

$$(2) \quad w(x) = \varphi\left(x_1^2 + x_2^2 - x_0^2, \frac{x_0x_1 \pm x_2\sqrt{x_1^2 + x_2^2 - x_0^2}}{x_1^2 + x_2^2}\right),$$

$$(4) \quad w(x) = \varphi\left(x_3, \frac{x_0x_1 \pm x_2\sqrt{x_1^2 + x_2^2 - x_0^2}}{x_1^2 + x_2^2}\right).$$

Provided the above Ansätze do not depend on the second argument, the usual Lie Ansätze are obtained which are invariant under some subgroups of the Poincaré group $P(1,3)$ [19]. Consequently, if we imagine invariant solutions as dots in a solution space of the nonlinear d'Alembert equation, then through some of them one can draw curves which are conditionally-invariant solutions. In this respect a number of interesting questions arise, let us mention two of these:

(1) Is any invariant solution of the nonlinear d'Alembert equation (33) a particular case of some more general conditionally-invariant solution?

(2) Does there exist such conditionally-invariant solution of Eq. (33) that all invariant solutions of Eq. (33) are its particular cases? (saying about invariant solutions we mean solutions invariant under some subgroup of the symmetry group of Eq. (33)).

An answer to the first question seems to be positive. A positive answer to the second one would provide us with a concept of a "general invariant solution". But so far this problem is completely open and needs further investigation.

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1. Kaluza Th., *Sitzungsber. Preuss. Akad. Wiss. Phys. Math. Kl.*, 1921, **996**.
2. Khodos A., *Uspekhi Fiz. Nauk*, 1985, **146**, 647.
3. Fushchych W., Krivsky I., *Nuclear Physics*, 1968, **17**, 79.
4. Fushchych W., Krivsky I., *Nuclear Physics*, 1969, **18**, 573.
5. Fushchych W., *Theor. Math. Phys.*, 1970, **4**, 360.
6. Fushchych W., Nikitin A., *J. Phys. A: Math. Gen.*, 1980, **13**, 2319.
7. Cieciura G., Grundland A., *J. Math. Phys.*, 1984, **25**, 3460.
8. Fushchych W., Zhdanov R., Yegorchenko I., *J. Math. Anal. Appl.*, 1991, **161**, 352.
9. Smirnov V., Sobolev S., *Trudy Seismol. Instituta AN SSSR*, 1932, **20**, 37.
10. Smirnov V., Sobolev S., *Trudy Seismol. Instituta AN SSSR*, 1933, **29**, 43.
11. Yerugin N.P., *Doklady AN SSSR*, 1944, **20**, 385.
12. Collins C.B., *J. Math. Phys.*, 1983, **24**, 22.
13. Bateman H., The mathematical analysis of electrical and optical wave motion of the basis of Maxwell's equations, Cambridge, Univ. Press, 1915.
14. Shulga M., in Group-Theoretical Studies of the Mathematical Physics Equations, Kiev, Inst. of Mathematics of Ukrain. Acad. of Sci., Kiev, 1985, 36.

15. Fushchych W., Zhdanov R., Revenko I., *Ukr. Math. J.*, 1991, **43**, 1364.
16. Fushchych W., Shtelen W., Serov M., Symmetry analysis and exact solutions of equations of nonlinear mathematical physics, Dordrecht, Kluwer Academic Publisher, 1992.
17. Fushchych W., Zhdanov R., *Phys. Reports*, 1989, **172** 123.
18. Collins C.B., *Math. Proc. Cambr. Phyl. Soc.*, 1976, **80**, 165.
19. Patera J., Sharp R.T., Winternitz P., Zassenhaus H., *J. Math. Phys.*, 1976, **17**, 977.
20. Fushchych W., *Ukr. Math. J.*, 1991, **43**, 1456.
21. Fushchych W., Zhdanov R., *Ukr. Math. J.*, 1992, **44**, 970.
22. Bluman G., Cole J., *J. Math. Mech.*, 1969, **18**, 1025.
23. Olver P., Rosenau Ph., *Phys. Lett. A*, 1987, **112**, 107.
24. Levi D., Winternitz P., *J. Phys. A: Math. Gen.*, 1989, **22**, 2922.
25. Clarkson P., Kruskal M., *J. Math. Phys.*, 1989, **30**, 2201.
26. Kamke E., Differentialgleichungen Lösungmethoden und Lösungen, Leipzig, Akademische Verlagsgesellschaft, 1976.
27. Winternitz P., Grundland A., Tuszyński J., *J. Math. Phys.*, 1987, **28**, 2194.
28. Fushchych W., Zhdanov R., Revenko I., in Symmetry Analysis of Equations of Mathematical Physics, Kiev, Inst. of Mathematics of Ukrain. Acad. of Sci., 1992, 68.

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