1 Introduction and tools

Scattering of acoustic or electromagnetic waves plays an important role in many fields of applied sciences. Acoustic and electromagnetic waves are used and investigated in such different areas as medical imaging, ultrasound tomography, material science, nondestructive testing, radar, remote sensing, aeronautics, and seismic exploration.

In the last twenty years the development of computational power has also had a strong impact on the classical fields of direct and inverse scattering. The computational simulation of scattering processes has become accessible using microcomputers and the field of inverse scattering problems arose. Inverse scattering is concerned with the reconstruction of scattering objects or their properties. It grew from its early beginnings in the middle of the century to a large and fast-developing area of applied mathematics and engineering.

In the first part of Chapter 1 we give a brief introduction into inverse scattering theory and outline main results. In the second part, we collect definitions and tools from functional analysis and the theory of integral equations, which are the basis for the further chapters.

1.1 A survey on inverse scattering theory

Scattering by obstacles and media. The classical area of acoustic and electromagnetic scattering is concerned mainly with two different problems, which are studied and applied in many different settings and applications.

The first problem is the scattering of acoustic or electromagnetic waves by an impenetrable scatterer, i.e., the waves do not significantly penetrate into the interior of the scattering obstacle D. In this case the scattering process is determined by the shape of D and and boundary conditions.

The second problem consists in the scattering of acoustic or electromagnetic waves by a penetrable scatterer, where the waves penetrate the obstacle and the interior structure of the obstacle strongly influences the scattering process. If the scatterer is homogeneous, the second problem leads to transmission problems; however if the scatterer is inhomogeneous, we speak of scattering by an inhomogeneous medium.

There exist two main approaches to the simulation of scattering processes. The first treats the problems in the time domain, the second uses a Fourier transform to map the differential equations into the frequency domain and investigate time-harmonic waves. We will follow the second approach.

We will use the letter $\mathcal{D}$ to denote the full scatterer with its physical properties and D to denote the interior of the support of the scatterer in

$I\!R^m$, $m = 2, 3$. We will always assume that the scatterer is bounded.

Mathematically the behavior of a time-harmonic acoustic wave

$$u(x)e^{-i\omega t}$$

in a homogeneous background medium is governed by the Helmholtz equation

$$\Delta u + \kappa^2 u = 0, \tag{1.1.1}$$

where $\kappa = \omega/c_0 > 0$ is the wave number of the acoustic wave, w its frequency and c_0 the speed of sound. For scattering of an incident wave u^i by an impenetrable scatterer $\mathcal{D}$ a mathematical model also needs to take into account the behavior of the total field

$$u = u^i + u^s \tag{1.1.2}$$

at the boundary ∂D of the scatterer. Here u^s denotes the scattered acoustic field. Different boundary conditions are used to model the underlying physical behavior. For a sound-soft scatterer the total field vanishes at the boundary, which leads to the Dirichlet boundary condition

$$u(x) \ = \ 0, \ \ x \in \partial D. \tag{1.1.3}$$

For a sound-hard scatterer the Neumann boundary condition

$$\frac{\partial u}{\partial \nu}(x) \ = \ 0, \ \ x \in \partial D, \tag{1.1.4}$$

is used, where ν denotes the exterior unit normal vector to the boundary ∂D. The model is completed by the Sommerfeld radiation condition

$$\lim_{r\to\infty} r^{\frac{m-1}{2}} \left(\frac{\partial u^s(x)}{\partial r} - i\kappa u^s(x) \right) \ = \ 0, \ \ r = |x|, \tag{1.1.5}$$

uniformly for all directions $x/|x|$ for the scattered field u^s. It physically implies that energy is transported to infinity and it is an important ingredient in ensuring that the physical correct solution of the scattering problem is selected.

To model scattering by inhomogeneous media the pairs of equations (1.1.1), (1.1.3) or (1.1.1), (1.1.4) are replaced by

$$\Delta u + \kappa^2 n(x) u = 0 \tag{1.1.6}$$

in the whole space or $I\!R^m \setminus \partial D$. Here

$$n(x) := \frac{c_0^2}{c(x)^2} + i\sigma(x)$$

is the refractive index. The function $c(x)$ is the sound speed in the inhomogeneous medium, the constant c_0 is the sound speed in the homogeneous host medium and the term $\sigma(x) \geq 0$ models absorption. We assume that $c(x) = c_0$ and $\sigma(x) = 0$ outside of the inhomogeneous region D.

For scattering of electromagnetic waves in $I\!R^3$ the corresponding governing equations are the time-harmonic Maxwell equations

$$\operatorname{curl} E - i\kappa H = 0, \qquad \operatorname{curl} H + i\kappa E = 0 \tag{1.1.7}$$

for the electric field E and the magnetic field H in a homogeneous medium, where $\kappa := \omega\sqrt{\epsilon_0\mu_0}$ is the wave number and ω the frequency of the time-harmonic wave, ϵ_0 the electric permittivity and μ_0 the magnetic permeability of the host medium. For scattering of an incident electromagnetic field E^i, H^i by a perfect conductor $\mathcal{D}$ the boundary condition

$$\nu(x) \times E(x) \;=\; 0, \;\; x \in \partial D, \tag{1.1.8}$$

on the total field

$$E := E^i + E^s \tag{1.1.9}$$

models the behavior of the electric field at the boundary ∂D of D. The tangential components of the electric field E vanish at the boundary ∂D of the perfect conductor $\mathcal{D}$. The appropriate radiation condition is the Silver-Müller radiation condition

$$\lim_{r\to\infty} \left(H^s \times x - rE^s\right) \;=\; 0 \tag{1.1.10}$$

for the scattered electromagnetic field E^s, H^s.

To describe scattering of electromagnetic waves by an inhomogeneous medium the equations (1.1.7) and (1.1.8) are replaced by

$$\operatorname{curl} E - i\kappa H = 0, \qquad \operatorname{curl} H + i\kappa n(x) E = 0, \tag{1.1.11}$$

where the refractive index

$$n(x) := \frac{1}{\epsilon_0}\left(\epsilon(x) + i\frac{\sigma(x)}{\omega}\right)$$

is defined in terms of the permittivity $\epsilon(x)$ of the inhomogeneous medium, the permittivity ϵ_0 of the homogeneous background medium, the conductivity $\sigma(x)$ and the frequency ω of the wave. The magnetic permeability μ is assumed to be constant. We will also investigate anisotropic problems, where n is replaced by a matrix N.

The radiation conditions (1.1.5) or (1.1.10) together with the governing equations (1.1.1) or (1.1.7) imply the behavior

$$u^s(x) \;=\; \frac{e^{i\kappa r}}{r^{\frac{m-1}{2}}}\left\{u^\infty(\hat{x}) \;+\; O\left(\frac{1}{r}\right)\right\}, \;\; r = |x| \to \infty, \tag{1.1.12}$$

where $\hat{x} := x/|x|$ and

$$E^s(x) \;=\; \frac{e^{i\kappa r}}{r}\;\left\{E^\infty(\hat{x}) \;+\; O\left(\frac{1}{r}\right)\right\},\;\; r = |x| \to \infty, \qquad (1.1.13)$$

of the scattered acoustic or electric field $u^s(x)$ or $E^s(x)$, respectively. Here u^∞ and E^∞ are known as the acoustic and electromagnetic far field pattern or scattering amplitude. In the acoustic case the far field pattern is a scalar function defined on the unit sphere or unit circle Ω, respectively. In the electromagnetic case the far field pattern is a tangential vector field on Ω.

Direct and inverse scattering problems. For a direct scattering problem the scatterer and the incident field are assumed to be given. The problem is to compute the scattered field or the far field pattern, respectively. Direct scattering problems have been studied for a long time and a number of different approaches for their solution have been developed (see [3], [7], [38], [57], [58], [62], [66], [80], [83], [90]). However, it is still an important problem of current research to develop efficient algorithms for the numerical computation of the scattered field, especially in three dimensions.

A whole range of different inverse problems are of interest in this framework. Given the far field pattern for scattering of plane waves we may try to reconstruct special properties of $\mathcal{D}$ or the full scatterer with all its properties. Different settings for the measurements lead to a variety of practically relevant mathematical and algorithmical problems.

In this work we will focus on the reconstruction of the shape D of the scatterer $\mathcal{D}$, i.e., the scattering domain for obstacle scattering and the support of the inhomogeneity for scattering by an inhomogeneous medium. For a wide range of applications it is not necessary to reconstruct the full behavior and actual values of the refractive index $n(x)$, but it is sufficient to approximately determine the support $\overline{D} := \mathrm{supp}(n-1)$. For example, in nondestructive testing often this information is all that is needed.

As scattering data we will suppose that the far field pattern $u^\infty(\cdot, d)$ or $E^\infty(\cdot, d, p)$ is measured. These notations denote, respectively, the acoustic or electromagnetic far field patterns of incident plane waves, the incident fields given by

$$u^i(x,d) := e^{i\kappa x\cdot d},\;\; x \in I\!R^m, \qquad (1.1.14)$$

in the acoustic case or

$$\begin{array}{rcl} E^i(x) &=& i\kappa(d\times p)\times d\, e^{i\kappa\cdot d} \\ H^i(x) &=& i\kappa d\times p\, e^{i\kappa x\cdot d} \end{array} \quad , x \in I\!R^3, \qquad (1.1.15)$$

in the electromagnetic case, where $d \in \Omega$ denotes the direction of incidence and $p \in \Omega$ the polarization of the electromagnetic wave. We will investigate the cases where the far field pattern is given for one, a finite number or all incident plane wave directions, with the far field pattern measured either on the whole unit sphere Ω or at a given finite number n of observation points $\hat{x}_j \in \Omega$, $j = 1, ..., n$.

Basic mathematical problems. The investigation of an inverse problem starts with several basic mathematical questions, which are strongly related to the inverse nature of the task.

1. *Uniqueness.* It has to be asked which data sets uniquely determine the object. Without at least local uniqueness there is no possibility to reconstruct unknown objects by stable algorithms.

2. *Existence.* One can ask whether a solution of the inverse problem exists for a given data set. Errors in the measurements may lead to data for which there is no solution to the inverse problem.

3. *Stability.* Because of errors in the measurements or in the numerical storage of data we need to investigate stability for the reconstruction of D from the given far field data. Mathematically this is the problem of continuity of the nonlinear inverse operator under appropriate assumptions.

4. *Algorithms.* Efficient and stable reconstruction algorithms need to be developed. This leads us to the numerical and algorithmical analysis.

We will now place important results in a historical context and give a brief introduction to the contributions to the above questions which are discussed later in the book in more detail. This includes a sketch of related approaches and a description of differences and similarities.

The ill-posedness of inverse scattering problems. First, consider one of the main features of inverse scattering problems. Let B denote a ball with fixed radius R_e around the origin. Given the *a priori* information that B contains the scatterer $\mathcal{D}$ in its interior, by

$$\mathcal{F} : u^s|_{\partial B} \mapsto u^\infty \tag{1.1.16}$$

we denote the operator which maps the scattered field $u^s(x)$, $x \in \partial B$, onto its far field pattern $u^\infty(\hat{x})$, $\hat{x} \in \Omega$. Computing $\mathcal{F}$ explicitly, it can be seen to be compact in any reasonable function space, for example from $C(\partial B)$ into $L^2(\Omega)$ (see Chapter 2 for further details). Thus, by functional analytic arguments, the range of the operator $\mathcal{F}$ cannot be the whole space and the inverse $\mathcal{F}^{-1}$ of the operator $\mathcal{F}$ cannot be bounded. This indicates that the inverse scattering problem is an ill-posed problem in the sense of

Hadamard [22], i.e., that in this case the demands of existence and stability are violated.

The problem of existence for given measured data. Consider the problem of existence. We already pointed out that, in general, we cannot expect existence, i.e., we cannot expect that a far field pattern, which has been polluted by measurement error, will any longer be the exact far field pattern corresponding to a scattering problem. The most we can expect is the denseness or completeness of the far field patterns for a given set of incident waves or of scatterers $\mathcal{D}$ in some measurement space, for example in $L^2(\Omega)$. Completeness of the set of far field patterns for the above acoustic and electromagnetic scattering problems for a set of incident plane waves $\{u^i(\cdot, d_n) : n \in I\!N\}$, where $\{d_n : n \in I\!N\}$ is dense in the unit sphere Ω, has been investigated in detail by Colton, Kirsch, Kress, Blöhbaum and Päivärinta between 1984 and 1990 (see [8] for further references). Necessary conditions for a function in $L^2(\Omega)$ to be a far field pattern can be given in terms of the decay of its Fourier coefficients with respect to spherical harmonics (see [8], Theorem 2.16). This can be obtained by an expansion of the scattered field outside of the ball B. Further necessary conditions have been given by Müller [63] (see also Colton and Kress [7]) using the theory of entire functions of exponential type. More recently Kirsch ([41], [42]) obtained a characterization of the set of far field patterns for a given scatterer in terms of its series representation with respect to the eigenfunctions of the corresponding far field operator

$$(F\varphi)(\hat{x}) := \int_\Omega u^\infty(\hat{x}, d)\, \varphi(d) ds(d), \quad \hat{x} \in \Omega. \tag{1.1.17}$$

We will describe these results in Section 7.2. A corresponding method for the reconstruction of the support D of the scatterer $\mathcal{D}$ is described below. To the author's knowledge no general characterization of the set of far field patterns for arbitrary scatterers $\mathcal{D}$ is known.

In this work, we will assume that the given data are either the exact far field data u^∞ for scattering by a scatterer $\mathcal{D}$ or some measured data

$$u_\delta^\infty(\cdot) \in L^2(\Omega) \quad \text{or} \quad u_\delta^\infty(\cdot,\cdot) \in L^2(\Omega \times \Omega)$$

with

$$||u^\infty(\cdot) - u_\delta^\infty(\cdot)||_{L^2(\Omega)} \le \delta \quad \text{or} \quad ||u^\infty(\cdot,\cdot) - u_\delta^\infty(\cdot,\cdot)||_{L^2(\Omega\times\Omega)} \le \delta. \tag{1.1.18}$$

We will also study the finite data case, where a finite number of measured data $u^\infty_{(n_o,n_i),\delta} \in L^2(\Omega_{n_o} \times \Omega_{n_i})$ are given with

$$\left(\frac{c_m}{n_o} \frac{c_m}{n_i} \sum_{\hat{x} \in \Omega_{n_o}} \sum_{d \in \Omega_{n_i}} \left| u^\infty(\hat{x}, d) - u^\infty_{(n_o,n_i),\delta}(\hat{x}, d) \right|^2 \right)^{\frac{1}{2}} \leq \delta \qquad (1.1.19)$$

for $\delta \geq 0$ with some constant c_m. Here we assume $(\Omega_n)_{n \in \mathbb{N}}$ to be a sequence of finite subsets Ω_n of Ω, such that Ω_n consists of n elements and for given ϵ we can find n such that the distance

$$d(\hat{x}, \Omega_n) := \inf_{d \in \Omega_n} |\hat{x} - d|$$

is smaller than ϵ for all $\hat{x} \in \Omega$. The left-hand side of (1.1.19) defines a norm $|| \cdot ||_{L^2(\Omega_{n_o} \times \Omega_{n_i})}$. The positive real number δ in (1.1.18) and (1.1.19) is referred to as the data error.

Uniqueness results for reconstructions. The origin of uniqueness results for inverse obstacle scattering problems can be found in the works of Rellich in the 40's. He proved that the far field pattern uniquely determines the (analytic) scattered field in the exterior of the scatterer $\mathcal{D}$ (which result we refer to as Rellich's Lemma). Then, Schiffer (see [57]) showed for the inverse acoustic obstacle scattering problem with Dirichlet boundary condition, that the far field pattern $u^\infty(\hat{x}, d)$, $\hat{x}, d \in \Omega$, for all incident plane waves and for one fixed wave number κ uniquely determines the domain of the scatterer. The corresponding result for the reconstruction of the acoustic refractive index in three dimensions was obtained by Nachman [65], Novikov [67] and Ramm [81], and considerably simplified by Hähner [23], (see also [24]), in 1996. Analogous results for the electromagnetic problems were first obtained by Colton and Päivärinta [11] in 1990, Colton and Kress [8] in 1993, and by Ola, Päivärinta and Somersalo [68] in 1993.

In 1983, Colton and Sleeman [15] investigated the case where the sound-soft scatterer is known to be a subset of a ball with given radius R_e. They showed that the support is determined by a finite number N of incident plane waves depending on R_e. If R_e is small enough, one wave is sufficient to determine the scatterer.

So far it has not been possible to extend Schiffer's approach or the ideas of Colton and Sleeman to the sound-hard scatterer or to the case of an inhomogeneous medium. In 1992, Isakov [34] obtained uniqueness results for penetrable obstacles using different techniques, which were simplified and applied to impenetrable sound-soft and sound-hard scatterers by Kirsch and Kress [47] in 1993. The results could also be successfully transferred to the case of electromagnetic obstacle scattering [8]. Since these ideas will be the starting point of a large part of this work (with contributions to uniqueness, stability, the finite data problem, and reconstruction algorithms), we briefly want to describe the main ingredients.

Consider the scattered acoustic field $\Phi^s(\cdot, z)$ for scattering of a point-source $\Phi(\cdot, z)$ with source point $z \in \mathbb{R}^m \setminus \overline{D}$, where Φ is the standard fundamental solution of the Helmholtz equation in dimension m, $m = 2, 3$. From the sound-soft boundary condition and the singularity of the incident point-source we derive that for a point $x \in \partial D$ we have

$$\Phi^s(x, z) \to \infty, \quad z \to x. \tag{1.1.20}$$

Kirsch and Kress used (1.1.20) to show that if the far field patterns of two scatterers $\mathcal{D}_1$ and $\mathcal{D}_2$ for scattering of plane waves coincide for all directions of incidence $d \in \Omega$, then the domains D_1 and D_2 are the same.

In Chapter 3 we will further develop the techniques of Kirsch and Kress to derive uniqueness of the support D of an inhomogeneous medium n, if n has a jump in one of its derivatives at the boundary of the scatterer $\mathcal{D}$. For the three-dimensional case this also can be obtained from the results of Nachman [65], Novikov [67], and Ramm [81] for the acoustic and from Colton and Päivärinta [12] in the electromagnetic case. In two dimensions, Sun and Uhlmann [86] proved uniqueness of the support of n, if n has a jump at the boundary. They use Fourier techniques, which are different from our approach. Our techniques will be constructive and allow the development of reconstruction methods in Chapter 6.

ϵ-Uniqueness for finite data. Uniqueness results for inverse scattering problems usually assume the full far field pattern on the unit sphere to be given. Often it is assumed that the far field pattern is known for all or a full open set of directions $d \in \Omega$ of the incident plane waves. Since most proofs use Rellich's lemma, the knowledge of the far field pattern at least in an open subset of Ω seems to be necessary to uniquely determine the scattered field u^s.

From a practical perspective it is reasonable to ask the question: What can be said if the far field pattern is given only at a finite number of measurement points and for a finite number of waves? In Chapter 3, we develop a technique to answer this question avoiding the use of Rellich's uniqueness results. It leads to a relaxed concept of uniqueness, a preliminary version of which was first proposed in 1998 (see [73]). We will prove ϵ-uniqueness for the reconstruction of the shape of a scatterer, i.e., that given $\epsilon > 0$ there are $n_o, n_i \in \mathbb{N}$ such that, if for two scatterers $\mathcal{D}_1$ and $\mathcal{D}_2$ the far field patterns for all n_i directions of incidence $d \in \Omega_{n_i}$ coincide at the n_o observations points $\hat{x} \in \Omega_{n_o}$, the Hausdorff distance

$$d(D_1, D_2) := \max\left\{\sup_{x \in D_1} d(x, D_2), \sup_{x \in D_2} d(x, D_1)\right\} \tag{1.1.21}$$

between the scatterers D_1 and D_2 satisfies the estimate

$$d(D_1, D_2) < \epsilon.$$

Since the concept of ϵ-uniqueness is close to stability, let us postpone further considerations until after discussing the stability question.

Stability estimates. We have already pointed out that inverse scattering problems are ill-posed problems, i.e., that for the inverse of the nonlinear scattering operator in general we do not have stability. There exist two main approaches to restore stability results for this ill-posed problem.

The first approach consists of a modification of the norm used for the far field pattern. In inverse scattering it was used for example in 1990 by Stefanov [85] to study stability for inverse scattering by a medium and has been extended by Hähner [24] in 1998 to electromagnetic and elastic wave scattering. In principle they consider a space X of functions on Ω with a very strong norm $||\cdot||_X$ involving all derivatives of functions $\varphi \in C^\infty(\Omega)$, such that the inverse $\mathcal{F}^{-1} : X \to C(\partial B)$ of (1.1.16) becomes a bounded operator. More recently these ideas have been extended by Hähner and Hohage to prove logarithmic stability estimates for the reconstruction of the refractive index with respect to L^2-norms in the space of the far field patterns, see [25]. But note that so far these results do not include the stability estimates for the support of inhomogeneous medium scatterers as discussed below.

Another approach is the use of *a priori* bounds on the set of fields or objects to be determined (see for example [37]). This approach has been applied to scattering theory by Isakov [34], [35] (see also [36]). We will use the well known fact that the inverse of a continuous mapping is continuous if it is defined on a compact subset of a Banach space. Thus with appropriate restrictions on the set of scatterers, stability can be restored and stability estimates obtained.

Isakov's restrictions mainly consist of a uniform bound on the $C^{2,\alpha}$-norm of all boundaries in a special parametrization. For the reconstruction of the shape of a sound-soft scatterer from the knowledge of the far field pattern for one incident wave, Isakov derives a double-logarithmic estimate

$$d(D_1, D_2) \ < \ C\Big(\ln \Big| \ln ||u_1^\infty(\cdot) - u_2^\infty(\cdot)||_{C(\Omega)} \Big| \Big)^{-\gamma} \tag{1.1.22}$$

for the Hausdorff distance $d(D_1, D_2)$ of the domains D_1, D_2 with positive constants C, γ depending on a bound for the $C^{2,\alpha}$-norm of the boundary. So far Isakov's techniques could not be used to treat other boundary conditions or electromagnetic scattering problems.

In Chapter 3 we will pursue the idea of imposing appropriate restrictions on the set of scatterers under consideration. With techniques different from Isakov we will be able to derive stability estimates for the reconstruction of the shape of either a penetrable or impenetrable scatterer from the knowledge of the far field patterns for all incident plane waves. More explicitly we prove an estimate of the form

$$d(D_1, D_2) \ \leq \ F\Big(||u_1^{\infty}(\cdot,\cdot) - u_2^{\infty}(\cdot,\cdot)||_{L^2(\Omega)\times L^2(\Omega)} \Big), \tag{1.1.23}$$

where F is a function with the property

$$F(\delta) \to 0, \ \ \delta \to 0, \tag{1.1.24}$$

which can be computed according to some *a priori* knowledge on the class of scatterers. For the convex hulls $\mathcal{H}(D)$ of the shape D of scatterers $\mathcal{D}$ we derive a logarithmic estimate

$$d\Big(\mathcal{H}(D_1), \mathcal{H}(D_2)\Big) \ \leq \ C \left| \ln ||u_1^{\infty}(\cdot,\cdot) - u_2^{\infty}(\cdot,\cdot)||_{L^2(\Omega)\times L^2(\Omega)} \right|^{-\gamma} \tag{1.1.25}$$

with constants C and γ.

Our technique is inspired by the uniqueness proof of Isakov, Kirsch and Kress. The first step is the explicit estimation of the behavior of $\Phi^s(z,z)$ for $z \to \partial D$. As a second step we develop a method for the approximate reconstruction of $\Phi^s(z,z)$ in the exterior of the domain

$$D_\rho := \{ y \in I\!\!R^m : d(y, D) < \rho \} \tag{1.1.26}$$

with some small parameter $\rho > 0$ from the scattering data $u^{\infty}(\hat{x}, d)$ for all $\hat{x}, d$ in Ω. Estimating the bound of the approximate reconstruction operator $Q : L^2(\Omega) \times L^2(\Omega) \to C(B \setminus D_\rho)$ and using the fact that $\Phi^s(z,z)$ is large only in a neighborhood of the boundary, stability estimates will be obtained in Chapter 3. For the acoustic sound-soft and sound-hard scatterer the results can be found in [70]. Similar stability results for the reconstruction of the support of media and for electromagnetic scattering problems will be derived in Chapter 3.

At this point we would like to relate these results to some demands on the degree of ill-posedness of an inverse problem formulated by Fritz John [37] in 1960. For purposes of computation John demands Hölder continuous dependence of a problem on the data. Here, for the reconstruction of the domains, we obtained logarithmic continuity, which is a typical type of estimate for continuing solutions of the wave equation in space-like directions. As shown in [70], for the reconstruction of the scattered field u^s on fixed compact subsets U of the open exterior of the convex hull $\mathcal{H}(D)$ of D

$$\Big|\Big| u_1^{\infty} - u_2^{\infty} \Big|\Big|_{L^2(\Omega)} \ \leq \ \delta$$

yields the estimate

$$\Big|u_1^s(x) - u_2^s(x)\Big| \le \alpha\delta^{\frac{\beta}{\lceil \ln(-\gamma \ln(\delta))\rceil}}, \ \ x \in U,$$

with constants $\alpha, \beta, \gamma > 0$. This can be proven with the same techniques which we will use in Chapter 3. These estimates come close to Hölder continuity demanded by John and are reflected by the numerical results of Chapters 5 and 6.

Stability for the case of finite data: ϵ-stability. With the help of the stability estimates it is not difficult to derive related statements for the case of finite data.

We will work with the same assumptions as for stability or ϵ-uniqueness, i.e., a uniform bound on the C^2-norm of the boundaries (and corresponding assumptions on the uniform smoothness of the refractive index n), to derive a uniform bound for the far field patterns in $C^1(\Omega \times \Omega)$. This bound can be used to relate the distance of two far field patterns at a finite number of points to the distance of the full far field patterns. Then for the case of finitely many measurements we derive ϵ-uniqueness and a modified stability statement, which we will refer to as ϵ-stability.

Consider a simple example for the derivation of ϵ-uniqueness. Given $\epsilon > 0$ we can use (1.1.25) to obtain a $\delta > 0$ such that

$$||u_1^\infty(\cdot,\cdot) - u_2^\infty(\cdot,\cdot)||_{L^2(\Omega)\times L^2(\Omega)} \le \delta \tag{1.1.27}$$

implies

$$d\Big(\mathcal{H}(D_1), \mathcal{H}(D_2)\Big) \le \epsilon. \tag{1.1.28}$$

Now given δ, we choose $n \in I\!N$ sufficiently large such that with the help of the bound on $||u_j^\infty(\cdot,\cdot)||_{C^1(\Omega\times\Omega)}$ for $j = 1, 2$, the equation

$$u_1^\infty(\hat{x}, d) = u^\infty(\hat{x}, d) \text{ for all } \hat{x}, d \in \Omega_n \tag{1.1.29}$$

yields (1.1.27). Thus given ϵ we may choose $n \in I\!N$ such that (1.1.29) implies (1.1.28), i.e., we have proven the statement of ϵ-uniqueness for the convex hulls of the scattering domains as a simple consequence of stability.

Now, we describe the concept of ϵ-stability. Since in general we do not have uniqueness, for a finite data set we will not be able to obtain stability. More explicitly, we cannot obtain a function $F(\delta)$ which satisfies (1.1.24) and an estimate of the type (1.1.23) when the data error is considered at finitely many points. But for given $\epsilon > 0$ it is possible to find $n_o, n_i \in I\!N$ and a function $F_{(n_o,n_i)}(\delta)$, such that F has the behavior

$$\limsup_{\delta\to 0} F_{(n_o,n_i)}(\delta) \le \epsilon \tag{1.1.30}$$

and the domains D_1 and D_2 satisfy the estimate

$$d(D_1, D_2) \leq F_{(n_o,n_i)}\Big(||u^\infty_{1,(n_o,n_i)} - u^\infty_{2,(n_o,n_i)}||_{L^2(\Omega_{n_o}\times\Omega_{n_i})}\Big) \qquad (1.1.31)$$

with $||\cdot||_{L^2(\Omega_{n_o}\times\Omega_{n_i})}$ given by (1.1.19) and

$$u^\infty_{j,(n_0,n_i)} := u^\infty_j\Big|_{\Omega_{n_o}\times\Omega_{n_i}} \in L^2(\Omega_{n_o}\times\Omega_{n_i})$$

for $j = 1, 2$. We call a statement of this form ϵ-stability. We will prove ϵ-stability for the reconstruction of the shape of the scatterer for the scattering problems described above and explicitly study the behavior of the function $F_{(n_o,n_i)}(\delta)$ for the reconstruction of the convex hull of a scatterer.

Three main categories of reconstruction methods. Consider now the problem of the actual reconstruction of the shape of the unknown scatterer. Basically three different types of reconstruction methods have been developed. We will summarize their main features and use them as a background to explain our results.

The first category and Newton's method. The first approach to consider is the inverse problem as a nonlinear ill-posed operator equation and adapt iterative methods of gradient- or Newton-type to solve this equation. For inverse obstacle scattering this approach mainly relies on the Fréchet differentiability or domain derivative of the scattered field with respect to variations of the boundary of the scatterer and a characterization of the derivative as a corresponding boundary value problem. Results for interior boundary value problems were obtained in 1980 by Simon [84] with the help of the implicit function theorem, for the scattering problems in 1993 by Kirsch [44] using variational methods, and in 1994 by the author [78] by means of integral equations.

For a discussion of the large number of papers on the numerical implementation of these type of methods, (for example Murch, Tan and Wall [64], Roger [82], Tobocman [88], Wang and Chen [89], Kirsch [44] and [45], Kress and Rundell [54], [55] and [56], Kress [50], [52] and [53], Mönch [61], Hohage [30], Hanke, Hettlich and Scherzer [26], and Hettlich [27]) we refer to [28] and [31], see also [8]. In Section 5.4 we will study the convergence of a regularized Newton scheme in inverse acoustic scattering and relate the Newton method to the point-source method, which belongs to the second category of inversion schemes.

The second category and the point-source method. The second category in principle splits the inverse scattering problem into a linear ill-posed part to reconstruct the scattered field in the exterior of the scatterer and a nonlinear well-posed part to find the boundary of the scatterer or the refractive index using the boundary condition or the partial differential equation, respectively.

In Chapter 5, with the point-source method, we describe a method of this second category developed by the author since 1995. Different steps in this development can be found in [74], [75] and [76]. Other examples of methods which belong to the second category are the optimization methods proposed by Colton and Monk in 1985 and by Kirsch and Kress in 1986, both described in [8].

The main aim of the point-source method is the explicit construction of a kernel $g(z,d)$, such that in the domain $B \setminus D_\rho$, where D_ρ is given by (1.1.26) and B by (1.1.16), the scattered field u^s is approximated in the form Au^∞ with a linear integral operator

$$(A\varphi)(z) := \int_\Omega g(z,\hat{x})\varphi(\hat{x})\, ds(\hat{x}), \quad z \in B. \tag{1.1.32}$$

To this end the far field pattern $\Phi^\infty(\cdot,z)$ for incident point-sources $\Phi(\cdot,z)$ with source-point z is considered. Given some *a priori* knowledge on the size of the scatterer, the kernel g will be constructed in the following three steps.

1. An approximation for a point-source by a superposition of plane waves

   $$\Phi(x,z) \approx \int_\Omega e^{i\kappa x\cdot d}\tilde{g}(z,d)\, ds(d), \quad x \in \overline{D},\ z \in B \setminus D_\rho \tag{1.1.33}$$

   is computed.

2. Passing to the far field patterns, an approximation for the far field pattern due to point-sources by a superposition of the far field patterns of plane waves

   $$\Phi^\infty(\hat{x},z) \approx \int_\Omega u^\infty(\hat{x},d)\tilde{g}(z,d)\, ds(d), \quad \hat{x} \in \Omega,\ z \in B \setminus D_\rho \tag{1.1.34}$$

   is obtained.

3. Using the mixed reciprocity relation $\Phi^\infty(\hat{x},z) = \gamma u^s(z,-\hat{x})$, the far field reciprocity relation $u^\infty(\hat{x},d) = u^\infty(-d,-\hat{x})$ and the substitution $d \to -d$, an approximation

   $$u^s(z,-\hat{x}) \approx \int_\Omega u^\infty(d,-\hat{x}) \left\{\frac{1}{\gamma}\tilde{g}(z,-d)\right\} ds(d), \quad \hat{x} \in \Omega,\ z \in B \setminus D_\rho \tag{1.1.35}$$

for the scattered field u^s from its far field pattern u^∞ is derived. Here, $u^s(\cdot, d)$ denotes the scattered field for an incident plane wave with direction of incidence $d \in \Omega$ and $u^\infty(\cdot, d)$ is the corresponding far field pattern.

Using rotations and translations of the approximating functions, the computation of g can be performed efficiently. Given the reconstruction of the scattered field u^s, parts of the unknown boundary ∂D of D can be found using the total field $u = u^i + u^s$ and the boundary condition. For electromagnetic waves a corresponding operator will be constructed (see also [75]).

Note that some similarities exist between the construction procedure of the point-source method and the Backus-Gilbert method or mollifier methods [16], [43], [59].

The third category, the method of singular sources and linear sampling methods. Since 1996 methods for the reconstruction of the shape of a scatterer have been developed, which are based on characterizations of the boundary of the scatterer independent of its physical properties. For an algorithm of category III the boundary condition or physical properties of the scatterer do not need to be known. The independence of a reconstruction method on the physical properties of the scatterers is of great practical importance, since in many cases knowledge about these properties of the objects to be reconstructed is not available. We will introduce two different approaches: the method of singular sources and linear sampling methods.

A linear sampling method was proposed in 1996 by Colton and Kirsch [6]; see also Colton and Monk [10] and Colton, Piana, and Potthast [13] for a formulation of the mathematical basis and numerical realization of the method. The idea is to characterize the boundary ∂D of a scatterer D by the behavior of the solution $g = g(z, \cdot) \in L^2(\Omega)$ of a linear integral equation of the first kind

$$(Fg)(\hat{x}) \;=\; e^{-i\kappa z \cdot \hat{x}}, \;\; \hat{x} \in \Omega, \tag{1.1.36}$$

for $z \in D$, where F is the far field operator

$$(Fg)(\hat{x}) := \int_\Omega u^\infty(\hat{x}, d) g(d)\, ds(d), \;\; \hat{x} \in \Omega. \tag{1.1.37}$$

Examining either an interior boundary value problem or an interior transmission problem, in [13] (see also [8]) it is shown that there exists an approximate solution of (1.1.36) with

$$||g(z, \cdot)||_{L^2(\Omega)} \to \infty \quad \text{and} \; ||v_g(z, \cdot)||_{L^2(D)} \to \infty \quad \text{for} \quad z \to \partial D, \tag{1.1.38}$$

where v_g denotes the Herglotz wave function

$$v_g(x) := \int_\Omega e^{i\kappa x\cdot d} g(d)\, ds(d), \quad x \in I\!R^m. \tag{1.1.39}$$

Colton and Kirsch propose to compute a regularized approximate solution of (1.1.36) on a grid containing D. The domain D then can be found as the set of points where $||g(z,\cdot)||_{L^2(\Omega)}$ is large. Numerical experiments can be found in [5], [6], [10], and [13].

Colton, Piana, and Potthast [13] also applied Morozov's discrepancy principle for the solution of (1.1.36) and used either $||g(z,\cdot)||$ or the values of the regularization parameter $\alpha(z)$ to determine the shape D of the scatterer $\mathcal{D}$. Numerical results using this method can be found in [5].

We will introduce the linear sampling method and present the main results in Chapter 7, studying the method for the reconstruction of impenetrable obstacles and inhomogeneous isotropic, orthotropic, and anisotropic media.

In 1998 and 1999, Kirsch [41], [42] was able to derive a characterization of the shape D of the scatterer $\mathcal{D}$ for acoustic scattering in the case of obstacles or non-absorbing media, i.e., a real-valued refractive index n. Kirsch showed that the domain D is the set of points z, where the equation

$$(F^*F)^{1/4} g(\hat{x}) = e^{-i\kappa z\cdot\hat{x}}, \quad \hat{x} \in \Omega, \tag{1.1.40}$$

is solvable. This characterization holds both for scattering by obstacles [42] or for a non-absorbing medium [41]. The support of the scatterer can then be found as above, computing approximate regularized solutions of (1.1.40) and using either the norm $||g(z,\cdot)||$ of the solution or the size of the regularization parameter $\alpha(z)$, which is chosen according to Morozov's discrepancy principle. A comparison of numerical results for the method of Colton and Kirsch (1.1.36) and the version proposed by Kirsch (1.1.40) can be found in [5]. More recently, Kirsch managed to extend his approach to more general refractive indices by a reformulation into an optimization problem, see [40]. We will introduce the main ideas of Kirsch in Section 7.2 and outline a proof for his result in the case of obstacle scattering.

In Chapter 6, we propose a method of singular sources for the reconstruction of the support of obstacles or scattering media from the knowledge of the far field pattern $u^\infty(\hat{x}, d)$, $\hat{x}, d \in \Omega$. It is based on the ideas used in the uniqueness and stability proofs. For the reconstruction of obstacles in the case of acoustic waves theoretical and numerical results can be found in [70].

We want to sketch the main ideas of this method. As described above, the boundary ∂D of an obstacle D is the set of points where the scattered field $\Phi^s(z,z)$ for incident point-sources becomes singular. We construct a kernel $g_\tau(x,d)g_\eta(z,\tilde{d})$, such that, for $x,z \in B \setminus D_\rho$, an approximation for the field $\Phi^s(x,z)$ is obtained in the form $(Qu^\infty)(x,z)$ with the bounded linear integral operator Q defined by

$$(Q\varphi)(x,z) := \int_\Omega \int_\Omega g_\tau(x,d)g_\eta(z,\tilde{d})\varphi(d,\tilde{d})\, ds(d)ds(\tilde{d}), \quad x,z \in B. \quad (1.1.41)$$

The boundary ∂D is found as the set of points where $(Qu^\infty)(z,z)$ is large. We investigate the method both for the reconstruction of obstacles and the support of media and give numerical examples.

The methods of the second and the third category have important differences at a fundamental level. Methods of the second category use the scattered field and the boundary condition to determine the scatterer. The boundary condition does not need to be known for the reconstructions with methods of the third category. The missing knowledge, as we show and discuss in Chapter 6, leads to a greater ill-posedness of the inverse scattering problem.

Contents. We split our presentation into seven chapters.

- Section 1.1 of this chapter has already been used for a survey about inverse acoustic and electromagnetic scattering theory. Section 1.2 serves to present basic definitions and tools for further use.

- In Chapter 2 we study the solutions to the direct scattering problems and derive properties on which our investigation of the inverse problems will be based. The main results of Chapter 2 will be uniform bounds for integral operators and scattering maps and estimates for the behavior of the scattered field $\Phi^s_{\mu,q}(z,z)$ for incident multipoles of order μ and polarization $q \in \Omega$.

- The themes of Chapter 3 are uniqueness and stability for the reconstruction of the shape of a scatterer. We first derive uniqueness results from the knowledge of the full far field patterns $u^\infty(\hat{x},d)$, $\hat{x},d \in \Omega$. In a second part we develop a technique to derive stability estimates for the reconstruction of the shape of both impenetrable and penetrable scatterers from the knowledge of the far field patterns for all incident plane waves.

- The finite data case is investigated in Chapter 4. We investigate the question of uniqueness if the far field patterns are known only for a finite number of observation points and a finite number of incident plane waves. For this situation we propose a concept which we call ϵ-uniqueness: given ϵ there are numbers n_o, n_i such that the far field patterns for n_i directions of incidence measured at n_o observations directions determine the unknown shape up to an error ϵ in the Hausdorff distance between the domains.

In a second part stability for a finite set of measured data is studied. For this case we propose a concept of ϵ-stability: given ϵ there are numbers n_o, n_i and a function $F_{(n_i,n_o)}$, such that (1.1.30) and (1.1.31) are satisfied.

- In Chapter 5 a point-source method is introduced for the reconstruction of a scattered field u^s from its far field pattern u^∞ and the construction of the shape of an unknown impenetrable scatterer $\mathcal{D}$. We explicitly construct a family of bounded linear integral operators (1.1.32) for the reconstruction of u^s and prove error estimates and convergence to the true scattered field. Numerical examples for reconstructions in two and three dimensions are given. We also relate the point-source method to Newton's method for domain reconstruction and prove convergence of a regularized Newton scheme.

- A method for the reconstruction of the shape of both impenetrable and penetrable scatterers is proposed in Chapter 6. We call it the method of singular sources, since it uses the singular behavior of the scattered fields $\Phi^s_{\mu,q}(z,z)$ of multipoles, if the source point z of the incident multipole $\Phi_{\mu,q}(\cdot,z)$ tends to the boundary of the scatterer. Some numerical examples in two dimensions demonstrate the applicability of the ideas.

- Chapter 7 is dedicated to linear sampling methods. We introduce the method including full proofs for the reconstruction of inhomogeneous isotropic, orthotropic and anisotropic media. In Section 7.2 we also present the spectral theory of the far field operator which was developed by Kirsch to prove convergence of a modified linear sampling method for domain reconstructions in inverse scattering.

1.2 Basic definitions and tools.

In this part we introduce basic definitions, notations, and theorems from analysis, functional analysis, scattering, and potential theory for later use. An introduction to these areas can be found in [8], [29], [43], [51], and [87].

Direct and inverse scattering theory investigates scattering of acoustic, electromagnetic or elastic waves from scatterers in $\mathbb{R}^m$, $m = 2, 3$. We will restrict our attention to bounded scatterers with a sufficiently smooth boundary to introduce and study properties of the solutions and inversion schemes.

We consider our assumptions of smoothness and boundedness as a first and important step towards more complicated and realistic problems. Here, we can study all characteristic problems and difficulties of inverse scattering problems in a well-defined setting. Note that even with rather strong assumptions we cover a wide range of applications.

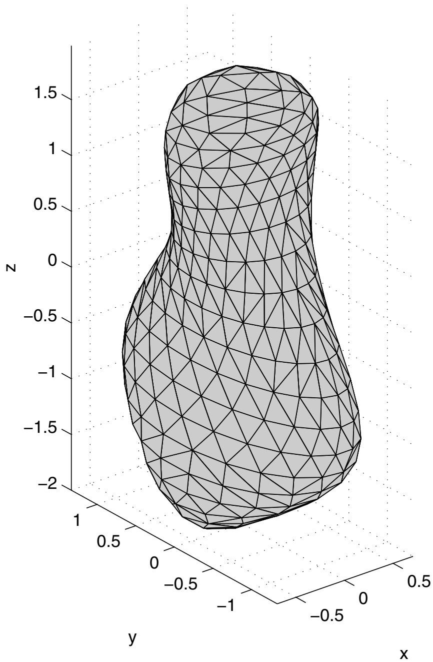

Figure 1.1: Triangulation of scatterer with smooth boundary.

Domains, balls, cylinders.
By D we denote a bounded open set in $\mathbb{R}^m$, $m \in \mathbb{N}$, with boundary ∂D and closure $\overline{D}$, such that the exterior of D in $\mathbb{R}^m$ is connected. $B_r(x)$ is the open ball with radius r and center x in $\mathbb{R}^m$. The lower half plane is given by

$$H := \{x = (x_1, ..., x_m) : x_m \leq 0\}$$

and we define $H_r := B_r(0) \cap H$. Let $Z_{a,r}$ be the open finite cylinder

$$Z_{a,r} := \left\{ x = (x_1, ..., x_m) \in \mathbb{R}^m, \sqrt{x_1^2 + ... + x_{m-1}^2} < r, \ |x_m| < a \right\}. \tag{1.2.1}$$

We use the notation $Z_{a,r}(x, p)$ for the cylinder defined by (1.2.1) in the coordinate system with origin x and the x_m-axis given by $p \in \Omega$.

Function spaces. For the treatment of scattering problems using boundary integral equations we will need to work both with Hölder spaces and with Sobolev spaces. Sobolev spaces provide a Hilbert-space setting which is of advantage for many parts of our analysis, whereas the Hölder-space analysis is a powerful tool for the treatment of boundary integral operators on the boundary of smooth scattering domains.

The spaces of continuous or l-times continuously differentiable functions on D or ∂D are denoted by $C(D)$, $C^l(D)$ or $C(\partial D)$, $C^l(\partial D)$, respectively. The space of l-times Hölder continuously differentiable functions with Hölder coefficient α is $C^{l,\alpha}(D)$ or $C^{l,\alpha}(\partial D)$, respectively. For a multi-index

$$\gamma := (\gamma_1, ..., \gamma_m)$$

we define

$$|\gamma| := \sum_{j=1}^{m} \gamma_j. \tag{1.2.2}$$

The l-th derivatives of a function $f \in C^l(D)$ are given by

$$f^{(\gamma)} := \frac{\partial^{|\gamma|} f}{\partial^{\gamma_1} x_1 ... \partial^{\gamma_m} x_m}$$

for all $\gamma \in I\!N_0^m$ with $|\gamma| = l$. We will need the space $L^2(D)$ of square-integrable functions on D and the Sobolev spaces $H^l(D)$, which are defined as the closure of $C^l(\overline{D})$ with respect to the norm

$$||f||_{H^l(D)} := \sum_{|\gamma| \leq l} ||f^{(\gamma)}||_{L^2(D)}. \tag{1.2.3}$$

Parametrizations of boundaries. Given $m \in I\!N_0$ and $\alpha \in [0,1]$ the boundary ∂D is said to be of class $C^{l,\alpha}$, if for each point $x \in \partial D$ there is an open set $V \in I\!R^m$ with $x \in V$ and a bijective mapping $\psi \in C^{l,\alpha}(B_1(0))$ such that $\psi(B_1(0)) = V$ and $\psi(H_1) = V \cap D$. Since ∂D is compact, we can always find a finite number of such domains V which cover ∂D. A parametrization of $V \cap \partial D$ is given by $\psi|_{U_1}$ with

$$U_1 := \{x \in B_1(0) : x_m = 0\}. \tag{1.2.4}$$

Then $\{U_1, \psi, V \cap \partial D\}$ is called a set of local coordinates for ∂D. For domains of class $C^{l,\alpha}$ with $l \geq 1$ and $\alpha \in [0,1]$ let $\nu(x)$ be the exterior unit normal vector to the boundary ∂D of D at the point $x \in \partial D$. We use special local coordinates for ∂D. For a point $x \in \partial D$ we can find a coordinate system K_x with origin x and the x_m-axis given by $\{x + h\nu(x) : h \in I\!R\}$. In this special coordinate system in a neighborhood $Z_{a,r}$ of 0 with $r, a > 0$, a parametrization of $\partial D \cap Z_{a,r}$ is given by

$$\partial D \cap Z_{a,r} \;=\; \Big\{ \Big(t_1, ..., t_{m-1}, f(t_1, ..., t_{m-1}) \Big) \,:\, (t_1, ..., t_{m-1}) \in B_r(0) \Big\} \tag{1.2.5}$$

with a mapping $f \in C^{l,\alpha}(B_r(0))$ defined on the open set $B_r(0) \subset I\!R^{m-1}$. The function $f \in C^{l,\alpha}(B_r(0))$ is uniquely determined up to rotation and we have

$$D \cap Z_{a,r} = \{ (t_1, ..., t_m) \in Z_{a,r} \,:\, t_m \leq f(t_1, ..., t_{m-1}) \}. \tag{1.2.6}$$

Classes of domains. In Chapters 3 and 4 we will investigate stability of reconstructions and the case of finite data. To this end we need various properties of special functions and integral operators to be valid uniformly for special sets of domains under consideration. Here, we specify classes of domains with mainly three different appropriate conditions. First, we need the domains to be a subset of a fixed large ball (boundedness). Second, we assume that the smoothness of the domain is uniform in the sense that we obtain uniform parametrizations. Third, we need an infinite cone condition which enables us to reach each point of the boundary ∂D of a scatterer by a fixed infinite cone which is a subset of the exterior of D. This cone will be used to obtain uniform reconstructions of the scattered fields and also excludes several bad cases of sequences of domains where different parts of the boundaries converge towards each other.

DEFINITION 1.2.1 **(Classes of domains.)** *Given constants R_e, r_0, a_0, $l \in I\!N$, $\alpha \in [0,1]$, $C_0 > 0$ and $\beta_e > 0$ we define the class*

$$\mathcal{A} = \mathcal{A}(R_e, r_0, a_0, l, \alpha, C_0) \tag{1.2.7}$$

of domains in $I\!R^m$, which satisfy the following conditions:

1. **(Boundedness.)** *For all $D \in \mathcal{A}$ we have*

$$D \subset B_{R_e}(0). \tag{1.2.8}$$

2. **(Uniform smoothness.)** *For each point $x \in \partial D$ and the special coordinate system K_x defined above there exists $r = r_0$, $a = a_0$ and some function $f \in C^{l,\alpha}(B_{r_0}(0))$, such that the equations (1.2.5) and (1.2.6), and the estimate*

$$||f||_{C^{l,\alpha}(B_r(0))} \;\leq\; C_0 \tag{1.2.9}$$

are satisfied. Without loss of generality we will assume that $a_0 \geq 2r_0$.

3. **(Exterior cone condition.)** *For each $x \in I\!R^m \setminus D$ there is a cone*

$$\mathrm{co}(x, p, \beta_e) := \left\{ y \in I\!R^m : \frac{y-x}{|y-x|} \cdot p \geq \cos(\beta_e) \right\} \tag{1.2.10}$$

with direction $p \in \Omega$, and opening angle β_e in the exterior $I\!R^m \setminus D$ of D.

In contrast to the theory of partial differential equations, where cone conditions are used to describe the regularity of the boundary of a domain, here the infinite cone condition is a geometrical condition.

It can be seen as a condition to limit non-convexity and, simultaneously, allow scatterers consisting of several separate components. It guarantees that each point x on the boundary ∂D can be reached by an exterior infinite cone $\mathrm{co}(x, p, \beta_e)$ with a direction $p \in \Omega$ depending on x and a fixed given opening angle β_e. We use the notation $B := B_{R_e}(0)$.

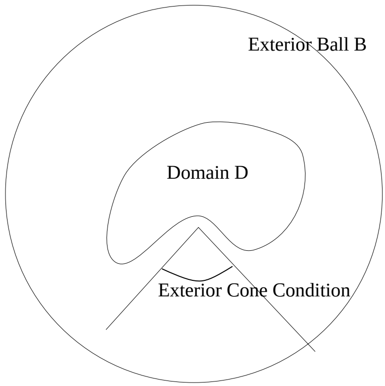

Figure 1.2: Boundedness and exterior cone condition.

To work with the class $\mathcal{A}$ of domains we need to note some of its properties.

THEOREM 1.2.2 *For parameters $R_e, r_0, a_0 > 0$, $l \geq 2$, $\alpha \in [0, 1]$ and $C_0 > 0$ we obtain for the class of domains $\mathcal{A} = \mathcal{A}(R_e, r_0, a_0, l, \alpha, C_0)$ the following properties.*

1. *There is a radius $r_i = r_i(r_0, a_0, C_0)$, such that for each domain $D \in \mathcal{A}$ and each point $x \in \partial D$ we have*

$$B_{r_i}(x + \nu(x) r_i) \subset I\!R^m \setminus \overline{D} \tag{1.2.11}$$

and

$$B_{r_i}(x - \nu(x) r_i) \subset D. \tag{1.2.12}$$

2. *Each domain $D \in \mathcal{A}$ is a union of balls with radius r_i.*

3. *For each domain $D \in \mathcal{A}$ and each point $x \in \partial D$ we have*

$$x + \nu(x)s \notin D, \quad s \in [0, r_i] \tag{1.2.13}$$

and

$$x - \nu(x)s \in D, \quad s \in (0, r_i]. \tag{1.2.14}$$

4. *There is a number $L_1 = L_1(R_e, r_0, a_0, C_0) \in I\!N$, such that for each domain D in $\mathcal{A}$ the boundary ∂D of D is piecewise parametrized by at most L_1 twice continuously differentiable injective mappings*

$$\psi_j : B_{r_0}(0) \to I\!R^m, \quad j = 1, ..., L_1, \tag{1.2.15}$$

of the form (1.2.5) with a norm (1.2.9) bounded uniformly for all domains $D \in \mathcal{A}$.

5. *There is a number $L_2 = L_2(R_e, r_0, a_0, C_0)$, such that for every scatterer D in $\mathcal{A}$ the domain D is parametrized by L_2 injective mappings*

$$\psi_j : Z_{a,r_0} \to I\!R^m, \quad j = 1, ..., L_2, \tag{1.2.16}$$

where ψ_j is an element of $C^{l,\alpha}(Z_{a,r_0})$, $j = 1, ..., L_2$, with a norm bounded uniformly for all domains $D \in \mathcal{A}$.

Proof. Given $x \in \partial D$ the property 1 is obtained with the help of the special coordinate system K_x and the bound C_0 on the norm

$$||f||_{C^{2,0}(B_{r_0}(0))}$$

by elementary calculations as follows. We only need to consider the line $x_2 = 0$ in a neighborhood of 0. Here the boundary of $B_{r_i}(0 + \nu(0)r_i)$ is given by

$$g(x_1) := r_i - \sqrt{r_i^2 - x_1^2}, \quad 0 \le x_1 < r_i.$$

For $r_i < \frac{1}{C_0}$ we estimate the derivative of g by

$$g'(x_1) = \frac{x_1}{\sqrt{r_i^2 - x_1^2}} \ge \frac{x_1}{r_i} > C_0 x_1, \quad 0 \le x_1 < r_i. \tag{1.2.17}$$

From

$$\frac{\partial f(t,0)}{\partial t}(x_1, 0) = \int_0^{x_1} \frac{\partial^2 f(t,0)}{\partial t^2} dt$$

for the derivative of f we derive

$$\left| \frac{\partial f(t,0)}{\partial t}(x_1, 0) \right| \le C_0 x_1, \quad 0 \le x_1 < r_i. \tag{1.2.18}$$

Thus for the derivatives of g and f from (1.2.17) and (1.2.18) we obtain the estimate

$$g'(x_1) > \left| \frac{\partial f(x_1,0)}{\partial x_1}(x_1) \right|, \quad 0 \le x_1 < r_i,$$

which yields $g(x_1) \; > \; f(x_1,0), \; 0 \le x_1 < r_i$, and thus property 1.

Properties 2 and 3 are immediate consequences of property 1. Property 4 can be obtained by compactness of

$$\overline{B_{R_e}(0)} \subset I\!R^m, \tag{1.2.19}$$

since there is a finite number L_1 of balls $B_{r_0}(x)$, $x \in B_{R_e}(0)$, which cover $B_{R_e}(0)$ and for each ball $B_{r_0}(x)$ the set $\partial D \cap B_{r_0}(x)$ is parametrized by special local coordinates of the form (1.2.5) with norm bounded by (1.2.9).

To prove 5, we first proceed as in 4 and obtain L_1 local coordinate systems K_{x_j} and corresponding local coordinates of the form (1.2.5) covering ∂D. For the local coordinates K_{x_j} we obtain a mapping $\psi_{1j} : Z_{a,r_0} \to Z_{a,r_0} \cap D$ by

$$(t_1, ..., t_m) \mapsto \Big(t_1, ..., t_{m-1}, \frac{t_m + a}{2a} \Big[f(t_1, ..., t_{m-1}) + a \Big] - a \Big).$$

For this mapping from (1.2.9) we derive $||\psi_{1j}||_{C^{l,\alpha}(Z_{a,r_o})} \le C$ with some constant C uniformly for $D \in \mathcal{A}$. The set

$$G := D \setminus \Big(\bigcup_{j=1}^{L_1} \psi_{1j}(Z_{a,r_0}) \Big)$$

is a compact subset of D with

$$d(G, \partial D) \ge \tau > 0 \tag{1.2.20}$$

uniformly for all domains $D \in \mathcal{A}$.

Second, there is a finite number $\tilde{L}$ of cylinders $Z_{\tau/3,\tau/3}(y_j, p_j)$, $y_j, p_j \in I\!R^m$, covering $B_{R_e}(0)$. We choose those cylinders which are contained in D. Because of (1.2.20) they cover G. Using translation, rotation and multiplication with a diagonal matrix with diagonal terms $\tau/(3r_0)$ and $\tau/(3a)$ we obtain continuously differentiable parametrizations $\psi_{2j} : Z_{a,r_0} \to Z_{\tau/3,\tau/3}(y_j, p_j)$, $j = 1, ..., \tilde{L}$, such that

$$G \subset \Big(\bigcup_{j=1}^{L_2} \psi_{2j}(Z_{a,r_0}) \Big).$$

The proof is now completed with $L_2 := L_1 + \tilde{L}$ by combining steps one and two. $\square$

When working with the exterior cone condition the following technical lemma will be useful.

LEMMA 1.2.3 *Given the class $\mathcal{A}$ of domains, there is $0 < \beta_0 \le \beta_e$ and $\rho_0 > 0$ such that all domains*

$$D_\rho := \{\, y \in I\!R^m : \ d(y, D) < \rho \,\} \tag{1.2.21}$$

with $D \in \mathcal{A}$ and $0 \le \rho \le \rho_0$ satisfy the exterior cone condition with angle β_0.

Proof. We will show that there is $0 < \beta_0 < \beta_e$, such that for each point x in $I\!R^m \setminus D_\rho \subset I\!R^m \setminus D$ and each cone $\mathrm{co}(x, p, \beta_e) \in I\!R^m \setminus D$ the cone $\mathrm{co}(x, p, \beta_0)$ satisfies

$$\mathrm{co}(x, p, \beta_0) \subset I\!R^m \setminus D_\rho.$$

According to Theorem 1.2.2 the domain D is the union of balls with radius r_i. Thus D is a subset of

$$G := \bigcup_{B_{r_i}(y) \subset I\!R^m \setminus (B_\rho(x) \cup \mathrm{co}(x,p,\beta_e))} B_{r_i}(y).$$

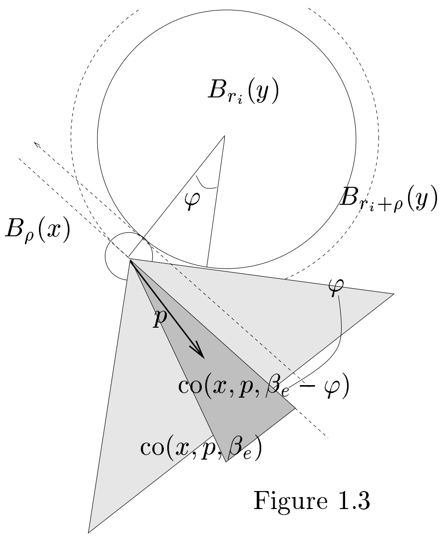

Figure 1.3

Considering balls $B_{r_i}(y)$, which touch both $B_\rho(x)$ and $\mathrm{co}(x, p, \beta_e)$, we obtain by elementary geometric arguments from Figure 1.3 that for angles

$$\varphi := \arccos\left(\frac{r_i}{r_i + \rho}\right) \tag{1.2.22}$$

with

$$\varphi < \beta_e \tag{1.2.23}$$

we have

$$B_{r_i+\rho}(y) \cap \mathrm{co}(x, p, \beta_e - \varphi) = \emptyset.$$

Thus we have proven

$$\mathrm{co}(x, p, \beta_e - \varphi) \cap D_\rho \subset \mathrm{co}(x, p, \beta_e - \varphi) \cap G_\rho = \emptyset,$$

i.e., we have

$$\mathrm{co}(x, p, \beta_e - \varphi) \subset I\!R^m \setminus D_\rho.$$

The proof is complete by observing that $\arccos(r_i/(r_i + \rho)) \to 0, \rho \to 0$, and choosing ρ_0, β_0 such that for all $\rho < \rho_0$ and φ defined by (1.2.22) the estimate (1.2.23) is satisfied. □

Spherical harmonics, Bessel and Hankel functions. Indispensable tools for the investigation of both direct and inverse scattering problems are special solutions to the Helmholtz equation. For later use we now introduce Legendre polynomials, associated Legendre polynomials, spherical harmonics, Bessel and Neumann functions and note some of their properties. If not pointed out otherwise, for a proof of these properties we refer to [8].

Let Y_n^l for $l = -n, ..., n$ and $n = 0, 1, 2, ...$ be a complete orthonormal system of spherical harmonics, as for example given by

$$Y_n^l(\theta,\varphi) := \sqrt{\frac{2n+1}{4\pi}\frac{(n-|l|)!}{(n+|l|)!}}\, P_n^{|l|}(\cos(\theta))e^{il\varphi} \tag{1.2.24}$$

for $l = -n, ..., n$, $n = 0, 1, 2, ...$ Here, P_n^l are the associated Legendre functions, which can be derived from the Legendre polynomials P_n by

$$P_n^l(t) := (1-t^2)^{l/2}\frac{d^l P_n(t)}{dt^l}, \;\; l = 0, 1, ..., n. \tag{1.2.25}$$

The Legendre polynomials

$$P_n(x) := \frac{1}{2^n\, n!}\frac{d^n}{dx^n}\Big\{(x^2-1)^n\Big\}, \; n = 0, 1, ... \tag{1.2.26}$$

form an orthogonal system in $L^2[-1,1]$; more explicitly we have

$$\int_{-1}^{1} P_n(t)P_l(t)dt \;=\; \frac{2}{2n+1}\,\delta_{nl}, \;\; n, l = 0, 1, 2, ... \tag{1.2.27}$$

They satisfy the inequality

$$|P_n(t)| \;\le\; 1, \;\; -1 \le t \le 1, \;\; n = 0, 1, 2, ... \tag{1.2.28}$$

For $2n+1$ orthonormal spherical harmonics of order n the addition theorem

$$\sum_{l=-n}^{n} Y_n^l(\hat{x})\overline{Y_n^l(\hat{y})} \;=\; \frac{2n+1}{4\pi}\, P_n(\cos(\theta)) \tag{1.2.29}$$

holds for $\hat{x}, \hat{y} \in \Omega$, where θ is the angle between $\hat{x}$ and $\hat{y}$. For the surface gradient of spherical harmonics we note the estimate

$$\Big|\text{Grad}\; Y_n^l(\hat{x})\Big| \;\le\; C\, n^{3/2}||Y_n^l||_{L^2(\Omega)}, \;\; \hat{x} \in \Omega \tag{1.2.30}$$

(see Section X, Lemma 6.1 of [60]).

The spherical Bessel functions and spherical Neumann functions of order n are given by the series

$$j_n(t) := \sum_{p=0}^{\infty} \frac{(-1)^p t^{n+2p}}{2^p p! 1 \cdot 3 \cdots (2n+2p+1)} \tag{1.2.31}$$

and

$$y_n(t) := -\frac{(2n)!}{2^n n!} \sum_{p=0}^{\infty} \frac{(-1)^p t^{2p-n-1}}{2^p p!(-2n+1)(-2n+3)\cdots(-2n+2p-1)}, \tag{1.2.32}$$

where the first coefficient in the series (1.2.32) has to be set equal to one. The linear combinations

$$h_n^{(1,2)} := j_n \pm i y_n \tag{1.2.33}$$

are known as spherical Hankel functions of the first and second kind of order n. By straightforward calculation from the series (1.2.31) and (1.2.32) it is possible to derive the differentiation formula

$$t^{n+1} f_{n-1}(t) = \frac{d}{dt}\left\{t^{n+1} f_n(t)\right\} \tag{1.2.34}$$

for both $f_n = j_n$ and $f_n = y_n$, and together with Stirlings formula

$$n! = \sqrt{2\pi n}\left(\frac{n}{e}\right)^n (1 + o(1)), \quad n \to \infty \tag{1.2.35}$$

we obtain the behavior

$$j_n(t) = \frac{1}{2n+1}\left(\frac{et}{2n}\right)^n \left(1 + O\left(\frac{1}{n}\right)\right), \quad n \to \infty, \tag{1.2.36}$$

and

$$h_n^{(1)}(t) = \frac{-\sqrt{2}}{t}\left(\frac{2n}{et}\right)^n \left(1 + O\left(\frac{1}{n}\right)\right), \quad n \to \infty, \tag{1.2.37}$$

uniformly on compact subsets of $(0, \infty)$.

Point-sources and multipoles. One can use the spherical harmonics and the spherical Bessel or Hankel functions to construct special solutions of the Helmholtz equation. Given a spherical harmonic Y_n of order n, the function

$$u_n(x) := j_n(\kappa|x|) Y_n(\hat{x}) \tag{1.2.38}$$

is an entire solution of the Helmholtz equation. The multipole of order n

$$v_n(x) := h_n^{(1)}(\kappa|x|) Y_n(\hat{x}) \tag{1.2.39}$$

is a radiating solution to the Helmholtz equation in $I\!R^m \setminus \{0\}$.

Modulo a constant, the three-dimensional multipole of order zero is the point-source

$$\Phi(x,z) := \frac{1}{4\pi}\frac{e^{i\kappa|x-z|}}{|x-z|}, \quad (1.2.40)$$

the multipole of order one is a dipole, the multipole of order two a quadrupole, etc.

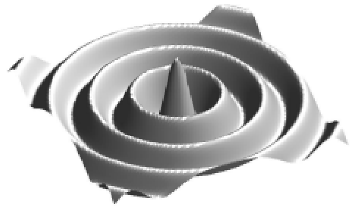

Figure 1.4

Multipole expansions, i.e., expansions of solutions of the Helmholtz equation with respect to the functions v_n, are used both in direct and inverse scattering. For the proof of explicit stability estimates in Chapter 3, we will need the multipole-expansion of the fundamental solution

$$\Phi(x,y) = i\kappa \sum_{n=0}^{\infty}\sum_{l=-n}^{n} h_n^{(1)}(\kappa|x|)Y_n^l(\hat{x})j_n(\kappa|y|)\overline{Y_n^l(\hat{y})}, \quad (1.2.41)$$

where $\hat{x} = x/|x|$ and $\hat{y} = y/|y|$. Here, the series and its term-by-term derivatives with respect to x and y are absolutely and uniformly convergent on compact subsets of $|x| > |y|$. Further tools are the Funk-Hecke formula

$$\int_{\Omega} e^{-i\kappa x\cdot\hat{z}}Y_n(\hat{z})\,ds(\hat{z}) = \frac{4\pi}{i^n}\,j_n(\kappa|x|)Y_n(\hat{x}), \quad x \in I\!R^m \quad (1.2.42)$$

for spherical harmonics Y_n of order n and the Jacobi-Anger expansion

$$e^{i\kappa x\cdot d} = \sum_{n=0}^{\infty} i^n(2n+1)j_n(\kappa|x|)P_n(\cos(\theta)), \quad x \in I\!R^m, \quad (1.2.43)$$

where d is a unit vector, θ denotes the angle between x and d and the convergence is uniform on compact subsets of $I\!R^m$.

The two-dimensional case. For scattering in $I\!R^2$ the multipoles and some constants have to be modified. The functions j_n and y_n are replaced by the Bessel function of order n

$$J_n(t) := \sum_{p=0}^{\infty}\frac{(-1)^p}{p!(n+p)!}\left(\frac{t}{2}\right)^{n+2p} \quad (1.2.44)$$

and the Neumann function of order n

$$\begin{aligned} Y_n(t) \quad &:= \frac{2}{\pi}\left\{\ln\frac{t}{2}+C\right\} J_n(t) - \frac{1}{\pi}\sum_{p=0}^{n-1}\frac{(n-1-p)!}{p!}\left(\frac{2}{t}\right)^{n-2p} \\ &-\frac{1}{\pi}\sum_{p=0}^{\infty}\frac{(-1)^p}{p!(n+p)!}\left(\frac{t}{2}\right)^{n+2p}\{\psi(n+p)+\psi(p)\} \end{aligned} \tag{1.2.45}$$

for $n = 0, 1, 2, \ldots$, where we define $\psi(0) := 0$,

$$\psi(p) := \sum_{l=1}^{p}\frac{1}{l}, \quad p = 1, 2, \ldots, \tag{1.2.46}$$

and

$$C := \lim_{p\to\infty}\left\{\sum_{l=1}^{p}\frac{1}{l} - \ln p\right\} \tag{1.2.47}$$

denotes Euler's constant, and if $n = 0$ the finite sum in (1.2.45) is set equal to zero. The linear combinations

$$H_n^{(1,2)} := J_n \pm iY_n \tag{1.2.48}$$

are called Hankel functions of the first and second kind of order n, respectively. The multipoles in $I\!R^2$ are given by the functions

$$V_n(x) := H_n^{(1)}(\kappa r)e^{\pm in\varphi} \tag{1.2.49}$$

with the polar coordinates (r, φ). For the two-dimensional fundamental solution

$$\Phi(x, z) := \frac{i}{4}H_0^{(1)}(\kappa|x-z|) \tag{1.2.50}$$

we obtain the multipole-expansion

$$\Phi(x, y) = \frac{i}{4}H_0^{(1)}(\kappa|x|)J_0(\kappa|y|) + \frac{i}{2}\sum_{n=1}^{\infty}H_n^{(1)}(\kappa|x|)J_n(\kappa|y|)\cos(n\theta), \tag{1.2.51}$$

where θ denotes the angle between x and y. It is valid uniformly on compact subsets of $|x| > |y|$. The Jacobi-Anger expansion (1.2.43) in $I\!R^2$ assumes the form

$$e^{i\kappa x\cdot d} = J_0(\kappa|x|) + 2\sum_{n=1}^{\infty} i^n J_n(\kappa|x|)\cos(n\theta), \quad x \in I\!R^2. \tag{1.2.52}$$

Single- and double-layer potentials, jump relations. We will use the boundary-layer approach to investigate the properties of the solutions to scattering problems for impenetrable scatterers. With the help of boundary-layer potentials the scattering problems are reduced to integral equations on the boundary of the scatterer.

For a domain $D \subset I\!R^m$ with boundary of class C^2 consider the single-layer potential

$$u(x) := \int_{\partial D} \Phi(x,y)\varphi(y)ds(y), \quad x \in I\!R^m, \tag{1.2.53}$$

and the double-layer potential

$$v(x) := \int_{\partial D} \frac{\partial \Phi(x,y)}{\partial \nu(y)}\varphi(y)ds(y), \quad x \in I\!R^m \setminus \partial D. \tag{1.2.54}$$

Later we will also use $P_1\varphi$ and $P_2\varphi$ for the single-layer or double-layer potential, respectively. The behavior of the single- and double-layer potentials at the boundary ∂D is described by the following jump relations.

THEOREM 1.2.4 **(Jump relations.)** *The single-layer potential u with continuous density φ is continuous throughout $I\!R^m$. On the boundary we have*

$$u(x) = \int_{\partial D} \Phi(x,y)\varphi(y)\, ds(y), \quad x \in \partial D, \tag{1.2.55}$$

and

$$\frac{\partial u_\pm}{\partial \nu}(x) = \int_{\partial D} \frac{\partial \Phi(x,y)}{\partial \nu(x)}\varphi(y)\, ds(y) \mp \frac{1}{2}\varphi(x), \quad x \in \partial D, \tag{1.2.56}$$

where

$$\frac{\partial u_\pm}{\partial \nu}(x) := \lim_{h\to+0} \nu(x)\cdot \mathrm{grad}\, u(x \pm h\nu(x)) \tag{1.2.57}$$

is to be understood in the sense of uniform convergence on ∂D. The double-layer potential v with density φ can be continuously extended from D to $\overline{D}$ and from $I\!R^m \setminus \overline{D}$ to $I\!R^m \setminus D$ with limiting values

$$\frac{\partial v_\pm}{\partial \nu}(x) = \int_{\partial D} \frac{\partial \Phi(x,y)}{\partial \nu(y)}\varphi(y)\, ds(y) \pm \frac{1}{2}\varphi(x), \quad x \in \partial D, \tag{1.2.58}$$

where

$$v_\pm(x) := \lim_{h\to+0} v(x \pm h\nu(x))$$

and where the integral exists as an improper integral. For a density $\varphi \in L^2(\partial D)$ the jump relations (1.2.55) to (1.2.58) have to be replaced by

$$\lim_{h\to+0} \int_{\partial D} \left| u(x \pm h\nu(x)) - \textstyle\int_{\partial D} \Phi(x,y)\varphi(y)ds(y) \right|^2 ds(x) = 0, \tag{1.2.59}$$

$$\lim_{h\to+0}\int_{\partial D}\left|\frac{\partial u}{\partial\nu}(x\pm h\nu(x))-\int_{\partial D}\frac{\partial\Phi(x,y)}{\partial\nu(x)}\varphi(y)ds(y)\pm\frac{1}{2}\varphi(x)\right|^2 ds(x) \;=\; 0 \tag{1.2.60}$$

and

$$\lim_{h\to+0}\int_{\partial D}\left|\frac{\partial u}{\partial\nu}(x\pm h\nu(x))-\int_{\partial D}\frac{\partial\Phi(x,y)}{\partial\nu(y)}\varphi(y)ds(y)\mp\frac{1}{2}\varphi(x)\right|^2 ds(x) = 0. \tag{1.2.61}$$

Proof. The proof for continuous densities can be found in [7], Theorems 2.12, 2.13 and 2.19; the proof for $\varphi\in L^2(\partial D)$ is due to Kersten [39]. □

For the treatment of point-sources with source-point on the boundary of a domain, in Chapter 5 jump-relations with L^p-densities will be investigated.

Riesz and Fredholm theory. With the help of boundary-layer potentials and jump relations the acoustic and electromagnetic scattering problems can be reduced to boundary integral equations of the second kind, i.e., operator equations of the form

$$(I-A)\varphi = f \tag{1.2.62}$$

with a compact linear operator $A : X\to X$ defined on a normed space X. Integral equations of this kind can be solved using the following theorem of Riesz.

THEOREM 1.2.5 **(Riesz Theorem.)** *Let X be a normed space and $A : X\to X$ a compact linear operator. If the homogeneous equation*

$$(I-A)\varphi = 0$$

only has the trivial solution $\varphi = 0$, then for all $f\in X$ the inhomogeneous equation

$$(I-A)\varphi = f$$

has a unique solution $\varphi\in X$ and this solution depends continuously on f.

Proof. See Corollary 1.17 of [7]. □

According to the Riesz theorem the injectivity of an operator $I-A$ yields its continuous invertibility. Usually the injectivity of an integral operator corresponding to a scattering problem is obtained from the uniqueness of the solution to this scattering problem.

For the investigation of special scattered fields for scattering from inhomogeneous medium scatterers we will need to study the integral equations of the scattering problems both in the spaces of continuous and square-integrable functions. The injectivity of the integral operators in $L^2(D)$ will be obtained from the results in $C(D)$ with the help of dual systems, defined on subspaces of $L^2(D)$ by the sesquilinear form

$$\langle \varphi, \psi \rangle := \int_D \varphi(y)\overline{\psi(y)}dy \tag{1.2.63}$$

for $\varphi, \psi \in L^2(D)$.

THEOREM 1.2.6 **(Fredholm Alternative Theorem)** *Let X and Y be normed spaces, $\langle X, Y \rangle$ a dual system and $A : X \to X$, $B : Y \to Y$ compact adjoint operators. We have either*

$$N(I - A) = \{0\} \qquad \text{and} \qquad N(I - B) = \{0\}$$

and

$$(I - A)(X) = X \qquad \text{and} \qquad (I - B)(Y) = Y$$

or

$$\dim N(I - A) = \dim N(I - B) \in \mathbb{N}$$

and

$$(I - A)(X) = N(I - B)^{\perp} \qquad \text{and} \qquad (I - B)(Y) = N(I - A)^{\perp}.$$

Proof. For a proof we refer the reader to [51]. □

For operators A with $||A|| < 1$ the invertibility of $I - A$ is obtained by more elementary arguments (generalizing the geometric series).

THEOREM 1.2.7 **(Neumann series.)** *Let $A : X \to X$ be a bounded linear operator on a Banach space X with $||A|| < 1$. Then $I - A$ has a bounded inverse operator on X which is given by the Neumann series*

$$(I - A)^{-1} \ = \ \sum_{j=0}^{\infty} A^k \tag{1.2.64}$$

and which satisfies

$$||(I - A)^{-1}|| \ \leq \ \frac{1}{1 - ||A||}. \tag{1.2.65}$$

Proof. We refer to [51]. □

Singular systems and Riesz basis. A main object of inverse scattering theory is the operator $\mathcal{F}$ which maps the incident waves u^i defined on the support of the scatterer onto the far field pattern of the scattered fields. It is a linear operator and it is compact in all reasonable spaces. An important part of the study of the inverse problems will be the spectral theory of compact linear operators.

Let $A : X \to Y$ be a linear compact operator. The nonnegative square roots of the eigenvalues of the self-adjoint compact operator $A^*A : X \to X$ are called singular values of A. It is well known from elementary functional analysis that the singular values form an at most countable set accumulating only at zero. All nonzero eigenvalues have finite multiplicity, i.e., the dimension of the nullspaces $N(\mu_j^2 I - A^*A)$ is finite. Eigenvectors corresponding to different eigenvalues are orthogonal.

THEOREM 1.2.8 **(Singular System.)** *Let (μ_j) denote the sequence of the nonzero singular values of the compact linear operator $A \neq 0$ ordered as a decreasing sequence and repeated according to their multiplicity. Then there exist orthonormal sequences φ_j in X and $\tilde{\varphi}_j$ in Y such that*

$$A\varphi_j = \mu_j \tilde{\varphi}_j, \qquad A^*\tilde{\varphi}_j = \mu_j \varphi_j \tag{1.2.66}$$

for all $j \in I\!N$. For each $\varphi \in X$ we have the singular value decomposition

$$\varphi = \sum_{j=1}^{\infty} (\varphi, \varphi_j)\varphi_j + Q\varphi \tag{1.2.67}$$

with the orthogonal projection operator $Q : X \to N(A)$ and

$$A\varphi = \sum_{j=1}^{\infty} \mu_j (\varphi, \varphi_j)\tilde{\varphi}_j. \tag{1.2.68}$$

Each system with these properties is called a singular system of A.

Proof. We refer to [8] or [51]. □

Orthonormal basis as obtained from the singular value decomposition form a general tool which is widely used in functional analysis and for the treatment of inverse problems. For the spectral theory of scattering problems we will need a more general type of expansion. A set $\{\varphi_j : j \in I\!N\}$ is called Riesz basis of a complex Hilbert space X, if every $\varphi \in X$ is of the form

$$\varphi = \sum_{j=1}^{\infty} \alpha_j \varphi_j \tag{1.2.69}$$

with coefficients α_j which satisfy

$$\sum_{j=1}^{\infty} |\alpha_j|^2 \; < \; \infty, \tag{1.2.70}$$

and every sequence $(\alpha_j), j \in \mathbb{N}$ in $\mathbb{C}$ with (1.2.70) converges in X towards an element φ such that equation (1.2.69) is true.

THEOREM 1.2.9 *Let X be a complex Hilbert space, $K : X \to X$ a linear compact operator with $\mathrm{Im}(K\varphi, \varphi) \neq 0$ for all $\varphi \in X, \varphi \neq 0$. Let $\{\varphi_j : j \in \mathbb{N}\}$ be a linearly independent and complete set in X and assume that we have the orthogonality relation*

$$((I + K)\varphi_j, \varphi_l) \; = \; s_j \delta_{jl}, \;\; j, l \in \mathbb{N}, \tag{1.2.71}$$

where $s_j \in \mathbb{C}$ satisfies the following conditions: There exists $r > 0$ with

$$|s_j| = r \; \forall j \in \mathbb{N}, \qquad \mathrm{Im}(s_j) \to 0, \; j \to \infty. \tag{1.2.72}$$

Then $\{\varphi_j : j \in \mathbb{N}\}$ is a Riesz basis of X.

Proof. We refer to [42]. □

Ill-posed problems and regularization. In general, inverse problems are ill-posed in the sense of Hadamard [22], i.e., the demands of uniqueness, existence and stability are violated. Ill-posed equations of the type

$$A(\varphi) = f \tag{1.2.73}$$

with a compact (linear or nonlinear) operator $A : X \to Y$ are usually solved approximately by a family of bounded (linear or nonlinear) regularization operators

$$R_\alpha : Y \to X, \;\; \alpha > 0, \tag{1.2.74}$$

with the property

$$\lim_{\alpha \to 0} R_\alpha(A(x)) = x \quad \text{for all} \;\; x \in X, \tag{1.2.75}$$

i.e., the operators $R_\alpha A$ converge pointwise to the identity for $\alpha \to 0$. Here, α is called the regularization parameter. If R_α satisfies (1.2.75), then the family of operators R_α is called a regularization strategy (see [43]). In the presence of data error of size δ we calculate the solution of (1.2.73) with f replaced by f^δ, i.e., we calculate

$$\varphi^\delta \; := \; R_{\alpha(\delta)} f^\delta \tag{1.2.76}$$

with a regularization parameter $\alpha(\delta)$ depending on $\delta > 0$. Of course, we would like to choose the regularization parameter in a way such that our approximate solution tends towards the true solution if the data error tends to zero. This motivates the following definition.

A strategy for the choice of the parameter α depending on the error level δ is called regular, if for all $f \in A(X)$ and all $f^\delta \in Y$ with $||f^\delta - f|| \leq \delta$ there holds

$$R_{\alpha(\delta)} f^\delta \to A^{-1} f, \;\; \delta \to 0. \tag{1.2.77}$$

We will call a set of regularization operators R_α with a regular strategy for the inversion of (1.2.73) a convergent regularization.

We already discussed the different aspects of the ill-posedness of an equation or problem in the preceding survey. For compact linear operators in Hilbert spaces these aspects can be investigated using the singular system.

THEOREM 1.2.10 **(Picard.)** *Let $A : X \to Y$ be a compact linear operator with singular system $(\mu_j, \varphi_j, \tilde{\varphi}_j)$. The equation of the first kind*

$$A\varphi = f \tag{1.2.78}$$

is solvable if and only if f belongs to the orthogonal complement $N(A^*)^\perp$ and satisfies

$$\sum_{j=1}^{\infty} \frac{1}{\mu_j^2} |(f, \tilde{\varphi}_j)|^2 \; < \; \infty. \tag{1.2.79}$$

In this case a solution is given by

$$\varphi \; = \; \sum_{j=1}^{\infty} \frac{1}{\mu_j} (f, \tilde{\varphi}_j) \varphi_j. \tag{1.2.80}$$

Proof. See [8]. □

In Picard's Theorem the ill-posedness of the inverse operator A^{-1} is due to the convergence $\mu_j \to 0$ of the sequence of singular values. The faster the decay of the singular values, the more ill-posed is the equation (1.2.78). A natural regularization method for A^{-1} is obtained by taking a finite sum instead of the series (1.2.80).

THEOREM 1.2.11 **(Spectral cut-off.)** *Let $A : X \to Y$ be an injective compact linear operator with singular system $(\mu_j, \varphi_j, \tilde{\varphi}_j)$. Then the spectral cut-off*

$$R_n f \; := \; \sum_{\mu_j \geq \mu_n} \frac{1}{\mu_j} (f, \tilde{\varphi}_j) \varphi_j \tag{1.2.81}$$

describes a regularization scheme with regularization parameter $m \to \infty$ and $||R_m|| = 1/\mu_n$. The spectral cut-off is regular if

$$\alpha(\delta) \to 0, \quad \frac{\delta^2}{\alpha(\delta)} \to 0, \quad \textit{for } \delta \to 0. \tag{1.2.82}$$

Proof. See [8] and [51]. □

Another well-known regularization strategy (compare for example [19]) for the approximate solution of equation (1.2.73) for linear operators $A : X \to Y$ on Hilbert spaces X, Y is given by the Tikhonov regularization scheme, which computes an approximate solution φ_α by

$$\varphi_\alpha := (\alpha I + A^* A)^{-1} A^* f. \tag{1.2.83}$$

We need to collect some of its properties.

THEOREM 1.2.12 **(Tikhonov regularization.)** *The Tikhonov regularization (1.2.83) defines a regularization scheme with $||R_\alpha|| \leq \frac{1}{2\alpha}$. It is regular provided the parameter $\alpha(\delta)$ is chosen such that (1.2.82) is satisfied.*

Proof. See [51]. □

A further possibility to approximately solve (1.2.73) is minimum norm solutions. For a bounded linear operator $A : X \to Y$ between two normed spaces X and Y, $\delta > 0$ and $f \in Y$ an element $\varphi_0 \in X$ is called minimum norm solution of $A\varphi = f$ with discrepancy δ, if $||A\varphi_0 - f|| \leq \delta$ and

$$||\varphi_0|| = \inf \{||\varphi|| : \ ||A\varphi - f|| \leq \delta\}.$$

THEOREM 1.2.13 **(Minimum norm solutions.)** *We consider two Hilbert spaces X and Y. If $A : X \to Y$ has dense range in Y, then for each $f \in Y$ there is a unique minimum norm solution φ_0 of $H\varphi = f$ with discrepancy δ. The minimum norm solution φ_0 can be calculated by*

$$\varphi_0 = (\alpha I + A^* A)^{-1} A^* f, \tag{1.2.84}$$

where α is a zero of the function

$$G(\alpha) := \left|\left|(\alpha I + A^* A)^{-1} A^* f - f\right|\right|^2 - \delta^2. \tag{1.2.85}$$

For an injective bounded linear operator A minimum norm solutions define a regular strategy for the choice of the regularization parameter α for the Tikhonov regularization.

Proof. A proof is given in [51]. $\square$

The preceding theorem provides an *a posteriori* strategy for the choice of the parameter α in the Tikhonov regularization scheme for the approximate solution of $A\varphi = f$.

Next, we will introduce a fourth well-known regularization method for later use. For a bounded linear injective operator $A : X \to Y$, $\rho > 0$ and $f \in Y$ we call an element $\varphi_0 \in X$ a quasi-solution of (1.2.73) with constraint ρ, if $||\varphi_0|| \leq \rho$ and

$$||A\varphi_0 - f|| \;=\; \inf\{||A\varphi - f|| : ||\varphi|| \leq \rho\}. \tag{1.2.86}$$

For quasi-solutions we obtain regularity only if the constraint is chosen such that $||A^{-1}f|| = \rho$ (see [51]). In Chapter 5 we will use quasi-solutions as smoothing operators for an application where we do not need regularity.

So far we only treated regularization for the case where the data have errors which might strongly influence the solution of an ill-posed equation. In Chapter 7 we will be faced with the case where the ill-posed operator itself is given by measured data. For this situation the Tikhonov-Morozov regularization is adequate.

THEOREM 1.2.14 **(Tikhonov-Morozov regularization.)** *Let $A_\delta : X \to Y$ be a family of injective and compact operators between Hilbert spaces X and Y with dense ranges $R(A_\delta)$ such that $||A_0 - A_\delta|| \leq \delta$ for all $\delta > 0$. Given $f \in Y$ with $f \notin N(A_\delta^*)$ for all $\delta \geq 0$ the Tikhonov-Morozov solution of the equation $A_\delta\varphi = r$ is the solution of the system*

$$(\alpha I + A_\delta^* A_\delta)\varphi_\alpha = A_\delta^* r, \tag{1.2.87}$$

where α is chosen such that

$$||A_\delta\varphi_\alpha - r|| \;=\; \delta||\varphi_\alpha||. \tag{1.2.88}$$

If the unperturbed equation is solvable by some $g \in X$, then

$$g_{\alpha(\delta)} \;\to\; g, \;\; \delta \to 0. \tag{1.2.89}$$

If the unperturbed equation $A_0 g = r$ is not solvable, then $||g_{\alpha(\delta)}|| \to \infty$ for $\delta \to 0$.

Proof. We refer to [42]. $\square$

Scatterers, domains, boundary conditions. For the different scattering objects under consideration we need to clarify our notation for scatterers and scattering domains. An impenetrable acoustic or electromagnetic scatterer is given by a domain D and a boundary condition. We will use the letter $\mathcal{D}$ for the full scatterer with all its properties. The type of a scatterer is either sound-soft or sound-hard for the acoustic problems or perfect-conductor for electromagnetic scattering. Thus, an impenetrable scatterer $\mathcal{D}$ can be viewed as a pair

$$\mathcal{D} = (D, type) \tag{1.2.90}$$

of its domain D and its boundary condition.

For penetrable scatterers the situation is slightly different. Again, we use $\mathcal{D}$ for the full scatterer. The scatterer $\mathcal{D}$ is located in a domain D, defined as the interior of the support of the inhomogeneity given by a refractive index n (or N for the cases of orthotropic or anisotropic media) with $n|_{\mathbb{R}^m \setminus \overline{D}} = 1$ and $n|_{\overline{D}} \in C^{0,\alpha}(\overline{D})$. We write the full scatterer $\mathcal{D}$ as a pair

$$\mathcal{D} = (D, n). \tag{1.2.91}$$

We will study uniqueness, stability and algorithms for the reconstruction of the domain D of impenetrable and penetrable scatterers $\mathcal{D}$ for both acoustic and electromagnetic scattering problems.