PH 349 : Methods of Mathematical Physics I - Lecture Notes

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0 Syllabus and Introductory Course Notes

0.1 Instructor

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0.2 References

1. *Methods of Mathematical Physics*, G.F. Arfken & B.L. Weber, Academic Press.
2. *Handbook of Mathematical Functions*, Abramowitz & Stegun, Dover.
3. *Mathematical Methods for Physicists*, Mathews & Walker, Benjamin.

0.3 Summary and Outline

This course introduces some of the mathematical techniques that are frequently used in the analysis of a variety of physical processes. These notes are a slightly expanded version of a one-semester course that I gave at Case Western Reserve University in the Fall of 1999. Although I chose several textbooks to accompany the course, much of the material is presented in the way that I was taught it. I learnt mathematical physics from many people at Cambridge University, but those whose lectures clearly influenced these notes are Alan Macfarlane, Stephen Siklos, and John Stewart. In places I have borrowed heavily from their presentations and so these notes are certainly not a wholly original effort. However, I have tried to synthesize the material in a useful and orderly way, and have contributed significant material where I felt that more clarity was needed. It is my hope that these notes will evolve into a better and better resource as time passes. Certainly there is much room for improvement, and I welcome comments and suggestions from anyone who reads them.

The basis for the techniques studied in this course is a firm understanding of functions of a complex variable and we shall begin by laying the groundwork for this. In particular, we will spend some time reviewing and gaining expertise in performing contour integration of complex functions.

The experience we build up with contour integration will be useful when we next turn to the evaluation of sums and integrals. We will investigate a number of techniques, but the focus of this part of the course will be the various methods of obtaining asymptotic expansions of integrals. While a full understanding of this technique is quite challenging, it forms the basis of many physical calculations across many subdisciplines; particle physics, statistical mechanics, condensed matter physics ...

The third part of the course concerns ordinary differential equations (ODEs). We will use everything we have learned up to this point so it is important to keep up with the material as the course progresses. We will investigate some classic techniques such as Green's functions and the WKB approximation, as well as introducing some of the special functions of mathematical physics that will be explored more fully in Physics 350.

In the final section of the course I will introduce Transform Calculus. I will cover the Fourier and Laplace transforms in some detail, and will make a number of remarks about transforms involving general kernel functions.

A more formal outline of the syllabus is:

1. Analysis of Complex Functions
   - (a) Singularities, Residues, and Laurent Series
   - (b) Jordan's Lemma and Morera's Theorem
   - (c) Cauchy's Theorem
   - (d) Contour Integration
2. Exact and Approximate Evaluation of Sums and Integrals
   - (a) Approximation of Sums by Integrals
   - (b) Laplace's Method
   - (c) Method of Stationary Phase
   - (d) Method of Steepest Descent
3. Exact and Approximate Solution of Ordinary Differential Equations
   - (a) Green's Functions

   (b) WKB Analysis
   (c) Special Functions

4. Transform Calculus.
   (a) General Picture
   (b) The Fourier Transform
   (c) Theory of Distributions
   (d) The Laplace Transform

5. Sturm-Liouville Theory
   (a) Eigenvalues and Eigenfunctions
   (b) Completeness
   (c) Green's Functions representations

6. The Calculus of Variations

0.4 Mathematical Notation

$\forall$	for all
$\exists$	there exists
$\neg$	not
$\equiv$	is equivalent to, or is defined as
$\sim$	is asymptotically equal to
$\approx$	is approximately equal to
$\propto$	is proportional to
$\Rightarrow$	implies
iff	if and only if; implies and is implied by
$\Leftrightarrow$	an alternative form of *iff*
$A \cup B$	the union of the sets (groups) A and B
$A \cap B$	the intersection of the sets (groups) A and B
$A \subset B$	the set (group) A is a subset of the set (group) B

$a \in A$	the element a is a member of the set (group) A
$A \ni a$	the set (group) A contains the element a
$a \notin A$	the element a is not a member of the set (group) A
$A \equiv \{z \mid \cdots\}$	the set (group) A consists of all z satisfying the condition $\cdots$
$\emptyset$	the empty set
$\sum$	the sum of
$\prod$	the product of
$\lim_{x \to x_0}$	the limit as the variable x approaches the value x_0
$\oint$	the integral around a closed curve of
∂B	the boundary of the region B
z^*	the complex conjugate of the complex variable z
$\bar{z}$	an alternative form of z^*
$\Re(z)$	the real part of the complex variable z
$\Im(z)$	the imaginary part of the complex variable z

1 Analysis of Complex Functions

Let's jump right in and begin with some definitions:

Definition 1.1 *The complex plane* $\mathcal{C}$ *is (a representation in* $\mathcal{R}^2$ *of) the set* $\{z : \ |z| < \infty\}$, *and the extended complex plane is* $\mathcal{C} \cup \{\infty\}$, *where* $\{\infty\}$ *is the single point at infinity corresponding to the origin under the transformation* $z \to z^{-1}$.

Definition 1.2 *An argument of* z *is any one of the numbers* $\theta + 2\pi m$, *where* $z = re^{i\theta}$ *and* m *is an integer. The principal value of* $\arg(z)$ *satisfies* $-\pi < \arg(z) \leq \pi$

Definition 1.3 *A region* $\mathcal{D} \subset \mathcal{C}$ *is a (non-empty) connected open subset of* $\mathcal{C}$.

Definition 1.4 $\Gamma \subset \mathcal{D}$ *is a simple closed curve (s.c.c.) if* Γ *is a continuous image of* $S^1 = \{z : \ |z| = 1\}$

Definition 1.5 *A function of a complex variable* z *is a mapping from* $\mathcal{D}$ *to* $\mathcal{C}$

Fundamental to a discussion of complex functions is the issue of differentiation. We say that a function $f : \mathcal{D} \to \mathcal{C}$ is *differentiable* at the point $z_0 \in \mathcal{D}$ if

$$f'(z_0) \equiv \lim_{z \to z_0} \left(\frac{f(z) - f(z_0)}{z - z_0} \right) \tag{1}$$

exists and is unique. We then say that a single-valued function $f(z)$ is *(complex) analytic, regular or holomorphic* in $\mathcal{D}$ if $f'(z)$ exists (and is continuous) at each point in $\mathcal{D}$. Finally, $f(z)$ is *entire* if it is holomorphic everywhere in $\mathcal{C}$, or *meromorphic* if its only singularities in $\mathcal{C}$ are poles (see later).

Theorem 1.1 *If $w = u + iv = f(x + iy)$ is differentiable at $z_0 = x_0 + iy_0$, then u and v are differentiable as functions of two real variables $u = u(x, y)$, $v = v(x, y)$ and:*

$$\begin{aligned} \frac{\partial u}{\partial x} &= \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} &= -\frac{\partial v}{\partial x} \ , \end{aligned} \tag{2}$$

and are known as the Cauchy-Riemann equations.

Corollary 1.1 *If f is at least twice differentiable in $\mathcal{D}$, then the real and imaginary parts are both harmonic functions:*

$$\begin{aligned} \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} &= 0 \\ \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} &= 0 \ . \end{aligned} \tag{3}$$

Thus, u and v cannot have maxima or minima in $\mathcal{D}$ and so any stationary points must be saddle points. Hence, their biggest and smallest values are attained only on the boundary $\partial\mathcal{D}$ of $\mathcal{D}$.

As with functions of a real variable, we may consider power series expansions of complex functions. For now we'll look at holomorphic functions, but later we'll extend our discussion to include functions with singularities. As with real functions, if $f(z) = \sum_{n=0}^{\infty} a_n z^n$, then the derivative of this function is given by $f(z) = \sum_{n=1}^{\infty} n a_n z^{n-1}$. Each power series has a *radius of convergence*, R, such that $\sum_{n=0}^{\infty} a_n z^n$ converges for $|z| < R$ and diverges for $|z| > R$.

We may use power series expansions to define complex versions of some of our favourite real functions:

$$\begin{aligned}
\exp(z) &= \sum_{n=0}^{\infty}\frac{z^n}{n!} = 1 + z + \frac{z^2}{2!} + \cdots \\
\sin(z) &= z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots \\
\cos(z) &= 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \cdots \ .
\end{aligned} \tag{4}$$

In all these cases the radius of convergence is $R = \infty$. Note that, with this definition of $\exp(z)$, the usual result; $\exp(z_1 + z_2) = \exp(z_1)\exp(z_2)$ holds.

As an introduction to more complicated functions, consider trying to define the complex logarithm, $\log(z)$. We define w to be a logarithm of z if

$$z = \exp(w) \tag{5}$$

If we write $w = u + iv$, then

$$z = e^u[\cos(v) + isin(v)] \tag{6}$$

and therefore, in particular, $|z| = e^u$. Formally, we can write

$$w = \log(z) = \log(|z|) + i\arg(z) + i(2\pi k)\ , \qquad k \in \mathcal{N} \tag{7}$$

where we have defined $-\pi < \arg(z) \leq \pi$. With $\arg(z)$ restricted to this region, each choice for the integer k specifies a *branch* of the logarithm function. The *principal value* of $\log(z)$ is obtained by setting $k = 0$.

Note that given some domain $\mathcal{D}$ (with $0 \notin \mathcal{D}$), then once we specify the value of $\log(z_0)$ for some $z_0 \in \mathcal{D}$, then $\log(z)$ is uniquely defined on $\mathcal{D}$. (This is equivalent to specifying the branch).

However, it is impossible to define $\log(z)$ in this way in any domain which contains a simple closed curve which encircles the origin, since $\log(0)$ is undefined.

For any holomorphic function, $f(z)$, which is many valued (i.e. has several branches), a point z_0 which behaves in the same way as the origin for $\log(z)$ is referred to as a *branch point*. In such a situation $f(z_0)$ may or may not be defined.

There exists a power series expansion for the logarithm, given by

$$\log(1+z) = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{z^n}{n} = z - \frac{z^2}{2} + \frac{z^3}{3} - \cdots\ , \tag{8}$$

with radius of convergence $R = 1$. In addition, all points on this circle also converge, except for $z = 1$.

Another example of a function with branches is provided by the function $w = z^a$, with $a \in \mathcal{C}$. This can be seen by

$$\begin{aligned} w = z^a &= \exp[a \log(z)] \\ &= \exp[a(\log(r) + i\theta)] \ . \end{aligned} \tag{9}$$

Here we have set $z = re^{i\theta}$. It is the choice of range for θ, for example $-\pi < \theta \leq \pi$, that specifies the branch. Note that the choice becomes unnecessary if $a \in \mathcal{Z}$, because in this case the choice of a different branch merely changes z^a by $e^{2\pi i m} = 1$. This fits with our elementary definition of z^n.

Example 1.1

$$\begin{aligned} i^i &= \exp[i \log(i)] \\ &= \exp\left[i\left(i\frac{\pi}{2} + i2\pi k\right)\right] \\ &= \exp\left[-\left(2k + \frac{1}{2}\right)\pi\right] \ . \end{aligned} \tag{10}$$

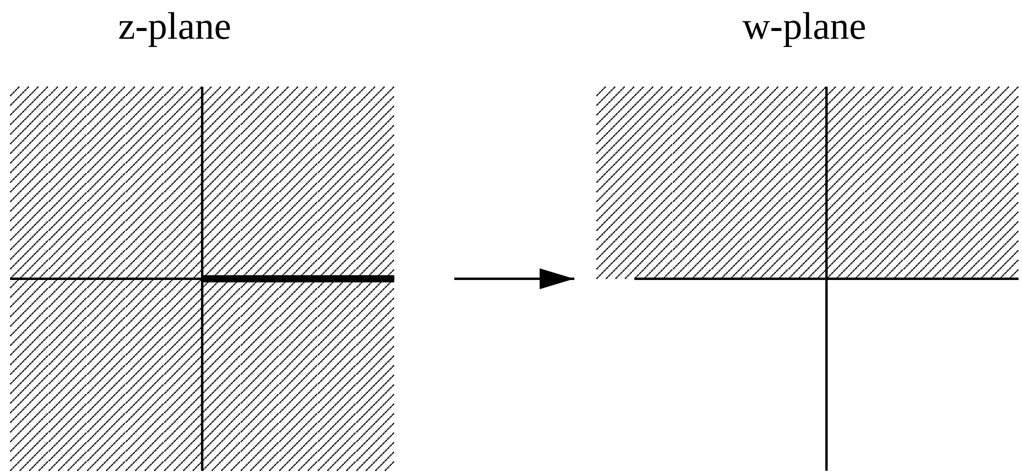
This takes infinitely many distinct **real** *values.*

Example 1.2 *$w = z^{p/q}$, where p/q is a rational number in lowest terms. In this case we obtain a function with q branches, having $z = 0$ as a branch point.*

Consider the special case $w = \sqrt{z}$. Write $z = re^{i\theta}$, $0 \leq \theta < 2\pi$ (note a different, more convenient choice of range here). Then $w = \rho e^{i\phi}$, with $0 \leq \phi < \pi$, where $\rho = +\sqrt{r}$ and $\phi = \theta/2$.

Geometrically, the whole complex plane with the positive real line removed is mapped to the upper half plane (see below)

As we have defined $f : w^2 = z$, it defines a branch of $\sqrt{z}$ with $z = 0$ as a branch point. We must have a cut along the positive real line because we cannot possibly have continuity there since $f(4) = 2$, but $f(4e^{i\theta}) \to -2$ as $\theta \to 2\pi$ from below.



1.1 Integration and Cauchy's Theorem

We shall consider curves C in the z-plane, parameterized by a single real variable t. that is to say,

$$C = \{\gamma(t) = \phi(t) + i\psi(t) \mid t \in [0, 1]\} \; , \tag{11}$$

where ϕ and ψ are continuously differentiable functions of t. A particularly useful example is when C is a union of line segment, or in other words, a polygon.

It is useful to be able to define the length of such a curve. In general, $\gamma(t)$ is called *rectifiable* if $\sup\{\sum \text{length of inscribed line segments}\}$ exists, and if so, then this quantity is referred to as the *length* of the curve. It is clear that a union of line elements is rectifiable and that the length is equal to the sum of the lengths of the included line segments.

Proposition 1.1 *Subject to the conditions introduced so far:*

$$\int_C f(z)\, dz = \int_0^1 (u + iv)(\phi' + i\psi')\, dt \tag{12}$$

Theorem 1.2 (Cauchy's Theorem) *Let $\mathcal{D}$ be a simply-connected domain in $\mathcal{C}$, and let C be a piecewise differentiable simple closed curve entirely contained in the interior of $\mathcal{D}$, then, if $f'(z)$ exists for each point $z \in \mathcal{D}$*

$$\int_C f(z)\, dz = 0 \; . \tag{13}$$

Note that simple-connectivity of $\mathcal{D}$ is used to ensure the existence of a domain $\mathcal{D}_1 \subset \mathcal{D}$ whose boundary is C.

From the point of view of physicists, this is one of the most important theorems of complex analysis. The proof is somewhat involved, and I will not go into it here. Rather, I'll just give a simple demonstration that the theorem is true under the stronger assumption that $f(z)$ is holomorphic in $\mathcal{D}$ (i.e. $f'(z)$ exists and is continuous in $\mathcal{D}$)

Proof 1.3

$$\int_C f(z)\,dz = \int_C (udx - vdy) + i\int_C (vdx + udy) \tag{14}$$

If C bounds the domain $\mathcal{D}_1$, we may apply Green's theorem:

$$\begin{aligned} &= -\int\int_{\mathcal{D}_1}\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right)dxdy + i\int\int_{\mathcal{D}_1}\left(\frac{\partial u}{\partial x}-\frac{\partial v}{\partial y}\right)dxdy \\ &= 0\ , \end{aligned} \tag{15}$$

by the Cauchy-Riemann equations.

Of equal importance to physicists is a result that allows us to calculate the values of integrals. As we'll see, Cauchy's theorem is required to prove this.

Theorem 1.4 (Cauchy's Integral Formula) *Let $f'(z)$ exist inside the open disc $B(z_0, r)$. Then for each point a with $|a - z_0| < r$, we have*

$$f(a) = \frac{1}{2\pi i}\int_{\partial B}\frac{f(z)}{z-a}\,dz\ . \tag{16}$$

(here $\partial B = \{z \mid |z - z_0| = r\}$).

Proof 1.5

$$\begin{aligned} \int_{\partial B}\frac{f(z)}{z-a}\,dz &= \int_{\partial B(a,\varepsilon)}\frac{f(z)}{z-a}\,dz \\ &= \lim_{\varepsilon\to 0}\int_{\partial B(a,\varepsilon)}\frac{f(z)-f(a)}{z-a}\,dz + \int_{\partial B(a,\varepsilon)}\frac{f(a)}{z-a}\,dz\ . \end{aligned} \tag{17}$$

The first term on the right hand side can be made arbitrarily small because the integrand is bounded and the length of the simple closed curve $\to 0$. As for the second term; substitute

$$\gamma(t) = a + \varepsilon e^{it} \qquad , \qquad t \in [0, 2\pi] \tag{18}$$

and the final answer then becomes

$$\begin{aligned} &= \int_0^{2\pi}\frac{f(a)}{\varepsilon e^{it}}i\varepsilon e^{it}\,dt \\ &= 2\pi i f(a)\ , \end{aligned} \tag{19}$$

as required.

This must seem like a lot of heavy-handed mathematical machinery, but I'd like to present one more important result before we get down to some examples of how these theorems are applied.

Theorem 1.6 (Power Series Expansions) *Let $f : U \to \mathcal{C}$ (U open), be complex differentiable, and let $B(z_0, \rho) \subseteq U$. Then, $\exists$ a unique power series*

$$f(z) = \sum_{n=0}^{\infty} c_n (z - z_0)^n \ , \tag{20}$$

with positive radius of convergence ($\geq \rho$), in some neighbourhood of z_0. Furthermore

$$c_n = \frac{1}{2\pi i} \int_{|z-z_0|=r} \frac{f(z)}{(z - z_0)^{n+1}} \, dz \ , \tag{21}$$

for $0 < r < \rho$.

This power series, if it exists, must be unique, since we can use Cauchy's integral formula to show that

$$c_n = \frac{1}{n!} f^{(n)}(z_0) \ . \tag{22}$$

Thus, we have formally shown the existence of a power series that we mentioned earlier.

1.2 Singularities and Laurent Series

Thus far we have been concerned with functions that are well-defined everywhere. We now turn to the various ways in which they can misbehave.

Definition 1.6 *A function $f(z)$ is singular at z_0 if there is no neighbourhood of z_0 in which it is holomorphic.*

$f(z)$ has a singularity at $z = \infty$ if $g(\eta)$, defined by $g(\eta) \equiv f(z)$ and $\eta \equiv z^{-1}$ has a singularity at $\eta = 0$.

Definition 1.7 *A singularity at z_0 is isolated if f is holomorphic on $B(z_0, \varepsilon) \backslash \{z_0\}$.*

There are three types of isolated singularities:

1. z_0 is *removable* if, with a suitable definition of $f(z_0)$, we obtain a function holomorphic on all of $B(z_0, \varepsilon)$.

2. z_0 is called a *pole* of f if , for some $m \geq 1$, the function $g(z) = (z - z_0)^m f(z)$ has a removable singularity at z_0. The smallest value of m for which this condition holds is called the *order* of the pole at z_0.

3. Otherwise, z_0 is called an *essential* singularity.

For an example of a non-isolated singularity see $f(z) = \csc(z^{-1})$.

Definition 1.8 *A function $f(z)$ is meromorphic on the domain $\mathcal{D}$ if all its singularities in $\mathcal{D}$ are isolated and non-essential.*

Lemma 1.1 *If $f(z)$ is meromorphic, then locally $f(z)$ is expressible as a quotient $g(z)/h(z)$, where g and h are holomorphic. Conversely, if g and h are holomorphic on a domain $\mathcal{D}$, and $h \not\equiv 0$, then $g(z)/h(z)$ is meromorphic (perhaps with removable singularities).*

Proof 1.7 ($\Rightarrow$) *is clear from definition, just take $h(z) = (z - z_0)^m$*
($\Leftarrow$) *: If $h(z_0) \neq 0$, then $g(z)/h(z)$ is holomorphic near $z = z_0$. If $h(z_0) = 0$, then, since h is holomorphic, we may use a power series expansion about z_0 to express $h(z) = (z - z_0)^m \tilde{h}(z_0)$, with $h(z_0) \neq 0$.*

Furthermore $g(z) = (z - z_0)^k \tilde{g}(z_0)$, for some $k > 0$ [it may be that g and h vanish at the same point z_0, in which case g/h has a removable singularity at z_0]. We can now define

$$\frac{g(z)}{h(z)} = (z - z_0)^{k-m} \frac{\tilde{g}(z)}{\tilde{h}(z)} \ . \tag{23}$$

Note how the following definition parallels that of a pole and its order:

Definition 1.9 *Let $f : U \to \mathcal{C}$ be a holomorphic function defined on some open subset $U \subset \mathcal{C}$, and let $f(z_0) = 0$. We say that z_0 is a zero of f, and define its order, k, by the condition*

$$f(z_0) = f'(z_0) = \cdots = f^{(k-1)}(z_0) = 0 \qquad , \qquad f^{(k)} \neq 0 \tag{24}$$

If no such finite value k exists (i.e. $f^{(i)}(z_0) = 0 \ \forall \ 0 \leq i < \infty$), we say that z_0 is a zero of infinite order. In this case, the assumption of holomorphy implies $f(z_0) \ \forall \ z$ in some neighborhood of z_0.

1.2.1 Laurent Series

We'll now generalize our power series discussion to functions with singularities. Recall the expansion we derived earlier for holomorphic functions, and compare to the following.

If $f(z)$ is holomorphic in the annulus $R_1 < |z - z_0| < R_2$ (note, f not required to be holomorphic at z_0) then we may express

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n \tag{25}$$

and this series is referred to as a *Laurent* series for f. This can be decomposed as

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=-\infty}^{-1} a_n (z - z_0)^n \tag{26}$$

where the first term is referred to as the *subsidiary part* and the second term as the *principal part.* Both parts are unique, and, as earlier,

$$a_n = \int_C \frac{f(z)}{(z - z_0)^{n+1}}\, dz \tag{27}$$

with C a simple closed curve containing z_0 in the domain of f.

The Laurent series converges absolutely, and uniformly in any closed subset of the annulus.

1.3 Calculus of Residues and Contour Integration

After all the waiting, we'll get down to many applications of what we've learned in this section.

Definition 1.10 (Residue) *The coefficient a_{-1} of $(z - z_0)^{-1}$ in the Laurent expansion is called the residue of $f(z)$ at $z = z_0$.*

Note that Cauchy's integral formula gives

$$c_{-1} = \frac{1}{2\pi i} \int_C f(z)\, dz \tag{28}$$

where $\mathcal{C}$ is a small circle of radius ε say, about z_0.

This leads us to the most important result of complex analysis (as far as we'll be concerned in this course)

Theorem 1.8 (Residue theorem (Cauchy)) *If $f(z)$ is meromorphic in some domain $\mathcal{D}$, which contains the subdomain $\mathcal{D}_1$, bounded by the simple closed curve C (with no singularities on C), then*

$$\int_C f(z)\,dz = 2\pi i\left(\sum_{j=1}^{N} R_j\right) , \tag{29}$$

where R_j is the residue at the isolated singularity at $z = z_j$ $j = 1, 2, \ldots, N$ in $\mathcal{D}_1$.

Proof 1.9 *Easy!!!!*

The problem with this result is actually to calculate the residues R_j. Two hints:

1. Use the Laurent expansion about z_j if this is easily calculated.

2. Assume that $z_j = 0$ is a pole of order m for $f(z)$. Then R_j is given by

$$R_j = \frac{1}{(m-1)!}\lim_{z\to 0}\frac{d^{m-1}}{dz^{m-1}}[z^m f(z)] \tag{30}$$

Note that for a simple pole $(m = 1)$ $R_j = \lim_{z\to 0}(zf(z))$.

Proof 1.10 *Assume that*

$$f(z) = \sum_{n=-m}^{\infty} c_n z^n , \tag{31}$$

and differentiate term by term.

Now, the main point of many of the recent results is that they allow us to analytically evaluate many definite integrals which would seem almost impossible without the techniques we'll learn here. Although these methods will apply to complex integrals, we'll see that they provide an excellent method for the evaluation of *real* integrals.

In all the examples that follow, the procedure is the same. We have a definite real integral to evaluate, and we do this by first making the integral complex and including the range of integration (e.g. $[-a, a]$, where a frequently tends to infinity), inside some suitable simple closed curve C in $\mathcal{C}$. With minor adaptation, we suppose that the complex integrand has no singular points on $\mathcal{C}$. We then apply the residue theorem and arrange that all contributions to the integral, other than that on the real axis, are vanishingly small.

Warning: Watch out for branching. Recall that this occurs, for example, when $f(x)$ contains a factor x^α, where α is non-integral, or $\log(x)$.

We shall consider various different cases:

1. Integrals of the form
$$\int_0^{2\pi} R(\cos(\theta), \sin(\theta))\, d\theta \tag{32}$$
Method; write
$$\cos(\theta) = \left(\frac{z+z^{-1}}{2}\right) \quad , \quad \sin(\theta) = \left(\frac{z-z^{-1}}{2i}\right) \tag{33}$$
(i.e. $z = e^{i\theta}$, $dz = iz\, d\theta$), and take C to be defined by $|z| = 1$.

2. $f(z)$ meromorphic with finite number of poles in the region $\Im(z) > 0$ and no poles on the real axis. Assume that $|z^2 f(z)| \le M$ whenever $\Im(z) \ge 0$ and $|z| > R$, say. Label the poles as $a_1, \ldots, a_k$. Then
$$\int_{-\infty}^{\infty} f(x)\, dx = 2\pi i \sum_{j=1}^{k} \mathrm{Res}_f(a_j) \tag{34}$$
To see this, integrate around a contour consisting of an upper semicircle of radius R together with the real interval $[-R, R]$. As $R \to \infty$, we obtain the above result.

3. The same general method applies to
$$\int_{-\infty}^{\infty} f(z) e^{imz}\, dz \ , \tag{35}$$
subject to the weaker assumption that $\lim_{z\to\infty} f(z) = 0$ for $\Im(z) \ge 0$, $m \in \mathcal{R}$.

Lemma 1.2 (Jordan's Lemma) *Let Γ be an upper semicircle, of radius R, centered at 0. Let $f(z)$ have a finite number of poles or removable singularities in the upper half plane ($\Im(z) > 0$), and let $\lim_{\substack{z\to\infty \\ \Im(z)\ge 0}} f(z) \to 0$. Then*
$$\int_\Gamma e^{imz} f(z)\, dz \to 0 \quad as \quad R \to \infty \ , \tag{36}$$
for $m > 0$.

Proof 1.11 *Choose R so large that $|f(z)| < \varepsilon\ \forall\ z$ on Γ. Now*
$$\begin{aligned} |e^{imz}| &= |\exp[imR(\cos\theta + i\sin\theta)]| \\ &= \exp(-mRsin\theta) \ . \end{aligned} \tag{37}$$

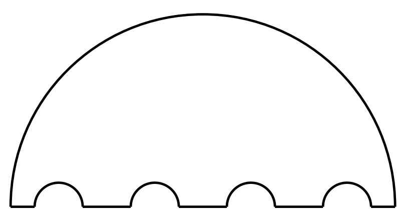
Therefore

$$\begin{aligned}
\left|\int_\Gamma f(z)e^{imz}\,dz\right| &= \left|\int_0^\pi f(Re^{i\theta})\exp(imRe^{i\theta})Re^{i\theta}\,d\theta\right| \\
&< \varepsilon\int_0^\pi e^{-mR\sin\theta}R\,d\theta \\
&= 2R\varepsilon\int_0^{\pi/2} e^{-mR\sin\theta}\,d\theta \\
&\le 2R\varepsilon\int_0^{\pi/2} e^{-2mR\theta/\pi}\,d\theta \\
&= \frac{\pi\varepsilon}{m}(1-e^{-mR}) \\
&< \frac{\pi\varepsilon}{m}\ , \qquad (38)
\end{aligned}$$

as required.

Note, that we can distinguish between real and imaginary parts, and thereby replace e^{imz} by $\cos(mz)$ or $\sin(mz)$.

4. We can adapt this general technique to allow for a finite number of singular points on the real axis. Diagramatically, we replace our original contour by



In this form, the following lemma can be useful

Lemma 1.3 *Suppose that $f(z)$ has a simple pole at $z = 0$, and we integrate round a circular arc of radius R between the angles θ_1 and θ_2. Then*

$$\int_{re^{i\theta_2}\to re^{i\theta_2}} f(z)\,dz = \text{residue at } 0 \times i(\theta_2-\theta_1) + \varepsilon R\ , \qquad (39)$$

where $\varepsilon R \to \infty$, as $R \to 0$.

Proof 1.12 *Expand $f(z)$ as (residue at $0)/z$ + (regular function), and use Jordan's lemma.*

5. Branching Problems: Let $R(z)$ be rational, and $|z^2 f(z)| \leq M$ outside a large semicircle. Assume that $R(z)$ has at worst a simple pole at $z = 0$, and that there are no other poles on the real axis. If $0 < \alpha < 1$, we can determine

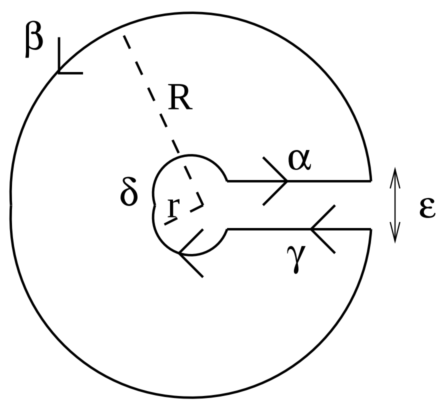
$$\int_0^\infty x^\alpha R(x)\,dx \tag{40}$$

as follows:

Cut $\mathcal{C}$ along the non-negative real axis. Then $z = e^{i\theta}$ is uniquely defined with $0 < r$ and $0 < \theta < 2\pi$. Write the logarithm over this domain as $z \to \log(r) + i\theta$, and define $z^\alpha = e^{\alpha \log z}$.

With these conventions we have fixed the branches of $\log(z)$ and z^α with which we work in this problem.

Choose $\varepsilon > 0$ so small and $R > 0$ so large that the contour encloses all poles except the one that may exist at $z = 0$.



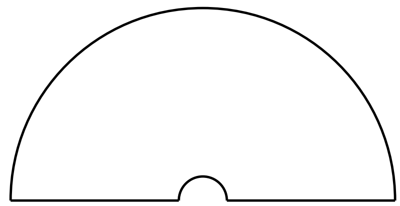
Then

$$\begin{aligned}\lim_{\varepsilon\to 0}\int_{\alpha_{\varepsilon,r}} z^\alpha R(z)\,dz &= \int_r^R x^\alpha R(x)\,dx \\ &= \lim_{\varepsilon\to 0}\int_{\gamma_{\varepsilon,r}} z^\alpha R(z)\,dz \\ &= -e^{2\pi i\alpha}\int_r^R x^\alpha R(x)\,dx\ ,\end{aligned} \tag{41}$$

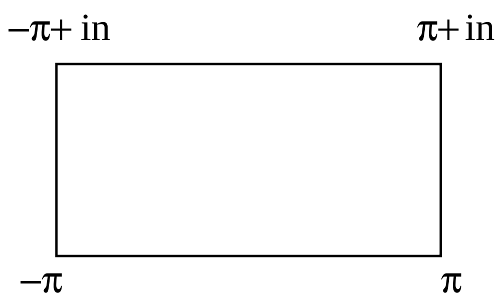
because on $\gamma_{\varepsilon,r}$ we are about to cross over to another branch of the integrand. Therefore,

$$\begin{aligned}\lim_{R\to\infty}\lim_{\varepsilon\to 0}\int_{\alpha\cup\beta\cup\delta\cup\gamma} z^\alpha R(z)\,dz &= (1 - e^{2\pi i\alpha})\int_0^\infty x^\alpha R(x)\,dx \\ &= 2\pi i\left(\sum_j \mathrm{Res}_j(z^\alpha R(z))\right)\ .\end{aligned} \tag{42}$$

6. With $\log(z)$ rather than z^α in the integrand, it may be more convenient to replace the given contour by



7. For some integrands it may be useful to use a large rectangle, for example if a trigonometric or hyperbolic function appears in the denominator.



1.4 Worked Examples: Contour Integration

1.4.1 Example 1

$$\int_0^\infty \frac{dx}{(x^2+1)^2(x^2+4)}$$

As with all our examples we choose a contour and extend the integral around the contour. Here consider

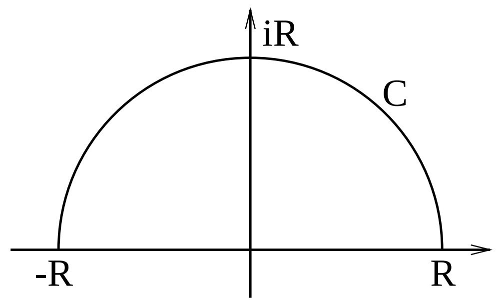
$$\oint_\Gamma \frac{dz}{(z^2+1)^2(z^2+4)} \;, \tag{43}$$

where Γ is defined to be the contour consisting of that part of the real axis from $-R$ to R $(R > 0)$, and the semicircle, radius R, center 0 in the upper half-plane.

Then

$$\oint_\Gamma \frac{dx}{(z^2+1)^2(z^2+4)} = \int_{-R}^{+R} \frac{dx}{(x^2+1)^2(x^2+4)} + \int_C \frac{dz}{(z^2+1)^2(z^2+4)}$$

Now let $R \to \infty$. Clearly the second term in the above goes to zero in this limit. We now evaluate the left hand side using the calculus of residues.



The integrand has a pole of order 2 at $z = i$, and a pole of order 1 at $z = 2i$, which lie within the contour Γ. Now;

$$\begin{aligned}
\text{Res}(z = i) &= \lim_{z\to i} \frac{d}{dz}\left[\frac{1}{(z+i)(z+i)(z^2+4)}\right] \\
&= \lim_{z\to i}\left[\frac{-2}{(z+i)^3(z^2+4)} - \frac{2z}{(z+i)^2(z^2+4)^2}\right] \\
&= \frac{-2}{3(2i)^3} - \frac{2i}{9(2i)^2} \\
&= -\frac{i}{36}\ . \qquad (44)
\end{aligned}$$

Similarly

$$\begin{aligned}
\text{Res}(z = 2i) &= \lim_{z\to 2i}\left[\frac{1}{(z^2+1)^2(z+2i)}\right] \\
&= -\frac{i}{36}\ . \qquad (45)
\end{aligned}$$

Therefore

$$\begin{aligned}
\oint_\Gamma \frac{dz}{(z^2+1)^2(z^2+4)} &= 2\pi i\left(-\frac{i}{36} - \frac{i}{36}\right) \\
&= \frac{\pi}{9}\ . \qquad (46)
\end{aligned}$$

So we obtain

$$\int_{-\infty}^{\infty} \frac{dx}{(x^2+1)^2(x^2+4)} = \frac{\pi}{9}\ . \qquad (47)$$

Finally, since the integrand is an even function of x, this implies

$$\int_0^{\infty} \frac{dx}{(x^2+1)^2(x^2+4)} = \frac{\pi}{18}\ . \qquad (48)$$

1.4.2 Example 2

$$\int_{-\infty}^{\infty} \frac{\cos(x)\,dx}{x^2+a^2} \qquad , \ a>0$$

This requires a little more cunning. Consider

$$\oint_\Gamma \frac{e^{iz}\,dz}{z^2+a^2}$$

with Γ the same contour as in example 1. We then obtain:

$$\oint_\Gamma \frac{e^{iz}\,dz}{z^2+a^2} = \int_{-R}^{R} \frac{\cos(x)\,dx}{x^2+a^2} + \int_{-R}^{R} \frac{i\sin(x)\,dx}{x^2+a^2} + \int_C \frac{e^{iz}\,dz}{z^2+a^2}$$

Obviously, the final term goes to zero as $R \to \infty$, thus

$$\oint_\Gamma \frac{e^{iz}\,dz}{z^2+a^2} = \int_{-\infty}^{\infty} \frac{\cos(x)\,dx}{x^2+a^2} + i\int_{-\infty}^{\infty} \frac{\sin(x)\,dx}{x^2+a^2}$$

The integrand on the left hand side has simple poles at $z = \pm ia$. However, since $a > 0$ we have chosen the above contour, and hence only require the pole at $z = +ia$ (enclosed by the contour). So;

$$\begin{aligned} \text{Res}(z=ia) &= \lim_{z\to ia}\left[\frac{e^{iz}}{z+ia}\right] \\ &= \frac{e^{-a}}{2ia} \ . \end{aligned} \tag{49}$$

Therefore

$$\int_{-\infty}^{\infty} \frac{\cos(x)\,dx}{x^2+a^2} + i\int_{-\infty}^{\infty} \frac{\sin(x)\,dx}{x^2+a^2} = 2\pi i\left(\frac{e^{-a}}{2ia}\right) = \frac{\pi}{a}e^{-a} \ . \tag{50}$$

Finally, taking the real parts of both sides we obtain

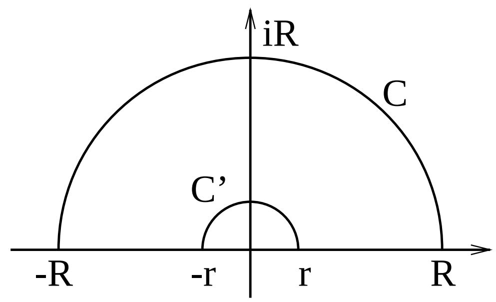
$$\int_{-\infty}^{\infty} \frac{\cos(x)\,dx}{x^2+a^2} = \frac{\pi}{a}e^{-a} \ . \tag{51}$$

1.4.3 Example 3

$$\int_{-\infty}^{\infty} \frac{x-\sin(x)}{x^3}\,dx \ . \tag{52}$$

We'll begin by integrating by parts twice, to put the integral into a form that is more simple to handle by complex methods.

$$\begin{aligned} \int_{-\infty}^{\infty} \frac{x-\sin(x)}{x^3}\,dx &= \left[-\frac{(x-\sin(x))}{2x^2}\right]_{-\infty}^{\infty} + \frac{1}{2}\int_{-\infty}^{\infty} \frac{1-\cos(x)}{x^2}\,dx \\ &= \left[-\frac{(1-\cos(x))}{2x}\right]_{-\infty}^{\infty} + \frac{1}{2}\int_{-\infty}^{\infty} \frac{\sin(x)}{x}\,dx \\ &= \frac{1}{2}\int_{-\infty}^{\infty} \frac{\sin(x)}{x}\,dx \ , \end{aligned} \tag{53}$$



since both terms in the square brackets vanish.

Now to use our complex variable machinery. Consider

$$\int_\Gamma \frac{e^{iz}}{z}\, dz \ , \tag{54}$$

where Γ is the contour shown

Now, this integral is zero, by Cauchy's theorem. We may write it as

$$\int_\Gamma \frac{e^{iz}}{z}\, dz = \int_C \frac{e^{iz}}{z}\, dz + \int_{-R}^{-r} \frac{e^{ix}}{x}\, dx + \int_{C'} \frac{e^{iz}}{z}\, dz + \int_r^R \frac{e^{ix}}{x}\, dx \ . \tag{55}$$

Now, as $R \to \infty$ the integral around the large semicircle, C becomes zero. Thus,

$$\int_{-\infty}^{-r} \frac{e^{ix}}{x}\, dx + \int_r^\infty \frac{e^{ix}}{x}\, dx = -\int_{C'} \frac{e^{iz}}{z}\, dz \ . \tag{56}$$

Now, since $(e^{iz} - 1)/z$ has a removable singularity at the origin, we have

$$\begin{aligned}\int_{-\infty}^{-r} \frac{e^{ix}}{x}\, dx + \int_r^\infty \frac{e^{ix}}{x}\, dx &= -\int_{C'} \frac{1}{z}\, dz \\ &= -\int_\pi^0 \frac{1}{re^{i\theta}}(ire^{i\theta})\, d\theta \\ &= \pi i \ . \end{aligned} \tag{57}$$

Letting $r \to 0$, we have

$$\int_{-\infty}^\infty \frac{e^{ix}}{x}\, dx = \pi i \ . \tag{58}$$

Taking imaginary parts we finally obtain

$$\int_{-\infty}^\infty \frac{\sin(x)}{x}\, dx = \pi \ , \tag{59}$$

so that by our initial integration by parts:

$$\int_{-\infty}^\infty \frac{x - \sin(x)}{x^3}\, dx = \frac{\pi}{2} \ . \tag{60}$$

2 Exact and Approximate Evaluation of Sums and Integrals

Definition 2.1 *An asymptotic sequence is a set of functions* $\{\phi_n(z)\}$ *such that*

$$\phi_{n+1}(z) = o(\phi_n(z)) \quad , \quad as \quad z \to z_0 \ . \tag{61}$$

(Usually we take $\phi_n = z^{-n}$*, and* $z_0 = \infty$*).*

Definition 2.2 *If* $\{\phi_n(z)\}$ *is an asymptotic sequence, then the asymptotic expansion for a function* $f(z)$ *is*

$$f(z) \sim \sum_{r=0}^{\infty} a_r \phi_r(z) \ , \tag{62}$$

provided that

$$\left| f(z) - \sum_{r=0}^{n-1} a_r \phi_r(z) \right| = O(\phi_n) \quad or \quad o(\phi_{n-1}) \ , \tag{63}$$

as $z \to z_0$*. (i.e. the remainder after n terms is smaller than the last included term, or the same order as the first neglected term)*

Some important properties of asymptotic expansions are (Here consider $f \sim \sum a_r z^{-r}$ always.):

1. Asymptotic expansions depend on the sector (i.e $\arg(z)$). For example,

$$e^{-z} + \frac{1}{z} \sim \frac{1}{z} \quad , \quad |\arg(z)| < \frac{\pi}{2} \ . \tag{64}$$

(no more terms, since e^{-z} is smaller than any power of z). But,

$$e^{-z} + \frac{1}{z} \not\sim \frac{1}{z} \quad , \quad \frac{3\pi}{2} < |\arg(z)| < \frac{\pi}{2} \ . \tag{65}$$

If the asymptotic expansion of $f(z)$ is different in different sectors, we say it exhibits *Stokes' phenomenon.*

Theorem 2.1 *If* $f(z)$ *is single-valued and holomorphic for* $|z| \geq a$*, and*

$$f(z) \sim \sum_{r=0}^{\infty} a_r z^{-r} \ , \tag{66}$$

is valid for all $\arg(z)$ *(i.e., doesn't exhibit Stokes' phenomenon), then the series is in fact convergent; i.e.*

$$f(z) = \sum_{r=0}^{\infty} a_r z^{-r} \ . \tag{67}$$

Proof 2.2 *f is single-valued, holomorphic for $|z| \geq a$, therefore $f(z) = \sum_{-\infty}^{\infty} c_n z^n$, with*

$$c_n = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z^{n+1}} \, dz \ . \tag{68}$$

Choose C to be a large circle, radius R. Then

$$|c_n| \leq \frac{1}{2\pi} \left| \int_0^{2\pi} f(Re^{i\theta}) \, d\theta \right| \frac{1}{R^n} \ . \tag{69}$$

Now, since $f(z) \sim \sum_0^{\infty} a_r z^{-r}$, $f \to a_0$ as $|z| \to \infty$. Therefore, we can find M such that $|f| < M$ for large enough $|z|$. This implies that

$$|c_n| < \frac{M}{R^n} \quad , \quad (n > 0) \ . \tag{70}$$

But R can be as large as we like, so $c_n = 0$ for $n > 0$. Also, $a_n = c_{-n}$, since asymptotic expansions are unique (see next property).

2. For a given range of $\arg(z)$, the asymptotic expansion of $f(z)$ is unique

   To see this, let $f(z) \sim \sum_0^{\infty} a_n z^{-n}$ as $z \to \infty$ in a given sector. Then $f \sim a_0$ as $z \to \infty$, and

   $$(f - a_0)z \to a_1 \ , \tag{71}$$

   as $z \to \infty$. Similarly

   $$\left(f - \sum_0^{n-1} a_r z^{-r} \right) z^n \to a_n \ , \tag{72}$$

   as $z \to \infty$. Thus, the coefficients $\{a_n\}$ are uniquely defined. (Note that the converse does not hold).

3. Asymptotic expansions can be added and multiplied as if they were convergent

Let's now see how we might calculate asymptotic expansions for several different classes of functions.

2.1 Watson's Lemma and Laplace's Method

Lemma 2.1 (Watson) *Let*

$$F(z) = \int_0^\infty e^{-zt}\phi(t)\,dt \quad , \quad \Re(z) \geq \delta > 0\ , \tag{73}$$

with $\phi = \sum_0^\infty b_n t^n$ *for* $|t| < R$. *Then*

$$F(z) \sim \sum_0^\infty \frac{b_n n!}{z^{n+1}}\ . \tag{74}$$

It is important to note here that the right hand side is merely the left hand side expanded and integrated term by term. However, it is the fact that the result is an asymptotic expansion that is nontrivial. This is because the summation need not converge uniformly in t for all t in the range of integration. Thus, it is not clear that we can interchange the order of integration and summation.

Under more restrictive circumstances we could just integrate by parts to show this. However, Watson's lemma works in more general situations, and a more subtle proof is required. I won't give the proof here, although if we have time I may come back and supply it later.

Laplace's method is a way to calculate asymptotic expansions for functions of the form

$$F(x) = \int_a^b e^{xh(u)}g(u)\,du\ , \tag{75}$$

as $x \to +\infty$ (x real).

The rough argument is that the largest contribution comes from the biggest value of $h(u)$, say $h(u_0)$, which is exponentially larger than any other contribution. We'll see how this works in 2 distinct situations. In both these, Watson's lemma is crucial to obtaining the final result.

1. <u>$h'(u_0) = 0$; (a calculus-type maximum)</u>

   Begin by taking Taylor series of h and f about u_0:

$$\begin{aligned} F(x) &= \int_a^b \exp\left\{x\left[h(u_0) + \frac{1}{2}(u-u_0)^2 h''(u_0) + \cdots\right]\right\}[g(u_0) + (u-u_0)g'(u_0) + \cdots]\,du \\ &\sim e^{xh(u_0)}\int_{-\infty}^{\infty} \exp\left[\frac{1}{2}x\tau^2 h''(u_0)\right][g(u_0) + \cdots]\,d\tau\ , \end{aligned} \tag{76}$$

where $\tau = u - u_0$ and we can extend the range of integration to $(-\infty, \infty)$ since any extra contributions are negligible (the dominant contribution comes from $\tau = 0$)

Now integrate term by term using Watson's lemma, to obtain

$$F(x) \sim e^{xh(u_0)} \left[g(u_0) \sqrt{\frac{2\pi}{-xh''(u_0)}} + O(x^{-3/2}) \right] . \tag{77}$$

2. $\underline{h'(u_0) \neq 0}$

In this case we have $u_0 = b$ (or a). Now a Taylor expansion about u_0 yields

$$\begin{aligned} F(x) &\sim e^{xh(u_0)} g(u_0) \int_{-\infty}^{0} e^{x\tau h'(u_0)}\, d\tau \\ &\sim e^{xh(u_0)} g(u_0) \frac{1}{xh'(u_0)} + O(x^{-2}) . \end{aligned} \tag{78}$$

Let's see immediately how this works by applying what we've just learned to an example that is well-known (the result at least) to some of you.

Consider expanding $\Gamma(x+1)$ as $x \to \infty$, for x real. If you know what the Γ-function is, you'll know that the answer we hope to get is known as *Stirling's formula*, and is very useful in all types of situations in physics. If you haven't heard of the Γ-function, then this will still be a good example of how to use Laplace's method.

The Γ-function has an integral expression given by

$$\Gamma(x+1) = \int_0^\infty e^{-t} t^x \, dt . \tag{79}$$

Although this appears to already be in the correct form to apply Laplace's method to, we must transform it because the largest value of the exponential occurs at $t = 0$, where t^x vanishes. Therefore, we'll write

$$\begin{aligned} \Gamma(x+1) &= \int_0^\infty e^{-t + x\log(t)}\, dt \\ &= x^{x+1} \int_0^\infty e^{x(-u + \log(u))}\, du , \end{aligned} \tag{80}$$

where we have made the change of variables $t = xu$ not because it is essential, but because it makes things neater because the position of the maximum stays at a fixed point and doesn't go to infinity as we take the asymptotic limit.

Now, $h(u) = -u + \log(u)$ has a maximum at $u = 1$. We will only be interested in this example in getting the leading term of the expansion. Therefore, we Taylor expand everything about $u = 1$ as far as the first non-constant term. This gives

$$\begin{aligned} \Gamma(x+1) &= x^{x+1} \int_0^\infty \exp\left\{ x \left[h(1) + \frac{1}{2}(u-1)^2 h''(1) + \cdots \right] \right\} du \\ &\sim x^{x+1} e^{-x} \int_{-1}^\infty e^{xs^2/2}\, ds \ , \end{aligned} \tag{81}$$

with $s = u - 1$. The $\sim$ here comes from Watson's lemma. We could have expanded to higher order but have chosen not to. We can extend the limit of integration since any contribution from the range $(-\infty, -1)$ is subdominant. Thus, to leading order we obtain

$$\Gamma(x+1) \sim x^{x+1} e^{-x} \left(\frac{2\pi}{x} \right)^{1/2} \ , \tag{82}$$

which you may recognize as the leading term in Stirling's formula.

2.2 Riemann-Lebesgue Lemma and Method of Stationary Phase

Lemma 2.2 (Riemann-Lebesgue) *Let $q(t)$ be piecewise continuous on the compact interval $[a, b]$. Then, for real x*

$$I(x) \equiv \int_a^b e^{ixt} q(t)\, dt = o(1) \ , \qquad \text{as} \quad x \to \infty \ . \tag{83}$$

Proof 2.3 *Assume w.l.o.g that $q(t)$ is continuous on $[a, b]$ so that for any given $\epsilon > 0$, the interval $[a, b]$ can be divided into $n - 1$ subintervals in each of which $q(t)$ varies by less than 2ϵ. Then, $\exists \{t_n\}$ such that $a = t_0 < t_1 < \ldots < t_n = b$, with $|q(t) - q(t_k)| < \epsilon$, for $t \in [t_{k-1}, t_k]$. Also, $q(t)$ is bounded in $[a, b]$, so $\exists Q$ such that $|q(t)| < Q \ \forall \ t \in [a, b]$. Then*

$$I(x) = \sum_1^n q(t_i) \int_{t_{i-1}}^{t_i} e^{ixt}\, dt + \sum_1^n q(t_i) \int_{t_{i-1}}^{t_i} [q(t) - q(t_i)] e^{ixt}\, dt \ . \tag{84}$$

Now,

$$\left| \int_{t_{i-1}}^{t_i} e^{ixt}\, dt \right| = \left| \frac{e^{ixt_i} - e^{ixt_{i-1}}}{ix} \right| \le \frac{2}{x} \ , \tag{85}$$

and

$$\left| \int_{t_{i-1}}^{t_i} [q(t) - q(t_i)] e^{ixt}\, dt \right| \le \epsilon (t_i - t_{i-1}) \ . \tag{86}$$

Putting these together, we obtain

$$|I(x)| \le Q \frac{2}{x} n + \epsilon (b - a) \ , \tag{87}$$

which can be made as small as you like by choosing ϵ small enough and/or choosing x large enough.

The method of stationary phase is a way to calculate asymptotic expansions for functions of the form

$$I(x) = \int_a^b e^{ixh(u)} g(u)\, du \ . \tag{88}$$

(with h twice differentiable and g once differentiable) as $x \to +\infty$ (x real).

The rough argument is that the largest contribution comes from the place where the integrand oscillates least, since where rapid oscillations occur, one expects cancellations to occur. More formally, making the substitution $h(u) = t$, and using the Riemann-Lebesgue lemma, we see that the above expression is $o(1)$ unless there's a place where $h' = 0$.

2.3 The Saddle-Point Method

This technique is used to find the asymptotic form of general functions of the type

$$I(z) = \int_A^B e^{zh(t)} g(t)\, dt \ , \tag{89}$$

where A and B are complex in general, and h is holomorphic, as $z \to \infty$.

Writing $h = \phi + i\psi$, note that if h is not holomorphic, the dominant contributions are hard to find. The Laplace contributions from the maximum value of ϕ may be cancelled by rapid oscillations of ψ, and vice-versa. However, as we'll see, for holomorphic h, the Cauchy-Riemann equations help us.

Note also in passing that the magnitude of e^{zh} is determined by ϕ via $|e^{zh}| = e^{z\phi}$.

2.3.1 Notes on Paths of Constant $\Im(h) = \psi$

Paths of constant ψ are paths along which ϕ increases/decreases most rapidly: To see this, note that the Cauchy-Riemann equations give

$$\frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y} \quad , \quad \frac{\partial \phi}{\partial y} = -\frac{\partial \psi}{\partial x} \ , \tag{90}$$

and therefore

$$\nabla\phi . \nabla\psi = 0 \ . \tag{91}$$

Now, $\nabla\phi$ is normal to the lines of constant ϕ, so paths of constant ϕ are orthogonal to paths of constant ψ. These paths form a grid over a region where $h(t)$ is holomorphic. Also, paths of constant ψ are paths of maximum descent/ascent of ϕ.

The biggest value of ϕ along a path of constant ψ occurs at a saddle point of ϕ (or at the endpoint). To see this, let γ be any path from A to B. Assume $\phi(t) > \phi(A)$, and $\phi(t) > \phi(B)$ for some t on γ.

Let $\phi(t_0)$ be the biggest value of ϕ on γ. Then $\nabla\phi.\mathbf{b} = 0$ at t_0, where $\mathbf{b}$ is tangent to γ. If γ is a path of constant ψ, then $\nabla\psi.\mathbf{b} = 0$ on γ. Together with the Cauchy-Riemann equations, these imply $\nabla\phi = \nabla\psi = 0$ at t_0, i.e. $h'(t_0) = 0$.

Thus, the stationary phase point and the Laplace points coincide. Therefore, t_0 is a *saddle point* of ϕ because the C-R equations imply $\nabla^2\phi = 0$, which means that the eigenvalues must have opposite signs.

2.3.2 Nature of Paths of $\psi =$ constant near a General Point

In general (i.e. $h' \neq 0$) there is exactly one path of constant ψ through each point t_0:

Near t_0, we can write $h(t) = h(t_0) + (t - t_0)h'(t_0) + \ldots$. Further, write $h'(t_0) = |h'(t_0)|e^{i\alpha}$, and $(t - t_0) = re^{i\theta}$. The path of constant $\Im(h)$ near t_0 is given by

$$\Im[(t - t_0)h'(t_0)] = 0 \ . \tag{92}$$

So $\Im(h(t)) = \Im(h(t_0))$. i.e. $|h'(t_0)|r\sin(\alpha + \theta) = 0$, and thus $\theta = -\alpha$. Hence, there is only one path of constant ψ, at angle $-\alpha$ ($\pi + \alpha$ is the same path).

2.3.3 Nature of Paths of $\psi =$ constant near Saddle Points

If $h'(t_0) = 0$, write $h(t) = h(t_0) + (t - t_0)^2 h''(t_0)/2 + \ldots$, near t_0. Further, write $h''(t_0) = |h''(t_0)|e^{i\alpha}$, and $(t - t_0) = re^{i\theta}$. Then, as above, we obtain $\sin(2\theta + \alpha) = 0$, which implies

$$\theta = -\alpha/2 \ , \qquad \text{or} \quad \theta = -\alpha/2 + \pi/2 \ . \tag{93}$$

So the two paths are at right angles. One is the steepest ascent of ϕ, the other is the steepest descent of ϕ. (Note, if $h''(t_0) = 0$ also, get three paths).

2.3.4 Method in Theory

1. If there exists a path γ, from A to B of constant ψ, then let γ be given by $t = t(s)$, for real s. Then

$$\int_A^B e^{zh(t)} g(t)\, dt = \int_\gamma \left(e^{z\phi} g(t) \frac{dt}{ds} \right) ds e^{i\psi(A)z} \ , \tag{94}$$

which can be evaluated by Laplace's method. If the original path of integration has to be deformed onto γ, there may be additional contributions due to, for example, poles between the two paths.

2. If there exists a path γ, from A to B of constant ϕ, then use the method of stationary phase.

3. In general, suppose $\phi(A) \geq \phi(B)$. Try to find a path from A to B consisting of curves of constant ϕ and curves of constant ψ.

2.3.5 Method in Practice

1. Find the saddle points (i.e. $h' = 0$)

2. Use the saddle point with largest $\Re(h)$

3. Find the steepest *descent* path across the saddle:

$$\begin{aligned} h(t) &= h(t_0) + (t - t_0)^2 h''(t_0)/2 + \dots \ , \\ h''(t_0) &= |h''(t_0)|e^{i\alpha}, and (t - t_0) = re^{i\theta} \ , \\ \phi(t) &= \phi(t_0) + |h''(t_0)|r^2 \cos(\alpha + 2\theta)/2 + \dots \ . \end{aligned} \tag{95}$$

   Then, maximum descent is when $\cos(\alpha + 2\theta) = -1$, i.e. for

$$\theta = \left(\frac{\pi - \alpha}{2}\right) \ . \tag{96}$$

4. Taylor expand and do the Gaussian integral:

$$\begin{aligned} \int_A^B e^{zh(t)} g(t)\, dt &= g(t_0) e^{zh(t_0)} \int_{-\infty}^{\infty} e^{zh''(t_0)(t-t_0)^2/2} \frac{dt}{dr}\, dr + \cdots \\ &\simeq g(t_0) e^{zh(t_0)} \int_{-\infty}^{\infty} e^{-z|h''(t_0)|r^2/2} e^{i\theta}\, dr \\ &\sim g(t_0) e^{zh(t_0)} \sqrt{\frac{2\pi}{z|h''(t_0)|}} e^{i(\pi-\alpha)/2} \ . \end{aligned} \tag{97}$$

Let's see how all this works in practice with an example. Consider the Bessel-type function $H_\nu^{(2)}(x)$, defined by

$$H_\nu^{(2)}(x) = -\frac{1}{\pi} \int_{-i\infty}^{\pi+i\infty} e^{i(\nu t - x\sin(t))}\, dt \ , \tag{98}$$

where x and ν are real. We wish to find the asymptotic behavior in the limit $x \to \infty$ and $\nu \to \infty$ in the case $x < \nu$.

The answer depends critically on the relative magnitudes of x and ν, and it is convenient to set $x = \nu/\cosh(\beta)$, with $\beta > 0$, to deal with the case $x < \nu$. We obtain

$$H_\nu^{(2)}\left(\frac{\nu}{\cosh(\beta)}\right) = -\frac{1}{\pi} \int_{-i\infty}^{\pi+i\infty} e^{i\nu(t - \sin t/\cosh\beta)}\, dt \ . \tag{99}$$

Now, let

$$h(t) = i\left(t - \frac{\sin t}{\cosh\beta}\right) \ . \tag{100}$$

The saddle points occur where $h'(t) = 0$, i.e. when $\cos t = \cosh\beta$, so there are saddle points at $t = \pm i\beta + 2n\pi$, for all integers n. The ones most relevant to the path of integration are at $\pm i\beta$. To see this, sketch the paths of constant $\Im(h)$ and verify that, if one kept on such paths, one could get from $-i\infty$ to $\pi + i\infty$ without crossing any other saddle points. First let us see which saddle point gives the dominant contribution.

Near $\pm i\beta$, we have

$$\begin{aligned} h(t) &= h(\pm i\beta) + \frac{1}{2!}(\pm i\beta - t)^2 h''(\pm i\beta) + \cdots \\ &= \pm(-\beta + \tanh\beta) + \frac{1}{2!}(\pm i\beta - t)^2(\mp\tanh\beta) + \cdots \ . \end{aligned} \tag{101}$$

Close to the saddle point, the magnitude of the integrand is determined by the first (i.e. the constant) term in the Taylor series. Since $\beta > \tanh\beta$, the contribution from $t = -i\beta$ is exponentially larger than the contribution from $+i\beta$. We would therefore expect the steepest descent paths to cross straight over the saddle at $t = -i\beta$, but to turn through a right-angle at $t = +i\beta$, so that $\Re[h(t)]$ continues to decrease.

The steepest descent paths are given by

$$\Im\left[i\left(t - \frac{\sin t}{\cosh t}\right)\right] = \text{constant} \ , \tag{102}$$

and for a path passing through t_0, the value of this constant is $\Im[i(t_0 - \sin t_0/\cosh t_0)]$, which vanishes in the case of $t_0 = \pm i\beta$. The steepest descent paths through $\pm i\beta$ are therefore given by

$$x\cosh\beta - \sin x\cosh y = 0 \ , \tag{103}$$

where $t = x + iy$. Clearly, one path is $x = 0$, but the other one can only be sketched roughly. It is a good idea to know roughly where these paths go, but it is not necessary to know them exactly, unless: either there are singularities in the integrand, so that there may be extra contributions from deforming the contour of integration to the path of constant $\Im[h]$ (e.g. a pole contribution); or further terms in the asymptotic expansion are required.

The required asymptotic behavior is the Laplace contribution from the path which crosses the dominant saddle in the direction corresponding to steepest descent. This direction can be found as follows. Near t_0 we write $(t - t_0) = re^{i\theta}$, then the direction of paths of constant $\Im[h]$ is given by $\sin(\alpha+2\theta)=0$, where α is the phase of $h''(t_0)$. To find which of these is the steepest descent path, look at the real part of h near t_0:

$$\Re[h(t)] = \Re[h(t_0)] + \frac{1}{2}r^2|h''(t_0)|\cos(\alpha+2\theta) \ , \tag{104}$$

so $\Re[h]$ decreases fastest when $\cos(\alpha+2\theta) = -1$; i.e. when $\theta = (\pi-\alpha)/2$. You will probably find it easiest to do the calculations afresh each time, rather than try to remember this formula.

Near $t = -i\beta$

$$h(t) = (\beta - \tanh\beta) + \frac{1}{2}(-i\beta - t)^2\tanh\beta + \cdots \ , \tag{105}$$

so that the path $t = -i\beta + ir$ (r real) corresponds to steepest descent.

The saddle point contribution is

$$\begin{aligned} H_\nu^{(2)}\left(\frac{\nu}{\cosh\beta}\right) &\sim -\frac{1}{\pi}e^{\nu(\beta-\tanh\beta)}\int_{-\infty}^{\infty} e^{-\frac{1}{2}r^2\nu\tanh\beta} i\,dr \\ &\sim -\frac{i}{\pi}e^{\nu(\beta-\tanh\beta)}\sqrt{\frac{2\pi}{\nu\tanh\beta}} \ . \end{aligned} \tag{106}$$

Note that we are guaranteed that this is the leading term in the asymptotic expansion by Watson's lemma.

3 Solution of Ordinary Differential Equations

Let I be an interval of the real line. $C^n(I)$ is the set of functions $f(x)$ defined on I such that

$$\frac{d^n f}{dx^n} \equiv D^n f \equiv f^{(n)} \tag{107}$$

exists and is continuous. If $f, g \in C^n(I)$, then so are $f+g$ and αf. Thus, *$C^n(I)$ is a vector space.*

Definition 3.1 *Suppose $a_i(x)$, $0 \le i \le n$, are defined and bounded on I ($a_n \neq 0$). Then $L : C^n(I) \to C^0(I)$, $f \to Lf$*

$$Lf(x) = \sum_{i=0}^{n} a_i(x)(D^i f)(x) \tag{108}$$

is a linear differential operator (LDO) of order n. If $a_n(x) \neq 0$ on I, then L is normal.

Definition 3.2 *Let L be a LDO of order n on I and let $f(x)$ be n-times differentiable on I. An equation*

$$Ly = f \tag{109}$$

is a linear differential equation (LDE) of order n on I. If $f \equiv 0$ on I, the LDE is homogeneous.

We refer to solutions of the homogeneous equation as *complementary functions* (CFs), and the set of CFs as the *Kernel* of the operator L. Specific solutions of the non-homogeneous equation we then refer to as *particular integrals* (PIs).

3.1 Normal LDEs of Order 1

The general form is

$$a_1(x)y'(x) + a_0(x)y(x) = f(x) \tag{110}$$

on I. Since $a_1(x) \neq 0$ on I, we divide by $a_1(x)$ and rewrite as

$$y'(x) + p(x)y(x) = r(x) \ . \tag{111}$$

We first consider the homogeneous equation

$$y'(x) + p(x)y(x) = 0 \ . \tag{112}$$

Define the *integrating factor* as e^P, where

$$P(x) = \int^x p(u)\, du \ , \tag{113}$$

which exists, since we assume that $p(x)$ is bounded. Then, $(ye^P)' = (y' + py)e^P$. Thus, if $y(x)$ satisfies the homogeneous equation, then $(ye^P)' = 0$, which implies

$$y(x) = Ce^{-P(x)} \ . \tag{114}$$

Now consider the non-homogeneous equation. Clearly $(ye^P)' = re^P$. Therefore the general solution is

$$y(x) = Ce^{-P(x)} + e^{-P(x)} \int^x r(u)e^{P(u)}\, du \ . \tag{115}$$

Example 3.1 *Show that*

$$(x^2 + 1)y'(x) - (1 - x)^2 y(x) = xe^{-x} \tag{116}$$

has solution

$$y(x) = \frac{Ce^x - \frac{1}{2}\left(x + \frac{1}{2}\right)e^{-x}}{x^2 + 1} \ . \tag{117}$$

3.2 Normal Linear Second Order LDEs with Constant Coefficients

The general form is

$$L[y] = y'' + p(x)y' + q(x)y = r(x) \ . \tag{118}$$

Once again, here are two sub-problems to solving this equation; determining the Kernel and the particular integrals. In general both are difficult.

Example 3.2

$$L[y] \equiv y'' + 3y' + 2y = x^2 e^x \ . \tag{119}$$

Consider first the CFs. Try $y = e^{cx}$.

$$\begin{aligned} \Rightarrow (c^2 + 3c + 2)e^{cx} &= 0 \\ \Rightarrow c &= -1 \ or \ -2 \ . \end{aligned} \tag{120}$$

Therefore, independent solutions are $y_1 = e^{-x}$ *and* $y_2 = e^{-2x}$*, and the CF is a linear combination of these.*

The fastest way to find a PI is to guess one! Guess $y = (ax^2 + bx + c)e^x$*. Then routine algebra gives* $a = 1/6$, $b = -5/18$, $c = 19/108$*. The general solution is therefore*

$$y(x) = c_1 e^{-x} + c_2 e^{-2x} + \frac{1}{108}(18x^2 - 30x + 19)e^x \ . \tag{121}$$

Example 3.3

$$L[y] \equiv y'' + 2y' + y = \cos(x) \ . \tag{122}$$

Consider first the CFs. Try $y = e^{cx}$*. Yields* $c = -1$ *(repeated). So there are not independent roots here. To find the other linearly independent CF, try* $y = u(x)e^{-x}$*. This then gives*

$$L[y] = u'' e^{-x} = 0 \ , \tag{123}$$

So $u = ax + b$*. Therefore, two linearly independent solutions are* $y_1 = e^{-x}$*, and* $y_2 = xe^{-x}$.

Now, guess a PI: $y = c\cos(x) + d\sin(x)$*. This gives* $c = 0$ *and* $d = 1/2$*. Therefore, the general solution is*

$$y(x) = (c_1 x + c_2)e^{-x} + \frac{1}{2}\sin(x) \ . \tag{124}$$

Why did this trick work in the latter example? This is aparticular case of *reduction of order*: If one solution of a nth order LDE is known, the equation can be converted to an order $n-1$ one. Let's verify this explicitly when $n=2$.

$$L[y] = y'' + p(x)y' + q(x)y = 0 \ . \tag{125}$$

Suppose $y = v(x)$ is a solution. Then try $y(x) = u(x)v(x)$. Then

$$L[uv] = u''v + 2u'v' + uL[v] + pu'v \ . \tag{126}$$

But $L[v] = 0$, so by writing $w = u'$ we obtain the first order equation

$$w' + \left(2\frac{v'}{v} + p\right) w = 0 \tag{127}$$

for w.

3.3 Green's Functions

This primarily concerns finding PIs for second order equations, although the concept can be generalized to higher order systems. The technique assumes that one can find the general CF. There are two standard cases in which to do this. First, consider

$$y'' + Ay' + By = 0 \ , \tag{128}$$

where A and B are constants. Let n_1 and n_2 be the roots of $n^2 + An + B = 0$. Then

$$y(x) = \begin{cases} C_1 e^{n_1 x} + C_2 e^{n_2 x} & n_1 \neq n_2 \\ (C_1 + C_2 x) e^{n_1 x} & n_1 = n_2 \end{cases} \ , \tag{129}$$

as we have seen.

Second, consider

$$y'' + \frac{A}{x}y' + \frac{B}{x^2}y = 0 \ . \tag{130}$$

Now n_1 and n_2 are roots of $n(n-1) + An + B = 0$. Then

$$y(x) = \begin{cases} C_1 x^{n_1} + C_2 x^{n_2} & n_1 \neq n_2 \\ (C_1 + C_2 \ln x) x^{n_1} & n_1 = n_2 \end{cases} \ , \tag{131}$$

where the second solution here can be found by reduction of order.

3.3.1 Initial value Problems

In this section, wlog, we will use $I = [0, \infty)$, and for appropriateness of notation, will use t (time) instead of x as our variable.

The general problem is

$$M[\mathbf{y}(t)] = \mathbf{f}(t) \ , \qquad \mathbf{y}(0) = \mathbf{y_0} \ . \tag{132}$$

If this solution exists it can be shown to be unique. By assumption we can solve the homogeneous problem (with the same initial condition). We therefore consider the standard problem

$$M[\mathbf{y}(t)] = \mathbf{f}(t) \ , \qquad \mathbf{y}(0) = 0 \ , \tag{133}$$

since a solution of the homogeneous problem, with the given boundary condition, added to a solution of this equation is the general solution to the equation.

Let us write the standard problem as

$$L[y] \equiv \ddot{y}(t) + p(t)\dot{y}(t) + q(t)y(t) = f(t) \ , \tag{134}$$

with $y(0) = \dot{y}(0) = 0$.

A heuristic approach is as follows. Suppose we can solve (134) when $f(t) = \delta(t - s)$, with s fixed. Let the associated solution be $G(t, s)$, i.e.

$$L[G] = \delta(t - s) \ , \tag{135}$$

with $G(0, s) = G_t(0, s) = 0$. Now consider

$$y(t) = \int_0^\infty G(t, s) f(s)\, ds \ . \tag{136}$$

Clearly $y(0) = 0$, and $y_t(0) = 0$. Also,

$$\begin{aligned} L[y] &= \int_0^\infty L[G(t, s)] f(s)\, ds \\ &= \int_0^\infty \delta(t - s) f(s)\, ds \\ &= f(t) \ . \end{aligned} \tag{137}$$

Thus, (136) is the solution of (134).

Now, what does (135) mean? Clearly, if $t \neq s$ we can assume that G is a smooth function of t. In addition, assume that G and G_t are bounded as $t \to s$. Now integrate (135) from $t = s - \epsilon$ to $s + \epsilon$, $\epsilon > 0$:

$$\int_{s-\epsilon}^{s+\epsilon} (G_{tt} + pG_t + qG)\, dt = \int_{s-\epsilon}^{s+\epsilon} \delta(t-s)\, dt = 1 \ . \tag{138}$$

Thus

$$[G_t]_{s-\epsilon}^{s+\epsilon} + \int_{s-\epsilon}^{s+\epsilon} (pG_t + qG)\, dt = 1 \ . \tag{139}$$

By assumption the integrand is bounded, and so the integral is $\mathcal{O}(\epsilon)$ as $\epsilon \to 0$, so in this limit we get

$$[G_t]_{s_-}^{s+} = 1 \ . \tag{140}$$

Thus, G_t is not continuous, but has a jump of 1 at $t = s$. Let's be a little more formal about all this.

Definition 3.3 *The Green's function for the initial value problem posed earlier is a function $G(t,s)$ satisfying*

1. *for $t \geq 0$, $s \geq 0$, $t \neq s$, G is smooth, and $L[G] = 0$, for fixed s.*
2. *$G(0,s) = G_t(0,s) = 0$, for $s > 0$*
3. *G is C^1 at $t = s$, but $[G_t]_{s_-}^{s+} = 1$*

Definition 3.4 *If $y_1(x)$ and $y_2(x)$ are lineearly independent solutions to a second order LDE. Then, the wronskian is*

$$W[y_1, y_2] \equiv y_1(x)y_2'(x) - y_2(x)y_1'(x) \ , \tag{141}$$

and can be shown to be nonzero on I.

Lemma 3.1 *G exists and is unique*

Proof 3.1 *By explicit construction. Let y_1, y_2 be two independent solutions of $L[y] = 0$, so that the wronskian is nonzero. Let*

$$G(t,s) = \begin{cases} 0 & 0 \leq t < s < \infty \\ c_1 t_1(t) + c_2 y_2(t) & 0 < s \leq t < \infty \end{cases} \ . \tag{142}$$

Clearly the first two conditions are satisfied. We need to impose continuity at $t = s$ and a jump of 1 *in G_t. These conditions read*

$$\begin{aligned} c_1 y_1(s) + c_2 y_2(s) &= 0 \\ c_1 \dot{y}_1(s) + c_2 \dot{y}_2(s) &= 1 \ . \end{aligned} \tag{143}$$

By the definition of the wronskian, there exists a unique solution

$$\begin{aligned} c_1 &= -\frac{y_2(s)}{w(s)} \\ c_2 &= \frac{y_1(s)}{w(s)} \ . \end{aligned} \tag{144}$$

So, given these definitions, the solution of the initial value problem is

$$y(t) = \int_0^\infty G(t,s) f(s)\, ds \ . \tag{145}$$

Example 3.4

$$\ddot{y} + \omega^2 y = e^{-t} \ , \tag{146}$$

with $t > 0$, $y(0) = \dot{y}(0) = 0$. The Green's function is $\sin(\omega(t-s))$. *(show this) Therefore, the solution is*

$$\begin{aligned} y(t) &= \int_0^t G(t,s) e^{-s}\, ds \\ &= \frac{1}{\omega} \int_0^t e^{-s} \sin(\omega(t-s))\, ds \\ &= \frac{1}{1+\omega^2} \left\{ e^{-t} + \frac{\sin(\omega t)}{\omega} - \cos(\omega t) \right\} \ . \end{aligned} \tag{147}$$

Example 3.5

$$t^2 \ddot{y} - (t^2 + 2t)\dot{y} + (t+2) y = f(t) \ , \tag{148}$$

with $y(0) = \dot{y}(t) = 0$. We have

$$L[y] \equiv \ddot{y} - \left(1 + \frac{2}{t}\right)\dot{y} + \left(\frac{1}{t} + \frac{2}{t^2}\right) y = \hat{f} \equiv \frac{f}{t^2} \ . \tag{149}$$

By inspection, one solution of $L[y] = 0$ is $y = t$. Using reduction of order, the second solution is te^{-t}. Set

$$G(t,s) = \begin{cases} 0 & t < s \\ \frac{c_1 t + c_2 t e^{t-s}}{s} & t > s \end{cases} \ , \tag{150}$$

Continuity at $t = s$ implies $c_1 + c_2 = 0$. A jump of 1 *in G_t implies $c_2 = 1/s = -c_1$. Therefore*

$$\begin{aligned} y(t) &= \int_0^t \frac{1}{s}[te^{t-s} - t]\hat{f}(s)\, ds \\ &= t \int_0^t \left[\frac{e^{t-s} - 1}{s^3}\right] f(s)\, ds \ . \end{aligned} \tag{151}$$

3.3.2 Two-Point Boundary Value Problems

Now set $I = [a, b]$. Consider a n-th order system. The kernel has dimension n. In a 2-point bvp, we impose $m > 0$ conditions at $x = a$, and $n - m > 0$ conditions at $x = b$, to fix a complementary function. Such a problem may have 0, 1 or infinitely many solutions.

Example 3.6 *Consider $y'' + y = 0$, for which the candidate functions are $\sin x$ and $\cos x$. Consider the following possibilities for boundary conditions (bcs).*

1. $y(0) = 1$, $y'(\pi) = 0$. *This has one solution;* $y(x) = \cos x$.
2. $y(0) = y(\pi) = 0$. *This has an infinite number of solutions;* $y(x) = \lambda \sin x$, *for arbitrary* λ.
3. $y(0) = 0$, $y'(\pi) = 0$. *This has no non-trivial solutions.*
4. $y(0) = 1$, y *bounded as* $x \to \infty$. *This has one solution;* $y(x) = e^{-x}$.
5. $y(0) = 0$, y *bounded as* $x \to \infty$. *This has no non-trivial solutions.*

Definition 3.5 *Suppose the problem $M[y] = 0$ with boundary values at $x = a$ and $x = b$ has no non-trivial solutions. Then a, b are conjugate points.*

Definition 3.6 *A boundary condition $C[y, a]$ is homogeneous if, whenever it is satisfied by y, it is also satisfied by λy, with λ an arbitrary constant.*

Homogeneous bcs usually come in the form

$$c_1 y(a) + c_2 y'(a) = 0 \ , \tag{152}$$

for example.

Definition 3.7 *Consider a 2-point bvp*

$$L[y] \equiv y'' + p(x)y' + q(x)y = f(x) \tag{153}$$

$x \in [a, b]$, with bcs $C_1(y, y', a)$, $C_2(y, y', b)$. The Green's function $G(x, \xi)$ satisfies

1. $G(x, \xi)$ *is smooth,* $L[G] = 0$ *for* $a \le x$, $\xi \le b$, $x \ne \xi$.

2. *Considered as a function of* x*,* G *satisfies the bcs.*

3. G *is* C^1 *at* $x = \xi$*, but* G_x *has a jump* $[G_x(x,\xi)]_{x=\xi_-}^{x=\xi_+} = 1$

I will state, but not prove, that if the bcs are homogeneous, and a, b are conjugate, then G exists and is unique.

It can then be shown that the solution of the problem (153) is

$$y(x) = \int_a^b G(x,\xi) f(\xi)\, d\xi \ . \tag{154}$$

Example 3.7

$$y''(x) + y(x) = f(x) \ , \tag{155}$$

on $[0,\pi]$*, with* $y(0) = 0$*,* $y'(\pi) = 0$*. It is easy to see that the homogeneous solution has solutions*

$$\begin{aligned} y_1(x) &= \sin x \qquad \textit{satisfying } y(0) = 0 \\ y_2(x) &= \cos x \qquad \textit{satisfying } y'(\pi) = 0 \ . \end{aligned} \tag{156}$$

The wronskian is then $w = -1$*, so that the Green's function is*

$$G(x,\xi) = \begin{cases} -\cos\xi \sin x & 0 \le x \le \xi \le \pi \\ -\sin\xi \cos x & 0 \le \xi \le x \le \pi \end{cases} \ , \tag{157}$$

and the final solution to the problem is

$$y(x) = -\cos x \int_0^x \sin\xi f(\xi)\, d\xi - \sin x \int_x^\pi \cos\xi f(\xi)\, d\xi \ . \tag{158}$$

3.4 The WKB Method

The WKB method deals with the matching of approximate solutions of a differential equation between two regions where the approximation is valid, which are separated by a region where the approximation is not valid. It is sufficient to consider the equation

$$\frac{d^2 w}{dx^2} + \lambda^2 f(x) w(x) = 0 \ , \tag{159}$$

where $f(x)$ is real and strictly monotonically increasing, with $f(0) = 0$. The parameter λ is real and positive, and large. This equation occurs frequently in physics; for example, the Schrödinger equation is of this form.

We will be guided by the fact that, when f is constant, solutions are of the form $e^{\lambda\sqrt{f}}$. In general, formally, we will look for asymptotic solutions of the form

$$w(x) = \exp\left\{\lambda\phi_0 + \phi_1 + \frac{\phi_2}{\lambda} + \cdots\right\} , \tag{160}$$

where $\{\lambda^{1-n}\phi_n\}$ is an asymptotic sequence. We find the ϕ_n by substituting into the differential equation and equating powers of λ.

To order λ^2, we find

$$\phi_0'^2 + f(x) = 0 , \tag{161}$$

which yields

$$\phi_0(x) = \begin{cases} \pm i \int_0^x \sqrt{f(u)}\, du & f > 0 \text{ i.e. } x > 0 \\ \pm \int_x^0 \sqrt{|f(u)|}\, du & f < 0 \text{ i.e. } x < 0 \end{cases} , \tag{162}$$

To order λ, we find

$$\phi_0'' + 2\phi_0'\phi_1' = 0 , \tag{163}$$

which yields

$$\begin{aligned} \phi_1(x) &= -\frac{1}{2}\log(\phi_0') \\ &= \log(f^{-1/4}) . \end{aligned} \tag{164}$$

(ignoring constants of integration). Thus the asymptotic solutions, valid when $|f|$ is not small, are exponential in form for $x < 0$ (i.e., for $f(x) < 0$), and oscillating in form for $x > 0$. We write them in the form

$$w_{\uparrow\downarrow}(x) = (-f)^{-1/4} \exp\left(\pm\lambda \int_x^0 \sqrt{-f(u)}\, du\right) \quad f < 0 \text{ i.e. } x < 0 , \tag{165}$$

and

$$w_{\pm}(x) = (f)^{-1/4} \exp\left(\pm i\lambda \int_0^x \sqrt{f(u)}\, du\right) \quad f > 0 \text{ i.e. } x > 0 . \tag{166}$$

These functions are called *Liouville-Green* (LG) functions for the differential equation.

The approximation used to obtain these asymptotic solutions is not valid when $|f|$ is small. Nevertheless, any given exact solution to the equation will be asymptotic to some linear combination of the exponential LG functions in $x < 0$ and to some linear combination of the oscillating LG functions in $x > 0$:

$$\begin{aligned} w(x) &\sim a w_{\uparrow}(x) + b w_{\downarrow}(x) \quad \text{as } x \to -\infty \\ w(x) &\sim c w_{+}(x) + d w_{-}(x) \quad \text{as } x \to +\infty . \end{aligned} \tag{167}$$

The idea of WKB is to find the relationship between a, b, c, and d. For example, if boundary conditions are given as $x \to -\infty$, thus determining a and b, one can use the WKB method to determine c and d from a and b, and hence to find the behavior of the solution as $x \to \infty$.

One might expect the relationship between a, b, c, and d to depend on the function $f(x)$ in the differential equation. It turns out that the relationship is in fact independent of $f(x)$ (provided, as is assumed here, $f'(0) \neq 0$). This is what makes the method so useful.

The relationship between a, b, c, and d cannot be completely deterministic; given c and d one could not hope to find b because the exponentially divergent solution (in $x < 0$) would swamp the bounded solution so that it would not feature at all in an asymptotic expansion. Therefore, just two sorts of matching can be achieved

1. given b, and given that $a = 0$, find c and d,
2. given c and d, find a

The relationship can easily be seen to be linear (since the differential equation is linear), and since it is independent of $f(x)$, it can be determined by reference to a particular $f(x)$, for which the solutions of the differential equation are known exactly for all x. The most convenient choice is to use $f(x) = x$, giving the *Airy equation.* This equation can be solved by integral representation, and asymptotic forms of the solution can easily be obtained for positive and negative x.

We will only have time to state the results. the asymptotic method for obtaining these is given, for example, in the textbook by Mathews and Walker. The results are

1. If $w(x) \sim w_\downarrow(x)$ as $x \to -\infty$, (i.e. the decaying solution, which is the usual boundary condition for a physical problem) then
$$w(x) \sim \frac{2}{f^{1/4}} \cos\left(\lambda \int_0^x f^{1/2} - \frac{\pi}{4}\right) , \tag{168}$$
as $x \to +\infty$.

2. If $w(x) \sim f^{-1/4} \cos\left(\lambda \int_0^x f^{1/2} + \alpha\right)$ as $x \to +\infty$ then
$$w(x) \sim \frac{1}{|f|^{1/4}} \sin\left(\alpha + \frac{\pi}{4}\right) \exp\left(\lambda \int_x^0 |f|^{1/2}\right) , \tag{169}$$
as $x \to -\infty$.

Example 3.8 (Harmonic Oscillator)

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + \frac{1}{2}m\omega^2 x^2 \psi = E\psi \ . \tag{170}$$

We'll rewrite this in the standard form

$$\psi'' + \lambda^2 f(x)\psi = 0 \ , \tag{171}$$

where $\lambda^2 \equiv 2m/\hbar^2$*, and* $f(x) \equiv -m\omega^2 x^2/2 + E$*.*

In this well-known example, $f > 0$ *for* $|x| < \sqrt{2E/m\omega^2} \equiv a$*, and* $\psi \to 0$ *as* $x \to \infty$*. Consider even solutions, and match at* $x = +a$ *and* $x = -a$*.*

For $x < -a$*, we obtain*

$$\psi = \frac{A}{f^{1/4}} \exp\left(-\int_x^{-a} \sqrt{|f|}\right) \ , \tag{172}$$

and for $x > -a$*, we obtain*

$$\psi = \frac{A}{f^{1/4}} \exp\left(-\int_a^{x} \sqrt{|f|}\right) \ . \tag{173}$$

(where we use the same a *because we are looking for even solutions).*

Now, matching at $-a$ *gives*

$$\psi \sim \frac{2A}{f^{1/4}} \cos\left(\lambda \int_{-a}^{x} \sqrt{f} - \frac{\pi}{4}\right) \ , \tag{174}$$

and matching at a *gives*

$$\psi \sim \frac{2A}{f^{1/4}} \cos\left(\lambda \int_{x}^{a} \sqrt{f} - \frac{\pi}{4}\right) \ , \tag{175}$$

as the solution in $-a < x < a$*. Therefore,*

$$\lambda \int_{-a}^{x} \sqrt{f} - \frac{\pi}{4} = \lambda \int_{x}^{a} \sqrt{f} - \frac{\pi}{4} + 2n\pi \ , \tag{176}$$

which gives

$$\lambda \int_{-a}^{a} \sqrt{f} = 2n\pi + \frac{\pi}{2} \ , \tag{177}$$

so that finally we obtain

$$2E - \frac{\pi}{2} = \left(2n + \frac{1}{2}\right)\hbar\omega\pi \ . \tag{178}$$

4 Transform Calculus

4.1 The Fourier Transform

I'll assume that you know something about Fourier series. Suppose $g(x)$ is continuous on $-\pi$ to π, and that $g(\pm\pi) = 0$. Then

$$g(y) = \sum_{-\infty}^{\infty} C_n e^{iny} \ , \tag{179}$$

where

$$C_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} g(y) e^{-iny} \, dy \ . \tag{180}$$

Now consider changing the interval to $[-L/2, L/2]$. Set $y = \omega x$, $\omega = 2\pi/L$, $g(y) = f(x)$, $C_n = Lc_n$. Then we have

$$f(x) = \frac{1}{L} \sum_{-\infty}^{\infty} c_n e^{i\omega n x} \ , \tag{181}$$

with

$$c_n = \int_{-L/2}^{L/2} f(x) e^{-i\omega n x} \, dx \ . \tag{182}$$

In the sum, $k \equiv \omega n$ changes by $\Delta k = 2\pi/L$ at each term.

$$f(x) = \frac{1}{2\pi} \sum_{-\infty}^{\infty} c_n e^{ikx} \Delta k \ . \tag{183}$$

We now take the limit as $L \to \infty$. The sum now samples points increasingly close together and in the limit becomes an integral. Denoting c_n by $\tilde{f}(k)$ we obtain the relationships

$$\begin{aligned} f(x) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{f}(k) e^{ikx} \, dk \\ \tilde{f}(k) &= \int_{-\infty}^{\infty} f(x) e^{-ikx} \, dx \ . \end{aligned} \tag{184}$$

Definition 4.1 *Suppose* $\int_{-\infty}^{\infty} |f(x)| dx < \infty$. *Then* $\tilde{f}(k)$ *defined by the above is the Fourier Transform, and the expression for* $f(x)$ *is the inversion formula.*

Note that I will use these definitions consistently, however, physicists often switch the signs in the exponents, and make this more symmetric by having a $1/\sqrt{2\pi}$ in front of each integral. These are just issues of convention.

4.1.1 Fundamental Relations

The so-called *shifting relations* are extremely useful

Lemma 4.1 (Shifting Relations) *Suppose $\tilde{f}(k)$ exists. Let $g(x) = f(x - x_0)$, and $h(x) = e^{i\lambda x}$, with λ and x_0 constant. Then*

$$\begin{aligned} \tilde{g}(k) &= e^{-ikx_0}\tilde{f}(k) \\ \tilde{h}(k) &= \tilde{f}(k-\lambda) \ . \end{aligned} \tag{185}$$

Proof 4.1 *Do it in class. Easy!!*

Lemma 4.2 *Suppose $\tilde{f}(k)$ exists. Let $g(x) = f(ax)$, with $a \neq 0$ real. Then*

$$\tilde{g}(k) = \frac{1}{|a|}\tilde{f}\left(\frac{k}{a}\right) \ . \tag{186}$$

Proof 4.2

$$\begin{aligned} \tilde{g}(k) &= \int_{-\infty}^{\infty} f(ax)e^{-ikx}\,dx \\ &= sign(a)\int_{-\infty}^{\infty} f(y)e^{-iky/a}\frac{dy}{a} \\ &= \frac{1}{|a|}\tilde{f}\left(\frac{k}{a}\right) \ . \end{aligned} \tag{187}$$

Lemma 4.3 *Suppose $g(x) = f'(x)$, $\tilde{h}(k) = d\tilde{f}/dk = \tilde{f}'(k)$. Then, assuming all the integrals converge,*

$$\begin{aligned} \tilde{g}(k) &= ik\tilde{f}(k) \\ h(x) &= -ixf(x) \ . \end{aligned} \tag{188}$$

Proof 4.3 *Easy!!*

Simple extensions of these results show a general trend that the faster $f(x)$ falls off as $x \to \pm\infty$, the smoother $\tilde{f}(k)$ is, and vice-versa.

An important point is that Fourier transforms can be used to solve differential equations with constant coefficients:

$$y''(x) + py'(x) + qy(x) = f(x) \ . \tag{189}$$

Because of linearity,

$$(ik)^2\tilde{y} + (ikp)\tilde{y} + q\tilde{y} = \tilde{f} \ , \tag{190}$$

and so

$$y(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\tilde{f}(k)e^{ikx}}{q + ikp - k^2}\,dk \ . \tag{191}$$

This is clearly related to the Green's function approach, and we shall return to it later.

4.1.2 A Digression on Distributions

The delta function, $\delta(x)$, is an example of a generalized function, or *distribution*: something which may fail to satisfy either smoothness, boundedness, or asymptotic properties required of a given class of functions, but which can still be manipulated like a function. For example, neither the delta function nor the function $f(x) = 1$ comes in the class of functions for which Fourier transforms are normally defined, but the results

$$\begin{aligned}\tilde{\delta}(k) &= 1 \\ \tilde{1} &= 2\pi\delta(k)\ ,\end{aligned} \tag{192}$$

are familiar. For these purposes, both these functions must be regarded as distributions.

Definition 4.2 *Let $\mathcal{F}$ be a class of "good" functions on $(-\infty, \infty)$; for example, C^∞ with exponential decay at $\pm\infty$. Then, $g(x)$ is a distribution with respect to $\mathcal{F}$ if $\langle f, g\rangle$, defined by*

$$\langle f, g\rangle \equiv \int_{-\infty}^{\infty} f(x)g(x)\,dx \tag{193}$$

is finite $\forall\ f \in \mathcal{F}$, i.e., for all test functions.

Note that a different definition of "good" would lead to a different class of distributions; but C^∞ is usually required because this implies that the derivative of a distribution (defined below) is also a distribution.

With the definition above, the space of distributions (which is dual to the space of test functions) has many of the nice properties of a space of functions (e.g. linearity). Each distribution is defined by its action on test functions. For example, $\delta(x)$ is the distribution defined by

$$\langle \delta, f\rangle = f(0) \qquad \forall f \in \mathcal{F}\ . \tag{194}$$

The derivative of a distribution g is defined by

$$\langle g', f\rangle = -\langle g, f'\rangle \qquad \forall f \in \mathcal{F}\ . \tag{195}$$

(i.e. by integration by parts).

The Fourier Transform of a distribution g is defined similarly:

$$\langle \tilde{g}, f\rangle = \langle g, \tilde{f}\rangle \qquad \forall f \in \mathcal{F}\ . \tag{196}$$

It is straightforward to show that most of the properties of Fourier transforms hold also for the Fourier transforms of distributions:

1. The Fourier transform of the delta function. By the above definition $\langle\tilde{\delta}, f\rangle = \langle\delta, \tilde{f}\rangle$. The RHS is

$$\int_{-\infty}^{\infty} \delta(k)\tilde{f}(k)\,dk = \tilde{f}(0) = \int_{-\infty}^{\infty} f(x)\,dx \equiv \int_{-\infty}^{\infty} f(k)\,dk \ , \tag{197}$$

and the LHS is

$$\int_{-\infty}^{\infty} \tilde{\delta}(k) f(k)\,dk \ . \tag{198}$$

Comparing these, which must be equal for all test functions $f(x)$, gives the result $\tilde{\delta}(k) = 1$.

2. The Fourier transform of a constant. By the above definition $\langle\tilde{1}, f\rangle = \langle 1, \tilde{f}\rangle$. The RHS is

$$\int_{-\infty}^{\infty} 1\tilde{f}(k)\,dk = \int_{-\infty}^{\infty} \tilde{f}(k)\,dk = 2\pi f(0) \ , \tag{199}$$

and the LHS is

$$\int_{-\infty}^{\infty} \tilde{1} f(k)\,dk \ . \tag{200}$$

These must be equal for all test functions, so $\tilde{1} = 2\pi\delta(k)$. This result is consistent with the Fourier inversion theorem, but the conditions of the theorem do not hold here.

3. The Fourier transform of $H(x)$ (the Heaviside function). A naive approach gives the wrong answer. One could argue that since $H'(x) = \delta(x)$, and, for any f $\tilde{f}'(k) = ik\tilde{f}(k)$, then since $\tilde{\delta}(k) = 1$, it follows that $ik\tilde{H}(k) = 1$, which is correct. However, it does not follow that $\tilde{H}(k) = 1/ik$ because when distributions are allowed, the full solution of the equation $1 = ik\tilde{H}(k)$ should be

$$\tilde{H}(k) = \frac{1}{ik} + A\delta(k) \ , \tag{201}$$

where A is a constant which is not determined by this method. Since $H(x) + H(-x) = 1$, the real part of the Fourier transform of H must be the Fourier transform of $1/2$. Therefore $A = \pi$.

4.1.3 Convolution Integrals

Unfortunately, there exists no simple formula relating $\tilde{fg}$ to $\tilde{f}$ and $\tilde{g}$. Instead, $\tilde{fg}$ need not exist. There is however another kind of multiplication which is physically very important and for which Fourier transforms are easy to evaluate.

Definition 4.3 *Suppose* $\int_{-\infty}^{\infty} |f|^2\,dx < \infty$ *and* $\int_{-\infty}^{\infty} |g|^2\,dx < \infty$. *The convolution of* $f(x)$ *and* $g(x)$ *is*

$$\begin{aligned}(f*g)(x) &= \int_{-\infty}^{\infty} f(y)g(x-y)\,dy \\ &= \int_{-\infty}^{\infty} g(u)f(x-u)\,du \\ &= (g*f)(x)\ . \end{aligned} \tag{202}$$

Note that $(f*\delta) = f$ for all f. Thus, δ is the identity for $(*)$ considered as multiplication.

Theorem 4.4 (The Convolution Theorem) *Suppose* $\tilde{f}$, $\tilde{g}$, *and* $f*g$ *exist. Then* $\widetilde{f*g}$ *exists and*

$$\widetilde{f*g} = \tilde{f}\tilde{g}\ . \tag{203}$$

Proof 4.5 *See homework.*

A useful result, obtained from the inversion theorem to be seen soon, is

$$\delta(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx}\,dk\ . \tag{204}$$

Theorem 4.6 (Rayleigh (1899) - Plancherel (1910)) *Suppose complex* $f(t)$ *is such that* $\tilde{f}(k)$, *and* $\int_{-\infty}^{\infty}|f|^2\,dx$ *both exist. Then*

$$\int_{-\infty}^{\infty} |f(x)|^2\,dx = \frac{1}{2\pi}\int_{-\infty}^{\infty} |\tilde{f}(k)|^2\,dk\ . \tag{205}$$

Proof 4.7

$$\begin{aligned} RHS &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{f}(k)\bar{\tilde{f}}(k)\,dk \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (f(x)e^{-ikx}\,dx)(\bar{f}(y)e^{iky}\,dy)\,dk \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x)\bar{f}(y)e^{ik(y-x)}\,dkdxdy \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x)\bar{f}(y)\delta(y-x)\,dxdy \\ &= \int_{-\infty}^{\infty} f(x)\bar{f}(x)\,dx \\ &= \int_{-\infty}^{\infty} |f(x)|^2\,dx\ . \end{aligned} \tag{206}$$

Theorem 4.8 (Parseval's Theorem) *If f, g, are real, and $\tilde{f}$, $\tilde{g}$ and $\int_{-\infty}^{\infty} fg\,dx$ all exist, then*

$$\int_{-\infty}^{\infty} f(x)g(x)\,dx = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{f}(k)\tilde{g}(-k)\,dk \ . \tag{207}$$

Theorem 4.9 *Suppose $f(x)$ is continuous and $\tilde{f}$ exists. Then*

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{f}(k)e^{ikx}\,dk \ . \tag{208}$$

Proof 4.10 *Let x be fixed, and let $g(t) = f(t+x)$. Then, taking Fourier transforms with respect to t we have*

$$\begin{aligned} \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{f}(k)e^{ikx}\,dk &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{g}(k)\,dk \\ &= \frac{1}{2\pi}\langle 1, \tilde{g}\rangle \\ &= \frac{1}{2\pi}\langle \tilde{1}, g\rangle \\ &= \frac{1}{2\pi}\langle 2\pi\delta, g\rangle \\ &= g(0) \\ &= f(x) \ . \end{aligned} \tag{209}$$

Note that if f is discontinuous at, say, $x = x_0$, then $\tilde{f}(k)$ is continuous and the inversion integral is also continuous. It can be rigorously shown that

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{f}(k)e^{ikx_0}\,dk = \frac{1}{2\pi}[f(x_0+0) + f(x_0-0)] \ . \tag{210}$$

4.2 The Laplace Transform

In this section I will use t as the independent variable and p as the transform variable. Define the *Laplace Transform* by

$$F(p) = \int_0^{\infty} e^{-pt}f(t)\,dt \equiv \mathcal{L}.f(t) \ . \tag{211}$$

The Laplace transform traditionally treats only $t \geq 0$. It is therefore conventional to regard $f(t) = 0$ for $t < 0$. In the Laplace transform, p may be complex, defined at first for $\Re(p) > \gamma$, where γ is as required for convergence of the transform. However, it is important to be aware that no such γ may exist. For example, this is true for $f(t) = e^{t^2}$.

Here are some examples, for which the Laplace integrals are easy to compute.

$$f(t) = 1 \quad , \quad F(p) = \frac{1}{p} \tag{212}$$

$$f(t) = e^{at} \quad , \quad F(p) = \frac{1}{p-a} \quad (\Re(p) > \Re(a)) \tag{213}$$

$$f(t) = \cos(\omega t) \quad , \quad F(p) = \frac{p}{p^2+\omega^2} \tag{214}$$

$$f(t) = \sin(\omega t) \quad , \quad F(p) = \frac{\omega}{p^2+\omega^2} \tag{215}$$

$$f(t) = \sinh(\omega t) \quad , \quad F(p) = \frac{\omega}{p^2-\omega^2} \tag{216}$$

$$f(t) = \cosh(\omega t) \quad , \quad F(p) = \frac{p}{p^2-\omega^2} \ , \tag{217}$$

$$f(t) = \delta(t-a) \quad , \quad F(p) = e^{-ap} \ , \tag{218}$$

$$f(t) = \Theta(t-a) \quad , \quad F(p) = \frac{e^{-ap}}{p} \ , \tag{219}$$

$$f(t) = t^n \quad , \quad F(p) = \frac{n!}{p^{n+1}} \ . \tag{220}$$

4.2.1 Properties of Laplace Transforms

The change of scale property

$$\mathcal{L}.f(\lambda t) = \frac{1}{\lambda} F\left(\frac{p}{\lambda}\right) \ . \tag{221}$$

The shift theorems

$$\mathcal{L}.e^{\lambda t} f(t) = F(p-\lambda) \ . \tag{222}$$

If $g(t) = f(t-a)$ for $t > a$ and is zero otherwise, then

$$\mathcal{L}.g(t) = e^{-ap} F(p) \ . \tag{223}$$

A very important result is that the Laplace transform of a derivative is given by

$$\mathcal{L}.\frac{df}{dt} = p.\mathcal{L}.f(t) - f(0) \ , \tag{224}$$

and, similarly, we obtain

$$\mathcal{L}.\frac{d^2 f}{dt^2} = p^2 \mathcal{L}.f(t) - pf(0) - f'(0) \ . \tag{225}$$

The Laplace transform of an integral:

$$\mathcal{L}.\int_0^t f(u)\, du = \frac{1}{p}\mathcal{L}.f(t) \ . \tag{226}$$

Theorem 4.11 (Initial Value Theorem)

$$\lim_{t\to 0} f(t) = \lim_{p\to\infty} pF(p) \ . \tag{227}$$

(provided both limits exist).

Proof 4.12

$$\begin{aligned} pF(p) &= \mathcal{L}.\frac{df}{dt} + f(0) \\ &= f(0) + \int_0^\infty e^{-pt}\frac{df}{dt}\,dt \ . \end{aligned} \tag{228}$$

As $p \to \infty$ the right hand side becomes $f(0)$ as required, provided $f(t)$ is bounded near $t = 0$.

There exists a similar Final Value Theorem

$$\lim_{t\to\infty} f(t) = \lim_{p\to 0} pF(p) \ . \tag{229}$$

Convolutions are also important for Laplace transforms. Recall that for Laplace transforms we assume that functions vanish for $t < 0$. Therefore, in a convolution integral $h(t) = \int_{-\infty}^{\infty} f(t-u)g(u)\,du$ the integrand is nonzero only for $t > u > 0$. Thus we have

$$h(t) = \int_0^t f(t-u)g(u)\,du \quad , \quad t > 0 \ , \tag{230}$$

$$h(t) = 0 \quad , \quad t < 0 \ . \tag{231}$$

Theorem 4.13 *If $h = f * g$, then $H(p) = F(p)G(p)$.*

Proof 4.14

$$\begin{aligned} F(p)G(p) &= \int_0^\infty e^{-ps} f(s)\,ds \int_0^\infty e^{-pu} g(u)\,du \\ &= \int_{-\infty}^\infty e^{-ps} f(s)\theta(s)\,ds \int_{-\infty}^\infty e^{-pu} g(u)\theta(u)\,du \\ &= \int_{-\infty}^\infty du\, g(u)\theta(u) \int_{-\infty}^\infty ds e^{-p(s+u)} f(s)\theta(s) \\ &= \int_{-\infty}^\infty du\, g(u)\theta(u) \int_{-\infty}^\infty dt e^{-pt} f(t-u)\theta(t-u) \\ &= \int_{-\infty}^\infty dt\, e^{-pt} \int_{-\infty}^\infty du g(u) f(t-u)\theta(t-u)\theta(u) \\ &= \int_{-\infty}^\infty dt\, e^{-pt}\theta(t) \int_0^t du f(t-u)g(u) \\ &= \int_0^\infty dt e^{-pt} \left[\int_0^t du\, f(t-u)g(u)\right] \\ &= \int_0^\infty dt\, e^{-pt} h(t) \\ &= H(p) \ . \end{aligned} \tag{232}$$

To illustrate the usefulness of the Laplace transform, we'll tackle an example of a differential equation with non-constant coefficients. The Bessel function, $J_0(t)$ obeys

$$\frac{d}{dt}\left(t\frac{dJ_0}{dt}\right) + tJ_0 = 0 \ , \tag{233}$$

with the boundary conditions $J_0(0) = 1$, $J_0'(0) = 0$. First rewrite the equation as

$$tJ_0'' + J_o' + tJ_0 = 0 \ . \tag{234}$$

Now Laplace transform and use the result $\mathcal{L}.t^n f(t) = (-d/dp)^n F(p)$ to get

$$-\frac{d}{dp}(\mathcal{L}.J_0'') + \mathcal{L}.J_0' - \frac{d}{dp}(\mathcal{L}.J_0) = 0 \ . \tag{235}$$

Next I'll write $K(p) \equiv \mathcal{L}.J_0$ and use our results on the Laplace transforms of derivatives to get

$$(p^2 + 1)K'(p) + pK = 0 \ . \tag{236}$$

This is now a simple first order equation that we can solve to give

$$K(p) = \frac{c}{p}\left(1 + \frac{1}{p^2}\right)^{-1/2} \ , \tag{237}$$

where c is a constant. We can now use the initial value theorem to fix c via $1 = J_0(0) = \lim_{p\to\infty} pK(p) = c$. So finally,

$$K(p) = \frac{1}{p}\left(1 + \frac{1}{p^2}\right)^{-1/2} \ . \tag{238}$$

This can be rewritten as (exercise!)

$$K(p) = \sum_{n=0}^{\infty} \frac{\alpha_n}{p^{2n+1}} \ , \tag{239}$$

where

$$\alpha_n = \frac{(-1)^n (2n)!}{2^{2n}(n!)^2} \ . \tag{240}$$

Finally, we can use that the Laplace transform of t^n is $n!/p^{n+1}$, to invert and get

$$\begin{aligned} J_0(t) &= \sum_{n=0}^{\infty} \frac{\alpha_n t^{2n}}{(2n)!} \\ &= \sum_{n=0}^{\infty} \frac{\left(-\frac{1}{4}t^2\right)^2}{(n!)^2} \ . \end{aligned} \tag{241}$$

In this example, we used a Laplace transform that we know to invert another Laplace transform. However, for general problems we'll need a general inversion formula, analogous to the one that we used for Fourier transforms. Remember that for Fourier transforms the transform variable was real, therefore the inversion formula was a real integral. However, as I mentioned, with Laplace transforms, the transform variable is complex in general. Therefore, we'll end up needing a complex (contour) integral to invert and recover $f(t)$.

To get to the appropriate inversion formula, we'll postulate a form for the integral, and then show how it can be an inversion. Remember that I said we'd need to require $\Re(p) > \gamma$ for some γ in order for the Laplace transform to converge. With this is mind, let's examine integrals of the form

$$\int_{\gamma-i\infty}^{\gamma+i\infty} dp\, e^{pt} F(p) \ , \tag{242}$$

where $F(p)$ is the Laplace transform of a function $f(t)$. We would like this integral to yield $f(t)$. Substituting in for the actual Laplace transform we get

$$\int_{\gamma-i\infty}^{\gamma+i\infty} dp\, e^{pt} \int_{-\infty}^{\infty} du\, e^{-pu} f(u)\theta(u) \ . \tag{243}$$

Set $p = \gamma + ik$. We then get

$$ie^{\gamma t} \int_{-\infty}^{\infty} dk e^{ikt} \int_{-\infty}^{\infty} du e^{-iku} [e^{-\gamma u} f(u)\theta(u)] \ . \tag{244}$$

But, by the Fourier transform and inversion formula, this is

$$2\pi i e^{\gamma t} [e^{-\gamma t} f(t)\theta(t)] \ . \tag{245}$$

So, we can finally rearrange things to get the *Bromwich Inversion Formula* for the Laplace transform:

$$f(t)\theta(t) = \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} dp\, e^{pt} F(p) \ . \tag{246}$$

Now, notice that, in deriving this, we have not specified what γ is. For a given $F(p)$, γ is not known a priori. In fact, γ must be chosen so that the right hand side of the inversion integral is zero for $t < 0$ (to match the left hand side).

To do this, start with $\gamma > 0$ and close the contour using a semicircle in $\Re(p) > \gamma > 0$ to form a closed contour C. Now, for $t < 0$, the factor of e^{pt} ensures that the contribution from the integral over the semicircle at infinity vanishes. Thus, for the integral around

C to yield zero , $F(p)$ must have no singularities inside C. Therefore, all singularities of $F(p)$ must lie to the left of the line $\Re(p) = \gamma$. This fixes γ.

For $t > 0$, we close the contour in $\Re(p) < \gamma$. This gives

$$f(t) = \frac{1}{2\pi i} \oint_C e^{pt} F(p)\, dp \quad , \quad t > 0 \; . \tag{247}$$

If the only singularities of $F(p)$ are isolated poles, the inversion integral can be performed by the calculus of residues

$$\begin{aligned} f(t) &= \sum_{\text{poles}} \text{Res}_i [e^{pt} F(p)] \\ &= \sum_j e^{p_j t} F(p_j) \; , \end{aligned} \tag{248}$$

for poles at $p = p_j$. Suppose the pole of $F(p)$ with largest real part is $p = p_j$. Then $f(t) \sim e^{p_j t}$ as $t \to \infty$, and therefore we require $\gamma > \Re(p_j)$.

Let me give some examples of how to use Laplace transforms to solve ordinary differential equations, in particular initial value problems. Consider

$$\ddot{x} + 2\dot{x} + x = e^{-t} \; , \tag{249}$$

with initial values $x(0) = 1$, $\dot{x}(0) = 0$. Write $\mathcal{L}.x(t) \equiv X(p)$. Laplace transforming the equation, and using our results about the Laplace transforms of derivatives, gives

$$[p^2 X(p) - px(0) - \dot{x}(0)] + 2[pX(p) - x(0)] + X(p) = \frac{1}{p+1} \; , \; , \tag{250}$$

which, after a little rearranging, implies that

$$X(p) = \frac{1}{p+1} + \frac{1}{(p+1)^2} + \frac{1}{(p+1)^3} \; . \tag{251}$$

But now, using our example from earlier, we can invert each of these term by term to obtain

$$x(t) = e^{-t} + te^{-t} + \frac{t^2}{2} e^{-t} \; . \tag{252}$$

Here's another example

$$\ddot{y} + \dot{y} - 2y = 0 \; , \tag{253}$$

subject to $y(0) = 1$ and $y \to 0$ as $t \to \infty$. Write $\mathcal{L}.y(t) \equiv Y(p)$. Transforming the equation we get

$$(p^2 + p - 2) Y(p) = p + \dot{y}(0) + 1 \; . \tag{254}$$

This is now easily inverted to give

$$y(t) = \frac{\dot{y}(0)+2}{3}e^{t} + \frac{1-\dot{y}(0)}{3}e^{-2t} \ , \tag{255}$$

Now, requiring $y \to 0$ as $t \to \infty$ implies that $\dot{y}(0)+2=0$, and therefore the solution is

$$y(t) = e^{-2t} \ . \tag{256}$$

What's interesting about this example is that we use the first boundary condition just after transforming, to dispose of one of the terms generated by Laplace transforming a derivative, but use the second boundary term only after inverting the transform.

The next example deals with a pair of coupled first order differential equations. Consider

$$\begin{aligned} \dot{x} + x + 2y &= e^{2t} \ , \\ 2\dot{x} + \dot{y} - x &= 0 \ , \end{aligned} \tag{257}$$

with $x(0) = y(0) = 0$. Laplace transforming gives

$$\begin{aligned} (p+1)X(p) + 2Y(p) &= \frac{1}{p-2} \ , \\ (2p-1)X(p) + pY(p) &= 0 \ . \end{aligned} \tag{258}$$

These can be trivially solved to give

$$\begin{aligned} X(p) &= \frac{p}{(p-1)(p-2)^2} \ , \\ Y(p) &= \frac{1-2p}{(p-1)(p-2)^2} \ . \end{aligned} \tag{259}$$

We could do this by partial fractions and using earlier results. However, instead we'll use this as our first Bromwich inversion formula example. Clearly we will need $\gamma > 2$. The right hand side of the equation for $X(p)$ has poles at $p = 1$ and $p = 2$, with

$$\begin{aligned} \mathrm{Res}(p=1) &= \lim_{p\to 1}(p-1)(X(p)e^{pt}) = e^{t} \ , \\ \mathrm{Res}(p=2) &= \lim_{p\to 2}\left(\frac{pe^{pt}}{p-1}\right) \ , \end{aligned} \tag{260}$$

which yields

$$x(t) = (2t-1)e^{2t} + e^{t} \ . \tag{261}$$

Similarly we obtain

$$y(t) = (1 - 3t)e^{2t} - e^t \ . \tag{262}$$

Sometimes, a convolution trick is useful. Consider

$$\ddot{x} + \omega^2 x = f(t) \ , \tag{263}$$

with $x(0) = \dot{x}(0) = 0$. Laplace transforming we get

$$X(p) = \frac{F(p)}{p^2 + \omega^2} \ . \tag{264}$$

Now, as we learned earlier, $G(p) \equiv 1/(p^2 + \omega^2)$ is the Laplace transform of $g(t) = (\sin(\omega t)/\omega)\theta(t)$. Thus, we can write

$$X(p) = G(p)F(p) \ , \tag{265}$$

and use the convolution theorem to tell us that $x(t) = (g * f)(t)$. This reads

$$x(t) = \int_0^t dt' \, \frac{\sin\omega(t - t')}{\omega} f(t') \ , \tag{266}$$

for $t > 0$. Therefore, $g(t)$ is the Green's function of the problem.

As a final example for Laplace transforms, consider the diffusion equation

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{k}\frac{\partial u}{\partial t} \ , \tag{267}$$

in $x \geq 0$, $t \geq 0$, subject to $U(0,t) = f(t)$, given, $u(x, 0) = 0$, and $u(x,t) \to 0$, as $x \to \infty$. This problem could describe, for example, a prescribed heating $(f(t))$ applied to the $x = 0$ end of a semi-infinite rod, initially unheated.

Perform the Laplace transform with respect to t:

$$U(x,p) = \int_0^\infty dt \, e^{-pt} u(x,t) \ . \tag{268}$$

Note that, evaluating this at $x = 0$ gives $U(0,p) = \mathcal{L}.f(t) \equiv F(p)$. Now, the diffusion equation, using $u(x, 0) = 0$, gives

$$\frac{\partial^2 U}{\partial x^2} = \frac{p}{k} U \ , \tag{269}$$

which is easily solved to give

$$U(x,p) = A(p)\exp(-\sqrt{p/k}x) + B(p)\exp(\sqrt{p/k}x) \ . \tag{270}$$

Now, since the x-dependence of U and u will be the same, we require $U(x,p) \to 0$, as $x \to \infty$, which gives $B(p) = 0$. So, we have $U(x,p) = A(p)\exp(-\sqrt{p/k}x)$. At $x = 0$, we have $U(0,p) = A(p)$, and so we write

$$\begin{aligned} U(x,p) &= U(0,p)\exp(-\sqrt{p/k}x) \ , \\ &= F(p)G(p) \ , \end{aligned} \tag{271}$$

with $G(p) = \exp(-\sqrt{p/k}x)$. Therefore, the convolution theorem tells us that $U(x,t) = (g * f)(x,t)$, and it remains to find $g(x,t)$. But from an earlier result we know this:

$$g(x,t) = \sqrt{\frac{x^2}{4\pi k}} t^{-3/2} \exp\left(-\frac{x^2}{4kt}\right) \ . \tag{272}$$

Therefore, the complete solution to the diffusion equation problem is

$$u(x,t) = \sqrt{\frac{x^2}{4\pi k}} \int_0^t dt' \, (t-t')^{-3/2} \exp\left(-\frac{x^2}{4k(t-t')}\right) f(t') \ . \tag{273}$$

I've given you a lot of examples using the Laplace transform. During this time you've had some time to get used to the Fourier transform. I'd now like to go back to the Fourier transform for one example that is of particular physical significance.

Consider the problem of finding the Green's function that satisfies

$$\left(-\frac{\partial^2}{\partial x^2} - q^2\right) G(x,x') = \delta(x-x') \ , \tag{274}$$

where q is a fixed, real, positive number, and $-\infty < x, x' < \infty$. This Green's function describes one-dimensional scattering in quantum mechanics. Set $x' = 0$ (w.l.o.g.), and then Fourier transform. We obtain that

$$(k^2 - q^2)\tilde{G}(k) = 1 \ , \tag{275}$$

with the function we're looking for given by the inversion formula

$$G(x) = \int_{-\infty}^{\infty} dk \, e^{-ikx} \tilde{G}(k) \ . \tag{276}$$

Now, we would like to solve for $\tilde{G}(k)$. However, $\tilde{G}(k) = 1/(k^2 - q^2)$ will not do, because it puts poles on the real k-axis, and this gives problems for the inversion integral.

To proceed, we will apply *Feynman's Rule*. This technique is extremely important in quantum mechanics and quantum field theory. Replace q^2 by $q^2 \pm i\epsilon$. This enables one to define two independent Green's functions $G_{\pm}(x)$ by taking the limit $\epsilon \to 0$ at an appropriate later stage. Consider

$$\begin{aligned} \tilde{G}_{+}(k) &= \frac{1}{k^2 - (q^2 + i\epsilon)} , \\ &= \frac{1}{[k - (q^2 + i\epsilon)^{1/2}][k + (q^2 + i\epsilon)^{1/2}]} , \\ &= \frac{1}{k - q - i\epsilon'} \cdot \frac{1}{k + q + i\epsilon'} , \end{aligned} \tag{277}$$

where $\epsilon' \equiv \epsilon/(2q^2)$, and $\lim_{\epsilon \to 0}$ and $\lim_{\epsilon' \to 0}$ are equivalent. So, we have

$$G_{+}(x) = \int_{-\infty}^{\infty} dx\, e^{-ikx} \frac{1}{k - q - i\epsilon'} \cdot \frac{1}{k + q + i\epsilon'} . \tag{278}$$

Now, for $x > 0$, we can evaluate this integral by closing the contour in the upper half k-plane ($\Im(k) > 0$), and for $x < 0$, we can evaluate this integral by closing the contour in the lower half k-plane ($\Im(k) < 0$). We use the residue theorem, and then take the limit $\lim_{\epsilon \to 0}$ at the very end. The result is

$$G_{+}(x) = \pi i \theta(x) \frac{e^{iqx}}{q} + \pi i \theta(-x) \frac{e^{-iqx}}{q} . \tag{279}$$

One can calculate $G_{-}(x)$ similarly. Given our technique, you should check that these Green's functions obey the differential equation.

5 Sturm-Liouville Theory

Let's begin with an example to get the feel of the kind of problems we'll tackle with these techniques.

Consider a uniform string with fixed ends. The displacement of this string obeys the wave equation

$$\frac{1}{c^2} \frac{\partial^2 y}{\partial t^2} = \frac{\partial^2 y}{\partial x^2} , \tag{280}$$

with boundary conditions $y = 0$ at $x = 0$ and at $x = l$ for all time. To start, we separate variables, making the ansatz $y(x,t) = X(x)T(t)$. This yields

$$-\frac{1}{c^2} \frac{\ddot{T}}{T} = -\frac{X''}{X} = \text{const} = \lambda , \tag{281}$$

say. Vibrations correspond to $T \sim e^{i\omega t}$, so $\lambda = \omega^2/c^2$. We seek solutions of

$$X'' = -\lambda X , \tag{282}$$

subject to $X = 0$ at $x = 0$ and at $x = l$. The solutions are of the form

$$X_n(x) = B_n \sin\left(\frac{n\pi x}{l}\right) , \tag{283}$$

with $n = 1, 2, 3, \ldots$.

Note that there exist solutions only for $\lambda \in S$, a discretely distributed set of real eigenvalues. S is referred to as the *spectrum* of eigenvalues. The corresponding eigenfunctions X_n are orthogonal on $[0, l]$. The equation of motion for $X(x)$ is typical of a wide class of eigenvalue problems which arise from some partial differential equations of physics.

5.1 General Remarks

We will study (at first) second order linear differential equations for $x \in I = [a, b]$. Restrict attention to linear differential operators of the form $L = -a_2 D^2 - a_1 D + a_0$ for given functions $a_i(x)$ with $a_2(x) > 0$ on I. Actually, it will prove sufficient to restrict our attention to L of the *self-adjoint* form:

$$L = -DpD + q , \tag{284}$$

with $p(x) > 0$, $q(x)$ given functions on I.

the *Sturm-Liouville Problem* is specified by the differential equation

$$Ly(x) = \lambda w(x) y(x) , \tag{285}$$

to be solved for $y(x)$ for $x \in I$, subject to the boundary conditions (to be specified), with $w(x) > 0$ on I, and λ an eigenvalue parameter. The solution has some general features

1. $\exists$ nontrivial solutions which obey the boundary conditions in use iff $\lambda \in S$, the spectrum of the problem. S is a monotonic set of discretely distributed real eigenvalues λ_n, so that $\lambda_0 < \lambda_1 < \lambda_2 < \cdots$, with $\lambda_n \to \infty$ like n^2 as $n \to \infty$.

2. The eigenfunctions y_n corresponding to the $\lambda_n \in S$ are unique to within normalization. Also, y_n and y_m are orthogonal in a sense that we shall see soon, if $n \neq m$.

3. The y_n provide a basis in the infinite dimensional vector space of functions on I, which obey the boundary conditions in use, and suitable smoothness properties.

5.2 Orthogonality and Boundary Conditions

Suppose y_1 and y_2 obey the Sturm-Liouville equation. Then

$$-(py_1')' + qy_1 = \lambda_1 w y_1 \ , \tag{286}$$

$$-(py_2')' + qy_2 = \lambda_2 w y_2 \ , \tag{287}$$

and suitable boundary conditions, for distinct values λ_1 and λ_2 of the eigenvalue parameter. Form the object

$$\int_a^b dx\, [y_2 \times (286) - y_1 \times (287)] \ . \tag{288}$$

This yields

$$\begin{aligned} (\lambda_1 - \lambda_2) \int_a^b dx\, w(x) y_1(x) y_2(x) &= \int_a^b dx\, [-(py_1')' y_2 + (py_2')' y_1] \ , \\ &= [-(py_1')y_2 + (py_2')y_1]_a^b \ . \end{aligned} \tag{289}$$

Now, appropriate boundary conditions are those that make this vanish. For then, since $\lambda_1 \neq \lambda_2$, we have

$$\int_a^b dx\, w(x) y_1(x) y_2(x) = 0 \ . \tag{290}$$

This is the sense in which y_1 and y_2 are *orthogonal* with respect to the *weight function* $w(x)$.

A good example is given by Legendre's equation and polynomials. This arises from the Laplace equation in cylindrical coordinates (r, θ, z). Writing $x \equiv \cos\theta$, the Legendre equation is

$$-\frac{d}{dx}\left[(1-x^2)\frac{d}{dx}P(x)\right] = \lambda P(x) \ , \tag{291}$$

with $I = (-1, 1)$. Note that this is a Sturm-Liouville problem with $w(x) = 1$, and $p(x) = (1 - x^2)$. The suitable boundary conditions are automatically imposed if $P(x)$ is finite at $x = \pm 1$, since $p(x) \to 0$ at the endpoints. The solutions are a set of polynomials, the first few of which are

$$P_0 = 1 \quad , \quad \lambda_0 = 0 \ , \tag{292}$$

$$P_1 = x \quad , \quad \lambda_1 = 2 \ , \tag{293}$$

$$P_2 = x^2 - \frac{1}{3} \quad , \quad \lambda_2 = 6 \ . \tag{294}$$

More generally, there exists a unique $P_n = x^n + \cdots$ with $\lambda_n = n(n+1)$. It can also be checked that the P_n are orthogonal on I.

5.3 Real Eigenvalues

Let us allow the possibility the λ_n and y_n are complex. Then, we may write two Sturm-Liouville equations as

$$-(py')' + qy = \lambda wy \ , \tag{295}$$

$$-(p(y^*)')' + qy^* = \lambda^* wy^* \ . \tag{296}$$

We now form the object

$$\int_a^b dx \, [y^* \times (295) - y \times (296)] \ . \tag{297}$$

This yields

$$\begin{aligned}(\lambda - \lambda^*) \int_a^b dx \, w(x)y(x)y(x)^* &= \int_a^b dx \, [-(p(y^*)')'y + (py')'y^*] \ , \\ &= [-(p(y^*)')y + (py')y^*]_a^b \ , \\ &= 0 \ ,\end{aligned} \tag{298}$$

for the suitable boundary conditions. Now, since $w(x) > 0$ on I, this implies that $\int_a^b dx \, w|y|^2$ is strictly positive on I. Therefore,

$$\lambda^* = \lambda \ . \tag{299}$$

i.e., λ is real.

5.4 Formal Vector Space View

Let us regard suitably behaved functions $f(x)$, $g(x)$, which obey the boundary conditions of our Sturm-Liouville problem as elements of an infinite dimensional vector space $\mathcal{V}$, spanned by the $y_n(x)$. Thus, we can write

$$f(x) = \sum_{n=0}^{\infty} c_n y_n(x) \ . \tag{300}$$

Define a scalar (inner) product on $\mathcal{V}$ by

$$(f, g) \equiv \int_a^b dx \, w(x) f(x) g(x) \ , \tag{301}$$

and note that "suitably well-behaved" requires that $||f|| = (f, f)^{1/2}$ exists. Then, $(y_n, y_m) = 0$ for $n \neq m$. Further, choose the scale of y_n to achieve orthonormality:

$$(y_n, y_n) = \delta_{nm} \ . \tag{302}$$

Also

$$\begin{aligned}(y_m, f) &= \sum_n c_n (y_n, y_m) \ , \\ &= \sum_n cm\delta_{nm} \ , \\ &= c_m \ . \end{aligned} \tag{303}$$

This is to be compared with the formulae for Fourier series. Thus, if we assume that the y_n are known, then for a given f we can obtain the c_m via

$$c_n = \int_a^b d\xi \, w(\xi) y_n(\xi) f(\xi) \ . \tag{304}$$

An important and useful result follows if we substitute this into the expansion for $f(x)$. We obtain

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} \left[\int_a^b d\xi \, w(\xi) y_n(\xi) f(\xi) \right] y_n(x) \ , \\ &= \int_a^b d\xi \left[\sum_{n=0}^{\infty} w(\xi) y_n(\xi) y_n(x) \right] f(\xi) \ . \end{aligned} \tag{305}$$

Since this is true $\forall \ f \in \mathcal{V}$, we may therefore infer

$$\sum_{n=0}^{\infty} w(\xi) y_n(\xi) y_n(x) = \delta(x - \xi) \ . \tag{306}$$

This is a formal *completeness relation* for the Sturm-Liouville problem. Note that we can check this if we assume that $\delta(x-\xi)$ obeys the boundary conditions for $a < \xi < b$. We can then expand the delta function as

$$\delta(x - \xi) = \sum_{n=0}^{\infty} c_n(\xi) y_n(\xi) \ . \tag{307}$$

Then, our expression for c_n gives

$$\begin{aligned} c_n(\xi) &= \int_a^b dx \, w(x) y_n(x) \delta(x - \xi) \ , \\ &= w(\xi) y_n(\xi) \ . \end{aligned} \tag{308}$$

5.5 Inhomogeneous Equations and Green's Functions

Suppose we have solved the Sturm-Liouville problem

$$L y_n(x) = \lambda_n w(x) y_n(x) \ , \tag{309}$$

where the y_n obey suitable boundary conditions on $I = [a, b]$, and $w(x) > 0$ on I. By *solved* I mean that the λ_n and y_n are determined, and $(y_n, y_m) = \delta_{nm}$ has been arranged.

We would like to solve the problem

$$[L - \lambda w(x)]y(x) = f(x) \ , \tag{310}$$

for $y(x)$, for $x \in I$, subject to the same boundary conditions. Here λ is a fixed real number, and $f(x)$ is a given function; naturally assumed to obey the same boundary conditions. To proceed, write

$$\begin{aligned} f(x) &= w(x)h(x) \ , \\ h(x) &= \sum_{n=0}^{\infty} h_n y_n(x) \ , \end{aligned} \tag{311}$$

where we can calculate the coefficients h_n when f (and hence h) is given, and the y_n are known.

We now posit the expansion

$$y(x) = \sum_{n=0}^{\infty} a_n y_n(x) \ , \tag{312}$$

and seek the unknowns a_n to complete the specification of the solution. Substituting in to the problem (310), the left hand side becomes

$$\begin{aligned} \sum_{n=0}^{\infty} a_n(L - \lambda w)y_n(x) &= \sum_{n=0}^{\infty} a_n(\lambda_n w - \lambda w)y_n(x) \ , \\ &= w(x)\sum_{n=0}^{\infty} a_n(\lambda_n - \lambda)y_n(x) \ . \ , \end{aligned}$$

Now the right hand side is

$$w(x)\sum_{n=0}^{\infty} h_n y_n(x) \ . \tag{313}$$

Finally, equating these, multiplying both sides by $y_m(x)$, and integrating over $a < x < b$, we obtain

$$a_m(\lambda_m - \lambda) = h_m \ . \tag{314}$$

Now, if $\lambda \notin S$, we have

$$a_n = \frac{h_n}{\lambda_n - \lambda} \ . \tag{315}$$

We now have the solution in terms of quantities calculated for the homogeneous equation. It is instructive to write this in another way. The solution is

$$\begin{aligned} y(x) &= \sum_{n=0}^{\infty} y_n(x) \frac{h_n}{\lambda_n - \lambda} , \\ &= \sum_{n=0}^{\infty} y_n(x) \frac{1}{\lambda_n - \lambda} \left[\int_a^b d\xi \, w(\xi) y_n(\xi) h(\xi) \right] , \\ &= \int_a^b d\xi \left[\sum_{n=0}^{\infty} \frac{y_n(x) y_n(\xi)}{\lambda_n - \lambda} \right] f(\xi) , \end{aligned} \tag{316}$$

In this form it is easy to identify the *Green's Function* in the form

$$G(x, \xi) = \sum_{n=0}^{\infty} \frac{y_n(x) y_n(\xi)}{\lambda_n - \lambda} , \tag{317}$$

We can check that this Green's function behaves as it should, by acting on it with the left hand side of (310):

$$\begin{aligned} [L - \lambda w(x)] G(x, \xi) &= \sum_{n=0}^{\infty} \frac{y_n(\xi)}{\lambda_n - \lambda} (L - \lambda w) y_n(x) , \\ &= \sum_{n=0}^{\infty} \frac{y_n(\xi)}{\lambda_n - \lambda} (\lambda_n - \lambda) w(x) y_n(x) , \\ &= \sum_{n=0}^{\infty} y_n(\xi) w(x) y_n(x) , \end{aligned}$$

which, using our earlier results, is

$$[L - \lambda w(x)] G(x, \xi) = \delta(x - \xi) , \tag{318}$$

as expected.

5.6 Self-Adjointness

It has probably not escaped your notice that the Sturm-Liouville problem and its solutions are related to things that you've seen in quantum mechanics. Here, we shall see precisely what this relation is. In particular, we shall compare the terms *self-adjoint* (in Sturm-Liouville theory) and *Hermitian* (in quantum mechanics).

I will begin by considering the case of $w(x) = 1$. Consider a typical Sturm-Liouville problem

$$\begin{aligned} Ly &= \lambda y , \\ L &= -DpD + q , \end{aligned} \tag{319}$$

with boundary conditions $y(a) = y(b) = 0$. In the vector space $\mathcal{V}$ of (possibly complex) functions f, g, obeying the boundary conditions, we will use the inner product

$$(f, g) = \int_a^b dx\, f(x)^* g(x)\ . \tag{320}$$

We are interested in operators A such that $Af \in \mathcal{V}$, $\forall f \in \mathcal{V}$. Define the *Hermitian conjugate* (or Hermitian adjoint) operator, $A^\dagger$ by

$$(A^\dagger f, g) = (f, Ag)\ , \tag{321}$$

$\forall\ f,\ g \in \mathcal{V}$. We say that the operator A is *hermitian* if $A^\dagger = A$, or

$$(Af, g) = (f, Ag)\ , \tag{322}$$

With this definition, note that our self-adjoint operator L is hermitian (for the special case $w = 1$). You can check this trivially using the definition of the inner product. Do this as a brief exercise.

Now turn to the case of $w \neq 1$. In this course, and in Sturm-Liouville theory in general, we refer to the operator L as self-adjoint, even when $w \neq 1$. we also use $Ly = \lambda w y$ and $(f, g) = \int_a^b dx\, w f g$, from which real λ_n and orthogonal y_n follow. However, with this definition, $(Lf, g) \neq (f, Lg)$, because of w. To understand the relationship to hermiticity, we define

$$M \equiv -w^{-1/2} D p D w^{1/2} + q\ , \tag{323}$$

which is hermitian. To see this:

$$\begin{aligned} (Mf, g) &= -\int_a^b dx\, w w^{-1/2} [D p D w^{1/2} f] g + \cdots\ , \\ &= \int_a^b dx\, (p D w^{1/2} f) D(w^{1/2} g) + \cdots\ , \\ &= (f, Mg)\ . \end{aligned} \tag{324}$$

Therefore, care is needed with terminology, even though either definition leads to real eigenvalues and orthogonal eigenfunctions.

6 The Calculus of Variations

The techniques we are about to discuss are of central importance in the theory of dynamics, and more generally, for finding the equations of motion for many physical systems.

A classic example is given by the Brachistochrone problem of John Bernoulli. Consider a particle, displaced by a horizontal distance a, and a vertical distance, from a particular point, A. We assume the particle begins at the origin, O. The problem is then: along which path from O to A will a particle slide (from rest at O), under gravity, with no friction, *in the least time.*

Begin by writing the energy of the particle at any time, as satisfying

$$\frac{1}{2}v^2 = gy \ , \tag{325}$$

(where I've set the mass to be unity). We may then write the time taken for a particular motion as

$$\begin{aligned} T &= \int_0^T dt \ , \\ &= \int \frac{ds}{v} \ , \\ &= \int_0^a \frac{(1+y'^2)^{1/2}}{(2gy)^{1/2}}\, dx \ . \end{aligned} \tag{326}$$

T is a *functional* of the path $y(x)$ in question. We write $T[y(x)]$. Given $y(x)$, we can calculate $T[y]$. The question is, which y minimizes T?

6.1 The Euler-Lagrange Equations

To answer this, let us consider a more general problem. Given a functional $F(x, y, y')$, where $y = y(x)$, form the functional

$$I[y] = \int_a^b dx\, F(x, y, y') \ . \tag{327}$$

We seek to find stationary values of I between fixed end points, such that $y(a)$, and $y(b)$ are given. Our aim is to choose y so that $I[y]$ is stationary; i.e.

$$\delta y = 0 \ . \tag{328}$$

More explicitly, we will require $\delta y = \varepsilon\eta(x)$, for ε small and $\eta(x)$ arbitrary except that $\eta(a) = \eta(b) = 0$.

We study

$$\delta I = \varepsilon \left. \frac{\partial I}{\partial \varepsilon} \right|_{\varepsilon \to 0} = 0 \; . \tag{329}$$

Proceed in the following way. Write

$$I(\varepsilon) = \int_a^b dx \, F(x, y + \varepsilon\eta, y' + \varepsilon\eta') \; . \tag{330}$$

Then

$$\begin{aligned} \frac{\partial I}{\partial \varepsilon} &= \int_a^b dx \left(\frac{\partial F}{\partial y}\eta + \frac{\partial F}{\partial y'}\eta' \right) \; , \\ &= \int_a^b dx \, \eta \left(\frac{\partial F}{\partial y} - \frac{d}{dx}\frac{\partial F}{\partial y'} \right) + \eta \left. \frac{\partial F}{\partial y'} \right|_a^b \; , \end{aligned} \tag{331}$$

by an integration by parts. Now, the last term is zero, since $\eta(a) = \eta(b) = 0$. Therefore, $\partial I/\partial\varepsilon = 0$ (evaluated at $\varepsilon = 0$) will hold for (otherwise) arbitrary $\eta(x)$ if y obeys

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\frac{\partial F}{\partial y'} = 0 \; , \tag{332}$$

which is the *Euler-Lagrange* equation.

Note that $\partial F/\partial x$, $\partial F/\partial y$, $\partial F/\partial y'$ have natural meanings for $F(x, y, y')$. However, dF/dx means the total derivative of F with respect to x, with F viewed as a function of x via $F(x, y(x), y'(x))$. i.e.

$$\frac{dF}{dx} = \frac{\partial F}{\partial x} + \frac{\partial F}{\partial y}\frac{dy}{dx} + \frac{\partial F}{\partial y'}\frac{d^2y}{dx^2} \; . \tag{333}$$

As a matter of notation, we define

$$\frac{\delta F}{\delta y} \equiv \frac{\partial F}{\partial y} - \frac{d}{dx}\frac{\partial F}{\partial y'} \; . \tag{334}$$

We refer to $\delta F/\delta y$ as the *functional derivative* of F with respect to y. Then, the Euler-Lagrange equation is

$$\frac{\delta F}{\delta y} = 0 \; . \tag{335}$$

An alternative form of the Euler-Lagrange equation is

$$\frac{\delta F}{\delta x} = \frac{d}{dx}\left(F - y\frac{\partial F}{\partial y'} \right) \; . \tag{336}$$

To prove this, look at the right hand side of the expression.

$$\begin{aligned}\text{RHS} &= \frac{\partial F}{\partial x} + y'\frac{\partial F}{\partial y} + y''\frac{\partial F}{\partial y'} - \left(y''\frac{\partial F}{\partial y'} + y'\frac{d}{dx}\frac{\partial F}{\partial y'}\right) \ , \\ &= \frac{\partial F}{\partial x} + y'\left(\frac{\partial F}{\partial y} - \frac{d}{dx}\frac{\partial F}{\partial y'}\right) \ , \\ &= \frac{\partial F}{\partial x} \ . \end{aligned} \tag{337}$$

In general, the Euler-Lagrange equations lead to a second order nonlinear ordinary differential equation. This is generally very hard to solve directly. However, we shall see that there can be a number of possible simplifications.

6.2 First Integrals

There are two particular simplifications of the Euler-Lagrange equations that are very easy to prove. I'll state them here, but you should prove them for yourselves.

1. Suppose $F = F(x, y')$. i.e., F does not depend on y. In this case, the Euler-Lagrange equations imply
$$\frac{\partial F}{\partial y'} = \text{const} \ . \tag{338}$$
This is a first order ordinary differential equation, and consequently much easier to deal with.

2. Suppose $F = F(y, y')$. i.e., F does not depend on x. In this case, the alternative form of the Euler-Lagrange equation implies
$$F - y'\frac{\partial F}{\partial y'} = \text{const} \ . \tag{339}$$

6.3 Some Examples

Let's see how this all works in some concrete physical examples.

6.3.1 The Brachistochrone

The relevant functional is

$$\sqrt{2g}T \equiv I = \int_0^a dx\, F(y, y') \ , \tag{340}$$

with

$$F(y,y') = \sqrt{\frac{1+y'^2}{y}} \ . \tag{341}$$

By our first integrals, the Euler-Lagrange equation implies

$$\sqrt{\frac{1+y'^2}{y}} - y'\sqrt{\frac{1}{y}}\frac{y'}{\sqrt{1+y'^2}} = K \ , \tag{342}$$

where K is a constant. This implies that

$$y' = \sqrt{\frac{2c}{y} - 1} \ , \tag{343}$$

where $2c = 1/K^2$. We solve this parametrically, by setting

$$\begin{aligned} y &= 2c\sin^2\theta \ , \\ &= c(1-\cos 2\theta) \ , \end{aligned} \tag{344}$$

so that $\theta = 0$ at the origin. Then

$$\begin{aligned} x &= \int \frac{\sqrt{2c}\sin\theta}{\sqrt{2c}\cos\theta} 2c\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right) d\theta \ , \\ &= c\int d\theta\,(1-\cos\theta) \ , \\ &= c\left(\theta - \frac{1}{2}\sin 2\theta\right) \ , \end{aligned} \tag{345}$$

These are the parametric equations of a cycloid.

6.3.2 Fermat's Principle of Least Time

This states that a ray of light between two fixed points in a medium travels along that path it can traverse in the least time. The refractive index, μ, in the medium is related to the speed of light, c, in the medium by

$$\mu = \frac{1}{c} \ . \tag{346}$$

To find the path, minimize

$$T = \int dt = \int \frac{ds}{c} = \int \mu\, ds \ . \tag{347}$$

Given $\mu(x,y)$ in two dimensions, we seek y to secure $\delta T = 0$, where

$$T = \int dx\, \sqrt{1+y'^2}\mu(x,y) \ . \tag{348}$$

For μ independent of y, our first integrals give

$$\frac{\mu y'}{\sqrt{1+y'^2}} = K \ , \tag{349}$$

with K a constant. If μ is also independent of x, then

$$y' = \text{constant} \ . \tag{350}$$

We can write $y' = \tan\theta$. Therefore,

$$y' = \frac{y'}{\sqrt{1+y'^2}} = \text{constant} \tag{351}$$

also. Looking at an interface, we then get

$$K = \mu_1 \sin\theta_1 = \mu_2 \sin\theta_2 \ , \tag{352}$$

which is *Snell's Law.*

6.3.3 Geodesics

A *geodesic* is the path of minimum length between two fixed points on some surface. As an example, consider the unit sphere.

$$S = \int ds \ , \tag{353}$$

where

$$ds^2 = d\theta^2 + \sin^2(\theta) d\phi^2 \ . \tag{354}$$

Let's choose θ as the independent variable, and therefore seek a solution of the form $\phi(\theta)$:

$$S = \int d\theta \, F(\theta, \phi, \phi') \ , \tag{355}$$

with

$$F = \sqrt{1 + \sin^2(\theta)\phi'^2} \ , \tag{356}$$

and a prime denotes differentiation with respect to θ. Now, F is independent of ϕ, and so this implies that $\partial F/\partial\phi'$ is a constant:

$$\frac{\sin^2(\theta)\phi'}{\sqrt{1+\sin^2(\theta)\phi'^2}} = K \ , \tag{357}$$

a constant. This equation yields the great circle paths.

6.4 More General Cases

Let's see how the ideas presented in this section can be extended in more general contexts.

6.5 $n > 1$ Dependent Variables, and 1 Independent Variable

We now consider the case $\mathbf{y} = (y_1(x), y_2(x), \ldots, y_n(x))$. The objective is to find $\mathbf{y}(x)$ such that $\delta I[\mathbf{y}] = 0$, where

$$I[\mathbf{y}] = \int_a^b F(x, \mathbf{y}, \mathbf{y}')\, dx \ , \tag{358}$$

for variations $\delta\mathbf{y}$ which leave the end points fixed, i.e. $\mathbf{y}(a)$ =fixed, $\mathbf{y}(b)$ =fixed. A trivial extension of our previous analysis yields a set of n Euler-Lagrange equations, given by

$$\frac{\delta F}{\delta y_i} \equiv \frac{\partial F}{\partial y_i} - \frac{d}{dx}\frac{\partial F}{\partial y_i'} = 0 \ . \tag{359}$$

Also, if F is independent of x, then we obtain a conserved quantity (first integral), given by

$$F - \sum_{i=1}^{n} y_i \frac{\partial F}{\partial y_i'} = \text{constant} \ . \tag{360}$$

6.5.1 Several Independent Variables, 1 Dependent Variable

Here, I is defined as an integral over a surface, volume, or higher-dimensional hypersurface. For example

$$I[u] = \int_V dV\, F(\mathbf{x}, u, \nabla u) \ . \tag{361}$$

We seek $\delta I = 0$ for variations of u that leave the values of u on the boundary, $S = \partial V$, fixed. Again, applying our variational machinery, we obtain n separate Euler-Lagrange equations:

$$\frac{\partial F}{\partial u} - \frac{d}{dx_i}\frac{\partial F}{\partial u_i} = 0 \ , \tag{362}$$

where $u_i \equiv \partial u/\partial x_i$.