INVARIANCE THEORY, THE HEAT EQUATION, AND THE ATIYAH-SINGER INDEX THEOREM

by Peter B. Gilkey

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INTRODUCTION

This book treats the Atiyah-Singer index theorem using heat equation methods. The heat equation gives a local formula for the index of any elliptic complex. We use invariance theory to identify the integrand of the index theorem for the four classical elliptic complexes with the invariants of the heat equation. Since the twisted signature complex provides a sufficiently rich family of examples, this approach yields a proof of the Atiyah-Singer theorem in complete generality. We also use heat equation methods to discuss Lefschetz fixed point formulas, the Gauss-Bonnet theorem for a manifold with smooth boundary, and the twisted eta invariant. We shall not include a discussion of the signature theorem for manifolds with boundary.

The first chapter reviews results from analysis. Sections 1.1 through 1.7 represent standard elliptic material. Sections 1.8 through 1.10 contain the material necessary to discuss Lefschetz fixed point formulas and other topics.

Invariance theory and differential geometry provide the necessary link between the analytic formulation of the index theorem given by heat equation methods and the topological formulation of the index theorem contained in the Atiyah-Singer theorem. Sections 2.1 through 2.3 are a review of characteristic classes from the point of view of differential forms. Section 2.4 gives an invariant-theoretic characterization of the Euler form which is used to give a heat equation proof of the Gauss-Bonnet theorem. Sections 2.5 and 2.6 discuss the Pontrjagin forms of the tangent bundle and the Chern forms of the coefficient bundle using invariance theory.

The third chapter combines the results of the first two chapters to prove the Atiyah-Singer theorem for the four classical elliptic complexes. We first present a heat equation proof of the Hirzebruch signature theorem. The twisted spin complex provides a unified way of discussing the signature, Dolbeault, and de Rham complexes. In sections 3.2–3.4, we discuss the half-spin representations, the spin complex, and derive a formula for the $\hat{A}$ genus. We then discuss the Riemann-Roch formula for an almost complex manifold in section 3.5 using the SPIN_c complex. In sections 3.6–3.7 we give a second derivation of the Riemann-Roch formula for holomorphic Kaehler manifods using a more direct approach. In the final two sections we derive the Atiyah-Singer theorem in its full generality.

The final chapter is devoted to more specialized topics. Sections 4.1–4.2 deal with elliptic boundary value problems and derive the Gauss-Bonnet theorem for manifolds with boundary. In sections 4.3–4.4 we discuss the twisted eta invariant on a manifold without boundary and we derive the Atiyah-Patodi-Singer twisted index formula. Section 4.5 gives a brief discussion of Lefschetz fixed point formulas using heat equation methods. In section 4.6 we use the eta invariant to calculate the K-theory of spherical space forms. In section 4.7, we discuss Singer's conjecture for the Euler form and related questions. In section 4.8, we discuss the local formulas for the invariants of the heat equation which have been derived by several authors, and in section 4.9 we apply these results to questions of spectral geometry.

The bibliography at the end of this book is not intended to be exhaustive but rather to provide the reader with a list of a few of the basic papers which have appeared. We refer the reader to the bibliography of Berger and Berard for a more complete list of works on spectral geometry.

This book is organized into four chapters. Each chapter is divided into a number of sections. Each Lemma or Theorem is indexed according to this subdivision. Thus, for example, Lemma 1.2.3 is the third Lemma of section 2 of Chapter 1.

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CHAPTER 1
PSEUDO-DIFFERENTIAL OPERATORS

Introduction

In the first chapter, we develop the analysis needed to define the index of an elliptic operator and to compute the index using heat equation methods. Sections 1.1 and 1.2 are brief reviews of Sobolev spaces and pseudo-differential operators on Euclidean spaces. In section 1.3, we transfer these notions to compact Riemannian manifolds using partition of unity arguments. In section 1.4 we review the facts concerning Fredholm operators needed in section 1.5 to prove the Hodge decomposition theorem and to discuss the spectral theory of self-adjoint elliptic operators. In section 1.6 we introduce the heat equation and in section 1.7 we derive the local formula for the index of an elliptic operator using heat equation methods. Section 1.8 generalizes the results of section 1.7 to find a local formula for the Lefschetz number of an elliptic complex. In section 1.9, we discuss the index of an elliptic operator on a manifold with boundary and in section 1.10, we discuss the zeta and eta invariants.

Sections 1.1 and 1.4 review basic facts we need, whereas sections 1.8 through 1.10 treat advanced topics which may be omitted from a first reading. We have attempted to keep this chapter self-contained and to assume nothing beyond a first course in analysis. An exception is the de Rham theorem in section 1.5 which is used as an example.

A number of people have contributed to the mathematical ideas which are contained in the first chapter. We were introduced to the analysis of sections 1.1 through 1.7 by a course taught by L. Nirenberg. Much of the organization in these sections is modeled on his course. The idea of using the heat equation or the zeta function to compute the index of an elliptic operator seems to be due to R. Bott. The functional calculus used in the study of the heat equation contained in section 1.7 is due to R. Seeley as are the analytic facts on the zeta and eta functions of section 1.10.

The approach to Lefschetz fixed point theorems contained in section 1.8 is due to T. Kotake for the case of isolated fixed points and to S. C. Lee and the author in the general case. The analytic facts for boundary value problems discussed in section 1.9 are due to P. Greiner and R. Seeley.

1.1. Fourier Transform, Schwartz Class, And Sobolev Spaces.

The Sobolev spaces and Fourier transform provide the basic tools we shall need in our study of elliptic partial differential operators. Let $x = (x_1, \dots, x_m) \in \mathbf{R}^m$. If $x, y \in \mathbf{R}^m$, we define:

$$x \cdot y = x_1 y_1 + \cdots + x_m y_m \qquad \text{and} \qquad |x| = (x \cdot x)^{1/2}$$

as the Euclicean dot product and length. Let $\alpha = (\alpha_1, \dots, \alpha_m)$ be a multi-index. The α_j are non-negative integers. We define:

$$|\alpha| = \alpha_1 + \cdots + \alpha_m\,, \qquad \alpha! = \alpha_1! \dots \alpha_m!\,, \qquad x^\alpha = x_1^{\alpha_1} \dots x_m^{\alpha_m}.$$

Finally, we define:

$$d_x^\alpha = \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \dots \left(\frac{\partial}{\partial x_m}\right)^{\alpha_m} \qquad \text{and} \qquad D_x^\alpha = (-i)^{|\alpha|} d_x^\alpha$$

as a convenient notation for multiple partial differentiation. The extra factors of $(-i)$ defining D_x^α are present to simplify later formulas. If $f(x)$ is a smooth complex valued function, then Taylor's theorem takes the form:

$$f(x) = \sum_{|\alpha| \le n} d_x^\alpha f(x_0) \frac{(x - x_0)^\alpha}{\alpha!} + O(|x - x_0|^{n+1}).$$

The Schwartz class $\mathcal{S}$ is the set of all smooth complex valued functions f on $\mathbf{R}^m$ such that for all α, β there are constants $C_{\alpha,\beta}$ such that

$$|x^\alpha D_x^\beta f| \le C_{\alpha,\beta}.$$

This is equivalent to assuming there exist estimates of the form:

$$|D_x^\beta f| \le C_{n,\beta}(1 + |x|)^{-n}$$

for all (n, β). The functions in $\mathcal{S}$ have all their derivatives decreasing faster at ∞ than the inverse of any polynomial.

For the remainder of Chapter 1, we let dx, dy, $d\xi$, etc., denote Lebesgue measure on $\mathbf{R}^m$ with an additional normalizing factor of $(2\pi)^{-m/2}$. With this normalization, the integral of the Gaussian distribution becomes:

$$\int e^{-\frac{1}{2}|x|^2}\, dx = 1.$$

We absorb the normalizing constant into the measure in order to simplify the formulas of the Fourier transform. If $C_0^\infty(\mathbf{R}^m)$ denotes the set of smooth functions of compact support on $\mathbf{R}^m$, then this is a subset of $\mathcal{S}$. Since $C_0(\mathbf{R}^m)$ is dense in $L^2(\mathbf{R}^m)$, $\mathcal{S}$ is dense in $L^2(\mathbf{R}^m)$.

We define the convolution product of two elements of $\mathcal{S}$ by:

$$(f * g)(x) = \int f(x - y) g(y)\, dy = \int f(y) g(x - y)\, dy.$$

This defines an associative and commutative multiplication. Although there is no identity, there do exist approximate identities:

LEMMA 1.1.1. *Let $f \in \mathcal{S}$ with $\int f(x)\,dx = 1$. Define $f_u(x) = u^{-m} f(\frac{x}{u})$. Then for any $g \in \mathcal{S}$, $f_u * g$ converges uniformly to g as $u \to 0$.*

PROOF: Choose C so $\int |f(x)|\,dx \le C$ and $|g(x)| \le C$. Because the first derivatives of g are uniformly bounded, g is uniformly continuous. Let $\varepsilon > 0$ and choose $\delta > 0$ so $|x-y| \le \delta$ implies $|g(x) - g(y)| \le \varepsilon$. Because $\int f_u(x)\,dx = 1$, we compute:

$$\begin{aligned}|f_u * g(x) - g(x)| &= \left| \int f_u(y)\{g(x-y) - g(x)\}\,dy \right| \\ &\le \int |f_u(y)\{g(x-y) - g(x)\}|\,dy.\end{aligned}$$

We decompose this integral into two pieces. If $|y| \le \delta$ we bound it by $C\varepsilon$. The integral for $|y| \ge \delta$ can be bounded by:

$$2C \int_{|y| \ge \delta} |f_u(y)|\,dy = 2C \int_{|y| \ge \delta/u} |f(y)|\,dy.$$

This converges to zero as $u \to 0$ so we can bound this by $C\varepsilon$ if $u < u(\varepsilon)$. This completes the proof.

A similar convolution smoothing can be applied to approximate any element of L^p arbitrarily well in the L^p norm by a smooth function of compact support.

We define the Fourier transform $\hat{f}(\xi)$ by:

$$\hat{f}(\xi) = \int e^{-ix\cdot\xi} f(x)\,dx \qquad \text{for } f \in \mathcal{S}.$$

For the moment $\xi \in \mathbf{R}^m$; when we consider operators on manifolds, it will be natural to regard ξ as an element of the fiber of the cotangent space. By integrating by parts and using Lebesgue dominated convergence, we compute:

$$D_\xi^\alpha\{\hat{f}(\xi)\} = (-1)^{|\alpha|}\{\widehat{x^\alpha f}\} \qquad \text{and} \qquad \xi^\alpha \hat{f}(\xi) = \{\widehat{D_x^\alpha f}\}.$$

This implies $\hat{f} \in \mathcal{S}$ so Fourier transform defines a map $\mathcal{S} \to \mathcal{S}$.

We compute the Fourier transform of the Gaussian distribution. Let $f_0(x) = \exp(-\frac{1}{2}|x|^2)$, then $f_0 \in \mathcal{S}$ and $\int f_0(x)\,dx = 1$. We compute:

$$\begin{aligned}\hat{f}_0(\xi) &= \int e^{-ix\cdot\xi} e^{-\frac{1}{2}|x|^2}\,dx \\ &= e^{-\frac{1}{2}|\xi|^2} \int e^{-(x+i\xi)\cdot(x+i\xi)/2}\,dx.\end{aligned}$$

We make a change of variables to replace $x + i\xi$ by x and to shift the contour in $\mathbf{C}^m$ back to the original contour $\mathbf{R}^m$. This shows the integral is 1 and $\hat{f}_0(\xi) = \exp\left(-\frac{1}{2}|\xi|^2\right)$ so the function f_0 is its own Fourier transform.

In fact, the Fourier transform is bijective and the Fourier inversion formula gives the inverse expressing f in terms of $\hat{f}$ by:

$$f(x) = \int e^{ix\cdot\xi}\,\hat{f}(\xi)\,d\xi = \hat{\hat{f}}(-x).$$

We define $T(f) = \hat{\hat{f}}(-x) = \int e^{ix\cdot\xi}\,\hat{f}(\xi)\,d\xi$ as a linear map from $\mathcal{S} \to \mathcal{S}$. We must show that $T(f) = f$ to prove the Fourier inversion formula.

Suppose first $f(0) = 0$. We expand:

$$f(x) = \int_0^1 \frac{d}{dt}\{f(tx)\}\,dt = \sum_j x_j \int_0^1 \frac{\partial f}{\partial x_j}(tx)\,dt = \sum_j x_j g_j$$

where the g_j are smooth. Let $\phi \in C_0^\infty(\mathbf{R}^m)$ be identically 1 near $x = 0$. Then we decompose:

$$f(x) = \phi f(x) + (1-\phi)f(x) = \sum_j x_j \phi g_j + \sum_j x_j \left\{\frac{x_j(1-\phi)f}{|x|^2}\right\}.$$

Since ϕg_j has compact support, it is in $\mathcal{S}$. Since ϕ is identically 1 near $x = 0$, $x_j(1-\phi)f/|x|^2 \in \mathcal{S}$. Thus we can decompose $f = \sum x_j h_j$ for $h_j \in \mathcal{S}$. We Fourier transform this identity to conclude:

$$\hat{f} = \sum_j \{\widehat{x_j h_j}\} = \sum_j i\frac{\partial \hat{h}_j}{\partial \xi_j}.$$

Since this is in divergence form, $T(f)(0) = \int \hat{f}(\xi)\,d\xi = 0 = f(0)$.

More generally, let $f \in \mathcal{S}$ be arbitrary. We decompose $f = f(0)f_0 + (f - f(0)f_0)$ for $f_0 = \exp(-\frac{1}{2}|x|^2)$. Since $\hat{f}_0 = f_0$ is an even function, $T(f_0) = f_0$ so that $T(f)(0) = f(0)f_0(0) + T(f - f(0)f_0) = f(0)f_0(0) = f(0)$ since $(f - f(0)f_0)(0) = 0$. This shows $T(f)(0) = f(0)$ in general.

We use the linear structure on $\mathbf{R}^m$ to complete the proof of the Fourier inversion formula. Let $x_0 \in \mathbf{R}^m$ be fixed. We let $g(x) = f(x + x_0)$ then:

$$\begin{aligned} f(x_0) = g(0) = T(g)(0) &= \int e^{-ix\cdot\xi}\,f(x + x_0)\,dx\,d\xi \\ &= \int e^{-ix\cdot\xi}\,e^{ix_0\cdot\xi}\,f(x)\,dx\,d\xi \\ &= T(f)(x_0). \end{aligned}$$

This shows the Fourier transform defines a bijective map $\mathcal{S} \to \mathcal{S}$. If we use the constants $C_{\alpha,\beta} = \sup_{x\in\mathbf{R}^m} |x^\alpha D_x^\beta f|$ to define a Frechet structure on $\mathcal{S}$, then the Fourier transform is a homeomorphism of topological vector spaces. It is not difficult to show $C_0^\infty(\mathbf{R}^m)$ is a dense subset of $\mathcal{S}$ in this topology. We can use either pointwise multiplication or convolution to define a multiplication on $\mathcal{S}$ and make $\mathcal{S}$ into a ring. The Fourier transform interchanges these two ring structures. We compute:

$$\begin{aligned}\hat f \cdot \hat g &= \int e^{-ix\cdot\xi} f(x) e^{-iy\cdot\xi} g(y)\,dx\,dy\\ &= \int e^{-i(x-y)\cdot\xi} f(x-y) e^{-iy\cdot\xi} g(y)\,dx\,dy\\ &= \int e^{-ix\cdot\xi} f(x-y) g(y)\,dx\,dy.\end{aligned}$$

The integral is absolutely convergent so we may interchange the order of integration to compute $\hat f \cdot \hat g = \widehat{(f * g)}$. If we replace f by $\hat f$ and g by $\hat g$ we see $(f\cdot g)(-x) = \widehat{(\hat f * \hat g)}$ using the Fourier inversion formula. We now take the Fourier transform and use the Fourier inversion formula to see $\widehat{(f\cdot g)}(-\xi) = (\hat f * \hat g)(-\xi)$ so that $\widehat{(f\cdot g)} = \hat f * \hat g$.

The final property we shall need of the Fourier transform is related to the L^2 inner product $(f,g) = \int f(x)\overline{g}(x)\,dx$. We compute:

$$\begin{aligned}(\hat f, g) &= \int f(x) e^{-ix\cdot\xi}\overline{g}(\xi)\,dx\,d\xi = \int f(x) e^{-ix\cdot\xi}\overline{g}(\xi)\,d\xi\,dx\\ &= (f, \hat g(-x)).\end{aligned}$$

If we replace g by $\hat g$ then $(\hat f, \hat g) = (f, \hat{\hat g}(-x)) = (f,g)$ so the Fourier transform is an isometry with respect to the L^2 inner product. Since $\mathcal{S}$ is dense in L^2, it extends to a unitary map $L^2(\mathbf{R}^m) \to L^2(\mathbf{R}^m)$. We summarize these properties of the Fourier transform as follows:

LEMMA 1.1.2. *The Fourier transform is a homeomorphism $\mathcal{S} \to \mathcal{S}$ such that:*
(a) $f(x) = \int e^{ix\cdot\xi}\hat f(\xi)\,d\xi = \int e^{i(x-y)\cdot\xi} f(y)\,dy\,d\xi$ (Fourier inversion formula);
(b) $D_x^\alpha f(x) = \int e^{ix\cdot\xi}\xi^\alpha \hat f(\xi)\,d\xi$ and $\xi^\alpha \hat f(\xi) = \int e^{-ix\cdot\xi} D_x^\alpha f(x)\,dx$;
(c) $\hat f \cdot \hat g = \widehat{(f * g)}$ and $\hat f * \hat g = \widehat{(f\cdot g)}$;
(d) The Fourier transform extends to a unitary map of $L^2(\mathbf{R}^m) \to L^2(\mathbf{R}^m)$ such that $(f,g) = (\hat f, \hat g)$. (Plancherel theorem).

We note that without the normalizing constant of $(2\pi)^{-m/2}$ in the definition of the measures dx and $d\xi$ there would be various normalizing constants appearing in these identities. It is property (b) which will be of the most interest to us since it will enable us to interchange differentiation and multiplication.

We define the Sobolev space $H_s(\mathbf{R}^m)$ to measure L^2 derivatives. If s is a real number and $f \in \mathcal{S}$, we define:

$$|f|_s^2 = \int (1 + |\xi|^2)^s |\hat{f}(\xi)|^2 \, d\xi.$$

The Sobolev space $H_s(\mathbf{R}^m)$ is the completion of $\mathcal{S}$ with respect to the norm $|_s$. The Plancherel theorem shows $H_0(\mathbf{R}^m)$ is isomorphic to $L^2(\mathbf{R}^m)$. More generally, $H_s(\mathbf{R}^m)$ is isomorphic to L^2 with the measure $(1 + |\xi|^2)^{s/2} \, d\xi$. Replacing $(1 + |\xi|^2)^s$ by $(1 + |\xi|)^{2s}$ in the definition of $|_s$ gives rise to an equivalent norm since there exist positive constants c_i such that:

$$c_1(1 + |\xi|^2)^s \leq (1 + |\xi|)^{2s} \leq c_2(1 + |\xi|^2)^s.$$

In some sense, the subscript "s" counts the number of L^2 derivatives. If $s = n$ is a positive integer, there exist positive constants c_1, c_2 so:

$$c_1(1 + |\xi|^2)^n \leq \sum_{|\alpha| \leq n} |\xi^\alpha|^2 \leq c_2(1 + |\xi|^2)^n.$$

This implies that we could define

$$|f|_n^2 = \sum_{|\alpha| \leq n} \int |\xi^\alpha \hat{f}|^2 \, d\xi = \sum_{|\alpha| \leq n} \int |D_x^\alpha f|^2 \, dx$$

as an equivalent norm for $H_n(\mathbf{R}^m)$. With this interpretation in mind, it is not surprising that when we extend D_x^α to H_s, that $|\alpha|$ L^2 derivatives are lost.

LEMMA 1.1.3. *D_x^α extends to define a continuous map $D_x^\alpha \colon H_s \to H_{s-|\alpha|}$.*

PROOF: Henceforth we will use C to denote a generic constant. C can depend upon certain auxiliary parameters which will usually be supressed in the interests of notational clarity. In this proof, for example, C depends on (s, α) but not of course upon f. The estimate:

$$|\xi^\alpha|^2 (1 + |\xi|^2)^{s-|\alpha|} \leq C(1 + |\xi|^2)^s$$

implies that:

$$|D_x^\alpha f|_{s-\alpha}^2 = \int |\xi^\alpha \hat{f}(\xi)|^2 (1 + |\xi|^2)^{s-|\alpha|} \, d\xi \leq C|f|_s^2$$

for $f \in \mathcal{S}$. Since H_s is the closure of $\mathcal{S}$ in the norm $|_s$, this completes the proof.

We can also use the sup norm to measure derivatives. If k is a non-negative integer, we define:

$$|f|_{\infty,k} = \sup_{x\in\mathbf{R}^m} \sum_{|\alpha|\le k} |D_x^\alpha f| \qquad \text{for} \qquad f \in \mathcal{S}.$$

The completion of $\mathcal{S}$ with respect to this norm is a subset of $C^k(\mathbf{R}^m)$ (the continuous functions on $\mathbf{R}^m$ with continuous partial derivatives up to order k). The next lemma relates the two norms $|_s$ and $|_{\infty,k}$. It will play an important role in showing the weak solutions we will construct to differential equations are in fact smooth.

Lemma 1.1.4. *Let k be a non-negative integer and let $s > k + \frac{m}{2}$. If $f \in H_s$, then f is C^k and there is an estimate $|f|_{\infty,k} \le C|f|_s$. (Sobolev Lemma).*

Proof: Suppose first $k = 0$ and $f \in \mathcal{S}$. We compute

$$\begin{aligned} f(x) &= \int e^{ix\cdot\xi}\, \hat f(\xi)\, d\xi \\ &= \int \{e^{ix\cdot\xi}\, \hat f(\xi)(1+|\xi|^2)^{s/2}\}\cdot\{(1+|\xi|^2)^{-s/2}\}\, d\xi. \end{aligned}$$

We apply the Cauchy-Schwarz inequality to estimate:

$$|f(x)|^2 \le |f|_s^2 \int (1+|\xi|^2)^{-s}\, d\xi.$$

Since $2s > m$, $(1+|\xi|^2)^{-s}$ is integrable so $|f(x)| \le C|f|_s$. We take the sup over $x \in \mathbf{R}^m$ to conclude $|f|_{\infty,0} \le C|f|_s$ for $f \in \mathcal{S}$. Elements of H_s are the limits in the $|_s$ norm of elements of $\mathcal{S}$. The uniform limit of continuous functions is continuous so the elements of H_s are continuous and the same norm estimate extends to H_s. If $k > 0$, we use the estimate:

$$|D_x^\alpha f|_{\infty,0} \le C|D_x^\alpha f|_{s-|\alpha|} \le C|f|_s \qquad \text{for } |\alpha| \le k \text{ and } s - k > \frac{m}{2}$$

to conclude $|f|_{\infty,k} \le C|f|_s$ for $f \in \mathcal{S}$. A similar argument shows that the elements of H_s must be C^k and that this estimate continues to hold.

If $s > t$, we can estimate $(1+|\xi|^2)^s \ge (1+|\xi|^2)^t$. This implies that $|f|_s \ge |f|_t$ so the identity map on $\mathcal{S}$ extends to define an injection of $H_s \to H_t$ which is norm non-increasing. The next lemma shows that this injection is compact if we restrict the supports involved.

Lemma 1.1.5. *Let $\{f_n\} \in \mathcal{S}$ be a sequence of functions with support in a fixed compact set K. We suppose there is a constant C so $|f_n|_s \le C$ for all n. Let $s > t$. There exists a subsequence f_{n_k} which converges in H_t. (Rellich lemma).*

Proof: Choose $g \in C_0(\mathbf{R}^m)$ which is identically 1 on a neighborhood of K. Then $gf_n = f_n$ so by Lemma 1.1.2(c) $\hat{f}_n = \hat{g} * \hat{f}_n$. We let $\partial_j = \dfrac{\partial}{\partial \xi_j}$ then $\partial_j(\hat{g} * \hat{f}_n) = \partial_j \hat{g} * \hat{f}_n$ so that:

$$|\partial_j \hat{f}_n(\xi)| \le \int |\{\partial_j \hat{g}(\xi - \zeta)\} \cdot \hat{f}_n(\zeta)|\, d\zeta.$$

We apply the Cauchy-Schwarz inequality to estimate:

$$|\partial_j \hat{f}_n(\xi)| \le |f_n|_s \cdot \left\{ \int |\partial_j \hat{g}(\xi - \zeta)|^2 (1 + |\zeta|^2)^{-s}\, d\zeta \right\}^{1/2} \le C \cdot h(\xi)$$

where h is some continuous function of ξ. A similar estimate holds for $|\hat{f}_n(\xi)|$. This implies that the $\{\hat{f}_n\}$ form a uniformly bounded equi-continuous family on compact ξ subsets. We apply the Arzela-Ascoli theorem to extract a subsequence we again label by f_n so that $\hat{f}_n(\xi)$ converges uniformly on compact subsets. We complete the proof by verifying that f_n converges in H_t for $s > t$. We compute:

$$|f_j - f_k|_t^2 = \int |\hat{f}_j - \hat{f}_k|^2 (1 + |\xi|^2)^t\, d\xi.$$

We decompose this integral into two parts, $|\xi| \ge r$ and $|\xi| \le r$. On $|\xi| \ge r$ we estimate $(1 + |\xi|^2)^t \le (1 + r^2)^{t-s}(1 + |\xi|^2)^s$ so that:

$$\begin{aligned} \int_{|\xi| \ge r} |\hat{f}_j - \hat{f}_k|^2 (1 + |\xi|^2)^t\, d\xi &\le (1 + r^2)^{t-s} \int |\hat{f}_j - \hat{f}_k|^2 (1 + |\xi|^2)^s\, d\xi \\ &\le 2C(1 + r^2)^{t-s}. \end{aligned}$$

If $\varepsilon > 0$ is given, we choose r so that $2C(1 + r^2)^{t-s} < \varepsilon$. The remaining part of the integral is over $|\xi| \le r$. The $\hat{f}_j$ converge uniformly on compact subsets so this integral can be bounded above by ε if $j, k > j(\varepsilon)$. This completes the proof.

The hypothesis that the supports are uniformly bounded is essential. It is easy to construct a sequence $\{f_n\}$ with $|f_n|_s = 1$ for all n and such that the supports are pair-wise disjoint. In this case we can find $\varepsilon > 0$ so that $|f_j - f_k|_t > \varepsilon$ for all (j, k) so there is no convergent subsequence.

We fix $\phi \in \mathcal{S}$ and let $\phi_\varepsilon(x) = \phi(\varepsilon x)$. We suppose $\phi(0) = 1$ and fix $f \in \mathcal{S}$. We compute:

$$D_x^\alpha(f - \phi_\varepsilon f) = (1 - \phi_\varepsilon)D_x^\alpha f + \text{terms of the form } \varepsilon^j D_x^\beta \phi(\varepsilon x) D_x^\gamma f.$$

As $\varepsilon \to 0$, these other terms go to zero in L^2. Since $\phi_\varepsilon \to 1$ pointwise, $(1-\phi_\varepsilon)D_x f$ goes to zero in L^2. This implies $\phi_\varepsilon f \to f$ in H_n for any $n \geq 0$ as $\varepsilon \to 0$ and therefore $\phi_\varepsilon f \to f$ in H_s for any s. If we take $\phi \in C_0^\infty(\mathbf{R}^m)$, this implies $C_0^\infty(\mathbf{R}^m)$ is dense in H_s for any s.

Each H_s space is a Hilbert space so it is isomorphic to its dual. Because there is no preferred norm for H_s, it is useful to obtain an invariant alternative characterization of the dual space H_s^*:

LEMMA 1.1.6. *The L^2 pairing which maps $\mathcal{S} \times \mathcal{S} \to \mathbf{C}$ extends to a map of $H_s \times H_{-s} \to \mathbf{C}$ which is a perfect pairing and which identifies H_{-s} with H_s^*. That is:*

(a) $|(f,g)| \leq |f|_s |g|_{-s}$ *for* $f, g \in \mathcal{S}$,
(b) *given $f \in \mathcal{S}$ there exists $g \in \mathcal{S}$ so $(f,g) = |f|_s|g|_{-s}$ and we can define*

$$|f|_s = \sup_{g \in \mathcal{S},\, g \neq 0} \frac{|(f,g)|}{|g|_{-s}}.$$

PROOF: This follows from the fact that H_s is L^2 with the weight function $(1+|\xi|^2)^s$ and H_{-s} is L^2 with the weight function $(1+|\xi|^2)^{-s}$. We compute:

$$(f,g) = (\hat{f}, \hat{g}) = \int \hat{f}(\xi)(1+|\xi|^2)^{s/2}\overline{\hat{g}}(\xi)(1+|\xi|^2)^{-s/2}\, d\xi$$

and apply the Cauchy-Schwartz inequality to prove (a).

To prove part (b), we note $|f|_s \geq \sup\limits_{g \in \mathcal{S},\, g \neq 0} \dfrac{|(f,g)|}{|g|_{-s}}$. We take g to be defined by:

$$\hat{g} = \hat{f}(1+|\xi|^2)^s \in \mathcal{S}$$

and note that $(f,g) = (\hat{f}, \hat{g}) = |f|_s^2$ and that $|g|_{-s}^2 = |f|_s^2$ to see that equality can occur in (a) which proves (b)

If $s > t > u$ then we can estimate:

$$(1+|\xi|)^{2t} \leq \varepsilon(1+|\xi|)^{2s} + C(\varepsilon)(1+|\xi|)^{2u}$$

for any $\varepsilon > 0$. This leads immediately to the useful estimate:

LEMMA 1.1.7. *Let $s > t > u$ and let $\varepsilon > 0$ be given. Then*

$$|f|_t \leq \varepsilon |f|_s + C(\varepsilon)|f|_u.$$

If V is a finite dimensional vector space, let $C^\infty(V)$ be the space of smooth complex valued maps of $\mathbf{R}^m \to V$. We choose a fixed Hermitian inner product on V and define $\mathcal{S}(V)$ and $H_s(V)$ as in the scalar case. If $\dim(V) = k$ and if we choose a fixed orthonormal basis for V, then $\mathcal{S}(V)$ and $H_s(V)$ become isomorphic to the direct sum of k copies of $\mathcal{S}$ and of H_s. Lemmas 1.1.1 through 1.1.7 extend in the obvious fashion.

We conclude this subsection with an extremely useful if elementary estimate:

LEMMA 1.1.8. (PEETRE'S INEQUALITY). *Let s be real and $x, y \in \mathbf{R}^m$. Then $(1 + |x + y|)^s \leq (1 + |y|)^s (1 + |x|)^{|s|}$.*

PROOF: We suppose first $s > 0$. We raise the triangle inequality:

$$1 + |x + y| < 1 + |x| + |y| \leq (1 + |y|)(1 + |x|)$$

to the s^{th} power to deduce the desired inequality. We now suppose $s < 0$. A similar inequality:

$$(1 + |y|)^{-s} \leq (1 + |x + y|)^{-s}(1 + |x|)^{-s}$$

yields immediately:

$$(1 + |x + y|)^s \leq (1 + |y|)^s (1 + |x|)^{-s}$$

to complete the proof.

1.2. Pseudo-Differential Operators on $\mathbf{R}^m$.

A linear partial differential operator of order d is a polynomial expression $P = p(x, D) = \sum_{|\alpha| \le d} a_\alpha(x) D_x^\alpha$ where the $a_\alpha(x)$ are smooth. The symbol $\sigma P = p$ is defined by:

$$\sigma P = p(x, \xi) = \sum_{|\alpha| \le d} a_\alpha(x) \xi^\alpha$$

and is a polynomial of order d in the dual variable ξ. It is convenient to regard the pair (x, ξ) as defining a point of the cotangent space $T^*(\mathbf{R}^m)$; we will return to this point again when we discuss the effect of coordinate transformations. The leading symbol $\sigma_L P$ is the highest order part:

$$\sigma_L P(x, \xi) = \sum_{|\alpha| = d} a_\alpha(x) \xi^\alpha$$

and is a homogeneous polynomial of order d in ξ.

We can use the Fourier inversion formula to express:

$$Pf(x) = \int e^{ix\cdot\xi} p(x, \xi) \hat{f}(\xi)\, d\xi = \int e^{i(x-y)\cdot\xi}\, p(x, \xi) f(y)\, dy\, d\xi$$

for $f \in \mathcal{S}$. We note that since the second integral does not converge absolutely, we cannot interchange the dy and $d\xi$ orders of integration. We use this formalism to define the action of pseudo-differential operators (ΨDO's) for a wider class of symbols $p(x, \xi)$ than polynomials. We make the following

DEFINITION. *$p(x, \xi)$ is a symbol of order d and we write $p \in S^d$ if*

(a) *$p(x, \xi)$ is smooth in $(x, \xi) \in \mathbf{R}^m \times \mathbf{R}^m$ with compact x support,*

(b) *for all (α, β) there are constants $C_{\alpha,\beta}$ such that*

$$|D_x^\alpha D_\xi^\beta p(x, \xi)| \le C_{\alpha,\beta}(1 + |\xi|)^{d - |\beta|}.$$

For such a symbol p, we define the associated operator $P(x, D)$ by:

$$P(x, D)(f)(x) = \int e^{ix\cdot\xi} p(x, \xi) \hat{f}(\xi)\, d\xi = \int e^{(x-y)\cdot\xi}\, p(x, \xi) f(y)\, dy\, d\xi$$

as a linear operator mapping $\mathcal{S} \to \mathcal{S}$.

A differential operator has as its order a positive integer. The order of a pseudo-differential operator is not necessarily an integer. For example, if $f \in C_0^\infty(\mathbf{R}^m)$, define:

$$p(x, \xi) = f(x)(1 + |\xi|^2)^{d/2} \in S^d \qquad \text{for any } d \in \mathbf{R}.$$

This will be a symbol of order d. If $p \in S^d$ for all d, then we say that $p \in S^{-\infty}$ is infinitely smoothing. We adopt the notational convention of letting p, q, r denote symbols and P, Q, R denote the corresponding ΨDO's.

Because we shall be interested in problems on compact manifolds, we have assumed the symbols have compact x support to avoid a number of technical complications. The reader should note that there is a well defined theory which does not require compact x support.

When we discuss the heat equation, we shall have to consider a wider class of symbols which depend on a complex parameter. We postpone discussion of this class until later to avoid unnecessarily complicating the discussion at this stage. We shall phrase the theorems and proofs of this section in such a manner that they will generalize easily to the wider class of symbols.

Our first task is to extend the action of P from $\mathcal{S}$ to H_s.

Lemma 1.2.1. *Let $p \in S^d$ then $|Pf|_{s-d} \le C|f|_s$ for $f \in \mathcal{S}$. P extends to a continuous map $P\colon H_s \to H_{s-d}$ for all s.*

Proof: We compute $Pf(x) = \int e^{ix\cdot\xi} p(x,\xi)\hat{f}(\xi)\,d\xi$ so that the Fourier transform is given by:

$$\widehat{Pf}(\zeta) = \int e^{ix\cdot(\xi-\zeta)} p(x,\xi)\hat{f}(\xi)\,d\xi\,dx.$$

This integral is absolutely convergent since p has compact x support so we may interchange the order of integration. If we define

$$q(\zeta,\xi) = \int e^{-ix\cdot\zeta} p(x,\xi)\,dx$$

as the Fourier transform in the x direction, then

$$\widehat{Pf}(\zeta) = \int q(\zeta-\xi,\xi)\hat{f}(\xi)\,d\xi.$$

By Lemma 1.1.6, $|Pf|_{s-d} = \sup_{g\in S} \dfrac{|(Pf,g)|}{|g|_{d-s}}$. We compute:

$$(Pf,g) = \int q(\zeta-\xi,\xi)\hat{f}(\xi)\bar{\hat{g}}(\zeta)\,d\xi\,d\zeta.$$

Define:

$$K(\zeta,\xi) = q(\zeta-\xi,\xi)(1+|\xi|)^{-s}(1+|\zeta|)^{s-d}$$

then:

$$(Pf,g) = \int K(\zeta,\xi)\hat{f}(\xi)(1+|\xi|)^{s}\bar{\hat{g}}(\zeta)(1+|\zeta|)^{d-s}\,d\xi\,d\zeta.$$

We apply the Cauchy-Schwarz inequality to estimate:

$$|(Pf,g)| \le \left\{ \int |K(\zeta,\xi)|\,|\hat f(\xi)|^2(1+|\xi|)^{2s}\,d\xi\,d\zeta \right\}^{1/2}$$
$$\times \left\{ \int |K(\zeta,\xi)|\,|\hat g(\zeta)|^2(1+|\zeta|)^{2d-2s}\,d\xi\,d\zeta \right\}^{1/2}.$$

We complete the proof by showing

$$\int |K(\zeta,\xi)|\,d\xi \le C \qquad \text{and} \qquad \int |K(\zeta,\xi)|\,d\zeta \le C$$

since then $|(Pf,g)| \le C|f|_s|g|_{d-s}$.

By hypothesis, p has suppport in a compact set $\mathcal{K}$ and we have estimates:

$$|D_x^\alpha p(x,\xi)| \le C_\alpha(1+|\xi|)^d.$$

Therefore:

$$|\zeta^\alpha q(\zeta,\xi)| = \left| \int e^{-ix\cdot\zeta} D_x^\alpha p(x,\xi)\,dx \right| \le C_\alpha(1+|\xi|)^d \operatorname{vol}(\mathcal{K}).$$

Therefore, for any integer k, $|q(\zeta,\xi)| \le C_k(1+|\xi|)^d(1+|\zeta|)^{-k}\operatorname{vol}(\mathcal{K})$ and:

$$|K(\zeta,\xi)| \le C_k(1+|\xi|)^{d-s}(1+|\zeta|)^{s-d}(1+|\zeta-\xi|)^{-k}\operatorname{vol}(\mathcal{K}).$$

We apply Lemma 1.1.8 with $x+y=\xi$ and $y=\zeta$ to estimate:

$$|K(\zeta,\xi)| \le C_k(1+|\zeta-\xi|)^{|d-s|-k}\operatorname{vol}(\mathcal{K}).$$

If we choose $k > \frac{m}{2}+|d-s|$, then this will be integrable and complete the proof.

Our next task is to show that the class of ΨDO's forms an algebra under the operations of composition and taking adjoint. Before doing that, we study the situation with respect to differential operators to motivate the formulas we shall derive. Let $P = \sum_\alpha p_\alpha(x)D_x^\alpha$ and let $Q = \sum_\alpha q_\alpha(x)D_x^\alpha$ be two differential operators. We assume p and q have compact x support. It is immediate that:

$$P^* = \sum_\alpha D_x^\alpha p_\alpha^* \qquad \text{and} \qquad PQ = \sum_{\alpha,\beta} p_\alpha(x)D_x^\alpha q_\beta(x)D_x^\beta$$

are again differential operators in our class. Furthermore, using Leibnitz's rule

$$D_x^\alpha(fg) = \sum_{\beta+\gamma=\alpha} D_x^\beta(f)\cdot D_x^\gamma(g)\cdot\frac{\alpha!}{\beta!\gamma!},$$
$$d_\xi^\beta(\xi^{\beta+\gamma}) = \xi^\gamma\cdot\frac{(\beta+\gamma)!}{\gamma!},$$

it is an easy combinatorial exercise to compute that:

$$\sigma(P^*) = \sum_\alpha d_\xi^\alpha D_x^\alpha p^*/\alpha! \qquad \text{and} \qquad \sigma(PQ) = \sum_\alpha d_\xi^\alpha p \cdot D_x^\alpha q/\alpha!\,.$$

The perhaps surprising fact is that these formulas remain true in some sense for ΨDO's, only the sums will become infinite rather than finite.

We introduce an equivalence relation on the class of symbols by defining $p \sim q$ if $p - q \in S^{-\infty}$. We note that if $p \in S^{-\infty}$ then $P\colon H_s \to H_t$ for all s and t by Lemma 1.2.1. Consequently by Lemma 1.1.4, $P\colon H_s \to C_0^\infty$ for all s so that P is infinitely smoothing in this case. Thus we mod out by infinitely smoothing operators.

Given symbols $p_j \in S^{d_j}$ where $d_j \to -\infty$, we write

$$p \sim \sum_{j=1}^\infty p_j$$

if for every d there is an integer $k(d)$ such that $k \geq k(d)$ implies that $p - \sum_{j=1}^k p_j \in S^d$. We emphasize that this sum does not in fact need to converge. The relation $p \sim \sum p_j$ simply means that the difference between P and the partial sums of the P_j is as smoothing as we like. It will turn out that this is the appropriate sense in which we will generalize the formulas for $\sigma(P^*)$ and $\sigma(PQ)$ from differential to pseudo-differential operators.

Ultimately, we will be interested in operators which are defined on compact manifolds. Consequently, it poses no difficulties to restrict the domain and the range of our operators. Let U be a open subset of $\mathbf{R}^m$ with compact closure. Let $p(x,\xi) \in S^d$ have x support in U. We restrict the domain of the operator P to $C_0^\infty(U)$ so $P\colon C_0^\infty(U) \to C_0^\infty(U)$. Let $\Psi_d(U)$ denote the space of all such operators. For $d \leq d'$, then $\Psi_d(U) \subseteq \Psi_{d'}(U)$. We define

$$\Psi(U) = \bigcup_d \Psi_d(U) \qquad \text{and} \qquad \Psi_{-\infty}(U) = \bigcap_d \Psi_d(U)$$

to be the set of all pseudo-differential operators on U and the set of infinitely smoothing pseudo-differential operators on U.

More generally, let $p(x,\xi)$ be a matrix valued symbol; we suppose the components of p all belong to S^d. The corresponding operator P is given by a matrix of pseudo-differential operators. P is a map from vector valued functions with compact support in U to vector valued functions with compact support in U. We shall not introduce separate notation for the shape of p and shall continue to denote the collection of all such operators by $\Psi_d(U)$. If p and q are matrix valued and of the proper shape, we define $p \cdot q$ and also the operator $P \cdot Q$ by matrix product and by composition. We also define p^* and P^* to be the matrix adjoint and the operator adjoint so that $(P^*f, g) = (f, P^*g)$ where f and g are vector valued and of compact support. Before studying the algebra structure on $\Psi(U)$, we must enlarge the class of symbols which we can admit:

LEMMA 1.2.2. *Let $r(x,\xi,y)$ be a matrix valued symbol which is smooth in (x,ξ,y). We suppose r has compact x support inside U and that there are estimates:*

$$|D_x^\alpha D_\xi^\beta D_y^\gamma r| \le C_{\alpha,\beta,\gamma}(1+|\xi|)^{d-|\beta|}$$

for all multi-indices (α,β,γ). If f is vector valued with compact support in U, we define:

$$Rf(x) = \int e^{i(x-y)\cdot\xi}\, r(x,\xi,y) f(y)\, dy\, d\xi.$$

Then this operator is in $\Psi_d(U)$ and the symbol is given by:

$$\sigma R(x,\xi) \sim \left\{ \sum_\alpha d_\xi^\alpha D_y^\alpha r(x,\xi,y)/\alpha! \right\}\Bigg|_{x=y}.$$

PROOF: We note that any symbol in S^d belongs to this class of operators if we define $r(x,\xi,y) = p(x,\xi)$. We restricted to vector valued functions with compact support in U. By multiplying r by a cut-off function in y with compact support which is 1 over U, we may assume without loss of generality the y support of r is compact as well. Define:

$$q(x,\xi,\zeta) = \int e^{-iy\cdot\zeta} r(x,\xi,y)\, dy$$

to be the Fourier transform of r in the y variable. Using Lemma 1.1.2 we see $\widehat{(rf)} = \hat{r} * \hat{f}$. This implies that:

$$\int e^{-iy\cdot\xi}\, r(x,\xi,y) f(y)\, dy = \int q(x,\xi,\xi-\zeta)\hat{f}(\zeta)\, d\zeta.$$

The argument given in the proof of Lemma 1.2.1 gives estimates of the form:

$$|q(x,\xi,\zeta)| \le C_k(1+|\xi|)^d(1+|\zeta|)^{-k} \qquad \text{and} \qquad |\hat{f}(\zeta)| \le C_k(1+|\zeta|)^{-k}$$

for any k. Consequently:

$$|q(x,\xi,\xi-\zeta)\hat{f}(\zeta)| \le C_k(1+|\xi|)^d(1+|\xi-\zeta|)^{-k}(1+|\zeta|)^{-k}.$$

We apply Lemma 1.1.8 to estimate:

$$|q(x,\xi,\xi-\zeta)\hat{f}(\zeta)| \le C_k(1+|\xi|)^{|d|-k}(1+|\zeta|)^{|d|-k}$$

so this is absolutely integrable. We change the order of integration and express:

$$Rf(x) = \int e^{ix\cdot\xi} q(x,\xi,\xi-\zeta)\hat{f}(\zeta)\,d\xi\,d\zeta.$$

We define:

$$p(x,\zeta) = \int e^{ix(\xi-\zeta)} q(x,\xi,\xi-\zeta)\,d\xi$$

and compute:

$$Rf(x) = \int e^{ix\cdot\zeta} p(x,\zeta)\hat{f}(\zeta)\,d\zeta$$

is a pseudo-differential operator once it is verified that $p(x,\zeta)$ is a symbol in the correct form.

We change variables to express:

$$p(x,\zeta) = \int e^{ix\cdot\xi} q(x,\xi+\zeta,\xi)\,d\xi$$

and estimate:

$$\begin{aligned} |q(x,\xi+\zeta,\xi)| &\le C_k(1+|\xi+\zeta|)^d(1+|\xi|)^{-k} \\ &\le C_k(1+|\zeta|)^d(1+|\xi|)^{|d|-k}. \end{aligned}$$

This is integrable so $|p(x,\zeta)| \le C'_k(1+|\zeta|)^d$. Similar estimates on $|D_x^\alpha D_\zeta^\beta$ $q(x,\xi+\zeta,\xi)|$ which arise from the given estimates for r show that $p \in S^d$ so that R is a pseudo-differential operator.

We use Taylor's theorem on the middle variable of $q(x,\xi+\zeta,\xi)$ to expand:

$$q(x,\xi+\zeta,\xi) = \sum_{|\alpha|\le k} \frac{d_\zeta^\alpha q(x,\zeta,\xi)\xi^\alpha}{\alpha!} + q_k(x,\zeta,\xi).$$

The remainder q_k decays to arbitrarily high order in (ξ,ζ) and after integration gives rise to a symbol in S^{d-k} which may therefore be ignored. We integrate to conclude

$$\begin{aligned} p(x,\zeta) &= \sum_{|\alpha|\le k} \int e^{ix\cdot\xi} \frac{d_\zeta^\alpha q(x,\zeta,\xi)\xi^\alpha}{\alpha!}\,d\xi + \text{remainder} \\ &= \sum_{|\alpha|\le k} \left.\frac{d_\zeta^\alpha D_y^\alpha r(x,\zeta,y)}{\alpha!}\right|_{x=y} + \text{a remainder} \end{aligned}$$

using Lemma 1.1.2. This completes the proof of the lemma.

We use this technical lemma to show that the pseudo-differential operators form an algebra:

LEMMA 1.2.3. *Let $P \in \Psi_d(U)$ and let $Q \in \Psi_e(U)$. Then:*
(a) If U' is any open set with compact closure containing $\overline{U}$, then $P^* \in \Psi_d(U')$ and $\sigma(P^*) \sim \sum_\alpha d_\xi^\alpha D_x^\alpha p^*/\alpha!$.
(b) Assume that P and Q have the proper shapes so PQ and pq are defined. Then $PQ \in \Psi_{d+e}(U)$ and $\sigma(PQ) \sim \sum_\alpha d_\xi^\alpha p \cdot D_x^\alpha q/\alpha!$.

PROOF: The fact that P^* lies in a larger space is only a slight bit of technical bother; this fact plays an important role in considering boundary value problems of course. Let $(f,g) = f \cdot g$ be the pointwise Hermitian inner product. Fix $\phi \in C_0^\infty(U')$ to be identically 1 on U and compute:

$$\begin{aligned}(Pf,g) &= \int e^{i(x-y)\cdot\xi}\, p(x,\xi)\phi(y)f(y)\cdot g(x)\,dy\,d\xi\,dx\\ &= \int f(y)\cdot e^{i(y-x)\cdot\xi}\, p^*(x,\xi)\phi(y)g(x)\,dy\,d\xi\,dx\end{aligned}$$

since the inner product is Hermitian. By approximating $p^*(x,\xi)$ by functions with compact ξ support, we can justify the use of Fubini's theorem to replace $dy\,d\xi\,dx$ by $dx\,d\xi\,dy$ and express:

$$\begin{aligned}(Pf,g) &= \int f(y)\cdot e^{i(y-x)\cdot\xi}\, p^*(x,\xi)\phi(y)g(x)\,dx\,d\xi\,dy\\ &= (f,P^*g)\end{aligned}$$

where we define:

$$P^*g(y) = \int e^{i(y-x)\cdot\xi}\, p^*(x,\xi)\phi(y)g(x)\,dx\,d\xi.$$

This is an operator of the form discussed in Lemma 1.2.2 so $P^* \in \Psi_d(U')$ and we compute:

$$\sigma(P^*) \sim \sum_\alpha d_\xi^\alpha D_x^\alpha p^*/\alpha!$$

since $\phi = 1$ on the support of p. This completes the proof of (a). We note that we can delete the factor of ϕ from the expression for P^*g since it was only needed to prove P^* was a ΨDO.

We use (a) to prove (b). Since:

$$Q^*g(y) = \int e^{i(y-x)\cdot\xi}\, q^*(x,\xi)g(x)\,dx\,d\xi$$

the Fourier inversion formula implies:

$$\widehat{(Q^*g)} = \int e^{-ix\cdot\xi}\, q^*(x,\xi)g(x)\,dx.$$

If $\tilde{q}$ is the symbol of Q^*, then we interchange the roles of Q and Q^* to see:

$$\widehat{(Qg)} = \int e^{-ix\cdot\xi}\,\tilde{q}^*(x,\xi)g(x)\,dx.$$

Therefore:

$$\begin{aligned} PQg(x) &= \int e^{ix\cdot\xi}\,p(x,\xi)\widehat{(Qg)}(\xi)\,d\xi \\ &= \int e^{i(x-y)\cdot\xi}\,p(x,\xi)\tilde{q}^*(y,\xi)g(y)\,dy\,d\xi \end{aligned}$$

which is an operator of the form discussed in Lemma 1.2.2 if $r(x,\xi,y) = p(x,\xi)\tilde{q}^*(y,\xi)$. This proves PQ is a pseudo-differential operator of the correct order. We compute the symbol of PQ to be:

$$\sim \sum_\alpha d_\xi^\alpha D_y^\alpha(p(x,\xi)\tilde{q}^*(y,\xi))/\alpha! \quad \text{evaluated at } x = y.$$

We use Leibnitz's formula and expand this in the form:

$$\sim \sum_{\beta,\gamma} d_\xi^\beta p(x,\xi) D_y^\beta d_\xi^\gamma D_y^\gamma \tilde{q}^*/\beta!\gamma!\,.$$

The sum over γ yields the symbol of $Q^{**} = Q$ so we conclude finally

$$\sigma(PQ) \sim \sum_\beta d_\xi^\beta p(x,\xi) D_x^\beta q(x,\xi)/\beta!$$

which completes the proof.

Let $K(x,y)$ be a smooth matrix valued function with compact x support in U. If f is vector valued with compact support in U, we define:

$$P(K)(f)(x) = \int K(x,y)f(y)\,dy.$$

LEMMA 1.2.4. *Let $K(x,y)$ be smooth with compact x support in U, then $P(K) \in \Psi_{-\infty}(U)$.*

PROOF: We let $\phi(\xi) \in C_0^\infty(\mathbf{R}^m)$ with $\int \phi(\xi)\,d\xi = 1$. Define:

$$r(x,\xi,y) = e^{i(y-x)\cdot\xi}\,\phi(\xi)K(x,y)$$

then this is a symbol in $S^{-\infty}$ of the sort discussed in Lemma 1.2.2. It defines an infinitely smoothing operator. It is immediate that:

$$P(K)(f)(x) = \int e^{i(x-y)\cdot\xi}\,r(x,\xi,y)f(y)\,dy\,d\xi.$$

Conversely, it can be shown that any infinitely smoothing map has a smooth kernel. In general, of course, it is not possible to represent an arbitrary pseudo-differential operator by a kernel. If P is smoothing enough, however, we can prove:

LEMMA 1.2.5. *Let r satisfy the hypothesis of Lemma 1.2.2 where $d < -m-k$. We define $K(x,y) = \int e^{i(x-y)\cdot\xi}\, r(x,\xi,y)\, d\xi$. Then K is C^k in (x,y) and $Rf(x) = \int K(x,y)f(y)\, dy$.*

PROOF: If we can show K is well defined, then the representation of R in terms of the kernel K will follow from Fubini's theorem. We estimate:

$$D_x^\alpha D_y^\beta K(x,y) = \sum_{\substack{\alpha=\alpha_1+\alpha_2\\ \beta=\beta_1+\beta_2}} \frac{\alpha!\,\beta!}{\alpha_1!\,\alpha_2!\,\beta_1!\,\beta_2!}(-1)^{|\beta_1|} \times \left\{\int e^{i(x-y)\cdot\xi}\,\xi^{\alpha_1+\beta_1} D_x^{\alpha_2} D_y^{\beta_2} r(x,\xi,y)\, d\xi\right\}.$$

Since we can estimate:

$$|\xi^{\alpha_1+\beta_1} D_x^{\alpha_2} D_y^{\beta_2} r(x,\xi,y)| \le C(1+|\xi|)^{d+|\alpha|+|\beta|}$$

this will be integrable for $|\alpha|+|\beta| \le k$. Thus K is C^k and the representation of R follows immediately.

In Lemma 1.2.2 we computed the symbol of the pseudo-differential operator defined by $r(x,\xi,y)$ in terms of $d_\xi^\alpha D_y^\alpha r$ when $x=y$. This implies the singular (i.e., the non-smoothing part) of R is concentrated near the diagonal $x=y$. We make this more precise:

LEMMA 1.2.6. *Let $r(x,\xi,y)$ satisfy the hypothesis of Lemma 1.2.2. Suppose the x support of r is disjoint from the y support of r, then R is infinitely smoothing and is represented by a smooth kernel function $K(x,y)$.*

PROOF: We would like to define $K(x,y) = \int e^{(x-y)\cdot\xi}\, r(x,\xi,y)\, d\xi$. Unfortunately, this integral need not converge in general. By hypothesis, $|x-y| \ge \varepsilon > 0$ on the support of r. We define the Laplacian $\Delta_\xi = \sum_\nu D_{\xi_\nu}^2$. Since $\Delta_\xi e^{i(x-y)\cdot\xi} = |x-y|^2 e^{i(x-y)\cdot\xi}$ we integrate by parts in a formal sense k times to express:

$$Rf(x) = \int e^{i(x-y)\cdot\xi}\,|x-y|^{-2k}\Delta_\xi^k r(x,\xi,y) f(y)\, dy\, d\xi.$$

This formal process may be justified by first approximating r by a function with compact ξ support. We now define

$$K(x,y) = \int e^{i(x-y)\cdot\xi}\,|x-y|^{-2k}\Delta_\xi^k r(x,\xi,y)\, d\xi$$

for any k sufficiently large. Since $\Delta_\xi^k r$ decays to arbitrarily high order in ξ, we use the same argument as that given in Lemma 1.2.5 to show that

$K(x, y)$ is arbitrarily smooth in (x, y) and hence is C^∞. This completes the proof.

We note that in general $K(x, y)$ will become singular at $x = y$ owing to the presence of the terms $|x - y|^{-2k}$ if we do not assume the support of x is disjoint from the support of y.

A differential operator P is local in the sense that if $f = 0$ on some open subset of U, then $Pf = 0$ on that same subset since differentiation is a purely local process. ΨDO's are not local in general since they are defined by the Fourier transform which smears out the support. Nevertheless, they do have a somewhat weaker property, they do not smear out the singular support of a distribution f. More precisely, let $f \in H_s$. If $\phi \in C_0^\infty(U)$, we define the map $f \mapsto \phi f$. If we take $r(x, \xi, y) = \phi(x)$ and apply Lemma 1.2.2, then we see that this is a pseudo-differential operator of order 0. Therefore $\phi f \in H_s$ as well. This gives a suitable notion of restriction. We say that f is smooth on an open subset U' of U if and only if $\phi f \in C^\infty$ for every such ϕ. An operator P is said to be pseudo-local if f is smooth on U' implies Pf is smooth on U'.

Lemma 1.2.7. *Pseudo-differential operators are pseudo-local.*

Proof: Let $P \in \Psi_d(U)$ and let $f \in H_s$. Fix $x \in U'$ and choose $\phi \in C_0^\infty(U')$ to be identically 1 near x. Choose $\psi \in C_0^\infty(U')$ with support contained in the set where ϕ is identically 1. We must verify that $\psi P f$ is smooth. We compute:

$$\psi P f = \psi P \phi f + \psi P(1 - \phi) f.$$

By hypothesis, ϕf is smooth so $\psi P \phi f$ is smooth. The operator $\psi P(1 - \phi)$ is represented by a kernel of the form $\psi(x) p(x, \xi)(1 - \phi(y))$ which has disjoint x and y support. Lemma 1.2.6 implies $\psi P(1 - \phi) f$ is smooth which completes the proof.

In Lemmas 1.2.2 and 1.2.3 we expressed the symbol of an operator as a infinite asymptotic series. We show that the algebra of symbols is complete in a certain sense:

Lemma 1.2.8. *Let $p_j \in S^{d_j}(U)$ where $d_j \to -\infty$. Then there exists $p \sim \sum_j p_j$ which is a symbol in our class. p is a unique modulo $S^{-\infty}$.*

Proof: We may assume without loss of generality that $d_1 > d_2 > \cdots \to -\infty$. We will construct $p \in S^{d_1}$. The uniqueness is clear so we must prove existence. The p_j all have support inside U; we will construct p with support inside U' where U' is any open set containing the closure of U.

Fix a smooth function ϕ such that:

$$0 \le \phi \le 1, \qquad \phi(\xi) = 0 \text{ for } |\xi| \le 1, \qquad \phi(\xi) = 1 \text{ for } |\xi| \ge 2.$$

We use ϕ to cut away the support near $\xi = 0$. Let $t_j \to 0$ and define:

$$p(x,\xi) = \sum_j \phi(t_j\xi)p_j(x,\xi).$$

For any fixed ξ, $\phi(t_j\xi) = 0$ for all but a finite number of j so this sum is well defined and smooth in (x,ξ). For $j > 1$ we have

$$|p_j(x,\xi)| \le C_j(1+|\xi|)^{d_j} = C_j(1+|\xi|)^{d_1}(1+|\xi|)^{d_j-d_1}.$$

If $|\xi|$ is large enough, $(1+|\xi|)^{d_j-d_1}$ is as small as we like and therefore by passing to a subsequence of the t_j we can assume

$$|\phi(t_j\xi)p_j(x,\xi)| \le 2^{-j}(1-|\xi|)^{d_1} \qquad \text{for } j > 1.$$

This implies that $|p(x,\xi)| \le (C_1+1)(1+|\xi|)^{d_1}$. We use a similar argument with the derivatives and use a diagonalization argument on the resulting subsequences to conclude $p \in S^d$. The supports of all the p_j are contained compactly in U so the support of p is contained in $\overline{U}$ which is contained in U'.

We now apply exactly the same argument to $p_{d_2} + \cdots$ to assume that $p_{d_2} + \cdots \in S^{d_2}$. We continue in this fashion and use a diagonalization argument on the resulting subsequences to conclude in the end that

$$\sum_{j=j_0}^{\infty} \phi(t_j\xi)p_j(x,\xi) \in S^k \qquad \text{for } k = d_{j_0}.$$

Since $p_j - \phi(t_j\xi)p_j \in S^{-\infty}$, this implies $p - \sum_{j=1}^{j_0} p_j \in S^k$ and completes the proof.

If $K(x,y)$ is smooth with compact x,y support in U, then $P(K) \in \Psi_{-\infty}(U)$ defines a continuous operator from $H_s \to H_t$ for any s,t. Let $|P|_{s,t}$ denote the operator norm so $|Pf|_t \le |P|_{s,t}|f|_s$ for any $f \in \mathcal{S}$. It will be convenient to be able to estimate $|K|_{\infty,k}$ in terms of these norms:

LEMMA 1.2.9. *Let $K(x,y)$ be a smooth kernel with compact x,y support in U. Let $P = P(K)$ be the operator defined by K. If k is a non-negative integer, then $|K|_{\infty,k} \le C(k)|P|_{-k,k}$*

PROOF: By arguing separately on each entry in the matrix K, we may reduce ourselves to the scalar case. Suppose first $k = 0$. Choose $\phi \in C_0^\infty(\mathbf{R}^m)$ positive with $\int \phi(x)\,dx = 1$. Fix points $(x_0,y_0) \in U \times U$ and define:

$$f_n(x) = n^m\phi(n(x-x_0)) \qquad \text{and} \qquad g_n(y) = n^m\phi(n(y-y_0)).$$

Then if n is large, f_n and g_n have compact support in U. Then:

$$K(x_0, y_0) = \lim_{n\to\infty} \int f_n(x)K(x,y)g_n(y)\,dy\,dx = \lim_{n\to\infty}(f_n, Pg_n)$$

by Lemma 1.1.1. We estimate

$$|(f_n, Pg_n)| \le |P|_{0,0}|f_n|_0|g_n|_0 = |P|_{0,0}|\phi|_0^2$$

to complete the proof in this case.

If $|\alpha| \le k, \ \ |\beta| \le k$ then:

$$\begin{aligned} D_x^\alpha D_y^\beta K(x,y) &= \lim_{n\to\infty} \int f_n(x)\big\{D_x^\alpha D_y^\beta K(x,y)\big\}g_n(y)\,dy\,dx \\ &= \lim_{n\to\infty} \int \big(D_x^\alpha f_n)K(x,y)(D_y^\beta g_n(y)\big)\,dy\,dx \\ &= \lim_{n\to\infty}(D_x^\alpha f_n, PD_y^\beta g_n). \end{aligned}$$

We use Lemma 1.1.6 to estimate this by

$$\begin{aligned} |D_x^\alpha f_n|_{-k}|PD_y^\beta g_n|_k &\le |f_n|_0|P|_{-k,k}|D_y^\beta g_n|_{-k} \\ &\le |f_n|_0|g_n|_0|P|_{-k,k} = |P|_{-k,k}|\phi|_0^2 \end{aligned}$$

to complete the proof.

1.3. Ellipticity and Pseudo-Differential Operators on Manifolds.

The norms we have given to define the spaces H_s depend upon the Fourier transform. In order to get a more invariant definition which can be used to extend these notions to manifolds, we must consider elliptic pseudo-differential operators.

Let $p \in S^d(U)$ be a square matrix and let U_1 be an open set with $\overline{U}_1 \subset U$. We say that p is elliptic on U_1 if there exists an open subset U_2 with $\overline{U}_1 \subset U_2 \subset \overline{U}_2 \subset U$ and if there exists $q \in S^{-d}$ such that $pq-I \in S^{-\infty}$ and $qp-I \in S^{-\infty}$ over U_2. (To say that $r \in S^{-\infty}$ over U_2 simply means the estimates of section 1.2 hold over U_2. Equivalently, we assume $\phi r \in S^{-\infty}$ for every $\phi \in C_0^\infty(U_2)$). This constant technical fuss over domains will be eliminated very shortly when we pass to considering compact manifolds; the role of U_2 is to ensure uniform estimates over U_1.

It is clear that p is elliptic over U_1 if and only if there exists constants C_0 and C_1 such that $p(x,\xi)$ is invertible for $|\xi| \geq C_0$ and

$$|p(x,\xi)^{-1}| \leq C_1(1+|\xi|)^{-d} \qquad \text{for } |\xi| \geq C_0,\ \ x \in U_2.$$

We define $q = \phi(\xi)p^{-1}(x,\xi)$ where $\phi(\xi)$ is a cut-off function identically 0 near $\xi = 0$ and identically 1 near $\xi = \infty$. We used similar cutoff functions in the proof of Lemma 1.2.8. Furthermore, if $p_0 \in S^{d-1}$, then p is elliptic if and only if $p+p_0$ is elliptic; adding lower order terms does not alter the ellipticity. If p is a polynomial and P is a differential operator, then p is elliptic if and only if the leading symbol $\sigma_L(p) = \sum_{|\alpha|=d} p_\alpha(x)\xi^\alpha$ is invertible for $\xi \neq 0$.

There exist elliptic operators of all orders. Let $\phi(x) \in C_0^\infty$ and define the symbol $p(x,\xi) = \phi(x)(1+|\xi|^2)^{d/2}I$, then this is an elliptic symbol of order d whenever $\phi(x) \neq 0$.

Lemma 1.3.1. *Let $P \in \Psi^d(U)$ be elliptic over U_1 then:*
(a) There exists $Q \in \Psi^{-d}(U)$ such that $PQ - I \sim 0$ and $QP - I \sim 0$ over U_1 (i.e., $\phi(PQ-I)$ and $\phi(QP-I)$ are infinitely smoothing for any $\phi \in C_0^\infty(U_2)$).
(b) P is hypo-elliptic over U_1, i.e., if $f \in H_s$ and if Pf is smooth over U_1 then f is smooth over U_1.
(c) There exists a constant C such that $|f|_d \leq C(|f|_0 + |Pf|_0)$ for $f \in C_0^\infty(U_1)$. (Gärding's inequality).

Proof: We will define Q to have symbol $q_0 + q_1 + \cdots$ where $q_j \in S^{-d-j}$. We try to solve the equation

$$\sigma(PQ-I) \sim \sum_{\alpha,j} d_\xi^\alpha p \cdot D_x^\alpha q_j/\alpha! - I \sim 0.$$

When we decompose this sum into elements of S^{-k}, we conclude we must solve

$$\sum_{|\alpha|+j=k} d_\xi^\alpha p \cdot D_x^\alpha q_j/\alpha! = \begin{cases} I & \text{if } k=0 \\ 0 & \text{if } k \neq 0. \end{cases}$$

We define $q_0 = q$ and then solve the equation inductively to define:

$$q_k = -q \cdot \sum_{\substack{|\alpha|+j=k \\ j<k}} d_\xi^\alpha p \cdot D_x^\alpha q_j/\alpha!\,.$$

This defines Q so $\sigma(PQ-I) \sim 0$ over U_2. Similarly we could solve $\sigma(Q_1P-I) \sim 0$ over U_2. We now compute $\sigma(Q-Q_1) = \sigma(Q-Q_1PQ)+\sigma(Q_1PQ-Q_1) = \sigma((I-Q_1P)Q)+\sigma(Q_1(QP-I)) \sim 0$ over U_2 so that in fact Q and Q_1 agree modulo infinitely smoothing operators. This proves (a).

Let $f \in H_s$ with Pf smooth over U_1, and choose $\phi \in C_0(U_1)$. We compute:

$$\phi f = \phi(I-QP)f + \phi QPf.$$

As $\phi(I-QP) \sim 0$, $\phi(I-QP)f$ is smooth. Since Pf is smooth over U_1, ϕQPf is smooth since Q is pseudo-local. Thus ϕf is smooth which proves (b).

Finally, we choose $\phi \in C_0^\infty(U_2)$ to be identically 1 on U_1. Then if $f \in C_0^\infty(U_1)$,

$$|f|_d = |\phi f|_d = |\phi(I-QP)f + \phi QPf|_d \le |\phi(I-QP)f|_d + |\phi QPf|_d.$$

We estimate the first norm by $C|f|_0$ since $\phi(I-QP)$ is an infinitely smoothing operator. We estimate the second norm by $C|Pf|_0$ since ϕQ is a bounded map from L^2 to H^d. This completes the proof.

We note (c) is immediate if $d<0$ since $|_d \le |_0$. If $d>0$, $|f|_0 + |Pf|_0 \le C(|f|_d)$ so this gives a equivalent norm on H^d.

We now consider the effect of changes of coordinates on our class of pseudo-differential operators. Let $h\colon U \to \widetilde{U}$ be a diffeomorphism. We define $h^*\colon C^\infty(\widetilde{U}) \to C^\infty(U)$ by $h^*f(x) = f(h(x))$. If P is a linear operator on $C^\infty(U)$, we define h_*P acting on $C^\infty(\widetilde{U})$ by $(h_*P)f = (h^{-1})^*P(h^*f)$. The fundamental lemma we shall need is the following:

LEMMA 1.3.2. *Let $h\colon U \to \widetilde{U}$ be a diffeomorphism. Then:*
(a) If $P \in \Psi^d(U)$ then $h_*P \in \Psi^d(\widetilde{U})$. Let $p=\sigma(P)$ and define $h(x) = x_1$ and $dh(x)^t\xi_1 = \xi$. Let $p_1(x_1,\xi_1) = p(x,\xi)$ then $\sigma(h_*P) - p_1 \in S^{d-1}$.
(b) Let U_1 be an open subset with $\overline{U}_1 \subset U$. There exists a constant C such that $|h^*f|_d \le C|f|_d$ for all $f \in C_0^\infty(h(U_1))$. In other words, the Sobolev spaces are invariant.

PROOF: The first step is to localize the problem. Let $\{\phi_i\}$ be a partition of unity and let $P_{ij} = \phi_i P \phi_j$ so $P = \sum_{i,j} P_{ij}$. If the support of ϕ_i is disjoint from the support of ϕ_j, then P_{ij} is an infinitely smoothing operator with a smooth kernel $K_{ij}(x,y)$ by Lemma 1.2.6. Therefore h_*P_{ij} is also given by a smooth kernel and is a pseudo-differential operator by Lemma 1.2.4. Consequently, we may restrict attention to pairs (i,j) such that the supports of ϕ_i and ϕ_j intersect. We assume henceforth P is defined by a symbol $p(x,\xi,y)$ where p has arbitrarily small support in (x,y).

We first suppose h is linear to motivate the constructions of the general case. Let $h(x) = hx$ where h is a constant matrix. We equate:

$$hx = x_1, \qquad hy = y_1, \qquad h^t\xi_1 = \xi$$

and define

$$p_1(x_1,\xi_1,y_1) = p(x,\xi,y).$$

(In the above, h^t denotes the matrix transpose of h). If $f \in C_0^\infty(\widetilde{U})$, we compute:

$$\begin{aligned}(h_*P)f(x_1) &= \int e^{i(x-y)\cdot\xi}\, p(x,\xi,y) f(hy)\, dy\, d\xi \\ &= \int e^{ih^{-1}(x_1-y_1)\cdot\xi}\, p(h^{-1}x_1,\xi,h^{-1}y_1) f(y_1) \\ &\qquad\qquad \times |\det(h)|^{-1}\, dy_1\, d\xi.\end{aligned}$$

We now use the identities $h^{-1}(x_1-y_1)\cdot\xi = (x_1-y_1)\cdot\xi_1$ and $|\det(h)|\, d\xi_1 = d\xi$ to write:

$$\begin{aligned}(h_*P)f(x_1) &= \int e^{i(x_1-y_1)\cdot\xi_1}\, p(h^{-1}x_1,h^t\xi_1,h^{-1}y_1) f(y_1)\, dy_1\, d\xi_1 \\ &= \int e^{i(x_1-y_1)\cdot\xi_1}\, p_1(x_1,\xi_1,y_1) f(y_1)\, dy_1\, d\xi_1.\end{aligned}$$

This proves that (h_*P) is a pseudo-differential operator on $\widetilde{U}$. Since we don't need to localize in this case, we compute directly that

$$\sigma(h_*P)(x_1,\xi_1) = p(h^{-1}x_1, h^t\xi_1).$$

We regard (x,ξ) as giving coordinates for T^*M when we expand any covector in the form $\sum \xi_i\, dx^i$. This is exactly the transformation for the cotangent space so we may regard σP as being invariantly defined on $T^*\mathbf{R}^m$.

If h is not linear, the situation is somewhat more complicated. Let

$$\begin{aligned}x - y = h^{-1}(x_1) - h^{-1}(y_1) &= \int_0^1 \frac{d}{dt}\{h^{-1}(tx_1+(1-t)y_1)\}\, dt \\ = \int_0^1 d(h^{-1})(tx_1+(1-t)y_1)&\cdot(x_1-y_1)\, dt = T(x_1,y_1)(x_1-y_1)\end{aligned}$$

where $T(x_1, y_1)$ is a square matrix. If $x_1 = y_1$, then $T(x_1, y_1) = d(h^{-1})$ is invertible since h is a diffeomorphism. We localize using a partition of unity to suppose henceforth the supports are small enough so $T(x_1, y_1)$ is invertible for all points of interest.

We set $\xi_1 = T(x_1, y_1)^t \xi$ and compute:

$$\begin{aligned}(h_* P)(f)(x_1) &= \int e^{i(x-y)\cdot\xi}\, p(x, \xi, y) f(hy)\, dy\, d\xi \\ &= \int e^{iT(x_1,y_1)(x_1-y_1)\cdot\xi}\, p(h^{-1}x_1, \xi, h^{-1}y_1) f(y_1) J\, dy_1\, d\xi \\ &= \int e^{i(x_1-y_1)\cdot\xi_1}\, p_1(x_1, \xi_1, y_1) f(y_1) \\ &\qquad\qquad \times J |\det T(x_1, y_1)|^{-1}\, dy_1\, d\xi_1\end{aligned}$$

where $J = |\det(dh^{-1})| = |\det T(y_1, y_1)|$. By Lemma 1.2.2, this defines a pseudo-differential operator of order d such that $\sigma(h_* P) = p_1$ modulo S^{d-1} which completes the proof of (a). Since $|dh|$ is uniformly bounded on U_1, $|f|_0 \le C|h^* f|_0$ and $|h^* f|_0 \le C|f|_0$. If P is elliptic, then $h_* P$ is elliptic of the same order. For $d > 0$, choose P elliptic of order d and compute:

$$|h^* f|_d \le C(|h^* f|_0 + |P h^* f|_0) \le C(|f|_0 + |(h_* P) f|_0) \le C|f|_d$$

which completes the proof of (b) if $d \ge 0$. The result for $d \le 0$ follows by duality using Lemma 1.1.6(b).

We introduce the spaces S^d/S^{d-1} and define $\sigma_L(P)$ to be the element defined by $\sigma(P)$ in this quotient. Let P and Q be pseudo-differential operators of order d_1 and d_2 Then PQ is a pseudo-differential operator of order $d_1 + d_2$ and $\sigma_L(PQ) = \sigma_L(P)\sigma_L(Q)$ since the remaining terms in the asymptotic series are of lower order. Similarly $\sigma_L(P^*) = \sigma_L(P)^*$. If we define (x, ξ) as coordinates for $T^*(\mathbf{R}^m)$ by representing a cotangent vector at a point x in the form $\sum \xi_i dx^i$, then Lemma 1.3.2 implies $\sigma_L(P)$ is invariantly defined on $T^*(\mathbf{R}^m)$. If $P = \sum_\alpha p_\alpha D_x^\alpha$ is a differential operator there is a natural identification of $\sum_{|\alpha|=d} p_\alpha \xi^\alpha$ with the image of p in S^d/S^{d-1} so this definition of the leading symbol agrees with that given earlier.

We now extend the results of section 1.2 to manifolds. Let M be a smooth compact Riemannian manifold without boundary. Let m be the dimension of M and let dvol or sometimes simply dx denote the Riemanian measure on M. In Chapter 2, we will use the notation $|\,\mathrm{dvol}\,|$ to denote this measure in order to distinguish between measures and m-forms, but we shall not bother with this degree of formalism here. We restrict to scalars first. Let $C^\infty(M)$ be the space of smooth functions on M and let $P\colon C^\infty(M) \to C^\infty(M)$ be a linear operator. We say that P is a pseudo-differential operator of order d and write $P \in \Psi_d(M)$ if for every open chart

U on M and for every $\phi, \psi \in C_0^\infty(U)$, the localized operator $\phi P \psi \in \Psi_d(U)$. We say that P is elliptic if $\phi P \psi$ is elliptic where $\phi\psi(x) \neq 0$. If $Q \in \Psi_d(U)$, we let $P = \phi Q \psi$ for $\phi, \psi \in C_0^\infty(U)$. Lemma 1.3.2 implies P is a pseudo-differential operator on M so there exists operators of all orders on M. We define:

$$\Psi(M) = \bigcup_d \Psi_d(M) \qquad \text{and} \qquad \Psi_{-\infty}(M) = \bigcap_d \Psi_d(M)$$

to be the set of all pseudo-differential operators on M and the set of infinitely smoothing operators on M.

In any coordinate system, we define $\sigma(P)$ to the symbol of the operator $\phi P \phi$ where $\phi = 1$ near the point in question; this is unique modulo $S^{-\infty}$. The leading symbol is invariantly defined on T^*M, but the total symbol changes under the same complicated transformation that the total symbol of a differential operator does under coordinate transformations. Since we shall not need this transformaton law, we omit the statement; it is implicit in the computations performed in Lemma 1.3.2.

We define $L^2(M)$ using the L^2 inner product

$$(f, g) = \int_M f(x)\overline{g}(x)\,dx, \qquad |f|_0^2 = (f, f).$$

We let $L^2(M)$ be the completion of $C^\infty(M)$ in this norm. Let $P\colon C^\infty(M) \to C^\infty(M)$. We let P^* be defined by $(Pf, g) = (f, P^*g)$ if such a P^* exists. Lemmas 1.3.2 and 1.2.3 imply that:

Lemma 1.3.3.
(a) If $P \in \Psi_d(M)$, then $P^* \in \Psi_d(M)$ and $\sigma_L(P^*) = \sigma_L(P)^*$. In any coordinate chart, $\sigma(P^*)$ has a asymptotic expansion given by Lemma 1.2.3(a).
(b) If $P \in \Psi_d(M)$ and $Q \in \Psi_e(M)$, then $PQ \in \Psi_{d+e}(M)$ and $\sigma_L(PQ) = \sigma_L(P)\sigma_L(Q)$. In any coordinate chart $\sigma(PQ)$ has an asymptotic expansion given in Lemma 1.2.3(b).

We use a partition of unity to define the Sobolev spaces $H_s(M)$. Cover M by a finite number of coordinate charts U_i with diffeomorphisms $h_i\colon O_i \to U_i$ where the O_i are open subsets of $\mathbf{R}^m$ with compact closure. If $f \in C_0^\infty(U_i)$, we define:

$$|f|_s^{(i)} = |h_i^* f|_s$$

where we shall use the superscript ${}^{(i)}$ to denote the localized norm. Let $\{\phi_i\}$ be a partition of unity subordinate to this cover and define:

$$|f|_s = \sum |\phi_i f|_s^{(i)}.$$

If $\psi \in C^\infty(M)$, we note that $|\phi_i \psi f|_s^{(i)} \le C|\phi_i f|_s^{(i)}$ since multiplication by ψ defines a ΨDO of order 0. Suppose $\{U_j', O_j', h_j', \phi_j'\}$ is another possible choice to define $|_s'$. We estimate:

$$|\phi_j' f|_s'^{(j)} \le \sum_i |\phi_j' \phi_i f|_s'^{(j)}.$$

Since $\phi_j' \phi_i f \in C_0^\infty(U_i \cap U_j')$, we can use Lemma 1.3.2 (b) to estimate $|_s'^{(j)}$ by $|_s^{(i)}$ so

$$|\phi_j' f|_s'^{(j)} \le \sum_i C|\phi_j' \phi_i f|_s^{(i)} \le C|\phi_i f|_s^{(i)} \le C|f|_s$$

so that $|f|_s' \le C|f|_s$. Similarly $|f|_s \le C|f|_s'$. This shows these two norms are equivalent so $H_s(M)$ is defined independent of the choices made.

We note that:

$$\sum_i \{|\phi_i f|_s^{(i)}\}^2 \le |f|_s^2 \le C \sum_i \{|\phi_i f|_s^{(i)}\}^2$$

so by using this equivalent norm we conclude the $H_s(M)$ are topologically Hilbert spaces. If $\{\psi_i\}$ are given subordinate to the cover U_i with $\psi_i \ge 0$ and $\psi = \sum_i \psi_i > 0$, we let $\phi_i = \psi_i/\psi$ and compute:

$$\begin{aligned} \sum_i |\psi_i f|_s^{(i)} &= \sum_i |\psi \phi_i f|_s^{(i)} \le C \sum_i |\phi_i f|_s^{(i)} = C|f|_s \\ \sum_i |\phi_i f|_s^{(i)} &= \sum_i |\psi^{-1} \psi_i f|_s^{(i)} \le C \sum_i |\psi_i f|_s^{(i)} \end{aligned}$$

to see the norm defined by $\sum_i |\psi_i f|_s^{(i)}$ is equivalent to the norm $|f|_s$ as well.

Lemma 1.1.5 implies that $|f|_t \le |f|_s$ if $t < s$ and that the inclusion of $H_s(M) \to H_t(M)$ is compact. Lemma 1.1.7 implies given $s > t > u$ and $\varepsilon > 0$ we can estimate:

$$|f|_t \le \varepsilon |f|_s + C(\varepsilon)|f|_u.$$

We assume the coordinate charts U_i are chosen so the union $U_i \cup U_j$ is also contained in a larger coordinate chart for all (i, j). We decompose $p \in \Psi_d(M)$ as $P = \sum_{i,j} P_{i,j}$ for $P_{i,j} = \phi_i P \phi_j$. By Lemma 1.2.1 we can estimate:

$$|\phi_i P \phi_j f|_s^{(i)} \le C|\phi_j f|_{s+d}^{(j)}$$

so $|Pf|_s \le C|f|_{s+d}$ and P extends to a continuous map from $H_{s+d}(M) \to H_s(M)$ for all s.

We define:

$$|f|_{\infty,k} = \sum_i |\phi_i f|^{(i)}_{\infty,k}$$

as a measure of the sup norm of the k^{th} derivatives of f. This is independent of particular choices made. Lemma 1.1.4 generalizes to:

$$|f|_{\infty,k} \le C|f|_s \qquad \text{for } s > \frac{m}{2} + k.$$

Thus $H_s(M)$ is a subset of $C^k(M)$ in this situation.

We choose $\psi_i \in C^\infty(U_i)$ with $\sum_i \psi_i^2 = 1$ then:

$$\begin{aligned} |(f,g)| = |\sum_i (\psi_i f, \psi_i g)| &\le \sum_i |(\psi_i f, \psi_i g)| \\ &\le C \sum_i |\psi_i f|^{(i)}_s |\psi_i g|^{(i)}_{-s} \le C|f|_s |g|_{-s}. \end{aligned}$$

Thus the L^2 inner product gives a continuous map $H_s(M) \times H_{-s}(M) \to \mathbf{C}$.

Lemma 1.3.4.
(a) The natural inclusion $H_s \to H_t$ is compact for $s > t$. Furthermore, if $s > t > u$ and if $\varepsilon > 0$, then $|f|_t \le \varepsilon |f|_s + C(\varepsilon)|f|_u$.
(b) If $s > k + \frac{m}{2}$ then $H_s(M)$ is contained in $C^k(M)$ and we can estimate $|f|_{\infty,k} \le C|f|_s$.
(c) If $P \in \Psi_d(M)$ then $P\colon H_{s+d}(M) \to H_s(M)$ is continuous for all s.
(d) The pairing $H_s(M) \times H_{-s}(M) \to \mathbf{C}$ given by the L^2 inner product is a perfect pairing.

Proof: We have proved every assertion except the fact (d) that the pairing is a perfect pairing. We postpone this proof briefly until after we have discussed elliptic ΨDO's.

The sum of two elliptic operators need not be elliptic. However, the sum of two elliptic operators with positive symbols is elliptic. Let P_i have symbol $(1+|\xi_i|^2)^{d/2}$ on U_i and let ϕ_i be a partition of unity. $P = \sum_i \phi_i P_i \phi_i$; this is an elliptic ΨDO of order d for any d, so elliptic operators exist. We let P be an elliptic ΨDO of order d and let ψ_i be identically 1 on the support of ϕ_i. We use these functions to construct $Q \in \Psi_{-d}(M)$ so $PQ - I \in \Psi_{-\infty}(M)$ and $QP - I \in \Psi_{-\infty}(M)$. In each coordinate chart, let $P_i = \psi_i P \psi_i$ then $P_i - P_j \in \Psi_{-\infty}$ on the support of $\phi_i\phi_j$. We construct Q_i as the formal inverse to P_i on the support of ϕ_i, then $Q_i - Q_j \in \Psi_{-\infty}$ on the support of $\phi_i\phi_j$ since the formal inverse is unique. Modulo $\Psi_{-\infty}$ we have $P = \sum_i \phi_i P \sim \sum_i \phi_i P_i$. We define $Q = \sum_j Q_j \phi_j$ and note Q has the desired properties.

It is worth noting we could also construct the formal inverse using a Neumann series. By hypothesis, there exists q so $qp - I \in S^{-1}$ and $pq - I \in$

S^{-1}, where $p = \sigma_L(P)$. We construct Q_1 using a partition of unity so $\sigma_L(Q_1) = q$. We let Q^r and Q^l be defined by the formal series:

$$Q^l = Q_1\Big\{\sum_k (-1)^k (PQ_1 - I)^k\Big\}$$
$$Q^r = \Big\{\sum_k (-1)^k (Q_1 P - I)^k\Big\} Q_1$$

to construct formal left and right inverses so $Q = Q^r = Q^l$ modulo $\Psi_{-\infty}$.

Let P be elliptic of order $d > 0$. We estimate:

$$|f|_d \le |(QP - I)f|_d + |QPf|_d \le C|f|_0 + C|Pf|_0 \le C|f|_d$$

so we could define H_d using the norm $|f|_d = |f|_0 + |Pf|_0$. We specialize to the following case. Let Q be elliptic of order $d/2$ and let $P = Q^*Q + 1$. Then Q^*Q is self-adjoint and non-negative so we can estimate $|f|_0 \le |Pf|_0$ and we can define $|f|_d = |Pf|_0$ in this case. Consequently:

$$|f|_d^2 = (Pf, Pf) = (f, P^*Pf) = (f, g),$$

for $g = P^*Pf$. Since $|P^*Pf|_{-d} \le C|f|_d$, we conclude:

$$|f|_d = (Pf, Pf)/|f|_d \le C(f, g)/|g|_{-d} \le C \sup_h |(f, h)|/|h|_{-d}.$$

Since the pairing of H_d with H_{-d} is continuous, this proves $H_d = H^*_{-d}$. Topologically these are Hilbert spaces so we see dually that $H^*_d = H_{-d}$. This completes the proof of Lemma 1.3.4. We also note that we have proved:

Lemma 1.3.5. *Let $P \in \Psi_d$ be elliptic. Then there exists $Q \in \Psi_{-d}$ so $PQ - I \in \Psi_{-\infty}$ and $QP - I \in \Psi_{-\infty}$. P is hypoelliptic. If $d > 0$, we can define h_d by using the norm $|f|_0 + |Pf|_0$ and define H_{-d} by duality.*

If V is a vector bundle, we cover M by coordinate charts U_i over which V is trivial. We use this cover to define $H_s(V)$ using a partition of unity. We shall always assume V has a given fiber metric so $L^2(V)$ is invariantly defined. $P\colon C^\infty(V) \to C^\infty(W)$ is a ΨDO of order d if $\phi P \psi$ is given by a matrix of d^{th} order ΨDO's for $\phi, \psi \in C_0^\infty(U)$ for any coordinate chart U over which V and W are trivial. Lemmas 1.3.4 and 1.3.5 generalize immediately to this situation.

1.4. Fredholm Operators and the Index of a Fredholm Operator.

Elliptic ΨDO's are invertible modulo $\Psi_{-\infty}$. Lemma 1.3.4 will imply that elliptic ΨDO's are invertible modulo compact operators and that such operators are Fredholm. We briefly review the facts we shall need concerning Fredholm and compact operators.

Let H be a Hilbert space and let $\mathrm{END}(H)$ denote the space of all bounded linear maps $T: H \to H$. There is a natural norm on $\mathrm{END}(H)$ defined by:

$$|T| = \sup_{x \in H} \frac{|Tx|}{|x|}$$

where the sup ranges over $x \neq 0$. $\mathrm{END}(H)$ becomes a Banach space under this norm. The operations of addition, composition, and taking adjoint are continuous. We let $\mathrm{GL}(H)$ be the subset of $\mathrm{END}(H)$ consisting of maps T which are 1-1 and onto. The inverse boundedness theorem shows that if $T \in \mathrm{END}(H)$ is 1-1 and onto, then there exists $\varepsilon > 0$ such that $|Tx| \geq \varepsilon|x|$ so T^{-1} is bounded as well. The Neuman series:

$$(1 - z)^{-1} = \sum_{k=0}^{\infty} z^k$$

converges for $|z| < 1$. If $|I - T| < 1$, we may express $T = I - (I - T)$. If we define

$$S = \sum_{k=0}^{\infty} (I - T)^k$$

then this converges in $\mathrm{END}(H)$ to define an element $S \in \mathrm{END}(H)$ so $ST = TS = I$. Furthermore, this shows $|T|^{-1} \leq (1 - |I - T|)^{-1}$ so $\mathrm{GL}(H)$ contains an open neighborhood of I and the map $T \to T^{-1}$ is continuous there. Using the group operation on $\mathrm{GL}(H)$, we see that $\mathrm{GL}(H)$ is a open subset of $\mathrm{END}(H)$ and is a topological group.

We say that $T \in \mathrm{END}(H)$ is compact if T maps bounded sets to precompact sets—i.e., if $|x_n| \leq C$ is a bounded sequence, then there exists a subsequence x_{n_k} so $Tx_{n_k} \to y$ for some $y \in H$. We let $\mathrm{COM}(H)$ denote the set of all compact maps.

Lemma 1.4.1. *$\mathrm{COM}(H)$ is a closed 2-sided $*$-ideal of* $\mathrm{END}(H)$.

Proof: It is clear the sum of two compact operators is compact. Let $T \in \mathrm{END}(H)$ and let $C \in \mathrm{COM}(H)$. Let $\{x_n\}$ be a bounded sequence in H then $\{Tx_n\}$ is also a bounded sequence. By passing to a subsequence, we may assume $Cx_n \to y$ and $CTx_n \to z$. Since $TCx_n \to Ty$, this implies CT and TC are compact so $\mathrm{COM}(H)$ is a ideal. Next let $C_n \to C$ in $\mathrm{END}(H)$, and let x_n be a bounded sequence in H. Choose a subsequence

x_n^1 so $C_1x_n^1 \to y^1$. We choose a subsequence of the x_n^1 so $C_2x_n^2 \to y^2$. By continuing in this way and then using the diagonal subsequence, we can find a subsequence we denote by x_n^n so $C_k(x_n^n) \to y^k$ for all k. We note $|Cx_n^n - C_kx_n^n| \le |C - C_k|c$. Since $|C - C_k| \to 0$ this shows the sequence Cx_n^n is Cauchy so C is compact and $\mathrm{COM}(H)$ is closed. Finally let $C \in \mathrm{COM}(H)$ and suppose $C^* \notin \mathrm{COM}(H)$. We choose $|x_n| \le 1$ so $|C^*x_n - C^*x_m| \ge \varepsilon > 0$ for all n, m. We let $y_n = C^*x_n$ be a bounded sequence, then $(Cy_n - Cy_m, x_n - x_m) = |C^*x_n - C^*x_m|^2 \ge \varepsilon^2$. Therefore $\varepsilon^2 \le |Cy_n - Cy_m||x_n - x_m| \le 2|Cy_n - Cy_m|$ so Cy_n has no convergent subsequence. This contradicts the assumption $C \in \mathrm{COM}(H)$ and proves $C^* \in \mathrm{COM}(H)$.

We shall assume henceforth that H is a separable infinite dimensional space. Although any two such Hilbert spaces are isomorphic, it is convenient to separate the domain and range. If E and F are Hilbert spaces, we define $\mathrm{HOM}(E,F)$ to be the Banach space of bounded linear maps from E to F with the operator norm. We let $\mathrm{ISO}(E,F)$ be the set of invertible maps in $\mathrm{HOM}(E,F)$ and let $\mathrm{COM}(E,F)$ be the closed subspace of $\mathrm{HOM}(E,F)$ of compact maps. If we choose a fixed isomorphism of E with F, we may identify $\mathrm{HOM}(E,F) = \mathrm{END}(E)$, $\mathrm{ISO}(E,F) = \mathrm{GL}(E)$, and $\mathrm{COM}(E,F) = \mathrm{COM}(E)$. $\mathrm{ISO}(E,F)$ is a open subset of $\mathrm{HOM}(E,F)$ and the operation of taking the inverse is a continuous map from $\mathrm{ISO}(E,F)$ to $\mathrm{ISO}(F,E)$. If $T \in \mathrm{HOM}(E,F)$, we define:

$$\mathrm{N}(T) = \{\, e \in E : T(E) = 0 \,\} \qquad \text{(the null space)}$$
$$\mathrm{R}(T) = \{\, f \in F : f = T(e) \text{ for some } e \in E \,\} \qquad \text{(the range)}.$$

$\mathrm{N}(T)$ is always closed, but $\mathrm{R}(T)$ need not be. If $\perp$ denotes the operation of taking orthogonal complement, then $\mathrm{R}(T)^\perp = \mathrm{N}(T^*)$. We let $\mathrm{FRED}(E,F)$ be the subset of $\mathrm{HOM}(E,F)$ consisting of operators invertible modulo compact operators:

$$\begin{aligned}\mathrm{FRED}(E,F) = \{\, T \in \mathrm{HOM}(E,F) : \exists S_1, S_2 \in \mathrm{HOM}(F,E) \text{ so}\\ S_1T - I \in \mathrm{COM}(E) \text{ and } TS_2 - I \in \mathrm{COM}(F) \,\}.\end{aligned}$$

We note this condition implies $S_1 - S_2 \in \mathrm{COM}(F,E)$ so we can assume $S_1 = S_2$ if we like. The following lemma provides another useful characterization of $\mathrm{FRED}(E,F)$:

Lemma 1.4.2. *The following are equivalent:*

(a) $T \in \mathrm{FRED}(E,F)$;
(b) $T \in \mathrm{END}(E,F)$ *has* $\dim \mathrm{N}(T) < \infty$, $\dim \mathrm{N}(T^*) < \infty$, $\mathrm{R}(T)$ *is closed, and* $\mathrm{R}(T^*)$ *is closed.*

Proof: Let $T \in \mathrm{FRED}(E,F)$ and let $x_n \in \mathrm{N}(T)$ with $|x_n| = 1$. Then $x_n = (I - S_1T)x_n = Cx_n$. Since C is compact, there exists a convegent subsequence. This implies the unit sphere in $\mathrm{N}(T)$ is compact so $\mathrm{N}(T)$ is finite dimensional. Next let $y_n = Tx_n$ and $y_n \to y$. We may assume without loss of generality that $x_n \in \mathrm{N}(T)^\perp$. Suppose there exists a constant C so $|x_n| \le C$. We have $x_n = S_1 y_n + (I - S_1 T)x_n$. Since $S_1 y_n \to S_1 y$ and since $(I - S_1 T)$ is compact, we can find a convergent subsequence so $x_n \to x$ and hence $y = \lim_n y_n = \lim_n Tx_n = Tx$ is in the range of T so $\mathrm{R}(T)$ will be closed. Suppose instead $|x_n| \to \infty$. If $x_n' = x_n/|x_n|$ we have $Tx_n' = y_n/|x_n| \to 0$. We apply the same argument to find a subsequence $x_n' \to x$ with $Tx = 0$, $|x| = 1$, and $x \in \mathrm{N}(T)^\perp$. Since this is impossible, we conclude $\mathrm{R}(T)$ is closed (by passing to a subsequence, one of these two possibilities must hold). Since $T^* S_1^* - I = C_1^*$ and $S_2^* T^* - I = C_2^*$ we conclude $T^* \in \mathrm{FRED}(F, E)$ so $\mathrm{N}(T^*)$ is finite dimensional and $\mathrm{R}(T^*)$ is closed. This proves (a) implies (b).

Conversely, suppose $\mathrm{N}(T)$ and $\mathrm{N}(T^*)$ are finite dimensional and that $\mathrm{R}(T)$ is closed. We decompose:

$$E = \mathrm{N}(T) \oplus \mathrm{N}(T)^\perp \qquad F = \mathrm{N}(T^*) \oplus \mathrm{R}(T)$$

where $T\colon \mathrm{N}(T)^\perp \to \mathrm{R}(T)$ is 1-1 and onto. Consequently, we can find a bounded linear operator S so $ST = I$ on $\mathrm{N}(T)^\perp$ and $TS = I$ on $\mathrm{R}(T)$. We extend S to be zero on $\mathrm{N}(T^*)$ and compute

$$ST - I = \pi_{\mathrm{N}(T)} \qquad \text{and} \qquad TS - I = \pi_{\mathrm{N}(T^*)}$$

where π denotes orthogonal projection on the indicated subspace. Since these two projections have finite dimensional range, they are compact which proves $T \in \mathrm{FRED}(E, F)$.

If $T \in \mathrm{FRED}(E, F)$ we shall say that T is Fredholm. There is a natural law of composition:

LEMMA 1.4.3.
(a) If $T \in \mathrm{FRED}(E, F)$ then $T^* \in \mathrm{FRED}(F, E)$.
(b) If $T_1 \in \mathrm{FRED}(E, F)$ and $T_2 \in \mathrm{FRED}(F, G)$ then $T_2 T_1 \in \mathrm{FRED}(E, G)$.

PROOF: (a) follows from Lemma 1.4.2. If $S_1 T_1 - I \in C(E)$ and $S_2 T_2 - I \in C(F)$ then $S_1 S_2 T_2 T_1 - I = S_1(S_2 T_2 - I)T_1 + (S_1 T_1 - I) \in C(E)$. Similarly $T_2 T_1 S_1 S_2 - I \in C(E)$.

If $T \in \mathrm{FRED}(E, F)$, then we define:

$$\mathrm{index}(T) = \dim \mathrm{N}(T) - \dim \mathrm{N}(T^*).$$

We note that $\mathrm{ISO}(E, F)$ is contained in $\mathrm{FRED}(E, F)$ and that $\mathrm{index}(T) = 0$ if $T \in \mathrm{ISO}(E, F)$.

Lemma 1.4.4.
(a) $\operatorname{index}(T) = -\operatorname{index}(T^*)$.
(b) If $T \in \mathrm{FRED}(E, F)$ *and* $S \in \mathrm{FRED}(F, G)$ *then*

$$\operatorname{index}(ST) = \operatorname{index}(T) + \operatorname{index}(S).$$

(c) $\mathrm{FRED}(E, F)$ *is a open subset of* $\mathrm{HOM}(E, F)$.
(d) $\operatorname{index}\colon \mathrm{FRED}(E, F) \to \mathbf{Z}$ *is continuous and locally constant.*

Proof: (a) is immediate from the definition. We compute

$$\begin{aligned} \mathrm{N}(ST) &= \mathrm{N}(T) \oplus T^{-1}(\mathrm{R}(T) \cap \mathrm{N}(S)) \\ \mathrm{N}(T^*S^*) &= \mathrm{N}(S^*) \oplus (S^*)^{-1}(\mathrm{R}(S^*) \cap \mathrm{N}(T^*)) \\ &= \mathrm{N}(S^*) \oplus (S^*)^{-1}(\mathrm{R}(T)^\perp \cap \mathrm{N}(S)^\perp) \end{aligned}$$

so that

$$\begin{aligned} \operatorname{index}(ST) &= \dim \mathrm{N}(T) + \dim(\mathrm{R}(T) \cap \mathrm{N}(S)) - \dim \mathrm{N}(S^*) \\ &\qquad - \dim(\mathrm{R}(T)^\perp \cap \mathrm{N}(S)^\perp) \\ &= \dim \mathrm{N}(T) + \dim(\mathrm{R}(T) \cap \mathrm{N}(S)) + \dim(\mathrm{R}(T)^\perp \cap \mathrm{N}(S)) \\ &\qquad - \dim \mathrm{N}(S^*) - \dim(\mathrm{R}(T)^\perp \cap \mathrm{N}(S)^\perp) \\ &\qquad\qquad - \dim(\mathrm{R}(T)^\perp \cap \mathrm{N}(S)) \\ &= \dim \mathrm{N}(T) + \dim \mathrm{N}(S) - \dim \mathrm{N}(S^*) - \dim(\mathrm{R}(T)^\perp) \\ &= \dim \mathrm{N}(T) + \dim \mathrm{N}(S) - \dim \mathrm{N}(S^*) - \dim \mathrm{N}(T^*) \\ &= \operatorname{index}(S) + \operatorname{index}(T) \end{aligned}$$

We prove (c) and (d) as follows. Fix $T \in \mathrm{FRED}(E, F)$. We decompose:

$$E = \mathrm{N}(T) \oplus \mathrm{N}(T)^\perp \qquad \text{and} \qquad F = \mathrm{N}(T^*) \oplus \mathrm{R}(T)$$

where $T\colon \mathrm{N}(T)^\perp \to \mathrm{R}(T)$ is 1-1 onto. We let $\pi_1\colon E \to \mathrm{N}(T)$ be orthogonal projection. We define $E_1 = \mathrm{N}(T^*) \oplus E$ and $F_1 = \mathrm{N}(T) \oplus F$ to be Hilbert spaces by requiring the decompositon to be orthogonal. Let $S \in \mathrm{HOM}(E, F)$, we define $S_1 \in \mathrm{HOM}(E_1, F_1)$ by:

$$S_1(f_0 \oplus e) = \pi_1(e) \oplus (f_0 + S_1(e)).$$

It is clear $|S_1 - S_1'| = |S - S'|$ so the map $S \to S_1$ defines a continuous map from $\mathrm{HOM}(E, F) \to \mathrm{HOM}(E_1, F_1)$. Let $i_1\colon E \to E_1$ be the natural inclusion and let $\pi_2\colon F_1 \to F$ be the natural projection. Since $\mathrm{N}(T)$ and $\mathrm{N}(T^*)$ are finite dimensional, these are Fredholm maps. It is immediate from the definition that if $S \in \mathrm{HOM}(E, F)$ then

$$S = \pi_2 S_1 i_1.$$

If we let $T = S$ and decompose $e = e_0 \oplus e_1$ and $f = f_0 \oplus f_1$ for $e_0 \in \mathrm{N}(T)$ and $f_0 \in \mathrm{N}(T^*)$, then:

$$T_1(f_0 \oplus e_0 \oplus e_1) = e_0 \oplus f_0 \oplus Te_1$$

so that $T_1 \in \mathrm{ISO}(E_1, F_1)$. Since $\mathrm{ISO}(E_1, F_1)$ is an open subset, there exists $\varepsilon > 0$ so $|T - S| < \varepsilon$ implies $S_1 \in \mathrm{ISO}(E_1, F_1)$. This implies S_1 is Fredholm so $S = \pi_2 S_1 i_1$ is Fredholm and $\mathrm{FRED}(E, F)$ is open. Furthermore, we can compute $\mathrm{index}(S) = \mathrm{index}(\pi_2) + \mathrm{index}(S_1) + \mathrm{index}(i_1) = \mathrm{index}(\pi_2) + \mathrm{index}(i_1) = \dim \mathrm{N}(T) - \dim \mathrm{N}(T^*) = \mathrm{index}(T)$ which proves index is locally constant and hence continuous. This completes the proof of the lemma.

We present the following example of an operator with index 1. Let ϕ_n be orthonormal basis for L^2 as $n \in \mathbf{Z}$. Define the one sided shift

$$T\phi_n = \begin{cases} \phi_{n-1} & \text{if } n > 0 \\ 0 & \text{if } n = 0 \\ \phi_n & \text{if } n < 0 \end{cases}$$

then T is surjective so $\mathrm{N}(T^*) = \{0\}$. Since $\mathrm{N}(T)$ is one dimensional, $\mathrm{index}(T) = 1$. Therefore $\mathrm{index}(T^n) = n$ and $\mathrm{index}((T^*)^n) = -n$. This proves $\mathrm{index}\colon \mathrm{FRED}(E, F) \to \mathbf{Z}$ is surjective. In the next chapter, we will give several examples of differential operators which have non-zero index.

If we specialize to the case $E = F$ then $\mathrm{COM}(E)$ is a closed two-sided ideal of $\mathrm{END}(E)$ so we can pass to the quotient algebra $\mathrm{END}(E)/\mathrm{COM}(E)$. If $\mathrm{GL}(\mathrm{END}(E)/\mathrm{COM}(E))$ denotes the group of invertible elements and if $\pi\colon \mathrm{END}(E) \to \mathrm{END}(E)/\mathrm{COM}(E)$ is the natural projection, then $\mathrm{FRED}(E)$ is π^{-1} of the invertible elements. If C is compact and T Fredholm, $T + tC$ is Fredholm for any t. This implies $\mathrm{index}(T) = \mathrm{index}(T + tC)$ so we can extend $\mathrm{index}\colon \mathrm{GL}(\mathrm{END}(E)/\mathrm{COM}(E)) \to \mathbf{Z}$ as a surjective group homomorphism.

Let $P\colon C^\infty(V) \to C^\infty(W)$ be a elliptic ΨDO of order d. We construct an elliptic ΨDO S of order $-d$ with $S\colon C^\infty(W) \to C^\infty(V)$ so that $SP - I$ and $PS - I$ are infinitely smoothing operators. Then

$$P\colon H_s(V) \to H_{d-s}(W) \qquad \text{and} \qquad S\colon H_{d-s}(W) \to H_s(V)$$

are continuous. Since $SP - I\colon H_s(V) \to H_t(V)$ is continuous for any t, it is compact. Similarly $PS - I$ is a compact operator so both P and S are Fredholm. If $f \in \mathrm{N}(P)$, then f is smooth by Lemma 1.3.5. Consequently, $\mathrm{N}(P)$ and $\mathrm{N}(P^*)$ are independent of the choice of s and $\mathrm{index}(P)$ is invariantly defined. Furthermore, if P_τ is a smooth 1-parameter family of such operators, then $\mathrm{index}(P_\tau)$ is independent of the parameter τ by Lemma 1.4.4. In particular, $\mathrm{index}(P)$ only depends on the homotopy type of the leading symbol of P. In Chapter 3, we will give a topological formula for $\mathrm{index}(P)$ in terms of characteristic classes.

We summarize our conclusions about $\mathrm{index}(P)$ in the following

LEMMA 1.4.5. *Let* $P: C^\infty(V) \to C^\infty(W)$ *be an elliptic* Ψ*DO of order* d *over a compact manifold without boundary. Then:*
(a) $\mathrm{N}(P)$ *is a finite dimensional subset of* $C^\infty(V)$.
(b) $P: H_s(V) \to H_{s-d}(W)$ *is Fredholm.* P *has closed range.* $\operatorname{index}(P)$ *does not depend on the particular* s *chosen.*
(c) $\operatorname{index}(P)$ *only depends on the homotopy type of the leading symbol of* P *within the class of elliptic* Ψ*DO's of order* d.

1.5. Elliptic Complexes, The Hodge Decomposition Theorem, And Poincaré Duality.

Let V be a graded vector bundle. V is a collection of vector bundles $\{V_j\}_{j\in\mathbf{Z}}$ such that $V_j \neq \{0\}$ for only a finite number of indices j. We let P be a graded ΨDO of order d. P is a collection of d^{th} order pseudo-differential operators $P_j\colon C^\infty(V_j) \to C^\infty(V_{j+1})$. We say that (P, V) is a complex if $P_{j+1}P_j = 0$ and $\sigma_L P_{j+1}\sigma_L P_j = 0$ (the condition on the symbol follows from $P^2 = 0$ for differential operators). We say that (P, V) is elliptic if:

$$\mathrm{N}(\sigma_L P_j)(x,\xi) = \mathrm{R}(\sigma_L P_{j-1})(x,\xi) \qquad \text{for } \xi \neq 0$$

or equivalently if the complex is exact on the symbol level.

We define the cohomology of this complex by:

$$H^j(V,P) = \mathrm{N}(P_j)/\,\mathrm{R}(P_{j-1}).$$

We shall show later in this section that $H^j(V,P)$ is finite dimensional if (P,V) is an elliptic complex. We then define

$$\operatorname{index}(P) = \sum_j (-1)^j \dim H^j(V,P)$$

as the Euler characteristic of this elliptic complex.

Choose a fixed Hermitian inner product on the fibers of V. We use that inner product together with the Riemannian metric on M to define $L^2(V)$. We define adjoints with respect to this structure. If (P,V) is an elliptic complex, we construct the associated self-adjoint Laplacian:

$$\Delta_j = (P^*P)_j = P_j^*P_j + P_{j-1}P_{j-1}^*\colon C^\infty(V_j) \to C^\infty(V_j).$$

If $p_j = \sigma_L(P_j)$, then $\sigma_L(\Delta_j) = p_j^*p_j + p_{j-1}p_{j-1}^*$. We can also express the condition of ellipticity in terms of Δ_j :

Lemma 1.5.1. *Let (P,V) be a d^{th} order partial differential complex. Then (P,V) is elliptic if and only if Δ_j is an elliptic operator of order $2d$ for all j.*

Proof: We suppose that (P,V) is elliptic; we must check $\sigma_L(\Delta_j)$ is non-singular for $\xi \neq 0$. Suppose $(p_j^*p_j + p_{j-1}p_{j-1}^*)(x,\xi)v = 0$. If we dot this equation with v, we see $p_jv\cdot p_jv + p_{j-1}^*v\cdot p_{j-1}^*v = 0$ so that $p_jv = p_{j-1}^*v = 0$. Thus $v \in \mathrm{N}(p_j)$ so $v \in \mathrm{R}(p_{j-1})$ so we can write $v = p_{j-1}w$. Since $p_{j-1}^*p_{j-1}w = 0$, we dot this equation with w to see $p_{j-1}w\cdot p_{j-1}w = 0$ so $v = p_{j-1}w = 0$ which proves Δ_j is elliptic. Conversely, let $\sigma_L(\Delta_j)$ be non-singular for $\xi \neq 0$. Since (P,V) is a complex, $\mathrm{R}(p_{j-1})$ is a subset of $\mathrm{N}(p_j)$.

Conversely, let $v \in \mathrm{N}(p_j)$. Since $\sigma_L(\Delta_j)$ is non-singular, we can express $v = (p_j^* p_j + p_{j-1} p_{j-1}^*)w$. We apply p_j to conclude $p_j p_j^* p_j w = 0$. We dot this equation with $p_j w$ to conclude $p_j p_j^* p_j w \cdot p_j w = p_j^* p_j w \cdot p_j^* p_j w = 0$ so $p_j^* p_j w = 0$. This implies $v = p_{j-1} p_{j-1}^* w \in \mathrm{R}(p_{j-1})$ which completes the proof.

We can now prove the following:

Theorem 1.5.2 (Hodge decomposition theorem). *Let (P, V) be a d^{th} order ΨDO elliptic complex. Then*
(a) We can decompose $L^2(V_j) = \mathrm{N}(\Delta_j) \oplus \mathrm{R}(P_{j-1}) \oplus \mathrm{R}(P_j^*)$ as an orthogonal direct sum.
(b) $\mathrm{N}(\Delta_j)$ is a finite dimensional vector space and there is a natural isomorphism of $H^j(P, V) \simeq \mathrm{N}(\Delta_j)$. The elements of $\mathrm{N}(\Delta_j)$ are smooth sections to V_j.

Proof: We regard $\Delta_j\colon H_{2d}(V_j) \to L^2(V_j)$. Since this is elliptic, it is Fredholm. This proves $\mathrm{N}(\Delta_j)$ is finite dimensional. Since Δ_j is hypoelliptic, $\mathrm{N}(\Delta_j)$ consists of smooth sections to V. Since Δ_j is self adjoint and Fredholm, $\mathrm{R}(\Delta_j)$ is closed so we may decompose $L^2(V_j) = \mathrm{N}(\Delta_j) \oplus \mathrm{R}(\Delta_j)$. It is clear $\mathrm{R}(\Delta_j)$ is contained in the span of $\mathrm{R}(P_{j-1})$ and $\mathrm{R}(P_j^*)$. We compute the L^2 inner product:

$$(P_j^* f, P_{j-1} g) = (f, P_j P_{j-1} g) = 0$$

since $P_j P_{j-1} = 0$. This implies $\mathrm{R}(P_{j-1})$ and $\mathrm{R}(P_j^*)$ are orthogonal. Let $f \in \mathrm{N}(\Delta_j)$, we take the L^2 inner product with f to conclude

$$0 = (\Delta_j f, f) = (P_j f, P_j f) + (P_{j-1}^* f, P_{j-1}^* f)$$

so $\mathrm{N}(\Delta_j) = \mathrm{N}(P_j) \cap \mathrm{N}(P_{j-1}^*)$. This implies $\mathrm{R}(\Delta_j)$ contains the span of $\mathrm{R}(P_{j-1})$ and $\mathrm{R}(P_j^*)$. Since these two subspaces are orthogonal and $\mathrm{R}(\Delta_j)$ is closed, $\mathrm{R}(P_{j-1})$ and $\mathrm{R}(P_j^*)$ are both closed and we have an orthogonal direct sum:

$$L^2(V_j) = \mathrm{N}(\Delta_j) \oplus \mathrm{R}(P_{j-1}) \oplus \mathrm{R}(P_j^*).$$

This proves (a).

The natural inclusion of $\mathrm{N}(\Delta_j)$ into $\mathrm{N}(P_j)$ defines a natural map of $\mathrm{N}(\Delta_j) \to H^j(P, V) = \mathrm{N}(P_j)/\mathrm{R}(P_{j-1})$. Since $\mathrm{R}(P_{j-1})$ is orthogonal to $\mathrm{N}(\Delta_j)$, this map is injective. If $f \in C^\infty(V_j)$ and $P_j f = 0$, we can decompose $f = f_0 \oplus \Delta_j f_1$ for $f_0 \in \mathrm{N}(\Delta_j)$. Since f and f_0 are smooth, $\Delta_j f_1$ is smooth so $f_1 \in C^\infty(V_j)$. $P_j f = P_j f_0 + P_j \Delta_j f_1 = 0$ implies $P_j \Delta_j f_1 = 0$. We dot this equation with $P_j f_1$ to conclude:

$$0 = (P_j f_1, P_j P_j^* P_j f_1 + P_j P_{j-1} P_{j-1}^* f_1) = (P_j^* P_j f_1, P_j^* P_j f_1)$$

so $P_j^* P_j f_1 = 0$ so $\Delta_j f_1 = P_{j-1} P_{j-1}^* f_1 \in \mathrm{R}(P_{j-1})$. This implies f and f_0 represent the same element of $H^j(P, V)$ so the map $\mathrm{N}(\Delta_j) \to H^j(P, V)$ is surjective. This completes the proof.

To illustrate these concepts, we discuss briefly the de Rham complex. Let T^*M be the cotangent space of M. The exterior algebra $\Lambda(T^*M)$ is the universal algebra generated by T^*M subject to the relation $\xi \wedge \xi = 0$ for $\xi \in T^*M$. If $\{e_1, \dots, e_m\}$ is a basis for T^*M and if $I = \{1 \le i_1 < i_2 < \cdots < i_p \le m\}$, we define $e_I = e_{i_1} \wedge \cdots \wedge e_{i_p}$. The $\{e_I\}$ form a basis for $\Lambda(T^*M)$ which has dimension 2^m. If we define $|I| = p$, then $\Lambda^p(T^*M)$ is the span of the $\{e_I\}_{|I|=p}$; this is the bundle of p-forms. A section of $C^\infty(\Lambda^p T^*M)$ is said to be a smooth p-form over M.

Let $x = (x_1, \dots, x_m)$ be local coordinates on M and let $\{dx_1, \dots, dx_m\}$ be the corresponding frame for T^*M. If $f \in C^\infty(M) = C^\infty(\Lambda^0(T^*M))$, we define:

$$df = \sum_k \frac{\partial f}{\partial x_k}\, dx_k.$$

If $y = (y_1, \dots, y_m)$ is another system of local coordinates on M, the identity:

$$dy_j = \sum_k \frac{\partial y_j}{\partial x_k}\, dx_k$$

means that d is well defined and is independent of the coordinate system chosen. More generally, we define $d(f\, dx_I) = df \wedge dx_I$ so that, for example,

$$d\Bigl(\sum_j f_j\, dx^j\Bigr) = \sum_{j<k} \left\{ \frac{\partial f_k}{\partial x_j} - \frac{\partial f_j}{\partial x_k} \right\} dx_j \wedge dx_k.$$

Again this is well defined and independent of the coordinate system. Since mixed partial derivatives commute, $d^2 = 0$ so

$$d\colon C^\infty(\Lambda^p(T^*M)) \to C^\infty(\Lambda^{p+1}(T^*M))$$

forms a complex.

Let $\xi \in T^*M$ and let $\mathrm{ext}(\xi)\colon \Lambda^p(T^*M) \to \Lambda^{p+1}(T^*M)$ be defined by exterior multiplication, i.e., $\mathrm{ext}(\xi)\omega = \xi \wedge \omega$. If we decompose $\xi = \sum \xi_j\, dx_j$ relative to a local coordinate frame, then $df = \sum_j \partial f/\partial x_j\, dx_j$ implies $\sigma(d) = i\ \mathrm{ext}(\xi)$; the symbol of exterior differentiation is exterior multiplication up to a factor of i. Fix $\xi \neq 0$ and choose a basis $\{e_1, \dots, e_m\}$ for T^*M such that $\xi = e_1$. Then

$$\mathrm{ext}(\xi) e_I = \begin{cases} 0 & \text{if } i_1 = 1 \\ e_J \text{ for } J = \{1, i_1, \dots, i_p\} & \text{if } i_1 > 1. \end{cases}$$

From this it is clear that $\mathrm{N}(\mathrm{ext}(\xi)) = \mathrm{R}(\mathrm{ext}(\xi))$ so the de Rham complex is an elliptic complex.

A Riemannian metric on M defines fiber metrics on $\Lambda^p(T^*M)$. If $\{e_i\}$ is an orthonormal local frame for TM we take the dual frame $\{e_i^*\}$ for T^*M. For notational simplicity, we will simply denote this frame again by $\{e_i\}$.

The corresponding $\{e_I\}$ define an orthonormal local frame for $\Lambda(T^*M)$. We define interior multiplication $\operatorname{int}(\xi)\colon \Lambda^p(T^*M) \to \Lambda^{p-1}(T^*M)$ to be the dual of exterior multiplication. Then:

$$\operatorname{int}(e_1)e_I = \begin{cases} e_j \quad \text{for } J = \{i_2, \dots, i_p\} & \text{id } i_1 = 1 \\ 0 & \text{if } i_1 > 1 \end{cases}$$

so

$$\operatorname{int}(e_1)\operatorname{ext}(e_1) + \operatorname{ext}(e_1)\operatorname{int}(e_1) = I.$$

If $|\xi|^2$ denotes the length of the covector ξ, then more generally:

$$(i \operatorname{ext}(\xi) - i \operatorname{int}(\xi))^2 = |\xi|^2 I.$$

If $\delta\colon C^\infty(\Lambda^p(T^*M)) \to C^\infty(\Lambda^{p-1}(T^*M))$ is the adjoint of d, then $\sigma_L\delta = i \operatorname{int}(\xi)$. We let $\delta d + d\delta = (d+\delta)^2 = \Delta$. $\sigma_L\Delta = |\xi|^2$ is elliptic. The de Rham theorem gives a natural isomorphism from the cohomology of M to $H^*(\Lambda, d)$:

$$H^p(M; \mathbf{C}) = \mathrm{N}(d_p)/\,\mathrm{R}(d_{p-1})$$

where we take closed modulo exact forms. The Hodge decomposition theorem implies these groups are naturally isomorphic to the harmonic p-forms $\mathrm{N}(\Delta_p)$ which are finite dimensional. The Euler-Poincaré characteristic $\chi(M)$ is given by:

$$\chi(M) = \sum(-1)^p \dim H^p(M; \mathbf{C}) = \sum(-1)^p \dim \mathrm{N}(\Delta_p) = \operatorname{index}(d)$$

is therefore the index of an elliptic complex.

If M is oriented, we let dvol be the oriented volume element. The Hodge $*$ operator $*\colon \Lambda^p(T^*M) \to \Lambda^{m-p}(T^*M)$ is defined by the identity:

$$\omega \wedge *\omega = (\omega \cdot \omega)\,\mathrm{dvol}$$

where "$\cdot$" denotes the inner product defined by the metric. If $\{e_i\}$ is an oriented local frame for T^*M, then $\mathrm{dvol} = e_1 \wedge \cdots \wedge e_m$ and

$$*(e_1 \wedge \cdots \wedge e_p) = e_{p+1} \wedge \cdots \wedge e_m.$$

The following identities are immediate consequences of Stoke's theorem:

$$** = (-1)^{p(m-p)} \qquad \text{and} \qquad \delta = (-1)^{mp+m+1} * d * .$$

Since $\Delta = (d+\delta)^2 = d\delta+\delta d$ we compute $*\Delta = \Delta*$ so $*\colon \mathrm{N}(\Delta_p) \to \mathrm{N}(\Delta_{m-p})$ is an isomorphism. We may regard $*$ as an isomorphism $*\colon H^p(M;\mathbf{C}) \to H^{m-p}(M;\mathbf{C})$; in this description it is Poincaré duality.

The exterior algebra is *not* very suited to computations owing to the large number of signs which enter in the discussion of $*$. When we discuss the signature and spin complexes in the third chapter, we will introduce Clifford algebras which make the discussion of Poincaré duality much easier.

It is possible to "roll up" the de Rham complex and define:

$$(d+\delta)_{\mathrm{e}}\colon C^\infty(\Lambda^{\mathrm{e}}(T^*M)) \to C^\infty(\Lambda^{\mathrm{o}}(T^*M))$$

where

$$\Lambda^{\mathrm{e}}(T^*M) = \bigoplus_{2k} \Lambda^{2k}(T^*M) \qquad \text{and} \qquad \Lambda^{\mathrm{o}}(T^*M) = \bigoplus_{2k+1} \Lambda^{2k+1}(T^*M)$$

denote the differential forms of even and odd degrees. $(d+\delta)$ is an elliptic operator since $(d+\delta)_{\mathrm{e}}^*(d+\delta)_{\mathrm{e}} = \Delta$ is elliptic since $\dim\Lambda^{\mathrm{e}} = \dim\Lambda^{\mathrm{o}}$. (In this representation $(d+\delta)_{\mathrm{e}}$ is not self-adjoint since the range and domain are distinct, $(d+\delta)_{\mathrm{e}}^* = (d+\delta)_{\mathrm{o}}$). It is clear $\operatorname{index}(d+\delta)_{\mathrm{e}} = \dim \mathrm{N}(\Delta^{\mathrm{e}}) = \dim \mathrm{N}(\Delta^{\mathrm{o}}) = \chi(M)$. We can always "roll up" any elliptic complex to form an elliptic complex of the same index with two terms. Of course, the original elliptic complex does not depend upon the choice of a fiber metric to define adjoints so there is some advantage in working with the full complex occasionally as we shall see later.

We note finally that if m is even, we can always find a manifold with $\chi(M)$ arbitrary so there exist lots of elliptic operators with non-zero index. We shall see that $\operatorname{index}(P) = 0$ if m is odd (and one must consider pseudo-differential operators to get a non-zero index in that case).

We summarize these computations for the de Rham complex in the following:

Lemma 1.5.3. *Let $\Lambda(M) = \Lambda(T^*M)$ be the complete exterior algebra.*
(a) $d\colon C^\infty(\Lambda^p(T^*M)) \to C^\infty(\Lambda^{p+1}(T^*M))$ is an elliptic complex. The symbol is $\sigma_L(d)(x,\xi) = i\ \mathrm{ext}(\xi)$.
(b) If δ is the adjoint, then $\sigma_L(\delta)(x,\xi) = -i\ \mathrm{int}(\xi)$.
(c) If $\Delta_p = (d\delta+\delta d)_p$ is the associated Laplacian, then $\mathrm{N}(\Delta_p)$ is finite dimensional and there are natural identifications:

$$\mathrm{N}(\Delta_p) \simeq \mathrm{N}(d_p)/\,\mathrm{R}(d_{p-1}) \simeq H^p(M;\mathbf{C}).$$

(d) $\operatorname{index}(d) = \chi(M)$ is the Euler-Poincaré characteristic of M.
(e) If M is oriented, we let $*$ be the Hodge operator. Then

$$** = (-1)^{p(m-p)} \qquad \text{and} \qquad \delta = (-1)^{mp+m+1} * d * .$$

Furthermore, $*\colon \mathrm{N}(\Delta_p) \simeq \mathrm{N}(\Delta_{m-p})$ gives Poincaré duality.

In the next chapter, we will discuss the Gauss-Bonnet theorem which gives a formula for $\operatorname{index}(d) = \chi(M)$ in terms of curvature.

1.6. The Heat Equation.

Before proceeding with our discussion of the index of an elliptic operator, we must discuss spectral theory. We restrict ourselves to the context of a compact self-adjoint operator to avoid unnecessary technical details. Let $T \in \mathrm{COM}(H)$ be a self-adjoint compact operator on the Hilbert space H. Let

$$\mathrm{spec}(T) = \{\, \lambda \in \mathbf{C} : (T - \lambda) \notin \mathrm{GL}(H) \,\}.$$

Since $\mathrm{GL}(H)$ is open, $\mathrm{spec}(T)$ is a closed subset of C. If $|\lambda| > |T|$, the series

$$g(\lambda) = \sum_{n=0}^{\infty} T^n/\lambda^{n+1}$$

converges to define an element of $\mathrm{END}(H)$. As $(T-\lambda)g(\lambda) = g(\lambda)(T-\lambda) = -I$, $\lambda \notin \mathrm{spec}(T)$. This shows $\mathrm{spec}(T)$ is bounded.

Since T is self-adjoint, $\mathrm{N}(T - \bar{\lambda}) = \{0\}$ if $\lambda \notin \mathbf{R}$. This implies $\mathrm{R}(T - \lambda)$ is dense in H. Since $Tx \cdot x$ is always real, $|(T - \lambda)x \cdot x| \geq \mathrm{im}(\lambda)|\,|x|^2$ so that $|(T - \lambda)x| \geq \mathrm{im}(\lambda)|\,|x|$. This implies $\mathrm{R}(T - \lambda)$ is closed so $T - \lambda$ is surjective. Since $T - \lambda$ is injective, $\lambda \notin \mathrm{spec}(T)$ if $\lambda \in \mathbf{C} - \mathbf{R}$ so $\mathrm{spec}(T)$ is a subset of $\mathbf{R}$. If $\lambda \in [-|T|, |T|]$, we define $E(\lambda) = \{\, x \in H : Tx = \lambda x \,\}$.

Lemma 1.6.1. *Let $T \in \mathrm{COM}(H)$ be self-adjoint. Then*

$$\dim\{E(-|T|) \oplus E(|T|)\} > 0.$$

Proof: If $|T| = 0$ then $T = 0$ and the result is clear. Otherwise choose $|x_n| = 1$ so that $|Tx_n| \to |T|$. We choose a subsequence so $Tx_n \to y$. Let $\lambda = |T|$, we compute:

$$\begin{aligned} |T^2 x_n - \lambda^2 x_n|^2 &= |T^2 x_n|^2 + |\lambda^2 x_n|^2 - 2\lambda^2 T^2 x_n \cdot x_n \\ &\leq 2\lambda^4 - 2\lambda^2 |Tx_n|^2 \to 0. \end{aligned}$$

Since $Tx_n \to y$, $T^2 x_n \to Ty$. Thus $\lambda^2 x_n \to Ty$. Since $\lambda^2 \neq 0$, this implies $x_n \to x$ for $x = Ty/\lambda^2$. Furthermore $|T^2 x - \lambda^2 x| = 0$. Since $(T^2 - \lambda^2) = (T - \lambda)(T + \lambda)$, either $(T + \lambda)x = 0$ so $x \in E(-\lambda) \neq \{0\}$ or $(T - \lambda)(y_1) = 0$ for $y_1 = (T + \lambda)x \neq 0$ so $E(\lambda) \neq \{0\}$. This completes the proof.

If $\lambda \neq 0$, the equation $Tx = \lambda x$ implies the unit disk in $E(\lambda)$ is compact and hence $E(\lambda)$ is finite dimensional. $E(\lambda)$ is T-invariant. Since T is self-adjoint, the orthogonal complement $E(\lambda)$ is also T-invariant. We take an orthogonal decomposition $H = E(|T|) \oplus E(-|T|) \oplus H_1$. T respects this decomposition; we let T_1 be the restriction of T to H_1. Clearly $|T_1| \leq |T|$. If we have equality, then Lemma 1.6.1 implies there exists $x \neq 0$ in H_1

so $T_1 x = \pm|T|x$, which would be false. Thus $|T_1| < |T|$. We proceed inductively to decompose:

$$H = E(\lambda_1) \oplus E(-\lambda_1) \oplus \cdots \oplus E(\lambda_n) \oplus E(-\lambda_n) \oplus H_n$$

where $\lambda_1 > \lambda_2 > \cdots > \lambda_n$ and where $|T_n| < \lambda_n$. We also assume $E(\lambda_n) \oplus E(-\lambda_n) \neq \{0\}$. We suppose that the λ_n do not converge to zero but converge to ε positive. For each n, we choose x_n so $|x_n| = 1$ and $Tx_n = \pm\lambda_n x_n$. Since T is self-adjoint, this is an orthogonal decomposition; $|x_j - x_k| = \sqrt{2}$. Since T is compact, we can choose a convergent subsequence $\lambda_n x_n \to y$. Since $\lambda_n \to \varepsilon$ positive, this implies $x_n \to x$. This is impossible so therefore the $\lambda_n \to 0$. We define $H_0 = \bigcap_n H_n$ as a closed subset of H. Since $|T| < \lambda_n$ for all n on H_0, $|T| = 0$ so $T = 0$ and $H_0 = E(0)$. This defines a direct sum decomposition of the form:

$$H = \bigoplus_k E(\mu_k) \oplus E(0)$$

where the $\mu_k \in \mathbf{R}$ are the non-zero subspace of the $E(\lambda_n)$ and $E(-\lambda_n)$. We construct a complete orthonormal system $\{\phi_n\}_{n=1}^{\infty}$ for H with either $\phi_n \in E(0)$ or $\phi_n \in E(\mu_k)$ for some k. Then $T\phi_n = \lambda_n \phi_n$ so ϕ_n is an eigenvector of T. This proves the spectral decomposition for self-adjoint compact operators:

Lemma 1.6.2. *Let $T \in \mathrm{COM}(H)$ be self-adjoint. We can find a complete orthonormal system for H consisting of eigenvectors of T.*

We remark that this need not be true if T is self-adjoint but not compact or if T is compact but not self-adjoint.

We can use this lemma to prove the following:

Lemma 1.6.3. *Let $P\colon C^\infty(V) \to C^\infty(V)$ be an elliptic self-adjoint ΨDO of order $d > 0$.*
(a) We can find a complete orthonormal basis $\{\phi_n\}_{n=1}^{\infty}$ for $L^2(V)$ of eigenvectors of P. $P\phi_n = \lambda_n \phi_n$.
(b) The eigenvectors ϕ_n are smooth and $\lim_{n\to\infty} |\lambda_n| = \infty$.
(c) If we order the eigenvalues $|\lambda_1| \le |\lambda_2| \le \cdots$ then there exists a constant $C > 0$ and an exponent $\delta > 0$ such that $|\lambda_n| \ge Cn^\delta$ if $n > n_0$ is large.
Remark: The estimate (c) can be improved to show $|\lambda_n| \sim n^{d/m}$ but the weaker estimate will suffice and is easier to prove.

Proof: $P\colon H_d(V) \to L^2(V)$ is Fredholm. $P\colon \mathrm{N}(P)^\perp \cap H_d(M) \to \mathrm{N}(P)^\perp \cap L^2(V)$ is 1-1 and onto; by Gärding's inequality $P\phi \in L^2$ implies $\phi \in H_d$. We let S denote the inverse of this map and extend S to be zero on the finite dimensional space $\mathrm{N}(P)$. Since the inclusion of $H_d(M)$ into $L^2(V)$ is compact, S is a compact self-adjoint operator. S is often referred to as the Greens operator. We find a complete orthonormal system $\{\phi_n\}$ of eigenvectors of S. If $S\phi_n = 0$ then $P\phi_n = 0$ since $\mathrm{N}(S) = \mathrm{N}(P)$. If $S\phi_n = \mu_n\phi_n \neq 0$ then $P\phi_n = \lambda_n\phi_n$ for $\lambda_n = \mu_n^{-1}$. Since the $\mu_n \to 0$, the $|\lambda_n| \to \infty$. If k is an integer such that $dk > 1$, then $P^k - \lambda_n^k$ is elliptic. Since $(P^k - \lambda_n^k)\phi_n = 0$, this implies $\phi_n \in C^\infty(V)$. This completes the proof of (a) and (b).

By replacing P with P^k we replace λ_n by λ_n^k. Since $kd > \frac{m}{2}$ if k is large, we may assume without loss of generality that $d > \frac{m}{2}$ in the proof of (c). We define:

$$|f|_{\infty,0} = \sup_{x\in M} |f(x)| \qquad \text{for } f \in C^\infty(V).$$

We estimate:

$$|f|_{\infty,0} \leq C|f|_d \leq C(|Pf|_0 + |f|_0).$$

Let $F(a)$ be the space spanned by the ϕ_j where $|\lambda_j| \leq a$ and let $n = \dim F(a)$. We estimate $n = n(a)$ as follows. We have

$$|f|_{\infty,0} \leq C(1+a)|f|_0 \qquad \text{on } F(a).$$

Suppose first $V = M \times \mathbf{C}$ is the trivial line bundle. Let c_j be complex constants, then this estimate shows:

$$\left|\sum_{j=1}^n c_j\phi_j(x)\right| \leq C(1+a)\left\{\sum_{j=1}^n |c_j|^2\right\}^{1/2}.$$

If we take $c_j = \bar{\phi}_j(x)$ then this yields the estimate:

$$\sum_{j=1}^n \phi_j(x)\bar{\phi}_j(x) \leq C(1+a)\left\{\sum_{j=1}^n \phi_j(x)\bar{\phi}_j(x)\right\}^{1/2},$$

i.e.,

$$\sum_{j=1}^n \phi_j(x)\bar{\phi}_j(x) \leq C^2(1+a)^2.$$

We integrate this estimate over M to conclude

$$n \leq C^2(1+a)^2 \operatorname{vol}(M)$$

or equivalently

$$C_1(n - C_2)^{1/2} \leq a = |\lambda_n|$$

from which the desired estimate follows.

If $\dim V = k$, we choose a local orthonormal frame for V to decompose ϕ_j into components ϕ_j^u for $1 \le u \le k$. We estimate:

$$\left|\sum_{j=1}^n c_j^u \phi_j^u(x)\right| \le C(1+a)\left\{\sum_{j=1}^n |c_j^u|^2\right\}^{1/2} \qquad \text{for } u = 1, \dots, k.$$

If we let $c_j^u = \bar\phi_j^u(x)$, then summing over u yields:

$$\left|\sum_{j=1}^n \phi_j \bar\phi_j(x)\right| \le kC(1+a)\left|\sum_{j=1}^n \phi_j \bar\phi_j(x)\right|^{1/2}$$

since each term on the left hand side of the previous inequality can be estimated seperately by the right hand side of this inequality. This means that the constants which arise in the estimate of (c) depend on k, but this causes no difficulty. This completes the proof of (c).

The argument given above for (c) was shown to us by Prof. B. Allard (Duke University) and it is a clever argument to avoid the use of the Schwarz kernel theorem.

Let $P: C^\infty(V) \to C^\infty(V)$ be an elliptic ΨDO of order $d > 0$ which is self-adjoint. We say P has positive definite leading symbol if there exists $p(x,\xi): T^*M \to \mathrm{END}(V)$ such that $p(x,\xi)$ is a positive definite Hermitian matrix for $\xi \ne 0$ and such that $\sigma P - p \in S^{d-1}$ in any coordinate system. The spectrum of such a P is not necessarily non-negative, but it is bounded from below as we shall show. We construct Q_0 with leading symbol $\sqrt{p}$ and let $Q = Q_0^* Q_0$. Then by hypothesis $P - Q \in S^{d-1}$. We compute:

$$\begin{aligned}
(Pf,f) &= ((P-Q)f,f) + (Qf,f)\\
|((P-Q)f,f)| &\le C|f|_{d/2}|(P-Q)f|_{d/2} \le C|f|_{d/2}|f|_{d/2-1}\\
(Qf,f) &= (Q_0f, Q_0f) \qquad \text{and} \qquad |f|_{d/2}^2 < C|f|_0^2 + C|Q_0 f|_0^2.
\end{aligned}$$

We use this to estimate for any $\varepsilon > 0$ that:

$$\begin{aligned}
|((P-Q)f,f)| &\le C|f|_{d/2}(|f|_{d/2-1}) \le \varepsilon|f|_{d/2}^2 + C(\varepsilon)|f|_{d/2}|f|_0\\
&\le 2\varepsilon|f|_{d/2}^2 + C(\varepsilon)|f|_0^2\\
&\le 2C\varepsilon|Q_0f|_0^2 + C(\varepsilon)|f|_0^2.
\end{aligned}$$

Choose ε so $2C\varepsilon \le 1$ and estimate:

$$(Pf,f) \ge (Q_0f, Q_0f) - |((P-Q)f,f)| \ge -C(\varepsilon)|f|_0^2.$$

This implies:

Lemma 1.6.4. *Let $P: C^\infty(V) \to C^\infty(V)$ be an elliptic ΨDO of order $d > 0$ which is self-adjoint with positive definite leading symbol. Then* $\operatorname{spec}(P)$ *is contained in $[-C, \infty)$ for some constant C.*

We fix such a P henceforth. The heat equation is the partial differential equation:

$$\left(\frac{d}{dt} + P\right) f(x,t) = 0 \qquad \text{for } t \geq 0 \text{ with } f(x,0) = f(x).$$

At least formally, this has the solution $f(x,t) = e^{-tP} f(x)$. We decompose $f(x) = \sum c_n \phi_n(x)$ for $c_n = (f, \phi_n)$ in a generalized Fourier series. The solution of the heat equation is given by:

$$f(x,t) = \sum_n e^{-t\lambda_n} c_n \phi_n(x).$$

Proceeding formally, we define:

$$K(t,x,y) = \sum_n e^{-t\lambda_n} \phi_n(x) \otimes \bar{\phi}_n(y): V_y \to V_x$$

so that:

$$\begin{aligned} e^{-tP} f(x) &= \int_M K(t,x,y) f(y) \operatorname{dvol}(y) \\ &= \sum_n e^{-t\lambda_n} \phi_n(x) \int_M f(y) \cdot \phi_n(y) \operatorname{dvol}(y). \end{aligned}$$

We regard $K(t,x,y)$ as an endomorphism from the fiber of V over y to the fiber of V over x.

We justify this purely formal procedure using Lemma 1.3.4. We estimate that:

$$|\phi_n|_{\infty,k} \leq C(|\phi_n|_0 + |P^j \phi|_0) = C(1 + |\lambda_n|^j) \qquad \text{where } jd > k + \tfrac{m}{2}.$$

Only a finite number of eigenvalues of P are negative by Lemma 1.6.4. These will not affect convergence questions, so we may assume $\lambda > 0$. Estimate:

$$e^{-t\lambda} \lambda^j \leq t^{-j} C(j) e^{-t\lambda/2}$$

to compute:

$$|K(t,x,y)|_{\infty,k} \leq t^{-j(k)} C(k) \sum_n e^{-t\lambda_n/2}.$$

Since $|\lambda| \geq C n^\delta$ for $\delta > 0$ and n large, the series can be bounded by

$$\sum_{n>0} e^{-tn^{\delta}/2}$$

which converges. This shows $K(t,x,y)$ is an infinitely smooth function of (t,x,y) for $t>0$ and justifies all the formal procedures involved. We compute

$$\mathrm{Tr}_{L^2}\, e^{-tP} = \sum e^{-t\lambda_n} = \int_M \mathrm{Tr}_{V_x} K(t,x,x)\,\mathrm{dvol}(x).$$

We can use this formula to compute the index of Q:

LEMMA 1.6.5. *Let $Q\colon C^\infty(V)\to C^\infty(W)$ be an elliptic ΨDO of order $d>0$. Then for $t>0$, e^{-tQ^*Q} and e^{-tQQ^*} are in $\Psi_{-\infty}$ with smooth kernel functions and*

$$\mathrm{index}(Q) = \mathrm{Tr}\, e^{-tQ^*Q} - \mathrm{Tr}\, e^{-tQQ^*} \qquad \text{for any } t>0.$$

PROOF: Since e^{-tQ^*Q} and e^{-tQQ^*} have smooth kernel functions, they are in $\Psi_{-\infty}$ so we must only prove the identity on index(Q). We define $E_0(\lambda) = \{\,\phi\in L^2(V) : Q^*Q\phi = \lambda\phi\,\}$ and $E_1(\lambda) = \{\,\phi\in L^2(W) : QQ^*\phi=\lambda\phi\,\}$. These are finite dimensional subspaces of smooth sections to V and W. Because $Q(Q^*Q) = (QQ^*)Q$ and $Q^*(QQ^*) = (Q^*Q)Q^*$, Q and Q^* define maps:

$$Q\colon E_0(\lambda)\to E_1(\lambda) \qquad \text{and} \qquad Q^*\colon E_1(\lambda)\to E_0(\lambda).$$

If $\lambda\neq 0$, $\lambda = Q^*Q\colon E_0(\lambda)\to E_1(\lambda)\to E_0(\lambda)$ is an isomorphism so $\dim E_0(\lambda) = \dim E_1(\lambda)$. We compute:

$$\begin{aligned}\mathrm{Tr}\, e^{-tQ^*Q} - \mathrm{Tr}\, e^{-tQQ^*} &= \sum_\lambda e^{-t\lambda}\{\dim E_0(\lambda) - \dim E_1(\lambda)\}\\ &= e^{-t0}\{\dim E_0(0) - \dim E_1(0)\}\\ &= \mathrm{index}(Q)\end{aligned}$$

which completes the proof.

If (Q,V) is a elliptic complex, we use the same reasoning to conclude $e^{-t\Delta_i}$ is in $\Psi_{-\infty}$ with a smooth kernel function. We define $E_i(\lambda) = \{\,\phi\in L^2(V_i) : \Delta_i\phi = \lambda\phi\,\}$. Then $Q_i\colon E_i(\lambda)\to E_{i+1}(\lambda)$ defines an acyclic complex (i.e., $\mathrm{N}(Q_i) = \mathrm{R}(Q_{i-1})$) if $\lambda\neq 0$ so that $\sum(-1)^i \dim E_i(\lambda) = 0$ for $\lambda\neq 0$. Therefore

$$\mathrm{index}(Q) = \sum_i (-1)^i\, \mathrm{Tr}(e^{-t\Delta_i})$$

and Lemma 1.6.5 generalizes to the case of elliptic complexes which are not just two term complexes.

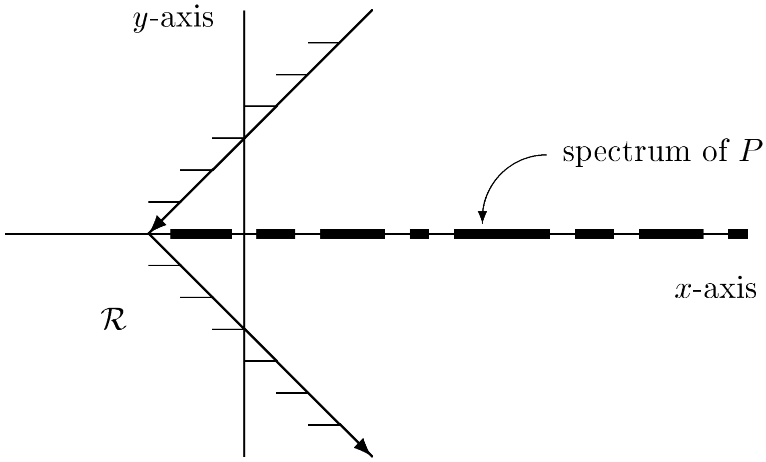
Let P be an elliptic ΨDO of order $d>0$ which is self-adjoint with positive definite leading symbol. Then spec(P) is contained in $[-C,\infty)$ for some constant C. For $\lambda\notin\mathrm{spec}(P)$, $(P-\lambda)^{-1}\in \mathrm{END}(L^2(V))$ satisfies:

$$|(P-\lambda)^{-1}| = \mathrm{dist}(\lambda,\mathrm{spec}(P))^{-1} = \inf_n |\lambda-\lambda_n|^{-1}.$$

The function $|(P-\lambda)^{-1}|$ is a continuous function of $\lambda \in \mathbf{C} - \operatorname{spec}(P)$. We use the Riemann integral with values in the Banach space $\operatorname{END}(L^2(V))$ to define:

$$e^{-tP} = \frac{1}{2\pi i}\int_\gamma e^{-t\lambda}(P-\lambda)^{-1}\,d\lambda$$

where γ is the path about $[-C,\infty)$ given by the union of the appropriate pieces of the straight lines $\operatorname{Re}(\lambda+C+1) = \pm\operatorname{Im}(\lambda)$ pictured below. We let $\mathcal{R}$ be the closed region of $\mathbf{C}$ consisting of γ together with that component of $\mathbf{C}-\gamma$ which does not contain $[-C,\infty)$.



We wish to extend $(P-\lambda)^{-1}$ to H_s. We note that

$$|\lambda-\mu|^{-1} \le C \qquad \text{for } \lambda \in \mathcal{R},\ \mu \in \operatorname{spec}(P)$$

so that $|(P-\lambda)^{-1}f|_0 < C|f|_0$. We use Lemma 1.3.5 to estimate:

$$\begin{aligned}|(P-\lambda)^{-1}f|_{kd} &\le C\{|P^k(P-\lambda)^{-1}f|_0 + |(P-\lambda)^{-1}f|_0\}\\ &\le C\{|P^{k-1}f|_0 + |\lambda P^{k-1}(P-\lambda)^{-1}f|_0 + |f|_0\}\\ &\le C\{|f|_{kd-d} + |\lambda|\,|(P-\lambda)^{-1}f|_{kd-d}\}.\end{aligned}$$

If $k=1$, this implies $|(P-\lambda)^{-1}f|_d \le C(1+|\lambda|)|f|_0$. We now argue by induction to estimate:

$$|(P-\lambda)^{-1}f|_{kd} \le C(1+|\lambda|)^{k-1}|f|_{kd-d}.$$

We now interpolate. If $s>0$, choose k so $kd \ge s > kd-d$ and estimate:

$$\begin{aligned}|(P-\lambda)^{-1}f|_s &\le C|(P-\lambda)^{-1}f|_{kd} \le C(1+|\lambda|)^{k-1}|f|_{kd-d}\\ &\le C(1+|\lambda|)^{k-1}|f|_s.\end{aligned}$$

Similarly, if $s < 0$, we use duality to estimate:

$$|((P-\lambda)^{-1}f,g)| = |(f,(P-\bar{\lambda})^{-1}g)| \le C(1+|\lambda|)^{k-1}|f|_s|g|_{-s}$$

which by Lemma 1.3.4 shows

$$|(P-\lambda)^{-1}f|_s < C(1+|\lambda|)^{k-1}|f|_s$$

in this case as well.

LEMMA 1.6.6. *Let P be a elliptic ΨDO of order $d > 0$ which is self-adjoint. We suppose the leading symbol of P is positive definite. Then:*
(a) Given s there exists $k = k(s)$ and $C = C(s)$ so that

$$|(P-\lambda)^{-1}f|_s \le C(1+|\lambda|)^k|f|_s$$

for all $\lambda \in \mathcal{R}$.
(b) Given j there exists $k = k(j,d)$ so $(P^k-\lambda)^{-1}$ represents a smoothing operator with C^j kernel function which is of trace class for any $\lambda \in \mathcal{R}$.

PROOF: (a) follows from the estimates previously. To prove (b) we let

$$K_k(x,y,\lambda) = \sum_n \frac{1}{\lambda_n^k-\lambda}\phi_n(x)\otimes\phi_n(y)$$

be the kernel of $(P^k-\lambda)^{-1}$. The region $\mathcal{R}$ was chosen so $|\tilde{\lambda}-\lambda| \ge \varepsilon|\tilde{\lambda}|$ for some $\varepsilon > 0$ and $\tilde{\lambda} \in \mathbf{R}^+$. Therefore $|(\lambda_n^k-\lambda)|^{-1} \le \varepsilon|\lambda_n^k|^{-1} \le \varepsilon^{-1}n^{-k\delta}$ by Lemma 1.6.3. The convergence of the sum defining K_k then follows using the same arguments as given in the proof that e^{-tP} is a smoothing operator.

This technical lemma will be used in the next subsection to estimate various error terms which occur in the construction of a parametrix.

1.7. Local Formula for The Index of an Elliptic Operator.

In this subsection, let $P: C^\infty(V) \to C^\infty(V)$ be a self-adjoint elliptic partial differential operator of order $d > 0$. Decompose the symbol $\sigma(P) = p_d + \cdots + p_0$ into homogeneous polymonials p_j of order j in $\xi \in T^*M$. We assume p_d is a positive definite Hermitian matrix for $\xi \neq 0$. Let the curve γ and the region $\mathcal{R}$ be as defined previously.

The operator $(P-\lambda)^{-1}$ for $\lambda \in \mathcal{R}$ is not a pseudo-differential operator. We will approximate $(P-\lambda)^{-1}$ by a pseudo-differential operator $R(\lambda)$ and then use that approximation to obtain properties of $\exp(-tP)$. Let U be an open subset of $\mathbf{R}^m$ with compact closure. Fix $d \in \mathbf{Z}$. We make the following definition to generalize that given in section 1.2:

Definition. *$q(x, \xi, \lambda) \in S^k(\lambda)(U)$ is a symbol of order k depending on the complex parameter $\lambda \in \mathcal{R}$ if*

(a) *$q(x, \xi, \lambda)$ is smooth in $(x, \xi, \lambda) \in \mathbf{R}^m \times \mathbf{R}^m \times \mathcal{R}$, has compact x-support in U and is holomorphic in λ.*

(b) *For all (α, β, γ) there exist constants $C_{\alpha,\beta,\gamma}$ such that:*

$$|D_x^\alpha D_\xi^\beta D_\lambda^\gamma q(x, \xi, \gamma)| \leq C_{\alpha,\beta,\gamma}(1 + |\xi| + |\lambda|^{1/d})^{k-|\beta|-d|\gamma|}.$$

We say that $q(x, \xi, \gamma)$ is homogeneous of order k in (ξ, λ) if

$$q(x, t\xi, t^d\lambda) = t^k q(x, \xi, \gamma) \qquad \text{for } t \geq 1.$$

We think of the parameter λ as being of order d. It is clear if q is homogeneous in (ξ, λ), then it satisfies the decay conditions (b). Since the p_j are polynomials in ξ, they are regular at $\xi = 0$ and define elements of $S^j(\lambda)$. If the p_j were only pseudo-differential, we would have to smooth out the singularity at $\xi = 0$ which would destroy the homogeneity in (ξ, λ) and they would not belong to $S^j(\lambda)$; equivalently, $d_\xi^\alpha p_j$ will not exhibit decay in λ for $|\alpha| > j$ if p_j is not a polynomial. Thus the restriction to differential operators is an essential one, although it is possible to discuss some results in the pseudo-differential case by taking more care with the estimates involved.

We also note that $(p_d - \lambda)^{-1} \in S^{-d}(\lambda)$ and that the spaces $S^*(\lambda)$ form a symbol class closed under differentiation and multiplication. They are suitable generalizations of ordinary pseudo-differential operators discussed earlier.

We let $\Psi_k(\lambda)(U)$ be the set of all operators $Q(\lambda): C_0^\infty(U) \to C_0^\infty(U)$ with symbols $q(x, \xi, \gamma)$ in $S^k(\lambda)$ having x-support in U; we let $q = \sigma Q$. For any fixed λ, $Q(\lambda) \in \Psi_k(U)$ is an ordinary pseudo-differential operator of order k. The new features arise from the dependence on the parameter λ. We extend the definition of $\sim$ given earlier to this wider class by:

$$q \sim \sum_j q_j$$

if for every $k > 0$ there exists $n(k)$ so $n \geq n(k)$ implies $q - \sum_{j \leq n} q_j \in S^{-k}(\lambda)$. Lemmas 1.2.1, 1.2.3(b), and 1.3.2 generalize easily to yield the following Lemma. We omit the proofs in the interests of brevity as they are identical to those previously given with only minor technical modifications.

LEMMA 1.7.1.
(a) Let $Q_i \in \Psi_{k_i}(\lambda)(U)$ with symbol q_i. Then $Q_1Q_2 \in \Psi_{k_1+k_2}(\lambda)(U)$ has symbol q where:

$$q \sim \sum_{\alpha} d_\xi^\alpha q_1 D_x^\alpha q_2/\alpha!\,.$$

(b) Given $n > 0$ there exists $k(n) > 0$ such that $Q \in \Psi_{-k(n)}(\lambda)(U)$ implies $Q(\lambda)$ defines a continuous map from $H_{-n} \to H_n$ and we can estimate the operator norm by:

$$|Q(\lambda)|_{-n,n} \leq C(1 + |\lambda|)^{-n}.$$

(c) If $h\colon U \to \widetilde{U}$ is a diffeomorphism, then $h_*\colon \Psi_k(\lambda)(U) \to \Psi_k(\lambda)(\widetilde{U})$ and

$$\sigma(h_*P) - p(h^{-1}x_1,\,(dh^{-1}(x_1))^t\xi_1,\lambda) \in S^{k-1}(\lambda)(\widetilde{U}).$$

As before, we let $\Psi(\lambda)(U) = \bigcup_n \Psi_n(\lambda)(U)$ be the set of all pseudo-differential operators depending on a complex parameter $\lambda \in \mathcal{R}$ defined over U. This class depends on the order d chosen and also on the region $\mathcal{R}$, but we supress this dependence in the interests of notational clarity. There is no analogue of the completeness of Lemma 1.2.8 for such symbols since we require analyticity in λ. Thus in constructing an approximation to the parametrix, we will always restrict to a finite sum rather than an infinite sum.

Using (c), we extend the class $\Psi(\lambda)$ to compact manifolds using a partition of unity argument. (a) and (b) generalize suitably. We now turn to the question of ellipticity. We wish to solve the equation:

$$\sigma(R(\lambda)(P - \lambda)) - I \sim 0.$$

Inductively we define $R(\lambda)$ with symbol $r_0 + r_1 + \cdots$ where $r_j \in S^{-d-j}(\lambda)$. We define $p'_j(x,\xi,\lambda) = p_j(x,\xi)$ for $j < d$ and $p'_d(x,\xi,\lambda) = p_d(x,\xi) - \lambda$. Then $\sigma(P - \lambda) = \sum_{j=0}^d p'_j$. We note that $p'_j \in S^j(\lambda)$ and that ${p'_d}^{-1} \in S^{-d}(\lambda)$ so that $(P - \lambda)$ is elliptic in a suitable sense. The essential feature of this construction is that the parameter λ is absorbed in the leading symbol and not treated as a lower order perturbation.

The equation $\sigma(R(\lambda)(P - \lambda)) \sim I$ yields:

$$\sum_{\alpha,j,k} d_\xi^\alpha r_j \cdot D_x^\alpha p'_k/\alpha! \sim I.$$

We decompose this series into terms homogeneous of order $-n$ to write:

$$\sum_n \sum_{|\alpha|+j+d-k=n} d_\xi^\alpha r_j \cdot D_x^\alpha p'_k/\alpha! \sim I$$

where $j, k \geq 0$ and $k \leq d$. There are no terms with $n < 0$, i.e., positive order. If we investigate the term with $n = 0$, we arrive at the condition $r_0 p'_d = I$ so $r_0 = (p_d - \lambda)^{-1}$ and inductively:

$$r_n = -r_0 \sum_{\substack{|\alpha|+j+d-k=n \\ j<n}} d_\xi^\alpha r_j D_x^\alpha p'_k/\alpha!\,.$$

If $k = d$ then $|\alpha| > 0$ so $D_x^\alpha p'_k = D_x^\alpha p_k$ in this sum. Therefore we may replace p'_k by p_k and write

$$r_n = -r_0 \sum_{\substack{|\alpha|+j+d-k=n \\ j<n}} d_\xi^\alpha r_j D_x^\alpha p_k/\alpha!\,.$$

In a similar fashion, we can define $\widetilde{R}(\lambda)$ so $\sigma((P-\lambda)\widetilde{R}(\lambda) - I) \sim 0$. This implies $\sigma(R(\lambda) - \widetilde{R}(\lambda)) \sim 0$ so $R(\lambda)$ provides a formal left and right inverse.

Since such an inverse is unique modulo lower order terms, $R(\lambda)$ is well defined and unique modulo lower order terms in any coordinate system. We define $R(\lambda)$ globally on M using a partition of unity argument. To avoid convergence questions, we shall let $R(\lambda)$ have symbol $r_0 + \cdots + r_{n_0}$ where n_0 is chosen to be very large. $R(\lambda)$ is unique modulo $\Psi_{-n_0-d}(\lambda)$. For notational convenience, we supress the dependence of $R(\lambda)$ upon n_0.

$R(\lambda)$ gives a good approximation to $(P-\lambda)^{-1}$ in the following sense:

LEMMA 1.7.2. *Let $k > 0$ be given. We can choose $n_0 = n_0(k)$ so that*

$$|\{(P-\lambda)^{-1} - R(\lambda)\}f|_k \leq C_k(1+|\lambda|)^{-k}|f|_{-k} \qquad \text{for } \lambda \in \mathcal{R},\ f \in C^\infty(V).$$

Thus $(P-\lambda)^{-1}$ is approximated arbitrarily well by the parametrix $R(\lambda)$ in the operator norms as $\lambda \to \infty$.

PROOF: We compute:

$$\begin{aligned} |\{(P-\lambda)^{-1} - R(\lambda)\}f|_k &= |(P-\lambda)^{-1}\{I - (P-\lambda)R(\lambda)\}f|_k \\ &\leq C_k(1+|\lambda|^\nu)|(I - (P-\lambda)R(\lambda))f|_k \end{aligned}$$

by Lemma 1.6.6. Since $I - (P-\lambda)R(\lambda) \in S^{-n_0}$, we use Lemma 1.7.1 to complete the proof.

We define $E(t) = \frac{1}{2\pi i}\int_\gamma e^{-t\lambda}R(\lambda)\,d\lambda$. We will show shortly that this has a smooth kernel $K'(t,x,y)$. Let $K(t,x,y)$ be the smooth kernel of e^{-tP}. We will use Lemma 1.2.9 to estimate the difference between these two kernels. We compute:

$$E(t) - e^{-tP} = \frac{1}{2\pi i}\int_\gamma e^{-t\lambda}(R(\lambda) - (P-\lambda)^{-1})\, d\lambda.$$

We assume $0 < t < 1$ and make a change of variables to replace $t\lambda$ by λ. We use Cauchy's theorem to shift the resulting path $t\gamma$ inside $\mathcal{R}$ back to the original path γ where we have uniform estimates. This expresses:

$$E(t) - e^{-tP} = \frac{1}{2\pi i}\int_\gamma e^{-\lambda}(R(t^{-1}\lambda) - (P - t^{-1}\lambda))t^{-1}\, d\lambda$$

We estimate therefore:

$$\begin{aligned} |E(t) - e^{-tP}|_{-k,k} &\le C_k \int_\gamma |e^{-\lambda}(1 + t^{-1}|\lambda|)^{-k}\, d\lambda| \\ &\le C_k t^k \end{aligned}$$

provided n_0 is large enough. Lemma 1.2.9 implies

Lemma 1.7.3. *Let k be given. If n_0 is large enough, we can estimate:*

$$|K(t,x,y) - K'(t,x,y)|_{\infty,k} \le C_k t^k \qquad \text{for } 0 < t < 1.$$

This implies that K' approximates K to arbitrarily high jets as $t \to 0$.

We now study the operator $E(t)$. We define

$$e_n(t,x,\xi) = \frac{1}{2\pi i}\int_\gamma e^{-t\lambda} r_n(x,\xi,\lambda)\, d\lambda,$$

then $E(t)$ is a ΨDO with symbol $e_0 + \cdots + e_{n_0}$. If we integrate by parts in λ we see that we can express

$$e_n(t,x,\xi) = \frac{1}{2\pi i}\frac{1}{t^k}\int_\gamma e^{-t\lambda}\frac{d^k}{d\lambda^k} r_n(x,\xi,\lambda)\, d\lambda.$$

Since $\dfrac{d^k}{d\lambda^k} r_n$ is homogeneous of degree $-d-n-kd$ in (ξ,λ), we see that $e_n(t,x,\xi) \in S^{-\infty}$ for any $t > 0$. Therefore we can apply Lemma 1.2.5 to conclude $E_n(t)$ which has symbol $e_n(t)$ is represented by a kernel function defined by:

$$K_n(t,x,y) = \int e^{i(x-y)\cdot\xi}\, e_n(t,x,\xi)\, d\xi.$$

We compute:

$$K_n(t,x,x) = \frac{1}{2\pi i}\iint_\gamma e^{-t\lambda} r_n(x,\xi,\lambda)\,d\lambda\,d\xi.$$

We make a change of variables to replace λ by $t^{-1}\lambda$ and ξ by $t^{-1/d}\xi$ to compute:

$$\begin{aligned} K_n(t,x,x) &= t^{-\frac{m}{d}-1}\frac{1}{2\pi i}\iint_\gamma e^{-\lambda} r_n(x,t^{\frac{-1}{d}}\xi,t^{-1}\lambda)\,d\lambda\,d\xi \\ &= t^{-\frac{m}{d}-1+\frac{n+d}{d}}\frac{1}{2\pi i}\iint_\gamma e^{-\lambda} r_n(x,\xi,\lambda)\,d\lambda\,d\xi \\ &= t^{\frac{n-m}{d}} e_n(x) \end{aligned}$$

where we let this integral define $e_n(x)$.

Since $p_d(x,\xi)$ is assumed to be positive definite, d must be even since p_d is a polynomial in ξ. Inductively we express r_n as a sum of terms of the form:

$$r_0^{j_1} q_1 r_0^{j_2} q_2 \dots r_0^{j_k}$$

where the q_k are polynomials in (x,ξ). The sum of the degrees of the q_k is odd if n is odd and therefore $r_n(x,-\xi,\lambda) = -r_n(x,\xi,\lambda)$. If we replace ξ by $-\xi$ in the integral defining $e_n(x)$, we conclude $e_n(x) = 0$ if n is odd.

LEMMA 1.7.4. *Let P be a self-adjoint elliptic partial differential operator of order $d > 0$ such that the leading symbol of P is positive definite for $\xi \neq 0$. Then:*
(a) If we choose a coordinate system for M near a point $x \in M$ and choose a local frame for V, we can define $e_n(x)$ using the complicated combinatorial recipe given above. $e_n(x)$ depends functorially on a finite number of jets of the symbol $p(x,\xi)$.
(b) If $K(t,x,y)$ is the kernel of e^{-tP} then

$$K(t,x,x) \sim \sum_{n=0}^{\infty} t^{\frac{n-m}{d}} e_n(x) \qquad \text{as } t \to 0^+$$

i.e., given any integer k there exists $n(k)$ such that:

$$\left| K(t,x,x) - \sum_{n \le n(k)} t^{\frac{n-m}{d}} e_n(x) \right|_{\infty,k} < C_k t^k \qquad \text{for } 0 < t < 1.$$

(c) $e_n(x) \in \mathrm{END}(V,V)$ is invariantly defined independent of the coordinate system and local frame for V.

(d) $e_n(x) = 0$ if n is odd.

PROOF: (a) is immediate. We computed that

$$K'(t,x,x) = \sum_{n=0}^{n_0} t^{\frac{n-m}{d}} e_n(x)$$

so (b) follows from Lemma 1.7.3. Since $K(t,x,x)$ does not depend on the coordinate system chosen, (c) follows from (b). We computed (d) earlier to complete the proof.

We remark that this asymptotic representation of $K(t,x,x)$ exists for a much wider class of operators P. We refer to the literature for further details. We shall give explicit formulas for e_0, e_2, and e_4 in section 4.8 for certain examples arising in geometry.

The invariants $e_n(x) = e_n(x,P)$ are sections to the bundle of endomorphisms, $\mathrm{END}(V)$. They have a number of functorial properties. We sumarize some of these below.

LEMMA 1.7.5.
(a) Let $P_i\colon C^\infty(V_i) \to C^\infty(V_i)$ be elliptic self-adjoint partial differential operators of order $d > 0$ with positive definite leading symbol. We form $P = P_1 \oplus P_2\colon C^\infty(V_1 \oplus V_2) \to C^\infty(V_1 \oplus V_2)$. Then P is an elliptic self-adjoint partial differential operator of order $d > 0$ with positive definite leading symbol and $e_n(x, P_1 \oplus P_2) = e_n(x,P_1) \oplus e_n(x,P_2)$.
(b) Let $P_i\colon C^\infty(V_i) \to C^\infty(V_i)$ be elliptic self-adjoint partial differential operators of order $d > 0$ with positive definite leading symbol defined over different manifolds M_i. We let

$$P = P_1 \otimes 1 + 1 \otimes P_2\colon C^\infty(V_1 \otimes V_2) \to C^\infty(V_1 \otimes V_2)$$

over $M = M_1 \times M_2$. Then P is an elliptic self-adjoint partial differential operator of order $d > 0$ with positive definite leading symbol over M and

$$e_n(x,P) = \sum_{p+q=n} e_p(x_1,P_1) \otimes e_q(x_2,P_2).$$

(c) Let $P\colon C^\infty(V) \to C^\infty(V)$ be an elliptic self-adjoint partial differential operator of order $d > 0$ with positive definite leading symbol. We decompose the total symbol of P in the form:

$$p(x,\xi) = \sum_{|\alpha| \le d} p_\alpha(x)\xi^\alpha.$$

Fix a local frame for V and a system of local coordinates on M. We let indices a, b index the local frame and let $p_\alpha = p_{ab\alpha}$ give the components

of the matrix p_α. *We introduce formal variables* $p_{ab\alpha/\beta} = D_x^\beta p_{ab\alpha}$ *for the jets of the symbol of* P. *Define* $\operatorname{ord}(p_{ab\alpha/\beta}) = |\beta| + d - |\alpha|$. *Then* $e_n(x,P)$ *can be expressed as a sum of monomials in the* $\{p_{ab\alpha/\beta}\}$ *variables which are homogeneous of order* n *in the jets of the symbol as discussed above and with coefficients which depend smoothly on the leading symbol* $\{p_{ab\alpha}\}_{|\alpha|=d}$.
(d) If the leading symbol of P *is scalar, then the invariance theory can be simplified. We do not need to introduce the components of the symbol explicitly and can compute* $e_n(x,P)$ *as a non-commutative polynomial which is homogeneous of order* n *in the* $\{p_{\alpha/\beta}\}$ *variables with coefficients which depend smoothly upon the leading symbol.*
Remark: The statement of this lemma is somewhat technical. It will be convenient, however, to have these functorial properties precisely stated for later reference. This result suffices to study the index theorem. We shall give one more result which generalizes this one at the end of this section useful in studying the eta invariant. The objects we are studying have a bigrading; one grading comes from counting the number of derivatives in the jets of the symbol, while the other measures the degree of homogeneity in the (ξ,λ) variables.

PROOF: (a) and (b) follow from the identities:

$$\begin{aligned} e^{-t(P_1\oplus P_2)} &= e^{-tP_1} \oplus e^{-tP_2} \\ e^{-t(P_1\otimes 1 + 1\otimes P_2)} &= e^{-tP_1} \otimes e^{-tP_2} \end{aligned}$$

so the kernels satisfy the identities:

$$\begin{aligned} K(t,x,x,P_1\oplus P_2) &= K(t,x,x,P_1)\oplus K(t,x,x,P_2) \\ K(t,x,x,P_1\otimes 1 + 1\otimes P_2) &= K(t,x_1,x_1,P_1)\otimes K(t,x_2,x_2,P_2). \end{aligned}$$

We equate equal powers of t in the asymptotic series:

$$\begin{aligned} &\sum t^{\frac{n-m}{d}} e_n(x,P_1\oplus P_2) \\ &\qquad \sim \sum t^{\frac{n-m}{d}} e_n(x,P_1) \oplus \sum t^{\frac{n-m}{d}} e_n(x,P_2) \\ &\sum t^{\frac{n-m}{d}} e_n(x,P_1\otimes 1 + 1\otimes P_2) \\ &\qquad \sim \left\{\sum t^{\frac{p-m_1}{d}} e_p(x_1,P_1)\right\} \otimes \left\{\sum t^{\frac{q-m_2}{d}} e_q(x_2,P_2)\right\} \end{aligned}$$

to complete the proof of (a) and (b). We note that the multiplicative property (b) is a direct consequence of the identity $e^{-t(a+b)} = e^{-ta}e^{-tb}$; it was for this reason we worked with the heat equation. Had we worked instead with the zeta function to study $\operatorname{Tr}(P^{-s})$ the corresponding multiplicative property would have been much more difficult to derive.

We prove (c) as follows: expand $p(x,\xi)=\sum_j p_j(x,\xi)$ into homogeneous polynomials where $p_j(x,\xi)=\sum_{|\alpha|=j} p_\alpha(x)\xi^\alpha$. Suppose for the moment that $p_d(x,\xi)=p_d(x,\xi)I_V$ is scalar so it commutes with every matrix. The approximate resolvant is given by:

$$\begin{aligned} r_0 &= (p_d(x,\xi)-\lambda)^{-1} \\ r_n &= -r_0 \sum_{\substack{|\alpha|+j+d-k=n \\ j<n}} d_\xi^\alpha r_j D_x^\alpha p_k/\alpha!\,. \end{aligned}$$

Since p_d is scalar, r_0 is scalar so it and all its derivatives commute with any matrix. If we assume inductively r_j is of order j in the jets of the symbol, then $d_\xi^\alpha r_j D_x^\alpha p_k/\alpha!$ is homogeneous of order $j+|\alpha|+d-k=n$ in the jets of the symbol. We observe inductively we can decompose r_n in the form:

$$r_n = \sum_{\substack{n=dj+d-|\alpha| \\ |\alpha|\le n}} r_0^j r_{n,j,\alpha}(x)\xi^\alpha$$

where the $r_{n,j,\alpha}(x)$ are certain non-commutative polynomials in the jets of the total symbol of P which are homogeneous of order n in the sense we have defined.

The next step in the proof of Lemma 1.7.4 was to define:

$$\begin{aligned} e_n(t,x,\xi) &= \frac{1}{2\pi i}\int_\gamma e^{-t\lambda} r_n(x,\xi,\lambda)\,d\lambda \\ &= \frac{1}{2\pi i}\sum_{j,\alpha} r_{n,j,\alpha}(x)\xi^\alpha \int_\gamma e^{-t\lambda} r_0^j\,d\lambda \\ e_n(x) &= \frac{1}{2\pi i}\sum_{j,\alpha} r_{n,j,\alpha}(x)\int\left(\int_\gamma e^{-\lambda} r_0^j\,d\lambda\right)\xi^\alpha\,d\xi. \end{aligned}$$

We note again that $e_n=0$ if n is odd since the resulting function of ξ would be odd. The remaining coefficients of $r_{n,j,\alpha}(x)$ depend smoothly on the leading symbol p_d. This completes the proof of the lemma.

If the leading symbol is not scalar, then the situation is more complicated. Choose a local frame to represent the $p_\alpha=p_{ab\alpha}$ as matrices. Let $h(x,\xi,\lambda)=\det(p_d(x,\xi)-\lambda)^{-1}$. By Cramer's rule, we can express $r_0=r_{0ab}=\{(p_d-\lambda)^{-1}\}_{ab}$ as *polynomials* in the $\{h,\lambda,\xi,p_{ab\alpha}(x)\}$ variables. We noted previously that r_n was a sum of terms of the form $r_0q_0r_0q_1\dots r_0q_kr_0$. The matrix components of such a product can in turn be decomposed as a sum of terms $h^v\tilde q_v$ where $\tilde q_v$ is a polynomial in the $(\lambda,\xi,p_{ab\alpha/\beta})$ variables. The same induction argument which was used in the scalar case shows the $\tilde q_v$ will be homogeneous of order n in the jets of the symbol. The remainder of the argument is the same; performing the $d\lambda\, d\xi$ integral yields a smooth function of the leading symbol as a coefficient of such a term, but this function is not in general rational of course. The scalar case is much simpler as we don't need to introduce the components of the matrices explicitly (although the frame dependence is still there of course) since r_0 can be commuted. This is a technical point, but one often useful in making specific calculations.

We define the scalar invariants

$$a_n(x,P) = \operatorname{Tr} e_n(x,P)$$

where the trace is the fiber trace in V over the point x. These scalar invariants $a_n(x,P)$ inherit suitable functorial properties from the functorial properties of the invariants $e_n(x,P)$. It is immediate that:

$$\begin{aligned}
\operatorname{Tr}_{L^2} e^{-tP} &= \int_M \operatorname{Tr}_{V_x} K(t,x,x)\,\mathrm{dvol}(x) \\
&\sim \sum_{n=0}^{\infty} t^{\frac{m-n}{d}} \int_M a_n(x,P)\,\mathrm{dvol}(x). \\
&\sim \sum_{n=0}^{\infty} t^{\frac{m-n}{d}} a_n(P)
\end{aligned}$$

where $a_n(P) = \int_M a_n(x,P)\,\mathrm{dvol}(x)$ is the integrated invariant. This is a spectral invariant of P which can be computed from local information about the symbol of P.

Let (P,V) be an elliptic complex of differential operators and let Δ_i be the associated Laplacians. We define:

$$a_n(x,P) = \sum_i (-1)^i \operatorname{Tr} e_n(x,\Delta_i)$$

then Lemma 1.6.5 and the remark which follows this lemma imply

$$\operatorname{index}(P) = \sum_i (-1)^i \operatorname{Tr} e^{-t\Delta_i} \sim \sum_{n=0}^{\infty} t^{\frac{n-m}{d}} \int_M a_n(x,P)\,\mathrm{dvol}(x).$$

Since the left hand side does not depend on the parameter t, we conclude:

THEOREM 1.7.6. *Let (P,V) be a elliptic complex of differential operators.*
(a) $a_n(x,P)$ can be computed in any coordinate system and relative to any local frames as a complicated combinatorial expression in the jets of P and of P^* up to some finite order. $a_n = 0$ if n is odd.

(b)

$$\int_M a_n(x,P)\,\mathrm{dvol}(x) = \begin{cases} \mathrm{index}(P) & \text{if } n=m \\ 0 & \text{if } n \neq m\,. \end{cases}$$

We note as an immediate consequence that $\mathrm{index}(P) = 0$ if m is odd. The local nature of the invariants a_n will play a very important role in our discussion of the index theorem. We will develop some of their functorial properties at that point. We give one simple example to illustrate this fact.

Let $\pi\colon M_1 \to M_2$ be a finite covering projection with fiber F. We can choose the metric on M_1 to be the pull-back of a metric on M_2 so that π is an isometry. Let d be the operator of the de Rham complex. $a_n(x,d) = a_n(\pi x, d)$ since a_n is locally defined. This implies:

$$\chi(M_1) = \int_{M_1} a_m(x,d)\,\mathrm{dvol}(x) = |F| \int_{M_2} a_m(x,d)\,\mathrm{dvol}(x) = |F|\chi(M_2)$$

so the Euler-Poincare characteristic is multiplicative under finite coverings. A similar argument shows the signature is multiplicative under orientation preserving coverings and that the arithmetic genus is multiplicative under holomorphic coverings. (We will discuss the signature and arithmetic genus in more detail in Chapter 3).

We conclude this section with a minor generalization of Lemma 1.7.5 which will be useful in discussing the eta invariant in section 1.10:

Lemma 1.7.7. *Let $P\colon C^\infty(V) \to C^\infty(V)$ be an elliptic self-adjoint partial differential operator of order $d > 0$ with positive definite leading symbol and let Q be an auxilary partial differential operator on $C^\infty(V)$ of order $a \geq 0$. Qe^{-tP} is an infinitely smoothing operator with kernel $QK(t,x,y)$. There is an asymptotic expansion on the diagonal:*

$$\{QK(t,x,y)\}|_{x=y} \sim \sum_{n=0}^{\infty} t^{(n-m-a)/d} e_n(x,Q,P).$$

The e_n are smooth local invariants of the jets of the symbols of P and Q and $e_n = 0$ if $n+a$ is odd. If we let $Q = \sum q_\alpha D_x^\alpha$, we define $\mathrm{ord}(q_\alpha) = a - |\alpha|$. If the leading symbol of P is scalar, we can compute $e_n(x,Q,P)$ as a non-commutative polynomial in the variables $\{q_\alpha, p_{\alpha/\beta}\}$ which is homogeneous of order n in the jets with coefficients which depend smoothly on the $\{p_\alpha\}_{|\alpha|=d}$ variables. This expression is <u>linear</u> *in the $\{q_\alpha\}$ variables and does not involve the higher jets of these variables. If the leading symbol of P is not scalar, there is a similar expression for the matrix components of e_n in the matrix components of these variables. $e_n(x,Q,P)$ is additive and multiplicative in the sense of Lemma 1.7.5(a) and (b) with respect to direct sums of operators and tensor products over product manifolds.*

Remark: In fact it is not necessary to assume P is self-adjoint to define e^{-tP} and to define the asymptotic series. It is easy to generalize the techniques we have developed to prove this lemma continues to hold true if we only asume that $\det(p_d(x,\xi)-\lambda)\neq 0$ for $\xi\neq 0$ and $\mathrm{Im}(\lambda)\leq 0$. This implies that the spectrum of P is pure point and contained in a cone about the positive real axis. We omit the details.

PROOF: Qe^{-tP} has smooth kernel $QK(t,x,y)$ where Q acts as a differential operator on the x variables. One representative piece is:

$$\begin{aligned} q_\alpha(x)D_x^\alpha K_n(t,x,y) &= q_\alpha(x)D_x^\alpha \int e^{i(x-y)\cdot\xi}\, e_n(t,x,\xi)\,d\xi \\ &= q_\alpha(x)\sum_{\beta+\gamma=\alpha}\frac{\alpha!}{\beta!\,\gamma!}\int e^{i(x-y)\cdot\xi}\,\xi^\beta D_x^\gamma e_n(t,x,\xi)\,d\xi. \end{aligned}$$

We define $q_{\beta,\gamma}(x)=q_\alpha(x)\dfrac{\alpha!}{\beta!\,\gamma!}$ where $\beta+\gamma=\alpha$, then a representative term of this kernel has the form:

$$q_{\beta,\gamma}(x)\int e^{i(x-y)\cdot\xi}\,\xi^\beta D_x^\gamma e_n(t,x,\xi)\,d\xi$$

where $q_{\beta,\gamma}(x)$ is homogeneous of order $a-|\beta|-|\gamma|$ in the jets of the symbol of Q.

We suppose the leading symbol is scalar to simplify the computations; the general case is handled similarly using Cramer's rule. We express

$$e_n=\frac{1}{2\pi i}\int e^{-t\lambda}r_n(x,\xi,\lambda)\,d\lambda$$

where r_n is a sum of terms of the form $r_{n,j,\alpha}(x)\xi^\alpha r_0^j$. When we differentiate such a term with respect to the x variables, we change the order in the jets of the symbol, but do not change the (ξ,λ) degree of homogeneity. Therefore $QK(t,x,y)$ is a sum of terms of the form:

$$q_{\beta,\gamma}(x)\int\left\{\int e^{-t\lambda}r_0^j(x,\xi,\lambda)\,d\lambda\right\}e^{i(x-y)\cdot\xi}\,\xi^\beta\xi^\delta\tilde r_{n,j,\delta}(x)\,d\xi$$

where:

$$\begin{aligned} &r_{n,j,\delta}\text{ is of order } n+|\gamma| \text{ in the jets of the symbol,}\\ &-d-n=-jd+|\delta|. \end{aligned}$$

We evaluate on the diagonal to set $x=y$. This expression is homogeneous of order $-d-n+|\beta|$ in (ξ,λ), so when we make the appropriate change of variables we calculate this term becomes:

$$t^{(n-m-|\beta|)/d}q_{\beta,\gamma}(x)\tilde r_{n,j,\delta}(x)\cdot\left\{\int\left\{e^{-\lambda}r_0^j(x,\xi,\lambda)\,d\lambda\right\}\xi^\beta\xi^\delta\,d\xi\right\}.$$

This is of order $n+|\gamma|+a-|\beta|-|\gamma| = n+a-|\beta| = \nu$ in the jets of the symbol. The exponent of t is $(n-m-|\beta|)/d = (\nu-a+|\beta|-m-|\beta|)/d = (\nu-a-m)/d$. The asymptotic formula has the proper form if we index by the order in the jets of the symbol ν. It is linear in q and vanishes if $|\beta| + |\delta|$ is odd. As d is even, we compute:

$$\nu + a = n + a - |\beta| + a = jd - d - |\delta| + 2a - |\beta| \equiv |\delta| + |\beta| \mod 2$$

so that this term vanishes if $\nu + a$ is odd. This completes the proof.

1.8. Lefschetz Fixed Point Theorems.

Let $T: M \to M$ be a continuous map. T^* defines a map on the cohomology of M and the Lefschetz number of T is defined by:

$$L(T) = \sum (-1)^p \operatorname{Tr}(T^* \text{ on } H^p(M; \mathbf{C})).$$

This is always an integer. We present the following example to illustrate the concepts involved. Let $M = S^1 \times S^1$ be the two dimensional torus which we realize as $\mathbf{R}^2$ modulo the integer lattice $\mathbf{Z}^2$. We define $T(x, y) = (n_1 x + n_2 y, n_3 x + n_4 y)$ where the n_i are integers. Since this preserves the integer lattice, this defines a map from M to itself. We compute:

$$\begin{aligned}
&T^*(1) = 1, \qquad T^*(dx \wedge dy) = (n_1 n_4 - n_2 n_3) dx \wedge dy \\
&T^*(dx) = n_1\, dx + n_2\, dy, \qquad T^*(dy) = n_3\, dx + n_4\, dy \\
&L(T) = 1 + (n_1 n_4 - n_2 n_3) - (n_1 + n_4).
\end{aligned}$$

Of course, there are many other interesting examples.

We computed $L(T)$ in the above example using the de Rham isomorphism. We let $T^* = \Lambda^p(dT): \Lambda^p(T^*M) \to \Lambda^p(T^*M)$ to be the pull-back operation. It is a map from the fiber over $T(x)$ to the fiber over x. Since $dT^* = T^* d$, T^* induces a map on $H^p(M, \mathbf{C}) = \ker(d_p)/\operatorname{image}(d_{p-1})$. If T is the identity map, then $L(T) = \chi(M)$. The perhaps somewhat surprising fact is that Lemma 1.6.5 can be generalized to compute $L(T)$ in terms of the heat equation.

Let $\Delta_p = (d\delta + \delta d)_p: C^\infty(\Lambda^p T^*M) \to C^\infty(\Lambda^p T^*M)$ be the associated Laplacian. We decompose $L^2(\Lambda^p T^*M) = \bigoplus_\lambda E_p(\lambda)$ into the eigenspaces of Δ_p. We let $\pi(p, \lambda)$ denote orthogonal projection on these subspaces, and we define $T^*(p, \lambda) = \pi(p, \lambda) T^*: E_p(\lambda) \to E_p(\lambda)$. It is immediate that:

$$\operatorname{Tr}(T^* e^{-t\Delta_p}) = \sum_\lambda e^{-t\lambda} \operatorname{Tr}(T^*(p, \lambda)).$$

Since $dT^* = T^* d$ and $d\pi = \pi d$, for $\lambda \neq 0$ we get a chain map between the exact sequences:

$$\begin{array}{ccccccc}
\cdots & E_{p-1}(\lambda) & \xrightarrow{d} & E_p(\lambda) & \xrightarrow{d} & E_{p+1}(\lambda) & \cdots \\
 & \downarrow{\scriptstyle T^*} & & \downarrow{\scriptstyle T^*} & & \downarrow{\scriptstyle T^*} & \\
\cdots & L^2(\Lambda^{p-1}) & \xrightarrow{d} & L^2(\Lambda^p) & \xrightarrow{d} & L^2(\Lambda^{p+1}) & \cdots \\
 & \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} & \\
\cdots & E_{p-1}(\lambda) & \xrightarrow{d} & E_p(\lambda) & \xrightarrow{d} & E_{p+1}(\lambda) & \cdots
\end{array}$$

Since this diagram commutes and since the two rows are long exact sequences of finite dimensional vector spaces, a standard result in homological algebra implies:

$$\sum_p (-1)^p \operatorname{Tr}(T^*(p,\lambda)) = 0 \qquad \text{for } \lambda \neq 0.$$

It is easy to see the corresponding sum for $\lambda = 0$ yields $L(T)$, so Lemma 1.6.5 generalizes to give a heat equation formula for $L(T)$.

This computation was purely formal and did not depend on the fact that we were dealing with the de Rham complex.

LEMMA 1.8.1. *Let (P,V) be an elliptic complex over M and let $T: M \to M$ be smooth. We assume given linear maps $V_i(T): V_i(Tx) \to V_i(x)$ so that $P_i(V_i(T)) = V_i(T)P_i$. Then T induces a map on $H^p(P,V)$. We define*

$$L(T)_P = \sum_p (-1)^p \operatorname{Tr}(T \text{ on } H^p(P,V)).$$

Then we can compute $L(T)_P = \sum_p \operatorname{Tr}(V_i(T)e^{-t\Delta_p})$ where $\Delta_p = (P^*P + PP^*)_p$ is the associated Laplacian.

The $V_i(T)$ are a smooth linear action of T on the bundles V_i. They will be given by the representations involved for the de Rham, signature, spin, and Dolbeault complexes as we shall discuss in Chapter 4. We usually denote $V_i(T)$ by T^* unless it is necessary to specify the action involved.

This Lemma implies Lemma 1.6.5 if we take $T = I$ and $V_i(T) = I$. To generalize Lemma 1.7.4 and thereby get a local formula for $L(T)_P$, we must place some restrictions on the map T. We assume the fixed point set of T consists of the finite disjoint union of smooth submanifolds N_i. Let $\nu(N_i) = T(M)/T(N_i)$ be the normal bundle over the submanifold N_i. Since dT preserves $T(N_i)$, it induces a map dT_ν on the bundle $\nu(N_i)$. We suppose $\det(I - dT_\nu) \neq 0$ as a non-degeneracy condition; there are no normal directions left fixed infinitesimally by T.

If T is an isometry, this condition is automatic. We can construct a non-example by defining $T(z) = z/(z+1): S^2 \to S^2$. The only fixed point is at $z = 0$ and $dT(0) = I$, so this fixed point is degenerate.

If K is the kernel of e^{-tP}, we pull back the kernel to define $T^*(K)(t,x,y) = T^*(x)K(t,Tx,y)$. It is immediate T^*K is the kernel of T^*e^{-tP}.

LEMMA 1.8.2. *Let P be an elliptic partial differential operator of order $d > 0$ which is self-adjoint and which has a positive definite leading symbol for $\xi \neq 0$. Let $T: M \to M$ be a smooth non-degenerate map and let $T^*: V_{T(x)} \to V_x$ be a smooth linear action. If K is the kernel of e^{-tP} then $\operatorname{Tr}(T^*e^{-tP}) = \int_M \operatorname{Tr}(T^*K)(t,x,x)\, \operatorname{dvol}(x)$. Furthermore:*
(a) If T has no fixed points, $|\operatorname{Tr}(T^*e^{-tP})| \leq C_n t^{-n}$ as $t \to 0^+$ for any n.

(b) If the fixed point set of T consists of submanifolds N_i of dimension m_i, we will construct scalar invariants $a_n(x)$ which depend functorially upon a finite number of jets of the symbol and of T. The $a_n(x)$ are defined over N_i and

$$\operatorname{Tr}(T^*e^{-tP}) \sim \sum_i \sum_{n=0}^{\infty} t^{\frac{n-m_i}{d}} \int_{N_i} a_n(x)\ \mathrm{dvol}_i(x).$$

$\mathrm{dvol}_i(x)$ *denotes the Riemannian measure on the submanifold.*
It follows that if T has no fixed points, then $L(T)_P = 0$.

PROOF: Let $\{e_n, r_n, K_n\}$ be as defined in section 1.7. The estimates of Lemma 1.7.3 show that $|T^*K - \sum_{n\le n_0} T^*K_n|_{\infty,k} \le C(k)t^{-k}$ as $t \to 0^+$ for any k if $n_0 = n_0(k)$. We may therefore replace K by K_n in proving (a) and (b). We recall that:

$$e_n(t,x,\xi) = \frac{1}{2\pi i}\int_\gamma e^{-t\lambda} r_n(x,\xi,\lambda)\, d\lambda.$$

We use the homogeneity of r_n to express:

$$\begin{aligned} e_n(t,x,\xi) &= \frac{1}{2\pi i}\int_\gamma e^{-\lambda} r_n(x,\xi,t^{-1}\lambda)t^{-1}\, d\lambda \\ &= t^{\frac{n}{d}}\frac{1}{2\pi i}\int_\gamma e^{-\lambda} r_n(x,t^{\frac{1}{d}}\xi,\lambda)\, d\lambda \\ &= t^{\frac{n}{d}} e_n(x,t^{\frac{1}{d}}\xi) \end{aligned}$$

where we define $e_n(x,\xi) = e_n(1,x,\xi) \in S^{-\infty}$. Then:

$$\begin{aligned} K_n(t,x,y) &= \int e^{i(x-y)\cdot\xi}\, e_n(t,x,\xi)\, d\xi \\ &= t^{\frac{n-m}{d}} \int e^{i(x-y)\cdot t^{-\frac{1}{d}}\xi}\, e_n(x,\xi)\, d\xi. \end{aligned}$$

This shows that:

$$T^*K_n(t,x,x) = t^{\frac{n-m}{d}} \int T^*(x) e^{i(Tx-x)\cdot t^{-1/d}\xi}\, e_n(Tx,\xi)\, d\xi.$$

We must study terms which have the form:

$$\int e^{i(Tx-x)\cdot t^{-1/d}\xi}\ \operatorname{Tr}(T^*(x)e_n(Tx,\xi))\, d\xi\, dx$$

where the integral is over the cotangent space $T^*(M)$. We use the method of stationary phase on this highly oscillatory integral. We first bound

$|Tx - x| \geq \varepsilon > 0$. Using the argument developed in Lemma 1.2.6, we integrate by parts to bound this integral by $C(n,k,\varepsilon)t^k$ as $t \to 0$ for any k. If T has no fixed points, this proves (a). There is a slight amount of notational sloppiness here since we really should introduce partitions of unity and coordinate charts to define $Tx - x$, but we supress these details in the interests of clarity.

We can localize the integral to an arbitrarily small neighborhood of the fixed point set in proving (b). We shall assume for notational simplicity that the fixed point set of T consists of a single submanifold N of dimension m_1. The map dT_ν on the normal bundle has no eigenvalue 1. We identify ν with the span of the generalized eigenvectors of dT on $T(M)|_N$ which correspond to eigenvalues not equal to 1. This gives a direct sum decomposition over N:

$$T(M) = T(N) \oplus \nu \qquad \text{and} \qquad dT = I \oplus dT_\nu.$$

We choose a Riemannian metric for M so this splitting is orthogonal. We emphasize that these are bundles over N and not over the whole manifold M.

We describe the geometry near the fixed manifold N using the normal bundle ν. Let $y = (y_1, \dots, y_{m_1})$ be local coordinates on N and let $\{\vec{s}_1, \dots, \vec{s}_{m-m_1}\}$ be a local orthonormal frame for ν. We use this local orthonormal frame to introduce fiber coordinates $z = (z_1, \dots, z_{m-m_1})$ for ν by decomposing any $\vec{s} \in \nu$ in the form:

$$\vec{s} = \sum_j z_j \vec{s}_j(y).$$

We let $x = (y, z)$ be local coordinates for ν. The geodesic flow identifies a neighborhood of the zero section of the bundle ν with a neighborhood of N in M so we can also regard $x = (y, z)$ as local coordinates on M.

We decompose

$$T(x) = (T_1(x), T_2(x))$$

into tangential and fiber coordinates. Because the Jacobian matrix has the form:

$$dT(y,0) = \begin{pmatrix} I & 0 \\ 0 & dT_\nu \end{pmatrix}$$

we conclude that $T_1(x) - y$ must vanish to second order in z along N.

We integrate $\mathrm{Tr}(T^* K_n)(t, x, x)$ along a small neighborhood of the zero section of ν. We shall integrate along the fibers first to reduce this to an integral along N. We decompose $\xi = (\xi_1, \xi_2)$ corresponding to the decomposition of $x = (y, z)$. Let

$$D(\nu) = \{\, (y, z) : |z| \leq 1 \,\}$$

be the unit disk bundle of the normal bundle. Let $U = T^*(D(\nu))$ be the cotangent bundle of the unit disk bundle of the normal bundle. We assume the metric chosen so the geodesic flow embeds $D(\nu)$ in M. We parametrize U by $\{\,(y, z, \xi_1, \xi_2) : |z| \le 1\,\}$. Let $s = t^{\frac{1}{d}}$. Modulo terms which vanish to infinite order in s, we compute:

$$\begin{aligned}\mathcal{I} &\stackrel{\text{def}}{=} \int_M \operatorname{Tr}((T^* K_n)(t, x, x))\, dx \\ &= s^{n-m} \int_U e^{i(T_1(y,z)-y)\cdot\xi_1 s^{-1}}\ e^{i(T_2(y,z)-z)\cdot\xi_2 s^{-1}} \\ &\qquad\qquad \times \operatorname{Tr}(T^*(x) e_n(Tx, \xi_1, \xi_2))\, d\xi_1\, d\xi_2\, dz\, dy.\end{aligned}$$

The non-degeneracy assumption on T means the phase function $\overline{w} = T_2(y, z) - z$ defines a non-degenerate change of variables if we replace (y, z) by $(y, \overline{w})$. This transforms the integral into the form:

$$\begin{aligned}\mathcal{I} = s^{n-m} &\int_{\overline{U}} e^{i(T_1(y,\overline{w})-y)\cdot\xi_1 s^{-1}}\ e^{i\overline{w}\cdot\xi_2 s^{-1}} \\ &\times |\det(I - dT_2)|^{-1} \operatorname{Tr}(T^*(y, \overline{w}) e_n(T(y, \overline{w}), \xi_1, \xi_2)\, d\xi_1\, d\xi_2\, d\overline{w}\, dy,\end{aligned}$$

where $\overline{U}$ is the image of U under this change. We now make another change of coordinates to let $w = s^{-1}\overline{w}$. As $s \to 0$, $s^{-1}\overline{U}$ will converge to $T^*(v)$. Since $d\overline{w} = s^{m-m_1}\, dw$ this transforms the integral I to the form:

$$\begin{aligned}\mathcal{I} = s^{n-m_1} \int_{s^{-1}\overline{U}} & e^{i(T_1(y,sw)-y)\cdot\xi_1 s^{-1}}\ e^{iw\cdot\xi_2}\, |\det(I - dT_2)^{-1}|(y, sw) \\ &\times \operatorname{Tr}(T^*(y, sw) e_n(T(y, sw), \xi_1, \xi_2))\, d\xi_1\, d\xi_2\, dw\, dy.\end{aligned}$$

Since the phase function $T_1(y, \overline{w}) - y$ vanishes to second order at $\overline{w} = 0$, the function $(T_1(y, sw) - y)s^{-1}$ is regular at $s = 0$. Define:

$$\begin{aligned}e_n'(y, w, \xi_1, \xi_2, s) = e^{i(T_1(y,sw)-y)\cdot\xi_1 s^{-1}}\ &|\det(I - dT_2)(y, sw)| \\ &\times \operatorname{Tr}(T^*(y, sw) e_n(T(y, sw), \xi_1, \xi_2)).\end{aligned}$$

This vanishes to infinite order in ξ_1, ξ_2 at ∞ and is regular at $s = 0$. To complete the proof of Lemma 1.8.2, we must evaluate:

$$s^{(n-m_1)} \int_{s^{-1}\overline{U}} e_n'(y, w, \xi_1, \xi_2, s) e^{iw\cdot\xi_2}\ d\xi_1\, d\xi_2\, dw\, dy.$$

We expand e_n' in a Taylor series in s centered at $s = 0$. If we differentiate e_n' with respect to s a total of k times and evaluate at $s = 0$, then the exponential term disappears and we are left with an expression which is polynomial in w and of degree at worst $2k$. It still vanishes to infinite order in (ξ_1, ξ_2). We decompose:

$$e'_n = \sum_{j \le j_0} s^j \sum_{|\alpha| \le 2j} c_{j,\alpha}(y, \xi_1, \xi_2) w^\alpha + s^{j_0} \varepsilon(y, w, \xi_1, \xi_2, s)$$

where ε is the remainder term.

We first study a term of the form:

$$s^{n+j-m_1} \int_{|w|<s^{-1}} c_{j,\alpha}(y, \xi_1, \xi_2) w^\alpha e^{i\xi_2 \cdot w} \, d\xi_1 \, d\xi_2 \, dw \, dy.$$

Since $c_{j,\alpha}$ vanishes to infinite order in ξ, the $d\xi$ integral creates a function which vanishes to infinite order in w. We can let $s \to 0$ to replace the domain of integration by the normal fiber. The error vanishes to infinite order in s and gives a smooth function of y. The dw integral just yields the inverse Fourier transform with appropriate terms and gives rise to asymptotics of the proper form. The error term in the Taylor series grows at worst polynomially in w and can be bounded similarly. This completes the proof. If there is more than one component of the fixed point set, we sum over the components since each component makes a separate contibution to the asymptotic series.

The case of isolated fixed points is of particular interest. Let $d = 2$ and $\sigma_L(P) = |\xi|^2 I_V$. We compute the first term at a fixed point:

$$r_0(x, \xi, \lambda) = (|\xi|^2 - \lambda)^{-1} \qquad e_0(x, \xi, t) = e^{-t|\xi|^2}$$

$$\int \operatorname{Tr}(T^* K_0)(t, x, x) \operatorname{dvol}(x) = \int \operatorname{Tr}(T^*(x)) e^{i(Tx-x)\cdot\xi} \, e^{-t|\xi|^2} \, d\xi \, dx.$$

We assume $T(0) = 0$ is an isolated non-degenerate fixed point. We let $y = Tx - x$ be a change of variables and compute the first term:

$$\int \operatorname{Tr}(T^*(y)) e^{iy\cdot\xi} \, |\det(I - dT(y))|^{-1} e^{-t|\xi|^2} \, d\xi \, dy.$$

We make a change of variables $\xi \to \xi t^{-1/2}$ and $y \to y t^{1/2}$ to express the first term:

$$\int \operatorname{Tr}(T^*(yt^{1/2})) e^{iy\cdot\xi} \, |\det(I - dT(yt^{1/2})|e^{-|\xi|^2} \, d\xi \, dy.$$

The $d\xi$ integral just gives $e^{-|y|^2}$ so this becomes

$$\int \operatorname{Tr}(T^*(yt^{1/2})) |\det(I - dT(yt^{1/2}))|^{-1} e^{-|y|^2} \, dy.$$

We expand this in a Taylor series at $t = 0$ and evaluate to get

$$\operatorname{Tr} T^*(0) \, |\det(I - dT(O))|^{-1}.$$

This proves:

LEMMA 1.8.3. *Let P be a second order elliptic partial differential operator with leading symbol $|\xi|^2 I$. Let $T\colon M \to M$ be smooth with non-degenerate isolated fixed points. Then:*

$$\operatorname{Tr} T^* e^{-tP} = \sum_i \operatorname{Tr}(T^*)\,|\det(I - dT)|^{-1}(x_i)$$

summed over the fixed point set.

We combine Lemmas 1.8.1 and 1.8.2 to constuct a local formula for $L(T)_P$ to generalize the local formula for index(P) given by Theorem 1.7.6; we will discuss this further in the fourth chapter.

We can use Lemma 1.8.3 to prove the classical Lefschetz fixed point formula for the de Rham complex. Let $T\colon V \to V$ be a linear map, then it is easily computed that:

$$\sum_p (-1)^p \operatorname{Tr} \Lambda^p(T) = \det(I - T).$$

We compute:

$$\begin{aligned} L(T) &= \sum_p (-1)^p \operatorname{Tr}(T^* e^{-t\Delta_p}) \\ &= \sum_{p,i} (-1)^p \operatorname{Tr}(\Lambda^p dT)\,|\det(I - dT)|^{-1}(x_i) \\ &= \sum_i \det(I - dT)\,|\det(I - dT)|^{-1}(x_i) \\ &= \sum_i \operatorname{sign}\det(I - dT)(x_i) \end{aligned}$$

summed over the fixed point set of T. This proves:

THEOREM 1.8.4 (CLASSICAL LEFSCHETZ FIXED POINT FORMULA). *Let $T\colon M \to M$ be smooth with isolated non-degenerate fixed points. Then:*

$$\begin{aligned} L(T) &= \sum_p (-1)^p \operatorname{Tr}(T^* \text{ on } H^p(M; \mathbf{C})) \\ &= \sum_i \operatorname{sign}\det(I - dT)(x_i) \end{aligned}$$

summed over the fixed point set.

Remark: We can generalize Lemma 1.8.2 to study $\operatorname{Tr}(T^* Q e^{-tP})$ where Q is an auxilary differential operator of order a. Just as in lemma 1.7.7 we may obtain an asymptotic series:

$$\operatorname{Tr}(T^* Q e^{-tP}) \sim \sum_i \sum_{n=0}^{\infty} t^{(n - m_i - a)/d}\, a_n(x, Q, P)\, \operatorname{dvol}_i(x).$$

We shall omit the details as the additional terms created by the operator Q are exactly the same as those given in the proof of Lemma 1.7.7. Each term a_n is homogeneous of order n in the jets of the symbols of (Q, P) and of the map T in a suitable sense.

1.9. Elliptic Boundary Value Problems.

In section 1.7 we derived a local formula for the index of an elliptic partial-differential complex using heat equation methods. This formula will lead to a heat equation proof of the Atiyah-Singer index theorem which we shall discuss later. Unfortunately, it is not known at present how to give a heat equation proof of the Atiyah-Bott index theorem for manifolds with boundary in full generality. We must adopt a much stronger notion of ellipticity to deal with the analytic problems involved. This will yield a heat equation proof of the Gauss-Bonnet theorem for manifolds with boundary which we shall discuss in the fourth chapter.

Let M be a smooth compact manifold with smooth boundary dM and let $P\colon C^\infty(V) \to C^\infty(V)$ be a partial differential operator of order $d > 0$. We let $p = \sigma_L(P)$ be the leading symbol of P. We assume henceforth that p is self-adjoint and elliptic—i.e., $\det p(x, \xi) \neq 0$ for $\xi \neq 0$. Let $\mathbf{R}_\pm$ denote the non-zero positive/negative real numbers. It is immediate that

$$\det\{p(x, \xi) - \lambda\} \neq 0 \qquad \text{for } (\xi, \lambda) \neq (0, 0) \in T^*(M) \times \{\mathbf{C} - \mathbf{R}_+ - \mathbf{R}_-\},$$

since p is self-adjoint and elliptic.

We fix a fiber metric on V and a volume element on M to define the global inner product $(\cdot, \cdot)$ on $L^2(V)$. We assume that P is formally self-adjoint:

$$(Pf, g) = (f, Pg)$$

for f and g smooth section with supports disjoint from the boundary dM. We must impose boundary conditions to make P self-adjoint. For example, if $P = -\partial^2/\partial x^2$ on the line segment $[0, A]$, then P is formally self-adjoint and elliptic, but we must impose Neumann or Dirichlet boundary conditions to ensure that P is self-adjoint with discrete spectrum.

Near dM we let $x = (y, r)$ where $y = (y_1, \dots, y_{m-1})$ is a system of local coordinates on dM and where r is the normal distance to the boundary. We assume $dM = \{x : r(x) = 0\}$ and that $\partial/\partial r$ is the inward unit normal. We further normalize the choice of coordinate by requiring the curves $x(r) = (y_0, r)$ for $r \in [0, \delta)$ are unit speed geodesics for any $y_0 \in dM$. The inward geodesic flow identifies a neighborhood of dM in M with the collar $dM \times [0, \delta)$ for some $\delta > 0$. The collaring gives a splitting of $T(M) = T(dM) \oplus T(\mathbf{R})$ and a dual splitting $T^*(M) = T^*(dM) \oplus T^*(\mathbf{R})$. We let $\xi = (\zeta, z)$ for $\zeta \in T^*(dM)$ and $z \in T^*(\mathbf{R})$ reflect this splitting.

It is convenient to discuss boundary conditions in the context of graded vector bundles. A graded bundle U over M is a vector bundle U together with a fixed decomposition

$$U = U_0 \oplus \cdots \oplus U_{d-1}$$

into sub-bundles U_j where $U_j = \{0\}$ is permitted in this decomposition. We let $W = V \otimes 1^d = V \oplus \cdots \oplus V$ restricted to dM be the bundle of

Cauchy data. If $W_j = V|_{dM}$ is the $(j+1)^{\text{st}}$ factor in this decomposition, then this defines a natural grading on W; we identify W_j with the bundle of $j^{\text{ th}}$ normal derivatives. The restriction map:

$$\underline{\gamma}: C^\infty(V) \to C^\infty(W)$$

defined by:

$$\underline{\gamma}(f) = (f_0, \dots, f_{d-1}) \qquad \text{where } f_j = D_r^j f|_{dM} = (-i)^j \left.\frac{\partial^j f}{\partial r^j}\right|_{dM}$$

assigns to any smooth section its Cauchy data.

Let W' be an auxiliary graded vector bundle over dM. We assume that $\dim W = d \cdot \dim V$ is even and that $2 \dim W' = \dim W$. Let $B: C^\infty(W) \to C^\infty(W')$ be a tangential differential operator over dM. Decompose $B = B_{ij}$ for

$$B_{ij}: C^\infty(W_i) \to C^\infty(W'_j)$$

and assume that

$$\operatorname{ord}(B_{ij}) \le j - i.$$

It is natural to regard a section to $C^\infty(W_i)$ as being of order i and to define the graded leading symbol of B by:

$$\sigma_g(B)_{ij}(y,\zeta) = \begin{cases} \sigma_L(B_{ij})(y,\zeta) & \text{if } \operatorname{ord}(B_{ij}) = j - i \\ 0 & \text{if } \operatorname{ord}(B_{ij}) < j - i. \end{cases}$$

We then regard $\sigma_g(B)$ as being of graded homogeneity 0.

We let P_B be the operator P restricted to those $f \in C^\infty(V)$ such that $B\underline{\gamma} f = 0$. For example, let $P = -\partial^2/\partial x^2$ on the interval $[0, A]$. To define Dirichlet boundary conditions at $x = 0$, we would set:

$$W' = \mathbf{C} \oplus 0 \qquad \text{and} \quad B_{0,0} = 1,\ B_{1,1} = B_{0,1} = B_{1,0} = 0$$

while to define Neumann boundary conditions at $x = 0$, we would set:

$$W' = 0 \oplus \mathbf{C} \qquad \text{and} \quad B_{1,1} = 1,\ B_{0,0} = B_{0,1} = B_{1,0} = 0.$$

To define the notion of ellipticity we shall need, we consider the ordinary differential equation:

$$p(y, 0, \zeta, D_r) f(r) = \lambda f(r) \qquad \text{with } \lim_{r \to \infty} f(r) = 0$$

where

$$(\zeta, \lambda) \ne (0,0) \in T^*(dM) \times \mathbf{C} - \mathbf{R}_+.$$

We say that (P, B) is elliptic with respect to $\mathbf{C}-\mathbf{R}_+$ if $\det(p(x,\xi)-\lambda)\neq 0$ on the interior for all $(\zeta,\lambda)\neq(0,0)\in T^*(M)\times\mathbf{C}-\mathbf{R}_+$ and if on the boundary there always exists a unique solution to this ordinary differential equation such that $\sigma_g(B)(y,\zeta)\underline{\gamma}f=f'$ for any prescribed $f'\in W'$. In a similar fashion, we define ellipticity with respect to $\mathbf{C}-\mathbf{R}_+-\mathbf{R}_-$ if these conditions hold for $\lambda\in\mathbf{C}-\mathbf{R}_+-\mathbf{R}_-$.

Again, we illustrate these notions for the operator $P=-\partial^2/\partial x^2$ and a boundary condition at $x=0$. Since $\dim M=1$, there is no dependence on ζ and we must simply study the ordinary differential equation:

$$-f''=\lambda f \qquad \text{with } \lim_{r\to\infty} f(r)=0 \quad (\lambda\neq 0).$$

If $\lambda\in\mathbf{C}-\mathbf{R}_+$, then we can express $\lambda=\mu^2$ for $\operatorname{Im}(\mu)>0$. Solutions to the equation $-f''=\lambda f$ are of the form $f(r)=ae^{i\mu r}+be^{-i\mu r}$. The decay at ∞ implies $b=0$ so $f(r)=ae^{i\mu r}$. Such a function is uniquely determined by either its Dirichlet or Neumann data at $r=0$ and hence P is elliptic with respect to either Neumann or Dirichlet boundary conditions.

We assume henceforth that P_B is self-adjoint and that (P,B) is elliptic with respect to either the cone $\mathbf{C}-\mathbf{R}_+$ or the cone $\mathbf{C}-\mathbf{R}_+-\mathbf{R}_-$. It is beyond the scope of this book to develop the analysis required to discuss elliptic boundary value problems; we shall simply quote the required results and refer to the appropriate papers of Seeley and Greiner for further details. Lemmas 1.6.3 and 1.6.4 generalize to yield:

Lemma 1.9.1. *Let $P\colon C^\infty(V)\to C^\infty(V)$ be an elliptic partial differential operator of order $d>0$. Let B be a boundary condition. We assume (P,B) is self-adjoint and elliptic with respect to $\mathbf{C}-\mathbf{R}_+-\mathbf{R}_-$.*
(a) We can find a complete orthonormal system $\{\phi_n\}_{n=1}^\infty$ for $L^2(V)$ with $P\phi_n=\lambda_n\phi_n$.
(b) $\phi_n\in C^\infty(V)$ and satisfy the boundary condition $B\underline{\gamma}\phi_n=0$.
(c) $\lambda_n\in\mathbf{R}$ and $\lim_{n\to\infty}|\lambda_n|=\infty$. If we order the λ_n so $|\lambda_1|\leq|\lambda_2|\cdots$ then there exists n_0 and $\delta>0$ so that $|\lambda_n|>n^\delta$ for $n>n_0$.
(d) If (P,B) is elliptic with respect to the cone $\mathbf{C}-\mathbf{R}_+$, then the λ_n are bounded from below and $\operatorname{spec}(P_B)$ is contained in $[-C,\infty)$ for some C.

For example, if $P=-\dfrac{\partial^2}{\partial x^2}$ on $[0,\pi]$ with Dirichlet boundary conditions, then the spectral resolution becomes $\left\{\sqrt{\frac{2}{\pi}}\sin(nx)\right\}_{n=1}^\infty$ and the corresponding eigenvalues are n^2.

If (P,B) is elliptic with respect to the cone $\mathbf{C}-\mathbf{R}_+$, then e^{-tP_B} is a smoothing operator with smooth kernel $K(t,x,x')$ defined by:

$$K(t,x,x')=\sum_n e^{-t\lambda_n}\phi_n(x)\otimes\bar{\phi}_n(x').$$

Lemma 1.7.4 generalizes to yield:

LEMMA 1.9.2. *Let (P, B) be elliptic with respect to the cone $\mathbf{C} - \mathbf{R}_+$, and be of order $d > 0$.*
(a) e^{-tP_B} is an infinitely smoothing operator with smooth kernel $K(t, x, x')$.
(b) On the interior we define $a_n(x, P) = \operatorname{Tr} e_n(x, P)$ be as in Lemma 1.7.4. On the boundary we define $a_n(y, P, B)$ using a complicated combinatorial recipe which depends functorially on a finite number of jets of the symbols of P and of B.
(c) As $t \to 0^+$ we have an asymptotic expansion:

$$\begin{aligned}\operatorname{Tr} e^{-tP_B} = \sum_n e^{-t\lambda_n} &= \int_M \operatorname{Tr} K(t, x, x) \operatorname{dvol}(x) \\ &\sim \sum_{n=0}^{\infty} t^{\frac{n-m}{d}} \int_M a_n(x, P) \operatorname{dvol}(x) \\ &\quad + \sum_{n=0}^{\infty} t^{\frac{n-m+1}{d}} \int_{dM} a_n(y, P, B) \operatorname{dvol}(y).\end{aligned}$$

(d) $a_n(x, P)$ and $a_n(y, P, B)$ are invariantly defined scalar valued functions which do not depend on the coordinate system chosen nor on the local frame chosen. $a_n(x, P) = 0$ if n is odd, but $a_n(y, P, B)$ is in general non-zero for all values of n.

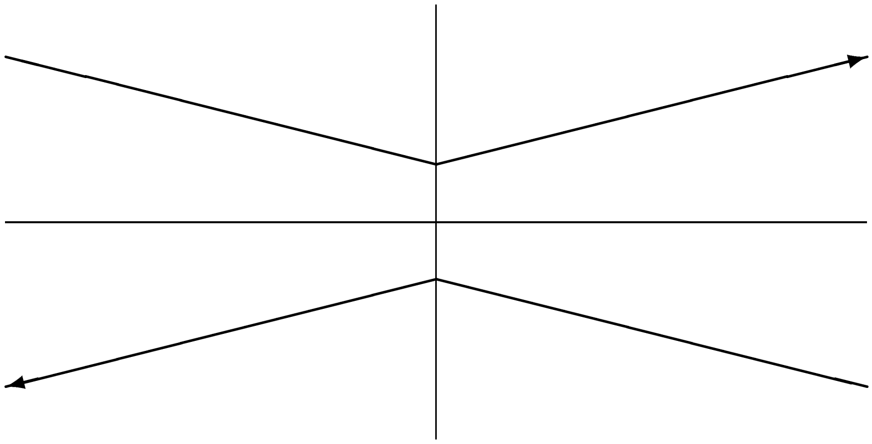
The interior term $a_n(x, P)$ arises from the calculus described previously. We first construct a parametrix on the interior. Since this parametrix will not have the proper boundary values, it is necessary to add a boundary correction term which gives rise to the additional boundary integrands $a_n(y, P, B)$.

To illustrate this asymptotic series, we let $P = -\dfrac{\partial^2}{\partial x^2} - e(x)$ on the interval $[0, A]$ where $e(x)$ is a real potential. Let B be the modified Neumann boundary conditions: $f'(0) + s(0)f(0) = f'(A) + s(A)f(A) = 0$. This is elliptic and self-adjoint with respect to the cone $\mathbf{C} - \mathbf{R}_+$. It can be computed that:

$$\begin{aligned}&\operatorname{Tr} e^{-tP_B} \\ &\quad\sim (4\pi t)^{-1/2} \int \{1 + t \cdot e(x) + t^2 \cdot (e''(x) + 3e(x)^2)/6 + \cdots\}\, dx \\ &\qquad + \frac{1}{2} + \left(\frac{t}{\pi}\right)^{1/2} \big(s(0) - s(A)\big) \\ &\qquad + \frac{t}{4} \cdot \big(e(0) + e(A) + 2s^2(0) + 2s^2(A)\big) + \cdots\end{aligned}$$

The terms arising from the interior increase by integer powers of t while the terms from the boundary increase by powers of $t^{1/2}$. In this integral, dx is ordinary unnormalized Lebesgue measure.

In practice, we shall be interested in first order operators which are elliptic with respect to the cone $\mathbf{C}-\mathbf{R}_+-\mathbf{R}_-$ and which have both positive and negative spectrum. An interesting measure of the spectral asymmetry of such an operator is obtained by studying $\mathrm{Tr}(P_B e^{-tP_B^2})$ which will be discussed in more detail in the next section. To study the index of an elliptic operator, however, it suffices to study $e^{-tP_B^2}$; it is necessary to work with P_B^2 of course to ensure that the spectrum is positive so this converges to define a smoothing operator. There are two approaches which are available. The first is to work with the function $e^{-t\lambda^2}$ and integrate over a path of the form:



to define $e^{-tP_B^2}$ directly using the functional calculus. The second approach is to define $e^{-tP_B^2}$ using the operator P^2 with boundary conditions $B\gamma f = B\gamma P f = 0$. These two approaches both yield the same operator and give an appropriate asymptotic expansion which generalizes Lemma 1.9.2.

We use the heat equation to construct a local formula for the index of certain elliptic complexes. Let $Q\colon C^\infty(V_1) \to C^\infty(V_2)$ be an elliptic differential operator of order $d > 0$. Let $Q^*\colon C^\infty(V_2) \to C^\infty(V_1)$ be the formal adjoint. Let $B_1\colon C^\infty(W_1) \to C^\infty(W_1')$ be a boundary condition for the operator Q. We assume there exists a boundary condition B_2 for Q^* of the same form so that

$$(Q_{B_1})^* = (Q^*)_{B_2}.$$

We form:

$$P = Q \oplus Q^* \qquad \text{and} \qquad B = B_1 \oplus B_2$$

so $P\colon C^\infty(V_1 \oplus V_2) \to C^\infty(V_1 \oplus V_2)$. Then P_B will be self-adjoint. We assume that (P, B) is elliptic with respect to the cone $\mathbf{C} - \mathbf{R}_+ - \mathbf{R}_-$.

This is a very strong assmption which rules out many interesting cases, but is necessary to treat the index theorem using heat equation methods. We emphasize that the Atiyah-Bott theorem in its full generality does not follow from these methods.

Since $P^2 = Q^*Q + QQ^*$, it is clear that P_B^2 decomposes as the sum of two operators which preserve $C^\infty(V_1)$ and $C^\infty(V_2)$. We let $\mathcal{S}$ be the endomorphism $+1$ on V_1 and -1 on V_2 to take care of the signs; it is clear $P_B^2\mathcal{S} = \mathcal{S}P_B^2$. The same cancellation lemma we have used previously yields:

$$\operatorname{index}(Q_1, B_1) = \dim \mathrm{N}(Q_{B_1}) - \dim \mathrm{N}(Q^*_{B_2}) = \operatorname{Tr}(\mathcal{S}e^{-tP_B^2}).$$

(The fact that this definition of the index agrees with the definition given in the Atiyah-Bott paper follows from the fact that $B\colon C^\infty(W) \to C^\infty(W_1)$ is surjective). Consequently, the application of Lemma 1.9.2 yields:

THEOREM 1.9.3. *Let $Q\colon C^\infty(V_1) \to C^\infty(V_2)$ be an elliptic differential operator of order $d > 0$. Let $Q^*\colon C^\infty(V_2) \to C^\infty(V_1)$ be the formal adjoint and define $P = Q \oplus Q^*\colon C^\infty(V_1 \oplus V_2) \to C^\infty(V_1 \oplus V_2)$. Let $B = B_1 \oplus B_2$ be a boundary condition such that (P, B) is elliptic with respect to the cone $\mathbf{C} - \mathbf{R}_+ - \mathbf{R}_-$ and so that P_B is self-adjoint. Define $\mathcal{S} = +1$ on V_1 and -1 on V_2 then:*
(a) $\operatorname{index}(Q_{B_1}) = \operatorname{Tr}(\mathcal{S}e^{-tP_B^2})$ *for all t.*
(b) There exist local invariants $a_n(x, Q)$ and $a_n(y, Q, B_1)$ such that:

$$\operatorname{Tr}(\mathcal{S}e^{-tP_B^2}) \sim \sum_{n=0}^{\infty} t^{\frac{n-m}{2d}} \int_M a_n(x, Q)\,\mathrm{dvol}(x) + \sum_{n=0}^{\infty} t^{\frac{n-m+1}{2d}} \int_{dM} a_n(y, Q, B_1)\,\mathrm{dvol}(y).$$

(c)

$$\int_M a_n(x, Q)\,\mathrm{dvol}(x) + \int_{dM} a_{n-1}(y, Q, B_1)\,\mathrm{dvol}(y) = \begin{cases} \operatorname{index}(Q_{B_1}) & \text{if } n = m \\ 0 & \text{if } n \neq m. \end{cases}$$

This gives a local formula for the index; there is an analogue of Lemma 1.7.5 giving various functorial properties of these invariants we will discuss in the fourth chapter where we shall discuss the de Rham complex and the Gauss-Bonnet theorem for manifolds with boundary.

We now specialize henceforth to first order operators. We decompose

$$p(x, \xi) = \sum_j e_j(x)\xi_j$$

and near the boundary express:

$$p(y, 0, \zeta, z) = e_0(y) \cdot z + \sum_{j=1}^{m-1} e_j(y)\zeta_j = e_0(y) \cdot z + p(y, 0, \zeta, 0).$$

We study the ordinary differential equation:

$$\{ip_0\partial/\partial r + p(y,0,\zeta,0)\}f(r) = \lambda f(r)$$

or equivalently:

$$-ip_0\{\partial/\partial r + ip_0^{-1}p(y,0,\zeta,0) - ip_0^{-1}\lambda)\}f(r) = 0.$$

With this equation in mind, we define:

$$\tau(y,\zeta,\lambda) = ip_0^{-1}(p(y,0,\zeta,0) - \lambda).$$

LEMMA 1.9.4. *Let $p(x,\xi)$ be self-adjoint and elliptic. Then $\tau(y,\zeta,\lambda)$ has no purely imaginary eigenvalues for $(\zeta,\lambda) \neq (0,0) \in T^*(dM) \times (\mathbf{C} - \mathbf{R}_+ - \mathbf{R}_-)$.*

PROOF: We suppose the contrary and set $\tau(y,\zeta,\lambda)v = izv$ where z is real and $v \neq 0$. This implies that:

$$(p(y,0,\zeta,0) - \lambda)v = p_0 zv$$

or equivalently that:

$$p(y,0,\zeta,-z)v = \lambda v.$$

Since p is self-adjoint, this implies $\lambda = 0$ so $\zeta \neq 0$ which contradicts the ellipticity of p.

We define bundles $\Pi_\pm(\tau)$ over $T^*(dM) \times \{\mathbf{C} - \mathbf{R}_+ - \mathbf{R}_-\} - (0,0)$ to be the span of the generalized eigenvectors of τ which correspond to eigenvalues with positive/negative real part; $\Pi_+ \oplus \Pi_- = V$. The differential equation has the form:

$$\{\partial/\partial r + \tau\}f = 0$$

so the condition $\lim_{r\to\infty} f(r) = 0$ implies that $f(0) \in \Pi_+(\tau)$. Thus $\Pi_+(\tau)$ is the bundle of Cauchy data corresponding to solutions to this ODE. Since $d = 1$, the boundary condition B is just an endomorphism:

$$B: V_{|dM} \to W'$$

and we conclude:

LEMMA 1.9.5. *Let P be a first order formally self-adjoint elliptic differential operator. Let B be a 0^{th} order boundary condition. Define*

$$\tau(y,\zeta,\lambda) = ip_0^{-1}(p(y,0,\zeta,0) - \lambda).$$

Then τ has no purely imaginary eigenvalues and we define $\Pi_\pm(\tau)$ to be bundles of generalized eigenvectors corresponding to eigenvalues with positive real and negative real parts. (P,B) is elliptic with respect to the cone

$\mathbf{C} - \mathbf{R}_+ - \mathbf{R}_-$ *if and only if* $B\colon \Pi_+(\tau) \to W'$ *is an isomorphism for all* $(\zeta, \lambda) \neq (0,0) \in T^*(dM) \times \{\mathbf{C} - \mathbf{R}_+ - \mathbf{R}_-\}$. P_B *is self-adjoint if and only if* $p_0\,\mathrm{N}(B)$ *is perpendicular to* $\mathrm{N}(B)$.

PROOF: We have checked everything except the condition that P_B be self-adjoint. Since P is formally self-adjoint, it is immediate that:

$$(Pf, g) - (f, Pg) = \int_{dM} (-ip_0 f, g).$$

We know that on the boundary, both f and g have values in $\mathrm{N}(B)$ since they satisfy the boundary condition. Thus this vanishes identically if and only if $(p_0 f, g) = 0$ for all $f, g \in \mathrm{N}(B)$ which completes the proof.

We emphasize that these boundary conditions are much stronger than those required in the Atiyah-Bott theorem to define the index. They are much more rigid and avoid some of the pathologies which can occur otherwise. Such boundary conditions do not necessarily exist in general as we shall see later.

We shall discuss the general case in more detail in Chapter 4. We complete this section by discussing the one-dimensional case case to illustrate the ideas involved. We consider $V = [0,1] \times \mathbf{C}^2$ and let P be the operator:

$$P = -i\frac{\partial}{\partial r}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

At the point 0, we have:

$$\tau(\lambda) = -i\lambda\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Thus if $\mathrm{Im}(\lambda) > 0$ we have $\Pi_+(\tau) = \left\{\binom{a}{a}\right\}$ while if $\mathrm{Im}(\lambda) < 0$ we have $\Pi_+(\tau) = \left\{\binom{a}{-a}\right\}$. We let B be the boundary projection which projects on the first factor—$\mathrm{N}(B) = \left\{\binom{0}{a}\right\}$; we take the same boundary condition at $x = 1$ to define an elliptic self-adjoint operator P_B. Let $P^2 = -\dfrac{\partial^2}{\partial x^2}$ with boundary conditions: Dirichlet boundary conditions on the first factor and Neumann boundary conditions on the second factor. The index of the problem is -1.

1.10. Eta and Zeta Functions.

We have chosen to work with the heat equation for various technical reasons. However, much of the development of the subject has centered on the zeta function so we shall briefly indicate the relationship between these two in this section. We shall also define the eta invariant which plays an important role in the Atiyah-Singer index theorem for manifolds with boundary.

Recall that Γ is defined by:

$$\Gamma(s) = \int_0^\infty t^{s-1} e^{-t} dt$$

for $\mathrm{Re}(s) > 0$. We use the functional equation $s\Gamma(s) = \Gamma(s+1)$ to extend Γ to a meromorphic function on C with isolated simple poles at $s = 0, -1, -2, \ldots$. Let $P\colon C^\infty(V) \to C^\infty(V)$ be an elliptic self-adjoint partial differential operator of order $d > 0$ with positive definite leading symbol. We showed in section 1.6 that this implies $\mathrm{spec}(P)$ is contained in $[-C, \infty)$ for some constant C. We now assume that P itself is positive definite—i.e., $\mathrm{spec}(P)$ is contained in $[\varepsilon, \infty)$ for some $\varepsilon > 0$.

Proceeding formally, we define P^s by:

$$P^s = \frac{1}{2\pi i} \int_\alpha \lambda^s (P - \lambda)^{-1}\, d\lambda$$

where α is a suitable path in the half-plane $\mathrm{Re}(\lambda) > 0$. The estimates of section 1.6 imply this is smoothing operator if $\mathrm{Re}(s) \ll 0$ with a kernel function given by:

$$L(s, x, y) = \sum_n \lambda_n^s \phi_n(x) \otimes \bar{\phi}_n(y);$$

this converges to define a C^k kernel if $\mathrm{Re}(s) < s_0(k)$.

We use the Mellin transform to relate the zeta and heat kernels.

$$\int_0^\infty t^{s-1} e^{-\lambda t} dt = \lambda^{-s} \int_0^\infty (\lambda t)^{s-1} e^{-\lambda t} d(\lambda t) = \lambda^{-s}\Gamma(s).$$

This implies that:

$$\Gamma(s) \operatorname{Tr}(P^{-s}) = \int_0^\infty t^{s-1} \operatorname{Tr} e^{-tP}\, dt.$$

We define $\zeta(s, P) = \operatorname{Tr}(P^{-s})$; this is holomorphic for $\mathrm{Re}(s) \gg 0$. We decompose this integral into $\int_0^1 + \int_1^\infty$. We have bounded the eigenvalues of

P away from zero. Using the growth estimates of section 1.6 it is immediate that:

$$\int_1^\infty t^{s-1} \operatorname{Tr} e^{-tP}\, dt = r(s,P)$$

defines an entire function of s. If $0 < t < 1$ we use the results of section 1.7 to expand

$$\operatorname{Tr} e^{-tP} = \sum_{n \le n_0} t^{\frac{n-m}{d}} a_n(P) + O\left(t^{\frac{n_0-m}{d}}\right)$$
$$a_n(P) = \int_M a_n(x,P)\,\mathrm{dvol}(x)$$

where $a_n(x,P)$ is a local scalar invariant of the jets of the total symbol of P given by Lemma 1.7.4.

If we integrate the error term which is $O\left(t^{\frac{n_0-m}{d}}\right)$ from 0 to 1, we define a holomorphic function of s for $\mathrm{Re}(s) + \frac{n_0-m}{d} > 0$. We integrate $t^{s-1}t^{\frac{(n-m)}{d}}$ between 0 and 1 to conclude:

$$\Gamma(s)\operatorname{Tr}(P^{-s}) = \Gamma(s)\zeta(s,P) = \sum_{n<n_0} d(sd+n-m)^{-1} a_n(P) + r_{n_0}$$

where r_{n_0} is holomorphic for $\mathrm{Re}(s) > -\frac{(n_0-m)}{d}$. This proves $\Gamma(s)\zeta(s,P)$ extends to a meromorphic function on C with isolated simple poles. Furthermore, the residue at these poles is given by a local formula. Since Γ has isolated simple poles at $s = 0, -1, -2, \dots$ we conclude that:

LEMMA 1.10.1. *Let P be a self-adjoint, positive, elliptic partial differential operator of order $d > 0$ with positive definite symbol. We define:*

$$\zeta(s,P) = \operatorname{Tr}(P^{-s}) = \Gamma(s)^{-1}\int_0^\infty t^{s-1}\operatorname{Tr} e^{-tP}\, dt.$$

This is well defined and holomorphic for $\mathrm{Re}(s) \gg 0$. It has a meromorphic extension to C with isolated simple poles at $s = (m-n)/d$ for $n = 0, 1, 2, \dots$. The residue of ζ at these local poles is $a_n(P)\Gamma((m-n)/d)^{-1}$. If $s = 0, -1, \dots$ is a non-positive integer, then $\zeta(s,P)$ is regular at this value of s and its value is given by $a_n(P)\operatorname{Res}_{s=(m-n)/d}\Gamma(s)$. $a_n(P)$ is the invariant given in the asymptotic expansion of the heat equation. $a_n(P) = \int_M a_n(x,P)\,\mathrm{dvol}(x)$ where $a_n(x,P)$ is a local invariant of the jets of the total symbol of P. a_n vanishes if n is odd.

Remark: If A is an auxilary differential operator of order a, we can define

$$\zeta(s,A,P) = \operatorname{Tr}(AP^{-s}) = \Gamma(s)^{-1}\int_0^\infty t^{s-1}\operatorname{Tr}(Ae^{-tP})\, dt.$$

We apply Lemma 1.7.7 to see this has a meromorphic extension to $\mathbf{C}$ with isolated simple poles at $s = (m + a - n)/d$. The residue at these poles is given by the generalized invariants of the heat equation

$$\frac{a_n(A, P)}{\Gamma\{(m + a - n)/d\}}.$$

There is a similar theorem if $dM \neq \emptyset$ and if B is an elliptic boundary condition as discussed in section 1.9.

If P is only positive semi-definite so it has a finite dimensional space corresponding to the eigenvalue 0, we define:

$$\zeta(s, P) = \sum_{\lambda_n > 0} \lambda_n^{-s}$$

and Lemma 1.10 extends to this case as well with suitable modifications. For example, if $M = S^1$ is the unit circle and if $P = -\partial^2/\partial\theta^2$ is the Laplacian, then:

$$\zeta(s, P) = 2\sum_{n>0} n^{-2s}$$

is essentially just the Riemann zeta function. This has a simple isolated pole at $s = 1/2$.

If $Q: C^\infty(V_1) \to C^\infty(V_2)$, then the cancellation lemma of section 1.6 implies that:

$$\varepsilon^{-s}\,\mathrm{index}(Q) = \zeta(s, Q^*Q + \varepsilon) - \zeta(s, QQ^* + \varepsilon)$$

for any $\varepsilon > 0$ and for any s. ζ is regular at $s = 0$ and its value is given by a local formula. This gives a local formula for $\mathrm{index}(Q)$. Using Lemma 1.10.1 and some functorial properties of the heat equation, it is not difficult to show this local formula is equivalent to the local formula given by the heat equation so no new information has resulted. The asymptotics of the heat equation are related to the values and residues of the zeta function by Lemma 1.10.1.

If we do not assume that P is positive definite, it is possible to define a more subtle invariant which measures the difference between positive and negative spectrum. Let P be a self-adjoint elliptic partial differential operator of order $d > 0$ which is not necessarily positive definite. We define the eta invariant by:

$$\begin{aligned}\eta(s, P) &= \sum_{\lambda_n > 0} \lambda_n^{-s} - \sum_{\lambda_n < 0} (-\lambda_n)^{-s} = \sum_{\lambda_n \neq 0} \mathrm{sign}(\lambda_n)|\lambda_n|^{-s} \\ &= \mathrm{Tr}(P \cdot (P^2)^{-(s+1)/2}).\end{aligned}$$

Again, this is absolutely convergent and defines a holomorphic function if $\mathrm{Re}(s) \gg 0$.

We can also discuss the eta invariant using the heat equation. The identity:

$$\int_0^\infty t^{(s-1)/2} \lambda e^{-\lambda^2 t}\, dt = \Gamma((s+1)/2)\, \mathrm{sign}(\lambda)|\lambda|^{-s}$$

implies that:

$$\eta(s,P) = \Gamma((s+1)/2)^{-1} \int_0^\infty t^{(s-1)/2}\, \mathrm{Tr}\{Pe^{-tP^2}\}\, dt.$$

Again, the asymptotic behavior at $t = 0$ is all that counts in producing poles since this decays exponentially at ∞ assuming P has no zero eigenvector; if $\dim \mathrm{N}(P) > 0$ a seperate argument must be made to take care of this eigenspace. This can be done by replacing P by $P + \varepsilon$ and letting $\varepsilon \to 0$.

Lemma 1.7.7 shows that there is an asymptotic series of the form:

$$\mathrm{Tr}(Pe^{-tP^2}) \sim \sum_{n=0}^\infty t^{\frac{n-m-d}{2d}} a_n(P, P^2)$$

for

$$a_n(P, P^2) = \int_M a_n(x, P, P^2)\, \mathrm{dvol}(x).$$

This is a local invariant of the jets of the total symbol of P.

We substitute this asymptotic expansion into the expression for η to see:

$$\eta(s,P)\Gamma((s+1)/2)^{-1} = \sum_{n \le n_0} \frac{2d}{ds + n - m} a_n(P, P^2) + r_{n_0}$$

where r_{n_0} is holomorphic on a suitable half-plane. This proves η has a suitable meromorphic extension to $\mathbf{C}$ with locally computable residues.

Unfortunately, while it was clear from the analysis that ζ was regular at $s = 0$, it is not immediate that η is regular at $s = 0$ since Γ does not have a pole at $\frac{1}{2}$ to cancel the pole which may be introduced when $n = m$. in fact, if one works with the local invariants involved, $a_n(x, P, P^2) \neq 0$ in general so the local poles are in fact present at $s = 0$. However, it is a fact which we shall discuss later that η is regular at $s = 0$ and we will define

$$\tilde{\eta}(p) = \frac{1}{2}\{\eta(0,p) + \dim \mathrm{N}(p)\} \mod \mathbf{Z}$$

(we reduce modulo $\mathbf{Z}$ since η has jumps in $2\mathbf{Z}$ as eigenvalues cross the origin).

We compute a specific example to illustrate the role of η in measuring spectral asymmetry. Let $P = -i\partial/\partial\theta$ on $C^\infty(S^1)$, then the eigenvalues of p are the integers so $\eta(s,p) = 0$ since the spectrum is symmetric about the origin. Let $a \in \mathbf{R}$ and define:

$$P_a = P - a, \qquad \eta(s, P_a) = \sum_{n\in\mathbf{Z}} \operatorname{sign}\{n-a\}|n-a|^{-s}.$$

We differentiate this with respect to the parameter a to conclude:

$$\frac{d}{da}\eta(s, P(a)) = \sum_{n\in\mathbf{Z}} s((n-a)^2)^{-(s+1)/2}.$$

If we compare this with the Riemann zeta function, then

$$\sum_{n\in\mathbf{Z}} ((n-a)^2)^{-(s+1)/2}$$

has a simple pole at $s = 0$ with residue 2 and therefore:

$$\left.\frac{d}{da}\eta(s, P(a))\right|_{s=0} = 2.$$

Since η vanishes when $a = 0$, we integrate this with respect to a to conclude:

$$\eta(0, P(a)) = 2a + 1 \mod 2\mathbf{Z} \qquad \text{and} \qquad \tilde\eta(P(a)) = a + \frac{1}{2}$$

is non-trivial. (We must work modulo $\mathbf{Z}$ since $\operatorname{spec} P(a)$ is periodic with period 1 in a).

We used the identity:

$$\frac{d}{da}\eta(s, P_a) = -s\operatorname{Tr}(\dot P_a (P_a^2)^{-(s+1)/2})$$

in the previous computation; it in fact holds true in general:

LEMMA 1.10.2. *Let $P(a)$ be a smooth 1-parameter family of elliptic self-adjoint partial differential operators of order $d > 0$. Assume $P(a)$ has no zero spectrum for a in the parameter range. Then if "$\cdot$" denotes differentiation with respect to the parameter a,*

$$\dot\eta(s, P(a)) = -s\operatorname{Tr}(\dot P(a)(P(a)^2)^{-(s+1)/2}).$$

If $P(a)$ has zero spectrum, we regard $\eta(s, P(a)) \in \mathbf{C}/\mathbf{Z}$.

PROOF: If we replace P by P^k for an odd positive integer k, then $\eta(s, P^k) = \eta(ks, P)$. This shows that it suffices to prove Lemma 1.10.2 for $d \gg 0$.

By Lemma 1.6.6, $(P-\lambda)^{-1}$ is smoothing and hence of trace class for d large. We can take trace inside the integral to compute:

$$\eta(s,P(a)) = \frac{1}{2\pi i}\left\{\int_{\gamma_1} \lambda^{-s}\operatorname{Tr}((P(a)-\lambda)^{-1})\,d\lambda \right.$$
$$\left. - \int_{\gamma_2} (-\lambda)^{-s}\operatorname{Tr}((P(a)-\lambda)^{-1}\,d\lambda\right\}$$

where γ_1 and γ_2 are paths about the positive/negative real axis which are oriented suitably. We differentiate with respect to the parameter a to express:

$$\frac{d}{da}((P(a)-\lambda)^{-1} = -(P(a)-\lambda)^{-1}\dot{P}(a)(P(a)-\lambda)^{-1}.$$

Since the operators involved are of Trace class,

$$\operatorname{Tr}\left(\frac{d}{da}(P(a)-\lambda)^{-1}\right) = -\operatorname{Tr}(\dot{P}(a)(P(a)-\lambda)^{-2}).$$

We use this identity and bring trace back outside the integral to compute:

$$\eta(s,P(a)) = \frac{1}{2\pi i}\operatorname{Tr}(\dot{P}(a))\left\{\int_{\gamma_1} -\lambda^{-s}(P(a)-\lambda)^{-2}\,d\lambda\right.$$
$$\left. + \int_{\gamma_2} (-\lambda)^{-s}(P(a)-\lambda)^{-2}\,d\lambda\right\}.$$

We now use the identity:

$$\frac{d}{d\lambda}((P(a)-\lambda)^{-1}) = (P(a)-\lambda)^{-2}$$

to integrate by parts in λ in this expression. This leads immediately to the desired formula.

We could also calculate using the heat equation. We proceed formally:

$$\begin{aligned}\frac{d\eta}{da}(s,P(a)) &= \Gamma\{(s+1)/2\}^{-1}\int_0^\infty t^{(s-1)/2}\operatorname{Tr}\left(\frac{d}{da}(P(a)e^{-tP(a)^2})\right)dt\\ &= \Gamma\{(s+1)/2\}^{-1}\int_0^\infty t^{(s-1)/2}\operatorname{Tr}\left(\frac{dP}{da}(1-2tP^2)e^{-tP^2}\right)dt\\ &= \Gamma\{(s+1)/2\}^{-1}\int_0^\infty t^{(s-1)/2}\left(1+2t\frac{d}{dt}\right)\operatorname{Tr}\left(\frac{dP}{da}e^{-tP^2}\right)dt.\end{aligned}$$

We now integrate by parts to compute:

$$\begin{aligned}
&= \Gamma\{(s+1)/2\}^{-1} \int_0^\infty -st^{(s-1)/2} \operatorname{Tr}\left(\frac{dP}{da} e^{-tP^2}\right) dt \\
&\sim -s\Gamma\{(s+1)/2\}^{-1} \sum_n \int_0^1 t^{(s-1)/2} t^{(n-m-d)/2d} a_n \left(\frac{dP}{da}, P^2\right) dt \\
&\sim -s\Gamma\{(s+1)/2\}^{-1} \sum_n 2/(s+(n-m)/d) a_n \left(\frac{dP}{da}, P^2\right).
\end{aligned}$$

In particular, this shows that $\dfrac{d\tilde{\eta}}{da}$ is regular at $s = 0$ and the value

$$-\Gamma\left(\tfrac{1}{2}\right)^{-1} a_m \left(\frac{dP}{da}, P^2\right)$$

is given by a local formula.

The interchange of order involved in using global trace is an essential part of this argument. $\operatorname{Tr}(\dot{P}(a)(P^2)^{-(s+1)/2})$ has a meromorphic extension to $\mathbf{C}$ with isolated simple poles. The pole at $s = -1/2$ can be present, but is cancelled off by the factor of $-s$ which multiplies the expression. Thus $\dfrac{d}{da} \operatorname{Res}_{s=0} \eta(s, P(a)) = 0$. This shows the global residue is a homotopy invariant; this fact will be used in Chapter 4 to show that in fact η is regular at the origin. This argument does not go through locally, and in fact it is possible to construct operators in any dimension $m \geq 2$ so that the local eta function is not regular at $s = 0$.

We have assumed that $P(a)$ has no zero spectrum. If we supress a finite number of eigenvalues which may cross the origin, this makes no contribution to the residue at $s = 0$. Furthermore, the value at $s = 0$ changes by jumps of an even integer as eigenvalues cross the origin. This shows $\tilde{\eta}$ is in fact smooth in the parameter a and proves:

LEMMA 1.10.3.
(a) Let P be a self-adjoint elliptic partial differential operator of positive order d. Define:

$$\eta(s, P) = \operatorname{Tr}(P(P^2)^{-(s+1)/2}) = \Gamma\{(s+1)/2\}^{-1} \int_0^\infty t^{(s-1)/2} \operatorname{Tr}(Pe^{-tP^2})\, dt.$$

This admits a meromorphic extension to $\mathbf{C}$ with isolated simple poles at $s = (m-n)/d$. The residue of η at such a pole is computed by:

$$\operatorname{Res}_{s=(m-n)/d} = 2\Gamma\{(m+d-n)/2d\}^{-1} \int_M a_n(x, P, P^2) \operatorname{dvol}(x)$$

where $a_n(x, P, P^2)$ is defined in Lemma 1.7.7; it is a local invariant in the jets of the symbol of P.

(b) Let $P(a)$ be a smooth 1-parameter family of such operators. If we assume eigenvalues do not cross the origin, then:

$$\begin{aligned}\frac{d}{da}\eta(s,P(a)) &= -s\operatorname{Tr}\left(\frac{dP}{da}\cdot(P(a)^2)^{-(s+1)/2}\right)\\ &= -2s\Gamma\{(s+1)/2\}^{-1}\int_0^\infty t^{(s-1)/2}\operatorname{Tr}\left(\frac{dP}{da}\cdot e^{-tP^2}\right)dt.\end{aligned}$$

Regardless of whether or not eigenvalues cross the origin, $\tilde\eta(P(a))$ is smooth in $\mathbf{R} \bmod \mathbf{Z}$ in the parameter a and

$$\begin{aligned}\frac{d}{da}\tilde\eta(P(a)) &= -\Gamma(\tfrac12)^{-1}a_m\left(\frac{dP}{da},P^2\right)\\ &= -\Gamma(\tfrac12)^{-1}\int_M a_m\left(x,\frac{dP}{da},P^2\right)\operatorname{dvol}(x)\end{aligned}$$

is the local invariant in the jets of the symbols of $\left(\frac{dP}{da},P^2\right)$ given by Lemma 1.7.7.

In the example on the circle, the operator $P(a)$ is locally isomorphic to P. Thus the value of η at the origin is *not* given by a local formula. This is a global invariant of the operator; only the derivative is locally given.

It is not necessary to assume that P is a differential operator to define the eta invariant. If P is an elliptic self-adjoint pseudo-differential operator of order $d > 0$, then the sum defining $\eta(s,P)$ converges absolutely for $s \gg 0$ to define a holomorphic function. This admits a meromorphic extension to the half-plane $\operatorname{Re}(s) > -\delta$ for some $\delta > 0$ and the results of Lemma 1.10.3 continue to apply. This requires much more delicate estimates than we have developed and we shall omit details. The reader is referred to the papers of Seeley for the proofs.

We also remark that it is not necessary to assume that P is self-adjoint; it suffices to assume that $\det(p(x,\xi)-it) \neq 0$ for $(\xi,t) \neq (0,0) \in T^*M\times\mathbf{R}$. Under this ellipticity hypothesis, the spectrum of P is discrete and only a finite number of eigenvalues lie on or near the imaginary axis. We define:

$$\begin{aligned}\eta(s,P) &= \sum_{\operatorname{Re}(\lambda_i)>0}(\lambda_i)^{-s} - \sum_{\operatorname{Re}(\lambda_i)<0}(-\lambda_i)^{-s}\\ \tilde\eta(P) &= \frac12\left\{\eta(s,P) - \frac1s\operatorname{Res}_{s=0}\eta(s,P) + \dim\mathrm{N}(i\mathbf{R})\right\}_{s=0} \quad \bmod \mathbf{Z}\end{aligned}$$

where $\dim\mathrm{N}(i\mathbf{R})$ is the dimension of the finite dimensional vector space of generalized eigenvectors of P corresponding to purely imaginary eigenvalues.

In section 4.3, we will discuss the eta invariant in further detail and use it to define an index with coefficients in a locally flat bundle using secondary characteristic classes.

If the leading symbol of P is positive definite, the asymptotics of $\mathrm{Tr}(e^{-tP})$ as $t \to 0^+$ are given by local formulas integrated over the manifold. Let $A(x) = a_0x^a + \cdots + a_a$ and $B(x) = b_0x^b + \cdots + b_b$ be polynomials where $b_0 > 0$. The operator $A(P)e^{-tB(P)}$ is infinitely smoothing. The asymptotics of $\mathrm{Tr}(A(P)e^{-tB(P)})$ are linear combinations of the invariants $a_N(P)$ giving the asymptotics of $\mathrm{Tr}(e^{-tP})$. Thus there is no new information which results by considering more general operators of heat equation type. If the leading symbol of P is indefinite, one must consider both the zeta and eta function or equivalently:

$$\mathrm{Tr}(e^{-tP^2}) \qquad \text{and} \qquad \mathrm{Tr}(Pe^{-tP^2}).$$

Then if $b_0 > 0$ is even, we obtain enough invariants to calculate the asymptotics of $\mathrm{Tr}(A(P)e^{-tB(P)})$. We refer to Fegan-Gilkey for further details.

CHAPTER 2

CHARACTERISTIC CLASSES

Introduction

In the second chapter, we develop the theory of characteristic classes. In section 2.1, we discuss the Chern classes of a complex vector bundle and in section 2.2 we discuss the Pontrjagin and Euler classes of a real vector bundle. We shall define the Todd class, the Hirzebruch L-polynomial, and the A-roof genus which will play a central role in our discussion of the index theorem. We also discuss the total Chern and Pontrjagin classes as well as the Chern character.

In section 2.3, we apply these ideas to the tangent space of a real manifold and to the holomorphic tangent space of a holomorphic manifold. We compute several examples defined by Clifford matrices and compute the Chern classes of complex projective space. We show that suitable products of projective spaces form a dual basis to the space of characteristic classes. Such products will be used in chapter three to find the normalizing constants which appear in the formula for the index theorem.

In section 2.4 and in the first part of section 2.5, we give a heat equation proof of the Gauss-Bonnet theorem. This is based on first giving an abstract characterization of the Euler form in terms of invariance theory. This permis us to identify the integrand of the heat equation with the Euler integrand. This gives a more direct proof of Theorem 2.4.8 which was first proved by Patodi using a complicated cancellation lemma. (The theorem for dimension $m = 2$ is due to McKean and Singer).

In the remainder of section 2.5, we develop a similar characterization of the Pontrjagin forms of the real tangent space. We shall wait until the third chapter to apply these results to obtain the Hirzebruch signature theorem. There are two different approaches to this result. We have presented both our original approach and also one modeled on an approach due to Atiyah, Patodi and Bott. This approach uses H. Weyl's theorem on the invariants of the orthogonal group and is not self-contained as it also uses facts about geodesic normal coordinates we shall not develop. The other approach is more combinatorial in flavor but is self-contained. It also generalizes to deal with the holomorphic case for a Kaehler metric.

The signature complex with coefficients in a bundle V gives rise to invariants which depend upon both the metric on the tangent space of M and on the connection 1-form of V. In section 2.6, we extend the results of section 2.5 to cover more general invariants of this type. We also construct dual bases for these more general invariants similar to those constructed in section 2.3 using products of projective spaces and suitable bundles over these spaces.

The material of sections 2.1 through 2.3 is standard and reviews the theory of characteristic classes in the context we shall need. Sections 2.4 through 2.6 deal with less standard material. The chapter is entirely self-contained with the exception of some material in section 2.5 as noted above. We have postponed a discussion of similar material for the holomorphic Kaehler case until sections 3.6 and 3.7 of chapter three since this material is not needed to discuss the signature and spin complexes.

2.1. Characteristic Classes Of a Complex Vector Bundle.

The characteristic classes are topological invariants of a vector bundle which are represented by differential forms. They are defined in terms of the curvature of a connection.

Let M be a smooth compact manifold and let V be a smooth complex vector bundle of dimension k over M. A connection ∇ on V is a first order partial differential operator $\nabla\colon C^\infty(V) \to C^\infty(T^*M \otimes V)$ such that:

$$\nabla(fs) = df \otimes s + f\nabla s \qquad \text{for } f \in C^\infty(M) \text{ and } s \in C^\infty(V).$$

Let $(s_1, \ldots, s_k)$ be a local frame for V. We can decompose any section $s \in C^\infty(V)$ locally in the form $s(x) = f_i(x)s_i(x)$ for $f_i \in C^\infty(M)$. We adopt the convention of summing over repeated indices unless otherwise specified in this subsection. We compute:

$$\nabla s = df_i \otimes s_i + f_i \nabla s_i = df_i \otimes s_i + f_i \omega_{ij} \otimes s_j \qquad \text{where } \nabla s_i = \omega_{ij} \otimes s_j.$$

The connection 1-form ω defined by

$$\omega = \omega_{ij}$$

is a matrix of 1-forms. The connection ∇ is uniquely determined by the derivation property and by the connection 1-form. If we specify ω arbitrarily locally, we can define ∇ locally. The convex combination of connections is again a connection, so using a partition of unity we can always construct connections on a bundle V.

If we choose another frame for V, we can express $s'_i = h_{ij}s_j$. If $h^{-1}_{ij}s'_j = s_i$ represents the inverse matrix, then we compute:

$$\begin{aligned} \nabla s'_i = \omega'_{ij} \otimes s'_j &= \nabla(h_{ik}s_k) = dh_{ik} \otimes s_k + h_{ik}\omega_{kl} \otimes s_l \\ &= (dh_{ik}h^{-1}_{kj} + h_{ik}\omega_{kl}h^{-1}_{lj}) \otimes s'_j. \end{aligned}$$

This shows ω' obeys the transformation law:

$$\omega'_{ij} = dh_{ik}h^{-1}_{kj} + h_{ik}\omega_{kl}h^{-1}_{lj} \qquad \text{i.e., } \omega' = dh \cdot h^{-1} + h\omega h^{-1}$$

in matrix notation. This is, of course, the manner in which the 0^{th} order symbol of a first order operator transforms.

We extend ∇ to be a derivation mapping

$$C^\infty(\Lambda^p T^*M \otimes V) \to C^\infty(\Lambda^{p+1} T^*M \otimes V)$$

so that:

$$\nabla(\theta_p \otimes s) = d\theta_p \otimes s + (-1)^p \theta_p \wedge \nabla s.$$

We compute that:

$$\nabla^2(fs) = \nabla(df \otimes s + f\nabla s) = d^2 f \otimes s - df \wedge \nabla s + df \wedge \nabla s + f\nabla^2 s = f\nabla^2 s$$

so instead of being a second order operator, ∇^2 is a 0^{th} order operator. We may therefore express

$$\nabla^2(s)(x_0) = \Omega(x_0)s(x_0)$$

where the curvature Ω is a section to the bundle $\Lambda^2(T^*M) \otimes \text{END}(V)$ is a 2-form valued endomorphism of V.

In local coordinates, we compute:

$$\Omega(s_i) = \Omega_{ij} \otimes s_j = \nabla(\omega_{ij} \otimes s_j) = d\omega_{ij} \otimes s_j - \omega_{ij} \wedge \omega_{jk} \otimes s_k$$

so that:

$$\Omega_{ij} = d\omega_{ij} - \omega_{ik} \wedge \omega_{kj} \qquad \text{i.e.,} \ \ \Omega = d\omega - \omega \wedge \omega.$$

Since ∇^2 is a 0^{th} order operator, Ω must transform like a tensor:

$$\Omega'_{ij} = h_{ik}\Omega_{kl}h^{-1}_{lj} \qquad \text{i.e.,} \ \ \Omega' = h\Omega h^{-1}.$$

This can also be verified directly from the transition law for ω'. The reader should note that in some references, the curvature is defined by $\Omega = d\omega + \omega \wedge \omega$. This sign convention results from writing $V \otimes T^*M$ instead of $T^*M \otimes V$ and corresponds to studying left invariant rather than right invariant vector fields on $\text{GL}(k, C)$.

It is often convenient to normalize the choice of frame.

Lemma 2.1.1. *Let ∇ be a connection on a vector bundle V. We can always choose a frame s so that at a given point x_0 we have*

$$\omega(x_0) = 0 \qquad \text{and} \qquad d\Omega(x_0) = 0.$$

Proof: We find a matrix $h(x)$ defined near x_0 so that $h(x_0) = I$ and $dh(x_0) = -\omega(x_0)$. If $s'_i = h_{ij}s_j$, then it is immediate that $\omega'(x_0) = 0$. Similarly we compute $d\Omega'(x_0) = d(d\omega' - \omega' \wedge \omega')(x_0) = \omega'(x_0) \wedge d\omega'(x_0) - d\omega'(x_0) \wedge \omega'(x_0) = 0$.

We note that as the curvature is invariantly defined, we cannot in general find a parallel frame s so ω vanishes in a neighborhood of x_0 since this would imply $\nabla^2 = 0$ near x_0 which need not be true.

Let A_{ij} denote the components of a matrix in $\text{END}(\mathbf{C}^k)$ and let $P(A) = P(A_{ij})$ be a polynomial mapping $\text{END}(\mathbf{C}^k) \to \mathbf{C}$. We assume that P is invariant—i.e., $P(hAh^{-1}) = P(A)$ for any $h \in \text{GL}(k, \mathbf{C})$. We define $P(\Omega)$ as an even differential form on M by substitution. Since P is invariant and since the curvature transforms tensorially, $P(\Omega) \stackrel{\text{def}}{=} P(\nabla)$ is defined independently of the particular local frame which is chosen.

There are two examples which are of particular interest and which will play an important role in the statement of the Atiyah-Singer index theorem. We define:

$$\begin{aligned} c(A) &= \det\left(I + \frac{i}{2\pi}A\right) \\ &= 1 + c_1(A) + \cdots + c_k(A) \qquad && \text{(the total Chern form)} \\ ch(A) = \operatorname{Tr} e^{iA/2\pi} &= \sum_j \operatorname{Tr}\left(\frac{iA}{2\pi}\right)^j \Big/ j! \qquad && \text{(the total Chern character)} \end{aligned}$$

The $c_j(A)$ represent the portion of $c(A)$ which is homogeneous of order j. $c_j(\nabla) \in \Lambda^{2j}(M)$. In a similar fashion, we decompose $ch(A) = \sum_j ch_j(A)$ for $ch_j(A) = \operatorname{Tr}(iA/2\pi)^j/j!$

Strictly speaking, $ch(A)$ is not a polynomial. However, when we substitute the components of the curvature tensor, this becomes a finite sum since $\operatorname{Tr}(\Omega^j) = 0$ if $2j > \dim M$. More generally, we can define $P(\Omega)$ if $P(A)$ is an invariant formal power series by truncating the power series appropriately.

As a differential form, $P(\nabla)$ depends on the connection chosen. We show $P(\nabla)$ is always closed. As an element of de Rham cohomology, $P(\nabla)$ is independent of the connection and defines a cohomology class we shall denote by $P(V)$.

LEMMA 2.1.2. *Let P be an invariant polynomial.*
(a) $dP(\nabla) = 0$ so $P(\nabla)$ is a closed differential form.
(b) Given two connections ∇_0 and ∇_1, we can define a differential form $TP(\nabla_0, \nabla_1)$ so that $P(\nabla_1) - P(\nabla_0) = d\{TP(\nabla_1, \nabla_0)\}$.

PROOF: By decomposing P as a sum of homogeneous polynomials, we may assume without loss of generality that P is homogeneous of order k. Let $P(A_1, \dots, A_k)$ denote the complete polarization of P. We expand $P(t_1A_1 + \cdots + t_kA_k)$ as a polynomial in the variables $\{t_j\}$. $1/k!$ times the coefficient of $t_1 \dots t_k$ is the polarization of P. P is a multi-linear symmetric function of its arguments. For example, if $P(A) = \operatorname{Tr}(A^3)$, then the polarization is given by $\frac{1}{2}\operatorname{Tr}(A_1A_2A_3 + A_2A_1A_3)$ and $P(A) = P(A, A, A)$.

Fix a point x_0 of M and choose a frame so $\omega(x_0) = d\Omega(x_0) = 0$. Then

$$dP(\Omega)(x_0) = dP(\Omega, \dots, \Omega)(x_0) = kP(d\Omega, \Omega, \dots, \Omega)(x_0) = 0.$$

Since x_0 is arbitrary and since $dP(\Omega)$ is independent of the frame chosen this proves $dP(\Omega) = 0$ which proves (a).

Let $\nabla_t = t\nabla_1 + (1-t)\nabla_0$ have connection 1-form $\omega_t = \omega_0 + t\theta$ where $\theta = \omega_1 - \omega_0$. The transformation law for ω implies relative to a new frame:

$$\begin{aligned}\theta' = \omega_1' - \omega_0' &= (dh \cdot h^{-1} + h\omega_1 h^{-1}) - (dh \cdot h^{-1} + h\omega_0 h^{-1}) \\ &= h(\omega_1 - \omega_0)h^{-1} = h\theta h^{-1}\end{aligned}$$

so θ transforms like a tensor. This is of course nothing but the fact that the difference between two first order operators with the same leading symbol is a 0^{th} order operator.

Let Ω_t be the curvature of the connection ∇_t. This is a matrix valued 2-form. Since θ is a 1-form, it commutes with 2-forms and we can define

$$P(\theta, \Omega_t, \dots, \Omega_t) \in \Lambda^{2k-1}(T^*M)$$

by substitution. Since P is invariant, its complete polarization is also invariant so $P(h^{-1}\theta h, h^{-1}\Omega_t h, \dots, h^{-1}\Omega_t h) = P(\theta, \Omega_t, \dots, \Omega_t)$ is invariantly defined independent of the choice of frame.

We compute that:

$$P(\nabla_1) - P(\nabla_0) = \int_0^1 \frac{d}{dt} P(\Omega_t, \dots, \Omega_t)\, dt = k \int_0^1 P(\Omega_t', \Omega_t, \dots, \Omega_t)\, dt.$$

We define:

$$TP(\nabla_1, \nabla_0) = k \int_0^1 P(\theta, \Omega_t, \dots, \Omega_t)\, dt.$$

To complete the proof of the Lemma, it suffices to check

$$dP(\theta, \Omega_t, \dots, \Omega_t) = P(\Omega_t', \Omega_t, \dots, \Omega_t).$$

Since both sides of the equation are invariantly defined, we can choose a suitable local frame to simplify the computation. Let $x_0 \in M$ and fix t_0. We choose a frame so $\omega_t(x_0, t_0) = 0$ and $d\Omega_t(x_0, t_0) = 0$. We compute:

$$\begin{aligned}\Omega_t' &= \{d\omega_0 + t d\theta - \omega_t \wedge \omega_t\}' \\ &= d\theta - \omega_t' \wedge \omega_t - \omega_t \wedge \omega_t' \\ \Omega_t'(x_0, t_0) &= d\theta\end{aligned}$$

and

$$\begin{aligned}dP(\theta, \Omega_t, \dots, \Omega_t)(x_0, t_0) &= P(d\theta, \Omega_t, \dots, \Omega_t)(x_0, t_0) \\ &= P(\Omega_t', \Omega_t, \dots, \Omega_t)(x_0, t_0)\end{aligned}$$

which completes the proof.

TP is called the transgression of P and will play an important role in our discussion of secondary characteristic classes in Chapter 4 when we discuss the eta invariant with coefficients in a locally flat bundle.

We suppose that the matrix $A = \operatorname{diag}(\lambda_1, \ldots, \lambda_k)$ is diagonal. Then modulo suitable normalizing constants $\left(\frac{i}{2\pi}\right)^j$, it is immediate that $c_j(A)$ is the j^{th} elementary symmetric function of the λ_i's since

$$\det(I + A) = \prod(1 + \lambda_j) = 1 + s_1(\lambda) + \cdots + s_k(\lambda).$$

If $P(\cdot)$ is any invariant polynomial, then $P(A)$ is a symmetric function of the λ_i's. The elementary symmetric functions form an algebra basis for the symmetric polynomials so there is a unique polynomial Q so that

$$P(A) = Q(c_1, \ldots, c_k)(A)$$

i.e., we can decompose P as a polynomial in the c_i's for diagonal matrices. Since P is invariant, this is true for diagonalizable matrices. Since the diagonalizable matrices are dense and P is continuous, this is true for all A. This proves:

LEMMA 2.1.3. *Let $P(A)$ be a invariant polynomial. There exists a unique polynomial Q so that $P = Q(c_1(A), \ldots, c_k(A))$.*

It is clear that any polynomial in the c_k's is invariant, so this completely characterizes the ring of invariant polynomials; it is a free algebra in k variables $\{c_1, \ldots, c_k\}$.

If $P(A)$ is homogeneous of degree k and is defined on $k \times k$ matrices, we shall see that if $P(A) \neq 0$ as a polynomial, then there exists a holomorphic manifold M so that if $T_c(M)$ is the holomorphic tangent space, then $\int_M P(T_c(M)) \neq 0$ where $\dim M = 2k$. This fact can be used to show that in general if $P(A) \neq 0$ as a polynomial, then there exists a manifold M and a bundle V over M so $P(V) \neq 0$ in cohomology.

We can apply functorial constructions on connections. Define:

$$\begin{array}{ccc} \nabla_1 \oplus \nabla_2 & \text{on} & C^\infty(V_1 \oplus V_2) \\ \nabla_1 \otimes \nabla_2 & \text{on} & C^\infty(V_1 \otimes V_2) \\ \nabla_1^* & \text{on} & C^\infty(V_1^*) \end{array}$$

by:

$$(\nabla_1 \oplus \nabla_2)(s_1 \oplus s_2) = (\nabla_1 s_1) \oplus (\nabla_2 s_2)$$
$$\text{with } \omega = \omega_1 \oplus \omega_2 \text{ and } \Omega = \Omega_1 \oplus \Omega_2$$
$$(\nabla_1 \otimes \nabla_2)(s_1 \otimes s_2) = (\nabla_1 s_1) \otimes (\nabla_2 s_2)$$
$$\text{with } \omega = \omega_1 \otimes 1 + 1 \otimes \omega_2 \text{ and } \Omega = \Omega_1 \otimes 1 + 1 \otimes \Omega_2$$
$$(\nabla_1 s_1, s_1^*) + (s_1, \nabla_1^* s_1^*) = d(s_1, s_1^*)$$
$$\text{relative to the dual frame } \omega = -\omega_1^t \text{ and } \Omega = -\Omega_1^t.$$

In a similar fashion we can define an induced connection on $\Lambda^p(V)$ (the bundle of p-forms) and $S^p(V)$ (the bundle of symmetric p-tensors). If V has a given Hermitian fiber metric, the connection ∇ is said to be unitary or Riemannian if $(\nabla s_1, s_2) + (s_1, \nabla s_2) = d(s_1, s_2)$. If we identify V with V^* using the metric, this simply means $\nabla = \overline{\nabla}^*$. This is equivalent to the condition that ω is a skew adjoint matrix of 1-forms relative to a local orthonormal frame. We can always construct Riemannian connections locally by taking $\omega = 0$ relative to a local orthonormal frame and then using a partition of unity to construct a global Riemannian connection.

If we restrict to Riemannian connections, then it is natural to consider polynomials $P(A)$ which are defined for skew-Hermitian matrices $A + A^* = 0$ and which are invariant under the action of the unitary group. Exactly the same argument as that given in the proof of Lemma 2.1.3 shows that we can express such a P in the form $P(A) = Q(c_1(A), \dots, c_k(A))$ so that the $\mathrm{GL}(\cdot, \mathbf{C})$ and the $\mathrm{U}(\cdot)$ characteristic classes coincide. This is not true in the real category as we shall see; the Euler form is a characteristic form of the special orthogonal group which can not be defined as a characteristic form using the general linear group $\mathrm{GL}(\cdot, \mathbf{R})$.

The Chern form and the Chern character satisfy certain identities with respect to functorial constructions.

LEMMA 2.1.4.
(a)

$$c(V_1 \oplus V_2) = c(V_1)c(V_2)$$
$$c(V^*) = 1 - c_1(V) + c_2(V) - \cdots + (-1)^k c_k(V).$$

(b)

$$ch(V_1 \oplus V_2) = ch(V_1) + ch(V_2)$$
$$ch(V_1) \oplus V_2) = ch(V_1)ch(V_2).$$

PROOF: All the identities except the one involving $c(V^*)$ are immediate from the definition if we use the direct sum and product connections. If we fix a Hermitian structure on V, the identification of V with V^* is conjugate linear. The curvature of ∇^* is $-\Omega^t$ so we compute:

$$c(V^*) = \det\left(I - \frac{i}{2\pi}\Omega^t\right) = \det\left(I - \frac{i}{2\pi}\Omega\right)$$

which gives the desired identity.

If we choose a Riemannian connection on V, then $\Omega + \bar{\Omega}^t = 0$. This immediately yields the identities:

$$\overline{ch(V)} = ch(V) \qquad \text{and} \qquad \overline{c(V)} = c(V)$$

so these are real cohomology classes. In fact the normalizing constants were chosen so $c_k(V)$ is an integral cohomology class—i.e., if N_{2k} is any oriented submanifold of dimension $2k$, then $\int_{N_{2k}} c_k(V) \in \mathbf{Z}$. The $ch_k(V)$ are not integral for $k > 1$, but they are rational cohomology classes. As we shall not need this fact, we omit the proof.

The characteristic classes give cohomological invariants of a vector bundle. We illustrate this by constructing certain examples over even dimensional spheres; these examples will play an important role in our later development of the Atiyah-Singer index theorem.

DEFINITION. *A set of matrices $\{e_0, \ldots, e_m\}$ are Clifford matrices if the e_i are self-adjoint and satisfy the commutation relations $e_ie_j + e_je_i = 2\delta_{ij}$ where δ_{ij} is the Kronecker symbol.*

For example, if $m = 2$ we can take:

$$e_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad e_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad e_2 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$$

to be the Dirac matrices. More generally, we can take the e_i's to be given by the spin representations. If $x \in \mathbf{R}^{m+1}$, we define:

$$e(x) = \sum_i x_ie_i \qquad \text{so} \quad e(x)^2 = |x|^2 I.$$

Conversely let $e(x)$ be a linear map from $\mathbf{R}^m$ to the set of self-adjoint matrices with $e(x) = |x|^2 I$. If $\{v_0, \ldots, v_m\}$ is any orthonormal basis for $\mathbf{R}^m$, $\{e(v_0), \ldots, e(v_m)\}$ forms a set of Clifford matrices.

If $x \in S^m$, we let $\Pi_\pm(x)$ be the range of $\frac{1}{2}(1 + e(x)) = \pi_\pm(x)$. This is the span of the ± 1 eigenvectors of $e(x)$. If $e(x)$ is a $2k \times 2k$ matrix, then $\dim \Pi_\pm(x) = k$. We have a decomposition $S^m \times \mathbf{C}^{2k} = \Pi_+ \oplus \Pi_-$. We project the flat connection on $S^m \times \mathbf{C}^{2k}$ to the two sub-bundles to define connections $\nabla_\pm$ on $\Pi_\pm$. If $e^0_\pm$ is a local frame for $\Pi_\pm(x_0)$, we define $e_\pm(x) = \pi_\pm e^0_\pm$ as a frame in a neighborhood of x_0. We compute

$$\nabla_\pm e_\pm = \pi_\pm \, d\pi_\pm e^0_\pm, \qquad \Omega_\pm e_\pm = \pi_\pm \, d\pi_\pm \, d\pi_\pm e^0_\pm.$$

Since $e^0_\pm = e_\pm(x_0)$, this yields the identity:

$$\Omega_\pm(x_0) = \pi_\pm \, d\pi_\pm \, d\pi_\pm(x_0).$$

Since Ω is tensorial, this holds for all x.

Let $m = 2j$ be even. We wish to compute ch_j. Suppose first $x_0 = (1, 0, \dots, 0)$ is the north pole of the sphere. Then:

$$\begin{aligned}
\pi_+(x_0) &= \frac{1}{2}(1 + e_0) \\
d\pi_+(x_0) &= \frac{1}{2}\sum_{i\geq 1} dx_i e_i \\
\Omega_+(x_0) &= \frac{1}{2}(1 + e_0)\left(\frac{1}{2}\sum_{i\geq 1} dx_i e_i\right)^2 \\
\Omega_+(x_0)^j &= \frac{1}{2}(1 + e_0)\left(\frac{1}{2}\sum_{i\geq 1} dx_i e_i\right)^{2j} \\
&= 2^{-m-1} m!(1 + e_0)(e_1 \dots e_m)(dx_1 \wedge \cdots \wedge dx_m).
\end{aligned}$$

The volume form at x_0 is $dx_1 \wedge \cdots \wedge dx_m$. Since e_1 anti-commutes with the matrix $e_1 \dots e_m$, this matrix has trace 0 so we compute:

$$ch_j(\Omega_+)(x_0) = \left(\frac{i}{2\pi}\right)^j 2^{-m-1} m! \operatorname{Tr}(e_0 \dots e_m) \operatorname{dvol}(x_0)/j!\,.$$

A similar computation shows this is true at any point x_0 of S^m so that:

$$\int_{S^m} ch_j(\Pi_+) = \left(\frac{i}{2\pi}\right)^j 2^{-m-1} m! \operatorname{Tr}(e_0 \dots e_m) \operatorname{vol}(s^m)/j!\,.$$

Since the volume of S^m is $j!2^{m+1}\pi^j/m!$ we conclude:

LEMMA 2.1.5. *Let $e(x)$ be a linear map from $\mathbf{R}^{m+1}$ to the set of self-adjoint matrices. We suppose $e(x)^2 = |x|^2 I$ and define bundles $\Pi_\pm(x)$ over S^m corresponding to the ± 1 eigenvalues of e. Let $m = 2j$ be even, then:*

$$\int_{S^m} ch_j(\Pi_+) = i^j 2^{-j} \operatorname{Tr}(e_0 \dots e_m).$$

In particular, this is always an integer as we shall see later when we consider the spin complex. If

$$e_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad e_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad e_2 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$$

then $\operatorname{Tr}(e_0 e_1 e_2) = -2i$ so $\int_{S^2} ch_1(\Pi_+) = 1$ which shows in particular Π_+ is a non-trivial line bundle over S^2.

There are other characteristic classes arising in the index theorem. These are most conveniently discussed using generating functions. Let

$$x_j = \frac{i}{2\pi}\lambda_j$$

be the normalized eigenvalues of the matrix A. We define:

$$c(A) = \prod_{j=1}^{k}(1 + x_j) \qquad \text{and} \qquad ch(A) = \sum_{j=1}^{k} e^{x_j}.$$

If $P(x)$ is a symmetric polynomial, we express $P(x) = Q(c_1(x), \dots, c_k(x))$ to show $P(A)$ is a polynomial in the components of A. More generally if P is analytic, we first express P in a formal power series and then collect homogeneous terms to define $P(A)$. We define:

$$\textit{Todd class}:\ Td(A) = \prod_{j=1}^{k} \frac{x_j}{1 - e^{-x_j}}$$
$$= 1 + \frac{1}{2}c_1(A) + \frac{1}{12}(c_1^2 + c_2)(A) + \frac{1}{24}c_1(A)c_2(A) + \cdots$$

The Todd class appears in the Riemann-Roch theorem. It is clear that it is multiplicative with respect to direct sum—$Td(V_1 \oplus V_2) = Td(V_1)\, Td(V_2)$. The Hirzebruch L-polynomial and the $\hat{A}$ polynomial will be discussed in the next subsection. These are real characteristic classes which also are defined by generating functions.

If V is a bundle of dimension k, then $V \oplus 1$ will be a bundle of dimension $k+1$. It is clear $c(V \oplus 1) = c(V)$ and $Td(V \oplus 1) = Td(V)$ so these are stable characteristic classes. $ch(V)$ on the other hand is *not* a stable characteristic class since $ch_0(V) = \dim V$ depends explicitly on the dimension of V and changes if we alter the dimension of V by adding a trivial bundle.

2.2. Characteristic Classes of a Real Vector Bundle. Pontrjagin and Euler Classes.

Let V be a real vector bundle of dimension k and let $V_c = V \otimes \mathbf{C}$ denote the complexification of V. We place a real fiber metric on V to reduce the structure group from $\mathrm{GL}(k, \mathbf{R})$ to the orthogonal group $\mathrm{O}(k)$. We restrict henceforth to Riemannian connections on V, and to local orthonormal frames. Under these assumptions, the curvature is a skew-symmetric matrix of 2-forms:

$$\Omega + \Omega^t = 0.$$

Since V_c arises from a real vector bundle, the natural isomorphism of V with V^* defined by the metric induces a complex linear isomorphism of V_c with V_c^* so $c_j(V_c) = 0$ for j odd by Lemma 2.1.4. Expressed locally,

$$\det\left(I + \frac{i}{2\pi}A\right) = \det\left(I + \frac{i}{2\pi}A^t\right) = \det\left(I - \frac{i}{2\pi}A\right)$$

if $A + A^t = 0$ so $c(A)$ is an even polynomial in A. To avoid factors of i we define the Pontrjagin form:

$$p(A) = \det\left(I + \frac{1}{2\pi}A\right) = 1 + p_1(A) + p_2(A) + \cdots$$

where $p_j(A)$ is homogeneous of order $2j$ in the components of A; the corresponding characteristic class $p_j(V) \in H^{4j}(M; \mathbf{R})$. It is immediate that:

$$p_j(V) = (-1)^j c_{2j}(V_c)$$

where the factor of $(-1)^j$ comes form the missing factors of i.

The set of skew-symmetric matrices is the Lie algebra of $\mathrm{O}(k)$. Let $P(A)$ be an invariant polynomial under the action of $\mathrm{O}(k)$. We define $P(\Omega)$ for ∇ a Riemannian connection exactly as in the previous subsections. Then the analogue of Lemma 2.1.3 is:

Lemma 2.2.1. *Let $P(A)$ be a polynomial in the components of a matrix A. Suppose $P(A) = P(gAg^{-1})$ for every skew-symmetric A and for every $g \in \mathrm{O}(k)$. Then there exists a polynomial $Q(p_1, \ldots)$ so $P(A) = Q(p_1(A), \ldots)$ for every skew-symmetric A.*

Proof: It is important to note that we are not asserting that we have $P(A) = Q(p_1(A), \ldots)$ for every matrix A, but only for skew-symmetric A. For example, $P(A) = \mathrm{Tr}(A)$ vanishes for skew symmetric A but does not vanish in general.

It is not possible in general to diagonalize a skew-symmetric real matrix. We can, however, put it in block diagonal form:

$$A = \begin{pmatrix} 0 & -\lambda_1 & 0 & 0 & \dots\dots \\ \lambda_1 & 0 & 0 & 0 & \dots\dots \\ 0 & 0 & 0 & -\lambda_2 & \dots\dots \\ 0 & 0 & \lambda_2 & 0 & \dots\dots \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \end{pmatrix}$$

If k is odd, then the last block will be a 1×1 block with a zero in it. We let $x_j = -\lambda_j/2\pi$; the sign convention is chosen to make the Euler form have the right sign. Then:

$$p(A) = \prod_j (1 + x_j^2)$$

where the product ranges from 1 through $\left[\frac{k}{2}\right]$.

If $P(A)$ is any invariant polynomial, then P is a symmetric function in the $\{x_j\}$. By conjugating A by an element of $\mathrm{O}(k)$, we can replace any x_j by $-x_j$ so P is a symmetric function of the $\{x_j^2\}$. The remainder of the proof of Lemma 2.2.1 is the same; we simply express $P(A)$ in terms of the elementary symmetric functions of the $\{x_j^2\}$.

Just as in the complex case, it is convenient to define additional characteristic classes using generating functions. The functions:

$$z/\tanh z \qquad \text{and} \qquad z\{2\sinh(z/2)\}$$

are both even functions of the parameter z. We define:

The Hirzebruch L-polynomial:

$$\begin{aligned} L(x) &= \prod_j \frac{x_j}{\tanh x_j} \\ &= 1 + \frac{1}{3}p_1 + \frac{1}{45}(7p_2 - p_1^2)\cdots \end{aligned}$$

The $\hat{A}$ (A-roof) genus:

$$\begin{aligned} A(x) &= \prod_j \frac{x_j}{2\sinh(x_j/2)} \\ &= 1 - \frac{1}{24}p_1 + \frac{1}{5760}(7p_1^2 - 4p_2) + \cdots . \end{aligned}$$

These characteristic classes appear in the formula for the signature and spin complexes.

For both the real and complex case, the characteristic ring is a pure polynomial algebra without relations. Increasing the dimension k just adds generators to the ring. In the complex case, the generators are the Chern classes $\{1, c_1, \dots, c_k\}$. In the real case, the generators are the Pontrjagin classes $\{1, p_1, \dots, p_{[k/2]}\}$ where $[k/2]$ is the greatest integer in $k/2$. There is one final structure group which will be of interest to us.

Let V be a vector bundle of real dimension k. V is orientable if we can choose a fiber orientation consistently. This is equivalent to assuming that the real line bundle $\Lambda^k(V)$ is trivial. We choose a fiber metric for V and an orientation for V and restrict attention to oriented orthonormal frames. This restricts the structure group from $\mathrm{O}(k)$ to the special orthogonal group $\mathrm{SO}(k)$.

If k is odd, no new characteristic classes result from the restriction of the fiber group. We use the final 1×1 block of 0 in the representation of A to replace any λ_i by $-\lambda_i$ by conjugation by an element of $\mathrm{SO}(k)$. If n is even, however, we cannot do this as we would be conjugating by a orientation reversing matrix.

Let $P(A)$ be a polymonial in the components of A. Suppose $P(A) = P(gAg^{-1})$ for every skew-symmetric A and every $g \in \mathrm{SO}(k)$. Let $k = 2\bar{k}$ be even and let $P(A) = P(x_1, \dots, x_{\bar{k}})$. Fix $g_0 \in \mathrm{O}(k) - \mathrm{SO}(k)$ and define:

$$P_0(A) = \frac{1}{2}(P(A) + P(g_0 A g_0^{-1})) \quad \text{and} \quad P_1(A) = \frac{1}{2}(P(A) - P(g_0 A g_0^{-1})).$$

Both P_0 and P_1 are $\mathrm{SO}(k)$ invariant. P_0 is $\mathrm{O}(k)$ invariant while P_1 changes sign under the action of an element of $\mathrm{O}(k) - \mathrm{SO}(k)$.

We can replace x_1 by $-x_1$ by conjugating by a suitable orientation reversing map. This shows:

$$P_1(x_1, x_2, \dots) = -P(-x_1, x_2, \dots)$$

so that x_1 must divide every monomial of P_0. By symmetry, x_j divides every monomial of P_1 for $1 \le j \le \bar{k}$ so we can express:

$$P_1(A) = x_1 \dots x_{\bar{k}} P_1'(A)$$

where $P_1'(A)$ is now invariant under the action of $\mathrm{O}(k)$. Since P_1' is polynomial in the $\{x_j^2\}$, we conclude that both P_0 and P_1' can be represented as polynomials in the Pontrjagin classes. We define:

$$e(A) = x_1 \dots x_k \qquad \text{so} \quad e(A)^2 = \det(A) = p_{\bar{k}}(A) = \prod_j x_j^2$$

and decompose:

$$P = P_0 + e(A) P_1'.$$

$e(A)$ is a square root of the determinant of A.

It is not, of course, immediate that $e(A)$ is a polynomial in the components of A. Let $\{v_i\}$ be an oriented orthonormal basis for $\mathbf{R}^k$. We let $Av_i = \sum_j A_{ij} v_j$ and define

$$\omega(A) = \frac{1}{2\pi} \sum_{i<j} A_{ij} v_i \wedge v_j \in \Lambda^2(\mathbf{R}^k).$$

We let $v_1 \wedge \cdots \wedge v_k$ be the orientation class of $\mathbf{R}^k$ and define:

$$e(A) = (\omega(A)^{\bar{k}}, v_1 \wedge \cdots \wedge v_k)/k!$$

where $(\ ,\ )$ denotes the natural inner product on $\Lambda^k \mathbf{R}^k \simeq \mathbf{R}$. It is clear from the definition that $e(A)$ is invariant under the action of $\mathrm{SO}(k)$ since $\omega(A)$ is invariantly defined for skew-symmetric matrices A. It is clear that $e(A)$ is polynomial in the components of A. If we choose a block basis so that:

$$Av_1 = \lambda_1 v_2, \quad Av_2 = -\lambda_1 v_1, \quad Av_3 = \lambda_2 v_4, \quad Av_4 = -\lambda_2 v_3, \ldots$$

then we compute:

$$\begin{aligned}
&\omega(A) = \{-\lambda_1 v_1 \wedge v_2 - \lambda_2 v_3 \wedge v_4 \ldots\}/2\pi = x_1 v_1 \wedge v_2 + x_2 v_3 \wedge v_4 \cdots \\
&e(A) = x_1 x_2 \ldots
\end{aligned}$$

This new characteristic class is called the Euler class. While the Pontrjagin classes can be computed from the curvature of an arbitrary connection, the Euler class can only be computed from the curvature of a Riemannian connection. If Ω_{ij} are the matrix components of the curvature of V relative to some oriented orthomormal basis, then:

$$e(\Omega) = (-4\pi)^{-\bar{k}}/k! \sum_\rho \mathrm{sign}(\rho) \Omega_{\rho(1)\rho(2)} \ldots \Omega_{\rho(k-1)\rho(k)} \in \Lambda^k(M)$$

for $2\bar{k} = k = \dim(V)$. The sum ranges over all possible permutations ρ.

We define $e(V) = 0$ if $\dim(V)$ is odd. It is immediate that $e(V_1 \oplus V_2) = e(V_1)e(V_2)$ if we give the natural orientation and fiber metric to the direct sum and if we use the direct sum connection.

We illustrate this formula (and check that we have the correct normalizing constants) by studying the following simple example. Let $m = 2$ and let $M = S^2$ be the unit sphere. We calculate $e(TS^2)$. Parametrize M using spherical coordinates $f(u,v) = (\cos u \sin v, \sin u \sin v, \cos v)$ for $0 \le u \le 2\pi$ and $0 \le v \le \pi$. Define a local orthonormal frame for $T(\mathbf{R}^3)$ over S^2 by:

$$\begin{aligned}
e_1 &= (\sin v)^{-1} \partial/\partial u = (-\sin u, \cos u, 0) \\
e_2 &= \partial/\partial v = (\cos u \cos v, \sin u \cos v, -\sin v) \\
e_3 &= N = (\cos u \sin v, \sin u \sin v, \cos v).
\end{aligned}$$

The Euclidean connection $\tilde\nabla$ is easily computed to be:

$$\begin{aligned}
&\tilde\nabla_{e_1}e_1 = (\sin v)^{-1}(-\cos u, -\sin u, 0) = -(\cos v/\sin v)e_2 - e_3\\
&\tilde\nabla_{e_1}e_2 = (\cos v/\sin v)(-\sin u, \cos u, 0) = (\cos v/\sin v)e_1\\
&\tilde\nabla_{e_2}e_1 = 0\\
&\tilde\nabla_{e_2}e_2 = (-\cos u\sin v, -\sin u\sin v, -\cos v) = -e_3.
\end{aligned}$$

Covariant differentiation ∇ on the sphere is given by projecting back to $T(S^2)$ so that:

$$\begin{aligned}
&\nabla_{e_1}e_1 = (-\cos v/\sin v)e_2 \qquad &&\nabla_{e_1}e_2 = (\cos v/\sin v)e_1\\
&\nabla_{e_2}e_1 = 0 &&\nabla_{e_2}e_2 = 0
\end{aligned}$$

and the connection 1-form is given by:

$$\begin{aligned}
&\nabla e_1 = (-\cos v/\sin v)e^1\otimes e_2 \quad &&\text{so } \omega_{11} = 0 \text{ and } \omega_{12} = -(\cos v/\sin v)e^1\\
&\nabla e_2 = (\cos v/\sin v)e^1\otimes e_1 &&\text{so } \omega_{22} = 0 \text{ and } \omega_{21} = (\cos v/\sin v)e^1.
\end{aligned}$$

As $du = e^1/\sin v$ we compute $\omega_{12} = -\cos v\,du$ so $\Omega_{12} = \sin v\,dv\wedge du = -e^1\wedge e^2$ and $\Omega_{21} = e^1\wedge e^2$. From this we calculate that:

$$e(\Omega) = -\frac{1}{4\pi}(\Omega_{12} - \Omega_{21}) = \frac{e^1\wedge e^2}{2\pi}$$

and consequently $\int_{S^2} E_2 = \mathrm{vol}(S^2)/2\pi = 2 = \chi(S^2)$.

There is a natural relation between the Euler form and $c_{\bar k}$. Let V be a complex vector space of dimension $\bar k$ and let V_r be the underlying real vector space of dimention $2\bar k = k$. If V has a Hermitian inner product, then V_r inherits a natural real inner product. V_r also inherits a natural orientation from the complex structure on V. If $\{e_j\}$ is a unitary basis for V, then $\{e_1, ie_1, e_2, ie_2, \dots\}$ is an oriented orthonormal basis for V_r. Let A be a skew-Hermitian matrix on V. The restriction of A to V_r defines a skew-symmetric matrix A_r on V_r. We choose a basis $\{e_j\}$ for V so $Ae_j = i\lambda_j e_j$. Then:

$$x_j = -\lambda_j/2\pi \qquad \text{and} \qquad c_{\bar k}(A) = x_1\dots x_{\bar k}$$

defines the $\bar k^{\text{th}}$ Chern class. If $v_1 = e_1$, $v_2 = ie_1$, $v_3 = e_2$, $v_4 = ie_2, \dots$ then:

$$A_r v_1 = \lambda_1 v_2,\quad A_r v_2 = -\lambda_1 v_1,\quad A_r v_3 = \lambda_2 v_4,\quad A_r v_4 = -\lambda_2 v_3, \dots$$

so that

$$e(A_r) = c_{\bar k}(A).$$

We summarize these results as follows:

LEMMA 2.2.2. *Let $P(A)$ be a polynomial with $P(A) = P(gAg^{-1})$ for every skew symmetric A and every $g \in \mathrm{SO}(k)$.*
(a) If k is odd, then $P(A)$ is invariant under $\mathrm{O}(k)$ and is expressible in terms of Pontrjagin classes.
(b) If $k = 2\bar{k}$ is even, then we can decompose $P(A) = P_0(A) + e(A)P_1(A)$ where $P_i(A)$ are $\mathrm{O}(k)$ invariant and are expressible in terms of Pontrjagin classes.
(c) $e(A)$ is defined by:

$$e(A) = (-4\pi)^{-\bar{k}} \sum_{\rho} \operatorname{sign}(\rho) A_{\rho(1)\rho(2)} \dots A_{\rho(k-1)\rho(k)} / k!$$

This satisfies the identity $e(A)^2 = p_{\bar{k}}(A)$. (We define $e(A) = 0$ if k is odd).

This completely describes the characteristic ring. We emphasize the conclusions are only applicable to skew-symmetric real matrices A. We have proved that $e(A)$ has the following functorial properties:

LEMMA 2.2.3.
(a) If we take the direct sum connection and metric, then $e(V_1 \oplus V_2) = e(V_1)e(V_2)$.
(b) If we take the metric and connection of V_r obtained by forgetting the complex structure on a complex bundle V then $e(V_r) = c_{\bar{k}}(V)$.

This lemma establishes that the top dimensional Chern class $c_{\bar{k}}$ does not really depend on having a complex structure but only depends on having an orientation on the underlying real vector bundle. The choice of the sign in computing $\det(A)^{1/2}$ is, of course, motivated by this normalization.

2.3. Characteristic Classes of Complex Projective Space.

So far, we have only discussed covariant differentiation from the point of view of the total covariant derivative. At this stage, it is convenient to introduce covariant differentiation along a direction. Let $X \in T(M)$ be a tangent vector and let $s \in C^\infty(V)$ be a smooth section. We define $\nabla_X s \in C^\infty(V)$ by:

$$\nabla_X s = X \cdot \nabla s$$

where "$\cdot$" denotes the natural pairing from $T(M) \otimes T^*(M) \otimes V \to V$. Let $[X, Y] = XY - YX$ denote the Lie-bracket of vector fields. We define:

$$\Omega(X, Y) = \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]}$$

and compute that:

$$\Omega(fX, Y)s = \Omega(X, fY)s = \Omega(X, Y)fs = f\Omega(X, Y)s$$
$$\text{for} \quad f \in C^\infty(M),\ s \in C^\infty(V),\ \text{and } X, Y \in C^\infty(TM).$$

If $\{e_i\}$ is a local frame for $T(M)$, let $\{e^i\}$ be the dual frame for $T^*(M)$. It is immediate that:

$$\nabla s = \sum_i e^i \otimes \nabla_{e_i}(s)$$

so we can recover the total covariant derivative from the directional derivatives. We can also recover the curvature:

$$\Omega s = \sum_{i<j} e^i \wedge e^j \otimes \Omega(e_i, e_j)s.$$

If $V = T(M)$, there is a special connection called the Levi-Cività connection on $T(M)$. It is the unique Riemannian connection which is torsion free—i.e.,

$$(\nabla_X Y, Z) + (Y, \nabla_X Z) = X(Y, Z) \qquad \text{(Riemannian)}$$
$$\nabla_X Y - \nabla_Y X - [X, Y] = 0 \qquad \text{(Torsion free).}$$

Let $x = (x_1, \ldots, x_m)$ be a system of local coordinates on M and let $\{\partial/\partial x_i\}$ be the coordinate frame for $T(M)$. The metric tensor is given by:

$$ds^2 = \sum_{i,j} g_{ij}\, dx^i \otimes dx^j \qquad \text{for} \quad g_{ij} = (\partial/\partial x_i, \partial/\partial x_j).$$

We introduce the Christoffel symbols Γ_{ij}^k and Γ_{ijk} by:

$$\nabla_{\partial/\partial x_i} \partial/\partial x_j = \sum_k \Gamma_{ij}^k \partial/\partial x_k$$
$$(\nabla_{\partial/\partial x_i} \partial/\partial x_j, \partial/\partial x_k) = \Gamma_{ijk}.$$

They are related by the formula:

$$\Gamma_{ijk} = \sum_l \Gamma_{ij}^l g_{lk}, \qquad \Gamma_{ij}^k = \sum_l \Gamma_{ijl} g^{lk}$$

where $g^{lk} = (dx_l, dx_k)$ is the inverse of the matrix g_{ij}. It is not difficult to compute that:

$$\Gamma_{ijk} = \tfrac{1}{2}\{g_{jk/i} + g_{ik/j} - g_{ij/k}\}$$

where we use the notation "/" to denote (multiple) partial differentiation.

The complete curvature tensor is defined by:

$$\Omega(\partial/\partial x_i, \partial/\partial x_j)\partial/\partial x_k = \sum_l R_{ijk}{}^l \partial/\partial x_l$$

or equivalently if we lower indices:

$$(\Omega(\partial/\partial x_i, \partial/\partial x_j)\partial/\partial x_k, \partial/\partial x_l) = R_{ijkl}.$$

The expression of the curvature tensor in terms of the derivatives of the metric is very complicated in general. By making a linear change of coordinates we can always normalize the metric so $g_{ij}(x_0) = \delta_{ij}$ is the Kronecker symbol. Similarly, by making a quadratic change of coordinates, we can further normalize the metric so $g_{ij/k}(x_0) = 0$. Relative to such a choice of coordinates:

$$\Gamma_{ij}^k(x_0) = \Gamma_{ijk}(x_0) = 0$$

and

$$R_{ijkl}(x_0) = R_{ijk}{}^l(x_0) = \tfrac{1}{2}\{g_{jl/ik} + g_{ik/lj} - g_{jk/il} - g_{il/jk}\}(x_0).$$

At any other point, of course, the curvature tensor is not as simply expressed. In general it is linear in the second derivatives of the metric, quadratic in the first derivatives of the metric with coefficients which depend smoothly on the g_{ij}'s.

We choose a local orthonormal frame $\{e_i\}$ for $T^*(M)$. Let $m = 2\bar{m}$ be even and let $\text{orn} = * \cdot 1 = \text{dvol}$ be the oriented volume. Let

$$e = E_m(g)\,\text{orn}$$

be the Euler form. If we change the choice of the local orientation, then e and orn both change signs so $E_m(g)$ is scalar invariant of the metric. In terms of the curvature, if $2\bar{m} = m$, then:

$$c(m) = (-1)^{\bar{m}}(8\pi)^{-\bar{m}}\frac{1}{(\bar{m})!}$$

$$E_m = c(m)\sum_{\rho,\tau} \operatorname{sign}(\rho)\operatorname{sign}(\tau) R_{\rho(1)\rho(2)\tau(1)\tau(2)} \ldots R_{\rho(m-1)\rho(m)\tau(m-1)\tau(m)}$$

where the sum ranges over all permutations ρ, τ of the integers 1 through m. For example:

$$\begin{aligned} E_2 &= -(2\pi)^{-1} R_{1212} \\ E_4 &= (2\pi)^{-2} \sum_{i,j,k,l} \{R_{ijij}R_{klkl} + R_{ijkl}R_{ijkl} - 4R_{ijik}R_{ljlk}\}/8. \end{aligned}$$

Let dvol be the Riemannian *measure* on M. If M is oriented, then

$$\int_M E_m \,\mathrm{dvol} = \int_M e$$

is independent of the orientation of M and of the metric. If M is not orientable, we pass to the double cover to see $\int_M E_m \,\mathrm{dvol}$ is a topological invariant of the manifold M. We shall prove later this integral is the Euler characteristic $\chi(M)$ but for the moment simply note it is not dependent upon a choice of orientation of M and is in fact defined even if M is not orientable.

It is worth computing an example. We let S^2 be the unit sphere in $\mathbf{R}^3$. Since this is homogeneous, E_2 is constant on S^2. We compute E_2 at the north pole $(0,0,1)$ and parametrize S^2 by $(u, v, (1-u^2-v^2)^{1/2})$. Then

$$\begin{aligned} \partial/\partial u &= (1, 0, -u/(1-u^2-v^2)^{1/2}), \quad \partial/\partial v = (0, 1, -v/(1-u^2-v^2)^{1/2}) \\ g_{11} &= 1 + u^2/(1-u^2-v^2) \\ g_{22} &= 1 + v^2/(1-u^2-v^2) \\ g_{12} &= uv/(1-u^2-v^2). \end{aligned}$$

It is clear $g_{ij}(0) = \delta_{ij}$ and $g_{ij/k}(0) = 0$. Therefore at $u = v = 0$,

$$\begin{aligned} E_2 &= -(2\pi)^{-1} R_{1212} = -(2\pi)^{-1}\{g_{11/22} + g_{22/11} - 2g_{12/12}\}/2 \\ &= -(2\pi)^{-1}\{0 + 0 - 2\}/2 = (2\pi)^{-1} \end{aligned}$$

so that

$$\int_{S^2} E_2 \,\mathrm{dvol} = (4\pi)(2\pi)^{-1} = 2 = \chi(S^2).$$

More generally, we can let $M = S^2 \times \cdots \times S^2$ where we have $\bar{m}$ factors and $2\bar{m} = m$. Since $e(V_1 \oplus V_2) = e(V_1)e(V_2)$, when we take the product metric we have $E_m = (E_2)^{\bar{m}} = (2\pi)^{-\bar{m}}$ for this example. Therefore:

$$\int_{S^2 \times \cdots \times S^2} E_m \,\mathrm{dvol} = 2^{\bar{m}} = \chi(S^2 \times \cdots \times S^2).$$

We will use this example later in this chapter to prove $\int_M E_m = \chi(M)$ in general. The natural examples for studying the Euler class are products of two dimensional spheres. Unfortunately, the Pontrjagin classes vanish identically on products of spheres so we must find other examples.

It is convenient at this point to discuss the holomorphic category. A manifold M of real dimension $m = 2\bar{m}$ is said to be holomorphic if we have local coordinate charts $z = (z_1, \dots, z_{\bar{m}})\colon M \to \mathbf{C}^{\bar{m}}$ such that two charts are related by holomorphic changes of coordinates. We expand $z_j = x_j + iy_j$ and define:

$$\begin{aligned} \partial/\partial z_j &= \tfrac{1}{2}(\partial/\partial x_j - i\partial/\partial y_j), & \partial/\partial \bar{z}_j &= \tfrac{1}{2}(\partial/\partial x_j + i\partial/\partial y_j) \\ dz_j &= dx_j + i\,dy_j, & d\bar{z}_j &= dx_j + i\,dy_j. \end{aligned}$$

We complexify $T(M)$ and $T^*(M)$ and define:

$$\begin{aligned} T_c(M) &= \operatorname{span}\{\partial/\partial z_j\}, \\ \Lambda^{1,0}(M) &= \operatorname{span}\{dz_j\}, \\ \Lambda^{0,1}(M) &= \operatorname{span}\{d\bar{z}_j\}. \end{aligned}$$

Then

$$\begin{aligned} \Lambda^1(M) &= \Lambda^{1,0}(M) \oplus \Lambda^{0,1}(M) \\ T_c(M)^* &= \Lambda^{1,0}(M). \end{aligned}$$

As complex bundles, $T_c(M) \simeq \Lambda^{0,1}(M)$. Define:

$$\begin{aligned} \partial(f) &= \sum_j \partial f/\partial z_j\, dz^j \colon C^\infty(M) \to C^\infty(\Lambda^{1,0}(M)) \\ \bar{\partial}(f) &= \sum_j \partial f/\partial \bar{z}_j\, d\bar{z}^j \colon C^\infty(M) \to C^\infty(\Lambda^{0,1}(M)) \end{aligned}$$

then the Cauchy-Riemann equations show f is holomorphic if and only if $\bar{\partial} f = 0$.

This decomposes $d = \partial + \bar{\partial}$ on functions. More generally, we define:

$$\begin{aligned} \Lambda^{p,q} &= \operatorname{span}\{dz_{i_1} \wedge \cdots \wedge dz_{i_p} \wedge d\bar{z}_{j_1} \wedge \cdots \wedge d\bar{z}_{j_q}\} \\ \Lambda^n &= \bigoplus_{p+q=n} \Lambda^{p,q}. \end{aligned}$$

This spanning set of $\Lambda^{p,q}$ is closed. We decompose $d = \partial + \bar{\partial}$ where

$$\partial\colon C^\infty(\Lambda^{p,q}) \to C^\infty(\Lambda^{p+1,q}) \qquad \text{and} \qquad \bar{\partial}\colon C^\infty(\Lambda^{p,q}) \to C^\infty(\Lambda^{p,q+1})$$

so that $\partial\partial = \bar{\partial}\bar{\partial} = \bar{\partial}\partial + \partial\bar{\partial} = 0$. These operators and bundles are all invariantly defined independent of the particular holomorphic coordinate system chosen.

A bundle V over M is said to be holomorphic if we can cover M by holomorphic charts U_α and find frames s_α over each U_α so that on $U_\alpha \cap U_\beta$ the transition functions $s_\alpha = f_{\alpha\beta} s_\beta$ define holomorphic maps to $\mathrm{GL}(n, \mathbf{C})$. For such a bundle, we say a local section s is holomorphic if $s = \sum_\nu a_\alpha^\nu s_\alpha^\nu$ for holomorphic functions a_α^ν. For example, $T_c(M)$ and $\Lambda^{p,0}(M)$ are holomorphic bundles over M; $\Lambda^{0,1}$ is anti-holomorphic.

If V is holomorphic, we use a partition of unity to construct a Hermitian fiber metric h on V. Let $h_\alpha = (s_\alpha, s_\alpha)$ define the metric locally; this is a positive definite symmetric matrix which satisfies the transition rule $h_\alpha = f_{\alpha\beta} h_\beta \bar{f}_{\alpha\beta}$. We define a connection 1-form locally by:

$$\omega_\alpha = \partial h_\alpha h_\alpha^{-1}$$

and compute the transition rule:

$$\omega_\alpha = \partial\{f_{\alpha\beta} h_\beta \bar{f}_{\alpha\beta}\} \bar{f}_{\alpha\beta}^{-1} h_\beta^{-1} f_{\alpha\beta}^{-1}.$$

Since $f_{\alpha\beta}$ is holomorphic, $\bar\partial f_{\alpha\beta} = 0$. This implies $df_{\alpha\beta} = \partial f_{\alpha\beta}$ and $\partial \bar{f}_{\alpha\beta} = 0$ so that:

$$\omega_\alpha = \partial f_{\alpha\beta} f_{\alpha\beta}^{-1} + f_{\alpha\beta} \partial h_\beta h_\beta^{-1} f_{\alpha\beta}^{-1} = df_{\alpha\beta} f_{\alpha\beta}^{-1} + f_{\alpha\beta} \omega_\beta f_{\alpha\beta}^{-1}.$$

Since this is the transition rule for a connection, the $\{\omega_\alpha\}$ patch together to define a connection ∇_h.

It is immediate from the definition that:

$$(\nabla_h s_\alpha, s_\alpha) + (s_\alpha, \nabla_h s_\alpha) = \omega_\alpha h_\alpha + h_\alpha \omega_\alpha^* = \partial h_\alpha + \bar\partial h_\alpha = dh_\alpha$$

so ∇_h is a unitary connection on V. Since $\nabla_h s_\alpha \in C^\infty(\Lambda^{1,0} \otimes V)$ we conclude ∇_h vanishes on holomorphic sections when differentiated in anti-holomorphic directions (i.e., ∇_h is a holomorphic connection). It is easily verified that these two properties determine ∇_h.

We shall be particularly interested in holomorphic line bundles. If L is a line bundle, then h_α is a positive function on U_α with $h_\alpha = |f_{\alpha\beta}|^2 h_\beta$. The curvature in this case is a 2-form defined by:

$$\Omega_\alpha = d\omega_\alpha - \omega_\alpha \wedge \omega_\alpha = d(\partial h_\alpha h_\alpha^{-1}) = \bar\partial \partial \log(h_\alpha) = -\partial\bar\partial \log(h_\alpha).$$

Therefore:

$$c_1(L) = \frac{1}{2\pi i} \partial\bar\partial \log(h)$$

is independent of the holomorphic frame chosen for evaluation. $c_1(L) \in C^\infty(\Lambda^{1,1}(M))$ and $dc_1 = \partial c_1 = \bar\partial c_1 = 0$ so $c_1(L)$ is closed in all possible senses.

Let $\mathbf{C}P_n$ be the set of all lines through 0 in $\mathbf{C}^{n+1}$. Let $\mathbf{C}^* = \mathbf{C} - 0$ act on $\mathbf{C}^{n+1} - 0$ by complex multiplication, then $\mathbf{C}P_n = (\mathbf{C}^{n+1} - 0)/\mathbf{C}^*$ and we give $\mathbf{C}P_n$ the quotient topology. Let L be the tautological line bundle over $\mathbf{C}P_n$:

$$L = \{\, (x, z) \in \mathbf{C}P_n \times \mathbf{C}^{n+1} : z \in x \,\}.$$

Let L^* be the dual bundle; this is called the hyperplane bundle.

Let $(z_0, \ldots, z_n)$ be the usual coordinates on $\mathbf{C}^{n+1}$. Let $U_j = \{\, z : z_j \neq 0 \,\}$. Since U_j is $\mathbf{C}^*$ invariant, it projects to define an open set on $\mathbf{C}P_n$ we again shall denote by U_j. We define

$$z_k^j = z_k / z_j$$

over U_j. Since these functions are invariant under the action of $\mathbf{C}^*$, they extend to continuous functions on U_j. We let $z^j = (z_0^j, \ldots, \widehat{z_j^j}, \ldots, z_n^j)$ where we have deleted $z_j^j = 1$. This gives local coordinates on U_j in $\mathbf{C}P_n$. The transition relations are:

$$z_\nu^j = (z_j^k)^{-1} z_\nu^k$$

which are holomorphic so $\mathbf{C}P_n$ is a holomorphic manifold.

We let $s^j = (z_0^j, \ldots, z_n^j)$ be a section to L over U_j. Then $s^j = (z_j^k)^{-1} s^k$ transform holomorphically so L is a holomorphic line bundle over $\mathbf{C}P_n$. The coordinates z_j on $\mathbf{C}^{n+1}$ give linear functions on L and represent global holomorphic sections to the dual bundle L^*. There is a natural inner product on the trivial bundle $\mathbf{C}P_n \times \mathbf{C}^{n+1}$ which defines a fiber metric on L. We define:

$$x = -c_1(L) = \frac{i}{2\pi} \partial \bar{\partial} \log(1 + |z^j|^2)$$

then this is a closed 2-form over $\mathbf{C}P_n$.

$\mathrm{U}(n+1)$ acts on $\mathbf{C}^{n+1}$; this action induces a natural action on $\mathbf{C}P_n$ and on L. Since the metric on L arises from the invariant metric on $\mathbf{C}^{n+1}$, the 2-form x is invariant under the action of $\mathrm{U}(n+1)$.

LEMMA 2.3.1. *Let $x = -c_1(L)$ over $\mathbf{C}P_n$. Then:*
(a) $\int_{\mathbf{C}P_n} x^n = 1$.
(b) $H^*(\mathbf{C}P_n; \mathbf{C})$ is a polynomial ring with additive generators $\{1, x, \ldots, x^n\}$.
(c) If $i \colon \mathbf{C}P_{n-1} \to \mathbf{C}P_n$ is the natural inclusion map, then $i^*(x) = x$.

PROOF: (c) is immediate from the naturality of the constructions involved. Standard methods of algebraic topology give the additive structure of $H^*(\mathbf{C}P_n; \mathbf{C}) = \mathbf{C} \oplus 0 \oplus \mathbf{C} \oplus 0 \cdots \oplus \mathbf{C}$. Since $x \in H^2(\mathbf{C}P_n; \mathbf{C})$ satisfies $x^n \neq 0$, (a) will complete the proof of (b). We fix a coordinate chart U_n. Since $\mathbf{C}P_n - U_n = \mathbf{C}P_{n-1}$, it has measure zero. It suffices to check that:

$$\int_{U_n} x^n = 1.$$

We identify $z \in U_n = \mathbf{C}^n$ with $(z,1) \in \mathbf{C}^{n+1}$. We imbed $\mathrm{U}(n)$ in $\mathrm{U}(n+1)$ as the isotropy group of the vector $(0,\dots,0,1)$. Then $\mathrm{U}(n)$ acts on $(z,1)$ in $\mathbf{C}^{n+1}$ exactly the same way as $\mathrm{U}(n)$ acts on z in $\mathbf{C}^n$. Let $\mathrm{dvol} = dx_1 \wedge dy_1 \wedge \cdots \wedge dx_n \wedge dy_n$ be ordinary Lebesgue measure on $\mathbf{C}^n$ without any normalizing constants of 2π. We parametrize $\mathbf{C}^n$ in the form (r,θ) for $0 \le r < \theta$ and $\theta \in S^{2n-1}$. We express $x^n = f(r,\theta)\,\mathrm{dvol}$. Since x is invariant under the action of $\mathrm{U}(n)$, $f(r,\theta) = f(r)$ is spherically symmetric and does not depend on the parameter θ.

We compute that:

$$\begin{aligned} x &= -\frac{1}{2\pi i}\partial\bar\partial \log\Big(1 + \sum z_j \bar z_j\Big) = -\frac{1}{2\pi i}\partial\Big(\sum_j z_j\, d\bar z_j/(1+|z|^2)\Big) \\ &= -\frac{1}{2\pi i}\Big\{\sum_j dz_j \wedge d\bar z_j/(1+r^2) - \sum_{j,k} z_j \bar z_k\, dz_k \wedge d\bar z_j/(1+r^2)^2\Big\}. \end{aligned}$$

We evaluate at the point $z = (r,0,\dots,0)$ to compute:

$$\begin{aligned} x &= -\frac{1}{2\pi i}\Big\{\sum_j dz_j\, d\bar z_j/(1+r^2) - r^2\, dz_1 \wedge d\bar z_1/(1+r^2)^2\Big\} \\ &= -\frac{1}{2\pi i}\Big\{dz_1 \wedge d\bar z_1/(1+r^2)^2 + \sum_{j>1} dz_j \wedge d\bar z_j/(1+r^2)\Big\}. \end{aligned}$$

Consequently at this point,

$$\begin{aligned} x^n &= \Big(-\frac{1}{2\pi i}\Big)^n n!(1+r^2)^{-n-1}\, dz_1 \wedge d\bar z_1 \wedge \cdots \wedge dz_n \wedge d\bar z_n \\ &= \pi^{-n} n!(1+r^2)^{-n-1}\, dx_1 \wedge dy_1 \wedge \cdots \wedge dx_n \wedge dy_n \\ &= \pi^{-n} n!(1+r^2)^{-n-1}\, \mathrm{dvol}\,. \end{aligned}$$

We use the spherical symmetry to conclude this identitiy holds for all z.

We integrate over $U_n = \mathbf{C}^n$ and use spherical coordinates:

$$\begin{aligned} \int x^n &= n!\,\pi^{-n}\int (1+r^2)^{-n-1}\,\mathrm{dvol} = n!\,\pi^{-n}\int (1+r^2)^{-n-1} r^{2n-1}\, dr\, d\theta \\ &= n!\,\pi^{-n}\,\mathrm{vol}(S^{2n-1})\int_0^\infty (1+r^2)^{-n-1} r^{2n-1}\, dr \\ &= \frac{1}{2} n!\,\pi^{-n}\,\mathrm{vol}(S^{2n-1})\int_0^\infty (1+t)^{-n-1} t^{n-1}\, dt. \end{aligned}$$

We compute the volume of S^{2n-1} using the identity $\sqrt{\pi} = \int_{-\infty}^{\infty} e^{-t^2}\, dt$. Thus:

$$\begin{aligned} \pi^n &= \int e^{-r^2}\,\mathrm{dvol} = \mathrm{vol}(S^{2n-1})\int_0^\infty r^{2n-1} e^{-r^2}\, dr \\ &= \frac{1}{2}\,\mathrm{vol}(S^{2n-1})\int_0^\infty t^{n-1} e^{-t}\, dt = \frac{(n-1)!}{2}\,\mathrm{vol}(S^{2n-1}). \end{aligned}$$

We solve this for $\mathrm{vol}(S^{2n-1})$ and substitute to compute:

$$\begin{aligned}\int x^n &= n\int_0^\infty (1+t)^{-n-1}t^{n-1}\,dt = (n-1)\int_0^\infty (1+t)^{-n}t^{n-2}\,dt = \cdots\\ &= \int_0^\infty (1+t)^{-2}\,dt = 1\end{aligned}$$

which completes the proof.

x can also be used to define a $\mathrm{U}(n+1)$ invariant metric on $\mathbf{C}P_n$ called the Fubini-Study metric. We shall discuss this in more detail in Chapter 3 as this gives a Kaehler metric for $\mathbf{C}P_n$.

There is a relation between L and $\Lambda^{1,0}(\mathbf{C}P_n)$ which it will be convenient to exploit:

Lemma 2.3.2. *Let $M = \mathbf{C}P_n$.*
(a) There is a short exact sequence of holomorphic bundles:

$$0 \to \Lambda^{1,0}(M) \to L \otimes 1^{n+1} \to 1 \to 0.$$

(b) There is a natural isomorphism of complex bundles:

$$T_c(M) \oplus 1 \simeq L^* \otimes 1^{n+1} = \underbrace{L^* \oplus \cdots \oplus L^*}_{n+1 \text{ times}}.$$

Proof: Rather than attempting to give a geometric proof of this fact, we give a combinatorial argument. Over U_k we have functions z_i^k which give coordinates $0 \le i \le n$ for $i \ne k$. Furthermore, we have a section s^k to L. We let $\{s_i^k\}_{i=0}^n$ give a frame for $L \otimes 1^{n+1} = L \oplus \cdots \oplus L$. We define $F\colon \Lambda^{1,0}(M) \to L \otimes 1^{n+1}$ on U_k by:

$$F(dz_i^k) = s_i^k - z_i^k s_k^k.$$

We note that $z_k^k = 1$ and $F(dz_k^k) = 0$ so this is well defined on U_k. On the overlap, we have the relations $z_i^k = (z_k^j)^{-1} z_i^j$ and:

$$s_i^k = (z_k^j)^{-1} s_i^j \qquad \text{and} \qquad dz_i^k = (z_k^j)^{-1}\,dz_i^j - (z_k^j)^{-2} z_i^j\,dz_k^j.$$

Thus if we compute in the coordinate system U_j we have:

$$\begin{aligned}F(dz_i^k) &= (z_k^j)^{-1}(s_i^j - z_i^j s_j^j) - (z_k^j)^{-2} z_i^j (s_k^j - z_k^j s_j^j)\\ &= s_i^k - z_i^j s_j^k - z_i^k s_k^k + z_i^j s_j^k = s_i^k - z_i^k s_k^k\end{aligned}$$

which agrees with the definition of F on the coordinate system U_k. Thus F is invariantly defined. It is clear F is holomorphic and injective. We let

$\nu = L \otimes 1^{n+1} / \operatorname{image}(F)$, and let π be the natural projection. It is clear that s_k^k is never in the image of F so $\pi s_k^k \neq 0$. Since

$$F(dz_j^k) = s_j^k - z_j^k s_k^k = z_j^k (s_j^j - s_k^k)$$

we conclude that $\pi s_j^j = \pi s_k^k$ so $s = \pi s_k^k$ is a globally defined non-zero section to ν. This completes the proof of (a). We dualize to get a short exact sequence:

$$0 \to 1 \to L^* \otimes 1^{n+1} \to T_c(M) \to 0.$$

These three bundles have natural fiber metrics. Any short exact sequence of vector bundles splits (although the splitting is not holomorphic) and this proves (b).

From assertion (b) it follows that the Chern class of $T_c(\mathbf{C}P_n)$ is given by:

$$c(T_c) = c(T_c \oplus 1) = c(L^* \oplus \cdots \oplus L^*) = c(L^*)^{n+1} = (1 + x)^{n+1}.$$

For example, if $m = 1$, then $c(T_c) = (1 + x)^2 = 1 + 2x$. In this case, $c_1(T_c) = e(T(S^2))$ and we computed $\int_{S^2} e(T(S)) = 2$ so $\int_{S^2} x = 1$ which checks with Lemma 2.3.1.

If we forget the complex structure on $T_c(M)$ when M is holomorphic, then we obtain the real tangent space $T(M)$. Consequently:

$$T(M) \otimes \mathbf{C} = T_c \oplus T_c^*$$

and

$$\begin{aligned} c(T(\mathbf{C}P_n) \otimes \mathbf{C}) &= c(T_c(\mathbf{C}P_n) \oplus T_c^*(\mathbf{C}P_n)) \\ &= (1 + x)^{n+1}(1 - x)^{n+1} = (1 - x^2)^{n+1}. \end{aligned}$$

When we take into account the sign changes involved in defining the total Pontrjagin form, we conclude:

Lemma 2.3.3.
(a) If $M = S^2 \times \cdots \times S^2$ has dimension $2n$, then $\int_M e(T(M)) = \chi(M) = 2^n$.
(b) If $M = \mathbf{C}P_n$ has dimension $2n$, then

$$c(T_c(M)) = (1 + x)^{n+1} \qquad \text{and} \qquad p(T(M)) = (1 + x^2)^{n+1}.$$

$x \in H^2(\mathbf{C}P_n; \mathbf{C})$ is the generator given by $x = c_1(L^*) = -c_1(L)$, L is the tautological line bundle over $\mathbf{C}P_n$, and L^* is the dual, the hyperplane bundle.

The projective spaces form a dual basis to both the real and complex characteristic classes. Let ρ be a partition of the positive integer k in the form $k = i_1 + \cdots + i_j$ where we choose the notation so $i_1 \ge i_2 \ge \cdots$. We let $\pi(k)$ denote the number of such partitions. For example, if $k = 4$ then $\pi(4) = 5$ and the possible partitions are:

$$4 = 4, \quad 4 = 3 + 1, \quad 4 = 2 + 2, \quad 4 = 2 + 1 + 1, \quad 4 = 1 + 1 + 1 + 1.$$

We define classifying manifolds:

$$M_\rho^c = \mathbf{C}P_{i_1} \times \cdots \times \mathbf{C}P_{i_j} \qquad \text{and} \qquad M_\rho^r = \mathbf{C}P_{2i_1} \times \cdots \times \mathbf{C}P_{2i_j}$$

to be real manifolds of dimension $2k$ and $4k$.

Lemma 2.3.4. *Let k be a positive integer. Then:*
(a) Let constants $c(\rho)$ be given. There exists a unique polynomial $P(A)$ of degree k in the components of a $k \times k$ complex matrix which is $\mathrm{GL}(k, \mathbf{C})$ *invariant such that the characteristic class defined by P satisfies:*

$$\int_{M_\rho^c} P(T_c(M_\rho^c)) = c(\rho)$$

for every such partition ρ of k.
(b) Let constants $c(\rho)$ be given. There exists a polynomial $P(A)$ of degree $2k$ in the components of a $2k \times 2k$ real matrix which is $\mathrm{GL}(2k, \mathbf{R})$ *invariant such that the characteristic class defined by P satisfies:*

$$\int_{M_\rho^r} P(T(M_\rho^r)) = c(\rho)$$

for every such partition ρ of k. If P' is another such polynomial, then $P(A) = P'(A)$ for every skew-symmetric matrix A.

In other words, the real and complex characteristic classes are completely determined by their values on the appropriate classifying manifolds.

Proof: We prove (a) first. Let $\mathcal{P}_k$ denote the set of all such polynomials $P(A)$. We define $c_\rho = c_{i_1} \ldots c_{i_j} \in \mathcal{P}_k$, then by Lemma 2.1.3 the $\{c_\rho\}$ form a basis for $\mathcal{P}_k$ so $\dim(\mathcal{P}_k) = \pi(k)$. The $\{c_\rho\}$ are not a very convenient basis to work with. We will define instead:

$$H_\rho = ch_{i_1} \ldots ch_{i_j} \in \mathcal{P}_k$$

and show that the matrix:

$$a(\rho, \tau) = \int_{M_\tau^c} H_\rho(T_c(M_\tau^c))$$

is a non-singular matrix. This will prove the H_ρ also form a basis for $\mathcal{P}_k$ and that the M_τ^c are a dual basis. This will complete the proof.

The advantage of working with the Chern character rather than with the Chern class is that:

$$ch_i(T_c(M_1 \times M_2)) = ch_i(T_cM_1 \oplus T_cM_2) = ch_i(T_cM_1) + ch_i(T_cM_2).$$

Furthermore, $ch_i(T_cM) = 0$ if $2i > \dim(M)$. We define the length $\ell(\rho) = j$ to be the number of elements in the partition ρ. Then the above remarks imply:

$$a(\rho, \tau) = 0 \qquad \text{if } \ell(\tau) > \ell(\rho).$$

Furthermore, if $\ell(\tau) = \ell(\rho)$, then $a(\rho, \tau) = 0$ unless $\tau = \rho$. We define the partial order $\tau > \rho$ if $\ell(\tau) > \ell(\rho)$ and extend this to a total order. Then $a(\rho, \tau)$ is a triangular matrix. To show it is invertible, it suffices to show the diagonal elements are non-zero.

We first consider the case in which $\rho = \tau = k$. Using the identity $T_c(\mathbf{C}P_k) \oplus 1 = (L^* \oplus \cdots \oplus L^*)$ $(k+1$ times), it is clear that $ch_k(T_c\mathbf{C}P_k) = (k+1)ch_k(L^*)$. For a line bundle, $ch_k(L^*) = c_1(L^*)^k/k!$. If $x = c_1(L^*)$ is the generator of $H^2(\mathbf{C}P_k; \mathbf{C})$, then $ch_k(T_c\mathbf{C}P_k) = (k+1)x^k/k!$ which does not integrate to zero. If $\rho = \tau = \{i_1, \dots, i_j\}$ then:

$$a(\rho, \rho) = c \prod_{\nu=1}^{\ell(\rho)} a(i_\nu, i_\nu)$$

where c is a positive constant related to the multiplicity with which the i_ν appear. This completes the proof of (a).

To prove (b) we replace M_ρ^c by $M_{2\rho}^c = M_\rho^r$ and H_ρ by $H_{2\rho}$. We compute

$$\begin{aligned} ch_{2i}(T(M)) \overset{\text{def}}{=} ch_{2i}(T(M) \otimes \mathbf{C}) &= ch_{2i}(T_cM \oplus T_c^*M) \\ &= ch_{2i}(T_cM) + ch_{2i}(T_c^*M) \\ &= 2ch_{2i}(T_cM). \end{aligned}$$

Using this fact, the remainder of the proof of (b) is immediate from the calculations performed in (a) and this completes the proof.

The Todd class and the Hirzebruch L-polynomial were defined using generating functions. The generating functions were chosen so that they would be particularly simple on the classifying examples:

Lemma 2.3.5.
(a) Let $x_j = -\lambda_j/2\pi i$ be the normalized eigenvalues of a complex matrix A. We define $Td(A) = Td(x) = \prod_j x_j/(1 - e^{-x_j})$ as the Todd class. Then:

$$\int_{M_\rho^c} Td(T_c(M_\rho^c)) = 1 \qquad \text{for all } \rho.$$

(b) Let $x_j = \lambda_j / 2\pi$ where the eigenvalues of the skew-symmetric real matrix A are $\{\pm\lambda_1, \ldots\}$. We define $L(A) = L(x) = \prod_j x_j / \tanh x_j$ as the Hirzebruch L-polynomial. Then $\int_{M_\rho^r} L(T(M^r)) = 1$ for all ρ.

We will use this calculation to prove the integral of the Todd class gives the arithmetic genus of a complex manifold and that the integral of the Hirzebruch L-polynomial gives the signature of an oriented real manifold. In each case, we only integrate the part of the total class which is of the same degree as the dimension of the manifold.

Proof: Td is a multiplicative class:

$$Td(T_c(M_1 \times M_2)) = Td(T_c(M_1) \oplus T_c(M_2)) = Td(T_c(M_1))\, Td(T_c(M_2)).$$

Similarly the Hirzebruch polynomial is multiplicative. This shows it suffices to prove Lemma 2.3.5 in the case $\rho = k$ so $M = \mathbf{C}P_k$ or $\mathbf{C}P_{2k}$.

We use the decomposition $T_c(\mathbf{C}P_k) \oplus 1 = L^* \oplus \cdots \oplus L^*$ $(k+1 \text{ times})$ to compute $Td(T_c(\mathbf{C}P_k)) = [\,Td(x)]^{k+1}$ where $x = c_1(L^*)$ is the generator of $H^2(\mathbf{C}P_k; \mathbf{R})$. Since x^k integrates to 1, it suffices to show the coefficient of x^k in $Td(x)^{k+1}$ is 1 or equivalently to show:

$$\operatorname{Res}_{x=0} x^{-k-1}[\,Td(x)^{k+1}] = \operatorname{Res}_{x=0}(1 - e^{-x})^{-k-1} = 1.$$

If $k = 0$, then:

$$(1 - e^{-x})^{-1} = \left(x - \frac{1}{2}x^2 + \cdots\right)^{-1} = x^{-1}\left(1 + \frac{1}{2}x + \cdots\right)$$

and the result follows. Similarly, if $k = 1$

$$(1 - e^{-x})^{-2} = x^{-2}(1 + x + \cdots)$$

and the result follows. For larger values of k, proving this directly would be a combinatorial nightmare so we use instead a standard trick from complex variables. If $g(x)$ is any meromorphic function, then $\operatorname{Res}_{x=0} g'(x) = 0$. We apply this to the function $g(x) = (1 - e^{-x})^{-k}$ for $k \geq 1$ to conclude:

$$\operatorname{Res}_{x=0}(1 - e^{-x})^{-k-1} e^{-x} = 0.$$

This implies immediately that:

$$\begin{aligned} \operatorname{Res}_{x=0}(1 - e^{-x})^{-k-1} &= \operatorname{Res}_{x=0}(1 - e^{-x})^{-k-1}(1 - e^{-x}) \\ &= \operatorname{Res}_{x=0}(1 - e^{-x})^{-k} = 1 \end{aligned}$$

by induction which completes the proof of assertion (a).

We now assume k is even and study $\mathbf{C}P_k$. Again, using the decomposition $T_c(\mathbf{C}P_k) \oplus 1 = L^* \oplus \cdots \oplus L^*$ it follows that

$$L(T(\mathbf{C}P_k)) = [L(x)]^{k+1} = \frac{x^{k+1}}{\tanh^{k+1} x}.$$

Since we are interested in the coefficient of x^k, we must show:

$$\mathrm{Res}_{x=0} \tanh^{-k-1} x = 1 \qquad \text{if } k \text{ is even}$$

or equivalently

$$\mathrm{Res}_{x=0} \tanh^{-k} x = 1 \qquad \text{if } k \text{ is odd.}$$

We recall that

$$\begin{aligned} \tanh x &= \frac{e^x - e^{-x}}{e^x + e^{-x}} \\ \tanh^{-1} x &= \frac{2 + x^2 + \cdots}{2x + \cdots} \end{aligned}$$

so the result is clear if $k = 1$. We now proceed by induction. We differentiate $\tanh^{-k} x$ to compute:

$$(\tanh^{-k} x)' = -k(\tanh^{-k-1} x)(1 - \tanh^2 x).$$

This implies

$$\mathrm{Res}_{x=0} \tanh^{-k+1} x = \mathrm{Res}_{x=0} \tanh^{-k-1} x$$

for any integer k. Consequently $\mathrm{Res}_{x=0} \tanh^{-k} x = 1$ for any odd integer k since these residues are periodic modulo 2. (The residue at k even is, of course, zero). This completes the proof.

2.4. The Gauss-Bonnet Theorem.

Let P be a self-adjoint elliptic partial differential operator of order $d > 0$. If the leading symbol of P is positive definite, we derived an asymptotic expansion for $\operatorname{Tr}\{e^{-tP}\}$ in section 1.7. This is too general a setting in which to work so we shall restrict attention henceforth to operators with leading symbol given by the metric tensor, as this is the natural category in which to work.

Let $P: C^\infty(V) \to C^\infty(V)$ be a second order operator. We choose a local frame for V. Let $x = (x_1, \dots, x_m)$ be a system of local coordinates. Let $ds^2 = g_{ij}\, dx^i\, dx^j$ be the metric tensor and let g^{ij} denote the inverse matrix; $dx^i \cdot dx^j = g^{ij}$ is the metric on the dual space $T^*(M)$. We assume P has the form:

$$P = -g^{ij} \frac{\partial^2}{\partial x_i \partial x_j} I + a_j \frac{\partial}{\partial x_j} + b$$

where we sum over repeated indices. The a_j and b are sections to $\operatorname{END}(V)$.

We introduce formal variables for the derivatives of the symbol of P. Let

$$g_{ij/\alpha} = d_x^\alpha g_{ij}, \qquad a_{j/\alpha} = d_x^\alpha a_j, \qquad b_{/\alpha} = d_x^\alpha b.$$

We will also use the notation $g_{ij/kl\dots}$, $a_{j/kl\dots}$, and $b_{/kl\dots}$ for multiple partial derivatives. We emphasize that these variables are not tensorial but depend upon the choice of a coordinate system (and local frame for V).

There is a natural grading on these variables. We define:

$$\operatorname{ord}(g_{ij/\alpha}) = |\alpha|, \qquad \operatorname{ord}(a_{j/\alpha}) = 1 + |\alpha|, \qquad \operatorname{ord}(b_{/\alpha}) = 2 + |\alpha|.$$

Let $\mathcal{P}$ be the non-commutative polynomial algebra in the variables of positive order. We always normalize the coordinate system so x_0 corresponds to the point $(0, \dots, 0)$ and so that:

$$g_{ij}(x_0) = \delta_{ij} \qquad \text{and} \qquad g_{ij/k}(x_0) = 0.$$

Let $K(t, x, x)$ be the kernel of e^{-tP}. Expand

$$K(t, x, x) \sim \sum_{n \ge 0} t^{\frac{n-m}{2}} e_n(x, P)$$

where $e_n(x, P)$ is given by Lemma 1.7.4. Lemma 1.7.5(c) implies:

LEMMA 2.4.1. *$e_n(x, P) \in \mathcal{P}$ is a non-commutative polynomial in the jets of the symbol of P which is homogeneous of order n. If $a_n(x, P) = \operatorname{Tr} e_n(x, P)$ is the fiber trace, then $a_n(x, P)$ is homogeneous of order n in* *the components (relative to some local frame) of the jets of the total symbol of P.*

We could also have proved Lemma 2.4.1 using dimensional analysis and the local nature of the invariants $e_n(x,P)$ instead of using the combinatorial argument given in Chapter 1.

We specialize to the case where $P = \Delta_p$ is the Laplacian on p-forms. If $p = 0$, then $\Delta_0 = -g^{-1}\partial/\partial x_i\{gg^{ij}\partial/\partial x_j\}$; $g = \det(g_{ij})^{1/2}$ defines the Riemannian volume $\operatorname{dvol} = g\,dx$. The leading symbol is given by the metric tensor; the first order symbol is linear in the 1-jets of the metric with coefficients which depend smoothly on the metric tensor. The 0^{th} order symbol is zero in this case. More generally:

LEMMA 2.4.2. *Let $x = (x_1, \ldots, x_m)$ be a system of local coordinates on M. We use the dx^I to provide a local frame for $\Lambda(T^*M)$. Relative to this frame, we expand $\sigma(\Delta_p) = p_2 + p_1 + p_0$. $p_2 = |\xi|^2 I$. p_1 is linear in the 1-jets of the metric with coefficients which depend smoothly on the g_{ij}'s. p_0 is the sum of a term which is linear in the 2-jets of the metric and a term which is quadratic in the 1-jets of the metric with coefficients which depend smoothly on the g_{ij}'s.*

PROOF: We computed the leading symbol in the first chapter. The remainder of the lemma follows from the decomposition $\Delta = d\delta + \delta d = \pm d * d * \pm * d * d$. In flat space, $\Delta = -\sum_i \partial^2/\partial x_i^2$ as computed earlier. If the metric is curved, we must also differentiate the matrix representing the Hodge $*$ operator. Each derivative applied to "$*$" reduces the order of differentiation by one and increases the order in the jets of the metric by one.

Let $a_n(x,P) = \operatorname{Tr} e_n(x,P)$. Then:

$$\operatorname{Tr} e^{-tP} \sim \sum_{n\geq 0} t^{\frac{n-m}{2}} \int_M a_n(x,P)\operatorname{dvol}(x).$$

For purposes of illustration, we give without proof the first few terms in the asymptotic expansion of the Laplacian. We shall discuss such formulas in more detail in the fourth chapter.

LEMMA 2.4.3.
(a) $a_0(x,\Delta_p) = (4\pi)^{-1}\dim(\Lambda^p) = (4\pi)^{-1}\binom{m}{p}$.
(b) $a_2(x,\Delta_0) = (4\pi)^{-1}(-R_{ijij})/6$.
(c)

$$a_4(x,\Delta_0) = \frac{1}{4\pi}\cdot\frac{-12R_{ijij;kk} + 5R_{ijij}R_{klkl} - 2R_{ijik}R_{ljlk} + 2R_{ijkl}R_{ijkl}}{360}.$$

In this lemma, "$;$" denotes multiple covariant differentiation. a_0, a_2, a_4, and a_6 have been computed for Δ_p for all p, but as the formulas are extremely complicated, we shall not reproduce them here.

We consider the algebra generated by the variables $\{g_{ij/\alpha}\}_{|\alpha|\geq 2}$. If X is a coordinate system and G a metric and if P is a polynomial in these variables, we define $P(X,G)(x_0)$ by evaluation. We always normalize the choice of X so $g_{ij}(X,G)(x_0) = \delta_{ij}$ and $g_{ij/k}(X,G)(x_0) = 0$, and we omit these variables from consideration. We say that P is invariant if $P(X,G)(x_0) = P(Y,G)(x_0)$ for any two normalized coordinate systems X and Y. We denote the common value by $P(G)(x_0)$. For example, the scalar curvature $K = -\frac{1}{2}R_{ijij}$ is invariant as are the $a_n(x,\Delta_p)$.

We let $\mathcal{P}_m$ denote the ring of all invariant polynomials in the derivatives of the metric for a manifold of dimension m. We defined $\operatorname{ord}(g_{ij/\alpha}) = |\alpha|$; let $\mathcal{P}_{m,n}$ be the subspace of invariant polynomials which are homogeneous of order n. This is an algebraic characterization; it is also useful to have the following coordinate free characterization:

LEMMA 2.4.4. *Let $P \in \mathcal{P}_m$, then $P \in \mathcal{P}_{m,n}$ if and only if $P(c^2G)(x_0) = c^{-n}P(G)(x_0)$ for every $c \neq 0$.*

PROOF: Fix $c = 0$ and let X be a normalized coordinate system for the metric G at the point x_0. We assume $x_0 = (0,\dots,0)$ is the center of the coordinate system X. Let $Y = cX$ be a new coordinate system, then:

$$\begin{aligned} \partial/\partial y_i &= c^{-1}\partial/\partial x_i \qquad & c^2G(\partial/\partial y_i, \partial/\partial y_j) &= G(\partial/\partial x_i, \partial/\partial x_j) \\ d_y^\alpha &= c^{-|\alpha|}d_x^\alpha & g_{ij/\alpha}(Y,c^2G) &= c^{-|\alpha|}g_{ij/\alpha}(X,G). \end{aligned}$$

This implies that if A is any monomial of P that:

$$A(Y,c^2G)(x_0) = c^{-\operatorname{ord}(A)}A(X,G)(x_0).$$

Since Y is normalized coordinate system for the metric c^2G, $P(c^2G)(x_0) = P(Y,c^2G)(x_0)$ and $P(G)(x_0) = P(X,G)(x_0)$. This proves the Lemma.

If $P \in \mathcal{P}_m$ we can always decompose $P = P_0 + \dots + P_n$ into homogeneous polynomials. Lemma 2.4.4 implies the P_j are all invariant separately. Therefore $\mathcal{P}_m$ has a direct sum decomposition $\mathcal{P}_m = \mathcal{P}_{m,0} \oplus \mathcal{P}_{m,1} \oplus \dots \oplus \mathcal{P}_{m,n} \oplus \cdots$ and has the structure of a graded algebra. Using Taylor's theorem, we can always find a metric with the $g_{ij/\alpha}(X,G)(x_0) = c_{ij,\alpha}$ arbitrary constants for $|\alpha| \geq 2$ and so that $g_{ij}(X,G)(x_0) = \delta_{ij}$, $g_{ij/k}(X,G)(x_0) = 0$. Consequently, if $P \in \mathcal{P}_m$ is non-zero as a polynomial, then we can always find G so $P(G)(x_0) \neq 0$ so P is non-zero as a formula. It is for this reason we work with the algebra of jets. This is a pure polynomial algebra. If we work instead with the algebra of covariant derivatives of the curvature tensor, we must introduce additional relations which correspond to the Bianchi identities as this algebra is not a pure polynomial algebra.

We note finally that $\mathcal{P}_{m,n}$ is zero if n is odd since we may take $c = -1$. Later in this chapter, we will let $\mathcal{P}_{m,n,p}$ be the space of p-form valued invariants which are homogeneous of order n. A similar argument will show $\mathcal{P}_{m,n,p}$ is zero if $n - p$ is odd.

Lemmas 2.4.1 and 2.4.2 imply:

LEMMA 2.4.5. *$a_n(x,\Delta_p)$ defines an element of $\mathcal{P}_{m,n}$.*

This is such an important fact that we give another proof based on Lemma 2.4.4 to illustrate the power of dimensional analysis embodied in this lemma. Fix $c>0$ and let $\Delta_p(c^2G)=c^{-2}\Delta_p(G)$ be the Laplacian corresponding to the new metric. Since $\operatorname{dvol}(c^2G)=c^m\operatorname{dvol}(G)$, we conclude:

$$\begin{aligned} e^{-t\Delta_p(c^2G)} &= e^{-tc^{-2}\Delta_p(G)}\\ K(t,x,x,\Delta_p(c^2G))\operatorname{dvol}(c^2G) &= K(c^{-2}t,x,x,\Delta_p(G))\operatorname{dvol}(G)\\ K(t,x,x,\Delta_p(c^2G)) &= c^{-m}K(c^{-2}t,x,x,\Delta_p(G))\\ \sum_n t^{\frac{n-m}{2}}a_n(x,\Delta_p(c^2G)) &\sim \sum_n c^{-m}c^mc^{-n}t^{\frac{n-m}{2}}a_n(x,\Delta_p(G))\\ a_n(x,\Delta_p(c^2G)) &= c^{-n}a_n(x,\Delta_p(G)). \end{aligned}$$

We expand $a_n(x,\Delta_p(G))=\sum_\nu a_{n,\nu,p}+r$ in a finite Taylor series about $g_{ij}=\delta_{ij}$ in the $g_{ij/\alpha}$ variables. Then if $a_{n,\nu,p}$ is the portion which is homogeneous of order ν, we use this identity to show $a_{n,\nu,p}=0$ for $n\neq\nu$ and to show the remainder in the Tayor series is zero. This shows a_n is a homogeneous polynomial of order n and completes the proof.

Since $a_n(x,\Delta_p)=0$ if n is odd, in many references the authors replace the asymptotic series by $t^{-\frac{m}{2}}\sum_n t^n a_n(x,\Delta_p)$, They renumber this sequence $a_0,a_1,\dots$ rather than $a_0,0,a_2,0,a_4,\dots$. We shall not adopt this notational convention as it makes dealing with boundary problems more cumbersome.

H. Weyl's theorem on the invariants of the orthogonal group gives a spanning set for the spaces $\mathcal{P}_{m,n}$:

LEMMA 2.4.6. *We introduce formal variables $R_{i_1i_2i_3i_4;i_5\dots i_k}$ for the multiple covariant derivatives of the curvature tensor. The order of such a variable is $k+2$. We consider the polynomial algebra in these variables and contract on pairs of indices. Then all possible such expressions generate $\mathcal{P}_m$. In particular:*

$$\begin{gathered}\{1\}\ \text{spans}\ \mathcal{P}_{m,0},\qquad \{R_{ijij}\}\ \text{spans}\ \mathcal{P}_{m,2}\\ \{R_{ijij;kk},R_{ijij}R_{klkl},R_{ijik}R_{ljlk},R_{ijkl}R_{ijkl}\}\ \text{spans}\ \mathcal{P}_{m,4}.\end{gathered}$$

This particular spanning set for $\mathcal{P}_{m,4}$ is linearly independent and forms a basis if $m\geq 4$. If $m=3$, $\dim(\mathcal{P}_{3,4})=3$ while if $m=2$, $\dim(\mathcal{P}_{2,4})=2$ so there are relations imposed if the dimension is low. The study of these additional relations is closely related to the Gauss-Bonnet theorem.

There is a natural restriction map

$$r\colon\mathcal{P}_{m,n}\to\mathcal{P}_{m-1,n}$$

which is defined algebraically as follows. We let

$$\deg_k(g_{ij/\alpha}) = \delta_{i,k} + \delta_{j,k} + \alpha(k)$$

be the number of times an index k appears in the variable $g_{ij/\alpha}$. Let

$$r(g_{ij/\alpha}) = \begin{cases} g_{ij/\alpha} \in \mathcal{P}_{m-1} & \text{if } \deg_m(g_{ij/\alpha}) = 0 \\ 0 & \text{if } \deg_m(g_{ij/\alpha}) > 0. \end{cases}$$

We extend $r\colon \mathcal{P}_m \to \mathcal{P}_{m-1}$ to be an algebra morphism; $r(P)$ is a polynomial in the derivatives of a metric on a manifold of dimension $m-1$. It is clear r preserves the degree of homogeneity.

r is the dual of a natural extension map. Let G' be a metric on a manifold M' of dimension $m-1$. We define $i(G') = G + d\theta^2$ on the manifold $M = M' \times S^1$ where S^1 is the unit circle with natural parameter θ. If X' is a local coordinate system on M', then $i(X') = (x, \theta)$ is a local coordinate system on M. It is clear that:

$$(rP)(X', G')(x'_0) = P(i(X'), i(G'))(x'_0, \theta_0)$$

for any $\theta_0 \in S^1$; what we have done by restricting to product manifolds $M' \times S^1$ with product metrics $G' + d\theta^2$ is to introduce the relation which says the metric is flat in the last coordinate. Restiction is simply the dual of this natural extension; rP is invariant if P is invariant. Therefore $r : \mathcal{P}_{m,n} \to \mathcal{P}_{m,n-1}$.

There is one final description of the restriction map which will be useful. In discussing a H. Weyl spanning set, the indices range from 1 through m. We define the restriction by letting the indices range from 1 through $m-1$. Thus $R_{ijij} \in \mathcal{P}_{m,2}$ is its own restriction in a formal sense; of course $r(R_{ijij}) = 0$ if $m = 2$ since there are no non-trivial local invariants over a circle.

THEOREM 2.4.7.
(a) $r\colon \mathcal{P}_{m,n} \to \mathcal{P}_{m-1,n}$ is always surjective.
(b) $r\colon \mathcal{P}_{m,n} \to \mathcal{P}_{m-1,n}$ is bijective if $n < m$.
(c) $r\colon \mathcal{P}_{m,m} \to \mathcal{P}_{m-1,m}$ has 1-dimensional kernel spanned by the Euler class E_m if m is even. If m is odd, $\mathcal{P}_{m,m} = \mathcal{P}_{m-1,m} = 0$.
(c) is an axiomatic characterization of the Euler form. It is an expression of the fact that the Euler form is a unstable characteristic class as opposed to the Pontrjagin forms which are stable characteristic classes.

PROOF: (a) is consequence of H. Weyl's theorem. If we choose a H. Weyl spanning set, we let the indices range from 1 to m instead of from 1 to $m-1$ to construct an element in the inverse image of r. The proof of (b) and of (c) is more complicated and will be postponed until the next subsection. Theorem 2.4.7 is properly a theorem in invariance theory, but we have stated it at this time to illustrate how invariance theory can be used to prove index theorems using heat equation methods:

THEOREM 2.4.8. *Let $a_n(x, d+\delta) = \sum_p (-1)^p a_n(x, \Delta_p) \in \mathcal{P}_{n,m}$ be the invariant of the de Rham complex. We showed in Lemma 1.7.6 that:*

$$\int_M a_n(x, d+\delta)\,\mathrm{dvol}(x) = \begin{cases} \chi(M) & \text{if } n = m \\ 0 & \text{if } n \neq m. \end{cases}$$

Then:
(a) $a_n(x, d+\delta) = 0$ if either m is odd or if $n < m$.
(b) $a_m(x, d+\delta) = E_m$ is m is even so $\chi(M) = \int_M E_m\,\mathrm{dvol}(x)$ (Gauss-Bonnet theorem).

PROOF: We suppose first that m is odd. Locally we can always choose an orientation for $T(M)$. We let $*$ be the Hodge operator then $*\Delta_p = \pm\Delta_{m-p}*$ locally. Since these operators are locally isomorphic, their local invariants are equal so $a_n(x, \Delta_p) = a_n(x, \Delta_{m-p})$. If m is odd, these terms cancel in the alternating sum to give $a_n(x, d+\delta) = 0$. Next we suppose m is even. Let $M = M' \times S^1$ with the product metric. We decompose any $\omega \in \Lambda(T^*M)$ uniquely in the form

$$\omega = \omega_1 + \omega_2 \wedge d\theta \qquad \text{for } \omega_i \in \Lambda(T^*M').$$

We define:

$$F(\omega) = \omega_1 \wedge d\theta + \omega_2$$

and compute easily that $F\Delta = \Delta F$ since the metric is flat in the S^1 direction. If we decompose $\Lambda(M) = \Lambda^{\mathrm{e}}(M) \oplus \Lambda^{\mathrm{o}}(M)$ into the forms of even and odd degree, then F interchanges these two factors. Therefore, $a_n(x, \Delta_e) = a_n(x, \Delta_0)$ so $a_n(x, d+\delta) = 0$ for such a product metric. This implies $r(a_n) = 0$. Therefore $a_n = 0$ for $n < m$ by Theorem 2.4.7. Furthermore:

$$a_m = c_m E_m$$

for some universal constant c_m. We show $c_m = 1$ by integrating over the classifying manifold $M = S^2 \times \cdots \times S^2$. Let $2\bar{m} = m$, then

$$2^{\bar{m}} = \chi(M) = \int_M a_m(x, d+\delta)\,\mathrm{dvol}(x) = \int_M E_m\,\mathrm{dvol}(x)$$

by Lemma 2.3.4. This completes the proof of the Gauss-Bonnet theorem.

2.5. Invariance Theory and the Pontrjagin Classes of the Tangent Bundle.

In the previous subsection, we gave in Theorem 2.4.7 an axiomatic characterization of the Euler class in terms of functorial properties. In this subsection we will complete the proof of Theorem 2.4.7. We will also give a similar axiomatic characterization of the Pontrjagin classes which we will use in our discussion of the signature complex in Chapter 3.

Let $T: \mathbf{R}^m \to \mathbf{R}^m$ be the germ of a diffeomorphism. We assume that:

$$T(0) = 0 \qquad \text{and} \qquad dT(x) = dT(0) + O(x^2) \qquad \text{for } dT(0) \in \mathrm{O}(k).$$

If X is any normalized coordinate system for a metric G, then TX is another normalized coordinate system for G. We define an action of the group of germs of diffeomorphisms on the polynomial algebra in the $\{g_{ij/\alpha}\},\ |\alpha| \geq 2$ variables by defining the evaluation:

$$(T^*P)(X,G)(x_0) = P(TX,G)(x_0).$$

Clearly P is invariant if and only if $T^*P = P$ for every such diffeomorphism T.

Let P be invariant and let A be a monomial. We let $c(A,P)$ be the coefficient of A in P; $c(A,P)$ defines a linear functional on $\mathcal{P}$ for any monomial A. We say A is a monomial of P if $c(A,P) \neq 0$. Let T_j be the linear transformation:

$$T_j(x_k) = \begin{cases} -x_j & \text{if } k = j \\ x_k & \text{if } k \neq j. \end{cases}$$

This is reflection in the hyperplane defined by $x_j = 0$. Then

$$T_j^*(A) = (-1)^{\deg_j(A)} A$$

for any monomial A. Since

$$T_j^* P = \sum (-1)^{\deg_j(A)} c(A,P) A = P = \sum c(A,P) A,$$

we conclude $\deg_j(A)$ must be even for any monomial A of P. If A has the form:

$$A = g_{i_1 j_1/\alpha_1} \dots g_{i_r j_r/\alpha_r}$$

we define the length of A to be:

$$\ell(A) = r.$$

It is clear $2\ell(A) + \mathrm{ord}(A) = \sum_j \deg_j(A)$ so $\mathrm{ord}(A)$ is necessarily even if A is a monomial of P. This provides another proof $\mathcal{P}_{m,n} = 0$ if n is odd.

In addition to the hyperplane reflections, it is convenient to consider coordinate permutations. If ρ is a permutation, then $T_\rho^*(A) = A^\rho$ is defined by replacing each index i by $\rho(i)$ in the variables $g_{ij/k\dots}$. Since $T_\rho^*(P) = P$, we conclude the form of P is invariant under coordinate permutations or equivalently $c(A, P) = c(A^\rho, P)$ for every monomial A of P.

We can use these two remarks to begin the proof of Theorem 2.4.7. Fix $P \neq 0$ with $r(P) = 0$ and $P \in \mathcal{P}_{m,n}$. Let A be a monomial of P. Then $r(P) = 0$ implies $\deg_m(A) > 0$ is even. Since P is invariant under coordinate permutations, $\deg_k(A) \geq 2$ for $1 \leq k \leq m$. We construct the chain of inequalities:

$$\begin{aligned}2m \leq \sum_k \deg_k(A) = 2\ell(A) + \operatorname{ord}(A)\\ = \sum_\nu (2 + |\alpha_\nu|) \leq \sum_\nu 2|\alpha_\nu| = 2\operatorname{ord}(A) = 2n.\end{aligned}$$

We have used the fact $|\alpha_\nu| \geq 2$ for $1 \leq \nu \leq \ell(A)$. This implies $m \leq n$ so in particular $P = 0$ if $n < m$ which proves assertion (b) of Theorem 2.4.7. If $n = m$, then all of the inequalities in this string must be equalities. This implies $2\ell(A) = m$, $\deg_k(A) = 2$ for all k, and $|\alpha_\nu| = 2$ for all ν. Thus the monomial A must have the form:

$$A = g_{i_1 j_1/k_1 l_1} \dots g_{i_r j_r/k_r l_r} \qquad \text{for } 2r = m$$

and in particular, P only involves the second derivatives of the metric.

In order to complete the proof of Theorem 2.4.7, we must use some results involving invariance under circle actions:

Lemma 2.5.1. *Parametrize the circle* $\mathrm{SO}(2)$ *by* $z = (a, b)$ *for* $a^2 + b^2 = 1$. *Let* T_z *be the coordinate transformation:*

$$y_1 = ax_1 + bx_2, \qquad y_2 = -bx_1 + ax_2, \qquad y_k = x_k \qquad \text{for } k > 2.$$

Let P *be a polynomial and assume* $T_z^* P = P$ *for all* $z \in S^1$. *Then:*
(a) If $g_{12/\alpha}$ *divides a monomial* A *of* P *for some* α, *then* $g_{11/\beta}$ *divides a monomial* B *of* P *for some* β.
(b) If $g_{ij/\alpha}$ *divides a monomial of* A *of* P *for some* (i, j), *then* $g_{kl/\beta}$ *divides a monomial* B *of* P *for some* β *and some* (k, l) *where* $\beta(1) = \alpha(1) + \alpha(2)$ *and* $\beta(2) = 0$.
Of course, the use of the indices 1 and 2 is for convenience only. This lemma holds true for any pair of indices under the appropriate invariance assumption.

We postpone the proof of this lemma for the moment and use it to complete the proof of Theorem 2.4.7. Let $P \neq 0 \in \mathcal{P}_{m,m}$ with $r(P) = 0$.

Let A be a monomial of P. We noted $\deg_k(A) = 2$ for $1 \le k \le m$ and A is a polynomial in the 2-jets of the metric. We decompose

$$A = g_{i_1 j_1/k_1 l_1} \dots g_{i_r j_r/k_r l_r} \qquad \text{for } 2r = m.$$

By making a coordinate permutation we can choose A so $i_1 = 1$. If $j_1 \ne 1$, we can make a coordinate permutation to assume $j_1 = 2$ and then apply Lemma 2.5.1(a) to assume $i_1 = j_1 = 1$. We let $P_1 = \sum c(A, P)A$ where the sum ranges over A of the form:

$$A = g_{11/k_1 l_1} \dots g_{i_r j_r/k_r l_r}$$

which are monomials of P. Then $P_1 \ne 0$ and P_1 is invariant under coordinate transformations which fix the first coordinate. Since $\deg_1(A) = 2$, the index 1 appears nowhere else in A. Thus k_1 is not 1 so we may make a coordinate permutation to choose A so $k_1 = 2$. If $l_2 \ne 2$, then $l_1 \ge 3$ so we may make a coordinate permutation to assume $l_1 = 3$. We then apply Lemma 2.5.1(b) to choose A a monomial of P_1 of the form:

$$A = g_{11/22} A'.$$

We have $\deg_1(A) = \deg_2(A) = 2$ so $\deg_1(A') = \deg_2(A') = 0$ so these indices do not appear in A'. We define $P_2 = \sum c(A, P)A$ where the sum ranges over those monomial A of P which are divisible by $g_{11/22}$, then $P_2 \ne 0$.

We proceed inductively in this fashion to show finally that

$$A_0 = g_{11/22} g_{33/44} \dots g_{m-1,m-1/mm}$$

is a monomial of P so $c(A_0, P) \ne 0$. The function $c(A_0, P)$ is a separating linear functional on the kernel of r in $\mathcal{P}_{m,m}$ and therefore

$$\dim(\{\, P \in \mathcal{P}_{m,m} : r(P) = 0 \,\}) \le 1.$$

We complete the proof of Theorem 2.4.7(c) by showing that $r(E_m) = 0$. If E_m is the Euler form and $M = M' \times S^1$, then:

$$E_m(M) = E_m(M' \times S^1) = E_{m-1}(M')E_1(S^1) = 0$$

which completes the proof; $\dim \mathrm{N}(r) = 1$ and $E_m \in \mathrm{N}(r)$ spans.

Before we begin the proof of Lemma 2.5.1, it is helpful to consider a few examples. If we take $A = g_{11/11}$ then it is immediate:

$$\partial/\partial y_1 = a\partial/\partial x_1 + b\partial/\partial x_2\,, \qquad \partial/\partial y_2 = -b\partial/\partial x_1 + a\partial/\partial x_2$$
$$\partial/\partial y_k = \partial/\partial x_k \qquad \text{for } k > 2.$$

We compute $T_z^*(A)$ by formally replacing each index 1 by $a1+b2$ and each index 2 by $-b1+a2$ and expanding out the resulting expression. Thus, for example:

$$\begin{aligned}T_z^*(g_{11/11}) &= g_{a1+b2,a1+b2/a1+b2,a1+b2}\\ &= a^4 g_{11/11} + 2a^3 b g_{11/12} + 2a^3 b g_{12/11}\\ &\quad + a^2 b^2 g_{11/22} + a^2 b^2 g_{22/11} + 4a^2 b^2 g_{12/12}\\ &\quad + 2ab^3 g_{12/22} + 2ab^3 g_{22/12} + b^4 g_{22/22}.\end{aligned}$$

We note that those terms involving $a^3 b$ arose from changing exactly one index to another index ($1 \to 2$ or $2 \to 1$); the coefficient reflects the multiplicity. Thus in particular this polynomial is not invariant.

In computing invariance under the circle action, all other indices remain fixed so

$$\begin{aligned}T_z^*(g_{34/11} + g_{34/22}) &= (a^2 g_{34/11} + b^2 g_{34/22} + 2ab g_{34/12}\\ &\qquad + a^2 g_{34/22} + b^2 g_{34/11} - 2ab g_{34/12})\\ &= (a^2+b^2)(g_{34/11} + g_{34/22})\end{aligned}$$

is invariant. Similarly, it is easy to compute:

$$T_z^*(g_{11/22} + g_{22/11} - 2g_{12/12}) = (a^2+b^2)^2 (g_{11/22} + g_{22/11} - 2g_{12/12})$$

so this is invariant. We note this second example is homogeneous of degree 4 in the (a,b) variables since $\deg_1(A) + \deg_2(A) = 4$.

With these examples in mind, we begin the proof of Lemma 2.5.1. Let P be invariant under the action of the circle acting on the first two coordinates. We decompose $P = P_0 + P_1 + \cdots$ where each monomial A of P_j satisfies $\deg_1(A) + \deg_2(A) = j$. If A is such a monomial, then $T_z^*(A)$ is a sum of similar monomials. Therefore each of the P_j is invariant separately so we may assume $P = P_n$ for some n. By setting $a = b = -1$, we see n must be even. Decompose:

$$T_z^*(P) = a^n P^{(0)} + ba^{n-1} P^{(1)} + \cdots + b^n P^{(n)}$$

where $P = P^{(0)}$. We use the assumption $T_z^*(P) = P$ and replace b by $-b$ to see:

$$0 = T_{(a,b)}^*(P) - T_{(a,-b)}^*(P) = 2ba^{n-1} P^{(1)} + 2b^3 a^{n-3} P^{(3)} + \cdots$$

We divide this equation by b and take the limit as $b \to 0$ to show $P^{(1)} = 0$. (In fact, it is easy to show $P^{(2j+1)} = 0$ and $P^{(2j)} = \binom{n/2}{j} P$ but as we shall not need this fact, we omit the proof).

We let A_0 denote a variable monomial which will play the same role as the generic constant C of Chapter 1. We introduce additional notation we shall find useful. If A is a monomial, we decompose $T^*_{(a,b)}A = a^nA + a^{n-1}bA^{(1)} + \cdots$. If B is a monomial of $A^{(1)}$ then $\deg_1(B) = \deg_1(A) \pm 1$ and B can be constructed from the monomial A by changing exactly one index $1 \to 2$ or $2 \to 1$. If $\deg_1(B) = \deg_1(A) + 1$, then B is obtained from A by changing exactly one index $2 \to 1$; $c(B, A^{(1)})$ is a negative integer which reflects the multiplicity with which this change can be made. If $\deg_1(B) = \deg_1(A) - 1$, then B is obtained from A by changing exactly one index $1 \to 2$; $c(B, A^{(1)})$ is a positive integer. We define:

$$\begin{aligned}
A(1 \to 2) &= \sum c(B, A^{(1)})B \qquad \text{summed over } \deg_1(B) = \deg_1(A) - 1\\
A(2 \to 1) &= \sum -c(B, A^{(1)})B \qquad \text{summed over } \deg_1(B) = \deg_1(A) + 1\\
A^{(1)} &= A(1 \to 2) - A(2 \to 1).
\end{aligned}$$

For example, if $A = (g_{12/33})^2 g_{11/44}$ then $n = 6$ and:

$$\begin{aligned}
A(1 \to 2) &= 2g_{12/33}g_{22/33}g_{11/44} + 2(g_{12/33})^2 g_{12/44}\\
A(2 \to 1) &= 2g_{12/33}g_{11/33}g_{11/44}.
\end{aligned}$$

It is immediate from the definition that:

$$\sum_B c(B, A(1 \to 2)) = \deg_1(A) \qquad \text{and} \qquad \sum_B c(B, A(2 \to 1)) = \deg_2(A).$$

Finally, it is clear that $c(B, A^{(1)}) \neq 0$ if and only if $c(A, B^{(1)}) \neq 0$, and that these two coefficients will be opposite in sign, and not necessarily equal in magnitude.

Lemma 2.5.2. *Let P be invariant under the action of* SO(2) *on the first two coordinates and let A be a monomial of P. Let B be a monomial of $A^{(1)}$. Then there exists a monomial of A_1 different from A so that*

$$c(B, A^{(1)})c(A, P)c(B, A_1^{(1)})c(A_1, P) < 0.$$

Proof: We know $P^{(1)} = 0$. We decompose

$$P^{(1)} = \sum_A c(A, P)A^{(1)} = \sum_B c(A, P)c(B, A^{(1)})B.$$

Therefore $c(B, P^{(1)}) = 0$ implies

$$\sum_A c(A, P)c(B, A^{(1)}) = 0$$

for all monomials B. If we choose B so $c(B, A^{(1)}) \neq 0$ then there must be some other monomial A_1 of P which helps to cancel this contribution. The signs must be opposite which proves the lemma.

This lemma is somewhat technical and formidable looking, but it is exactly what we need about orthogonal invariance. We can now complete the proof of Lemma 2.5.1. Suppose first that $A = (g_{12/\alpha})^k A_0$ for $k > 0$ where $g_{12/\alpha}$ does not divide A_0. Assume $c(A, P) \neq 0$. Let $B = g_{11/\alpha}(g_{12/\alpha})^{k-1} A_0$ then $c(B, A^{(1)}) = -k \neq 0$. Choose $A_1 \neq A$ so $c(A_1, P) \neq 0$. Then $c(A_1, B^{(1)}) \neq 0$. If $c(A_1, B(2 \to 1)) \neq 0$ then A_1 is constructed from B by changing a $2 \to 1$ index so $A_1 = g_{11/\beta} \dots$ has the desired form. If $c(A_1, B(1 \to 2)) \neq 0$, we expand $B(1 \to 2) = A +$ terms divisible by $g_{11/\beta}$ for some β. Since $A_1 \neq A$, again A_1 has the desired form which proves (a).

To prove (b), we choose $g_{ij/\alpha}$ dividing some monomial of P. Let β be chosen with $\beta(1)+\beta(2) = \alpha(1)+\alpha(2)$ and $\beta(k) = \alpha(k)$ for $k > 2$ with $\alpha(1)$ maximal so $g_{uv/\beta}$ divides some monomial of P for some (u, v). Suppose $\beta(2) \neq 0$, we argue for a contradiction. Set $\gamma = (\beta(1)+1, \beta(2)-1, \beta(3), \dots, \beta(m))$. Expand $A = (g_{uv/\beta})^k A_0$ and define $B = g_{uv/\gamma}(g_{uv/\beta})^{k-1} A_0$ where $g_{uv/\beta}$ does not divide A_0. Then $c(B, A^{(1)}) = -\beta(2)k \neq 0$ so we may choose $A_1 \neq A$ so $c(A_1, B^{(1)}) \neq 0$. If $c(A_1, B(2 \to 1)) \neq 0$ then either A_1 is divisible by $g_{u'v'/\gamma}$ or by $g_{uv/\gamma'}$ where $\gamma'(1) = \gamma(1) + 1$, $\gamma'(2) = \gamma(2) - 1$, and $\gamma'(j) = \alpha(j)$ for $j > 2$. Either possibility contradicts the choice of β as maximal so $c(A_1, B(1 \to 2)) \neq 0$. However, $B(1 \to 2) = \beta(1)A +$ terms divisible by $g_{u'v'/\gamma}$ for some (u', v'). This again contradicts the maximality as $A \neq A_1$ and completes the proof of Lemma 2.5.1 and thereby of Theorems 2.4.7 and 2.4.8.

If $I = \{1 \leq i_1 \leq \cdots \leq i_p \leq m\}$, let $|I| = p$ and $dx^I = dx_{i_1} \wedge \cdots \wedge dx_{i_p}$. A p-form valued polynomial is a collection $\{P_I\} = P$ for $|I| = p$ of polynomials P_I in the derivatives of the metric. We will also sometimes write $P = \sum_{|I|=p} P_I \, dx^I$ as a formal sum to represent P. If all the $\{P_I\}$ are homogeneous of order n, we say P is homogeneous of order n. We define:

$$P(X, G)(x_0) = \sum_I P_I(X, G) \, dx^I \in \Lambda^p(T^*M)$$

to be the evaluation of such a polynomial. We say P is invariant if $P(X, G)(x_0) = P(Y, G)(x_0)$ for every normalized coordinate systems X and Y; as before we denote the common value by $P(G)(x_0)$. In analogy with Lemma 2.4.4 we have:

Lemma 2.5.3. *Let P be p-form valued and invariant. Then P is homogeneous of order n if and only if $P(c^2G)(x_0) = c^{p-n} P(G)(x_0)$ for every $c \neq 0$.*

The proof is exactly the same as that given for Lemma 2.4.4. The only new feature is that $dy^I = c^p \, dx^I$ which contributes the extra feature of c^p in this equation.

We define $\mathcal{P}_{m,n,p}$ to be the vector space of all p-form valued invariants which are homogeneous of order n on a manifold of dimension m. If $\mathcal{P}_{m,*,p}$ denotes the vector space of all p-form valued invariant polynomials, then we have a direct sum decomposition $\mathcal{P}_{m,*,p} = \bigoplus_n \mathcal{P}_{m,n,p}$ exactly as in the scalar case $p = 0$.

We define $\deg_k(I)$ to be 1 if k appears in I and 0 if k does not appear in I.

LEMMA 2.5.4. *Let $P \in \mathcal{P}_{m,n,p}$ with $P \neq 0$. Then $n - p$ is even. If A is a monomial then A is a monomial of at most one P_I. If $c(A, P_I) \neq 0$ then:*

$$\deg_k(A) + \deg_k(I) \qquad \text{is always even.}$$

PROOF: Let T be the coordinate transformation defined by $T(x_k) = -x_k$ and $T(x_j) = x_j$ for $j \neq k$. Then

$$P = T^*(P) = \sum_{I,A} (-1)^{\deg_k(A)+\deg_k(I)} c(A, P_I)\, dx^I$$

which implies $\deg_k(A) + \deg_k(I)$ is even if $c(A, P_I) \neq 0$. Therefore

$$\sum_k \deg_k(A) + \deg_k(I) = 2\ell(A) + \operatorname{ord}(A) + p = 2\ell(A) + n + p$$

must be even. This shows $n+p$ is even if $P \neq 0$. Furthermore, if $c(A, P_I) \neq 0$ then I is simply the ordered collection of indices k so $\deg_k(A)$ is odd which shows A is a monomial of at most one P_I and completes the proof.

We extend the Riemannian metric to a fiber metric on $\Lambda(T^*M) \otimes \mathbf{C}$. It is clear that $P \cdot P = \sum_I P_I \bar{P}_I$ since the $\{dx^I\}$ form an orthonormal basis at x_0. $P \cdot P$ is a scalar invariant. We use Lemma 2.5.1 to prove:

LEMMA 2.5.5. *Let P be p-form valued and invariant under the action of* $\mathrm{O}(m)$. *Let A be a monomial of P. Then there is a monomial A_1 of P with $\deg_k(A_1) = 0$ for $k > 2\ell(A)$.*

PROOF: Let $r = \ell(A)$ and let $P_r' = \sum_{\ell(B)=r} c(B, P_I)\, dx^I \neq 0$. Since this is invariant under the action of $\mathrm{O}(m)$, we may assume without loss of generality $P = P_r$. We construct a scalar invariant by taking the inner product $Q = (P, P)$. By applying Lemma 2.5.1(a) and making a coordinate permutation if necessary, we can assume g_{11/α_1} divides some monomial of Q. We apply 2.5.1(b) to the indices > 1 to assume $\alpha_1(k) = 0$ for $k > 2$. g_{11/α_1} must divide some monomial of P. Let

$$P_1 = \sum_{A_0, I} c(g_{11/\alpha_1} A_0, P_I) g_{11/\alpha_1} A_0\, dx^I \neq 0.$$

This is invariant under the action of $\mathrm{O}(m-2)$ on the last $m-2$ coordinates. We let $Q_1=(P_1,P_1)$ and let $g_{i_2j_2/\alpha_2}$ divide some monomial of Q_1. If both i_2 and j_2 are ≤ 2 we leave this alone. If $i_2\le 2$ and $j_2\ge 3$ we perform a coordinate permutation to assume $j_2=3$. In a similar fashion if $i_2\ge 3$ and $j_2\le 2$, we choose this variable so $i_2=3$. Finally, if both indices ≥ 3, we apply Lemma 2.5.1(a) to choose this variable so $i_2=j_2=3$. We apply Lemma 2.5.1(b) to the variables $k\ge 4$ to choose this variable so $\alpha_2(k)=0$ for $k>4$. If $A_2=g_{11/\alpha_1}g_{i_2j_2/\alpha_2}$ then:

$$\deg_k(A_2)=0 \text{ for } k>4 \text{ and } A_2 \text{ divides some monomial of } P.$$

We continue inductively to construct $A_r=g_{11/\alpha_1}\dots g_{i_rj_r/\alpha_r}$ so that

$$\deg_k(A_r)=0 \text{ for } k>2r \text{ and } A_r \text{ divides some monomial of } P.$$

Since every monomial of P has length r, this implies A_r itself is a monomial of P and completes the proof.

Let $P_j(G)=p_j(TM)$ be the j^{th} Pontrjagin form computed relative to the curvature tensor of the Levi-Civita connection. If we expand p_j in terms of the curvature tensor, then p_j is homogeneous of order $2j$ in the $\{R_{ijkl}\}$ tensor so p_j is homogeneous of order $4j$ in the jets of the metric. It is clear p_j is an invariantly defined $4j$-form so $p_j\in\mathcal{P}_{m,4j,4j}$. The algebra generated by the p_j is called the algebra of the Pontrjagin forms. By Lemma 2.2.2, this is also the algebra of real characteristic forms of $T(M)$. If ρ is a partition of $k=i_1+\dots+i_j$ we define $p_\rho=p_{i_1}\dots p_{i_j}\in\mathcal{P}_{m,4k,4k}$. The $\{p_\rho\}$ form a basis of the Pontrjagin $4k$ forms. By Lemma 2.3.4, these are linearly independent if $m=4k$ since the matrix $\int_{M_\tau^r}p_\rho$ is non-singular. By considering products of these manifolds with flat tori T^{m-4k} we can easily show that the $\{p_\rho\}$ are linearly independent in $\mathcal{P}_{m,4k,4k}$ if $4k\le m$. We let $\pi(k)$ be the number of partitions of k; this is the dimension of the Pontrjagin forms.

The axiomatic characterization of the real characteristic forms of the tangent space which is the analogue of the axiomatic characterization of the Euler class given in Theorem 2.4.7 is the following:

Lemma 2.5.6.
(a) $\mathcal{P}_{m,n,p}=0$ *if* $n<p$.
(b) $\mathcal{P}_{m,n,n}$ *is spanned by the Pontrjagin forms—i.e.,*

$$\mathcal{P}_{m,n,n}=0 \text{ if } n \text{ is not divisible by } 4k,$$
$$\mathcal{P}_{m,4k,4k}=\operatorname{span}\{p_\rho\} \text{ for } 4k\le m \text{ has dimension } \pi(k).$$

Proof: By decomposing P into its real and imaginary parts, it suffices to prove this lemma for polynomials with real coefficients. Let $0\ne P\in\mathcal{P}_{m,n,p}$

and let A be a monomial of P. Use Lemma 2.5.5 to find a monomial A_1 of some P_I where $I = \{1 \le i_1 < \cdots < i_p \le m\}$ so $\deg_k(A) = 0$ for $k > 2\ell(A)$. Since $\deg_{i_p}(A)$ is odd (and hence non-zero) and since $2\ell(A) \le \sum |\alpha_\nu| = n$ as A is a polynomial in the jets of order 2 and higher, we estimate:

$$p \le i_p \le 2\ell(A) \le n.$$

This proves $P = 0$ if $n < p$, which proves (a).

If $n = p$, then all of these inequalities must have been equalities. This shows that the higher order jets of the metric do not appear in P so P is a polynomial in the $\{g_{ij/kl}\}$ variables. Furthermore, $i_p = p$ and there is some monomial A so that

$$\deg_k(A) = 0 \text{ for } k > p \quad \text{and} \quad A\,dx^1 \wedge \cdots \wedge dx^p \text{ appears in } P.$$

There is a natural restriction map $r\colon \mathcal{P}_{m,n,p} \to \mathcal{P}_{m-1,n,p}$ defined in the same way as the restriction map $r\colon \mathcal{P}_{m,n,0} \to \mathcal{P}_{m-1,n,0}$ discussed earlier. This argument shows $r\colon \mathcal{P}_{m,n,n} \to \mathcal{P}_{m-1,n,n}$ is injective for $n < m$ since $r(A\,dx^1 \wedge \cdots \wedge dx^p) = A\,dx^1 \wedge \cdots \wedge dx^p$ appears in $r(P)$. The Pontrjagin forms have dimension $\pi(k)$ for $n = 4k$. By induction, $r^{m-n}\colon \mathcal{P}_{m,n,n} \to \mathcal{P}_{n,n,n}$ is injective so $\dim \mathcal{P}_{m,n,n} \le \dim \mathcal{P}_{n,n,n}$.

We shall prove Lemma 2.5.6(b) for the special case $n = m$. If n is not divisible by $4k$, then $\dim \mathcal{P}_{n,n,n} = 0$ which implies $\dim \mathcal{P}_{m,n,n} = 0$. If $n = 4k$, then $\dim \mathcal{P}_{n,n,n} = \pi(k)$ implies that $\pi(k) \le \dim \mathcal{P}_{m,n,n} \le \dim \mathcal{P}_{n,n,n} \le \pi(k)$ so $\dim \mathcal{P}_{m,n,n} = \pi(k)$. Since the Pontrjagin forms span a subspace of exactly dimension $\pi(k)$ in $\mathcal{P}_{m,n,n}$ this will complete the proof of (b).

This lemma is at the heart of our discussion of the index theorem. We shall give two proofs for the case $n = m = p$. The first is based on H. Weyl's theorem for the orthogonal group and follows the basic lines of the proof given in Atiyah-Bott-Patodi. The second proof is purely combinatorial and follows the basic lines of the original proof first given in our thesis. The H. Weyl based proof has the advantage of being somewhat shorter but relies upon a deep theorem we have not proved here while the second proof although longer is entirely self-contained and has some additional features which are useful in other applications.

We review H. Weyl's theorem briefly. Let V be a real vector space with a fixed inner product. Let $\mathrm{O}(V)$ denote the group of linear maps of $V \to V$ which preserve this inner product. Let $\bigotimes^k(V) = V \otimes \cdots \otimes V$ denote the k^{th} tensor prodect of V. If $g \in \mathrm{O}(V)$, we extend g to act orthogonally on $\bigotimes^k(V)$. We let $z \mapsto g(z)$ denote this action. Let $f\colon \bigotimes^k(V) \to \mathbf{R}$ be a multi-linear map, then we say f is $\mathrm{O}(V)$ invariant if $f(g(z)) = f(z)$ for every $g \in \mathrm{O}(V)$. By letting $g = -1$, it is easy to see there are no $\mathrm{O}(V)$ invariant maps if k is odd. We let $k = 2j$ and construct a map

$f_0 : \bigotimes^k(V) = (V \otimes V) \otimes (V \otimes V) \otimes \cdots \otimes (V \otimes V) \to \mathbf{R}$ using the metric to map $(V \otimes V) \to \mathbf{R}$. More generally, if ρ is any permutation of the integers 1 through k, we define $z \mapsto z_\rho$ as a map from $\bigotimes^k(V) \to \bigotimes^k(V)$ and let $f_\rho(z) = f_0(z_\rho)$. This will be $\mathrm{O}(V)$ invariant for any permutation ρ. H. Weyl's theorem states that the maps $\{f_\rho\}$ define a spanning set for the collection of $\mathrm{O}(V)$ invariant maps.

For example, let $k = 4$. Let $\{v_i\}$ be an orthonormal basis for V and express any $z \in \bigotimes^4(V)$ in the form $a_{ijkl} v_i \otimes v_j \otimes v_k \otimes v_l$ summed over repeated indices. Then after weeding out duplications, the spanning set is given by:

$$f_0(z) = a_{iijj}, \qquad f_1(z) = a_{ijij}, \qquad f_2(z) = a_{ijji}$$

where we sum over repeated indices. f_0 corresponds to the identity permutation; f_1 corresponds to the permutation which interchanges the second and third factors; f_2 corresponds to the permutation which interchanges the second and fourth factors. We note that these need not be linearly independent; if $\dim V = 1$ then $\dim(\bigotimes^4 V) = 1$ and $f_1 = f_2 = f_3$. However, once $\dim V$ is large enough these become linearly independent.

We are interested in p-form valued invariants. We take $\bigotimes^k(V)$ where $k - p$ is even. Again, there is a natural map we denote by

$$f^p(z) = f_0(z_1) \wedge \Lambda(z_2)$$

where we decompose $\bigotimes^k(V) = \bigotimes^{k-p}(V) \otimes \bigotimes^p(V)$. We let f_0 act on the first $k - p$ factors and then use the natural map $\bigotimes^p(V) \xrightarrow{\Lambda} \Lambda^p(V)$ on the last p factors. If ρ is a permutation, we set $f^p_\rho(z) = f^p(z_\rho)$. These maps are equivariant in the sense that $f^p_\rho(gz) = g f^p_\rho(z)$ where we extend g to act on $\Lambda^p(V)$ as well. Again, these are a spanning set for the space of equivariant multi-linear maps from $\bigotimes^k(V)$ to $\Lambda^p(V)$. If $k = 4$ and $p = 2$, then after eliminating duplications this spanning set becomes:

$$\begin{aligned} &f_1(z) = a_{iijk} v_j \wedge v_k\,, \quad f_2(z) = a_{ijik} v_j \wedge v_k\,, \quad f_3(z) = a_{ijki} v_j \wedge v_k \\ &f_4(z) = a_{jiki} v_j \wedge v_k\,, \quad f_5(z) = a_{jiik} v_i \wedge v_k\,, \quad f_6(z) = a_{jkii} v_j \wedge v_k\,. \end{aligned}$$

Again, these are linearly independent if $\dim V$ is large, but there are relations if $\dim V$ is small. Generally speaking, to construct a map from $\bigotimes^k(V) \to \Lambda^p(V)$ we must alternate p indices (the indices j, k in this example) and contract the remaining indices in pairs (there is only one pair i, i here).

THEOREM 2.5.7 (H. WEYL'S THEOREM ON THE INVARIANTS OF THE ORTHOGONAL GROUP). *The space of maps $\{f^p_\rho\}$ constructed above span the space of equivariant multi-linear maps from $\bigotimes^k V \to \Lambda^p V$.*

The proof of this theorem is beyond the scope of the book and will be omitted. We shall use it to give a proof of Lemma 2.5.6 along the lines of the proof given by Atiyah, Bott, and Patodi. We will then give an independent proof of Lemma 2.5.6 by other methods which does not rely on H. Weyl's theorem.

We apply H. Weyl's theorem to our situation as follows. Let $P \in \mathcal{P}_{n,n,n}$ then P is a polynomial in the 2-jets of the metric. If we let X be a system of geodesic polar coordinates centered at x_0, then the 2-jets of the the metric are expressible in terms of the curvature tensor so we can express P as a polynomial in the $\{R_{ijkl}\}$ variables which is homogeneous of order $n/2$. The curvature defines an element $R \in \bigotimes^4(T(M))$ since it has 4 indices. There are, however, relations among the curvature variables:

$$R_{ijkl} = R_{klij}, \qquad R_{ijkl} = -R_{jikl}, \qquad \text{and} \qquad R_{ijkl} + R_{iklj} + R_{iljk} = 0.$$

We let V be the sub-bundle of $\bigotimes^4(T(M))$ consisting of tensors satisfying these 3 relations.

If $n = 2$, then $P\colon V \to \Lambda^2(T(M))$ is equivariant while more generally, $P\colon \bigotimes^{n/2}(V) \to \Lambda^n(T(M))$ is equivariant under the action of $\mathrm{O}(T(M))$. (We use the metric tensor to raise and lower indices and identify $T(M) = T^*(M)$). Since these relations define an $\mathrm{O}(T(M))$ invariant subspace of $\bigotimes^{2n}(T(M))$, we extend P to be zero on the orthogonal complement of $\bigotimes^{n/2}(V)$ in $\bigotimes^{2n}(T(M))$ to extend P to an equivariant action on the whole tensor algebra. Consequently, we can use H. Weyl's theorem to span $\mathcal{P}_{n,n,n}$ by expressions in which we alternate n indices and contract in pairs the remaining n indices.

For example, we compute:

$$p_1 = C \cdot R_{ijab} R_{ijcd}\, dx^a \wedge dx^b \wedge dx^c \wedge dx^d$$

for a suitable normalizing constant C represents the first Pontrjagin form. In general, we will use letters a, b, $c, \dots$ for indices to alternate on and indices i, j, $k, \dots$ for indices to contract on. We let P be such an element given by H. Weyl's theorem. There are some possibilities we can eliminate on a priori grounds. The Bianchi identity states:

$$R_{iabc}\, dx^a \wedge dx^b \wedge dx^c = \frac{1}{3}(R_{iabc} + R_{ibca} + R_{icab})\, dx^a \wedge dx^b \wedge dx^c = 0$$

so that three indices of alternation never appear in any $R_{\dots}$ variable. Since there are $n/2$ R variables and n indices of alternation, this implies each R variable contains exactly two indices of alternation. We use the Bianchi identity again to express:

$$\begin{aligned} R_{iajb}\, dx^a \wedge dx^b &= \frac{1}{2}(R_{iajb} - R_{ibja})\, dx^a \wedge dx^b \\ &= \frac{1}{2}(R_{iajb} + R_{ibaj})\, dx^a \wedge dx^b \\ &= -\frac{1}{2} R_{ijba}\, dx^a \wedge dx^b = \frac{1}{2} R_{ijab}\, dx^a \wedge dx^b. \end{aligned}$$

This together with the other curvature identities means we can express P in terms of $R_{ijab}\,dx^a \wedge dx^b = \Omega_{ij}$ variables. Thus $P = P(\Omega_{ij})$ is a polynomial in the components of the curvature matrix where we regard $\Omega_{ij} \in \Lambda^2(T^*M)$. (This differs by various factors of 2 from our previous definitions, but this is irrelevant to the present argument). $P(\Omega)$ is an $\mathrm{O}(n)$ invariant polynomial and thus is a real characteristic form. This completes the proof.

The remainder of this subsection is devoted to giving a combinatorial proof of this lemma in the case $n = m = p$ independent of H. Weyl's theorem. Since the Pontrjagin forms span a subspace of dimension $\pi(k)$ if $n = 4k$ we must show $\mathcal{P}_{n,n,n} = 0$ if n is not divisible by 4 and that $\dim \mathcal{P}_{4k,4k,4k} \le \pi(k)$ since then equality must hold.

We showed $n = 2j$ is even and any polynomial depends on the 2-jets of the metric. We improve Lemma 2.5.1 as follows:

Lemma 2.5.8. *Let P satisfy the hypothesis of Lemma 2.5.1 and be a polynomial in the 2-jets of the metric. Then:*
(a) Let $A = g_{12/\alpha}A_0$ be a monomial of P. Either by interchanging 1 and 2 indices or by changing two 2 indices to 1 indices we can construct a monomial A_1 of the form $A_1 = g_{11/\beta}A_0'$ which is a monomial of P.
(b) Let $A = g_{ij/12}A_0$ be a monomial of P. Either by interchanging 1 and 2 indices or by changing two 2 indices to 1 indices we can construct a monomial of A_1 of the form $A_1 = g_{i'j'/11}A_0'$ which is a monomial of P.
(c) The monomial $A_1 \ne A$. If $\deg_1(A_1) = \deg_1(A)+2$ then $c(A,P)c(A_1,P) > 0$. Otherwise $\deg_1(A_1) \ne \deg_1(A)$ and $c(A,P)c(A_1,P) < 0$.

Proof: We shall prove (a) as (b) is the same; we will also verify (c). Let $B = g_{11/\alpha}A_0$ so $c(B, A(2 \to 1)) \ne 0$. Then $\deg_1(B) = \deg_1(A) + 1$. Apply Lemma 2.5.2 to find $A_1 \ne A$ so $c(B, A_1^{(1)}) \ne 0$. We noted earlier in the proof of 2.5.1 that A_1 must have the desired form. If $c(A_1, B(2 \to 1)) \ne 0$ then $\deg_1(A_1) = \deg_1(B)+1 = \deg_1(A)+2$ so A_1 is constructed from A by changing two 2 to 1 indices. Furthermore, $c(B, A_1^{(1)}) > 0$ and $c(B, A^{(1)}) < 0$ implies $c(A,P)c(A_1,P) > 0$. If, on the other hand, $c(A_1, B(1 \to 2)) \ne 0$ then $\deg_1(A_1) = \deg_1(B) - 1 = \deg_1(A)$ and A changes to A_1 by interchanging a 2 and a 1 index. Furthermore, $c(B, A_1^{(1)}) < 0$ and $c(B, A^{(1)}) < 0$ implies $c(A,P)c(A_1,P) < 0$ which completes the proof.

Let $P \in \mathcal{P}_{n,n,n}$ and express $P = P'\,dx^1 \wedge \cdots \wedge dx^n$. $P' = *P$ is a scalar invariant which changes sign if the orientation of the local coordinate system is reversed. We identify P with P' for notational convenience and henceforth regard P as a skew-invariant scalar polynomial. Thus $\deg_k(A)$ is odd for every k and every monomial A of P.

The indices with $\deg_k(A) = 1$ play a particularly important role in our discussion. We say that an index i touches an index j in the monomial A

if A is divisible by a variable $g_{ij/..}$ or $g_{../ij}$ where ".." indicate two indices which are not of interest.

LEMMA 2.5.9. *Let $P \in \mathcal{P}_{n,n,n}$ with $P \neq 0$. Then there exists a monomial A of P so*
(a) $\deg_k(A) \leq 3$ *all k.*
(b) If $\deg_k(A) = 1$ *there exists an index $j(k)$ which touches k in A. The index $j(k)$ also touches itself in A.*
(c) Let $\deg_j A + \deg_k A = 4$. *Suppose the index j touches itself and the index k in A. There is a unique monomial A_1 different from A which can be formed from A by interchanging a j and k index.* $\deg_j A_1 + \deg_k A_1 = 4$ *and the index j touches itself and the index k in A_1.* $c(A,P) + c(A_1,P) = 0$.

PROOF: Choose A so the number of indices with $\deg_k(A) = 1$ is minimal. Among all such choices, we choose A so the number of indices which touch themselves is maximal. Let $\deg_k(A) = 1$ then k touches some index $j = j(k) \neq k$ in A. Suppose A has the form $A = g_{jk/..}A_0$ as the other case is similar. Suppose first that $\deg_j(A) \geq 5$. Use lemma 2.5.8 to find $A_1 = g_{kk/..}A_0$. Then $\deg_k(A_1) \neq \deg_k(A)$ implies $\deg_k(A_1) = \deg_k(A) + 2 = 3$. Also $\deg_j(A_1) = \deg_j(A) - 2 \geq 3$ so A_1 has one less index with degree 1 which contradicts the minimality of the choice of A. We suppose next $\deg_j(A) = 1$. If T is the coordinate transformation interchanging x_j and x_k, then T reverses the orientation so $T^*P = -P$. However $\deg_j(A) = \deg_k(A) = 1$ implies $T^*A = A$ which contradicts the assumption that A is a monomial of P. Thus $\deg_j(A) = 3$ which proves (a).

Suppose j does not touch itself in A. We use Lemma 2.5.8 to construct $A_1 = g_{kk/..}A_0'$. Then $\deg_k(A_1) = \deg_k(A) + 2 = 3$ and $\deg_j(A_1) = \deg_j(A) - 2 = 1$. This is a monomial with the same number of indices of degree 1 but which has one more index (namely k) which touches itself. This contradicts the maximality of A and completes the proof of (b).

Finally, let $\mathcal{A}_\nu = \{\, k : \deg_k(A) = \nu \,\}$ for $\nu = 1,\ 3$. The map $k \mapsto j(k)$ defines an injective map from $\mathcal{A}_1 \to \mathcal{A}_3$ since no index of degree 3 can touch two indices of degree 1 as well as touching itself. The equalities:

$$n = \operatorname{card}(\mathcal{A}_1) + \operatorname{card}(\mathcal{A}_3) \quad \text{and} \quad 2n = \sum_k \deg_k(A) = \operatorname{card}(\mathcal{A}_1) + 3\operatorname{card}(\mathcal{A}_3)$$

imply $2\operatorname{card}(\mathcal{A}_3) = n$. Thus $\operatorname{card}(\mathcal{A}_1) = \operatorname{card}(\mathcal{A}_3) = n/2$ and the map $k \mapsto j(k)$ is bijective in this situation.

(c) follows from Lemma 2.5.8 where $j = 1$ and $k = 2$. Since $\deg_k(A) = 1$, A_1 cannot be formed by transforming two k indices to j indices so A_1 must be the unique monomial different from A obtained by interchanging these indices. For example, if $A = g_{jj/ab}g_{jk/cd}A_0$, then $A_1 = g_{jk/ab}g_{jj/cd}A_0$. The multiplicities involved are all 1 so we can conclude $c(A,P) + c(A_1,P) = 0$ and not just $c(A,P)c(A_1,P) < 0$.

Before further normalizing the choice of the monomial A, we must prove a lemma which relies on cubic changes of coordinates:

LEMMA 2.5.10. *Let P be a polynomial in the 2-jets of the metric invariant under changes of coordinates of the form $y_i = x_i + c_{jkl}x_jx_kx_l$. Then:*
(a) $g_{12/11}$ and $g_{11/12}$ divide no monomial A of P.
(b) Let $A = g_{23/11}A_0$ and $B = g_{11/23}A_0$ then A is a monomial of P if and only if B is a monomial of P. Furthermore, $c(A,P)$ and $c(B,P)$ have the same sign.
(c) Let $A = g_{11/22}A_0$ and $B = g_{22/11}A_0$, then A is a monomial of P if and only if B is a monomial of P. Furthermore, $c(A,P)$ and $c(B,P)$ have the same sign.

PROOF: We remark the use of the indices 1, 2, 3 is for notational convenience only and this lemma holds true for any triple of distinct indices. Since $g_{ij}(X,G) = \delta_{ij} + O(x^2)$, $g^{ij}{}_{/kl}(X,G)(x_0) = -g_{ij/kl}(X,G)(x_0)$. Furthermore, under changes of this sort, $d_y^\alpha(x_0) = d_x^\alpha(x_0)$ if $|\alpha| = 1$. We consider the change of coordinates:

$$\begin{aligned} y_2 &= x_2 + cx_1^3, & y_k &= k_x \quad \text{otherwise} \\ dy_2 &= dx_2 + 3cx_1^2\,dx_1, & dy_k &= dx_k \quad \text{otherwise} \end{aligned}$$

with

$$\begin{aligned} g^{12}(Y,G) &= g^{12}(X,G) + 3cx_1^2 + O(x^4), \\ g^{ij}(Y,G) &= g^{ij}(X,G) + O(x^4) \quad \text{otherwise} \\ g_{12/11}(Y,G)(x_0) &= g_{12/11}(X,G)(x_0) - 6c \\ g_{ij/kl}(Y,G)(x_0) &= g_{ij/kl}(X,G)(x_0) \quad \text{otherwise.} \end{aligned}$$

We decompose $A = (g_{12/11})^\nu A_0$. If $\nu > 0$, $T^*(A) = A - 6\nu c(g_{12/11})^{\nu-1}A_0 + O(c^2)$. Since $T^*(P) = P$, and since there is no way to cancel this additional contribution, A cannot be a monomial of P so $g_{12/11}$ divides no monomial of P.

Next we consider the change of coordinates:

$$\begin{aligned} y_1 &= x_1 + cx_1^2x_2, & y_k &= x_k \quad \text{otherwise} \\ dy_1 &= dx_1 + 2cx_1x_2\,dx_1 + cx_1^2\,dx_2, & dy_k &= dx_k \quad \text{otherwise} \end{aligned}$$

with

$$\begin{aligned} g_{11/12}(Y,G)(x_0) &= g_{11/12}(X,G)(x_0) - 4c \\ g_{12/11}(Y,G)(x_0) &= g_{12/11}(X,G)(x_0) - 2c \\ g_{ij/kl}(Y,G)(x_0) &= g_{ij/kl}(X,G)(x_0) \qquad \text{otherwise.} \end{aligned}$$

We noted $g_{12/11}$ divides no monomial of P. If $A = (g_{11/12})^\nu A_0$, then $\nu > 0$ implies $T^*(A) = A - 4cv(g_{11/12})^{\nu-1}A_0 + O(c^2)$. Since there would be no way to cancel such a contribution, A cannot be a monomial of P which completes the proof of (a).

Let A_0' be a monomial not divisible by any of the variables $g_{23/11}$, $g_{12/13}$, $g_{13/12}$, $g_{11/23}$ and let $A(p,q,r,s) = (g_{23/11})^p (g_{12/13})^q (g_{13/12})^r (g_{11/23})^s A_0'$. We set $c(p,q,r,s) = c(A(p,q,r,s),P)$ and prove (b) by establishing some relations among these coefficients. We first consider the change of coordinates,

$$\begin{aligned} y_2 &= x_2 + cx_1^2 x_3, & y_k &= x_k \quad \text{otherwise} \\ dy_2 &= dx_2 + 2cx_1x_3\,dx_1 + cx_1^2\,dx_3, & dy_k &= dx_k \quad \text{otherwise} \end{aligned}$$

with

$$\begin{aligned} g_{12/13}(Y,G)(x_0) &= g_{12/13}(X,G)(x_0) - 2c \\ g_{23/11}(Y,G)(x_0) &= g_{23/11}(X,G)(x_0) - 2c \\ g_{ij/kl}(Y,G)(x_0) &= g_{ij/kl}(X,G)(x_0) \qquad \text{otherwise.} \end{aligned}$$

We compute that:

$$\begin{aligned} &T^*(A(p,q,r,s)) \\ &\quad = A(p,q,r,s) + c\{-2qA(p,q-1,r,s) - 2pA(p-1,q,r,s)\} + O(c^2). \end{aligned}$$

Since $T^*(P) = P$ is invariant, we conclude

$$pc(p,q,r,s) + (q+1)c(p-1,q+1,r,s) = 0.$$

By interchanging the roles of 2 and 3 in the argument we also conclude:

$$pc(p,q,r,s) + (r+1)c(p-1,q,r+1,s) = 0.$$

(We set $c(p,q,r,s) = 0$ if any of these integers is negative.)

Next we consider the change of coordinates:

$$\begin{aligned} y_1 &= x_1 + cx_1x_2x_3, & y_k &= x_k \quad \text{otherwise} \\ dy_1 &= dx_1 + cx_1x_2\,dx_3 + cx_1x_3\,dx_2 + cx_2x_3\,dx_1, & dy_k &= dx_k \quad \text{otherwise} \end{aligned}$$

with

$$\begin{aligned} g_{11/23}(Y,G)(x_0) &= g_{11/23}(X,G)(x_0) - 2c \\ g_{12/13}(Y,G)(x_0) &= g_{12/13}(X,G)(x_0) - c \\ g_{13/12}(Y,G)(x_0) &= g_{13/12}(Y,G)(x_0) - c \end{aligned}$$

so that

$$\begin{aligned} T^*(A(p,q,r,s)) = A(p,q,r,s) &+ c\{-2sA(p,q,r,s-1) \\ &- rA(p,q,r-1,s) - qA(p,q-1,r,s)\} + O(c^2). \end{aligned}$$

This yields the identities

$$(q+1)c(p,q+1,r,s-1)+(r+1)c(p,q,r+1,s-1)+2sc(p,q,r,s)=0.$$

Let $p \neq 0$ and let $A = g_{23/11}A_0'$ be a monomial of P. Then $c(p-1,q+1,r,s)$ and $c(p-1,q,r+1,s)$ are non-zero and have the opposite sign as $c(p,q,r,s)$. Therefore $(q+1)c(p-1,q+1,r,s)+(r+1)c(p-1,q,r+1,s)$ is non-zero which implies $c(p-1,q,r,s+1)(s+1)$ is non-zero and has the same sign as $c(p,q,r,s)$. This shows $g_{11/23}A_0'$ is a monomial of P. Conversely, if A is not a monomial of P, the same argument shows $g_{11/23}A_0'$ is not a monomial of P. This completes the proof of (b).

The proof of (c) is essentially the same. Let A_0' be a monomial not divisible by the variables $\{g_{11/22}, g_{12/12}, g_{22/11}\}$ and let

$$A(p,q,r) = (g_{11/22})^p (g_{12/12})^q (g_{22/11})^r.$$

Let $c(p,q,r) = c(A(p,q,r),P)$. Consider the change of coordinates:

$$\begin{aligned} y_1 &= x_1 + cx_1x_2^2, & y_k &= x_k \quad \text{otherwise}\\ dy_1 &= dx_1 + cx_2^2\,dx_1 + 2cx_1x_2\,dx_2, & dy_k &= dx_k \quad \text{otherwise} \end{aligned}$$

with

$$\begin{aligned} g_{11/22}(Y,g)(x_0) &= g_{11/22}(X,G)(x_0) - 4c\\ g_{12/12}(Y,G)(x_0) &= g_{12/12}(X,G)(x_0) - 2c\\ g_{ij/kl}(Y,G)(x_0) &= g_{ij/kl}(X,G)(x_0) \qquad \text{otherwise.} \end{aligned}$$

This yields the relation $2pc(p,q,r)+(q+1)c(p-1,q+1,r)=0$. By interchanging the roles of 1 and 2 we obtain the relation $2rc(p,q,r)+(q+1)c(p,q+1,r-1)=0$ from which (c) follows.

This step in the argument is functionally equivalent to the use made of the $\{R_{ijkl}\}$ variables in the argument given previously which used H. Weyl's formula. It makes use in an essential way of the invariance of P under a wider group than just first and second order transformations. For the Euler form, by contrast, we only needed first and second order coordinate transformations.

We can now construct classifying monomials using these lemmas. Fix $n = 2n_1$ and let A be the monomial of P given by Lemma 2.5.9. By making a coordinate permutation, we may assume $\deg_k(A) = 3$ for $k \le n_1$ and $\deg_k(A) = 1$ for $k > n_1$. Let $x(i) = i + n_1$ for $1 \le i \le n_0$; we may assume the index I touches itself and $x(i)$ in A for $i \le n_0$.

We further normalize the choice of A as follows. Either $g_{11/ij}$ or $g_{ij/11}$ divides A. Since $\deg_1(A) = 3$ this term is not $g_{11/11}$ and by Lemma 2.5.10 it is not $g_{11/1x(1)}$ nor $g_{1x(1)/11}$. By Lemma 2.5.10(b) or 2.5.10(c), we may assume $A = g_{11/ij}\dots$ for $i,\,j \ge 2$. Since not both i and j can have degree

1 in A, by making a coordinate permutation if necessary we may assume that $i = 2$. If $j = 2$, we apply Lemma 2.5.9(c) to the indices 2 and $x(2)$ to perform an interchange and assume $A = g_{11/2x(2)}A_0$. The index 2 must touch itself elsewhere in A. We apply the same considerations to choose A in the form $A = g_{11/2x(2)}g_{22/ij}A_0$. If i or j is 1, the cycle closes and we express $A = g_{11/2x(2)}g_{11/1x(1)}A_0$ where $\deg_k(A_0) = 0$ for $k = 1, 2, 1+n_1, 2+n_1$. If $i, j > 2$ we continue this argument until the cycle closes. This permits us to choose A to be a monomial of P in the form:

$$A = g_{11/2x(2)}g_{22/3x(3)}\cdots g_{j-1,j-1/jx(j)}g_{jj/1x(1)}A_0$$

where $\deg_k(A_0) = 0$ for $1 \le k \le j$ and $n_1 + 1 \le k \le n_1 + j$.

We wish to show that the length of the cycle involved is even. We apply Lemma 2.5.10 to show

$$B = g_{2x(2)/11}g_{3x(3)/22}\cdots g_{1x(1)/jj}A_0$$

satisfies $c(A,P)c(B,P) > 0$. We apply Lemma 2.5.9 a total of j times to see

$$C = g_{22,1x(1)}g_{33/2x(2)}\cdots g_{11/jx(j)}A_0$$

satisfies $c(B,P)c(C,P)(-1)^j > 0$. We now consider the even permutation:

$$\begin{aligned} \rho(1) &= j \\ \rho(k) &= \begin{cases} k-1, & 2 \le k \le j \\ k, & j < k \le n_1 \end{cases} \\ \rho(k(j)) &= x(\rho(j)) \qquad \text{for } 1 \le j \le n_1 \end{aligned}$$

to see that $c(C_\rho, P)c(C,P) > 0$. However $A = C_\rho$ so $c(A,P)^2(-1)^j > 0$ which shows j is necessarily even. (This step is formally equivalent to using the skew symmetry of the curvature tensor to show that only polynomials of even degree can appear to give non-zero real characteristic forms).

We decompose A_0 into cycles to construct A inductively so that A has the form:

$$\begin{aligned} A = &\{g_{11/2x(2)}\cdots g_{i_1i_1/1x(1)}\} \\ &\quad\{g_{i_1+1,i_1+1/i_1+2,x(i_1+2)}\cdots g_{i_1+i_2,i_1+i_2/i_1,x(i_1)}\}\cdots \end{aligned}$$

where we decompose A into cycles of length $i_1, i_2, \ldots$ with $n_1 = i_1 + \cdots + i_j$. Since all the cycles must have even length, $\ell(A) = n/2$ is even so n is divisible by 4.

We let $n = 4k$ and let ρ be a partition of $k = k_1 + \cdots + k_j$. We let A_ρ be defined using the above equation where $i_1 = 2k_1$, $i_2 = 2k_2, \ldots$. By making a coordinate permutation we can assume $i_1 \ge i_2 \ge \cdots$. We have shown that if $P = 0$, then $c(A_\rho, P) = 0$ for some ρ. Since there are exactly $\pi(k)$ such partitions, we have constructed a family of $\pi(k)$ linear functionals on $\mathcal{P}_{n,n,n}$ which form a seperating family. This implies $\dim \mathcal{P}_{n,n,n} < \pi(k)$ which completes the proof.

We conclude this subsection with a few remarks on the proofs we have given of Theorem 2.4.7 and Lemma 2.5.6. We know of no other proof of Theorem 2.4.7 other than the one we have given. H. Weyl's theorem is only used to prove the surjectivity of the restriction map r and is inessential in the axiomatic characterization of the Euler form. This theorem gives an immediate proof of the Gauss-Bonnet theorem using heat equation methods. It is also an essential step in settling Singer's conjecture for the Euler form as we shall discuss in the fourth chapter. The fact that $r(E_m) = 0$ is, of course, just an invariant statement of the fact E_m is an unstable characteristic class; this makes it difficult to get hold of axiomatically in contrast to the Pontrjagin forms which are stable characteristic classes.

We have discussed both of the proofs of Lemma 2.5.6 which exist in the literature. The proof based on H. Weyl's theorem and on geodesic normal coordinates is more elegant, but relies heavily on fairly sophisticated theorems. The second is the original proof and is more combinatorial. It is entirely self-contained and this is the proof which generalizes to Kaehler geometry to yield an axiomatic characterization of the Chern forms of $T_c(M)$ for a holomorphic Kaehler manifold. We shall discuss this case in more detail in section 3.7.

2.6. Invariance Theory and Mixed Characteristic Classes of the Tangent Space and of a Coefficient Bundle.

In the previous subsection, we gave in Lemma 2.5.6 an axiomatic characterization of the Pontrjagin forms in terms of functorial properties. In discussing the Hirzebruch signature formula in the next chapter, it will be convenient to have a generalization of this result to include invariants which also depend on the derivatives of the connection form on an auxilary bundle.

Let V be a complex vector bundle. We assume V is equipped with a Hermitian fiber metric and let ∇ be a Riemannian or unitary connection on V. We let $\vec{s} = (s_1, \dots, s_a, \dots, s_v)$ be a local orthonormal frame for V and introduce variables ω_{abi} for the connection 1-form;

$$\nabla(s_a) = \omega_{abi}\, dx^i \otimes s_b, \qquad \text{i.e., } \nabla\vec{s} = \omega \otimes \vec{s}.$$

We introduce variables $\omega_{abi/\alpha} = d_x^\alpha(\omega_{abi})$ for the partial derivatives of the connection 1-form. We shall also use the notation $\omega_{abi/jk\dots}$. We use indices $1 \le a, b, \dots \le v$ to index the frame for V and indices $1 \le i, j, k \le m$ for the tangent space variables. We define:

$$\operatorname{ord}(\omega_{abi/\alpha}) = 1 + |\alpha| \qquad \text{and} \qquad \deg_k(\omega_{abi/\alpha}) = \delta_{i,k} + \alpha(k).$$

We let $\mathcal{Q}$ be the polynomial algebra in the $\{\omega_{abi/\alpha}\}$ variables for $|\alpha| \ge 1$; if $Q \in \mathcal{Q}$ we define the evaluation $Q(X, \vec{s}, \nabla)(x_0)$. We normalize the choice of frame $\vec{s}$ by requiring $\nabla(\vec{s})(x_0) = 0$. We also normalize the coordinate system X as before so $X(x_0) = 0$, $g_{ij}(X, G)(x_0) = \delta_{ij}$, and $g_{ij/k}(X, G)(x_0) = 0$. We say Q is invariant if $Q(X, \vec{s}, \nabla)(x_0) = Q(Y, \vec{s}\,', \nabla)(x_0)$ for any normalized frames $\vec{s}$, $\vec{s}\,'$ and normalized coordinate systems X, Y; we denote this common value by $Q(\nabla)$ (although it also depends in principle on the metric tensor and the 1-jets of the metric tensor through our normalization of the coordinate system X). We let $\mathcal{Q}_{m,p,v}$ denote the space of all invariant p-form valued polynomials in the $\{\omega_{abi/\alpha}\}$ variables for $|\alpha| \ge 1$ defined on a manifold of dimension m and for a vector bundle of complex fiber dimension v. We let $\mathcal{Q}_{m,n,p,v}$ denote the subspace of invariant polynomials homogeneous of order n in the jets of the connection form. Exactly as was done for the $\mathcal{P}_*$ algebra in the jets of the metric, we can show there is a direct sum decomposition

$$\mathcal{Q}_{m,p,v} = \bigoplus_n \mathcal{Q}_{m,n,p,v} \qquad \text{and} \qquad \mathcal{Q}_{m,n,p,v} = 0 \qquad \text{for } n - p \text{ odd.}$$

Let $Q(gAg^{-1}) = Q(A)$ be an invariant polynomial of order q in the components of a $v \times v$ matrix. Then $Q(\Omega)$ defines an element of $\mathcal{Q}_{m,2q,q,v}$ for any $m \geq 2q$. By taking $Q \cdot \bar{Q}$ we can define scalar valued invariants and by taking $\delta(Q)$ we can define other form valued invariants in $\mathcal{Q}_{m,2q+1,2q-1,v}$. Thus there are a great many such form valued invariants.

In addition to this algebra, we let $\mathcal{R}_{m,n,p,v}$ denote the space of p-form valued invariants which are homogeneous of order n in the $\{g_{ij/\alpha}, \omega_{abk/\beta}\}$ variables for $|\alpha| \geq 2$ and $|\beta| \geq 1$. The spaces $\mathcal{P}_{m,n,p}$ and $\mathcal{Q}_{m,n,p,v}$ are both subspaces of $\mathcal{R}_{m,n,p,v}$. Furthermore wedge product gives a natural map $\mathcal{P}_{m,n,p} \otimes \mathcal{Q}_{m,n',p',v} \rightarrow \mathcal{R}_{m,n+n',p+p',v}$. We say that $R \in \mathcal{R}_{m,p,p,v}$ is a characteristic form if it is in the linear span of wedge products of Pontrjagin forms of $T(M)$ and Chern forms of V. The characteristic forms are characterized abstractly by the following Theorem. This is the generalization of Lemma 2.5.6 which we shall need in discussing the signature and spin complexes.

Theorem 2.6.1.
(a) $\mathcal{R}_{m,n,p,v} = 0$ if $n < p$ or if $n = p$ and n is odd.
(b) If $R \in \mathcal{R}_{m,n,n,v}$ then R is a characteristic form.

Proof: The proof of this fact relies heavily on Lemma 2.5.6 but is much easier. We first need the following generalization of Lemma 2.5.1:

Lemma 2.6.2. *Using the notation of Lemma 2.5.1 we define $T_z^*(R)$ if R is a scalar invariant in the $\{g_{ij/\alpha}, \omega_{abj/\beta}\}$ variables.*
(a) Let $g_{12/\alpha}$ divide some monomial of R, then $g_{../\beta}$ divides some other monomial of R.
(b) Let $g_{../\alpha}$ divide some monomial of R, then $g_{../\beta}$ divides some other monomial of R where $\beta(1) = \alpha(1) + \alpha(2)$ and $\beta(2) = 0$.
(c) Let $\omega_{abi/\alpha}$ divide some monomial of R for $i > 2$, then $\omega_{abi/\beta}$ divides some other monomial of R where $\beta(1) = \alpha(1) + \alpha(2)$ and $\beta(2) = 0$.

The proof of this is exactly the same as that given for Lemma 2.5.1 and is therefore omitted.

The proof of Theorem 2.6.1 parallels the proof of Lemma 2.5.6 for a while so we summarize the argument briefly. Let $0 \neq R \in \mathcal{R}_{m,n,p,v}$. The same argument given in Lemma 2.5.3 shows an invariant polynomial is homogeneous of order n if $R(c^2G, \nabla) = c^{p-n}R(G, \nabla)$ which gives a invariant definition of the order of a polynomial. The same argument as given in Lemma 2.5.4 shows $n - p$ must be even and that if A is a monomial of R, A is a monomial of exactly one of the R_I. If $c(A, R_I) = 0$ then $\deg_k(A) + \deg_k(I)$ is always even. We decompose A in the form:

$$A = g_{i_1 j_1/\alpha_1} \dots g_{i_q j_q/\alpha_q} \omega_{a_1 b_1 k_1/\beta_1} \dots \omega_{a_r b_r k_r/\beta_r} = A^g A^\omega$$

and define $\ell(A) = q + r$ to the length of A. We argue using Lemma 2.6.2 to choose A so $\deg_k(A^g) = 0$ for $k > 2q$. By making a coordinate permutation we can assume that the $k_\nu \leq 2q + r$ for $1 \leq \nu \leq r$. We apply Lemma 2.6.2(c) a total of r times to choose the β_i so $\beta_1(k) = 0$ for $k > 2q+r+1$, $\beta_2(k) = 0$ for $k > 2q+r+2, \ldots$, $\beta_r(k) = 0$ for $k > 2q+2r$. This chooses A so $\deg_k(A) = 0$ for $k > 2\ell(A)$. If A is a monomial of R_I for $I = \{1 \le i_1 < \cdots < i_p \le m\}$ then $\deg_{i_p}(A)$ is odd. We estimate $p \le i_p \le 2\ell(A) \le \sum |\alpha_\nu| + \sum(|\beta_\mu|+1) = n$ so that $\mathcal{P}_{m,n,p,v} = \{0\}$ if $n < p$ or if $n-p$ is odd which proves (a) of Theorem 2.6.1.

In the limiting case, we must have equalities so $|\alpha_\nu| = 2$ and $|\beta_\mu| = 1$. Furthermore, $i_p = p$ so there is some monomial A so $\deg_k(A) = 0$ for $k > p = n$ and $A\,dx^1 \wedge \cdots \wedge dx^p$ appears in R. There is a natural restriction map

$$r\colon \mathcal{R}_{m,n,p,v} \to \mathcal{R}_{m-1,n,p,v}$$

and our argument shows $r\colon \mathcal{R}_{m,n,n,v} \to \mathcal{R}_{m-1,n,n,v}$ is injective for $n < m$. Since the restriction of a characteristic form is a characteristic form, it suffices to prove (b) of Theorem 2.6.1 for the case $m = n = p$.

Let $0 \ne R \in \mathcal{R}_{n,n,n,v}$ then R is a polynomial in the $\{g_{ij/kl}, \omega_{abi/j}\}$ variables. The restriction map r was defined by considering products $M_1 \times S^1$ but there are other functorial constructions which give rise to useful projections. Fix non-negative integers (s,t) so that $n = s+t$. Let M_1 be a Riemannian manifold of dimension s. Let M_2 be the flat torus of dimension t and let V_2 be a vector bundle with connection ∇_2 over M_2. Let $M = M_1 \times M_2$ with the product metric and let V be the natural extension of V_2 to M which is flat in the M_1 variables. More exactly, if $\pi_2\colon M \to M_2$ is a projection on the second factor, then $(V,\nabla) = \pi_2^*(V_2,\nabla_2)$ is the pull back bundle with the pull back connection. We define

$$\pi_{(s,t)}(R)(G_1,\nabla_2) = R(G_1 \times 1, \nabla).$$

Using the fact that $\mathcal{P}_{s,n_1,p_1} = 0$ for $s < p_1$ or $n_1 < p_1$ and the fact $\mathcal{Q}_{t,n_2,p_2,k} = 0$ for $t < p_2$ or $n_2 < p_2$ it follows that $\pi_{(s,t)}$ defines a map

$$\pi_{(s,t)}\colon \mathcal{R}_{n,n,v} \to \mathcal{P}_{s,s,s} \otimes \mathcal{Q}_{t,t,t,v}.$$

More algebraically, let $A = A^g A^\omega$ be a monomial, then we define:

$$\pi_{(s,t)}(A) = \begin{cases} 0 & \text{if } \deg_k(A^g) > 0 \text{ for } k > s \text{ or } \deg_k(A^\omega) > 0 \text{ for } k \le s \\ A & \text{otherwise.} \end{cases}$$

The only additional relations imposed are to set $g_{ij/kl} = 0$ if any of these indices exceeds s and to set $\omega_{abi/j} = 0$ if either i or j is less than or equal to s.

We use these projections to reduce the proof of Theorem 2.6.1 to the case in which $R \in \mathcal{Q}_{t,t,t,v}$. Let $0 \ne R \in \mathcal{R}_{n,n,n,v}$ and let $A = A^g A^\omega$ be a monomial of R. Let $s = 2\ell(A^g) = \operatorname{ord}(A^g)$ and let $t = n - s = 2\ell(A^\omega) =$

$\operatorname{ord}(A^\omega)$. We choose A so $\deg_k(A^g) = 0$ for $k > s$. Since $\deg_k(A) \geq 1$ must be odd for each index k, we can estimate:

$$t \leq \sum_{k>s} \deg_k(A) = \sum_{k>s} \deg_k(A^\omega) \leq \sum_k \deg_k(A^\omega) = \operatorname{ord}(A^\omega) = t.$$

As all these inequalities must be equalities, we conclude $\deg_k(A^\omega) = 0$ for $k \leq s$ and $\deg_k(A^\omega) = 1$ for $k > s$. This shows in particular that $\pi_{(s,t)}(R) \neq 0$ for some (s,t) so that

$$\bigoplus_{s+t=n} \pi_{(s,t)} \colon \mathcal{R}_{n,n,n,v} \to \bigoplus_{s+t=n} \mathcal{P}_{s,s,s} \otimes \mathcal{Q}_{t,t,t,v}$$

is injective.

We shall prove that $\mathcal{Q}_{t,t,t,v}$ consists of characteristic forms of V. We showed earlier that $\mathcal{P}_{s,s,s}$ consists of Pontrjagin forms of $T(M)$. The characteristic forms generated by the Pontrjagin forms of $T(M)$ and of V are elements of $\mathcal{R}_{n,n,n,v}$ and $\pi_{(s,t)}$ just decomposes such products. Thererfore π is surjective when restricted to the subspace of characteristic forms. This proves π is bijective and also that $\mathcal{R}_{n,n,n,v}$ is the space of characteristic forms. This will complete the proof of Theorem 2.6.1.

We have reduced the proof of Theorem 2.6.1 to showing $\mathcal{Q}_{t,t,t,v}$ consists of the characteristic forms of V. We noted that $0 \neq Q \in \mathcal{Q}_{t,t,t,v}$ is a polynomial in the $\{\omega_{abi/j}\}$ variables and that if A is a monomial of Q, then $\deg_k(A) = 1$ for $1 \leq k \leq t$. Since $\operatorname{ord}(A) = t$ is even, we conclude $\mathcal{Q}_{t,t,t,v} = 0$ if t is odd.

The components of the curvature tensor are given by:

$$\Omega_{abij} = \omega_{abi/j} - \omega_{abi/j} \qquad \text{and} \qquad \Omega_{ab} = \sum \Omega_{abij}\, dx_i \wedge dx_j$$

up to a possible sign convention and factor of $\frac{1}{2}$ which play no role in this discussion. If A is a monomial of P, we decompose:

$$A = \omega_{a_1 b_1 i_1/i_2} \dots \omega_{a_u b_u i_{t-1}/i_t} \qquad \text{where } 2u = t.$$

All the indices i_ν are distinct. If ρ is a permutation of these indices, then $c(A,P) = \operatorname{sign}(\rho) c(A^\rho, P)$. This implies we can express P in terms of the expressions:

$$\begin{aligned}\bar{A} &= (\omega_{a_1 b_1 i_1/i_2} - \omega_{a_1 b_1 i_2/i_1}) \dots (\omega_{a_u b_u i_{t-1}/i_t} - \omega_{a_u b_u i_t/i_{t-1}}) \\ &\qquad\qquad\qquad\qquad dx_{i_1} \wedge \dots \wedge dx_{i_t} \\ &= \Omega_{a_1 b_1 i_1 i_2} \dots \Omega_{a_u b_u i_{t-1} i_t}\, dx_{i_1} \wedge \dots \wedge dx_{i_t}\end{aligned}$$

Again, using the alternating nature of these expression, we can express P in terms of expressions of the form:

$$\Omega_{a_1 b_1} \wedge \dots \wedge \Omega_{a_u b_u}$$

so that $Q = Q(\Omega)$ is a polynomial in the components Ω_{ab} of the curvature. Since the value of Q is independent of the frame chosen, Q is the invariant under the action of $\mathrm{U}(v)$. Using the same argument as that given in the proof of Lemma 2.1.3 we see that in fact Q is a characteristic form which completes the proof.

We conclude this subsection with some uniqueness theorems regarding the local formulas we have been considering. If M is a holomorphic manifold of real dimension m and if V is a complex vector bundle of fiber dimension v, we let $\mathcal{R}^{ch}_{m,m,p,v}$ denote the space of p-form valued invariants generated by the Chern forms of V and of $T_c(M)$. There is a suitable axiomatic characterization of these spaces using invariance theory for Kaehler manifolds which we shall discuss in section 3.7. The uniqueness result we shall need in proving the Hirzebruch signature theorem and the Riemann-Roch theorem is the following:

LEMMA 2.6.3.
(a) Let $0 \neq R \in \mathcal{R}_{m,m,m,v}$ then there exists (M,V) so M is oriented and $\int_M R(G,\nabla) \neq 0$.
(b) Let $0 \neq R \in \mathcal{R}^{ch}_{m,m,m,v}$ then there exists (M,V) so M is a holomorphic manifold and $\int_M R(G,\nabla) \neq 0$.

PROOF: We prove (a) first. Let $\rho = \{1 \leq i_1 \leq \cdots \leq i_\nu\}$ be a partition $k(\rho) = i_1 + \cdots + i_\nu$ for $4k = s \leq m$. Let $\{M^r_\rho\}$ be the collection of manifolds of dimension s discussed in Lemma 2.3.4. Let P_ρ be the corresponding real characteristic form so

$$\int_{M^r_\rho} P_\tau(G) = \delta_{\rho,\tau}.$$

The $\{P_\rho\}$ forms a basis for $\mathcal{P}_{m,s,s}$ for any $s \leq m$. We decompose

$$R = \sum_\rho P_\rho Q_\rho \qquad \text{for } Q_\rho \in \mathcal{Q}_{m,t,t,k}, \text{ where } t + s = m.$$

This decomposition is, of course, nothing but the decomposition defined by the projections $\pi_{(s,t)}$ discussed in the proof of the previous lemma.

Since $R \neq 0$, at least one of the $Q_\rho \neq 0$. We choose ρ so $k(\rho)$ is maximal with $Q_\rho \neq 0$. We consider $M = M^r_\rho \times M_2$ and $(V,\nabla) = \pi_2^*(V_2,\nabla_2)$ where (V_2,∇_2) is a bundle over M_2 which will be specified later. Then we compute:

$$\int_M P_\tau(G) Q_\tau(\nabla) = 0$$

unless $\operatorname{ord}(Q_\tau) \leq \dim(M_2)$ since ∇ is flat along M^r_ρ. This implies $k(\tau) \geq k(\rho)$ so if this integral is non-zero $k(\tau) = k(\rho)$ by the maximality of ρ. This implies

$$\int_M P_\tau(G) Q_\tau(\nabla) = \int_{M^r_\rho} P_\tau(G) \cdot \int_{M_2} Q_\tau(\nabla_2) = \delta_{\rho,\tau} \int_{M_2} Q_\tau(\nabla_2).$$

This shows that

$$\int_M R(G,\nabla) = \int_{M_2} Q_\rho(\nabla_2)$$

and reduces the proof of this lemma to the special case $Q \in \mathcal{Q}_{m,m,m,v}$.

Let A be a $v \times v$ complex matrix and let $\{x_1, \dots, x_v\}$ be the normalized eigenvalues of A. If $2k = m$ and if ρ is a partition of k we define

$$x_\rho = x_1^{i_1} \dots x_\nu^{i_\nu},$$

then x_ρ is a monomial of $Q(A)$ for some ρ. We let $M = M_1 \times \cdots \times M_\nu$ with $\dim(M_j) = 2i_j$ and let $V = L_1 \oplus \cdots \oplus L_\nu \oplus 1^{v-\nu}$ where the L_j are line bundles over M_j. If $c(x_\rho, Q)$ is the coefficient of x in Q, then:

$$\int_M Q(\nabla) = c(x_\rho, Q) \prod_{j=1}^{\nu} \int_{M_j} c_1(L_j)^{i_j}.$$

This is, of course, nothing but an application of the splitting principle. This reduces the proof of this lemma to the special case $Q \in \mathcal{Q}_{m,m,m,1}$.

If $Q = c_1^k$ we take $M = S^2 \times \cdots \times S^2$ to be the k-fold product of two dimensional spheres. We let L_j be a line bundle over the j $^{\text{th}}$ factor of S^2 and let $L = L_1 \otimes \cdots \otimes L_k$. Then $c_1(L) = c_1(L_j)$ so

$$\int_M c_1(L)^k = k! \prod_{j=1}^{k} \int_{S^2} c_1(L_j)$$

which reduces the proof to the case $m = 2$ and $k = 1$. We gave an example in Lemma 2.1.5 of a line bundle over S^2 so $\int_{S^2} c_1(L) = 1$. Alternatively, if we use the Gauss-Bonnet theorem with $L = T_c(S^2)$, then $\int_{S^2} c_1(T_c(S^2)) = \int_{S^2} e_2(T(S^2)) = \chi(S^2) = 2 \neq 0$ completes the proof of (a). The proof of (b) is the same where we replace the real manifolds M_ρ^r by the corresponding manifolds M_ρ^c and the real basis P_ρ by the corresponding basis P_ρ^c of characteristic forms of $T_c(M)$. The remainder of the proof is the same and relies on Lemma 2.3.4 exactly as for (a) and is therefore omitted in the interests of brevity.

CHAPTER 3

THE INDEX THEOREM

Introduction

In this the third chapter, we complete the proof of the index theorem for the four classical elliptic complexes. We give a proof of the Aityah-Singer theorem in general based on the Chern isomorphism between K-theory and cohomology (which is not proved). Our approach is to use the results of the first chapter to show there exists a suitable formula with the appropriate functorial properties. The results of the second chapter imply it must be a characteristic class. The normalizing constants are then determined using the method of universal examples.

In section 3.1, we define the twisted signature complex and prove the Hirzebruch signature theorem. We shall postpone until section 3.4 the determination of all the normalizing constants if we take coefficients in an auxilary bundle. In section 3.2, we introduce spinors as a means of connecting the de Rham, signature and Dolbeault complexes. In section 3.3, we discuss the obstruction to putting a spin structure on a real vector bundle in terms of Stieffel-Whitney classes. We compute the characteristic classes of spin bundles.

In section 3.4, we discuss the spin complex and the $\hat{A}$ genus. In section 3.5, we use the spin complex together with the spin_c representation to discuss the Dolbeault complex and to prove the Riemann-Roch theorem for almost complex manifolds. In sections 3.6 and 3.7 we give another treatment of the Riemann-Roch theorem based on a direct approach for Kaehler manifolds. For Kaehler manifolds, the integrands arising from the heat equation can be studied directly using an invariant characterization of the Chern forms similar to that obtained for the Euler form. These two subsections may be deleted by a reader not interested in Kaehler geometry.

In section 3.8, we give the preliminaries we shall need to prove the Atiyah-Singer index theorem in general. The only technical tool we will use which we do not prove is the Chern isomorphism between rational cohomology and K-theory. We give a discussion of Bott periodicity using Clifford algebras. In section 3.9, we show that the index can be treated as a formula in rational K-theory. We use constructions based on Clifford algebras to determine the normalizing constants involved. For these two subsections, some familarity with K-theory is helpful, but not essential.

Theorems 3.1.4 and 3.6.10 were also derived by V. K. Patodi using a complicated cancellation argument as a replacement of the invariance theory presented in Chapter 2. A similar although less detailed discussion may also be found in the paper of Atiyah, Bott and Patodi.

3.1. The Hirzebruch Signature Formula.

The signature complex is best described using Clifford algebras as these provide a unified framework in which to discuss (and avoid) many of the $\pm$ signs which arise in dealing with the exterior algebra directly. The reader will note that we are choosing the opposite sign convention for our discussion of Clifford algebras from that adopted in the example of Lemma 2.1.5. This change in sign convention is caused by the $\sqrt{-1}$ present in discussing the symbol of a first order operator.

Let V be a real vector with a positive definite inner product. The Clifford algebra $\operatorname{CLIF}(V)$ is the universal algebra generated by V subject to the relations

$$v * v + (v, v) = 0 \text{ for } v \in V.$$

If the $\{e_i\}$ are an orthonormal basis for V and if $I = \{1 \le i_1 < \cdots < i_p \le \dim(V)\}$ then $e_i * e_j + e_j * e_i = -2\delta_{ij}$ is the Kronecker symbol and

$$e_I = e_{i_1} * \cdots * e_{i_p}$$

is an element of the Clifford algebra. $\operatorname{CLIF}(V)$ inherits a natural inner product from V and the $\{e_I\}$ form an orthonormal basis for V.

If $\Lambda(V)$ denotes the exterior algebra of V and if $\operatorname{END}(\Lambda(V))$ is the algebra of linear endomorphisms of $\Lambda(V)$, there is a natural representation of $\operatorname{CLIF}(V)$ into $\operatorname{END}(\Lambda(V))$ given by Clifford multiplication. Let $\operatorname{ext}: V \to \operatorname{END}(\Lambda(V))$ be exterior multiplication on the left and let $\operatorname{int}(v)$ be interior multiplication, the adjoint. For example:

$$\operatorname{ext}(e_1)(e_{i_1} \wedge \cdots \wedge e_{i_p}) = \begin{cases} e_1 \wedge e_{i_1} \wedge \cdots \wedge e_{i_p} & \text{if } i_1 > 1 \\ 0 & \text{if } i_1 = 1 \end{cases}$$

$$\operatorname{int}(e_1)(e_{i_1} \wedge \cdots \wedge e_{i_p}) = \begin{cases} e_{i_2} \wedge \cdots \wedge e_{i_p} & \text{if } i_1 = 1 \\ 0 & \text{if } i_1 > 1. \end{cases}$$

We define

$$c(v) = \operatorname{ext}(v) - \operatorname{int}(v): V \to \operatorname{END}(\Lambda(V)).$$

It is immediate from the definition that

$$c(v)^2 = -(\operatorname{ext}(v)\operatorname{int}(v) + \operatorname{int}(v)\operatorname{ext}(v)) = -|v|^2 I$$

so that c extends to define an algebra morphism

$$c: \operatorname{CLIF}(V) \to \operatorname{END}(\Lambda(V)).$$

Furthermore:

$$c(e_I)1 = e_{i_1} \wedge \cdots \wedge e_{i_p}$$

so the map $w \mapsto c(w)1$ defines a vector space isomorphism (which is not, of course an algebra morphism) between $\mathrm{CLIF}(V)$ and $\Lambda(V)$. Relative to an orthonormal frame, we simply replace Clifford multiplication by exterior multiplication.

Since these constructions are all independent of the basis chosen, they extend to the case in which V is a real vector bundle over M with a fiber metric which is positive definite. We define $\mathrm{CLIF}(V)$, $\Lambda(V)$, and $c\colon \mathrm{CLIF}(V) \rightarrow \mathrm{END}(\Lambda(V))$ as above. Since we only want to deal with complex bundles, we tensor with $\mathbf{C}$ at the end to enable us to view these bundles as being complex. We emphasize, however, that the underlying constructions are all real.

Clifford algebras provide a convenient way to describe both the de Rham and the signature complexes. Let $(d+\delta)\colon C^\infty(\Lambda(T^*M)) \rightarrow C^\infty(\Lambda(T^*M))$ be exterior differentiation plus its adjoint as discussed earlier. The leading symbol of $(d+\delta)$ is $\sqrt{-1}(\mathrm{ext}(\xi)-\mathrm{int}(\xi)) = \sqrt{-1}c(\xi)$. We use the following diagram to define an operator A; let ∇ be covariant differentiation. Then:

$$A\colon C^\infty(\Lambda(T^*M)) \xrightarrow{\nabla} C^\infty(T^*M \otimes \Lambda(T^*M)) \xrightarrow{c} C^\infty(\Lambda(T^*M)).$$

A is invariantly defined. If $\{e_i\}$ is a local orthonormal frame for $T^*(M)$ which we identify with $T(M)$, then:

$$A(\omega) = \sum_i (\mathrm{ext}(e_i) - \mathrm{int}(e_i))\nabla_{e_i}(\omega).$$

Since the leading symbol of A is $\sqrt{-1}c(\xi)$, these two operators have the same leading symbol so $(d+\delta) - A = A_0$ is an invariantly defined 0^{th} order operator. Relative to a coordinate frame, we can express A_0 as a linear combination of the 1-jets of the metric with coefficients which are smooth in the $\{g_{ij}\}$ variables. Given any point x_0, we can always choose a frame so the $g_{ij/k}$ variables vanish at x_0 so $A_0(x_0) = 0$ so $A_0 \equiv 0$. This proves $A = (d+\delta)$ is defined by this diagram which gives a convenient way of describing the operator $(d+\delta)$ in terms of Clifford multiplication.

This trick will be useful in what follows. If A and B are natural first order differential operators with the same leading symbol, then $A = B$ since $A - B$ is a 0^{th} order operator which is linear in the 1-jets of the metric. This trick does not work in the holomorphic category unless we impose the additional hypothesis that M is Kaehler. This makes the study of the Riemann-Roch theorem more complicated as we shall see later since there are many natural operators with the same leading symbol.

We let $\alpha \in \mathrm{END}(\Lambda(T^*M))$ be defined by:

$$\alpha(\omega_p) = (-1)^p \omega_p \qquad \text{for } \omega_p \in \Lambda^p(T^*M).$$

It is immediate that

$$\operatorname{ext}(\xi)\alpha = -\alpha \operatorname{ext}(\xi) \qquad \text{and} \qquad \operatorname{int}(\xi)\alpha = -\alpha \operatorname{int}(\xi)$$

so that $-\alpha(d+\delta)\alpha$ and $(d+\delta)$ have the same leading symbol. This implies

$$\alpha(d+\delta) = -(d+\delta)\alpha.$$

We decompose $\Lambda(T^*M) = \Lambda^{\mathrm{e}}(T^*M) \oplus \Lambda^{\mathrm{o}}(T^*M)$ into the differential forms of even and odd degree. This decomposes $\Lambda(T^*M)$ into the ± 1 eigenspaces of α. Since $(d+\delta)$ anti-commutes with α, we decompose

$$(d+\delta)_{\mathrm{e,o}}\colon C^\infty(\Lambda^{\mathrm{e,o}}(T^*M)) \to C^\infty(\Lambda^{\mathrm{e,o}}(T^*M))$$

where the adjoint of $(d+\delta)_{\mathrm{e}}$ is $(d+\delta)_{\mathrm{o}}$. This is, of course, just the de Rham complex, and the index of this elliptic operator is $\chi(M)$.

If $\dim(M) = m$ is even and if M is oriented, there is another natural endomorphism $\tau \in \mathrm{END}(\Lambda(T^*M))$. It can be used to define an elliptic complex over M called the signature complex in just the same way that the de Rham complex was defined. Let $\mathrm{dvol} \in \Lambda^m(T^*M)$ be the volume form. If $\{e_i\}$ is an oriented local orthonormal frame for T^*M, then $\mathrm{dvol} = e_1 \wedge \cdots \wedge e_m$. We can also regard $\mathrm{dvol} = e_1 * \cdots * e_m \in \mathrm{CLIF}(T^*M)$ and we define

$$\tau = (\sqrt{-1})^{m/2} c(\mathrm{dvol}) = (\sqrt{-1})^{m/2} c(e_1)\ldots c(e_m).$$

We compute:

$$\begin{aligned}\tau^2 &= (-1)^{m/2} c(e_1 * \cdots * e_m * e_1 * \cdots * e_m)\\ &= (-1)^{m/2}(-1)^{m+(m-1)+\cdots+1}\\ &= (-1)^{(m+(m+1)m)/2} = 1.\end{aligned}$$

Because m is even, $c(\xi)\tau = -\tau c(\xi)$ so τ anti-commutes with the symbol of $(d+\delta)$. If we decompose $\Lambda(T^*M) = \Lambda^+(T^*M) \oplus \Lambda^-(T^*M)$ into the ± 1 eigenvalues of τ, then $(d+\delta)$ decomposes to define:

$$(d+\delta)_\pm\colon C^\infty(\Lambda^\pm(T^*M)) \to C^\infty(\Lambda^\mp(T^*M))$$

where the adjoint of $(d+\delta)_+$ is $(d+\delta)_-$. We define:

$$\mathrm{signature}(M) = \mathrm{index}(d+\delta)_+$$

to be the signature of M. (This is also often refered to as the index of M, but we shall not use this notation as it might be a source of some confusion).

We decompose the Laplacian $\Delta = \Delta^+ \oplus \Delta^-$ so

$$\operatorname{signature}(M) = \dim \mathrm{N}(\Delta^+) - \dim \mathrm{N}(\Delta^-).$$

Let $a_n^s(x, \mathrm{orn}) = a_n(x, \Delta^+) - a_n(x, \Delta^-)$ be the invariants of the heat equation; they depend on the orientation orn chosen. Although we have complexified the bundles $\Lambda(T^*M)$ and $\mathrm{CLIF}(T^*M)$, the operator $(d+\delta)$ is real. If $m \equiv 2$ (4), then τ is pure imaginary. Complex conjugation defines an isomorphism

$$\Lambda^+(T^*M) \overset{\approx}{\to} \Lambda^-(T^*M) \qquad \text{and} \qquad \Delta^+ \overset{\approx}{\to} \Delta^-.$$

This implies $\operatorname{signature}(M) = 0$ and $a_n^s(x, \mathrm{orn}) = 0$ in this case. We can get a non-zero index if $m \equiv 2$ (4) if we take coefficients in some auxiliary bundle as we shall discuss shortly.

If $m \equiv 0$ (4), then τ is a real endomorphism. In general, we compute:

$$\begin{aligned}\tau(e_1 \wedge \cdots \wedge e_p) &= (\sqrt{-1})^{m/2} e_1 * \cdots * e_m * e_1 * \cdots * e_p \\ &= (\sqrt{-1})^{m/2}(-1)^{p(p-1)/2}\, e_{p+1} \wedge \cdots \wedge e_m.\end{aligned}$$

If "$*$" is the Hodge star operator discussed in the first sections, then

$$\tau_p = (\sqrt{-1})^{m/2+p(p-1)} *_p$$

acting on p-forms. The spaces $\Lambda^p(T^*M) \oplus \Lambda^{m-p}(T^*M)$ are invariant under τ. If $p \neq m-p$ there is a natural isomorphism

$$\Lambda^p(T^*M) \overset{\approx}{\to} (\Lambda^p(T^*M) \oplus \Lambda^{m-p}(T^*M))^{\pm} \qquad \text{by} \quad \omega_p \mapsto \tfrac{1}{2}(\omega_p \pm \omega_p).$$

This induces a natural isomorphism from

$$\Delta_p \overset{\approx}{\to} \Delta^{\pm} \text{ on } (\Lambda^p(T^*M) \oplus \Lambda^{m-p}(T^*M))^{\pm}$$

so these terms all cancel off in the alternating sum and the only contribution is made in the middle dimension $p = m - p$.

If $m = 4k$ and $p = 2k$ then $\tau = *$. We decompose $\mathrm{N}(\Delta_p) = \mathrm{N}(\Delta_p^+) \oplus \mathrm{N}(\Delta_p^-)$ so $\operatorname{signature}(M) = \dim \mathrm{N}(\Delta_p^+) - \dim \mathrm{N}(\Delta_p^-)$. There is a natural symmetric bilinear form on $H^{2k}(T^*M; \mathbf{C}) = \mathrm{N}(\Delta_p)$ defined by

$$I(\alpha_1, \alpha_2) = \int_M \alpha_1 \wedge \alpha_2.$$

If we use the de Rham isomorphism to identify de Rham and simplicial cohomology, then this bilinear form is just the evaluation of the cup product of two cohomology classes on the top dimensional cycle. This shows I can be defined in purely topological terms.

The index of a real quadratic form is just the number of $+1$ eigenvalues minus the number of -1 eigenvalues when it is diagonalized over $\mathbf{R}$. Since

$$I(\alpha,\beta)=\int_M \alpha\wedge\beta=\int_M(\alpha,*\beta)\,\mathrm{dvol}=(\alpha,*\beta)_{L^2},$$

we see

$$\begin{aligned}\operatorname{index}(I)&=\dim\{+1 \text{ eigenspace of } * \text{ on } H^p\}\\&\qquad-\dim\{-1 \text{ eigenspace of } * \text{ on } H^p\}\\&=\dim \mathrm{N}(\Delta^+)-\dim\mathrm{N}(\Delta^-)=\operatorname{signature}(M).\end{aligned}$$

This gives a purely topological definition of the signature of M in terms of cup product. We note that if we reverse the orientation of M, then the signature changes sign.

Example: Let $M=\mathbf{C}P_{2k}$ be complex projective space. Let $x\in H^2(M;\mathbf{C})$ be the generator. Since x^k is the generator of $H^{2k}(M;\mathbf{C})$ and since $x^k\wedge x^k=x^{2k}$ is the generator of $H^{4k}(M;\mathbf{C})$, we conclude that $*x^k=x^k$. $\dim\mathrm{N}(\Delta^+_{2k})=1$ and $\dim\mathrm{N}(\Delta^-_{2k})=0$ so $\operatorname{signature}(\mathbf{C}P_{2k})=1$.

An important tool in the study of the de Rham complex was its multiplicative properties under products. Let M_i be oriented even dimensional manifolds and let $M=M_1\times M_2$ with the induced orientation. Decompose:

$$\begin{aligned}\Lambda(T^*M)&=\Lambda(T^*M_1)\otimes\Lambda(T^*M_2)\\ \mathrm{CLIF}(T^*M)&=\mathrm{CLIF}(T^*M_1)\otimes\mathrm{CLIF}(T^*M_2)\end{aligned}$$

as graded non-commutative algebras—i.e.,

$$(\omega_1\otimes\omega_2)\circ(\omega_1'\otimes\omega_2')=(-1)^{\deg\omega_2\cdot\deg\omega_1'}(\omega_1\circ\omega_1')\otimes(\omega_2\circ\omega_2')$$

for $\circ=$ either $\wedge$ or $*$. Relative to this decomposition, we have:

$$\tau=\tau_1\otimes\tau_2 \text{ where the } \tau_i \text{ commute.}$$

This implies that:

$$\begin{aligned}\Lambda^+(M)&=\Lambda^+(T^*M_1)\otimes\Lambda^+(T^*M_2)\ \oplus\ \Lambda^-(T^*M_1)\otimes\Lambda^-(T^*M_2)\\ \Lambda^-(M)&=\Lambda^-(T^*M_1)\otimes\Lambda^+(T^*M_2)\ \oplus\ \Lambda^+(T^*M_1)\otimes\Lambda^-(T^*M_2)\\ \mathrm{N}(\Delta^+)&=\mathrm{N}(\Delta_1^+)\otimes\mathrm{N}(\Delta_2^+)\ \oplus\ \mathrm{N}(\Delta_1^-)\otimes\mathrm{N}(\Delta_2^-)\\ \mathrm{N}(\Delta^-)&=\mathrm{N}(\Delta_1^-)\otimes\mathrm{N}(\Delta_2^+)\ \oplus\ \mathrm{N}(\Delta_1^+)\otimes\mathrm{N}(\Delta_2^-)\\ \operatorname{signature}(M)&=\operatorname{signature}(M_1)\operatorname{signature}(M_2).\end{aligned}$$

Example: Let ρ be a partition of $k = i_1 + \cdots + i_j$ and $M_\rho^r = \mathbf{C}P_{2i_1} \times \cdots \times \mathbf{C}P_{2i_j}$. Then signature$(M_\rho^r) = 1$. Therefore if L_k is the Hirzebruch L-polynomial,

$$\operatorname{signature}(M_\rho^r) = \int_{M_\rho^p} L_k(T(M_\rho^r))$$

by Lemma 2.3.5.

We can now begin the proof of the Hirzebruch signature theorem. We shall use the same argument as we used to prove the Gauss-Bonnet theorem with suitable modifications. Let $m = 4k$ and let $a_n^s(x, \mathrm{orn}) = a_n(x, \Delta^+) - a_n(x, \Delta^-)$ be the invariants of the heat equation. By Lemma 1.7.6:

$$\int_M a_n^s(x, \mathrm{orn}) = \begin{cases} 0 & \text{if } n \neq m \\ \operatorname{signature}(M) & \text{if } n = m \end{cases}$$

so this gives a local formula for signature(M). We can express τ functorially in terms of the metric tensor. We can find functorial local frames for $\Lambda^\pm$ relative to any oriented coordinate system in terms of the coordinate frames for $\Lambda(T^*M)$. Relative to such a frame, we express the symbol of $\Delta^\pm$ functorially in terms of the metric. The leading symbol is $|\xi|^2 I$; the first order symbol is linear in the 1-jets of the metric with coefficients which depend smoothly on the $\{g_{ij}\}$ variables; the 0^{th} order symbol is linear in the 2-jets of the metric and quadratic in the 1-jets of the metric with coefficients which depend smoothly on the $\{g_{ij}\}$ variables. By Lemma 2.4.2, we conclude $a_n^s(x, \mathrm{orn})$ is homogeneous of order n in the jets of the metric.

It is worth noting that if we replace the metric G by $c^2 G$ for $c > 0$, then the spaces $\Lambda^\pm$ are not invariant. On Λ^p we have:

$$\tau(c^2 G)(\omega_p) = c^{2p-m} \tau(G)(\omega_p).$$

However, in the middle dimension we have τ is invariant as are the spaces $\Lambda_p^\pm$ for $2p = m$. Clearly $\Delta_p^\pm(c^2 G) = c^{-2} \Delta_p^\pm(G)$. Since $a_n^s(x, \mathrm{orn})$ only depends on the middle dimension, this provides another proof that a_n^s is homogeneous of order n in the derivatives of the metric since

$$\begin{aligned} a_n^s(x, \mathrm{orn})(c^2 G) &= a_n(x, c^{-2}\Delta_p^+) - a_n(x, c^{-2}\Delta_p^-) \\ &= c^{-n} a_n(x, \Delta_p^+) - c^{-n} a_n(x, \Delta_p^-) = c^{-n} a_n^s(x, \mathrm{orn})(G). \end{aligned}$$

If we reverse the orientation, we interchange the roles of Δ^+ and Δ^- so a_n^s changes sign if we reverse the orientation. This implies a_n^s can be regarded as an invariantly defined m-form; $a_n^s(x) = a_n^s(x, \mathrm{orn})\,\mathrm{dvol} \in \mathcal{P}_{m,n,m}$.

Theorem 3.1.1. *Let* $a_n^s = \{a_n(x,\Delta^+) - a_n(x,\Delta^-)\}\,\mathrm{dvol} \in \mathcal{P}_{m,n,m}$ *then:*
(a) $a_n^s = 0$ *if either* $m \equiv 2\ (4)$ *or if* $n < m$.
(b) If $m = 4k$, *then* $a_{4k}^s = L_k$ *is the Hirzebruch polynomial so*

$$\mathrm{signature}(M) = \int_M L_k.$$

(Hirzebruch Signature Theorem).

Proof: We already noted that Δ^+ is naturally isomorphic to Δ^- if $m \equiv 2\ (4)$ so $a_n^s = 0$ in that case. Lemma 2.5.6 implies $a_n^s = 0$ for $n < m$. If $m = 4k$, then a_m^s is a characteristic form of $T(M)$ by Lemma 2.5.6. We know

$$\int_{M_\rho^r} a_m^s = \mathrm{signature}(M_\rho^r) = 1 = \int_{M_\rho^r} L_k$$

so Lemma 2.3.4 implies $a_m^s = L_k$. Since $\mathrm{signature}(M) = \int_M a_m^s$ for any manifold M, we conclude $\mathrm{signature}(M) = \int_M L_k$ in general which completes the proof of (b).

If $\omega \in \Lambda(T^*M)$, we define $\int_M \omega = \int_M \omega_m$ of the top degree form. With this notational convention, we can also express

$$\mathrm{signature}(M) = \int_M L$$

which is a common form in which the Hirzebruch signature theorem appears.

It is worth making a few remarks about the proof of this result. Just as in the case of the de Rham complex, the heat equation furnishes us with the a priori local formula for the signature of M. The invariance theory of the second chapter identifies this local formula as a characteristic class. We evaluate this local formula on a sufficient number of classifying examples to determine the normalizing constants to prove $a_m^s = L_k$.

There are a great many consequences of this theorem and of the Gauss-Bonnet theorem. We present just a few to illustrate some of the applications:

Corrolary 3.1.2.
(a) Let $F \to M_1 \to M_2$ *be a finite covering projection. Then* $\chi(M_1) = \chi(M_2)|F|$. *If* M_2 *is orientable, then* M_1 *is orientable and we give it the natural orientation inherited from* M_2. *Then* $\mathrm{signature}(M_1) = \mathrm{signature}(M_2)|F|$.
(b) If M_1 *and* M_2 *are manifolds of dimension* m, *we let* $M_1 \# M_2$ *be the connected sum. This is defined by punching out disks in both manifolds and gluing along the common resulting boundaries. Then* $\chi(M_1 \# M_2) + \chi(S^m) = \chi(M_1) + \chi(M_2)$. *If* M_1 *and* M_2 *are oriented by some orientation*

on $M_1 \# M_2$ *then* $\operatorname{signature}(M_1 \# M_2) = \operatorname{signature}(M_1) + \operatorname{signature}(M_2)$, *if* $m \equiv 0\ (4)$.

PROOF: (a) is an immediate consequence of the fact we have local formulas for the Euler characteristic and for the signature. To prove (b), we note that the two disks we are punching out glue together to form a sphere. We use the additivity of local formulas to prove the assertion about X. The second assertion follows similarly if we note $\operatorname{signature}(S^m) = 0$.

This corollary has topological consequences. Again, we present just one to illustrate the methods involved:

COROLLARY 3.1.3. *Let* $F \to \mathbf{C}P_{2j} \to M$ *be a finite covering. Then* $|F| = 1$ *and* $M = \mathbf{C}P_{2j}$.

PROOF: $\chi(\mathbf{C}P_{2j}) = 2j+1$ so as $2j+1 = |F|\chi(M)$, we conclude $|F|$ must be odd. Therefore, this covering projection is orientation preserving so M is orientable. The identity $1 = \operatorname{signature}(\mathbf{C}P_{2j}) = |F| \operatorname{signature}(M)$ implies $|F| = 1$ and completes the proof.

If $m \equiv 2\ (4)$, the signature complex does not give a non-trivial index. We twist by taking coefficients in an auxiliary complex vector bundle V to get a non-trivial index problem in any even dimension m.

Let V be a smooth complex vector bundle of dimension v equipped with a Riemannian connection ∇. We take the Levi-Civita connection on $T^*(M)$ and on $\Lambda(T^*M)$ and let ∇ be the tensor product connection on $\Lambda(T^*M) \otimes V$. We define the operator $(d+\delta)_V$ on $C^\infty(\Lambda(T^*M) \otimes V)$ using the diagram:

$$(d+\delta)_V\colon C^\infty(\Lambda(T^*M) \otimes V) \xrightarrow{\nabla} C(T^*M \otimes \Lambda(T^*M) \otimes V) \xrightarrow{c\otimes 1} C^\infty(\Lambda(T^*M) \otimes V).$$

We have already noted that if $V = 1$ is the trivial bundle with flat connection, then the resulting operator is $(d+\delta)$.

We define $\tau_V = \tau \otimes 1$, then a similar argument to that given for the signature complex shows $\tau_V^2 = 1$ and τ_V anti-commutes with $(d+\delta)_V$. The ± 1 eigenspaces of τ_V are $\Lambda^\pm(T^*M) \otimes V$ and the twisted signature complex is defined by the diagram:

$$(d+\delta)_V^\pm\colon C^\infty(\Lambda^\pm(T^*M) \otimes V) \to C^\infty(\Lambda^\mp(T^*M) \otimes V)$$

where as before $(d+\delta)_V^-$ is the adjoint of $(d+\delta)_V^+$. We let $\Delta_V^\pm$ be the associated Laplacians and define:

$$\begin{gathered}
\operatorname{signature}(M,V) = \operatorname{index}((d+\delta)_V^+) = \dim \mathrm{N}(\Delta_V^+) - \dim \mathrm{N}(\Delta_V^-) \\
a_n^s(x,V) = \{a_n(x,\Delta_V^+) - a_n(x,\Delta_V^-)\}\,\mathrm{dvol} \in \Lambda^m \\
\int_M a_n^s(x,V) = \operatorname{signature}(M,V).
\end{gathered}$$

The invariance of the index under homotopies shows signature(M,V) is independent of the metric on M, of the fiber metric on V, and of the Riemannian connection on V. If we do *not* choose a Riemannian connection on V, we can still compute signature$(M,V) = \operatorname{index}((d+\delta)_V^+)$, but then $(d+\delta)_V^-$ is not the adjoint of $(d+\delta)_V^+$ and a_n^s is not an invariantly defined m form.

Relative to a functorial coordinate frame for $\Lambda^\pm$, the leading symbol of $\Delta^\pm$ is $|\xi|^2 I$. The first order symbol is linear in the 1-jets of the metric and the connection form on V. The 0^{th} order symbol is linear in the 2-jets of the metric and the connection form and quadratic in the 1-jets of the metric and the connection form. Thus $a_n^s(X,V) \in \mathcal{R}_{m,n,m,v}$. Theorem 2.6.1 implies $a_n^s = 0$ for $n < m$ while a_m^s is a characteristic form of $T(M)$ and of V.

If $m = 2$ and if $v = 1$, then $\mathcal{R}_{2,2,2,1}$ is one dimensional and is spanned by the first Chern class $c_1(V) = ch(V) = \frac{i}{2\pi}\Omega$. Consequently $a_2^s = cc_1$ in this case. We shall show later that this normalizing constant $c = 2$—i.e.,

Lemma 3.1.4. *Let $m = 2$ and let V be a line bundle over M_2. Then:*

$$\operatorname{signature}(M,V) = 2\int_M c_1(L).$$

We postpone the proof of this lemma until later in this chapter.

With this normalizing constant established, we can compute a formula for signature(M,V) in general:

Theorem 3.1.5. *Let L be the total L-polynomial and let $ch(V)$ be the Chern character. Then:*
(a) $a_n^s(x,V) = 0$ for $n < m$.
(b) $a_m^s(x,V) = \sum_{4s+2t=m} L_s(TM) \wedge 2^t ch_t(V)$ so that:

$$\operatorname{signature}(M,V) = \sum_{4s+2t=m} \int_M L_s(TM) \wedge 2^t ch_t(V).$$

The factors of 2^t are perhaps a bit mysterious at this point. They arise from the normalizing constant of Lemma 3.1.4 and will be explained when we discuss the spin and Dolbeault complexes.

Proof: We have already proved (a). We know $a_m^s(x,V)$ is a characteristic form which integrates to signature(M,V) so it suffices to verify the formula of (b). If V_1 and V_2 are bundles, we let $V = V_1 \oplus V_2$ with the direct sum connection. Since $\Delta_V^\pm = \Delta_{V_1}^\pm \oplus \Delta_{V_2}^\pm$ we conclude

$$\operatorname{signature}(M, V_1 \oplus V_2) = \operatorname{signature}(M,V_1) + \operatorname{signature}(M,V_2).$$

Since the integrals are additive, we apply the uniqueness of Lemma 2.6.3 to conclude the local formulas must be additive. This also follows from Lemma 1.7.5 so that:

$$a_n^s(x, V_1 \oplus V_2) = a_n^s(x, V_1) + a_n^s(x, V_2).$$

Let $\{P_\rho\}_{|\rho|=s}$ be the basis for $\mathcal{P}_{m,4s,4s}$ and expand:

$$a_m^s(x, V) = \sum_{4|\rho|+2t=m} P_\rho \wedge Q_{m,t,v,\rho} \qquad \text{for} \quad Q_{m,t,v,\rho} \in \mathcal{Q}_{m,2t,2t,v}$$

a characteristic form of V. Then the additivity under direct sum implies:

$$Q_{m,t,v,\rho}(V_1 \oplus V_2) = Q_{m,t,v_1,\rho}(V_1) + Q_{m,t,v_2,\rho}(V_2).$$

If $v = 1$, then $Q_{m,t,1,\rho}(V_1) = c \cdot c_1(V)^t$ since $\mathcal{Q}_{m,2t,2t,1}$ is one dimensional. If A is diagonal matrix, then the additivity implies:

$$Q_{m,t,v,\rho}(A) = Q_{m,t,v,\rho}(\lambda) = c \cdot \sum_j \lambda_j^t = c \cdot ch_t(A).$$

Since Q is determined by its values on diagonal matrices, we conclude:

$$Q_{m,t,v,\rho}(V) = c(m,t,\rho) ch_t(V)$$

where the normalizing constant does not depend on the dimension v. Therefore, we expand a_m^s in terms of $ch_t(V)$ to express:

$$a_m^s(x, V) = \sum_{4s+2t=m} P_{m,s} \wedge 2^t ch_t(V) \qquad \text{for } P_{m,s} \in \mathcal{P}_{m,4s,4s}.$$

We complete the proof of the theorem by identifying $P_{m,s} = L_s$; we have reduced the proof of the theorem to the case $v = 1$.

We proceed by induction on m; Lemma 3.1.4 establishes this theorem if $m = 2$. Suppose $m \equiv 0$ (4). If we take V to be the trivial bundle, then if $4k = m$,

$$a_m^s(x, 1) = L_k = P_{m,k}$$

follows from Theorem 3.1.1. We may therefore assume $4s < m$ in computing $P_{m,s}$. Let $M = M_1 \times S^2$ and let $V = V_1 \otimes V_2$ where V_1 is a line bundle over M_1 and where V_2 is a line bundle over S^2 so $\int_{S^2} c_1(V_2) = 1$. (We constructed such a line bundle in section 2.1 using Clifford matrices). We take the product connection on $V_1 \otimes V_2$ and decompose:

$$\begin{aligned} \Lambda^+(V) &= \Lambda^+(V_1) \otimes \Lambda^+(V_2) \ \oplus \ \Lambda^-(V_1) \otimes \Lambda^-(V_2) \\ \Lambda^-(V) &= \Lambda^-(V_1) \otimes \Lambda^+(V_2) \ \oplus \ \Lambda^+(V_1) \otimes \Lambda^-(V_2). \end{aligned}$$

A similar decomposition of the Laplacians yields:

$$\begin{aligned}\operatorname{signature}(M,V) &= \operatorname{signature}(M_1,V_1)\operatorname{signature}(M_2,V_2)\\ &= 2\operatorname{signature}(M_1,V_1)\end{aligned}$$

by Lemma 3.1.4. Since the signatures are multiplicative, the local formulas are multiplicative by the uniqueness assertion of Lemma 2.6.3. This also follows by using Lemma 1.7.5 that

$$a_m^s(x,V) = \sum_{p+q=m} a_p^s(x_1,V_1)a_q^s(x_2,V_2)$$

and the fact $a_p = 0$ for $p < m_1$ and $a_q = 0$ for $q < m_2$. Thus we conclude:

$$a_m^s(x,V) = a_{m_1}^s(x_1,V_1)a_{m_2}^s(x_2,V_2)$$

where, of course, $m_2 = 2$ and $m_1 = m-2$.

We use the identity:

$$ch(V_1 \otimes V_2) = ch(V_1)ch(V_2)$$

to conclude therefore:

$$\begin{aligned}&\operatorname{signature}(M_1,V_1)\\ &\qquad= \frac{1}{2}\operatorname{signature}(M,V)\\ &\qquad= \frac{1}{2}\Big\{\sum_{4s+2t=m-2}\int_{M_I} P_{m,s}\wedge 2^t ch_t(V_1)\Big\}\int_{M_2} 2ch_1(V_2)\\ &\qquad= \sum_{4s+2t=m-2}\int_{M_1} P_{m,s}\wedge 2^t ch_t(V_1).\end{aligned}$$

We apply the uniqueness assertion of Lemma 2.6.3 to conclude $P_{m,s} = P_{m-2,s}$ for $4s \le m-2$. Since by induction, $P_{m-2,s} = L_s$ this completes the proof of the theorem.

We note that the formula is non-zero, so Lemma 2.6.3 implies that in any even dimension m, there always exist (M,V) so $\operatorname{signature}(M,V) \neq 0$. In fact, much more is true. Given any orientable manifold M, we can find V over M so $\operatorname{signature}(M,V) \neq 0$ if $\dim(M)$ is even. Since the proof of this assertion relies on the fact $ch\colon K(M)\otimes \mathbf{Q} \overset{\simeq}{\to} H^{2*}(M;\mathbf{Q})$ we postpone a discussion of this fact until we discuss the index theorem in general.

3.2. Spinors and their Representations.

To define the signature complex, we needed to orient the manifold M. For any Riemannian manifold, by restricting to local orthonormal frames, we can always assume the transition functions of $T(M)$ are maps $g_{\alpha\beta}: U_\alpha \cap U_\beta \to \mathrm{O}(m)$. If M is oriented, by restricting to local orthonormal frames, we can choose the transition functions of $T(M)$ to lie in $\mathrm{SO}(m)$ and to reduce the structure group from $\mathrm{GL}(m,R)$ to $\mathrm{SO}(m)$. The signature complex results from the represntation $\Lambda^\pm$ of $\mathrm{SO}(m)$; it cannot be defined in terms of $\mathrm{GL}(m,R)$ or $\mathrm{O}(m)$. By contrast, the de Rham complex results from the representation $\Lambda^{\mathrm{e,o}}$ which is a representation of $\mathrm{GL}(m,R)$ so the de Rham complex is defined for non-orientable manifolds.

To define the spin complex, which is in some sense a more fundamental elliptic complex than is either the de Rham or signature complex, we will have to lift the transition functions from $\mathrm{SO}(m)$ to $\mathrm{SPIN}(m)$. Just as every manifold is not orientable, in a similar fashion there is an obstruction to defining a spin structure.

If $m \geq 3$, then $\pi_1(\mathrm{SO}(m)) = \mathbf{Z}_2$. Abstractly, we define $\mathrm{SPIN}(m)$ to be the universal cover of $\mathrm{SO}(m)$. (If $m = 2$, then $\mathrm{SO}(2) = S^1$ and we let $\mathrm{SPIN}(2) = S^1$ with the natural double cover $\mathbf{Z}_2 \to \mathrm{SPIN}(2) \to \mathrm{SO}(2)$ given by $\theta \mapsto 2\theta$). To discuss the representations of $\mathrm{SPIN}(m)$, it is convenient to obtain a more concrete representation of $\mathrm{SPIN}(m)$ in terms of Clifford algebras.

Let V be a real vector space of dimension $v \equiv 0\ (2)$. Let

$$\bigotimes V = \mathbf{R} \oplus V \oplus (V \otimes V) \oplus \cdots \oplus \bigotimes\nolimits^k V \oplus \cdots$$

be the complete tensor algebra of V. We assume V is equipped with a symmetric positive definite bilinear form. Let I be the two-sided ideal of $\bigotimes V$ generated by $\{v \otimes v + |v|^2\}_{v \in V}$, then the real Clifford algebra $\mathrm{CLIF}(V) = \bigotimes V \bmod I$. (Of course, we will always construct the corresponding complex Clifford algebra by tensoring $\mathrm{CLIF}(V)$ with the complex numbers). There is a natural transpose defined on $\bigotimes V$ by:

$$(v_1 \otimes \cdots \otimes v_k)^t = v_k \otimes \cdots \otimes v_1.$$

Since this preserves the ideal I, it extends to $\mathrm{CLIF}(V)$. If $\{e_1, \dots, e_v\}$ are the orthonormal basis for V, then $e_i * e_j + e_j * e_i = -2\delta_{ij}$ and

$$(e_{i_1} * \cdots * e_{i_p})^t = (-1)^{p(p-1)/2} e_{i_1} * \cdots * e_{i_p}.$$

If V is oriented, we let $\{e_i\}$ be an oriented orthonormal basis and define:

$$\tau = (\sqrt{-1})^{v/2} e_1 * \cdots * e_v.$$

We already computed that:

$$\begin{aligned}\tau^2 &= (-1)^{v/2} e_1 * \cdots * e_v * e_1 * \ldots e_v \\ &= (-1)^{v(v-1)/2} e_1 * \cdots * e_v * e_1 * \cdots * e_v \\ &= e_1 * \cdots * e_v * e_v * \cdots * e_1 = (-1)^v = 1.\end{aligned}$$

We let $\mathrm{SPIN}(V)$ be the set of all $w \in \mathrm{CLIF}(V)$ such that w can be decomposed as a formal product $w = v_1 * \cdots * v_{2j}$ for some j where the $v_i \in V$ are elements of length 1. It is clear that w^t is such an element and that $ww^t = 1$ so that $\mathrm{SPIN}(V)$ forms a group under Clifford multiplication. We define

$$\rho(w)x = wxw^t \qquad \text{for } x \in \mathrm{CLIF}, w \in \mathrm{CLIF}(V).$$

For example, if $v_1 = e_1$ is the first element of our orthonormal basis, then:

$$e_1 e_i e_1 = \begin{cases} -e_1 & i = 1 \\ e_i & i \neq 1. \end{cases}$$

The natural inclusion of V in $\bigotimes V$ induces an inclusion of V in $\mathrm{CLIF}(V)$. $\rho(e_1)$ preserves V and is reflection in the hyperplane defined by e_1. If $w \in \mathrm{SPIN}(V)$, then $\rho(w)\colon V \to V$ is a product of an even number of hyperplane reflections. It is therefore in $\mathrm{SO}(V)$ so

$$\rho\colon \mathrm{SPIN}(V) \to \mathrm{SO}(V)$$

is a group homomorphism. Since any orthogonal transformation of determinant one can be decomposed as a product of an even number of hyperplane reflections, ρ is subjective.

If $w \in \mathrm{CLIF}(V)$ is such that

$$wvw^t = v \text{ all } v \in V \text{ and } ww^t = 1$$

then $wv = vw$ for all $v \in V$ so w must be in the center of $\mathrm{CLIF}(V)$.

LEMMA 3.2.1. *If* $\dim V = v$ *is even, then the center of* $\mathrm{CLIF}(V)$ *is one dimensional and consists of the scalars.*

PROOF: Let $\{e_i\}$ be an orthonormal basis for V and let $\{e_I\}$ be the corresponding orthonormal basis for $\mathrm{CLIF}(V)$. We compute:

$$e_i * e_{i_1} * \cdots * e_{i_p} * e_i = e_{i_1} * \cdots * e_{i_p} \qquad \text{for } p \geq 1,$$

if (a) p is even and i is one of the i_j or (b) p is odd and i is not one of the i_j. Thus given I we can choose i so $e_i * e_I = -e_I * e_i$. Thus

$e_i * (\sum_I c_I e_I) = \sum_I c_I e_I * e_i$ for all i implies $c_I = 0$ for $|I| > 0$ which completes the proof.

(We note that this lemma fails if $\dim V$ is odd since the center in that case consists of the elements $a + be_1 * \cdots * e_v$ and the center is two dimensional).

If $\rho(w) = 1$ and $w \in \mathrm{SPIN}(V)$, this implies w is scalar so $w = \pm 1$. By considering the arc in $\mathrm{SPIN}(V)$ given by $w(\theta) = ((\cos\theta)e_1 + (\sin\theta)e_2) * (-(\cos\theta)e_1 + (\sin\theta)e_2)$, we note $w(0) = 1$ and $w(\frac{\pi}{2}) = -1$ so $\mathrm{SPIN}(V)$ is connected. This proves we have an exact sequence of groups in the form:

$$\mathbf{Z}_2 \to \mathrm{SPIN}(V) \to \mathrm{SO}(V)$$

and shows that $\mathrm{SPIN}(V)$ is the universal cover of $\mathrm{SO}(V)$ for $v > 2$.

We note that is it possible to define $\mathrm{SPIN}(V)$ using Clifford algebras even if v is odd. Since the center of $\mathrm{CLIF}(V)$ is two dimensional for v odd, more care must be used with the relevant signs which arise. As we shall not need that case, we refer to Atiyah-Bott-Shapiro (see bibliography) for further details.

$\mathrm{SPIN}(V)$ acts on $\mathrm{CLIF}(V)$ from the left. This is an orthogonal action.

Lemma 3.2.2. *Let* $\dim V = 2v_1$. *We complexify* $\mathrm{CLIF}(V)$. *As a left* $\mathrm{SPIN}(V)$ *module, this is not irreducible. We can decompose* $\mathrm{CLIF}(V)$ *as a direct sum of left* $\mathrm{SPIN}(V)$ *modules in the form*

$$\mathrm{CLIF}(V) = 2^{v_1}\Delta.$$

This representation is called the spin representation. It is not irreducible but further decomposes in the form

$$\Delta = \Delta^+ \oplus \Delta^-.$$

If we orient V *and let* τ *be the orientation form discussed earlier, then left multiplication by* τ *is* ± 1 *on* $\Delta^\pm$ *so these are inequivalent representations. They are irreducible and act on a representation space of dimension* $2^{v_1 - 1}$ *and are called the half-spin representations.*

Proof: Fix an oriented orthonormal basis $\{e_i\}$ for V and define:

$$\alpha_1 = \sqrt{-1}\, e_1 e_2, \quad \alpha_2 = \sqrt{-1}\, e_3 e_4, \quad \ldots, \quad \alpha_{v_1} = \sqrt{-1}\, e_{v-1} e_v$$

as elements of $\mathrm{CLIF}(V)$. It is immediate that $\tau = \alpha_1 \ldots \alpha_{v_1}$ and:

$$\alpha_i^2 = 1 \qquad \text{and} \qquad \alpha_i \alpha_j = \alpha_j \alpha_i.$$

We let the $\{\alpha_i\}$ act on $\mathrm{CLIF}(V)$ from the right and decompose $\mathrm{CLIF}(V)$ into the 2^{v_1} simultaneous eigenspaces of this action. Since right and left multiplication commute, each eigenspace is invariant as a left $\mathrm{SPIN}(V)$

module. Each eigenspace corresponds to one of the 2^{v_1} possible sequences of $+$ and $-$ signs. Let ε be such a string and let Δ_ε be the corresponding representation space.

Since $e_1\alpha_1 = -\alpha_1 e_1$ and $e_1\alpha_i = \alpha_i e_1$ for $i > 1$, multiplication on the right by e_1 transforms $\Delta_\varepsilon \to \Delta_{\varepsilon'}$, where $\varepsilon(1) = -\varepsilon'(1)$ and $\varepsilon(i) = \varepsilon'(i)$ for $i > 1$. Since this map commutes with left multiplication by $\mathrm{SPIN}(V)$, we see these two representations are isomorphic. A similar argument on the other indices shows all the representation spaces Δ_ε are equivalent so we may decompose $\mathrm{CLIF}(V) = 2^{v_1}\Delta$ as a direct sum of 2^{v_1} equivalent representation spaces Δ. Since $\dim(CLIF(V)) = 2^v = 4^{v_1}$, the representation space Δ has dimension 2^{v_1}. We decompose Δ into ± 1 eigenspaces under the action of τ to define $\Delta^\pm$. Since $e_1 * \cdots * e_v$ is in the center of $\mathrm{SPIN}(V)$, these spaces are invariant under the left action of $\mathrm{SPIN}(V)$. We note that elements of V anti-commute with τ so Clifford multiplication on the left defines a map:

$$cl: V \otimes \Delta^\pm \to \Delta^\mp$$

so both of the half-spin representations have the same dimension. They are clearly inequivalent. We leave the proof that they are irreducible to the reader as we shall not need this fact.

We shall use the notation $\Delta^\pm$ to denote both the representations and the corresponding representation spaces. Form the construction we gave, it is clear the $\mathrm{CLIF}(V)$ acts on the left to preserve the space Δ so Δ is a representation space for the left action by the whole Clifford algebra. Since $v\tau = -\tau v$ for $v \in V$, Clifford multiplication on the left by an element of V interchanges Δ^+ and Δ^-.

LEMMA 3.2.3. *There is a natural map given by Clifford multiplication of $V \otimes \Delta^\pm \to \Delta^\mp$. This map induces a map on the representations involved: $\rho \otimes \Delta^\pm \mapsto \Delta^\mp$. If this map is denoted by $v * w$ then $v * v * w = -|v|^2 w$.*

PROOF: We already checked the map on the spaces. We check:

$$wvw^t \otimes wx \mapsto wvw^t wx = wvx$$

to see that the map preserves the relevant representation. We emphasize that $\Delta^\pm$ are complex representations since we must complexify $\mathrm{CLIF}(V)$ to define these representation spaces ($ww^t = 1$ for $w \in \mathrm{SPIN}$).

There is a natural map $\rho: \mathrm{SPIN}(V) \to \mathrm{SO}(V)$. There are natural representations Λ, $\Lambda^\pm$, $\Lambda^{e,o}$ of $\mathrm{SO}(V)$ on the subspaces of $\Lambda(V) = \mathrm{CLIF}(V)$. We use ρ to extend these representations of $\mathrm{SPIN}(V)$ as well. They are related to the half-spin representations as follows:

LEMMA 3.2.4.
(a) $\Lambda = \Delta \otimes \Delta$.
(b) $(\Lambda^+ - \Lambda^-) = (\Delta^+ - \Delta^-) \otimes (\Delta^+ + \Delta^-)$.

(c) $(\Lambda^{\mathrm{e}} - \Lambda^{\mathrm{o}}) = (\Delta^+ - \Delta^-) \otimes (\Delta^+ - \Delta^-)(-1)^{v/2}$.
These identities are to be understood formally (in the sense of K-theory). They are shorthand for the identities:

$$\Lambda^+ = (\Delta^+ \otimes \Delta^+) \oplus (\Delta^+ \otimes \Delta^-) \quad \text{and} \quad \Delta^- = (\Delta^- \otimes \Delta^+) \oplus (\Delta^- \otimes \Delta^-)$$

and so forth.

PROOF: If we let $\mathrm{SPIN}(V)$ act on $\mathrm{CLIF}(V)$ by right multiplication by $w^t = w^{-1}$, then this representation is equivalent to left multiplication so we get the same decomposition of $\mathrm{CLIF}(V) = 2^{v_1}\Delta$ as right spin modules. Since $\rho(w)x = wxw^t$, it is clear that $\Lambda = \Delta\otimes\Delta$ where one factor is viewed as a left and the other as a right spin module. Since $\Lambda^\pm$ is the decomposition of $\mathrm{SPIN}(V)$ under the action of τ from the left, (b) is immediate. (c) follows similarly once the appropriate signs are taken into consideration; $w \in \mathrm{CLIF}(V)^{\mathrm{even}}$ if and only if

$$\tau w \tau^t = (-1)^{v/2} w$$

which proves (c).

It is helpful to illustrate this for the case $\dim V = 2$. Let $\{e_1, e_2\}$ be an oriented orthonormal basis for V. We compute that:

$$\begin{aligned}\big((\cos\alpha)e_1 + (\sin\alpha)e_2\big)\big((\cos\beta)e_1 + (\sin\beta)e_2\big) \\ &= \big(-\cos\alpha\cos\beta - \sin\alpha\sin\beta\big) \\ &\quad + \big(\cos\alpha\sin\beta - \sin\alpha\cos\beta\big)e_1e_2\end{aligned}$$

so elements of the form $\cos\gamma + (\sin\gamma)e_1e_2$ belong to $\mathrm{SPIN}(V)$. We compute:

$$\big(\cos\alpha + (\sin\alpha)e_1e_2\big)\big(\cos\beta + (\sin\beta)e_1e_2\big) = \cos(\alpha+\beta) + \sin(\alpha+\beta)e_1e_2$$

so $\mathrm{spin}(V)$ is the set of all elements of this form and is naturally isomorphic to the circle $S^1 = [0, 2\pi]$ with the endpoints identified. We compute that

$$\begin{aligned}\rho(w)(e_1) &= \big(\cos\theta + (\sin\theta)e_1e_2\big)e_1\big(\cos\theta + (\sin\theta)e_2e_1\big) \\ &= \big(\cos^2\theta - \sin^2\theta\big)e_1 + 2(\cos\theta\sin\theta)e_2 \\ &= \cos(2\theta)e_1 + \sin(2\theta)e_2 \\ \rho(w)(e_2) &= \cos(2\theta)e_2 - \sin(2\theta)e_1\end{aligned}$$

so the map $\rho\colon S^1 \to S^1$ is the double cover $\theta \mapsto 2\theta$.

We construct the one dimensional subspaces V_i generated by the elements:

$$\begin{array}{llll} v_1 = 1+\tau, & v_2 = 1-\tau, & v_3 = (1+\tau)e_1, & v_4 = (1-\tau)e_1, \\ \tau v_1 = v_1, & \tau v_2 = -v_2, & \tau v_3 = v_3, & \tau v_4 = -v_4, \\ v_1\tau = v_1, & v_2\tau = -v_2, & v_3\tau = -v_3, & v_4\tau = v_4. \end{array}$$

When we decompose $\mathrm{CLIF}(V)$ under the left action of $\mathrm{SPIN}(V)$,

$$V_1 \simeq V_3 \simeq \Delta^+ \qquad \text{and} \qquad V_2 \simeq V_4 \simeq \Delta^-.$$

When we decompose $\mathrm{CLIF}(V)$ under the right action of $\mathrm{SPIN}(V)$, and replace τ by $-\tau = \tau^t$ acting on the right:

$$V_2 \simeq V_3 \simeq \Delta^+ \qquad \text{and} \qquad V_1 \simeq V_4 \simeq \Delta^-.$$

From this it follows that as $\mathrm{SO}(V)$ modules we have:

$$V_1 = \Delta^+ \otimes \Delta^-, \quad V_2 = \Delta^- \otimes \Delta^+, \quad V_3 = \Delta^+ \otimes \Delta^+, \quad V_4 = \Delta^- \otimes \Delta^-$$

from which it is immediate that:

$$\begin{aligned}
\Lambda^{\mathrm{e}} &= V_1 \oplus V_2 = \Delta^+ \otimes \Delta^- \ \oplus \ \Delta^- \otimes \Delta^+ \\
\Lambda^{\mathrm{o}} &= V_3 \oplus V_4 = \Delta^+ \otimes \Delta^+ \ \oplus \ \Delta^- \otimes \Delta^- \\
\Lambda^+ &= V_1 \oplus V_3 = \Delta^+ \otimes \Delta^- \ \oplus \ \Delta^+ \otimes \Delta^+ \\
\Lambda^- &= V_2 \oplus V_4 = \Delta^- \otimes \Delta^+ \ \oplus \ \Delta^- \otimes \Delta^-.
\end{aligned}$$

If V is a one-dimensional complex vector space, we let V_r be the underlying real vector space. This defines a natural inclusion of $\mathrm{U}(1) \to \mathrm{SO}(2)$. If we let $J : V \to V$ be complex multiplication by $\sqrt{-1}$ then $J(e_1) = e_2$ and $J(e_2) = -e_1$ so J is equivalent to Clifford multiplication by $e_1 e_2$ on the left. We define a complex linear map from V to $\Delta^- \otimes \Delta^-$ by:

$$T(v) = v - iJ(v) \in V_4$$

by computing $T(e_1) = (e_1 - ie_2) = (1 - ie_1e_2)(e_1)$ and $T(e_2) = e_2 + ie_1 = (i)(e_1 - ie_2)$.

LEMMA 3.2.5. *Let* $\mathrm{U}(1)$ *be identified with* $\mathrm{SO}(2)$ *in the usual manner. If* V *is the underlying complex* 1*-dimensional space corresponding to* V_r *then* $V \simeq \Delta^- \otimes \Delta^-$ *and* $V^* \simeq \Delta^+ \otimes \Delta^+$ *as representation spaces of* $\mathrm{SPIN}(2)$.

PROOF: We have already verified the first assertion. The second follows from the fact that V^* was made into a complex space using the map $-J$ instead of J on V_r. This takes us into V_3 instead of into V_4. It is also clear that $\Delta^- = (\Delta^+)^*$ if $\dim V = 2$. Of course, all these statements are to be interpreted as statements about representations since they are trivial as statements about vector spaces (since any vector spaces of the same dimension are isomorphic).

3.3. Spin Structures on Vector Bundles.

We wish to apply the constructions of 3.2 to vector bundles over manifolds. We first review some facts regarding principal bundles and Stieffel-Whitney classes which we shall need.

Principal bundles are an extremely convenient bookkeeping device. If G is a Lie group, a principal G-bundle is a fiber space $\pi: P_G \to M$ with fiber G such that the transition functions are elements of G acting on G by left multiplication in the group. Since left and right multiplication commute, we can define a right action of G on P_G which is fiber preserving. For example, let $\mathrm{SO}(2k)$ and $\mathrm{SPIN}(2k)$ be the groups defined by $\mathbf{R}^{2k}$ with the cannonical inner product. Let V be an oriented Riemannian vector bundle of dimension $2k$ over M and let P_{SO} be the bundle of oriented frames of V. P_{SO} is an $\mathrm{SO}(2k)$ bundle and the natural action of $\mathrm{SO}(2k)$ from the right which sends an oriented orthonormal frame $s = (s_1, \ldots, s_{2k})$ to $s \cdot g = (s'_1, \ldots, s'_{2k})$ is defined by:

$$s'_i = s_1 g_{1,i} + \cdots + s_{2k} g_{2k,i}.$$

The fiber of P_{SO} is $\mathrm{SO}(V_x)$ where V_x is the fiber of V over the point x. This isomorphism is not natural but depends upon the choice of a basis.

It is possible to define the theory of characteristic classes using principal bundles rather than vector bundles. In this approach, a connection is a splitting of $T(P_G)$ into vertical and horizontal subspaces in an equivariant fashion. The curvature becomes a Lie algebra valued endomorphism of $T(P_G)$ which is equivariant under the right action of the group. We refer to Euguchi, Gilkey, Hanson for further details.

Let $\{U_\alpha\}$ be a cover of M so V is trivial over U_α and let $\vec{s}_\alpha$ be local oriented orthonormal frames over U_α. On the overlap, we express $\vec{s}_\alpha = g_{\alpha\beta} \vec{s}_\beta$ where $g_{\alpha\beta}: U_\alpha \cap U_\beta \to \mathrm{SO}(2k)$. These satisfy the cocycle condition:

$$g_{\alpha\beta} g_{\beta\gamma} g_{\gamma\alpha} = I \qquad \text{and} \qquad g_{\alpha\alpha} = I.$$

The principal bundle P_{SO} of oriented orthonormal frames has transition functions $g_{\alpha\beta}$ acting on $\mathrm{SO}(2k)$ from the left.

A spin structure on V is a lifting of the transition functions to $\mathrm{SPIN}(2k)$ preserving the cocyle condition. If the lifting is denoted by g', then we assume:

$$\rho(g'_{\alpha\beta}) = g_{\alpha\beta}\,, \qquad g'_{\alpha\beta} g'_{\beta\gamma} g'_{\gamma\alpha} = I\,, \qquad \text{and} \qquad g'_{\alpha\alpha} = I.$$

This is equivalent to constructing a principal $\mathrm{SPIN}(2k)$ bundle P_{SPIN} together with a double covering map $\rho: P_{\mathrm{SPIN}} \to P_{\mathrm{SO}}$ which preserves the group action—i.e.,

$$\rho(x \cdot g') = \rho(x) \cdot \rho(g').$$

The transition functions of P_{SPIN} are just the $g'_{\alpha\beta}$ acting on the left.

Attempting to find a spin structure on V is equivalent to taking a square root in a certain sense which we will make clear later. There is an obstruction to defining a spin structure which is similar to that to defining an orientation on V. These obstructions are $\mathbf{Z}_2$ characteristic classes called the Stieffel-Whitney classes and are most easily defined in terms of Čech cohomology. To understand these obstructions better, we review the construction briefly.

We fix a Riemannian structure on $T(M)$ to define a notion of distance. Geodesics on M are curves which locally minimize distance. A set U is said to be geodesically convex if (a) given $x, y \in U$ there exists a unique geodesic in M joining x to y with $d(x, y) = \text{length}(\gamma)$ and (b) if γ is any such geodesic then γ is actually contained in U. It is immediate that the intersection of geodesically convex sets is again geodesically convex and that every geodesically convex set is contractible.

It is a basic theorem of Riemannian geometry that there exist open covers of M by geodesically convex sets. A cover $\{U_\alpha\}$ is said to be *simple* if the intersection of any number of sets of the cover is either empty or is contractible. A cover of M by open geodesically convex sets is a simple cover.

We fix such a simple cover hence forth. Since U_α is contractible, any vector bundle over M is trivial over U_α. Let $\mathbf{Z}_2$ be the multiplicative group $\{\pm 1\}$. A Čech j-cochain is a function $f(\alpha_0, \dots, \alpha_j) \in \mathbf{Z}_2$ defined for $j+1$-tuples of indices where $U_{\alpha_0} \cap \cdots \cap U_{\alpha_j} \neq \emptyset$ which is totally symmetric—i.e.,

$$f(\alpha_{\sigma(0)}, \dots, \alpha_{\sigma(j)}) = f(\alpha_0, \dots, \alpha_j)$$

for any permutation σ. If $C^j(M; \mathbf{Z}_2)$ denotes the multiplicative group of all such functions, the coboundary $\delta\colon C^j(M, \mathbf{Z}_2) \to C^{j+1}(M; \mathbf{Z}_2)$ is defined by:

$$(\delta f)(\alpha_0, \dots, \alpha_{j+1}) = \prod_{i=0}^{j+1} f(\alpha_0, \dots, \widehat{\alpha_i}, \dots, \alpha_{j+1}).$$

The multiplicative identity of $C^j(M; \mathbf{Z}_2)$ is the function 1 and it is an easy combinatorial exercise that $\delta^2 f = 1$. For example, if $f_0 \in C^0(M; \mathbf{Z}_2)$ and if $f_1 \in C^1(M; \mathbf{Z}_2)$, then:

$$\begin{aligned} \delta(f_0)(\alpha_0, \alpha_1) &= f(\alpha_1) f(\alpha_0) \\ \delta(f_1)(\alpha_0, \alpha_1, \alpha_2) &= f(\alpha_1, \alpha_2) f(\alpha_0, \alpha_1) f(\alpha_o, \alpha_2). \end{aligned}$$

$\mathbf{Z}_2$ is a particularly simple coefficient group to work with since every element is its own inverse; in defining the Čech cohomology with coefficients in other abelian groups, more care must be taken with the signs which arise.

Let $H^j(M;\mathbf{Z}_2) = \mathrm{N}(\delta_j)/\,\mathrm{R}(\delta_{j-1})$ be the cohomology group; it is an easy exercise to show these groups are independent of the particular simple cover chosen. There is a ring structure on $H^*(M;\mathbf{Z}_2)$, but we shall only use the "additive" group structure (which we shall write multiplicatively).

Let V be a real vector bundle, not necessarily orientable. Since the U_α are contractible, V is trivial over the U and we can find a local orthonormal frame $\vec{s}_\alpha$ for V over U_α. We let $\vec{s}_\alpha = g_{\alpha\beta}\vec{s}_\beta$ and define the 1-cochain:

$$f(\alpha\beta) = \det(g_{\alpha\beta}) = \pm 1.$$

This is well defined since $U_\alpha \cap U_\beta$ is contractible and hence connected. Since $f(\alpha,\beta) = f(\beta,\alpha)$, this defines an element of $C^1(M;\mathbf{Z}_2)$. Since the $\{g_{\alpha\beta}\}$ satisfy the cocycle condition, we compute:

$$\delta f(\alpha,\beta,\gamma) = \det(g_{\alpha\beta}g_{\beta\gamma}g_{\gamma\alpha}) = \det(I) = 1$$

so $\delta(f) = 1$ and f defines an element of $H^1(M;\mathbf{Z}_2)$. If we replace $\vec{s}_\alpha$ by $\vec{s}\,'_\alpha = h_\alpha \vec{s}_\alpha$ the new transition functions become $g'_{\alpha\beta} = h_\alpha g_{\alpha\beta} h_\beta^{-1}$ so if $f_0(\alpha) = \det(h_\alpha)$,

$$f'(\alpha,\beta) = \det(h_\alpha g_{\alpha\beta} h_\beta^{-1}) = \det(h_\alpha) f(\alpha,\beta)\det(h_\beta) = \delta(f_0)f$$

and f changes by a coboundary. This proves the element in cohomology defined by f is independent of the particular frame chosen and we shall denote this element by $w_1(V) \in H^1(M;\mathbf{Z}_2)$.

If V is orientable, we can choose frames so $\det(g_{\alpha\beta}) = 1$ and thus $w_1(V) = 1$ represents the trivial element in cohomology. Conversely, if $w_1(V)$ is trivial, then $f = \delta f_0$. If we choose h_α so $\det(h_\alpha) = f_0(\alpha)$, then the new frames $\vec{s}\,'_\alpha = h_\alpha \vec{s}_\alpha$ will have transition functions with $\det(g'_{\alpha\beta}) = 1$ and define an orientation of V. Thus V is orientable if and only if $w_1(V)$ is trivial and $w_1(V)$, which is called the first Stieffel-Whitney class, measures the obstruction to orientability.

If V is orientable, we restrict henceforth to oriented frames. Let $\dim V = 2k$ be even and let $g_{\alpha\beta} \in \mathrm{SO}(2k)$ be the transition functions. We choose any lifting $\tilde{g}_{\alpha\beta}$ to $\mathrm{SPIN}(2k)$ so that:

$$\rho(\tilde{g}_{\alpha\beta}) = g_{\alpha\beta} \qquad \text{and} \qquad \tilde{g}_{\alpha\beta}\tilde{g}_{\beta\alpha} = I;$$

since the U_α are contractible such lifts always exist. We have $g_{\alpha\beta}g_{\beta\gamma}g_{\gamma\alpha} = I$ so $\rho(\tilde{g}_{\alpha\beta}\tilde{g}_{\beta\gamma}\tilde{g}_{\gamma\alpha}) = I$ and hence $\tilde{g}_{\alpha\beta}\tilde{g}_{\beta\gamma}\tilde{g}_{\gamma\alpha} = \pm I = f(\alpha,\beta,\gamma)I$ where $f(\alpha,\beta,\gamma) \in \mathbf{Z}_2$. V admits a spin structure if and only if we can choose the lifting so $f(\alpha,\beta,\gamma) = 1$. It is an easy combinatorial exercise to show that f is symmetric and that $\delta f = 0$. Furthermore, if we change the choice of the $\vec{s}_\alpha$ or change the choice of lifts, then f changes by a coboundary. This implies f defines an element $w_2(V) \in H^2(M; \mathbf{Z}_2)$ independent of the choices made. w_2 is called the second Stieffel-Whitney class and is trivial if and only if V admits a spin structure.

We suppose $w_1(V) = w_2(V) = 1$ are trivial so V is orientable and admits a spin structure. Just as there are two orientations which can be chosen on V, there can be several possible inequivalent spin structures (i.e., several non-isomorphic principal bundles P_{SPIN}). It is not difficult to see that inequivalent spin structures are parametrized by representations of the fundamental group $\pi_1(M) \to \mathbf{Z}_2$ just as inequivalent orientations are parametrized by maps of the components of M into $\mathbf{Z}_2$.

To illustrate the existence and non-existence of spin structures on bundles, we take $M = S^2$. $S^2 = \mathbf{C}P_1$ is the Riemann sphere. There is a natural projection from $\mathbf{C}^2 - 0$ to S^2 given by sending $(x, w) \mapsto z/w$; S^2 is obtained by identifying $(z, w) = (\lambda z, \lambda w)$ for $(z, w) \neq (0, 0)$ and $\lambda \neq 0$. There are two natural charts for S^2:

$$U_1 = \{z : |z| \leq 1\} \qquad \text{and} \qquad U_2 = \{z : |z| \geq 1\} \cup \{\infty\}$$

where $w = 1/z$ gives coordinates on U_2.

$\mathbf{C}P_1$ is the set of lines in $\mathbf{C}^2$; we let L be the natural line bundle over S^2; this is also refered to as the tautological line bundle. We have natural sections to L over U_i defined by:

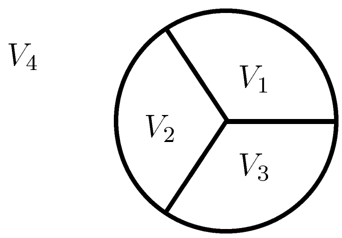
$$s_1 = (z, 1) \text{ over } U_1 \qquad \text{and} \qquad s_2 = (1, w) \text{ over } U_2.$$

these are related by the transition function:

$$s_1 = z s_2$$

so on $U_1 \cap U_2$ we have $g_{12} = e^{i\theta}$. The double cover $\mathrm{SPIN}(2) \to \mathrm{SO}(2)$ is defined by $\theta \mapsto 2\theta$ so this bundle does not have a spin structure since this transition function cannot be lifted to SPIN(2).

This cover is not a simple cover of S^2 so we construct the cover given in the following



(where we should "fatten up" the picture to have an open cover). If we have sections $\vec{s}_i$ to L over V_i, then we can choose the transition functions so $\vec{s}_j = e^{i\theta}\vec{s}_4$ for $1 \le j \le 3$ and $\vec{s}_1 = \vec{s}_2 = \vec{s}_3$. We let

$$\begin{aligned} 0 \le \theta \le 2\pi/3 &\qquad \text{parametrize} \quad V_1 \cap V_4 \\ 2\pi/3 \le \theta \le 4\pi/3 &\qquad \text{parametrize} \quad V_2 \cap V_4 \\ 4\pi/3 \le \theta \le 2\pi &\qquad \text{parametrize} \quad V_3 \cap V_4 \end{aligned}$$

and define:

$$\tilde{g}_{14} = e^{i\theta/2}, \quad \tilde{g}_{24} = e^{i\theta/2}, \quad \tilde{g}_{34} = e^{i\theta/2}$$

and all the $\tilde{g}_{jk} = 1$ for $1 \le j,\, k \le 3$ then we compute:

$$\tilde{g}_{13}\tilde{g}_{34}\tilde{g}_{41} = e^0 e^{i\pi} = -1$$

so the cochain defining w_2 satisfies:

$$f(1,3,4) = -1 \quad \text{with} \quad f(i,j,k) = 1 \quad \text{if } i \text{ or } j \text{ or } k{=}2.$$

We compute this is non-trivial in cohomology as follows: suppose $f = \delta f_1$. Then:

$$\begin{aligned} 1 &= f(1,2,3) = f_1(1,2)f_1(1,3)f_1(2,3) \\ 1 &= f(1,2,4) = f_1(1,2)f_1(1,4)f_1(2,4) \\ 1 &= f(2,3,4) = f_1(2,3)f_1(2,4)f_1(3,4) \end{aligned}$$

and multiplying together:

$$\begin{aligned} 1 &= f_1(1,2)^2 f_1(2,3)^2 f_1(2,4)^2 f_1(1,4)f_1(1,3)f_1(3,4) \\ &= f_1(1,4)f_1(1,3)f_1(3,4) = f(1,3,4) = -1 \end{aligned}$$

which is a contradiction. Thus we have computed combinatorially that $w_2 \ne 1$ is non-trivial.

Next we let $V = T(M)$ be the real tangent bundle. We identify U(1) with SO(2) to identify $T(M)$ with $T_C(M)$. Since $w = 1/z$ we have:

$$\frac{d}{dz} = \frac{dw}{dz}\frac{d}{dw} = -z^{-2}\frac{d}{dw}$$

so the transition function on the overlap is $-e^{-2i\theta}$. The minus sign can be eliminated by using $-\dfrac{d}{dw}$ instead of $\dfrac{d}{dw}$ as a section over U_2 to make the transition function be $e^{-2i\theta}$. Since the double cover of SPIN(2) $\to$ SO(2) is given by $\theta \mapsto 2\theta$, this transition function lifts and $T(M)$ has a spin structure. Since S^2 is simply connected, the spin structure is unique. This also proves $T_C(M) = L^* \otimes L^*$ since the two bundles have the same transition functions.

There is a natural inclusion of $\mathrm{SPIN}(V)$ and $\mathrm{SPIN}(W)$ into subgroups of $\mathrm{SPIN}(V \oplus W)$ which commute. This induces a map from $\mathrm{SPIN}(V) \times \mathrm{SPIN}(W) \to \mathrm{SPIN}(V \oplus W)$. Using this map, it is easy to compute $w_2(V \oplus W) = w_2(V)w_2(W)$. To define $w_2(V)$, we needed to choose a fixed orientation of V. It is not difficult to show that $w_2(V)$ is independent of the orientation chosen and of the fiber metric on V. (Our definition does depend upon the fact that V is orientable. Although it is possible to define $w_2(V)$ more generally, the identity $w_2(V \oplus W) = w_2(V)w_2(W)$ fails if V and W are not orientable in general). We summarize the properties of w_2 we have derived:

Lemma 3.3.1. *Let V be a real oriented vector bundle and let $w_2(V) \in H^2(M; \mathbf{Z}_2)$ be the second Stieffel-Whitney class. Then:*
(a) $w_2(V) = 1$ represents the trivial cohomology class if and only if V admits a spin structure.
(b) $w_2(V \oplus W) = w_2(V)w_2(W)$.
(c) If L is the tautological bundle over S^2 then $w_2(L)$ is non-trivial.

We emphasize that (b) is written in multiplicative notation since we have chosen the multiplicative version of $\mathbf{Z}_2$. w_2 is also functorial under pull-backs, but as we have not defined the Čech cohomology as a functor, we shall not discuss this property. We also note that the Stieffel-Whitney classes can be defined in general; $w(V) = 1 + w_1(V) + \cdots \in H^*(M; \mathbf{Z}_2)$ is defined for any real vector bundle and has many of the same properties that the Chern class has for complex bundles.

We can use Lemma 3.3.1 to obtain some other results on the existence of spin structures:

Lemma 3.3.2.
(a) If $M = S^m$ then any bundle over S^m admits a spin structure for $m \neq 2$.
(b) If $M = \mathbf{C}P_k$ and if L is the tautological bundle over M, then L does not admit a spin structure.
(c) If $M = \mathbf{C}P_k$ and if $V = T(M)$ is the tangent space, then V admits a spin structure if and only if k is odd.
(d) If $M = \mathbf{Q}P_k$ is quaternionic projective space, then any bundle over M admits a spin structure.

Proof: $H^2(M; \mathbf{Z}_2) = \{1\}$ in (a) and (d) so w_2 must represent the trivial element. To prove (b), we suppose L admits a spin structure. S^2 is embedded in $\mathbf{C}P_k$ for $k \geq 2$ and the restriction of L to S^2 is the tautological bundle over S^2. This would imply L admits a spin structure over S^2 which is false. Finally, we use the representation:

$$T_c(\mathbf{C}P_j) \oplus 1 = L^* \oplus \cdots \oplus L^*$$