Operator Algebras and Topology

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Abstract

These notes, based on three lectures on operator algebras and topology at the "School on High Dimensional Manifold Theory" at the ICTP in Trieste, introduce a new set of tools to high dimensional manifold theory, namely techniques coming from the theory of operator algebras, in particular C^*-algebras. These are extensively studied in their own right. We will focus on the basic definitions and properties, and on their relevance to the geometry and topology of manifolds.

A central pillar of work in the theory of C^*-algebras is the Baum-Connes conjecture. This is an isomorphism conjecture, as discussed in the talks of Lück, but with a certain special flavor. Nevertheless, it has important direct applications to the topology of manifolds, it implies e.g. the Novikov conjecture. In the first chapter, the Baum-Connes conjecture will be explained and put into our context.

Another application of the Baum-Connes conjecture is to the positive scalar curvature question. This will be discussed by Stephan Stolz. It implies the so-called "stable Gromov-Lawson-Rosenberg conjecture". The unstable version of this conjecture said that, given a closed spin manifold M, a certain obstruction, living in a certain (topological) K-theory group, vanishes if and only M admits a Riemannian metric with positive scalar curvature. It turns out that this is wrong, and counterexamples will be presented in the second chapter.

The third chapter introduces another set of invariants, also using operator algebra techniques, namely L^2-cohomology, L^2-Betti numbers and other L^2-invariants. These invariants, their basic properties, and the central questions about them, are introduced in the third chapter.

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Contents

Chapter 1

Index theory and the Baum-Connes conjecture

1.1 Index theory

The Atiyah-Singer index theorem is one of the great achievements of modern mathematics. It gives a formula for the index of a differential operator (the index is by definition the dimension of the space of its solutions minus the dimension of the solution space for its adjoint operator) in terms only of topological data associated to the operator and the underlying space. There are many good treatments of this subject available, apart from the original literature (most found in [2]). Much more detailed than the present notes can be, because of constraints of length and time, are e.g. [44, 7, 32].

1.1.1 Elliptic operators and their index

We quickly review what type of operators we are looking at.

1.1.1. Definition. Let M be a smooth manifold of dimension m; E, F smooth (complex) vector bundles on M. A *differential operator* (of order d) from E to F is a $\mathbb{C}$-linear map from the space of smooth sections $C^\infty(E)$ of E to the space of smooth sections of F:

$$D\colon C^\infty(E) \to C^\infty(F),$$

such that in local coordinates and with local trivializations of the bundles it can be written in the form

$$D = \sum_{|\alpha|\le d} A_\alpha(x) \frac{\partial^{|\alpha|}}{\partial x^\alpha}.$$

Here $A_\alpha(x)$ is a matrix of smooth complex valued functions, $\alpha = (\alpha_1, \dots, \alpha_m)$ is an m-tuple of non-negative integers and $|\alpha| = \alpha_1 + \cdots + \alpha_m$. $\partial^{|\alpha|}/\partial x^\alpha$ is an abbreviation for $\partial^{|\alpha|}/\partial x_1^{\alpha_1} \cdots \partial x_m^{\alpha_m}$. We require that $A_\alpha(x) \neq 0$ for some α with $|\alpha| = d$ (else, the operator is of order strictly smaller than d).

Let $\pi\colon T^*M \to M$ be the bundle projection of the cotangent bundle of M. We get pull-backs π^*E and π^*F of the bundles E and F, respectively, to T^*M.

The *symbol* $\sigma(D)$ of the differential operator D is the section of the bundle $\mathrm{Hom}(\pi^*E, \pi^*F)$ on T^*M defined as follows:

In the above local coordinates, using $\xi = (\xi_1, \dots, \xi_m)$ as coordinate for the cotangent vectors in T^*M, in the fiber of (x, ξ), the symbol $\sigma(D)$ is given by multiplication with

$$\sum_{|\alpha|=m} A_\alpha(x)\xi^\alpha.$$

Here $\xi^\alpha = \xi_1^{\alpha_1} \cdots \xi_m^{\alpha_m}$.

The operator D is called *elliptic*, if $\sigma(D)_{(x,\xi)}\colon \pi^*E_{(x,\xi)} \to \pi^*F_{(x,\xi)}$ is invertible outside the zero section of T^*M, i.e. in each fiber over $(x, \xi) \in T^*M$ with $\xi \neq 0$. Observe that elliptic operators can only exist if the fiber dimensions of E and F coincide.

In other words, the symbol of an elliptic operator gives us two vector bundles over T^*M, namely π^*E and π^*F, together with a choice of an isomorphism of the fibers of these two bundles outside the zero section. If M is compact, this gives an element of the relative K-theory group $K^0(DT^*M, ST^*M)$, where DT^*M and ST^*M are the disc bundle and sphere bundle of T^*M, respectively (with respect to some arbitrary Riemannian metric).

Recall the following definition:

1.1.2. Definition. Let X be a compact topological space. We define the K-theory of X, $K^0(X)$, to be the Grothendieck group of (isomorphism classes of) complex vector bundles over X (with finite fiber dimension). More precisely, $K^0(X)$ consists of equivalence classes of pairs (E, F) of (isomorphism classes of) vector bundles over X, where $(E, F) \sim (E', F')$ if and only if there exists another vector bundle G on X such that $E \oplus F' \oplus G \cong E' \oplus F \oplus G$. One often writes $[E] - [F]$ for the element of $K^0(X)$ represented by (E, F).

Let Y now be a closed subspace of X. The *relative K-theory* $K^0(X, Y)$ is given by equivalence classes of triples (E, F, ϕ), where E and F are complex vector bundles over X, and $\phi\colon E|_Y \to F|_Y$ is a given isomorphism between the restrictions of E and F to Y. Then (E, F, ϕ) is isomorphic to (E', F', ϕ')

if we find isomorphisms $\alpha\colon E \to E'$ and $\beta\colon F \to F'$ such that the following diagram commutes.

$$\begin{array}{ccc} E|_Y & \xrightarrow{\phi} & F|_Y \\ \big\downarrow \alpha & & \big\downarrow \beta \\ E'|_Y & \xrightarrow{\phi'} & F'|_Y \end{array}$$

Two pairs (E, F, ϕ) and (E', F', ϕ') are equivalent, if there is a bundle G on X such that $(E \oplus G, F \oplus G, \phi \oplus \mathrm{id})$ is isomorphic to $(E' \oplus G, F' \oplus G, \phi' \oplus \mathrm{id})$.

1.1.3. Example. The element of $K^0(DT^*M, ST^*M)$ given by the symbol of an elliptic differential operator D mentioned above is represented by the restriction of the bundles π^*E and π^*F to the disc bundle DT^*M, together with the isomorphism $\sigma(D)_{(x,\xi)}\colon E_{(x,\xi)} \to F_{(x,\xi)}$ for $(x, \xi) \in ST^*M$.

1.1.4. Example. Let $M = \mathbb{R}^m$ and $D = \sum_{i=1}^m (\partial/\partial_i)^2$ be the Laplace operator on functions. This is an elliptic differential operator, with symbol $\sigma(D) = \sum_{i=1}^m \xi_i^2$.

More generally, a second-order differential operator $D\colon C^\infty(E) \to C^\infty(E)$ on a Riemannian manifold M is a *generalized Laplacian*, if $\sigma(D)_{(x,\xi)} = |\xi|^2 \cdot \mathrm{id}_{E_x}$ (the norm of the cotangent vector $|\xi|$ is given by the Riemannian metric).

Notice that all generalized Laplacians are elliptic.

1.1.5. Definition. *(Adjoint operator)*
Assume that we have a differential operator $D\colon C^\infty(E) \to C^\infty(F)$ between two Hermitian bundles E and F on a Riemannian manifold (M, g). We define an L^2-inner product on $C^\infty(E)$ by the formula

$$\langle f, g\rangle_{L^2(E)} := \int_M \langle f(x), g(x)\rangle_{E_x}\, d\mu(x) \qquad \forall f, g \in C_0^\infty(E),$$

where $\langle\cdot,\cdot\rangle_{E_x}$ is the fiber-wise inner product given by the Hermitian metric, and $d\mu$ is the measure on M induced from the Riemannian metric. Here C_0^∞ is the space of smooth section with compact support. The Hilbert space completion of $C_0^\infty(E)$ with respect to this inner product is called $L^2(E)$.

The *formal adjoint* D^* of D is then defined by

$$\langle Df, g\rangle_{L^2(F)} = \langle f, D^*g\rangle_{L^2(E)} \qquad \forall f \in C_0^\infty(E),\ g \in C_0^\infty(F).$$

It turns out that exactly one operator with this property exists, which is another differential operator, and which is elliptic if and only if D is elliptic.

1.1.6. Remark. The class of differential operators is quite restricted. Many constructions one would like to carry out with differential operators automatically lead out of this class. Therefore, one often has to use *pseudodifferential operators*. Pseudodifferential operators are defined as a generalization of differential operators. There are many well written sources dealing with the theory of pseudodifferential operators. Since we will not discuss them in detail here, we omit even their precise definition and refer e.g. to [44] and [78]. What we have done so far with elliptic operators can all be extended to pseudodifferential operators. In particular, they have a symbol, and the concept of ellipticity is defined for them. When studying elliptic differential operators, pseudodifferential operators naturally appear and play a very important role. An pseudodifferential operator P (which could e.g. be a differential operator) is elliptic if and only if a pseudodifferential operator Q exists such that $PQ - \mathrm{id}$ and $QP - \mathrm{id}$ are so called *smoothing* operators, a particularly nice class of pseudodifferential operators. For many purposes, Q can be considered to act like an inverse of P, and this kind of invertibility is frequently used in the theory of elliptic operators. However, if P happens to be an elliptic differential operator of positive order, then Q necessarily is not a differential operator, but only a pseudodifferential operator.

It should be noted that almost all of the results we present here for differential operators hold also for pseudodifferential operators, and often the proof is best given using them.

We now want to state several important properties of elliptic operators.

1.1.7. Theorem. *Let M be a smooth manifold, E and F smooth finite dimensional vector bundles over M. Let $P\colon C^\infty(E) \to C^\infty(F)$ be an elliptic operator.*

Then the following holds.

(1) Elliptic regularity:
If $f \in L^2(E)$ is weakly in the null space of P, i.e. $\langle f, P^*g\rangle_{L^2(E)} = 0$ for all $g \in C_0^\infty(F)$, then $f \in C^\infty(E)$.

(2) Decomposition into finite dimensional eigenspaces:
Assume M is compact and $P = P^*$ (in particular, $E = F$). Then the set $s(P)$ of eigenvalues of P (P acting on $C^\infty(E)$) is a discrete subset of $\mathbb{R}$, each eigenspace e_λ ($\lambda \in s(P)$) is finite dimensional, and $L^2(E) = \oplus_{\lambda\in s(P)} e_\lambda$ (here we use the completed direct sum in the sense of Hilbert spaces, which means by definition that the algebraic direct sum is dense in $L^2(E)$).

(3) If M is compact, then $\ker(P)$ *and* $\ker(P^*)$ *are finite dimensional, and*

then we define the index of P

$$\operatorname{ind}(P) := \dim_{\mathbb{C}} \ker(P) - \dim_{\mathbb{C}} \ker(P^*).$$

(Here, we could replace $\ker(P^*)$ *by* $\operatorname{coker}(P)$*, because these two vector spaces are isomorphic).*

1.1.2 Statement of the Atiyah-Singer index theorem

There are different variants of the Atiyah-Singer index theorem. We start with a cohomological formula for the index.

1.1.8. Theorem. *Let M be a compact oriented manifold of dimension m, and $D\colon C^\infty(E) \to C^\infty(F)$ an elliptic operator with symbol $\sigma(D)$. There is a characteristic (inhomogeneous) cohomology class* $\operatorname{Td}(M) \in H^*(M;\mathbb{Q})$ *of the tangent bundle of M (called the complex Todd class of the complexified tangent bundle). Moreover, to the symbol is associated a certain (inhomogeneous) cohomology class* $\pi_!\operatorname{ch}(\sigma(D)) \in H^*(M;\mathbb{Q})$ *such that*

$$\operatorname{ind}(D) = (-1)^{m(m+1)/2} \langle \pi_! \operatorname{ch}(\sigma(D)) \cup \operatorname{Td}(M), [M] \rangle.$$

The class $[M] \in H_m(M;\mathbb{Q})$ is the fundamental class of the oriented manifold M, and $\langle \cdot, \cdot \rangle$ is the usual pairing between homology and cohomology.

If we start with specific operators given by the geometry, explicit calculation usually give more familiar terms on the right hand side.

For example, for the signature operator we obtain Hirzebruch's signature formula expressing the signature in terms of the L-class, for the Euler characteristic operator we obtain the Gauss-Bonnet formula expressing the Euler characteristic in terms of the Pfaffian, and for the spin or spin^c Dirac operator we obtain an $\hat{A}$-formula. For applications, these formulas prove to be particularly useful.

We give some more details about the signature operator, which we are going to use later again. To define the signature operator, fix a Riemannian metric g on M. Assume $\dim M = 4k$ is divisible by four.

The signature operator maps from a certain subspace Ω^+ of the space of differential forms to another subspace Ω^-. These subspaces are defined as follows. Define, on p-forms, the operator $\tau := i^{p(p-1)+2k}*$, where $*$ is the Hodge-$*$ operator given by the Riemannian metric, and $i^2 = -1$. Since $\dim M$ is divisible by 4, an easy calculation shows that $\tau^2 = \mathrm{id}$. We then define $\Omega^\pm$ to be the ± 1 eigenspaces of τ.

The signature operator D_{sig} is now simply defined to by $D_{sig} := d + d^*$, where d is the exterior derivative on differential forms, and $d^* = \pm * d *$ is its formal adjoint. We restrict this operator to Ω^+, and another easy calculation shows that Ω^+ is mapped to Ω^-. D_{sig} is elliptic, and a classical calculation shows that its index is the signature of M given by the intersection form in middle homology.

1.1.9. Definition. The *Hirzebruch L-class* as normalized by Atiyah and Singer is an inhomogeneous characteristic class, assigning to each complex vector bundle E over a space X a cohomology class $L(E) \in H^*(X;\mathbb{Q})$. It is characterized by the following properties:

(1) Naturality: for any map $f\colon Y \to X$ we have $L(f^*E) = f^*L(E)$.

(2) Normalization: If L is a complex line bundle with first Chern class x, then

$$L(E) = \frac{x/2}{\tanh(x/2)} = 1 + \frac{1}{12}x^2 - \frac{1}{720}x^4 + \cdots \in H^*(X;\mathbb{Q}).$$

(3) Multiplicativity: $L(E \oplus F) = L(E)L(F)$.

It turns out that L is a *stable* characteristic class, i.e. $L(E) = 1$ if E is a trivial bundle. This implies that L defines a map from the K-theory $K^0(X) \to H^*(X;\mathbb{Q})$.

The Atiyah-Singer index theorem now specializes to

$$\operatorname{sign}(M) = \operatorname{ind}(D_{sig}) = \langle 2^{2k} L(TM), [M]\rangle,$$

with $\dim M = 4k$ as above.

1.1.10. Remark. One direction to generalize the Atiyah-Singer index theorem is to give an index formula for manifolds with boundary. Indeed, this is achieved in the Atiyah-Patodi-Singer index theorem. However, these results are much less topological than the results for manifolds without boundary. They are not discussed in these notes.

Next, we explain the K-theoretic version of the Atiyah-Singer index theorem. It starts with the element of $K^0(DT^*M, ST^*M)$ given by the symbol of an elliptic operator. Given any compact manifold M, there is a well defined homomorphism

$$K^0(DT^*M, ST^*M) \to K^0(*) = \mathbb{Z},$$

constructed by embedding T^*M into high dimensional Euclidean space, then using a transfer map and Bott periodicity. The image of the symbol element under this homomorphism is denoted the *topological index* $\operatorname{ind}_t(D) \in K^0(*) = \mathbb{Z}$. The reason for the terminology is that it is obtained from the symbol only, using purely topological constructions. Now, the Atiyah-Singer index theorem states

1.1.11. Theorem. $\mathrm{ind}_t(D) = \mathrm{ind}(D)$.

1.1.3 The G-index

Let G be a finite group, or more generally a compact Lie group. The representation ring RG of G is defined to be the Grothendieck group of all finite dimensional complex representations of G, i.e. an element of RG is a formal difference $[V] - [W]$ of two finite dimensional G-representations V and W, and we have $[V] - [W] = [X] - [Y]$ if and only if $V \oplus Y \cong W \oplus X$ (strictly speaking, we have to pass to isomorphism classes of representations to avoid set theoretical problems). The direct sum of representations induces the structure of an abelian group on RG, and the tensor product makes it a commutative unital ring (the unit given by the trivial one-dimensional representation). More about this representation ring can be found e.g. in [11].

Assume now that the manifold M is a compact smooth manifold with a smooth G-action, and let E, F be complex G-vector bundles on M (this means that G acts on E and F by vector bundle automorphisms (i.e. carries fibers to fibers linearly), and the bundle projection maps are G-equivariant).

Let $D\colon C^\infty(E) \to C^\infty(F)$ be a G-equivariant elliptic differential operator.

This implies that $\ker(D)$ and $\operatorname{coker}(D)$ inherit a G-action by restriction, i.e. are finite dimensional G-representations. We define the (analytic) G-index of D to be

$$\mathrm{ind}^G(D) := [\ker(D)] - [\operatorname{coker}(D)] \in RG.$$

If G is the trivial group then $RG \cong \mathbb{Z}$ in a canonical way, and then $\mathrm{ind}^G(D)$ coincides with the usual index of D.

We can also define a *topological* equivariant index similar to the non-equivariant topological index, using transfer maps and Bott periodicity. This topological index lives in the G-equivariant K-theory of a point, which is canonically isomorphic to the representation ring RG. Again, the Atiyah-Singer index theorem says

1.1.12. Theorem. $\mathrm{ind}^G(D) = \mathrm{ind}^G_t(D) \in K^0_G(*) = RG$.

1.1.4 Families of operators and their index

Another generalization is given if we don't look at one operator on one manifold, but a family of operators on a family of manifolds. More precisely, let X be any compact topological space, $Y \to X$ a locally trivial fiber bundle with fiber M a smooth compact manifold, and structure group the diffeomorphisms of M. Let E, F be families of smooth vector bundles on Y (i.e. vector bundles which are fiber-wise smooth), and $C^\infty(E)$, $C^\infty(F)$ the continuous sections which are smooth along the fibers. Assume that $D\colon C^\infty(E) \to C^\infty(F)$ is a family $\{D_x\}$ of elliptic differential operator along the fiber $Y_x \cong M$ $(x \in X)$, i.e., in local coordinates D becomes

$$\sum_{|\alpha|\le m} A_\alpha(y,x)\frac{\partial^{|\alpha|}}{\partial y^\alpha}$$

with $y \in M$ and $x \in X$ such that $A_\alpha(y,x)$ depends continuously on x, and each D_x is an elliptic differential operator on Y_x.

If $\dim_{\mathbb{C}} \ker(D_x)$ is independent of $x \in X$, then all of these vector spaces patch together to give a vector bundle called $\ker(D)$ on X, and similarly for the (fiber-wise) adjoint D^*. This then gives a K-theory element $[\ker(D)] - [\ker(D^*)] \in K^0(X)$.

Unfortunately, it does sometimes happen that these dimensions jump. However, using appropriate perturbations, one can always define the K-theory element

$$\operatorname{ind}(D) := [\ker(D)] - [\ker(D^*)] \in K^0(X),$$

the analytic index of the family of elliptic operators D.

There is also a family version of the construction of the topological index, giving $\operatorname{ind}_t(D) \in K^0(X)$. The Atiyah-Singer index theorem for families states:

1.1.13. Theorem. $\operatorname{ind}(D) = \operatorname{ind}_t(D) \in K^0(X)$.

The upshot of the discussion of this and the last section (for the details the reader is referred to the literature) is that the natural receptacle for the index of differential operators in various situations are appropriate K-theory groups, and much of todays index theory deals with investigating these K-theory groups.

1.2 Survey on C^*-algebras and their K-theory

More detailed references for this section are, among others, [88], [32], and [8].

1.2.1 C^*-algebras

1.2.1. Definition. A *Banach algebra* A is a complex algebra which is a complete normed space, and such that $|ab| \le |a|\,|b|$ for each $a, b \in A$.

A $*$-*algebra* A is a complex algebra with an anti-linear involution $*\colon A \to A$ (i.e. $(\lambda a)^* = \overline{\lambda} a^*$, $(ab)^* = b^* a^*$, and $(a^*)^* = a$ for all $a, b \in A$).

A *Banach* $*$-*algebra* A is a Banach algebra which is a $*$-algebra such that $|a^*| = |a|$ for all $a \in A$.

A C^*-*algebra* A is a Banach $*$-algebra which satisfies $|a^*a| = |a|^2$ for all $a \in A$.

Alternatively, a C^*-algebra is a Banach $*$-algebra which is isometrically $*$-isomorphic to a norm-closed subalgebra of the algebra of bounded operators on some Hilbert space H (this is the Gelfand-Naimark representation theorem, compare e.g. [32, 1.6.2]).

A C^*-algebra A is called separable if there exists a countable dense subset of A.

1.2.2. Example. If X is a compact topological space, then $C(X)$, the algebra of complex valued continuous functions on X, is a commutative C^*-algebra (with unit). The adjoint is given by complex conjugation: $f^*(x) = \overline{f(x)}$, the norm is the supremum-norm.

Conversely, it is a theorem that every abelian unital C^*-algebra is isomorphic to $C(X)$ for a suitable compact topological space X [32, Theorem 1.3.12].

Assume X is locally compact, and set

$$C_0(X) := \{f\colon X \to \mathbb{C} \mid f \text{ continuous}, f(x) \xrightarrow{x\to\infty} 0\}.$$

Here, we say $f(x) \to 0$ for $x \to \infty$, or$f$ *vanishes at infinity*, if for all $\epsilon > 0$ there is a compact subset K of X with $|f(x)| < \epsilon$ whenever $x \in X - K$. This is again a commutative C^*-algebra (we use the supremum norm on $C_0(X)$), and it is unital if and only if X is compact (in this case, $C_0(X) = C(X)$).

1.2.2 K_0 of a ring

Suppose R is an arbitrary ring with 1 (not necessarily commutative). A module M over R is called finitely generated projective, if there is another R-module N and a number $n \ge 0$ such that

$$M \oplus N \cong R^n.$$

This is equivalent to the assertion that the matrix ring $M_n(R) = End_R(R^n)$ contains an idempotent e, i.e. with $e^2 = e$, such that M is isomorphic to the image of e, i.e. $M \cong eR^n$.

1.2.3. Example. Description of projective modules.

(1) If R is a field, the finitely generated projective R-modules are exactly the finite dimensional vector spaces. (In this case, every module is projective).

(2) If $R = \mathbb{Z}$, the finitely generated projective modules are the free abelian groups of finite rank

(3) Assume X is a compact topological space and $A = C(X)$. Then, by the Swan-Serre theorem [84], M is a finitely generated projective A-module if and only if M is isomorphic to the space $\Gamma(E)$ of continuous sections of some complex vector bundle E over X.

1.2.4. Definition. Let R be any ring with unit. $K_0(R)$ is defined to be the Grothendieck group of finitely generated projective modules over R, i.e. the group of equivalence classes $[(M,N)]$ of pairs of (isomorphism classes of) finitely generated projective R-modules M, N, where $(M,N) \equiv (M',N')$ if and only if there is an $n \geq 0$ with

$$M \oplus N' \oplus R^n \cong M' \oplus N \oplus R^n.$$

The group composition is given by

$$[(M,N)] + [(M',N')] := [(M \oplus M', N \oplus N')].$$

We can think of (M,N) as the formal difference of modules $M - N$.

Any unital ring homomorphism $f\colon R \to S$ induces a map

$$f_*\colon K_0(R) \to K_0(S)\colon [M] \mapsto [S \otimes_R M],$$

where S becomes a right R-module via f. We obtain that K_0 is a covariant functor from the category of unital rings to the category of abelian groups.

1.2.5. Example. Calculation of K_0.

- If R is a field, then $K_0(R) \cong \mathbb{Z}$, the isomorphism given by the dimension: $\dim_R(M,N) := \dim_R(M) - \dim_R(N)$.
- $K_0(\mathbb{Z}) \cong \mathbb{Z}$, given by the rank.
- If X is a compact topological space, then $K_0(C(X)) \cong K^0(X)$, the topological K-theory given in terms of complex vector bundles. To each vector bundle E one associates the $C(X)$-module $\Gamma(E)$ of continuous sections of E.

- Let G be a discrete group. The group algebra $\mathbb{C}G$ is a vector space with basis G, and with multiplication coming from the group structure, i.e. given by $g \cdot h = (gh)$.

  If G is a finite group, then $K_0(\mathbb{C}G)$ is the complex representation ring of G.

1.2.3 K-Theory of C^*-algebras

1.2.6. Definition. Let A be a unital C^*-algebra. Then $K_0(A)$ is defined as in Definition 1.2.4, i.e. by forgetting the topology of A.

1.2.3.1 K-theory for non-unital C^*-algebras

When studying (the K-theory of) C^*-algebras, one has to understand morphisms $f\colon A \to B$. This necessarily involves studying the kernel of f, which is a closed ideal of A, and hence a *non-unital* C^*-algebra. Therefore, we proceed by defining the K-theory of C^*-algebras without unit.

1.2.7. Definition. To any C^*-algebra A, with or without unit, we assign in a functorial way a new, unital C^*-algebra A_+ as follows. As $\mathbb{C}$-vector space, $A_+ := A \oplus \mathbb{C}$, with product

$$(a,\lambda)(b,\mu) := (ab + \lambda a + \mu b, \lambda\mu) \qquad \text{for } (a,\lambda),(b,\mu) \in A \oplus \mathbb{C}.$$

The unit is given by $(0,1)$. The star-operation is defined as $(a,\lambda)^* := (a^*, \overline{\lambda})$, and the new norm is given by

$$|(a,\lambda)| = \sup\{|ax + \lambda x| \mid x \in A \text{ with } |x| = 1\}$$

1.2.8. Remark. A is a closed ideal of A_+, the kernel of the canonical projection $A_+ \twoheadrightarrow \mathbb{C}$ onto the second factor. If A itself is unital, the unit of A is of course different from the unit of A_+.

1.2.9. Example. Assume X is a locally compact space, and let $X_+ := X \cup \{\infty\}$ be the one-point compactification of X. Then

$$C_0(X)_+ \cong C(X_+).$$

The ideal $C_0(X)$ of $C_0(X)_+$ is identified with the ideal of those functions $f \in C(X_+)$ such that $f(\infty) = 0$.

1.2.10. Definition. For an arbitrary C^*-algebra A (not necessarily unital) define

$$K_0(A) := \ker(K_0(A_+) \to K_0(\mathbb{C})).$$

Any C^*-algebra homomorphisms $f\colon A \to B$ (not necessarily unital) induces a unital homomorphism $f_+\colon A_+ \to B_+$. The induced map

$$(f_+)_*\colon K_0(A_+) \to K_0(B_+)$$

maps the kernel of the map $K_0(A_+) \to K_0(\mathbb{C})$ to the kernel of $K_0(B_+) \to K_0(\mathbb{C})$. This means it restricts to a map $f_*\colon K_0(A) \to K_0(B)$. We obtain a covariant functor from the category of (not necessarily unital) C^*-algebras to abelian groups.

Of course, we need the following result.

1.2.11. Proposition. *If A is a unital C^*-algebra, the new and the old definition of $K_0(A)$ are canonically isomorphic.*

1.2.3.2 Higher topological K-groups

We also want to define higher topological K-theory groups. We have an ad hoc definition using suspensions (this is similar to the corresponding idea in topological K-theory of spaces). For this we need the following.

1.2.12. Definition. Let A be a C^*-algebra. We define the cone CA and the suspension SA as follows.

$$\begin{aligned} CA &:= \{f\colon [0,1] \to A \mid f(0) = 0\} \\ SA &:= \{f\colon [0,1] \to A \mid f(0) = 0 = f(1)\}. \end{aligned}$$

These are again C^*-algebras, using pointwise operations and the supremum norm.

Inductively, we define

$$S^0A := A \qquad S^nA := S(S^{n-1}A) \quad \text{for } n \geq 1.$$

1.2.13. Definition. Assume A is a C^*-algebra. For $n \geq 0$, define

$$K_n(A) := K_0(S^nA).$$

These are the *topological K-theory groups of A*. For each $n \geq 0$, we obtain a functor from the category of C^*-algebras to the category of abelian groups.

For unital C^*-algebras, we can also give a more direct definition of higher K-groups (in particular useful for K_1, which is then defined in terms of (classes of) invertible matrices). This is done as follows:

1.2.14. Definition. Let A be a unital C^*-algebra. Then $Gl_n(A)$ becomes a topological group, and we have continuous embeddings

$$Gl_n(A) \hookrightarrow Gl_{n+1}(A)\colon X \mapsto \begin{pmatrix} X & 0 \\ 0 & 1 \end{pmatrix}.$$

We set $Gl_\infty(A) := \lim_{n\to\infty} Gl_n(A)$, and we equip $Gl_\infty(A)$ with the direct limit topology.

1.2.15. Proposition. *Let A be a unital C^*-algebra. If $k \geq 1$, then*

$$K_k(A) = \pi_{k-1}(Gl_\infty(A))(\cong \pi_k(BGl_\infty(A))).$$

Observe that any unital morphism $f\colon A \to B$ of unital C^*-algebras induces a map $Gl_n(A) \to Gl_n(B)$ and therefore also between $\pi_k(Gl_\infty(A))$ and $\pi_k(Gl_\infty(B))$. This map coincides with the previously defined induced map in topological K-theory.

1.2.16. Remark. Note that the topology of the C^*-algebra enters the definition of the higher topological K-theory of A, and in general the topological K-theory of A will be vastly different from the algebraic K-theory of the algebra underlying A. For connections in special cases, compare [83].

1.2.17. Example. It is well known that $Gl_n(\mathbb{C})$ is connected for each $n \in \mathbb{N}$. Therefore

$$K_1(\mathbb{C}) = \pi_0(Gl_\infty(\mathbb{C})) = 0.$$

A very important result about K-theory of C^*-algebras is the following long exact sequence. A proof can be found e.g. in [32, Proposition 4.5.9].

1.2.18. Theorem. *Assume I is a closed ideal of a C^*-algebra A. Then, we get a short exact sequence of C^*-algebras $0 \to I \to A \to A/I \to 0$, which induces a long exact sequence in K-theory*

$$\to K_n(I) \to K_n(A) \to K_n(A/I) \to K_{n-1}(I) \to \cdots \to K_0(A/I).$$

1.2.4 Bott periodicity and the cyclic exact sequence

One of the most important and remarkable results about the K-theory of C^*-algebras is Bott periodicity, which can be stated as follows.

1.2.19. Theorem. *Assume A is a C^*-algebra. There is a natural isomorphism, called the Bott map*

$$K_0(A) \to K_0(S^2A),$$

which implies immediately that there are natural isomorphism

$$K_n(A) \cong K_{n+2}(A) \qquad \forall n \geq 0.$$

1.2.20. Remark. Bott periodicity allows us to define $K_n(A)$ for each $n \in \mathbb{Z}$, or to regard the K-theory of C^*-algebras as a $\mathbb{Z}/2$-graded theory, i.e. to talk of $K_n(A)$ with $n \in \mathbb{Z}/2$. This way, the long exact sequence of Theorem 1.2.18 becomes a (six-term) cyclic exact sequence

$$\begin{array}{ccccc} K_0(I) & \longrightarrow & K_0(A) & \longrightarrow & K_0(A/I) \\ \uparrow & & & & \downarrow{\scriptstyle \mu_*} \\ K_1(A/I) & \longleftarrow & K_1(A) & \longleftarrow & K_1(I). \end{array}$$

The connecting homomorphism μ_* is the composition of the Bott periodicity isomorphism and the connecting homomorphism of Theorem 1.2.18.

1.2.5 The C^*-algebra of a group

Let Γ be a discrete group. Define $l^2(\Gamma)$ to be the Hilbert space of square summable complex valued functions on Γ. We can write an element $f \in l^2(\Gamma)$ as a sum $\sum_{g\in\Gamma} \lambda_g g$ with $\lambda_g \in \mathbb{C}$ and $\sum_{g\in\Gamma} |\lambda_g|^2 < \infty$.

We defined the *complex group algebra* (often also called the *complex group ring*) $\mathbb{C}\Gamma$ to be the complex vector space with basis the elements of Γ (this can also be considered as the space of complex valued functions on Γ with finite support, and as such is a subspace of $l^2(\Gamma)$). The product in $\mathbb{C}\Gamma$ is induced by the multiplication in Γ, namely, if $f = \sum_{g\in\Gamma} \lambda_g g, u = \sum_{g\in\Gamma} \mu_g g \in \mathbb{C}\Gamma$, then

$$(\sum_{g\in\Gamma} \lambda_g g)(\sum_{g\in\Gamma} \mu_g g) := \sum_{g,h\in\Gamma} \lambda_g \mu_h (gh) = \sum_{g\in\Gamma} \left(\sum_{h\in\Gamma} \lambda_h \mu_{h^{-1}g} \right) g.$$

This is a convolution product.

We have the *left regular representation* λ_Γ of Γ on $l^2(\Gamma)$, given by

$$\lambda_\Gamma(g) \cdot (\sum_{h\in\Gamma} \lambda_h h) := \sum_{h\in\Gamma} \lambda_h gh$$

for $g \in \Gamma$ and $\sum_{h\in\Gamma} \lambda_h h \in l^2(\Gamma)$.

This unitary representation extends linearly to $\mathbb{C}\Gamma$.

The *reduced C^*-algebra* $C_r^*\Gamma$ of Γ is defined to be the norm closure of the image $\lambda_\Gamma(\mathbb{C}\Gamma)$ in the C^*-algebra of bounded operators on $l^2(\Gamma)$.

1.2.21. Remark. It's no surprise that there is also a *maximal C^*-algebra* $C_{max}^*\Gamma$ of a group Γ. It is defined using not only the left regular representation of Γ, but simultaneously all of its representations. We will not make use of $C_{max}^*\Gamma$ in these notes, and therefore will not define it here.

Given a topological group G, one can define C^*-algebras C_r^*G and C_{max}^*G which take the topology of G into account. They actually play an important role in the study of the Baum-Connes conjecture, which can be defined for (almost arbitrary) topological groups, but again we will not cover this subject here. Instead, we will throughout stick to discrete groups.

1.2.22. Example. If Γ is finite, then $C_r^*\Gamma = \mathbb{C}\Gamma$ is the complex group ring of Γ.

In particular, in this case $K_0(C_r^*\Gamma) \cong R(\Gamma)$ coincides with the (additive group of) the complex representation ring of Γ.

1.3 The Baum-Connes conjecture

The Baum-Connes conjecture relates an object from algebraic topology, namely the K-homology of the classifying space of a given group Γ, to representation theory and the world of C^*-algebras, namely to the K-theory of the reduced C^*-algebra of Γ.

Unfortunately, the material is very technical. Because of lack of space and time we can not go into the details (even of some of the definitions). We recommend the sources [86], [87], [32], [4], [58] and [8].

1.3.1 The Baum-Connes conjecture for torsion-free groups

1.3.1. Definition. Let X be any CW-complex. $K_*(X)$ is the K-homology of X, where K-homology is the homology theory dual to topological K-theory. If BU is the spectrum of topological K-theory, and X_+ is X with a disjoint basepoint added, then

$$K_n(X) := \pi_n(X_+ \wedge BU).$$

1.3.2. Definition. Let Γ be a discrete group. A classifying space $B\Gamma$ for Γ is a CW-complex with the property that $\pi_1(B\Gamma) \cong \Gamma$, and $\pi_k(B\Gamma) = 0$ if $k \neq 1$. A classifying space always exists, and is unique up to homotopy equivalence. Its universal covering $E\Gamma$ is a contractible CW-complex with a free cellular Γ-action, the so called *universal space for Γ-actions.*

1.3.3. Remark. In the literature about the Baum-Connes conjecture, one will often find the definition

$$RK_n(X) := \varinjlim K_n(Y),$$

where the limit is taken over all finite subcomplexes Y of X. Note, however, that K-homology (like any homology theory in algebraic topology) is compatible with direct limits, which implies $RK_n(X) = K_n(X)$ as defined above. The confusion comes from the fact that operator algebraists often use Kasparov's bivariant KK-theory to define $K_*(X)$, and this coincides with the homotopy theoretic definition only if X is compact.

Recall that a group Γ is called torsion-free, if $g^n = 1$ for $g \in \Gamma$ and $n > 0$ implies that $g = 1$.

We can now formulate the Baum-Connes conjecture for torsion-free discrete groups.

1.3.4. Conjecture. *Assume Γ is a torsion-free discrete group. It is known that there is a particular homomorphism, the assembly map*

$$\overline{\mu}_*\colon K_*(B\Gamma) \to K_*(C_r^*\Gamma) \tag{1.3.5}$$

(which will be defined later). The Baum-Connes conjecture *says that this map is an isomorphism.*

1.3.6. Example. The map $\overline{\mu}_*$ of Equation (1.3.5) is also defined if Γ is not torsion-free. However, in this situation it will in general not be an isomorphism. This can already be seen if $\Gamma = \mathbb{Z}/2$. Then $C_r^*\Gamma = \mathbb{C}\Gamma \cong \mathbb{C}\oplus\mathbb{C}$ as a $\mathbb{C}$-algebra. Consequently,

$$K_0(C_r^*\Gamma) \cong K_0(\mathbb{C}) \oplus K_0(\mathbb{C}) \cong \mathbb{Z} \oplus \mathbb{Z}. \tag{1.3.7}$$

On the other hand, using the homological Chern character,

$$K_0(B\Gamma) \otimes_{\mathbb{Z}} \mathbb{Q} \cong \oplus_{n=0}^{\infty} H_{2n}(B\Gamma; \mathbb{Q}) \cong \mathbb{Q}. \tag{1.3.8}$$

(Here we use the fact that the rational homology of every finite group is zero in positive degrees, which follows from the fact that the transfer homomorphism $H_k(B\Gamma; \mathbb{Q}) \to H_k(\{1\}; \mathbb{Q})$ is (with rational coefficients) up to a factor

$|\Gamma|$ a left inverse to the map induced from the inclusion, and therefore is injective.)

The calculations (1.3.7) and (1.3.8) prevent μ_0 of (1.3.5) from being an isomorphism.

1.3.2 The Baum-Connes conjecture in general

To account for the problem visible in Example 1.3.6 if we are dealing with groups with torsion, one replaces the left hand side by a more complicated gadget, the equivariant K-homology of a certain Γ-space $E(\Gamma, fin)$, the classifying space for proper actions. We will define all of this later. Then, the Baum-Connes conjecture says the following.

1.3.9. Conjecture. *Assume Γ is a discrete group. It is known that there is a particular homomorphism, the assembly map*

$$\mu_*\colon K_*^\Gamma(E(\Gamma, fin)) \to K_*(C_r^*\Gamma) \tag{1.3.10}$$

(we will define it later). The conjecture says that this map is an isomorphism.

1.3.11. Remark. If Γ is torsion-free, then $K_*(B\Gamma) = K_*^\Gamma(E(\Gamma, fin))$, and the assembly maps $\overline{\mu}$ of Conjectures 1.3.4 and μ of 1.3.9 coincide (see Proposition 1.3.29).

Last, we want to mention that there is also a *real version* of the Baum-Connes conjecture, where on the left hand side the K-homology is replaced by KO-homology, i.e. the homology dual to the K-theory of real vector spaces (or an equivariant version hereof), and on the right hand side $C_r^*\Gamma$ is replaced by the real reduced C^*-algebra $C_{r,\mathbb{R}}^*\Gamma$.

1.3.3 Consequences of the Baum-Connes conjecture

1.3.3.1 Idempotents in $C_r^*\Gamma$

The connection between the Baum-Connes conjecture and idempotents is best shown via Atiyah's L^2-index theorem, which we discuss first.

Given a closed manifold M with an elliptic differential operator $D\colon C^\infty(E) \to C^\infty(F)$ between two bundles on M, and a normal covering $\tilde{M} \to M$ (with deck transformation group Γ, normal means that $M = \tilde{M}/\Gamma$), we can lift E, F and D to $\tilde{M}$, and get an elliptic Γ-equivariant differential operator $\tilde{D}\colon C^\infty(\tilde{E}) \to C^\infty(\tilde{F})$. If Γ is not finite, we can not use the equivariant index of Section 1.1.3. However, because the action is free, it is possible to define an equivariant analytic index

$$\operatorname{ind}_\Gamma(\tilde{D}) \in K_{\dim M}(C_r^*\Gamma).$$

This is described in Example 1.3.37.

Atiyah used a certain real valued homomorphism, the Γ-dimension

$$\dim_\Gamma \colon K_0(C_r^*\Gamma) \to \mathbb{R},$$

to define the L^2-index of $\tilde{D}$ (on an even dimensional manifold):

$$L^2\text{-}\operatorname{ind}(\tilde{D}) := \dim_\Gamma(\operatorname{ind}_\Gamma(\tilde{D})).$$

The L^2-index theorem says

$$L^2\text{-}\operatorname{ind}(\tilde{D}) = \operatorname{ind}(D),$$

in particular, it follows that the L^2-index is an integer. For a different point of view of the L^2-index theorem, compare Section 3.1.

An alternative description of the left hand side of (1.3.5) and (1.3.10) shows that, as long as Γ is torsion-free, the image of μ_0 coincides with the subset of $K_0(C_r^*\Gamma)$ consisting of $\operatorname{ind}_\Gamma(\tilde{D})$, where $\tilde{D}$ is as above. In particular, if μ_0 is surjective (and Γ is torsion-free), for each $x \in K_0(C_r^*\Gamma)$ we find a differential operator D such that $x = \operatorname{ind}_\Gamma(\tilde{D})$. As a consequence, $\dim_\Gamma(x) \in \mathbb{Z}$, i.e. the range of $\dim_\Gamma$ is contained in $\mathbb{Z}$. This is the statement of the so called *trace conjecture.*

1.3.12. Conjecture. *Assume Γ is a torsion-free discrete group. Then*

$$\dim_\Gamma(K_0(C_r^*\Gamma)) \subset \mathbb{Z}.$$

On the other hand, if $x \in K_0(C_r^*\Gamma)$ is represented by a projection $p = p^2 \in C_r^*\Gamma$, then elementary properties of $\dim_\Gamma$ (monotonicity and faithfulness) imply that $0 \leq \dim_\Gamma(p) \leq 1$, and $\dim_\Gamma(p) \notin \{0,1\}$ if $p \neq 0, 1$.

Therefore, we have the following consequence of the Baum-Connes conjecture. If Γ is torsion-free and the Baum-Connes map μ_0 is surjective, then $C_r^*\Gamma$ does not contain any projection different from 0 or 1.

This is the assertion of the Kadison-Kaplansky conjecture:

1.3.13. Conjecture. *Assume Γ is torsion-free. Then $C_r^*\Gamma$ does not contain any non-trivial projections.*

The following consequence of the Kadison-Kaplansky conjecture deserves to be mentioned:

1.3.14. Proposition. *If the Kadison-Kaplansky conjecture is true for a group Γ, then the spectrum $s(x)$ of every self adjoint element $x \in C_r^*\Gamma$ is connected. Recall that the spectrum is defined in the following way:*

$$s(x) := \{\lambda \in \mathbb{C} \mid (x - \lambda \cdot 1) \text{ not invertible}\}.$$

If Γ is not torsion-free, it is easy to construct non-trivial projections, and it is clear that the range of ind_Γ is not contained in $\mathbb{Z}$. Baum and Connes originally conjectured that it is contained in the abelian subgroup $Fin^{-1}(\Gamma)$ of $\mathbb{Q}$ generated by $\{1/|F| \mid F \text{ finite subgroup of } \Gamma\}$. This conjecture is not correct, as is shown by an example of Roy [67]. In [52], Lück proves that the Baum-Connes conjecture implies that the range of $\dim_\Gamma$ is contained in the *subring* of $\mathbb{Q}$ generated by $\{1/|F| \mid F \text{ finite subgroup of } \Gamma\}$.

1.3.3.2 Obstructions to positive scalar curvature

The Baum-Connes conjecture implies the so called "stable Gromov-Lawson-Rosenberg" conjecture. This implication is a theorem due to Stephan Stolz. The details of this will be discussed in the lectures of Stephan Stolz, therefore we can be very brief. We just mention the result.

1.3.15. Theorem. *Fix a group Γ. Assume that μ in the real version of* (1.3.10) *discussed in Section 1.4 is injective (which follows e.g. if μ in* (1.3.10) *is an isomorphism), and assume that M is a closed spin manifold with $\pi_1(M) = \Gamma$. Assume that a certain (index theoretic) invariant $\alpha(M) \in K_{\dim M}(C_{\mathbb{R},r}^*\Gamma)$ vanishes. Then there is an $n \geq 0$ such that $M \times B^n$ admits a metric with positive scalar curvature.*

Here, B is any simply connected 8-dimensional spin manifold with $\hat{A}(M) = 1$. Such a manifold is called a *Bott manifold*.

The converse of Theorem 1.3.15, i.e. positive scalar curvature implies vanishing of $\alpha(M)$, is true for arbitrary groups and without knowing anything about the Baum-Connes conjecture.

1.3.3.3 The Novikov conjecture about higher signatures

Direct approach The original form of the Novikov conjecture states that higher signatures are homotopy invariant.

More precisely, let M be an (even dimensional) closed oriented manifold with fundamental group Γ. Let $B\Gamma$ be a classifying space for Γ. There is a unique (up to homotopy) *classifying map* $u\colon M \to B\Gamma$ which is defined by the property that it induces an isomorphism on π_1. Equivalently, u classifies a universal covering of M.

Let $L(M) \in H^*(M;\mathbb{Q})$ be the Hirzebruch L-class (as normalized by Atiyah and Singer). Given any cohomology class $a \in H^*(B\Gamma,\mathbb{Q})$, we define the higher signature

$$\sigma_a(M) := \langle L(M) \cup u^*a, [M]\rangle \in \mathbb{Q}.$$

Here $[M] \in H_{\dim M}(M;\mathbb{Q})$ is the fundamental class of the oriented manifold M, and $\langle\cdot,\cdot\rangle$ is the usual pairing between cohomology and homology.

Recall that the Hirzebruch signature theorem states that $\sigma_1(M)$ is the signature of M, which evidently is an oriented homotopy invariant.

The Novikov conjecture generalizes this as follows.

1.3.16. Conjecture. *Assume $f\colon M \to M'$ is an oriented homotopy equivalence between two even dimensional closed oriented manifolds, with (common) fundamental group π. "Oriented" means that $f_*[M] = [M']$. Then all higher signatures of M and M' are equal, i.e.*

$$\sigma_a(M) = \sigma_a(M') \qquad \forall a \in H^*(B\Gamma,\mathbb{Q}).$$

There is an equivalent reformulation of this conjecture in terms of K-homology. To see this, let D be the signature operator of M. (We assume here that M is smooth, and we choose a Riemannian metric on M to define this operator. It is an elliptic differential operator on M.) The operator D defines an element in the K-homology of M, $[D] \in K_{\dim M}(M)$. Using the map u, we can push $[D]$ to $K_{\dim M}(B\Gamma)$. We define the higher signature $\sigma(M) := u_*[D] \in K_{\dim M}(B\Gamma)\otimes\mathbb{Q}$. It turns out that

$$2^{\dim M/2}\sigma_a(M) = \langle a, ch(\sigma(M))\rangle \qquad \forall a \in H^*(B\Gamma;\mathbb{Q}),$$

where $ch\colon K_*(B\Gamma)\otimes\mathbb{Q} \to H_*(B\Gamma,\mathbb{Q})$ is the homological Chern character (an isomorphism).

Therefore, the Novikov conjecture translates to the statement that $\sigma(M) = \sigma(M')$ if M and M' are oriented homotopy equivalent.

Now one can show *directly* that

$$\overline{\mu}(\sigma(M)) = \overline{\mu}(\sigma(M')) \in K_*(C_r^*\Gamma),$$

if M and M' are oriented homotopy equivalent. Consequently, rational injectivity of the Baum-Connes map $\overline{\mu}$ immediately implies the Novikov conjecture. If Γ is torsion-free, this is part of the assertion of the Baum-Connes conjecture. Because of this relation, injectivity of the Baum-Connes map μ is often called the "analytic Novikov conjecture".

L-theory approach There is a more obvious connection between the Baum-Connes isomorphism conjecture and the L-theory isomorphism conjecture (discussed in other lectures).

Namely, the L-theory isomorphism conjecture is concerned with a certain assembly map

$$A_\Gamma\colon H_*(B\Gamma, \mathbb{L}_\bullet(\mathbb{Z})) \to L_*(\mathbb{Z}[\Gamma]).$$

Here, the left hand side is the homology of $B\Gamma$ with coefficients the algebraic surgery spectrum of $\mathbb{Z}$, and the right hand side is the free quadratic L-group of the ring with involution $\mathbb{Z}[\Gamma]$.

The Novikov conjecture is equivalent to the statement that this map is rationally injective, i.e. that

$$A_\Gamma \otimes \mathrm{id}_\mathbb{Q}\colon H_*(B\Gamma, \mathbb{L}_\bullet(\mathbb{Z})) \otimes \mathbb{Q} \to L_*(\mathbb{Z}[\Gamma]) \otimes \mathbb{Q}$$

is an injection. This formulation has the advantage that, tensored with $\mathbb{Q}$, all the different flavors of L-theory are isomorphic (therefore, we don't have to and we won't discuss these distinctions here).

Now, we get a commutative diagram

$$\begin{array}{ccc} H_*(B\Gamma, \mathbb{L}_\bullet(\mathbb{Z})) \otimes \mathbb{Q} & \xrightarrow{A_\Gamma} & L_*(\mathbb{Z}[\Gamma]) \otimes \mathbb{Q} \\ \downarrow & & \downarrow \\ H_*(B\Gamma, \mathbb{L}_\bullet(\mathbb{C})) \otimes \mathbb{Q} & \xrightarrow{A_{\Gamma,\mathbb{C}}} & L_*(\mathbb{C}[\Gamma]) \otimes \mathbb{Q} \\ \downarrow & & \downarrow \\ K_*(B\Gamma) \otimes \mathbb{Q} & \xrightarrow{\overline{\mu}} & K_*(C_r^*\Gamma) \otimes \mathbb{Q} = L_*(C_r^*\Gamma) \otimes \mathbb{Q}. \end{array} \tag{1.3.17}$$

The maps on the left hand side are given by natural transformations of homology theories with values in rational vector spaces. These transformations are easily seen to be injective for the coefficients. Since we deal with *rational* homology theories, they are injective in general.

The maps on the right hand side are the maps in L-theory induced by the obvious ring homomorphisms $\mathbb{Z}\Gamma \to \mathbb{C}\Gamma \to C_r^*\Gamma$. Then we use the "folk theorem" that, for C^*-algebras, K-theory and L-theory are canonically isomorphic (even non-rationally). Of course, it remains to establish commutativity of the diagram (1.3.17). For more details, we refer to [66]. Using all these facts and the diagram (1.3.17), we see that for torsion-free groups, rational injectivity of the Baum-Connes map μ implies rational injectivity of the L-theory assembly A_Γ, i.e. the Novikov conjecture.

Groups with torsion For an arbitrary group Γ, we have a factorization of $\overline{\mu}$ as follows:

$$K_*(B\Gamma) \xrightarrow{f} K_*^\Gamma(E(\Gamma, fin)) \xrightarrow{\mu} K_*(C_r^*\Gamma).$$

One can show that f is rationally injective, so that rational injectivity of the Baum-Connes map μ implies the Novikov conjecture also in general.

1.3.4 The universal space for proper actions

1.3.18. Definition. Let Γ be a discrete group and X a Hausdorff space with an action of Γ. We say that the action is *proper*, if for all $x, y \in X$ there are open neighborhood $U_x \ni x$ and $U_y \ni y$ such that $gU_x \cap U_y$ is non-empty only for finitely many $g \in \Gamma$ (the number depending on x and y).

The action is said to be *cocompact*, if X/Γ is compact.

1.3.19. Lemma. *If the action of Γ on X is proper, then for each $x \in X$ the* isotropy group $\Gamma_x := \{g \in \Gamma \mid gx = x\}$ *is finite.*

1.3.20. Definition. Let Γ be a discrete group. A CW-complex X is a Γ-*CW-complex*, if X is a CW-complex with a cellular action of Γ with the additional property that, whenever $g(D) \subset D$ for a cell D of X and some $g \in \Gamma$, then $g|_D = \mathrm{id}_D$, i.e. g doesn't move D at all.

1.3.21. Remark. There exists also the notion of G-CW-complex for topological groups G (taking the topology of G into account). These have to be defined in a different way, namely by gluing together G-equivariant cells $D^n \times G/H$. In general, such a G-CW-complex is not an ordinary CW-complex.

1.3.22. Lemma. *The action of a discrete group Γ on a Γ-CW-complex is proper if and only if every isotropy group is finite.*

1.3.23. Definition. A proper Γ-CW-complex X is called *universal*, or more precisely *universal for proper actions*, if for every proper Γ-CW-complex Y there is a Γ-equivariant map $f\colon Y \to X$ which is unique up to Γ-equivariant homotopy. Any such space is denoted $E(\Gamma, fin)$ or $\underline{E}\Gamma$.

1.3.24. Proposition. *A Γ-CW-complex X is universal for proper actions if and only if the fixed point set*

$$X^H := \{x \in X \mid hx = x \quad \forall h \in H\}$$

is empty whenever H is an infinite subgroup of Γ, and is contractible (and in particular non-empty) if H is a finite subgroup of Γ.

1.3.25. Proposition. *If Γ is a discrete group, then $E(\Gamma, fin)$ exists and is unique up to Γ-homotopy equivalence.*

1.3.26. Remark. The general context for this discussion are actions of a group Γ where the isotropy belongs to a fixed family of subgroups of Γ (in our case, the family of all finite subgroups). For more information, compare [85].

1.3.27. Example.

- If Γ is torsion-free, then $E(\Gamma, fin) = E\Gamma$, the universal covering of the classifying space $B\Gamma$. Indeed, Γ acts freely on $E\Gamma$, and $E\Gamma$ is contractible.
- If Γ is finite, then $E(\Gamma, fin) = \{*\}$.
- If G is a connected Lie group with maximal compact subgroup K, and Γ is a discrete subgroup of G, then $E(\Gamma, fin) = G/K$ [4, Section 2].

1.3.28. Remark. In the literature (in particular, in [4]), also a slightly different notion of universal spaces is discussed. One allows X to be any proper metrizable Γ-space, and requires the universal property for all proper metrizable Γ-spaces Y. For discrete groups (which are the only groups we are discussing here), a universal space in the sense of Definition 1.3.23 is universal in this sense.

However, for some of the proofs of the Baum-Connes conjecture (for special groups) it is useful to use certain models of $E(\Gamma, fin)$ (in the broader sense) coming from the geometry of the group, which are not Γ-CW-complexes.

1.3.5 Equivariant K-homology

Let Γ be a discrete group. We have seen that, if Γ is not torsion-free, the assembly map (1.3.5) is not an isomorphism. To account for that, we replace $K_*(B\Gamma)$ by the equivariant K-theory of $E(\Gamma, fin)$. Let X be any proper Γ-CW complex. The original definition of equivariant K-homology is due to Kasparov, making ideas of Atiyah precise. In this definition, elements of $K_*^\Gamma(X)$ are equivalence classes of generalized elliptic operators. In [14], a more homotopy theoretic definition of $K_*^\Gamma(X)$ is given, which puts the Baum-Connes conjecture in the context of other isomorphism conjectures.

1.3.5.1 Homotopy theoretic definition of equivariant K-homology

The details of this definition are quite technical, using spaces and spectra over the orbit category of the discrete group Γ. The objects of the orbit category are the orbits Γ/H, H any subgroup of Γ. The morphisms from Γ/H to

Γ/K are simply the Γ-equivariant maps. In this setting, any spectrum over the orbit category gives rise to an equivariant homology theory. The decisive step is then the construction of a (periodic) topological K-theory spectrum $\mathbf{K}^\Gamma$ over the orbit category of Γ. This gives us then a functor from the category of (arbitrary) Γ-CW-complexes to the category of (graded) abelian groups, the *equivariant K-homology* $K_*^\Gamma(X)$ (X any Γ-CW-complex).

The important property (which justifies the name "topological K-theory spectrum) is that

$$K_k^\Gamma(\Gamma/H) = \pi_k(\mathbf{K}^\Gamma(\Gamma/H)) \cong K_k(C_r^*H)$$

for every subgroup H of Γ. In particular,

$$K_k^\Gamma(\{*\}) \cong K_k(C_r^*\Gamma).$$

Moreover, we have the following properties:

1.3.29. Proposition. *(1) Assume Γ is the trivial group. Then*

$$K_*^\Gamma(X) = K_*(X),$$

i.e. we get back the ordinary K-homology introduced above.

(2) If $H \leq \Gamma$ and X is an H-CW-complex, then there is a natural isomorphism

$$K_*^H(X) \cong K_*^\Gamma(\Gamma \times_H X).$$

Here $\Gamma \times_H X = \Gamma \times H/\sim$, where we divide out the equivalence relation generated by $(gh, x) \sim (g, hx)$ for $g \in \Gamma$, $h \in H$ and $x \in X$. This is in the obvious way a left Γ-space.

(3) Assume X is a free Γ-CW-complex. Then there is a natural isomorphism

$$K_*(\Gamma\backslash X) \to K_*^\Gamma(X).$$

In particular, using the canonical Γ-equivariant map $E\Gamma \to E(\Gamma, fin)$, we get a natural homomorphism

$$K_*(B\Gamma) \xrightarrow{\cong} K_*^\Gamma(E\Gamma) \to K_*^\Gamma(E(\Gamma, fin)).$$

1.3.5.2 Analytic definition of equivariant K-homology

Here we will give the original definition, which embeds into the powerful framework of equivariant KK-theory, and which is used for almost all proofs of special cases of the Baum-Connes conjecture. However, to derive some of the consequences of the Baum-Connes conjecture, most notably about the positive scalar curvature question —this is discussed in one of the lectures of Stephan Stolz— the homotopy theoretic definition is used.

1.3.30. Definition. A Hilbert space H is called *($\mathbb{Z}/2$)-graded*, if H comes with an orthogonal sum decomposition $H = H_0 \oplus H_1$. Equivalently, a unitary operator ϵ with $\epsilon^2 = 1$ is given on H. The subspaces H_0 and H_1 can be recovered as the $+1$ and -1 eigenspaces of ϵ, respectively.

A bounded operator $T\colon H \to H$ is called *even* (with respect to the given grading), if T commutes with ϵ, and *odd*, if ϵ and T anti-commute, i.e. if $T\epsilon = -\epsilon T$. An even operator decomposes as $T = \left(\begin{smallmatrix} T_0 & 0 \\ 0 & T_1 \end{smallmatrix}\right)$, an odd one as $T = \left(\begin{smallmatrix} 0 & T_0 \\ T_1 & 0 \end{smallmatrix}\right)$ in the given decomposition $H = H_0 \oplus H_1$.

1.3.31. Definition. A *generalized elliptic* Γ*-operator on* X, or a *cycle for* Γ*-K-homology of the* Γ*-space* X, simply a *cycle* for short, is a triple (H, π, F), where

- $H = H_0 \oplus H_1$ is a $\mathbb{Z}/2$-graded Γ-Hilbert space (i.e. the direct sum of two Hilbert spaces with unitary Γ-action)
- π is a Γ-equivariant $*$-representation of $C_0(X)$ on even bounded operators of H (equivariant means that $\pi(fg^{-1}) = g\pi(f)g^{-1}$ for all $f \in C_0(X)$ and all $g \in \Gamma$.
- $F\colon H \to H$ is a bounded, Γ-equivariant, self adjoint operator such that $\pi(f)(F^2 - 1)$ and $[\pi(f), F] := \pi(f)F - F\pi(f)$ are compact operators for all $f \in C_0(X)$. Moreover, we require that F is odd, i.e. $F = \left(\begin{smallmatrix} 0 & D^* \\ D & 0 \end{smallmatrix}\right)$ in the decomposition $H = H_0 \oplus H_1$.

1.3.32. Remark. There are many different definitions of cycles, slightly weakening or strengthening some of the conditions. Of course, this does not effect the equivariant K-homology groups which are eventually defined using them.

1.3.33. Definition. We define the direct sum of two cycles in the obvious way.

1.3.34. Definition. Assume $\alpha = (H, \pi, F)$ and $\alpha' = (H', \pi', F')$ are two cycles.

(1) They are called (isometrically) isomorphic, if there is a Γ-equivariant grading preserving isometry $\Psi\colon H \to H'$ such that $\Psi\circ\pi(f) = \pi'(f)\circ\Psi$ for all $f \in C_0(X)$ and $\Psi \circ F = F' \circ \Psi$.

(2) They are called *homotopic* (or *operator homotopic*) if $H = H'$, $\pi = \pi'$, and there is a norm continuous path $(F_t)_{t\in[0,1]}$ of operators with $F_0 = F$ and $F_1 = F'$ and such that (H, π, F_t) is a cycle for each $t \in [0, 1]$.

(3) (H, π, F) is called *degenerate*, if $[\pi(f), F] = 0$ and $\pi(f)(F^2 - 1) = 0$ for each $f \in C_0(X)$.

(4) The two cycles are called equivalent if there are degenerate cycles β and β' such that $\alpha \oplus \beta$ is operator homotopic to a cycle isometrically isomorphic to $\alpha' \oplus \beta'$.

The set of equivalence classes of cycles is denoted $KK_0^\Gamma(X)$. (Caution, this is slightly unusual, mostly one will find the notation $K^\Gamma(X)$ instead of $KK^\Gamma(X)$).

1.3.35. Proposition. *Direct sum induces the structure of an abelian group on* $KK_0^\Gamma(X)$.

1.3.36. Proposition. *Any proper* Γ*-equivariant map* $\phi\colon X \to Y$ *between two proper* Γ*-*CW*-complexes induces a homomorphism*

$$KK_0^\Gamma(X) \to KK_0^\Gamma(Y)$$

by $(H, \pi, F) \mapsto (H, \pi \circ \phi^*, F)$, *where* $\phi^*\colon C_0(Y) \to C_0(X)\colon f \mapsto f \circ \phi$ *is defined since* ϕ *is a proper map (else* $f \circ \phi$ *does not necessarily vanish at infinity).*

Recall that a continuous map $\phi\colon X \to Y$ is called *proper* if the inverse image of every compact subset of Y is compact .

It turns out that the analytic definition of equivariant K-homology is quite flexible. It is designed to make it easy to construct elements of these groups —in many geometric situations they automatically show up. We give one of the most typical examples of such a situation, which we will need later.

1.3.37. Example. Assume that M is a compact even dimensional Riemannian manifold. Let $X = \overline{M}$ be a normal covering of M with deck transformation group Γ (normal means that $X/\Gamma = M$). Of course, the action is free, in particular, proper. Let $E = E_0 \oplus E_1$ be a graded Hermitian vector bundle on M, and

$$D\colon C^\infty(E) \to C^\infty(E)$$

an odd elliptic self adjoint differential operator (odd means that D maps the subspace $C^\infty(E_0)$ to $C^\infty(E_1)$, and vice versa). If M is oriented, the signature operator on M is such an operator, if M is a spin-manifold, the same is true for its Dirac operator.

Now we can pull back E to a bundle $\overline{E}$ on $\overline{M}$, and lift D to an operator $\overline{D}$ on $\overline{E}$. The assumptions imply that $\overline{D}$ extends to an unbounded self adjoint operator on $L^2(\overline{E})$, the space of square integrable sections of $\overline{E}$. This space is the completion of $C_c^\infty(\overline{E})$ with respect to the canonical inner product (compare Definition 3.1.1). (The subscript c denotes sections with compact support). Using the functional calculus, we can replace $\overline{D}$ by

$$F := (\overline{D}^2 + 1)^{-1/2}\overline{D} \colon L^2(\overline{E}) \to L^2(\overline{E}).$$

Observe that

$$L^2(\overline{E}) = L^2(\overline{E}_0) \oplus L^2(\overline{E}_1)$$

is a $\mathbb{Z}/2$-graded Hilbert space with a unitary Γ-action, which admits an (equivariant) action π of $C_0(\overline{M}) = C_0(X)$ by fiber-wise multiplication. This action preserves the grading. Moreover, $\overline{D}$ as well as F are odd, Γ-equivariant, self adjoint operators on $L^2(\overline{E})$ and F is a bounded operator. From ellipticity it follows that

$$\pi(f)(F^2 - 1) = -\pi(f)(\overline{D}^2 + 1)^{-1}$$

is compact for each $f \in C_0(\overline{M})$ (observe that this is not true for $(\overline{D}^2 + 1)^{-1}$ itself, if $\overline{M}$ is not compact). Consequently, $(L^2(E), \pi, F)$ defines an (even) cycle for Γ-K-homology, i.e. it represents an element in $KK_0^\Gamma(X)$.

One can slightly reformulate the construction as follows: $\overline{M}$ is a principal Γ-bundle over M, and $l^2(\Gamma)$ has a (unitary) left Γ-action. We therefore can construct the associated flat bundle

$$L := l^2(\Gamma) \times_\Gamma \overline{M}$$

on M with fiber $l^2(\Gamma)$. Now we can twist D with this bundle L, i.e. define

$$\overline{D} := \nabla_L \otimes \mathrm{id} + \mathrm{id} \otimes D \colon C^\infty(L \otimes E) \to C^\infty(L \otimes E),$$

using the given flat connection ∇_L on L. Again, we can complete to $L^2(L \otimes E)$ and define

$$F := (\overline{D}^2 + 1)^{-1/2}\overline{D}.$$

The left action of Γ on $l^2\Gamma$ induces an action of Γ on L and then a unitary action on $L^2(L \otimes E)$. Since ∇_L preserves the Γ-action, $\overline{D}$ is Γ-equivariant. There is a canonical Γ-isometry between $L^2(L \otimes E)$ and $L^2(\overline{E})$ which identifies the two versions of $\overline{D}$ and F. The action of $C_0(\overline{M})$ on $L^2(L \otimes E)$ can be described by identifying $C_0(\overline{M})$ with the continuous sections of M on the associated bundle

$$C_0(\Gamma) \times_\Gamma \overline{M},$$

where $C_0(\Gamma)$ is the C^*-algebra of functions on Γ vanishing at infinity, and then using the obvious action of $C_0(\Gamma)$ on $l^2(\Gamma)$.

It is easy to see how this examples generalizes to Γ-equivariant elliptic differential operators on manifolds with a proper, but not necessarily free, Γ-action (with the exception of the last part, of course).

Work in progress of Baum and Schick [5] suggests the (somewhat surprising) fact that, given any proper Γ-CW-complex Y, we can, for each element $y \in KK_0^\Gamma(Y)$, find such a proper Γ-manifold X, together with a Γ-equivariant map $f\colon X \to Y$ and an elliptic differential operator on X giving an element $x \in KK_0^\Gamma(X)$ as in the example, such that $y = f_*(x)$.

Analytic K-homology is homotopy invariant, a proof can be found in [8].

1.3.38. Theorem. *If $\phi_1, \phi_2\colon X \to Y$ are proper Γ-equivariant maps which are homotopic through proper Γ-equivariant maps, then*

$$(\phi_1)_* = (\phi_2)_*\colon KK_*^\Gamma(X) \to KK_*^\Gamma(Y).$$

1.3.39. Theorem. *If Γ acts freely on X, then*

$$KK_*^\Gamma(X) \cong K_*(\Gamma\backslash X),$$

where the right hand side is the ordinary K-homology of $\Gamma\backslash X$.

1.3.40. Definition. Assume Y is an arbitrary proper Γ-CW-complex. Set

$$RK_*^\Gamma(Y) := \varinjlim KK_*^\Gamma(X),$$

where we take the direct limit over the direct system of Γ-invariant subcomplexes of Y with compact quotient (by the action of Γ).

1.3.41. Definition. To define higher (analytic) equivariant K-homology, there are two ways. The short one only works for complex K-homology. One considers cycles and an equivalence relation exactly as above — with the notable exception that one does not require any grading! This way, one defines $KK_1^\Gamma(X)$. Because of Bott periodicity (which has period 2), this is enough to define all K-homology groups ($KK_n^\Gamma(X) = KK_{n+2k}^\Gamma(X)$ for any $k \in \mathbb{Z}$).

A perhaps more conceptual approach is the following. Here, one generalizes the notion of a graded Hilbert space by the notion of a p-multigraded Hilbert space ($p \geq 0$). This means that the graded Hilbert space comes with p unitary operators $\epsilon_1, \dots, \epsilon_p$ which are odd with respect to the grading, which satisfy $\epsilon_i^2 = -1$ and $\epsilon_i\epsilon_j + \epsilon_j\epsilon_i = 0$ for all i and j with $i \neq j$. An operator $T\colon H \to H$ on a p-multigraded Hilbert space is called *multigraded* if it commutes with $\epsilon_1, \dots, \epsilon_p$. Such operators can (in addition) be even or odd.

This definition can be reformulated as saying that a multigraded Hilbert space is a (right) module over the Clifford algebra Cl_p, and a multigraded operator is a module map.

We now define $KK_p^\Gamma(X)$ using cycles as above, with the additional assumption that the Hilbert space is p-graded, that the representation π takes values in π-multigraded even operators, and that the operator F is an odd p-multigraded operator. Isomorphism and equivalence of these multigraded cycles is defined as above, requiring that the multigradings are preserved throughout.

This definition gives an equivariant homology theory if we restrict to *proper* maps. Moreover, it satisfies Bott periodicity. The period is two for the (complex) K-homology we have considered so far. All results mentioned in this section generalize to higher equivariant K-homology.

If X is a proper Γ-CW-complex, the analytically defined representable equivariant K-homology groups $RK_p^\Gamma(X)$ are canonically isomorphic to the equivariant K-homology groups $K_p^\Gamma(X)$ defined by Davis and Lück in [14] as described in Section 1.3.5.1.

1.3.6 The assembly map

Here, we will use the homotopy theoretic description of equivariant K-homology due to Davis and Lück [14] described in Section 1.3.5.1. The assembly map then becomes particularly convenient to describe. From the present point of view, the main virtue is that they define a functor from *arbitrary*, not necessarily proper, Γ-CW-complexes to abelian groups.

The Baum-Connes assembly map is now simply defined using the equivariant collapse $E(\Gamma, fin) \to *$:

$$\mu\colon K_k^\Gamma(E(\Gamma, fin)) \to K_k^\Gamma(*) = K_k(C_r^*\Gamma). \tag{1.3.42}$$

If Γ is torsion-free, then $E\Gamma = E(\Gamma, fin)$, and the assembly map of (1.3.5) is defined as the composition of (1.3.42) with the appropriate isomorphism in Proposition 1.3.29.

1.3.7 Survey of KK-theory

The analytic definition of Γ-equivariant K-homology can be extended to a bivariant functor on Γ-C^*-algebras. Here, a Γ-C^*-algebra is a C^*-algebra A with an action (by C^*-algebra automorphisms) of Γ. If X is a proper Γ-space, $C_0(X)$ is such a Γ-C^*-algebra.

Given two Γ-C^*-algebras A and B, Kasparov defines the bivariant KK-groups $KK_*^\Gamma(A,B)$. The most important property of this bivariant KK-theory is that it comes with a (composition) product, the *Kasparov product*. This can be stated most conveniently as follows:

Given a discrete group Γ, we have a category KK^Γ whose objects are Γ-C^*-algebras (we restrict here to separable C^*-algebras). The morphisms in this category between two Γ-C^*-algebras A and B are called $KK_*^\Gamma(A,B)$. They are $\mathbb{Z}/2$-graded abelian groups, and the composition preserves the grading, i.e. if $\phi \in KK_i^\Gamma(A,B)$ and $\psi \in KK_j^\Gamma(B,C)$ then $\psi\phi \in KK_{i+j}^\Gamma(A,C)$.

There is a functor from the category of separable Γ-C^*-algebras (where morphisms are Γ-equivariant $*$-homomorphisms) to the category KK_*^Γ which maps an object A to A, and such that the image of a morphism $\phi\colon A \to B$ is contained in $KK_0^\Gamma(A,B)$.

If X is a proper cocompact Γ-CW-complex then (by definition)

$$KK_p^\Gamma(C_0(X), \mathbb{C}) = KK_{-p}^\Gamma(X).$$

Here, $\mathbb{C}$ has the trivial Γ-action.

On the other hand, for any C^*-algebra A without a group action (i.e. with trivial action of hte trivial group $\{1\}$), $KK_*^{\{1\}}(\mathbb{C}, A) = K_*(A)$.

There is a functor from KK^Γ to $KK^{\{1\}}$, called *descent*, which assigns to every Γ-C^*-algebra A the *reduced crossed product* $C_r^*(\Gamma, A)$. The crossed product has the property that $C_r^*(\Gamma, \mathbb{C}) = C_r^*\Gamma$.

1.3.8 KK assembly

We now want to give an account of the analytic definition of the assembly map, which was the original definition. The basic idea is that the assembly map is given by taking an index. To start with, assume that we have an even generalized elliptic Γ-operator (H, π, F), representing an element in $K_0^\Gamma(X)$, where X is a proper Γ-space such that $\Gamma\backslash X$ is compact. The index of this operator should give us an element in $K_0(C_r^*\Gamma)$. Since the cycle is even, H

split as $H = H_0 \oplus H_1$, and $F = \left(\begin{smallmatrix} 0 & P \\ P^* & 0 \end{smallmatrix}\right)$ with respect to this splitting. Indeed, now, the kernel and cokernel of P are modules over $\mathbb{C}\Gamma$, and should, in most cases, give modules over $C_r^*\Gamma$.

If Γ is finite, the latter is indeed the case (since $C_r^*\Gamma = \mathbb{C}\Gamma$). Moreover, since $\Gamma\backslash X$ is compact and Γ is finite, X is compact, which implies that $C_0(X)$ is unital. We may then assume that π is unital (switching to an equivalent cycle with Hilbert space $\pi(1)H$, if necessary). But then the axioms for a cycle imply that $F^2 - 1$ is compact, i.e. that F is invertible modulo compact operators, or that F is Fredholm, which means that $\ker(P)$ and $\ker(P^*)$ are finite dimensional. Since Γ acts on them, $[\ker(P)] - [\ker(P^*)]$ defines an element of the representation ring $R\Gamma = K_0(C_r^*\Gamma)$ for the finite group Γ. It remains to show that this map respects the equivalence relation defining $K_0^\Gamma(X)$.

However, if Γ is not finite, the modules $\ker(P)$ and $\ker(P^*)$, even if they are $C_r^*\Gamma$-modules, are in general not finitely generated projective.

To grasp the difficulty, consider Example 1.3.37. Using the description where F acts on a bundle over the base space M with infinite dimensional fiber $L \otimes E$, we see that loosely speaking, the null space of F should rather "contain" certain copies of $l^2\Gamma$ than copies of $C_r^*\Gamma$ (for finite groups, "accidentally" these two are the same!). However, in general $l^2\Gamma$ is not projective over $C_r^*\Gamma$ (although it is a module over this algebra). To be specific, assume that M is a point, $E_0 = \mathbb{C}$ and $E_1 = 0$, and $D = 0$. Here we obtain, $L^2(E_0) = l^2\Gamma$, $L^2(E_1) = 0$, $F = 0$, and indeed, $\ker(P) = l^2\Gamma$.

In the situation of our example, there is a way around this problem: Instead of twisting the operator D with the flat bundle $l^2(\Gamma) \times_\Gamma \overline{M}$, we twist with $C_r^*(\Gamma) \times_\Gamma \overline{M}$, to obtain an operator D' acting on a bundle with fiber $C_r^*\Gamma \otimes \mathbb{C}^{\dim E}$. This way, we replace $l^2\Gamma$ by $C_r^*\Gamma$ throughout. Still, it is not true in general that the kernels we get in this way are finitely generated projective modules over $C_r^*\Gamma$. However, it is a fact that one can always add to the new F' an appropriate compact operator such that this is the case. Then the obvious definition gives an element

$$\operatorname{ind}(D') \in K_0(C_r^*\Gamma).$$

This is the Mishchenko-Fomenko index of D' which does not depend on the chosen compact perturbation. Mishchenko and Fomenko give a formula for this index extending the Atiyah-Singer index formula.

One way to get around the difficulty in the general situation (not necessarily studying a lifted differential operator) is to deform (H, π, F) to an equivalent (H, π, F') which is better behaved (reminiscent to the compact perturbation above). This allows to proceeds with a rather elaborate generalization of the Mishchenko-Fomenko example we just considered, essentially replacing $l^2(\Gamma)$ by $C_r^*\Gamma$ again. In this way, one defines an index as an element of $K_*(C_r^*\Gamma)$.

This gives a homomorphism $\mu^\Gamma\colon KK_*^\Gamma(C_0(X)) \to K_*(C_r^*\Gamma)$ for each proper Γ-CW-complex X where $\Gamma\backslash X$ is compact. This passes to direct limits and defines, in particular,

$$\mu_*\colon RK_*^\Gamma(E(\Gamma, fin)) \to K_*(C_r^*\Gamma).$$

Next, we proceed with an alternative definition of the Baum-Connes map using KK-theory and the Kasparov product. The basic observation here is that, given any proper Γ-CW-space X, there is a specific projection $p \in C_r^*(\Gamma, C_0(X))$ (unique up to an appropriate type of homotopy) which gives rise to a canonical element $[L_X] \in K_0(C_r^*(\Gamma, C_0(X))) = KK_0(\mathbb{C}, C_r^*(\Gamma, C_0(X)))$. This defines by composition the homomorphism

$$\begin{aligned} KK_*^\Gamma(X) = KK_*^\Gamma(C_0(X), \mathbb{C}) &\xrightarrow{\text{descent}} KK_*(C_r^*(\Gamma, C_0(X)), C_r^*\Gamma) \\ &\xrightarrow{[L_X]\circ\cdot} KK_*(\mathbb{C}, C_r^*\Gamma) = K_*(C_r^*\Gamma). \end{aligned}$$

Again, this passes to direct limits and defines as a special case the Baum-Connes assembly map

$$\mu\colon RK_*^\Gamma(E(\Gamma, fin)) \to K_*(C_r^*\Gamma).$$

1.3.43. Remark. It is a non-trivial fact (due to Hambleton and Pedersen [28]) that this assembly map coincides with the map μ of (1.3.10).

Almost all positive results about the Baum-Connes have been obtained using the powerful methods of KK-theory, in particular the so called Dirac-dual Dirac method, compare e.g. [86].

1.3.9 The status of the conjecture

The Baum-Connes conjecture is known to be true for the following classes of groups.

(1) discrete subgroups of $SO(n,1)$ and $SU(n,1)$ [37]

(2) Groups with the *Haagerup property*, sometimes called *a-T-menable groups*, i.e. which admit an isometric action on some affine Hilbert H space which is proper, i.e. such that $g_n v \xrightarrow{n\to\infty} \infty$ for every $v \in H$ whenever $g_n \xrightarrow{n\to\infty} \infty$ in G [29]. Examples of groups with the Haagerup property are amenable groups, Coxeter groups, groups acting properly on trees, and groups acting properly on simply connected CAT(0) cubical complexes

(3) One-relator groups, i.e. groups with a presentation $G = \langle g_1, \dots, g_n \mid r \rangle$ with only one defining relation r [6].

(4) Cocompact lattices in $Sl_3(\mathbb{R})$, $Sl_3(\mathbb{C})$ and $Sl_3(\mathbb{Q}_p)$ ($\mathbb{Q}_p$ denotes the p-adic numbers) [43]

(5) Word hyperbolic groups in the sense of Gromov [57].

(6) Artin's full braid groups B_n [73].

Since we will encounter amenability later on, we recall the definition here.

1.3.44. Definition. A finitely generated discrete group Γ is called amenable, if for any given finite set of generators S (where we require $1 \in S$ and require that $s \in S$ implies $s^{-1} \in S$) there exists a sequence of finite subsets X_k of Γ such that

$$\frac{|SX_k := \{sx \mid s \in S, x \in X_k\}|}{|X_k|} \xrightarrow{k \to \infty} 1.$$

$|Y|$ denotes the number of elements of the set Y.

An arbitrary discrete group is called amenable, if each finitely generated subgroup is amenable.

Examples of amenable groups are all finite groups, all abelian, nilpotent and solvable groups. Moreover, the class of amenable groups is closed under taking subgroups, quotients, extensions, and directed unions.

The free group on two generators is not amenable. "Most" examples of non-amenable groups do contain a non-abelian free group.

There is a certain stronger variant of the Baum-Connes conjecture, the *Baum-Connes conjecture with coefficients.* It has the following stability properties:

(1) If a group Γ acts on a tree such that the stabilizer of every edge and every vertex satisfies the Baum-Connes conjecture with coefficients, the same is true for Γ [61].

(2) If a group Γ satisfies the Baum-Connes conjecture with coefficients, then so does every subgroup of Γ [61]

(3) If we have an extension $1 \to \Gamma_1 \to \Gamma_2 \to \Gamma_3 \to 1$, Γ_3 is torsion-free and Γ_1 as well as Γ_3 satisfy the Baum-Connes conjecture with coefficients, then so does Γ_2.

It should be remarked that in the above list, all groups except for word hyperbolic groups, and cocompact subgroups of Sl_3 actually satisfy the Baum-Connes conjecture with coefficients.

The Baum-Connes assembly map μ of (1.3.10) is known to be rationally injective for considerably larger classes of groups, in particular the following.

(1) Discrete subgroups of connected Lie groups [38]

(2) Discrete subgroups of p-adic groups [39]

(3) Bolic groups (a certain generalization of word hyperbolic groups) [40].

(4) Groups which admit an amenable action on some compact space [31].

Last, it should be mentioned that recent constructions of Gromov show that certain variants of the Baum-Connes conjecture, among them the Baum-Connes conjecture with coefficients, and an extension called the *Baum-Connes conjecture for groupoids*, are false [30]. At the moment, no counterexample to the Baum-Connes conjecture 1.3.9 seems to be known. However, there are many experts in the field who think that such a counterexample eventually will be constructed [30].

1.4 Real C^*-algebras and K-theory

1.4.1 Real C^*-algebras

The applications of the theory of C^*-algebras to geometry and topology we present here require at some point that we work with real C^*-algebras. Most of the theory is parallel to the theory of complex C^*-algebras.

1.4.1. Definition. A unital real C^*-algebra is a Banach-algebra A with unit over the real numbers, with an isometric involution $*\colon A \to A$, such that

$$|x|^2 = |x^*x| \qquad \text{and } 1 + x^*x \text{ is invertible} \quad \forall x \in A.$$

It turns out that this is equivalent to the existence of a $*$-isometric embedding of A as a closed subalgebra into $\mathcal{B}H_{\mathbb{R}}$, the bounded operators on a suitable real Hilbert space (compare [62]).

1.4.2. Example. If X is a compact topological space, then $C(X;\mathbb{R})$, the algebra of real valued continuous function on X, is a real C^*-algebra with unit (and with trivial involution).

More generally, if X comes with an involution $\tau\colon X \to X$ (i.e. $\tau^2 = \mathrm{id}_X$), then $C_\tau(X) := \{f\colon X \to \mathbb{C} \mid f(\tau x) = \overline{f(x)}\}$ is a real C^*-algebra with involution $f^*(x) = \overline{f(\tau x)}$.

Conversely, every commutative unital real C^*-algebra is isomorphic to some $C_\tau(X)$.

If X is only locally compact, we can produce examples of non-unital real C^*-algebras as in Example 1.2.2.

Essentially everything we have done for (complex) C^*-algebras carries over to real C^*-algebras, substituting $\mathbb{R}$ for $\mathbb{C}$ throughout. In particular, the definition of the K-theory of real C^*-algebras is literally the same as for complex C^*-algebras (actually, the definitions make sense for even more general topological algebras), and a short exact sequence of real C^*-algebras gives rise to a long exact K-theory sequence.

The notable exception is Bott periodicity. We don't get the period 2, but the period 8.

1.4.3. Theorem. *Assume that A is a real C^*-algebra. Then we have a Bott periodicity isomorphism*

$$K_0(A) \cong K_0(S^8 A).$$

This implies

$$K_n(A) \cong K_{n+8}(A) \qquad \text{for } n \geq 0.$$

1.4.4. Remark. Again, we can use Bott periodicity to define $K_n(A)$ for arbitrary $n \in \mathbb{Z}$, or we may view $K_n(A)$ as an 8-periodic theory, i.e. with $n \in \mathbb{Z}/8$.

The long exact sequence of Theorem 1.2.18 becomes a 24-term cyclic exact sequence.

The *real reduced C^*-algebra* of a group Γ, denoted $C^*_{\mathbb{R},r}\Gamma$, is the norm closure of $\mathbb{R}\Gamma$ in the bounded operators on $l^2\Gamma$.

1.4.2 Real K-homology and Baum-Connes

A variant of the cohomology theory given by complex vector bundles is KO-theory, which is given by real vector bundles. The homology theory dual to this is KO-homology. If KO is the spectrum of topological KO-theory, then $KO_n(X) = \pi_n(X_+ \wedge KO)$.

The homotopy theoretic definition of equivariant K-homology can be varied easily to define equivariant KO-homology. The analytic definition can also be adapted easily, replacing $\mathbb{C}$ by $\mathbb{R}$ throughout, using in particular real Hilbert spaces. However, we have to stick to n-multigraded cycles to define $KK_n^\Gamma(X)$, it is not sufficient to consider only even and odd cycles.

All the constructions and properties translate appropriately from the complex to the real situation, again with the notable exception that Bott periodicity does not give the period 2, but 8. The upshot of all of this is that we get a real version of the Baum-Connes conjecture, namely

1.4.5. Conjecture. *The real Baum-Connes assembly map*

$$\mu_n\colon KO_n^\Gamma(E(\Gamma, fin)) \to KO_n(C^*_{\mathbb{R},r}\Gamma),$$

is an isomorphism.

It should be remarked that all known results about injectivity or surjectivity of the Baum-Connes map can be proved for the real version as well as for the complex version, since each proof translates without too much difficulty. Moreover, it is known that the complex version of the Baum-Connes conjecture for a group Γ implies the real version (for this abstract result, the isomorphism is needed as input, since this is based on the use of the five-lemma at a certain point).

Chapter 2

A counterexample to the Gromov-Lawson-Rosenberg conjecture

The Gromov-Lawson-Rosenberg conjecture is discussed in the notes by Stephan Stolz. As a reminder, we quickly recall the problem:

2.0.1. Question. Given a compact smooth spin-manifold M without boundary, when does M admit a Riemannian metric with positive scalar curvature?

Recall that a spin-manifold is a manifold for which the first and second Stiefel-Whitney class of the tangent bundle vanish. The spin condition can be compared to the condition that a manifold is orientable. Indeed, every spin-manifold is orientable. But the spin condition is considerably stronger (it is like orientability "squared").

The reason that we concentrate on spin-manifolds is that powerful obstructions to the existence of a metric with positive scalar curvature have been developed for them.

2.1 Obstructions to positive scalar curvature

2.1.1 Index theoretic obstructions

We start with a discussion of the index obstruction for spin manifolds to admit a metric with scal > 0, constructed by Lichnerowicz [45], Hitchin [33] and in the following refined version due to Rosenberg [64].

2.1.1. Theorem. *One can construct a homomorphism, called index, from the singular spin bordism group $\Omega_*^{spin}(B\pi)$ to the (real) KO-theory of the reduced real C^*-algebra of π:*

$$\mathrm{ind}\colon \Omega_*^{spin}(B\pi) \to KO_*(C^*_{\mathbb{R},r}\pi)$$

(this homomorphism is often called α instead of ind*). Assume $f\colon N \to B\pi$ represents an element of $\Omega_m^{spin}(B\pi)$. If N admits a metric with positive scalar curvature, then*

$$\mathrm{ind}([f\colon N \to B\pi]) = 0 \in KO_m(C^*_{\mathbb{R},r}\pi)$$

The converse of this theorem is the content of the following *Gromov-Lawson-Rosenberg conjecture.*

2.1.2. Conjecture. *Let M be a compact spin-manifold without boundary, $\pi = \pi_1(M)$, and $u\colon M \to B\pi$ be the classifying map for a universal covering of M. Assume that $m = \dim(M) \geq 5$.*

Then M admits a metric with $\mathrm{scal} > 0$ *if and only if*

$$\mathrm{ind}[u\colon M \to B\pi] = 0 \in KO_m(C^*_{\mathbb{R},r}\pi).$$

This conjecture was developed in [27] and [63].

The restriction to dimensions ≥ 5 comes from the observation that in these dimensions (and not below) the question of existence of metrics with $\mathrm{scal} > 0$ in a certain sense is a bordism invariant, which of course fits with the structure of the obstruction described in Theorem 2.1.1. Failure of this bordism invariance in dimension 4 is also reflected by the fact that for 4-dimensional manifolds, the *Seiberg-Witten invariants* provide additional obstructions to the existence of a metric with positive scalar curvature, which show in particular that the conjecture is not true if $m = 4$.

The conjecture was proved by Stefan Stolz [80] for $\pi = 1$, and subsequently by him and other authors also for some other groups [63, 42, 10, 65, 34].

2.1.2 Minimal surface obstructions

In dimension ≥ 5 there is only one known additional obstruction for positive scalar curvature metrics, the minimal surface method of Schoen and Yau, which we will recall now. (In dimension 4, the Seiberg-Witten theory yields additional obstructions).

The first theorem is the differential geometrical backbone for the application of minimal surfaces to the positive scalar curvature problem:

2.1.3. Theorem. *Let (M^m, g) be a manifold with* scal > 0, $\dim M = m \geq 3$. *If V is a smooth $(m-1)$-dimensional submanifold of M with trivial normal bundle, and if V is a local minimum of the volume functional, then V admits a metric of positive scalar curvature, too. "Local minimum" means that for any small deformation of the hypersurface, the $(m-1)$-volume of the surface increases.*

Actually, V can be a "minimal hypersurfaces" in the sense of differential geometry, defined in terms of curvature and second fundamental form of the hypersurface. Every local minimum for the $(m-1)$-volume is such a minimal hypersurface; the converse is not true.

Proof. Schoen/Yau: [75, 5.1] for $m = 3$, [76, proof of Theorem 1] for $m > 3$. We outline the proof, following closely [76, Theorem 1].

Given V, since its normal bundle is trivial, any smooth function ϕ on V gives rise to a variation (if ν is the unit normal vector field, pushing V in normal direction $\phi\nu$). Lt $\mathrm{Ric} \in \Gamma(\mathrm{End}(TM))$ be the Ricci curvature, considered as an operator on each fiber of TM. Let l be the second fundamental form of V. It is well known that minimality implies $\mathrm{tr}(l) = 0$. Moreover, the second variation of the area is non-negative. It is given by (see [13])

$$-\int_V \left(\langle \mathrm{Ric}(\nu), \nu\rangle + |l|^2\right)\phi^2 + \int_V |\nabla\phi|^2 \geq 0. \tag{2.1.4}$$

We now use the Gauss curvature equation (the "theorema egregium") to relate this to the scalar curvature of the submanifold. Taking appropriate traces of the Gauss equations, we obtain

$$\mathrm{scal}_V = \mathrm{scal}_M - 2\langle \mathrm{Ric}(\nu), \nu\rangle + (\mathrm{tr}\, l)^2 - |l|^2\,, \tag{2.1.5}$$

where scal_V is the scalar curvature of V with the induced Riemannian metric and scal_M the scalar curvature of M. Putting Equation (2.1.5) into Inequality (2.1.4), we have

$$\int_V \mathrm{scal}_M\,\phi^2 - \int_V \mathrm{scal}_V\,\phi^2 + \int_V |l|^2\,\phi^2 \leq 2\int_V |\nabla\phi|^2 \tag{2.1.6}$$

for all smooth functions $\phi\colon V \to \mathbb{R}$. We assume that the scalar curvature of M is everywhere strictly positive. Hence (2.1.6) implies

$$-\int_V \mathrm{scal}_V\,\phi^2 < 2\int_V |\nabla\phi|^2\,,$$

as long as ϕ is not identically zero.

Consider the *conformal Laplacian* $\Delta_c := \Delta + \frac{n-3}{4(n-2)} \operatorname{scal}_V$ on V (where Δ is the positive Laplacian on functions). Then all eigenvalues of Δ_c are strictly positive. Assume, otherwise, that ϕ is an eigenfunction to the eigenvalue $\lambda \leq 0$, i.e.

$$\Delta\phi = -\frac{m-3}{4(m-2)} \operatorname{scal}_V \phi + \lambda\phi.$$

Taking the L^2-inner product of this equation with ϕ (and integration by parts) gives

$$\int_V |\nabla\phi|^2 = -\frac{m-3}{4(m-2)} \int_V \operatorname{scal}_V \phi^2 + \lambda \int_V \phi^2 < \frac{m-3}{2(m-2)} \int_V |\nabla\phi|^2,$$

which is a contradiction. Now it's a standard fact in conformal geometry that, if the conformal Laplacian has only positive eigenvalues, then one can conformally deform the metric to a metric with positive scalar curvature (compare [41]). This is done in two steps: a generalized maximum principle implies that we can find an eigenfunction f to the first eigenvalues of Δ_c which is strictly positive everywhere. Then, explicit formulas for the scalar curvature of a conformally changed metric show that $f^{4/(m-3)}g$ indeed has a metric with $\operatorname{scal} > 0$.

Hence, on V there exists a metric with $\operatorname{scal} > 0$ (observe, however, that it is not necessarily the metric induced from M, but only conformally equivalent to this metric). □

The next statement due to Simons and Smale (special cases due to Fleming and Almgren) from geometric measure theory implies applicability of the previous theorem if $\dim(M) \leq 8$.

2.1.7. Theorem. *Suppose M is an orientable Riemannian manifold of dimension $\dim M = m \leq 8$. Furthermore let $\alpha \in H^1(M, \mathbb{Z})$. Then*

$$x := \alpha \cap [M] \in H_{m-1}(M, \mathbb{Z})$$

can be represented by an embedded hypersurface V with trivial normal bundle which is a local minimum for $(m-1)$-volume (if $m = 8$ with respect to suitable metrics arbitrarily close in C^3 to the metric we started with).

Proof. For $m \leq 7$ this is a classical result of geometric measure theory (cf. [59, Chapter 8]) and references therein, in particular [23, 5.4.18].

The case $m = 8$ follows from the following result of Nathan Smale [79]: the set of C^k-metrics for which the regularity statement holds is open and dense in the set of all C^k-metrics ($k \geq 3$ and with the usual Banach-space topology). We are only interested in C^∞-metrics. But these are dense in the set of C^k-metrics, which concludes the proof. $\square$

Unfortunately, the proofs of the theorems we have cited are very involved and require a lot of technical work. Therefore, we don't attempt to indicate the arguments here.

Recall that if we are given a class $\alpha \in H^1(M, \mathbb{Z})$ we may represent it by a map $f\colon M \to S^1$ being transverse to $1 \in S^1$. Then $V = f^{-1}(1) \subset M$ represents $\alpha \cap [M]$ (and conversely, every hypersurface representing $\alpha \cap [M]$ is obtained in this way). Furthermore, if $f'\colon M \to S^1$ is a second map as above and $V' = f'^{-1}(1)$ then f and f' are homotopic, and a homotopy $H\colon f \simeq f'$ being transverse to $1 \in S^1$ provides a bordism $W = H^{-1}(1)\colon V \sim V'$ embedded in $M \times [0,1]$. Since the normal bundle of $V \subset M$ and $W \subset M \times [0,1]$,respectively, is trivial, the manifolds and bordisms we construct this way belong to the spin-category, if we start with a spin-manifold M.

We want to use these ideas to construct, for an arbitrary space X, a map

$$\cap\colon H^1(X, \mathbb{Z}) \times \Omega_m^{spin}(X) \to \Omega_{m-1}^{spin}(X). \tag{2.1.8}$$

To do this, let $\phi\colon M \to X$ be a singular spin manifold for X, representing an element in $\Omega_m^{spin}(X)$. If $f\colon X \to S^1$ represents an element $\alpha \in H^1(X, \mathbb{Z})$, then $f \circ \phi$ is homotopic to a map $\psi\colon M \to S^1$ which is transverse to $1 \in S^1$.

Restricting ϕ to $V := \psi^{-1}(1)$ then gives a singular spin manifold $\phi|_V\colon V \to X$, which by definition represents $\alpha \cap [\phi\colon M \to X] \in \Omega_{m-1}^{spin}(X)$.

We have to check that this is well defined. To do this, let $\Phi\colon W \to X$ be a spin-bordism between $\phi\colon M \to X$ and $\phi'\colon M' \to X$. Then $f \circ \Phi$ is homotopic to a map $\Psi\colon W \to S^1$ with Ψ and $\Psi|\partial W$ being transverse to $1 \in S^1$; moreover, the map $\Psi|\partial W$ with the corresponding properties may be given in advance. Then $\Psi^{-1}(1) \subset W$ is a spin-bordism between the hypersurfaces $V = (\Psi)^{-1}(1) \cap M \subset M$ and $\Psi^{-1}(1) \cap M' \subset M'$, and restricting Φ to $\Psi^{-1}(1)$ now yields a singular spin-bordism between singular spin-hypersurfaces into X.
If $f'\colon X \to S^1$ is homotopic to f, a similar construction gives a singular spin-bordism between the resulting singular spin-hypersurfaces into X. Together, this implies that our map indeed is well defined.

The two theorems above now imply

2.1.9. Theorem. *Let X be a space and let $3 \leq m \leq 8$. Then* (2.1.8) *restricts to a homomorphism:*

$$\cap\colon H^1(X, \mathbb{Z}) \times \Omega_m^{spin,+}(X) \to \Omega_{m-1}^{spin,+}(X) \tag{2.1.10}$$

where $\Omega_*^{spin,+}(X) \subset \Omega_*^{spin}(X)$ *is the subgroup of bordism classes which can be represented by singular manifolds which admit a metric with* $\mathrm{scal} > 0$ *(observe, only one representative with* $\mathrm{scal} > 0$ *is required).*

Proof. Theorem 2.1.7 implies that, given $f\colon M \to S^1$ (dual to a given class in $H_{m-1}(M,\mathbb{Z})$) we find a homotopic map $g\colon M \to S^1$ which is transverse to 1 and such that the hypersurface $V = g^{-1}(1)$ is minimal for the $(m-1)$-volume (in dimension 8 we replace the given metric by one which is C^3-close). In any case, since the scalar curvature is continuous with respect to the C^3-topology on the space of all Riemannian metrics, V is volume minimizing with respect to a metric with positive scalar curvature whenever we start with such a metric. By Theorem 2.1.3, it admits a metric with $\mathrm{scal} > 0$. $\square$

To be honest, this does not quite give an obstruction, but rather a method to produce counterexamples. Namely, if we know $\Omega_{n-1}^{spin,+}(B\pi)$, and also the cap-product of (2.1.8) well enough, we can get information about $\Omega_n^{spin,+}(B\pi)$ (with $n \le 8$).

Obviously, one does need some information to start with. This can be obtained in dimension 2 using the Gauss-Bonnet theorem.

2.1.3 Gauss-Bonnet obstruction in dimension 2

2.1.11. Theorem. *Let G be a discrete group. Then*

$$\Omega_2^{spin,+}(BG) := \left\{ \begin{array}{c} \textit{bordism classes } [M \to BG] \in \Omega_2^{spin}(BG), \\ \textit{where } M \textit{ admits a metric with } \mathrm{scal} > 0 \end{array} \right\} = 0. \tag{2.1.12}$$

Proof. By the Gauss-Bonnet theorem there is only one orientable 2-manifold with positive (scalar) curvature, namely S^2. On the other hand, S^2 is a spin-manifold with a unique spin-structure, and is spin-bordant to zero, being the boundary of D^3. Since $\pi_2(BG)$ is trivial, up to homotopy only the trivial map from S^2 to BG exists. Therefore only the trivial element in $\Omega_2^{spin}(BG)$ can be represented by a manifold with positive scalar curvature. $\square$

2.2 Construction of the counterexample

2.2.1 Application of the minimal hypersurface obstruction

Now, we will construct a particular example of a manifold which does not admit a metric with positive scalar curvature, using the minimal hypersurface obstruction.

Let $p\colon S^1 \to B\mathbb{Z}/3$ be a map so that $\pi_1(p)$ is surjective and equip S^1 with the spin structure induced from D^2. Consider the singular manifold

$$f = \mathrm{id} \times p\colon T^5 = S^1 \overbrace{\times \cdots \times}^{4} S^1 \times S^1 \to S^1 \overbrace{\times \cdots \times}^{4} S^1 \times B\mathbb{Z}/3 = B\pi,$$

where $\pi = \mathbb{Z}^4 \times \mathbb{Z}/3$. This represents a certain element $x \in \Omega_5^{spin}(B\pi)$.

We have four distinguished maps from $B\pi$ to S^1, given by the projections $p_i\colon B\pi \to S^1$ onto each of the first four factors. Let $a_1, \dots, a_4 \in H^1(B\pi)$ be the corresponding elements in cohomology. Using the description of the cap-product given before (2.1.8), in our situation it is easy to find a representative for

$$z := a_1 \cap (a_2 \cap (a_3 \cap w)) \in \Omega_2^{spin}(B\pi).$$

Namely, taking inverse images of the base point, this z is given by

$$g = * \times * \times * \times \mathrm{id} \times p\colon T^2 = * \times * \times * \times S^1 \times S^1 \to S^1 \times S^1 \times S^1 \times S^1 \times B\mathbb{Z}/3 = B\pi.$$

We want to show that $z \notin \Omega_2^{spin,+}(B\pi)$, because then, by (2.1.10), $x \notin \Omega_5^{spin,+}(B\pi)$, i.e. whenever we find a representative $[f\colon M \to B\pi] = x$, then M does not admit a Riemannian metric with scal > 0 (in particular, this follows then for T^5, which, however, is not the manifold we are interested in here).

Because of Theorem 2.1.11 we only have to show that z is a non-trivial element of $\Omega_2^{spin}(B\pi)$. We have the natural homomorphism $\Omega_*^{spin}(B\pi) \to H_*(B\pi, \mathbb{Z})$, which maps $[f\colon M \to B\pi]$ to $f_*[M]$, i.e. to the image of the fundamental class of M, and the Künneth theorem implies immediately that the image of z under this homomorphism in $H_2(B\pi)$ is non-trivial, therefore the same is true for z.

2.2.2 Calculation of the index obstruction

We proceed by proving that the index obstruction 2.1.1 does vanish for the example constructed in Subsection 2.2.1.

This index obstruction is an element of $KO_5(C^*_{\mathbb{R},r}\pi)$, where $\pi = \mathbb{Z}^4 \times \mathbb{Z}/3$. First, we compute this K-theory group to the extent needed here. By a Künneth theorem for the K-theory of C^*-algebras, the K-theory of $C^*_{\mathbb{R},r}(G \times \mathbb{Z})$ can easily be computed from the K-theory of $C^*_{\mathbb{R},r}G$. Namely, by [77, p. 14–15 and 1.5.4]

$$KO_n(C^*_{\mathbb{R},r}(\mathbb{Z}^4 \times \mathbb{Z}/3)) \cong \bigoplus_{i=1}^{16} KO_{n-n_i}(C^*_{\mathbb{R},r}(\mathbb{Z}/3)); \qquad \text{for suitable } n_i \in \mathbb{N}.$$

For a finite group G, it is well known that $KO_*(C^*_{\mathbb{R},r}(G))$ is a direct sum of copies of the (known) KO-theories of $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{H}$. In particular, it is a direct sum of copies of $\mathbb{Z}$ and $\mathbb{Z}/2$. Therefore, the same is true for π. This implies the following Proposition.

2.2.1. Proposition. *$KO_*(C^*_{\mathbb{R},r}\pi)$ is a direct sum of copies of $\mathbb{Z}$ and $\mathbb{Z}/2$. In particular, its torsion is only 2-torsion.*

Let $p\colon S^1 \to B\mathbb{Z}/3$ be the map of Subsection 2.2.1 so that $\pi_1(p)$ is surjective (and S^1 is equipped with the spin structure induced from D^2). This represents a 3-torsion element y in $\Omega_1^{spin}(B\mathbb{Z}/3)$ since

$$\tilde{\Omega}_1^{spin}(B\mathbb{Z}/3) \cong H_1(B\mathbb{Z}/3, \mathbb{Z}) \cong \mathbb{Z}/3$$

(using e.g. the Atiyah-Hirzebruch spectral sequence).

It follows that

$$x = [\mathrm{id}_{(S^1)^4} \times p\colon T^5 \to B(\mathbb{Z}^4 \times \mathbb{Z}/3)]$$

is also 3-torsion (a zero bordism for $3x$ is obtained as the product of a zero bordism for $3y$ with $\mathrm{id}_{(S^1)^4}$).

Since $\mathrm{ind}\colon \Omega_5^{spin}(B\pi) \to KO_5(C^*_{\mathbb{R},r}\pi)$ is a group homomorphism,

$$3 \cdot \mathrm{ind}(x) = 0 \in KO_5(C^*_{\mathbb{R},r}\pi).$$

But for $\pi = \mathbb{Z}^4 \times \mathbb{Z}/3$, by Proposition 2.2.1 this implies that $\mathrm{ind}(x) = 0$, i.e. the index obstruction vanishes.

2.2.3 Surgery to produce the counterexample

So far, we have found a bordism class $x \in \Omega_5^{spin}(B\pi)$ ($\pi = \mathbb{Z}^4 \times \mathbb{Z}/3$) such that the index obstruction 2.1.1 vanishes for x, but on the other hand no representative $[f\colon M \to B\pi]$ can be found such that M has a metric with positive scalar curvature. To give a counterexample to Conjecture 2.1.2, we have to find a representative such that f induces an isomorphism on fundamental groups (i.e. is the classifying map for the universal covering of M). This is not the case for the tori we have explicitly constructed so far (and indeed, for tori one can use the *index method* to show that they do not admit a metric with positive scalar curvature).

But adjusting the fundamental group is easy. We only have to perform surgery on our explicitly given torus T^5. That is, we have to choose an embedded $S^1 \to T^5$ which represents the kernel of $\pi_1(f)\colon \pi_1(T^5) \to \pi_1(B\pi)$

(observe that in this situation, the kernel actually is cyclic) and which has a trivial normal bundle. Then a tubular neighborhood of S^1 is diffeomorphic to $S^1 \times D^4$, with boundary $S^1 \times S^3$. We can now cut away this tubular neighborhood and glue in $D^2 \times S^3$ (also with boundary $S^1 \times S^3$) instead. The fundamental group of the new manifold M' is the quotient of the fundamental group of the old manifold by the (normal) subgroup generated by the loop we started with, i.e. is isomorphic to π. Let $u\colon M' \to B\pi$ be the classifying map for the universal covering. Using classical "surgery below the middle dimension", we can arrange all this in such a way that

$$[f : T^5 \to B\pi] = [u\colon M' \to B\pi] \in \Omega_5^{spin}(B\pi)$$

(compare [82, Lemma 5.6]). Consequently, M' is a counterexample to the Gromov-Lawson-Rosenberg conjecture 2.1.2.

2.3 Other questions, other examples

The index map of Theorem 2.1.1 admits a factorization

$$\mathrm{ind}\colon \Omega_*^{spin}(B\pi) \xrightarrow{D} ko_*(B\pi) \xrightarrow{p} KO_*(B\pi) \xrightarrow{\mu} KO_*(C^*_{r,\mathbb{R}}\pi).$$

Here, ko_* is connective real K-homology, KO_* the periodic real K-homology we have considered so far, D is the ko-theoretic orientation, p the canonical map between the connective and the periodic theory, and μ the assembly map in topological K-theory. Note that for torsion free groups, this μ is the Baum-Connes map, and the Baum-Connes conjecture states that this map is an isomorphism.

The original conjecture of Gromov and Lawson asserted that the vanishing of the image of $[u\colon M \to B\pi_1(M)]$ in $KO_m(B\pi)$ decides whether M admits a metric with positive scalar curvature. Rosenberg observed that there are manifolds with scal > 0 for which this element does not vanish, and proposed to modify the conjecture as stated in Conjecture 2.1.2. We adopt the convention that $u\colon M \to B\pi_1(M)$ denotes the classifying map for the universal covering of M.

However, the following question remains.

2.3.1. Question. Is the stronger vanishing condition that

$$pD[u\colon M \to B\pi_1(M)] = 0$$

sufficient for the existence of metrics with positive scalar curvature?

If even $D[u\colon M \to B\pi_1(M)] = 0$, then M admits a metric with positive scalar curvature by a result of Stephan Stolz [81] (as usual, we have to assume that $\dim(M) \geq 5$).

In [34], a counterexample to question 2.3.1 is given. The first step to construct the counterexample is to find a group such that

$$p\colon ko_*(B\pi) \to KO_*(B\pi)$$

has a kernel, and since we want to use the minimal surface method, this kernel should be given for $* = 2$. In [34], this is done using explicit K-homology calculations for finite groups. The remaining proof is very much along the lines of the proof we have given above.

One of the virtues of the example we have given is that we avoid the calculation of the index. This is replaced by some (easy) considerations about torsion. To be able to do this, we used a fundamental group π with torsion. Dwyer and Stolz (unpublished) have constructed a counterexample to the Gromov-Lawson-Rosenberg conjecture with torsion-free fundamental group. In [74] a refinement of this is given where the classifying space $B\pi$ is a manifold with negative curvature. The first key idea is the same as in the example in [34] just described, namely to find an element in the kernel of $ko_2(B\pi) \to KO_2(B\pi)$. To find a π such that $B\pi$ is particularly nice (e.g. finite dimensional, which implies that π is torsion-free, or even a manifold of negative curvature) one uses asphericalization procedures of Baumslag, Dyer and Heller, or Charney, Davis, and Januszkiewicz, which produce nice $B\pi$ with certain prescribed homological properties (starting with (worse) spaces which have these same homological properties). More constructions of this kind are described in the lectures of Mike Davis.

The positive scalar curvature question makes sense also for manifolds which are not spin manifolds. There are "twisted" index obstructions as long as the universal covering is a spin manifold, and one can formulate an appropriate "twisted Gromov-Lawson-Rosenberg" conjecture. In [34], counterexamples to this twisted conjecture are given, as well.

Chapter 3

L^2-cohomology and the conjectures of Atiyah, Singer, and Hopf

L^2-cohomology and L^2-Betti numbers are certain "higher invariants" of manifolds and more general spaces. They were introduced 1976 by Michael Atiyah in [1], and since then have proved to be useful invariants with connections and applications in many other mathematical fields, from differential geometry to group theory and algebra. Apart from the original literature, there exists Lücks informative survey article [50], and also Eckmann lecture notes [20]. Moreover, at the time of writing of this article, Lück's very comprehensive textbook/research monograph [51] is almost finished, and the current version is available from the author's homepage. This chapter is a survey style article which focuses on the main points of the very extensive subject, leaving out many of the less illuminating details, which can be found e.g. in [51].

3.1 Analytic L^2-Betti numbers

3.1.1. Definition. Let $\overline{M}$ be a (not necessarily compact) Riemannian manifold without boundary, which is complete as a metric space. Define

$$L^2\Omega^p(\overline{M}) := \{\omega \text{ measurable } p\text{-form on } M \mid \int_{\overline{M}} |\omega(x)|_x^2 \, d\mu(x) < \infty\}.$$

Here, $|\omega(x)|_x$ is the pointwise norm (at $x \in \overline{M}$) of $\omega(x)$, which is given by the Riemannian metric, and $d\mu(x)$ is the measure induced by the Riemannian metric.

$L^2\Omega^p(\overline{M})$ can be considered as the Hilbert space completion of the space of compactly supported p-forms on $\overline{M}$. The inner product is given by integrating the pointwise inner product, i.e.

$$\langle \omega, \eta \rangle_{L^2} := \int_{\overline{M}} \langle \omega(x), \eta(x) \rangle_x \; d\mu(x).$$

3.1.2. Definition. Let M be a smooth compact Riemannian manifold without boundary, with Riemannian metric g. Let $\overline{M}$ be a normal covering of M, i.e. if Γ is the deck transformation group, then $M = \overline{M}/\Gamma$. Lift the metric g to $\overline{M}$. Then Γ acts isometrically on $\overline{M}$. Let $\overline{\Delta}_p$ be the Laplacian on p-forms on $\overline{M}$. This gives rise to an unbounded operator

$$\overline{\Delta}_p \colon L^2\Omega^p(\overline{M}) \to L^2\Omega^p(\overline{M}).$$

This operator is an elliptic differential operator (but on the not necessarily compact manifold $\overline{M}$) Let

$$\mathrm{pr}_p \colon L^2\Omega^p(\overline{M}) \to L^2\Omega^p(\overline{M})$$

be the orthogonal projection onto $\ker(\overline{\Delta}_p)$. Ellipticity of $\overline{\Delta}_p$ implies that pr_p has a smooth integral kernel, i.e. that there is a smooth section $\mathrm{pr}_p(x,y)$ over $\overline{M} \times \overline{M}$ (of the bundle with fiber $\mathrm{Hom}(\Lambda^p T_y^* M, \Lambda^p T_x^* M)$ over $(x,y) \in \overline{M} \times \overline{M}$), such that

$$\mathrm{pr}_p\, \omega(x) = \int_{\overline{M}} \mathrm{pr}_p(x,y) \omega(y) \; d\mu(y)$$

for every L^2-p-form on $\overline{M}$.

We define the L^2-cohomology of $\overline{M}$ by

$$H^p_{(2)}(M) := \ker(\overline{\Delta}^p) = \mathrm{im}(\mathrm{pr}_p).$$

It is an easy observation that, given a projection $P \colon V \to V$ on a finite dimensional vector space V, then $\dim(\mathrm{im}(P)) = \mathrm{tr}(P)$. On the other hand, the trace of an operator with a smooth integral kernel can be computed by integration over the diagonal.

We want to use these ideas to define a useful dimension for $\ker(\Delta_p)$. Note, however, that the Laplacian $\overline{\Delta}_p$ is defined using the Riemannian metric on

$\overline{M}$. It follows that it commutes with the induced action of Γ on $L^2\Omega^p(\overline{M})$. Consequently, $\ker(\overline{\Delta}_p)$ is Γ-invariant, and

$$\operatorname{tr}_{\overline{x}} \operatorname{pr}_p(\overline{x},\overline{x}) = \operatorname{tr}_{g\overline{x}} \operatorname{pr}_p(g\overline{x},g\overline{x}) \qquad \forall \overline{x} \in \overline{M}, g \in \Gamma.$$

Observe that $\operatorname{pr}_p(\overline{x},\overline{x}) \in \operatorname{End}(\Lambda^p T_x^* M)$ is an endomorphism of a finite dimensional vector space for each $\overline{x} \in \overline{M}$, and $\operatorname{tr}_{\overline{x}}$ is the usual trace of such endomorphisms. If $\overline{M}$ is not compact (i.e. if Γ is infinite), it follows that

$$\int_{\overline{M}} \operatorname{tr}_{\overline{x}} \operatorname{pr}_p(\overline{x},\overline{x})\; d\mu(x)$$

does not converge, and indeed, in general $\ker(\overline{\Delta}_p)$ is not a finite dimensional $\mathbb{C}$-vector space.

On the other hand, because of the Γ-invariance, the function

$$\overline{x} \mapsto \operatorname{tr}_{\overline{x}} \operatorname{pr}_p(\overline{x},\overline{x})$$

"contains the same information many times", and it doesn't make sense to try to compute the integral over all of $\overline{M}$. We therefore adopt the notion of "dimension per (unit) volume".

More concretely, because of Γ-invariance, the function $\overline{x} \mapsto \operatorname{tr}_{\overline{x}} \operatorname{pr}_p(\overline{x},\overline{x})$ descents to a smooth function on the quotient $\overline{M}/\Gamma = M$. We now define the L^2-Betti numbers

$$b^p_{(2)}(\overline{M},\Gamma) := \dim_\Gamma \ker(\overline{\Delta}_p) := \int_M \operatorname{tr}_x \operatorname{pr}_p(x,x)\; d\mu(x) \in [0,\infty).$$

This number is a non-negative real number. However, a priori no further restrictions appear for these values.

Extensions of all these definitions to manifolds with boundary are possible, compare e.g. [69].

The definition given here is the original definition of L^2-Betti numbers as given by Atiyah. Of course, the same construction can be applied to any elliptic differential operator D on M. If D is such an elliptic differential operator, and D^* its formal adjoint, Atiyah defined in this way the Γ-index of the lift $\overline{D}$ of D to $\overline{M}$ by

$$\operatorname{ind}_\Gamma(\overline{D}) := \dim_\Gamma(\ker \overline{D}) - \dim_\Gamma(\ker \overline{D^*}).$$

Atiyah's celebrated L^2-index theorem now states

3.1.3. Theorem.

$$\operatorname{ind}_\Gamma(\overline{D}) = \operatorname{ind}(D).$$

Here, recall that ellipticity of D and compactness of M imply that $\ker(D)$ and $\ker(D^*)$ are finite dimensional $\mathbb{C}$-vector spaces, and

$$\operatorname{ind}(D) = \dim_{\mathbb{C}}(\ker D) - \dim_{\mathbb{C}}(\ker D^*).$$

It should be observed that it is far from true in general that $\dim_\Gamma(\ker \overline{D}) = \dim_{\mathbb{C}}(\ker D)$. In particular, the L^2-Betti numbers and the ordinary Betti numbers usually are quite different from each other. However, Atiyah's L^2-index theorem has the following consequence for the L^2-Betti numbers:

$$\chi(M) = \sum_{p=0}^{\dim M} (-1)^p b^p_{(2)}(\overline{M}, \Gamma). \tag{3.1.4}$$

An extension of Atiyah's L^2-index theorem to manifolds with boundary can be found in [72], which provides one way to prove a corresponding result for the Euler characteristic of manifolds with boundary.

3.1.5. Example. Assume Γ is finite. Then $\overline{M}$ itself is a compact manifold and, by the above considerations, we get for the ordinary Betti numbers of $\overline{M}$:

$$b^p(\overline{M}) = \int_{\overline{M}} \operatorname{tr}_{\overline{x}} \operatorname{pr}_p(\overline{x}, \overline{x})\; d\mu(\overline{x}).$$

Because of Γ-invariance,

$$\int_{\overline{M}} \operatorname{tr}_{\overline{x}} \operatorname{pr}_p(\overline{x}, \overline{x})\; d\mu(\overline{x}) = |\Gamma| \cdot \int_M \operatorname{tr}_x \operatorname{pr}_p(x, x)\; d\mu(x),$$

in other words,

$$b^p_{(2)}(\overline{M}, \Gamma) = \frac{b^p(\overline{M})}{|\Gamma|}$$

3.1.6. Example. Let $p = 0$, and assume that $\overline{M}$ is connected. Integration by parts shows that $f \in L^2\Omega^0(\overline{M})$ belongs to $\ker(\overline{\Delta}_0)$ if and only if f is constant (recall that $L^2\Omega^0(\overline{M}) = L^2(\overline{M})$ is the space of L^2-functions on $\overline{M}$). If $\operatorname{vol}(\overline{M}) = \infty$, or equivalently $|\Gamma| = \infty$ then an L^2-function f is constant if and only if it is zero, i.e. $\ker(\overline{\Delta}_0) = 0$. Therefore, $\operatorname{pr}_p(x, y) = 0$ for all $x, y \in \overline{M}$ and $b^0_{(2)}(\overline{M}, \Gamma) = 0$.

Note that the 0-th ordinary Betti number never vanishes. Many of the applications of L^2-Betti numbers rely on such vanishing results, which don't hold for ordinary Betti numbers.

3.1.7. Theorem. *Assume $\overline{M}$ is orientable. Then, the Hodge-$*$ operator is defined and intertwines p-forms and $(\dim M - p)$-forms on $\overline{M}$. Since this is an isometry which commutes with the Laplace operators, it induces an isometry between $H^p_{(2)}(\overline{M})$ and $H^{\dim M - p}_{(2)}(\overline{M})$. Moreover, this isometry is compatible with the action of Γ and, in particular extends to the integral kernel of pr_p. As a consequence, we have Poincaré duality for L^2-Betti numbers:*

$$b^p_{(2)}(\overline{M}, \Gamma) = b^{\dim M - p}_{(2)}(\overline{M}, \Gamma).$$

3.1.1 The conjectures of Hopf and Singer

3.1.8. Example. In general, it will be almost impossible to compute the L^2-Betti numbers using the Riemannian metric and the integral kernel of pr_p. For very nice metrics, however, this is can be done, in particular if $(\overline{M}, \overline{g})$ is a symmetric space. One obtains e.g.

(1) If $M = T^n$ is a flat torus, $\overline{M} = \mathbb{R}^n$ is flat Euclidean space, then

$$b^p_{(2)}(\mathbb{R}^n, \mathbb{Z}^n) = 0 \qquad \forall p \in \mathbb{Z}.$$

(2) If (M, g) has constant sectional curvature $K = -1$, $\Gamma = \pi_1(M)$ and $\overline{M} = \mathbb{H}^m$ is the hyperbolic m-plane, then

$$b^p_{(2)}(\overline{M}, \Gamma) = 0 \qquad \text{if } p \neq m/2,$$

and if m is even and $p = m/2$, then

$$b^{m/2}_{(2)}(\overline{M}, \Gamma) > 0.$$

In particular, we conclude, using (3.1.4), that in this situation

$$(-1)^{m/2}\chi(M) > 0.$$

(3) If, more generally, (M, g) is a connected, locally symmetric space with strictly negative sectional curvature, $\overline{M}$ is its universal covering (a symmetric space) and $\Gamma = \pi_1(M)$, then

$$b^p_{(2)}(\overline{M}, \Gamma) = 0 \qquad \text{if } p \neq \dim M/2,$$

and if $\dim M/2$ is an integer, then

$$b^{\dim M/2}_{(2)}(\overline{M}, \Gamma) > 0.$$

In particular, if $\dim M$ is even we have again

$$(-1)^{\dim M/2}\chi(M) > 0.$$

(4) If (M,g) is a connected locally symmetric space with non-positive, but not strictly negative, sectional curvature, then

$$b^p_{(2)}(\overline{M},\Gamma)=0 \qquad \forall p\in\mathbb{Z},$$

with $\overline{M}$ and Γ as above. In particular $\chi(M)=0$.

Proof. These calculations are carried out in [9], another account can be found in [60], using the "representation theory of symmetric spaces". A more geometric proof of the hyperbolic case (i.e. constant curvature -1) is given in [17]. □

Given any compact Riemannian manifold without boundary, we can compute the Euler characteristic using the Pfaffian and the Gauss-Bonnet formula in higher dimensions. This gives rise to another argument for the inequality

$$(-1)^{\dim M/2}\chi(M)>0,$$

if M is an even dimensional manifold with constant negative sectional curvature, and this has been known for a long time. It has lead to the following conjecture, which is attributed to Hopf.

3.1.9. Conjecture. *Assume (M,g) is a compact Riemannian $2n$-dimensional manifold without boundary, and with strictly negative sectional curvature. Then*

$$(-1)^n\chi(M)>0.$$

If the sectional curvature is non-positive, then

$$(-1)^n\chi(M)\geq 0.$$

3.1.10. Remark. The flat torus, or the product of any negatively curved manifold with a flat torus, shows that $\chi(M)=0$ is possible if M is a manifold with non-positive sectional curvature.

In view of Atiyah's formula (3.1.4), and because of the calculations of Jozef Dodziuk and the others in Example 3.1.8, Singer proposed to use the L^2-Betti numbers to prove the Hopf conjecture 3.1.9. More precisely, he made the following stronger conjecture.

3.1.11. Conjecture. *If (M,g) is a compact Riemannian manifold without boundary and with non-positive sectional curvature, then*

$$b^p_{(2)}(\tilde{M},\pi_1(M))=0, \qquad \textit{if } p\neq\dim M/2.$$

If $\dim M = 2n$ *is even and the sectional curvature is strictly negative, then*

$$b_{(2)}^{\dim M/2}(\tilde{M}, \pi_1(M)) > 0.$$

Because of (3.1.4), the Singer conjecture 3.1.11 implies the Hopf conjecture 3.1.9.

Using estimates for the Laplacians and their spectrum, Ballmann and Brüning [3] prove (improving earlier results of Donnelly and Xavier [19] and Jost and Xin [35]) part of the Singer conjecture for very negative sectional curvatures:

3.1.12. Theorem. *If* (M, g) *is a closed Riemannian manifold of even dimension* $2n$*, such that its sectional curvature* K *satisfies* $-1 \leq K \leq -(a_n)^2$*, with* $1 \geq a_n > 1 - 1/n$*, then*

$$b_{(2)}^{p}(\tilde{M}, \pi_1(M)) = 0; \qquad \text{if } p \neq \dim M/2.$$

Therefore

$$(-1)^{\dim M/2}\chi(M) \geq 0.$$

Proof. This result is not explicitly stated in [3]. However, it follows from their [3, Theorem 5.3] in the same way in which [19, Theorem 3.2] of Donnelly and Xavier follows from the corresponding [19, Theorem 2.2]. $\square$

3.1.13. Remark. Classical estimates of the Gauss-Bonnet integrand imply that
$(-1)^{\dim M/2}\chi(M) > 0$ if $-1 \leq K \leq b_n$ with $1 \geq b_n > 1 - 3/(\dim M + 1)$.

This gives strict positivity, but the curvature bound of Theorem 3.1.12 is weaker.

3.1.2 Hodge decomposition

In the classical situation, (de Rham) cohomology is of course not defined as the kernel of the Laplacian, as is suggested at the beginning of Section 3.1, but as the cohomology of the de Rham cochain complex. Only afterwards, the Hodge de Rham theorem shows that these de Rham cohomology groups are canonically isomorphic to the space of harmonic forms. The picture is parallel for L^2-cohomology. Whenever $(\overline{M}, g)$ is a complete Riemannian manifold, we can define the L^2-de Rham complex

$$\to L^2\Omega^{p-1}(\overline{M}) \xrightarrow{d} L^2\Omega^{p}(\overline{M}) \xrightarrow{d} L^2\Omega^{p+1}(\overline{M}) \to$$

where d is the exterior differential considered as an unbounded operator on the Hilbert space $L^2\Omega^p(\overline{M})$. (The fact that $(\overline{M}, g)$ is complete implies that d has a unique self adjoint extension. Usually, we work with this self adjoint extension instead of d itself).

We then define the L^2-cohomology as

$$H^p_{(2)}(\overline{M}) := \ker(d)/\overline{\operatorname{im}(d)}.$$

Observe that we divide through the *closure* of the image of d. This way, we stay in the category of Hilbert spaces. Sometimes, the L^2-cohomology groups obtained this way are called the *reduced* L^2-cohomology groups.

We have to check that this definition coincides with the one given in Definition 3.1.2. Now, if $(\overline{M}, g)$ is a complete Riemannian manifold, then we have the following Hodge decomposition:

$$L^2\Omega^p(M) = \ker(\overline{\Delta}_p) \oplus \overline{\operatorname{im} d} \oplus \overline{\operatorname{im} d^*}, \tag{3.1.14}$$

where d^* is the formal adjoint of d, and where the sum is an orthogonal direct sum. This implies that we also have an orthogonal decomposition

$$\ker(d|_{L^2\Omega^p(\overline{M})}) = \ker(\overline{\Delta}_p) \oplus \overline{\operatorname{im} d}.$$

Of course this implies immediately that the inclusion of $\ker(\overline{\Delta}_p)$ into $L^2\Omega^p(\overline{M})$ induces an isomorphism between $\ker(\overline{\Delta}_p)$ and $\ker(d)/\overline{\operatorname{im} d}$.

3.1.3 The Singer conjecture and Kähler manifolds

The use of the Singer conjecture to prove the Hopf conjecture in the examples presented so far is probably not very impressive. In this section we will discuss a much more striking result, which makes use of additional structure, namely the presence of a Kähler metric. The idea to do this is due to Gromov [26].

3.1.15. Definition. A Riemannian manifold (M, g) is called a Kähler manifold, if the (real) tangent bundle TM comes with the structure of a complex vector bundle (i.e. M is an almost complex manifold) with a Hermitian metric $h\colon TM \times TM \to \mathbb{C}$ (for the given complex structure on TM) such that the following conditions are satisfied:

- g is the real part of the Hermitian metric h.
- The 2-form ω associated to the Hermitian metric by

  $$\omega(v, w) = -\frac{1}{2} Im(h(v, w)),$$

  the so called *Kähler form*, is closed, i.e. $d\omega = 0$.

3.1.16. Remark. Under these conditions, the almost complex structure on M is integrable, i.e. M is a complex manifold. Moreover, ω is non-degenerate, i.e. $\omega^{\dim M/2}$ is a nowhere vanishing multiple of the volume form of (M, g).

3.1.17. Definition. A compact Kähler manifold M is called *Kähler hyperbolic*, if we find a 1-form η on the universal covering $\tilde{M}$ such that $\tilde{\omega} = d\eta$, where $\tilde{\omega}$ is the pullback of the Kähler form ω to $\tilde{M}$, and such that $|\eta|_\infty := \sup_{x\in\tilde{M}} |\eta(x)|_x < \infty$.

3.1.18. Example. The following manifolds are Kähler hyperbolic:

(1) closed Kähler manifold which are homotopy equivalent to a Riemannian manifold with negative sectional curvature.

(2) closed Kähler manifolds with word-hyperbolic fundamental group, provided the second homotopy group vanishes

(3) Complex submanifolds of Kähler hyperbolic manifolds, or the product of two Kähler hyperbolic manifolds.

Proof. Compare [26, Example 0.3]. □

From the point of view of the Hopf conjecture and the Singer conjecture, the first example is the most relevant: manifolds with negative sectional curvature, which also admit a Kähler structure, are covered by the results of this section.

Gromov proved the Singer conjecture for Kähler hyperbolic manifolds. More precisely, he proved in [26] the following.

3.1.19. Theorem. *Let M be a closed Kähler hyperbolic manifold of real dimension $2n$. Then*

$$b^p_{(2)}(\tilde{M}, \pi_1(M)) = 0 \text{ if } p \neq n$$
$$b^n_{(2)}(\tilde{M}, \pi_1(M)) > 0.$$

In particular, because of (3.1.4), $(-1)^n\chi(M) > 0$.

Proof. The proof splits into two rather different parts. On one side, we have to show vanishing outside the middle dimension. This is done using a Lefschetz theorem, which in particular implies that the cup-product with the lift of the Kähler form $\tilde{\omega}$ induces a bounded injective map

$$L\colon \ker(\tilde{\Delta}_r) \to \ker(\tilde{\Delta}_{r+2}) \qquad \text{for } r < n.$$

This is a classical fact from complex geometry in the compact case, and it extends rather easily to our non-compact situation.

But, by assumption, $\tilde{\omega} = d\eta$ where η is an L^∞-bounded one form. The cup product of a closed form f with a boundary $d\eta$ is always a boundary, which shows that (on a compact manifold) cup product with a boundary $d\eta$ induces the zero map on de Rham cohomology.

The extra boundedness condition on η allows us to deduce the same for the L^2-de Rham cohomology groups, i.e. cup product with $\tilde{\omega}$ induces the zero map on $H^p_{(2)}(\tilde{M})$ if M is Kähler hyperbolic. Since the Lefschetz theorem implies that this map is also injective, as long as $p < n$, $H^p_{(2)}(\tilde{M}) = 0$ for $p < n$. Because of the Poincaré duality theorem 3.1.7, the same holds for $p > n$, thus establishing the vanishing part of the theorem.

The first step to prove the non-vanishing of the L^2-cohomology in the middle dimension is the following result: Using similar, but more delicate, arguments as the one above, one shows that $\tilde{\Delta}_r \colon L^2\Omega^r(\tilde{M}) \to L^2\Omega^r(\tilde{M})$ is invertible with a bounded inverse when restricted to the orthogonal complement of its null space. (If $r \neq n$, this kernel is 0, but we later have to prove that $\tilde{\Delta}_n$ has a non-trivial null space.)

The key point now is that one can construct a continuous family of twisted L^2-de Rham complexes, indexed by $\lambda \in \mathbb{R}$, such that for $\lambda = 0$ we obtain the original L^2-de Rham complex we are interested in. An extension of Atiyah's L^2-index theorem 3.1.3 (a proof can be found e.g. in [56, Theorem 3.6]) implies that the L^2-Euler characteristics of these twisted complexes are a non-trivial polynomial $p(\lambda)$. Here, the twisted L^2-Euler characteristic is defined as the alternating sum of the L^2-dimension of $\ker(\Delta_p(\lambda))$, where the twisted Laplacian $\Delta_p(\lambda)$ is obtained from the twisted de Rham complex. Of course, $\Delta_p(0) = \tilde{\Delta}_0$, and $p(0) = \chi(M)$.

On the other hand, if $\ker(\tilde{\Delta}_n)$ was zero, then all the operators $\overline{\Delta}_r$ would be invertible, with bounded inverse. Because of the properties of the perturbation, the twisted Laplacian $\Delta_r(\lambda)$ would also be invertible for λ sufficiently close to zero, which would imply that $p(\lambda) = 0$ for λ sufficiently close to 0. This is a contradiction to the result that $p(\lambda)$ is a non-trivial polynomial. $\square$

3.1.20. Remark. The vanishing part of Theorem 3.1.19 definitely is the easier part. To obtain the corresponding statement, the assumptions on the Kähler form can be weakened a little bit, namely, it suffices that the lift $\tilde{\omega}$ satisfies $\tilde{\omega} = d\eta$ for a one form η which has at most linear growth (such manifolds are called *Kähler non-elliptic*). This is carried out independently in [36] and [12]. The most important example of Kähler non-elliptic manifolds are closed Riemannian manifolds with non-positive sectional curvature which admit a Kähler structure. In particular, for such a manifold the assertions of the Singer conjecture and of the Hopf conjecture are true.

3.2 Combinatorial L^2-Betti numbers

So far, we have defined L^2-cohomology and L^2-Betti numbers only for coverings of Riemannian manifolds, and we have not even shown that they do not depend on the chosen Riemannian metric. Here, we will extend the definition to coverings of arbitrary CW-complexes. Moreover, we will show that L^2-cohomology is an equivariant homotopy invariant.

From now on, therefore, assume that X is a compact CW-complex, and that $\overline{X}$ is a regular covering of X, with deck transformation group Γ (in particular, Γ acts freely on $\overline{X}$, and $X = \overline{X}/\Gamma$). We use the induced structure of a CW-complex on $\overline{X}$, and Γ acts by cellular homeomorphisms.

Then, the cellular chain complex of $\overline{X}$ is a chain complex of finitely generated free $\mathbb{Z}\Gamma$-modules, with $\mathbb{Z}\Gamma$-basis given by lifts of cells of X . This way, the basis is unique up to permutation and multiplication with $\pm g \in \mathbb{Z}\Gamma$ for $g \in \Gamma$.

The cellular L^2-cochain complex is defined by

$$C^*_{(2)}(\overline{X}, \Gamma) := \mathrm{Hom}_{\mathbb{Z}\Gamma}(C^{cell}_*(\overline{X}), l^2(\Gamma)).$$

We assume that Γ acts on X from the right, and therefore consider homomorphisms of right $\mathbb{Z}\Gamma$-modules. Γ still acts on $C^*_{(2)}(\overline{\Gamma})$ by isometries, with

$$(fg)(x) := g^{-1} \cdot (f(x)) \text{ for } \in \Gamma,\ f \in C^*_{(2)}(\overline{X}) \text{ and } x \in C^{cell}_*(\overline{X}).$$

The choice of a cellular $\mathbb{Z}\Gamma$-basis for $C^{cell}_*(\overline{X})$ identifies $C^p_{(2)}(\overline{X})$ with $(l^2\Gamma)^n$ (n being the number of p-cells in X), which induces in particular the structure of a Hilbert space on the cochain groups (which does not depend on the particular cellular basis).

The L^2-cochain maps

$$d_p \colon C^p_{(2)}(\overline{X}, \Gamma) \to C^{p+1}_{(2)}(\overline{X}, \Gamma)$$

are bounded equivariant linear maps. Let d_p^* be the adjoint operator, and set

$$\Delta_p := d_p^* d_p + d_{p-1} d_{p-1}^*.$$

This is the *cellular Laplacian*, a bounded self adjoint equivariant operator on $C^p_{(2)}(\overline{X})$.

3.2.1. Remark. We can also define the cellular L^2-chain complex by

$$C_*^{(2)}(\overline{X}) := C_*^{cell}(\overline{X}) \otimes_{\mathbb{Z}\Gamma} l^2(\Gamma).$$

The self duality of the Hilbert space $l^2(\Gamma)$ induces a duality between $C_*^{(2)}$ and $C^*_{(2)}$ which gives a chain isometry between the L^2-cochain and the L^2-chain complex. In particular, in all the definitions we are going to make, there will be a canonical isomorphism between the cohomological and homological version. Because for manifolds de Rham cohomology seems to be more natural to use, we will stick to the cohomological version throughout.

3.2.1 Hilbert modules

3.2.2. Definition. A (finitely generated) Hilbert $\mathcal{N}\Gamma$-module is a Hilbert space V with a right Γ-action which admits an equivariant isometric embedding into $l^2(\Gamma)^n$ for some n.

3.2.3. Remark. To explain this notation, we remark that an isometric action of Γ on a (complex) Hilbert space by linearity extends to an "action" of the integral group ring $\mathbb{Z}[\Gamma]$ and the complex group ring $\mathbb{C}[\Gamma]$ (recall that, given a ring R, the group ring $R\Gamma$ is defined to consist of finite formal linear combination $\sum_{g\in G} r_g g$ with $r_g \in R$, with component-wise addition, and multiplication defined by $(r_g g)(r_h h) = (r_g r_h)(gh)$). We can embed $\mathbb{C}\Gamma$ into a certain completion, the reduced C^*-algebra $C_r^*\Gamma$.

3.2.4. Definition. The *group von Neumann algebra* $\mathcal{N}\Gamma$ is an even bigger completion of $\mathbb{C}\Gamma$. It can be defined to consists of those bounded operators on $l^2\Gamma$ which commute with the *right* action of Γ on $l^2\Gamma$, i.e.

$$\mathcal{N}\Gamma := \mathcal{B}(l^2\Gamma)^\Gamma.$$

The right action is given by

$$(\sum_{g\in\Gamma} \lambda_g g) \cdot v := \sum_{g\in\Gamma} \lambda_g (gv) \text{ for } v \in G \text{ and } \sum_{g\in\Gamma} \lambda_g g \in l^2\Gamma.$$

Then $\mathcal{N}\Gamma$ is a ring which acts on the left on $l^2\Gamma$, and $\mathbb{C}\Gamma$ (actually $C_r^*\Gamma$) is contained in $\mathcal{N}\Gamma$.

Equivalently, one can define $\mathcal{N}\Gamma$ to be the closure of $\mathbb{C}\Gamma$ (with the left action) in $\mathcal{B}\Gamma$ with respect to the weak topology. This is a consequence of von Neumann's bicommutant theorem.

The action of Γ on a Hilbert $\mathcal{N}\Gamma$-module V extends to $\mathbb{C}\Gamma$ and then to $\mathcal{N}\Gamma$, making V indeed a module over $\mathcal{N}\Gamma$. Observe, however, that we don't get arbitrary algebraic modules, but modules with a topology and of a rather special kind. This additional $\mathcal{N}\Gamma$-module structure is underlying many of the definitions and proofs in L^2-cohomology. However, in the sequel we will omit explicitly using this, and instead work with the (for our purposes equivalent) unitary action of Γ and existence of the embedding into $(l^2\Gamma)^n$.

Given a Hilbert $\mathcal{N}\Gamma$-module V, let $\mathrm{pr}\colon l^2(\Gamma)^n \to l^2(\Gamma)^n$ be the orthogonal projection onto the image of any such embedding. We define the Γ-dimension of V by

$$\dim_\Gamma(V) := \mathrm{tr}_\Gamma(\mathrm{pr}) := \sum_{i=1}^{n} \langle \mathrm{pr}(e_i), e_i\rangle_{l^2(\Gamma)^n}. \tag{3.2.5}$$

Here, $e_i = (0, \dots, \delta_1, \dots, 0)$ is the standard basis vector of $l^2(\Gamma)^n$ with i-th entry being the characteristic function of the unit of Γ, and all other entries being zero.

Observe that Γ-invariance of V implies that pr is Γ-equivariant, i.e.

$$\langle \mathrm{pr}(e_i g), e_i g\rangle = \langle \mathrm{pr}(e_i), e_i\rangle$$

for all $g \in \Gamma$. If we would like to compute the $\mathbb{C}$-dimension of V, and therefore take the ordinary trace of pr (as endomorphism of $\mathbb{C}$-vector spaces) we would have to sum over $\langle \mathrm{pr}(e_i g), e_i g\rangle$ for all $g \in \Gamma$. Of course, in general this doesn't make sense since pr is not of trace class. As in the Definition 3.1.2, we pick the relevant part of this trace in (3.2.5), summing over a "fundamental domain" for the Γ-action on $l^2(\Gamma)^n$.

It is not hard to check that the above definition is independent of the choice of the embedding of V into $l^2(\Gamma)^n$.

3.2.6. Example. If Γ is finite, then every finitely generated Hilbert $\mathcal{N}\Gamma$-module V is a finite dimensional vector space over $\mathbb{C}$, and

$$\dim_\Gamma(V) = \frac{1}{|\Gamma|}\dim_{\mathbb{C}}(V).$$

3.2.7. Example. A more interesting example is given by free abelian groups. Assume that $\Gamma = \mathbb{Z}$. Then Fourier transform provides an isometric isomorphism between $l^2(\Gamma)$ and $L^2(S^1)$. Under this isomorphism, the subspace $\mathbb{C}[\Gamma]$ corresponds to the space of trigonometric polynomials in $L^2(S^1)$, which act by pointwise multiplication. The reduced C^*-algebra $C^*_r\mathbb{Z}$ becomes $C(S^1)$, and the von Neumann algebra $\mathcal{N}\mathbb{Z}$ becomes $L^\infty(S^1)$, also acting by pointwise multiplication. A projection P of $L^2(\Gamma)$ which commutes with all these trigonometric polynomials is itself given by multiplication with a measurable function f, and being a projection translates to the fact that f only takes the values 0 and 1 (up to a set of measure zero).

The image of P is the set of functions in $L^2(S^1)$ which vanish on the zero set of f. The Γ-trace of P is the constant term in the Fourier expansion of f, which can be computed by integration over S^1, i.e.

$$\operatorname{tr}_\Gamma(P) = \int_{S^1} f = \operatorname{vol}(\operatorname{supp}(f)),$$

which here is of course just the volume of the support of f, i.e. the set of all $x \in S^1$ with $f(x) = 1$ We use the standard measure on S^1, normalized in such a way that $\operatorname{vol}(S^1) = 1$.

It should be observed that one can obtain any real number between 0 and 1 in this way.

The Γ-dimension has the following useful properties, which in particular justify the term "dimension".

3.2.8. Proposition. *Let U, V, W be finitely generated Hilbert $\mathcal{N}\Gamma$-modules.*

(1) Faithfulness: $\dim_\Gamma(U) = 0$ *if and only if* $U = 0$.

(2) Additivity: If we have a weakly exact sequence of Hilbert $\mathcal{N}\Gamma$-modules

$$0 \to U \to W \to V \to 0$$

then

$$\dim_\Gamma(W) = \dim_\Gamma(U) + \dim_\Gamma(V).$$

Weakly exact means that the kernel of the outgoing map coincides with the closure *of the image of the incoming map, i.e.*

$$\cdots \xrightarrow{\phi_1} X \xrightarrow{\phi_2} \cdots$$

is weakly exact at X if and only if $\ker(\phi_2) = \overline{\operatorname{im}(\phi_1)}$.

(3) Monotonicity: If $U \subset V$ then $\dim_\Gamma(U) \leq \dim_\Gamma(V)$, *and* $\dim_\Gamma(U) = \dim_\Gamma(V)$ *if and only if* $U = V$.

(4) Normalization: $\dim_\Gamma(l^2(\Gamma)) = 1$.

(5) If H is a subgroup of finite index d in Γ, then every finitely generated Hilbert $\mathcal{N}\Gamma$-module V becomes by restriction of the action a finitely generated Hilbert $\mathcal{N}H$-module. Then

$$\dim_H(V) = d \cdot \dim_\Gamma(V).$$

(Note that Γ finite and H trivial is a special case of this situation.)

3.2.2 Cellular L^2-cohomology

3.2.9. Definition. We define the cellular L^2-cohomology by

$$H^p_{(2)}(\overline{X},\Gamma) := \ker(d^p)/\overline{\mathrm{im}}(d^{p-1}).$$

We have a Hodge decomposition

$$C^p_{(2)}(\overline{M},\Gamma) = \ker(\Delta_p) \oplus \overline{\mathrm{im}\, d} \oplus \overline{\mathrm{im}\, d^*}.$$

This is similar to Hodge decomposition for differential forms on Riemannian manifolds, but, since all operators involved here are bounded, is a much more elementary result. From this it follows that we have an isometric Γ-isomorphism

$$H^p_{(2)}(\overline{X},\Gamma) \cong \ker(\Delta_p)$$

We define the L^2-Betti numbers

$$b^p_{(2)}(\overline{X},\Gamma) := \dim_\Gamma(\ker(\Delta_p)).$$

Observe again the important fact that we divide by the closure of the image of the differential, such as to remain in the category of Hilbert spaces. This is the decisive difference to the equivariant cohomology with values in the $\mathbb{Z}\Gamma$-module $l^2\Gamma$. However, in [54, 55], Lück generalized the concept of L^2-Betti numbers from normal coverings of finite CW-complexes to arbitrary spaces with group action, using the usual twisted cohomology (with coefficients the group von Neumann algebra $\mathcal{N}\Gamma$). The starting point, however, is also in this treatment the theory of Hilbert $\mathcal{N}\Gamma$-modules.

3.2.2.1 Matrices over the group ring

Given a compact CW-complex, we can explicitly compute it's cohomology using a cellular basis and solving certain systems of linear equations. A similar approach is possible here (leading to more complicated, "non-commutative", equations in this situation).

In the compact case, the choice of an orientation for each cell identifies $C_p^{cell}(X)$ with $\mathbb{Z}^{c_p}$, where c_p is the number of p-cells of X, and this identification is well defined up to permutation of the basis, and multiplication of basis elements with ± 1. The boundary map, in this representation, is given by multiplication with an appropriate matrix with integral entries.

To proceed in the L^2-case, observe that each cell of the finitely many cells of X has as inverse image a free Γ-orbit of cells in $\tilde{X}$. We can choose one cell in each orbit. Together with the choice of an orientation, this identifies the $\mathbb{Z}\Gamma$-module $C_p^{cell}(\overline{X})$ with $(\mathbb{Z}\Gamma)^{c_p}$, and this identification is unique up to multiplication of the basis elements with $\pm g$ $(g \in \Gamma)$ and permutation. In this realization, the boundary maps of $C_p^{cell}(\overline{X})$ are given by multiplication with matrices A_p over the integral group ring.

The chosen $\mathbb{Z}\Gamma$-module isomorphism of $C_p^{cell}(\overline{X})$ with $(\mathbb{Z}\Gamma)^{c_p}$ induces an isomorphism of $C^*_{(2)}(\overline{X},\Gamma)$ with $(l^2\Gamma)^{c_p}$. An easy calculation shows that the coboundary maps are given by multiplication with the adjoint matrices A_p^* (extending the multiplication of elements of $\mathbb{C}\Gamma$ with elements of $l^2\Gamma$ used before). Here, if

$$u = \sum_{g\in\Gamma} \lambda_g g \in \mathbb{C}\Gamma \text{ then } u^* := \sum_{g\in\Gamma} \overline{\lambda_g} g^{-1},$$

and obviously $u^* \in \mathbb{Z}\Gamma$ if $u \in \mathbb{Z}\Gamma$. If $A = (A_{ij}) \in M(d_1 \times d_2, \mathbb{C}\Gamma)$, then

$$A^* := (A^*_{ji}) \in M(d_2 \times d_1, \mathbb{C}\Gamma),$$

and again this restricts to an operation on matrices over $\mathbb{Z}\Gamma$.

To finish the picture, the combinatorial Laplacian $\Delta_p = d_p^* d_p + d_{p-1} d_{p-1}^*$ is given by the matrix

$$\Delta := A_p^* A_p + A_{p-1}^* A_{p-1} \in M(c_p \times c_p, \mathbb{Z}\Gamma).$$

To understand the L^2-cohomology we therefore have to understand the kernel of such matrices, acting on $(l^2\Gamma)^{c_p}$.

This gives an algebraic way of studying questions about L^2-cohomology —they translate to questions about matrices over $\mathbb{Z}\Gamma$.

Actually, if Γ is finitely presented (i.e. has a presentation with finitely many generators and finitely many relations), given any matrix $A \in M(d \times d, \mathbb{Z}\Gamma)$, a standard construction provides us with a compact CW-complex X with $\pi_1(X) = \Gamma$ such that the kernel of the combinatorial Laplacian Δ_3 becomes the kernel of A, acting on $l^2(\Gamma)^d$. Therefore, we can also translate questions about matrices over $\mathbb{Z}\Gamma$ to questions in L^2-cohomology.

3.2.2.2 Properties of L^2-Betti numbers

3.2.10. Theorem. *L^2-cohomology and in particular L^2-Betti numbers have the following basic properties. Here, let $\overline{X}$ be a normal covering of a finite CW-complex X with covering group Γ.*

(1) *Let (M,g) be a closed Riemannian manifold, and equip it with the CW-structure coming from a smooth triangulation. Let $(\overline{M},\overline{g})$ be a normal*

covering of M with covering group Γ. Then integration of forms over simplices (the de Rham map) defines a Γ-isomorphism

$$\ker(\overline{\Delta}_p(\overline{g})) \to H^p_{(2),cell}(\overline{M}, \Gamma).$$

In particular, the L^2-Betti numbers defined using the Riemannian metric and using the Γ-CW-structure coincide.

(2) *Let Y be another finite CW-complexes and $f\colon Y \to X$ a homotopy equivalence. Let $\overline{Y}$ be the pullback of $\overline{X}$ along f (this means that the deck transformation group for $\overline{Y}$ is also Γ). Then*

$$b^p_{(2)}(\overline{X}, \Gamma) = b^p_{(2)}(\overline{Y}, \Gamma) \qquad \forall p \geq 0.$$

(3) *For the Euler characteristic of X, we get*

$$\chi(X) = \sum_{p=0}^{\infty} (-1)^p b^p_{(2)}(\overline{X}, \Gamma). \tag{3.2.11}$$

(4) *Assume Γ is finite. Then $\overline{X}$ is itself a finite CW-complex, and its (ordinary) Betti number $b^p(\overline{X})$ are defined. They satisfy*

$$b^p_{(2)}(\overline{X}, \Gamma) = \frac{1}{|\Gamma|} b^p(\overline{X}).$$

(5) *If Γ is infinite, then for the zeroth L^2-Betti number we get*

$$b^0_{(2)}(\overline{X}, \Gamma) = 0.$$

(6) *Let $H \leq \Gamma$ be a subgroup of finite index d, and set $X_1 := \overline{X}/H$. This is a finite d-sheeted covering of X, and $\overline{X}$ can be considered to be a normal covering of X_1 with covering group H. Then*

$$b^p_{(2)}(\overline{X}, H) = d \cdot b^p_{(2)}(\overline{X}, \Gamma) \qquad \forall p \geq 0.$$

Note that the Euler characteristic is multiplicative under finite covering, in the situation of (6) this means that $\chi(X_1) = d \cdot \chi(X)$. In view of (6) and the Euler characteristic formula (3), the L^2-Betti numbers are the appropriate refinement of the Euler characteristic which (unlike the ordinary Betti numbers) remain multiplicative under finite coverings.

Proof of Theorem 3.2.10. We only indicate reference and the main points.

(1) This was a classical question of Atiyah, proved by Dodziuk in [16].

(2) f is covered by a Γ-homotopy equivalence $\overline{f}\colon \overline{Y} \to \overline{X}$. One easily checks that such a map induces a map on L^2-cohomology, and that two Γ-homotopic maps induce the same map. The claim follows.

(3) The Euler characteristic formula follows exactly as in the classical situation, using additivity of the Γ-dimension and the normalization

$$\dim_\Gamma(l^2\Gamma) = 1.$$

(4) If Γ is finite, all the Hilbert spaces in question are finite dimensional. Consequently, $\mathrm{im}(d)$ is automatically closed, and there is no difference between L^2-cohomology and ordinary cohomology with complex coefficients of $\overline{X}$. In particular,

$$b^p(\overline{X}) = \dim_{\mathbb{C}} H^p_{(2)}(\overline{X},\Gamma) = |\Gamma| \cdot \dim_\Gamma(H^p_{(2)}(\overline{X},\Gamma)) = |\Gamma| \cdot b^p_{(2)}(\overline{X},\Gamma).$$

(6) This follows immediately from the corresponding formula in Proposition 3.2.8.

□

3.3 Approximating L^2-Betti numbers

As mentioned in Section 3.1, there are almost no relations between the ordinary Betti numbers of a space X and the L^2-Betti numbers of a covering $\tilde{X}$ of X. However, if we have a whole sequence of nested coverings $X \leftarrow X_1 \leftarrow X_2 \leftarrow \cdots$, "converging" to $\tilde{X}$, in many cases we can approximate the L^2-Betti numbers of $\tilde{X}$ in terms of this sequence. More precisely, let $\tilde{X}$ be a Γ-covering of X. Assume that there is a nested sequence of normal subgroups $\Gamma \geq \Gamma_1 \geq \Gamma_2 \geq \cdots$ (each Γ_k normal in Γ) such that $\bigcap_{k\geq 1}\Gamma_k = \{1\}$. Then $X_k := \Gamma_k\backslash\tilde{X}$ is a normal covering of X, with covering group Γ/Γ_k.

3.3.1. Conjecture. *In this situation,*

$$b^p_{(2)}(\tilde{X},\Gamma) = \lim_{k\to\infty} b^p_{(2)}(X_k,\Gamma/\Gamma_k). \tag{3.3.2}$$

Observe that convergence is not clear, but part of the statement.

This question was first asked by Gromov if all the groups Γ/Γ_k are finite. In this case, the conjecture is true, as proved by Lück:

3.3.3. Theorem. *Equation* (3.3.2) *is correct if* Γ/Γ_k *is finite for each* $k \in \mathbb{N}$.

Observe that, in this setting, X_k is a finite covering of X. Consequently, we can express the L^2-Betti numbers in terms of ordinary Betti numbers and obtain, in the setting of Conjecture 3.3.1,

$$b^p_{(2)}(\tilde{X}, \Gamma) = \lim_{k \to \infty} \frac{b^p(X_k)}{|\Gamma/\Gamma_k|}.$$

In general, Conjecture 3.3.1 is still open. However, in [71] (with an improvement in [18]) a quite large class $\mathcal{G}$ of groups is constructed for which the conjecture is true. $\mathcal{G}$ contains all amenable and all free groups, and is closed under taking subgroups, extensions with amenable quotients, directed unions, and inverse limits (therefore, it contains e.g. all residually finite groups).

More precisely

3.3.4. Definition. Let $\mathcal{G}$ be the smallest class of groups which contains the trivial group and is closed under the following processes:

- If $H \in \mathcal{G}$ and G is a generalized amenable extension of H, then $G \in \mathcal{G}$.
- If $G = \lim_{i \in I} G_i$ is the direct or inverse limit of a directed system of groups $G_i \in \mathcal{G}$, then $G \in \mathcal{G}$.
- If $H \in \mathcal{G}$ and $U \leq H$, then $U \in \mathcal{G}$.

Here, the notion of *generalized amenable extension* is defined as follows:

3.3.5. Definition. Assume that G is a finitely generated discrete group with a finite symmetric set of generators S (i.e. $s \in S$ implies $s^{-1} \in S$), and let H be an arbitrary discrete group. We say that G *is a generalized amenable extension of* H, if there is a set X with a free G-action (from the left) and a commuting free H-action (from the right), such that a sequence of H-subsets $X_1 \subset X_2 \subset X_3 \subset \cdots \subset X$ exists with $\bigcup_{k \in \mathbb{N}} X_k = X$, and with $|X_k/H| < \infty$ for every $k \in \mathbb{N}$, and such that

$$\frac{|(S \cdot X_k - X_k)/H|}{|X_k/H|} \xrightarrow{k \to \infty} 0.$$

In [71] and [18], the following theorem is proved.

3.3.6. Theorem. *Equation* (3.3.2) *is correct if* Γ/Γ_k *belongs to* $\mathcal{G}$ *for each* $k \in \mathbb{N}$.

In particular, we obtain the following corollary which we will use later.

3.3.7. Corollary. *Equation* (3.3.2) *is true if* Γ/Γ_k *is amenable, e.g. solvable or nilpotent, or virtually solvable, for each* $k \in \mathbb{N}$. *Recall that a group* G *has* virtually *a certain property* P, *if it contains a subgroup of finite index which has property* P.

3.3.8. Remark. There are generalizations of the above approximation results to other L^2-invariants, in particular to the L^2-signature, compare [53].

3.4 The Atiyah conjecture

Fix a discrete group Γ. The L^2-Betti numbers $b^p_{(2)}(\overline{X}, \Gamma)$ of a Γ-covering of a finite CW-complex X are the Γ-dimensions of certain Hilbert $\mathcal{N}\Gamma$-modules. In Example 3.2.7 we have seen that a priori arbitrary non-negative real numbers could occur, even for groups as nice as $\mathbb{Z}$. However, the Euler characteristic formula (3.2.11) shows that certain combinations ofL^2-Betti numbers are always integers.

The Atiyah conjecture predicts a certain amount of integrality for the individual L^2-Betti numbers.

3.4.1. Conjecture. *Fix a discrete group* Γ. *Let* $Fin^{-1}(\Gamma)$ *be the additive subgroup of* $\mathbb{Q}$ *generated by*

$$\{\frac{1}{|F|} \mid F \textit{ finite subgroup of } \Gamma\}.$$

Let X *be a finite CW-complex or a compact manifold,* $\overline{X}$ *a* Γ*-covering of* X.

(1) If Γ *is torsion-free, then*

$$b^p_{(2)}(\overline{X}, \Gamma) \in \mathbb{Z}.$$

(2) Assume there is a bound on the finite subgroups of Γ *(observe that this is equivalent to* $Fin^{-1}(\Gamma)$ *being a discrete subset of* $\mathbb{R}$*). Then*

$$b^p_{(2)}(\overline{X}, \Gamma) \in Fin^{-1}(\Gamma).$$

(3) Without any assumption on Γ,

$$b^p_{(2)}(\overline{X}, \Gamma) \in \mathbb{Q}.$$

For a while, also the following conjecture was around:

3.4.2. Conjecture. *Without any assumption on* Γ, $b^p_{(2)}(\overline{X}, \Gamma) \in Fin^{-1}(\Gamma)$.

This last conjecture is singled out here because it is wrong. In [24], a smooth 7-dimensional Riemannian manifold M is constructed such that every finite subgroup of $\pi_1(M)$ is an elementary abelian 2-group, but $b^3_{(2)}(\tilde{M}, \pi_1(M)) = \frac{1}{3}$. This example is based on the explicit calculation of the eigenspaces and their L^2-dimensions of a certain operator in [25], using in particular the methods of the proof of Theorem 3.3.3. A more direct and slightly more general computation for such eigenspaces is carried out in [15].

It should be remarked that none of the above conjectures were formulated by Atiyah as stated here, although he makes some remarks which show that he was interested in the question of the possible values of L^2-Betti numbers.

Statement (3) of Conjecture 3.4.1, which is the oldest version of the Atiyah conjecture, is also quite unlikely to hold in general. In [15], for each $r, s \in \mathbb{N}$ with $r, s \geq 2$, a manifold $M_{r,s}$ is constructed such that

$$b^3_{(2)}(\tilde{M}, \pi_1(M)) = \alpha_{r,s} := (r-1)^2(s-1)^2 \cdot \sum_{n=2}^{\infty} \frac{\phi(n)}{(r^n-1)(s^n-1)},$$

where $\phi(n)$ is Euler's phi-function, i.e. the number of primitive n-th roots of unity. At the moment, it is unknown whether any of the numbers $\alpha_{r,s}$ is irrational. However, the use of computer algebra shows e.g. that, if $\alpha_{2,2}$ is rational, both the numerator and denominator exceed 10^{100}. It seems reasonable to assert that $\alpha_{2,2}$ is not obviously rational.

3.4.1 Combinatorial reformulation of the Atiyah Conjecture

The following assertion is equivalent to Conjecture 3.4.1.

3.4.3. Conjecture. *Let* Γ *be a discrete group, and assume that* $A \in M(d \times d, \mathbb{Z}\Gamma)$. *Consider* A *to be a bounded operator on* $l^2(\Gamma)^d$, *as in Section 3.2.2.1.*

(1) If Γ *is torsion-free, then* $\dim_\Gamma(\ker(A)) \in \mathbb{Z}$.

(2) If $Fin^{-1}(\Gamma)$ *is a discrete subset of* $\mathbb{R}$, *then*

$$\dim_\Gamma(\ker(A)) \in Fin^{-1}(\Gamma).$$

Without any assumption on Γ, $\dim_\Gamma(\ker(A)) \in \mathbb{Q}$.

3.4.4. Remark. It is equivalent to require the assertions of Conjecture 3.4.3 for all matrices over $\mathbb{Z}\Gamma$, or for all square matrices, or for all self-adjoint matrices (these are automatically square matrices), or for all matrices of the form $A = B^*B$ (these are automatically self-adjoint). This is the case since $\ker(A) = \ker(A^*A)$. Moreover, we have the weakly exact sequence

$$0 \to \ker(A) \hookrightarrow l^2(\Gamma)^d \xrightarrow{A} \overline{\operatorname{im} A} \to 0.$$

Because of additivity and normalization of the Γ-dimension (Proposition 3.2.8), we could replace the kernel of A by the closure of the image, throughout.

The equivalence of Conjecture 3.4.1 and Conjecture 3.4.3 follows immediately from the principle described at the end of Section 3.2.2.1.

3.4.2 Atiyah conjecture and non-commutative algebraic geometry — Generalizations

As an illustration, we now want to study the Atiyah conjecture for the group $\Gamma = \mathbb{Z}$, which we understand particularly well because of Example 3.2.7.We look at the algebraic reformulation. For simplicity, assume first that $d = 1$. Then $A \in \mathbb{Z}[\mathbb{Z}] = \mathbb{Z}[z, z^{-1}]$ is a Laurent polynomial with $\mathbb{Z}$-coefficients. Under Fourier transform we get the commutative diagram

$$\begin{array}{ccc} l^2(\mathbb{Z}) & \xrightarrow{A} & l^2(\mathbb{Z}) \\ \downarrow\cong & & \downarrow\cong \\ L^2(S^1) & \xrightarrow{A(z)} & L^2(S^1), \end{array}$$

i.e. the action of A translates to multiplication with the function $A(z)$, ($z \in S^1 \subset \mathbb{C}$). Now,

$$\dim_{\mathbb{Z}}(\ker(A)) = \mu(\{z \in S^1 \mid A(z) = 0\})$$

is the volume of the set of zeros of the Laurant polynomial $A(z)$ on S^1. But the Laurant polynomial $A(z)$ has, if it is not identically zero, only finitely many zero. Therefore

$$\dim_{\mathbb{Z}}(\ker(A)) = 0 \text{ if } A \neq 0,$$

and of course

$$\dim_{\mathbb{Z}}(\ker(A)) = 1 \text{ if } A = 0.$$

Since $\mathbb{Z}$ is commutative, every matrix A can be replaced by a diagonal matrix without changing the dimension of the kernel, and this way the above calculation proves the Atiyah conjecture for $\mathbb{Z}$.

Similar considerations for $\Gamma = \mathbb{Z}^n$ show that the Atiyah conjecture here amounts to understanding the zeros of polynomials in several variables. This created the slogan that the Atiyah conjecture (and more generally L^2-cohomology) in a certain sense is non-commutative algebraic geometry.

Exactly the same proof works if we replace the coefficient ring $\mathbb{Z}$ by $\mathbb{C}$, or by any subring of $\mathbb{C}$.

This leads to the following algebraic generalization of the Atiyah conjecture.

3.4.5. Conjecture. *Fix a discrete group* Γ. *Let* K *be any subring of* $\mathbb{C}$ *which is closed under complex conjugation. Let* Γ *be a discrete group, and assume that* $a \in M(d \times d, K\Gamma)$.

If $Fin^{-1}(\Gamma)$ *is discrete, then* $\dim_\Gamma(\ker(A)) \in Fin^{-1}(\Gamma)$.

Since we can multiply any matrix with a non-zero constant (e.g. a common denominator of the finitely many non-zero coefficients), the assertion of Conjecture 3.4.5 for a ring $K \subset \mathbb{C}$ and its field of fractions is equivalent. In the sequel, we will therefore usually assume that the subring K is a field.

It is not clear, however, whether Conjecture 3.4.5 is equivalent to the original geometric Atiyah conjecture 3.4.1 if $K \not\subset \mathbb{Q}$.

Observe that the proof of the Atiyah conjecture for $\mathbb{Z}$ extends from the ring of Laurent polynomials (with complex coefficients) to the ring of meromorphic functions $\mathbb{C}$ without poles on S^1, which is also a subring of $L^\infty(S^1) \cong \mathcal{N}\mathbb{Z}$. The question arises whether there are reasonable generalizations similar to this ring for other groups. One possibility would be to look at infinite sums $a = \sum_{g\in\Gamma} \lambda_g g$, where the coefficients λ_g very rapidly tend to zero as $g \to \infty$ (with respect to a suitable word length metric). (Observe that this is the case for the coefficients of the Laurent expansion of a meromorphic function on $\mathbb{C}$). Under suitable circumstances, (convolution) multiplication with such an a will indeed give rise to a (very special) Γ-equivariant operator on $l^2\Gamma$, and the question arises whether for the dimension of its kernel the statement of the Atiyah conjecture holds. This was suggested by Nigel Higson. It is quite distinct from Conjecture 3.4.5 in that it is analytic in flavor, and no longer algebraic.

3.4.3 Atiyah conjecture and zero divisors

Among the most interesting observations about the Atiyah conjecture are its strong connections to questions in algebra, in particular to group rings.

Here, we address the following conjecture, the so called *zero divisor conjecture.*

3.4.6. Conjecture. *Let Γ be a torsion-free discrete group and K a subring of the complex numbers. Then, there are no non-trivial zero divisors in the group ring $K\Gamma$, i.e. if $a, b \in K\Gamma$ with $ab = 0$ then either $a = 0$ or $b = 0$.*

This is one of the longstanding questions in the theory of group rings (which of course makes also sense for other coefficients rings K and is studied also in this broader generality by ring theorists).

Observe that, if $g \in \Gamma$ is a torsion element, i.e. $g \neq 1$ but $g^n = 1$ for some $n > 0$, then $a = (1 - g)$ and $b = 1 + g + \cdots + g^{n-1}$ are two non-zero elements of $\mathbb{Z}\Gamma$ with $ab = 0$.

It now turns out that the Atiyah conjecture implies the zero divisor conjecture. More precisely:

3.4.7. Theorem. *Assume Γ is a torsion-free discrete group, and $K \subset \mathbb{C}$ is a ring (closed under complex conjugation).*

If the statement of the algebraic Atiyah conjecture 3.4.5 is true for every $A \in M(1 \times 1, K\Gamma)$, then there are no non-trivial zero divisors in $K\Gamma$.

Proof. Fix $a, b \in K\Gamma$ with $ab = 0$. We have to show that either $a = 0$ or $b = 0$. Now observe that

$$a \in K\Gamma = M(1 \times 1, K\Gamma)$$

is a 1-by-1 matrix over $K\Gamma$. On the other hand,

$$b \in K\Gamma \subset l^2\Gamma$$

can be considered to be an element of $l^2\Gamma$. And $0 = ab$ is just the result of the action of the matrix a on the l^2-function b. Therefore, $b \in \ker(a)$. Now we know that $\dim_\Gamma(\ker(a)) \in \mathbb{Z}$ because the Atiyah conjecture is true for the torsion-free group Γ. Evidently,

$$\{0\} \subset \ker(a) \subset l^2\Gamma,$$

with

$$0 = \dim_\Gamma(\{0\}) \quad \text{and} \quad 1 = \dim_\Gamma(l^2\Gamma).$$

Because of monotonicity, either $\dim_\Gamma(\ker(a)) = 0$ or $\dim_\Gamma(\ker(a)) = 1$. In the first case, because of faithfulness $\ker(a) = \{0\}$ which implies $b = 0$. In the second case, $\ker(a) = l^2\Gamma$ (again because of faithfulness) which means that $a = 0$. This proves the statement.

Here, we used some of the properties of $\dim_\Gamma$ developed in Proposition 3.2.8. ☐

In Theorem 3.4.18, we will see that there is actually an even stronger relation between the Atiyah conjecture and the zero divisor conjecture for a torsion free group Γ.

3.4.4 Atiyah conjecture and calculations

A second possible application of the Atiyah conjecture could be the explicit calculation of L^2-Betti numbers.

Here one would use that by now there are several approximation formulas for L^2-Betti numbers. In particular, we have discussed one of these in Section 3.3. Obviously, if we know in advance that the limit has to be an integer, this can make it much easier to exactly compute the limit, in particular if (as is the case for some of the approximation results) error bounds are available. Up to now, however, to the authors knowledge this idea has not been used anywhere.

3.4.5 The status of the Atiyah conjecture

By now, the Atiyah conjecture is known for a reasonably large class of groups. We use the following definitions.

3.4.8. Definition. The class of *elementary amenable groups* is the smallest class of groups which contains all abelian and all finite groups and is closed under extensions and directed unions. It is denoted $\mathcal{Y}$. Obviously, every elementary amenable group is amenable. Moreover, every nilpotent and every solvable group is elementary amenable, as well as every group which is virtually solvable. (A group has *virtually* a property P, if it contains a subgroup of finite index which actually has property P).

3.4.9. Definition. Let $\mathcal{D}$ be the smallest non-empty class of groups such that:

(1) If G is torsion-free and A is elementary amenable, and we have a projection $p\colon G \to A$ such that $p^{-1}(E) \in \mathcal{D}$ for every finite subgroup E of A, then $G \in \mathcal{D}$.

(2) $\mathcal{D}$ is subgroup closed.

(3) Let $G_i \in \mathcal{D}$ be a directed system of groups and G its (direct or inverse) limit. Then $G \in \mathcal{D}$.

3.4.10. Definition. A *directed system* of groups is a system of groups G_i, indexed by an index set I with a partial ordering $<$, and either with a homomorphism $\phi_{ij}\colon G_i \to G_j$ if $i < j$ such that $\phi_{jk} \circ \phi_{ij} = \phi_{ik}$ if $i < j <$

k (then we take the direct limit), or with homomorphisms the other way around, i.e. $\phi_{ij}\colon G_j \to G_i$ if $i < j$, with the corresponding compatibility condition (then we take the inverse limit).

Directed means that to each $i, j \in I$ exists $k \in I$ with $i < k$ and $j < k$. The most obvious examples are systems of groups indexed by $\mathbb{N}$ with its usual ordering.

Observe that $\mathcal{D}$ contains only torsion-free groups.

3.4.11. Example. The class $\mathcal{D}$ contains all torsion-free elementary amenable groups. It also contains all free groups and all braid groups (compare Section 3.4.5.4). Moreover, $\mathcal{D}$ is closed under direct sums, direct products, and free products.

To see this, observe that clearly $\mathcal{D}$ contains all elementary amenable groups, as long as they are torsion-free. Moreover, if Γ contains a sequence of normal subgroups $\Gamma \geq \Gamma_1 \geq \Gamma_2 \geq \cdots$ with $\bigcap_{k\in\mathbb{N}} \Gamma_k = \{1\}$ and such that Γ/Γ_k is torsion-free elementary amenable, then Γ is a subgroup of the inverse limit of the sequence of quotients, and consequently belongs to $\mathcal{D}$.

In particular, every free group admits such a sequence in such a way that the quotients are torsion-free nilpotent, and every braid group admits such a sequence where the quotients are torsion-free and virtually nilpotent.

The following theorem is proved in [70] and [18].

3.4.12. Theorem. *Set $K := \overline{\mathbb{Q}}$, the field of algebraic numbers (over $\mathbb{Q}$) in $\mathbb{C}$.*

If $\Gamma \in \mathcal{D}$, then the Atiyah conjecture 3.4.5 is true for $K\Gamma$.

To prove this, one only has to prove that the Atiyah conjecture is preserved when passing to subgroups, doing extensions with torsion-free elementary amenable quotients, and under direct and inverse limits of directed systems of groups in $\mathcal{D}$. The former (almost) translates to directed unions of groups, and the latter to the case where a group Γ has the nested sequence of normal subgroups Γ_k with trivial intersection we have discussed previously.

We want to start with two rather elementary observations, which are not very useful without examples of groups for which the Atiyah conjecture is true, but which give rise to part of the statements of Theorem 3.4.12.

3.4.13. Proposition. *If G fulfills the Atiyah conjecture, and if U is a subgroup of G with $Fin^{-1}(U) = Fin^{-1}(G)$, then U fulfills the Atiyah conjecture.*

Proof. This follows from the fact that the U-dimension of the kernel of a matrix over KU acting on $l^2(U)^n$ coincides with the G-dimension of the same matrix, considered as an operator on $l^2(G)^n$ (compare e.g. [71, 3.1]), which follows from a simply diagonal decomposition argument. $\square$

3.4.14. Proposition. *Let G be the directed union of groups $\{G_i\}_{i\in I}$ and assume that each G_i fulfills the Atiyah conjecture. Then G fulfills the Atiyah conjecture.*

Proof. A matrix over KG, having only finitely many non-trivial coefficients, already is a matrix over KG_i for some i. The G_i-dimension and the G-dimension of the kernel of the matrix coincide, as in Proposition 3.4.13, compare e.g. [71, 3.1]. Note that $Fin^{-1}(G_i)$ is contained in $Fin^{-1}(G)$ for each $i \in I$. $\square$

Besides of these two results, at the moment three rather different methods are known to prove the Atiyah conjecture for certain groups.

3.4.5.1 Fredholm modules and the Atiyah conjecture

One of these methods, which one could call the method of the "finite rank Fredholm module", was developed by Peter Linnell in [48] to prove the Atiyah conjecture for free groups. It extends in an ingenious way the method used by Connes to prove the trace conjecture 1.3.12 (compare e.g. [21]) for the free group, and therefore is related to the Baum-Connes conjecture. However, whereas the KK-theory methods for Baum-Connes turn out to be quite flexible and generalize to many other groups, compare e.g. [73], nobody so far was able to find a corresponding generalized approach to the Atiyah conjecture. Since the Atiyah conjecture for free groups follows also from one of the other methods to be described later, we don't discuss Linnell's original approach but refer instead to the original article [48] and the review article [49]. (Note, however, that the generalization to products of free groups does not work as described in [49], since the proof of the basic Lemma [49, Lemma 11.7] has a gap. One has to rely on a different method for the proof, instead.)

3.4.5.2 Atiyah conjecture and algebra

We have shown in Theorem 3.4.7 that the Atiyah conjecture for a torsion-free group implies the zero divisor conjecture. The ring $K\Gamma$ evidently has no zero divisors, if it can be embedded into a skew field, i.e. a (not necessary commutative) ring where every non-zero element has a multiplicative inverse. The optimal solution to the zero divisor question therefore is to construct exactly this. It turns out that the Atiyah conjecture provides us with such an embedding. We need the following definitions.

3.4.15. Definition. Given the group von Neumann algebra $\mathcal{N}\Gamma$, we define $\mathcal{U}\Gamma$ to be the algebra of all unbounded operators affiliated to $\mathcal{N}\Gamma$ (compare e.g. [49, Section 8]). That means a densely defined unbounded operator D on $l^2\Gamma$ belongs to $\mathcal{U}\Gamma$ if and only if all its spectral projections belong to $\mathcal{N}\Gamma$. It is a classical fact that the ring $\mathcal{U}\Gamma$ is a (non-commutative) localization of $\mathcal{N}\Gamma$, which here means it is obtained from $\mathcal{N}\Gamma$ by inverting all non-zero divisors of $\mathcal{N}\Gamma$.

Fix a subfield $K \subset \mathbb{C}$ which is closed under complex conjugation. We have to consider $K\Gamma \subset \mathcal{N}\Gamma \subset \mathcal{U}\Gamma$. Define $D_K\Gamma$ as the *division closure* of $K\Gamma$ in $\mathcal{U}\Gamma$. By definition, this is the smallest subring of $\mathcal{U}\Gamma$ which contains $K\Gamma$ and which has the property that, whenever $x \in D_K\Gamma$ is invertible in $\mathcal{U}\Gamma$, then $x^{-1} \in D_K\Gamma$.

3.4.16. Remark. Since the operators which belong to $\mathcal{U}\Gamma$ are only densely defined, one has to be careful when defining the sum or product of two such operators. This is done by first defining these operators on the obvious (common) domain, but then taking there closure, i.e. to extend the domain of definition as far as possible. One has to check that this indeed gives a reasonable ring. This is a classical result which uses the Γ-dimension.

3.4.17. Example. Again, we turn to the example $\Gamma = \mathbb{Z}$. We have seen that, via Fourier transform, $\mathcal{N}\Gamma$ becomes $L^\infty(S^1)$ acting on $L^2(S^1)$ by pointwise multiplication. The ring $\mathcal{U}\mathbb{Z}$ becomes the ring of *all* measurable functions on S^1, still acting by pointwise multiplication. These operators are in general unbounded and not defined on all of $L^2(S^1)$, because the product of an L^2-function with an arbitrary measurable function belongs not necessarily to $L^2(S^1)$. It is not hard to show that every measurable function f on S^1 is the quotient of two bounded functions $g, h \in L^\infty(S^1)$, $f = g/h$, where the set of zeros of h has measure zero. This reflects the fact that $\mathcal{U}\Gamma$ is a localization of $\mathcal{N}\Gamma$.

If $K \subset \mathbb{C}$, then $K\mathbb{Z}$ are the Laurent polynomials $K[z, z^{-1}]$ identified with functions on S^1 (by substituting $z \in S^1$ for the variable). In the same way, $D_K\Gamma$ is the field of rational functions $K(z)$, identifies with functions on S^1 by substituting $z \in S^1$ for the variable.

The (very strong) connection between the Atiyah conjecture and ring theoretic properties of $D_K\Gamma$ is given by the following theorem.

3.4.18. Theorem. *Let Γ be a torsion-free group, and let K be a subfield of $\mathbb{C}$ which is closed under complex conjugation.*

$K\Gamma$ fulfills the strong Atiyah conjecture in the sense of Conjecture 3.4.5 if and only if the division closure $D_k\Gamma$ of $K\Gamma$ in $\mathcal{U}\Gamma$ is a skew field.

In other words, we have a canonical candidate $D_K\Gamma$ for a skew field, into which $K\Gamma$ embeds, and this ring is a skew field if and only if $K\Gamma$ satisfies the Atiyah conjecture.

Proof of Theorem 3.4.18. If $D_K\Gamma$ is a skew field, then each matrix $A \in M(d \times d, K\Gamma)$ acts on $(D_K\Gamma)^d$, and its kernel is a finite dimensional vector space over the field $D_K\Gamma$. In particular, its $D_K\Gamma$-dimension of course is an integer. Then, one can establish that the Γ-dimension of $\ker(A\colon (l^2\Gamma)^d \to (l^2\Gamma)^d)$ coincides with this dimension. Details are given in [70, Lemma 3].

For the converse, given an element $0 \neq a \in D_K\Gamma$ one can, using a matrix trick for division closures due to Cohn, produce a $d \times d$-matrix A over $K\Gamma$ such that the Γ-dimension of $\ker(A)$ is evidently strictly smaller than 1, and which is non-zero if and only if a is invertible in $\mathcal{U}\Gamma$ (slogan: "a non-trivial kernel of a gives rise to a non-trivial kernel of A"). Because $\dim_\Gamma(\ker(A)) \in \mathbb{Z}$ by assumption, $\dim_\Gamma(\ker(A)) = 0$, i.e. a is invertible in $\mathcal{U}\Gamma$. Since $D_K\Gamma$ is division closed, a is invertible in $D_K\Gamma$, as well. $\square$

This property allows to prove the Atiyah conjecture for the first interesting class of groups (containing non-abelian groups), namely the class of elementary amenable groups.

3.4.19. Theorem. *Fix a subfield $K \subset \mathbb{C}$ which is closed under complex conjugation.*

Let $1 \to H \to G \to A \to 1$ be an exact sequence of groups. Assume that G is torsion free and A is elementary amenable. For every finite subgroup $E \leq A$ let H_E be the inverse image of E in G. Assume for all finite subgroups $E \leq G$ that KH_E fulfills the Atiyah conjecture 3.4.5. Then KG fulfills also the Atiyah conjecture.

Proof. The proof is given in [48] for $K = \mathbb{Q}$. Essentially the same proof works for arbitrary K, compare [70, Proposition 3.1]. $\square$

3.4.20. Corollary. *Fix a subfield $K = \overline{K} \subset \mathbb{C}$.*

Suppose H is torsion-free and KH fulfills the Atiyah conjecture. If G is an extension of H with elementary amenable torsion-free quotient then KG fulfills the Atiyah conjecture.

In particular (with $H = 1$) if G is a torsion-free elementary amenable group then KG satisfies the Atiyah conjecture.

Proof. By assumption, the only finite subgroup of G/H is the trivial group and the Atiyah conjecture is true for its inverse image H. $\square$

3.4.5.3 Atiyah conjecture and approximation

Here, we describe the last method of proof for the Atiyah conjecture. It is based on the approximation results of Section 3.3.

3.4.21. Theorem. *Assume Γ is a torsion-free discrete group with a nested sequence of normal subgroups $\Gamma \geq \Gamma_1 \geq \Gamma_2 \geq \cdots$ such that $\bigcap_{k\in\mathbb{N}} \Gamma/\Gamma_k = \{1\}$ and such that $\Gamma/\Gamma_k \in \mathcal{G}$ for each $k \in \mathbb{N}$. Moreover, assume that all the quotient groups Γ/Γ_k are torsion-free and satisfy the Atiyah conjecture 3.4.1.*

For example, all the groups Γ/Γ_k might be torsion-free elementary amenable.

Then Γ also satisfies the Atiyah conjecture 3.4.1.

Proof. Given any Γ-covering $\overline{X} \to X$ (with a finite CW-complex X), we have to prove that $b^p_{(2)}(\overline{X}, \Gamma) \in \mathbb{Z}$ for each p.

Now, the sequence of normal subgroups Γ_k provides us with a sequences of normal coverings $X_k := \overline{X}/\Gamma_k$ of X, with covering group Γ/Γ_k. Since $\Gamma/\Gamma_k \in \mathcal{G}$ for each $k \in \mathbb{N}$, by Theorem 3.3.6

$$b^p_{(2)}(\overline{X}, \Gamma) = \lim_{k\to\infty} b^p_{(2)}(X_k, \Gamma/\Gamma_k).$$

By assumption, each term on the right hand side is an integer, since the Atiyah conjecture holds for Γ/Γ_k. Since $\mathbb{Z}$ is discrete in $\mathbb{R}$, the same will be true for its limit, and this is exactly what we have to prove. □

As observed above, this translates to a statement about the integral group ring of Γ. To extend this result from the integral group ring to more general coefficient rings, which is interesting because of the algebraic consequences, we have to generalize the approximation results of Section 3.3 to algebraic approximating results for more general coefficient rings. In fact, we have the following result of [18].

3.4.22. Theorem. *Assume Γ is a discrete group with a nested sequence of normal subgroups $\Gamma \geq \Gamma_1 \geq \Gamma_2 \geq \cdots$ such that $\bigcap_{k\in\mathbb{N}} \Gamma/\Gamma_k = \{1\}$ and such that $\Gamma/\Gamma_k \in \mathcal{G}$ for each $k \in \mathbb{N}$.*

For example, all the groups Γ/Γ_k might be elementary amenable, e.g. finite.

Let $\overline{\mathbb{Q}}$ be the field of algebraic numbers (over $\mathbb{Q}$) in $\mathbb{C}$. Assume $A \in M(d \times d, \overline{\mathbb{Q}}\Gamma)$.

The projection $\Gamma \to \Gamma/\Gamma_k$ extends canonically to the group rings and to matrix rings over the group rings. Let $A_k \in M(d \times d, \overline{\mathbb{Q}}\Gamma/\Gamma_k)$ be the image of A under this induced homomorphism.

Then A acts on $(l^2\Gamma)^d$ and A_k acts on $(l^2\Gamma/\Gamma_k)^d$.

The following approximation result for the kernels of these operators holds:

$$\dim_\Gamma(\ker(A)) = \lim_{k\to\infty} \dim_{\Gamma/\Gamma_k}(\ker(A_k)).$$

In particular, if all the groups Γ/Γ_k are torsion-free and $\overline{\mathbb{Q}}[\Gamma/\Gamma_k]$ satisfies the algebraic Atiyah conjecture 3.4.5, then, using the same argument as above, $\overline{\mathbb{Q}}\Gamma$ also satisfies the Atiyah conjecture 3.4.5.

3.4.23. Remark. The approximation result 3.3.6 is a special case, since if Δ is a matrix representative for a combinatorial Laplacian for $\overline{X}$, then Δ_k, constructed as in Theorem 3.4.22, is a matrix representative for the corresponding combinatorial Laplacian of X_k.

The proof uses the fact that one is working with algebraic coefficients. So far, no generalization to $\mathbb{C}\Gamma$ has been obtained.

However, it is a well known fact that, if $\overline{\mathbb{Q}}\Gamma$ has no non-trivial zero divisors, then the same is true for $\mathbb{C}\Gamma$ (compare e.g. [18]). Therefore, from the point of view of the zero divisor conjecture 3.4.6, there is no need to generalize Theorem 3.4.22.

3.4.24. Remark. It is not hard to see that $\mathcal{D}$ is contained in $\mathcal{G}$. Therefore, Theorem 3.4.22 provides the last step for the proof of Theorem 3.4.12.

3.4.5.4 Atiyah conjecture for braid groups

3.4.25. Definition. Fix $n \in \mathbb{N}$. A *braid with n-strings* is an embedding

$$\phi\colon \{1, \dots, n\} \times [0,1] \to \mathbb{C} \times [0,1]$$

such that $\phi(p,0) = (p,0)$ and $\phi(p,1) \in \{1,\dots,n\} \times \{1\}$ for $p = 1,\dots,n$. Two braid are considered equal if they are isotopic where the isotopy fixes the top and the bottom.

Isotopy classes of braids from a group by stacking two braids together, the so called *Artin braid group* B_n. Note that the p-th string is not necessarily stacked on the p-th string, since the p-th string might lead from $(p,0)$ to $(\sigma(p),1)$ for some permutation σ of $\{1,\dots,n\}$. We have to account for this when we define the "stacked" map $\{1,\dots,n\} \times [0,1] \to \mathbb{C} \times [0,1]$.

The braid group B_n contains a normal subgroup P_n, the *pure braid group*, where we require $\phi(p,1) = (p,1)$ for each $p \in \{1,\dots,n\}$. The quotient B_n/P_n is the symmetric group S_n of permutations of $\{1,\dots,n\}$, where the image permutation is given as above.

In Example 3.4.11, we assert that all the braid groups belong to $\mathcal{D}$. Indeed, every braid group B_n has a nested sequence of normal subgroups $B_n \geq P_{1,n} \geq P_{2,n} \geq \cdots$ with $B_n/P_{k,n}$ torsion-free elementary amenable for each k and such that $\bigcap_{k\in\mathbb{Z}} P_{k,n} = \{1\}$. The corresponding result for the pure braid groups is proved in [22, Theorem 2.6]. However, to extend such a result from a subgroup of finite index to a bigger group is highly non-trivial and in general not possible. Indeed, for the full braid group it was conjectured for a while that it has no non-trivial torsion-free quotients at all, opposite to what we need. Using certain totally disconnected completions of the groups involved, and cohomology of these completions (Galois cohomology, which takes the topology of the completions into account) the above result is proved in [46]. Actually, it is proved there that every torsion-free finite extension of P_n has a sequence of subgroups as above, and therefore belongs to $\mathcal{D}$. This paper also contains generalizations, where the pure braid groups are replaced by other kinds of groups, still with the result that the property to belong to $\mathcal{D}$ passes to finite extensions (as long as they are torsion-free). In [47], it is shown how this applies to fundamental groups of certain complements of links in $\mathbb{R}^3$ (a link is an embedding of the disjoint union of finitely many circles).

The proof of the Baum-Connes conjecture for the full braid group [68] mentioned in Section 1.3.9 is based on the same results.

3.4.6 Atiyah conjecture for groups with torsion

The Atiyah conjecture has also been obtained for many groups Γ with torsion, as long as $Fin^{-1}(\Gamma)$ is a discrete subset of $\mathbb{R}$. We are not discussing them here because of lack of space and time, and because there is no zero divisor conjecture for groups with torsion. The ring $D_K\Gamma$ also exists for groups with torsion. It can not be a skew field but, under the assumption that $Fin^{-1}(\Gamma)$ is discrete, it often turns out to be a semi-simple Artinian ring. More details can be found e.g. in the original sources [49, 48, 70].

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