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Uniqueness Theorems for Variational Problems by the Method of Transformation Groups

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To Mirjam

Preface

A classical problem in the calculus of variations is the investigation of critical points of a C^1-functional $\mathcal{L} : V \to \mathbb{R}$ on a normed space V. Typical examples are $\mathcal{L}[u] = \int_\Omega L(x, u, \nabla u)\,dx$ with $\Omega \subset \mathbb{R}^n$ and V a space of admissible functions $u : \Omega \to \mathbb{R}^k$. A large variety of methods has been invented to obtain existence of critical points of $\mathcal{L}$. The present work addresses a different question:

> *Under what conditions on the Lagrangian L, the domain Ω and the set of admissible functions V does $\mathcal{L}$ have at most one critical point?*

The following sufficient condition for uniqueness is presented in this work: the functional $\mathcal{L}$ has at most one critical point u_0 if a differentiable one-parameter group $G = \{g_\epsilon\}_{\epsilon\in\mathbb{R}}$ of transformations $g_\epsilon : V \to V$ exists, which strictly reduces the values of $\mathcal{L}$, i.e. $\mathcal{L}[g_\epsilon u] < \mathcal{L}[u]$ for all $\epsilon > 0$ and all $u \in V \setminus \{u_0\}$. If G is not differentiable the uniqueness result is recovered under the extra assumption that the Lagrangian is a convex function of ∇u (ellipticity condition). This approach to uniqueness is called "the method of transformation groups".

The interest for uniqueness results in the calculus of variations comes from two sources:

1) In applications to physical problems uniqueness is often considered as supporting the validity of a model.
2) For semilinear boundary value problems like $\Delta u + \lambda u + |u|^{p-1}u = 0$ in Ω with $u = 0$ on $\partial\Omega$ uniqueness means that $u \equiv 0$ is the only solution. Conditions on Ω, p, λ ensuring uniqueness may be compared with those conditions guaranteeing the existence of nontrivial solutions. E.g., if Ω is bounded and $1 < p < \frac{n+2}{n-2}$, then nontrivial solutions exist for all λ. If, in turn, one can prove uniqueness for $p \geq \frac{n+2}{n-2}$ and certain λ and Ω, then the restriction on p made for existence is not only sufficient but also necessary.

A very important uniqueness theorem for semilinear problems was found in 1965 by S.I. Pohožaev [75]. If Ω is star-shaped with respect to the origin,

$p \geq \frac{n+2}{n-2}$ and $\lambda \leq 0$, then uniqueness of the trivial solution follows. In his proof Pohožaev tested the equation with $x \cdot \nabla u$ and u. The resulting integral identity admits only the zero-solution. A crucial role is played by the vector-field x. The motivation of the present work was to exhibit arguments within the calculus of variations which explain Pohožaev's result and, in particular, explain the role of the vector-field x.

Chapter 1 provides two examples illustrating the method of transformation groups in an elementary way.

In Chapter 2 we develop the general theory of uniqueness of critical points for abstract functionals $\mathcal{L} : V \to \mathbb{R}$ on a normed space V. The notion of a differentiable one-parameter transformation group $g_\epsilon : \operatorname{dom} g_\epsilon \subset V \to V$ is developed and the following fundamental uniqueness result is shown: if $\mathcal{L}[g_\epsilon u] < \mathcal{L}[u]$ for all $\epsilon > 0$ and all $u \in V \setminus \{u_0\}$ then u_0 is the only possible critical point of $\mathcal{L}$. We mention two applications: 1) a strictly convex functional has at most one critical point and 2) the first eigenvalue of a linear elliptic divergence-operator with zero Dirichlet or Neumann boundary conditions is simple.

As a generalization the concept of non-differentiable one-parameter transformation groups is developed in Chapter 3. Its interaction with first order variational functionals $\mathcal{L}[u] = \int_\Omega L(x, u, \nabla u)\, dx$ is studied. Under the extra assumption of rank-one convexity of L w.r.t. ∇u, a uniqueness result in the presence of energy reducing transformation groups is proved, which is a suitable generalization of the one in Chapter 2. In particular, Pohožaev's identity will emerge as two ways of computing the rate of change of the functional $\mathcal{L}$ under the action of the one-parameter transformation group.

In Chapter 4 the semilinear Dirichlet problem $\Delta u + \lambda u + |u|^{p-1}u = 0$ in Ω, $u = 0$ on $\partial\Omega$ is treated, where Ω is a domain on a Riemannian manifold M. An exponent $p^* \geq \frac{n+2}{n-2}$ is associated with Ω such that $u \equiv 0$ is the only solution provided $p \geq p^*$ and λ is sufficiently small. On more special manifolds better results can be achieved. If M possesses a one-parameter group $\{\Phi_t\}_{t\in\mathbb{R}}$ of conformal self-maps $\Phi_t : M \to M$, then a complete analogue of the Euclidean vector-field x is given by the so-called conformal vector-field $\boldsymbol{\xi}(x) := \frac{d}{dt}\Phi_t(x)|_{t=0}$. In the presence of conformal vector-fields one can show that the critical Sobolev exponent $\frac{n+2}{n-2}$ is the true barrier for existence/non-existence of non-trivial solutions. Generalizations of the semilinear Dirichlet problem to nonlinear Neumann boundary value problems are also considered.

In Chapter 5 and 6 we study variational problems in Euclidean $\mathbb{R}^n$. Examples of non-starshaped domains are given, for which Pohožaev's original result still holds. A number of boundary value problems for semilinear and quasilinear equations is studied. Uniqueness results for trivial/non-trivial solutions of supercritical problems as well as L^∞-bounds from below for solutions of subcritical problems are investigated. Uniqueness questions from the theory of elasticity (boundary displacement problem) and from geometry (surfaces of prescribed mean curvature) are treated as examples.

It is my great pleasure to thank friends, colleagues and co-authors, who helped me to achieve a better understanding of uniqueness questions in the calculus of variations. First among all is Catherine Bandle, who encouraged me to write this monograph, read the manuscript carefully and with great patience and suggested numerous improvements. I am indebted to Joachim von Below, Miro Chlebík, Marek Fila, Edward Fraenkel, Hubert Kalf, Bernd Kawohl, Moshe Markus, Joe McKenna, Peter Olver, Pavol Quittner, James Serrin, Michael Struwe, John Toland, Alfred Wagner and Hengui Zou for valuable discussions (some of them took place years ago), which laid the foundation for this work, and for pointing out references to the literature. My thanks also go to Springer Verlag for publishing this manuscript in their Lecture Note Series. Finally I express my admiration to S.I. Pohožaev for his mathematical work.

Basel,
February 2004

Wolfgang Reichel

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1

Introduction

We begin this study with two well known examples to illustrate our point of view to uniqueness questions in the calculus of variations. We look for conditions such that a Fréchet differentiable functional $\mathcal{L} : V \to \mathbb{R}$ defined on a normed space V possesses at most one critical point. Here $u \in V$ is called a critical point if

$$\frac{d}{dt}\mathcal{L}[u+tv]\big|_{t=0} = 0 \text{ for all } v \in V.$$

1.1 A convex functional

Probably the best known uniqueness result in the calculus of variations states that a Fréchet differentiable, strictly convex functional $\mathcal{L} : V \to \mathbb{R}$ defined on a normed space V possesses at most one critical point. Consider the following uniqueness proof: suppose u_0 is a critical point of $\mathcal{L}$. For a fixed element $u \in V$ we define a transformed element

$$\tilde{u} = e^{-\epsilon}(u-u_0) + u_0.$$

For each ϵ the operator

$$g_\epsilon : u \mapsto g_\epsilon u := \tilde{u}$$

maps V into itself. We want to compare $\mathcal{L}[g_\epsilon u]$ with $\mathcal{L}[u]$. The simplest way to do this is by differentiation with respect to ϵ. If one takes strict convexity of $\mathcal{L}$ into account then

$$\begin{aligned} \frac{d}{d\epsilon}\mathcal{L}[g_\epsilon u] &= \mathcal{L}'[g_\epsilon u]e^{-\epsilon}(u_0 - u) \\ &= (\mathcal{L}'[g_\epsilon u] - \mathcal{L}'[u_0])(u_0 - g_\epsilon u) \\ &< 0 \text{ unless } u \equiv u_0. \end{aligned} \tag{1.1}$$

If additionally u is a critical point of $\mathcal{L}$ then the rate of change can be computed by the chain rule as $\frac{d}{d\epsilon}\mathcal{L}[g_\epsilon u]|_{\epsilon=0} = \mathcal{L}'[u](u_0 - u) = 0$. By (1.1) this implies uniqueness $u \equiv u_0$.

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For $\epsilon > 0$ the transformation g_ϵ has the property $\mathcal{L}[g_\epsilon u] < \mathcal{L}[u]$ if $u \neq u_0$, i.e., it strictly reduces the energy. In analogy to variational symmetries (cf. Olver [71]), which leave the energy invariant, the transformation g_ϵ is called a *strict variational sub-symmetry w.r.t.* u_0. The presence of a strict variational sub-symmetry is responsible for the uniqueness of the critical point of $\mathcal{L}$.

1.2 A functional with supercritical growth

As a second example consider the functional $\mathcal{L}[u] = \int_\Omega \frac{1}{2}|\nabla u|^2 - \frac{1}{p+1}|u|^{p+1}\,dx$ for $p > 1$ defined on the space $V = C^1_0(\overline{\Omega})$ of C^1-functions u vanishing on $\partial\Omega$, where $\Omega \subset \mathbb{R}^n$, $n \geq 3$ is a bounded domain. Sufficiently smooth critical points of $\mathcal{L}$ are classical solutions of

$$\Delta u + |u|^{p-1}u = 0 \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega.$$

We assume that $p > \frac{n+2}{n-2}$ is strictly supercritical. Suppose also that the domain Ω is star-shaped with respect to $0 \in \Omega$. For $\epsilon > 0$ we define the following transformation of a fixed function u:

$$\tilde{u}(\tilde{x}) = e^{\frac{n-2}{2}\epsilon}u(e^\epsilon \tilde{x}) \text{ for } \tilde{x} \in e^{-\epsilon}\Omega = \{e^{-\epsilon}x : x \in \Omega\}$$

and extend $\tilde{u}$ outside $e^{-\epsilon}\Omega$ by zero. Due to the star-shapedness of Ω the function $\tilde{u}$ is a well-defined function with a fold (german: “Knick”) at $e^{-\epsilon}\partial\Omega = \{e^{-\epsilon}x : x \in \partial\Omega\}$. For each $\epsilon > 0$ the operator

$$g_\epsilon : u \mapsto g_\epsilon u := \tilde{u}.$$

is a well defined selfmap of the space $C^{0,1}_0(\overline{\Omega})$ of Lipschitz-functions vanishing on $\partial\Omega$. It will again be useful to write both $g_\epsilon u$ and $\tilde{u}(\tilde{x})$ for the transformed function. Notice that in contrast to our first example the transformation not only changes the dependent variable u but also the independent variable x.

As we will show later, strictly supercritical $p > \frac{n+2}{n-2}$ implies that $\mathcal{L}[g_\epsilon u]$ is strictly decreasing in $\epsilon > 0$, i.e.

$$\mathcal{L}[g_\epsilon u] < \mathcal{L}[u] \text{ for } \epsilon > 0 \text{ if } u \neq 0.$$

As in Section 1.1 such a transformation is called a *strict variational sub-symmetry w.r.t.* 0.

A heuristic argument for uniqueness

Before we give a rigorous uniqueness proof let consider a heuristic argument why the presence of a variational sub-symmetry is responsible for the absence of non-trivial critical points of $\mathcal{L}$. It is easy to verify that $u_0 = 0$ is a local minimum of $\mathcal{L}[u] = \int_\Omega \frac{1}{2}|\nabla u|^2 - \frac{1}{p+1}|u|^{p+1}\,dx$. Moreover, $\mathcal{L}$ cannot have any other local minimum $u_1 \neq 0$ since $g_\epsilon u_1$ produces for $\epsilon > 0$ a nearby function with strictly lower energy $\mathcal{L}[g_\epsilon u_1] < \mathcal{L}[u_1]$. We show that the same property excludes the existence of critical points of mountain-pass type: since $\mathcal{L}$ has a local minimum at $u_0 = 0$ and since $\mathcal{L}[t\phi] \to -\infty$ as $t \to \infty$ for any $\phi \neq 0$, one could expect a critical point of mountain-pass type, i.e. a critical point u with $\mathcal{L}[u] = c$ where the energy-level c is given by

$$c = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} \mathcal{L}[\gamma(t)]$$

and Γ is the set of all continuous paths $\gamma : [0,1] \to C_0^{0,1}(\overline{\Omega})$ with $\gamma(0) = 0, \gamma(1) = \bar{u}$ and $\bar{u}$ such that $\mathcal{L}[\bar{u}] < 0$. Let us assume that γ is a minimizing path (assuming the existence of such a minimizing path makes the argument heuristic). We define a deformed path $g_\epsilon \circ \gamma$, which now connects 0 to $g_\epsilon \bar{u}$. By connecting $g_\epsilon \bar{u}$ linearly to $\bar{u}$ and composing both paths we obtain a new path $\gamma_\epsilon \in \Gamma$, see Figure 1.1. By choosing $\epsilon > 0$ small enough we may achieve that $\mathcal{L}[\gamma_\epsilon(t)] < 0$ on the linear "tail". Therefore, by the strict sub-symmetry property of g_ϵ we find

$$\max_{t \in [0,1]} \mathcal{L}[\gamma_\epsilon(t)] = \max_{t \in [0,1]} \mathcal{L}[g_\epsilon \circ \gamma(t)] < \max_{t \in [0,1]} \mathcal{L}[\gamma(t)]$$

which contradicts the optimality of the path γ. Therefore no mountain-pass type critical point exists, and similar arguments show heuristically that there are no other min-max type critical points of $\mathcal{L}$.

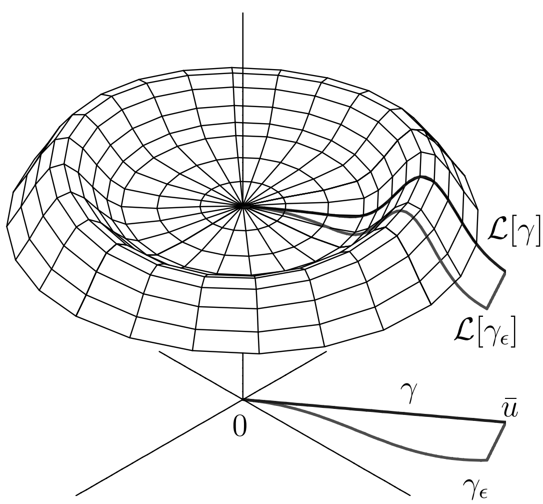

Fig. 1.1. Applying the sub-symmetry to a minimizing path

A rigorous uniqueness proof

To evaluate the functional on $g_\epsilon u$ it is sufficient to integrate over $e^{-\epsilon}\Omega$, since the support of $g_\epsilon u$ is contained in $e^{-\epsilon}\Omega$. By the assumption of strictly supercritical growth one finds

$$\begin{aligned}
\mathcal{L}[g_\epsilon u] &= \int_{e^{-\epsilon}\Omega} \frac{1}{2}|\nabla\tilde{u}|^2 - \frac{1}{p+1}|\tilde{u}|^{p+1}\,d\tilde{x} \\
&= \int_{e^{-\epsilon}\Omega} \frac{1}{2}e^{n\epsilon}|\nabla u|^2(e^\epsilon\tilde{x}) - \frac{1}{p+1}e^{\frac{(p+1)(n-2)}{2}\epsilon}|u(e^\epsilon\tilde{x})|^{p+1}\,d\tilde{x} \quad (1.2)\\
&= \int_\Omega \frac{1}{2}|\nabla u|^2(x) - \frac{1}{p+1}e^{\frac{(p+1)(n-2)-2n}{2}\epsilon}|u(x)|^{p+1}\,dx \quad (1.3)\\
&< \mathcal{L}[u] \text{ if } u \not\equiv 0 \text{ and } \epsilon > 0.
\end{aligned}$$

We will calculate the rate of change of $\mathcal{L}$ under the action of the sub-symmetry. It follows from (1.3) that

$$\frac{d}{d\epsilon}\mathcal{L}[g_\epsilon u]|_{\epsilon=0} = \int_\Omega -\frac{(p+1)(n-2)-2n}{2}|u(x)|^{p+1}\,dx < 0 \text{ if } u \not\equiv 0. \quad (1.4)$$

On the other hand we can directly differentiate (1.2) by using the formula $\frac{d}{d\epsilon}|_{\epsilon=0}\int_{e^{-\epsilon}\Omega} h(x)\,dx = -\int_{\partial\Omega}(x\cdot\nu)h(x)\,d\sigma$. This results in

$$\begin{aligned}
\frac{d}{d\epsilon}\mathcal{L}[g_\epsilon u]|_{\epsilon=0} &= \int_\Omega \frac{n}{2}|\nabla u|^2 + x^T D^2 u\nabla u - \frac{n-2}{2}|u|^{p+1}\,dx \\
&\quad - \int_\Omega |u|^{p-1}u\nabla u\cdot x\,dx - \int_{\partial\Omega}(x\cdot\nu)(\frac{1}{2}|\nabla u|^2 - \frac{1}{p+1}|u|^{p+1})\,d\sigma \\
&= \int_\Omega (\frac{2-n}{2}u - x\cdot\nabla u)|u|^{p-1}u - \nabla(\frac{2-n}{2}u - x\cdot\nabla u)\cdot\nabla u\,dx \\
&\quad -\frac{1}{2}\int_{\partial\Omega}(x\cdot\nu)|\nabla u|^2\,d\sigma.
\end{aligned}$$

If we introduce the Euler-Lagrange operator $\mathcal{E}[u] = \Delta u + |u|^{p-1}u$ and assume that u is a C^2-function we can rewrite the volume integral in the previous formula:

$$\begin{aligned}
\frac{d}{d\epsilon}\mathcal{L}[g_\epsilon u]|_{\epsilon=0} &= \int_\Omega (\frac{2-n}{2}u - x\cdot\nabla u)\mathcal{E}[u]\,dx \\
&\quad - \int_\Omega \operatorname{div}\left(\frac{2-n}{2}u\nabla u - (x\cdot\nabla u)\nabla u\right)dx - \frac{1}{2}\int_{\partial\Omega}(x\cdot\nu)|\nabla u|^2\,d\sigma.
\end{aligned}$$

Using $u = 0$ on $\partial\Omega$ we get $(x\cdot\nabla u)\nabla u = (x\cdot\nu)|\nabla u|^2$, which implies

$$\frac{d}{d\epsilon}\mathcal{L}[g_\epsilon u]|_{\epsilon=0} = \int_\Omega (\frac{2-n}{2}u - x\cdot\nabla u)\mathcal{E}[u]\,dx + \frac{1}{2}\int_{\partial\Omega}(x\cdot\nu)|\nabla u|^2\,d\sigma. \quad (1.5)$$

If u is a critical point which satisfies $\mathcal{E}[u] = 0$ then (1.5) and the star-shapedness of Ω imply $\frac{d}{d\epsilon}\mathcal{L}[g_\epsilon u]|_{\epsilon=0} \geq 0$. Together with (1.4) this implies $u \equiv 0$ and finishes our uniqueness proof.

The two ways of calculating $\frac{d}{d\epsilon}\mathcal{L}[g_\epsilon u]|_{\epsilon=0}$ give the equality of (1.4) and (1.5). This identity is called Pohožaev's identity, since it was first discovered by S.I. Pohožaev [75] in 1965 using different means. In Section 5.1 we show how the proof extends to the exactly critical case $p = \frac{n+2}{n-2}$.

Remark 1.1. (i) In the first example a variational sub-symmetry was obtained by a transformation of the dependent variable only. In the second example both the dependent and the independent variable were transformed.
(ii) Let us point out an important difference in the two examples. In our first example the transformation $g_\epsilon : u \mapsto g_\epsilon u$ was differentiable in ϵ in the sense $\frac{d}{d\epsilon}g_\epsilon u = e^{-\epsilon}(u_0 - u) \in V$. Therefore $\frac{d}{d\epsilon}\mathcal{L}[g_\epsilon u]|_{\epsilon=0} = \mathcal{L}'[u](u_0 - u) = 0$ by the chain rule for any critical point u. However, for the support-shrinking transformations of our second example we find from the definition of $g_\epsilon u$

$$\frac{d}{d\epsilon}g_\epsilon(\tilde{x})\Big|_{\epsilon=0} = \frac{n-2}{2}u(x) + x \cdot \nabla u(x).$$

Since $\frac{n-2}{2}u + x \cdot \nabla u \notin C_0^{0,1}(\overline{\Omega})$ unless $u \equiv 0$, we obtain that $g_\epsilon u$ is in general not differentiable in the space $C_0^{0,1}(\overline{\Omega})$. Thus $\frac{d}{d\epsilon}\mathcal{L}[g_\epsilon u]|_{\epsilon=0}$ does not vanish by application of the chain rule.

1.3 Construction of the transformations

The transformations $u \mapsto g_\epsilon u$ in the previous two examples can be constructed from differential equations.

First example. Consider the differential equation

$$\frac{dU}{d\epsilon} = u_0 - U, \qquad U(0) = u. \tag{1.6}$$

Solutions are given by $g_\epsilon u := e^{-\epsilon}(u - u_0) + u_0$.

Second example. Consider the system

$$\frac{dX}{d\epsilon} = -X, \qquad \frac{dU}{d\epsilon} = \frac{n-2}{2}U, \qquad X(0) = x, U(0) = u. \tag{1.7}$$

Solutions are $(e^{-\epsilon}x, e^{\frac{n-2}{2}\epsilon}u)$. If the initial state (x, u) lies on the graph of a function u, i.e. $u = u(x)$ then after time ϵ the dynamical system has evolved to a new point $(\tilde{x}, \tilde{u})$, which lies on the graph of a new function $\tilde{u}(\tilde{x})$ given by

$$\tilde{u}(\tilde{x}) = e^{\frac{n-2}{2}\epsilon}u(e^\epsilon \tilde{x}).$$

Both systems (1.6), (1.7) give rise to the transformations $g_\epsilon : u \mapsto \tilde{u}$ sending a initial element $u \in V$ to a new element $g_\epsilon u \in V$. These transformations have the group property $g_{\epsilon_1+\epsilon_2} = g_{\epsilon_1} \circ g_{\epsilon_2}$ and are called *one-parameter transformation groups* .

We summarize the main features of these examples:

(i) A transformation group acts as a strict variational sub-symmetry on $\mathcal{L}[u]$, i.e. $\frac{d}{d\epsilon}\mathcal{L}[g_\epsilon u]|_{\epsilon=0} < 0$ for all $u \neq u_0$ by the convexity assumption in the first example and by the supercritical growth in the second example.

(ii) The rate of change can also be computed in terms of the Euler-Lagrange operator. If u is a critical point then in the first example $\frac{d}{d\epsilon}\mathcal{L}[g_\epsilon u]|_{\epsilon=0} = 0$ by the differentiability of the group orbit g_ϵ. In the second example the star-shapedness of Ω makes $\frac{d}{d\epsilon}\mathcal{L}[g_\epsilon u]|_{\epsilon=0} \geq 0$ for any critical point. In both cases (i) and (ii) imply uniqueness of the critical point of $\mathcal{L}$.

(iii) The variational sub-symmetries are generated by differential equations.

(iv) In both examples Lipschitz functions supported in Ω are mapped to other Lipschitz functions supported in Ω. In the first example this is trivial; in the second example this is simply expressed by $x \cdot \nu \geq 0$ on $\partial\Omega$ and by the restriction to $\epsilon \geq 0$.

Remark 1.2. In both our examples the orbits $\{g_\epsilon u\}_{\epsilon\geq 0}$ are non-compact subsets of the underlying function space. In contrast, if in the second example Ω is an annulus then the group of rotations $\tilde{u}(\tilde{x}) = u(R_\epsilon \tilde{x})$ with a parameterized rotation matrix R_ϵ acts as an exact *variational symmetry* of the functional $\mathcal{L}[u]$. Now the group orbits are compact. This rotation group is acting in favor of non-trivial critical points rather than preventing them.

The method of transformation groups

In Chapter 2 we develop a general theory of differentiable transformation groups acting monotonically on C^1-functionals $\mathcal{L} : V \to \mathbb{R}$ in the sense that $\mathcal{L}[g_\epsilon u] \leq \mathcal{L}[u]$ for $\epsilon \geq 0$. Various examples generalizing the basic uniqueness result for strictly convex functionals are given including uniqueness of saddle points, uniqueness of positive solutions to sublinear problems and simplicity of first eigenvalues.

Chapter 3 contains the general theory for non-differentiable transformation groups. Here we restrict attention to first-order functionals $\mathcal{L}[u] = \int_\Omega L(x, u, \nabla u)\, dx$. The main uniqueness result is proved under the assumption that the transformation group acts monotonically in the sense that $\mathcal{L}[g_\epsilon u] \leq \mathcal{L}[u]$ for $\epsilon \geq 0$. We will give an infinitesimal criterion for such groups, which allows a computationally easy verification. Under suitable structural assumptions on the functional and geometric assumptions on the underlying domain Ω uniqueness of critical points of $\mathcal{L}$ will follow.

The standard theory for uniqueness of critical points relies on testing the Euler-Lagrange equation with suitable test-functions as in Pohožaev [75] or on an ad-hoc divergence-identity as in Pucci, Serrin [77]. Both lead to the well known Pohožaev-identity. In contrast, our approach remains as closely as possible within the calculus of variations. All our uniqueness results follow via the method of transformation groups, i.e. the calculation of the rate of change of a functional $\mathcal{L}$ under the action of a transformation group. Pohožaev's identity itself emerges as a side-result.

In Chapters 4, 5, 6 the uniqueness results of Chapter 3 are applied to many specific problems including *semilinear and quasilinear boundary value problems on bounded domains on Riemannian manifolds, some special aspects of harmonic maps between Riemannian manifolds, boundary displacement problems in nonlinear elasticity and a non-existence result for parametric closed surfaces of prescribed mean curvature.* In a number of cases the transformation groups and their generating differential equations allow some geometric insight into old and new results. A set of applications is centered around semilinear Dirichlet problems

$$\Delta u + f(x,u) = 0 \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega,$$

where Ω is a bounded subdomain of a Riemannian manifold M. A second set of applications is developed around the nonlinear Neumann boundary value problem

$$\Delta u + f(x,u) = 0 \text{ in } \Omega, \quad u = 0 \text{ on } \Gamma_D, \quad \partial_\nu u - g(x,u) = 0 \text{ on } \Gamma_N,$$

where $\partial\Omega = \overline{\Gamma}_D \cup \overline{\Gamma}_N$ is decomposed into two parts. The results depend on the structure of the nonlinearities $f(x,u), g(x,u)$ and the amount of symmetry of the underlying manifold M and the domains Ω.

Every work has its temporal and spatial limitations. In the selection of the presented material the following areas are not considered: *higher order variational problems, semilinear problems on manifolds without boundary and on unbounded domains.* Uniqueness results for all three problems may be given by the method of transformation groups. Some notes on these problems are given in Section 5.6.

Convention on monotonicity

A function $\varphi(\tau)$ of a real variable is called "increasing" in τ if $\varphi(\tau_1) \le \varphi(\tau_2)$ for all $\tau_1 < \tau_2$ and "strictly increasing" if $\varphi(\tau_1) < \varphi(\tau_2)$ for all $\tau_1 < \tau_2$. Similarly the words "decreasing" and "strictly decreasing" are used.

2

Uniqueness of critical points (I)

2.1 One-parameter transformation groups

A one-parameter transformation group on a normed space V is a family of maps $g_\epsilon : \operatorname{dom} g_\epsilon \subset V \to V$ which obey the group laws

$$\text{(a)} \quad g_{\epsilon_1} \circ g_{\epsilon_2} = g_{\epsilon_1+\epsilon_2}, \qquad \text{(b)} \quad g_0 = \mathrm{Id}, \qquad \text{(c)} \quad g_{-\epsilon} \circ g_\epsilon = \mathrm{Id}$$

on their respective domain of definition. For general references to one-parameter transformation groups we refer to Olver [71]. The precise definition using the map $G(\epsilon, u) := g_\epsilon u$ is the following; see also Fig. 2.1:

Definition 2.1. *Let V be a normed vector space. A one-parameter transformation group on V is given by an open set $\mathcal{W} \subset \mathbb{R} \times V$ and a smooth map $G : \mathcal{W} \to V$ with the following properties:*

(a) if $(\epsilon_1, u), (\epsilon_2, G(\epsilon_1, u)), (\epsilon_1 + \epsilon_2, u) \in \mathcal{W}$ then

$$G(\epsilon_2, G(\epsilon_1, u)) = G(\epsilon_1 + \epsilon_2, u),$$

(b) $(0, u) \in \mathcal{W}$ for all $u \in V$ and $G(0, u) = u$,
(c) if $(\epsilon, u) \in \mathcal{W}$ then $(-\epsilon, u) \in \mathcal{W}$ and

$$G(-\epsilon, G(\epsilon, u)) = u.$$

It is convenient to write $\operatorname{dom} g_\epsilon := \{u \in V : (\epsilon, u) \in \mathcal{W}\}$ and to refer to the group $G = \{g_\epsilon\}_{\epsilon\in\mathbb{R}}$ as the collection of the group-elements g_ϵ.

Definition 2.2. *A one-parameter transformation group $G = \{g_\epsilon\}_{\epsilon\in\mathbb{R}}$ on a normed vector space V is called differentiable if $\frac{d}{d\epsilon} g_\epsilon u\big|_{\epsilon=0} \in V$ for all $u \in V$.*

Examples of differentiable groups arise from ordinary differential equations

$$\frac{dU}{d\epsilon} = \boldsymbol{\phi}(U), \tag{2.1}$$

W. Reichel: LNM 1841, pp. 9–26, 2004.

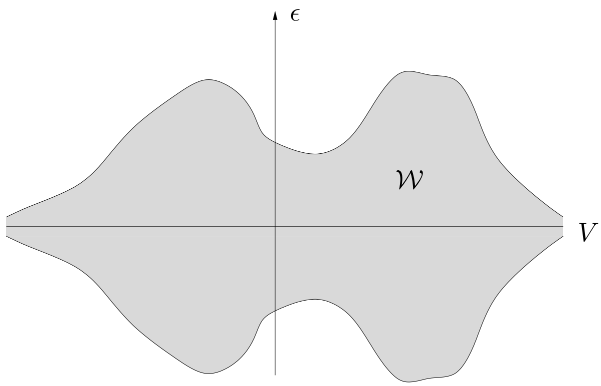

Fig. 2.1. Domain of definiton of G

if one assumes that the map $\boldsymbol{\phi} : V \to V$ is locally Lipschitz continuous (for convenience assume that $\boldsymbol{\phi}$ is continuously differentiable). We denote by $g_\epsilon u$ the unique local solution of (2.1) at time ϵ with initial condition $U(0) = u \in V$. We assume that $g_\epsilon u$ is maximally extended in time. The map g_ϵ is called the *flow-map at time* ϵ of the *flow* given by (2.1). By continuous dependence on initial conditions the family $G = \{g_\epsilon\}_{\epsilon\in\mathbb{R}}$ forms a one-parameter transformation group. The function $\boldsymbol{\phi}$ is called the *infinitesimal generator* of the group G.

We assume in this chapter that all one-parameter transformation groups G on V are differentiable and given through (2.1).

2.2 Variational sub-symmetries and uniqueness of critical points

A one-parameter transformation group, which leaves the values of a functional $\mathcal{L} : V \to \mathbb{R}$ invariant, is called a *variational symmetry.* If the values of $\mathcal{L}$ are reduced, we speak of a *variational sub-symmetry.* For our purpose of finding uniqueness of critical points of functionals, the notion of a variational sub-symmetry is most important. Precise definitions are given next.

Definition 2.3 (Variational symmetry/sub-symmetry). *Let $\mathcal{L} : V \to \mathbb{R}$ be a functional on a normed space V. Consider a one-parameter transformation group G on V.*

(i) G is called a variational symmetry if for all $(\epsilon, u) \in \mathcal{W}$

$$\mathcal{L}[g_\epsilon u] = \mathcal{L}[u]. \tag{2.2}$$

(ii)G is called a variational sub-symmetry if for all $(\epsilon, u) \in \mathcal{W}$ with $\epsilon \geq 0$

$$\mathcal{L}[g_\epsilon u] \leq \mathcal{L}[u]. \tag{2.3}$$

The restriction to $\epsilon \geq 0$ in (ii) cannot be avoided, since clearly for $\epsilon \leq 0$ the group elements will increase the values of $\mathcal{L}$.

Proposition 2.4. *Let $\mathcal{L} : V \to \mathbb{R}$ be a C^1-functional and let G be a one-parameter transformation group with infinitesimal generator $\boldsymbol{\phi}$.*

(i) G is a variational symmetry for $\mathcal{L}$ if and only if

$$\mathcal{L}'[u]\boldsymbol{\phi}(u) = 0 \tag{2.4}$$

holds for every $u \in V$.

(ii)G is a variational sub-symmetry for $\mathcal{L}$ if and only if

$$\mathcal{L}'[u]\boldsymbol{\phi}(u) \leq 0 \tag{2.5}$$

holds for every $u \in V$.

Proof. We only show (ii), since (i) follows by applying (ii) to $\mathcal{L}$ and $-\mathcal{L}$. If G is a variational sub-symmetry then (2.5) follows from (2.3) by differentiation. Reversely let us assume (2.5). We show that $\frac{d}{d\epsilon}\mathcal{L}[g_\epsilon u] \leq 0$ for all ϵ. To do this notice that $\frac{d}{d\epsilon}\mathcal{L}[g_\epsilon u] = \frac{d}{dt}\mathcal{L}[g_t \circ g_\epsilon u]|_{t=0}$, and since $g_\epsilon u$ is differentiable w.r.t. ϵ we get

$$\frac{d}{d\epsilon}\mathcal{L}[g_\epsilon u] = \frac{d}{dt}\mathcal{L}[g_t \circ g_\epsilon u]|_{t=0} = \mathcal{L}'[g_\epsilon u]\boldsymbol{\phi}(g_\epsilon u),$$

which is non-positive by our hypotheses. □

Definition 2.5. *Let $\mathcal{L} : V \to \mathbb{R}$ be a C^1-functional on a normed space V and let G be a one-parameter transformation group with infinitesimal generator $\boldsymbol{\phi}$. The group G is called a strict variational sub-symmetry w.r.t. $u_0 \in V$ provided*

$$\mathcal{L}'[u]\boldsymbol{\phi}(u) < 0 \textit{ for all } u \in V \setminus \{u_0\}.$$

Now we can state the main result of this chapter.

Theorem 2.6. *Let $\mathcal{L} : V \to \mathbb{R}$ be a C^1-functional and let G be a one-parameter transformation group defined on V. If G is a strict variational sub-symmetry w.r.t. u_0 then the only possible critical point of $\mathcal{L}$ is u_0.*

Proof. Assume $u \in V$ is a critical point. Then $\mathcal{L}'[u] = 0$. The definition of a strict variational sub-symmetry implies $u = u_0$. □

Remark 2.7. While the idea of a variational symmetry is due to Sophus Lie, it was Emmy Noether's breakthrough paper [70] on conservation laws induced by variational symmetries which showed the importance of the concept. E.g., consider the Lagrange-functional $\int_0^T L(t, q, \dot{q})\, dt$ of a particle $q : [0, T] \to \mathbb{R}^3$. If L is time-independent then $(g_\epsilon q)(t) = q(t+\epsilon)$ is a variational symmetry, which implies conservation of energy $E = L - \sum_{i=1}^3 \dot{q}^i \frac{\partial L}{\partial \dot{q}^i}$. Likewise, if for example L is independent of the x^1-direction in space then $g_\epsilon q(t) = q(t) + \epsilon(1, 0, 0)$ is a variational symmetry which generates the conservation of momentum $P_1 = \frac{\partial L}{\partial \dot{q}^1}$ in direction x^1. And similarly, if L is invariant under a rotation then angular-momentum is the conserved quantity.

2.3 Uniqueness results for critical points of constrained functionals

A similar uniqueness result holds for critical points of functionals $\mathcal{L} : V \to \mathbb{R}$ subject to a functional constraint $\mathcal{N}[u] = 0$. We assume the non-degeneracy hypotheses $\mathcal{N}'[u] \neq 0$ for all critical points u of $\mathcal{L}$ subject to $\mathcal{N}[u] = 0$.

Theorem 2.8. *Let $\mathcal{L}, \mathcal{N} : V \to \mathbb{R}$ be C^1-functionals and let G be a one-parameter transformation group defined on V. If G is a variational symmetry for $\mathcal{N}$ and a strict variational sub-symmetry for $\mathcal{L}$ w.r.t. $u_0 \in V$ then the only possible critical point of $\mathcal{L}$ w.r.t. the constraint $\mathcal{N}[u] = 0$ is u_0.*

Proof. Assume $u \in V$ is a critical point on $\mathcal{L}$ subject to $\mathcal{N}[u] = 0$. By the non-degeneracy assumption there exists a Lagrange-multiplier $\lambda \in \mathbb{R}$ such that $\mathcal{L}'[u] + \lambda\mathcal{N}'[u] = 0$. Moreover $\mathcal{N}'[u]\boldsymbol{\phi}(u) = 0$ by Proposition 2.4. Hence

$$\frac{d}{d\epsilon}\mathcal{L}[g_\epsilon u]\Big|_{\epsilon=0} = \mathcal{L}'[u]\boldsymbol{\phi}(u) = (\mathcal{L}'[u] + \lambda\mathcal{N}'[u])\boldsymbol{\phi}(u) = 0.$$

One the other hand, since G is a strict variational sub-symmetry for $\mathcal{L}$ w.r.t. u_0, the previous expression is strictly negative if $u \neq u_0$. Hence necessarily $u = u_0$. □

2.4 First order variational integrals

The present theory will to a large part be applied to first order variational integrals $\mathcal{L}[u] = \int_\Omega L(x, u, \nabla u)\, dx$ for real- or vectorvalued functions $u = (u^1, \ldots, u^k) : \Omega \to \mathbb{R}^k$ on a bounded domain $\Omega \subset \mathbb{R}^n$. We assume that the Lagrangian $L : \mathbb{R}^n \times \mathbb{R}^k \times \mathbb{R}^{nk} \to \mathbb{R}$ is continuous and that $L(x, u, \mathbf{p})$ is continuously differentiable w.r.t. $(u, \mathbf{p}) = (u^1, \ldots, u^k, \mathbf{p}^1, \ldots, \mathbf{p}^k)$.

Theorem 2.9 (Rate of change formula). *Let $G = \{g_\epsilon\}_{\epsilon\in\mathbb{R}}$ be a one-parameter transformation group on $V = C^1(\overline{\Omega})^k$ with infinitesimal generator $\boldsymbol{\phi} : \mathbb{R}^k \to \mathbb{R}^k$. Define the formal differential operator*

$$\mathbf{w}^{(1)} = \sum_{\alpha=1}^{k} \phi^\alpha(u)\partial_{u^\alpha} + \sum_{\alpha=1}^{k} \nabla\phi^\alpha(u) \cdot \nabla_{\mathbf{p}^\alpha}.$$

Then the rate of change of $\mathcal{L}$ under the action of G is given by

$$\frac{d}{d\epsilon}\mathcal{L}[g_\epsilon u]\Big|_{\epsilon=0} = \int_\Omega \mathbf{w}^{(1)} L(x, u, \nabla u)\, dx.$$

Proof. The result is evident since $\mathcal{L}'[u]\boldsymbol{\phi}(u) = \int_\Omega \mathbf{w}^{(1)} L(x, u, \nabla u)\, dx$. □

The rate of change formula will be generalized to non-differentiable transformation groups in Chapter 3, Theorem 3.13.

Remark 2.10. If $L = L(x, u)$ only depends on x and u then the rate of change is given by $\frac{d}{d\epsilon}\mathcal{L}[g_\epsilon u]|_{\epsilon=0} = \int_\Omega \mathbf{w}L(x,u)\,dx$ with $\mathbf{w} = \sum_{\alpha=1}^k \phi^\alpha(u)\partial_{u^\alpha}$. The operator $\mathbf{w}^{(1)}$ is called the *prolongation* of $\mathbf{w}$.

2.5 Classical uniqueness results

2.5.1 Convex functionals

In the first example of Section 1.1 we have already seen the well known result that a strictly convex functional $\mathcal{L}$ has at most one critical point.

Example 2.11. Let $\mathcal{L}[u] = \int_\Omega L(x, u, \nabla u)\,dx$ with a first order Lagrangian $L : \mathbb{R}^n \times \mathbb{R}^k \times \mathbb{R}^{nk} \to \mathbb{R}$ for vector-valued functions $u : \overline{\Omega} \to \mathbb{R}^k$ on a bounded domain $\Omega \subset \mathbb{R}^n$. Consider $\mathcal{L}$ on the normed space $V = C_0^1(\overline{\Omega})^l \times C^1(\overline{\Omega})^{k-l}$ for $l \in \{0, 1, \ldots, k\}$. The following is evident: if for fixed $x \in \mathbb{R}^n$ the Lagrangian $L(x, u, \mathbf{p})$ is continuously differentiable and strictly convex in $(u^1, \ldots, u^k, \mathbf{p}^1 \ldots, \mathbf{p}^k) \in \mathbb{R}^k \times \mathbb{R}^{nk}$ then $\mathcal{L}$ has a most one critical point $u_0 \in V$.

E.g., consider the case $k = 1$. Let $F(s)$ be a continuously differentiable function. Define the functional $\mathcal{L}[u] = \int_\Omega \frac{1}{2}|\nabla u|^2 - F(u)\,dx$ on the space $C_0^1(\overline{\Omega})$ ($l = 1$) or $C^1(\overline{\Omega})$ ($l = 0$). Critical points are weak solutions of

$$\Delta u + F'(u) = 0 \text{ in } \Omega$$

with either $u = 0$ on $\partial\Omega$ or $\frac{\partial u}{\partial \nu} = 0$ on $\partial\Omega$. If $F(s)$ is concave in $s \in \mathbb{R}$ then $\mathcal{L}$ has a unique critical point.

Example 2.12. Let H be a Hilbert space and $\mathcal{L}[u] = \frac{1}{2}\|u\|^2 + C[u] + \lambda D[u]$ with C^1-functionals $C, D : H \to \mathbb{R}$. Suppose that C is convex and $D'[u] : H \to H$ globally Lipschitz-continuous w.r.t. u with Lipschitz-constant $\operatorname{Lip} D' = \sup_{v \neq w} \frac{\|D'[v] - D'[w]\|}{\|v - w\|}$. Then $\mathcal{L}$ is strictly convex for $|\lambda| < 1/\operatorname{Lip} D'$.

Example 2.13. Suppose $F'(t) = f_1(t) + \lambda t$ with f_1 decreasing and $\lambda < \lambda_1$, the first Dirichlet-eigenvalue of $-\Delta$ on Ω. Then the functional $\mathcal{L}[u] = \int_\Omega \frac{1}{2}|\nabla u|^2 - F(u)\,dx$ has at most one critical point $u_0 \in W_0^{1,2}(\Omega)$. With the help of Poincaré's inequality the functional $D[u] = \frac{1}{2}\int_\Omega |u|^2\,dx$ has a derivative D' with Lipschitz-constant $\operatorname{Lip} D' = 1/\lambda_1$ w.r.t. the norm $\|u\| = (\int_\Omega |\nabla u|^2\,dx)^{1/2}$. Hence the previous example applies.

Example 2.14 (Contraction mapping principle). Let H be Hilbert-space and $\mathcal{L}[u] = \frac{1}{2}\|u\|^2 - K[u]$ with $\operatorname{Lip} K' < 1$. Then $\mathcal{L}$ has at most one critical point by Example 2.12. Since the Euler-Lagrange equation is $u - K'[u] = 0$ uniqueness also follows from the contraction mapping principle.

2.5.2 Uniqueness of a saddle point

In the previous examples the unique critical point of a functional was the global minimizer. One might therefore get the impression that the concept of variational sub-symmetries will only work under circumstances where a unique global minimizer exists. The following examples show that it works equally well to show uniqueness of saddle points. For illustration we begin with a simplified example.

Example 2.15. Consider $\mathcal{L} : \mathbb{R}^2 \to \mathbb{R}$ given by $\mathcal{L}[x, y] = -(x-x_0)^2 + (y-y_0)^2$, where $(x_0, y_0) \in \mathbb{R}^2$ is fixed. Clearly $\mathcal{L}$ has the unique critical point (x_0, y_0), which is a saddle point. The one-parameter group

$$g_\epsilon(x, y) := \big(e^\epsilon(x - x_0), e^{-\epsilon}(y - y_0)\big) + (x_0, y_0)$$

is a strict variational sub-symmetry w.r.t. (x_0, y_0). Indeed,

$$\frac{d}{d\epsilon}\mathcal{L}[x, y]\Big|_{\epsilon=0} = -2\big((x - x_0)^2 + (y - y_0)^2\big) < 0 \text{ unless } (x, y) = (x_0, y_0).$$

The main feature of this example is that for $\epsilon > 0$ the group G enhances the direction $x - x_0$ where $\mathcal{L}$ is decreasing and damps the direction $y - y_0$ where $\mathcal{L}$ is increasing. The following is an infinite dimensional analogue of Example 2.15. Consider the boundary value problem

$$\Delta u + f(x, u) = 0 \text{ in } \Omega, \qquad u = 0 \text{ on } \partial\Omega \tag{2.6}$$

on a bounded domain $\Omega \subset \mathbb{R}^n$ with a Carathéodory-function $f : \Omega \times \mathbb{R} \to \mathbb{R}$, which means that $f(x, s)$ is measurable in x and continuous in s.

Theorem 2.16 (Dolph [23]). *Counting multiplicities let* $0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \ldots$ *be the Dirichlet eigenvalues of* $-\Delta$ *on* Ω. *Let* $f : \Omega \times \mathbb{R} \to \mathbb{R}$ *and suppose that there exists an eigenvalue-index* i_0 *such that*

$$\lambda_{i_0} < \sup_{x\in\Omega, s\neq t} \frac{f(x, s) - f(x, t)}{s - t} < \lambda_{i_0+1}. \tag{2.7}$$

Then (2.6) *has a unique solution, which is a saddle point of* $\mathcal{L}[u] = \int_\Omega \frac{1}{2}|\nabla u|^2 - F(x, u)\,dx$, *where* $F(x, s) = \int_0^s f(x, t)\,dt$.

Remark 2.17. This result is a generalization of Example 2.13.

Usually this is proved by a contraction-mapping argument, see Lazer, McKenna [60]. With the method of transformation groups we will show how uniqueness in Dolph's result is a special case of the following abstract result.

For the rest of this section we assume that $(H, \langle\cdot,\cdot\rangle)$ is a real Hilbert-space and $A : \mathcal{D} \subset H \to H$ a selfadjoint, strictly positive definite, densely defined linear operator. Let H_A be the completion of $\mathcal{D}$ w.r.t. norm $\|\cdot\|_A$ generated by the inner product $\langle u, v\rangle_A := \langle Au, v\rangle$.

Definition 2.18. *Let $f \in H$. An element $u \in H_A$ is called a weak solution of*

$$Au = f$$

provided $\langle u, v\rangle_A = \langle f, v\rangle$ for all $v \in H_A$.

This is equivalent to saying that u is a critical point of the functional $\mathcal{J} : H_A \to \mathbb{R}$ with $\mathcal{J}[u] := \frac{1}{2}\langle u, u\rangle_A - \langle f, u\rangle$.

Theorem 2.19. *Assume that the selfadjoint linear operator $A : \mathcal{D} \subset H \to H$ has discrete spectrum σ consisting of the eigenvalues $0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \dots$ including multiplicities. Let $K : H_A \to \mathbb{R}$ be a C^2-functional and for $u \in H_A$ let $k(u) \in H$ be the Riesz-representation of $K'(u)$ w.r.t. $\langle\cdot,\cdot\rangle$. The equation*

$$Au = k(u) \tag{2.8}$$

has at most one weak solution provided there exists an index $i_0 \in \mathbb{N}$ such that

$$\operatorname{Lip}\big(k - \frac{\lambda_{i_0} + \lambda_{i_0+1}}{2}\operatorname{Id}\big) < \frac{\lambda_{i_0+1} - \lambda_{i_0}}{2},$$

where the Lipschitz constant is computed w.r.t. the norm $\|\cdot\|$ of H.

Proof. The proof consists in the construction of a suitable strict variational sub-symmetry as in Example 2.15. The eigenvectors ϕ_i corresponding to the eigenvalues λ_i form an orthonormal Fourier-basis in H, i.e. for every $u \in H$

$$u = \sum_{i=1}^{\infty} u_i \phi_i \text{ with } u_i = \langle u, \phi_i\rangle.$$

Recall that H_A is the completion of $\mathcal{D}$ w.r.t. the norm $\|u\|_A := \langle u, u\rangle_A^{1/2}$. Since $\|u\|_A \geq \lambda_1 \|u\|$ we can consider H_A as closed subspace of H. Weak solutions of (2.8) are critical points of the functional $\mathcal{L} : H_A \to \mathbb{R}$ given by

$$\mathcal{L}[u] = \frac{1}{2}\|u\|_A^2 - K(u).$$

Suppose u_0 is a critical point of $\mathcal{L}$. Define the one-parameter group $G = \{g_\epsilon : H_A \to H_A\}_{\epsilon\in\mathbb{R}}$ as follows

$$g_\epsilon u := \sum_{i=1}^{i_0} e^{\epsilon}(u - u_0)_i \phi_i + \sum_{i=i_0+1}^{\infty} e^{-\epsilon}(u - u_0)_i \phi_i + u_0.$$

We show that g_ϵ strictly reduces $\mathcal{L}$ unless $u = u_0$. Using the orthogonality relations of the eigenvectors we get

$$\begin{aligned}\frac{d}{d\epsilon}\mathcal{L}[g_\epsilon u]|_{\epsilon=0} &= \langle u, \frac{d}{d\epsilon}g_\epsilon u\big|_{\epsilon=0}\rangle_A - K'(u)\frac{d}{d\epsilon}g_\epsilon u\big|_{\epsilon=0} \\ &= \langle u, \frac{d}{d\epsilon}g_\epsilon u\big|_{\epsilon=0}\rangle_A - \langle k(u), \frac{d}{d\epsilon}g_\epsilon u\big|_{\epsilon=0}\rangle \\ &= \sum_{i=1}^{i_0}\lambda_i(u-u_0)_i u_i - \sum_{i=i_0+1}^{\infty}\lambda_i(u-u_0)_i u_i \\ &\quad -\big\langle k(u), \sum_{i=1}^{i_0}(u-u_0)_i\phi_i - \sum_{i=i_0+1}^{\infty}(u-u_0)_i\phi_i\big\rangle.\end{aligned}$$

Next one uses that u_0 weakly solves (2.8), i.e. for each i it holds that $\lambda_i u_{0,i} = \langle k(u_0), \phi_i\rangle$. Inserting this in the rate-of-change formula one obtains

$$\begin{aligned}\frac{d}{d\epsilon}\mathcal{L}[g_\epsilon u]|_{\epsilon=0} &= \sum_{i=1}^{i_0}\lambda_i(u-u_0)_i^2 - \sum_{i=i_0+1}^{\infty}\lambda_i(u-u_0)_i^2 \\ &+ \big\langle k(u_0)-k(u), \sum_{i=1}^{i_0}(u-u_0)_i\phi_i\big\rangle - \big\langle k(u_0)-k(u), \sum_{i=i_0+1}^{\infty}(u-u_0)_i\phi_i\big\rangle.\end{aligned}$$

With the definition

$$l(u) := k(u) - \frac{\lambda_{i_0}+\lambda_{i_0+1}}{2}u$$

one can replace $k(u)$ by $\frac{\lambda_{i_0}+\lambda_{i_0+1}}{2}u + l(u)$. Estimation of λ_i by λ_{i_0} in the first sum and of $-\lambda_i$ by $-\lambda_{i_0+1}$ in the second sum implies

$$\begin{aligned}&\frac{d}{d\epsilon}\mathcal{L}[g_\epsilon u]|_{\epsilon=0} \\ &\le (\lambda_{i_0} - \frac{\lambda_{i_0}+\lambda_{i_0+1}}{2})\sum_{i=1}^{i_0}(u-u_0)_i^2 - (\lambda_{i_0+1} - \frac{\lambda_{i_0}+\lambda_{i_0+1}}{2})\sum_{i=i_0+1}^{\infty}(u-u_0)_i^2 \\ &\quad +\big\langle l(u_0)-l(u), \sum_{i=1}^{i_0}(u-u_0)_i\phi_i\big\rangle - \big\langle l(u_0)-l(u), \sum_{i=i_0}^{\infty}(u-u_0)_i\phi_i\big\rangle \\ &= \frac{\lambda_{i_0}-\lambda_{i_0+1}}{2}\|u-u_0\|^2 + \langle l(u_0)-l(u), \sum_{i=1}^{\infty}\pm(u-u_0)_i\phi_i\rangle,\end{aligned}$$

where $+$ is used for $i = 1\ldots i_0$ and $-$ for $i = i_0+1\ldots\infty$. By the Cauchy-Schwarz inequality we conclude

$$\frac{d}{d\epsilon}\mathcal{L}[g_\epsilon u]|_{\epsilon=0} \le \big(\frac{\lambda_{i_0}-\lambda_{i_0+1}}{2} + \text{Lip } l\big)\|u-u_0\|^2,$$

which is strictly negative by our assumption on Lip l unless $u = u_0$. Thus g_ϵ is a strict variational sub-symmetry w.r.t. u_0. □

Remark 2.20. Note that $g_\epsilon u$ is the flow map of the infinite system of differential equations in the Fourier-coefficients

$$\begin{aligned} \dot U_i &= (U-u_0)_i, && 1 \le i \le i_0, \\ \dot U_i &= -(U-u_0)_i, && i_0+1 \le i < \infty. \end{aligned}$$

with initial condition $U(0) = u$. Compare with Example 2.15 where

$$\dot X = X - X_0, \qquad \dot Y = -(Y - Y_0),$$

and with the example in Section 1.1, where for a convex functional we used

$$\dot U = -(U - u_0).$$

Example 2.21 (Uniqueness part of the Fredholm-alternative). For a given $b \in H$ the linear equation

$$Au = b + \lambda u, \qquad b \in H$$

has exactly one weak solution provided $\lambda_{i_0} < \lambda < \lambda_{i_0+1}$. Theorem 2.19 applies, since the Lipschitz-condition for $b + (\lambda - \frac{\lambda_{i_0}+\lambda_{i_0+1}}{2})u$ amounts to $|\lambda - \frac{\lambda_{i_0}+\lambda_{i_0+1}}{2}| < \frac{\lambda_{i_0+1}-\lambda_{i_0}}{2}$.

Example 2.22 (Uniqueness part of Theorem 2.16). Consider the operator $-\Delta$ on the Hilbert-space $H = L^2(\Omega)$ with domain $\mathcal{D}_0 = W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$. Let $A : \mathcal{D} \subset H \to H$ be a self-adjoint extension of $-\Delta$. If for $u \in H$ the map K is given by $K(u) = \int_\Omega F(x,u)\,dx$ then the Riesz-representation of the Fréchet derivative is $k : u \mapsto f(x,u)$. Let $l : u \mapsto f(x,u) - \frac{\lambda_{i_0+1}+\lambda_{i_0}}{2}u$. To find the Lipschitz-constant of l we calculate

$$\begin{aligned} \|l(v) - l(w)\|^2 &= \int_\Omega \Big(f(x,v) - f(x,w) - \frac{\lambda_{i_0+1}+\lambda_{i_0}}{2}(v-w)\Big)^2\,dx \\ &\le \Big(\frac{\lambda_{i_0+1}-\lambda_{i_0}}{2} - \delta\Big)^2 \|v-w\|^2, \end{aligned}$$

with a suitable $\delta > 0$ by assumption (2.7). Therefore Lip $l \le \frac{\lambda_{i_0+1}-\lambda_{i_0}}{2} - \delta$ and Theorem 2.19 shows uniqueness. □

2.5.3 Strict variational sub-symmetry w.r.t. an affine subspace

The notion of a strict variational sub-symmetry can be suitably weakened if instead of uniqueness one wants to localize the critical points of a functional $\mathcal{L} : V \to \mathbb{R}$ within an affine subspace of V.

Definition 2.23. *Let $\mathcal{L} : V \to \mathbb{R}$ be a functional on a normed space V. Suppose $u_0 \in V$ is given and let $V_1 \le V$ be a linear subspace. The one-parameter transformation group G defined on V is called a strict variational sub-symmetry w.r.t. the affine space $u_0 \oplus V_1$ provided*

$$\frac{d}{d\epsilon}\mathcal{L}[g_\epsilon u]\Big|_{\epsilon=0} < 0 \text{ for all } u \in V \setminus (u_0 \oplus V_1).$$

Note that for $V_1 = \{0\}$ we recover the notion of a strict variational sub-symmetry w.r.t. u_0 from Definition 2.5. As shown in the next theorem the notion of a strict variational sub-symmetry w.r.t. an affine subspace localizes the critical points in that subspace. The proof is the same as for Theorem 2.6.

Theorem 2.24. *Let $\mathcal{L} : V \to \mathbb{R}$ be a C^1-functional on a normed space V and let G be a one-parameter transformation group defined on V. Suppose $u_0 \in V$ is given and let $V_1 \leq V$ be a linear subspace. If G is a strict variational sub-symmetry w.r.t. the affine subspace $u_0 \oplus V_1$ then all critical points of $\mathcal{L}$ belong to $u_0 \oplus V_1$.*

As a first application of this concept consider on the unit ball $B_1(0) \subset \mathbb{R}^n$ the boundary value problem

$$\Delta u + f(x,u) = 0 \text{ in } B_1(0), \quad u = 0 \text{ on } \partial B_1(0). \tag{2.9}$$

Denote by $0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \ldots$ the Dirichlet eigenvalues of $-\Delta$ including multiplicities with corresponding eigenfunctions ϕ_i. It is well known that λ_1 is simple and ϕ_1 radially symmetric, whereas λ_2 is not simple and ϕ_2 not radially symmetric. Let us denote by L^2_{rad}, $W^{1,2}_{0,rad}$ the respective subspaces of $L^2(B_1(0))$, $W^{1,2}_0(B_1(0))$ consisting of radially symmetric functions and by $\mu_1 \leq \mu_2 \leq \mu_3 \leq \ldots$ the eigenvalues of $-\Delta$ corresponding to non-radial eigenfunctions $\psi_1, \psi_2, \psi_3, \ldots$. In this notation $\mu_1 = \lambda_2$.

Theorem 2.25. *Assume that*

(a) $\exists u_0 \in W^{1,2}_0(B_1(0))$ *s.t.* $\int_{B_1(0)} \nabla u_0 \nabla v - f(x,u_0)v\,dx = 0\ \forall v \in (W^{1,2}_{0,rad})^\perp$
(b) $f(x,s)$ *is locally Lipschitz continuous in s uniformly w.r.t. x and*

$$\sup_{x \in B_1(0), s \neq t} \frac{f(x,s) - f(x,t)}{s-t} < \lambda_2.$$

Then every weak solution u of (2.9) *belongs to* $u_0 \oplus W^{1,2}_{0,rad}$.

Proof. For $s,t \in [-M,M]$ we may assume

$$-L_M < \sup_{x \in B_1(0), s \neq t} \frac{f(x,s) - f(x,t)}{s-t} < \lambda_2 = \mu_1 \tag{2.10}$$

for a large $L_M > 0$. First, this restricts the result to all those solutions which attain values in $[-M,M]$. But since M can be taken arbitrarily large the full statement is recovered.

For $u \in W^{1,2}_0(B_1(0))$ let $u_j := \int_{B_1(0)} u\psi_j\,dx$ for $j \geq 1$. Let $g_\epsilon u := \sum_{j=1}^\infty e^{-\epsilon}(u-u_0)_j\psi_j + P(u-u_0) + u_0$, where $P : L^2(B_1(0)) \to L^2_{rad}$ is the orthogonal projection. Consider the functional $\mathcal{L}[u] = \int_{B_1(0)} \frac{1}{2}|\nabla u|^2 - F(x,u)\,dx$ on $W^{1,2}_0(B_1(0))$ with $F(x,s) = \int_0^s f(x,t)\,dt$. One obtains

$$\frac{d}{d\epsilon}\mathcal{L}[g_\epsilon u]\big|_{\epsilon=0} = -\sum_{j=1}^{\infty}\mu_j(u-u_0)_j u_j + \sum_{j=1}^{\infty}\int_{B_1(0)} f(x,u)(u-u_0)_j\psi_j\,dx.$$

By *(a)* we know $\mu_j u_{0,j} = \int_{B_1(0)} f(x,u_0)\psi_j\,dx$ for $j \geq 1$. Therefore

$$\frac{d}{d\epsilon}\mathcal{L}[g_\epsilon u]\big|_{\epsilon=0} = -\sum_{j=1}^{\infty}\mu_j(u-u_0)_j^2 + \sum_{j=1}^{\infty}\int_{B_1(0)}(f(x,u)-f(x,u_0))(u-u_0)_j\psi_j\,dx.$$

By adding and subtracting $\frac{\mu_1-L_M}{2}\sum_{j=1}^{\infty}(u-u_0)_j^2$ we obtain

$$\begin{aligned}
&\frac{d}{d\epsilon}\mathcal{L}[g_\epsilon u]\big|_{\epsilon=0}\\
&= -\sum_{j=1}^{\infty}\big(\mu_j - \frac{\mu_1-L_M}{2}\big)(u-u_0)_j^2\\
&\quad + \sum_{j=1}^{\infty}\int_{B_1(0)}\big(f(x,u)-f(x,u_0)-\frac{\mu_1-L_M}{2}(u-u_0)\big)(u-u_0)_j\psi_j\,dx\\
&\leq \Big(-\frac{\mu_1+L_M}{2} + \mathrm{Lip}\,\big(f(x,s)-\frac{\mu_1-L_M}{2}s\big)\Big)\sum_{j=1}^{\infty}(u-u_0)_j^2.
\end{aligned}$$

By (2.10) the Lipschitz-constant of $f(x,s)-\frac{\mu_1-L_M}{2}s$ is strictly smaller than $\frac{\mu_1+L_M}{2}$. Hence $\frac{d}{d\epsilon}\mathcal{L}[g_\epsilon u]|_{\epsilon=0} < 0$ if $u-u_0 \notin W^{1,2}_{0,rad}$, i.e., g_ϵ is a strict variational sub-symmetry w.r.t. $u_0 \oplus W^{1,2}_{0,rad}$. Theorem 2.24 applies and proves the result. □

Corollary 2.26. *Suppose only condition (b) of Theorem* 2.25 *holds.*

(i) If u, v *are two solutions of* (2.9) *then* $u - v \in W^{1,2}_{0,rad}$.
(ii) If $f(x,s) = f(y,s)$ *whenever* $|x| = |y|$ *then every solution of* (2.9) *is radially symmetric.*

Remark 2.27. Part *(ii)* was obtained by Lazer, McKenna [59]. It should be compared with Example 2.13.

Proof. (i) Let v be a solution of (2.9). With $u_0 := v$ Theorem 2.25 implies that $u \in v \oplus W^{1,2}_{0,rad}$ for every other solution u. *(ii)* Let $u_0 = 0$. Condition *(a)* in Theorem 2.25 holds since $f(x,0)$ is a radial function. As a result every solution belongs to $W^{1,2}_{0,rad}$. □

As another example we have the following generalization of Theorem 2.19. Again we assume that $(H, \langle\cdot,\cdot\rangle)$ is a real Hilbert-space and $A : \mathcal{D} \subset H \to H$ a selfadjoint, strictly positive definite, densely defined linear operator. By H_A we denote the completion of $\mathcal{D}$ w.r.t. the inner product $\langle u, v\rangle_A := \langle Au, v\rangle$.

Theorem 2.28. *Assume that the selfadjoint linear operator* $A : \mathcal{D} \subset H \to H$ *has discrete spectrum* σ *consisting of the eigenvalues* $0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \dots$ *including multiplicities. Assume further that there exists a closed subspace* $W \subset H$ *such that* $A : W \cap \mathcal{D} \to W \cap \mathcal{D}$ *is self-adjoint with spectrum* σ_1*. Then* $\sigma \setminus \sigma_1$ *consist of the eigenvalues* $0 < \mu_1 \leq \mu_2 \leq \mu_3 \leq \dots$ *including multiplicities. Let* $K : H_A \to \mathbb{R}$ *be a* C^2*-functional and for each* $u \in H_A$ *let* $k(u) \in H$ *be the Riesz-representation of* $K'(u)$ *w.r.t.* $\langle \cdot, \cdot \rangle$*. All weak solutions of the equation*

$$Au = k(u) \tag{2.11}$$

belong to the affine subspace $u_0 \oplus W$ *provided*

(a) $u_0 \in H_A$ *is such that* $\langle u_0, v \rangle_A - \langle k(u_0), v \rangle = 0$ *for all* $v \in (W \cap H_A)^\perp$*,*
(b) $\exists j_0 \in \mathbb{N}$ *such that* $\operatorname{Lip}\left(k - \frac{\mu_{j_0} + \mu_{j_0+1}}{2} \operatorname{Id}\right) < \frac{\mu_{j_0+1} - \mu_{j_0}}{2}$*, where the Lipschitz constant is computed w.r.t. to the norm* $\| \cdot \|$ *of* H*.*

Remark 2.29. Theorems of the same spirit were first found by Lazer, McKenna [59] and subsequently by Fečkan [30] and Mawhin, Walter [62].

Proof. Recall that H_A is the completion of $\mathcal{D}$ w.r.t. the inner product $\langle u, v \rangle_A := \langle Au, v \rangle$. We have the following decomposition

$$H_A = \underbrace{(W \cap H_A)}_{=:V_1} \oplus \underbrace{(W \cap H_A)^\perp}_{=:V_0}.$$

We need to show that Theorem 2.24 applies. As in Theorem 2.19 we have the Fourier-decomposition of H with respect to the full spectrum σ of A. Moreover we have the spectrum σ_1 of A on $W \cap \mathcal{D}$ and the remaining eigenvalues $0 < \mu_1 \leq \mu_2 \leq \mu_3 \leq \dots$ from $\sigma \setminus \sigma_1$ with corresponding eigenvectors $\psi_1, \psi_2, \psi_3, \dots$. We use the following notation

$$u = \sum_{j=1}^{\infty} u_j \psi_j + Pu,$$

where as usual $u_j = \langle u, \psi_j \rangle$ and P is the orthogonal projector from H onto W. This leads to the following definition of a one-parameter transformation group

$$g_\epsilon u := \sum_{j=1}^{j_0} e^\epsilon (u - u_0)_j \psi_j + \sum_{j=j_0+1}^{\infty} e^{-\epsilon} (u - u_0)_j \psi_j + P(u - u_0) + u_0.$$

The same computation as in Theorem 2.19 leads to

$$\begin{aligned} \frac{d}{d\epsilon} \mathcal{L}[g_\epsilon u]|_{\epsilon=0} = &\sum_{j=1}^{j_0} \mu_j (u - u_0)_j u_j - \sum_{j=j_0+1}^{\infty} \mu_j (u - u_0)_j u_j \\ &- \Big\langle k(u), \sum_{j=1}^{j_0} (u - u_0)_j \psi_j - \sum_{j=j_0+1}^{\infty} (u - u_0)_j \phi_j \Big\rangle. \end{aligned}$$

Next one uses condition *(a)*, i.e., for each j it holds that $\mu_j u_{0,j} = \langle k(u_0), \psi_j \rangle$. Inserting this in the rate-of-change formula one obtains

$$\frac{d}{d\epsilon}\mathcal{L}[g_\epsilon u]|_{\epsilon=0} = \sum_{j=1}^{j_0} \mu_j (u-u_0)_j^2 - \sum_{j=j_0+1}^{\infty} \mu_j (u-u_0)_j^2$$
$$+ \bigl\langle k(u_0) - k(u), \sum_{j=1}^{j_0} (u-u_0)_j \psi_j \bigr\rangle - \bigl\langle k(u_0) - k(u), \sum_{j=j_0+1}^{\infty} (u-u_0)_j \psi_j \bigr\rangle.$$

Using the Lipschitz properties of $l(u) := k(u) - \frac{\mu_{j_0}+\mu_{j_0+1}}{2} u$ we obtain as in Theorem 2.19

$$\frac{d}{d\epsilon}\mathcal{L}[g_\epsilon u]|_{\epsilon=0} \le \bigl(\frac{\mu_{j_0} - \mu_{j_0+1}}{2} + \text{Lip } l\bigr) \|Q(u-u_0)\|^2,$$

where $Q : H \to W^\perp$ is the orthogonal projection onto $W^\perp$. By assumption *(b)* we find $\frac{d}{d\epsilon}\mathcal{L}[g_\epsilon u]|_{\epsilon=0} < 0$ for all $u \in H_A$ such that $u - u_0 \notin V_1$. Hence g_ϵ is a strict variational sub-symmetry w.r.t. the affine space $u_0 \oplus V_1$. Hence Theorem 2.24 applies and proves the claim. □

If one applies the previous theorem to problem (2.9) then one finds the following result, which complements Theorem 2.25 and Corollary 2.26. It was essentially shown by Lazer, McKenna [59].

Theorem 2.30. *Counting multiplicities let* $0 < \lambda_1 < \lambda_2 \le \lambda_3 \le \dots$ *be the Dirichlet eigenvalues of* $-\Delta$ *on* $B_1(0)$ *and let* $\mu_1 \le \mu_2 \le \mu_3 \le \dots$ *be the eigenvalues of* $-\Delta$ *corresponding to non-radial eigenfunctions* $\psi_1, \psi_2, \psi_3, \dots$. *Let* $f : B_1(0) \times \mathbb{R} \to \mathbb{R}$. *Suppose there exists an eigenvalue-index* j_0 *such that*

$$\mu_{j_0} < \sup_{x\in\Omega, s\neq t} \frac{f(x,s) - f(x,t)}{s-t} < \mu_{j_0+1}.$$

Then the following holds:

(i) if u, v *are two solutions of* (2.9) *then* $u - v \in W^{1,2}_{0,rad}$,
(ii) if $f(x,s) = f(y,s)$ *whenever* $|x| = |y|$ *then every solution of* (2.9) *is radially symmetric.*

2.5.4 Uniqueness of positive solutions for sublinear problems

So far the one-parameter transformation group $G = \{g_\epsilon\}_{\epsilon\in\mathbb{R}}$ was defined on a normed vector space V. If instead we have that $g_\epsilon : \operatorname{dom} g_\epsilon \subset \mathcal{O} \to \mathcal{O}$ for an open subset $\mathcal{O}$ of V then again a strict variational sub-symmetry w.r.t. $u_0 \in \mathcal{O}$ implies that every critical point of $\mathcal{L} : V \to \mathbb{R}$ in $\mathcal{O}$ coincides with u_0.

This observation is applied to the following problem on a bounded smooth domain $\Omega \subset \mathbb{R}^n$

$$\Delta u + f(x,u) = 0 \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega, \tag{2.12}$$

where $f : \Omega \times [0,\infty) \to [0,\infty)$ is sublinear, i.e.,

(i) $f(x,s)/s$ is strictly decreasing for $s \in (0,\infty)$,
(ii) $\exists C > 0$ such that $0 \le f(x,s) \le C(1+s)$ for all $s \in [0,\infty)$ and all $x \in \Omega$.

A prototype sublinear function is $f(x,u) = u^p$ with $0 < p < 1$. Existence and uniqueness results for positive solutions of (2.12) are due to Krasnoselskii [56], Keller, Cohen [54] and Laetsch [58]. The methods are based on maximum principles. Later Brezis, Oswald [10] and Ambrosetti, Brezis, Cerami [1] avoided the maximum principle and obtained uniqueness of the positive solution by testing (2.12) with suitable test-functions. We will show how the uniqueness of positive solutions fits well in the framework of the method of transformation-groups provided we restrict attention to weak solutions in $C_0^1(\overline{\Omega})$. Belloni and Kawohl [8] obtained uniqueness in the pure $W_0^{1,2}(\Omega)$-context.

We start with a purely formal calculation, which will be made rigorous later. Let $\mathcal{L}[u] = \int_\Omega \frac{1}{2}|\nabla u|^2 - F(x,u)\,dx$ with $F(x,s) = \int_0^s f(x,t)\,dt$. Let $u_0 \in C_0^1(\overline{\Omega})$ be a positive weak solution of (2.12). For the infinitesimal generator of a variational sub-symmetry we set $\boldsymbol{\phi}(x,u) = (-u + \frac{u_0(x)^2}{u})\partial_u$, where for the moment we ignore the fact that $\boldsymbol{\phi}$ is discontinuous at $u=0$. Then we find the prolongation

$$\mathbf{w}^{(1)} = (-u + \frac{u_0(x)^2}{u})\partial_u + (-\nabla u + 2\nabla u_0(x)\frac{u_0(x)}{u} - \nabla u\frac{{u_0(x)}^2}{u^2}) \cdot \nabla_{\mathbf{p}}.$$

For verification of the sub-symmetry criterion we use the rate of change formula from Theorem 2.9 and calculate

$$\begin{aligned}&\int_\Omega \mathbf{w}^{(1)} L(x,u,\nabla u)\,dx \\ &\qquad = \int_\Omega f(x,u)u - \frac{f(x,u)}{u}u_0^2 - |\nabla u|^2 + 2\nabla u \cdot \nabla u_0 \frac{u_0}{u} - |\nabla u|^2 \frac{{u_0}^2}{u^2}\,dx.\end{aligned}$$

Using the fact that u_0 solves (2.12) this yields

$$\begin{aligned}\int_\Omega \mathbf{w}^{(1)} L(x,u,\nabla u)\,dx &= \int_\Omega \Big(\frac{f(x,u)}{u} - \frac{f(x,u_0)}{u_0}\Big)(u-u_0)(u+u_0)\,dx \\ &\quad + \int_\Omega -\Big|\nabla u - \frac{u}{u_0}\nabla u_0\Big|^2 - \Big|\nabla u_0 - \frac{u_0}{u}\nabla u\Big|^2\,dx.\end{aligned}$$

Since $f(x,s)/s$ is strictly decreasing for $s > 0$ we find $\int_\Omega \mathbf{w}^{(1)} L(x,u,\nabla u)\,dx < 0$ unless $u \equiv u_0$. Hence $\boldsymbol{\phi}$ generates a strict variational sub-symmetry w.r.t. u_0 and uniqueness follows by Theorem 2.6.

Now we need to address the question how to justify rigorously the previous steps. Although the infinitesimal generator $\boldsymbol{\phi}$ is singular at $u=0$ the ordinary differential equation

$$\dot{U} = -U + u_0^2/U, \quad U(0) = u$$

generated by $\boldsymbol{\phi}$ has the unique solution

$$g_\epsilon u = \sqrt{u_0^2 + e^{-2\epsilon}(u^2 - u_0^2)},$$

which is well defined in $C_0^1(\overline{\Omega})$.

Lemma 2.31. *Let $u_0 \in C_0^1(\overline{\Omega})$ be a fixed positive weak solution of* (2.12). *If $u \in C_0^1(\overline{\Omega})$ is an arbitrary positive weak solution then u/u_0 and u_0/u are bounded on $\overline{\Omega}$.*

Proof. Since $\Delta u \le 0$ weakly on Ω and $u = 0$ on $\partial\Omega$ the strong maximum principle implies that $\nabla u \cdot \nu < 0$ everywhere on $\partial\Omega$, and hence $\nabla u \cdot \nu \le -\delta < 0$ on $\partial\Omega$. Since the same holds for u_0 the claim follows. □

We proceed with the uniqueness proof as follows. For $k > 1$ define $\mathcal{O}_k := \{u \in C_0^1(\overline{\Omega}) : u_0(x)/k < u(x) < u_0(x)\}$. By Lemma 2.31 we know that any positive critical points of $\mathcal{L}$ belongs to $\mathcal{O}_k$ for $k \ge k_0 = k_0(u)$. Given a fixed $u \in \mathcal{O}_k$ with $k \ge k_0$ there exists $\epsilon_0 = \epsilon_0(u) > 0$ such that $g_\epsilon u \in \mathcal{O}_k$ for all $\epsilon \in [-\epsilon_0, \epsilon_0]$. This means that for small ϵ the group operation $g_\epsilon u$ is well defined for $u \in \mathcal{O}_k$. The vector-field $\mathbf{w}$ generates a transformation group for the functional $\mathcal{L}$ if the group-operation is restricted to initial functions belonging $\mathcal{O}_k$. Since we have already verified that the variational sub-symmetry is strict w.r.t. u_0, the uniqueness proof is complete.

Example 2.32. The functional $\mathcal{L}[u] = \int_\Omega \frac{a(x)}{2}|\nabla u|^2 - F(x,u)\,dx$ has at most one positive critical point in $C_0^1(\overline{\Omega})$ provided $a > 0$ in $\overline{\Omega}$ and $f(x,s)/s$ is strictly decreasing for $s \in (0,\infty)$.

Example 2.33. For $0 < p < 1$ the Neumann problem

$$\Delta u - u + u^p = 0 \text{ in } \Omega, \quad \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial\Omega, \tag{2.13}$$

on a bounded domain has only the positive solution $u \equiv 1$ in the class of weak C^1-solutions. Positive solutions are critical points of the functional $\mathcal{L}[u] = \int_\Omega \frac{1}{2}|\nabla u|^2 + \frac{1}{2}u^2 - \frac{1}{p+1}u^{p+1}\,dx$ on the space $C^1(\overline{\Omega})$. The result follows if one applies the same variational sub-symmetry as for the Dirichlet-problem (2.12).

Example 2.34. For $0 < p < 1$ the nonlinear Neumann problem

$$\Delta u - u = 0 \text{ in } \Omega, \quad \frac{\partial u}{\partial \nu} = u^p \text{ on } \partial\Omega, \tag{2.14}$$

on a bounded domain has only one positive weak solution in the class $C^1(\overline{\Omega})$. Positive solutions are now given as critical points of the functional $\mathcal{L}[u] = \int_\Omega \frac{1}{2}|\nabla u|^2 + \frac{1}{2}u^2\,dx - \int_{\partial\Omega} \frac{1}{p+1}u^{p+1}\,d\sigma$ with $u \in C^1(\overline{\Omega})$. By choosing again $\mathbf{w} = (-u + \frac{u_0(x)^2}{u})\partial_u$ one can prove uniqueness.

Example 2.35 (q-Laplacian). For $1 < q < \infty$ critical points in $W_0^{1,q}(\Omega)$ of the functional $\mathcal{L}[u] = \int_\Omega \frac{1}{q}|\nabla u|^q - F(x,u)\,dx$ weakly satisfy

$$\operatorname{div}(|\nabla u|^{q-2}\nabla u) + f(x,u) = 0 \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega, \tag{2.15}$$

provided $F(s)$ satisfies a subcritical growth condition. The operator $\Delta_q u = \operatorname{div}(|\nabla u|^{q-2}\nabla u)$ is called the q-Laplacian. For $q \neq 2$ the operator Δ_q is not uniformly elliptic near those points x where $|\nabla u(x)| = 0$ or ∞. Weak solutions of (2.15) are therefore in general not classical but only $C^{1,\alpha}$-regular, cf. DiBenedetto [20], Lieberman [61]. Like in the sublinear case, uniqueness of positive solutions holds in the class of weak $C_0^1(\overline{\Omega})$-solutions provided

(i) $f(x,s)/s^{q-1}$ is strictly decreasing for $s \in (0,\infty)$,
(ii) $\exists C > 0$ such that $0 \le f(x,s) \le C(1+s^{q-1})$ for all $s \in [0,\infty)$.

This result goes back do Díaz, Saa [22] and was recently sharpened by Belloni, Kawohl [8]. For a proof via transformation groups we may use the uniqueness principle of strict variational sub-symmetries, provided we restrict to the class of weak solution in $C_0^1(\overline{\Omega})$. We follow the lines of the sublinear case. The strong maximum principle used in Lemma 2.31 has its analogue for the p-Laplacian, cf. Vazquez [90]. Next, one has to show that the transformation group generated by $\mathbf{w}(x,u) = (-u + \frac{u_0(x)^q}{u^{q-1}})\partial_u$ is a strict variational sub-symmetry w.r.t. u_0. The proof requires the inequality

$$\begin{aligned}|\nabla u|^q + |\nabla u_0|^q + (q-1)\frac{u^q}{u_0^q}|\nabla u_0|^q + (q-1)\frac{u_0^q}{u^q}|\nabla u|^q \\ \ge q\frac{u_0^{q-1}}{u^{q-1}}\nabla u \cdot \nabla u_0 |\nabla u|^{q-2} + q\frac{u^{q-1}}{u_0^{q-1}}\nabla u \cdot \nabla u_0 |\nabla u_0|^{q-2}\end{aligned}$$

with equality if and only if u, u_0 are linearly dependent. For a proof one uses the strict convexity of $a \mapsto |a|^q$ for $a \in \mathbb{R}^n$ to show the following three inequalities:

$$\begin{aligned}1 + (q-1)s^q &\ge qs^{q-1} \text{ for all } s > 0,\\ |a|^q + (q-1)t^q|b|^q &\ge qt^{q-1}a \cdot b|b|^{q-2} \text{ for all } a,b \in \mathbb{R}^n, t > 0,\\ |b|^q + (q-1)t^{-q}|a|^q &\ge qt^{1-q}a \cdot b|a|^{q-2} \text{ for all } a,b \in \mathbb{R}^n, t > 0.\end{aligned}$$

2.5.5 Simplicity of the first eigenvalue

A simple variant of the uniqueness proof for sublinear problems shows that the first eigenvalue of second-order divergence type operators is simple. Suppose again that $\Omega \subset \mathbb{R}^n$ is a bounded smooth domain.

Theorem 2.36. *Let*

$$\lambda_1 = \inf_{u \in C_0^1(\overline{\Omega})} \frac{\int_\Omega a(x)|\nabla u|^2 + b(x)u^2\,dx}{\int_\Omega c(x)u^2\,dx}$$

with bounded measurable functions $a, b, c : \Omega \to \mathbb{R}$ such that $a(x) \geq \delta > 0$ and $c(x) \geq 0$ for all $x \in \Omega$. If the Dirichlet eigenvalue λ_1 is attained in $C_0^1(\overline{\Omega})$ then λ_1 is simple. The same holds for Neumann eigenvalues if $C_0^1(\overline{\Omega})$ is replaced by $C^1(\overline{\Omega})$.

Proof. The proof is done for the Dirichlet case only. For the Neumann case the proof is identically the same. We consider λ_1 as a critical value of $\mathcal{L}[u] = \int_\Omega a(x)|\nabla u|^2 + b(x)u^2\,dx$ over the constraint $S = \{u \in C_0^1(\overline{\Omega}) : \mathcal{N}[u] = \int_\Omega c(x)u^2\,dx = 1\}$. We fix some positive first-eigenfunction u_0 with $\mathcal{N}[u_0] = 1$ and use the same transformation group as in the previous section restricted to the set $\mathcal{O}_k := \{(u \in C_0^1(\overline{\Omega}) : u_0(x)/k < u(x) < ku_0(x)\}$. The same reasoning as in Lemma 2.31 shows that for sufficiently large k any positive first-eigenfunction lies in $\mathcal{O}_k$. Let us check that the functional constraint S is left invariant: by the rate of change-formula from Theorem 2.9 we find

$$\begin{aligned}
\frac{d}{d\epsilon}\mathcal{N}[u]|_{\epsilon=0} &= \int_\Omega \mathbf{w}N\,dx \\
&= \int_\Omega 2c(x)u(-u + \frac{u_0(x)^2}{u})\,dx \\
&= 0.
\end{aligned}$$

Hence the functional constraint S is left invariant. We proceed to check the criterion for a strict variational sub-symmetry. As for the sublinear case we obtain

$$\begin{aligned}
&\int_\Omega \mathbf{w}^{(1)}L(x,u,\nabla u)\,dx \\
&= \int_\Omega -a(x)\left|\nabla u - \frac{u}{u_0(x)}\nabla u_0(x)\right|^2 - a(x)\left|\nabla u_0(x) - \frac{u_0(x)}{u}\nabla u\right|^2 dx \\
&\leq 0
\end{aligned}$$

and in fact the last inequality is strict unless $u(x)\nabla u_0(x) = u_0(x)\nabla u(x)$ for almost all $x \in \Omega$, i.e. unless $u(x)/u_0(x) \equiv$ const.. Therefore Theorem 2.8 applies and proves that every positive first-eigenfunction lying in S must be equal to u_0, or in other words, any positive first-eigenfunction must be a multiple of u_0. □

Example 2.37. The first eigenvalue of the modified Stekloff-problem $\Delta u - u = 0$ in Ω with $\frac{\partial u}{\partial \nu} = \lambda u$ on $\partial\Omega$ is simple.

Example 2.38. For $q > 1$ the first eigenvalue of the q-Laplacian

$$\lambda_1 = \inf_{u \in C_0^1(\overline{\Omega})} \frac{\int_\Omega |\nabla u|^{q+1}\, dx}{\int_\Omega |u|^{q+1}\, dx}$$

is simple. This result (and a generalization of it) was proved by Anane [2] and then by Belloni, Kawohl [8]. A proof can be given with the same transformation group as in Example 2.35.

Example 2.39. For $q > 1$ the functional $\mathcal{L}[u] = \int_\Omega \frac{1}{q}|\nabla u|^q - F(u)\, dx$ has at most one critical point $u_0 \in W^{2,1}(\Omega) \cap C_0^1(\overline{\Omega})$ provided $F'(t) = f_1(t) + \lambda |t|^{q-1} t$ with f_1 decreasing and $\lambda < \lambda_1$. Here λ_1 denotes the first Dirichlet-eigenvalue of q-Laplacian on Ω. Examples 2.13 and 2.38 are relevant.

3

Uniqueness of critical points (II)

In the previous chapter we discussed one-parameter transformation groups arising from ordinary differential equations in a normed space. For the prototype functional $\mathcal{L}[u] = \int_\Omega L(x, u, \nabla u)\, dx$ this means that only those transformations were considered where the dependent variable u was transformed and the independent variable x was left untouched. In this chapter we extend the theory of one-parameter transformation groups to cases where the dependent and the independent variable are simultaneously transformed. Now the structure of the underlying domain Ω will be important. We assume throughout that Ω is subset of an n-dimensional Riemannian manifold M with metric g. First we recall the basic concepts of Riemannian manifolds.

3.1 Riemannian manifolds

A collection is presented of those differential geometric concepts, which are useful in the calculus of variations on manifolds.

Manifolds. A manifold M of dimension n is a topological Hausdorff space, such that for each point $x \in M$ there exists an open neighbourhood $\mathcal{U} \subset M$ of x and a homeomorphism $h : \mathcal{U} \to \mathbb{R}^n$. The pair $(\mathcal{U}, h)$ is called a local chart, the functions $(x^1(x), \dots, x^n(x)) = h(x)$ are called local coordinates. The manifold M is called smooth if a collection of local charts $(\mathcal{U}_\iota, h_\iota)_{\iota \in I}$ exists, such that $\bigcup_{\iota \in I} \mathcal{U}_\iota = M$ and whenever $\mathcal{U}_{\iota_1} \cap \mathcal{U}_{\iota_2} \neq \emptyset$ then $h_{\iota_1} \circ h_{\iota_2}^{-1}$ and $h_{\iota_2} \circ h_{\iota_1}^{-1}$ are smooth maps between neighbourhoods of $\mathbb{R}^n$.

Differentiable functions. A function $f : \mathcal{U} \subset M \to \mathbb{R}$ is called differentiable at x if for a local chart $(\mathcal{U}, h)$ at x the function $f \circ h^{-1} : h(\mathcal{U}) \to \mathbb{R}$ is differentiable at $h(x)$. We write $\partial_{x^i} f(x)$ or simply $f_{,i}(x)$ for $\frac{\partial}{\partial x^i} f \circ h^{-1}|_{h(x)}$.

Tangent vectors, tangent space. A tangent vector $\mathbf{w}$ at $x \in M$ is a map $\mathbf{w} : f \mapsto \mathbf{w}(f) \in \mathbb{R}$ defined for functions f, which are differentiable in a neighbourhood of x such that:

W. Reichel: LNM 1841, pp. 27–57, 2004.

(a) $\mathbf{w}(\alpha f + \beta g) = \alpha \mathbf{w}(f) + \beta \mathbf{w}(g)$ for all $\alpha, \beta \in \mathbb{R}$,
(b) $\mathbf{w}(fg) = f(x)\mathbf{w}(g) + g(x)\mathbf{w}(f)$.

The tangent space T_xM is the set of all tangent vectors at x. In local coordinates the vectors $\partial_{x^i}|_x$, $i = 1, \ldots, n$ with $\partial_{x^i}|_x f := \partial_{x^i} f(x)$ are a basis of T_xM. A vector $\mathbf{w}$ can be written as $\mathbf{w} = \sum_{i=1}^n w^i \partial_{x^i}|_x$.

Summation convention. Indices occurring twice are automatically summed, e.g. $\mathbf{w} = w^i \partial_{x^i}|_x$.

Vector fields, tangent bundle. The tangent bundle $TM = \bigcup_{x \in M} T_xM$ is a $2n$-dimensional manifold. A vector-field $\mathbf{w} : M \to TM$ is a smooth map assigning to each $x \in M$ a vector $\mathbf{w}(x) \in T_xM$. In local coordinates $\mathbf{w}(x) = w^i(x)\partial_{x^i}|_x$ or simply $\mathbf{w} = w^i \partial_{x^i}$.

Covectors, cotangent space. The space of linear functionals on T_xM, i.e. the dual space T_x^*M is called the cotangent space; its elements are called covectors. In local coordinates the dual-basis of $\partial_{x^i}|_x$ is called $dx^i|_x$ with $dx^i|_x(\partial_{x^j}|_x) = \delta^i_j$. A covector $\boldsymbol{\omega}$ at x may be written as $\boldsymbol{\omega} = \omega_i dx^i|_x$.

One forms, cotangent bundle. The cotangent bundle $T^*M = \bigcup_{x \in M} T_x^*M$ is also a $2n$-dimensional manifold. A one-form $\boldsymbol{\omega} : M \to T^*M$ is a smooth map assigning to each $x \in M$ a covector $\boldsymbol{\omega}(x) \in T^*M$. In local coordinates $\boldsymbol{\omega}(x) = \omega_i(x) dx^i|_x$ or simply $\boldsymbol{\omega} = \omega_i dx^i$.

The differential of a map. If $\tau : M \to N$ is a smooth map between two manifolds then its differential $d\tau : TM \to TN$ is a linear map defined pointwise for fixed x as follows: let $\mathbf{w} \in T_xM$ be an arbitrary vector and $h : N \to \mathbb{R}$ an arbitrary smooth function. Then a new vector $(d\tau \mathbf{w})|_{\tau(x)} \in T_{\tau(x)}N$ is defined by

$$(d\tau \mathbf{w})|_{\tau(x)} h := \mathbf{w}(h \circ \tau)(x).$$

Thus $d\tau|_x : T_xM \to T_{\tau(x)}N$. For vector-fields $\mathbf{w}$ on M the definition $(d\tau \mathbf{w})h := \mathbf{w}(h \circ \tau)$ defines a new vector field $d\tau \mathbf{w}$ on N. In local coordinates one finds for $\mathbf{w} = w^i \partial_{x^i}$ that $d\tau \mathbf{w} = \tau^k_{,j} w^j \partial_{y^k}$. If $u : M \to \mathbb{R}$ is a function, then its differential du is a one-form on M given by $du = u_{,i} dx^i$.

Tensors, tensor fields. For our purposes we only need the notion of a 2-tensor. A tensor $\mathbf{v}$ of type $(0, 2)$ at a point x is a bilinear map $\mathbf{v} : T_xM \times T_xM \to \mathbb{R}$. A basis of $(0, 2)$-tensors is given by $dx^i|_x \otimes dx^j|_x$ with $dx^i|_x \otimes dx^j|_x(\partial_{x^k}|_x, \partial_{x^l}|_x) = \delta^i_k \delta^j_l$. Thus $\mathbf{v} = a_{ij} dx^i|_x \otimes dx^j|_x$. A tensor $\mathbf{B}$ of type $(2, 0)$ at x is a bilinear map $T_x^*M \times T_x^*M \to \mathbb{R}$, a basis is given by $\partial_{x^i}|_x \otimes \partial_{x^j}|_x$. Hence $\mathbf{B} = b^{ij} \partial_{x^i}|_x \otimes \partial_{x^j}|_x$. Finally a tensor of type $(1, 1)$ is a bilinear map $\mathbf{C} : T_x^*M \times T_xM$ with basis $\partial_{x^i}|_x \otimes dx^j|_x$, i.e. $\mathbf{C} = c^i_j \partial_{x^i}|_x \otimes dx^j|_x$. A simple way to construct a $(1, 1)$-tensor from a vector $\mathbf{w}$ and a covector $\boldsymbol{\omega}$ is by defining the tensor $\mathbf{w} \otimes \boldsymbol{\omega}$ pointwise for $\boldsymbol{\eta} \in T_x^*M$ and $\mathbf{z} \in T_xM$ through the formula $(\mathbf{w} \otimes \boldsymbol{\omega})(\boldsymbol{\eta}, \mathbf{z}) := \boldsymbol{\eta}(\mathbf{w})\boldsymbol{\omega}(\mathbf{z})$. Tensor fields arise by smoothly assigning each point $x \in M$ a tensor at x.

Tensors of type $(1,1)$ *as self-maps of* T_xM. If $\mathbf{C} : T_x^*M \times T_xM \to \mathbb{R}$ is a $(1,1)$-tensor then we may interpret $\mathbf{C}$ as a self-map of T_xM. In local coordinates $\mathbf{C}\mathbf{w} = c^i_j w^j \partial_{x^i}|_x$ for every vector $\mathbf{w} = w^i \partial_{x^i}|_x$, and in abstract form

$$(\mathbf{C}\mathbf{w})f := \mathbf{C}(df, \mathbf{w}) \text{ for every smooth function} f.$$

Riemannian manifolds. A Riemannian manifold (M, g) is a manifold M together with a smooth $(0,2)$-tensor field $g = g_{ij} dx^i \otimes dx^j$ such that for each fixed $x \in M$ the map $g|_x : T_xM \times T_xM$ is a positive definite and symmetric bilinear form. On T_xM the tensor g induces a scalar product $\mathbf{v} \cdot \mathbf{w} = g|_x(\mathbf{v}, \mathbf{w})$, and likewise for vector-fields. By forming the inverse matrix $g^{ij}(x)$ one defines a $(2,0)$-tensor field $g^{ij} \partial_{x^i} \otimes \partial_{x^j}$.

Raising and lowering indices. Type conversion. With the metric tensor g we can define the type conversion, e.g. if $\mathbf{v} = a_{ij} dx^i \otimes dx^j$ is a $(0,2)$-tensor-field then $c^i_j = g^{il} a_{lj}$ defines the coefficients of a $(1,1)$-tensor $\mathbf{C} = c^i_j \partial_{x^i} \otimes dx^j$. This operation is called *raising an index*. Similarly, indices can be lowered by multiplication with g_{ij}.

Covariant differentiation, Christoffel symbols. For the general theory of covariant differentiation see Aubin [4]. We only need the following facts: for a smooth function $u : M \to \mathbb{R}$ the gradient ∇u is the unique vector-field such that $g(\nabla u, \mathbf{w}) = du(\mathbf{w}) = w^i u_{,i}$. In local coordinates $\nabla u = g^{ij} u_{,j} \partial_{x^i}$. We use the notation $\nabla u = u^{;i} \partial_{x^i}$ with $u^{;i} = g^{ij} u_{,j}$. For the definition of the covariant derivative (in the sense of Levi-Civita) of a vector-field we need to define the Christoffel-symbols $\Gamma^i_{jk} : M \to \mathbb{R}$

$$\Gamma^i_{jk} = \frac{1}{2} g^{il}(g_{jl,k} + g_{kl,j} - g_{jk,l}), \quad i, j, k = 1, \ldots, n.$$

For a smooth vector-field $\mathbf{w} = w^i \partial_{x^i}$ the covariant derivative is defined as the $(1,1)$-tensor $D\mathbf{w} = w^i_{;j} dx^j \otimes \partial_{x^i}$ with

$$w^i_{;j} = w^i_{,j} + \Gamma^i_{jk} w^k.$$

The function $\operatorname{div} \mathbf{w} = \operatorname{trace} D\mathbf{w} = w^i_{;i}$ is called the divergence of $\mathbf{w}$.

Covariant derivative along paths. Let $\gamma : (-1,1) \to M$ be a smooth path. A vector-field $\boldsymbol{\zeta}$ along γ is a map $\boldsymbol{\zeta} : (-1,1) \to TM$ such that $\boldsymbol{\zeta}(t) \in T_{\gamma(t)}M$. The covariant derivative of $\boldsymbol{\zeta}$ along γ is the vector-field $\boldsymbol{\zeta}' : (-1,1) \to TM$ with

$$\boldsymbol{\zeta}' = (\dot{\zeta}^i + \Gamma^i_{jk} \zeta^j \dot{\gamma}^k) \partial_{x^i}.$$

Here we use the notation $\dot{\zeta}^i(t) = \frac{d}{dt} \zeta^i(t)$ and $\dot{\gamma}^i(t) = \frac{d}{dt} \gamma^i(t)$.

Hessian and Laplace-Beltrami. Let $u : M \to \mathbb{R}$ be a smooth function. The $(1,1)$-tensor $D\nabla u$ is called the Hessian of u. It has the symmetry property $g(D\nabla u \mathbf{w}, \mathbf{z}) = g(\mathbf{w}, D\nabla u \mathbf{z})$ for any two vector fields $\mathbf{w}, \mathbf{z}$. The function $\Delta u = \operatorname{trace} D\nabla u = u^{;i}_{;i}$ is called the Laplace-Beltrami of u.

3.2 The total space $M \times \mathbb{R}^k$

Let (M, g) be an n-dimensional smooth Riemannian manifold with metric g and without boundary. On M we consider a subset Ω and $\mathbb{R}^k$-valued functions $u : \Omega \subset M \to \mathbb{R}^k$. The graph of such a function u is a subset of $M \times \mathbb{R}^k$.

The *total space* $M \times \mathbb{R}^k$ is an $(n+k)$-dimensional smooth manifold. Each tangent space has the simple structure $T_{(x,u)}(M \times \mathbb{R}^k) = T_xM \times \mathbb{R}^k$. Vector fields $\mathbf{w}$ on $M \times \mathbb{R}^k$ are written in local coordinates as $\mathbf{w} = \xi^i(x,u)\partial_{x^i} + \phi^\alpha(x,u)\partial_{u^\alpha}$. We use the notation $\mathbf{w} = \boldsymbol{\xi}(x,u) + \boldsymbol{\phi}(x,u)$ with $\boldsymbol{\xi}(x,u) \in T_xM$ and $\boldsymbol{\phi}(x,u) = (\phi^1(x,u), \dots, \phi^k(x,u)) \in \mathbb{R}^k$.

Partial derivatives

Consider a smooth function $f : M \times \mathbb{R}^k \to \mathbb{R}$. Partial derivatives of f are defined as follows: for fixed x partial derivatives of the function $f(x,\cdot) : \mathbb{R}^k \to \mathbb{R}$ with respect to u^α, $\alpha = 1, \dots, k$ are denoted by $\partial_{u^\alpha} f(x,u)$. Likewise, for fixed $u \in \mathbb{R}^k$ the function $f(\cdot,u) : M \to \mathbb{R}$ has a partial gradient with respect to x denoted by $\nabla_x f(x,u)$, and written in coordinates as $f(x,u)^{;x^i}\partial_{x^i}$.

Consider now a smooth vector field $\boldsymbol{\xi} : M \times \mathbb{R}^k \to TM$. By fixing x and differentiating w.r.t. u^α we obtain a new vector field $\partial_{u^\alpha}\boldsymbol{\xi}(x,u)$ for each $\alpha = 1, \dots, k$. By fixing $u \in \mathbb{R}^k$ we consider $\boldsymbol{\xi}(\cdot,u) : M \to TM$. The partial covariant derivative with respect to x gives the $(1,1)$-tensor $D_x\boldsymbol{\xi}(x,u)$. In local coordinates we write $\boldsymbol{\xi}(x,u)^i_{;x^j}\partial_{x^i} \otimes dx^j$. Similarly the partial divergence $\operatorname{div}_x \boldsymbol{\xi}(x,u)$ is defined.

Total derivatives

Suppose $u : M \to \mathbb{R}^k$ is a smooth $\mathbb{R}^k$-valued function. For $f : M \times \mathbb{R}^k \to \mathbb{R}$ we consider the new real-valued function $f(\mathrm{Id} \times u)(x) := f(x, u(x))$. For the total gradient of $f(\mathrm{Id} \times u) : M \to \mathbb{R}$ we have the formula

$$\nabla(f(x,u(x))) = \nabla_x f(x,u(x)) + \partial_{u^\alpha} f(x,u(x))\nabla u^\alpha,$$

written in local coordinates as $f^{;i}\partial_{x^i} = (f^{;x^i} + f_{,u^\alpha}u^{\alpha;i})\partial_{x^i}$. A similar constructions leads to the total differential of $f(\mathrm{Id} \times u)$:

$$d(f(x,u(x))) = d_x f(x,u(x)) + \partial_{u^\alpha} f(x,u(x))du^\alpha$$

written in local coordinates as $f_{,i}dx^i = (f_{,x^i} + f_{,u^\alpha}u^\alpha_{,i})dx^i$. If $\boldsymbol{\xi} : M \times \mathbb{R}^k \to TM$ is a vector-field we can consider the new vector-field $\boldsymbol{\xi}(\mathrm{Id} \times u) : M \to TM$. For the total covariant derivative we find

$$D\boldsymbol{\xi}(x,u(x)) = D_x\boldsymbol{\xi}(x,u(x)) + \partial_{u^\alpha}\boldsymbol{\xi}(x,u(x)) \otimes du^\alpha(x)$$

and in local coordinates we use the following notation

$$D\boldsymbol{\xi}(x,u(x)) = \xi^i_{;j}\partial_{x^i} \otimes dx^j = (\xi^i_{;x^j} + \xi^i_{,u^\alpha} u^\alpha_{,j})\partial_{x^i} \otimes dx^j.$$

For the total divergence we have

$$\operatorname{Div}\boldsymbol{\xi}(x,u(x)) = \operatorname{div}_x \boldsymbol{\xi}(x,u(x)) + \partial_{u^\alpha}\boldsymbol{\xi}(x,u(x)) \cdot \nabla u^\alpha(x)$$

with a similar expression in local coordinates.

3.3 One-parameter transformation groups on $M \times \mathbb{R}^k$

We start with the definition of a one-parameter transformation group on the total space $M \times \mathbb{R}^k$. It is a variant of Definition 2.1. For general references to one-parameter transformation groups see Olver [71].

Definition 3.1. *A one-parameter transformation group on $M \times \mathbb{R}^k$ is given by an open set $\mathcal{W} \subset \mathbb{R} \times M \times \mathbb{R}^k$ and a smooth map $G : \mathcal{W} \to M \times \mathbb{R}^k$ with the following properties:*

(a) if $(\epsilon_1, x, u), (\epsilon_2, G(\epsilon_1, x, u)), (\epsilon_1 + \epsilon_2, x, u) \in \mathcal{W}$ then

$$G(\epsilon_2, G(\epsilon_1, x, u)) = G(\epsilon_1 + \epsilon_2, x, u),$$

(b) $(0, x, u) \in \mathcal{W}$ for all $(x, u) \in M \times \mathbb{R}^k$ and $G(0, x, u) = (x, u)$,
(c) if $(\epsilon, x, u) \in \mathcal{W}$ then $(-\epsilon, x, u) \in \mathcal{W}$ and

$$G(-\epsilon, G(\epsilon, x, u)) = (x, u).$$

As before we write $g_\epsilon(x,u) := G(\epsilon, x, u)$ and $\operatorname{dom} g_\epsilon := \{(x,u) \in M \times \mathbb{R}^k : (\epsilon, x, u) \in \mathcal{W}\}$. We refer to the group $G = \{g_\epsilon\}_{\epsilon \in \mathbb{R}}$ as the collection of the group-elements g_ϵ.

An example of such a group can be constructed by a system of ordinary differential equations

$$\frac{dX}{d\epsilon} = \boldsymbol{\xi}(X, U), \qquad \frac{dU^\alpha}{d\epsilon} = \boldsymbol{\phi}^\alpha(X, U), \quad \alpha = 1, \ldots, k \tag{3.1}$$

with a smooth [1] vector field $\mathbf{w} = \boldsymbol{\xi} + \boldsymbol{\phi}$ on $M \times \mathbb{R}^k$. In local coordinates we have $\mathbf{w} = \xi^i(X,U)\partial_{x^i} + \phi^\alpha(X,U)\partial_{u^\alpha}$ and (3.1) is then the system of ordinary differential equations

$$\frac{dX^i}{d\epsilon} = \xi^i(X, U), \quad i = 1, \ldots, n, \quad \frac{dU^\alpha}{d\epsilon} = \phi^\alpha(X, U), \quad \alpha = 1, \ldots, k. \tag{3.2}$$

[1] continuous in (x, u) and locally Lipschitz continuous in u uniformly w.r.t. x are minimal requirements

We denote by $(\chi_\epsilon(x,u), \psi_\epsilon(x,u))$ the solution of (3.1) at time ϵ with initial condition $(x,u) \in M \times \mathbb{R}^k$ at $\epsilon = 0$, and we assume that the solution is maximally extended in time. The map

$$g_\epsilon(x,u) = (\chi_\epsilon(x,u), \psi_\epsilon(x,u)) \text{ for every } (x,u) \in M \times \mathbb{R}^k$$

is called the *flow-map at time* ϵ of the *flow* given by the system (3.1). The family $G = \{g_\epsilon\}_{\epsilon\in\mathbb{R}}$ forms a one-parameter transformation group. The vector field $\mathbf{w}$ is called the *infinitesimal generator* of the group G. By continuous dependence on parameters and initial conditions we have the following result:

Proposition 3.2. *For every compact subset $K \subset M \times \mathbb{R}^k$ there exists a maximal interval I_K containing* 0 *such that for each initial condition $(x,u) \in K$ the flow-map $g_\epsilon(x,u)$ is well defined for all $\epsilon \in I_K$.*

In the following we assume that all one-parameter transformation groups on $M \times \mathbb{R}^k$ are given through (3.1).

3.4 Action of transformation groups on functions

So far the group elements g_ϵ are defined for points (x,u) of the total space $M \times \mathbb{R}^k$. We want to generalize the group G such that it acts on $\mathbb{R}^k$-valued globally Lipschitz-continuous functions $u : \Omega \subset M \to \mathbb{R}^k$ where Ω is open and bounded. We fix such a function u. The graph $\Gamma_u = \{(x, u^1, \ldots, u^k) \in M \times \mathbb{R}^k : u^\alpha = u^\alpha(x), \alpha = 1, \ldots, k\}$ lies in a compact subset of $M \times \mathbb{R}^k$, and by Proposition 3.2 the group elements g_ϵ are well defined on Γ_u for ϵ in an open interval around 0. For ϵ sufficiently close to 0 the transformed graph $g_\epsilon(\Gamma_u)$ represents the graph of a new Lipschitz-continuous function $\tilde{u} = \tilde{u}(\tilde{x})$ defined for $\tilde{x} \in \tilde{\Omega} = \{\chi_\epsilon(x, u(x)) : x \in \Omega\}$ [2]. To find the expression for the new function $\tilde{u} = \tilde{u}(\tilde{x})$ we take a point $(\tilde{x}, \tilde{u})$ on $g_\epsilon(\Gamma_u)$, which means $(\tilde{x}, \tilde{u}) = (\chi_\epsilon(x, u(x)), \psi_\epsilon(x, u(x)))$ for some $x \in \Omega$. To write $\tilde{u}$ as an expression of $\tilde{x}$ we invert the expression $\tilde{x} = \chi_\epsilon(\mathrm{Id} \times u)(x)$ as shown

$$x \xrightarrow{\chi_\epsilon(\mathrm{Id}\times u)} \tilde{x}, \qquad x \xleftarrow{[\chi_\epsilon(\mathrm{Id}\times u)]^{-1}} \tilde{x}.$$

This implies the following formula for the transformed function $\tilde{u}$

$$\tilde{u}(\tilde{x}) = \psi_\epsilon(\mathrm{Id} \times u)[\chi_\epsilon(\mathrm{Id} \times u)]^{-1}(\tilde{x}), \quad \tilde{x} \in \tilde{\Omega}. \tag{3.3}$$

As an extension of the action of $G = \{g_\epsilon\}_{\epsilon\in\mathbb{R}}$ on points of $M \times \mathbb{R}^k$ we can now define the map

$$g_\epsilon : u \mapsto \tilde{u}$$

[2] The restriction to small ϵ is necessary, since for large ϵ the set $g_\epsilon(\Gamma_u)$ will no longer be the graph of a single-valued function.

for Lipschitz-functions $u : \Omega \to \mathbb{R}^k$ and for all $\epsilon \in (-\epsilon_0(u), \epsilon_0(u))$. The transformed function $\tilde{u}$ is defined on the transformed domain $\tilde{\Omega}$. For the transformed function we use the notation $g_\epsilon u$ as well as $\tilde{u}(\tilde{x})$, and for its domain of definition we write $g_\epsilon \Omega$ as well as $\tilde{\Omega}$. The following theorem, which is proved in Appendix B, states the mapping-properties of the group-elements $g_\epsilon : u \mapsto g_\epsilon u$.

Proposition 3.3. *For a Lipschitz domain Ω let $C^{0,1}(\overline{\Omega})$, $C^1(\overline{\Omega})$ be the space of Lipschitz continuous, continuously differentiable functions, respectively. Let $u \in C^{0,1}(\overline{\Omega})$ or $u \in C^1(\overline{\Omega})$. Then there exists $\epsilon_0 = \epsilon_0(u)$ such that for all $\epsilon \in (-\epsilon_0, \epsilon_0)$ we have*

(i) $g_\epsilon u$ belongs to $C^{0,1}(g_\epsilon \Omega)$ or $C^1(g_\epsilon \Omega)$, respectively,
(ii)$g_\epsilon \Omega$ is a Lipschitz domain.

Finally, it is easy to verify that $(g_{\epsilon_1} \circ g_{\epsilon_2})u = g_{\epsilon_1+\epsilon_2} u$. Thus, the transformation group G is now extended to act on globally Lipschitz-continuous functions defined on bounded subsets of M. The following figure sums up our definitions:

Differential equation	Initial condition	Solution
$\dot{X} = \boldsymbol{\xi}(X,U)$	$X(0) = x$	$\chi_\epsilon(x,u)$
$\dot{U} = \boldsymbol{\phi}(X,U)$	$U(0) = u$	$\psi_\epsilon(x,u)$

Function u	Transformed function $g_\epsilon u$	Transformed domain
$u : \begin{cases} \Omega \to \mathbb{R}^k \\ x \mapsto u(x) \end{cases}$	$\tilde{u} : \begin{cases} \tilde{\Omega} \to \mathbb{R}^k \\ \tilde{x} \mapsto \psi_\epsilon(\mathrm{Id} \times u) \circ [\chi_\epsilon(\mathrm{Id} \times u)]^{-1}(\tilde{x}) \end{cases}$	$g_\epsilon \Omega = \tilde{\Omega} = \chi_\epsilon(\mathrm{Id} \times u)\Omega$

The definition of $g_\epsilon u$ can be equivalently obtained by solving the transport equation

$$\partial_t U = \boldsymbol{\phi}(x,U) - \boldsymbol{\xi}(x,U) \cdot \nabla U, \quad U(0) = u \tag{3.4}$$

with an initial function $u \in C^{0,1}(\overline{\Omega})$ and setting $g_\epsilon u = U(\epsilon)$. In fact the system (3.1) is exactly the system of characteristic equations for the partial differential equation (3.4). We do not follows this approach to transformation groups through the transport equation.

Remark 3.4. (i) If $\boldsymbol{\xi} = 0$ and $\boldsymbol{\phi} = \boldsymbol{\phi}(u)$ then there is no change in the independent variable. In this case $G = \{g_\epsilon\}_{\epsilon \in \mathbb{R}}$ is indeed a one-parameter transformation group on e.g. $C^1(\overline{\Omega})$ or $C^{0,1}(\overline{\Omega})$. The group coincides with a differentiable group introduced in Chapter 2.

(ii) If $\boldsymbol{\xi}$ is non-tangential to $\partial\Omega$ then the spatial domain of definition of $g_\epsilon u$ varies with ϵ. Therefore $G = \{g_\epsilon\}_{\epsilon\in\mathbb{R}}$ does not map the function space like $C^1(\overline{\Omega})$ or $C^{0,1}(\overline{\Omega})$ for a fixed domain Ω into itself. Instead it maps $\mathcal{C}^1 = \{u \in C^1(\Omega) : \Omega \subset M,$ open and bounded$\}$ or $\mathcal{C}^{0,1} = \{u \in C^{0,1}(\Omega), \Omega \subset M$ open and bounded$\}$ into itself. These spaces are not vector spaces. Hence G is not a one-parameter transformation group in the strict sense of Definition 2.1. This problem will be resolved in Section 3.7. We still call $G = \{g_\epsilon\}_{\epsilon\in\mathbb{R}}$ a one-parameter transformation group.

Fixed points of the group-action

Definition 3.5. *Let G be a one-parameter transformation group on a function space V with infinitesimal generator $\mathbf{w} = \boldsymbol{\xi} + \boldsymbol{\phi}$. A function $u_0 \in V$ is a fixed point of G if $g_\epsilon u_0 = u_0$ for all $\epsilon \in \mathbb{R}$.*

Lemma 3.6. *A C^1-function u_0 is fixed point of a one-parameter transformation group G generated by $\mathbf{w} = \boldsymbol{\xi} + \boldsymbol{\phi}$ if and only if*

$$\boldsymbol{\phi}(x, u_0(x)) - \boldsymbol{\xi}(x, u_0(x)) \cdot \nabla u_0(x) = 0 \text{ for all } x \in \Omega. \tag{3.5}$$

Proof. Assume that u_0 is a fixed point, i.e. $(g_\epsilon u_0)(\tilde{x}) = u_0(\tilde{x})$. This is equivalent to

$$\psi_\epsilon(x, u_0(x)) = u_0(\chi_\epsilon(x, u_0(x))). \tag{3.6}$$

Differentiation w.r.t. ϵ at $\epsilon = 0$ leads to

$$\boldsymbol{\phi}(x, u_0(x)) = \nabla u_0 \cdot \boldsymbol{\xi}(x, u_0(x)).$$

Reversely, assume that (3.5) holds. Denote by $\chi_\epsilon^0(x)$ the solution of $\dot{X} = \boldsymbol{\xi}(X, u_0(X))$ with $X(0) = x$ and define $\psi_\epsilon^0(x) := u_0(\chi_\epsilon^0(x))$. For fixed x the two functions $\chi_\epsilon^0, \psi_\epsilon^0$ as functions of ϵ satisfy

$$\begin{aligned}
\dot{\chi}_\epsilon^0 &= \boldsymbol{\xi}(\chi_\epsilon^0, \psi_\epsilon^0), \\
\dot{\psi}_\epsilon^0 &= \nabla u_0(\chi_\epsilon^0) \cdot \boldsymbol{\xi}(\chi_\epsilon^0, \psi_\epsilon^0) = \boldsymbol{\phi}(\chi_\epsilon^0, \psi_\epsilon^0)
\end{aligned}$$

by assumption (3.5). Hence, as functions of ϵ the pair $(\chi_\epsilon^0(x), \psi_\epsilon^0(x))$ solves the same basic differential equation as $(\chi_\epsilon(x, u_0(x)), \psi_\epsilon(x, u_0(x)))$. Since the two pairs have the same initial conditions at $\epsilon = 0$ uniqueness of the solution guarantees

$$\chi_\epsilon^0(x) = \chi_\epsilon(x, u_0(x)), \quad \psi_\epsilon^0(x) = \psi_\epsilon(x, u_0(x)).$$

The latter of the two equations and the definition of ψ_ϵ^0 imply (3.6). Hence u_0 is a fixed point of G. ☐

Rate of change formula – a simplified case

We finish this section by showing that the interpretation of vector-fields as derivations is a useful concept. Let $L : M \times \mathbb{R}^k \to \mathbb{R}$ be a smooth function and suppose $u \in C^1(M; \mathbb{R}^k)$. Then we have in local coordinates

$$\frac{d}{d\epsilon} L(\tilde{x}, \tilde{u}(\tilde{x})) = \Big(\xi^i \partial_{x^i} L + \phi^\alpha \partial_{u^\alpha} L\Big)(\tilde{x}, \tilde{u}(\tilde{x})), \tag{3.7}$$

which can be written as

$$\frac{d}{d\epsilon} L(\tilde{x}, \tilde{u}(\tilde{x})) = (\mathbf{w}L)(\tilde{x}, \tilde{u}(\tilde{x})) \tag{3.8}$$

with the infinitesimal generator $\mathbf{w} = \boldsymbol{\xi} + \boldsymbol{\phi}$ of the flow (3.1). The proof of (3.7) is an immediate consequence of the chain-rule and of the definition of $\tilde{x}$ and $\tilde{u}(\tilde{x})$. If we recall that $\mathbf{w}$ acts as a derivation on L then (3.8) follows.

The example gives a formula for the rate of change of functions under the action of the group G. Our goal is to find a formula for the rate of change of functionals $\int_\Omega L(x, u, \nabla u)\, dx$ under the action of G. We will see in Section 3.6 that in local coordinates

$$\begin{aligned} &\frac{d}{d\epsilon} \int_{\tilde{\Omega}} L(\tilde{x}, \tilde{u}(\tilde{x}), \tilde{\nabla}\tilde{u}(\tilde{x}))\, d\tilde{x}\Big|_{\epsilon=0} \\ &= \int_\Omega \xi^i \partial_{x^i} L + \phi^\alpha \partial_{u^\alpha} L + (\phi^{\alpha;i} - \xi^j_{,k} g^{ki} g_{jl} u^{\alpha;l} - g^{ij} g_{kj,l} \xi^l u^{\alpha;k}) \partial_{p^{\alpha;i}} L\, dx \\ &\quad + \int_\Omega L \operatorname{Div} \boldsymbol{\xi}\, dx. \end{aligned}$$

The nontrivial formula arises because already the derivatives $\partial\tilde{u}/\partial\tilde{x}^i$ depend on ϵ in a complicated way. Also the domain of definition of the integral depends on ϵ. It is therefore necessary to find a systematic approach to the rate-of-change formula. In the following section we show how the derivatives of $\tilde{u}$ and the volume-form $d\tilde{x}$ change with ϵ.

3.5 Rate of change of derivatives and volume-forms

Derivatives of the initial function u will be transformed to derivatives of the new function $g_\epsilon u = \tilde{u}$. For the gradient of $\tilde{u}$ with respect to $\tilde{x} \in \tilde{\Omega}$ we write $\tilde{\nabla}\tilde{u}$. The next Proposition shows how $\tilde{\nabla}\tilde{u}(\tilde{x})$ changes with ϵ. To formulate it the concept of the adjoint of a linear map is needed.

Definition 3.7 (Adjoint map). *Let $A : T_xM \to T_yM$ be linear. Then its adjoint map $\operatorname{Adj} A : T_yM \to T_xM$ is defined by the relation*

$$g|_y(A\mathbf{v}, \mathbf{w}) = g|_x(\mathbf{v}, \operatorname{Adj} A\mathbf{w}) \textit{ for every } \mathbf{v} \in T_xM, \mathbf{w} \in T_yM.$$

In local coordinates, if $A\mathbf{v} = a^i_j v^j \partial_{y^i}|_y$ then $\operatorname{Adj} A\mathbf{w}|_x = {a^*}^i_j w^j \partial_{x^i}|_x$ with ${a^*}^i_j = g^{is}(x) g_{jr}(y) a^r_s$. In Euclidean $\mathbb{R}^n$ we have ${a^*}^i_j = a^j_i$.

Proposition 3.8. *Let G be a one-parameter transformation group with infinitesimal generator* $\mathbf{w}$. *For $u \in C^1(\overline{\Omega})$ let $\tilde{u}(\tilde{x})$ be the transformed function. For fixed $x \in M$ let $\frac{d}{d\epsilon}\tilde{\nabla}\tilde{u}(\tilde{x})$ be the covariant derivative of $\tilde{\nabla}\tilde{u}$ along the path $\gamma : \epsilon \mapsto \chi_\epsilon(x, u(x))$. Then*

$$\frac{d}{d\epsilon}\tilde{\nabla}\tilde{u}(\tilde{x})|_{\epsilon=0} = \nabla\boldsymbol{\phi}(x, u(x)) - \mathrm{Adj}[D\boldsymbol{\xi}(x, u(x))]\nabla u.$$

Proof. Recall the definition of $\tilde{u}(\tilde{x})$:

$$\tilde{u}(\tilde{x}) = \psi_\epsilon(\mathrm{Id} \times u) \circ [\chi_\epsilon(\mathrm{Id} \times u)]^{-1}(\tilde{x})$$

for $\tilde{x} \in \tilde{\Omega} = \chi_\epsilon(\mathrm{Id} \times u)\Omega$. The differential of the new function is given by

$$\tilde{d}\tilde{u}(\tilde{x}) = d\psi_\epsilon(\mathrm{Id} \times u)|_{\chi_\epsilon(\mathrm{Id} \times u)^{-1}(\tilde{x})} \circ [d\chi_\epsilon(\mathrm{Id} \times u)]^{-1}|_{\tilde{x}}.$$

Inserting the definition $\tilde{x} = \chi_\epsilon(x, u(x))$ in the above formula yields

$$\tilde{d}\tilde{u}(\tilde{x}) = d\psi_\epsilon(x, u(x)) \circ [d\chi_\epsilon(x, u(x))|_x]^{-1}.$$

By forming the metrically equivalent vector-field we obtain

$$\tilde{\nabla}\tilde{u}(\tilde{x}) = \mathrm{Adj}[d\chi_\epsilon(x, u(x))]^{-1}\nabla\psi_\epsilon(x, u(x)). \tag{3.9}$$

For a fixed base point $x \in M$ the path γ is defined as $\gamma : \epsilon \mapsto \chi_\epsilon(x, u(x))$. Let $\boldsymbol{\zeta}(\epsilon) := \tilde{\nabla}\tilde{u}^\alpha(\tilde{x})$ for a fixed $\alpha \in \{1, \ldots, k\}$. Then $\boldsymbol{\zeta}$ is a vector-field along the path γ. Thus

$$\frac{d}{d\epsilon}\boldsymbol{\zeta}(\epsilon)|_{\epsilon=0} = \big(\dot{\zeta}^i(0) + \Gamma^i_{jk}\dot{\gamma}^j(0)\zeta^k(0)\big)\partial_{x^i}|_x = \big(\dot{\zeta}^i(0) + \Gamma^i_{jk}\xi^j\zeta^k(0)\big)\partial_{x^i}|_x. \tag{3.10}$$

It remains to compute $\dot{\zeta}^i$. By (3.9), $\boldsymbol{\zeta}(\epsilon) = \mathrm{Adj}[d\chi_\epsilon(x, u(x))]^{-1}\nabla\psi^\alpha_\epsilon(x, u(x))$, which implies

$$\begin{aligned}
\zeta^i(\epsilon) &= \mathrm{Adj}[d\chi_\epsilon(x, u(x))]^{-1}{}^i_j g^{lj}(x)\psi^\alpha_\epsilon(x, u(x))_{,l} \\
&= [d\chi_\epsilon(x, u(x)))^{-1}]^r_s g^{is}(\chi_\epsilon(x, u(x))g_{rj}(x)g^{lj}(x)\psi^\alpha_\epsilon(x, u(x))_{,l} \\
&= [d\chi_\epsilon(x, u(x)))^{-1}]^l_s g^{is}(\chi_\epsilon(x, u(x))\psi^\alpha_\epsilon(x, u(x))_{,l}.
\end{aligned}$$

For the differentiation w.r.t. ϵ notice the law $\frac{d}{d\epsilon}A^{-1}_\epsilon = -A^{-1}_\epsilon \circ \frac{d}{d\epsilon}A_\epsilon \circ A^{-1}_\epsilon$ for the operator-family $A_\epsilon = d\chi_\epsilon(x, u(x)))^{-1}$. Since $A_0 = \mathrm{Id}$ we find

$$\begin{aligned}
\dot{\zeta}(\epsilon)^i|_{\epsilon=0} &= -\frac{d}{d\epsilon}\chi_\epsilon(x, u(x))^l_{,s}|_{\epsilon=0}g^{is}u^\alpha_{,l} + \delta^l_s g^{is}_{,k}\xi^k u^\alpha_{,l} + \delta^l_s g^{is}\phi^\alpha_{,l} \\
&= -\xi^l_{,s}g^{is}u^\alpha_{,l} + g^{il}_{,k}\xi^k u^\alpha_{,l} + g^{il}\phi^\alpha_{,l}.
\end{aligned} \tag{3.11}$$

Combining (3.10) and (3.11) we get

$$\begin{aligned}\frac{d}{d\epsilon}\tilde{\nabla}\tilde{u}^\alpha(\tilde{x})|_{\epsilon=0} &= (-\xi^l_{,s}g^{is}u^\alpha_{,l} + g^{il}_{,k}\xi^k u^\alpha_{,l} + g^{il}\phi^\alpha_{,l} + \Gamma^i_{jk}\xi^j u^{\alpha;k})\partial_{x^i} \\ &= (g^{il}\phi^\alpha_{,l} + (g^{il}_{,r}\xi^r g_{lk} - \xi^l_{,s}g^{is}g_{lk} + \Gamma^i_{jk}\xi^j)u^{\alpha;k})\partial_{x^i}.\end{aligned}$$

By a short calculation this simplifies to

$$\begin{aligned}\frac{d}{d\epsilon}\tilde{\nabla}\tilde{u}^\alpha(\tilde{x})|_{\epsilon=0} &= \nabla\boldsymbol{\phi}^\alpha - \Big((D\boldsymbol{\xi})^l_s g^{is}g_{lk}(\nabla u^\alpha)^k\Big)\partial_{x^i} \\ &= \nabla\boldsymbol{\phi}^\alpha - \mathrm{Adj}(D\boldsymbol{\xi})\nabla u^\alpha\end{aligned}$$

which establishes the claim. □

The rate of change of volume

Consider an ordinary differential equation on $\mathbb{R}^n$ given by $\dot{X} = \boldsymbol{\xi}(X)$. The solution with initial condition $X(0) = x$ is written as $\chi_\epsilon(x)$. The infinitesimal rate of change of the volume is given by

$$\frac{d}{d\epsilon}\int_{\tilde{\Omega}} d\tilde{x} = \int_\Omega \frac{d}{d\epsilon}\det d\chi_\epsilon(x)\,dx.$$

Differentiation yields $\frac{d}{d\epsilon}\det\chi_\epsilon(x) = \operatorname{div}\boldsymbol{\xi}|_{\chi_\epsilon(x)}\det d\chi_\epsilon(x)$ and hence

$$\frac{d}{d\epsilon}\int_{\tilde{\Omega}} d\tilde{x} = \int_{\tilde{\Omega}}(\operatorname{div}\boldsymbol{\xi})(\tilde{x})\,d\tilde{x}.$$

This statement in the context for transformation groups on manifolds is given below in Proposition 3.9. It requires a few preparations.

Volume forms. We will use the following facts, see Aubin [4]: a differential k-form $\boldsymbol{\omega}$ assigns to each $x \in M$ smoothly a multilinear, alternating map from $T_xM \times \ldots \times T_xM \to \mathbb{R}$. The space of n-forms on a n-dimensional manifold is one-dimensional. The manifold M is called *orientable* if there exists an n-form which never vanishes. An *orientation* is a selection of a non-vanishing n-form $\boldsymbol{\omega}$. A basis $\{\mathbf{w}^1, \ldots, \mathbf{w}^n\}$ of vector-fields of TM is called positively oriented if $\boldsymbol{\omega}(\mathbf{w}^1, \ldots, \mathbf{w}^n) > 0$. On an oriented manifold the unique n-form dx such that $dx(\mathbf{w}^1, \ldots, \mathbf{w}^n) = 1$ for every positively oriented orthonormal basis of vector-fields is called the *volume-form.* In local coordinates $dx = \sqrt{\det(g_{ij}(x))}dx^1 \wedge dx^2 \wedge \ldots \wedge dx^n$.

Jacobian determinant of a diffeomorphism. Let $\tau : \Omega \subset M \to M$ be a diffeomorphism with differential

$$d\tau|_x : T_xM \to T_{\tau(x)}M.$$

For a given n-form $\boldsymbol{\omega}$ we can define a new n-form $\tau^*\boldsymbol{\omega}$ by

$$(\tau^*\boldsymbol{\omega})(\mathbf{z}_1, \ldots, \mathbf{z}_n) := \boldsymbol{\omega}(d\tau\mathbf{z}_1, \ldots, d\tau\mathbf{z}_n) \text{ for vector fields } \mathbf{z}_1, \ldots, \mathbf{z}_n.$$

Since the space of n-forms is one-dimensional there exists a unique function $\det d\tau$ defined on Ω such that

$$\tau^* dx = \det d\tau\, dx.$$

The function $\det d\tau$ is call the *Jacobian determinant* of τ. In local coordinates $d\tau|_x \mathbf{z} = \tau(x)^i_{,j} z^j \partial_{x^i}|_{\tau(x)}$, which leads to the familiar formula $\det d\tau|_x = \det(\tau(x)^i_{,j})_{i,j=1,\ldots,n}$.

Proposition 3.9. *Let G be a one-parameter transformation group with infinitesimal generator $\mathbf{w} = \boldsymbol{\xi} + \boldsymbol{\phi}$. Then we find*

$$\frac{d}{d\epsilon} \det d\chi_\epsilon(x, u(x)) = (\operatorname{Div} \boldsymbol{\xi})(\tilde{x}, \tilde{u}(\tilde{x}), \tilde{\nabla}\tilde{u}(\tilde{x})) \det d\chi_\epsilon(x, u(x)) \text{ in } \Omega.$$

Proof. Suppose $x \in M$ and $\mathbf{z} \in T_x X$ are fixed. The map $\epsilon \mapsto d\chi_\epsilon(x, u(x))\mathbf{z}$ is a vector-field along the path $\gamma : \epsilon \mapsto \chi_\epsilon(x, u(x))$. We find $d\chi_\epsilon(x, u(x))\mathbf{z} = \chi_\epsilon(x, u(x))^i_{,j} z^j \partial_{x^i}|_{\tau_\epsilon(x)}$. We abbreviate $\chi_\epsilon = \chi_\epsilon(x, u(x))$ and $\psi_\epsilon = \psi_\epsilon(x, u(x))$. By the formula for the covariant derivative along γ we have

$$\begin{aligned}
&\frac{d}{d\epsilon}(d\chi_\epsilon \mathbf{z}) \\
&= \Big(\frac{d\chi^i_{\epsilon,j}}{d\epsilon} z^j + \Gamma^i_{jk} \chi^j_{\epsilon,l} z^l \dot{\gamma}^k\Big) \partial_{x^i}|_{\chi_\epsilon} \\
&= \Big(\xi(\chi_\epsilon, \psi_\epsilon)^i_{,j} z^j + \Gamma^i_{jk} \chi^j_{\epsilon,l} z^l \xi^k(\chi_\epsilon, \psi_\epsilon)\Big) \partial_{x^i}|_{\chi_\epsilon} \\
&= \Big(\xi^i_{,x^l}(\chi_\epsilon, \psi_\epsilon)\chi^l_{\epsilon,j} z^j + \xi^i_{,u^\alpha}(\chi_\epsilon, \psi_\epsilon)\psi^\alpha_{\epsilon,j} z^j + \Gamma^i_{jk} \chi^j_{\epsilon,l} z^l \xi^k(\chi_\epsilon, \psi_\epsilon)\Big) \partial_{x^i}|_{\chi_\epsilon}.
\end{aligned}$$

Collecting terms this can be written as

$$\begin{aligned}
\frac{d}{d\epsilon}(d\chi_\epsilon \mathbf{z}) &= (D_x \boldsymbol{\xi})(\chi_\epsilon, \psi_\epsilon) d\chi_\epsilon \mathbf{z} + (\partial_{u^\alpha} \boldsymbol{\xi})(\chi_\epsilon, \psi_\epsilon) \otimes d\psi^\alpha_\epsilon \mathbf{z} \\
&= \Big((D_x \boldsymbol{\xi})(\chi_\epsilon, \psi_\epsilon) + (\partial_{u^\alpha} \boldsymbol{\xi})(\chi_\epsilon, \psi_\epsilon) \otimes d\psi^\alpha_\epsilon \circ d\chi_\epsilon^{-1}\Big) d\chi_\epsilon \mathbf{z}.
\end{aligned}$$

And if we recall from Proposition 3.8 that $\tilde{d}\tilde{u} = d\psi^\alpha_\epsilon \circ d\chi_\epsilon^{-1}$ we finally obtain

$$\frac{d}{d\epsilon}(d\chi_\epsilon \mathbf{z}) = \Big(\underbrace{(D_x \boldsymbol{\xi})(\chi_\epsilon, \psi_\epsilon) + (\partial_{u^\alpha} \boldsymbol{\xi})(\chi_\epsilon, \psi_\epsilon) \otimes \tilde{d}\tilde{u}}_{=: A_\epsilon}\Big) d\chi_\epsilon \mathbf{z}. \tag{3.12}$$

Recall now the definition of the Jacobian determinant:

$$(\det d\chi_\epsilon) dx(\mathbf{z}_1, \ldots, \mathbf{z}_n) = dx(d\chi_\epsilon \mathbf{z}_1, \ldots, d\chi_\epsilon \mathbf{z}_n) \tag{3.13}$$

for all $\mathbf{z}_1, \ldots, \mathbf{z}_n \in T_x M$. Differentiation of (3.13) w.r.t. ϵ and use of (3.12) yield

$$
\begin{aligned}
&\Big(\frac{d}{d\epsilon}\det d\chi_\epsilon\Big)dx(\mathbf{z}_1,\dots,\mathbf{z}_n)\\
&=\sum_{i=1}^{n} dx(d\chi_\epsilon\mathbf{z}_1,\dots,d\chi_\epsilon\mathbf{z}_{i-1},A_\epsilon d\chi_\epsilon\mathbf{z}_i,d\chi_\epsilon\mathbf{z}_{i+1},\dots,d\chi_\epsilon\mathbf{z}_n)\\
&=\operatorname{trace}A_\epsilon\; dx(d\chi_\epsilon\mathbf{z}_1,\dots,d\chi_\epsilon\mathbf{z}_n)\\
&=(\operatorname{Div}\boldsymbol{\xi})(\tilde{x},\tilde{u}(\tilde{x}),\tilde{\nabla}\tilde{u}(\tilde{x}))\det d\chi_\epsilon\, dx(\mathbf{z}_1,\dots,\mathbf{z}_n).
\end{aligned}
$$

The claim is now proved. □

3.6 Rate of change of first-order variational functionals

Our goal is the rate of change formula for functionals $\int_\Omega L(x,u,\nabla u)\,dx$ under the action of a one-parameter transformation group G. To achieve this we need a precise notion of the partial derivatives of the Lagrangian.

3.6.1 Partial derivatives of Lagrangians

Consider a Lagrangian $L(x,u,\nabla u)$ in Euclidean space. If u is $\mathbb{R}^k$-valued then L is defined on $\mathbb{R}^{n+k+nk}$ and partial derivatives can be defined by freezing all but one variable and differentiating with respect to the remaining variable. For example

$$\nabla_x L(x,u,\mathbf{p})\cdot\mathbf{z} := \lim_{t\to 0}\frac{1}{t}\Big(L(x+t\mathbf{z},u,\mathbf{p})-L(x,u,\mathbf{p})\Big)$$

is the partial derivative of L w.r.t. to x in direction $\mathbf{z}$. On a Riemannian manifold a Lagrangian $L(x,u,\mathbf{p})$ is defined on $\mathbb{R}^k\times\bigcup_{y\in M}(T_yM)^k$, which means $(x,\mathbf{p})\in\bigcup_{y\in M}(T_yM)^k$ and $u\in\mathbb{R}^k$. The set $\bigcup_{y\in M}(T_yM)^k$ is an $n+nk$-dimensional manifold. It consists of points $(x,\mathbf{p}^1,\dots,\mathbf{p}^k)$ with $x\in M$ and $\mathbf{p}^1,\dots,\mathbf{p}^k\in T_xM$. The construction of varying x while freezing u and $\mathbf{p}$ is no longer possible, since the tangent space T_xM changes with x. Instead a construction based on the notion of parallel translation is necessary.

Parallel translation

Let $\gamma:(-1,1)\to M$ be a smooth path on M. If $\boldsymbol{\zeta}:(-1,1)\to TM$ is a vector-field along γ such that $\boldsymbol{\zeta}'=0$ then $\boldsymbol{\zeta}$ is called *parallel.* If $a\in(-1,1)$ and $\mathbf{z}\in T_{\gamma(a)}M$ are both fixed, then there exists a unique parallel vector-field $\boldsymbol{\zeta}$ with $\boldsymbol{\zeta}(a)=\mathbf{z}$. For two fixed values $a,b\in(-1,1)$ the *parallel translation* from $\gamma(a)$ to $\gamma(b)$ along γ is defined as the map

$$P_{a,b}:\begin{cases} T_{\gamma(a)}M & \to T_{\gamma(b)}M\\ \mathbf{z} & \mapsto \boldsymbol{\zeta}(b)\end{cases}$$

where $\boldsymbol{\zeta}$ is the unique parallel vector-field with $\boldsymbol{\zeta}(a) = \mathbf{z}$. Notice that $P_{a,b} = P_{b,a}^{-1}$. We will need the formula

$$\frac{d}{dt}(P_{0,t}\mathbf{z})^i = -\Gamma^i_{jk}(\gamma(t))\dot{\gamma}^j(t)z^k$$

which follows from the fact that $P_{0,t}\mathbf{z}$ is parallel along γ.

Definition 3.10 (Partial derivatives of a Lagrangian). *Let $L : \mathbb{R}^k \times \bigcup_{y\in M}(T_yM)^k \to \mathbb{R}$ be C^1. At the fixed point $(x, u, \mathbf{p}) \in \mathbb{R}^k \times \bigcup_{y\in M}(T_yM)^k$ we define:*

(i) the partial derivative of L w.r.t. x in direction $\mathbf{z} \in T_xM$ by

$$\nabla_x L(x, u, \mathbf{p}) \cdot \mathbf{z} := \frac{d}{dt}L(\gamma(t), u, P_{0,t}\mathbf{p})|_{t=0}.$$

Here $P_{0,t}$ is the parallel translation along a curve $\gamma : (-1, 1) \to M$ with $\gamma(0) = x$ and $\dot{\gamma}(0) = \mathbf{z}$. The value is independent of the curve γ.

(ii) the partial derivative of L w.r.t. u^α by

$$\partial_{u^\alpha} L(x, u, \mathbf{p}) := \frac{d}{dt}L(x, u^1, \dots, u^{\alpha-1}, u^\alpha + t, u^{\alpha+1} \dots, u^k, \mathbf{p})|_{t=0}.$$

(iii) the partial derivative of L w.r.t. $\mathbf{p}^\alpha$ in direction $\mathbf{z} \in T_xM$ by

$$\nabla_{\mathbf{p}^\alpha} L(x, u, \mathbf{p}) \cdot \mathbf{z} := \frac{d}{dt}L(x, u, \mathbf{p}^1, \dots, \mathbf{p}^{\alpha-1}, \mathbf{p}^\alpha + t\mathbf{z}, \mathbf{p}^{\alpha+1}, \dots, \mathbf{p}^k)|_{t=0}.$$

Proposition 3.11. *Let $L : \mathbb{R}^k \times \bigcup_{y\in M}(T_yM)^k \to \mathbb{R}$ be C^1. Then in local coordinates we have*

$$\nabla_x L(x, u, \mathbf{p}) \cdot \mathbf{z} = \Big(\frac{\partial L}{\partial x^i}(x, u, \mathbf{p}) - \Gamma^l_{ik}(x)p^{\alpha,k}\frac{\partial L}{\partial p^{\alpha,l}}(x, u, \mathbf{p})\Big)z^i \quad (3.14)$$

$$\nabla_{\mathbf{p}^\alpha} L(x, u, \mathbf{p}) \cdot \mathbf{z} = \frac{\partial L}{\partial p^{\alpha,i}}(x, u, \mathbf{p})z^i, \quad \alpha = 1, \dots, k. \quad (3.15)$$

In vector notation we have

$$\nabla_x L(x, u, \mathbf{p}) = g^{ij}(x)\Big(\frac{\partial L}{\partial x^j}(x, u, \mathbf{p}) - \Gamma^l_{jk}p^{\alpha,k}\frac{\partial L}{\partial p^{\alpha,l}}(x, u, \mathbf{p})\Big)\partial_{x^i} \quad (3.16)$$

$$\nabla_{\mathbf{p}^\alpha} L(x, u, \mathbf{p}) = g^{ij}(x)\frac{\partial L}{\partial p^{\alpha,j}}(x, u, \mathbf{p})\partial_{x^i}, \quad \alpha = 1, \dots, k. \quad (3.17)$$

Proof. (3.15) follows directly from Definition 3.10. To prove (3.14) we apply the definition of ∇_x and the properties of parallel translation:

$$\begin{aligned}
\nabla_x L(x, u, \mathbf{p}) \cdot \mathbf{z} &= \frac{d}{dt}L(\gamma(t), u, P_{0,t}\mathbf{p})\ |_{t=0} \\
&= \frac{\partial L}{\partial x^i}(x, u, \mathbf{p})z^i + \frac{\partial L}{\partial p^{\alpha,l}}(x, u, \mathbf{p})\frac{d}{dt}(P_{0,t}\mathbf{p}^\alpha)^l\ |_{t=0} \\
&= \frac{\partial L}{\partial x^i}(x, u, \mathbf{p})z^i - \frac{\partial L}{\partial p^{\alpha,l}}(x, u, \mathbf{p})\Gamma^l_{ik}(x)z^i p^{\alpha,k}.
\end{aligned}$$

This proves (3.14). □

Example 3.12. (i) The definition of the partial derivatives implies the following chain-rule for a C^2-function $u : M \to \mathbb{R}^k$:

$$\nabla L(x,u,\nabla u) = \\ \nabla_x L(x,u,\nabla u) + \partial_{u^\alpha} L(x,u,\nabla u)\nabla u^\alpha + (D\nabla u^\alpha)\, \nabla_{\mathbf{p}^\alpha} L(x,u,\nabla u).$$

The proof is straight-forward and uses the fact that $D\nabla u^\alpha$ is self-adjoint.
(ii) For the case of a single dependent variable u consider $L(x,u,\mathbf{p}) = |\mathbf{p}|^2 = g_{rs}p^r p^s$. Then $\nabla_x L = 0$. This is easily seen from $\frac{d}{dt}L(\gamma(t),u,P_{0,t}\mathbf{p})|_{t=0} = 0$ by the isometry property of parallel translation, which is equivalent to $\nabla_x g = 0$. It is also easy so see that $\nabla_{\mathbf{p}} L = 2\mathbf{p}$.

3.6.2 The rate of change formula

We are now ready to state and prove the main theorem in this section. In the Euclidean case this theorem can be obtained in full generality for Lagrangians of arbitrary order from results of Olver [71], Section 2.3 and 4.2.

Theorem 3.13. *Let $L : \mathbb{R}^k \times \bigcup_{y\in M}(T_yM)^k \to \mathbb{R}$ be a C^1-Lagrangian and let G be a transformation group with infinitesimal generator $\mathbf{w} = \boldsymbol{\xi} + \boldsymbol{\phi}$. If $u \in C^1(\overline{\Omega})$ then we find that*

$$\frac{d}{d\epsilon}\int_{\tilde{\Omega}} L(\tilde{x},\tilde{u}(\tilde{x}),\tilde{\nabla}\tilde{u}(\tilde{x}))\,d\tilde{x}\bigg|_{\epsilon=0} \\ = \int_{\Omega} \boldsymbol{\xi}\cdot\nabla_x L + \phi^\alpha\partial_{u^\alpha}L + (\nabla\phi^\alpha - \mathrm{Adj}(D\boldsymbol{\xi})\nabla u^\alpha)\cdot\nabla_{\mathbf{p}^\alpha}L + L\,\mathrm{Div}\,\boldsymbol{\xi}\,dx,$$

where L and its derivatives are evaluated at $(x,u,\nabla u(x))$ and $\boldsymbol{\xi}$, $\boldsymbol{\phi}^\alpha$ and their derivatives are evaluated at $(x,u(x))$.

Remark 3.14. If $\boldsymbol{\xi} = 0$ and $\boldsymbol{\phi} = \boldsymbol{\phi}(u)$ then Theorem 3.13 reduces to the result of Theorem 2.9.

Proof. We begin by applying the change of variables formula

$$\int_{\tilde{\Omega}} L(\tilde{x},\tilde{u}(\tilde{x}),\tilde{\nabla}\tilde{u}(\tilde{x}))\,d\tilde{x} \\ = \int_{\Omega} L(\chi_\epsilon(x,u(x)),\psi_\epsilon(x,u(x)),\tilde{\nabla}\tilde{u}|_{\chi_\epsilon(x,u(x))})\det d\chi_\epsilon(x,u(x))\,dx. \quad (3.18)$$

For the purpose of finding the ϵ-derivative at $\epsilon = 0$ of the integrand we need to calculate difference quotients. By adding and subtracting L at suitable intermediate points we find

$$\begin{aligned}
&L(\chi_\epsilon(x,u(x)),\psi_\epsilon(x,u(x)),\tilde\nabla\tilde u|_{\chi_\epsilon(x,u(x))})-L(x,u(x),\nabla u(x))\\
&\quad=L(\chi_\epsilon(x,u(x)),\psi_\epsilon(x,u(x)),\tilde\nabla\tilde u|_{\chi_\epsilon(x,u(x))})\\
&\qquad-L(\chi_\epsilon(x,u(x)),u(x),\tilde\nabla\tilde u|_{\chi_\epsilon(x,u(x))})\\
&\qquad+L(\chi_\epsilon(x,u(x)),u(x),\tilde\nabla\tilde u|_{\chi_\epsilon(x,u(x))})-L(x,u(x),P_{\epsilon,0}\tilde\nabla\tilde u|_{\chi_\epsilon(x,u(x))})\\
&\qquad+L(x,u(x),P_{\epsilon,0}\tilde\nabla\tilde u|_{\chi_\epsilon(x,u(x))})-L(x,u(x),\nabla u(x)).
\end{aligned}$$

By dividing through ϵ and taking the limit $\epsilon\to 0$ the difference of the first pair of terms converges to $\boldsymbol{\phi}^\alpha(x,u(x))\partial_{u^\alpha}L$. Similarly the second pair yields $\boldsymbol{\xi}(x,u(x))\cdot\nabla_x L$. By the definition of the partial derivative of L w.r.t. $\mathbf{p}^\alpha$ the third pair leads to $\nabla_{\mathbf{p}^\alpha}L\cdot\frac{d}{d\epsilon}P_{\epsilon,0}\tilde\nabla\tilde u|_{\chi_\epsilon(x,u(x))}|_{\epsilon=0}$. The definition of the parallel translation implies that $\frac{d}{d\epsilon}P_{\epsilon,0}\boldsymbol{\zeta}(\epsilon)=P_{\epsilon,0}\boldsymbol{\zeta}'(\epsilon)$. If we recall the rate-of-change of $\tilde\nabla\tilde u$ from Proposition 3.8 then the third pair is seen to converge to $\nabla_{\mathbf{p}^\alpha}L\cdot(\nabla\boldsymbol{\phi}^\alpha-\operatorname{Adj}(D\boldsymbol{\xi})\nabla u^\alpha)$. Altogether we get

$$\begin{aligned}
&\frac{d}{d\epsilon}L(\chi_\epsilon(x,u(x)),\psi_\epsilon(x,u(x)),\tilde\nabla\tilde u|_{\chi_\epsilon(x,u(x))})\Big|_{\epsilon=0}\\
&\quad=\partial_{u^\alpha}L\boldsymbol{\phi}^\alpha(x,u(x))+\nabla_x L\cdot\boldsymbol{\xi}(x,u(x))+\nabla_{\mathbf{p}^\alpha}L\cdot(\nabla\boldsymbol{\phi}^\alpha-\operatorname{Adj}(D\boldsymbol{\xi})\nabla u^\alpha).
\end{aligned}$$

Finally, differentiation w.r.t. ϵ under the integral in (3.18) produces the claim if we observe that by the product rule we also have to differentiate $\det d\chi_\epsilon(x,u(x))$ at $\epsilon=0$. By Proposition 3.9 we pick up the term $L\operatorname{Div}\boldsymbol{\xi}$. □

The prolongation of $\mathbf{w}$

Theorem 3.13 is a generalization of the rate of change formula of Theorem 2.9. This similarity becomes more evident if we introduce the formal differential operator $\mathbf{w}^{(1)}=\boldsymbol{\xi}\cdot\nabla_x+\phi^\alpha\partial_{u^\alpha}+(\nabla\phi^\alpha-\operatorname{Adj}(D\boldsymbol{\xi})\nabla u^\alpha)\cdot\nabla_{\mathbf{p}^\alpha}$. Then the rate-of-change formula of Theorem 3.13 can be written as

$$\frac{d}{d\epsilon}\int_{\tilde\Omega}L(\tilde x,\tilde u(\tilde x))\,d\tilde x\Big|_{\epsilon=0}=\int_\Omega \mathbf{w}^{(1)}L+L\operatorname{Div}\boldsymbol{\xi}\,dx. \tag{3.19}$$

The differential operator $\mathbf{w}^{(1)}$ is called the *prolongation* of $\mathbf{w}$, cf. Olver [71]. If the Lagrangian $L=L(x,u)$ does not depend on ∇u then $\mathbf{w}^{(1)}L=\mathbf{w}L$ in the sense that $\mathbf{w}=\boldsymbol{\xi}+\boldsymbol{\phi}$ acts through $\boldsymbol{\xi}$ as a derivation in x and through $\boldsymbol{\phi}$ as a derivation in u.

Corollary 3.15. *In explicit terms we have*

$$\begin{aligned}
\mathbf{w}^{(1)}&=\boldsymbol{\xi}\cdot\nabla_x+\phi^\alpha\partial_{u^\alpha}\\
&\qquad+\Big(\nabla_x\phi^\alpha+(\partial_{u^\beta}\phi^\alpha)\nabla u^\beta-\operatorname{Adj}(D_x\boldsymbol{\xi}+(\partial_{u^\beta}\boldsymbol{\xi})\otimes du^\beta)\nabla u^\alpha\Big)\cdot\nabla_{\mathbf{p}^\alpha}
\end{aligned}$$

and in terms of local coordinates we have

$$\mathbf{w}^{(1)} = \xi^i \partial_{x^i} + \phi^\alpha \partial_{u^\alpha}$$
$$+ (\phi^{\alpha;x^i} + \phi^\alpha_{,u^\beta} u^{\beta;i} - \xi^j_{,x^k} g^{ki} g_{jl} u^{\alpha;l} - \xi^j_{,u^\beta} g_{jk} u^{\beta;i} u^{\alpha;k} - g^{ij} g_{kj,l} \xi^l u^{\alpha;k}) \partial_{p_{\alpha,i}}.$$

Remark 3.16. Notice that $\boldsymbol{\xi} L = \boldsymbol{\xi} \cdot \nabla_x L$, since $\boldsymbol{\xi}$ acts as a derivation on the function L. Hence, in a strict sense, $\mathbf{w}^{(1)} = \boldsymbol{\xi} + \phi^\alpha \partial_{u^\alpha} + (\ldots) \cdot \nabla_{\mathbf{p}^\alpha}$ would be correct. However, we prefer $\mathbf{w}^{(1)} = \boldsymbol{\xi} \cdot \nabla_x + \phi^\alpha \partial_{u^\alpha} + (\ldots) \cdot \nabla_{\mathbf{p}^\alpha}$ since then $\mathbf{w}^{(1)} L$ resembles better the term in the rate of change formula of Theorem 3.13.

3.6.3 Noether's formula and Pohožaev's identity

Our previous consideration led to a formula of the rate of change of first order variational functionals under the action of a one-parameter group of transformations. In a celebrated paper [70], Emmy Noether realized that the volume-integrand in the rate-of-change formula of Theorem 3.13 can be rewritten as a multiple of the associated Euler-Lagrange operator plus a divergence term. This observation was the key-stone for Emmy Noether's famous theorem on symmetry induced conservation laws, see Olver [71], Section 4.4.

Theorem 3.17 (Noether's formula). *Let $L : \mathbb{R}^k \times \bigcup_{y \in M} (T_y M)^k \to \mathbb{R}$ be a C^1-Lagrangian with Euler-Lagrange operator $\mathcal{E}_\alpha[u] = -\operatorname{Div}(\nabla_{\mathbf{p}^\alpha} L) + \partial_{u^\alpha} L$ and let G be a transformation group with infinitesimal generator $\mathbf{w}$. If $u \in C^2(\Omega)$ then*

$$\mathbf{w}^{(1)} L + L \operatorname{Div} \boldsymbol{\xi} = (\phi^\alpha - \boldsymbol{\xi} \cdot \nabla u^\alpha) \mathcal{E}_\alpha[u] + \operatorname{Div}(\boldsymbol{\xi} L + (\phi^\alpha - \boldsymbol{\xi} \cdot \nabla u^\alpha) \nabla_{\mathbf{p}^\alpha} L). \quad (3.20)$$

If $u \in W^{2,1}(\Omega)$ then the formula holds almost everywhere.

Proof. We need the formula $\nabla(\boldsymbol{\xi} \cdot \nabla u) = (\operatorname{Adj} D\boldsymbol{\xi}) \nabla u + (\operatorname{Adj} D\nabla u) \boldsymbol{\xi}$. We prove (3.20) by direct calculation of the right-hand-side:

$$\begin{aligned}
&(\phi^\alpha - \boldsymbol{\xi} \cdot \nabla u^\alpha) \mathcal{E}_\alpha[u] + \operatorname{Div}\Big(\boldsymbol{\xi} L + (\phi^\alpha - \boldsymbol{\xi} \cdot \nabla u^\alpha) \nabla_{\mathbf{p}^\alpha} L\Big) \\
&= (\phi^\alpha - \boldsymbol{\xi} \cdot \nabla u^\alpha) \partial_{u^\alpha} L + L \operatorname{Div} \boldsymbol{\xi} + \boldsymbol{\xi} \cdot \nabla L + \nabla(\phi^\alpha - \boldsymbol{\xi} \cdot \nabla u^\alpha) \cdot \nabla_{\mathbf{p}^\alpha} L \\
&= (\phi^\alpha - \boldsymbol{\xi} \cdot \nabla u^\alpha) \partial_{u^\alpha} L + L \operatorname{Div} \boldsymbol{\xi} + \boldsymbol{\xi} \cdot \Big(\nabla_x L + \partial_{u^\alpha} L \nabla u^\alpha + (D\nabla u^\alpha) \nabla_{\mathbf{p}^\alpha} L\Big) \\
&\quad + \Big(\nabla \phi^\alpha - \operatorname{Adj}(D\boldsymbol{\xi}) \nabla u^\alpha - \operatorname{Adj}(D\nabla u^\alpha) \boldsymbol{\xi}\Big) \cdot \nabla_{\mathbf{p}^\alpha} L.
\end{aligned}$$

Since $D\nabla u^\alpha$ is a self-adjoint $(1,1)$-tensor, a cancellation happens, and the left-hand-side of (3.20) follows. □

For the next result we briefly discuss the divergence theorem on oriented Riemannian manifolds. Suppose that Ω is a bounded open subset of the manifold M such that $\partial\Omega$ is Lipschitz. Clearly $\partial\Omega$ inherits the metric, and becomes itself a Riemannian manifold. Let $\boldsymbol{\nu}(x)$ denote the exterior normal-field on $\partial\Omega$. If dx denotes the volume-form on M and $d\sigma$ the volume-form on $\partial\Omega$ then we have the formula

$$\int_\Omega \operatorname{div}\mathbf{z}\,dx = \int_{\partial\Omega} \mathbf{z}\cdot\boldsymbol{\nu}\,d\sigma$$

for every C^1-vector-field $\mathbf{z}$ on Ω.

Theorem 3.18 (Pohožaev's identity). *Let $L:\mathbb{R}^k\times\bigcup_{y\in M}(T_yM)^k\to\mathbb{R}$ be a C^1-Lagrangian with Euler-Lagrange operator $\mathcal{E}_\alpha[u]=-\operatorname{Div}(\nabla_{\mathbf{p}^\alpha}L)+\partial_{u^\alpha}L$ and let G be a transformation group with infinitesimal generator $\mathbf{w}$. If $u\in C^1(\overline{\Omega})\cap C^2(\Omega)$ then the following identity holds:*

$$\begin{aligned}
&\frac{d}{d\epsilon}\int_{\tilde\Omega} L(\tilde x,\tilde u(\tilde x),\tilde\nabla\tilde u(\tilde x))\,d\tilde x\Big|_{\epsilon=0}\\
&=\int_\Omega(\mathbf{w}^{(1)}L+L\operatorname{Div}\boldsymbol{\xi})(x,u(x),\nabla u(x))\,dx && (3.21)\\
&=\int_\Omega(\boldsymbol{\phi}^\alpha-\boldsymbol{\xi}\cdot\nabla u^\alpha)\mathcal{E}_\alpha[u]\,dx && (3.22)\\
&\quad+\int_{\partial\Omega}\boldsymbol{\nu}\cdot\boldsymbol{\xi}L+(\phi^\alpha-\boldsymbol{\xi}\cdot\nabla u^\alpha)\boldsymbol{\nu}\cdot\nabla_{\mathbf{p}^\alpha}L\,d\sigma_x
\end{aligned}$$

If $u\in C^{0,1}(\overline{\Omega})\cap W^{2,1}(\Omega)$ then the same formula holds.

Proof. The proof follows from integrating Noether's formula (3.20) and applying the divergence theorem. □

3.7 Admissible transformation groups

So far we have investigated how $\int_{\tilde\Omega} L(\tilde x,\tilde u(\tilde x),\tilde\nabla\tilde u(\tilde x))\,d\tilde x$ changes with ϵ. Our primary goal is however to obtain information on the critical points of the functional $\mathcal{L}[u]=\int_\Omega L(x,u(x),\nabla u(x))\,dx$. In order to link the two objects we will have to look out for those transformation groups which have the property

$$\mathcal{L}[g_\epsilon u]=\int_{\tilde\Omega} L(\tilde x,\tilde u(\tilde x),\tilde\nabla\tilde u(\tilde x))\,d\tilde x. \tag{3.23}$$

Notice that $g_\epsilon u$ is defined on $\tilde\Omega=g_\epsilon\Omega$, whereas in order to insert it in the functional $\mathcal{L}$ the transformed function $g_\epsilon u$ must be defined (or at least definable) on Ω. In order to achieve (3.23) we consider special classes of transformation groups. Their definition depends on the underlying function space.

Choice of the function spaces

The following two function spaces are frequently used.

The space $C_0^{0,1}(\overline{\Omega})$. This is the space of Lipschitz functions with zero-boundary conditions. The functional $\mathcal{L}$ is Fréchet-differentiable in the space $C_0^{0,1}(\overline{\Omega})$ provided $L(x,u,\mathbf{p})$ is a C^1-Lagrangian, cf. Appendix A.

The space $C_{\Gamma_D}^{0,1}(\overline{\Omega})$. Suppose $\partial\Omega$ splits into $\overline{\Gamma}_D \cup \overline{\Gamma}_N$ with two relatively open, disjoint sets $\Gamma_D, \Gamma_N \subset \partial\Omega$. The space $C_{\Gamma_D}^{0,1}(\overline{\Omega})$ is defined as the space of all Lipschitz functions vanishing on $\overline{\Gamma}_D$. Notice that for $\Gamma_N = \emptyset$ we recover the previous situation.

Since $\partial\Omega$ is assumed to be Lipschitz we can consider Γ_N as a subset of the $n-1$-dimensional Lipschitz manifold $\partial\Omega$ with inherited metric g and volume form $d\sigma$. Likewise we assume that Γ_N has a relative boundary $\partial\Gamma_N$ which is Lipschitz. Thus $\partial\Gamma_N$ with the inherited metric g becomes a $n-2$-dimensional Riemannian manifold with volume form $d\lambda$.

Definition 3.19. *Let $G = \{g_\epsilon\}_{\epsilon\in\mathbb{R}}$ be a one-parameter transformation group with infinitesimal generator $\mathbf{w} = \boldsymbol{\xi} + \boldsymbol{\phi}$ defined on a function space V.*

(a) If $V = C_0^{0,1}(\overline{\Omega})$ then the group G is called domain contracting if $g_\epsilon\Omega \subset \Omega$ for all $\epsilon \geq 0$.
(b) If $V = C_{\Gamma_D}^{0,1}(\overline{\Omega})$ then the group G is called domain contracting if $g_\epsilon\Omega \subset \Omega$ and $g_\epsilon\Gamma_N \subset \Gamma_N$ for all $\epsilon \geq 0$.

This definition immediately leads to the following characterization of the domain contraction property.

Lemma 3.20. *Let G be a one-parameter transformation group with infinitesimal generator $\mathbf{w} = \boldsymbol{\xi} + \boldsymbol{\phi}$ defined on the function space V. The group G is domain contracting provided*

(a) $\boldsymbol{\xi}(x,u(x)) \cdot \boldsymbol{\nu}(x) \leq 0$ on $\partial\Omega$ for all $u \in V$ if $V = C_0^{0,1}(\overline{\Omega})$,
(b) $\boldsymbol{\xi}(x,u(x)) \cdot \boldsymbol{\nu}(x) \leq 0$ on Γ_D, $\boldsymbol{\xi}(x,u(x)) \cdot \boldsymbol{\nu}(x) = 0$ on Γ_N and $\boldsymbol{\xi}(x,u(x)) \cdot \boldsymbol{\nu}_{\Gamma_N}(x) \leq 0$ on $\partial\Gamma_N$ for all $u \in V$ if $V = C_{\Gamma_D}^{0,1}(\overline{\Omega})$.

Here $\boldsymbol{\nu}(x)$ is the exterior normal on $\partial\Omega$ and $\boldsymbol{\nu}_{\Gamma_N}(x)$ is the relative exterior normal on $\partial\Gamma_N$. Notice that for $x \in \Gamma_N$ one has $\boldsymbol{\xi}(x,u(x)) \in T_x\Gamma_N$ since $\boldsymbol{\xi}(x,u(x)) \in T_xM$ and $\boldsymbol{\xi}(x,\nu(x)) \cdot \boldsymbol{\nu}(x) = 0$.

Fixed points and extension of $g_\epsilon u$ to all of Ω

Assume now that the domain contracting transformation group G is defined on the function space $V = C_{\Gamma_D}^{0,1}(\overline{\Omega})$. If G has a fixed point $u_0 \in V$ then the extension

$$g_\epsilon u = u_0 \text{ on } \overline{\Omega} \setminus g_\epsilon\Omega$$

is well defined for $\epsilon \geq 0$. It has the following implication, which is an extension of Proposition 3.3.

Lemma 3.21. *Let G be domain contracting one-parameter transformation group on $V = C^{0,1}_{\Gamma_D}(\overline{\Omega})$. If $u \in V$ then $g_\epsilon u \in V$ provided $\epsilon \geq 0$.*

The lemma shows that for $\epsilon \geq 0$ the "semi-group" $G = \{g_\epsilon\}_{\epsilon \geq 0}$ satisfies the Definition 2.1 of a "proper" one-parameter transformation group except (c). A transformation group $G = \{g_\epsilon\}_{\epsilon \geq 0}$ with the property stated in Lemma 3.21 is called an *admissible transformation group.*

Proof. Fix $u \in C^{0,1}_{\Gamma_D}(\overline{\Omega})$. Recall that $(\chi_\epsilon(x,u), \psi_\epsilon(x,u))$ denotes the solution of $\dot{X} = \boldsymbol{\xi}(X,U)$, $\dot{U} = \boldsymbol{\phi}(X,U)$ with initial condition (x,u) at $\epsilon = 0$. If $x \in \overline{\Gamma_N}$ then $\tilde{x} = \chi_\epsilon(x,u(x)) \in \overline{\Gamma_N}$ for all $\epsilon \geq 0$. If $x \in \overline{\Gamma_D}$ then $\tilde{x} = \chi_\epsilon(x,u(x)) \in g_\epsilon\overline{\Gamma_D}$. By the fact that u_0 is a fixed point of G we have $\psi_\epsilon(x,u_0(x)) = u_0(\chi_\epsilon(x,u_0(x)))$ for all $x \in \overline{\Gamma_D}$. Now recall that $\tilde{u}(\tilde{x}) = \psi_\epsilon(x,u(x))$, where $\tilde{x} = \chi_\epsilon(x,u(x))$. If we take $\tilde{x} \in g_\epsilon\overline{\Gamma_D}$ then the corresponding x lies in $\overline{\Gamma_D}$ and thus $\tilde{u}(\tilde{x}) = \psi_\epsilon(x,u(x)) = \psi_\epsilon(x,u_0(x)) = u_0(\chi_\epsilon(x,u_0(x)))$ since $u = u_0 = 0$ on $\overline{\Gamma_D}$. Hence $\tilde{u}(\tilde{x}) = u_0(\tilde{x})$ for $x \in g_\epsilon\overline{\Gamma_D}$. The function $g_\epsilon u$ is then continuously extended by u_0 into the region $\Omega \setminus g_\epsilon\Omega$. Since $g_\epsilon\Omega$ is a Lipschitz domain by Proposition 3.3 we see that the extension of $g_\epsilon u$ onto Ω is Lipschitz. □

Remark 3.22. We know from Proposition 3.3 that $g_\epsilon u$ on $g_\epsilon\Omega$ is as smooth as u on Ω. Since $g_\epsilon\Omega$ is a Lipschitz-domain the extension of $g_\epsilon u$ to all of Ω is again Lipschitz. However, even if $u \in C^1(\overline{\Omega})$ one has in general that $g_\epsilon u \notin C^1(\overline{\Omega})$ since a fold (german "Knick") occurs at those parts of $g_\epsilon\Gamma_D$ where $g_\epsilon\Gamma_D \not\subset \Gamma_D$. Therefore the natural function space to work in is the Lipschitz-space $V = C^{0,1}_{\Gamma_D}(\overline{\Omega})$.

3.8 Rate of change formula for solutions

Next we compute the rate of change $\frac{d}{d\epsilon}\mathcal{L}[u]|_{\epsilon=0}$ at critical points of functionals $\mathcal{L}[u] = \int_\Omega L(x,u,\nabla u)\,dx$. With out loss of generality we may assume the normalization $L(x,u_0(x),\nabla u_0(x)) = 0$ for all $x \in \Omega$ where $u_0 \in V$ is a fixed point of a one-parameter transformation group on the function space V. Then (3.23) holds due to the way we extended $g_\epsilon u$ outside $g_\epsilon\Omega$.

Theorem 3.23 (Pohožaev's identity for solutions I). *Let $L : \mathbb{R}^k \times \bigcup_{y\in M}(T_yM)^k \to \mathbb{R}$ be a C^1-Lagrangian with Euler-Lagrange operator $\mathcal{E}_\alpha[u] = -\operatorname{Div}(\nabla_{\mathbf{p}^\alpha}L) + \partial_{u^\alpha}L$. Let G defined on $C^{0,1}_0(\overline{\Omega})$ be an admissible domain contracting transformation group with infinitesimal generator $\mathbf{w}$. Let u_0 be a fixed point of G and assume $L(x,u_0,\nabla u_0) = 0$. If $u \in C^2(\Omega) \cap C^1_0(\overline{\Omega})$ satisfies the Euler-Lagrange equation $\mathcal{E}_\alpha[u] = 0$ in Ω with $u = 0$ on $\partial\Omega$ then the following identity holds:*

$$\frac{d}{d\epsilon}\mathcal{L}[g_\epsilon u]|_{\epsilon=0} = \int_\Omega (\mathbf{w}^{(1)}L + L\operatorname{Div}\boldsymbol{\xi})(x,u(x),\nabla u(x))\,dx \tag{3.24}$$

$$= \int_{\partial\Omega} \boldsymbol{\nu}\cdot\boldsymbol{\xi}(L + (\nabla u_0^\alpha - \nabla u^\alpha)\cdot\nabla_{\mathbf{p}^\alpha}L)\,d\sigma_x. \tag{3.25}$$

For $u \in C_0^{0,1}(\overline{\Omega}) \cap W^{2,1}(\Omega)$ *satisfying the Euler-Lagrange equation* $\mathcal{E}_\alpha[u] = 0$ *almost everywhere the identity* (3.24)-(3.25) *still holds.*

Proof. The proof is based on (3.21)-(3.22) in Theorem 3.18. Since u_0 is a fixed point of G we have by Lemma 3.6 $\phi^\alpha(x,0) = \boldsymbol{\xi}(x,0) \cdot \nabla u_0^\alpha(x)$ for all $x \in \partial\Omega$. Therefore the surface-integrand in (3.22) becomes

$$\boldsymbol{\xi}(x,0) \cdot \boldsymbol{\nu} L + \boldsymbol{\xi}(x,0) \cdot (\nabla u_0^\alpha(x) - \nabla u^\alpha(x))\boldsymbol{\nu} \cdot \nabla_{\mathbf{p}^\alpha} L. \tag{3.26}$$

Due to the Dirichlet boundary-conditions one finds $\nabla u_0^\alpha - \nabla u^\alpha = ((\nabla u_0^\alpha - \nabla u^\alpha) \cdot \boldsymbol{\nu})\boldsymbol{\nu}$. Inserting this into (3.26) we obtain the boundary-integrand of (3.25). □

Remark 3.24. In the literature the equality between the volume integral (3.24) and the surface integral (3.25) is sometimes called Pohožaev's identity. It has a long history. Clearly Noether [70] is an early source of reference. Subsequently, Finkelstein [32] used it in the derivation of the virial theorem of quantum mechanics (see Kalf [52] for a rigorous proof). Rellich [81] used it similarly as Finkelstein to show the absence of eigenvalues of linear differential operators. In a truly nonlinear context of elliptic equations the identity was first observed by Pohožaev [75] and later proved in full generality by Pucci, Serrin [77]. Hulshof, vanderVorst [48] made an attempt to link Pohožaev's identity to transformation groups in the case of exact variational symmetries rather than sub-symmetries. Wagner [91] derived a version of Pohožaev's identity based on domain-variation formulas of Hadamard.

Next an extension of the previous theorem is given for the case where

$$\mathcal{L}[u] = \int_\Omega L(x,u,\nabla u)\,dx + \int_{\Gamma_N} K(x,u)\,d\sigma_x.$$

is defined on $V = C^{0,1}_{\Gamma_D}(\overline{\Omega})$. The Euler-Lagrange equation satisfied by a critical point of $\mathcal{L}$ is

$$-\operatorname{Div}(\nabla_{\mathbf{p}^\alpha} L) + \partial_{u^\alpha} L \text{ in } \Omega, \quad u = 0 \text{ on } \Gamma_D, \quad \nabla_{\mathbf{p}^\alpha} L \cdot \nu + \partial_{u^\alpha} K = 0 \text{ on } \Gamma_N.$$

Theorem 3.25 (Pohožaev's identity for solutions II). *Let* $L : \mathbb{R}^k \times \bigcup_{y\in M}(T_yM)^k \to \mathbb{R}$ *and* $K : M \times \mathbb{R}^k \to \mathbb{R}$ *be* C^1*-Lagrangians with associated Euler-Lagrange operator* $\mathcal{E}_\alpha[u] = -\operatorname{Div}(\nabla_{\mathbf{p}^\alpha} L) + \partial_{u^\alpha} L$ *in* Ω*. Let* G *defined on* $C^{0,1}_{\Gamma_D}(\overline{\Omega})$ *be an admissible domain contracting transformation group with infinitesimal generator* $\mathbf{w}$*. Let* u_0 *be a fixed point of* G *and assume* $L(x,u_0,\nabla u_0) = 0$ *in* Ω *and* $K(x,u_0) = 0$ *on* Γ_N*. If* $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$ *satisfies the Euler-Lagrange equation* $\mathcal{E}_\alpha[u] = 0$ *in* Ω *with* $u = 0$ *on* Γ_D *and* $\nabla_{\mathbf{p}^\alpha} L \cdot \nu + \partial_{u^\alpha} K = 0$ *on* Γ_N *then the following identity holds:*

$$\frac{d}{d\epsilon}\mathcal{L}[g_\epsilon u]|_{\epsilon=0} = \int_\Omega (\mathbf{w}^{(1)}L + L\,\mathrm{Div}\,\boldsymbol{\xi})(x,u(x),\nabla u(x))\,dx \qquad (3.27)$$
$$+\int_{\Gamma_N}(\mathbf{w}K + K\,\mathrm{Div}_{\Gamma_N}\boldsymbol{\xi})(x,u(x))\,d\sigma_x$$
$$= \int_{\Gamma_D}\boldsymbol{\nu}\cdot\boldsymbol{\xi}(L + (\nabla u_0^\alpha - \nabla u^\alpha)\cdot\nabla_{\mathbf{p}^\alpha}L)\,d\sigma_x. \qquad (3.28)$$

For $u \in C^{0,1}_{\Gamma_D}(\overline{\Omega}) \cap W^{2,1}(\Omega)$ *satisfying the Euler-Lagrange equation and the boundary condition pointwise almost everywhere the identity* (3.27)-(3.28) *still holds.*

Remark 3.26. For the case $\Gamma_N = \emptyset$ the above reduces to Theorem 3.23.

Proof. We consider $\int_{\Gamma_N} K(x,u)\,d\sigma_x$ as a functional-integral on a subset Γ_N of the $n-1$-dimensional manifold $\partial\Omega$. By applying (3.21) separately to the two parts of the functional $\mathcal{L}$ we get

$$\frac{d}{d\epsilon}\mathcal{L}[g_\epsilon u]\Big|_{\epsilon=0} = \int_\Omega \mathbf{w}^{(1)}L + L\,\mathrm{Div}\,\boldsymbol{\xi}\,dx + \int_{\Gamma_N}\mathbf{w}K + K\,\mathrm{Div}_{\Gamma_N}\boldsymbol{\xi}\,d\sigma_x.$$

This is (3.27). Next one uses (3.22) again separately for the two integrals. Notice that the “Euler-Lagrange operator” for the surface integral $\int_{\Gamma_N} K(x,u(x))\,\sigma_x$ is $\mathcal{F}_\alpha[u] = \partial_{u^\alpha}K$. Thus one finds

$$\frac{d}{d\epsilon}\mathcal{L}[g_\epsilon u]\Big|_{\epsilon=0}$$
$$= \int_{\Gamma_D}\nu\cdot\boldsymbol{\xi}(L + (\nabla u_0^\alpha - \nabla u^\alpha)\cdot\nabla_{\mathbf{p}^\alpha}L)d\sigma_x + \int_{\Gamma_N}(\boldsymbol{\phi}^\alpha - \boldsymbol{\xi}\cdot\nabla u^\alpha)\nabla_{\mathbf{p}^\alpha}L\cdot\boldsymbol{\nu}\,d\sigma_x$$
$$+\int_{\Gamma_N}(\boldsymbol{\phi}^\alpha - \boldsymbol{\xi}\cdot\nabla u^\alpha)\partial_{u^\alpha}K\,d\sigma_x + \int_{\partial\Gamma_N}\boldsymbol{\nu}_{\Gamma_N}\cdot\boldsymbol{\xi}K\,d\lambda_x.$$

The second and third integral add up to zero due to the boundary condition on Γ_N. For the fourth integral notice that $\partial\Gamma_N = \partial\Gamma_D$ and hence $u = u_0 = 0$ on $\partial\Gamma_N$. Due to the normalization $K(x,u_0) = 0$ the fourth integral vanishes. This establishes (3.28). □

3.9 Variational sub-symmetries

In order to prove a global uniqueness theorem for critical points of $\mathcal{L}[u] = \int_\Omega L(x,u,\nabla u)\,dx$ on the function space $C^{0,1}_0(\overline{\Omega})$ we will look again for variational sub-symmetries. Recall from Definition 2.3 that a one-parameter transformation group G on a space V is called a variational sub-symmetry provided

$$\mathcal{L}[g_\epsilon u] \le \mathcal{L}[u] \quad \forall u\in V \text{ and } \forall\epsilon\ge 0. \qquad (3.29)$$

In our context this makes sense for admissible one-parameter transformation groups.

Proposition 3.27. *Let $\mathcal{L}$ be a functional with associated C^1-Lagrangian $L : \mathbb{R}^k \times \bigcup_{y\in M}(T_yM)^k \to \mathbb{R}$. Let G be an admissible domain contracting transformation group on $V = C_0^{0,1}(\overline{\Omega})$ with infinitesimal generator $\mathbf{w} = \boldsymbol{\xi} + \boldsymbol{\phi}$. Let u_0 be a fixed point of G and assume $L(x, u_0, \nabla u_0) = 0$ in Ω. Then G is a variational sub-symmetry for $\mathcal{L}$ if and only if*

$$\int_\Omega (\mathbf{w}^{(1)}L + L\,\mathrm{Div}\,\boldsymbol{\xi})(x, u(x), \nabla u(x))\,dx \le 0 \tag{3.30}$$

holds for every function $u \in C_0^1(\overline{\Omega})$. Likewise, if a surface integral of the form $\int_{\Gamma_N} K(x,u)\,d\sigma_x$ with $K(x, u_0) = 0$ on Γ_N is added to $\mathcal{L}$ and $\mathcal{L}$ is defined on $V = C_{\Gamma_D}^{0,1}(\overline{\Omega})$ then the corresponding condition is

$$\int_\Omega (\mathbf{w}^{(1)}L + L\,\mathrm{Div}\,\boldsymbol{\xi})(x, u(x), \nabla u(x))\,dx + \int_{\Gamma_N} (\mathbf{w}K + K\,\mathrm{Div}_{\Gamma_N}\,\boldsymbol{\xi})(x, u(x))\,d\sigma_x \le 0 \tag{3.31}$$

for every function $u \in C_{\Gamma_D}^1(\overline{\Omega})$.

Notice that (3.30), (3.31) are formulated only for C^1-functions. Since domain contracting transformation groups are defined on the Lipschitz-space $C_0^{0,1}(\overline{\Omega})$, it takes an extra step in the proof to show that the validity of (3.30) for C_0^1-functions implies (3.29) for all $u \in C_0^{0,1}(\overline{\Omega})$.

Proof. The proof is given only for the first part of the proposition. First we consider a C^1-function u and we want to show that $\frac{d}{d\epsilon}\mathcal{L}[g_\epsilon u] \le 0$ for $\epsilon \ge 0$. Since $g_\epsilon u$ is a C^1-function on $g_\epsilon \Omega$ we can apply the rate-of-change formula (3.19) with $g_\epsilon\Omega$ as underlying domain and t as a parameter and we find

$$\frac{d}{dt}\mathcal{L}[g_t \circ g_\epsilon u]|_{t=0} = \int_{g_\epsilon\Omega} (\mathbf{w}^{(1)}L + L\,\mathrm{Div}\,\boldsymbol{\xi})(\tilde{x}, g_\epsilon u(\tilde{x}), \nabla g_\epsilon u(\tilde{x}))\,d\tilde{x}.$$

In the last integral we can replace $g_\epsilon\Omega$ by Ω. To see this one needs to realize that $g_\epsilon u = u_0$ on $\Omega\backslash g_\epsilon\Omega$ and that the integrand $\mathbf{w}^{(1)}L + L\,\mathrm{Div}\,\boldsymbol{\xi}$ vanishes pointwise if the fixed-point u_0 is inserted; a fact that is best seen from Noether's formula (3.20) if one recalls $\boldsymbol{\phi}(x, u_0) - \boldsymbol{\xi}(x, u_0)\cdot\nabla u_0 = 0$ from Lemma 3.6 and $L(x, u_0, \nabla u_0) = 0$. After the replacement of $g_\epsilon\Omega$ by Ω and the re-naming $\tilde{x}$ by x we obtain

$$\frac{d}{d\epsilon}\mathcal{L}[g_\epsilon u] = \int_\Omega (\mathbf{w}^{(1)}L + L\,\mathrm{Div}\,\boldsymbol{\xi})(x, g_\epsilon u(x), \nabla g_\epsilon u(x))\,dx, \tag{3.32}$$

which holds for every C^1-function u on Ω. We would like to use hypothesis (3.30) to see from (3.32) that $\frac{d}{d\epsilon}\mathcal{L}[g_\epsilon u] \le 0$. However, since $g_\epsilon u$ is only Lipschitz continuous on $\overline{\Omega}$ and not C^1 we cannot directly use hypotheses (3.30). Instead, we need to approximate $g_\epsilon u \in C_0^{0,1}(\overline{\Omega})$ by a sequence $v_n \in C_0^1(\overline{\Omega})$ such that $v_n \to g_\epsilon u$, $\nabla v_n \to \nabla g_\epsilon u$ pointwise almost everywhere in Ω as $n \to \infty$. For each n the quantity $\int_\Omega (\mathbf{w}^{(1)} L + L \,\mathrm{Div}\, \boldsymbol{\xi})(x, v_n(x), \nabla v_n(x))\, dx$ is non-positive by hypotheses (3.30) and converges by the dominated convergence theorem to the right-hand side in (3.32). As a consequence we have $\frac{d}{d\epsilon}\mathcal{L}[g_\epsilon u] \leq 0$ for every $u \in C_0^1(\overline{\Omega})$, i.e. the group-elements $\{g_\epsilon\}_{\epsilon \geq 0}$ reduce the values of $\mathcal{L}$ when applied to C_0^1-functions.

In order to show that G is a variational sub-symmetry, it remains to demonstrate the energy-reduction for every function $u \in C_0^{0,1}(\overline{\Omega})$. This is again done by approximation: for $u \in C_0^{0,1}(\overline{\Omega})$ let $u_n \in C_0^1(\overline{\Omega})$ be a sequence such that $u_n \to u$ and $\nabla u_n \to \nabla u$ pointwise almost everywhere in Ω as $n \to \infty$. By definition $g_\epsilon u_n$ and $g_\epsilon u$ arise from solutions of ordinary differential equations. Therefore the standard continuous-dependence on initial data shows that $g_\epsilon u_n(x) \to g_\epsilon u(x)$ and $\nabla g_\epsilon u_n(x) \to \nabla g_\epsilon u(x)$ as $n \to \infty$ pointwise a.e. in Ω. By the dominated convergence theorem $\mathcal{L}[g_\epsilon u_n] \to \mathcal{L}[g_\epsilon u]$ and since $\mathcal{L}[g_\epsilon u_n]$ is decreasing in ϵ for $\epsilon \geq 0$ the same is true for $\mathcal{L}[g_\epsilon u]$. □

Definition 3.28. *Let u_0 be the fixed point of an admissible domain contracting transformation group. We say that G is a strict variational sub-symmetry w.r.t. u_0, if there exists a C^1-function $u_0 \in V$ such that* (3.30) *or* (3.31) *holds with strict inequality for every $u \neq u_0$.*

3.10 Uniqueness of critical points

The following two definitions provide structural assumptions on the Lagrangian L of the functional $\mathcal{L}$ which allow to prove uniqueness of critical points in the presence of variational sub-symmetries.

Definition 3.29 (Rank-one-convexity). *A function $F : (T_xM)^k \to \mathbb{R}$ is called rank-one-convex at the point $\mathbf{p} = (\mathbf{p}^1, \ldots, \mathbf{p}^k) \in (T_xM)^k$ provided for all $a \in \mathbb{R}^k$ and all $q \in T_xM$ the function $F(\mathbf{p} + ta \otimes q)$ is a convex scalar function of $t \in \mathbb{R}$. Here $a \otimes q$ stand for the set of vectors $(a^1 q, \ldots, a^k q)$ which has at most rank 1.*

If F is a C^1-function then rank-one-convexity at $\mathbf{p}$ implies

$$F(\mathbf{p} + a \otimes q) \leq F(\mathbf{p}) + a^\alpha\, q \cdot \nabla_{\mathbf{p}^\alpha} F(\mathbf{p} + a \otimes q)$$

for all $q \in T_xM$ and $a \in \mathbb{R}^k$.

Example 3.30. The function $F(\mathbf{p}^1, \ldots, \mathbf{p}^n) = \det(\mathbf{p}^1, \ldots, \mathbf{p}^n)$ maps $\mathbb{R}^{n \times n}$ to $\mathbb{R}$. It is not convex but rank-one convex. In fact even $-F$ is rank-one convex, because $F(\mathbf{p} + ta \otimes q)$ is an affine function of t. Further examples of rank-one convex functions can be constructed by $\mathbf{p} \mapsto h(\det(\mathbf{p})))$ with a convex function $h : \mathbb{R} \to \mathbb{R}$. The function $F : \mathbb{R}^{2n} \to \mathbb{R}$ with $F(\mathbf{p}, \mathbf{q}) = \mathbf{p} \cdot \mathbf{q}$ is not rank-one convex.

For our applications we will use rank-one-convexity for a Lagrangian $L : \mathbb{R}^k \times \bigcup_{y\in M}(T_yM)^k \to \mathbb{R}$ in a restricted way, namely we require $L(x, u_0(x), \nabla u_0(x) + \mathbf{p})$ to be rank-one-convex in $\mathbf{p} = (\mathbf{p}^1, \dots, \mathbf{p}^k) \in (T_xM)^k$ at $\mathbf{p} = 0$. If L is C^1 and $L(x, u_0(x), \nabla u_0(x)) = 0$ for all $x \in M$ then this implies

$$L(x, u_0(x), \nabla u_0(x) + a \otimes q) \leq a^\alpha\, q \cdot \nabla_{\mathbf{p}^\alpha} L(x, u_0(x), \nabla u_0(x) + a \otimes q) \quad (3.33)$$

for all $x \in M$, $a \in \mathbb{R}^k$ and $q \in T_xM$.

Remark 3.31. In case where u_0 is a strong local minimizer of $\mathcal{L}$ in the sense that $\mathcal{L}[u_0] \leq \mathcal{L}[u]$ if $\|u - u_0\|_\infty < \delta$ and if we assume the normalization $L(x, u_0, \nabla u_0) = 0$ then the necessary condition of Weierstrass states that

$$L(x, u_0, \nabla u_0 + a \otimes q) \geq a^\alpha\, q \cdot \nabla_{\mathbf{p}^\alpha} L(x, u_0, \nabla u_0). \quad (3.34)$$

Conditions (3.33) and (3.34) are independent, but related in the sense that both follow from rank-one-convexity of $L(x, u_0(x), \nabla u_0(x) + \mathbf{p})$ at $\mathbf{p} = 0$.

Example 3.32. (i) For the functional

$$\mathcal{L}[u] = \int_\Omega \frac{1}{q+1}(|\nabla u^1(x)|^2 + \ldots + |\nabla u^k(x)|^2)^{\frac{q+1}{2}} - F(x, u(x))\, dx$$

with $q > 1$ the Lagrangian $L(x, u, \mathbf{p})$ is rank-one-convex in $\mathbf{p}$ because it is convex in $\mathbf{p}$.
(ii) For functions $u : \Omega \subset \mathbb{R}^2 \to \mathbb{R}^3$ consider the functional

$$\mathcal{L}[u] = \int_\Omega \frac{1}{2}(|\nabla u^1(x,y)|^2 + |\nabla u^2(x,y)|^2 + |\nabla u^3(x,y)|^2) + \frac{2}{3}(u \cdot u_x \wedge u_y)\, d(x,y),$$

where u_x, u_y stands for the partial derivatives in x, y and $u_x \wedge u_y$ is the exterior product of u_x and u_y. The Lagrangian is not rank-one convex in (u_x, u_y) because of the term $u \cdot (u_x \wedge u_y)$. However, for $u_0 \equiv 0$ the restricted Lagrangian $L(x, 0, \mathbf{p}^1, \mathbf{p}^2, \mathbf{p}^3) = \frac{1}{2}(|\mathbf{p}^1|^2 + |\mathbf{p}^2|^2 + |\mathbf{p}^3|^2)$ is convex and hence rank-one-convex in $\mathbf{p}$.

Definition 3.33 (Unique continuation property). *The Euler-Lagrange operator $\mathcal{E}[u]$ of a C^1-Lagrangian $L : \mathbb{R}^k \times \bigcup_{y\in M}(T_yM)^k \to \mathbb{R}$ has the unique continuation property w.r.t. $u_0 \in V$ if the following holds: if $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$ solves $\mathcal{E}[u] = 0$ in Ω, $u = 0$ on a non-empty, relatively open set $\Gamma_D \subset \partial\Omega$ and*

$$(\boldsymbol{\nu} \cdot \boldsymbol{\xi})\big(L(x, 0, \nabla u) - L(x, 0, \nabla u_0) + (\nabla u_0^\alpha - \nabla u^\alpha) \cdot \nabla_{\mathbf{p}^\alpha} L(x, 0, \nabla u)\big) = 0 \text{ on } \Gamma_D$$

then $u \equiv u_0$ in Ω.

Remark 3.34. In the following application one also requires the normalization $L(x, u_0, \nabla u_0) = 0$ in $\overline{\Omega}$. Since $u_0 = 0$ on Γ_D the term $L(x, 0, \nabla u_0)$ drops out in the above equation on Γ_D.

Example 3.35. Suppose Ω is a bounded, piecewise smooth domain, cf. Definition 4.7. Consider the Lagrangian $L(x,u,\mathbf{p}) = \frac{1}{2}|\mathbf{p}|^2 - F(u)$ for a scalar function $u : \Omega \to \mathbb{R}$ with $f(0) = 0$ and f Lipschitz. Take $u_0 \equiv 0$. Then $(\boldsymbol{\nu}\cdot\boldsymbol{\xi})(L(x,0,\nabla u(x)) - \nabla u(x)\cdot\nabla_{\mathbf{p}}L(x,0,\nabla u(x))) = \frac{-1}{2}(\boldsymbol{\nu}\cdot\boldsymbol{\xi})|\nabla u|^2$. If this quantity vanishes everywhere on Γ_D, and if $\boldsymbol{\nu}\cdot\boldsymbol{\xi} < 0$ on a relatively open subset $T \subset \Gamma_D$ then $|\nabla u| = 0$ and $u = 0$ on T. Moreover, u satisfies the linear equation $\Delta u + a(x)u = 0$ in Ω with $a(x) = f(u(x))/u(x) \in L^\infty(\Omega)$. In local coordinates the equation takes the form $1/\sqrt{g}\partial_{x^i}(\sqrt{g}g^{ij}\partial_{x^j}u) + a(x)u = 0$ with $g = \det(g_{ij})$. By the unique continuation principle for linear, uniformly elliptic equations, cf. Miranda [65], we conclude that $u \equiv 0$ in Ω, i.e. the unique continuation property holds for such a Lagrangian provided $\boldsymbol{\nu}\cdot\boldsymbol{\xi} < 0$ on a relatively open subset of Γ_D.

The following is the main result of this section. It is the basis for all further applications. We state it in a form which applies to $\mathcal{L}[u] = \int_\Omega L(x,u,\nabla u)\,dx + \int_{\Gamma_N} K(x,u(x))\,d\sigma_x$ on the space $C^{0,1}_{\Gamma_D}(\overline{\Omega})$. The case $\Gamma_N = \emptyset$ is included.

Theorem 3.36 (Uniqueness result). *Suppose $L : \mathbb{R}^k \times \bigcup_{y\in M}(T_yM)^k \to \mathbb{R}$ and $K : \mathbb{R}^k \times M \to \mathbb{R}$ are C^1-Lagrangians for the functional $\mathcal{L}[u] = \int_\Omega L(x,u,\nabla u)\,dx + \int_{\Gamma_N} K(x,u)\,d\sigma_x$. Let G defined on $C^{0,1}_{\Gamma_D}(\overline{\Omega})$ be an admissible domain contracting transformation group with infinitesimal generator $\mathbf{w} = \boldsymbol{\xi} + \boldsymbol{\phi}$. Let u_0 be a fixed point of G and assume $L(x,u_0,\nabla u_0) = 0$ in Ω and $K(x,u_0) = 0$ on Γ_N. If furthermore $L(x,u_0,\nabla u_0 + \mathbf{p})$ is rank-one-convex in $\mathbf{p}$ at $\mathbf{p} = 0$ then either of the following two conditions implies the uniqueness of the critical point $u_0 \in C^2(\Omega)\cap C^1_{\Gamma_D}(\overline{\Omega})$:*

(i) G is a strict variational sub-symmetry w.r.t. u_0,
(ii) G is a variational sub-symmetry and the unique continuation property at u_0 holds.

Part (i) of the uniqueness result remains true for critical points in $C^{0,1}_{\Gamma_D}(\overline{\Omega}) \cap W^{2,1}(\Omega)$ satisfying the Euler-Lagrange equation $\mathcal{E}[u] = 0$ and the boundary condition $\partial_{\mathbf{p}^\alpha}L\cdot\nu + \partial_{u^\alpha}K = 0$ on Γ_N almost everywhere.

Proof. Let $u \in C^2(\Omega)\cap C^1_{\Gamma_D}(\overline{\Omega})$ be a critical point of $\mathcal{L}$. By (3.28) in Theorem 3.25 we have

$$\frac{d}{d\epsilon}\mathcal{L}[g_\epsilon u]|_{\epsilon=0} = \int_{\Gamma_D}(\boldsymbol{\xi}\cdot\boldsymbol{\nu})\big(L + (\nabla u_0^\alpha - \nabla u^\alpha)\cdot\nabla_{\mathbf{p}^\alpha}L\big)\,d\sigma_x,$$

where $\boldsymbol{\xi}, L$ and $\nabla_{\mathbf{p}^\alpha}L$ are evaluated at $(x,0,\nabla u(x))$. On Γ_D the solution u attains zero boundary-values, which implies $\nabla u^\alpha = a^\alpha\boldsymbol{\nu}$ for some scalar values a^α. Hence rank-one-convexity of $L(x,0,\nabla u_0(x) + \mathbf{p})$ w.r.t. $\mathbf{p}$ at $\mathbf{p} = 0$ implies $L + (\nabla u_0^\alpha - \nabla u^\alpha)\cdot\nabla_{\mathbf{p}^\alpha}L \le 0$ on Γ_D, where L and $\nabla_{\mathbf{p}^\alpha}L$ are evaluated at $(x,0,\nabla u_0(x) + a\otimes\boldsymbol{\nu})$. Since furthermore $\boldsymbol{\xi}\cdot\boldsymbol{\nu} \le 0$ on Γ_D by the domain contracting property of G we see altogether that $\frac{d}{d\epsilon}\mathcal{L}[g_\epsilon u]|_{\epsilon=0} \ge 0$. On the other hand G is a variational sub-symmetry. Hence it follows that $\frac{d}{d\epsilon}\mathcal{L}[g_\epsilon u]|_{\epsilon=0} = 0$. In case *(i)*, where G is a strict variational sub-symmetry, this immediately implies $u \equiv u_0$. In case *(ii)* we first deduce from $\frac{d}{d\epsilon}\mathcal{L}[g_\epsilon u]|_{\epsilon=0} = 0$ and from the non-negativity of the surface-integrand in (3.25) that $(\boldsymbol{\nu} \cdot \boldsymbol{\xi})(L + (\nabla u_0^\alpha - \nabla u^\alpha) \cdot \nabla_{\mathbf{p}^\alpha} L)(x, 0, \nabla u(x)) = 0$ everywhere on Γ_D. The unique-continuation property from condition *(ii)* now implies $u \equiv u_0$. □

3.11 Uniqueness of critical points for constrained functionals

The theory of Section 3.10 deals with the critical points of a functional $\mathcal{L}[u] = \int_\Omega L(x, u, \nabla u)\, dx + \int_{\Gamma_N} K(x, u)\, d\sigma_x$ defined on the entire function space $V = C^{0,1}_{\Gamma_D}(\overline{\Omega})$. Often in the calculus of variations one is interested in critical points of $\mathcal{L}[u]$ restricted by a constraint. We will consider two types of constraints:

(1) functional constraints, where $\mathcal{L}$ is restricted to the set of $u \in V$ which lie on the "hypersurface" $\mathcal{N}[u] = \int_\Omega N(x, u, \nabla u)\, dx + \int_{\Gamma_N} M(x, u(x))\, d\sigma_x = 0$
(2) pointwise constraints, where $\mathcal{L}$ is restricted to the set of $u \in V$ with $N(x, u(x)) = 0$ for all $x \in \overline{\Omega}$.

Multiple constraints of the above type can be considered similarly as soon as the case of one constraint is clarified.

3.11.1 Functional constraints

Assume the non-degeneracy hypothesis $\mathcal{N}'[u] \neq 0$ for all critical points u of $\mathcal{L}$ in $S = \{u \in V : \mathcal{N}[u] = 0\}$. Here $\mathcal{N}'[u]h = \int_\Omega \nabla_{p^\alpha} N(x, u, \nabla u) \cdot \nabla h^\alpha + \partial_{u^\alpha} N(x, u, \nabla u) h^\alpha\, dx + \int_{\Gamma_N} \partial_{u^\alpha} M(x, u) h^\alpha\, d\sigma_x$ is the Fréchet-derivative of $\mathcal{N}$ at u. Under this condition it is known that for every extremal u of $\mathcal{L}$ over S there exists a Lagrange-multiplier $\lambda \in \mathbb{R}$ such that u is a critical point of the functional $\mathcal{L} + \lambda\mathcal{N}$ over the entire space V. Therefore the Euler-Lagrange equation in its weak form holds: $\mathcal{L}'[u] + \lambda\mathcal{N}'[u] = 0$. We want to use the method of transformation groups to such constrained variational problems. So let $G = \{g_\epsilon\}_{\epsilon \geq 0}$ be an admissible, domain contracting transformation group with fixed point u_0. We need to require that whenever $u \in S$ then also $g_\epsilon u \in S$. This is the case if and only if

$$0 = \frac{d}{d\epsilon}\mathcal{N}[g_\epsilon u] = \int_\Omega \mathbf{w}^{(1)} N + N \operatorname{div} \boldsymbol{\xi}\, dx + \int_{\Gamma_N} \mathbf{w} M + M \operatorname{div}_{\Gamma_N} \boldsymbol{\xi}\, d\sigma_x$$

for all C^1-functions $u \in V$. Here $\mathbf{w}$ is the infinitesimal generator of $\{g_\epsilon\}_{\epsilon \in \mathbb{R}}$. Moreover we need to require the normalization $N(x, u_0, \nabla u_0) = 0$ in Ω and $M(x, u_0) = 0$ on Γ_N.

As a consequence, a transformation group which leaves S invariant, is a (strict) variational sub-symmetry for $\mathcal{L}$ if and only if it is a (strict) variational sub-symmetry for the free functional $\mathcal{L} + \lambda\mathcal{N}$. In order to find uniqueness of the critical points of $\mathcal{L}$ over S we only need to reformulate the hypotheses of Theorem 3.36 to the extended function $\mathcal{L} + \lambda\mathcal{N}$.

Theorem 3.37. *Suppose the transformation group G generated by $\mathbf{w} = \boldsymbol{\xi} + \boldsymbol{\phi}$ with fixed point $u_0 \in S$ leaves S invariant. Theorem 3.36 remains true for constrained critical points of $\mathcal{L}$ provided G is a variational sub-symmetry for $\mathcal{L}$, $L(x, u_0, \nabla u_0) = N(x, u_0, \nabla u_0) = 0$ in Ω, $K(x, u_0) = M(x, u_0) = 0$ on Γ_N and $(L + \lambda N)(x, u_0, \nabla u_0 + \mathbf{p})$ is rank-one-convex in $\mathbf{p}$ at $\mathbf{p} = 0$ and either of the following two conditions hold*

(i) the variational sub-symmetry is strict w.r.t. u_0,
(ii) the unique continuation property of Theorem 3.36 holds for $L + \lambda N$.

Here λ is the (in general unknown) Lagrange-multiplier of an arbitrary critical point.

Remark 3.38. The verification of the rank-one-convexity condition and the unique continuation property for $L + \lambda N$ requires either structural assumptions on L and N (e.g. N is gradient-independent) or a-priori knowledge of the non-negativity of the Lagrange multiplier λ if e.g. L and N individually satisfy the rank-one-convexity condition. Both situations will be illustrated by some applications in Section 5.

3.11.2 Pointwise constraints

Now we look at critical points of $\mathcal{L}$ subject to some pointwise condition $N(x, u(x)) = 0$ in $\overline{\Omega}$. Since we explicitly require that N does not depend on $\nabla u(x)$, we consider N as a smooth map from $M \times \mathbb{R}^k \to \mathbb{R}$. The constraint on which we seek critical points of $\mathcal{L}$ is given by $S = \{u \in V : N(x, u(x)) = 0 \text{ for all } x \in \overline{\Omega}\}$. We assume the usual non-degeneracy condition: for all $(x, v) \in \overline{\Omega} \times \mathbb{R}^k$ with $N(x, v) = 0$ there is an index $\alpha \in \{1, \dots, k\}$ with $\partial_{v^\alpha} N(x, v) \neq 0$. Then for every critical point u of $\mathcal{L}$ over S there exists a Lagrange-multiplier $\lambda(x) \in \mathbb{R}$ such that u is a critical point of the functional $\mathcal{L}[u] + \int_\Omega \lambda(x) N(x, u(x))\, dx$ over the entire space V, see Giaquinta-Hildebrandt [36], Chapter 2.2. The method of transformation groups can be successfully applied to such problems provided the transformation group leaves the set S invariant, i.e. whenever $N(x, u(x)) = 0$ for all $x \in \Omega$ we have that $N(\chi_\epsilon(x, u(x)), \psi_\epsilon(x, u(x))) = 0$ for $\epsilon \in \mathbb{R}$.

Remark 3.39. Suppose G is a domain contracting transformation group generated by $\mathbf{w} = \boldsymbol{\xi} + \boldsymbol{\phi}$ with fixed point $u_0 \in S$. A criterion for invariance of S is given by

$$\nabla_x N(x, u) \cdot \boldsymbol{\xi}(x, u) + \partial_{u^\alpha} N(x, u) \phi^\alpha(x, u) = 0$$

for all $x \in \Omega$, $u \in \mathbb{R}^k$, cf. Olver [71] Section 2.1.

Lemma 3.40. *Suppose G is a transformation group with infinitesimal generator $\mathbf{w}$, which leaves the set $S = \{u \in V : N(x, u(x)) = 0 \; \forall x \in \overline{\Omega}\}$ invariant. Then*

$$\partial_{u^\alpha} N(x, u(x))(\phi^\alpha(x, u(x)) - \boldsymbol{\xi}(x, u(x)) \cdot \nabla u^\alpha(x)) = 0 \text{ on } \Omega$$

for every element $u \in S$.

Proof. Differentiation of $N(\chi_\epsilon(x, u(x)), \psi_\epsilon(x, u(x))) = 0$ with respect to ϵ at $\epsilon = 0$ yields

$$\nabla_x N(x, u(x)) \cdot \boldsymbol{\xi}(x, u(x)) + \partial_{u^\alpha} N(x, u(x))\phi^\alpha(x, u(x)) = 0 \text{ in } \Omega,$$

and differentiation of $N(x, u(x)) = 0$ w.r.t. x yields

$$\nabla_x N(x, u(x)) + \partial_{u^\alpha} N(x, u(x))\nabla u^\alpha(x) = 0 \text{ in } \Omega.$$

Inserting the second into the first expression gives the result. □

As a simple corollary we find:

Corollary 3.41. *Suppose G is an admissible, domain contracting transformation group with infinitesimal generator $\mathbf{w}$ and fixed point u_0. Suppose G leaves the set $S = \{u \in V : N(x, u(x)) = 0 \; \forall x \in \overline{\Omega}\}$ invariant. Let $\mathcal{E}_{\alpha,\mathcal{L}} = -\operatorname{div}(\nabla_{\mathbf{p}^\alpha} L) + \partial_{u^\alpha} L$ be the Euler-Lagrange operator associated with $\mathcal{L}$. If u is a critical point of the functional $\mathcal{L}[u] = \int_\Omega L(x, u, \nabla u)\, dx$ on the constraint S then*

$$\mathcal{E}_{\alpha,\mathcal{L}}[u]\Big(\phi^\alpha(x, u(x)) - \boldsymbol{\xi}(x, u(x)) \cdot \nabla u^\alpha(x)\Big) = 0 \text{ in } \Omega.$$

Proof. Recall that u is a critical point of the unconstrained functional $\mathcal{L}[u] + \int_\Omega \lambda(x) N(x, u(x))\, dx$. Hence u satisfies the Euler-Lagrange equation $\mathcal{E}_{\alpha,\mathcal{L}}[u] + \lambda(x)\partial_{u^\alpha} N(x, u(x)) = 0$ in Ω. If we multiply the Euler-Lagrange equation with $\phi^\alpha(x, u(x)) - \boldsymbol{\xi}(x, u(x)) \cdot \nabla u^\alpha(x)$ and sum over α, then the term involving N and its derivatives vanishes by Lemma 3.40. The remaining identity proves the claim. □

As a conclusion we see the following: u satisfies an Euler-Lagrange equation with additional terms coming from the constraint. But still a suitably weighted sum of the Euler-Lagrange operator $\mathcal{E}_{\alpha,\mathcal{L}}$ associated with $\mathcal{L}$ is annihilated. Therefore the critical point u still satisfies Pohožaev's identity for the functional $\mathcal{L}$ as stated in Theorem 3.25 since the volume term involving the Euler-Lagrange operator in Theorem 3.18 vanishes by Corollary 3.41. As a consequence we have

Theorem 3.42. *Theorem* 3.36 *remains valid for constrained critical points of the functional $\mathcal{L}$ provided the transformation group G generated by $\mathbf{w} = \boldsymbol{\xi} + \boldsymbol{\phi}$ with fixed point $u_0 \in S$ leaves S invariant and acts as a variational subsymmetry for $\mathcal{L}$.*

Remark 3.43. The requirement that a pointwise constraint is left invariant is much more restrictive than the requirement that a functional constraint is left invariant. However, as a bonus we get that in the presence of pointwise constraints the uniqueness theorems hold with no further assumption or a-priori knowledge of the Lagrange-multiplier.

3.12 Differentiability of the group orbits

We finish the discussion of the method of transformation groups with the question whether the groups constructed in this section are differentiable in the sense of Definition 2.2, i.e., if for given $u \in V$ there exists $h \in V$ with

$$\lim_{\epsilon \to 0+} \|\frac{g_\epsilon u - u}{\epsilon} - h\|_V = 0.$$

We assume that G is domain contracting with fixed point u_0.

Let us begin with the special case, where $\boldsymbol{\xi} = 0$ and $\boldsymbol{\phi} = \boldsymbol{\phi}(u)$. In this case the group generated by $\mathbf{w} = \boldsymbol{\xi} + \boldsymbol{\phi}$ is the same as the one generated by the ordinary differential equation $\dot{U} = \Phi(U)$, where $\Phi : V \to V$ is given by $\Phi(U)(x) := \boldsymbol{\phi}(u(x))$. Thus, if e.g. $V = C^1(\overline{\Omega})$ then G is differentiable. If, however, $V = C_0^1(\overline{\Omega})$ then G is only differentiable if $\boldsymbol{\phi}(0) = 0$ since only then is Φ as self-map of V.

More generally, consider now a *domain preserving* group, i.e., a group with the property $\boldsymbol{\xi}(x, u) \cdot \boldsymbol{\nu}(x) = 0$ for all $x \in \partial\Omega$ and all $u \in \mathbb{R}$. Recalling the definition (3.3) of $\tilde{u}(\tilde{x}) = \psi_\epsilon(\mathrm{Id} \times u)[\chi_\epsilon(\mathrm{Id} \times u)]^{-1}(\tilde{x})$ we find that for fixed $x \in \Omega$ the first order Taylor-expansion of $\tilde{u}(\tilde{x})$ with respect to ϵ is given by

$$\tilde{u}(\tilde{x}) = u(x) + \epsilon(\boldsymbol{\phi}(x, u(x)) - \boldsymbol{\xi}(x, u(x)) \cdot \nabla u(x)) + O(\epsilon^2). \tag{3.35}$$

The first order-term represents a function in $L^\infty(\Omega)$ and, under higher differentiability of $\mathbf{w}$ and u, it represents a function in $C^1(\overline{\Omega})$. This leads to the following differentiability properties of the group-orbits.

Proposition 3.44. *Suppose G is a domain-preserving transformation group defined on $C_0^1(\overline{\Omega})$ or $C^1(\overline{\Omega})$ with infinitesimal generator $\mathbf{w} = \boldsymbol{\xi} + \boldsymbol{\phi}$. If G is defined on the space $C_0^1(\overline{\Omega})$ assume furthermore $\boldsymbol{\phi}(x, 0) = 0$ for all $x \in \partial\Omega$, whereas if G is defined on $C^1(\overline{\Omega})$ we make no further hypotheses. With $h = \boldsymbol{\phi}(x, u(x)) - \boldsymbol{\xi}(x, u(x)) \cdot \nabla u(x)$ we get*

$$\lim_{\epsilon \to 0} \|\frac{g_\epsilon u - u}{\epsilon} - h\|_\infty = 0.$$

If the infinitesimal generator $\mathbf{w}$ is C^1 and $u \in C^2(\Omega)$ then $h \in C^1(\overline{\Omega})$. If $u \in C^2(\Omega) \cap C_0^1(\overline{\Omega})$ then $h \in C_0^1(\overline{\Omega})$. In both cases we have

$$\lim_{\epsilon \to 0} \|\frac{g_\epsilon u - u}{\epsilon} - h\|_{C^1} = 0.$$

Remark 3.45. (i) The last relation shows that the group-orbit is differentiable at least for certain $u \in V$. This is equivalent to the fact that in (3.22) in Theorem 3.18 there is no boundary integral since $\boldsymbol{\xi} \cdot \boldsymbol{\nu} = 0$ and either $\nabla u \cdot \boldsymbol{\xi} = 0$ (Dirichlet boundary condition) or $\nabla_{\mathbf{p}^\alpha} L \cdot \boldsymbol{\nu} = 0$ (natural Neumann boundary condition).
(ii) The condition $\boldsymbol{\phi}(x, 0) = 0$ on $\partial\Omega$ is nothing but the condition that $u_0 = 0$ is a fixed point of the group G. The condition has only to hold on $\partial\Omega$ since the group is domain preserving, i.e., $g_\epsilon \Omega = \Omega$.

The situation for domain-contracting transformation groups is different.

Proposition 3.46. *Let G be domain-contracting transformation group with infinitesimal generator $\mathbf{w} = \boldsymbol{\xi} + \boldsymbol{\phi}$. Let u_0 be a fixed point of G, i.e. $\boldsymbol{\phi}(x, u_0(x)) - \boldsymbol{\xi}(x, u_0(x)) \cdot \nabla u_0(x) = 0$ for all $x \in \Omega$. For $u \in C_0^{0,1}(\overline{\Omega})$ we define $h = \boldsymbol{\phi}(x, u(x)) - \boldsymbol{\xi}(x, u(x)) \cdot \nabla u(x) \in L^\infty(\Omega)$ and get*

$$\lim_{\epsilon \to 0+} \| \frac{g_\epsilon u - u}{\epsilon} - h \|_\infty = 0.$$

If the infinitesimal generator $\mathbf{w}$ is C^1 and $u \in C^2(\Omega) \cap C_0^1(\overline{\Omega})$ then

$$\lim_{\epsilon \to 0+} \| \frac{g_\epsilon u - u}{\epsilon} - h \|_{C^1} = 0.$$

On $\partial\Omega$ the function h becomes $\boldsymbol{\phi}(x, 0) - \boldsymbol{\xi}(x, 0) \cdot \nabla u(x)$. Using the fixed point property of u_0 this implies $h = \boldsymbol{\xi}(x, 0) \cdot (\nabla u_0(x) - \nabla u(x))$ on $\partial\Omega$. Since u, u_0 attain zero-Dirichlet boundary values on $\partial\Omega$ it follows that $\nabla u - \nabla u_0$ is pointing in the normal direction on $\partial\Omega$. Hence h does in general not vanish on $\partial\Omega$ and the group is not differentiable in the space $C_0^1(\overline{\Omega})$. Since differentiability of the group-orbit does not hold a boundary-integral occurs in the identity of Theorem 3.18 and further structural conditions on the Lagrangian were needed to succeed with a uniqueness result.

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5

Scalar problems in Euclidean space

5.1 Extensions of Pohožaev's result to more general domains

In this section we consider the boundary value problem

$$\Delta u + f(x,u) = 0 \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega, \tag{5.1}$$

for a bounded, piecewise smooth domain $\Omega \subset \mathbb{R}^n$ which is conformally contractible. Solutions are critical points $u \in C_0^{0,1}(\overline{\Omega})$ of the functional $\mathcal{L}[u] = \int_\Omega \frac{1}{2}|\nabla u|^2 - F(x,u)\,dx$, where $F(x,s) = \int_0^s f(x,t)\,dt$.

In the case when Ω is star-shaped with respect to $0 \in \Omega$ and for the model nonlinearity $f(x,s) = |x|^\sigma |s|^{p-1}s$ it is known from the work of Ni [69] and Egnell [24] that for $\sigma > -1$ and $p \geq \frac{n+2+2\sigma}{n-2}$ no nontrivial solutions in the class $W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$ exist.

By the method of transformation groups we consider extensions of this result to conformally contractible domains. Recall the classification of conformal vector-fields $\boldsymbol{\xi}$ in $\mathbb{R}^n$ from Lemma 4.33:

(a) If $n = 2$ then $\boldsymbol{\xi} = a(x,y)\mathbf{e}_1 + b(x,y)\mathbf{e}_2$ where $w = a + ib$ is a holomorphic function of $z = x + iy$.
(b) If $n \geq 3$ then (up to a constant shift) $\boldsymbol{\xi}$ is a linear combination of the vector-fields

$$\begin{aligned}
\mathbf{X} &= x^i \mathbf{e}_i, \\
\mathbf{Y}_{ij} &= x^j \mathbf{e}_i - x^i \mathbf{e}_j \textit{ for } 1 \leq i < j \leq n \\
\mathbf{Z}_i &= (x^i x^i - \textstyle\sum_{j \neq i}^n x^j x^j)\mathbf{e}_i + 2\sum_{j \neq i}^n x^i x^j \mathbf{e}_j \textit{ for } i = 1 \dots n.
\end{aligned}$$

Additionally all conformal vector-fields in $\mathbb{R}^n$ satisfy $\Delta \operatorname{div} \boldsymbol{\xi} = 0$.

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Remark 5.1. In Euclidean $\mathbb{R}^n$ the standard basis is $\mathbf{e}_1, \ldots, \mathbf{e}_n$. In this case we find it more convenient to write $\boldsymbol{\xi} = \xi^i \mathbf{e}_i$ instead of $\boldsymbol{\xi} = \xi^i \partial_{x^i}$. Thus, we write $\mathbf{w} = \boldsymbol{\xi} \cdot \nabla_x + \phi \partial_u$ for the infinitesimal generator of a one-parameter transformation group.

The fact that $\Delta \operatorname{div} \boldsymbol{\xi} = 0$ is special for Euclidean $\mathbb{R}^n$. It is verified directly or through Proposition 4.28. It was shown in Section 4.4 that conformal vector-fields correspond to one-parameter groups of conformal self-maps of the underlying manifold. The groups can be computed explicitly in $\mathbb{R}^n$ and are shown in Table 5.1.

vectorfield	one-parameter group	
$\mathbf{X}$	$x \mapsto e^t x$	dilation
$\mathbf{Y}_{ij}$	$x \mapsto (x^i \cos t + x^j \sin t)\mathbf{e}_i + (-x^i \sin t + x^j \cos t)\mathbf{e}_j$	rotation
$\mathbf{Z}_i$	$x \mapsto (I \circ S_i(t) \circ I)x$ with $I : x \mapsto x/\lvert x\rvert^2$	inversion
	and $S_i(t) : x \mapsto x - t\mathbf{e}_i$	shift

Table 5.1. One-parameter groups of conformal maps in $\mathbb{R}^n$

For the following uniqueness result one should consider the nonlinearity $f(x,s) = |x|^\sigma |s|^{p-1} s$ with $p \geq \frac{n+2+2\sigma}{n-2}$ and $\sigma > -1$ as a guiding example, since due to the possibly negative σ it explains the need for hypotheses (b).

Theorem 5.2. *Let $\Omega \subset \mathbb{R}^n$ be a bounded, piecewise smooth, conformally contractible domain with associated vector-field $\boldsymbol{\xi}$ such that* $\operatorname{div} \boldsymbol{\xi} \leq 0$ *in Ω and* $0 \in \overline{\Omega}, \boldsymbol{\xi}(0) = 0$. *Suppose moreover that*

(a) $f : \overline{\Omega} \setminus \{0\} \times \mathbb{R} \to \mathbb{R}$ *is locally Lipschitz continuous in the second variable and* $f(x,0) = 0$,
(b) there exists $q > n$ such that $f(\cdot, s) \in L^q(\Omega)$ with L^q-norm uniformly bounded for s in bounded intervals
(c) the following function is increasing in $\epsilon > 0$ *for all* $(x,u) \in \Omega \setminus \{0\} \times \mathbb{R}$

$$\frac{F(\chi_\epsilon(x), \psi_\epsilon(x,u))}{\psi_\epsilon(x,u)^{\frac{2n}{n-2}}},$$

where $\chi_\epsilon(x)$ is the solution of $\dot{X} = \boldsymbol{\xi}(X)$, $X(0) = x$ *and $\psi_\epsilon(u)$ is the solution of* $\dot{U} = \frac{2-n}{2n}(\operatorname{div} \boldsymbol{\xi})U$, $U(0) = u$.

Then (5.1) *has only the zero-solution in the class of weak* $L^\infty(\Omega) \cap C_0^1(\overline{\Omega})$-*solutions.*

Remark 5.3. Condition *(c)* is equivalent to

$$\boldsymbol{\xi}(x)\cdot\nabla_x F(x,u)+\frac{2-n}{2n}(\operatorname{div}\boldsymbol{\xi}(x))uf(x,u)+F(x,u)\operatorname{div}\boldsymbol{\xi}(x)\geq 0$$

for all $(x,u)\in\Omega\setminus\{0\}\times\mathbb{R}$.

Proof. Every bounded weak solution in $L^\infty(\Omega)\cap C^1(\overline{\Omega})$ is a strong $W^{2,q}$-solution, since by hypothesis *(b)* the right-hand side $-f(x,u(x))\in L^q(\Omega)$ for some $q>n$. We can therefore work with critical points of $\mathcal{L}$ in the class of strong $W^{2,1}(\Omega)\cap C_0^1(\overline{\Omega})$-solutions. The uniqueness proof is done through Theorem 3.36 by finding a variational sub-symmetry w.r.t. 0. Like in the previous chapter we set

$$\mathbf{w}=\boldsymbol{\xi}\cdot\nabla_x+\frac{2-n}{2n}(\operatorname{div}\boldsymbol{\xi})u\partial_u.$$

This corresponds to the choice $\boldsymbol{\phi}(x,u)=\frac{2-n}{2n}(\operatorname{div}\boldsymbol{\xi})u$. For the prolongation we find

$$\begin{aligned}\mathbf{w}^{(1)}&=\boldsymbol{\xi}\cdot\nabla_x+\frac{2-n}{2n}(\operatorname{div}\boldsymbol{\xi})u\partial_u\\&\qquad+\Big(\frac{2-n}{2n}u\nabla\operatorname{div}\boldsymbol{\xi}+\frac{2-n}{2n}(\operatorname{div}\boldsymbol{\xi})\nabla u-D\boldsymbol{\xi}^*\nabla u\Big)\cdot\nabla_{\mathbf{p}}.\end{aligned}$$

To verify that G is a group of variational sub-symmetries we use the infinitesimal criterion of Proposition 3.27 and calculate

$$\begin{aligned}&\mathbf{w}^{(1)}L+L\operatorname{Div}\boldsymbol{\xi}\\&=-\boldsymbol{\xi}\cdot\nabla_x F(x,u)-\frac{2-n}{2n}(\operatorname{div}\boldsymbol{\xi})uf(x,u)-F\operatorname{div}\boldsymbol{\xi}+\frac{2-n}{2n}\operatorname{div}\Big(\frac{u^2}{2}\nabla\operatorname{div}\boldsymbol{\xi}\Big).\end{aligned}$$

Integration over Ω shows that G generates a variational sub-symmetry w.r.t. 0 since the integral over $-\boldsymbol{\xi}\cdot\nabla_x F(x,u)-\frac{2-n}{2n}(\operatorname{div}\boldsymbol{\xi})uf(x,u)-F\operatorname{div}\boldsymbol{\xi}$ is non-positive, as seen from the differentiated version of hypothesis *(c)*, see Remark 5.3. Also, since $f(x,s)$ is locally Lipschitz in s and $f(x,0)=0$ the unique continuation property at 0 holds, cf. Definition 3.33. Moreover the Lagrangian $L(x,u,\mathbf{p})=|\mathbf{p}|^2/2-F(x,u)$ is convex in $\mathbf{p}$. Therefore Theorem 3.36(ii) applies and shows uniqueness of the critical point $u\equiv 0$ in the class $W^{2,1}(\Omega)\cap C_0^1(\overline{\Omega})$. □

Corollary 5.4. *Let $\Omega\subset\mathbb{R}^n$ be a bounded, piecewise smooth, conformally contractible domain with associated vector-field $\boldsymbol{\xi}$ such that $\operatorname{div}\boldsymbol{\xi}\leq 0$ in Ω and $0\in\overline{\Omega}$, $\boldsymbol{\xi}(0)=0$. Let $f(u)=\lambda u+s(x)^\sigma|u|^{p-1}u$ with $s(x)=|x|$ or $|\boldsymbol{\xi}(x)|$.*

(i) If $p\geq\frac{n+2+2\sigma}{n-2}$, $\sigma>-1$ and $\lambda\leq 0$ then (c) of Theorem 5.2 holds.
(ii) If moreover $\lambda=0$ and $p=\frac{n+2+2\sigma}{n-2}$, $\sigma>-1$ (exact critical growth) then the additional assumption $\operatorname{div}\boldsymbol{\xi}\leq 0$ is not necessary.

Proof. Only *(i)* with $s(x)=|\boldsymbol{\xi}(x)|$ is not completely obvious. To see its validity one shows by a computation that $\boldsymbol{\xi}\cdot x=\frac{1}{n}|x|^2\operatorname{div}\boldsymbol{\xi}$ for every conformal vector-field with $\boldsymbol{\xi}(0)=0$. □

The pure critical exponent case

For completeness we mention that in the case of exactly critical growth $f(x,s) = |\boldsymbol{\xi}|^\sigma |s|^{\frac{4+2\sigma}{n-2}} s$ with a conformal vector-field $\boldsymbol{\xi}$ with $\boldsymbol{\xi}(0) = 0$ more information about the nature of critical points is available. Let us consider the related minimization problem

$$\inf_{u\neq 0} I[u] \quad \text{with} \quad I[u] = \frac{\int_\Omega |\nabla u|^2\,dx}{\left(\int_\Omega |\boldsymbol{\xi}(x)|^\sigma |u|^{\frac{2n+2\sigma}{n-2}}\,dx\right)^{\frac{n-2}{n+\sigma}}}.$$

Suitably rescaled critical points of I are in one-to-one correspondence to the critical points of the "free" functional $\mathcal{L}$. An interesting fact is that I possesses no minimizer on any bounded domain Ω with $0 \in \overline{\Omega}$ regardless of the geometry or topology of Ω, cf. Struwe [87], Chapter I Section 4.5 and 4.7, where a proof for $\sigma = 0$ is readily adapted to $\sigma > -1$.

Dancer [16] remarked that the autonomous pure critical exponent problem, i.e. (5.1) with $f(u) = |u|^{4/(n-2)}u$, is invariant under conformal maps of $\mathbb{R}^n$. Hence, it follows for any bounded domain Ω, which is the conformal image of a bounded star-shaped domain, that only the zero-solution of (5.1) exists. In [16] Dancer gave an analytic characterization of such domains. As we shall see in part *(i)* of Proposition 5.5 Dancer's observation is contained in Corollary 5.4*(ii)*.

Examples of conformally contractible domains

We give two examples of domains in $\mathbb{R}^3$, which are not star-shaped but conformally contractible, and the associated vector-field $\boldsymbol{\xi}$ satisfies $\operatorname{div}\boldsymbol{\xi} \leq 0$.

Example domain 1: Let $\boldsymbol{\xi} = (-x+y, -y-x, -z)$. Then $\operatorname{div}\boldsymbol{\xi} = -3$. The vector-field $\boldsymbol{\xi}$ generates a composition of a dilation and a rotation in the x,y-plane. We construct a conformally contractible domain by extending a 2d-domain cylindrically in the z-direction. Both the 2d-cut and the 3d-domain are shown in Figure 5.1. In the 2d-domain the trajectories of the flow $(\dot{x},\dot{y}) = (-x+y, -y-x)$ starting from the boundary are shown. The 2d-domain is positively-invariant under the flow, i.e. the transformation group is domain contracting. By the cylindrical extension this remains true for the 3d-domain.

Example domain 2: For $\boldsymbol{\xi} = (-2xz, -2yz, -z^2+x^2+y^2)$ one has $\operatorname{div}\boldsymbol{\xi} = -6z$. The vector-field $\boldsymbol{\xi}$ is the infinitesimal generator of a one-parameter group of conformal maps involving inversions. We construct a conformally contractible domain by rotating a planar domain around the z-axis. The flow $(\dot{x},\dot{y},\dot{z}) = \boldsymbol{\xi}(x,y,z)$ is also rotationally symmetric around the z-axis. In Figure 5.2 the 2d-cut and the trajectories starting from the boundary are visualized. Again the 2d-domain is positively-invariant, and due to the rotation-symmetry this remains true for the 3d-domain. Hence our transformation group is contracting, and since the domain lies in the region $z \geq 0$ we have $\operatorname{div}\boldsymbol{\xi} \leq 0$. In Figure 5.3 the 3d-domain is visualized from above and from below.

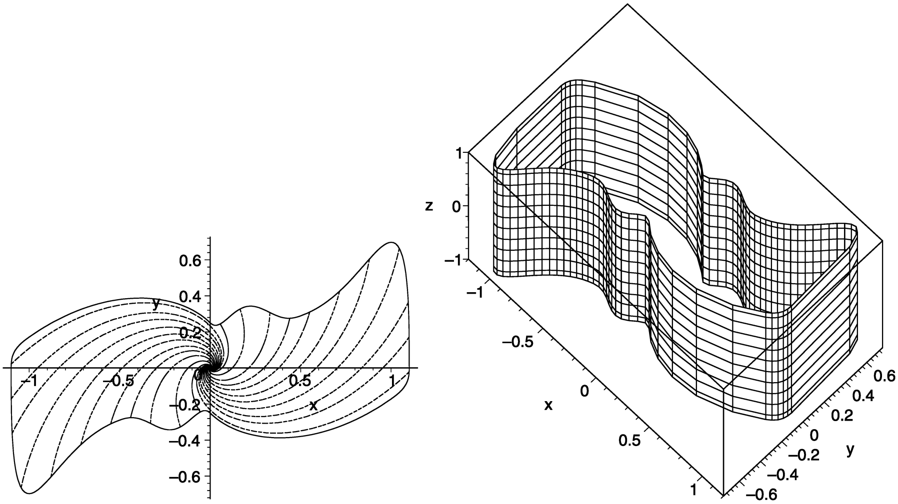

Fig. 5.1. Example domain 1. 2d-cut and 3d-view

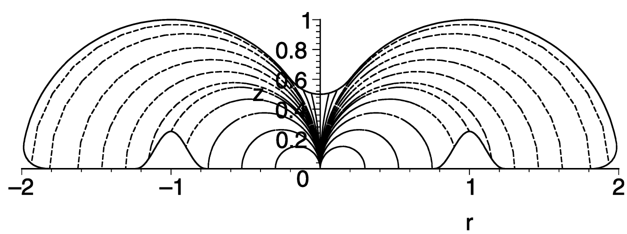

Fig. 5.2. Example domain 2. 2d-cut and trajectories

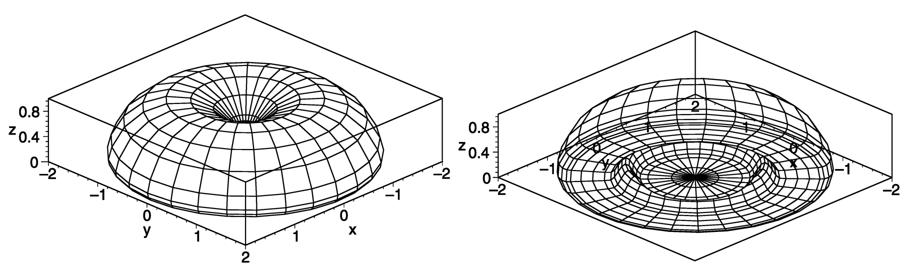

Fig. 5.3. Example domain 2. 3d-view from above and below

The set of conformally contractible domains

Let us denote by $\mathcal{CC}$ the set of all bounded, conformally contractible domains in $\mathbb{R}^n$, $n \geq 3$. By $\mathcal{CC}_-$ we denote all members of $\mathcal{CC}$ such that $\operatorname{div} \boldsymbol{\xi} \leq 0$ for one associated conformal vector-field. Finally consider the set $\mathcal{CI}$ of all bounded domains, which are conformal images of bounded star-shaped domains. Then we have the following relations:

Proposition 5.5. *(i)* $\mathcal{CI} \subset \mathcal{CC}$, *(ii)* $\mathcal{CC}_- \subsetneq \mathcal{CC}$.

Proof. *(i)* Let Ω belong to $\mathcal{CI}$, i.e. $\Omega = h(\Omega')$ where h is a conformal map and Ω' is bounded star-shaped w.r.t. 0. If S_t denotes the map $x \mapsto e^{-t}x$ then the one-parameter group hS_th^{-1} of conformal maps contracts the domain Ω. Hence Ω belongs to $\mathcal{CC}$.
(ii) By taking the 2d-domain of Example 2 and running it backwards in time under the flow, we obtain a new domain shown in Figure 5.4. Rotating the new 2d-domain around the z-axis we find an example of a domain which is conformally contractible with associated vector-field $\boldsymbol{\xi} = (-2xz, -2yz, -z^2 + x^2 + y^2)$ with divergence $-6z$. Clearly $\operatorname{div} \boldsymbol{\xi}$ now also attains positive values, since the domain extends into the region $z < 0$. The domain is therefore not in $\mathcal{CC}_-$. $\square$

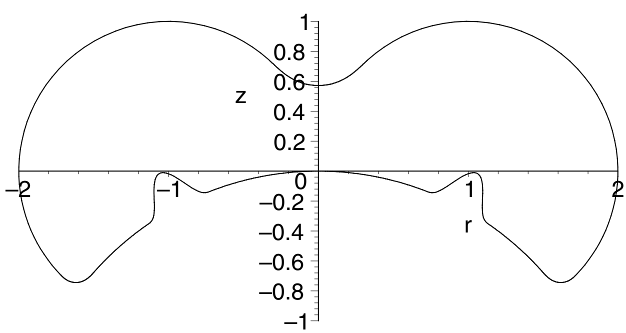

Fig. 5.4. Rotating this 2d-domain produces a set not in $\mathcal{CC}_-$

Remark 5.6. We do not know if $\mathcal{CI} = \mathcal{CC}$ or if $\mathcal{CI} = \mathcal{CC}_-$. A result of Dancer [16] characterizes $\mathcal{CI}$ as follows: there exists a point $\overline{x} \in \overline{\Omega}$ and a point $b \in \mathbb{R}^n \setminus \overline{\Omega}$ such that every planar circular arc with endpoints $\overline{x}$ and b intersects Ω in a connected set.

5.1.1 Nonlinear Neumann boundary conditions

If $\partial\Omega$ is decomposed into $\overline{\Gamma}_D \cup \overline{\Gamma}_N$ then the natural extension of (5.1) is

$$\begin{aligned} \Delta u + f(x,u) &= 0 \text{ in } \Omega, \quad u = 0 \text{ on } \Gamma_D \\ \partial_\nu u - g(x,u) &= 0 \text{ on } \Gamma_N. \end{aligned} \tag{5.2}$$

We consider solutions which are critical points $u \in C^{0,1}_{\Gamma_D}(\overline{\Omega})$ of the functional $\mathcal{L}[u] = \int_\Omega \frac{1}{2}|\nabla u|^2 - F(x,u)\,dx - \int_{\Gamma_N} G(x,u)\,d\sigma_x$, where $F(x,s) = \int_0^s f(x,t)\,dt$ and $G(x,s) = \int_0^s g(x,t)\,dt$.

Theorem 5.7. *Let $\Omega \subset \mathbb{R}^n$ be bounded and piecewise smooth. Assume that (Ω, Γ_N) is conformally contractible domain with associated vector-field $\boldsymbol{\xi}$ such that* $\operatorname{div}\boldsymbol{\xi} \le 0$ *in Ω and $0 \in \Gamma_N$, $\boldsymbol{\xi}(0) = 0$. Suppose moreover that*

(a) $f : \overline{\Omega} \setminus \{0\} \times \mathbb{R} \to \mathbb{R}$ is locally Lipschitz continuous in the second variable and $f(x,0) = 0$,
(b) there exists $q_1 > n$ and $q_2 > n-1$ such that $f(\cdot,s) \in L^{q_1}(\Omega)$, $g(\cdot,s) \in L^{q_2}(\Gamma_N)$ with L^{q_1}, L^{q_2}-norms uniformly bounded for s in bounded intervals,
(c) the following functions are increasing in $\epsilon > 0$ for all $(x,u) \in \Omega \setminus \{0\} \times \mathbb{R}$

$$\frac{F(\chi_\epsilon(x), \psi_\epsilon(x,u))}{\psi_\epsilon(x,u)^{\frac{2n}{n-2}}}, \qquad \frac{G(\chi_\epsilon(x), \psi_\epsilon(x,u)) + \frac{n-2}{4} H(\chi_\epsilon(x))\psi_\epsilon(x,u)^2}{\psi_\epsilon(x,u)^{\frac{2n-2}{n-2}}},$$

where $\chi_\epsilon(x)$ is the solution of $\dot{X} = \boldsymbol{\xi}(X)$, $X(0) = x$, $\psi_\epsilon(u)$ is the solution of $\dot{U} = \frac{2-n}{2n}(\operatorname{div}\boldsymbol{\xi})U$, $U(0) = u$ and H is the mean-curvature of Γ_N.

Then (5.2) *has only the zero-solution in the class of weak $L^\infty(\Omega) \cap C^1_0(\overline{\Omega})$-solutions.*

Remark 5.8. The second part of condition *(c)* is equivalent to

$$\begin{aligned} \boldsymbol{\xi}(x) \cdot \nabla_x G(x,u) &+ \frac{2-n}{2n}(\operatorname{div}\boldsymbol{\xi}(x))ug(x,u) \\ &+ \frac{n-1}{n} G(x,u) \operatorname{div}\boldsymbol{\xi}(x) + \frac{n-2}{4n} u^2 (H \operatorname{div}\boldsymbol{\xi} + n\nabla H \cdot \boldsymbol{\xi}) \ge 0 \end{aligned}$$

for all $(x,u) \in \overline{\Omega} \setminus \{0\} \times \mathbb{R}$.

Proof. The uniqueness proof is very similar to the proof of Theorem 5.2. Consider the group generated by $\mathbf{w} = \boldsymbol{\xi} \cdot \nabla_x + \frac{2-n}{2n}(\operatorname{div}\boldsymbol{\xi})u\partial_u$. The underlying functional $\mathcal{L}$ decomposes into two parts $\mathcal{L}[u] = \int_\Omega L(x,u,\nabla u)\,dx + \int_{\Gamma_N} G(x,u)\,d\sigma_x$. From the rate of change formula one finds as before

$$\begin{aligned} \int_\Omega \mathbf{w}^{(1)} L + L \operatorname{Div}\boldsymbol{\xi}\,dx = -\int_\Omega \boldsymbol{\xi} \cdot \nabla_x F + \frac{2-n}{2n}(\operatorname{div}\boldsymbol{\xi})uf + F \operatorname{div}\boldsymbol{\xi}\,dx \\ + \int_{\Gamma_N} \frac{2-n}{2n} \frac{u^2}{2} \partial_\nu \operatorname{div}\boldsymbol{\xi}\,d\sigma_x, \end{aligned} \tag{5.3}$$

and the first part of condition *(c)* guarantees that the volume-integral is non-positive. Applying the rate of change formula to $\int_{\Gamma_N} G(x,u)\,d\sigma_x$ one finds

$$\int_{\Gamma_N} \mathbf{w}G(x,u) + G(x,u)\,\mathrm{div}_{\Gamma_N}\,\boldsymbol{\xi}\,d\sigma_x$$
$$= \int_{\Gamma_N} \boldsymbol{\xi}\cdot\nabla_x G(x,u) + \frac{2-n}{2n}(\mathrm{div}\,\boldsymbol{\xi})ug(x,u) + \frac{n-1}{n}G\,\mathrm{div}\,\boldsymbol{\xi}\,d\sigma_x. \quad (5.4)$$

Recall from Lemma 4.24 that $\mathrm{div}_{\Gamma_N}\,\boldsymbol{\xi} = \frac{n-1}{n}\,\mathrm{div}\,\boldsymbol{\xi}$ and from Proposition 4.28 that $\partial_\nu\,\mathrm{div}\,\boldsymbol{\xi} = H\,\mathrm{div}\,\boldsymbol{\xi} + n\nabla H\cdot\boldsymbol{\xi}$. Hence, after adding the two surface integrals from (5.3) and (5.4), we find that the rate of change $\frac{d}{d\epsilon}\mathcal{L}[g_\epsilon u]|_{\epsilon=0}$ is non-positive due to the differentiated version of condition *(c)*, see Remark 5.8. As in the proof of Theorem 5.2 uniqueness of the critical point $u\equiv 0$ in the class $W^{2,1}(\Omega)\cap C^1_0(\overline{\Omega})$ follows from Theorem 3.36(ii). □

In all of the following example domains we have $H\equiv 0$ on Γ_N.

Example 1: Let $\mathcal{H}_+ = \{x\in\mathbb{R}^n : x_1 > 0\}$ be a halfspace and $\partial\mathcal{H}_+ = \{x\in\mathbb{R}^n : x_1 = 0\}$ be its boundary. Let $\Omega\subset\mathbb{R}^n$ be star-shaped w.r.t. $0\in\Omega$ and let $\Omega_+ = \Omega\cap\mathcal{H}_+$. Let $\Gamma_N = \Omega\cap\partial\mathcal{H}_+$ and $\Gamma_D = \partial\Omega\cap\mathcal{H}_+$. The vector-field $\boldsymbol{\xi} = -x$ verifies the conditions of Theorem 5.7. E.g., a half-ball $B_1(0)^+ = \{x\in\mathbb{R}^n : |x|<1, x_1>0\}$ with $\Gamma_N = \{x\in\mathbb{R}^n : |x|\le 1 : x_1 = 0\}$ is such a domain.

Example 2: Consider the two-dimensional domain $\omega\subset\mathbb{R}^2$ depicted in Figure 5.1. Let $\Omega = \omega\times[0,1]^{n-2}$ and set $\Gamma_N = \omega\times\{0\}$ and Γ_D the remaining part of the boundary. The vector-field $\boldsymbol{\xi} = (-x_1+x_2, -x_1-x_2, -x_3, \dots, -x_n)$ satisfies the requirements of Theorem 5.7.

Example 3: The three-dimensional example depicted in Figure 5.5 with $\Gamma_N = \partial\Omega\cap\{y=0\}$ shows a domain similar to a quarter-ball, but which is non star-shaped. The corresponding vector-field is $\boldsymbol{\xi} = (-2xz, -2yz, -z^2+x^2+y^2)$.

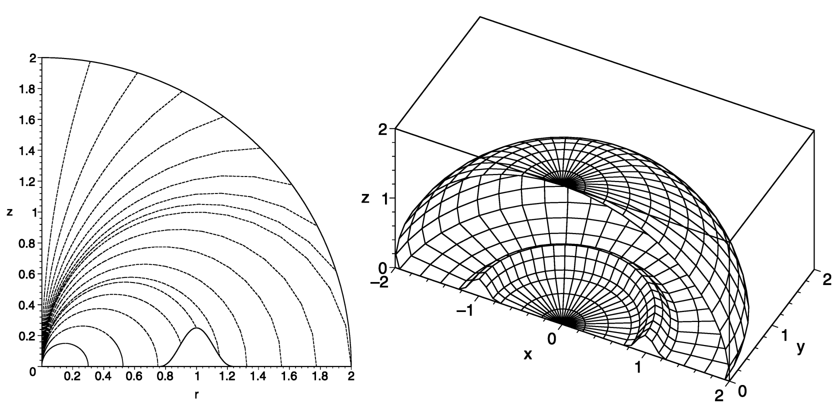

Fig. 5.5. Example domain 3. 2d-cut and 3d-view

Corollary 5.9. *Let $\Omega \subset \mathbb{R}^n$ be bounded and piecewise smooth. Assume that (Ω, Γ_N) is conformally contractible domain with associated vector-field $\boldsymbol{\xi}$ such that* $\operatorname{div} \boldsymbol{\xi} \leq 0$ *in Ω and* $0 \in \Gamma_N$, $\boldsymbol{\xi}(0) = 0$. *Let $f(u) = \lambda u + s(x)^{\sigma_1}|u|^{p-1}u$ and $g(u) = \mu u + t(x)^{\sigma_2}|u|^{q-1}u$ with $s(x), t(x) = |x|$ or $|\boldsymbol{\xi}(x)|$.*

(i) Suppose $\partial_{\boldsymbol{\nu}} \operatorname{div} \boldsymbol{\xi} \geq 0$ on Γ_N. If $p \geq \frac{n+2+2\sigma_1}{n-2}$, $q \geq \frac{n+2\sigma_2}{n-2}$, $\sigma_1, \sigma_2 > -1$ and $\lambda \leq 0$, $\mu \leq 0$ then (c) of Theorem 5.7 holds.

(ii) If no condition on $\partial_{\boldsymbol{\nu}} \operatorname{div} \boldsymbol{\xi}$ on Γ_N is assumed then suppose instead that $\boldsymbol{\xi} \cdot \nabla H \geq 0$ on Γ_N. With all conditions of (i) kept except that $\mu \leq \frac{2-n}{2} \max_{\Gamma_N} H$ it follows that (c) of Theorem 5.7 holds.

(iii) If moreover $\lambda = \mu = 0$ and $p = \frac{n+2+2\sigma_1}{n-2}$, $q = \frac{n+2\sigma_2}{n-2}$, $\sigma_1, \sigma_2 > -1$ then the assumption $\operatorname{div} \boldsymbol{\xi} \leq 0$ *is not necessary provided $H = 0$ on Γ_N.*

5.1.2 Extension to operators of q-Laplacian type

In Example 2.35 of Chapter 2 boundary value problems for the q-Laplacian were introduced. For simplicity we restrict attention to autonomous problems. Recall that for $1 < q < \infty$ critical points in $W_0^{1,q}(\Omega)$ of the functional $\mathcal{L}[u] = \int_\Omega \frac{1}{q}|\nabla u|^q - F(u)\,dx$ weakly satisfy

$$\operatorname{div}(|\nabla u|^{q-2}\nabla u) + f(u) = 0 \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega, \tag{5.5}$$

provided $F(s)$ satisfies a subcritical growth condition. For $q \neq 2$ the operator Δ_q is not uniformly elliptic near those points x where $\nabla u(x) = 0$. Therefore the solutions of (5.5) are typically not classical. E.g., if f is continuous and $u \in L^\infty(\Omega) \cap W_0^{1,q}(\overline{\Omega})$ then it is known from DiBenedetto [20], Lieberman [61] that u is $C^{1,\alpha}$-regular – but examples exist, where the regularity cannot be improved.

To overcome the regularity problems the following regularization was introduced for $\epsilon > 0$:

$$\Delta_{q,\epsilon} u = \operatorname{div}((|\nabla u|^2 + \epsilon)^{\frac{q-2}{2}} \nabla u).$$

Concerning uniqueness of the trivial solutions of supercritical problems

$$Lu + f(u) = 0 \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega \tag{5.6}$$

we show that the two operators $L = \Delta_q$ and $L = \Delta_{q,\epsilon}$ behave similarly with only minor differences.

In the following we assume that f is continuous. A solution of (5.6) always means a weak $C_0^{1,\alpha}(\overline{\Omega})$-solution.

Theorem 5.10. *Let $\Omega \subset \mathbb{R}^n, n > q$ be a bounded $C^{1,\alpha}$-smooth conformally contractible domain with associated conformal vector-field $\boldsymbol{\xi} = -\mathbf{X} + \boldsymbol{\zeta}$ s.t. $\boldsymbol{\zeta} \in \operatorname{span}[x^i \mathbf{e}_j - x^j \mathbf{e}_i, i, j = 1, \ldots, n]$.*

(i) *Problem* (5.6) *has no non-trivial solution if* $F(tu)/t^{\frac{2q}{n-q}}$ *is strictly increasing in* $t>0$ *for all* $u \in \mathbb{R}\backslash\{0\}$. *For* $f(u) = |u|^{p-1}u + \lambda |u|^{q-2}u$ *this amounts to* $p > \frac{nq-n+q}{n-q}$, $\lambda \le 0$ *or* $p \ge \frac{nq-n+q}{n-q}$, $\lambda < 0$.

(ii) *In the special case* $f(u) = |u|^{\frac{nq-2n-2q}{n-q}}u$ *there are no non-trivial solutions of* (5.6) *for* $L = \Delta_{q,\epsilon}$. *For* $L = \Delta_q$ *we can only conclude that* (5.6) *has no non-trivial solution of one sign* $u \ge 0$ *or* $u \le 0$.

Proof. The two operators $L = \Delta_q, \Delta_{q,\epsilon}$ are variational. In both cases $C^{1,\alpha}(\overline{\Omega})$-solutions to (5.6) satisfy the Euler-equation pointwise almost everywhere, see Tolksdorf [88]. For $1 < q \le 2$ they belong to $W^{2,q}(\Omega)$ and for $q \ge 2$ to $W^{2,2}(\Omega)$, see Díaz [21], Section 4.1c. Hence the theory of transformation groups as in Theorem 3.36 is available.

Part (i): First we give the proof for $\Delta_{q,\epsilon}$. Solutions are critical points of the functional

$$\mathcal{L}[u] = \int_\Omega \frac{1}{q}(|\nabla u|^2 + \epsilon)^{q/2} - F(u)\,dx.$$

With the vector-field $\mathbf{w} = \boldsymbol{\xi}\cdot\nabla_x + a(x)u\partial_u$ one obtains

$$\begin{aligned}
&\int_\Omega \mathbf{w}^{(1)}L + L\operatorname{div}\boldsymbol{\xi}\,dx \\
&= \int_\Omega -a(x)uf(u) - F(u)\operatorname{div}\boldsymbol{\xi} + (a(x) - \frac{\operatorname{div}\boldsymbol{\xi}}{n})(|\nabla u|^2+\epsilon)^{\frac{q-2}{2}}|\nabla u|^2\,dx \\
&\quad + \int_\Omega (|\nabla u|^2+\epsilon)^{\frac{q-2}{2}} u\nabla a(x)\cdot\nabla u + \frac{\operatorname{div}\boldsymbol{\xi}}{q}(|\nabla u|^2+\epsilon)^{\frac{q}{2}}\,dx.
\end{aligned}$$

By our hypotheses $\boldsymbol{\xi}$ is a conformal vector field with constant divergence $-n$. We will choose $a(x) = a = \text{const.} > 0$ with the value of a determined a-posteriori. Therefore $\nabla a(x)$ vanishes and we find

$$\begin{aligned}
&\int_\Omega \mathbf{w}^{(1)}L + L\operatorname{div}\boldsymbol{\xi}\,dx \\
&= \int_\Omega -auf(u) + nF(u) + (a+1)(|\nabla u|^2+\epsilon)^{\frac{q-2}{2}}|\nabla u|^2 - \frac{n}{q}(|\nabla u|^2+\epsilon)^{\frac{q}{2}}\,dx \\
&< \int_\Omega -auf(u) + nF(u) + (a+1-\frac{n}{q})(|\nabla u|^2+\epsilon)^{\frac{q}{2}}\,dx \qquad (5.7)
\end{aligned}$$

unless $u \equiv 0$. The strict inequality comes from the fact that $\epsilon > 0$. If we choose $a = (n-q)/q$ then we obtain that $\mathbf{w}$ generates a strict variational sub-symmetry w.r.t. 0.

The proof for the q-Laplacian Δ_q is the same proof as above with $\epsilon = 0$. Now the vector-field $\mathbf{w}$ produces equality in (5.7). But since $F(tu)/t^{\frac{2q}{n-q}}$ is strictly increasing we still have the strict variational sub-symmetry. Uniqueness follows by Theorem 3.36.

Part (ii): If $f(u)$ is exactly the critical-power function, then $\mathbf{w}$ still generates a strict variational sub-symmetry for $\Delta_{q,\epsilon}$ if $\epsilon > 0$. Uniqueness follows as in *Part (i)*. This is different for the q-Laplacian Δ_q. Now we get $\int_\Omega \mathbf{w}^{(1)} L + L \operatorname{div} \boldsymbol{\xi}\, dx = 0$ and hence by Pohožaev's identity of Theorem 3.23 we obtain

$$0 = \int_{\partial\Omega} \boldsymbol{\nu} \cdot \boldsymbol{\xi}(L - \nabla u \cdot \nabla_{\mathbf{p}} L)\, d\sigma = \int_{\partial\Omega} \boldsymbol{\nu} \cdot \boldsymbol{\xi} \frac{1-q}{q} |\nabla u|^q\, d\sigma.$$

This implies that $\nabla u = 0$ on a subset of positive measure of $\partial\Omega$. However, if u is either entirely positive or negative in Ω this contradicts Hopf's maximum principle (cf. Vazquez [90]) for non-trivial, one-signed solutions of $\Delta_q u + |u|^{\frac{nq}{n-q}-2} u = 0$ with zero Dirichlet conditions on $\partial\Omega$. □

Remark 5.11. 1) The result of Theorem 5.10 is sharp in the sense that for $1 < p < \frac{nq-n+q}{n-q}$ $(n > q)$ or $1 < p < \infty$ $(n \leq q)$ standard variational methods show existence of positive solutions for all $\lambda < \lambda_1$.
2) In difference to the Laplacian we cannot admit quadratic conformal vector-fields of type $\mathbf{Z}_i$, see Section 5.1, because $\mathbf{Z}_i$ has non-constant divergence and hence the choice of $a =$ const. in the above proof does no longer work.
3) In order to obtain the result of Theorem 5.10 *(ii)* for arbitrary solutions one would need the unique continuation principle for the q-Laplacian.

Example 5.12. If $\partial\Omega$ is decomposed into $\overline{\Gamma}_D \cup \overline{\Gamma}_N$ then one can consider

$$\begin{aligned} \Delta_q u + f(u) = 0 \text{ in } \Omega, \quad u = 0 \text{ on } \Gamma_D \\ |\nabla u|^{q-2} \partial_\nu u - g(u) = 0 \text{ on } \Gamma_N. \end{aligned} \tag{5.8}$$

Suppose $\Omega \subset \mathbb{R}^n, n > q$ is a bounded and smooth. Let (Ω, Γ_N) be conformally contractible with associated conformal vector-field $\boldsymbol{\xi} = -\mathbf{X} + \boldsymbol{\zeta}$ s.t. $\boldsymbol{\zeta} \in \operatorname{span}[x^i e_j - x^j e_i, i, j = 1, \ldots, n]$. Then (5.8) has no non-trivial $W^{2,1}(\Omega) \cap C^1(\overline{\Omega})$-solution provided $F(tu)/t^{\frac{nq}{n-q}}$, $G(tu)/t^{\frac{(n-1)q}{n-q}}$ is strictly increasing in $t > 0$ for all $u \in \mathbb{R} \setminus \{0\}$. The proof uses explicitly that $\operatorname{div} \boldsymbol{\xi} =$ const. and hence $\partial_\nu \operatorname{div} \boldsymbol{\xi} = 0$ on Γ_N.

5.1.3 Extension to the mean-curvature operator

In their work on generalized versions of Pohožaev's identity Pucci and Serrin [77] also studied a nonlinear boundary-value problem related to the mean-curvature operator

$$\operatorname{div} \Big(\frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} \Big) + f(u) = 0 \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega. \tag{5.9}$$

For the case of star-shaped domains, they obtained a uniqueness result if f has supercritical growth. We state the following extension:

Theorem 5.13. *Let $\Omega \subset \mathbb{R}^n$, ≥ 3 is a bounded Lipschitz domain, which is conformally contractible with associated vector-field $\boldsymbol{\xi} = -\mathbf{X} + \boldsymbol{\zeta}$ s.t. $\boldsymbol{\zeta} \in$* $\mathrm{span}[x^i e_j - x^j e_i, i, j = 1, \ldots, n]$. *If $F(tu)/t^{\frac{2}{n-2}}$ is increasing then* (5.9) *has no non-trivial $C^2(\Omega) \cap C^1(\overline{\Omega})$-solution. If $f(u) = |u|^{p-1}u + \lambda u$ then* (5.9) *has no non-trivial solution if $p \geq \frac{n+2}{n-2}$ and $\lambda \leq 0$.*

The proof is similar to the proof of Theorem 5.10. One applies the transformation group generated by $\mathbf{w} = \boldsymbol{\xi} \cdot \nabla_x + \frac{n-2}{2} u \partial_u$ to the functional $\mathcal{L}[u] = \int_\Omega \sqrt{1 + |\nabla u|^2} - 1 - F(u)\, dx$. One finds in all cases that $\mathbf{w}$ generates a strict variational sub-symmetry w.r.t. 0.

Example 5.14. The nonlinear Neumann problem for the mean-curvature operator is given by

$$\begin{aligned} \operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) + f(u) = 0 \text{ in } \Omega, \quad u = 0 \text{ on } \Gamma_D \\ \frac{\partial_\nu u}{\sqrt{1+|\nabla u|^2}} - g(u) = 0 \text{ on } \Gamma_N, \end{aligned} \tag{5.10}$$

where as usual $\partial\Omega$ is decomposed into $\overline{\Gamma_D} \cup \overline{\Gamma_N}$. Let $\Omega \subset \mathbb{R}^n, n \geq 3$ be bounded. If (Ω, Γ_N) is conformally contractible with associated conformal vector-field $\boldsymbol{\xi} = -\mathbf{X} + \boldsymbol{\zeta}$ s.t. $\boldsymbol{\zeta} \in \mathrm{span}[x^i e_j - x^j e_i, i, j = 1, \ldots, n]$ then (5.8) has no non-trivial $C^2(\Omega) \cap C^1(\overline{\Omega})$-solution provided $F(tu)/t^{\frac{2n}{n-2}}$, $G(tu)/t^{\frac{2n-2}{n-2}}$ is strictly increasing in $t > 0$ for all $u \in \mathbb{R} \setminus \{0\}$. Again the proof uses explicitly that $\operatorname{div} \boldsymbol{\xi} = \text{const.}$ and hence $\partial_\nu \operatorname{div} \boldsymbol{\xi} = 0$ on Γ_N.

5.2 Uniqueness of non-zero solutions

In the previous section examples for supercritical problems were presented where $u \equiv 0$ was the unique solution. Consider now for $p > 1$ the model problem

$$\Delta u + \lambda u^p + 1 = 0 \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega. \tag{5.11}$$

We are interested in positive solutions. For $\lambda > 0$ and close to zero the implicit function theorem shows that (5.11) has a positive solution-curve u_λ, with $u_0 = \lim_{\lambda \to 0} u_\lambda$ being the solution of the torsion-problem $\Delta u_0 + 1 = 0$ in Ω with $u_0 = 0$ on $\partial\Omega$. The next theorem shows that for strictly supercritical exponents $p > \frac{n+2}{n-2}$ global uniqueness of this solution curve holds for small positive λ. It applies more generally to problems of the form

$$\Delta u + f(\lambda, u) = 0 \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega. \tag{5.12}$$

Theorem 5.15. *Let $\Omega \subset \mathbb{R}^n, n \geq 3$ be a bounded Lipschitz domain, which is conformally contractible with associated vector-field $\boldsymbol{\xi}$ such that $\operatorname{div} \boldsymbol{\xi} \leq 0$ in*

Ω. Suppose $f : [0,\infty)\times\mathbb{R}\to\mathbb{R}$ is a C^2-function w.r.t. the second variable and assume that $\partial_s f(\lambda,s), \partial^2_{ss} f(\lambda,s)$ are continuous in (λ,s). Moreover suppose that

(i) $f(0,s) = \text{const.}$ *for all $s\in\mathbb{R}$,*
(ii) $\exists p > \frac{n+2}{n-2}$ *and $s_0,\lambda_0 > 0$ such that $F(\lambda,s)/|s|^{p+1}$ is increasing in s for $|s|\geq s_0$ and $\lambda\in[0,\lambda_0]$.*

Then the following uniqueness result holds:

(a) If $\operatorname{div}\boldsymbol{\xi} < 0$ *in $\overline{\Omega}$ then there exists $\bar\lambda > 0$ such that* (5.12) *has a unique solution for all $\lambda\in[0,\bar\lambda]$.*
(b) If $\operatorname{div}\boldsymbol{\xi}\leq 0$ *in Ω, $f(\lambda,0) > 0$ for $\lambda > 0$ and if $f(\lambda,s)$ is convex in s then there exists $\bar\lambda > 0$ such that* (5.12) *has a unique positive solution for all $\lambda\in(0,\bar\lambda]$.*

Remark 5.16. Examples of convex nonlinearities which satisfy *(i)* and *(ii)* are $f(\lambda,s) = \lambda|s|^{p-1}s + 1$, $\lambda(|s|^{p-1}s+1)$ for $p > \frac{n+2}{n-2}$ and λe^s.

The proof uses the following weighted Poincaré inequality on conformally contractible domains with the weight-function $-\operatorname{div}\boldsymbol{\xi}\geq 0$. It states that the best constant $\tilde\lambda_1$ is strictly positive.

Lemma 5.17. *Let $\Omega\subset\mathbb{R}^n, n\geq 2$ be a bounded Lipschitz domain, which is conformally contractible domain with associated vector-field $\boldsymbol{\xi}$ such that* $\operatorname{div}\boldsymbol{\xi}\leq 0$ *in Ω. Then there exists a value $\tilde\lambda_1$ such that*

$$\int_\Omega (-\operatorname{div}\boldsymbol{\xi})|\nabla u|^2\,dx \geq \tilde\lambda_1\int_\Omega(-\operatorname{div}\boldsymbol{\xi})u^2\,dx \tag{5.13}$$

for all $u\in C^1_0(\overline{\Omega})$. If λ_1 denotes the first Dirichlet eigenvalue of $-\Delta$ then the optimal value $\tilde\lambda_1$ in (5.13) *satisfies $\tilde\lambda\leq\lambda_1$. If* $\operatorname{div}\boldsymbol{\xi} = \text{const.} < 0$ *then clearly $\tilde\lambda_1 = \lambda_1$.*

Proof. For $n\geq 3$ the function $-\operatorname{div}\boldsymbol{\xi}$ is linear and non-negative. Hence we may suppose after a rotation of the coordinate system that $-\operatorname{div}\xi = a+bx_1\geq 0$ in Ω. To avoid trivialities assume $b<0$ and $x_1\leq -a/b$ for $x\in\Omega$ (a similar proof holds if $b>0$ and $x_1\geq -a/b$). Let C denote a generic constant. First, we find

$$\begin{aligned}\int_\Omega(a+bx_1)u^2\,dx \leq C\int_\Omega u^2\,dx &= \frac{-C}{b}\int_\Omega(a+bx_1)\partial_{x_1}(u^2)\,dx\\ &\leq C\Big(\int_\Omega(a+bx_1)u^2\,dx\Big)^{1/2}\Big(\int_\Omega(a+bx_1)|\partial_{x_1}u|^2\,dx\Big)^{1/2},\end{aligned}$$

i.e., $\int_\Omega(a+bx_1)u^2\,dx\leq C\int_\Omega(a+bx_1)|\partial_{x_1}u|^2\,dx$. Likewise, for $i=2,\ldots,n$ we find

$$\int_\Omega (a+bx_1)u^2\,dx = -\int_\Omega x_i\partial_{x_i}\Big((a+bx_1)u^2\Big)\,dx \le C\int_\Omega (a+bx_1)u\partial_{x_i}u\,dx$$
$$\le C\Big(\int_\Omega (a+bx_1)u^2\,dx\Big)^{1/2}\Big(\int_\Omega (a+bx_1)|\partial_{x_i}u|^2\,dx\Big)^{1/2}.$$

Hence $\int_\Omega (a+bx_1)u^2\,dx \le C\int_\Omega (a+bx_1)|\partial_{x_i}u|^2\,dx$ for $i=2,\dots,n$. The result in (5.13) of Lemma 5.17 now follows by summation. To find the relation of the best constant $\tilde\lambda$ in (5.13) and λ_1 let ϕ_1 be the first Dirichlet eigenfunction of $-\Delta$. Then

$$\int_\Omega (-\operatorname{div}\boldsymbol{\xi})\nabla\phi_1\nabla\phi_1\,dx = \int_\Omega \nabla\big((-\operatorname{div}\boldsymbol{\xi})\phi_1\big)\nabla\phi_1\,dx + \int_\Omega \nabla\operatorname{div}\boldsymbol{\xi}\nabla\frac{\phi_1^2}{2}\,dx.$$

Integration by parts and using $\Delta\operatorname{div}\boldsymbol{\xi} = 0$ shows that the second integral vanishes. Hence integration by parts of the first integral yields

$$\int_\Omega (-\operatorname{div}\boldsymbol{\xi})\nabla\phi_1\nabla\phi_1\,dx = \lambda_1\int_\Omega (-\operatorname{div}\boldsymbol{\xi})\phi_1^2\,dx,$$

which shows that the optimal constant $\tilde\lambda_1$ is smaller or equal to λ_1. Finally, in the case $n=2$ the function $-\operatorname{div}\boldsymbol{\xi}\ge 0$ is harmonic. Hence it also has at most simple zeroes on $\partial\Omega$ and we can estimate it from above and below by a linear function. A similar proof as above works and shows also in this case that $\tilde\lambda_1 \ge \lambda_1$. $\square$

Proof (of Theorem 5.15). For $\lambda\in[0,\tilde\lambda]$ let u_λ be the locally unique solution of (5.12) obtained from the implicit function theorem. In case of *(b)* this solution is the positive minimal solution since 0 is a strict subsolution for $\lambda>0$. Any solution u of (5.12) is a critical point of

$$\mathcal{L}[u] = \int_\Omega \frac{1}{2}|\nabla u - \nabla u_\lambda|^2 - F(\lambda,u) + F(\lambda,u_\lambda) + f(\lambda,u_\lambda)(u-u_\lambda)\,dx.$$

Then $L(x,u_\lambda,\nabla u_\lambda)=0$. We want to show $u=u_\lambda$. Clearly $\mathcal{L}$ is convex in the gradient variable ∇u. We set $\mathbf{w} = \boldsymbol{\xi}\cdot\nabla_x + \big(a\operatorname{div}\boldsymbol{\xi}(u-u_\lambda) + \boldsymbol{\xi}\cdot\nabla u_\lambda\big)\partial_u$. By Lemma 3.6 the function u_λ is a fixed point of the group generated by $\mathbf{w}$. For the prolongation of $\mathbf{w}$ we get

$$\mathbf{w}^{(1)} = \mathbf{w} + \Big(a(u-u_\lambda)\nabla(\operatorname{div}\boldsymbol{\xi}) + a\operatorname{div}\boldsymbol{\xi}(\nabla u - \nabla u_\lambda)$$
$$+ D\boldsymbol{\xi}\nabla u_\lambda + D^2u_\lambda\boldsymbol{\xi} - D\boldsymbol{\xi}^T\nabla u\Big)\cdot\nabla_{\mathbf{p}}.$$

Applying the infinitesimal sub-symmetry criterion to the Lagrangian of $\mathcal{L}$ we find

$$
\begin{aligned}
&\mathbf{w}^{(1)}L + L\,\mathrm{Div}\,\boldsymbol{\xi}\\
&= \mathrm{div}\,\boldsymbol{\xi}(a+\frac{n-2}{2n})|\nabla u-\nabla u_\lambda|^2 + \frac{a}{2}\,\mathrm{div}\,\big(\nabla(\mathrm{div}\,\boldsymbol{\xi})(u-u_\lambda)^2\big)\\
&\quad+\big(f_s(\lambda,u_\lambda)(u-u_\lambda)-f(\lambda,u)+f(\lambda,u_\lambda)\big)\boldsymbol{\xi}\cdot\nabla u_\lambda\\
&\quad+a\,\mathrm{div}\,\boldsymbol{\xi}\big(-f(\lambda,u)+f(\lambda,u_\lambda)\big)(u-u_\lambda)\\
&\quad+\mathrm{div}\,\boldsymbol{\xi}\big(-F(\lambda,u)+F(\lambda,u_\lambda)+f(\lambda,u_\lambda)(u-u_\lambda)\big).
\end{aligned}
\tag{5.14}
$$

We choose $a \in (\frac{2-n}{2n}, -\frac{1}{p+1})$ and define two functions $h_1, h_2 : [0,\tilde{\lambda}]\times\Omega\times\mathbb{R}\to\mathbb{R}$ by

$$
\begin{aligned}
h_1(\lambda,x,s) &= \big(f_s(\lambda,u_\lambda)(s-u_\lambda)-f(\lambda,s)+f(\lambda,u_\lambda)\big)\boldsymbol{\xi}\cdot\nabla u_\lambda\\
h_2(\lambda,x,s) &= a\,\mathrm{div}\,\boldsymbol{\xi}\big(-f(\lambda,s)+f(\lambda,u_\lambda)\big)(s-u_\lambda)\\
&\qquad+\mathrm{div}\,\boldsymbol{\xi}\big(-F(\lambda,s)+F(\lambda,u_\lambda)+f(\lambda,u_\lambda)(s-u_\lambda)\big).
\end{aligned}
$$

Then (5.14) can be integrated to

$$
\begin{aligned}
&\int_\Omega \mathbf{w}^{(1)}L + L\,\mathrm{Div}\,\boldsymbol{\xi}\,dx\\
&=\int_\Omega \mathrm{div}\,\boldsymbol{\xi}(a+\frac{n-2}{2n})|\nabla u-\nabla u_\lambda|^2 + h_1(\lambda,x,u(x))+h_2(\lambda,x,u(x))\,dx.
\end{aligned}
\tag{5.15}
$$

We discuss the behaviour of h_1, h_2 depending on the different types of hypotheses.

Case (a). By *(ii)* we know that $f(\lambda,s)$ grows superlinearly and thus

$$
|h_1(\lambda,x,s)| \le C(1+|f(\lambda,s)|)(-\,\mathrm{div}\,\boldsymbol{\xi}) \text{ for all } (\lambda,x,s)\in[0,\tilde{\lambda}]\times\Omega\times\mathbb{R} \tag{5.16}
$$

and by Taylor's theorem $|h_1(\lambda,x,s)|/(s-u_\lambda(x))^2$ is bounded for s in bounded intervals. Moreover, by using $\partial^2_{ss}f(0,u_0(x)) = 0$ and the continuity of the second derivative, one has

$$
|h_1(\lambda,x,s)|/(s-u_\lambda(x))^2 \to 0 \text{ as } \lambda\to 0 \tag{5.17}
$$

uniformly for s in bounded intervals and $x\in\overline{\Omega}$. The same reasoning shows that (5.17) also holds for h_2. By using the growth assumption *(ii)* we get for a large constant $C>0$ and all $(\lambda,x,s)\in[0,\tilde{\lambda}]\times\Omega\times\mathbb{R}$ the estimate

$$
h_2(\lambda,x,s) \le \Big(\underbrace{(a+\frac{1}{p+1})f(\lambda,s)s + C}_{\to-\infty \text{ as } |s|\to\infty}\Big)(-\,\mathrm{div}\,\boldsymbol{\xi}). \tag{5.18}
$$

For $|s|\to\infty$ we see that h_2 grows fast enough to $-\infty$ to dominate $|h_1|$. Altogether this means for all $(\lambda,x,s)\in[0,\tilde{\lambda}]\times\Omega\times\mathbb{R}$ that

$$h_1(\lambda, x, s) + h_2(\lambda, x, s) \le o(1)|s - u_\lambda(x)|^2(-\operatorname{div}\boldsymbol{\xi}) \tag{5.19}$$

where $o(1) \to 0$ as $\lambda \to 0$.

Case (b). Here can restrict attention to positive solutions. The discussion of h_2 is exactly the same as before. Only for the discussion of h_1 there are differences. We split the domain $\Omega = D_1 \cup D_2$ where D_1 is a compact subset of Ω and D_2 a neighbourhood of $\partial\Omega$ such that $\boldsymbol{\xi} \cdot \nabla u_\lambda \ge 0$ in D_2. By convexity of $f(\lambda, s)$ in s we find

$$h_1(\lambda, x, s) \le 0 \text{ for all } (\lambda, x, s) \in [0, \tilde{\lambda}] \times D_2 \times \mathbb{R}. \tag{5.20}$$

Moreover, since $-\operatorname{div}\boldsymbol{\xi} > 0$ in D_1 there exists a constant $C > 0$ such that

$$h_1(\lambda, x, s) \le C(1+|f(\lambda, s)|)(-\operatorname{div}\boldsymbol{\xi}) \text{ for all } (\lambda, x, s) \in [0, \tilde{\lambda}] \times D_1 \times \mathbb{R}. \tag{5.21}$$

Both (5.20) and (5.21) together yield the same estimate (5.16) as in *Case (a).* Hence also in this case we reach the same conclusion (5.19).

In both cases we may estimate (5.15) by

$$\begin{aligned}\int_\Omega &\mathbf{w}^{(1)} L + L \operatorname{Div}\boldsymbol{\xi}\, dx \\ &\le \int_\Omega \operatorname{div}\boldsymbol{\xi}(a + \frac{n-2}{2n})|\nabla u - \nabla u_\lambda|^2 + o(1)|u - u_\lambda|^2(-\operatorname{div}\boldsymbol{\xi})\, dx \end{aligned} \tag{5.22}$$

where $o(1) \to 0$ as $\lambda \to 0$. Since $a > \frac{2-n}{2n}$ we can apply the weighted Poincaré inequality of Lemma 5.17 and obtain

$$\int_\Omega \mathbf{w}^{(1)} L + L \operatorname{Div}\boldsymbol{\xi}\, dx \le \int_\Omega \operatorname{div}\boldsymbol{\xi}(u - u_\lambda)^2\big(\tilde{\lambda}_1(a + \frac{n-2}{2n}) + o(1)\big)\, dx$$

which shows that for $\lambda > 0$ sufficiently small the vector-field $\mathbf{w}$ generates a strict variational sub-symmetry w.r.t. u_λ. By Theorem 3.36 we know that $u \equiv u_\lambda$ is the only critical point of $\mathcal{L}$ for sufficiently small λ. □

Remark 5.18. 1) For the nonlinearities $f(\lambda, s) = \lambda(1 + |s|^{p-2}s)$, $\lambda|s|^{p-2}s + 1$ with $1 < p < \frac{n+2}{n-2}$, $n \ge 3$ and for arbitrary $p > 1$, $n = 1, 2$ there are at least two positive solutions of (5.12) for small positive λ. This was shown for smooth bounded domains by Crandall, Rabinowitz [14] and for a problem similar to (5.11) on balls by Joseph, Lundgren [50].

2) For $f(\lambda, s) = \lambda e^s$ (5.12) is known as the Gelfand problem. For $n \ge 3$ there exists $\bar{\lambda} > 0$ such that (5.12) has a unique positive solution exists for $\lambda \in [0, \bar{\lambda}]$. For star-shaped domains such a theorem can be found in Schmitt [83], Theorem 2.6.6. For the construction in Theorem 5.15 one needs a negative value $0 > a > \frac{2-n}{2n}$ to obtain a variational sub-symmetry by $\mathbf{w} = \boldsymbol{\xi} \cdot \nabla x + (a \operatorname{div}\boldsymbol{\xi}(u - u_\lambda) + \boldsymbol{\xi} \cdot \nabla u_\lambda)\partial_u$. Therefore $n \ge 3$ is needed for

the proof. Indeed, the condition $n \geq 3$ is also necessary, since for $n = 1, 2$ the Gelfand-problem has at least two positive solutions for small positive λ. This was shown by Gelfand [35] for intervals and disks and by Crandall, Rabinowitz [14] for bounded smooth planar domains.

Example 5.19. For a nonlinear Neumann boundary value problem let us suppose that $\partial\Omega$ is decomposed into $\overline{\Gamma}_D \cup \overline{\Gamma}_N$ and that $\Omega \subset \mathbb{R}^n, n \geq 3$ is bounded. Let (Ω, Γ_N) be conformally contractible with associated conformal vector-field $\boldsymbol{\xi}$ such that $\operatorname{div} \boldsymbol{\xi} \leq 0$ in Ω. In analogy to the weighted Poincareé-inequality of Lemma 5.17 we consider the best constant $\tilde{\lambda}_1$ of

$$\int_\Omega (-\operatorname{div} \boldsymbol{\xi})|\nabla u|^2\, dx \geq \tilde{\lambda}_1 \Big(\int_\Omega (-\operatorname{div} \boldsymbol{\xi}) u^2\, dx + \int_{\Gamma_N} (-\operatorname{div} \boldsymbol{\xi}) u^2\, d\sigma \Big) \quad (5.23)$$

for all $u \in C^1_{\Gamma_D}(\overline{\Omega})$. We conjecture that always $\tilde{\lambda}_1 > 0$, but so far we can only prove this for a restricted class of conformally contractible domains Ω, see Lemma 5.20 below. In the following we suppose that Ω is such a domain with $\tilde{\lambda}_1 > 0$.

Consider the problem

$$\begin{aligned} \Delta u + f(\lambda, u) &= 0 \text{ in } \Omega, \quad u = 0 \text{ on } \Gamma_D \\ \partial_\nu u - g(\mu, u) &= 0 \text{ on } \Gamma_N. \end{aligned} \quad (5.24)$$

suppose $\partial_\nu \operatorname{div} \boldsymbol{\xi} = (H \operatorname{div} \boldsymbol{\xi} + n\boldsymbol{\xi}\nabla H) \geq 0$ on Γ_N. If f, g satisfy the smoothness assumption of Theorem 5.15 together with

(i) $f(0, s) = \text{const.}$, $g(0, s) = \text{const.}$ for all $s \in \mathbb{R}$,
(ii) $\exists p > \frac{n+2}{n-2}, q > \frac{n}{n-2}$ and $s_0, \sigma_0 > 0$ such that $F(\lambda, s)/|s|^{p+1}$, $G(\mu, s)/|s|^{q+1}$ are increasing in s for $|s| \geq s_0$ and $\lambda, \mu \in [0, \sigma_0]$

then the following uniqueness result holds: if $f(\lambda, 0), g(\mu, 0) > 0$ for $\lambda, \mu > 0$ and if $f(\lambda, s), g(\mu, s)$ are convex in s then there exists $\bar{\sigma} > 0$ such that (5.24) has a unique positive solution for all $\lambda, \mu \in (0, \bar{\sigma}]$.

The weighted Poincaré-inequality for a restricted class of domains is given next.

Lemma 5.20. *Assume that $\partial\Omega$ is decomposed into $\overline{\Gamma}_D \cup \overline{\Gamma}_N$ and that $\Omega \subset \mathbb{R}^n, n \geq 2$ is bounded and piecewise smooth. Let (Ω, Γ_N) be conformally contractible with associated conformal vector-field $\boldsymbol{\xi}$ such that* $\operatorname{div} \boldsymbol{\xi} \leq 0$ *in Ω. Let $\tilde{\lambda}_1$ be the best constant in the inequality* (5.23)*. In each of the following cases $\tilde{\lambda}_1 > 0$:*

(a) $\operatorname{div} \boldsymbol{\xi} < 0$ *in $\overline{\Omega}$*
(b) $\operatorname{div} \boldsymbol{\xi} = 0$ *on a hyperplane E with $\Gamma_N \subset E$.*

Examples for *(a)*, *(b)* are given found in Example 1 - 3 after Theorem 5.7.

Proof. Case *(a)* is a consequence of the uniform positivity of the weight. In case *(b)* the surface integral in (5.23) vanishes and the same proof as in Lemma 5.17 works. □

5.3 The subcritical case

Our attention so far was set on supercritical variational problems. We show in this section that also for subcritical problems information can be obtained by the method of transformation groups. For simplicity we restrict ourselves to the bifurcation problem

$$\Delta u + \lambda u + |u|^{p-1}u = 0 \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega. \tag{5.25}$$

As usual we suppose that $\Omega \subset \mathbb{R}^n$ is a bounded conformally contractible domain with associated vector-field $\boldsymbol{\xi}$. The main result is that any solution with sufficiently small $\|\cdot\|_\infty$-norm must be the zero-solution. Reversely the $\|\cdot\|_\infty$-norm of non-trivial solutions must be sufficiently large.

Theorem 5.21. *Consider a bounded Lipschitz domain $\Omega \subset \mathbb{R}^n$ which is conformally contractible domain with associated vector-field $\boldsymbol{\xi}$ such that* $\operatorname{div}\boldsymbol{\xi} \le 0$ *in Ω. Let $\tilde{\lambda}_1$ be the weighted Poincaré constant from Lemma* 5.17*. Let u be a nontrivial solution of* (5.25)*.*

(i) If $1 < p < \infty$ and $\lambda < \tilde{\lambda}_1$ then

$$\|u\|_\infty \ge \tilde{\lambda}_1 - \lambda.$$

For those domains, where $\tilde{\lambda}_1 = \lambda_1$ the estimate shows how the solution branch bifurcating at $\lambda = \lambda_1$ leaves the trivial solution.

(ii) If $1 < p < \frac{n+2}{n-2}$ and $\lambda < 0$ then

$$\|u\|_\infty^{p-1} \ge -\lambda \frac{2(p+1)}{2n-(n-2)(p+1)}$$

In the case $n > 2$, $\lambda < 0$ the L^∞-norm of any nontrivial solution blows up as $p \nearrow \frac{n+2}{n-2}$.

Proof. Let $\mathcal{L}[u] = \int_\Omega \frac{1}{2}|\nabla u|^2 - \frac{\lambda}{2}u^2 - \frac{1}{p+1}|u|^{p+1}\,dx$. We will show that every solution u of (2.12) with L^∞-norm less than the bound in *(i), (ii)* is trivial. Let $\mathbf{w} = \boldsymbol{\xi}\cdot\nabla_x - \alpha \operatorname{div}\boldsymbol{\xi} u\partial_u$. We begin to verify the infinitesimal sub-symmetry criterion for the vector-field $\mathbf{w}$:

$$\begin{aligned}\int_\Omega \mathbf{w}^{(1)}L + L \operatorname{Div}\boldsymbol{\xi}\,dx &= \int_\Omega (\alpha - \frac{n-2}{2n})(-\operatorname{div}\boldsymbol{\xi})|\nabla u|^2\,dx \\ &\quad + \int_\Omega (-\operatorname{div}\boldsymbol{\xi})(\frac{1}{p+1} - \alpha)|u|^{p+1} + (-\operatorname{div}\boldsymbol{\xi})(\frac{1}{2} - \alpha)\lambda u^2\,dx.\end{aligned}$$

We choose $\alpha \le \min\{\frac{1}{p+1}, \frac{n-2}{2n}\}$. Thus the coefficient of $(-\operatorname{div}\boldsymbol{\xi})|\nabla u|^2$ is non-positive and the coefficient of $(-\operatorname{div}\boldsymbol{\xi})|u|^{p+1}$ is non-negative. By applying the weighted Poincaré inequality from Lemma 5.17 we get

$$\int_\Omega \mathbf{w}^{(1)} L + L \operatorname{Div} \boldsymbol{\xi}\, dx \le \int_\Omega \Big[\tilde{\lambda}_1(\alpha - \frac{n-2}{2n}) + (\frac{1}{2} - \alpha)\lambda\Big](-\operatorname{div} \boldsymbol{\xi}) u^2\, dx$$
$$+ \int_\Omega (\frac{1}{p+1} - \alpha)\|u\|_\infty^{p-1}(-\operatorname{div} \boldsymbol{\xi}) u^2\, dx$$

where $\tilde{\lambda}_1$ is the Poincaré constant. Thus we have a strict variational sub-symmetry w.r.t. 0 provided

$$\|u\|_\infty^{p-1} < \frac{\tilde{\lambda}_1(\frac{n-2}{2n} - \alpha) - \lambda(\frac{1}{2} - \alpha)}{\frac{1}{p+1} - \alpha}. \tag{5.26}$$

In order to apply the uniqueness principle of Theorem 3.36 one needs to notice that for $u \in C_0^{0,1}(\overline{\Omega})$ with $\|u\|_\infty < M$ there exists $\epsilon_0 = \epsilon_0(u) > 0$ such that $\|g_\epsilon u\|_\infty < M$ for all $\epsilon \in [0, \epsilon_0]$. In other words, open L^∞-norm balls are left locally invariant by the variational sub-symmetry G. Hence any solution satisfying (5.26) is trivial by the uniqueness principle of Theorem 3.36. In turn, any non-trivial solution has to satisfy the reverse inequality in (5.26). If we let $\alpha \to -\infty$ then we obtain part *(i)* of the theorem. Part *(ii)* follows if we take $\alpha = \frac{n-2}{2n}$. □

Example 5.22. If $\partial\Omega = \overline{\Gamma}_D \cup \overline{\Gamma}_N$ then the corresponding nonlinear Neumann boundary value problem is

$$\begin{aligned} &\Delta u + \lambda u + |u|^{p-1} u = 0 \text{ in } \Omega, \quad u = 0 \text{ on } \Gamma_D \\ &\partial_\nu u - \mu u - |u|^{q-1} u = 0 \text{ on } \Gamma_N. \end{aligned} \tag{5.27}$$

Suppose that $\Omega \subset \mathbb{R}^n$ is bounded. Let (Ω, Γ_N) be conformally contractible with associated conformal vector-field $\boldsymbol{\xi}$ such that $\operatorname{div} \boldsymbol{\xi} \le 0$ in Ω. Then the following result holds: let $1 < p < \frac{n+2}{n-2}$, $1 < q < \frac{n}{n-2}$ and $\lambda < 0$.

(i) Suppose $\partial_\nu \operatorname{div} \boldsymbol{\xi} \ge 0$ on Γ_N and $\mu < 0$. Then at least one of the following two estimates holds:

$$\|u\|_\infty^{p-1} \ge \frac{-2\lambda(p+1)}{2n - (n-2)(p+1)}, \|u\|_{\infty,\Gamma_N}^{q-1} \ge \frac{-\mu(q+1)}{2n - 2 - (n-2)(q+1)}.$$

(ii) If no condition on $\partial_\nu \operatorname{div} \boldsymbol{\xi}$ on Γ_N is assumed then suppose instead that $\boldsymbol{\xi} \cdot \nabla H \ge 0$ on Γ_N and $\mu < \frac{2-n}{2} \max_{\Gamma_N} H$. Then at least one of the following two estimates holds:

$$\|u\|_\infty^{p-1} \ge \frac{-2\lambda(p+1)}{2n - (n-2)(p+1)}, \|u\|_{\infty,\Gamma_N}^{q-1} \ge \frac{-(\mu + \frac{n-2}{2} \max_{\Gamma_N} H)(q+1)}{2n - 2 - (n-2)(q+1)}.$$

5.4 Perturbations of conformally contractible domains

The following definition suggests a value which measures how far a way a given domain Ω is from being conformally contractible. It is inspired by Dancer and Zhang's definition of ϵ-starshaped domains, cf. [17]. We use the notation $f^+(x) = \max\{f(x), 0\}$ for the positive part of a function f.

Definition 5.23. *Let $\Omega \subset \mathbb{R}^n$ be a bounded $C^{2,\alpha}$-domain and let $\boldsymbol{\xi}_0$ be a fixed conformal vector-field. The value*

$$c_0(\boldsymbol{\xi}_0, \Omega) = \max_{x \in \partial\Omega} (\boldsymbol{\xi}_0(x) \cdot \nu(x))^+$$

is called the contractibility-defect with respect to $\boldsymbol{\xi}_0$.

If $c_0(\boldsymbol{\xi}_0, \Omega)$ is small then the vector-field $\boldsymbol{\xi}_0$ is almost pointing inward on $\partial\Omega$ – up to a small correction term. For such a correction term Dancer, Zhang suggested to consider the solution $h : \overline{\Omega} \to \mathbb{R}$ of

$$\Delta h = \frac{|\partial\Omega|}{\operatorname{vol}\Omega} \text{ in } \Omega, \qquad \frac{\partial h}{\partial \nu} = 1 \text{ on } \partial\Omega.$$

Then $(\boldsymbol{\xi}_0 - c_0(\boldsymbol{\xi}_0, \Omega)\nabla h) \cdot \nu \leq 0$ on $\partial\Omega$. The value $c_0(\boldsymbol{\xi}_0, \Omega)$ is not scaling-invariant. But if we introduce

$$\|D^2 h\|_{\boldsymbol{\xi}_0} := \sup_{x \in \Omega} \frac{n\mu_{\max} D^2 h(x)}{-\operatorname{div} \boldsymbol{\xi}_0},$$

where $\mu_{\max} D^2 h$ is the largest eigenvalue of the Hessian of h, then

$$c_0(\boldsymbol{\xi}_0, \Omega)\|D^2 h\|_{\boldsymbol{\xi}_0}$$

is scaling-invariant.

If one takes $\boldsymbol{\xi}_0 = -x$ then $c_0(-x, \Omega)$ is called the star-shapedness defect, and $c_0(-x, \Omega)\|D^2 h\|_x$ is a scale-invariant measure of how far away Ω is from being star-shaped. Based on this value Dancer, Zhang [17] proved the following theorem.

Theorem 5.24. *(Dancer, Zhang) Let $\Omega \subset \mathbb{R}^n$ be a bounded $C^{2,\alpha}$-domain with star-shapedness defect $c_0 = c_0(-x, \Omega)$. If $p > \frac{n+2}{n-2}$ and if*

$$c_0\|D^2 h\|_x < \frac{p(n-2) - (n+2)}{n + 2 + 2p}$$

then

$$\Delta u + |u|^{p-1}u = 0 \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega \tag{5.28}$$

has only the trivial solution.

Our next result generalizes the above theorem of Dancer and Zhang.

Theorem 5.25. *Let $\Omega \subset \mathbb{R}^n$ be a bounded $C^{2,\alpha}$-domain with contractibility-defect $c_0 = c_0(\boldsymbol{\xi}_0, \Omega) > 0$ in direction of a conformal vector-field $\boldsymbol{\xi}_0$. Suppose* $\operatorname{div} \boldsymbol{\xi}_0 \leq 0$ *in Ω. If $p > \frac{n+2}{n-2}$ and if*

$$c_0\|D^2h\|_{\boldsymbol{\xi}_0} \le \frac{p(n-2)-(n+2)}{n+2+2p}$$
$$\lambda \le \frac{2\tilde{\lambda}_1}{n(p-1)} \cdot \frac{p(n-2)-(n+2)-(n+2+2p)c_0\|D^2h\|_{\boldsymbol{\xi}_0}}{2+c_0\|D^2h\|_{\boldsymbol{\xi}_0}}$$

then

$$\Delta u + \lambda u + |u|^{p-1}u = 0 \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega \tag{5.29}$$

has only the trivial solution.

Proof. The vector-field $\boldsymbol{\xi} = \boldsymbol{\xi}_0 - c_0\nabla h$ is not conformal, but points inward on $\partial\Omega$. Moreover, $\operatorname{div}\boldsymbol{\xi} < 0$ in $\overline{\Omega}$ which implies that $\boldsymbol{\xi}\cdot\boldsymbol{\nu} < 0$ on a subset of $\partial\Omega$ of positive measure. Recall from Theorem 4.8(c)–(d) that uniqueness of the trivial solution holds provided

$$\begin{gathered} -\operatorname{div}\boldsymbol{\xi} - 2M_\infty \ge 0 \text{ in } \Omega, \quad p \ge p^* = \sup_\Omega \frac{-\operatorname{div}\boldsymbol{\xi} + 2M_\infty}{-\operatorname{div}\boldsymbol{\xi} - 2M_\infty}, \\ \lambda \le \tilde{\lambda}_1 \frac{2(p-p^*)}{(p-1)(p^*+1)} + \frac{\Delta \operatorname{div}\boldsymbol{\xi}}{(p-1)\operatorname{div}\boldsymbol{\xi}}, \end{gathered} \tag{5.30}$$

where $M_\infty = M_\infty(x;\boldsymbol{\xi}) := \sup_{|\mathbf{b}|=1} -\mathbf{b}^T D\boldsymbol{\xi}(x)\mathbf{b}^T$. The proof of the present theorem consists in verification of (5.30) for the given vector-field $\boldsymbol{\xi}$. First we compute

$$M_\infty \le \frac{-1}{n}\operatorname{div}\boldsymbol{\xi}_0 + \frac{c_0}{n}\|D^2h\|_{\boldsymbol{\xi}_0}(-\operatorname{div}\boldsymbol{\xi}_0).$$

By using the hypotheses on $c_0\|D^2h\|_{\boldsymbol{\xi}_0}$ we find

$$\begin{aligned} -\operatorname{div}\boldsymbol{\xi} - 2M_\infty &\ge -\operatorname{div}\boldsymbol{\xi}_0\frac{n-2}{n} + c_0\frac{|\partial\Omega|}{\operatorname{vol}\Omega} - \frac{2c_0}{n}\|D^2h\|_{\boldsymbol{\xi}_0}(-\operatorname{div}\boldsymbol{\xi}_0) \\ &> \frac{-\operatorname{div}\boldsymbol{\xi}_0}{n}\Big(n-2-\frac{2p(n-2)-2(n+2)}{n+2+2p}\Big) = \frac{-\operatorname{div}\boldsymbol{\xi}_0(n+2)}{n+2+2p} \ge 0, \end{aligned}$$

which establishes the first part of (5.30). Since the definition of p^* in (5.30) is monotone w.r.t. to the upper estimate of M_∞ we find

$$p^* \le \sup_\Omega \frac{-\operatorname{div}\boldsymbol{\xi}_0(1+2/n) + c_0\frac{|\partial\Omega|}{\operatorname{vol}\Omega} + \frac{2c_0}{n}\|D^2h\|_{\boldsymbol{\xi}_0}(-\operatorname{div}\boldsymbol{\xi}_0)}{-\operatorname{div}\boldsymbol{\xi}_0(1-2/n) + c_0\frac{|\partial\Omega|}{\operatorname{vol}\Omega} - \frac{2c_0}{n}\|D^2h\|_{\boldsymbol{\xi}_0}(-\operatorname{div}\boldsymbol{\xi}_0)}.$$

Since $|\partial\Omega|/\operatorname{vol}\Omega = \Delta h \le n\|D^2h\|_\infty \le \|D^2h\|_{\boldsymbol{\xi}_0}(-\operatorname{div}\boldsymbol{\xi}_0)$ the above estimate simplifies to

$$p^* \le \frac{(n+2)(1+c_0\|D^2h\|_{\boldsymbol{\xi}_0})}{n-2-2c_0\|D^2h\|_{\boldsymbol{\xi}_0}}$$

and by a direct computation the inequality $p^* \le p$ holds under the given hypotheses on $c_0\|D^2h\|_{\boldsymbol{\xi}_0}$. It remains to verify the λ-part of (5.30). Note that $\Delta\operatorname{div}\boldsymbol{\xi} = 0$. Moreover using the above estimate for p^* one finds

$$\frac{2\tilde{\lambda}_1(p-p^*)}{(p-1)(p^*+1)} \geq \frac{2\tilde{\lambda}_1}{(p-1)n} \cdot \frac{p(n-2)-(n+2)-(n+2+2p)c_0\|D^2h\|_{\boldsymbol{\xi}_0}}{2+c_0\|D^2h\|_{\boldsymbol{\xi}_0}}.$$

For any λ less than the above quantity, (5.30) and hence the uniqueness statement of the theorem holds. □

Remark 5.26. If we let $c_0\|D^2h\| \to 0$ in Theorem 5.25 then we recover the result of Theorem 4.35 in the Euclidean case. Moreover, for large $p >> \frac{n+2}{n-2}$ the admissible range for $c_0\|D^2h\|$ stays bounded but becomes larger. This shows that the more "supercritical" the boundary-value problem (5.29) becomes the more robust the uniqueness result is towards domain perturbation.

Problem 5.27. Find an appropriate version of Theorem 5.25 for nonlinear Neumann boundary value problems.

5.5 Uniqueness in the presence of radial symmetry

In the previous sections we have obtained uniqueness results for entire classes of domains, e.g., for the class of conformally contractible domains. Now we take a look at very special domains, namely balls $B(0)$ in $\mathbb{R}^n$. If u is a positive solution of

$$\Delta u + f(|x|, u) = 0 \text{ in } B(0), \quad u = 0 \text{ on } \partial B(0) \tag{5.31}$$

and $f(r,u)$ is locally Lipschitz continuous in u and increasing in $r = |x|$ then the well known symmetry theorem of Gidas, Ni and Nirenberg [37] shows that u is radially symmetric; similar symmetry theorems for positive solutions of boundary value problems on geodesic balls in $\mathbb{S}^n$ and $\mathbb{H}^n$ are due to Kumaresan, Prajapat [57] and Padilla [74]. One may therefore ask under what conditions radially symmetric solutions of (5.31) are unique/trivial. Radially symmetric solutions of (5.31) are critical points of the functional $\int_0^1 (\frac{1}{2}|u'|^2 - F(r, u(r)))r^{n-1}\,dr$ with $F(r,s) = \int_0^s f(r,t)\,dt$. More generally we consider the uniqueness question for critical points of

$$\mathcal{L}[u] = \int_0^{r_0} L(r, u, u')\,dr$$

with $u \in C_0^{0,1}[0, r_0] = \{u : [0,1] \to \mathbb{R} : u, u' \in L^\infty[0, r_0], u(r_0) = 0, u'(0) \text{ exists and } = 0\}$. The function space has a built-in Dirichlet condition at $r = r_0$ and a Neumann condition at $r = 0$. Our uniqueness results of Section 3.10 do not directly apply to functionals on such spaces with boundary conditions varying from Dirichlet at one endpoint to Neumann at the other. A few extra considerations need to be done:

Transformation groups acting on this function space are generated by

$$\mathbf{w}(r,u) = \xi(r,u)\partial_r + \phi(r,u)\partial_u.$$

We only consider those transformation groups which satisfy

(i) $\xi(0,u) = 0$ for all $u \in \mathbb{R}$,
(ii) $\xi(r_0,u) \le 0$ for all $u \in \mathbb{R}$,
(iii) $\xi(r,0) = 0$ for all $r \in [0,r_0]$.

Conditions (i) and (ii) guarantee that $[0,r_0]$ is mapped onto $[0,r_\epsilon] \subset [0,r_0]$ for $\epsilon > 0$, whereas (iii) ensures that $u = 0$ is a fixed point of the flow, i.e. it guarantees that the Dirichlet-boundary condition at the right-endpoint is preserved. A transformation group satisfying (i)–(iii) is called admissible. The following uniqueness theorem covers the radially symmetric case of Theorem 3.36 for domain-contracting groups (we only consider the scalar case).

Theorem 3.36'. *Suppose $L : [0,r_0] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a C^1-Lagrangian for the functional $\mathcal{L}[u] = \int_0^{r_0} L(r,u,u')\,dr$. Let G defined on $C_0^{0,1}([0,r_0])$ be an admissible transformation group with infinitesimal generator $\mathbf{w}(r,u) = \boldsymbol{\xi}(r,u)\partial_r + \boldsymbol{\phi}(r,u)\partial_u$. Let u_0 be a fixed point of G and assume $L(r,u_0,u_0') = 0$ in $[0,r_0]$ and $\partial_{\mathbf{p}} L(0,u,0) = 0$ for all $u \in \mathbb{R}$. If furthermore $L(r,u_0,u_0'+\mathbf{p})$ is convex in $\mathbf{p}$ at $\mathbf{p} = 0$ then either of the following two conditions implies the uniqueness of the critical point $u_0 \in C^2(0,r_0) \cap C^1([0,r_0])$:*

(i) G is a strict variational sub-symmetry w.r.t. u_0,
(ii) G is a variational sub-symmetry and the unique continuation property at u_0 holds.

For ordinary differential equations the unique continuation property at u_0 means the following: if u is a C^2-solution of the Euler-equation with $u(r_0) = 0$, $u'(0) = 0$ and if $\xi(r_0,0)\Big(L(r_0,0,u'(r_0)) + (u_0'(r_0) - u'(r_0))\partial_{\mathbf{p}} L(r_0,0,u'(r_0))\Big) = 0$ holds then $u \equiv u_0$.

The proof of Theorem 3.36' differs only in one detail from the proof of Theorem 3.36: Since the boundary integral in Pohožaev's identity (3.23) contributes both at $r = 0$ and at $r = r_0$ we have

$$
\begin{aligned}
&\frac{d}{d\epsilon}\mathcal{L}[g_\epsilon u]|_{\epsilon=0} \\
&= \xi(r_0,0)L(r_0,0,u'(r_0)) + \Big(\underbrace{\phi(r_0,0)}_{=\xi(r_0,0)u_0'(r_0)} - \xi(r_0,0)u'(r_0)\Big)\partial_{\mathbf{p}} L(r_0,0,u'(r_0)) \\
&- \underbrace{\xi(0,u(0))}_{=0} L(0,u(0),0) - \Big(\phi(0,u(0)) - \underbrace{\xi(0,u(0))u'(0)}_{=0}\Big) \underbrace{\partial_{\mathbf{p}} L(0,u(0),0)}_{=0}.
\end{aligned}
$$

By our hypothesis this implies

$$
\frac{d}{d\epsilon}\mathcal{L}[g_\epsilon u]|_{\epsilon=0} = \xi(r_0,0)\Big(L(r_0,0,u'(r_0)) + (u_0'(r_0) - u'(r_0))\partial_{\mathbf{p}} L(r_0,0,u'(r_0))\Big),
$$

i.e. only the boundary $r = r_0$ contributes to the rate of change. The proof now continues exactly as in Theorem 3.36.

5.5.1 Radially symmetric problems on $\mathbb{R}^n, \mathbb{S}^n, \mathbb{H}^n$

We apply Theorem 3.36' to

$$\Delta_B u + \lambda u + |u|^{p-1}u = 0 \text{ in } B_{R_0}, \quad u = 0 \text{ on } \partial B_{R_0} \tag{5.32}$$

where Δ_B is the Laplace-Beltrami operator on either Euclidean $\mathbb{R}^n$, the n-sphere $\mathbb{S}^n$ or the hyperbolic space $\mathbb{H}^n$ and B_{R_0} is a geodesic ball of geodesic radius R_0. Recall from Section 4.1 the representations of $\mathbb{S}^n$ and $\mathbb{H}^n$:

Spherical space: Let Y be the south-pole of $\mathbb{S}^n$. Then $\mathbb{S}^n \setminus \{Y\}$ can be represented as $(\mathbb{R}^n, (\frac{2}{1+|x|^2})^2\delta_{ij})$. Geodesic balls of radius $R_0 \in (0,\pi)$ around the north-pole $-Y$ are given as Euclidean balls $B_{r_0}(0)$ with $r_0 = \tan(R_0/2)$.

Hyperbolic space: $\mathbb{H}^n$ can be represented as $(B_1(0) \subset \mathbb{R}^n, (\frac{2}{1-|x|^2})^2\delta_{ij})$. Geodesic balls of radius $R_0 \in (0,\infty)$ are given as Euclidean balls $B_{r_0}(0)$ with $r_0 = \tanh(R_0/2) \in (0,1)$.

space	$\mathbb{R}^n$	$\mathbb{S}^n$	$\mathbb{H}^n$
metric $\rho^2\delta_{ij}$	$\rho = 1$	$\rho = \frac{2}{1+r^2}$	$\rho = \frac{2}{1-r^2}$
radius of a geodesic ball	$R_0 \in (0,\infty)$	$R_0 \in (0,\pi)$	$R_0 \in (0,\infty)$
corresponding Euclidean radius	R_0	$\tan(R_0/2)$	$\tanh(R_0/2)$
scalar curvature	0	$n(n-1)$	$-n(n-1)$

Table 5.2. The model of $\mathbb{R}^n, \mathbb{S}^n, \mathbb{H}^n$

All three models will be treated in the same way. We assume that the metric is given by $g_{ij} = \rho^2(r)\delta_{ij}$ with ρ as in Table 5.2. Radially symmetric solutions of (5.32) satisfy

$$\frac{(\rho^{n-2}r^{n-1}u')'}{r^{n-1}\rho^n} + \lambda u + |u|^{p-1}u = 0 \text{ in } (0,r_0), \quad u'(0) = 0, u(r_0) = 0 \tag{5.33}$$

and they are found as critical points of the functional

$$\mathcal{L}[u] = \int_0^{r_0} r^{n-1}\rho^{n-2}\frac{u'^2}{2} - r^{n-1}\rho^n\Big(\lambda\frac{u^2}{2} + \frac{|u|^{p+1}}{p+1}\Big)\,dr$$

on the space $C_0^{0,1}[0,r_0]$. Such a Lagrangian satisfies the hypothesis of Theorem 3.36'. We investigate when the more special vector-field

$$\mathbf{w}(r,u) = \xi(r)\partial_r + a(r)u\partial_u.$$

generates a variational sub-symmetry. The prolongation of $\mathbf{w}$ is given by

$$\mathbf{w}^{(1)}(r,u) = \xi(r)\partial_r + a(r)u\partial_u + \Big(a'(r)u + (a(r) - \xi'(r))u'\Big)\partial_{\mathbf{p}}.$$

Lemma 5.28. *Let G be an admissible transformation group on $C_0^{0,1}[0,r_0]$ with infinitesimal generator $\mathbf{w} = \xi(r)\partial_r + a(r)u\partial_u$. Let the functional $\mathcal{L}[u] = \int_0^{r_0} r^{n-1}\rho^{n-2}\frac{u'^2}{2} - r^{n-1}\rho^n(\lambda\frac{u^2}{2} + \frac{|u|^{p+1}}{p+1})\,dr$ be defined on $C_0^{0,1}[0,r_0]$. Then the rate of change of $\mathcal{L}$ under the action of G at $\epsilon = 0$ is given by*

$$\begin{aligned}\frac{d}{d\epsilon}\mathcal{L}[g_\epsilon u]|_{\epsilon=0} &= \int_0^{r_0} \mathbf{w}^{(1)}L + L\xi'(r)\,dr\\ &= \int_0^{r_0} r^{n-1}\rho^{n-2}b_1(r)\frac{u'^2}{2} - r^{n-1}\rho^n\Big(b_2(r)\frac{u^2}{2} + b_3(r)\frac{|u|^{p+1}}{p+1}\Big)\,dr,\end{aligned}$$

where

$$\begin{aligned}b_1(r) &= 2a(r) - \xi' + \xi(\frac{n-1}{r} + \frac{(n-2)\rho'}{\rho}),\\ b_2(r) &= \lambda\Big(\xi' + \xi(\frac{n-1}{r} + \frac{n\rho'}{\rho}) + 2a\Big) + \Delta_B a,\\ b_3(r) &= \xi' + \xi(\frac{n-1}{r} + \frac{n\rho'}{\rho}) + (p+1)a.\end{aligned}$$

Our strategy for the choice of the unknown functions $\xi(r), a(r)$ is to set

$$a(r) = \frac{1}{2}\Big(\xi' - \xi(\frac{n-1}{r} + \frac{(n-2)\rho'}{\rho})\Big).$$

and determine $\xi(r)$ such that $b_2(r), b_3(r) \geq 0$. Once we have achieved the variational sub-symmetry, Theorem 3.36' shows uniqueness of the zero-solution.

The supercritical case for $n \geq 3$

If one sets $\xi(r) = -r$, which corresponds in the multi-dimensional case to $\xi(x) = -x$, one obtains the following result already contained in Theorem 4.35.

Proposition 5.29. *Consider a geodesic ball B or radius R_0 in $\mathbb{R}^n, \mathbb{S}^n, \mathbb{H}^n$ for $n \geq 3$. In the case of $\mathbb{S}^n$ suppose furthermore that B is contained in a half-sphere. Then* (5.32) *has only the trivial solution for $p \geq \frac{n+2}{n-2}$ provided*

$$\lambda \leq \begin{cases} 0 & \text{in case of } \mathbb{R}^n,\\ \frac{-n(n-2)}{4} & \text{in case of } \mathbb{S}^n,\\ \frac{n(n-2)}{4} & \text{in case of } \mathbb{H}^n.\end{cases}$$

Remark. All three results are sharp for $n \geq 4$ and $p = (n+2)/(n-2)$. In the Euclidean case Brezis, Nirenberg [9] showed that in dimension $n \geq 4$ positive solutions exist for $\lambda \in (0, \lambda_1)$ on any bounded domain. In case of $\mathbb{S}^n$ the above uniqueness result was obtained by Bandle, Brillard, Flucher [7]. Existence for $\lambda \in (-n(n-2)/4, \lambda_1)$ follows from the methods used by Bandle, Benguria [6]. In case of $\mathbb{H}^n$ the corresponding existence and non-existence results were obtained Stapelkamp [85].

The supercritical case for $n = 3$

In the case $n = 3$ the situation is not accurately described by Proposition 5.29. If the exponent $p = 5$ is exactly critical it is known that positive solution exist for $\lambda \in (\lambda^*, \lambda_1)$ and only the trivial solution exists for $\lambda \le \lambda^*$ with

$$\lambda^* = \begin{cases} \pi^2/4 & \text{in case of } \mathbb{R}^3, \\ \frac{\pi^4/4 - R_0^2}{R_0^2} & \text{for a geodesic ball } B_{R_0} \subset \mathbb{S}^n, R_0 < \pi/2 \\ \frac{\pi^4/4 + R_0^2}{R_0^2} & \text{for a geodesic ball } B_{R_0} \subset \mathbb{H}^n. \end{cases}$$

Brezis, Nirenberg [9] proved the Euclidean case, Bandle, Benguria [6] the spherical and Stapelkamp [85] the hyperbolic case. This exceptional behaviour of dimension $n = 3$ leads to the notion of a *critical dimension*, cf. Pucci, Serrin [78] and Janelli [49].

We will investigate the case $n = 3$ for supercritical $p \ge 5$ explicitly in the three model cases $\mathbb{R}^3, \mathbb{S}^3$ and $\mathbb{H}^3$ by one of the following two options:

(a) solve $b_2(r) \equiv 0$ and verify the sub-symmetry criterion via $b_3(r) \ge 0$,
(b) solve $b_3(r) \equiv 0$ and verify the sub-symmetry criterion via $b_2(r) \le 0$.

The best uniqueness results are always obtained by strategy (a).

The 3-dimensional Euclidean ball

In the Euclidean case $\rho \equiv 1$. Without loss of generality we may assume that the geodesic ball has radius $r_0 = 1$.

Theorem 5.30. *Let $p \ge 5$ be a given value and let $\lambda^*(p)$ be the largest value of $\lambda > 0$ such that*

$$\frac{\frac{\sqrt{\lambda} r}{2}(\cot^2(\sqrt{\lambda} r) - 1) - \cot(\sqrt{\lambda})\sqrt{\lambda} r \cot(\sqrt{\lambda} r)}{\cot(\sqrt{\lambda} r) - \cot(\sqrt{\lambda})} \le \frac{p-1}{p+3} \quad \forall r \in (0,1). \quad (5.34)$$

Then the zero solution of (5.32) *is unique for $\lambda \le \lambda^*(p)$. In particular, for $p = 5$ we find that there is no nontrivial solution for $\lambda \le \pi^2/4$.*

In Figure 5.6 the numerical evaluation of $\lambda^*(p)$ as a function of $p \ge 5$ is displayed. One can notice that $\lambda^*(5) = \pi^2/4$ and $\lambda^*(\infty) = \pi^2 = \lambda_1$ which means that asymptotically the values $\lambda^*(p)$ are best possible.

Proof. The proof consists in choosing $\xi(r)$ and $a(r) = \frac{1}{2}(\xi'(r) - \frac{2}{r}\xi(r))$ such that the coefficients $b_2(r)$ and $b_3(r)$ from Lemma 5.28 are non-positive. One finds

$$b_2(r) = \frac{1}{2}\xi''' + 2\lambda\xi', \qquad b_3(r) = \frac{3+p}{2}\xi' + \xi\frac{(1-p)}{r}$$

and solves $b_2(r) \equiv 0$ by choosing $\xi(r) = -\alpha\frac{\sin(2\sqrt{\lambda} r)}{2\sqrt{\lambda}} + \beta\left(\frac{\cos(2\sqrt{\lambda} r) - 1}{2\sqrt{\lambda}}\right)$, where α, β need to be found according to $b_3(r) \ge 0$ for $r \in (0,1)$ and $\xi(1) \le 0$. By

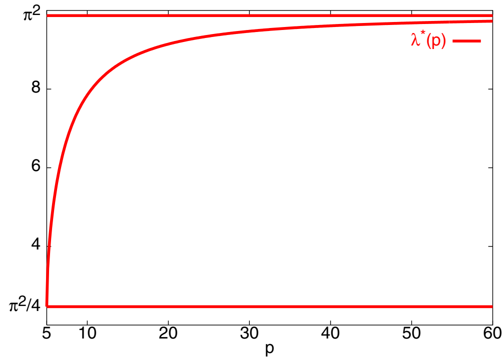

Fig. 5.6. The curve λ^* as a function of $p \geq 5$

expanding $b_3(r)$ again around $r = 0$ we find that necessarily $\alpha > 0$. Thus, we may assume $\alpha = 1$. The relevant equations $b_3(r) \geq 0$ and $\xi(1) \leq 0$ become

$$\cos(2\sqrt{\lambda}r) + \beta \sin(2\sqrt{\lambda}r) \leq \frac{2(p-1)}{p+3}\Big(\frac{\sin(2\sqrt{\lambda}r)}{2\sqrt{\lambda}r} - \beta\frac{\cos(2\sqrt{\lambda}r)-1}{2\sqrt{\lambda}r}\Big),$$
$$\frac{\sin(2\sqrt{\lambda})}{2\sqrt{\lambda}} - \beta\frac{\cos(2\sqrt{\lambda})-1}{2\sqrt{\lambda}} \geq 0.$$

Since we know a priori that $\lambda \leq \lambda_1 = \pi^2$ the terms $\sin(\sqrt{\lambda}r)$ are non-negative. Thus, after some trigonometry this set of equations simplifies to

$$\frac{\sqrt{\lambda}r}{2}(\cot^2(\sqrt{\lambda}r)-1) + \beta\sqrt{\lambda}r\cot(\sqrt{\lambda}r) \leq \frac{p-1}{p+3}(\cot(\sqrt{\lambda}r)+\beta), \quad (5.35)$$
$$\beta + \cot(\sqrt{\lambda}) \geq 0. \quad (5.36)$$

In the case $p = 5$ it is simple to find the optimal λ-interval: choose $\beta = 0$. Then (5.36) holds for $0 \leq \lambda \leq \pi^2/4$ and (5.35) reduces to

$$\frac{z(\cot^2 z - 1)}{\cot z} \leq 1 \text{ for } z = \sqrt{\lambda}r$$

which holds provided $0 \leq z \leq \pi/2$, i.e., $0 \leq \lambda \leq \pi^2/4$.

The case $p > 5$ is more involved. By (5.36) the right hand side of (5.35) is non-negative. Therefore the optimal value $\lambda^*(p)$ is determined as the biggest value for which both of the following inequalities hold:

$$\frac{\frac{\sqrt{\lambda}r}{2}(\cot^2(\sqrt{\lambda}r)-1) + \beta\sqrt{\lambda}r\cot(\sqrt{\lambda}r)}{\cot(\sqrt{\lambda}r)+\beta} \leq \frac{p-1}{p+3} \text{ for } r \in (0,1), \quad (5.37)$$
$$\beta + \cot(\sqrt{\lambda}) \geq 0. \quad (5.38)$$

Since (5.37) is monotone increasing in β we need to choose β as small as possible to obtain the largest λ-interval of validity of (5.37)-(5.38). Thus the choice

$$\beta = -\cot(\sqrt{\lambda})$$

is best possible. This reduces the determination of $\lambda^*(p)$ to the single inequality (5.34). □

The 3-dimensional spherical ball

In this case $\rho(r) = 2/(1+r^2)$. The result now depends on the size of the geodesic ball. For $p = 5$ the result was obtained by Bandle, Benguria [6].

Theorem 5.31. *Let $p \geq 5$ be a given value and let $s_0 = 2\arctan(r_0)$ be the geodesic radius of a spherical ball.*

(a) If $0 < s_0 \leq \pi/2$ then there exists a value $\lambda^*(p, s_0) \in (-1, \lambda_1)$ such that (5.32) *has only the zero-solution for $\lambda \leq \lambda^*$.*
(b) If $\pi/2 < s_0 < \pi$ then there exists values $\lambda^*(p, s_0)$ and $\lambda_*(p, s_0)$ such that (5.32) *has only the zero-solution for $\lambda_* \leq \lambda \leq \lambda^*$.*

For $p = 5$ one has $\lambda^* = \frac{\pi^2/4 - s_0^2}{s_0^2}$.

Sketch of proof. We need to choose $\xi(r)$ and $a(r) = \frac{1}{2}(\xi'(r) - \xi(r)(\frac{n-1}{r} + \frac{(n-2)\rho'}{r}))$ such that the coefficients $b_2(r)$ and $b_3(r)$ from Lemma 5.28 are non-positive. For $r \in (0, r_0)$ one finds

$$b_2(r) = \frac{1}{8}\xi'''(1+r^2)^2 + \xi'(2\lambda + \frac{3}{2}) - \xi\frac{(3+4\lambda)r}{1+r^2},$$
$$b_3(r) = \xi'(\frac{3+p}{2} - 6\frac{r}{1+r^2}) + \xi\frac{1-p+2r^2}{r(1+r^2)}.$$

A special solution of $b_2(r) = 0$ is found by $\xi(r) = (1+r^2)$. Further solutions are found by setting $\xi(r) = (1+r^2)k(s)$ and $s = 2\arctan(r)$, s is then the geodesic radius. The resulting form of $b_2(s) = 0$ is

$$k'''(s) + 4k'(s)(1+\lambda) = 0,$$

which can be solved explicitly. The complete solution ξ is written as $\xi(s) = (1 + \tan(s/2)^2)l(s)$, with

$$\begin{aligned} l(s) &= \frac{-\alpha}{2}\sin(2\sqrt{1+\lambda}s) - \frac{\beta}{2}\Big(1 - \cos(2\sqrt{1+\lambda}s)\Big), && \text{if } \lambda > -1,\\ l(s) &= \frac{-\alpha}{2}\sinh(2\sqrt{-1-\lambda}s) - \frac{\beta}{2}\Big(1 - \cosh(2\sqrt{-1-\lambda}s)\Big), && \text{if } \lambda < -1,\\ l(s) &= -\alpha s^2 - \beta s^3, && \text{if } \lambda = -1. \end{aligned}$$

The formula for b_3 now looks like follows:

$$b_3(s) = \dot{l} - \frac{2(p-1)}{p+3}\cot(s)l \geq 0 \quad \text{for } s \in (0, s_0).$$

Expanding around $s = 0$ quickly shows that necessarily $\alpha > 0$, which implies that we may take $\alpha = 1$. The goal is to find for a given value of s_0 the best choice of β giving the largest possible λ-interval for uniqueness. A detailed analysis of the case $p = 5$ was given in Bandle, Benguria [6]. □

We illustrate Theorem 5.31 by plotting some numerically determined curves λ^*, λ_* together with the first eigenvalue $\lambda_1(s_0)$ as a function of the geodesic radius $s_0 = 2\arctan r_0$. Figure 5.7 shows the case of exponent $p = 5$ and $p = 5, 10$. The diagram indicates that both λ^* and λ_* are monotone increasing with respect to the exponent p.

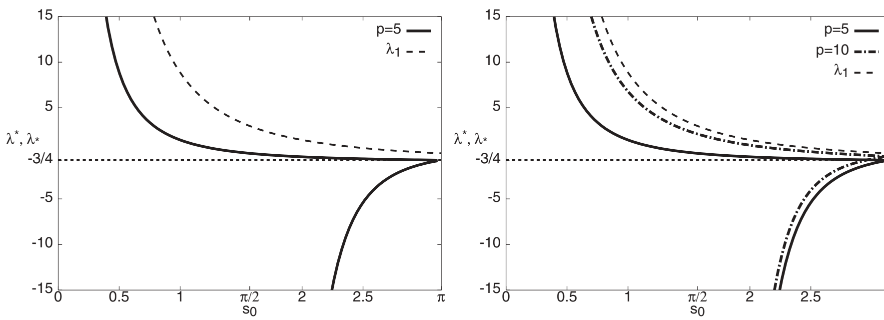

Fig. 5.7. λ^*, λ_* for $p = 5$ (left) and $p = 5, 10$ (right)

Remark 5.32. Theorem 5.31 is sharp for the case $p = 5$ in the following sense: Bandle, Benguria [6] showed that positive solutions exist for $\lambda \in (\lambda^*, \lambda_1)$. In the case of a geodesic ball lager than the half-sphere, it is very surprising to have uniqueness only for a finite λ-interval between λ_* and λ^*. This is in striking difference to the Euclidean case. Bandle, Benguria [6] and Stingelin [86] found strong numerical evidence that solutions exist for all $\lambda < \lambda_*$.

The 3-dimensional hyperbolic ball

The hyperbolic conformal factor is $\rho(r) = 2/(1-r^2)$. The result again depends on the size of the geodesic ball. It is illustrated by numerically determined curves in Figure 5.8. For $p = 5$ the result can be found in Stapelkamp [85].

Theorem 5.33. *Let $p \geq 5$ be a given value and let $t_0 = 2\,\mathrm{artanh}(r_0)$ be the geodesic radius of a hyperbolic ball. Then there exists a value $\lambda^*(p, t_0) \in (1, \lambda_1)$ such that* (5.32) *has only the zero-solution for $\lambda \leq \lambda^*$. For $p = 5$ one has $\lambda^* = \frac{\pi^2/4 + t_0^2}{t_0^2}$.*

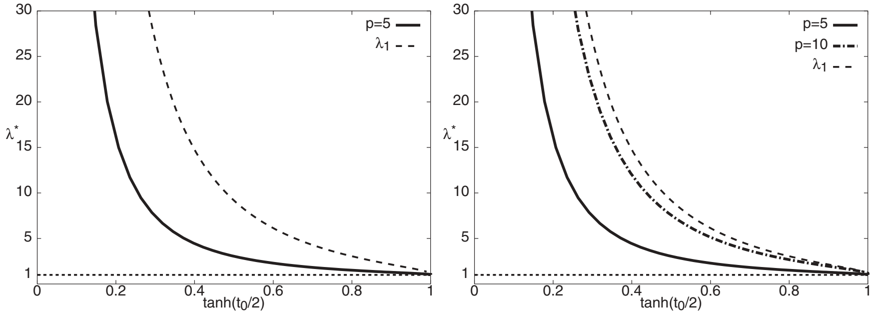

Fig. 5.8. λ^* for $p = 5$ (left) and $p = 5, 10$ (right)

Sketch of proof. As in the spherical case we need to choose $\xi(r)$ and $a(r) = \frac{1}{2}(\xi'(r) - \xi(r)(\frac{n-1}{r} + \frac{(n-2)\rho'}{r}))$ such that the coefficients $b_2(r)$ and $b_3(r)$ from Lemma 5.28 are non-positive. For $r \in (0, r_0)$ we obtain

$$b_2(r) = \frac{1}{8}\xi'''(1-r^2)^2 + \xi'(2\lambda - \frac{3}{2}) - \xi\frac{(3-4\lambda)r}{1-r^2},$$
$$b_3(r) = \xi'(\frac{3+p}{2} + 6\frac{r}{1-r^2}) + \xi\frac{1-p-2r^2}{r(1-r^2)}.$$

A special solution of $b_2(r) = 0$ is found by $\xi(r) = (1-r^2)$. Further solutions are found by setting $\xi(r) = (1-r^2)k(t)$ and $t = 2\,\mathrm{artanh}(r)$, t is then the geodesic radius. The resulting form of $b_2(t) = 0$ is

$$k'''(t) + 4k'(t)(-1+\lambda) = 0,$$

which can be solved explicitly. Thus the complete solution ξ is written as $\xi(t) = (1 - \tanh(t/2)^2)l(t)$, with

$$\begin{aligned} l(t) &= \frac{-\alpha}{2}\sin(2\sqrt{\lambda-1}t) - \frac{\beta}{2}\Big(1 - \cos(2\sqrt{\lambda-1}t)\Big), && \text{if } \lambda > 1,\\ l(t) &= \frac{-\alpha}{2}\sinh(2\sqrt{1-\lambda}t) - \frac{\beta}{2}\Big(1 - \cosh(2\sqrt{1-\lambda}t)\Big), && \text{if } \lambda < 1,\\ l(t) &= -\alpha t^2 - \beta t^3, && \text{if } \lambda = 1. \end{aligned}$$

The formula for $b_3(t)$ takes the form:

$$a_1(t) = \dot{l} - \frac{2(p-1)}{p+3}\coth(t)l \geq 0 \quad \text{for } t \in (0, t_0).$$

Expanding around $t = 0$ shows that necessarily $\alpha > 0$, which implies that we may take $\alpha = 1$. The goal is to find for a given value of s_0 the best choice of β giving the largest possible λ-interval for uniqueness. □

Remark 5.34. For $p = 5$ the result is again sharp, since Stapelkamp [85] showed existence of positive solutions for $\lambda \in (\lambda^*, \lambda_1)$.

5.5.2 The radially symmetric q-Laplacian

As a further application of Theorem 3.36' we consider

$$\Delta_q u + \lambda |u|^{q-2} u + |u|^{p-1} u = 0 \text{ in } B_1(0), \quad u = 0 \text{ on } \partial B_1(0) \tag{5.39}$$

where $\Delta_q u = \operatorname{div}(|\nabla u|^{q-2} \nabla u)$ is the Euclidean q-Laplacian for $1 < q < \infty$, cf. Section 5.1.2. Radially symmetric solutions satisfy

$$\begin{aligned} r^{1-n}(r^{n-1}|u'|^{q-2}u')' + \lambda |u|^{q-2}u + |u|^{p-1}u = 0 \text{ in } (0,1), \\ u'(0) = u(1) = 0 \end{aligned} \tag{5.40}$$

and they are found as critical points of the functional

$$\mathcal{L}[u] = \int_0^1 r^{n-1} \Big(\frac{|u'|^q}{q} - \lambda \frac{|u|^q}{q} - \frac{|u|^{p+1}}{p+1} \Big) \, dr$$

on the space $C_0^{0,1}(0,1)$. Let us write $q^* = nq/(n-q)$ if $q < n$ and $q^* = \infty$ if $q \geq n$. Then the embedding $W_0^{1,q}$ into L^s is continuous for all finite s with $1 \leq s \leq q^*$ and compact for all s with $1 \leq s < q^*$. The following results are known:

(a) In the subcritical case $1 < p < q^* - 1$ solutions of (5.39) exist for all values of $\lambda \in \mathbb{R}$, and positive solutions for all $\lambda < \lambda_1$, see del Pino, Manásevich [19].
(b) In the supercritical case $p \geq q^* - 1$ there exist positive solutions of (5.39) for λ in an interval $(\lambda^*, \lambda_1) \subset (0, \lambda_1)$, see del Pino, Manásevich [19]. For $\lambda \leq 0$ only the zero-solution exists, see Theorem 5.10.
(c) In the critical case $p = q^* - 1$, two subcases occur
   (i) For $n \geq q^2$ Guedda and Veron [41] showed that positive solutions of (5.39) exist for all $\lambda \in (0, \lambda_1)$.
   (ii) For $q < n < q^2$ there are no nontrivial solutions in a right-neighbourhood of $\lambda = 0$, as proved by Egnell [25] and as we show next.

In the critical-exponent case Egnell's result shows that the dimensions n strictly between q and q^2 play the same exceptional role as dimension $n = 3$ in the Laplace or Laplace-Beltrami equations of the previous Section. Theses dimensions are therefore called the *critical dimensions* for the q-Laplacian.

Theorem 5.35. *For a critical dimension $n \in (q, q^2)$ and for $p \geq q^* - 1$ the only solution of* (5.40) *for $\lambda \leq (q^2 - n) n^q q^{-1-q}$ is $u \equiv 0$.*

Proof. We already know from Theorem 5.10 that there are no non-trivial solutions for $\lambda \leq 0$. So suppose $\lambda > 0$. Consider an admissible transformation group G generated by the vector-field $\mathbf{w} = \xi(r)\partial_r + a(r)u\partial_u$. For the functional $\mathcal{L}[u] = \int_0^1 (\frac{1}{q}|u'|^q - F(u)) r^{n-1} \, dr$ we find

$$\begin{aligned}
&\int_0^1 \mathbf{w}^{(1)}L + L\xi'\,dr \\
&= \int_0^1 (a'u|u'|^{q-2}u')r^{n-1} + \Big(a - \xi' + \frac{1}{q}(\xi' + \frac{n-1}{r}\xi)\Big)|u'|^q r^{n-1}\,dr \\
&\quad - \int_0^1 af(u)ur^{n-1} + (\xi' + \frac{n-1}{r}\xi)F(u)r^{n-1}\,dr.
\end{aligned}$$

The simplest way to handle the first term is to take $a = \text{const.}$, and by the previous experience from Theorem 5.10 we take $a = \frac{n-q}{q}$. Egnell suggested $\xi(r) = -r + \zeta(r)$, where one can think of ζ as a perturbation of the previous choice $-r$ from Theorem 5.10. If we furthermore insert $f(u) = \lambda|u|^{q-2}u + |u|^{p-1}u$ then we obtain

$$\begin{aligned}
&\int_0^1 \mathbf{w}^{(1)}L + L\xi'\,dr \\
&= \int_0^1 \Big(\zeta'\frac{1-q}{q} + \frac{(n-1)\zeta}{rq}\Big)|u'|^q r^{n-1} + \Big(1 - \frac{1}{q}(\zeta' + \frac{n-1}{r}\zeta)\Big)\lambda|u|^q r^{n-1}\,dr \\
&\quad + \int_0^1 \Big(\frac{q-n}{q} + \frac{n}{p+1} - \frac{1}{p+1}(\zeta' + \frac{n-1}{r}\zeta)\Big)|u|^{p+1}r^{n-1}\,dr.
\end{aligned}$$

Egnell used the first integral over $|u'|^q$ to estimate the integral over $|u|^q$. For this purpose he used Hardy's inequality, see Hardy, Littlewood, Polya [42], 9.9.10:

$$\int_0^\infty |v|^q r^{n-1}\,dr \le \Big(\frac{q}{n}\Big)^q \int_0^\infty |v'|^q r^{n+q-1}\,dr$$

valid for C^1-functions on $[0,\infty)$, which vanish for large r. To obtain the right power of r choose $\zeta = r^{q+1}$. Then the vector-field $\mathbf{w} = (-r + r^{q+1})\partial_r + \frac{n-q}{q}u\partial_u$ generates an admissible group of transformations and we obtain

$$\begin{aligned}
\int_0^1 \mathbf{w}^{(1)}L + L\xi'\,dr &= \int_0^1 \frac{n-q^2}{q}|u'|^q r^{n+q-1} + \Big(\underbrace{1 - \frac{q+n}{q}r^q}_{\ge 0}\Big)\lambda|u|^q r^{n-1}\,dr \\
&\quad + \int_0^1 \Big(\frac{q-n}{q} + \frac{n}{p+1} - \underbrace{\frac{q+n}{p+1}r^q}_{\ge 0}\Big)|u|^{p+1}r^{n-1}\,dr.
\end{aligned}$$

For the critical dimensions $q < n < q^2$ we may use Hardy's inequality and drop some negative terms to find

$$\begin{aligned}
&\int_0^1 \mathbf{w}^{(1)}L + L\xi'\,dr \\
&\quad < \int_0^1 \Big(\lambda + \frac{n-q^2}{q}\Big(\frac{n}{q}\Big)^q\Big)|u|^q r^{n-1} + \Big(\frac{q-n}{q} + \frac{n}{p+1}\Big)|u|^{p+1}r^{n-1}\,dr,
\end{aligned}$$

unless $u \equiv 0$. For values $\lambda \le (q^2-n)n^q q^{-1-q}$ and supercritical $p \ge q^*-1$ the vector-field $\mathbf{w}$ generates a strict variational sub-symmetry w.r.t. 0, and hence we have uniqueness of the zero-solution by Theorem 3.36'. □

5.5.3 Partial radial symmetry

Consider a half-ball $B_1^+(0) = \{x \in \mathbb{R}^n : |x| \le 1, x_n > 0\}$ and the boundary value problem

$$\Delta u + \lambda u + |u|^{p-1}u = 0 \text{ in } B_1^+(0), \quad u = 0 \text{ on } \partial B_1^+(0). \tag{5.41}$$

Since $B_1^+(0)$ is star-shaped w.r.t. 0 it is easy to see that (5.41) has no non-trivial solution if $p \ge \frac{n+2}{n-2}$ and $\lambda \le 0$. For $p = \frac{n+2}{n-2}$ and $n \ge 4$ this is a sharp result, since for $0 < \lambda < \lambda_1$ positive solutions exist, cf. Brezis, Nirenberg [9]. However, the situation is different for $n = 3$, since it is known that a value $\lambda^* \in (0, \lambda_1)$ exists such (5.41) has positive solutions for every $\lambda \in (\lambda^*, \lambda_1)$ and no positive solution for $\lambda \le \lambda^*$. We want to show how the method of transformation groups can be used to get lower bounds for λ^*.

It follows from the symmetry theorem of Gidas, Ni and Nirenberg [37] that every positive solution is symmetric in the variable $r = \sqrt{x^2+y^2}$, i.e., $u = u(r,z)$. By using this symmetry we can show the following result.

Theorem 5.36. *For $\lambda \le \lambda_* = 1/8$ there is no positive solution of* (5.41) *for $n = 3$ and $p = 5$.*

Proof. Since the domain $\Omega = B_1^+(0)$ is star-shaped the result needs to be proved only for $\lambda > 0$. We recall the following result from the proof of Theorem 4.8: if $L(x,u,\mathbf{p}) = \frac{1}{2}|\mathbf{p}^2| - \frac{\lambda}{2}u^2 - \frac{1}{6}u^6$ and $\mathbf{w} = \boldsymbol{\xi} \cdot \nabla x + \alpha u \partial_u$ is the generator of a one-parameter transformation group then

$$\begin{aligned}\mathbf{w}^{(1)}L + L\,\mathrm{Div}\,\boldsymbol{\xi} &= |\nabla u|^2(\alpha + \frac{1}{2}\operatorname{div}\boldsymbol{\xi}) - (\mathrm{Adj}\, D\boldsymbol{\xi}\nabla u)\cdot\nabla u + \frac{1}{2}\nabla\alpha\cdot\nabla(u^2)\\ &\quad + u^2(-\alpha\lambda - \frac{\lambda}{2}\operatorname{div}\boldsymbol{\xi}) + |u|^6(-\alpha - \frac{1}{6}\operatorname{div}\boldsymbol{\xi}).\end{aligned}$$

We take $\boldsymbol{\xi}(r,z) = (a(r,z)\cos t, a(r,z)\sin t, b(r,z))$, where we use cylindrical coordinates $x = r\cos t$, $y = r\sin t$ and z. Due to the symmetry of u we have $\nabla u(r,z) = (u_r \cos t, u_r \sin t, u_z)$ and hence

$$\nabla u^T D\boldsymbol{\xi}\nabla u = u_r^2 a_r + u_z u_r(a_z + b_r) + u_z^2 b_z. \tag{5.42}$$

Let us choose $a(r,z), b(r,z)$ as real- and imaginary part of the holomorphic function $p(w) = a(r,z) + ib(r,z)$ w.r.t. the variable $w = r + iz$. Then we find

$$\nabla u^T D\boldsymbol{\xi}\nabla u = a_r|\nabla u|^2 \text{ and } \operatorname{div}\boldsymbol{\xi} = 2a_r + \frac{a}{r}.$$

Hence

$$\int_\Omega \mathbf{w}^{(1)}L + L\,\mathrm{div}\,\boldsymbol{\xi}\,dx$$
$$= \int_\Omega |\nabla u|^2(\alpha + \frac{a}{2r}) + u^2\big(\lambda(-\alpha - a_r - \frac{a}{2r}) - \frac{1}{2}\Delta\alpha\big) + u^6(-\alpha - \frac{a_r}{3} - \frac{a}{6r})\,dx.$$

We choose $\alpha = -a/(2r)$. Then $\mathbf{w}$ generates a variational sub-symmetry provided the following conditions hold

$$\text{(i)}\quad \boldsymbol{\xi}\cdot\boldsymbol{\nu} \le 0 \text{ on } \partial\Omega, \qquad \text{(ii)}\quad \Delta\frac{a}{r} \le 4\lambda a_r \text{ in } \Omega, \qquad \text{(iii)}\quad \frac{a}{r} \le a_r \text{ in } \Omega,$$

with a strict inequality on a subset of $\partial\Omega$ of positive measure in (i). Now we choose the holomorphic function $p(w) = -w + \alpha w^3$ with α to be determined. In this case $a(r,z) = -r + \alpha r^3 - 3\alpha r z^2$ and $b(r,z) = -z + 3\alpha r^2 z - \alpha z^3$. First we look at condition (iii) and find that it is satisfied for $\alpha \ge 0$. Next we consider condition (i). On the flat part of $\partial\Omega$ we have $\boldsymbol{\nu} = (0,0,-1)$ and $\boldsymbol{\xi}\cdot\boldsymbol{\nu} = -b(r,0) = 0$. On the curved part of $\partial\Omega$ we have $\boldsymbol{\nu} = (r\cos t, r\sin t, z)$ and hence for $r^2 + z^2 = 1, r, z > 0$ we compute

$$\boldsymbol{\xi}\cdot\boldsymbol{\nu} = -1 + \alpha(-1 + 2r^2).$$

For $\alpha \ge 0$ the latter takes its maximum at $r = 1$, and hence we need to restrict to $0 \le \alpha \le 1$. It remains to consider condition (ii). First we need to compute $\Delta(a/r)$. Using the fact that $a_{rr} + a_{zz} = 0$ one finds

$$\Delta\frac{a}{r} = -\frac{1}{r^2}(a_r - \frac{a}{r})$$

and by using the specific form of a condition (i) amounts to

$$-\alpha \le 2\lambda(-1 + 3\alpha r^2 - 3\alpha z^2). \tag{5.43}$$

Since $\alpha \ge 0$ and $z^2 \le 1 - r^2$ it is necessary and sufficient for (5.43) that $-\alpha \le 2\lambda(-1 + 6\alpha r^2 - 3\alpha)$. The right-hand side attains its infimum at $r = 0$ and hence we obtain that

$$\lambda \le \frac{\alpha}{2(1+3\alpha)} \text{ for } 0 \le \alpha \le 1.$$

The optimal value $\alpha = 1$ maximizes the right-hand side and shows that for $\lambda \le 1/8$ we have a variational sub-symmetry. Theorem 3.36*(ii)* together with the unique continuation property at $u_0 \equiv 0$ shows that (5.41) has only the trivial solution $u \equiv 0$ for $\lambda \le 1/8$. □

Remark 5.37. The idea of using a vector-field $(a(r,z)\cos t, b(r,z)\sin t, z)$ with a holomorphic function $p(w) = a(r,z) + ib(r,z)$ of the variable $w = r + iz$ goes back to Chlebík, Fila and Reichel [11], where results similar to Theorem 5.36 were obtained for nonlinear Neumann problems. E.g., if $\Omega = B_1^+(0) \subset \mathbb{R}^3$ and $\Gamma_D = \{(x,y,z) : x^2 + y^2 + z^2 = 1, z \ge 0\}$, $\Gamma_N = \{(x,y,0) : x^2 + y^2 \le 1\}$ then the boundary value problem

$$\Delta u + \lambda u = 0 \text{ in } B_1^+(0), \quad u = 0 \text{ on } \Gamma_D, \quad \partial_\nu u - u^4 = 0 \text{ on } \Gamma_N$$

has no positive solution if $\lambda \le \lambda_* = 1/8$.

5.6 Notes on further results

There is a large number of results related to uniqueness/nonexistence of solutions to nonlinear elliptic equations on domains in $\mathbb{R}^n$ which we did not cover. Some of them are listed here.

1. Cooperative systems. Consider the (possibly non-variational) cooperative system

$$-\Delta u_i = f_i(u) \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega$$

for $i = 1, \dots, N$. If $\Omega \subset \mathbb{R}^n$ is bounded, star-shaped and $f_i(tu)/t^{\frac{n+2}{2-n}}$ is increasing in $t > 0$ then no positive solution exists as shown by Reichel, Zou [80]. The method is based on the maximum principle. Corresponding results on complements of bounded star-shaped domains are also shown without decay assumptions at infinity.

2. Quasilinear elliptic problems. Problems of the type

$$\operatorname{div}(g(x, \nabla u)) + \lambda f(x, u) = 0 \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega$$

are studied by McGough, Schmitt [63] and McGough, Mortensen [64]. There the ideas of Schaaf [82] and Dancer, Zhang [17] find a natural extension. The method is the same as in Chapter 4. The notion of h-starlike domains $\Omega \subset \mathbb{R}^n$ is introduced where a vector-field $h : \Omega \to \mathbb{R}^n$ exists with

$$-\mathbf{b}^T Dh\mathbf{b} \le \big(\frac{1}{2}(-\operatorname{div} h) - c\big)|\mathbf{b}|^2, \qquad h \cdot \boldsymbol{\nu} \le 0 \text{ on } \partial\Omega,$$

where $c > 0$ is a constant. Depending on the size of the constant c a non-existence exponent p^* is found as in Theorem 4.8. Various examples of h-starlike domains are discussed. As an application the question of a-priori bounds is treated. E.g., based on Pohožaev-type identities $W^{1,2}$-bounds for solutions of the Gelfand-problem (see also Section 5.2) $\Delta u + \lambda e^u = 0$ on an h-starlike domain with zero Dirichlet data is proved. The interest in this result lies in the fact that the L^∞-norm on the branch of minimal positive solutions is unbounded.

3. Anisotropic quasilinear elliptic problems. On a bounded domain $\Omega \subset \mathbb{R}^n$ consider the anisotropic quasilinear degenerate boundary value problem

$$\sum_{i=1}^n \partial_i(|\partial_i u|^{m_i-2}\partial_i u) + u^p = 0, u \ge 0 \text{ in } \Omega, \qquad u = 0 \text{ on } \partial\Omega.$$

Fragalà, Gazzola and Kawohl [33] obtained uniqueness of the trivial solution in a supercritical setting. Due to the anisotropic nature the correct generalization of starshapedness is α-starshapedness, where $\alpha = (\alpha_1, \dots, \alpha_n)$ is a multi-index of positive numbers and the domain Ω is positively invariant under the group-action generated by $\dot{X}_i = -\alpha_i X_i, X(0) = x$. The correct critical exponent in this setting is

$$m^* := \frac{n}{\sum_{i=1}^n \frac{1}{m_i} - 1}$$

and indeed the authors prove that for $p > m^* - 1$ the above problem has only the zero-solution provided the α-starshapedness holds with $\alpha_i := n(\frac{1}{m_i} - \frac{1}{m^*})$, $m_i \geq 2$ and $\sup_i m_i / \inf_i m_i < (n+2)/n$.

4. Polyharmonic Dirichlet problems. For the polyharmonic Dirichlet problem

$$(-\Delta)^m u = f(x,u) \text{ in } \Omega, \quad u = \ldots = D^{m-1}u = 0 \text{ on } \partial\Omega,$$

with $m \geq 1$ on a bounded domain $\Omega \subset \mathbb{R}^n$ similar uniqueness results as for the case $m = 1$ are known. Here the expression $D^l u$ stands for the $n^{|l|}$-tensor consisting of the $|l|$-th order derivatives of u for a multi-index l with $1 \leq |l| \leq m-1$. E.g., for $f(x,u) = |u|^{p-1}u$, Pucci and Serrin [77] proved uniqueness of the trivial solution on star-shaped domains if $p > \frac{n+2m}{n-2m}$ and $n > 2m$. Admitting $p = \frac{n+2m}{n-2m}$ is a delicate case because here the corresponding unique continuation property as in Definition 3.33 is not available. Results were found if u is positive and $\Omega = B_1(0)$ by Osvald [73] and if u is radial and $m = 2, 3$ by Grunau [39].

The basic uniqueness result of Pucci, Serrin [77] was generalized to conformally contractible domains by Reichel [79]. In the same paper the lower L^∞-bounds in the subcritical case (Theorem 5.21) and the uniqueness result for non-trivial solutions for supercritical nonlinearities of the type $f(s) = \lambda(1 + |s|^{p-1}s)$ (Theorem 5.15) were generalized to polyharmonic operators.

In the radially symmetric case $\Omega = B_1(0)$ a similar phenomenon of *critical dimensions* as for the case $m = 1$ in Theorem 5.30, Section 5.5.1 is known. It was conjectured by Pucci, Serrin [78] that for the polyharmonic problem with $f(x,u) = \lambda u + u^{\frac{n+2m}{n-2m}}$ the dimensions $n = 2m+1, \ldots, 4m-1$ are critical dimensions in the following sense: there exists $\lambda_* > 0$ such that no non-trivial solution exists for $\lambda \leq \lambda_*$. This conjecture was proved by Pucci and Serrin [78] for $m = 2$. In a weaker version, the full conjecture was proved by Grunau [40]. He showed that for $n = 2m+1, \ldots, 4m-1$ there is a $\lambda_* > 0$ such that no non-trivial *radial* solution exists for $\lambda \leq \lambda_*$. In its full version the conjecture on critical dimension seems still open.

As a variant of the Dirichlet problem the biharmonic Navier problem

$$(-\Delta)^2 u = \lambda u + |u|^{p-1}u \text{ in } \Omega, \quad u = \Delta u = 0 \text{ on } \partial\Omega$$

was studied. For $\lambda \leq 0$ Mitidieri [66] and vanderVorst [89] showed that on bounded star-shaped domains there are no positive solutions if $p \geq \frac{n+4}{n-4}$ and $n > 4$. The case $\lambda > 0$ with $p = \frac{n+4}{n-4}$ on a ball was studied by Gazzola, Grunau, Squassina [34], where in the critical dimensions $n = 5, 6, 7$ it was shown that no positive radial solution exists for $\lambda \leq \lambda_*$. An extension of the above results to solutions, which are not a-priori assumed to be positive is not available so far.

5. Unbounded domains. All of the above examples may be considered on unbounded instead of bounded conformally contractible domains. In such a situation uniqueness results can still be obtained via the method of transformation groups. The results are dual to the results on bounded domains. We explain this in the polyharmonic case on an unbounded conformally contractible domain, cf. Reichel [79]: if $f(s) = \lambda s + |s|^{p-1}s$ and $1 < p < \frac{n+2m}{n-2m}$ if $n > 2m$ and $1 < p < \infty$ if $n \leq 2m$, $\lambda \geq 0$ then the zero-solution is unique provided one has a growth assumption $u \in H^{2m-1}(\Omega)$. This is needed to control the boundary contributions on $\partial\Omega$ intersected with a large ball $B_R(0)$ if one lets $R \to \infty$. Notice that the assumptions on the exponent p and the coefficient λ are reversed if compared to the case of a bounded domain. This observation was already made by Pucci and Serrin [77] for the complements of star-shaped domains (under weaker growth conditions than above). Related results for solutions with pre-assumed decay at infinity are found in Esteban and Lions [28]. Reichel and Zou [80] obtained uniqueness results for nonlinear Laplace equations with no decay assumptions at infinity.

6

Vector problems in Euclidean space

6.1 The Emden-Fowler system

The scalar Dirichlet problem for the Emden-Fowler equation $\Delta u + \lambda u + |u|^{p-1}u = 0$ can be generalized to a system as follows:

$$\begin{aligned} \Delta u + \mu v + |v|^{q-1}v = 0 \text{ in } \Omega, &\quad u = 0 \text{ on } \partial\Omega \\ \Delta v + \lambda u + |u|^{p-1}u = 0 \text{ in } \Omega, &\quad v = 0 \text{ on } \partial\Omega, \end{aligned} \tag{6.1}$$

where $\Omega \subset \mathbb{R}^n$ and $p, q \geq 1$. Solutions correspond to critical points of the functional

$$\mathcal{L}[u,v] = \int_\Omega \nabla u \cdot \nabla v - \lambda \frac{u^2}{2} - \mu \frac{v^2}{2} - \frac{|u|^{p+1}}{p+1} - \frac{|v|^{q+1}}{q+1}\, dx.$$

For finding weak solutions one would look for $(u,v) \in V_\alpha = W_0^{1,\alpha}(\Omega) \times W_0^{1,\alpha'}(\Omega)$ with $1 < \alpha, \alpha' < \infty$ and $\frac{1}{\alpha} + \frac{1}{\alpha'} = 1$. For this space we have the embedding

$$V_\alpha = W_0^{1,\alpha}(\Omega) \times W_0^{1,\alpha'}(\Omega) \to L^\beta(\Omega) \times L^\gamma(\Omega)$$

with β, γ such that

$$\begin{aligned} &\alpha \leq \beta \leq \tfrac{n\alpha}{n-\alpha} \text{ if } n > \alpha, && \alpha \leq \gamma \leq \tfrac{n\alpha}{n(\alpha-1)-\alpha} \text{ if } n > \tfrac{\alpha}{\alpha-1}, \\ &\alpha \leq \beta < \infty \quad \text{ if } n \leq \alpha, && \alpha \leq \gamma < \infty \qquad\qquad \text{ if } n \leq \tfrac{\alpha}{\alpha-1}. \end{aligned}$$

Hence, for a given pair (p,q) the functional $\mathcal{L}$ is well defined on V_α provided there exists a value α such that

$$(n-\alpha)(p+1) \leq n\alpha, \quad \Big(n(\alpha-1) - \alpha\Big)(q+1) \leq n\alpha.$$

This is equivalent to

$$\frac{1}{p+1} + \frac{1}{q+1} \geq \frac{n-2}{n}, \tag{6.2}$$

W. Reichel: LNM 1841, pp. 127–138, 2004.

and the embedding $V_\alpha \to L^{p+1}(\Omega) \times L^{q+1}(\Omega)$ is even compact on bounded domains Ω if (6.2) holds with strict inequality. Based on the compactness of the embedding, Hulshof, vanderVorst [47] and Felmer, de Figueiredo [31] proved existence of non-trivial solutions provided (p, q) satisfy (6.2) strictly and $\lambda\mu \le \lambda_1^2$.

The natural compactness barrier turns out to be indeed the barrier for the existence of nontrivial critical points, as the following theorem of Mitidieri [66] and vanderVorst [89] shows.

Theorem 6.1 (Mitidieri, vanderVorst). *Suppose* (p, q) *satisfy*

$$\frac{1}{p+1} + \frac{1}{q+1} \le \frac{n-2}{n} \tag{6.3}$$

and suppose $\lambda, \mu \le 0$. *If* Ω *is a bounded, piecewise smooth, star-shaped domain then* (6.1) *has no solution* (u, v) *with* $u, v \ge 0$ *or* $u, v \le 0$ *other than* $u, v \equiv 0$.

Proof. For any $p, q > 1$ the functional $\mathcal{L}[u, v]$ is well defined for $u, v \in C_0^{0,1}(\overline{\Omega})$. Consider the Lagrangian

$$L(x, u, v, \mathbf{p}_1, \mathbf{p}_2) = \mathbf{p}_1 \cdot \mathbf{p}_2 - \lambda\frac{u^2}{2} - \mu\frac{v^2}{2} - \frac{|u|^{p+1}}{p+1} - \frac{|v|^{q+1}}{q+1}.$$

The function $L(x, 0, 0, \mathbf{p}_1, \mathbf{p}_2) = \mathbf{p}_1 \cdot \mathbf{p}_2$ is not rank-one-convex at $(\mathbf{p}_1, \mathbf{p}_2) = (0, 0)$. To overcome this obstacle we have to restrict our attention to functions u, v both attaining non-negative values (or non-positive values) on $\overline{\Omega}$. As a consequence we have that $\nabla u\cdot\boldsymbol{\nu} \le 0$, $\nabla v\cdot\boldsymbol{\nu} \le 0$ (or both inequalities reversed) on $\partial\Omega$, and in particular $\nabla u\cdot\nabla v \ge 0$ on $\partial\Omega$. It is therefore sufficient to consider the rank-one convexity of the function $F(\mathbf{p}_1, \mathbf{p}_2) = \mathbf{p}_1 \cdot \mathbf{p}_2$ at $\mathbf{p}_1 = \mathbf{p}_2 = 0$ with respect to rank-one matrices $a \otimes \mathbf{q}$, $a = (a_1, a_2) \in \mathbb{R}^2$, $\mathbf{q} \in \mathbb{R}^n$ such that $a_1 a_2 \ge 0$. Under this extra condition we have indeed that $F(ta \otimes \mathbf{q})$ is convex as a function of $t \in \mathbb{R}$. Therefore Theorem 3.36 is applicable once we find a variational sub-symmetry.

As the generator of a transformation-group we take the vector-field

$$\mathbf{w} = -x \cdot \nabla_x + \alpha u\partial_u + \beta v\partial_v$$

with constants α, β. Its prolongation is given by

$$\mathbf{w}^{(1)} = -x \cdot \nabla_x + \alpha u\partial_u + (\alpha + 1)\nabla u \cdot \nabla_{\mathbf{p}_1} + \beta v\partial_v + (\beta + 1)\nabla v \cdot \nabla_{\mathbf{p}_2}.$$

For the verification of the infinitesimal sub-symmetry criterion of Proposition 3.27 we compute

$$\begin{aligned}\mathbf{w}^{(1)}L + L\operatorname{div}\boldsymbol{\xi} = (\alpha + \beta + 2 - n)\nabla u \cdot \nabla v + (\frac{n}{p+1} - \alpha)u^{p+1} + \lambda(\frac{n}{2} - \alpha)u^2 \\ +(\frac{n}{q+1} - \beta)v^{q+1} + \mu(\frac{n}{2} - \beta)v^2.\end{aligned}$$

The choice of $\alpha + \beta = n - 2$ shows that $\mathbf{w}$ generates indeed a variational sub-symmetry provided the two inequalities

$$\alpha \geq \frac{n}{p+1}, \quad \beta \geq \frac{n}{q+1}$$

hold. This amounts exactly to condition (6.3). Uniqueness of the critical point $(u, v) = 0$ follows from Theorem 3.36. □

Those points $(\tilde{p}, \tilde{q})$, which satisfy (6.2) with equality are said to lie on the "critical hyperbola". A pair (p, q) produces a subcritical/supercritical variational problem if (p, q) lies above/below the critical hyperbola.

Remark 6.2. 1) In the critical case, where (p, q) satisfy (6.3) with equality, and for dimensions $n \geq 4$ Hulshof, Mitidieri and vanderVorst [46] obtained existence of non-trivial solutions for certain non-negative values of λ, μ.
2) For non star-shaped, conformally contractible domains our technique fails. We explain briefly why this is the case: if we had used a vector-field $\boldsymbol{\xi}(x)$ instead of $-x$ to define $\mathbf{w} = \boldsymbol{\xi} \cdot \nabla_x + \alpha u \partial_u + \beta v \partial_v$ then

$$\mathbf{w}^{(1)} L = (\alpha + \beta) \nabla u \cdot \nabla v - 2 \nabla u D\boldsymbol{\xi}(x) \nabla v - \alpha u^{p+1} - \lambda \alpha u^2 - \beta v^{q+1} - \mu \beta v^2.$$

We would need $\boldsymbol{\xi}$ such that $\nabla u D\boldsymbol{\xi}(x) \nabla v = M(x) \nabla u \cdot \nabla v$. Since $\nabla u, \nabla v$ may be arbitrary this amounts to

$$\mathbf{a} D\boldsymbol{\xi}(x) \mathbf{b} = M(x) \mathbf{a} \cdot \mathbf{b} \text{ for all } \mathbf{a}, \mathbf{b} \in \mathbb{R}^n$$

which is only possible is $\boldsymbol{\xi}(x)$ is a constant multiple of x. Therefore Theorem 6.1 is proved for star-shaped domains only.

Remark 6.3. For points (p, q) strictly above the critical hyperbola (subcritical case) lower bounds for the L^∞-norm of nontrivial solutions of (6.1) hold in the spirit of Theorem 5.21.

Problem 6.4. For the system $-\Delta u = \lambda v^q + 1$, $-\Delta v = \mu u^p + 1$ with zero Dirichlet conditions and (p, q) strictly below the critical hyperbola (supercritical case) uniqueness of the positive solution for small positive values of λ, μ is an open problem. So far no result in the spirit of Theorem 5.15 is available.

Example 6.5. There are various ways to consider a system similar to (6.1) with nonlinear Neumann boundary conditions. E.g. suppose Ω is star-shaped w.r.t. $0 \in \partial\Omega$ and $\partial\Omega = \overline{\Gamma_D} \cup \overline{\Gamma_N}$, where Γ_N is part of a hyperplane through 0. Then the boundary value problem

$$\begin{aligned}
&\Delta u + \mu v + |v|^{q-1} v = 0 \text{ in } \Omega,\ \partial_\nu u - \tilde{\mu} v - |v|^{\tilde{q}-1} v = 0 \text{ on } \Gamma_N,\ u = 0 \ \text{ on } \Gamma_D, \\
&\Delta v + \lambda u + |u|^{p-1} u = 0 \text{ in } \Omega,\ \partial_\nu v - \tilde{\lambda} u - |u|^{\tilde{p}-1} u = 0 \text{ on } \Gamma_N,\ v = 0 \text{ on } \Gamma_D,
\end{aligned}$$

has no non-trivial solution provided $\lambda, \tilde{\lambda}, \mu, \tilde{\mu} \leq 0$ and

$$\frac{1}{p+1} + \frac{1}{q+1} \leq \frac{n-2}{n}, \quad \frac{1}{\tilde{p}+1} + \frac{1}{\tilde{q}+1} \leq \frac{n-2}{n-1}.$$

6.2 Boundary displacement problem in nonlinear elasticity

For the formulation of nonlinear elasticity we follow Ball [5]. Consider an elastic body, which occupies the domain $\Omega \subset \mathbb{R}^n$ in its reference position. A deformation $h : \overline{\Omega} \to \mathbb{R}^n$ describes the deviation of the body from its reference position. We assume that $h \in C^1(\overline{\Omega})$. For compressible materials one requires $\det \nabla h > 0$ and for incompressible materials $\det \nabla h = 1$.

Total energy and equilibria

In the so called *hyperelastic theory* one assumes that for a deformation h the elastic energy per unit volume of the body is given by $L(x, h, \nabla h) = L_1(x, \nabla h) + L_2(x, h)$. The function $L_1 : \mathbb{R}^n \times \mathbb{R}^{n\times n} \to \mathbb{R}$ is called the stored-energy density and $L_2 : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is called the body force potential. The total energy of the deformation h is given by

$$\mathcal{L}[h] = \int_\Omega L(x, h, \nabla h)\, dx, \quad h \in C^1(\overline{\Omega}).$$

Let us suppose that there are no body forces (like gravity), which means $L_2(x, h) \equiv 0$. If one assumes moreover that the body is homogeneous then the stored-energy function takes the form $L(\nabla h)$. For a compressible material a deformation h is called an *equilibrium deformation* if h is a critical point of $\mathcal{L}$. For sufficiently smooth h this means that the Euler-Lagrange-equations hold

$$\operatorname{div} \nabla_{\mathbf{p}^\alpha} L(\nabla h) = 0 \text{ in } \Omega \text{ for } \alpha = 1, \ldots, n.$$

For an incompressible material a deformation h is an *equilibrium deformation* if h is a critical point of $\mathcal{L}$ subject to the constraint $\det h = 1$. If h is sufficiently smooth this means that

$$\operatorname{div} \nabla_{\mathbf{p}^\alpha} L(\nabla h) - (\operatorname{co} \nabla h)^T \nabla q = 0 \text{ in } \Omega \text{ for } \alpha = 1, \ldots, n$$

where $q : \Omega \to \mathbb{R}$ is a Lagrange-multiplier with the meaning of a pressure and $\operatorname{co} \nabla h$ is the matrix of the co-factors of ∇h, i.e., $(\operatorname{co} \nabla h)_{i\alpha} = (-1)^{i+\alpha} \det(\nabla h)_{i\alpha}$ and $\det(\nabla h)_{i\alpha}$ is the determinant of ∇h after deletion of the i^{th}-row and the α^{th}-column. In this notation recall that

$$(\operatorname{co} \nabla h)^T \nabla q = \frac{d}{dt} \det(\nabla(h + tq))|_{t=0}.$$

Frame-indifference, isotropy, quasi-convexity

As a slight generalization let us consider "deformations" $h : \Omega \subset \mathbb{R}^n \to \mathbb{R}^k$. From the point of view of elasticity only the case $k = n$ is of interest. The generalized "boundary displacement problem" is then formulated with a $k \times n$-matrix $A = (a_{\alpha i})$, $\alpha = 1, \ldots, k$, $i = 1, \ldots, n$ and a k-vector b.

The following definitions are well established in the theory of elasticity and vector-valued variational problems, see Morrey [67] for quasi-convexity, Ball [5], Dacorogna [15] and Evans [29] for general reference.

We use the notation $e_i, i = 1, \dots, n$ for the i-th standard unit normal vector in $\mathbb{R}^n$. For each $x \in \Omega$ the deformation gradient $\mathbf{P} = \nabla h(x)$ is an $n \times k$-matrix. We write $\mathbf{P} = (\mathbf{p}^1, \dots, \mathbf{p}^k)$, where each $\mathbf{p}^\alpha = \nabla h^\alpha(x)$ is a column-vector in $\mathbb{R}^n$. When $\mathbf{P}$ is used in matrix-multiplication we have $\mathbf{P}_{i\alpha} = p_i^\alpha = \frac{\partial h^\alpha}{\partial x^i}$ with $\alpha = 1, \dots, k$ and $i = 1, \dots, n$. In this sense $\nabla(Ax + b) = A^T$. Recall the definition of $SO(n) = \{Q \in \mathbb{R}^{n \times n} : \det Q = 1, Q^T = Q^{-1}\}$.

Definition 6.6. *Consider a Lagrangian $L : \mathbb{R}^{n\times k} \to \mathbb{R}$.*

(i) L is called frame-indifferent if $L(Q\mathbf{P}) = L(\mathbf{P})$ for every matrix $\mathbf{P} \in \mathbb{R}^{n\times k}$ and all orthogonal matrices $Q \in SO(n)$.
(ii) L is called isotropic if $L(\mathbf{P}Q) = L(\mathbf{P})$ for every matrix $\mathbf{P} \in \mathbb{R}^{n\times k}$ and all orthogonal matrices $Q \in SO(k)$.

Interpretation. Frame-indifference means the following: if $Q \in SO(n)$ generates an isometric change of variables in the reference space $\mathbb{R}^n$ then a deformation h is replaced by $\tilde{h} = h \circ Q^T$. Also $\nabla\tilde{h}(x) = (Q\nabla h)(Q^T x)$ but $L(\nabla\tilde{h}(x)) = L((\nabla h)(Q^T x))$, i.e. the energy per unit-volume is unchanged.

Isotropy means the following: if $Q \in SO(k)$ generates an isometric change of variables in the state space $\mathbb{R}^k$ then h is replaced by $\tilde{h} = Q^T h$. Also $\nabla\tilde{h} = \nabla h \circ Q$ but $L(\tilde{\nabla} h) = L(\nabla h)$ is unchanged.

Lemma 6.7. *Let $L : \mathbb{R}^{n\times k} \to \mathbb{R}$ be frame-indifferent. If i_0, j_0 are two indices between 1 and n then $\partial_{p^\alpha_{i_0}} L(\mathbf{P}) p^\alpha_{j_0} = \partial_{p^\alpha_{j_0}} L(\mathbf{P}) p^\alpha_{i_0}$.*

Proof. Let $Q_t := \cos t\,\mathrm{Id} + \sin t(e_{i_0} \otimes e_{j_0} - e_{j_0} \otimes e_{i_0})$. By frame-indifference $L(Q_t\mathbf{P}) = L(\mathbf{P})$ and hence

$$\begin{aligned} 0 = \frac{d}{dt} L(Q_t\mathbf{P})|_{t=0} &= \partial_{p_i^\alpha} L(\mathbf{P})(e_{i_0} \otimes e_{j_0}\mathbf{P} - e_{j_0} \otimes e_{i_0}\mathbf{P})_{i\alpha} \\ &= \partial_{p_i^\alpha} L(\mathbf{P})(\delta_{i_0 i}\delta_{j_0 m} - \delta_{j_0 i}\delta_{i_0 m}) p_m^\alpha = \partial_{p_i^\alpha} L(\mathbf{P})(\delta_{i_0 i} p^\alpha_{j_0} - \delta_{j_0 i} p^\alpha_{i_0}), \end{aligned}$$

as claimed. □

Definition 6.8. *Consider a Lagrangian $L : \mathbb{R}^{n\times k} \to \mathbb{R}$. L is called (strictly) quasi-convex on the domain Ω if for every $\mathbf{P} \in \mathbb{R}^{n\times k}$ and every $\mathbb{R}^k$-valued function $\psi \in C_0^\infty(\overline{\Omega})$ with $\psi \not\equiv 0$ we have $\int_\Omega L(\mathbf{P})\,dx \le (<) \int_\Omega L(\mathbf{P} + \nabla\psi)\,dx$.*

Every quasi-convex function is rank-one convex, see Evans [29].

Example 6.9. Convex functions $L : \mathbb{R}^{n\times k} \to \mathbb{R}$ are quasi-convex, because by convexity

$$L(\mathbf{P}) + \nabla_{\mathbf{p}^\alpha} L(\mathbf{P}) \cdot \nabla\psi^\alpha \le L(\mathbf{P} + \nabla\psi) \tag{6.4}$$

for every ψ. Since $\mathbf{P}$ is a constant one has $\nabla_{\mathbf{p}^\alpha} L(\mathbf{P})\cdot\nabla\psi^\alpha = \mathrm{div}(\nabla_{\mathbf{p}^\alpha} L(\mathbf{P})\psi^\alpha)$. Integration of (6.4) and using $\psi \in C_0^\infty(\overline{\Omega})$ yields

$$\int_\Omega L(\mathbf{P})\,dx \le \int_\Omega L(\mathbf{P}+\nabla\psi)\,dx.$$

Other quasi-convex functions are $\det\nabla h$ if $h:\mathbb{R}^n \to \mathbb{R}^n$, or more generally the so-called polyconvex functions, i.e. convex functions of the determinants of the quadratic sub-matrices of ∇h, see Ball [5], Evans [29].

6.2.1 Uniqueness for the boundary displacement problem (compressible case)

We consider an compressible, homogeneous material with no body forces. In the boundary displacement problem one assumes that $\partial\Omega$ is displaced by an affine map $Ax+b$, with a $k\times n$-matrix A and a k-vector b. Therefore only those deformations h with $h = Ax+b$ on $\partial\Omega$ are considered. Equilibrium deformations must satisfy

$$\mathrm{div}(\nabla_{\mathbf{p}^\alpha} L(\nabla h)) = 0 \text{ in } \Omega \text{ for } \alpha = 1,\dots n, \qquad h = Ax+b \text{ on } \partial\Omega.$$

Notice that $h \equiv Ax+b$ is always an equilibrium solution. The goal of this section is to determine conditions on L and Ω such that $Ax+b$ is the only equilibrium solution of the boundary displacement problem. Such uniqueness conditions were obtained by Knops and Stuart [55].

Theorem 6.10. *Suppose $\Omega \subset \mathbb{R}^n$ is a bounded Lipschitz domain and let $L: \mathbb{R}^{n\times k} \to \mathbb{R}$ be a strictly quasi-convex Lagrangian. Then in the class of $C^2(\Omega)\cap C^1(\overline{\Omega})$-solutions the unique equilibrium solution of the boundary-displacement problem (compressible case) is $h = Ax+b$ if one of the following holds*

(i) Ω is star-shaped,
(ii) L is frame-indifferent and Ω is conformally contractible with associated vector-field $\boldsymbol{\xi} = -\mathbf{X}+\boldsymbol{\zeta}$ such that $\boldsymbol{\zeta} \in \mathrm{span}[x^i e_j - x^j e_i, i,j = 1,\dots,n]$ lies in the null-space of the matrix A.

Proof. In order to apply the uniqueness theory of Chapter 3, we transform to zero-boundary conditions, i.e. $u = h - Ax - b$. If h is a critical point of $\mathcal{L}[h]$ with $h = Ax+b$ on $\partial\Omega$ then equivalently u with $u = 0$ on $\partial\Omega$ is a critical point of $\tilde{\mathcal{L}}[u] = \int_\Omega \tilde{L}(\nabla u)\,dx$ with $\tilde{L}(\mathbf{P}) = L(\mathbf{P}+A^T)$. Since L is quasi-convex, $\tilde{L}$ is also quasi-convex, and hence rank-one convex.

Part *(i)*: Without loss of generality we assume $\tilde{L}(0) = 0$. By the strict quasi-convexity we find $0 = \tilde{\mathcal{L}}[0] = \int_\Omega \tilde{L}(0)\,dx < \int_\Omega \tilde{L}(\nabla u)\,dx = \tilde{\mathcal{L}}[u]$ for all $u \in C_0^1(\overline{\Omega})$ with $u \not\equiv 0$. Next we will show that by a suitable choice of coefficients

$c^\alpha(x)$ and $\boldsymbol{\xi}$ we can make $\mathbf{w} = \boldsymbol{\xi} \cdot \nabla_x + c^\alpha(x) u^\alpha \partial_{u^\alpha}$ a strict variational sub-symmetry w.r.t. 0 for the functional $\tilde{\mathcal{L}}$. Since Ω is a star-shaped domain we choose $\boldsymbol{\xi}(x) = -x$. For the prolongation $\mathbf{w}^{(1)}$ we find

$$\mathbf{w}^{(1)} = -x \cdot \nabla_x + c^\alpha(x) u^\alpha \partial_{u^\alpha} + (c^\alpha(x)_{,i} u^\alpha + c^\alpha(x) u^\alpha_{,i} + \delta^j_i u^\alpha_{,j}) \partial_{p^\alpha_i}.$$

For the infinitesimal sub-symmetry criterion we calculate

$$\int_\Omega \mathbf{w}^{(1)} \tilde{L} + L \operatorname{div} \boldsymbol{\xi}\, dx = \int_\Omega \Big(c^\alpha(x)_{,i} u^\alpha + (c^\alpha(x) + 1) u^\alpha_{,i} \Big) \partial_{p^\alpha_i} \tilde{L}\, dx - n \tilde{\mathcal{L}}[u].$$

Choose $c^\alpha(x) = -1$. Together with our previous observation $\tilde{\mathcal{L}}[u] > 0$ for $u \not\equiv 0$ this shows that $\mathbf{w}$ generates a strict variational sub-symmetry w.r.t. 0. Clearly $\mathbf{w}$ generates a domain-contracting admissible transformation group and hence we obtain uniqueness of the critical point $u \equiv 0$, i.e. $h \equiv Ax + b$ by Theorem 3.36.

Part *(ii)*: We choose $\boldsymbol{\xi} = -x + \boldsymbol{\zeta}$ as the associated conformal vector-field to the conformally contractible domain Ω. We may assume that $\boldsymbol{\zeta}$ has the special form $\boldsymbol{\zeta} = x^{i_0} e_{j_0} - x^{j_0} e_{i_0}$. The general case, where ζ is a linear combination of such vector-fields follows in an obvious manner. Let $\mathbf{w} = \boldsymbol{\xi} \cdot \nabla_x - u^\alpha \partial_{u^\alpha}$. To show that $\mathbf{w}$ generates a strict variational sub-symmetry w.r.t. 0 we can adopt the proof of part *(i)* if we can show that $-\xi^j_{,i} p^\alpha_j \partial_{p^\alpha_i} \tilde{L}(\mathbf{P}) = p^\alpha_i \partial_{p^\alpha_i} \tilde{L}(\mathbf{P})$ for all $x \in \Omega$ and all $\mathbf{P} \in \mathbb{R}^{n \times k}$. Since $\boldsymbol{\xi} = -x + \boldsymbol{\zeta}$ this means

$$\zeta^j_{,i} p^\alpha_j \partial_{p^\alpha_i} \tilde{L}(\mathbf{P}) = 0 \tag{6.5}$$

and since $\boldsymbol{\zeta}$ has the special form $x^{i_0} e_{j_0} - x^{j_0} e_{i_0}$ the condition to be verified simplifies to

$$p^\alpha_{j_0} \partial_{p^\alpha_{i_0}} \tilde{L}(\mathbf{P}) = p^\alpha_{i_0} \partial_{p^\alpha_{j_0}} \tilde{L}(\mathbf{P}). \tag{6.6}$$

For the proof of (6.6) we first use Lemma 6.7 for the frame-indifferent function L and rewrite the result for $\tilde{L}(\mathbf{P}) = L(\mathbf{P} + A^T)$ as follows:

$$\partial_{p^\alpha_{i_0}} \tilde{L}(\mathbf{P})(p^\alpha_{j_0} + a_{\alpha j_0}) = \partial_{p^\alpha_{j_0}} \tilde{L}(\mathbf{P})(p^\alpha_{i_0} + a_{\alpha i_0}). \tag{6.7}$$

Since $\boldsymbol{\zeta} = x^{i_0} e_{j_0} - x^{j_0} e_{i_0}$ lies in the null-space of A for all x in Ω it follows that both e_{j_0} and e_{i_0} are in the null-space of A. Therefore both $a_{\alpha j_0}$ and $a_{\alpha i_0}$ vanish for all $\alpha = 1, \dots, k$. Together with (6.7) this proves (6.6) and the claim of Part *(ii)* is verified. □

Remark 6.11. Part *(i)* of the theorem for $n = k$ was proved by Knops and Stuart in [55]. Part *(ii)* is new to the best of our knowledge. An example, where Part *(ii)* applies for the physically interesting case $n = 3$ is given by the domain of Example 1 in Section 5.1: the domain is conformally contractible with associated conformal vector-field $(-x + y, -y - x, -z)$. If the matrix A is a 3×3-matrix such that $Ax = \alpha x_3 e_3$, i.e. A is a dilation in the z-direction, then $\boldsymbol{\zeta} = (y, -x, 0)$ lies in the null-space of A.

6.2.2 Uniqueness for the boundary displacement problem (incompressible case)

Here we consider an incompressible, homogeneous material with no body forces and deformations $h : \Omega \subset \mathbb{R}^n \to \mathbb{R}^n$. Now the boundary displacement problem consists of finding the solution (h, q) of

$$\operatorname{div} \nabla_{\mathbf{p}^\alpha} L(\nabla h) - (\operatorname{co} \nabla h)^T \nabla q = 0 \text{ in } \Omega \text{ for } \alpha = 1, \ldots, n, \quad h = Ax + b \text{ on } \partial\Omega,$$

where $\det A, \det h = 1$ and $q : \Omega \to \mathbb{R}$ is the pressure. Again $h \equiv Ax + b$ and $q = \text{const.}$ is a solution. Knops and Stuart [55] obtained the following result:

Theorem 6.12 (Knops, Stuart). *Suppose $\Omega \subset \mathbb{R}^n$ is a bounded Lipschitz domain and let $L : \mathbb{R}^{n\times k} \to \mathbb{R}$ be a strictly quasi-convex Lagrangian. Then in the class of $C^2(\Omega) \cap C^1(\overline{\Omega})$-solutions the unique equilibrium solution of the boundary-displacement problem (incompressible case) is $h = Ax + b$ if Ω is star-shaped.*

Proof. The proof is almost identical with the one of Theorem 6.10. We use the transformation $u = h - Ax - b$. Then we look for critical points of $\tilde{\mathcal{L}}[u] = \int_\Omega \tilde{L}(\nabla u)\, dx$ with $\tilde{L}(\mathbf{P}) = L(\mathbf{P} + A^T)$ subject to the pointwise constraint $\det(\nabla u + A^T) = 1$. We choose $\mathbf{w} = -x \cdot \nabla_x + c^\alpha u^\alpha \partial_{u^\alpha}$. With $c^\alpha = -1$ the prolongation $\mathbf{w}^{(1)}$ becomes

$$\mathbf{w}^{(1)} = -x \cdot \nabla_x + c^\alpha(x) u^\alpha \partial_{u^\alpha}$$

since the factor $c^\alpha = -1$ was chosen such that the coefficient of $\partial_{p_i^\alpha}$ vanishes. Since the constraint $\det(\nabla u + A^T) = 1$ only depends on ∇u the transformation group is an exact symmetry and leaves it invariant. As before we have seen that $\mathbf{w}$ acts as a strict variational sub-symmetry w.r.t. 0 for the functional $\tilde{\mathcal{L}}$. Hence Theorem 3.42 (i.e. a suitably extended version to cover constraints $N(x, u(x), \nabla u(x)) = 0$) gives the uniqueness result. □

Remark 6.13. An extension of Theorem 6.12 in the spirit of Theorem 6.10(ii) is not possible, since $\det A = 1$ implies that the null-space of A is trivial.

6.3 A uniqueness result in dimension two

So far the aspects of conformally contractible domains in dimension $n = 2$ have been left aside. Recall from Definition 4.20 and Lemma 4.33 that a bounded domain in $\mathbb{R}^2$ is called conformally contractible if there exists a vector-field $\boldsymbol{\xi} = a(x, y)\partial_x + b(x, y)\partial_y$ such that $a(x, y) + ib(x, y)$ is holomorphic in $z = x + iy$ and $\boldsymbol{\xi} \cdot \boldsymbol{\nu} \le 0$ on $\partial\Omega$ with strict inequality on a subset of positive measure. The large number of such domains in $\mathbb{R}^2$ is linked to the complex structure of the space.

A uniqueness theorem on two-dimensional domains is given next for functionals $\mathcal{L}[u] = \int_\Omega L(u, \nabla u)\, dx$ with vector-valued functions $u : \Omega \subset \mathbb{R}^2 \to \mathbb{R}^k$. An application will be given in the next section.

Theorem 6.14. *Let $\Omega \subset \mathbb{R}^2$ be a bounded, piecewise smooth, conformally contractible domain. Consider the functional*

$$\mathcal{L}[u] = \int_\Omega L(u, \nabla u)\, dx, \quad u \in C_0^{0,1}(\overline{\Omega})$$

with a Lagrangian $L : \mathbb{R}^k \times (\mathbb{R}^2)^k \to \mathbb{R}$ such that $L(0, \mathbf{P})$ is rank-one convex in $\mathbf{P}$ at $\mathbf{P} = 0$ (cf. Definition (3.29)) and L has the unique continuation property w.r.t. 0 (cf. Definition (3.33)). If $L(u, \mathbf{P})$ is frame-indifferent w.r.t. $\mathbf{P}$ and if $L(u, t\mathbf{P}) = t^2 L(u, \mathbf{P})$ for all $u \in \mathbb{R}^k$ and all $\mathbf{P} \in (\mathbb{R}^2)^k$ then $u \equiv 0$ is the unique critical point of $\mathcal{L}$ in the class $C^2(\Omega) \cap C_0^1(\overline{\Omega})$.

Proof. By homogeneity we have $L(0, 0) = 0$. Let $\mathbf{w} = \boldsymbol{\xi} \cdot \nabla_{(x,y)}$ where $\boldsymbol{\xi} = (a(x, y), b(x, y))$ is the conformal vector-field associated to Ω. We will show that $\mathbf{w}$, which is independent of u, generates a variational sub-symmetry for the functional $\mathcal{L}$. For the prolongation we find $\mathbf{w}^{(1)} = \boldsymbol{\xi} \cdot \nabla_{(x,y)} - \xi^j_{,i} u^\alpha_{,j} \partial_{p^\alpha_i}$. For the sub-symmetry criterion we calculate

$$\begin{aligned}
&\mathbf{w}^{(1)} L + L \operatorname{div} \boldsymbol{\xi} \\
&= -\xi^j_{,i} u^\alpha_{,j} \partial_{p^\alpha_i} L + L \xi^i_{,i} \\
&= -\xi^1_{,x} u^\alpha_x \partial_{p^\alpha_1} L - \xi^1_{,y} u^\alpha_{,x} \partial_{p^\alpha_2} L - \xi^2_{,x} u^\alpha_{,y} \partial_{p^\alpha_1} L - \xi^2_{,y} u^\alpha_{,y} \partial_{p^\alpha_2} L + L(\xi^1_{,x} + \xi^2_{,y}).
\end{aligned}$$

Since $\boldsymbol{\xi}$ is conformal we have $\xi^1_{,x} = \xi^2_{,y}$, $\xi^1_{,y} = -\xi^2_{,x}$ and hence

$$\begin{aligned}
&\mathbf{w}^{(1)} L + L \operatorname{div} \boldsymbol{\xi} \\
&\qquad = -\xi^1_{,x}(u^\alpha_{,x} \partial_{p^\alpha_1} L + u^\alpha_{,y} \partial_{p^\alpha_2} L - 2L) - \xi^1_{,y}(u^\alpha_{,x} \partial_{p^\alpha_2} L - u^\alpha_{,y} \partial_{p^\alpha_1} L). \quad (6.8)
\end{aligned}$$

By the frame-indifference of L and Lemma 6.7 the second bracket in (6.8) vanishes. Differentiation of the homogeneity assumption $L(u, t\mathbf{P}) = t^2 L(u, \mathbf{P})$ with respect to t at $t = 1$ gives $p^\alpha_i \partial_{p^\alpha_i} L(u, \mathbf{P}) = 2L(u, \mathbf{P})$, i.e. $p^\alpha_1 \partial_{p^\alpha_1} L(u, \mathbf{P}) + p^\alpha_2 \partial_{p^\alpha_2} L(u, \mathbf{P}) - 2L(u, \mathbf{P}) = 0$. This shows that the first bracket in (6.8) also vanishes. Hence $\mathbf{w}$ generates an exact variational symmetry. All the hypotheses of Theorem 3.36, Part *(ii)* are now verified, and hence uniqueness of the critical point $u \equiv 0$ follows. □

Example 6.15. Let $\Omega \subset \mathbb{R}^2$ be bounded, piecewise smooth, conformally contractible and let (N, h) be a k-dimensional Riemannian manifold. Then the harmonic-mapping problem $u : \Omega \to N$ with $u = \text{const.} = u_0 \in N$ on $\partial\Omega$ has the unique solution $u \equiv u_0$. The Lagrangian $L(u, \mathbf{P}) = h_{\alpha\beta}(u) \mathbf{p}^\alpha \cdot \mathbf{p}^\beta$ has all the properties of Theorem 6.14.

Conformally contractible versus simply connected domains

Lemma 6.16. *Let $\Omega \subset \mathbb{R}^2$ be a bounded, simply-connected $C^{1,\alpha}$-domain. Then the following holds:*

(i) Ω is conformally contractible. Moreover, there exists an associated conformal vector-field $\boldsymbol{\xi}$ with $\boldsymbol{\xi}\cdot\boldsymbol{\nu}<0$.
(ii) Whenever $\boldsymbol{\xi}$ is an associated vector-field with $\boldsymbol{\xi}\cdot\boldsymbol{\nu}<0$ on $\partial\Omega$ then $\boldsymbol{\xi}$ has precisely one zero in $\overline{\Omega}$.

Proof. (i) Let $\phi:\Omega\to\mathbb{D}$ be an orientation preserving Riemann map onto the open unit disk $\mathbb{D}$. Then ϕ extends as a C^1-function with non-degenerate Jacobian onto $\overline{\Omega}$, cf. Pommerenke [76], Theorem 3.5. By setting $\boldsymbol{\xi}(z)=-\phi(z)\phi'(z)$ the vector-field $\boldsymbol{\xi}$ is continuous on $\overline{\Omega}$ and analytic in Ω. Let us denote by $\boldsymbol{\nu}(z)$ the exterior unit-normal to $z\in\partial\Omega$ and by $\tilde{\boldsymbol{\nu}}(\phi(z))$ the exterior unit-normal to $\mathbb{D}$ at $\phi(z)$, i.e. $\tilde{\boldsymbol{\nu}}(\phi(z))=\phi(z)$. By conformality of the map ϕ we have $\boldsymbol{\nu}(z)=\frac{\phi'(z)}{|\phi'(z)|}\tilde{\boldsymbol{\nu}}(\phi(z))=\frac{\phi'(z)}{|\phi'(z)|}\phi(z)$. Therefore it follows that $\boldsymbol{\nu}(z)\cdot\boldsymbol{\xi}(z)=\operatorname{Re}\boldsymbol{\nu}(z)\overline{\boldsymbol{\xi}}(z)=-|\phi'(z)|<0$.

(ii) Suppose $\boldsymbol{\xi}$ is a conformal vector-field on Ω. Let $\phi:\Omega\to\mathbb{D}$ be the Riemann map. Then we construct by $\tilde{\boldsymbol{\xi}}(z)=\boldsymbol{\xi}(\phi^{-1}(z))/\phi'(\phi^{-1}(z))$ a conformal vector-field on $\mathbb{D}$ with $\tilde{\boldsymbol{\xi}}(z)\cdot z=|\phi'(\phi^{-1}(z))|^{-1}\boldsymbol{\xi}(\phi^{-1}(z))\cdot\boldsymbol{\nu}(\phi^{-1}(z))<0$ on $\partial\mathbb{D}$. Clearly the number of zeroes of $\boldsymbol{\xi}$ on Ω is the same as the number of zeroes of $\tilde{\boldsymbol{\xi}}$ on $\mathbb{D}$. By choosing a suitably large number $K>0$ we can achieve $|\tilde{\boldsymbol{\xi}}(z)|^2<-2K\tilde{\boldsymbol{\xi}}(z)\cdot z$ on $\partial\mathbb{D}$ and hence $|\tilde{\boldsymbol{\xi}}(z)+Kz|<|Kz|$ on $\partial\mathbb{D}$. By Rouché's Theorem the two functions $\tilde{\boldsymbol{\xi}}(z)$ and Kz have the same number of zeroes in $\mathbb{D}$. □

If $\phi:\Omega\to\mathbb{R}^2$ is a conformal map and Ω conformally contractible then $\phi(\Omega)$ is also conformally contractible. This raises the question whether the converse of Lemma 6.16 *(i)* hold:

Problem 6.17. Are conformally contractible domains simply-connected?

We do not know any counter-example. One might think that a path in a conformally contractible domain should be deformed to a point via the differential equation $(\dot{x},\dot{y})=\boldsymbol{\xi}(x,y)$. Clearly the deformed path exists for all $t\geq 0$ and stays inside Ω. However, difficulties arise since the process might stop due to equilibria (=zeroes of $\boldsymbol{\xi}$) or limit cycles.

Example 6.18. Suppose a Lagrangian $L:\mathbb{R}^k\times(\mathbb{R}^2)^k\to\mathbb{R}$ is frame-indifferent in the ∇u-variable and degree-2-homogeneous $L(u,t\mathbf{P})=t^2L(u,\mathbf{P})$. Then on two-dimensional domains Ω the functional $\mathcal{L}[u]=\int_\Omega L(u,\nabla u)\,d(x,y)$ is invariant under conformal transformations $\phi:\tilde{\Omega}\to\Omega$, i.e. for $\tilde{u}(\tilde{x},\tilde{y})=u(\phi(\tilde{x},\tilde{y}))$ we have

$$\int_\Omega L(u,\nabla u)\,d(x,y)=\int_{\tilde{\Omega}} L(\tilde{u},\nabla\tilde{u})\,d(\tilde{x},\tilde{y}).$$

Hence, if the open problem of simple-connectedness were solved, then Theorem 6.14 could be proved by transforming Ω conformally to the unit-disc, where the standard conformal vector-field $\boldsymbol{\xi} = -(x, y)$ is available.

6.4 H. Wente's uniqueness result for closed surfaces of prescribed mean curvature

A two-dimensional parametric surface of mean-curvature H is represented in isothermal coordinates as a map $u : \Omega \subset \mathbb{R}^2 \to \mathbb{R}^3$ such that

$$\Delta u = 2H u_x \wedge u_y \text{ in } \Omega.$$

Typical cases are $H \equiv 0$ (minimal surfaces) or $H \equiv$ const. (soap bubbles). The case where $H = H(u)$ is called the prescribed mean-curvature problem. The Dirichlet boundary-value problem for surfaces of prescribed mean curvature is given by

$$\Delta u = 2H(u) u_x \wedge u_y \text{ in } \Omega, \qquad u = u_0 \text{ on } \partial\Omega, \tag{6.9}$$

where u_0 is a given smooth function $\partial\Omega \to \mathbb{R}^3$, which may be understood as a smooth "curve" in $\mathbb{R}^3$ bounding the surface. Solutions are supposed to be in $C^2(\Omega) \cap C(\overline{\Omega})$. To solve (6.9) Hildebrandt [44] introduced the functional

$$\mathcal{L}[u] = \int_\Omega \frac{1}{2}|\nabla u|^2 + \frac{2}{3} M(u) u_x \wedge u_y \, dx$$

where $M : \mathbb{R}^3 \to \mathbb{R}^3$ is such that $\operatorname{div} M(z) = 3H(z)$. A straight-forward calculation shows that critical points of $\mathcal{L}$ weakly solve (6.9). The following result was observed by H. Wente [92] in the case $H =$ const.:

Theorem 6.19 (Wente). *If $\Omega \subset \mathbb{R}^2$ is smoothly bounded and simply connected then any critical point $u \in H_0^{1,2}(\Omega)$ of $\mathcal{L}$ vanishes identically.*

By a regularity result Wente immediately concluded that u is $C^2(\Omega) \cap C(\overline{\Omega})$. He then used conformal maps and a unique continuation principle to prove his theorem.

Interpretation. Critical points of $\mathcal{L}$ with constant zero boundary data $u_0 = 0$ represent closed surfaces, since the bounding "curve" has now shrunk to a single point in space. The uniqueness result then means that is it impossible to represent a closed surface of prescribed mean-curvature parametrically over a bounded, simply connected domain Ω. Consider for example the closed constant-mean-curvature surface $\hat{\mathbb{S}}^2 \subset \mathbb{R}^3$ and its stereographic projection $\Pi : \hat{\mathbb{S}}^2$ to $\mathbb{R}^2$. Then $\Pi^{-1} : \mathbb{R}^2 \to \hat{\mathbb{S}}^2$ represents $\hat{\mathbb{S}}^2$ in parametric form, but one needs all of $\mathbb{R}^2$ to achieve this.

It was already mentioned in Struwe [87] that Wente's uniqueness result may be understood as a companion of Pohožaev's uniqueness result. Indeed, based on the method of transformation groups we can prove the following slightly different version of Wente's result.

Theorem 6.20. *Let $\Omega \subset \mathbb{R}^2$ be a bounded, piecewise smooth, conformally contractible domain. Then any critical point $u \in C^2(\Omega) \cap C_0^1(\overline{\Omega})$ of $\mathcal{L}$ vanishes identically.*

Proof. The Lagrangian $L(u, \mathbf{P}) = \frac{1}{2}|\mathbf{P}|^2 + \frac{2}{3}M(u)\mathbf{p}^1 \wedge \mathbf{p}^2$ is both frame-invariant and degree-2-homogeneous with respect to $\mathbf{P}$. We may assume $M(0) = 0$. Therefore, the restricted Lagrangian $L(0, \mathbf{P}) = \frac{1}{2}|\mathbf{P}|^2$ is convex in $\mathbf{P}$, see also Example 3.32(ii). Moreover, if $u = 0$ and $\nabla u = 0$ on a smooth piece of $\partial\Omega$ then the unique continuation principle of Hartman, Wintner [43], Corollary 1, applies and shows $u \equiv 0$. Hence we can use the two-dimensional uniqueness Theorem 6.14 from the previous section to see that $u \equiv 0$ is the only critical point of $\mathcal{L}$. $\square$

A

Fréchet-differentiability

In this section we discuss the Fréchet-differentiability of the functional $\mathcal{L}[u] = \int_\Omega L(x,u,\nabla u)\,dx$ on the space $C^{0,1}(\overline{\Omega})$ of Lipschitz-functions $u : \overline{\Omega} \to \mathbb{R}^k$.

Proposition A.1. *Suppose $L(x,u,\mathbf{p})$ is measurable in $x \in \overline{\Omega}$ and has partial derivatives $\partial_{u^\alpha} L(x,u,\mathbf{p})$ and $\nabla_{\mathbf{p}^\alpha} L(x,u,\mathbf{p})$, which are continuous in u and $\mathbf{p}$ for fixed x. Suppose moreover that $\partial_{u^\alpha} L(x,u,\mathbf{p})$ and $\nabla_{\mathbf{p}^\alpha} L(x,u,\mathbf{p})$ are bounded if $(x,u,\mathbf{p})$ is in bounded subsets of $\mathbb{R}^k \times \bigcup_{y\in M}(T_yM)^k$. Then $\mathcal{L}[u] = \int_\Omega L(x,u,\nabla u)\,dx$ is Fréchet-differentiable in $C^{0,1}(\overline{\Omega})$ with derivative*

$$\mathcal{L}'[u]h = \int_\Omega \partial_{u^\alpha} L(x,u,\nabla u)h^\alpha + \nabla_{\mathbf{p}^\alpha} L(x,u,\nabla u)\cdot \nabla h^\alpha\,dx$$

for every function $h \in C^{0,1}(\overline{\Omega})$.

The hypotheses of Proposition A.1 are fulfilled if $L : \mathbb{R}^k \times \bigcup_{y\in M}(T_yM)^k$ is continuously differentiable with respect to x, u and $\mathbf{p}$.

Proof. For $u \in C^{0,1}(\overline{\Omega})$ we write $L(x,u,\nabla u)$ for $L(x,u(x),\nabla u(x))$. Define $Ah := \int_\Omega \partial_{u^\alpha} L(x,u,\nabla u)h^\alpha + \nabla_{\mathbf{p}^\alpha} L(x,u,\nabla u)\cdot\nabla h^\alpha\,dx$. Since the derivatives $\partial_{u^\alpha} L(x,u,\nabla u)$ and $\nabla_{\mathbf{p}^\alpha} L(x,u,\nabla u)$ are L^∞-functions one finds that the functional A is continuous on $C^{0,1}(\overline{\Omega})$. $h \in C^{0,1}(\overline{\Omega})$ then for almost all $x \in \Omega$ it holds that

$$\begin{aligned} L(x,u{+}h,\nabla u{+}\nabla h) - L(x,u,\nabla u) - h^\alpha\partial_{u^\alpha} L(x,u,\nabla u) - \nabla h^\alpha\cdot\nabla_{\mathbf{p}^\alpha} L(x,u,\nabla u) \\ = \int_0^1 \Big(\partial_{u^\alpha} L(x,u+th,\nabla u+t\nabla h) - \partial_{u^\alpha} L(x,u,\nabla u)\Big)h^\alpha\,dt \\ + \int_0^1 \Big(\nabla_{\mathbf{p}^\alpha} L(x,u+th,\nabla u+t\nabla h) - \nabla_{\mathbf{p}^\alpha} L(x,u,\nabla u)\Big)\cdot\nabla h^\alpha\,dt. \end{aligned}$$

Integration over the domain Ω yields

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$$\begin{aligned}&\Big|\mathcal{L}[u+h]-\mathcal{L}[u]-Ah\Big|\\&\quad\le\|h\|_\infty\int_\Omega\int_0^1\sum_{\alpha=1}^k\Big|\partial_{u^\alpha}L(x,u+th,\nabla u+t\nabla h)-\partial_{u^\alpha}L(x,u,\nabla u)\Big|\,dt\,dx\\&+\|\nabla h\|_\infty\int_\Omega\int_0^1\sum_{\alpha=1}^k\Big|\nabla_{\mathbf{p}^\alpha}L(x,u+th,\nabla u+t\nabla h)-\nabla_{\mathbf{p}^\alpha}L(x,u,\nabla u)\Big|\,dt\,dx.\end{aligned}$$

By the dominated convergence theorem the integrals on the right hand side converge to 0 as $\|h\|_{C^{0,1}}\to 0$. Hence

$$\Big|\mathcal{L}[u+h]-\mathcal{L}[u]-Ah\Big|/\|h\|_{C^{0,1}}\to 0\text{ as }\|h\|_{C^{0,1}}\to 0.$$

This shows the Fréchet-differentiability of $\mathcal{L}$. $\square$

B

Lipschitz-properties of g_ϵ and Ω_ϵ

We recall the following versions of the inverse and implicit function theorem. Let X, Y, Z be Banach-spaces and let $B_\rho(x_0)$ be the open norm-ball of radius ρ around x_0.

Inverse function theorem. *Let $f : B_\rho(x_0) \subset X \to Y$ satisfy*

$$\|f(x_1) - f(x_2) - L(x_1 - x_2)\| \le K\|x_1 - x_2\| \quad \forall x \in B_\rho(x_0)$$

for a bounded linear homeomorphism $L : X \to Y$ with $\|L^{-1}\|K < 1$. Then there exists $\rho_1 \in (0, \rho]$ such that $f : B_{\rho_1}(x_0) \to f(B_{\rho_1}(x_0))$ has a Lipschitz inverse with

$$\begin{aligned} \|f^{-1}(y_1) - f^{-1}(y_2)\| &\le \frac{\|L^{-1}\|}{1 - \|L^{-1}\|K}\|y_1 - y_2\| \\ \|f^{-1}(y_1) - f^{-1}(y_2) - L^{-1}(y_1 - y_2)\| &\le \frac{\|L^{-1}\|^2 K}{1 - \|L^{-1}\|K}\|y_1 - y_2\| \end{aligned}$$

for all $y_1, y_2 \in f(B_{\rho_1})$.

Implicit function theorem. *Let $f : B_r(x_0) \times B_s(y_0) \subset X \times Y \to Z$ with $f(x_0, y_0)$ be a Lipschitz function and suppose that*

$$\|f(x, y_1) - f(x, y_2) - L(y_1 - y_2)\| \le K\|y_1 - y_2\| \quad \forall x \in B_r(x_0), \forall y_1, y_2 \in B_s(y_0)$$

for a bounded linear homeomorphism $L : Y \to Z$ with $\|L^{-1}\|K < 1$. Then there exist $r_1 \in (0, r]$ and $s_1 \in (0, s]$ and a Lipschitz function $g : B_{r_1}(x_0) \to Y$ with $g(x_0) = y_0$ such that the unique solution of $f(x, y) = 0$ in $B_{r_1}(x_0) \times B_{s_1}(y_0)$ is given by $y = g(x)$.

The proof of both theorems relies on the contraction mapping principle applied to $x - L^{-1}(f(x) - y) = x$ for the inverse function theorem and $y - L^{-1}f(x, y) = y$ for the implicit function theorem. Details can be found in Deimling [18], Chapter 4 and Hildebrandt, Graves [45].

We will use the inverse and implicit function theorems to prove Proposition 3.3 used in Chapter 3. If u is a Lipschitz-function we denote by Lip u the best Lipschitz constant.

Proposition 3.3 *Let Ω be a Lipschitz domain and let $u \in C^{0,1}(\overline{\Omega})$ or $u \in C^1(\overline{\Omega})$, respectively. Then there exists $\epsilon_0 = \epsilon_0(u)$ such that for all $\epsilon \in (-\epsilon_0, \epsilon_0)$ we have*

(i) $g_\epsilon u$ belongs to $C^{0,1}(g_\epsilon \Omega)$ or $C^1(g_\epsilon \Omega)$, respectively.
(ii)$g_\epsilon \Omega$ is a Lipschitz domain.

Proof. Part *(i)*: For simplicity we give the proof only for $k = 1$. Since $\boldsymbol{\xi}$, $\boldsymbol{\phi}$ are C^1-functions we know that $\chi_\epsilon(x,u), \Psi_\epsilon(x,u)$ are C^1-functions w.r.t. ϵ and w.r.t. the initial conditions $(x,u) \in \overline{\Omega} \times \mathbb{R}$. Fix a local coordinate system at $x_0 \in \overline{\Omega}$ and let $B(x_0)$ be a ball in $\mathbb{R}^n$ around x_0. Moreover, let $j \in \{1, \dots, n\}$ be fixed. For $x_1, x_2 \in B(x_0)$ and $u_1, u_2 \in \mathbb{R}$ we find by the mean-value theorem that there exists vectors $\bar{x}_i$ for $i = 1, \dots, n$ on the straight-line between x_1 and x_2 and a value $\bar{u}$ between u_1, u_2 such that

$$
\begin{aligned}
&|\chi^j_\epsilon(x_1,u_1) - \chi^j_\epsilon(x_2,u_2) - (x^j_1 - x^j_2)| \\
&\le |(\chi^j_{\epsilon,i}(\bar{x}_i,u_1) - \delta^j_i)(x^i_1 - x^i_2)| + |\chi^j_{\epsilon,u}(x_2,\bar{u})(u_1 - u_2)| \\
&\le |(\epsilon\xi^j_{,i}(\bar{x}_i,u_1) + O(\epsilon^2))(x^i_1 - x^i_2)| + |(\epsilon\xi^j_{,u}(x_2,\bar{u}) + O(\epsilon^2))(u_1 - u_2)|,
\end{aligned}
$$

where $O(\epsilon^2)$ is uniform w.r.t. $x_1, x_2 \in B(x_0)$ and u_1, u_2 in compact intervals K. Let $|\cdot|_\infty$ be the maximum-norm in $\mathbb{R}^n$. Let $u : \overline{\Omega} \to \mathbb{R}$ be in $C^{0,1}(\overline{\Omega})$. For $x_1, x_2 \in B(x_0)$ we obtain

$$
\begin{aligned}
&|\chi_\epsilon(x_1,u(x_1)) - \chi_\epsilon(x_2,u(x_2)) - (x_1 - x_2)|_\infty \\
&\le \epsilon\Big(\max_{i,j=1,\dots,n} \|\xi^j_{,i}\|_\infty + \text{Lip}\ u \max_{j=1,\dots,n} \|\xi^j_{,u}\|_\infty\Big)|x_1 - x_2|_\infty + O(\epsilon^2)|x_1 - x_2|_\infty,
\end{aligned}
$$

where $\|\xi^j_{,i}\|_\infty$ and $\|\xi^j_{,u}\|_\infty$ are taken over $\overline{\Omega} \times K$ and K is a compact interval which contains $u(\overline{\Omega})$. Here $O(\epsilon^2)$ is uniform w.r.t. $x_1, x_2 \in B(x_0)$. This implies that $\chi_\epsilon(\text{Id} \times u) \to \text{Id}$ uniformly on $\overline{\Omega}$ as $\epsilon \to 0$, and furthermore that for ϵ sufficiently close to 0 the map $\chi_\epsilon(\text{Id} \times u) : \overline{\Omega} \to \overline{\Omega}_\epsilon$ is a homeomorphism. And finally, by the inverse function theorem we find that $\chi_\epsilon(\text{Id} \times u)^{-1}$ is a Lipschitz-function with $\text{Lip}(\chi_\epsilon(\text{Id} \times u)^{-1})$ bounded in ϵ and $\text{Lip}(\chi_\epsilon(\text{Id} \times u)^{-1} - \text{Id}) = O(\epsilon)$ in every local coordinate system. This proves part *(i)* of the Proposition in the case u is Lipschitz on Ω. If u is C^1 on Ω the same proof with the conventional implicit function for C^1-functions can be used.

Part *(ii)*: We study the boundary of Ω_ϵ. Suppose that a portion of $\partial\Omega$ around the point $\bar{x} \in \partial\Omega$ is given in local coordinates by $x^n = f(x')$, $x' = (x^1, \dots, x^{n-1})$ with a Lipschitz continuous function $f : U \subset \mathbb{R}^{n-1} \to \mathbb{R}$. The corresponding point $\bar{x}_\epsilon = \chi_\epsilon(\bar{x}, u(\bar{x}))$ is on $\partial\Omega_\epsilon$. To find the defining equation for $\partial\Omega_\epsilon$ let us define the coordinate projections $\Pi_n(y) = y^n$ and

$\Pi_{1\ldots n-1}(y) = y' = (y^1, \ldots, y^{n-1})$. Then in a small neighborhood of $\bar{x}_\epsilon$ we have

$$\tilde{x} \in \partial\Omega_\epsilon \Leftrightarrow \underbrace{\Pi_n([\chi_\epsilon(\mathrm{Id}\times u)]^{-1}\tilde{x})}_{=x^n} = f\Big(\underbrace{\Pi_{1\ldots n-1}([\chi_\epsilon(\mathrm{Id}\times u)]^{-1}\tilde{x})}_{=(x^1,\ldots,x^{n-1})}\Big). \tag{B.1}$$

To solve (B.1) implicitly we define

$$H(\tilde{x}) = \Big(\Pi_n \circ [\chi_\epsilon(\mathrm{Id}\times u)]^{-1} - f \circ \Pi_{1\ldots n-1} \circ [\chi_\epsilon(\mathrm{Id}\times u)]^{-1}\Big)\tilde{x}.$$

Then we need to find a Lipschitz function h such that the solution of $H(\tilde{x}) = 0$ is given by $\tilde{x}^n = h(\tilde{x}^1, \ldots, \tilde{x}^{n-1})$. A sufficient condition to apply the above version of the implicit function theorem is

$$|H(\tilde{x}', \tilde{x}^n) - H(\tilde{x}', \hat{x}^n) - (\tilde{x}^n - \hat{x}^n)| \leq K_\epsilon |\tilde{x}^n - \hat{x}^n| \tag{B.2}$$

locally around $\bar{x}_\epsilon$ with $K_\epsilon \to 0$ as $\epsilon \to 0$. To verify (B.2) we use the definition of H and the properties of $[\chi_\epsilon(\mathrm{Id}\times u)]^{-1}$ from Part *(i)*:

$$\begin{aligned}
&\Big|\Pi_n\Big([\chi_\epsilon(\mathrm{Id}\times u)]^{-1}(\tilde{x}', \tilde{x}^n) - [\chi_\epsilon(\mathrm{Id}\times u)]^{-1}(\tilde{x}', \hat{x}^n)\Big) - (\tilde{x}^n - \hat{x}^n) \\
&\quad - f \circ \Pi_{1\ldots n-1} \circ [\chi_\epsilon(\mathrm{Id}\times u)]^{-1}(\tilde{x}', \tilde{x}^n) + f \circ \Pi_{1\ldots n-1} \circ [\chi_\epsilon(\mathrm{Id}\times u)]^{-1}(\tilde{x}', \hat{x}^n)\Big| \\
&\quad \leq \ \mathrm{Lip}\Big([\chi_\epsilon(\mathrm{Id}\times u)]^{-1} - \mathrm{Id}\Big)|\tilde{x}^n - \hat{x}^n| \\
&\qquad + \mathrm{Lip}\ f\Big|\Pi_{1\ldots n-1}\Big([\chi_\epsilon(\mathrm{Id}\times u)]^{-1}(\tilde{x}', \tilde{x}^n) - [\chi_\epsilon(\mathrm{Id}\times u)]^{-1}(\tilde{x}', \hat{x}^n)\Big)\Big| \\
&\quad = O(\epsilon)|\tilde{x}^n - \hat{x}^n| + \mathrm{Lip}\ f\Big|\Pi_{1\ldots n-1}\Big((\mathrm{Id} + O(\epsilon))(0, \tilde{x}^n - \hat{x}^n)\Big)\Big| \\
&\quad = O(\epsilon)|\tilde{x}^n - \hat{x}^n|
\end{aligned}$$

uniformly for $(\tilde{x}', \tilde{x}^n), (\tilde{x}', \hat{x}^n)$ in a small ball around $\bar{x}'_\epsilon$. This shows that the above version of the implicit function theorem is applicable, and hence $\partial\Omega_\epsilon$ is Lipschitz for sufficiently small ϵ. □

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