INTRODUCTION TO LEBESGUE INTEGRATION

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Chapter 1

THE REAL NUMBERS AND COUNTABILITY

1.1. Introduction

We shall only give a brief introduction of the basic properties of the real numbers, and denote the set of all real numbers by $\mathbb{R}$.

The first set of properties of $\mathbb{R}$ is generally known as the Field axioms. We offer no proof of these properties, and simply accept them as given.

FIELD AXIOMS.

(A1) *For every* $a, b \in \mathbb{R}$, *we have* $a + b \in \mathbb{R}$.
(A2) *For every* $a, b, c \in \mathbb{R}$, *we have* $a + (b + c) = (a + b) + c$.
(A3) *For every* $a \in \mathbb{R}$, *we have* $a + 0 = a$.
(A4) *For every* $a \in \mathbb{R}$, *there exists* $-a \in \mathbb{R}$ *such that* $a + (-a) = 0$.
(A5) *For every* $a, b \in \mathbb{R}$, *we have* $a + b = b + a$.
(M1) *For every* $a, b \in \mathbb{R}$, *we have* $ab \in \mathbb{R}$.
(M2) *For every* $a, b, c \in \mathbb{R}$, *we have* $a(bc) = (ab)c$.
(M3) *For every* $a \in \mathbb{R}$, *we have* $a1 = a$.
(M4) *For every* $a \in \mathbb{R}$ *such that* $a \neq 0$, *there exists* $a^{-1} \in \mathbb{R}$ *such that* $aa^{-1} = 1$.
(M5) *For every* $a, b \in \mathbb{R}$, *we have* $ab = ba$.
(D) *For every* $a, b, c \in \mathbb{R}$, *we have* $a(b + c) = ab + ac$.

Remark. The properties (A1)–(A5) concern the operation addition, while the properties (M1)–(M5) concern the operation multiplication. In the terminology of group theory, we say that the set $\mathbb{R}$ forms

† This chapter was first used in lectures given by the author at Imperial College, University of London, in 1982 and 1983.

an abelian group under addition, and that the set of all non-zero real numbers forms an abelian group under multiplication. We also say that the set $\mathbb{R}$ forms a field under addition and multiplication.

The set of all real numbers also possesses an ordering relation, so we have the Order Axioms.

ORDER AXIOMS.
(O1) *For every $a, b \in \mathbb{R}$, exactly one of $a < b$, $a = b$, $a > b$ holds.*
(O2) *For every $a, b, c \in \mathbb{R}$ satisfying $a > b$ and $b > c$, we have $a > c$.*
(O3) *For every $a, b, c \in \mathbb{R}$ satisfying $a > b$, we have $a + c > b + c$.*
(O4) *For every $a, b, c \in \mathbb{R}$ satisfying $a > b$ and $c > 0$, we have $ac > bc$.*

An important subset of the set $\mathbb{R}$ of all real numbers is the set

$$\mathbb{N} = \{1, 2, 3, \ldots\}$$

of all natural numbers. However, this definition does not bring out some of the main properties of the set $\mathbb{N}$ in a natural way. The following more complicated definition is therefore sometimes preferred.

DEFINITION. The set $\mathbb{N}$ of all natural numbers is defined by the following four conditions:
(N1) $1 \in \mathbb{N}$.
(N2) If $n \in \mathbb{N}$, then the number $n + 1$, called the successor of n, also belongs to $\mathbb{N}$.
(N3) Every $n \in \mathbb{N}$ other than 1 is the successor of some number in $\mathbb{N}$.
(WO) Every non-empty subset of $\mathbb{N}$ has a least element.

REMARK. The condition (WO) is called the Well-ordering principle.

To explain the significance of each of these four requirements, note that the conditions (N1) and (N2) together imply that $\mathbb{N}$ contains $1, 2, 3, \ldots$. However, these two conditions alone are insufficient to exclude from $\mathbb{N}$ numbers such as 5.5. Now, if $\mathbb{N}$ contained 5.5, then by condition (N3), $\mathbb{N}$ must also contain $4.5, 3.5, 2.5, 1.5, 0.5, -0.5, -1.5, -2.5, \ldots$, and so would not have a least element. We therefore exclude this possibility by stipulating that $\mathbb{N}$ has a least element. This is achieved by the condition (WO).

It can be shown that the condition (WO) implies the Principle of induction. The following two forms of the Principle of induction are particularly useful.

PRINCIPLE OF INDUCTION (WEAK FORM). *Suppose that the statement $p(.)$ satisfies the following conditions:*
(PIW1) $p(1)$ is true; and
(PIW2) $p(n + 1)$ is true whenever $p(n)$ is true.
Then $p(n)$ is true for every $n \in \mathbb{N}$.

PRINCIPLE OF INDUCTION (STRONG FORM). *Suppose that the statement $p(.)$ satisfies the following conditions:*
(PIS1) $p(1)$ is true; and
(PIS2) $p(n + 1)$ is true whenever $p(m)$ is true for all $m \leq n$.
Then $p(n)$ is true for every $n \in \mathbb{N}$.

1.2. Completeness of the Real Numbers

The set $\mathbb{Z}$ of all integers is an extension of the set $\mathbb{N}$ of all natural numbers to include 0 and all numbers of the form $-n$, where $n \in \mathbb{N}$. The set $\mathbb{Q}$ of all rational numbers is the set of all real numbers of the form pq^{-1}, where $p \in \mathbb{Z}$ and $q \in \mathbb{N}$.

We see that the Field axioms and Order axioms hold good if the set $\mathbb{R}$ is replaced by the set $\mathbb{Q}$. On the other hand, the set $\mathbb{Q}$ is incomplete. A good illustration is the following result.

THEOREM 1A. *No rational number $x \in \mathbb{Q}$ satisfies $x^2 = 2$.*

PROOF. Suppose that pq^{-1} has square 2, where $p \in \mathbb{Z}$ and $q \in \mathbb{N}$. We may assume, without loss of generality, that p and q have no common factors apart from ± 1. Then $p^2 = 2q^2$ is even, so that p is even. We can write $p = 2r$, where $r \in \mathbb{Z}$. Then $q^2 = 2r^2$ is even, so that q is even, contradicting that assumption that p and q have no common factors apart from ± 1. ♣

It follows that the real number we know as $\sqrt{2}$ does not belong to $\mathbb{Q}$. We shall now discuss a property that distinguishes the set $\mathbb{R}$ from the set $\mathbb{Q}$.

DEFINITION. A non-empty set S of real numbers is said to be bounded above if there exists a number $K \in \mathbb{R}$ such that $x \leq K$ for every $x \in S$. The number K is called an upper bound of the set S. A non-empty set S of real numbers is said to be bounded below if there exists a number $k \in \mathbb{R}$ such that $x \geq k$ for every $x \in S$. The number k is called a lower bound of the set S. Furthermore, a non-empty set S of real numbers is said to be bounded if it is bounded above and below.

EXAMPLE 1.2.1. The set $\mathbb{N}$ is bounded below but not bounded above. See Section 1.3 for further discussion.

EXAMPLE 1.2.2. The set $\mathbb{Q}$ is neither bounded above nor bounded below.

EXAMPLE 1.2.3. The set $\{x \in \mathbb{R} : -1 < x < 1\}$ is bounded.

The axiom that distinguishes the set $\mathbb{R}$ from the set $\mathbb{Q}$ is the Completeness axiom. It can be stated in many equivalent forms. Here we state it as the Axiom of bound.

AXIOM OF BOUND (UPPER BOUND). *Suppose that S is a non-empty set of real numbers and S is bounded above. Then there is a number $M \in \mathbb{R}$ such that*
(B1) M is an upper bound of S; and
(B2) given any $\epsilon > 0$, there exists $s \in S$ such that $s > M - \epsilon$.

REMARK. Note that (B2) essentially says that any real number less than M cannot be an upper bound of S. In other words, M is the least upper bound of S. Note the important point here that the number M is a real number.

The axiom can be stated in the obvious alternative form below.

AXIOM OF BOUND (LOWER BOUND). *Suppose that S is a non-empty set of real numbers and S is bounded below. Then there is a number $m \in \mathbb{R}$ such that*
(b1) m is a lower bound of S; and
(b2) given any $\epsilon > 0$, there exists $s \in S$ such that $s < m + \epsilon$.

DEFINITION. The real number M satisfying (B1) and (B2) is called the supremum (or least upper bound) of S and denoted by $M = \sup S$. The real number m satisfying (b1) and (b2) is called the infimum (or greatest lower bound) of S and denoted by $m = \inf S$.

DEFINITION. Any number in $\mathbb{R} \setminus \mathbb{Q}$ is called an irrational number.

We now show that $\sqrt{2}$ is a real number.

THEOREM 1B. *There is a positive real number M satisfying $M^2 = 2$.*

PROOF. Let $S = \{x \in \mathbb{R} : x^2 < 2\}$. Since $0 \in S$, the set S is non-empty. On the other hand, it is easy to see that 2 is an upper bound of S; for if $x > 2$, then $x^2 > 4$. Hence S is bounded above. By the Axiom of bound, S has a supremum $M \in \mathbb{R}$. Clearly $M > 0$, since $1 \in S$. It remains to show that $M^2 = 2$. Suppose on the contrary that $M^2 \neq 2$. Then by Axiom (O1), we must have $M^2 < 2$ or $M^2 > 2$. Suppose first of all that $M^2 < 2$. Then

$$(M+\epsilon)^2 = M^2 + 2M\epsilon + \epsilon^2 < M^2 + (2M+1)\epsilon < 2 \qquad \text{if} \qquad 0 < \epsilon < \min\left\{1, \frac{2-M^2}{2M+1}\right\},$$

contradicting that M is an upper bound of S. Suppose next that $M^2 > 2$, then

$$(M-\epsilon)^2 = M^2 - 2M\epsilon + \epsilon^2 > M^2 - 2M\epsilon > 2 \qquad \text{if} \qquad 0 < \epsilon < \frac{M^2-2}{2M},$$

contradicting that M is the least upper bound of S. We must therefore have $M^2 = 2$. ♣

REMARK. The above argument can be adapted to prove the following more general result: Suppose that $n \in \mathbb{N}$, $c \in \mathbb{R}$ and $c > 0$. Then the equation $x^n = c$ has a unique solution for $x \in \mathbb{R}$ and $x > 0$.

1.3. Consequences of the Completeness Axiom

In this section, we shall prove two simple consequences of the Completeness axiom. The first of these shows that there are arbitrarily large natural numbers, while the second shows that rational numbers and irrational numbers are everywhere along the real line.

THEOREM 1C. (ARCHIMEDEAN PROPERTY) *For every $x \in \mathbb{R}$, there exists $n \in \mathbb{N}$ such that $n > x$.*

PROOF. Suppose that $x \in \mathbb{R}$, and suppose on the contrary that $n \leq x$ for every $n \in \mathbb{N}$. Then x is an upper bound of $\mathbb{N}$, so that $\mathbb{N}$ is bounded above. By the Axiom of bound, the set $\mathbb{N}$ has a supremum, M say. Then

$$M \geq n \qquad \text{for every } n = 1, 2, 3, \ldots.$$

In particular, dropping the case $n = 1$, we have

$$M \geq n \qquad \text{for every } n = 2, 3, 4, \ldots.$$

Now every $n = 2, 3, 4, \ldots$ can be written as $k+1$, where $k = 1, 2, 3, \ldots$ respectively. Hence

$$M \geq k+1 \qquad \text{for every } k = 1, 2, 3, \ldots,$$

so that

$$M - 1 \geq k \qquad \text{for every } k = 1, 2, 3, \ldots;$$

in other words, $M-1$ is an upper bound of $\mathbb{N}$. This contradicts the assumption that M is the supremum of $\mathbb{N}$. ♣

We are now in a position to prove the following important result.

THEOREM 1D. *The rational and irrational numbers are dense in $\mathbb{R}$. More precisely, between any two distinct real numbers, there exist a rational number and an irrational number.*

PROOF. Suppose that $x, y \in \mathbb{R}$ and $x < y$.

(a) We shall show that there exists $r \in \mathbb{Q}$ such that $x < r < y$. Suppose first of all that $x > 0$. By the Archimedean property, there exists $q \in \mathbb{N}$ such that $q > 1/(y-x)$, so that $q(y-x) > 1$. Consider the positive real number qx. By the Archimedean property, there exists $n \in \mathbb{N}$ such that $n > qx$. It follows that $S = \{n \in \mathbb{N} : n > qx\}$ is a non-empty set of natural numbers, and so has a least element p, in view of (WO). We now claim that $p-1 \leq qx$. To see this, note that if $p=1$, then $p-1=0<qx$. On the other hand, if $p \neq 1$, then $p-1>qx$ would contradict the definition of p. It now follows that

$$qx < p = (p-1)+1 < qx + q(y-x) = qy,$$

so that

$$x < \frac{p}{q} < y.$$

Suppose now that $x \leq 0$. Then by the Archimedean property, there exists $k \in \mathbb{N}$ such that $k > -x$, so that $k+x>0$. Then there exists $s \in \mathbb{Q}$ such that $x+k<s<y+k$, so that

$$x < s-k < y.$$

Clearly $r = s-k \in \mathbb{Q}$.

(b) We shall now show that there exists $z \in \mathbb{R} \setminus \mathbb{Q}$ such that $x<z<y$. By (a), there exist $r_1, r_2 \in \mathbb{Q}$ such that $x < r_1 < r_2 < y$. The number

$$z = r_1 + \frac{r_2 - r_1}{\sqrt{2}}$$

is clearly irrational and satisfies $r_1 < z < r_2$. ♣

1.4. Countability

In this account, we treat intuitively the distinction between finite and infinite sets. A set is finite if it contains a finite number of elements. To treat infinite sets, our starting point is the set $\mathbb{N}$ of all natural numbers, an example of an infinite set.

Definition. A set X is said to be countably infinite if there exists a bijective mapping from X to $\mathbb{N}$. A set X is said to be countable if it is finite or countably infinite.

Remark. Suppose that X is countably infinite. Then we can write

$$X = \{x_1, x_2, x_3, \ldots\}.$$

Here we understand that there is a bijective mapping $\phi : X \to \mathbb{N}$ where $\phi(x_n) = n$ for every $n \in \mathbb{N}$.

THEOREM 1E. *A countable union of countable sets is countable.*

Proof. Let I be a countable index set, where for each $i \in I$, the set X_i is countable. Either (a) I is finite; or (b) I is countably infinite. We shall only consider (b), since (a) needs only minor modification. Since I is countably infinite, there exists a bijective mapping from I to $\mathbb{N}$. We may therefore assume, without loss of generality, that $I = \mathbb{N}$. For each $n \in \mathbb{N}$, since X_n is countable, we may write

$$X_n = \{a_{n1}, a_{n2}, a_{n3}, \ldots\},$$

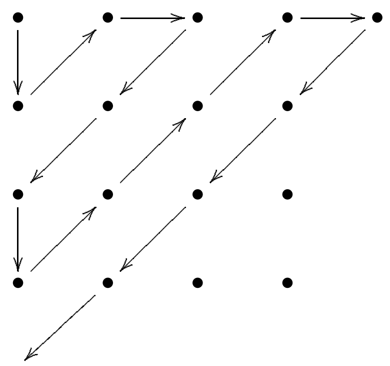
with the convention that if X_n is finite, then the sequence $a_{n1}, a_{n2}, a_{n3}, \ldots$ is constant from some point onwards. Hence we have a doubly infinite array

$$\begin{array}{cccc} a_{11} & a_{12} & a_{13} & \cdots \\ a_{21} & a_{22} & a_{23} & \cdots \\ a_{31} & a_{32} & a_{33} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{array}$$

of elements of the set

$$X = \bigcup_{n \in \mathbb{N}} X_n.$$

We now list these elements in the order indicated by



but discarding duplicates. If X is infinite, the above clearly gives rise to a bijection from X to $\mathbb{N}$. ♣

Example 1.4.1. The set $\mathbb{Z}$ is countable; simply note that $\mathbb{Z} = \mathbb{N} \cup \{0\} \cup \{-1, -2, -3, \ldots\}$.

THEOREM 1F. *The set $\mathbb{Q}$ is countable.*

Proof. Any $x \in \mathbb{Q}$ can be written in the form p/q, where $p \in \mathbb{Z}$ and $q \in \mathbb{N}$. For every $n \in \mathbb{N}$, the set $Q_n = \{p/n : p \in \mathbb{Z}\}$ is countable (why?). Clearly

$$\mathbb{Q} = \bigcup_{n \in \mathbb{N}} Q_n.$$

The result follows from Theorem 1E. ♣

Suppose that two sets X_1 and X_2 are both countably infinite. Since both can be mapped to $\mathbb{N}$ bijectively, it follows that each can be mapped to the other bijectively. In this case, we say that the two sets X_1 and X_2 have the same cardinality. Cardinality can be considered as a way of measuring size. If there exists a one-to-one mapping from X_1 to X_2 and no one-to-one mapping from X_2 to X_1, then we say that X_2 has greater cardinality than X_1. For example, $\mathbb{N}$ and $\mathbb{Q}$ have the same cardinality. We shall now show that $\mathbb{R}$ has greater cardinality than $\mathbb{Q}$.

We shall first of all need an intermediate result.

THEOREM 1G. *Any subset of a countable set is countable.*

Proof. Let X be a countable set. If X is finite, then the result is trivial. We therefore assume that X is countably infinite, so that we can write

$$X = \{x_1, x_2, x_3, \ldots\}.$$

Let Y be a subset of X. If Y is finite, then the result is trivial. If Y is countably infinite, then we can write

$$Y = \{x_{n_1}, x_{n_2}, x_{n_3}, \ldots\},$$

where

$$n_1 = \min\{n \in \mathbb{N} : x_n \in Y\},$$

and where, for every $p \geq 2$,

$$n_p = \min\{n > n_{p-1} : x_n \in Y\}.$$

The result follows. ♣

THEOREM 1H. *The set $\mathbb{R}$ is not countable.*

PROOF. In view of Theorem 1G, it suffices to show that $[0,1)$ is not countable. Suppose on the contrary that $[0,1)$ is countable. Then we can write

$$[0,1) = \{x_1, x_2, x_3, \ldots\}. \tag{1}$$

For each $n \in \mathbb{N}$, we express x_n in decimal notation in the form

$$x_n = .x_{n1}x_{n2}x_{n3}\ldots,$$

where for each $k \in \mathbb{N}$, the digit $x_{nk} \in \{0,1,2,\ldots,9\}$. Note that this expression may not be unique, but it does not matter, as we simply choose one. We now have

$$\begin{aligned} x_1 &= .x_{11}x_{12}x_{13}\ldots, \\ x_2 &= .x_{21}x_{22}x_{23}\ldots, \\ x_3 &= .x_{31}x_{32}x_{33}\ldots, \\ &\vdots \end{aligned}$$

Let $y = .y_1y_2y_3\ldots$, where for each $n \in \mathbb{N}$, $y_n \in \{0,1,2,\ldots,9\}$ and $y_n \equiv x_{nn} + 5 \pmod{10}$. Then clearly $y \neq x_n$ for any $n \in \mathbb{N}$. But $y_n \in [0,1)$, contradicting (1). ♣

Note that the set $\mathbb{R} \setminus \mathbb{Q}$ of all irrational numbers is not countable. It follows that in the sense of cardinality, there are far more irrational numbers than rational numbers.

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INTRODUCTION TO LEBESGUE INTEGRATION

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Chapter 2

THE RIEMANN INTEGRAL

2.1. Riemann Sums

Suppose that a function $f(x)$ is bounded on the interval $[A, B]$, where $A, B \in \mathbb{R}$ and $A < B$. Suppose further that

$$\Delta: \ A = x_0 < x_1 < x_2 < \ldots < x_n = B$$

is a dissection of the interval $[A, B]$.

Definition. The sum

$$s(f, \Delta) = \sum_{i=1}^{n} (x_i - x_{i-1}) \inf_{x \in [x_{i-1}, x_i]} f(x)$$

is called the lower Riemann sum of $f(x)$ corresponding to the dissection Δ.

Definition. The sum

$$S(f, \Delta) = \sum_{i=1}^{n} (x_i - x_{i-1}) \sup_{x \in [x_{i-1}, x_i]} f(x)$$

is called the upper Riemann sum of $f(x)$ corresponding to the dissection Δ.

Example 2.1.1. Consider the function $f(x) = x^2$ in the interval $[0, 1]$. Suppose that $n \in \mathbb{N}$ is given and fixed. Let us consider a dissection

$$\Delta_n: \ 0 = x_0 < x_1 < x_2 < \ldots < x_n = 1$$

† This chapter was written at Macquarie University in 1996.

of the interval $[0,1]$, where $x_i = i/n$ for every $i = 0,1,2,\ldots,n$. It is easy to see that for every $i = 1,2,\ldots,n$, we have

$$\inf_{x\in[x_{i-1},x_i]} f(x) = \inf_{\frac{i-1}{n}\le x\le\frac{i}{n}} x^2 = \frac{(i-1)^2}{n^2}$$

and

$$\sup_{x\in[x_{i-1},x_i]} f(x) = \sup_{\frac{i-1}{n}\le x\le\frac{i}{n}} x^2 = \frac{i^2}{n^2}.$$

It follows that

$$s(f,\Delta_n) = \sum_{i=1}^{n}(x_i - x_{i-1}) \inf_{x\in[x_{i-1},x_i]} f(x) = \sum_{i=1}^{n} \frac{(i-1)^2}{n^3} = \frac{(n-1)n(2n-1)}{6n^3}$$

and

$$S(f,\Delta_n) = \sum_{i=1}^{n}(x_i - x_{i-1}) \sup_{x\in[x_{i-1},x_i]} f(x) = \sum_{i=1}^{n} \frac{i^2}{n^3} = \frac{n(n+1)(2n+1)}{6n^3}.$$

Note that $s(f,\Delta_n) \le S(f,\Delta_n)$, and that both terms converge to $1/3$ as $n\to\infty$.

THEOREM 2A. *Suppose that a function $f(x)$ is bounded on the interval $[A,B]$, where $A,B\in\mathbb{R}$ and $A<B$. Suppose further that Δ' and Δ are dissections of the interval $[A,B]$, and that $\Delta'\subseteq\Delta$. Then*

$$s(f,\Delta') \le s(f,\Delta) \qquad \text{and} \qquad S(f,\Delta) \le S(f,\Delta').$$

PROOF. Suppose that $x' < x''$ are consecutive dissection points of Δ', and suppose that

$$x' = y_0 < y_1 < \ldots < y_m = x''$$

are all the dissection points of Δ in the interval $[x',x'']$. Then it is easy to see that

$$\sum_{i=1}^{m}(y_i - y_{i-1}) \inf_{x\in[y_{i-1},y_i]} f(x) \ge \sum_{i=1}^{m}(y_i - y_{i-1}) \inf_{x\in[x',x'']} f(x) = (x''-x') \inf_{x\in[x',x'']} f(x)$$

and

$$\sum_{i=1}^{m}(y_i - y_{i-1}) \sup_{x\in[y_{i-1},y_i]} f(x) \le \sum_{i=1}^{m}(y_i - y_{i-1}) \sup_{x\in[x',x'']} f(x) = (x''-x') \sup_{x\in[x',x'']} f(x).$$

The result follows on summing over all consecutive points of the dissection Δ' (the reader is advised to draw a few pictures if in doubt). ♣

THEOREM 2B. *Suppose that a function $f(x)$ is bounded on the interval $[A,B]$, where $A,B\in\mathbb{R}$ and $A<B$. Suppose further that Δ' and Δ'' are dissections of the interval $[A,B]$. Then*

$$s(f,\Delta') \le S(f,\Delta'').$$

PROOF. Consider the dissection $\Delta = \Delta'\cup\Delta''$ of $[A,B]$. Then it follows from Theorem 2A that

$$s(f,\Delta') \le s(f,\Delta) \qquad \text{and} \qquad S(f,\Delta) \le S(f,\Delta''). \tag{1}$$

On the other hand, it is easy to check that

$$s(f,\Delta) \le S(f,\Delta). \tag{2}$$

The result follows on combining (1) and (2). ♣

2.2. Lower and Upper Integrals

Suppose that a function $f(x)$ is bounded on the interval $[A,B]$, where $A,B \in \mathbb{R}$ and $A<B$.

Definition. The real number

$$I^-(f,A,B) = \sup_{\Delta} s(f,\Delta),$$

where the supremum is taken over all dissections Δ of $[A,B]$, is called the lower integral of $f(x)$ over $[A,B]$.

Definition. The real number

$$I^+(f,A,B) = \inf_{\Delta} S(f,\Delta),$$

where the infimum is taken over all dissections Δ of $[A,B]$, is called the upper integral of $f(x)$ over $[A,B]$.

Remark. Since $f(x)$ is bounded on $[A,B]$, it follows that $s(f,\Delta)$ and $S(f,\Delta)$ are bounded above and below. This guarantees the existence of $I^-(f,A,B)$ and $I^+(f,A,B)$.

THEOREM 2C. *Suppose that a function $f(x)$ is bounded on the interval $[A,B]$, where $A,B \in \mathbb{R}$ and $A<B$. Then $I^-(f,A,B) \le I^+(f,A,B)$.*

Proof. Suppose that Δ' is a dissection of $[A,B]$. Then it follows from Theorem 2A that $s(f,\Delta') \le S(f,\Delta)$ for every dissection Δ of $[A,B]$. Taking the infimum over all dissections Δ of $[A,B]$, we conclude that

$$s(f,\Delta') \le \inf_{\Delta} S(f,\Delta) = I^+(f,A,B).$$

Taking the supremum over all dissections Δ' of $[A,B]$, we conclude that

$$I^+(f,A,B) \ge \sup_{\Delta'} s(f,\Delta') = I^-(f,A,B).$$

The result follows. ♣

Example 2.2.1. Consider again the function $f(x)=x^2$ in the interval $[0,1]$. Recall from Example 2.1.1 that both $s(f,\Delta_n)$ and $S(f,\Delta_n)$ converge to $1/3$ as $n\to\infty$. It follows that

$$I^-(f,0,1) \ge \frac{1}{3} \qquad \text{and} \qquad I^+(f,0,1) \le \frac{1}{3}.$$

In view of Theorem 2C, we must have

$$I^-(f,0,1) = I^+(f,0,1) = \frac{1}{3}.$$

2.3. Riemann Integrability

Suppose that a function $f(x)$ is bounded on the interval $[A,B]$, where $A,B \in \mathbb{R}$ and $A<B$.

Definition. Suppose that $I^-(f,A,B) = I^+(f,A,B)$. Then we say that the function $f(x)$ is Riemann integrable over $[A,B]$, denoted by $f \in \mathcal{R}([A,B])$, and write

$$\int_A^B f(x)\,\mathrm{d}x = I^-(f,A,B) = I^+(f,A,B).$$

EXAMPLE 2.3.1. Let us return to our Example 2.2.1, and consider again the function $f(x) = x^2$ in the interval $[0, 1]$. We have shown that

$$I^-(f, 0, 1) = I^+(f, 0, 1) = \frac{1}{3}.$$

It now follows that

$$\int_0^1 x^2 \, \mathrm{d}x = \frac{1}{3}.$$

THEOREM 2D. *Suppose that a function $f(x)$ is bounded on the interval $[A, B]$, where $A, B \in \mathbb{R}$ and $A < B$. Then the following two statements are equivalent:*
(a) $f \in \mathcal{R}([A, B])$.
(b) Given any $\epsilon > 0$, there exists a dissection Δ of $[A, B]$ such that

$$S(f, \Delta) - s(f, \Delta) < \epsilon. \tag{3}$$

PROOF. ((a)⇒(b)) If $f \in \mathcal{R}([A, B])$, then

$$\sup_\Delta s(f, \Delta) = \inf_\Delta S(f, \Delta), \tag{4}$$

where the supremum and infimum are taken over all dissections Δ of $[A, B]$. For every $\epsilon > 0$, there exist dissections Δ_1 and Δ_2 of $[A, B]$ such that

$$s(f, \Delta_1) > \sup_\Delta s(f, \Delta) - \frac{\epsilon}{2} \qquad \text{and} \qquad S(f, \Delta_2) < \inf_\Delta S(f, \Delta) + \frac{\epsilon}{2}. \tag{5}$$

Let $\Delta = \Delta_1 \cup \Delta_2$. Then by Theorem 2A, we have

$$s(f, \Delta) \geq s(f, \Delta_1) \qquad \text{and} \qquad S(f, \Delta) \leq S(f, \Delta_2). \tag{6}$$

The inequality (3) now follows on combining (4)–(6).

((b)⇒(a)) Suppose that $\epsilon > 0$ is given. We can choose a dissection Δ of $[A, B]$ such that (3) holds. Clearly

$$s(f, \Delta) \leq I^-(f, A, B) \leq I^+(f, A, B) \leq S(f, \Delta). \tag{7}$$

Combining (3) and (7), we conclude that $0 \leq I^+(f, A, B) - I^-(f, A, B) < \epsilon$. Note now that $\epsilon > 0$ is arbitrary, and that $I^+(f, A, B) - I^-(f, A, B)$ is independent of ϵ. It follows that we must have $I^+(f, A, B) - I^-(f, A, B) = 0$. ♣

2.4. Further Properties of the Riemann Integral

In this section, we shall use some of our earlier results to study some simple but useful properties of the Riemann integral. First of all, we shall study the arithmetic of Riemann integrals.

THEOREM 2E. *Suppose that $f, g \in \mathcal{R}([A, B])$, where $A, B \in \mathbb{R}$ and $A < B$. Then*

(a) $f + g \in \mathcal{R}([A, B])$ and $\displaystyle\int_A^B (f(x) + g(x)) \, \mathrm{d}x = \int_A^B f(x) \, \mathrm{d}x + \int_A^B g(x) \, \mathrm{d}x$;

(b) for every $c \in \mathbb{R}$, $cf \in \mathcal{R}([A, B])$ and $\displaystyle\int_A^B cf(x) \, \mathrm{d}x = c \int_A^B f(x) \, \mathrm{d}x$;

(c) if $f(x) \geq 0$ for every $x \in [A, B]$, then $\displaystyle\int_A^B f(x) \, \mathrm{d}x \geq 0$; and

(d) if $f(x) \leq g(x)$ for every $x \in [A, B]$, then $\displaystyle\int_A^B f(x) \, \mathrm{d}x \leq \int_A^B g(x) \, \mathrm{d}x$.

PROOF. (a) Since $f, g \in \mathcal{R}([A, B])$, it follows from Theorem 2D that for every $\epsilon > 0$, there exist dissections Δ_1 and Δ_2 of $[A, B]$ such that

$$S(f, \Delta_1) - s(f, \Delta_1) < \frac{\epsilon}{2} \qquad \text{and} \qquad S(g, \Delta_2) - s(g, \Delta_2) < \frac{\epsilon}{2}.$$

Let $\Delta = \Delta_1 \cup \Delta_2$. Then in view of Theorem 2A, we have

$$S(f, \Delta) - s(f, \Delta) < \frac{\epsilon}{2} \qquad \text{and} \qquad S(g, \Delta) - s(g, \Delta) < \frac{\epsilon}{2}. \tag{8}$$

Suppose that the dissection Δ is given by $\Delta : \; A = x_0 < x_1 < x_2 < \ldots < x_n = B$. It is easy to see that for every $i = 1, \ldots, n$, we have

$$\sup_{x \in [x_{i-1}, x_i]} (f(x) + g(x)) \le \sup_{x \in [x_{i-1}, x_i]} f(x) + \sup_{x \in [x_{i-1}, x_i]} g(x)$$

and

$$\inf_{x \in [x_{i-1}, x_i]} (f(x) + g(x)) \ge \inf_{x \in [x_{i-1}, x_i]} f(x) + \inf_{x \in [x_{i-1}, x_i]} g(x).$$

It follows that

$$S(f + g, \Delta) \le S(f, \Delta) + S(g, \Delta) \qquad \text{and} \qquad s(f + g, \Delta) \ge s(f, \Delta) + s(g, \Delta). \tag{9}$$

Combining (8) and (9), we have

$$S(f + g, \Delta) - s(f + g, \Delta) \le (S(f, \Delta) - s(f, \Delta)) + (S(g, \Delta) - s(g, \Delta)) < \epsilon.$$

It now follows from Theorem 2D that $f + g \in \mathcal{R}([A, B])$. Suppose now that Δ_1 and Δ_2 are any two dissections of $[A, B]$. Let $\Delta = \Delta_1 \cup \Delta_2$. Then in view of Theorem 2A and (9), we have

$$S(f, \Delta_1) + S(g, \Delta_2) \ge S(f, \Delta) + S(g, \Delta) \ge S(f + g, \Delta) \ge I^+(f + g, A, B),$$

so that

$$S(g, \Delta_2) \ge I^+(f + g, A, B) - S(f, \Delta_1).$$

Keeping Δ_1 fixed and taking the infimum over all Δ_2, we have $I^+(g, A, B) \ge I^+(f + g, A, B) - S(f, \Delta_1)$, so that

$$S(f, \Delta_1) \ge I^+(f + g, A, B) - I^+(g, A, B).$$

Taking the infimum over all Δ_1, we have $I^+(f, A, B) \ge I^+(f + g, A, B) - I^+(g, A, B)$, so that

$$I^+(f + g, A, B) \le I^+(f, A, B) + I^+(g, A, B). \tag{10}$$

Similarly, in view of Theorem 2A and (9), we have

$$s(f, \Delta_1) + s(g, \Delta_2) \le s(f, \Delta) + s(g, \Delta) \le s(f + g, \Delta) \le I^-(f + g, A, B),$$

so that

$$s(g, \Delta_2) \le I^-(f + g, A, B) - s(f, \Delta_1).$$

Keeping Δ_1 fixed and taking the supremum over all Δ_2, we have $I^-(g, A, B) \le I^-(f+g, A, B) - s(f, \Delta_1)$, so that

$$s(f, \Delta_1) \le I^-(f + g, A, B) - I^-(g, A, B).$$

Taking the supremum over all Δ_1, we have $I^-(f, A, B) \le I^-(f + g, A, B) - I^-(g, A, B)$, so that

$$I^-(f, A, B) + I^-(g, A, B) \le I^-(f + g, A, B). \tag{11}$$

Combining (10) and (11), we have

$$I^-(f,A,B)+I^-(g,A,B)\le I^-(f+g,A,B)=I^+(f+g,A,B)\le I^+(f,A,B)+I^+(g,A,B). \qquad (12)$$

Clearly $I^-(f,A,B)=I^+(f,A,B)$ and $I^-(g,A,B)=I^+(g,A,B)$, and so equality must hold everywhere in (12). In particular, we have $I^+(f,A,B)+I^+(g,A,B)=I^+(f+g,A,B)$.

(b) The case $c=0$ is trivial. Suppose now that $c>0$. Since $f\in\mathcal{R}([A,B])$, it follows from Theorem 2D that for every $\epsilon>0$, there exists a dissection Δ of $[A,B]$ such that

$$S(f,\Delta)-s(f,\Delta)<\frac{\epsilon}{c}.$$

It is easy to see that

$$S(cf,\Delta)=cS(f,\Delta) \qquad \text{and} \qquad s(cf,\Delta)=cs(f,\Delta). \qquad (13)$$

Hence

$$S(cf,\Delta)-s(cf,\Delta)<\epsilon.$$

It follows from Theorem 2D that $cf\in\mathcal{R}([A,B])$. Also, (13) clearly implies $I^+(cf,A,B)=cI^+(f,A,B)$. Suppose next that $c<0$. Since $f\in\mathcal{R}([A,B])$, it follows from Theorem 2D that for every $\epsilon>0$, there exists a dissection Δ of $[A,B]$ such that

$$S(f,\Delta)-s(f,\Delta)<-\frac{\epsilon}{c}.$$

It is easy to see that

$$S(cf,\Delta)=cs(f,\Delta) \qquad \text{and} \qquad s(cf,\Delta)=cS(f,\Delta). \qquad (14)$$

Hence

$$S(cf,\Delta)-s(cf,\Delta)<\epsilon.$$

It follows from Theorem 2D that $cf\in\mathcal{R}([A,B])$. Also, (14) clearly implies $I^+(cf,A,B)=cI^-(f,A,B)$.

(c) Note simply that

$$\int_A^B f(x)\,\mathrm{d}x\ge(B-A)\inf_{x\in[A,B]}f(x),$$

where the right hand side is the lower sum corresponding to the trivial dissection.

(d) Note that $g-f\in\mathcal{R}([A,B])$ in view of (a) and (b). We now apply (c) to the function $g-f$. ♣

Next, we investigate the question of breaking up the interval $[A,B]$ of integration.

THEOREM 2F. *Suppose that $f\in\mathcal{R}([A,B])$, where $A,B\in\mathbb{R}$ and $A<B$. Then for every real number $C\in(A,B)$, $f\in\mathcal{R}([A,C])$ and $f\in\mathcal{R}([C,B])$. Furthermore,*

$$\int_A^B f(x)\,\mathrm{d}x=\int_A^C f(x)\,\mathrm{d}x+\int_C^B f(x)\,\mathrm{d}x. \qquad (15)$$

Proof. We shall show that for every $C',C''\in\mathbb{R}$ satisfying $A\le C'<C''\le B$, we have $f\in\mathcal{R}([C',C''])$. Since $f\in\mathcal{R}([A,B])$, it follows from Theorem 2D that given any $\epsilon>0$, there exists a dissection Δ^* of $[A,B]$ such that

$$S(f,\Delta^*)-s(f,\Delta^*)<\epsilon.$$

It follows from Theorem 2A that the dissection $\Delta=\Delta^*\cup\{C',C''\}$ of $[A,B]$ satisfies

$$S(f,\Delta)-s(f,\Delta)<\epsilon. \qquad (16)$$

Suppose that the dissection Δ is given by $\Delta:\ A=x_0<x_1<x_2<\ldots<x_n=B$. Then there exist $k',k''\in\{0,1,2,\ldots,n\}$ satisfying $k'<k''$ such that $C'=x_{k'}$ and $C''=x_{k''}$. It follows that

$$\Delta_0:\ C'=x_{k'}<x_{k'+1}<x_{k'+2}<\ldots<x_{k''}=C''$$

is a dissection of $[C', C'']$. Furthermore,

$$\begin{aligned} S(f,\Delta_0) - s(f,\Delta_0) &= \sum_{i=k'+1}^{k''} (x_i - x_{i-1}) \left(\sup_{x\in[x_{i-1},x_i]} f(x) - \inf_{x\in[x_{i-1},x_i]} f(x) \right) \\ &\le \sum_{i=1}^{n} (x_i - x_{i-1}) \left(\sup_{x\in[x_{i-1},x_i]} f(x) - \inf_{x\in[x_{i-1},x_i]} f(x) \right) \\ &= S(f,\Delta) - s(f,\Delta) < \epsilon, \end{aligned}$$

in view of (16). It now follows from Theorem 2D that $f \in \mathcal{R}([C', C''])$. To complete the proof of Theorem 2F, it remains to establish (15). By definition, we have

$$\int_A^B f(x)\,\mathrm{d}x = \inf_{\Delta} S(f,\Delta), \tag{17}$$

while

$$\int_A^C f(x)\,\mathrm{d}x = \inf_{\Delta_1} S(f,\Delta_1) \qquad \text{and} \qquad \int_C^B f(x)\,\mathrm{d}x = \inf_{\Delta_2} S(f,\Delta_2). \tag{18}$$

Here Δ, Δ_1 and Δ_2 run over all dissections of $[A, B]$, $[A, C]$ and $[C, B]$ respectively. (15) will follow from (17) and (18) if we can show that

$$\inf_{\Delta} S(f,\Delta) = \inf_{\Delta_1} S(f,\Delta_1) + \inf_{\Delta_2} S(f,\Delta_2). \tag{19}$$

Suppose first of all that Δ is a dissection of $[A, B]$. Then we can write $\Delta \cup \{C\} = \Delta' \cup \Delta''$, where Δ' and Δ'' are dissections of $[A, C]$ and $[C, B]$ respectively. By Theorem 2A, we have

$$S(f,\Delta) \ge S(f,\Delta \cup \{C\}) = S(f,\Delta') + S(f,\Delta'').$$

Clearly

$$S(f,\Delta') + S(f,\Delta'') \ge \inf_{\Delta_1} S(f,\Delta_1) + \inf_{\Delta_2} S(f,\Delta_2).$$

Hence

$$S(f,\Delta) \ge \inf_{\Delta_1} S(f,\Delta_1) + \inf_{\Delta_2} S(f,\Delta_2).$$

Taking the infimum over all dissections Δ of $[A, B]$, we conclude that

$$\inf_{\Delta} S(f,\Delta) \ge \inf_{\Delta_1} S(f,\Delta_1) + \inf_{\Delta_2} S(f,\Delta_2). \tag{20}$$

Suppose next that Δ_1 and Δ_2 are dissections of $[A, C]$ and $[C, B]$ respectively. Then $\Delta_1 \cup \Delta_2$ is a dissection of $[A, B]$, and

$$S(f,\Delta_1) + S(f,\Delta_2) = S(f,\Delta_1 \cup \Delta_2) \ge \inf_{\Delta} S(f,\Delta).$$

This implies that

$$S(f,\Delta_1) \ge \inf_{\Delta} S(f,\Delta) - S(f,\Delta_2).$$

Keeping Δ_2 fixed and taking the infimum over all Δ_1, we have

$$\inf_{\Delta_1} S(f,\Delta_1) \ge \inf_{\Delta} S(f,\Delta) - S(f,\Delta_2),$$

and so

$$S(f,\Delta_2) \ge \inf_{\Delta} S(f,\Delta) - \inf_{\Delta_1} S(f,\Delta_1).$$

Taking the infimum over all Δ_2, we have

$$\inf_{\Delta_2} S(f,\Delta_2) \ge \inf_{\Delta} S(f,\Delta) - \inf_{\Delta_1} S(f,\Delta_1),$$

and so

$$\inf_{\Delta_1} S(f,\Delta_1) + \inf_{\Delta_2} S(f,\Delta_2) \ge \inf_{\Delta} S(f,\Delta). \tag{21}$$

The equality (19) now follows on combining (20) and (21). ♣

THEOREM 2G. *Suppose that $A,B,C \in \mathbb{R}$ and $A<C<B$. Suppose further that $f \in \mathcal{R}([A,C])$ and $f \in \mathcal{R}([C,B])$. Then $f \in \mathcal{R}([A,B])$. Furthermore,*

$$\int_A^B f(x)\,\mathrm{d}x = \int_A^C f(x)\,\mathrm{d}x + \int_C^B f(x)\,\mathrm{d}x.$$

Proof. Since $f \in \mathcal{R}([A,C])$ and $f \in \mathcal{R}([C,B])$, it follows from Theorem 2D that given any $\epsilon>0$, there exist dissections Δ_1 and Δ_2 of $[A,C]$ and $[C,B]$ respectively such that

$$S(f,\Delta_1) - s(f,\Delta_1) < \frac{\epsilon}{2} \qquad \text{and} \qquad S(f,\Delta_2) - s(f,\Delta_2) < \frac{\epsilon}{2}. \tag{22}$$

Clearly $\Delta = \Delta_1 \cup \Delta_2$ is a dissection of $[A,B]$. Furthermore,

$$S(f,\Delta) = S(f,\Delta_1) + S(f,\Delta_2) \qquad \text{and} \qquad s(f,\Delta) = s(f,\Delta_1) + s(f,\Delta_2).$$

It follows that

$$S(f,\Delta) - s(f,\Delta) = (S(f,\Delta_1) - s(f,\Delta_1)) + (S(f,\Delta_2) - s(f,\Delta_2)) < \epsilon,$$

in view of (22). It now follows from Theorem 2D that $f \in \mathcal{R}([A,B])$. The proof can now be completed as in the proof of Theorem 2F. ♣

Finally, we consider the question of altering the value of the function at a finite number of points. The following theorem may be applied a finite number of times.

THEOREM 2H. *Suppose that $f \in \mathcal{R}([A,B])$, where $A,B \in \mathbb{R}$ and $A<B$. Suppose further that the real number $C \in [A,B]$, and that $f(x)=g(x)$ for every $x \in [A,B]$ except possibly at $x=C$. Then $g \in \mathcal{R}([A,B])$, and*

$$\int_A^B f(x)\,\mathrm{d}x = \int_A^B g(x)\,\mathrm{d}x.$$

Proof. Write $h(x)=f(x)-g(x)$ for every $x \in [A,B]$. We shall show that

$$\int_A^B h(x)\,\mathrm{d}x = 0.$$

Note that $h(x)=0$ whenever $x \ne C$. The case $h(C)=0$ is trivial, so we assume, without loss of generality, that $h(C) \ne 0$. Given any $\epsilon>0$, we shall choose a dissection Δ of $[A,B]$ such that C is not one of the dissection points and such that the subinterval containing C has length less than $\epsilon/|h(C)|$. Since $-|h(C)| \le h(C) \le |h(C)|$, it is easy to check that

$$S(h,\Delta) \le |h(C)|\frac{\epsilon}{|h(C)|} < \epsilon \qquad \text{and} \qquad s(h,\Delta) \ge -|h(C)|\frac{\epsilon}{|h(C)|} > -\epsilon.$$

It follows that

$$-\epsilon < I^-(h,A,B) \le I^+(h,A,B) < \epsilon.$$

Note now that $\epsilon > 0$ is arbitrary, and the terms $I^-(h, A, B)$ and $I^+(h, A, B)$ are independent of ϵ. It follows that we must have $I^-(h, A, B) = I^+(h, A, B) = 0$. This completes the proof. ♣

2.5. An Important Example

In this section, we shall find a function that is not Riemann integrable. Consider the function

$$g(x) = \begin{cases} 0 & (x \text{ is rational}), \\ 1 & (x \text{ is irrational}). \end{cases}$$

We know from Theorem 1D that in any open interval, there are rational numbers and irrational numbers. It follows that in any interval $[\alpha, \beta]$, where $\alpha < \beta$, we have

$$\inf_{x\in[\alpha,\beta]} g(x) = 0 \qquad \text{and} \qquad \sup_{x\in[\alpha,\beta]} g(x) = 1.$$

It follows that for every dissection Δ of $[0, 1]$, we have

$$s(g, \Delta) = 0 \qquad \text{and} \qquad S(g, \Delta) = 1,$$

so that

$$I^-(g, 0, 1) = 0 \neq 1 = I^+(g, 0, 1).$$

It follows that $g(x)$ is not Riemann integrable over the closed interval $[0, 1]$.

Note, on the other hand, that the rational numbers in $[0, 1]$ are countable, while the irrational numbers in $[0, 1]$ are not countable. In the sense of cardinality, there are far more irrational numbers than rational numbers in $[0, 1]$. However, the definition of the Riemann integral does not highlight this inequality.

We wish therefore to develop a theory of integration more general than Riemann integration.

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INTRODUCTION TO LEBESGUE INTEGRATION

W W L CHEN

Chapter 3

POINT SETS

3.1. Open and Closed Sets

To study a Riemann integral, one needs to subdivide the interval of integration into a finite number of subintervals. In Lebesgue's approach, the interval is subdivided into more general sets called measurable sets. In 1902, Lebesgue gave a definition of measure for point sets and used this to develop his integral.

Since then, measure theory and integration theory have both been generalized and modified. It is now possible to introduce the Lebesgue integral with very little reference to measure theory, but focusing directly on functions and their integrals instead.

We shall attempt here to give an account of this approach. The only concept from measure theory that we shall need is that of sets of measure zero. In this chapter, we shall cover some basic results on point sets for later use.

DEFINITION. Suppose that $S \subseteq \mathbb{R}$ is given. A point $x \in S$ is said to be an interior point of S if there exists $\epsilon > 0$ such that the open interval $(x-\epsilon, x+\epsilon) \subseteq S$.

DEFINITION. A set $G \subseteq \mathbb{R}$ is said to be open if every point of G is an interior point of G.

REMARK. It is quite common to denote open sets by G after the German word "Gebiet".

EXAMPLE 3.1.1. The interval $(0,1)$ is open. For any given $x \in (0,1)$, we can choose $\epsilon = \min\{x, 1-x\}$. Then $\epsilon \le x$ and $\epsilon \le 1-x$, so that $0 \le x-\epsilon < x+\epsilon \le 1$, whence $(x-\epsilon, x+\epsilon) \subseteq (0,1)$.

EXAMPLE 3.1.2. The interval $[0,1]$ is not open, since clearly the point 0 is not an interior point of $[0,1]$.

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† This chapter was first used in lectures given by the author at Imperial College, University of London, in 1983.

EXAMPLE 3.1.3. The sets $\emptyset$ and $\mathbb{R}$ are both open.

We have the following two simple results.

THEOREM 3A. *The union of any collection of open sets in $\mathbb{R}$ is open.*

THEOREM 3B. *The intersection of any finite collection of open sets in $\mathbb{R}$ is open.*

REMARK. Note that Theorem 3B cannot be extended to infinite collections. Note, for example, that $G_n = (-1/n, 1/n)$ is open for every $n \in \mathbb{N}$. On the other hand,

$$\bigcap_{n=1}^{\infty} G_n = \{0\}$$

is not open. The reader is advised to study the proof of Theorem 3B below and try to pinpoint where the proof fails when the collection is infinite.

PROOF OF THEOREM 3A. Suppose that $\mathcal{G}$ is a collection of open sets in $\mathbb{R}$. Denote by U their union. Suppose that $x \in U$. Then $x \in G$ for some $G \in \mathcal{G}$. Since G is open, it follows that x is an interior point of G, and so there exists $\epsilon > 0$ such that

$$(x - \epsilon, x + \epsilon) \subseteq G \subseteq U.$$

It follows that x is an interior point of U. ♣

PROOF OF THEOREM 3B. Suppose that the open sets are $G_1, \ldots, G_n$. Denote by V their intersection. Suppose that $x \in V$. Then $x \in G_k$ for every $k = 1, \ldots, n$. Since G_k is open, it follows that x is an interior point of G_k, and so there exists $\epsilon_k > 0$ such that

$$(x - \epsilon_k, x + \epsilon_k) \subseteq G_k.$$

Now let $\epsilon = \min\{\epsilon_1, \ldots, \epsilon_n\} > 0$. Then for every $k = 1, \ldots, n$, we have

$$(x - \epsilon, x + \epsilon) \subseteq (x - \epsilon_k, x + \epsilon_k) \subseteq G_k,$$

so that

$$(x - \epsilon, x + \epsilon) \subseteq G_1 \cap \ldots \cap G_n = V.$$

It follows that x is an interior point of V. ♣

The following result gives a characterization of all open sets in $\mathbb{R}$.

THEOREM 3C. *Every open set $G \in \mathbb{R}$ is a countable union of pairwise disjoint open intervals in $\mathbb{R}$.*

PROOF. For every $x \in G$, let I_x denote the largest open interval in $\mathbb{R}$ satisfying $x \in I_x \subseteq G$. Suppose now that $x, y \in G$ and $I_x \cap I_y \neq \emptyset$. Then $I_x \cup I_y$ is also an open interval. Furthermore,

$$I_x \subseteq I_x \cup I_y \subseteq G \qquad \text{and} \qquad I_y \subseteq I_x \cup I_y \subseteq G.$$

From the definition of I_x and I_y, we must have $I_x = I_x \cup I_y$ and $I_y = I_x \cup I_y$, so that $I_x = I_y$. We therefore conclude that I_x and I_y are either disjoint or equal. It follows that G is a union of disjoint open intervals in $\mathbb{R}$. Write

$$G = \bigcup_{I \in \mathcal{C}} I.$$

It remains to show that the collection $\mathcal{C}$ is countable. Note that every interval $I \in \mathcal{C}$ contains a rational number x_I. We can now construct a bijective mapping $\phi : \mathcal{C} \to \{x_I : I \in \mathcal{C}\}$ by writing $\phi(I) = x_I$

for every $I \in \mathcal{C}$; in other words, we identify each interval I with a rational number it contains. Clearly $\{x_I : I \in \mathcal{C}\} \subseteq \mathbb{Q}$, and so must be countable. It follows that $\mathcal{C}$ is countable. ♣

DEFINITION. Suppose that $S \subseteq \mathbb{R}$ is given. A point $x \in \mathbb{R}$ is said to be a limit point of S if it is the limit of a sequence in S.

DEFINITION. A set $F \subseteq \mathbb{R}$ is said to be closed if it contains all its limit points.

REMARK. It is quite common to denote closed sets by F after the French word "fermé".

EXAMPLE 3.1.4. The interval $(0, 1)$ is not closed. The sequence $1/n$ is in $(0, 1)$, but its limit 0 is not.

EXAMPLE 3.1.5. The interval $[0, 1]$ is closed. If x_n is a convergent sequence in $[0, 1]$, then its limit x must satisfy $0 \le x \le 1$, so that $x \in [0, 1]$.

EXAMPLE 3.1.6. The sets $\emptyset$ and $\mathbb{R}$ are both closed. These are examples of sets which are both open and closed.

We have the following useful result on open and closed sets.

THEOREM 3D. *A set $F \subseteq \mathbb{R}$ is closed if and only if its complement $F' = \mathbb{R} \setminus F$ is open.*

PROOF. ($\Rightarrow$) Suppose that F is closed. For every $x \in F'$, x is not a limit point of F, so that no sequence in F converges to x. Hence there exists $\epsilon > 0$ such that $(x - \epsilon, x + \epsilon) \cap F = \emptyset$, so that $(x - \epsilon, x + \epsilon) \subseteq F'$.

($\Leftarrow$) Suppose that $x \notin F$. Then $x \in F'$. Since F' is open, it follows that there exists $\epsilon > 0$ such that $(x - \epsilon, x + \epsilon) \subseteq F'$, so that $(x - \epsilon, x + \epsilon) \cap F = \emptyset$. Hence no sequence in F converges to x, and so x is not a limit point of F. It now follows that F must contain all its limit points. ♣

Using Theorem 3D, the following two results follow immediately from Theorems 3A and 3B respectively.

THEOREM 3E. *The intersection of any collection of closed sets in $\mathbb{R}$ is closed.*

THEOREM 3F. *The union of any finite collection of closed sets in $\mathbb{R}$ is closed.*

PROOF OF THEOREMS 3E AND 3F. Note simply De Morgan's law, that

$$\bigcap_{F \in \mathcal{F}} F = \mathbb{R} \setminus \left(\bigcup_{F \in \mathcal{F}} (\mathbb{R} \setminus F) \right)$$

for any collection $\mathcal{F}$ of sets in $\mathbb{R}$. ♣

Our aim is to establish the following important result.

THEOREM 3G. (CANTOR INTERSECTION THEOREM) *Suppose that the sequence of sets $F_n \subseteq \mathbb{R}$ satisfies the following conditions:*
(a) For every $n \in \mathbb{N}$, $F_n \neq \emptyset$.
(b) For every $n \in \mathbb{N}$, $F_{n+1} \subseteq F_n$.
(c) For every $n \in \mathbb{N}$, F_n is closed.
(d) F_1 is bounded.
Then the intersection

$$\bigcap_{n=1}^{\infty} F_n$$

is closed and non-empty.

To prove Theorem 3G, we need some results on real sequences.

Definition. A sequence $x_n \in \mathbb{R}$ is said to be increasing if $x_{n+1} \ge x_n$ for every $n \in \mathbb{N}$. A sequence $x_n \in \mathbb{R}$ is said to be decreasing if $x_{n+1} \le x_n$ for every $n \in \mathbb{N}$. A sequence $x_n \in \mathbb{R}$ is said to be monotonic if it is increasing or decreasing.

THEOREM 3H. Consider a sequence $x_n \in \mathbb{R}$.
(a) Suppose that x_n is increasing and bounded above. Then x_n is convergent.
(b) Suppose that x_n is decreasing and bounded below. Then x_n is convergent.

Proof. We shall only prove (a), as the proof of (b) is similar. Since x_n is bounded above, let

$$M = \sup\{x_n : n \in \mathbb{N}\}.$$

We shall show that $x_n \to M$ as $n \to \infty$. Given any $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $x_N > M - \epsilon$. Since x_n is increasing, it follows that for every $n > N$, we have

$$M - \epsilon < x_N \le x_n \le M < M + \epsilon,$$

so that $|x_n - M| < \epsilon$. The result follows. ♣

Definition. Consider a sequence $x_n \in \mathbb{R}$. Suppose that $n_k \in \mathbb{N}$ for every $k \in \mathbb{N}$. Suppose further that

$$1 \le n_1 < n_2 < n_3 < \ldots < n_k < \ldots.$$

Then the sequence x_{n_k} is called a subsequence of the sequence x_n.

Example 3.1.7. The sequence of all even natural numbers is a subsequence of the sequence of all natural numbers. Here, note that $x_n = n$ for every $n \in \mathbb{N}$ and $n_k = 2k$ for every $k \in \mathbb{N}$.

Example 3.1.8. The sequence $3, 5, 7, 11, \ldots$ of all odd primes is a subsequence of the sequence $1, 3, 5, 7, \ldots$ of all odd natural numbers. Here $x_n = 2n - 1$ for every $n \in \mathbb{N}$. Also $x_{n_1} = 3 = x_2$, $x_{n_2} = 5 = x_3$, $x_{n_3} = 7 = x_4$, and so on, so that $n_1 = 2$, $n_2 = 3$, $n_3 = 4$, and so on.

THEOREM 3J. Any sequence $x_n \in \mathbb{R}$ has a monotonic subsequence.

Proof. We shall call $n \in \mathbb{N}$ a "peak" point if $x_m \le x_n$ for every $m \ge n$. Then there are two cases:

(a) There are infinitely many peak points $n_1 < n_2 < \ldots < n_k < \ldots$. Then clearly

$$x_{n_1} \ge x_{n_2} \ge \ldots \ge x_{n_k} \ge \ldots,$$

and we have a decreasing subsequence.

(b) There are finitely many or no peak points. In this case, let $n_1 = N + 1$ where N is the largest peak point, or $n_1 = 1$ if there are no peak points. Then n_1 is not a peak point, so there exists $n_2 > n_1$ such that $x_{n_2} > x_{n_1}$. Then n_2 is not a peak point, so there exists $n_3 > n_2$ such that $x_{n_3} > x_{n_2}$. Proceeding inductively, we conclude that there exists a sequence

$$n_1 < n_2 < \ldots < n_k < \ldots$$

of natural numbers such that

$$x_{n_1} < x_{n_2} < \ldots < x_{n_k} < \ldots,$$

and we have an increasing subsequence. ♣

THEOREM 3K. (BOLZANO-WEIERSTRASS THEOREM) *Any bounded sequence $x_n \in \mathbb{R}$ has a convergent subsequence.*

PROOF. By Theorem 3J, the sequence x_n has a monotonic subsequence. Clearly this subsequence is bounded. The result now follows from Theorem 3H. ♣

We can now prove the Cantor intersection theorem.

PROOF OF THEOREM 3G. The set

$$F = \bigcap_{n=1}^{\infty} F_n$$

is closed, in view of Theorem 3E. It remains to find a point $x \in F$. For every $n \in \mathbb{N}$, choose a point $x_n \in F_n$. The sequence x_n is clearly bounded, so it follows from the Bolzano-Weierstrass theorem that it has a convergent subsequence x_{n_k}. Suppose that $x_{n_k} \to x$ as $k \to \infty$. To show that $x \in F$, it suffices to show that $x \in F_n$ for every $n \in \mathbb{N}$. Note that in view of hypothesis (b), we have, for every $n \in \mathbb{N}$, that

$$x_n, x_{n+1}, x_{n+2}, \ldots \in F_n.$$

It follows that x is a limit point of F_n. Since F_n is closed, it follows that $x \in F_n$. This completes the proof of Theorem 3G. ♣

3.2. Sets of Measure Zero

Our study of the Lebesgue integral will depend crucially on the notion of sets of measure zero in $\mathbb{R}$.

DEFINITION. A set $S \subseteq \mathbb{R}$ is said to have measure zero if, for every $\epsilon > 0$, there exists a countable collection $\mathcal{C}$ of open intervals I such that

$$S \subseteq \bigcup_{I \in \mathcal{C}} I \qquad \text{and} \qquad \sum_{I \in \mathcal{C}} \mu(I) < \epsilon,$$

where, for every $I \in \mathcal{C}$, $\mu(I)$ denotes the length of the interval I. In other words, the set S can be covered by a countable union of open intervals of arbitrarily small total length.

REMARK. The argument in the remainder of this section depends on the use of a convergent series of positive terms. For the sake of convenience, we have chosen the series

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = 1.$$

In fact, the argument will work with any convergent series of positive terms. We do not even need to know its sum, except for the fact that it is finite and positive.

EXAMPLE 3.2.1. We shall show that the set $\mathbb{Q}$ has measure zero. Note that $\mathbb{Q}$ is countable, so that we can write

$$\mathbb{Q} = \{x_1, x_2, x_3, \ldots\}.$$

Let $\epsilon > 0$ be given. For every $n \in \mathbb{N}$, let

$$I_n = \left(x_n - \frac{\epsilon}{2^{n+2}}, x_n + \frac{\epsilon}{2^{n+2}} \right).$$

Then clearly

$$\mathbb{Q} \subseteq \bigcup_{n=1}^{\infty} I_n \qquad \text{and} \qquad \sum_{n=1}^{\infty} \mu(I_n) = \frac{\epsilon}{2} \sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{\epsilon}{2} < \epsilon.$$

In fact, we have all but proved the following result.

THEOREM 3L. *Every countable set in $\mathbb{R}$ has measure zero.*

A similar idea enables us to prove the following result.

THEOREM 3M. *A countable union of sets of measure zero in $\mathbb{R}$ has measure zero.*

PROOF. We shall show that a countably infinite union of sets of measure zero in $\mathbb{R}$ has measure zero. The case of a finite union needs only minor modification. Suppose that for every $n \in \mathbb{N}$, the set $S_n \subseteq \mathbb{R}$ has measure zero. Given any $\epsilon > 0$, there exists a countable collection $\mathcal{C}_n$ of open intervals I such that

$$S_n \subseteq \bigcup_{I \in \mathcal{C}_n} I \qquad \text{and} \qquad \sum_{I \in \mathcal{C}_n} \mu(I) < \frac{\epsilon}{2^n}.$$

Let

$$\mathcal{C} = \bigcup_{n=1}^{\infty} \mathcal{C}_n.$$

Then $\mathcal{C}$ is countable by Theorem 1E. Clearly

$$\bigcup_{n=1}^{\infty} S_n \subseteq \bigcup_{n=1}^{\infty} \bigcup_{I \in \mathcal{C}_n} I = \bigcup_{I \in \mathcal{C}} I \qquad \text{and} \qquad \sum_{I \in \mathcal{C}} \mu(I) \le \sum_{n=1}^{\infty} \sum_{I \in \mathcal{C}_n} \mu(I) < \epsilon \sum_{n=1}^{\infty} \frac{1}{2^n} = \epsilon.$$

The result follows. ♣

DEFINITION. A property $P(x)$ is said to hold for almost all $x \in S$ if $P(x)$ fails to hold for at most a set of measure zero in S.

3.3. Compact Sets

DEFINITION. A set $S \subseteq \mathbb{R}$ is said to be compact if and only if, for every collection $\mathcal{C}$ of open intervals I such that

$$S \subseteq \bigcup_{I \in \mathcal{C}} I,$$

there exists a finite subcollection $\mathcal{C}_0 \subseteq \mathcal{C}$ such that

$$S \subseteq \bigcup_{I \in \mathcal{C}_0} I.$$

In other words, every open covering of S can be achieved by a finite subcovering.

Our main task in this section is to establish the following important result.

THEOREM 3N. (HEINE-BOREL THEOREM) *Suppose that $F \subseteq \mathbb{R}$ is bounded and closed. Then F is compact.*

PROOF. We need to show that for every collection $\mathcal{C}$ of open intervals I such that

$$F \subseteq \bigcup_{I \in \mathcal{C}} I,$$

there exists a finite subcollection $\mathcal{C}_0 \subseteq \mathcal{C}$ such that

$$F \subseteq \bigcup_{I \in \mathcal{C}_0} I.$$

We shall achieve this by first (a) reducing $\mathcal{C}$ to a countable subcollection $\mathcal{C}' \subseteq \mathcal{C}$; and then (b) reducing $\mathcal{C}'$ to a finite subcollection $\mathcal{C}_0 \subseteq \mathcal{C}'$.

(a) Let $\mathcal{Q}$ denote the collection of all open intervals in $\mathbb{R}$ with rational midpoints and lengths. Then $\mathcal{Q}$ is countable (why?), so that we can write

$$\mathcal{Q} = \{J_1, J_2, J_3, \ldots\}.$$

Suppose that $x \in F$. Then there exists $I \in \mathcal{C}$ such that $x \in I$. It is easy to see that we can find an interval $J_{n(x)} \in \mathcal{Q}$ such that

$$x \in J_{n(x)} \subseteq I. \tag{1}$$

Clearly

$$F \subseteq \bigcup_{\substack{n=1 \\ n=n(x) \text{ for some } x \in F}}^{\infty} J_n.$$

For every $n \in \mathbb{N}$ for which $n = n(x)$ for some $x \in F$, we now find an interval $I_n \in \mathcal{C}$ for which $J_n \subseteq I_n$; this is possible in view of (1). Then

$$F \subseteq \bigcup_{\substack{n=1 \\ n=n(x) \text{ for some } x \in F}}^{\infty} I_n.$$

(b) Suppose that

$$F \subseteq \bigcup_{I \in \mathcal{C}'} I.$$

The result is immediate if $\mathcal{C}'$ is finite, so we assume, without loss of generality, that $\mathcal{C}'$ is countably infinite. We can therefore write

$$\mathcal{C}' = \{I_1, I_2, I_3, \ldots\},$$

so that

$$F \subseteq \bigcup_{k=1}^{\infty} I_k.$$

For every $n \in \mathbb{N}$, the set

$$G_n = \bigcup_{k=1}^{n} I_k$$

is open, in view of Theorem 3A. We shall show that there exists $n \in \mathbb{N}$ such that $F \subseteq G_n$. For every $n \in \mathbb{N}$, consider the set

$$F_n = F \cap (\mathbb{R} \setminus G_n).$$

To complete the proof, it clearly suffices to show that $F_n = \emptyset$ for some $n \in \mathbb{N}$. Suppose, on the contrary, that $F_n \neq \emptyset$ for every $n \in \mathbb{N}$. Note that for every $n \in \mathbb{N}$, the set F_n is closed and bounded. Furthermore, $F_{n+1} \subseteq F_n$ for every $n \in \mathbb{N}$. It follows from the Cantor intersection theorem that

$$\bigcap_{n=1}^{\infty} F_n \neq \emptyset.$$

Hence there exists $x \in F$ such that $x \notin I_k$ for every $k \in \mathbb{N}$, clearly a contradiction. ♣

Remark. Part (a) of Theorem 3N is sometimes known as the Lindelöf covering theorem.

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INTRODUCTION TO LEBESGUE INTEGRATION

W W L CHEN

Chapter 4

THE LEBESGUE INTEGRAL

4.1. Step Functions on an Interval

The first step in our definition of the Lebesgue integral concerns step functions. In this section, we formulate a definition of the Lebesgue integral for step functions in terms of Riemann integrals, and study some of its properties.

DEFINITION. Suppose that $A, B \in \mathbb{R}$ and $A < B$. A function $s : [A, B] \to \mathbb{R}$ is called a step function on $[A, B]$ if there exist a dissection $A = x_0 < x_1 < \ldots < x_n = B$ of $[A, B]$ and numbers $c_1, \ldots, c_n \in \mathbb{R}$ such that for every $k = 1, \ldots, n$, we have $s(x) = c_k$ for every $x \in (x_{k-1}, x_k)$.

REMARK. Note that we have not imposed any conditions on $s(x_k)$ for any $k = 0, 1, \ldots, n$, except that they are real-valued. This is in view of the fact that a Riemann integral is unchanged if we alter the value of the function at a finite number of points.

For every $k = 1, \ldots, n$, the integral

$$\int_{x_{k-1}}^{x_k} s(x)\,\mathrm{d}x = c_k(x_k - x_{k-1})$$

in the sense of Riemann. Also the integral

$$\int_A^B s(x)\,\mathrm{d}x = \sum_{k=1}^n c_k(x_k - x_{k-1}) \tag{1}$$

in the sense of Riemann, and is in fact independent of the choice of the dissection of $[A, B]$, provided that $s(x)$ is constant in any open subinterval arising from the dissection.

† This chapter was written at Imperial College, University of London, in 1977 while the author was an undergraduate.

We now make a simple generalization.

DEFINITION. Suppose that $I \subseteq \mathbb{R}$ is an interval. We say that a function $s : I \to \mathbb{R}$ is a step function on I, denoted by $s \in \mathcal{S}(I)$, if there exists a finite subinterval $(A, B) \subseteq I$ such that $s : [A, B] \to \mathbb{R}$ is a step function on $[A, B]$ and $s(x) = 0$ for every $x \in I \setminus [A, B]$. Furthermore, the integral

$$\int_I s(x)\,\mathrm{d}x \tag{2}$$

is defined by the integral of s over $[A, B]$ given by (1).

REMARKS. (1) Note that in the above definition, the function $s : I \to \mathbb{R}$ may not be defined at $x = A$ and/or $x = B$. In this case, we may assign $s(A)$ and $s(B)$ arbitrary finite values, and note that (1) is not affected by this process.

(2) Of course, the choice of the interval $[A, B]$ may not be unique. However, in view of the requirement that $s(x) = 0$ for every $x \in I \setminus [A, B]$, it is not difficult to see that the value of the integral (2) is independent of the choice of such $[A, B]$.

The following theorem can be deduced directly from the definitions.

THEOREM 4A. *Suppose that $I \subseteq \mathbb{R}$ is an interval, and that $s, t \in \mathcal{S}(I)$. Then*

(a) $s + t \in \mathcal{S}(I)$ and $\displaystyle\int_I (s(x) + t(x))\,\mathrm{d}x = \int_I s(x)\,\mathrm{d}x + \int_I t(x)\,\mathrm{d}x$;

(b) for every $c \in \mathbb{R}$, $cs \in \mathcal{S}(I)$ and $\displaystyle\int_I cs(x)\,\mathrm{d}x = c\int_I s(x)\,\mathrm{d}x$; and

(c) if $s(x) \le t(x)$ for every $x \in I$, then $\displaystyle\int_I s(x)\,\mathrm{d}x \le \int_I t(x)\,\mathrm{d}x$.

PROOF. (a) From the definition, there exist intervals $(A_1, B_1) \subseteq I$ and $(A_2, B_2) \subseteq I$ such that s and t are step functions on $[A_1, B_1]$ and $[A_2, B_2]$ respectively,

$$\int_I s(x)\,\mathrm{d}x = \int_{A_1}^{B_1} s(x)\,\mathrm{d}x \qquad \text{and} \qquad \int_I t(x)\,\mathrm{d}x = \int_{A_2}^{B_2} t(x)\,\mathrm{d}x,$$

and that $s(x) = 0$ for every $x \in I \setminus [A_1, B_1]$ and $t(x) = 0$ for every $x \in I \setminus [A_2, B_2]$. Furthermore, the integrals

$$\int_{A_1}^{B_1} s(x)\,\mathrm{d}x \qquad \text{and} \qquad \int_{A_2}^{B_2} t(x)\,\mathrm{d}x$$

are in the sense of Riemann. Now let $A = \min\{A_1, A_2\}$ and $B = \max\{B_1, B_2\}$. Then

$$(A_1, B_1) \subseteq (A, B) \subseteq I \qquad \text{and} \qquad (A_2, B_2) \subseteq (A, B) \subseteq I.$$

Furthermore, it is easy to see that both s and t are step functions in $[A, B]$, and that $s(x) = t(x) = 0$ for every $x \in I \setminus [A, B]$. Hence

$$\int_I s(x)\,\mathrm{d}x = \int_A^B s(x)\,\mathrm{d}x \qquad \text{and} \qquad \int_I t(x)\,\mathrm{d}x = \int_A^B t(x)\,\mathrm{d}x. \tag{3}$$

Note also that the integrals

$$\int_A^B s(x)\,\mathrm{d}x \qquad \text{and} \qquad \int_A^B t(x)\,\mathrm{d}x$$

are in the sense of Riemann. On the other hand, it is easily checked that $s + t$ is a step function on $[A, B]$, and that $s(x) + t(x) = 0$ for every $x \in I \setminus [A, B]$. By definition, we have

$$\int_I (s(x) + t(x))\,\mathrm{d}x = \int_A^B (s(x) + t(x))\,\mathrm{d}x. \tag{4}$$

Note, however, that

$$\int_A^B (s(x)+t(x))\,\mathrm{d}x = \int_A^B s(x)\,\mathrm{d}x + \int_A^B t(x)\,\mathrm{d}x, \tag{5}$$

where the integrals in (5) are in the sense of Riemann. The result now follows on combining (3)–(5).

(b) From the definition, there exists an interval $(A,B)\subseteq I$ such that s is a step function on $[A,B]$,

$$\int_I s(x)\,\mathrm{d}x = \int_A^B s(x)\,\mathrm{d}x, \tag{6}$$

and that $s(x)=0$ for every $x\in I\setminus[A,B]$. Furthermore, the integral

$$\int_A^B s(x)\,\mathrm{d}x$$

is in the sense of Riemann. It is easy to see that cs is a step function on $[A,B]$, and that $cs(x)=0$ for every $x\in I\setminus[A,B]$. By definition, we have

$$\int_I cs(x)\,\mathrm{d}x = \int_A^B cs(x)\,\mathrm{d}x. \tag{7}$$

Note, however, that

$$\int_A^B cs(x)\,\mathrm{d}x = c\int_A^B s(x)\,\mathrm{d}x, \tag{8}$$

where the integrals in (8) are in the sense of Riemann. The result now follows on combining (6)–(8).

(c) We follow the argument in part (a) and note, instead, that

$$\int_A^B s(x)\,\mathrm{d}x \le \int_A^B t(x)\,\mathrm{d}x, \tag{9}$$

where the integrals in (9) are in the sense of Riemann. The result now follows on combining (3) and (9). ♣

THEOREM 4B. *Suppose that the interval $I\subseteq\mathbb{R}$ can be written in the form $I=I_1\cup I_2$, where the intervals I_1 and I_2 have no interior points in common. Suppose further that $s\in\mathcal{S}(I)$. Then*

$$\int_I s(x)\,\mathrm{d}x = \int_{I_1} s(x)\,\mathrm{d}x + \int_{I_2} s(x)\,\mathrm{d}x.$$

PROOF. For $j=1,2$, let $\chi_j : I\to\mathbb{R}$ denote the characteristic function of the interval I_j. Then $s(x)=s(x)\chi_1(x)+s(x)\chi_2(x)$ for every $x\in I$, apart from possibly a finite number of exceptions (which do not affect the values of the integrals). Note now that $s(x)\chi_j(x)$ is a step function on I_1, I_2 and I, and that $s(x)\chi_j(x)=0$ for every $x\in I\setminus I_j$. Furthermore, $s(x)\chi_j(x)=s(x)$ for every $x\in I_j$. It follows that

$$\int_I s(x)\,\mathrm{d}x = \int_I (s(x)\chi_1(x)+s(x)\chi_2(x))\,\mathrm{d}x = \int_I s(x)\chi_1(x)\,\mathrm{d}x + \int_I s(x)\chi_2(x)\,\mathrm{d}x = \int_{I_1} s(x)\,\mathrm{d}x + \int_{I_2} s(x)\,\mathrm{d}x$$

as required. ♣

4.2. Upper Functions on an Interval

The second step in our definition of the Lebesgue integral concerns extending the definition of the Lebesgue integral for step functions to a larger collection which we shall call the upper functions. In this section, we formulate a definition of the Lebesgue integral for upper functions by studying sequences of step functions, and study some of its properties.

Definition. Suppose that $S \subseteq \mathbb{R}$. A sequence of functions $f_n : S \to \mathbb{R}$ is said to be increasing on S if $f_{n+1}(x) \geq f_n(x)$ for every $n \in \mathbb{N}$ and every $x \in S$. A sequence of functions $f_n : S \to \mathbb{R}$ is said to be decreasing on S if $f_{n+1}(x) \leq f_n(x)$ for every $n \in \mathbb{N}$ and every $x \in S$.

Definition. Suppose that $u : I \to \mathbb{R}$ is a function defined on an interval $I \subseteq \mathbb{R}$. Suppose further that there exists a sequence of step functions $s_n \in \mathcal{S}(I)$ satisfying the following conditions:
(a) The sequence $s_n : I \to \mathbb{R}$ is increasing on I.
(b) $s_n(x) \to u(x)$ as $n \to \infty$ for almost all $x \in I$.
(c) $\displaystyle\lim_{n\to\infty} \int_I s_n(x)\,\mathrm{d}x$ exists.

Then we say that the sequence of step functions $s_n \in \mathcal{S}(I)$ generates u, and that u is an upper function on I, denoted by $u \in \mathcal{U}(I)$. Furthermore, we define the integral of u over I by

$$\int_I u(x)\,\mathrm{d}x = \lim_{n\to\infty} \int_I s_n(x)\,\mathrm{d}x. \tag{10}$$

The validity of the definition is justified by the following result.

THEOREM 4C. *Suppose that $I \subseteq \mathbb{R}$ is an interval, and that $u \in \mathcal{U}(I)$. Suppose further that both sequences $s_n \in \mathcal{S}(I)$ and $t_n \in \mathcal{S}(I)$ generate u. Then*

$$\lim_{n\to\infty} \int_I s_n(x)\,\mathrm{d}x = \lim_{n\to\infty} \int_I t_n(x)\,\mathrm{d}x.$$

Theorem 4C is a simple consequence of the following result on step functions.

THEOREM 4D. *Suppose that $I \subseteq \mathbb{R}$ is an interval. Suppose further that the sequence $t_n \in \mathcal{S}(I)$ satisfies the following conditions:*
(a) The sequence $t_n : I \to \mathbb{R}$ is increasing on I.
(b) There exists a function $u : I \to \mathbb{R}$ such that $t_n(x) \to u(x)$ as $n \to \infty$ for almost all $x \in I$.
(c) $\displaystyle\lim_{n\to\infty} \int_I t_n(x)\,\mathrm{d}x$ exists.

Then for any $t \in \mathcal{S}(I)$ satisfying $t(x) \leq u(x)$ for almost all $x \in I$, we have

$$\int_I t(x)\,\mathrm{d}x \leq \lim_{n\to\infty} \int_I t_n(x)\,\mathrm{d}x.$$

Proof of Theorem 4C. Note that the sequence of step functions $t_n : I \to \mathbb{R}$ satisfies hypotheses (a) and (c) of Theorem 4D. Furthermore, since this sequence generates u, it follows that hypothesis (b) of Theorem 4D is satisfied. On the other hand, for every $m \in \mathbb{N}$, it is easy to see that $s_n(x) \leq u(x)$ for almost all $x \in I$. It now follows from Theorem 4D that for every $m \in \mathbb{N}$, we have

$$\int_I s_m(x)\,\mathrm{d}x \leq \lim_{n\to\infty} \int_I t_n(x)\,\mathrm{d}x,$$

and so on letting $m \to \infty$, we have

$$\lim_{m\to\infty} \int_I s_m(x)\,\mathrm{d}x \leq \lim_{n\to\infty} \int_I t_n(x)\,\mathrm{d}x$$

(note here that m and n are "dummy" variables). Reversing the roles of the two sequences, the opposite inequality

$$\lim_{n\to\infty}\int_I t_n(x)\,\mathrm{d}x \le \lim_{m\to\infty}\int_I s_m(x)\,\mathrm{d}x$$

can be established by a similar argument. The result follows immediately. ♣

The main part of the proof of Theorem 4D can be summarized by the following result.

THEOREM 4E. *Suppose that $I \subseteq \mathbb{R}$ is an interval. Suppose further that the sequence $s_n \in \mathcal{S}(I)$ satisfies the following conditions:*
(a) The sequence $s_n : I \to \mathbb{R}$ is decreasing on I.
(b) $s_n(x) \ge 0$ for every $n \in \mathbb{N}$ and every $x \in I$.
(c) $s_n(x) \to 0$ as $n \to \infty$ for almost all $x \in I$.
Then

$$\lim_{n\to\infty}\int_I s_n(x)\,\mathrm{d}x = 0.$$

Proof. Since $s_1 \in \mathcal{S}(I)$, there exists $(A,B) \subseteq I$ such that $s_1(x) = 0$ for every $x \in I \setminus [A,B]$. For every $n \in \mathbb{N}$ and every $x \in I$, we clearly have $0 \le s_n(x) \le s_1(x)$, and so $s_n(x) = 0$ for every $x \in I \setminus [A,B]$. Since $s_n \in \mathcal{S}(I)$, it is a step function on $[A,B]$, and

$$\int_I s_n(x)\,\mathrm{d}x = \int_A^B s_n(x)\,\mathrm{d}x, \qquad (11)$$

where the integral on the right hand side is in the sense of Riemann. Furthermore, there exists a dissection Δ_n of $[A,B]$ such that $s_n(x)$ is constant in any open subinterval arising from Δ_n. Let

$$\mathcal{D} = \bigcup_{n=1}^{\infty} \Delta_n$$

represent the collection of all dissection points. Since Δ_n is a finite set for every $n \in \mathbb{N}$, it follows that $\mathcal{D}$ is countable, and so has measure 0. Next, let

$$\mathcal{E} = \{x \in I : s_n(x) \not\to 0 \text{ as } n \to \infty\}$$

denote the set of exceptional points of non-convergence. By (c), $\mathcal{E}$ also has measure 0, so that the set

$$\mathcal{F} = \mathcal{D} \cup \mathcal{E}$$

has measure 0. Let $\epsilon > 0$ be given and fixed. Then there exists a countable collection of open intervals $\mathcal{F}_k$, where $k \in \mathcal{K}$, of total length less than ϵ, such that

$$\mathcal{F} \subseteq \bigcup_{k\in\mathcal{K}} \mathcal{F}_k.$$

Suppose now that $y \in [A,B] \setminus \mathcal{F}$. On the one hand, since $y \notin \mathcal{E}$, it follows that $s_n(y) \to 0$ as $n \to \infty$, so that there exists $N = N(y)$ such that $s_N(y) < \epsilon$. On the other hand, since $y \notin \mathcal{D}$, it follows that there is an open interval $\mathcal{I}(y)$ such that $y \in \mathcal{I}(y)$ and $s_N(x)$ is constant in $\mathcal{I}(y)$, so that $s_N(x) < \epsilon$ for every $x \in \mathcal{I}(y)$. Clearly the open intervals $\mathcal{I}(y)$, as y runs over $[A,B] \setminus \mathcal{F}$, together with the open intervals $\mathcal{F}_k$, where $k \in \mathcal{K}$, form an open covering of $[A,B]$. Since $[A,B]$ is compact, there is a finite subcovering

$$[A,B] \subseteq \left(\bigcup_{i=1}^{p} \mathcal{I}(y_i)\right) \cup \left(\bigcup_{j=1}^{q} \mathcal{F}_j\right).$$

Let $N_0 = \max\{N(y_1), \ldots, N(y_p)\}$. In view of (a), we clearly have

$$s_n(x) < \epsilon \qquad \text{for every } n > N_0 \text{ and } x \in \bigcup_{i=1}^{p} \mathcal{I}(y_i). \tag{12}$$

Write

$$\mathcal{T}_1 = \bigcup_{j=1}^{q} \mathcal{F}_j \qquad \text{and} \qquad \mathcal{T}_2 = [A, B] \setminus \mathcal{T}_1,$$

and note that both can be written as finite unions of disjoint intervals. For every $n \in \mathbb{N}$, since $s_n(x) = 0$ outside $[A, B]$, it follows that

$$\int_A^B s_n(x)\,\mathrm{d}x = \int_{\mathcal{T}_1} s_n(x)\,\mathrm{d}x + \int_{\mathcal{T}_2} s_n(x)\,\mathrm{d}x, \tag{13}$$

where all the integrals are in the sense of Riemann. We now estimate each of the integrals on the right hand side of (13). To estimate the integral over $\mathcal{T}_1$, let M denote an upper bound of $s_1(x)$ on $[A, B]$. Then $s_n(x) \le M$ for every $x \in \mathcal{T}_1$ (why?). On the other hand, note that the intervals $\mathcal{F}_k$ have total length less than ϵ. Hence

$$\int_{\mathcal{T}_1} s_n(x)\,\mathrm{d}x \le M\epsilon. \tag{14}$$

To estimate the integral over $\mathcal{T}_2$, note that

$$\mathcal{T}_2 \subseteq \bigcup_{i=1}^{p} \mathcal{I}(y_i).$$

It follows from (12) that $s_n(x) < \epsilon$ for every $n > N_0$ and $x \in \mathcal{T}_2$. On the other hand, note that $\mathcal{T}_2 \subseteq [A, B]$. Hence for every $n > N_0$,

$$\int_{\mathcal{T}_2} s_n(x)\,\mathrm{d}x \le \epsilon(B - A). \tag{15}$$

Combining (11) and (13)–(15), we conclude that for every $n > N_0$,

$$\int_I s_n(x)\,\mathrm{d}x \le (M + B - A)\epsilon.$$

The result follows. ♣

Proof of Theorem 4D. For every $n \in \mathbb{N}$ and every $x \in I$, write $s_n(x) = \max\{t(x) - t_n(x), 0\}$. Clearly $s_n(x) \ge 0$ for every $x \in I$. Since $t, t_n \in \mathcal{S}(I)$, it follows that $s_n \in \mathcal{S}(I)$. Since the sequence t_n is increasing on I, it follows that the sequence s_n is decreasing on I. Finally, since $t_n(x) \to u(x)$ as $n \to \infty$ for almost all $x \in I$, it follows that $s_n(x) \to \max\{t(x) - u(x), 0\}$ for almost all $x \in I$. It now follows from Theorem 4E that

$$\lim_{n \to \infty} \int_I s_n(x)\,\mathrm{d}x = 0. \tag{16}$$

On the other hand, clearly $s_n(x) \ge t(x) - t_n(x)$ for every $n \in \mathbb{N}$ and $x \in I$. It follows from Theorem 4A that

$$\int_I s_n(x)\,\mathrm{d}x \ge \int_I t(x)\,\mathrm{d}x - \int_I t_n(x)\,\mathrm{d}x. \tag{17}$$

The result now follows on letting $n \to \infty$ in (17) and combining with (16). ♣

Corresponding to Theorem 4A, we have the following result.

THEOREM 4F. *Suppose that $I \subseteq \mathbb{R}$ is an interval, and that $u, v \in \mathcal{U}(I)$. Then*
(a) $u+v \in \mathcal{U}(I)$ *and* $\displaystyle\int_I (u(x)+v(x))\,\mathrm{d}x = \int_I u(x)\,\mathrm{d}x + \int_I v(x)\,\mathrm{d}x$;
(b) *for every non-negative* $c \in \mathbb{R}$, $cu \in \mathcal{U}(I)$ *and* $\displaystyle\int_I cu(x)\,\mathrm{d}x = c\int_I u(x)\,\mathrm{d}x$;
(c) *if* $u(x) \le v(x)$ *for almost all* $x \in I$, *then* $\displaystyle\int_I u(x)\,\mathrm{d}x \le \int_I v(x)\,\mathrm{d}x$; *and*
(d) *if* $u(x) = v(x)$ *for almost all* $x \in I$, *then* $\displaystyle\int_I u(x)\,\mathrm{d}x = \int_I v(x)\,\mathrm{d}x$.

PROOF. Since $u, v \in \mathcal{U}(I)$, there exist increasing sequences $s_n \in \mathcal{S}(I)$ and $t_n \in \mathcal{S}(I)$ of step functions such that $s_n(x) \to u(x)$ and $t_n(x) \to v(x)$ as $n \to \infty$ for almost all $x \in I$, and that

$$\int_I u(x)\,\mathrm{d}x = \lim_{n\to\infty}\int_I s_n(x)\,\mathrm{d}x \qquad \text{and} \qquad \int_I v(x)\,\mathrm{d}x = \lim_{n\to\infty}\int_I t_n(x)\,\mathrm{d}x. \tag{18}$$

It follows that s_n+t_n and cs_n for any $c \ge 0$ are increasing sequences of step functions on I. Furthermore, $s_n(x)+t_n(x) \to u(x)+v(x)$ and $cs_n(x) \to cu(x)$ as $n \to \infty$ for almost all $x \in I$. By definition, we have

$$\int_I (u(x)+v(x))\,\mathrm{d}x = \lim_{n\to\infty}\int_I (s_n(x)+t_n(x))\,\mathrm{d}x \qquad \text{and} \qquad \int_I cu(x)\,\mathrm{d}x = \lim_{n\to\infty}\int_I cs_n(x)\,\mathrm{d}x, \tag{19}$$

provided that the limits exist. In view of Theorem 4A, we have, for every $n \in \mathbb{N}$, that

$$\int_I (s_n(x)+t_n(x))\,\mathrm{d}x = \int_I s_n(x)\,\mathrm{d}x + \int_I t_n(x)\,\mathrm{d}x \qquad \text{and} \qquad \int_I cs_n(x)\,\mathrm{d}x = c\int_I s_n(x)\,\mathrm{d}x. \tag{20}$$

(a) and (b) now follow on letting $n \to \infty$ in (20) and combining with (18) and (19). To prove (c), note that for every $m \in \mathbb{N}$, we have

$$s_m(x) \le u(x) \le v(x) = \lim_{n\to\infty} t_n(x)$$

for almost all $x \in I$. It follows from Theorem 4D that

$$\int_I s_m(x)\,\mathrm{d}x \le \lim_{n\to\infty}\int_I t_n(x)\,\mathrm{d}x = \int_I v(x)\,\mathrm{d}x.$$

(c) now follows on letting $m \to \infty$. To prove (d), note that we clearly have $u(x) \le v(x)$ and $v(x) \le u(x)$ for almost all $x \in I$. It follows from (c) that

$$\int_I u(x)\,\mathrm{d}x \le \int_I v(x)\,\mathrm{d}x \qquad \text{and} \qquad \int_I v(x)\,\mathrm{d}x \le \int_I u(x)\,\mathrm{d}x.$$

Equality therefore must hold. ♣

DEFINITION. Suppose that $S \subseteq \mathbb{R}$. For functions $f : S \to \mathbb{R}$ and $g : S \to \mathbb{R}$, we define the maximum and minimum functions $\max\{f,g\} : S \to \mathbb{R}$ and $\min\{f,g\} : S \to \mathbb{R}$ by writing

$$\max\{f,g\}(x) = \max\{f(x),g(x)\} \qquad \text{and} \qquad \min\{f,g\}(x) = \min\{f(x),g(x)\}$$

for every $x \in S$.

THEOREM 4G. *Suppose that $I \subseteq \mathbb{R}$ is an interval, and that $u, v \in \mathcal{U}(I)$. Then so are* $\max\{u,v\}$ *and* $\min\{u,v\}$.

PROOF. Since $u, v \in \mathcal{U}(I)$, there exist increasing sequences $s_n \in \mathcal{S}(I)$ and $t_n \in \mathcal{S}(I)$ of step functions such that $s_n(x) \to u(x)$ and $t_n(x) \to v(x)$ as $n \to \infty$ for almost all $x \in I$. It is easy to see that

$a_n = \max\{s_n, t_n\}$ and $b_n = \min\{s_n, t_n\}$ are increasing sequences of step functions on I, and that $a_n(x) \to \max\{u, v\}(x)$ and $b_n(x) \to \min\{u, v\}(x)$ as $n \to \infty$ for almost all $x \in I$. It remains to show that both sequences

$$\int_I a_n(x)\,\mathrm{d}x \qquad \text{and} \qquad \int_I b_n(x)\,\mathrm{d}x$$

are convergent. To establish the convergence of the sequence

$$\int_I b_n(x)\,\mathrm{d}x, \tag{21}$$

note that it is increasing. On the other hand, for every $n \in \mathbb{N}$, we have $b_n(x) \le s_n(x) \le u(x)$ for almost all $x \in I$. It follows from Theorem 4F(c) that

$$\int_I b_n(x)\,\mathrm{d}x \le \int_I u(x)\,\mathrm{d}x,$$

so that (21) is bounded above. Finally, it is not difficult to check that for every $n \in \mathbb{N}$, we have $a_n + b_n = s_n + t_n$, so that $a_n = s_n + t_n - b_n$. It follows from Theorem 4A that

$$\int_I a_n(x)\,\mathrm{d}x = \int_I s_n(x)\,\mathrm{d}x + \int_I t_n(x)\,\mathrm{d}x - \int_I b_n(x)\,\mathrm{d}x. \tag{22}$$

The convergence of the left hand side of (22) follows immediately from the convergence of the right hand side. ♣

Corresponding to Theorem 4B, we have the following result.

THEOREM 4H. *Suppose that the interval $I \subseteq \mathbb{R}$ can be written in the form $I = I_1 \cup I_2$, where the intervals I_1 and I_2 have no interior points in common. Suppose further that $u \in \mathcal{U}(I)$, and that $u(x) \ge 0$ for almost all $x \in I$. Then $u \in \mathcal{U}(I_1)$ and $u \in \mathcal{U}(I_2)$, and*

$$\int_I u(x)\,\mathrm{d}x = \int_{I_1} u(x)\,\mathrm{d}x + \int_{I_2} u(x)\,\mathrm{d}x.$$

This is complemented by the following result.

THEOREM 4J. *Suppose that the interval $I \subseteq \mathbb{R}$ can be written in the form $I = I_1 \cup I_2$, where the intervals I_1 and I_2 have no interior points in common. Suppose further that $u_1 \in \mathcal{U}(I_1)$ and $u_2 \in \mathcal{U}(I_2)$. Define the function $u : I \to \mathbb{R}$ by*

$$u(x) = \begin{cases} u_1(x) & \text{if } x \in I_1, \\ u_2(x) & \text{if } x \in I \setminus I_1. \end{cases}$$

Then $u \in \mathcal{U}(I)$, and

$$\int_I u(x)\,\mathrm{d}x = \int_{I_1} u_1(x)\,\mathrm{d}x + \int_{I_2} u_2(x)\,\mathrm{d}x.$$

Proof of Theorem 4H. Since $u \in \mathcal{U}(I)$, there exists an increasing sequence $s_n \in \mathcal{S}(I)$ of step functions such that $s_n(x) \to u(x)$ as $n \to \infty$ for almost all $x \in I$. Since $u(x) \ge 0$ for almost all $x \in I$, it is easy to see that $s_n^+ = \max\{s_n, 0\}$ is an increasing sequence of step functions on I, and that $s_n^+(x) \to u(x)$ for almost all $x \in I$. It follows that for every subinterval $J \subseteq I$, s_n^+ is an increasing sequence of step functions on J, and $s_n^+(x) \to u(x)$ for almost all $x \in J$. To show that $u \in \mathcal{U}(J)$, it remains to show that the sequence

$$\int_J s_n^+(x)\,\mathrm{d}x \tag{23}$$

is convergent. This follows easily on noting that the sequence (23) is increasing, and that

$$\int_J s_n^+(x)\,\mathrm{d}x \le \int_I s_n^+(x)\,\mathrm{d}x \le \int_I u(x)\,\mathrm{d}x,$$

so that it is bounded above. This proves that $u \in \mathcal{U}(I_1)$ and $u \in \mathcal{U}(I_2)$. To complete the proof, note that for every $n \in \mathbb{N}$, we have

$$\int_I s_n^+(x)\,\mathrm{d}x = \int_{I_1} s_n^+(x)\,\mathrm{d}x + \int_{I_2} s_n^+(x)\,\mathrm{d}x,$$

in view of Theorem 4B. The result now follows on letting $n \to \infty$. ♣

Proof of Theorem 4J. Since $u_1 \in \mathcal{U}(I_1)$, there exists an increasing sequence s_n of step functions on I_1 such that $s_n(x) \to u_1(x)$ as $n \to \infty$ for almost all $x \in I_1$. Since $u_2 \in \mathcal{U}(I_2)$, there exists an increasing sequence t_n of step functions on I_2 such that $t_n(x) \to u_2(x)$ as $n \to \infty$ for almost all $x \in I_2$. For every $n \in \mathbb{N}$, define the function $a_n : I \to \mathbb{R}$ by writing

$$a_n(x) = \begin{cases} s_n(x) & \text{if } x \in I_1, \\ t_n(x) & \text{if } x \in I \setminus I_1. \end{cases}$$

It is easy to see that a_n is an increasing sequence of step functions on I, and that $a_n(x) \to u(x)$ as $n \to \infty$ for almost all $x \in I$. This proves that $u \in \mathcal{U}(I)$. To complete the proof, note that for every $n \in \mathbb{N}$, we have

$$\int_I a_n(x)\,\mathrm{d}x = \int_{I_1} s_n(x)\,\mathrm{d}x + \int_{I_2} t_n(x)\,\mathrm{d}x,$$

noting that $a_n(x) = t_n(x)$ for almost all $x \in I_2$. The result now follows on letting $n \to \infty$. ♣

4.3. Lebesgue Integrable Functions on an Interval

The final step in our definition of the Lebesgue integral concerns extending the definition of the Lebesgue integral for upper functions to a larger collection which we shall call the Lebesgue integrable functions.

Definition. Suppose that $f : I \to \mathbb{R}$ is a function defined on an interval $I \subseteq \mathbb{R}$. Suppose further that there exist upper functions $u : I \to \mathbb{R}$ and $v : I \to \mathbb{R}$ on I such that $f(x) = u(x) - v(x)$ for all $x \in I$. Then we say that f is a Lebesgue integrable function on I, denoted by $f \in \mathcal{L}(I)$. We also say that f is Lebesgue integrable over I, and define the integral of f over I by

$$\int_I f(x)\,\mathrm{d}x = \int_I u(x)\,\mathrm{d}x - \int_I v(x)\,\mathrm{d}x.$$

The validity of the definition is justified by the following simple result. The proof is left as an exercise.

THEOREM 4K. *Suppose that $I \subseteq \mathbb{R}$ is an interval. Suppose further that $u_1, v_1, u_2, v_2 \in \mathcal{U}(I)$, and that $u_1(x) - v_1(x) = u_2(x) - v_2(x)$ for every $x \in I$. Then*

$$\int_I u_1(x)\,\mathrm{d}x - \int_I v_1(x)\,\mathrm{d}x = \int_I u_2(x)\,\mathrm{d}x - \int_I v_2(x)\,\mathrm{d}x.$$

Corresponding to Theorems 4A and 4F, we have the following result. The proof is left as an exercise.

THEOREM 4L. *Suppose that $I \subseteq \mathbb{R}$ is an interval, and that $f, g \in \mathcal{L}(I)$. Then*

(a) $f + g \in \mathcal{L}(I)$ *and* $\displaystyle\int_I (f(x) + g(x))\,\mathrm{d}x = \int_I f(x)\,\mathrm{d}x + \int_I g(x)\,\mathrm{d}x$;

(b) *for every* $c \in \mathbb{R}$, $cf \in \mathcal{L}(I)$ *and* $\displaystyle\int_I cf(x)\,\mathrm{d}x = c\int_I f(x)\,\mathrm{d}x$;

(c) *if* $f(x) \geq 0$ *for almost all* $x \in I$, *then* $\displaystyle\int_I f(x)\,\mathrm{d}x \geq 0$;

(d) *if* $f(x) \geq g(x)$ *for almost all* $x \in I$, *then* $\displaystyle\int_I f(x)\,\mathrm{d}x \geq \int_I g(x)\,\mathrm{d}x$; *and*

(e) *if* $f(x) = g(x)$ *for almost all* $x \in I$, *then* $\displaystyle\int_I f(x)\,\mathrm{d}x = \int_I g(x)\,\mathrm{d}x$.

We now investigate some further properties of the Lebesgue integral.

THEOREM 4M. *Suppose that $I \subseteq \mathbb{R}$ is an interval, and that $f \in \mathcal{L}(I)$. Then so are $f^+ = \max\{f, 0\}$, $f^- = \max\{-f, 0\}$ and $|f|$. Furthermore,*

$$\left|\int_I f(x)\,\mathrm{d}x\right| \leq \int_I f(x)\,\mathrm{d}x. \tag{24}$$

PROOF. There exist $u, v \in \mathcal{U}(I)$ such that $f(x) = u(x) - v(x)$ for all $x \in I$. Then

$$f^+ = \max\{u - v, 0\} = \max\{u, v\} - v.$$

By Theorem 4G, $\max\{u, v\} \in \mathcal{U}(I)$. It follows that $f^+ \in \mathcal{L}(I)$. By Theorem 4L(a)(b), we also have $f^- = f^+ - f \in \mathcal{L}(I)$ and $|f| = f^+ + f^- \in \mathcal{L}(I)$. On the other hand, we have $-|f(x)| \leq f(x) \leq |f(x)|$ for every $x \in I$. The inequality (24) now follows from Theorem 4L(d). ♣

THEOREM 4N. *Suppose that $I \subseteq \mathbb{R}$ is an interval, and that $f, g \in \mathcal{L}(I)$. Then so are $\max\{f, g\}$, $\min\{f, g\}$.*

PROOF. Note that

$$\max\{f, g\} = \frac{f + g + |f - g|}{2} \qquad \text{and} \qquad \min\{f, g\} = \frac{f + g - |f - g|}{2}.$$

The result now follows from Theorem 4L(a)(b). ♣

Corresponding to Theorems 4H and 4J, we have the following two results. The proofs are left as exercises.

THEOREM 4P. *Suppose that the interval $I \subseteq \mathbb{R}$ can be written in the form $I = I_1 \cup I_2$, where the intervals I_1 and I_2 have no interior points in common. Suppose further that $f \in \mathcal{L}(I)$. Then $f \in \mathcal{L}(I_1)$ and $f \in \mathcal{L}(I_2)$, and*

$$\int_I f(x)\,\mathrm{d}x = \int_{I_1} f(x)\,\mathrm{d}x + \int_{I_2} f(x)\,\mathrm{d}x.$$

THEOREM 4Q. *Suppose that the interval $I \subseteq \mathbb{R}$ can be written in the form $I = I_1 \cup I_2$, where the intervals I_1 and I_2 have no interior points in common. Suppose further that $f_1 \in \mathcal{L}(I_1)$ and $f_2 \in \mathcal{L}(I_2)$. Define the function $f : I \to \mathbb{R}$ by*

$$f(x) = \begin{cases} f_1(x) & \text{if } x \in I_1, \\ f_2(x) & \text{if } x \in I \setminus I_1. \end{cases}$$

Then $f \in \mathcal{L}(I)$, and

$$\int_I f(x)\,\mathrm{d}x = \int_{I_1} f_1(x)\,\mathrm{d}x + \int_{I_2} f_2(x)\,\mathrm{d}x.$$

We conclude this section by proving the following two results which are qualitative statements concerning the approximation of a Lebesgue integrable function by an upper function and by a step function respectively.

THEOREM 4R. *Suppose that $I \subseteq \mathbb{R}$ is an interval, and that $f \in \mathcal{L}(I)$. Then for every $\epsilon > 0$, there exist $u, v \in \mathcal{U}(I)$ satisfying the following conditions:*
(a) $f(x) = u(x) - v(x)$ for every $x \in I$;
(b) $v(x) \geq 0$ for almost all $x \in I$; and
(c) $\displaystyle\int_I v(x)\,\mathrm{d}x < \epsilon$.

PROOF. There exist $u_1, v_1 \in \mathcal{U}(I)$ such that $f = u_1 - v_1$ on I. Suppose that v_1 is generated by the sequence of step functions $t_n \in \mathcal{S}(I)$. Since

$$\int_I v_1(x)\,\mathrm{d}x = \lim_{n\to\infty} \int_I t_n(x)\,\mathrm{d}x,$$

it follows that there exists $N \in \mathbb{N}$ such that

$$0 \leq \int_I (v_1(x) - t_N(x))\,\mathrm{d}x = \left| \int_I v_1(x)\,\mathrm{d}x - \int_I t_N(x)\,\mathrm{d}x \right| < \epsilon.$$

Let $u = u_1 - t_N$ and $v = v_1 - t_N$ on I. Then it is easy to see that $u, v \in \mathcal{U}(I)$. Also (a) and (c) follow immediately. To show (b), note that the sequence t_n is increasing, and that $t_n(x) \to v_1(x)$ as $n \to \infty$ for almost all $x \in I$. ♣

THEOREM 4S. *Suppose that $I \subseteq \mathbb{R}$ is an interval, and that $f \in \mathcal{L}(I)$. Then for every $\epsilon > 0$, there exist $s \in \mathcal{S}(I)$ and $g \in \mathcal{L}(I)$ satisfying the following conditions:*
(a) $f(x) = s(x) + g(x)$ for every $x \in I$;and
(b) $\displaystyle\int_I |g(x)|\,\mathrm{d}x < \epsilon$.

PROOF. By Theorem 4R, there exist $u, v \in \mathcal{U}(I)$ such that $f = u - v$ on I, $v(x) \geq 0$ for almost all $x \in I$, and

$$0 \leq \int_I v(x)\,\mathrm{d}x < \frac{\epsilon}{2}. \tag{25}$$

Suppose that u is generated by the sequence of step functions $s_n \in \mathcal{S}(I)$. Since

$$\int_I u(x)\,\mathrm{d}x = \lim_{n\to\infty} \int_I s_n(x)\,\mathrm{d}x,$$

it follows that there exists $N \in \mathbb{N}$ such that

$$0 \leq \int_I (u(x) - s_N(x))\,\mathrm{d}x = \left| \int_I u(x)\,\mathrm{d}x - \int_I s_N(x)\,\mathrm{d}x \right| < \frac{\epsilon}{2}. \tag{26}$$

Let $s = s_N$ and $g = u - (v + s_N)$ on I. Clearly $s \in \mathcal{S}(I)$ and $g = \mathcal{L}(I)$. Also (a) follows immediately. On the other hand, we have

$$|g(x)| \leq |u(x) - s_N(x)| + |v(x)| = (u(x) - s_N(x)) + v(x)$$

for almost all $x \in I$. It follows from Theorem 4L, (25) and (26) that

$$\int_I |g(x)| \,\mathrm{d}x \leq \int_I (u(x) - s_N(x) + v(x)) \,\mathrm{d}x = \int_I (u(x) - s_N(x)) \,\mathrm{d}x + \int_I v(x) \,\mathrm{d}x < \epsilon.$$

This gives (b). ♣

4.4. Sets of Measure Zero

In this section, we shall show that the behaviour of a Lebesgue integrable function on a set of measure zero does not affect the integral. More precisely, we prove the following result.

THEOREM 4T. *Suppose that $I \subseteq \mathbb{R}$ is an interval, and that $f \in \mathcal{L}(I)$. Suppose further that the function $g : I \to \mathbb{R}$ is such that $f(x) = g(x)$ for almost all $x \in I$. Then $g \in \mathcal{L}(I)$, and*

$$\int_I f(x) \,\mathrm{d}x = \int_I g(x) \,\mathrm{d}x.$$

Example 4.4.1. Consider the function $g : [0, 1] \to \mathbb{R}$, defined by

$$g(x) = \begin{cases} 0 & \text{if } x \text{ is rational,} \\ 1 & \text{if } x \text{ is irrational.} \end{cases}$$

Let $f(x) = 1$ for every $x \in [0, 1]$. Then $f \in \mathcal{L}([0, 1])$, and

$$\int_{[0,1]} f(x) \,\mathrm{d}x = 1.$$

Note next that the set of rational numbers in $[0, 1]$ is a set of measure zero. It follows from Theorem 4T that $g \in \mathcal{L}([0, 1])$, and

$$\int_{[0,1]} g(x) \,\mathrm{d}x = 1.$$

Recall, however, that the function g is not Riemann integrable over $[0, 1]$.

The proof of Theorem 4T depends on the following intermediate result.

THEOREM 4U. *Suppose that $I \subseteq \mathbb{R}$ is an interval. Suppose further that the function $f : I \to \mathbb{R}$ is such that $f(x) = 0$ for almost all $x \in I$. Then $f \in \mathcal{L}(I)$, and*

$$\int_I f(x) \,\mathrm{d}x = 0.$$

Proof. Let $s_n : I \to \mathbb{R}$ satisfy $s_n(x) = 0$ for all $x \in I$. Then s_n is an increasing sequence of step functions which converges to 0 everywhere in I. It follows that $s_n(x) = f(x)$ for almost all $x \in I$. Furthermore, it is clear that

$$\lim_{n\to\infty} \int_I s_n(x) \,\mathrm{d}x = 0.$$

It follows that $f \in \mathcal{U}(I)$, and

$$\int_I f(x)\,\mathrm{d}x = \lim_{n\to\infty} \int_I s_n(x)\,\mathrm{d}x = 0$$

as required. ♣

Proof of Theorem 4T. In view of Theorem 4U, we have $g - f \in \mathcal{L}(I)$, and

$$\int_I (g(x) - f(x))\,\mathrm{d}x = 0.$$

Note next that $g = f + (g - f)$, and the result follows from Theorem 4L(a). ♣

4.5. Relationship with Riemann Integration

We conclude this chapter by showing that Lebesgue integration is indeed a generalization of Riemann integration. We prove the following result. Suppose that $A, B \in \mathbb{R}$ and $A < B$ throughout this section.

THEOREM 4V. *Suppose that the function $f : [A, B] \to \mathbb{R}$ is bounded. Suppose further that f is Riemann integrable over $[A, B]$.*
(a) Then the set $\mathcal{D}$ of discontinuities of f in $[A, B]$ has measure zero.
(b) Furthermore, $f \in \mathcal{U}([A, B])$, and the Lebesgue integral of f over $[A, B]$ is equal to the Riemann integral of f over $[A, B]$.

Remarks. (1) In fact, it can be shown that for any bounded function $f : [A, B] \to \mathbb{R}$, the condition (a) is equivalent to the condition that f is Riemann integrable over $[A, B]$.

(2) Note that if f is Riemann integrable over $[A, B]$, then it is an upper function on $[A, B]$. We shall show in the proof that the step functions generating f arise from some lower Riemann sums.

Proof of Theorem 4V. (a) For every $x \in [A, B]$, write

$$\omega(x) = \lim_{h\to 0+} \sup_{y\in[A,B]\cap(x-h,x+h)} |f(y) - f(x)|.$$

It can be shown that $\omega(x_0) = 0$ if and only if f is continuous at x_0. It follows that we can write

$$\mathcal{D} = \bigcup_{k=1}^{\infty} \mathcal{D}_k,$$

where, for every $k \in \mathbb{N}$,

$$\mathcal{D}_k = \left\{ x \in [A, B] : \omega(x) \geq \frac{1}{k} \right\}.$$

Suppose on the contrary that $\mathcal{D}$ does not have measure zero. Then by Theorem 3M, there exists $k_0 \in \mathbb{N}$ such that $\mathcal{D}_{k_0}$ does not have measure zero, so that there exists $\epsilon_0 > 0$ such that every countable collection of open intervals covering $\mathcal{D}_{k_0}$ has a sum of lengths at least ϵ_0. Suppose that

$$\Delta :\ A = x_0 < x_1 < x_2 < \ldots < x_n = B$$

is a dissection of the interval $[A, B]$. Then

$$S(f, \Delta) - s(f, \Delta) = \sum_{i=1}^{n} (x_i - x_{i-1}) \left(\max_{x\in[x_{i-1},x_i]} f(x) - \min_{x\in[x_{i-1},x_i]} f(x) \right).$$

Write
$$S(f,\Delta)-s(f,\Delta)=T_1+T_2\ge T_1, \tag{27}$$
where
$$T_1=\sum_{\substack{i=1\\(x_{i-1},x_i)\cap\mathcal{D}_{k_0}\ne\emptyset}}^{n}(x_i-x_{i-1})\left(\max_{x\in[x_{i-1},x_i]}f(x)-\min_{x\in[x_{i-1},x_i]}f(x)\right) \tag{28}$$
and
$$T_2=\sum_{\substack{i=1\\(x_{i-1},x_i)\cap\mathcal{D}_{k_0}=\emptyset}}^{n}(x_i-x_{i-1})\left(\max_{x\in[x_{i-1},x_i]}f(x)-\min_{x\in[x_{i-1},x_i]}f(x)\right).$$
Note that the open intervals in T_1 cover $\mathcal{D}_{k_0}$, with the possible exception of a finite number of points (which has total measure zero). It follows that the total length of the intervals in T_1 is at least ϵ_0. In other words,
$$\sum_{\substack{i=1\\(x_{i-1},x_i)\cap\mathcal{D}_{k_0}\ne\emptyset}}^{n}(x_i-x_{i-1})\ge\epsilon_0. \tag{29}$$
On the other hand,
$$\max_{x\in[x_{i-1},x_i]}f(x)-\min_{x\in[x_{i-1},x_i]}f(x)\ge\frac{1}{k_0} \tag{30}$$
whenever $(x_{i-1},x_i)\cap\mathcal{D}_{k_0}\ne\emptyset$. Combining (28)–(30), we conclude that
$$T_1\ge\frac{\epsilon_0}{k_0}. \tag{31}$$
It now follows from (27) and (31) that
$$S(f,\Delta)-s(f,\Delta)\ge\frac{\epsilon_0}{k_0}. \tag{32}$$
Note finally that (32) holds for every dissection Δ of $[A,B]$. It follows that f is not Riemann integrable over $[A,B]$.

(b) For every $n\in\mathbb{N}$, consider the dissection
$$\Delta_n:\ A=x_0<x_1<x_2<\ldots<x_{2^n}=B$$
of the interval $[A,B]$ into 2^n equal subintervals of length $(B-A)/2^n$, and note that the subintervals of Δ_{n+1} can be obtained by bisecting the subintervals of Δ_n. For every $i=1,\ldots,2^n$, let
$$m_i=\min\{f(x):x\in[x_{i-1},x_i]\}, \tag{33}$$
and define a step function $s_n:[A,B]\to\mathbb{R}$ by
$$s_n(x)=\begin{cases}m_i & \text{if } x\in(x_{i-1},x_i],\\ m_1 & \text{if } x=x_0.\end{cases} \tag{34}$$
It is easy to check (the reader is advised to draw a picture) that
$$s_n(x)\le f(x) \tag{35}$$
for every $x\in[A,B]$, and that the sequence s_n is increasing on $[A,B]$. To show that $f\in\mathcal{U}([A,B])$, it remains to show that $s_n(x)\to f(x)$ as $n\to\infty$ for almost all $x\in[A,B]$, and that the sequence
$$\int_{[A,B]}s_n(x)\,\mathrm{d}x=s(f,\Delta_n) \tag{36}$$

is convergent. Since the set of discontinuities of f in $[A, B]$ has measure zero, to show that $s_n(x) \to f(x)$ as $n \to \infty$ for almost all $x \in [A, B]$, it suffices to show that $s_n(x_0) \to f(x_0)$ as $n \to \infty$ at every point x_0 of continuity of f. Suppose now that f is continuous at x_0. Then given any $\epsilon > 0$, there exists $\delta > 0$ such that

$$f(x_0) - \epsilon < f(x) < f(x_0) + \epsilon \qquad \text{for every } x \in (x_0 - \delta, x_0 + \delta).$$

Let

$$m(\delta) = \inf\{f(x) : x \in (x_0 - \delta, x_0 + \delta)\}. \tag{37}$$

Then $f(x_0) - \epsilon \leq m(\delta)$, and so

$$f(x_0) \leq m(\delta) + \epsilon. \tag{38}$$

On the other hand, there clearly exists $N \in \mathbb{N}$ large enough such that an interval $[x_{i-1}, x_i]$ in the dissection Δ_N contains x_0 and lies inside $(x_0 - \delta, x_0 + \delta)$; in other words,

$$x_0 \in [x_{i-1}, x_i] \subset (x_0 - \delta, x_0 + \delta) \tag{39}$$

(the reader is advised to draw a picture). Then, in view of (33)–(35) and (37)–(39), we have

$$S_N(x_0) \leq f(x_0) \leq m(\delta) + \epsilon \leq m_i + \epsilon = S_N(x_0) + \epsilon. \tag{40}$$

Since the sequence s_n is increasing on $[A, B]$, it follows from (35) and (40) that for every $n > N$, we have

$$s_n(x_0) \leq f(x_0) \leq S_N(x_0) + \epsilon \leq S_n(x_0) + \epsilon.$$

Hence $|s_n(x_0) - f(x_0)| < \epsilon$ for every $n > N$, whence $s_n(x_0) \to f(x_0)$ as $n \to \infty$. Finally, note that the sequence (36) is increasing and bounded above. Clearly it converges to the Riemann integral of f over $[A, B]$. ♣

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INTRODUCTION TO LEBESGUE INTEGRATION

W W L CHEN

Chapter 5

MONOTONE CONVERGENCE THEOREM

5.1. Step Functions on an Interval

In this chapter, we shall study the question of term-by-term integration of sequences of functions. In particular, we shall establish a theorem of Levi, the simplest version of which, for step functions, is stated below.

THEOREM 5A. *Suppose that $I \subseteq \mathbb{R}$ is an interval. Suppose further that the sequence of step functions $s_n \in \mathcal{S}(I)$ satisfies the following conditions:*
(a) The sequence $s_n : I \to \mathbb{R}$ is increasing on I.
(b) $\lim_{n\to\infty} \int_I s_n(x)\,\mathrm{d}x$ exists.
Then the sequence s_n converges almost everywhere on I to a limit function $u \in \mathcal{U}(I)$, and

$$\int_I u(x)\,\mathrm{d}x = \lim_{n\to\infty} \int_I s_n(x)\,\mathrm{d}x. \tag{1}$$

PROOF. We may assume, without loss of generality, that the sequence s_n is non-negative, and shall show that the set

$$\mathcal{D} = \{x \in I : s_n(x) \text{ diverges as } n \to \infty\}$$

has measure zero. In other words, we shall show that given any $\epsilon > 0$, the set $\mathcal{D}$ can be covered by a countable collection of intervals of total length less than ϵ. In view of our assumption, the sequence

$$\int_I s_n(x)\,\mathrm{d}x$$

† This chapter was written at Imperial College, University of London, in 1977 while the author was an undergraduate.

is non-negative. Also, since it converges, it is bounded above by some positive constant M, say. For every $n \in \mathbb{N}$ and $x \in I$, let

$$t_n(x) = \left[\frac{\epsilon}{2M} s_n(x)\right],$$

where, for every $\beta \in \mathbb{R}$, $[\beta]$ denotes the greatest integer not exceeding β. It is easy to see that t_n is a non-negative increasing sequence of integer-valued step functions. Note next that the sequence $s_n : I \to \mathbb{R}$ is increasing on I. If $s_n(x)$ converges, then it is bounded, so that $t_n(x)$ is also bounded, whence $t_{n+1}(x) = t_n(x)$ for all sufficiently large $n \in \mathbb{N}$. If $s_n(x)$ diverges, then it is not bounded, so that $t_n(x)$ is also not bounded, whence $t_{n+1}(x) - t_n(x) \geq 1$ for infinitely many $n \in \mathbb{N}$. For every $n \in \mathbb{N}$, let

$$\mathcal{D}_n = \{x \in I : t_{n+1}(x) - t_n(x) \geq 1\}.$$

Clearly

$$\mathcal{D} \subseteq \bigcup_{n=1}^{\infty} \mathcal{D}_n.$$

Furthermore, each $\mathcal{D}_n$ is a union of a finite number of intervals, since $t_{n+1} - t_n$ is a step function on I. If we denote by $|\mathcal{D}_n|$ the total length of the intervals in $\mathcal{D}_n$, then it suffices to show that

$$\sum_{n=1}^{\infty} |\mathcal{D}_n| < \epsilon.$$

Note now that

$$|\mathcal{D}_n| = \int_{\mathcal{D}_n} \mathrm{d}x \leq \int_{\mathcal{D}_n} (t_{n+1}(x) - t_n(x)) \,\mathrm{d}x \leq \int_I (t_{n+1}(x) - t_n(x)) \,\mathrm{d}x.$$

It follows that for every $N \in \mathbb{N}$,

$$\begin{aligned}\sum_{n=1}^{N} |\mathcal{D}_n| &\leq \sum_{n=1}^{N} \int_I (t_{n+1}(x) - t_n(x)) \,\mathrm{d}x = \int_I t_{N+1}(x) \,\mathrm{d}x - \int_I t_1(x) \,\mathrm{d}x \\ &\leq \int_I t_{N+1}(x) \,\mathrm{d}x \leq \frac{\epsilon}{2M} \int_I s_{N+1}(x) \,\mathrm{d}x \leq \frac{\epsilon}{2} < \epsilon.\end{aligned}$$

Hence $\mathcal{D}$ has measure zero, so that s_n converges almost everywhere on I. Now define $u : I \to \mathbb{R}$ by

$$u(x) = \begin{cases} 0 & \text{if } x \in \mathcal{D}, \\ \lim_{n\to\infty} s_n(x) & \text{if } x \in I \setminus \mathcal{D}. \end{cases}$$

Then $s_n(x) \to u(x)$ as $n \to \infty$ for almost all $x \in I$. Clearly $u \in \mathcal{U}(I)$. The equality (1) now follows immediately from the definition of the integral of upper functions. ♣

5.2. Upper Functions on an Interval

In this section, we prove the following generalization of Theorem 5A.

THEOREM 5B. *Suppose that $I \subseteq \mathbb{R}$ is an interval. Suppose further that the sequence of upper functions $u_n \in \mathcal{U}(I)$ satisfies the following conditions:*
(a) The sequence $u_n : I \to \mathbb{R}$ is increasing almost everywhere on I.
(b) $\displaystyle\lim_{n\to\infty} \int_I u_n(x) \,\mathrm{d}x$ exists.
Then the sequence u_n converges almost everywhere on I to a limit function $u \in \mathcal{U}(I)$, and

$$\int_I u(x) \,\mathrm{d}x = \lim_{n\to\infty} \int_I u_n(x) \,\mathrm{d}x. \tag{2}$$

Proof. For every fixed $k \in \mathbb{N}$, suppose that the upper function $u_k \in \mathcal{U}(I)$ is generated by the sequence of step functions $s_{nk} \in \mathcal{S}(I)$, so that the sequence $s_{nk} : I \to \mathbb{R}$ is increasing on I, $s_{nk}(x) \to u_k(x)$ as $n \to \infty$ for almost all $x \in I$, and

$$\int_I u_k(x)\,\mathrm{d}x = \lim_{n\to\infty} \int_I s_{nk}(x)\,\mathrm{d}x.$$

The first step in our argument is to use Theorem 5A to find a function $u : I \to \mathbb{R}$ which will ultimately be the required limit function. However, in order to use Theorem 5A, we need some step functions. We therefore need to define a new sequence of step functions in terms of the step functions s_{nk}. Accordingly, for every $n \in \mathbb{N}$, we define the function $t_n : I \to \mathbb{R}$ by writing

$$t_n(x) = \max\{s_{n1}(x), \ldots, s_{nn}(x)\} \tag{3}$$

for every $x \in I$. Clearly $t_n \in \mathcal{S}(I)$. Note also that for every $x \in I$, we have

$$\begin{aligned} t_{n+1}(x) &= \max\{s_{(n+1)1}(x), \ldots, s_{(n+1)(n+1)}(x)\} \geq \max\{s_{(n+1)1}(x), \ldots, s_{(n+1)n}(x)\} \\ &\geq \max\{s_{n1}(x), \ldots, s_{nn}(x)\} = t_n(x), \end{aligned}$$

so that the sequence t_n is increasing on I. In order to use Theorem 5A, we need to show next that the sequence

$$\int_I t_n(x)\,\mathrm{d}x \tag{4}$$

is convergent. Clearly this sequence is increasing, so it suffices to show that it is bounded above. Note that $s_{nk}(x) \leq u_k(x)$ for almost all $x \in I$. It follows from (3) that $t_n(x) \leq \max\{u_1(x), \ldots, u_n(x)\}$ for almost all $x \in I$. Note next that the sequence u_n is increasing almost everywhere on I. Hence

$$t_n(x) \leq u_n(x) \tag{5}$$

for almost all $x \in I$. It follows from Theorem 4F(c) that

$$\int_I t_n(x)\,\mathrm{d}x \leq \int_I u_n(x)\,\mathrm{d}x. \tag{6}$$

By our hypotheses, the sequence on the right hand side of (6) is convergent and so bounded above. It follows that the sequence (4) is bounded above. We can now apply Theorem 5A to the sequence t_n, and conclude that t_n converges almost everywhere on I to a limit function $u \in \mathcal{U}(I)$, and

$$\int_I u(x)\,\mathrm{d}x = \lim_{n\to\infty} \int_I t_n(x)\,\mathrm{d}x.$$

Having established the existence of this function $u \in \mathcal{U}(I)$, we show next that $u_n(x) \to u(x)$ for almost all $x \in I$. For every $k \leq n$ and every $x \in I$, it is easy to see from (3) that $s_{nk}(x) \leq t_n(x)$. Since u_n is increasing and $t_n \to u$ as $n \to \infty$ almost everywhere on I, it follows that $s_{nk}(x) \leq u(x)$ for almost all $x \in I$. On the other hand, for any fixed $k \in \mathbb{N}$, since s_{nk} is increasing and $s_{nk} \to u_k$ as $n \to \infty$ almost everywhere on I, it follows that

$$u_k(x) \leq u(x) \tag{7}$$

for almost all $x \in I$. Hence the sequence u_k is increasing and bounded above by u almost everywhere on I, and so converges to some limit function v almost everywhere on I. Clearly, for any $k \in \mathbb{N}$,

$$u_k(x) \leq v(x) \tag{8}$$

for almost all $x \in I$. Furthermore, $v(x) \leq u(x)$ for almost all $x \in I$. To show that $u_n(x) \to u(x)$ for almost all $x \in I$, it remains to show that $u(x) \leq v(x)$ for almost all $x \in I$. To do this, note from (5) and (8) that $t_n(x) \leq v(x)$ for almost all $x \in I$. Since the sequence t_n generates u, it follows that $u(x) \leq v(x)$ for almost all $x \in I$. Hence $u_n(x) \to u(x)$ for almost all $x \in I$. To complete the proof of Theorem 5B, it remains to establish (2). It is easy to see from (5), Theorem 4F(c) and our hypotheses on the sequence u_n that

$$\int_I t_n(x)\,\mathrm{d}x \le \int_I u_n(x)\,\mathrm{d}x \le \lim_{m\to\infty}\int_I u_m(x)\,\mathrm{d}x.$$

Letting $n\to\infty$, we have

$$\int_I u(x)\,\mathrm{d}x \le \lim_{m\to\infty}\int_I u_m(x)\,\mathrm{d}x. \tag{9}$$

On the other hand, it follows from (7) and Theorem 4F(c) that

$$\int_I u_k(x)\,\mathrm{d}x \le \int_I u(x)\,\mathrm{d}x.$$

Letting $k\to\infty$, we have

$$\lim_{k\to\infty}\int_I u_k(x)\,\mathrm{d}x \le \int_I u(x)\,\mathrm{d}x. \tag{10}$$

The equality (2) now follows on combining (9) and (10). ♣

5.3. Lebesgue Integrable Functions on an Interval

In this section, we extend Theorem 5B to Lebesgue integrable functions. The result can be stated in the following two equivalent forms.

THEOREM 5C. (LEVI'S THEOREM FOR A SEQUENCE) *Suppose that $I\subseteq\mathbb{R}$ is an interval. Suppose further that the sequence of functions $f_n\in\mathcal{L}(I)$ satisfies the following conditions:*
(a) The sequence $f_n : I\to\mathbb{R}$ is increasing almost everywhere on I.
(b) $\displaystyle\lim_{n\to\infty}\int_I f_n(x)\,\mathrm{d}x$ exists.

Then the sequence f_n converges almost everywhere on I to a limit function $f\in\mathcal{L}(I)$, and

$$\int_I f(x)\,\mathrm{d}x = \lim_{n\to\infty}\int_I f_n(x)\,\mathrm{d}x.$$

THEOREM 5D. (LEVI'S THEOREM FOR A SERIES) *Suppose that $I\subseteq\mathbb{R}$ is an interval. Suppose further that the sequence of functions $g_n\in\mathcal{L}(I)$ satisfies the following conditions:*
(a) The sequence $g_n : I\to\mathbb{R}$ is non-negative almost everywhere on I.
(b) $\displaystyle\sum_{n=1}^{\infty}\int_I g_n(x)\,\mathrm{d}x$ converges.

Then the series $\displaystyle\sum_{n=1}^{\infty} g_n$ converges almost everywhere on I to a sum function $g\in\mathcal{L}(I)$, and

$$\int_I g(x)\,\mathrm{d}x = \int_I \sum_{n=1}^{\infty} g_n(x)\,\mathrm{d}x = \sum_{n=1}^{\infty}\int_I g_n(x)\,\mathrm{d}x. \tag{11}$$

Remark. To see the equivalence of the two versions, simply take $g_1=f_1$ and $g_n=f_n-f_{n-1}$ for every $n\ge 2$.

PROOF OF THEOREM 5D. Since every Lebesgue integrable function can be written as the difference of two upper functions, the idea is to use Theorem 5B on these upper functions. However, it is technically convenient to choose these upper functions rather carefully. For every $n \in \mathbb{N}$, we use Theorem 4R to find two sequences $u_n, v_n \in \mathcal{U}(I)$ such that $g_n(x) = u_n(x) - v_n(x)$ for every $x \in I$, $v_n(x) \geq 0$ for almost all $x \in I$, and

$$\int_I v_n(x)\,\mathrm{d}x < \frac{1}{2^n},$$

so that

$$\sum_{n=1}^{\infty} \int_I v_n(x)\,\mathrm{d}x < 1. \tag{12}$$

Clearly $u_n(x) = g_n(x) + v_n(x) \geq 0$ for almost all $x \in I$. Hence the partial sums

$$U_N(x) = \sum_{n=1}^{N} u_n(x)$$

give rise to a sequence of upper functions $U_N \in \mathcal{U}(I)$ which is increasing almost everywhere on I, so that the sequence

$$\int_I U_N(x)\,\mathrm{d}x \tag{13}$$

is increasing. On the other hand, we have

$$\int_I U_N(x)\,\mathrm{d}x = \int_I \sum_{n=1}^{N} u_n(x)\,\mathrm{d}x = \sum_{n=1}^{N} \int_I u_n(x)\,\mathrm{d}x = \sum_{n=1}^{N} \int_I g_n(x)\,\mathrm{d}x + \sum_{n=1}^{N} \int_I v_n(x)\,\mathrm{d}x,$$

so that (13) is bounded above, in view of (b) and (12), and so converges. It follows from Theorem 5B that the sequence U_N converges almost everywhere on I to a limit function $U \in \mathcal{U}(I)$, and

$$\int_I U(x)\,\mathrm{d}x = \lim_{N\to\infty} \int_I U_N(x)\,\mathrm{d}x. \tag{14}$$

Note, however, that

$$\int_I U_N(x)\,\mathrm{d}x = \sum_{n=1}^{N} \int_I u_n(x)\,\mathrm{d}x.$$

Letting $N \to \infty$ and combining with (14), we conclude that

$$\int_I U(x)\,\mathrm{d}x = \sum_{n=1}^{\infty} \int_I u_n(x)\,\mathrm{d}x. \tag{15}$$

A similar argument shows that the partial sums

$$V_N(x) = \sum_{n=1}^{N} v_n(x)$$

give rise to a sequence of upper functions $V_N \in \mathcal{U}(I)$ which converges almost everywhere on I to a limit function $V \in \mathcal{U}(I)$, and

$$\int_I V(x)\,\mathrm{d}x = \sum_{n=1}^{\infty} \int_I v_n(x)\,\mathrm{d}x. \tag{16}$$

Clearly $U - V \in \mathcal{L}(I)$, and the sequence $U_N - V_N$ converges almost everywhere on I to $U - V$. Let $g(x) = U(x) - V(x)$ for every $x \in I$. Then $g \in \mathcal{L}(I)$, and it follows from (15) and (16) that

$$\int_I g(x)\,\mathrm{d}x = \int_I U(x)\,\mathrm{d}x - \int_I V(x)\,\mathrm{d}x = \sum_{n=1}^{\infty} \int_I u_n(x)\,\mathrm{d}x - \sum_{n=1}^{\infty} \int_I v_n(x)\,\mathrm{d}x = \sum_{n=1}^{\infty} \int_I g_n(x)\,\mathrm{d}x$$

as required. ♣

Note that condition (a) of Theorem 5D is rather restrictive. In fact, we have the following version of Levi's theorem.

THEOREM 5E. *Suppose that $I \subseteq \mathbb{R}$ is an interval. Suppose further that the sequence of functions $g_n \in \mathcal{L}(I)$ satisfies*

$$\sum_{n=1}^{\infty} \int_I |g_n(x)| \, \mathrm{d}x$$

is convergent. Then the series $\displaystyle\sum_{n=1}^{\infty} g_n$ converges almost everywhere on I to a sum function $g \in \mathcal{L}(I)$, and

$$\int_I g(x) \, \mathrm{d}x = \int_I \sum_{n=1}^{\infty} g_n(x) \, \mathrm{d}x = \sum_{n=1}^{\infty} \int_I g_n(x) \, \mathrm{d}x.$$

PROOF. For every $n \in \mathbb{N}$, we can write $g_n = g_n^+ - g_n^-$. By Theorem 4M, we have $g_n^+, g_n^- \in \mathcal{L}(I)$. We can now apply Theorem 5D to each of these two sequences to conclude that there exist $g_+, g_- \in \mathcal{L}(I)$ such that the series

$$\sum_{n=1}^{\infty} g_n^+ \qquad \text{and} \qquad \sum_{n=1}^{\infty} g_n^-$$

converge almost everywhere on I to g_+ and g_- respectively, and that

$$\int_I g_+(x) \, \mathrm{d}x = \int_I \sum_{n=1}^{\infty} g_n^+(x) \, \mathrm{d}x = \sum_{n=1}^{\infty} \int_I g_n^+(x) \, \mathrm{d}x$$

and

$$\int_I g_-(x) \, \mathrm{d}x = \int_I \sum_{n=1}^{\infty} g_n^-(x) \, \mathrm{d}x = \sum_{n=1}^{\infty} \int_I g_n^-(x) \, \mathrm{d}x.$$

Clearly $\displaystyle\sum_{n=1}^{\infty} g_n$ converges almost everywhere on I to $g = g_+ - g_-$, and

$$\begin{aligned}\int_I g(x) \, \mathrm{d}x &= \int_I g_+(x) \, \mathrm{d}x - \int_I g_-(x) \, \mathrm{d}x = \int_I \sum_{n=1}^{\infty} g_n^+(x) \, \mathrm{d}x - \int_I \sum_{n=1}^{\infty} g_n^-(x) \, \mathrm{d}x \\ &= \int_I \left(\sum_{n=1}^{\infty} g_n^+(x) - \sum_{n=1}^{\infty} g_n^-(x) \right) \mathrm{d}x = \int_I \sum_{n=1}^{\infty} g_n(x) \, \mathrm{d}x\end{aligned}$$

and

$$\begin{aligned}\int_I g(x) \, \mathrm{d}x &= \int_I g_+(x) \, \mathrm{d}x - \int_I g_-(x) \, \mathrm{d}x = \sum_{n=1}^{\infty} \int_I g_n^+(x) \, \mathrm{d}x - \sum_{n=1}^{\infty} \int_I g_n^-(x) \, \mathrm{d}x \\ &= \sum_{n=1}^{\infty} \left(\int_I g_n^+(x) \, \mathrm{d}x - \int_I g_n^-(x) \, \mathrm{d}x \right) = \sum_{n=1}^{\infty} \int_I g_n(x) \, \mathrm{d}x.\end{aligned}$$

The result follows immediately. ♣

EXAMPLE 5.3.1. Define $f : [0,1] \to \mathbb{R}$ by

$$f(x) = \begin{cases} x^s & \text{if } x > 0, \\ 0 & \text{if } x = 0, \end{cases}$$

where $s \in \mathbb{R}$ is fixed. If $s \geq 0$, then f is bounded on $[0,1]$. Furthermore, the Riemann integral

$$\int_0^1 f(x)\,\mathrm{d}x = \frac{1}{s+1}.$$

However, if $s < 0$, then f is unbounded on $[0,1]$, and so the Riemann integral does not exist. Consider now the sequence of functions $f_n : [0,1] \to \mathbb{R}$, defined for each $n \in \mathbb{N}$ by

$$f_n(x) = \begin{cases} x^s & \text{if } x \geq 1/n, \\ 0 & \text{if } 0 \leq x < 1/n. \end{cases}$$

Clearly f_n is increasing on $[0,1]$, and $f_n(x) \to f(x)$ for every $x \in [0,1]$. Each f_n is Riemann integrable over $[0,1]$, and so Lebesgue integrable over $[0,1]$, and

$$\int_0^1 f_n(x)\,\mathrm{d}x = \int_{1/n}^1 x^s\,\mathrm{d}x = \frac{1}{s+1}\left(1 - \frac{1}{n^{s+1}}\right).$$

Note now that if $s + 1 > 0$, then the sequence

$$\int_0^1 f_n(x)\,\mathrm{d}x \to \frac{1}{s+1}$$

as $n \to \infty$. It follows from Theorem 5C that if $s > -1$, then

$$\int_0^1 f(x)\,\mathrm{d}x$$

exists as a Lebesgue integral, and has value $1/(s+1)$.

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INTRODUCTION TO LEBESGUE INTEGRATION

W W L CHEN

Chapter 6

DOMINATED CONVERGENCE THEOREM

6.1. Lebesgue's Theorem

In this section, we shall deduce the following result from the Monotone convergence theorem studied in the last chapter. The result below is usually considered the cornerstone of Lebesgue integration theory.

THEOREM 6A. (LEBESGUE'S THEOREM) *Suppose that $I \subseteq \mathbb{R}$ is an interval. Suppose further that the sequence of functions $f_n \in \mathcal{L}(I)$ satisfies the following conditions:*
(a) The sequence $f_n : I \to \mathbb{R}$ converges almost everywhere to a limit function $f : I \to \mathbb{R}$.
(b) There exists a non-negative function $F \in \mathcal{L}(I)$ such that for every $n \in \mathbb{N}$, $|f_n(x)| \le F(x)$ for almost all $x \in I$.
Then the limit function $f \in \mathcal{L}(I)$, the sequence

$$\int_I f_n(x)\,\mathrm{d}x$$

is convergent, and

$$\int_I f(x)\,\mathrm{d}x = \lim_{n\to\infty} \int_I f_n(x)\,\mathrm{d}x. \tag{1}$$

Remark. Note condition (b) that the sequence f_n is dominated by F almost everywhere.

† This chapter was written at Imperial College, University of London, in 1977 while the author was an undergraduate.

PROOF OF THEOREM 6A. We shall construct two sequences $g_n, h_n \in \mathcal{L}(I)$ such that

$$g_n(x) \le f_n(x) \le h_n(x) \tag{2}$$

for every $x \in I$, and where g_n is increasing and h_n is decreasing on I, and both converge to the limit function f almost everywhere on I. Clearly the sequence

$$\int_I g_n(x)\,\mathrm{d}x$$

is increasing and bounded above by

$$\int_I F(x)\,\mathrm{d}x,$$

so that

$$\lim_{n\to\infty} \int_I g_n(x)\,\mathrm{d}x$$

exists. It follows from Theorem 5C that $f \in \mathcal{L}(I)$ and

$$\int_I f(x)\,\mathrm{d}x = \lim_{n\to\infty} \int_I g_n(x)\,\mathrm{d}x. \tag{3}$$

On the other hand, the sequence

$$\int_I h_n(x)\,\mathrm{d}x$$

is decreasing and bounded below by

$$-\int_I F(x)\,\mathrm{d}x,$$

so that

$$\lim_{n\to\infty} \int_I h_n(x)\,\mathrm{d}x$$

exists. It follows from Theorem 5C (applied to the sequence $-h_n$) that

$$\int_I f(x)\,\mathrm{d}x = \lim_{n\to\infty} \int_I h_n(x)\,\mathrm{d}x. \tag{4}$$

Combining (3) and (4), we obtain

$$\int_I f(x)\,\mathrm{d}x = \lim_{n\to\infty} \int_I g_n(x)\,\mathrm{d}x = \lim_{n\to\infty} \int_I h_n(x)\,\mathrm{d}x. \tag{5}$$

On the other hand, it follows from (2) that for every $n \in \mathbb{N}$,

$$\int_I g_n(x)\,\mathrm{d}x \le \int_I f_n(x)\,\mathrm{d}x \le \int_I h_n(x)\,\mathrm{d}x. \tag{6}$$

The equality (1) follows on letting $n \to \infty$ in (6) and combining with (5). It remains to establish the existence of the sequences g_n and h_n. For every $n \in \mathbb{N}$, write

$$h_n(x) = \sup\{f_n(x), f_{n+1}(x), f_{n+2}(x), \ldots\}$$

for every $x \in I$. Clearly $f_n(x) \le h_n(x)$ for every $x \in I$, and h_n is decreasing on I. Suppose that $x \in I$ and $f_n(x) \to f(x)$ as $n \to \infty$. Then given any $\epsilon > 0$, there exists N such that for every $n > N$,

$$f(x) - \epsilon < f_n(x) < f(x) + \epsilon.$$

It follows that for all $n > N$,
$$f(x) - \epsilon < h_n(x) \le f(x) + \epsilon,$$
so that $h_n(x) \to f(x)$ as $n \to \infty$. Since $f_n \to f$ as $n \to \infty$ almost everywhere on I, it follows that $h_n \to f$ as $n \to \infty$ almost everywhere on I. Unfortunately, we also need to show that $h_n \in \mathcal{L}(I)$. Here, the difficulty arises since $h_n(x)$ is defined as the supremum of a collection which may be finite or infinite. This difficulty would not have arisen if the collection were finite, since the supremum of such a collection would then be equal to its maximum, and we could then use Theorem 4N repeatedly. However, the finite case suggests the following approach. For every $m, n \in \mathbb{N}$ with $m > n$, write
$$h_{nm}(x) = \max\{f_n(x), f_{n+1}(x), \ldots, f_m(x)\}$$
for every $x \in I$. Then by repeated application of Theorem 4N, we have $h_{nm} \in \mathcal{L}(I)$. For every fixed $n \in \mathbb{N}$, the sequence h_{nm} (in counting variable $m > n$) is increasing on I. On the other hand, clearly $|h_{nm}(x)| \le F(x)$ for almost all $x \in I$. It follows that
$$\left|\int_I h_{nm}(x)\,\mathrm{d}x\right| \le \int_I |h_{nm}(x)|\,\mathrm{d}x \le \int_I F(x)\,\mathrm{d}x.$$
Hence the sequence
$$\int_I h_{nm}(x)\,\mathrm{d}x$$
is increasing and bounded above and so converges. It follows from Theorem 5C that h_{nm} converges almost everywhere as $m \to \infty$ to a limit function in $\mathcal{L}(I)$. Clearly $h_{nm} \to h_n$ as $m \to \infty$. This proves that $h_n \in \mathcal{L}(I)$. Similarly, write
$$g_n(x) = \inf\{f_n(x), f_{n+1}(x), f_{n+2}(x), \ldots\}$$
for every $x \in I$. Clearly $g_n(x) \le f_n(x)$ for every $x \in I$, and g_n is increasing on I. Suppose that $x \in I$ and $f_n(x) \to f(x)$ as $n \to \infty$. Then given any $\epsilon > 0$, there exists N such that for every $n > N$,
$$f(x) - \epsilon < f_n(x) < f(x) + \epsilon.$$
It follows that for all $n > N$,
$$f(x) - \epsilon \le g_n(x) < f(x) + \epsilon,$$
so that $g_n(x) \to f(x)$ as $n \to \infty$. Since $f_n \to f$ as $n \to \infty$ almost everywhere on I, it follows that $g_n \to f$ as $n \to \infty$ almost everywhere on I. To show that $g_n \in \mathcal{L}(I)$, for every $m, n \in \mathbb{N}$ with $m > n$, write
$$g_{nm}(x) = \min\{f_n(x), f_{n+1}(x), \ldots, f_m(x)\}$$
for every $x \in I$. Then by repeated application of Theorem 4N, we have $g_{nm} \in \mathcal{L}(I)$. For every fixed $n \in \mathbb{N}$, the sequence g_{nm} (in counting variable $m > n$) is decreasing on I. On the other hand, clearly $|g_{nm}(x)| \le F(x)$ for almost all $x \in I$. It follows that
$$\left|\int_I g_{nm}(x)\,\mathrm{d}x\right| \le \int_I |g_{nm}(x)|\,\mathrm{d}x \le \int_I F(x)\,\mathrm{d}x.$$
Hence the sequence
$$\int_I g_{nm}(x)\,\mathrm{d}x$$
is decreasing and bounded below and so converges. It follows from Theorem 5C (applied to the sequence $-g_{nm}$) that g_{nm} converges almost everywhere as $m \to \infty$ to a limit function in $\mathcal{L}(I)$. Clearly $g_{nm} \to g_n$ as $m \to \infty$. This proves that $g_n \in \mathcal{L}(I)$. The proof of Theorem 6A is now complete. ♣

The following version for a series can be deduced easily from Theorem 6A.

THEOREM 6B. *Suppose that $I \subseteq \mathbb{R}$ is an interval. Suppose further that the sequence of functions $g_n \in \mathcal{L}(I)$ satisfies the following conditions:*
(a) The sequence $g_n : I \to \mathbb{R}$ is non-negative almost everywhere on I.
(b) $\displaystyle\sum_{n=1}^{\infty} g_n$ converges almost everywhere on I to a sum function $g : I \to \mathbb{R}$.
(c) There exists a non-negative function $G \in \mathcal{L}(I)$ such that $|g(x)| \le G(x)$ for almost all $x \in I$.
Then $g \in \mathcal{L}(I)$, the series

$$\sum_{n=1}^{\infty} \int_I g_n(x)\,\mathrm{d}x$$

converges, and

$$\int_I g(x)\,\mathrm{d}x = \int_I \sum_{n=1}^{\infty} g_n(x)\,\mathrm{d}x = \sum_{n=1}^{\infty} \int_I g_n(x)\,\mathrm{d}x.$$

Proof. Write $f_1 = g_1$ and $f_n = \displaystyle\sum_{m=1}^{n} g_m$. It is easy to check that the sequence f_n satisfies the hypotheses of Theorem 6A with $f = g$ and $F = G$. The result follows easily. ♣

6.2. Consequences of Lebesgue's Theorem

The following result is sometimes called the Bounded convergence theorem.

THEOREM 6C. *Suppose that $I \subseteq \mathbb{R}$ is a bounded interval. Suppose further that the sequence of functions $f_n \in \mathcal{L}(I)$ satisfies the following conditions:*
(a) The sequence $f_n : I \to \mathbb{R}$ converges almost everywhere to a limit function $f : I \to \mathbb{R}$.
(b) There exists $M \in \mathbb{R}$ such that for every $n \in \mathbb{N}$, $|f_n(x)| \le M$ for almost all $x \in I$.
Then the limit function $f \in \mathcal{L}(I)$, and

$$\int_I f(x)\,\mathrm{d}x = \lim_{n\to\infty} \int_I f_n(x)\,\mathrm{d}x.$$

Remark. In view of conditions (a) and (b), we say that the sequence f_n is boundedly convergent almost everywhere on I.

Proof of Theorem 6C. Let $F(x) = M$ for every $x \in I$, and note that since I is a bounded interval, we have $F \in \mathcal{L}(I)$. The result now follows from Theorem 6A. ♣

The last result in this section is sometimes useful in establishing Lebesgue integrability.

THEOREM 6D. *Suppose that $I \subseteq \mathbb{R}$ is an interval. Suppose further that the sequence of functions $f_n \in \mathcal{L}(I)$ satisfies the following conditions:*
(a) The sequence $f_n : I \to \mathbb{R}$ converges almost everywhere to a limit function $f : I \to \mathbb{R}$.
(b) There exists a non-negative function $F \in \mathcal{L}(I)$ such that $|f(x)| \le F(x)$ for almost all $x \in I$.
Then the limit function $f \in \mathcal{L}(I)$.

PROOF. For every $n \in \mathbb{N}$, write

$$g_n(x) = \max\{\min\{f_n(x), F(x)\}, -F(x)\}$$

for every $x \in I$ (the reader is advised to draw a picture). Then $g_n \in \mathcal{L}(I)$ by Theorem 4N. It is easy to see that $|g_n(x)| \leq F(x)$ for almost all $x \in I$, and that $g_n \to f$ as $n \to \infty$ almost everywhere on I. The result follows from Theorem 6A. ♣

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INTRODUCTION TO LEBESGUE INTEGRATION

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Chapter 7

LEBESGUE INTEGRALS ON UNBOUNDED INTERVALS

7.1. Some Limiting Cases

We begin by considering the following result which extends Lebesgue integrals on finite intervals to infinite intervals.

THEOREM 7A. *Suppose that $I = [A, \infty)$, where $A \in \mathbb{R}$. Suppose further that the function $f : I \to \mathbb{R}$ satisfies the following conditions:*
(a) $f \in \mathcal{L}([A, B])$ for every real number $B \geq A$.
(b) There exists a constant M such that $\displaystyle\int_A^B |f(x)|\,\mathrm{d}x \leq M$ for every real number $B \geq A$.
Then $f \in \mathcal{L}(I)$, the limit

$$\lim_{B\to\infty} \int_A^B f(x)\,\mathrm{d}x$$

exists, and

$$\int_A^\infty f(x)\,\mathrm{d}x = \lim_{B\to\infty} \int_A^B f(x)\,\mathrm{d}x. \tag{1}$$

† This chapter was written at Imperial College, University of London, in 1977 while the author was an undergraduate.

PROOF. Let $B_n \in \mathbb{R}$ be an increasing sequence satisfying $B_n \geq A$ for every $n \in \mathbb{N}$ and $B_n \to \infty$ as $n \to \infty$. For every $n \in \mathbb{N}$, define $f_n : I \to \mathbb{R}$ by writing

$$f_n(x) = \begin{cases} f(x) & \text{if } x \in [A, B_n], \\ 0 & \text{otherwise.} \end{cases}$$

Clearly $f_n \in \mathcal{L}(I)$, in view of Theorem 4J. Furthermore, $f_n(x) \to f(x)$ as $n \to \infty$ for every $x \in I$, and so $|f_n(x)| \to |f(x)|$ as $n \to \infty$ for every $x \in I$. It is not difficult to see that the sequence $|f_n|$ is increasing on I, so that

$$\int_I |f_n(x)| \, \mathrm{d}x$$

is an increasing sequence, bounded above by M in view of (b), and so converges as $n \to \infty$. It follows from the Monotone convergence theorem (Theorem 5C) that $|f| \in \mathcal{L}(I)$. Note also that $|f_n(x)| \leq |f(x)|$ for every $x \in I$. It follows from the Dominated convergence theorem (Theorem 6A) that $f \in \mathcal{L}(I)$, and that

$$\int_I f(x) \, \mathrm{d}x = \lim_{n \to \infty} \int_I f_n(x) \, \mathrm{d}x = \lim_{n \to \infty} \int_A^{B_n} f(x) \, \mathrm{d}x.$$

Note that this holds for every increasing sequence $B_n \to \infty$ as $n \to \infty$, and so the equality (1) follows immediately. ♣

We also have the following two corresponding results. The proofs are technically similar.

THEOREM 7B. *Suppose that $I = (-\infty, B]$, where $B \in \mathbb{R}$. Suppose further that the function $f : I \to \mathbb{R}$ satisfies the following conditions:*
(a) $f \in \mathcal{L}([A, B])$ for every real number $A \leq B$.
(b) There exists a constant M such that $\int_A^B |f(x)| \, \mathrm{d}x \leq M$ for every real number $A \leq B$.
Then $f \in \mathcal{L}(I)$, the limit

$$\lim_{A \to -\infty} \int_A^B f(x) \, \mathrm{d}x$$

exists, and

$$\int_{-\infty}^B f(x) \, \mathrm{d}x = \lim_{A \to -\infty} \int_A^B f(x) \, \mathrm{d}x.$$

THEOREM 7C. *Suppose that the function $f : \mathbb{R} \to \mathbb{R}$ satisfies the following conditions:*
(a) $f \in \mathcal{L}([A, B])$ for every $A, B \in \mathbb{R}$ satisfying $A \leq B$.
(b) There exists a constant M such that $\int_A^B |f(x)| \, \mathrm{d}x \leq M$ for every $A, B \in \mathbb{R}$ satisfying $A \leq B$.
Then $f \in \mathcal{L}(\mathbb{R})$, the limit

$$\lim_{\substack{A \to -\infty \\ B \to \infty}} \int_A^B f(x) \, \mathrm{d}x$$

exists, and

$$\int_{-\infty}^{\infty} f(x) \, \mathrm{d}x = \lim_{\substack{A \to -\infty \\ B \to \infty}} \int_A^B f(x) \, \mathrm{d}x.$$

EXAMPLE 7.1.1. Consider the function $f : \mathbb{R} \to \mathbb{R}$, given by

$$f(x) = \frac{1}{1+x^2}$$

for every $x \in \mathbb{R}$. It is easy to check that for every $A, B \in \mathbb{R}$ satisfying $A \le B$, we have

$$\int_A^B |f(x)|\,\mathrm{d}x = \int_A^B f(x)\,\mathrm{d}x = \tan^{-1} B - \tan^{-1} A \le \pi.$$

It follows from Theorem 7C that $f \in \mathcal{L}(\mathbb{R})$, and that

$$\int_{-\infty}^{\infty} f(x)\,\mathrm{d}x = \lim_{\substack{A\to-\infty\\B\to\infty}} \int_A^B f(x)\,\mathrm{d}x = \pi.$$

EXAMPLE 7.1.2. We shall demonstrate the importance of condition (b) in Theorem 7A. Define the function $f : [0,\infty) \to \mathbb{R}$ as follows: For every $n \in \mathbb{N}$, we write $f(x) = n^{-1}\sin \pi x$ for every $x \in [n-1, n)$. It is easy to check that for every real number $B \ge 0$, we have

$$\int_0^B f(x)\,\mathrm{d}x = \int_0^{[B]} f(x)\,\mathrm{d}x + \int_{[B]}^B f(x)\,\mathrm{d}x,$$

where $[B]$ denotes the greatest integer not exceeding B. Then

$$\begin{aligned}\int_0^B f(x)\,\mathrm{d}x &= \sum_{n=1}^{[B]} \int_{n-1}^n f(x)\,\mathrm{d}x + \int_{[B]}^B f(x)\,\mathrm{d}x = \sum_{n=1}^{[B]} \int_{n-1}^n \frac{\sin \pi x}{n}\,\mathrm{d}x + \int_{[B]}^B \frac{\sin \pi x}{[B]+1}\,\mathrm{d}x \\ &= \frac{2}{\pi}\sum_{n=1}^{[B]} \frac{(-1)^{n-1}}{n} + \frac{(-1)^{[B]} - \cos \pi B}{\pi([B]+1)},\end{aligned}$$

so that

$$\lim_{B\to\infty} \int_0^B f(x)\,\mathrm{d}x = \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} + \lim_{B\to\infty} \frac{(-1)^{[B]} - \cos \pi B}{\pi([B]+1)} = \frac{2\log 2}{\pi}.$$

On the other hand, note that

$$\int_0^B |f(x)|\,\mathrm{d}x \ge \int_0^{[B]} |f(x)|\,\mathrm{d}x = \sum_{n=1}^{[B]} \int_{n-1}^n |f(x)|\,\mathrm{d}x = \sum_{n=1}^{[B]} \int_{n-1}^n \left|\frac{\sin \pi x}{n}\right|\mathrm{d}x = \frac{2}{\pi}\sum_{n=1}^{[B]} \frac{1}{n}$$

is not bounded above as $B \to \infty$, so that condition (b) fails. We shall show that $f \notin \mathcal{L}([0,\infty))$. Suppose on the contrary that $f \in \mathcal{L}([0,\infty))$. For every $N \in \mathbb{N}$, define $f_N : [0,\infty) \to \mathbb{R}$ by writing

$$f_N(x) = \begin{cases} |f(x)| & \text{if } x < N, \\ 0 & \text{if } x \ge N. \end{cases}$$

It is not difficult to see that the sequence of functions $f_N \in \mathcal{L}([0,\infty))$ is increasing on $[0,\infty)$, and that $f_N(x) \to |f(x)|$ as $N \to \infty$ for every $x \in [0,\infty)$. On the other hand, it follows from Theorem 4M that $|f| \in \mathcal{L}([0,\infty))$; also $|f_N(x)| \le |f(x)|$ for every $x \in [0,\infty)$. Hence by the Dominated convergence theorem (Theorem 6A), the sequence

$$\int_0^\infty f_N(x)\,\mathrm{d}x$$

is convergent. Note, however, that

$$\int_0^\infty f_N(x)\,\mathrm{d}x = \int_0^N |f(x)|\,\mathrm{d}x = \frac{2}{\pi}\sum_{n=1}^{N} \frac{1}{n} \to \infty$$

as $N \to \infty$, a contradiction.

7.2. Improper Riemann Integrals

We now study Lebesgue integrals from the viewpoint of improper integrals.

DEFINITION. Suppose that $A \in \mathbb{R}$. Suppose further that the function $f : [A, \infty) \to \mathbb{R}$ satisfies the following conditions:
(a) $f \in \mathcal{R}([A, B])$ for every real number $B \geq A$.
(b) $\displaystyle\lim_{B \to \infty} \int_A^B f(x)\,\mathrm{d}x$ exists.
Then we say that f is improper Riemann integrable on $[A, \infty)$, and define the improper integral of f over $[A, \infty)$ by

$$\int_A^\infty f(x)\,\mathrm{d}x = \lim_{B \to \infty} \int_A^B f(x)\,\mathrm{d}x.$$

If we look at the Example 7.1.2, then we see that the existence of the improper integral does not imply the existence of the Lebesgue integral. Corresponding to Theorem 7A, we have the following result.

THEOREM 7D. *Suppose that $A \in \mathbb{R}$. Suppose further that the function $f : [A, \infty) \to \mathbb{R}$ satisfies the following conditions:*
(a) $f \in \mathcal{R}([A, B])$ for every real number $B \geq A$.
(b) There exists a constant M such that $\displaystyle\int_A^B |f(x)|\,\mathrm{d}x \leq M$ for every real number $B \geq A$.
Then both f and $|f|$ are improper Riemann integrable on $[A, \infty)$. Furthermore, $f \in \mathcal{L}([A, \infty))$, and the Lebesgue integral of f over $[A, \infty)$ is equal to the improper Riemann integral of f over $[A, \infty)$.

PROOF. Clearly

$$\int_A^B |f(x)|\,\mathrm{d}x$$

is an increasing function of B and is bounded above, so that it converges as $B \to \infty$, so that $|f|$ is improper Riemann integrable on $[A, \infty)$. On the other hand, clearly $0 \leq |f(x)| - f(x) \leq 2|f(x)|$ for every $x \in [A, \infty)$. It follows that

$$\int_A^B (|f(x)| - f(x))\,\mathrm{d}x$$

is also an increasing function of B and is bounded above, so that it also converges as $B \to \infty$. Hence

$$\int_A^B f(x)\,\mathrm{d}x$$

converges as $B \to \infty$, so that f is improper Riemann integrable on $[A, \infty)$. To complete the proof of Theorem 7D, we note that by Theorem 4V, $f \in \mathcal{L}([A, B])$ for every real number $B \geq A$, and that the Lebesgue integral of f over $[A, B]$ is equal to the Riemann integral of f over $[A, B]$. The result now follows from Theorem 7A. ♣

We also have results corresponding to Theorems 7B and 7C.

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INTRODUCTION TO LEBESGUE INTEGRATION

W W L CHEN

Chapter 8

MEASURABLE FUNCTIONS AND MEASURABLE SETS

8.1. Measurable Functions

Suppose that $I \subseteq \mathbb{R}$ is an interval. For every function $f \in \mathcal{L}(I)$, there exist functions $u, v \in \mathcal{U}(I)$ such that $f(x) = u(x) - v(x)$ for all $x \in I$. There exist sequences of step functions $s_n, t_n \in \mathcal{S}(I)$ such that $s_n(x) \to u(x)$ and $t_n(x) \to v(x)$ as $n \to \infty$ for almost all $x \in I$. It follows that $s_n(x) - t_n(x) \to f(x)$ as $n \to \infty$ for almost all $x \in I$. Clearly $s_n - t_n \in \mathcal{S}(I)$ for every $n \in \mathbb{N}$. Hence every Lebesgue integrable function on I is the limit almost everywhere on I of a sequence of step functions on I.

Let us examine the converse. The function $f(x) = 1$ for every $x \in \mathbb{R}$ is clearly not Lebesgue integrable on $\mathbb{R}$. For every $n \in \mathbb{N}$, define $f_n : \mathbb{R} \to \mathbb{R}$ by writing

$$f_n(x) = \begin{cases} 1 & \text{if } x \in [-n, n], \\ 0 & \text{otherwise.} \end{cases}$$

Then $f_n(x) \to f(x)$ as $n \to \infty$ for all $x \in \mathbb{R}$. Also $f_n \in \mathcal{S}(\mathbb{R})$ for every $n \in \mathbb{N}$. It follows that the limit of a sequence of step functions is not necessarily Lebesgue integrable.

We shall study the class of functions which are the limits of step functions.

Definition. Suppose that $I \subseteq \mathbb{R}$ is an interval. A function $f : I \to \mathbb{R}$ is said to be measurable on I, denoted by $f \in \mathcal{M}(I)$, if there exists a sequence of step functions $s_n \in \mathcal{S}(I)$ such that $s_n(x) \to f(x)$ as $n \to \infty$ for almost all $x \in I$.

Remark. It is clear that for any interval $I \subseteq \mathbb{R}$, we have $\mathcal{L}(I) \subseteq \mathcal{M}(I)$. In fact, it can be shown that $\mathcal{L}(I) \neq \mathcal{M}(I)$.

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† This chapter was written at Imperial College, University of London, in 1977 while the author was an undergraduate.

We have the following partial result.

THEOREM 8A. *Suppose that $I \subseteq \mathbb{R}$ is an interval, and that $f \in \mathcal{M}(I)$. Suppose further that there exists a non-negative function $F \in \mathcal{L}(I)$ such that $|f(x)| \leq F(x)$ for almost all $x \in I$. Then $f \in \mathcal{L}(I)$.*

PROOF. Since $f \in \mathcal{M}(I)$, there exists a sequence $f_n \in \mathcal{S}(I) \subseteq \mathcal{L}(I)$ such that $f_n(x) \to f(x)$ as $n \to \infty$ for almost all $x \in I$. The result now follows from Theorem 6D. ♣

The following two results are simple consequences of Theorem 8A.

THEOREM 8B. *Suppose that $I \subseteq \mathbb{R}$ is an interval, and that $f \in \mathcal{M}(I)$. Suppose further that $|f| \in \mathcal{L}(I)$. Then $f \in \mathcal{L}(I)$.*

THEOREM 8C. *Suppose that $I \subseteq \mathbb{R}$ is a bounded interval, and that $f \in \mathcal{M}(I)$. Suppose further that f is bounded on I. Then $f \in \mathcal{L}(I)$.*

8.2. Further Properties of Measurable Functions

First of all, we shall construct more measurable functions.

THEOREM 8D. *Suppose that $I \subseteq \mathbb{R}$ is an interval, and that $f, g \in \mathcal{M}(I)$. Then so are $f \pm g$, fg, $|f|$, $\max\{f, g\}$ and $\min\{f, g\}$. Also, $1/f \in \mathcal{M}(I)$ provided that $f(x) \neq 0$ for almost all $x \in I$.*

PROOF. We shall prove the more general result that if a function $\phi : \mathbb{R}^2 \to \mathbb{R}$ is continuous on $\mathbb{R}^2$, then any function $h : I \to \mathbb{R}$, defined by

$$h(x) = \phi(f(x), g(x))$$

for every $x \in I$, is measurable on I. Suppose that the sequences of step functions $s_n, t_n \in \mathcal{S}(I)$ satisfy $s_n(x) \to f(x)$ and $t_n(x) \to g(x)$ as $n \to \infty$ for almost all $x \in I$. For every $n \in \mathbb{R}$, the function $u_n : I \to \mathbb{R}$, defined by

$$u_n(x) = \phi(s_n(x), t_n(x))$$

for every $x \in I$, is a step function on I. Clearly $u_n(x) \to \phi(f(x), g(x))$ as $n \to \infty$ for almost all $x \in I$. ♣

Next, we show that the limits of sequences of measurable functions are also measurable.

THEOREM 8E. *Suppose that $I \subseteq \mathbb{R}$ is an interval, and that $f : I \to \mathbb{R}$ is given. Suppose further that there exists a sequence of functions $f_n \in \mathcal{M}(I)$ such that $f_n(x) \to f(x)$ as $n \to \infty$ for almost all $x \in I$. Then $f \in \mathcal{M}(I)$.*

PROOF. Choose any positive function $g \in \mathcal{L}(I) \subseteq \mathcal{M}(I)$ and keep it fixed. For every $n \in \mathbb{N}$, write

$$F_n(x) = g(x)\frac{f_n(x)}{1 + |f_n(x)|}$$

for every $x \in I$. Then

$$F_n(x) \to g(x)\frac{f(x)}{1 + |f(x)|}$$

as $n \to \infty$ for almost all $x \in I$. For every $x \in I$, write

$$F(x) = g(x)\frac{f(x)}{1 + |f(x)|}.$$

Note that $F_n \in \mathcal{M}(I)$ by Theorem 8D. Also $|F_n(x)| < g(x)$ for all $x \in I$. It follows from Theorem 8A that $F_n \in \mathcal{L}(I)$. On the other hand, $|F(x)| < g(x)$ for all $x \in I$, so it follows from Theorem 6D that $F \in \mathcal{L}(I) \subseteq \mathcal{M}(I)$. Finally, it is easily checked that

$$f(x) = \frac{F(x)}{g(x) - |F(x)|}$$

for every $x \in I$. The result now follows from Theorem 8D on noting that $g(x) - |F(x)| > 0$ for every $x \in I$. ♣

8.3. Measurable Sets

In this section, we shall develop the notion of the Lebesgue measure of sets of real numbers. We shall do this by using the characteristic function. Recall that for any subset $S \subseteq \mathbb{R}$, the characteristic function $\chi_S : \mathbb{R} \to \mathbb{R}$ of the set S satisfies $\chi_S(x) = 1$ if $x \in S$ and $\chi_S(x) = 0$ if $x \notin S$.

THEOREM 8F. *Suppose that $S \subseteq \mathbb{R}$. Then S has measure zero if and only if the following two conditions are satisfied:*
(a) $\chi_S \in \mathcal{L}(\mathbb{R})$*; and*
(b) $\displaystyle\int_{\mathbb{R}} \chi_S(x)\,\mathrm{d}x = 0.$

PROOF. ($\Rightarrow$) Suppose that S has measure zero. Then $\chi_S(x) = 0$ for almost all $x \in \mathbb{R}$. The result now follows from Theorem 4U.

($\Leftarrow$) For every $n \in \mathbb{N}$, let $f_n(x) = \chi_S(x)$ for every $x \in \mathbb{R}$. Then $f_n \in \mathcal{L}(\mathbb{R})$ and

$$\sum_{n=1}^{\infty} \int_{\mathbb{R}} |f_n(x)|\,\mathrm{d}x = \sum_{n=1}^{\infty} \int_{\mathbb{R}} \chi_S(x)\,\mathrm{d}x = 0.$$

By the Monotone convergence theorem (Theorem 5E), the series

$$\sum_{n=1}^{\infty} f_n(x) \tag{1}$$

converges almost everywhere on $\mathbb{R}$. If $x \in S$, then $f_n(x) = 1$ for every $n \in \mathbb{N}$, so that the series (1) diverges. If $x \notin S$, then $f_n(x) = 0$ for every $n \in \mathbb{N}$, so that the series (1) converges. It follows that the series (1) diverges if and only if $x \in S$. Hence S has measure zero. ♣

DEFINITION. A set $S \subseteq \mathbb{R}$ is said to be measurable if the characteristic function $\chi_S \in \mathcal{M}(\mathbb{R})$. If $\chi_S \in \mathcal{L}(\mathbb{R})$, then the measure $\mu(S)$ of the set S is given by

$$\mu(S) = \int_{\mathbb{R}} \chi_S(x)\,\mathrm{d}x.$$

If $\chi_S \in \mathcal{M}(\mathbb{R})$ but $\chi_S \notin \mathcal{L}(\mathbb{R})$, then we define $\mu(S) = \infty$.

REMARKS. (1) The function μ is sometimes called the Lebesgue measure.

(2) By Theorem 8F, a set $S \subseteq \mathbb{R}$ of measure zero is measurable and $\mu(S) = 0$.

(3) Every interval $I \subseteq \mathbb{R}$ is measurable. If I is bounded with endpoints $A \le B$, then $\mu(I) = B - A$. If I is unbounded, then $\mu(I) = \infty$.

(4) Suppose that $S, T \subseteq \mathbb{R}$ are measurable. If $S \subseteq T$, then $\mu(S) \le \mu(T)$.

The next two results give rise to more measurable sets.

THEOREM 8G. *Suppose that $S,T\subseteq\mathbb{R}$ are measurable. Then so is $S\setminus T$.*

PROOF. Note that the characteristic function of $S\setminus T$ is $\chi_S-\chi_S\chi_T$. The result now follows from Theorem 8D. ♣

THEOREM 8H. *Suppose that for every $k\in\mathbb{N}$, the set $S_k\subseteq\mathbb{R}$ is measurable. Then so are*

$$U=\bigcup_{k=1}^{\infty}S_k\qquad\text{and}\qquad V=\bigcap_{k=1}^{\infty}S_k.$$

PROOF. For every $n\in\mathbb{N}$, let

$$U_n=\bigcup_{k=1}^{n}S_k\qquad\text{and}\qquad V_n=\bigcap_{k=1}^{n}S_k.$$

Then it follows from Theorem 8D that the characteristic functions

$$\chi_{U_n}=\max\{\chi_{S_1},\ldots,\chi_{S_n}\}\qquad\text{and}\qquad \chi_{V_n}=\min\{\chi_{S_1},\ldots,\chi_{S_n}\}$$

are measurable on $\mathbb{R}$. On the other hand,

$$\chi_U(x)=\lim_{n\to\infty}\chi_{U_n}(x)\qquad\text{and}\qquad \chi_V(x)=\lim_{n\to\infty}\chi_{V_n}(x)$$

for every $x\in\mathbb{R}$. It follows from Theorem 8E that $\chi_U,\chi_V\in\mathcal{M}(\mathbb{R})$. ♣

8.4. Additivity of Measure

In this section, we study the important question of adding measure. Given a collection of pairwise disjoint measurable sets, we would like to find the measure of their union in terms of the measure of each of the sets in the collection.

We first study the case of the union of two measurable sets.

THEOREM 8J. *Suppose that $S_1,S_2\subseteq\mathbb{R}$ are measurable. Suppose further that $S_1\cap S_2=\emptyset$. Then $\mu(S_1\cup S_2)=\mu(S_1)+\mu(S_2)$.*

PROOF. Write $U=S_1\cup S_2$. Clearly $\chi_U(x)=\chi_{S_1}(x)+\chi_{S_2}(x)$ for every $x\in\mathbb{R}$. Suppose that $\chi_U\in\mathcal{L}(\mathbb{R})$. Then since $\chi_{S_1},\chi_{S_2}\in\mathcal{M}(\mathbb{R})$ and satisfy

$$0\le\chi_{S_1}(x)\le\chi_U(x)\qquad\text{and}\qquad 0\le\chi_{S_2}(x)\le\chi_U(x)$$

for every $x\in\mathbb{R}$, it follows from Theorem 8A that $\chi_{S_1},\chi_{S_2}\in\mathcal{L}(\mathbb{R})$. Clearly

$$\mu(U)=\int_{\mathbb{R}}\chi_U(x)\,\mathrm{d}x=\int_{\mathbb{R}}\chi_{S_1}(x)\,\mathrm{d}x+\int_{\mathbb{R}}\chi_{S_2}(x)\,\mathrm{d}x=\mu(S_1)+\mu(S_2).$$

Suppose next that $\chi_U\not\in\mathcal{L}(\mathbb{R})$. Then $\mu(U)=\infty$. On the other hand, we have, by Theorem 4L, that $\chi_{S_1}\not\in\mathcal{L}(\mathbb{R})$ or $\chi_{S_2}\not\in\mathcal{L}(\mathbb{R})$. It follows that $\mu(S_1)+\mu(S_2)=\infty$. ♣

Using Theorem 8J and induction, we can show that Lebesgue measure is finitely additive.

THEOREM 8K. *Suppose that $S_1,\ldots,S_n\subseteq\mathbb{R}$ are measurable. Suppose further that $S_i\cap S_j=\emptyset$ whenever $i\ne j$. Then*

$$\mu\left(\bigcup_{k=1}^{n}S_k\right)=\sum_{k=1}^{n}\mu(S_k).$$

We now extend Theorem 8K to show that Lebesgue measure is countably additive.

THEOREM 8L. *Suppose that for every $k \in \mathbb{N}$, the set $S_k \subseteq \mathbb{R}$ is measurable. Suppose further that $S_i \cap S_j = \emptyset$ whenever $i \neq j$. Then*

$$\mu\left(\bigcup_{k=1}^{\infty} S_k\right) = \sum_{k=1}^{\infty} \mu(S_k). \tag{2}$$

PROOF. For every $n \in \mathbb{N}$, let

$$U_n = \bigcup_{k=1}^{n} S_k \qquad \text{and} \qquad U = \bigcup_{k=1}^{\infty} S_k.$$

By Theorem 8K, we have

$$\mu(U_n) = \sum_{k=1}^{n} \mu(S_k).$$

We need to show that $\mu(U_n) \to \mu(U)$ as $n \to \infty$. Suppose that $\mu(U)$ is finite. Then $\chi_U \in \mathcal{L}(\mathbb{R})$. On the other hand, $0 \le \chi_{U_n}(x) \le \chi_U(x)$ for every $n \in \mathbb{N}$ and $x \in \mathbb{R}$, so it follows from Theorem 8A that $\chi_{U_n} \in \mathcal{L}(\mathbb{R})$ for every $n \in \mathbb{N}$. Note that the sequence

$$\mu(U_n) = \int_{\mathbb{R}} \chi_{U_n}(x)\,\mathrm{d}x$$

is increasing and bounded above by $\mu(U)$, and so converges. It follows from the Monotone convergence theorem (Theorem 5C) that

$$\lim_{n\to\infty} \mu(U_n) = \lim_{n\to\infty} \int_{\mathbb{R}} \chi_{U_n}(x)\,\mathrm{d}x = \int_{\mathbb{R}} \lim_{n\to\infty} \chi_{U_n}(x)\,\mathrm{d}x = \int_{\mathbb{R}} \chi_U(x)\,\mathrm{d}x = \mu(U).$$

Suppose next that $\mu(U) = \infty$. Then $\chi_U \notin \mathcal{L}(\mathbb{R})$. It follows from Theorem 5C that either $\chi_{U_n} \notin \mathcal{L}(\mathbb{R})$ for some $n \in \mathbb{N}$ or $\mu(U_n) \to \infty$ as $n \to \infty$. In either case, both sides of (2) are infinite. ♣

8.5. Lebesgue Integrals over Measurable Sets

In this section, we extend the definition of the Lebesgue integral to include all measurable sets.

DEFINITION. Suppose that the set $S \subseteq \mathbb{R}$ is measurable. For any function $f : S \to \mathbb{R}$, define a function $f^* : \mathbb{R} \to \mathbb{R}$ by writing

$$f^*(x) = \begin{cases} f(x) & \text{if } x \in S, \\ 0 & \text{otherwise.} \end{cases}$$

Suppose that $f^* \in \mathcal{L}(\mathbb{R})$. Then we say that f is Lebesgue integrable on S, denoted by $f \in \mathcal{L}(S)$. Furthermore, the integral of f over S is defined by

$$\int_S f(x)\,\mathrm{d}x = \int_{\mathbb{R}} f^*(x)\,\mathrm{d}x.$$

REMARK. Note that if $\mu(S)$ is finite, then $\mu(S) = \displaystyle\int_S \mathrm{d}x$.

Corresponding to Theorem 8L, we have the following two results.

THEOREM 8M. *Suppose that $f : U \to \mathbb{R}$ is given, where*

$$U = \bigcup_{k=1}^{\infty} S_k,$$

where $S_k \subseteq \mathbb{R}$ is measurable for every $k \in \mathbb{N}$ and $S_i \cap S_j = \emptyset$ whenever $i \neq j$. Suppose further that $f \in \mathcal{L}(S_k)$ for every $k \in \mathbb{N}$, and that

$$\sum_{k=1}^{\infty} \int_{S_k} f(x)\,\mathrm{d}x \tag{3}$$

is convergent. Then $f \in \mathcal{L}(U)$, and

$$\int_U f(x)\,\mathrm{d}x = \sum_{k=1}^{\infty} \int_{S_k} f(x)\,\mathrm{d}x.$$

THEOREM 8N. *Suppose that*

$$U = \bigcup_{k=1}^{\infty} S_k,$$

where $S_k \subseteq \mathbb{R}$ is measurable for every $k \in \mathbb{N}$ and $S_i \cap S_j = \emptyset$ whenever $i \neq j$. Suppose further that $f \in \mathcal{L}(U)$. Then $f \in \mathcal{L}(S_k)$ for every $k \in \mathbb{N}$, and

$$\int_U f(x)\,\mathrm{d}x = \sum_{k=1}^{\infty} \int_{S_k} f(x)\,\mathrm{d}x.$$

PROOF OF THEOREM 8M. Writing $f = f^+ - f^-$ and studying f^+ and f^- separately, we may assume, without loss of generality, that f is non-negative. For every $n \in \mathbb{N}$, let

$$U_n = \bigcup_{k=1}^{n} S_k.$$

It is not difficult to see that $f \in \mathcal{L}(U_n)$, and

$$\int_{U_n} f(x)\,\mathrm{d}x = \sum_{k=1}^{n} \int_{S_k} f(x)\,\mathrm{d}x.$$

We need to prove that $f \in \mathcal{L}(U)$ and

$$\lim_{n\to\infty} \int_{U_n} f(x)\,\mathrm{d}x = \int_U f(x)\,\mathrm{d}x. \tag{4}$$

Now define $f^* : \mathbb{R} \to \mathbb{R}$ by writing

$$f^*(x) = \begin{cases} f(x) & \text{if } x \in U, \\ 0 & \text{otherwise.} \end{cases} \tag{5}$$

To prove (4), it suffices to prove that

$$\lim_{n\to\infty} \int_{\mathbb{R}} f^*(x)\chi_{U_n}(x)\,\mathrm{d}x = \int_{\mathbb{R}} f^*(x)\,\mathrm{d}x. \tag{6}$$

Clearly f^* is non-negative. It follows that the sequence $f^*\chi_{U_n} \in \mathcal{L}(\mathbb{R})$ is increasing everywhere on $\mathbb{R}$, and $f^*(x)\chi_{U_n}(x) \to f^*(x)$ as $n \to \infty$ for every $x \in \mathbb{R}$. It follows from the convergence of the sequence (3) and the Monotone convergence theorem (Theorem 5C) that $f^* \in \mathcal{L}(\mathbb{R})$ and that (6) holds. ♣

PROOF OF THEOREM 8N. Since S_k is measurable, it follows that $\chi_{S_k} \in \mathcal{M}(\mathbb{R})$. On the other hand, define $f^* : \mathbb{R} \to \mathbb{R}$ by (5). Since $f \in \mathcal{L}(U)$, we have $f^* \in \mathcal{L}(\mathbb{R}) \subseteq \mathcal{M}(\mathbb{R})$. Also $|f^*| \in \mathcal{L}(\mathbb{R})$. By Theorem 8D, we also have $f^*\chi_{S_k} \in \mathcal{M}(\mathbb{R})$. Clearly $|f^*(x)\chi_{S_k}(x)| \leq |f^*(x)|$ for every $x \in \mathbb{R}$. It follows from Theorem 8A that $f^*\chi_{S_k} \in \mathcal{L}(\mathbb{R})$, so that $f \in \mathcal{L}(S_k)$. We can now complete the proof as for Theorem 8M. ♣

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INTRODUCTION TO LEBESGUE INTEGRATION

W W L CHEN

Chapter 9

CONTINUITY AND DIFFERENTIABILITY OF LEBESGUE INTEGRALS

9.1. Continuity

Many functions in analysis are of the form

$$F(y) = \int_X f(x,y)\,\mathrm{d}x$$

for some function $f : X \times Y \to \mathbb{R}$, where $X, Y \subseteq \mathbb{R}$ are intervals. Our task is to study the possible transfer of properties from the integrand f to the integral F.

We have the following result on continuity, that under suitable conditions, the continuity property can be transferred from the integrand to the integral through Lebesgue integration.

THEOREM 9A. *Suppose that $X, Y \subseteq \mathbb{R}$ are intervals. Suppose further that the function $f : X \times Y \to \mathbb{R}$ satisfies the following conditions:*
(a) For every fixed $y \in Y$, the function $f_y : X \to \mathbb{R}$, defined by $f_y(x) = f(x,y)$ for every $x \in X$, is measurable on X.
(b) There exists a non-negative function $g \in \mathcal{L}(X)$ such that for every $y \in Y$, $|f(x,y)| \leq g(x)$ for almost all $x \in X$.
(c) For every fixed $y \in Y$, $f(x,t) \to f(x,y)$ as $t \to y$ for almost all $x \in X$.
Then for every $y \in Y$, the Lebesgue integral

$$\int_X f(x,y)\,\mathrm{d}x$$

† This chapter was written at Imperial College, University of London, in 1977 while the author was an undergraduate.

exists. Furthermore, the function $F : Y \to \mathbb{R}$, defined by

$$F(y) = \int_X f(x, y)\,\mathrm{d}x$$

for every $y \in Y$, is continuous on Y; in other words,

$$\lim_{t \to y} \int_X f(x, t)\,\mathrm{d}x = \int_X f(x, y)\,\mathrm{d}x. \tag{1}$$

REMARK. Note that condition (c) states that the function $f(x, y)$ is a continuous function on Y for almost all $x \in X$, whereas (1) states that the integral $F(y)$ is a continuous function on Y.

PROOF OF THEOREM 9A. Conditions (a) and (b), together with Theorem 8A, show that for every $y \in Y$, we have $f_y \in \mathcal{L}(X)$. It remains to prove (1). Suppose that $y \in Y$ is fixed. Let $y_n \in Y$ be a sequence such that $y_n \to y$ as $n \to \infty$. We shall prove that $F(y_n) \to F(y)$ as $n \to \infty$. Note that

$$F(y_n) = \int_X f_{y_n}(x)\,\mathrm{d}x,$$

where $f_{y_n} \in \mathcal{L}(X)$, $f_{y_n}(x) \to f_y(x)$ as $n \to \infty$ for almost all $x \in X$, and $|f_{y_n}(x)| \le g(x)$ for almost all $x \in X$. It follows from the Dominated convergence theorem (Theorem 6A) that the sequence $F(y_n)$ converges, and

$$\int_X f_y(x)\,\mathrm{d}x = \lim_{n \to \infty} \int_X f_{y_n}(x)\,\mathrm{d}x.$$

This gives $F(y_n) \to F(y)$ as $n \to \infty$, and completes the proof of Theorem 9A. ♣

REMARK. In the above proof, we have used the following result: A function $F : Y \to \mathbb{R}$ is continuous at $y \in Y$ if and only if $F(y_n) \to F(y)$ as $n \to \infty$ for every sequence $y_n \in Y$ satisfying $y_n \to y$ as $n \to \infty$.

EXAMPLE 9.1.1. Consider the function $F : (0, \infty) \to \mathbb{R}$, given by

$$F(y) = \int_0^\infty \mathrm{e}^{-xy} k(x)\,\mathrm{d}x,$$

where

$$k(x) = \begin{cases} x^{-1} \sin x & \text{if } x > 0, \\ 1 & \text{if } x = 0. \end{cases}$$

Here we write $X = [0, \infty)$ and $Y = (0, \infty)$. Using Theorem 7D, we can easily check that condition (a) of Theorem 9A holds. On the other hand, condition (c) is easily checked. We now check condition (b). Note that for every fixed $y \in Y$, $\mathrm{e}^{-xy} \to \mathrm{e}^{-1}$ as $x \to 0$. Hence the best we can hope for an upper bound for e^{-xy} is e^{-1}. On the other hand, we may use the inequality

$$\left| \frac{\sin x}{x} \right| \le \min \left\{ 1, \frac{1}{x} \right\}.$$

Hence

$$|\mathrm{e}^{-xy} k(x)| \le \mathrm{e}^{-1} \min \left\{ 1, \frac{1}{x} \right\},$$

but the function on the right hand side is not Lebesgue integrable on X. Instead, for every fixed $A > 0$, we shall apply Theorem 9A in the case $X = [0, \infty)$ and $Y = Y_A = [A, \infty)$. Then conditions (a) and (c) still hold. Also, for every $y \in Y_A$, we have

$$|\mathrm{e}^{-xy} k(x)| \le \mathrm{e}^{-Ax}.$$

It is now easily checked that the function e^{-Ax} is Lebesgue integrable on X. We now conclude that the function $F(y)$ is continuous at every $y > A$. Since $A > 0$ is arbitrary, we can conclude that the function $F(y)$ is continuous at every $y > 0$.

Example 9.1.2. In this example, we continue our investigation of the function $F : (0, \infty) \to \mathbb{R}$ in Example 9.1.1 by showing that $F(y) \to 0$ as $y \to \infty$. Suppose that $y_n \in [1, \infty)$ is an increasing sequence satisfying $y_n \to \infty$ as $n \to \infty$. We shall prove that $F(y_n) \to 0$ as $n \to \infty$. For every $n \in \mathbb{N}$, define $f_n : X \to \mathbb{R}$ by writing

$$f_n(x) = \mathrm{e}^{-xy_n} k(x)$$

for every $x \geq 0$. Clearly $f_n(x) \to 0$ as $n \to \infty$ for every $x > 0$. On the other hand, since $y_n \geq 1$, we must have $|f_n(x)| \leq \mathrm{e}^{-x}$ for every $x \geq 0$. Suppose that $B \in \mathbb{R}$ is positive and fixed. It is easy to check that for every $n \in \mathbb{N}$, $f_n \in \mathcal{R}([0, B])$ and

$$\int_0^B |f_n(x)| \,\mathrm{d}x \leq \int_0^B \mathrm{e}^{-x} \,\mathrm{d}x < 1.$$

It follows from Theorem 7D that $f_n \in \mathcal{L}(X)$. Furthermore, it is easy to check that the function e^{-x} is Lebesgue integrable on X. It now follows from the Dominated convergence theorem (Theorem 6A) that

$$\lim_{n\to\infty} F(y_n) = \lim_{n\to\infty} \int_0^\infty f_n(x) \,\mathrm{d}x = \int_0^\infty \lim_{n\to\infty} f_n(x) \,\mathrm{d}x = 0.$$

Note that we have used the following result: A function $F : Y \to \mathbb{R}$ satisfies $F(y) \to L$ as $y \to \infty$ if and only if $F(y_n) \to L$ as $n \to \infty$ for every sequence $y_n \in Y$ satisfying $y_n \to \infty$ as $n \to \infty$.

9.2. Differentiability

We have the following result on differentiability, that under suitable conditions, we can differentiate under the integral sign.

THEOREM 9B. *Suppose that $X, Y \subseteq \mathbb{R}$ are intervals. Suppose further that the function $f : X \times Y \to \mathbb{R}$ satisfies the following conditions:*
(a) For every fixed $y \in Y$, the function $f_y : X \to \mathbb{R}$, defined by $f_y(x) = f(x, y)$ for every $x \in X$, is measurable on X.
(b) The partial derivative $\dfrac{\partial}{\partial y} f(x, y)$ exists for every interior point $(x, y) \in X \times Y$.
(c) There exists a non-negative function $G \in \mathcal{L}(X)$ such that $\left|\dfrac{\partial}{\partial y} f(x, y)\right| \leq G(x)$ for every interior point $(x, y) \in X \times Y$.
(d) There exists $y_0 \in Y$ such that $f_{y_0} \in \mathcal{L}(X)$.
Then for every $y \in Y$, the Lebesgue integral

$$\int_X f(x, y) \,\mathrm{d}x$$

exists. Furthermore, the function $F : Y \to \mathbb{R}$, defined by

$$F(y) = \int_X f(x, y) \,\mathrm{d}x$$

for every $y \in Y$, is differentiable at every interior point of Y, and the derivative $F'(y)$ satisfies

$$F'(y) = \int_X \frac{\partial}{\partial y} f(x, y) \,\mathrm{d}x. \tag{2}$$

Remark. Note that condition (b) states that the function $f(x, y)$ is differentiable with respect to y, whereas (1) states that the integral $F(y)$ is differentiable.

Proof of Theorem 9B. Suppose that $(x, y) \in X \times Y$ is an interior point, and that $y \neq y_0$. By the Mean value theorem, there exists ξ between y_0 and y such that

$$f(x, y) - f(x, y_0) = (y - y_0)\frac{\partial f}{\partial y}(x, \xi),$$

so that in view of (c), we have

$$|f(x, y)| \leq |f(x, y_0)| + |y - y_0|G(x).$$

Note that the function on the right hand side is $|f_{y_0}| + |y - y_0|G \in \mathcal{L}(X)$. It follows that the measurable function $f_y \in \mathcal{M}(X)$ is dominated almost everywhere on X by a non-negative function in $\mathcal{L}(X)$. Hence $f_y \in \mathcal{L}(X)$ by Theorem 8A. It remains to prove (2). Suppose that $y \in Y$ is fixed. Let $y_n \in Y \setminus \{y\}$ be a sequence such that $y_n \to y$ as $n \to \infty$. For every $n \in \mathbb{N}$, consider the function $h_n : X \to \mathbb{R}$, defined by

$$h_n(x) = \frac{f(x, y_n) - f(x, y)}{y_n - y}$$

for every $x \in X$. Clearly $h_n \in \mathcal{L}(X)$, and

$$h_n(x) \to \frac{\partial}{\partial y} f(x, y)$$

as $n \to \infty$ at every interior point $x \in X$. By the Mean value theorem, there exists ξ_n between y_n and y such that

$$|h_n(x)| = \left|\frac{\partial f}{\partial y}(x, \xi_n)\right| \leq G(x)$$

for almost all $x \in X$. It follows from the Dominated convergence theorem (Theorem 6A) that

$$\int_X \frac{\partial}{\partial y} f(x, y)\, \mathrm{d}x$$

exists, the sequence

$$\int_X \frac{f(x, y_n) - f(x, y)}{y_n - y}\, \mathrm{d}x$$

converges, and

$$\lim_{n\to\infty} \int_X \frac{f(x, y_n) - f(x, y)}{y_n - y}\, \mathrm{d}x = \int_X \frac{\partial}{\partial y} f(x, y)\, \mathrm{d}x. \tag{3}$$

Note now that

$$\int_X \frac{f(x, y_n) - f(x, y)}{y_n - y}\, \mathrm{d}x = \frac{1}{y_n - y} \int_X (f(x, y_n) - f(x, y))\, \mathrm{d}x = \frac{F(y_n) - F(y)}{y_n - y}.$$

But

$$\lim_{n\to\infty} \frac{F(y_n) - F(y)}{y_n - y} = F'(y). \tag{4}$$

The equality (2) now follows on combining (3) and (4). ♣

EXAMPLE 9.2.1. We continue our investigation of the function $F : (0,\infty) \to \mathbb{R}$, given by

$$F(y) = \int_0^\infty \mathrm{e}^{-xy} k(x)\,\mathrm{d}x,$$

where

$$k(x) = \begin{cases} x^{-1}\sin x & \text{if } x > 0, \\ 1 & \text{if } x = 0. \end{cases}$$

For every $A > 0$, we apply Theorem 9B with $X = [0,\infty)$ and $Y = Y_A = [A,\infty)$ to conclude that

$$F'(y) = -\int_0^\infty \mathrm{e}^{-xy} \sin x\,\mathrm{d}x \tag{5}$$

for every $y > A$. Since $A > 0$ is arbitrary, it follows that (5) holds for every $y > 0$. Suppose that $B \in \mathbb{R}$ is positive and fixed. It is easy to check that the Riemann integral

$$\int_0^B \mathrm{e}^{-xy} \sin x\,\mathrm{d}x = \frac{1}{1+y^2} - \frac{\mathrm{e}^{-By}(y\sin B + \cos B)}{1+y^2}$$

for every $y > 0$. In view of Theorem 7D, we can let $B \to \infty$ and obtain

$$F'(y) = -\int_0^\infty \mathrm{e}^{-xy} \sin x\,\mathrm{d}x = -\frac{1}{1+y^2}$$

for every $y > 0$. For any $y > 0$, choose $B \in \mathbb{R}$ such that $B > y$. Then

$$F(B) - F(y) = -\int_y^B \frac{\mathrm{d}t}{1+t^2} = \tan^{-1} y - \tan^{-1} B.$$

Letting $B \to \infty$, we obtain

$$-F(y) = \tan^{-1} y - \frac{\pi}{2}.$$

In other words,

$$\int_0^\infty \mathrm{e}^{-xy} k(x)\,\mathrm{d}x = \frac{\pi}{2} - \tan^{-1} y$$

for every $y > 0$.

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INTRODUCTION TO LEBESGUE INTEGRATION

W W L CHEN

Chapter 10

DOUBLE LEBESGUE INTEGRALS

10.1. Introduction

The purpose of this chapter is to extend the idea of the Lebesgue integral to functions of more than one real variable. Here we shall restrict ourselves to the case of functions of two real variables. The definitions and results here can be generalized in a very natural way to the case of functions of more than two real variables.

As in the one-variable case, Lebesgue integration for functions of two variables again is a generalization of Riemann integration. However, a new feature here is a result of Fubini which reduces the problem of calculating a two-dimensional integral to the problem of calculating one-dimensional integrals.

Again, our approach is via step functions and upper functions. Many of the details are similar to the one-variable case, and we shall omit some of the details.

We shall first of all make a few remarks on the problem of extending a number of definitions and results on point sets in $\mathbb{R}$ to point sets in $\mathbb{R}^2$. The reader may wish to provide the proofs by generalizing those in Chapter 3.

Remarks. (1) We shall measure distance in $\mathbb{R}^2$ by euclidean distance; in other words, the distance between two points $\mathbf{x}, \mathbf{y} \in \mathbb{R}^2$ is given by $|\mathbf{x} - \mathbf{y}|$, the modulus of the vector $\mathbf{x} - \mathbf{y}$.

(2) We can define interior points in terms of open discs

$$D(\mathbf{x}, \epsilon) = \{\mathbf{y} \in \mathbb{R}^2 : |\mathbf{x} - \mathbf{y}| < \epsilon\}$$

instead of open intervals $(x - \epsilon, x + \epsilon) = \{y \in \mathbb{R} : |x - y| < \epsilon\}$. Then open sets can be defined in the same way as before. Theorems 3A and 3B generalize easily.

(3) The limit of sequences in $\mathbb{R}^2$ can be defined in terms of the euclidean distance discussed in Remark (1). Then closed sets can be defined in the same way as before. Theorems 3D, 3E and 3F generalize easily.

† This chapter was written at Imperial College, University of London, in 1977 while the author was an undergraduate.

(4) The Cantor intersection theorem in $\mathbb{R}^2$ can also be established via the Bolzano-Weierstrass theorem in $\mathbb{R}^2$, a simple consequence of the Bolzano-Weierstrass theorem in $\mathbb{R}$.

(5) An interval (resp. an open interval, a closed interval) in $\mathbb{R}^2$ is defined to be the cartesian product of two intervals (resp. open intervals, closed intervals) in $\mathbb{R}$. If I is an interval in $\mathbb{R}^2$, then $\mu(I)$ denotes its area, the product of the lengths of the two intervals in $\mathbb{R}$ making up the cartesian product. Sets of measure zero and compact sets in $\mathbb{R}^2$ can be defined in the same way as before. Theorems 3L and 3M generalize easily. The Heine-Borel theorem can also be generalized: Any closed and bounded set in $\mathbb{R}^2$ is compact.

(6) As before, a property $P(\mathbf{x})$ is said to hold for almost all $\mathbf{x} \in S$ if $P(\mathbf{x})$ fails to hold for at most a set of measure zero in S.

We next make a few remarks on the problem of extending a number of definitions and results on Riemann integration of functions of one variable to Riemann integration of functions of two variables. The reader may wish to provide the proofs by generalizing those in Chapter 2.

REMARKS. (1) Suppose that a function $f(x,y)$ is bounded on the interval $[A_1,B_1]\times[A_2,B_2]$, where $A_1,B_1,A_2,B_2 \in \mathbb{R}$ satisfy $A_1 < B_1$ and $A_2 < B_2$. Suppose further that

$$\Delta_1 : \ A_1 = x_0 < x_1 < x_2 < \ldots < x_n = B_1 \qquad \text{and} \qquad \Delta_2 : \ A_2 = y_0 < y_1 < y_2 < \ldots < y_m = B_2$$

are dissections of the intervals $[A_1,B_1]$ and $[A_2,B_2]$ respectively. We consider $\Delta = \Delta_1 \times \Delta_2$ to be a dissection of $[A_1,B_1]\times[A_2,B_2]$.

(2) The sums

$$s(f,\Delta) = \sum_{i=1}^{n}\sum_{j=1}^{m}(x_i - x_{i-1})(y_j - y_{j-1}) \min_{\substack{x\in[x_{i-1},x_i]\\ y\in[y_{j-1},y_j]}} f(x,y)$$

and

$$S(f,\Delta) = \sum_{i=1}^{n}\sum_{j=1}^{m}(x_i - x_{i-1})(y_j - y_{j-1}) \max_{\substack{x\in[x_{i-1},x_i]\\ y\in[y_{j-1},y_j]}} f(x,y)$$

are then the lower and upper Riemann sums respectively of $f(x,y)$ corresponding to the dissection Δ.

(3) As before, we define the lower integral by taking the supremum of the lower sums over all dissections Δ of $[A_1,B_1]\times[A_2,B_2]$. Similarly, we define the upper integral by taking the infimum of the upper sums over all dissections Δ of $[A_1,B_1]\times[A_2,B_2]$. If the lower and upper integrals have the same value, then their common value is taken to be the Riemann integral

$$\int_{[A_1,B_1]\times[A_2,B_2]} f(x,y)\,\mathrm{d}(x,y).$$

(4) All the results in Chapter 2 can be extended to the case of functions of two variables, and the proofs are similar but perhaps technically slightly more complicated in a few cases. Note also the very restrictive nature of the generalizations of Theorems 2F and 2G. Also, try to find the strongest generalization of Theorem 2H.

(5) Note at this point that we have not established any criteria for the existence of the integrals

$$\int_{A_1}^{B_1}\left(\int_{A_2}^{B_2} f(x,y)\,\mathrm{d}y\right)\mathrm{d}x \qquad \text{and} \qquad \int_{A_2}^{B_2}\left(\int_{A_1}^{B_1} f(x,y)\,\mathrm{d}x\right)\mathrm{d}y.$$

Finally, we make a few remarks on the problem of extending the idea of Lebesgue integration of functions of one variable to Lebesgue integration of functions of two variables. Our main task is to make suitable generalizations of our definitions. If we follow our approach below, then all the results in

Chapters 4, 5, 6 and 8 admit generalizations to the two-variable case. The reader may wish to provide the detailed proofs.

REMARKS. (1) Suppose that $A_1, B_1, A_2, B_2 \in \mathbb{R}$ satisfy $A_1 < B_1$ and $A_2 < B_2$. We make the following natural extension of the notion of a step function. A function $s : [A_1, B_1] \times [A_2, B_2] \to \mathbb{R}$ is called a step function on $[A_1, B_1] \times [A_2, B_2]$ if there exist dissections $A_1 = x_0 < x_1 < x_2 < \ldots < x_n = B_1$ and $A_2 = y_0 < y_1 < y_2 < \ldots < y_m = B_2$ of $[A_1, B_1]$ and $[A_2, B_2]$ respectively, and numbers $c_{ij} \in \mathbb{R}$ such that for every $i = 1, \ldots, n$ and $j = 1, \ldots, m$, we have $s(x, y) = c_{ij}$ for every $x \in (x_{i-1}, x_i)$ and $y \in (y_{j-1}, y_j)$. For every $i = 1, \ldots, n$ and $j = 1, \ldots, m$, the integral

$$\int_{[x_{i-1},x_i]\times[y_{j-1},y_j]} s(x, y)\, \mathrm{d}(x, y) = c_{ij}(x_i - x_{i-1})(y_j - y_{j-1})$$

is in the sense of Riemann. Also the integral

$$\int_{[A_1,B_1]\times[A_2,B_2]} s(x, y)\, \mathrm{d}(x, y) = \sum_{i=1}^{n}\sum_{j=1}^{m} c_{ij}(x_i - x_{i-1})(y_j - y_{j-1})$$

is in the sense of Riemann, and is in fact independent of the choice of the dissection of $[A_1, B_1] \times [A_2, B_2]$, provided that $s(x, y)$ is constant in any open subinterval arising from the dissections.

(2) We next generalize the definition of step functions to arbitrary intervals $I \subseteq \mathbb{R}^2$ by using a finite subinterval $(A_1, B_1) \times (A_2, B_2) \subseteq I$ such that $s : [A_1, B_1] \times [A_2, B_2] \to \mathbb{R}$ is a step function on $[A_1, B_1] \times [A_2, B_2]$ and $s(x, y) = 0$ for every $x \in I \setminus [A_1, B_1] \times [A_2, B_2]$. The integral over I is then defined as the integral over $[A_1, B_1] \times [A_2, B_2]$. This establishes the collection $\mathcal{S}(I)$ of all step functions on I.

(3) The collections $\mathcal{U}(I)$, $\mathcal{L}(I)$ and $\mathcal{M}(I)$ respectively of all upper functions, all Lebesgue integrable functions and all measurable functions on I are now obtained from the collection $\mathcal{S}(I)$ in a similar way as before.

(4) We can also define measure on $\mathbb{R}^2$ in terms of the characteristic function. Lebesgue integrals over arbitrary measurable sets in $\mathbb{R}^2$ can also be defined in a similar way as before.

10.2. Decomposition into Squares

The following result can be viewed as a generalization of Theorem 3C.

THEOREM 10A. *Every open set $G \subseteq \mathbb{R}^2$ can be expressed as a countable union of disjoint squares whose closures are contained in G.*

REMARK. The closure of a set is the union of the set with the collection of all its limit points. The closure of a square is therefore the square together with its edges.

THEOREM 10B. *Every open set $G \subseteq \mathbb{R}^2$ is measurable. Furthermore, if G is bounded, then $\mu(G)$ is finite.*

THEOREM 10C. *Every closed set $F \subseteq \mathbb{R}^2$ is measurable. Furthermore, if F is bounded, then $\mu(F)$ is finite.*

PROOF OF THEOREM 10A. For every $m \in \mathbb{N}$ and $k_1, k_2 \in \mathbb{Z}$, consider the square

$$S(m, k_1, k_2) = \left[\frac{k_1}{2^m}, \frac{k_1 + 1}{2^m}\right) \times \left[\frac{k_2}{2^m}, \frac{k_2 + 1}{2^m}\right),$$

with closure

$$\overline{S(m, k_1, k_2)} = \left[\frac{k_1}{2^m}, \frac{k_1 + 1}{2^m}\right] \times \left[\frac{k_2}{2^m}, \frac{k_2 + 1}{2^m}\right].$$

It is easy to see that for every $m \in \mathbb{N}$, the collection

$$\mathcal{Q}_m = \{S(m, k_1, k_2) : k_1, k_2 \in \mathbb{Z}\}$$

is pairwise disjoint and countable. Suppose now that $G \subseteq \mathbb{R}^2$ is a given open set. Let

$$S_1 = \bigcup_{\substack{k_1,k_2 \in \mathbb{Z} \\ \overline{S(1,k_1,k_2)} \subseteq G}} S(1, k_1, k_2).$$

For every $m \in \mathbb{N}$ satisfying $m \geq 2$, let

$$S_m = \left(\bigcup_{\substack{k_1,k_2 \in \mathbb{Z} \\ \overline{S(m,k_1,k_2)} \subseteq G}} S(m, k_1, k_2) \right) \setminus (S_1 \cup \ldots \cup S_{m-1}).$$

Finally, let

$$S = \bigcup_{m=1}^{\infty} S_m.$$

Note that for each $m \in \mathbb{N}$, the set S_m is the union of a countable number of squares in $\mathcal{Q}_m$. Also, the sets $S_1, S_2, S_3, \ldots$ are pairwise disjoint. It follows from Theorem 1E that S is a countable union of disjoint squares whose closures are contained in G. Clearly $S \subseteq G$. To prove Theorem 10A, it suffices to prove that $G \subseteq S$, so that $G = S$. Suppose that $(x, y) \in G$. Since G is open, it follows that there exists $\epsilon > 0$ such that

$$(x - \epsilon, x + \epsilon) \times (y - \epsilon, y + \epsilon) \subseteq G.$$

Now choose $m \in \mathbb{N}$ so that $2^{-m} < \epsilon$. Then (the reader is advised to draw a picture) there exist $k_1, k_2 \in \mathbb{Z}$ such that

$$\frac{k_1}{2^m} \leq x < \frac{k_1 + 1}{2^m} \qquad \text{and} \qquad \frac{k_2}{2^m} \leq y < \frac{k_2 + 1}{2^m},$$

so that $(x, y) \in S(m, k_1, k_2)$. It is easy to see that

$$\overline{S(m, k_1, k_2)} = \left[\frac{k_1}{2^m}, \frac{k_1 + 1}{2^m} \right] \times \left[\frac{k_2}{2^m}, \frac{k_2 + 1}{2^m} \right] \subseteq (x - \epsilon, x + \epsilon) \times (y - \epsilon, y + \epsilon) \subseteq G.$$

In other words, there exist $m \in \mathbb{N}$ and $k_1, k_2 \in \mathbb{Z}$ such that

$$(x, y) \in S(m, k_1, k_2) \qquad \text{and} \qquad \overline{S(m, k_1, k_2)} \subseteq G. \tag{1}$$

Let m_0 be the smallest value of $m \in \mathbb{N}$ such that there exist $k_1, k_2 \in \mathbb{Z}$ for which (1) holds. It is easy to see that $(x, y) \in S_{m_0} \subseteq S$. Hence $G \subseteq S$. ♣

Proof of Theorems 10B and 10C. Clearly each square is measurable. The first assertion of Theorem 10B follows from Theorem 10A and the two-dimensional analogue of Theorem 8H. To prove the first assertion of Theorem 10C, note that the set $G = \mathbb{R}^2 \setminus F$ is open and so measurable, so that $\chi_G \in \mathcal{M}(\mathbb{R}^2)$. But $\chi_F = 1 - \chi_G$. Hence $\chi_F \in \mathcal{M}(\mathbb{R}^2)$, whence F is measurable. To complete the proof, note that a bounded measurable set S is contained in a square of finite area $\mu(T)$. Clearly $\mu(S) \leq \mu(T)$. ♣

10.3. Fubini's Theorem for Step Functions

A useful result of Fubini reduces the problem of calculating a two-dimensional integral to the problem of calculating one-dimensional integrals. In this section and the next two, we shall establish this result. The special case for step functions is summarized by the following theorem.

THEOREM 10D. *Suppose that $s \in \mathcal{S}(\mathbb{R}^2)$.*

(a) For each fixed $y \in \mathbb{R}$, the integral $\int_{\mathbb{R}} s(x,y)\,\mathrm{d}x$ exists and, as a function of y, is Lebesgue integrable on $\mathbb{R}$. Furthermore,

$$\int_{\mathbb{R}^2} s(x,y)\,\mathrm{d}(x,y) = \int_{\mathbb{R}} \left(\int_{\mathbb{R}} s(x,y)\,\mathrm{d}x \right) \mathrm{d}y.$$

(b) For each fixed $x \in \mathbb{R}$, the integral $\int_{\mathbb{R}} s(x,y)\,\mathrm{d}y$ exists and, as a function of x, is Lebesgue integrable on $\mathbb{R}$. Furthermore,

$$\int_{\mathbb{R}^2} s(x,y)\,\mathrm{d}(x,y) = \int_{\mathbb{R}} \left(\int_{\mathbb{R}} s(x,y)\,\mathrm{d}y \right) \mathrm{d}x.$$

(c) In particular, we have

$$\int_{\mathbb{R}} \left(\int_{\mathbb{R}} s(x,y)\,\mathrm{d}x \right) \mathrm{d}y = \int_{\mathbb{R}} \left(\int_{\mathbb{R}} s(x,y)\,\mathrm{d}y \right) \mathrm{d}x.$$

Proof. To prove (a), note that there exist $A_1, B_1, A_2, B_2 \in \mathbb{R}$ satisfying $A_1 < B_1$ and $A_2 < B_2$ such that $s : [A_1, B_1] \times [A_2, B_2] \to \mathbb{R}$ is a step function on $[A_1, B_1] \times [A_2, B_2]$ and $s(x,y) = 0$ for every $(x,y) \notin [A_1, B_1] \times [A_2, B_2]$. Hence there exist dissections $A_1 = x_0 < x_1 < x_2 < \ldots < x_n = B_1$ and $A_2 = y_0 < y_1 < y_2 < \ldots < y_m = B_2$ of $[A_1, B_1]$ and $[A_2, B_2]$ respectively, and numbers $c_{ij} \in \mathbb{R}$ such that for every $i = 1, \ldots, n$ and $j = 1, \ldots, m$, we have $s(x,y) = c_{ij}$ for every $x \in (x_{i-1}, x_i)$ and $y \in (y_{j-1}, y_j)$. For every $i = 1, \ldots, n$ and $j = 1, \ldots, m$, the integral

$$\int_{[x_{i-1},x_i]\times[y_{j-1},y_j]} s(x,y)\,\mathrm{d}(x,y) = c_{ij}(x_i - x_{i-1})(y_j - y_{j-1}) = \int_{[y_{j-1},y_j]} \left(\int_{[x_{i-1},x_i]} s(x,y)\,\mathrm{d}x \right) \mathrm{d}y.$$

Summing over $i = 1, \ldots, n$ and $j = 1, \ldots, m$, we obtain

$$\int_{[A_1,B_1]\times[A_2,B_2]} s(x,y)\,\mathrm{d}(x,y) = \int_{[A_2,B_2]} \left(\int_{[A_1,B_1]} s(x,y)\,\mathrm{d}x \right) \mathrm{d}y.$$

Since $s(x,y) = 0$ whenever $(x,y) \notin [A_1, B_1] \times [A_2, B_2]$, the result follows. Part (b) is similar. Part (c) follows immediately on combining (a) and (b). ♣

10.4. Sets of Measure Zero

The generalization of Theorem 10D to upper functions and Lebesgue integrable functions depends on the following result on sets of measure zero.

Definition. Suppose that $S \subseteq \mathbb{R}^2$. For every $y \in \mathbb{R}$, we write $S_1(y) = \{x \in \mathbb{R} : (x,y) \in S\}$. For every $x \in \mathbb{R}$, we write $S_2(x) = \{y \in \mathbb{R} : (x,y) \in S\}$.

THEOREM 10E. *Suppose that $S \subseteq \mathbb{R}^2$, and that $\mu(S) = 0$. Then*
(a) $\mu(S_1(y)) = 0$ for almost all $y \in \mathbb{R}$; and
(b) $\mu(S_2(x)) = 0$ for almost all $x \in \mathbb{R}$.

The proof of Theorem 10E depends on the following equivalent formulation for sets of measure zero.

THEOREM 10F.
(a) *Suppose that* $S \subseteq \mathbb{R}$. *Then* $\mu(S)=0$ *if and only if there exists a sequence of intervals* $I_n \in \mathbb{R}$ *such that*

$$\sum_{n=1}^{\infty} \mu(I_n)$$

is finite and every $x \in S$ *belongs to infinitely many* I_n.
(b) *Suppose that* $S \subseteq \mathbb{R}^2$. *Then* $\mu(S)=0$ *if and only if there exists a sequence of intervals* $J_n \in \mathbb{R}^2$ *such that*

$$\sum_{n=1}^{\infty} \mu(J_n)$$

is finite and every $(x,y) \in S$ *belongs to infinitely many* J_n.

PROOF. The proofs of the two parts are similar, so we shall only prove (a).
($\Rightarrow$) For every $m \in \mathbb{N}$, there exists a sequence of intervals $I_n^{(m)} \subseteq \mathbb{R}$ such that

$$S \subseteq \bigcup_{n=1}^{\infty} I_n^{(m)} \qquad \text{and} \qquad \sum_{n=1}^{\infty} \mu(I_n^{(m)}) < 2^{-m}.$$

Then the collection $\mathcal{Q} = \{I_n^{(m)} : m,n \in \mathbb{N}\}$ is countable and

$$\sum_{I \in \mathcal{Q}} \mu(I) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \mu(I_n^{(m)}) < 1.$$

Clearly every $x \in S$ belongs to infinitely many intervals in $\mathcal{Q}$.
($\Leftarrow$) Given any $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$\sum_{n=N}^{\infty} \mu(I_n) < \epsilon.$$

Since every $x \in S$ belongs to infinitely many I_n, it follows that

$$x \in \bigcup_{n=N}^{\infty} I_n.$$

Hence

$$S \subseteq \bigcup_{n=N}^{\infty} I_n \qquad \text{and} \qquad \sum_{n=N}^{\infty} \mu(I_n) < \epsilon.$$

The result follows. ♣

PROOF OF THEOREM 10E. We shall only prove (a), since (b) is similar. Since $\mu(S)=0$, it follows from Theorem 10F(b) that there exists a sequence of intervals $J_n \in \mathbb{R}^2$ such that

$$\sum_{n=1}^{\infty} \mu(J_n)$$

is finite and every $(x,y) \in S$ belongs to infinitely many J_n. For every $n \in \mathbb{N}$, write $J_n = X_n \times Y_n$, where the intervals $X_n, Y_n \subseteq \mathbb{R}$. Then (note that we slightly abuse notation and use μ to denote measure both in $\mathbb{R}$ and in $\mathbb{R}^2$)

$$\mu(J_n) = \mu(X_n)\mu(Y_n) = \mu(X_n)\int_{\mathbb{R}} \chi_{Y_n}(y)\,\mathrm{d}y,$$

where χ_{Y_n} denotes the characteristic function of Y_n. Consider now the function $g_n : \mathbb{R} \to \mathbb{R}$, defined by $g_n(y) = \mu(X_n)\chi_{Y_n}(y)$ for every $y \in \mathbb{R}$. Clearly $g_n \in \mathcal{L}(\mathbb{R})$, is non-negative, and

$$\sum_{n=1}^{\infty} \int_{\mathbb{R}} g_n(y)\,\mathrm{d}y$$

converges. It follows from the Monotone convergence theorem (Theorem 5D) that

$$\sum_{n=1}^{\infty} g_n(y)$$

converges for almost all $y \in \mathbb{R}$. In other words,

$$\sum_{n=1}^{\infty} \mu(X_n)\chi_{Y_n}(y) \tag{2}$$

converges for almost all $y \in \mathbb{R}$. Suppose now that $y \in \mathbb{R}$ and (2) converges. We shall show that $\mu(S_1(y)) = 0$. We may assume, without loss of generality, that $S_1(y) \neq \emptyset$. Clearly

$$\mathcal{Q}(y) = \{X_n : n \in \mathbb{N} \text{ and } y \in Y_n\}$$

is a countable collection of intervals in $\mathbb{R}$, of total length (2). Furthermore, if $x \in S_1(y)$, then $(x, y) \in S$, so that (x, y) belongs to infinitely many J_n, whence x belongs to infinitely X_n in $\mathcal{Q}(y)$. The result now follows from Theorem 10F(a). ♣

10.5. Fubini's Theorem for Lebesgue Integrable Functions

We complete this chapter by proving the following result.

THEOREM 10G. *Suppose that $f \in \mathcal{L}(\mathbb{R}^2)$. Then*

(a) the Lebesgue integral $\int_{\mathbb{R}} f(x, y)\,\mathrm{d}x$ exists for almost all $y \in \mathbb{R}$, the function $G : \mathbb{R} \to \mathbb{R}$, defined by

$$G(y) = \begin{cases} \int_{\mathbb{R}} f(x, y)\,\mathrm{d}x & \text{if the integral exists,} \\ 0 & \text{otherwise,} \end{cases}$$

is Lebesgue integrable on $\mathbb{R}$, and

$$\int_{\mathbb{R}^2} f(x, y)\,\mathrm{d}(x, y) = \int_{\mathbb{R}} G(y)\,\mathrm{d}y;$$

(b) the Lebesgue integral $\int_{\mathbb{R}} f(x, y)\,\mathrm{d}y$ exists for almost all $x \in \mathbb{R}$, the function $H : \mathbb{R} \to \mathbb{R}$, defined by

$$H(x) = \begin{cases} \int_{\mathbb{R}} f(x, y)\,\mathrm{d}y & \text{if the integral exists,} \\ 0 & \text{otherwise,} \end{cases}$$

is Lebesgue integrable on $\mathbb{R}$, and

$$\int_{\mathbb{R}^2} f(x, y)\,\mathrm{d}(x, y) = \int_{\mathbb{R}} H(x)\,\mathrm{d}y;$$

and

(c) *we have*

$$\int_{\mathbb{R}}\left(\int_{\mathbb{R}} f(x,y)\,\mathrm{d}x\right)\mathrm{d}y = \int_{\mathbb{R}}\left(\int_{\mathbb{R}} f(x,y)\,\mathrm{d}y\right)\mathrm{d}x.$$

PROOF. If $f \in \mathcal{S}(\mathbb{R}^2)$, then the result is given by Theorem 10D. To prove Theorem 10G, we shall first consider the special case when $f \in \mathcal{U}(\mathbb{R}^2)$. If $f \in \mathcal{U}(\mathbb{R}^2)$, then there exists an increasing sequence of step functions $s_n \in \mathcal{S}(\mathbb{R}^2)$ such that $s_n(x,y) \to f(x,y)$ as $n \to \infty$ for all $(x,y) \in \mathbb{R}^2 \setminus S$, where $\mu(S) = 0$, and

$$\lim_{n\to\infty} \int_{\mathbb{R}^2} s_n(x,y)\,\mathrm{d}(x,y) = \int_{\mathbb{R}^2} f(x,y)\,\mathrm{d}(x,y). \tag{3}$$

Note that $(x,y) \in S$ if and only if $x \in S_1(y)$, and that $\mu(S_1(y)) = 0$ in view of Theorem 10E. It follows that for every fixed $y \in \mathbb{R}$, $s_n(x,y) \to f(x,y)$ as $n \to \infty$ for all $x \in \mathbb{R} \setminus S_1(y)$. Note that by Theorem 10D, the integral

$$t_n(y) = \int_{\mathbb{R}} s_n(x,y)\,\mathrm{d}x$$

exists for every $y \in \mathbb{R}$ and, as a function of y, is integrable on $\mathbb{R}$. Furthermore,

$$\int_{\mathbb{R}} t_n(y)\,\mathrm{d}y = \int_{\mathbb{R}}\left(\int_{\mathbb{R}} s_n(x,y)\,\mathrm{d}x\right)\mathrm{d}y = \int_{\mathbb{R}^2} s_n(x,y)\,\mathrm{d}(x,y) \le \int_{\mathbb{R}^2} f(x,y)\,\mathrm{d}(x,y). \tag{4}$$

It is easy to see that t_n is an increasing sequence on $\mathbb{R}$, so that the left hand side of (4) is increasing and bounded above, and so converges. It follows from the Monotone convergence theorem (Theorem 5C) that there exists a function $t \in \mathcal{L}(\mathbb{R})$ such that $t_n(y) \to t(y)$ as $n \to \infty$ for all $y \in \mathbb{R} \setminus T$, where $\mu(T) = 0$, and

$$\int_{\mathbb{R}} t(y)\,\mathrm{d}y = \lim_{n\to\infty} \int_{\mathbb{R}} t_n(y)\,\mathrm{d}y. \tag{5}$$

We also have

$$t_n(y) = \int_{\mathbb{R}} s_n(x,y)\,\mathrm{d}x \le t(y)$$

for all $y \in \mathbb{R} \setminus T$. For any $y \in \mathbb{R} \setminus T$, it follows again from the Monotone convergence theorem (Theorem 5C) that there exists a function $g \in \mathcal{L}(\mathbb{R})$ such that $s_n(x,y) \to g(x,y)$ as $n \to \infty$ for almost all $x \in \mathbb{R} \setminus W_y$, where $\mu(W_y) = 0$, and

$$\int_{\mathbb{R}} g(x,y)\,\mathrm{d}x = \lim_{n\to\infty} \int_{\mathbb{R}} s_n(x,y)\,\mathrm{d}x.$$

This means that

$$f(x,y) = g(x,y) \qquad \text{for all } y \in \mathbb{R} \setminus T \text{ and } x \in \mathbb{R} \setminus (S_1(y) \cup W_y).$$

It follows that for all $y \in \mathbb{R} \setminus T$, the integral

$$\int_{\mathbb{R}} f(x,y)\,\mathrm{d}x$$

exists, and

$$\int_{\mathbb{R}} f(x,y)\,\mathrm{d}x = \int_{\mathbb{R}} g(x,y)\,\mathrm{d}x = \lim_{n\to\infty} \int_{\mathbb{R}} s_n(x,y)\,\mathrm{d}x = t(y). \tag{6}$$

Since $t \in \mathcal{L}(\mathbb{R})$, it follows that $G(y)$ is Lebesgue integrable on $\mathbb{R}$. Combining (3)–(5), we obtain

$$\int_{\mathbb{R}} t(y)\,\mathrm{d}y = \int_{\mathbb{R}^2} f(x,y)\,\mathrm{d}(x,y). \tag{7}$$

Also, (6) gives

$$\int_{\mathbb{R}} t(y)\,\mathrm{d}y = \int_{\mathbb{R}} \left(\int_{\mathbb{R}} f(x,y)\,\mathrm{d}x \right) \mathrm{d}y. \tag{8}$$

We can now combine (7) and (8) to complete the proof of (a) when $f \in \mathcal{U}(\mathbb{R}^2)$. Suppose now that $f \in \mathcal{L}(\mathbb{R}^2)$. Then $f = u - v$, where $u, v \in \mathcal{U}(\mathbb{R}^2)$. Hence

$$\begin{aligned}
\int_{\mathbb{R}^2} f(x,y)\,\mathrm{d}(x,y) &= \int_{\mathbb{R}^2} u(x,y)\,\mathrm{d}(x,y) - \int_{\mathbb{R}^2} v(x,y)\,\mathrm{d}(x,y) \\
&= \int_{\mathbb{R}} \left(\int_{\mathbb{R}} u(x,y)\,\mathrm{d}x \right) \mathrm{d}y - \int_{\mathbb{R}} \left(\int_{\mathbb{R}} v(x,y)\,\mathrm{d}x \right) \mathrm{d}y \\
&= \int_{\mathbb{R}} \left(\int_{\mathbb{R}} (u(x,y) - v(x,y))\,\mathrm{d}x \right) \mathrm{d}y = \int_{\mathbb{R}} \left(\int_{\mathbb{R}} f(x,y)\,\mathrm{d}x \right) \mathrm{d}y.
\end{aligned}$$

This completes the proof of (a). Part (b) is similar. Part (c) is a simple consequence of (a) and (b). ♣

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