LECTURES ON ANALYTIC DIFFERENTIAL EQUATIONS [1]

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Henri Poincaré (April 29, 1854–July 17, 1912)

David Hilbert (January 23, 1862–February 14, 1943)

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Chapter 1

Normal forms and desingularization

1. Holomorphic differential equations

1.1. Differential equations, solutions, initial value problems. Let $U \subseteq \mathbb{C} \times \mathbb{C}^n$ be an open domain and $F = (F_1, \dots, F_n)\colon U \to \mathbb{C}^n$ a holomorphic map (vector function). An *analytic ordinary differential equation* defined by F on U is the vector equation (or the *system* of n scalar equations)

$$\frac{dx}{dt} = F(t,x), \qquad (t,x) \in U \subseteq \mathbb{C} \times \mathbb{C}^n, \quad F \in \mathcal{O}^n(U). \tag{1.1}$$

Solution of this equation is a parameterized holomorphic curve, the holomorphic map $\varphi = (f_1, \dots, f_n)\colon V \to \mathbb{C}^n$, defined in an open subset $V \subseteq \mathbb{C}$, whose graph $\{(t, \varphi(t))\colon t \in V\}$ belongs to U and whose complex "velocity vector" $\frac{d\varphi}{dt} = \bigl(\frac{df_1}{dt}, \dots, \frac{df_n}{dt}\bigr) \in \mathbb{C}^n$ at each point t coincides with the vector $F(t, \varphi(t)) \in \mathbb{C}^n$.

The graph of φ in U is called the *integral curve*. From the real point of view it is a 2-dimensional smooth surface in $\mathbb{R}^{2n+2}$. Note that from the beginning we consider only holomorphic solutions which may be, however, defined on domains of different size.

The equation is *autonomous*, if F is independent of t. In this case the image $\varphi(V) \subseteq \mathbb{C}^n$ is called the *phase curve*. Any differential equation (1.1) can be made autonomous by introducing a fictitious variable $z \in \mathbb{C}$ governed by the equation $\dot z = 1$.

If $(t_0, x_0) = (t_0, x_{0,1}, \dots, x_{0,n}) \in U$ is a specified point, the *initial value problem*, sometimes also called the *Cauchy problem*, is to find an integral curve of the differential equation (1.1) passing through the point (t_0, x_0), i.e., a solution satisfying the condition

$$\varphi\colon V \to \mathbb{C}^n, \qquad \varphi(t_0) = x_0 \in \mathbb{C}^n. \tag{1.2}$$

In what follows we will often denote by dot the derivative with respect to the complex variable t, $\dot x(t) = \frac{dx}{dt}(t)$.

The first fundamental result is the local existence and uniqueness theorem.

Theorem 1.1. *For any holomorphic differential equation* (1.1) *every point* $(t_0, x_0) \in U$ *there exists a sufficiently small polydisk* $D_\varepsilon = \{|t - t_0| < \varepsilon,\ |x_j - x_{0,j}| < \varepsilon,\ j = 1, \dots, n\} \subseteq U$, *such that solution of the initial value problem* (1.2) *exists and is unique in this polydisk.*

This solution depends holomorphically on the initial value $x_0 \in \mathbb{C}^n$ *and on any additional parameters, provided that the vector function* F *depends holomorphically on these parameters.*

From the real point of view, Theorem 1.1 asserts existence of $2n$ functions of *two* independent real variables whose graph is a *surface* in $\mathbb{C}^{n+1} \simeq \mathbb{R}^{2n+2}$, with the tangent plane spanned by two real vectors $\operatorname{Re} F, \operatorname{Im} F$. To derive this theorem from the standard results on existence, uniqueness and differentiability of solutions of smooth ordinary differential equations in the real domain, one should use rather deep results on *integrability of distributions*, see Remark 2.10 below. Rather unexpectedly, the direct proof is *simpler* than in the real case in the part concerning dependence on initial conditions. This proof is given in the next section.

1.2. Contracting map principle. Consider the linear space $\mathcal{A}(D_\rho)$ of functions holomorphic in the polydisk D_ρ and continuous on its closure,

$$\mathcal{A}(D_\rho) = \{f\colon D_\rho \to \mathbb{C} \text{ holomorphic in } D_\rho \text{ and continuous on } \overline{D_\rho}\}. \quad (1.3)$$

This space is naturally equipped with the supremum-norm,

$$\|f\|_\rho = \max_{z \in D_\rho} |f(z)|, \qquad z = (z_1, \dots, z_n) \in \mathbb{C}^n, \quad (1.4)$$

and thus naturally a subspace of the *complete normed* (Banach) space $\mathcal{C}(\overline{D_\rho})$ of continuous complex-valued functions. Though holomorphic functions may have very complicated boundary behavior and thus $\mathcal{A}(U) \subsetneq \mathcal{O}(U)$, they are continuous and therefore for any smaller domain U' relatively compact in U (i.e., when $\overline{U'} \Subset U$), there is an obvious inclusion $\mathcal{A}(U') \subset \mathcal{O}(U)$.

Theorem 1.2. *The space $\mathcal{A}(D_\rho)$ and its vector counterparts $\mathcal{A}^m(D_\rho)$ are complete (Banach) spaces.*

Proof. Any fundamental sequence in $\mathcal{A}(D_\rho)$ is by definition fundamental in the Banach space $\mathcal{C}(\overline{D_\rho})$ and has a uniform limit in the latter space. By the Compactness principle (Theorem 0.5), this limit is again holomorphic in D_ρ, i.e., belongs to $\mathcal{A}(D_\rho)$. □

A map F of a metric space $\mathcal{M}$ into itself is called *contracting*, if $\operatorname{dist}(F(u), F(v)) \leqslant \lambda \operatorname{dist}(u, v)$ for some positive real number $\lambda < 1$ and all $u, v \in \mathcal{M}$.

Theorem 1.3 (Contracting map principle)**.** *Any contracting map $F\colon M \to M$ of a complete metric space M has a unique fixed point $w \in \mathcal{M}$ such that $F(w) = w$.*

This fixed point is the limit of any sequence of iterations $u_{k+1} = F(u_k)$, $k = 0, 1, 2, \dots$ beginning with an arbitrary initial point $u_0 \in M$.

Proof. For any initial point $u_0 \in M$, the sequence u_k, $k = 1, 2, \dots$ is fundamental, since $\operatorname{dist}(u_k, u_{k+1}) \leqslant \lambda^k \operatorname{dist}(u_0, u_1)$ and by the triangle inequality $\operatorname{dist}(u_k, u_l) \leqslant \operatorname{dist}(u_0, u_1)\lambda^k/(1-\lambda)$ for any $k < l$. By completeness assumption, the sequence u_k converges to a limit $w \in M$. Since F is continuous, passing to the limit in the identity $u_{k+1} = F(u_k)$ yields $w = F(w)$.

If w_1, w_2 are two fixed points, then $\operatorname{dist}(w_1, w_2) \leqslant \lambda \operatorname{dist}(F(w_1), F(w_2)) = \lambda \operatorname{dist}(w_1, w_2)$ which is possible only if $\operatorname{dist}(w_1, w_2) = 0$, i.e., when $w_1 = w_2$. □

1.3. Picard operators and their contractivity. Denote by $D_\varepsilon = \{|z - z_0| < \varepsilon,\ |t - t_0| < \varepsilon\} \subset \mathbb{C}^{n+1}$ a polydisk centered at the point $(t_0, x_0) \in U$ and sufficiently small to belong to U.

Definition 1.4. The *Picard operator* $\mathcal{P}$ associated with the differential equation (1.1) and the initial value $(t_0, x_0) \in U$, is the operator

$$\begin{gathered} \mathcal{P}\colon \mathcal{A}^n(D_\varepsilon) \to \mathcal{A}^n(D_\varepsilon), \\ \mathcal{P}f(t,v) = v + \int_{t_0}^{t} F(t, f(z,v))\,dz. \end{gathered} \tag{1.5}$$

Denote by L_0 and L_1 the bounds for the magnitude of F and its Lipschitz constant in U: for any $(t, x), (t, x') \in U$,

$$|F(t,x)| \leqslant L_0, \qquad |F(t,x) - F(t,x')| \leqslant L_1\,|x - x'|. \tag{1.6}$$

Lemma 1.5. *If the polydisk D_ε is sufficiently small, the Picard operator $\mathcal{P}$ (1.5) restricted on $\mathcal{A}^n(D_\varepsilon)$ is well defined and contracting. More precisely, for sufficiently small ε its contraction factor λ does not exceed εL_1, where L_1 is the Lipschitz constant for F in U.*

Proof. Explicit majorizing of the integral shows that

$$|\mathcal{P}f(t,v) - v| \leqslant L_0 \int_0^{|t-t_0|} |dz| \leqslant L_0\varepsilon,$$

so if ε is chosen sufficiently small, the operator $\mathcal{P}$ is well defined on $\mathcal{A}^n(D_\varepsilon)$ and maps this space into itself. For any two vector functions f, f' defined on such small polydisk D_ε, we have by virtue of the same estimate

$$\|\mathcal{P}f - \mathcal{P}f'\| = \sup_{|t-t_0|<\varepsilon} \int_0^{|t-t_0|} L_1\,|f(z,v) - f'(z,v)|\,|dz| \leqslant \varepsilon L_1\,\|f - f'\|.$$

If $\varepsilon L_1 < 1$, the operator $\mathcal{P}$ is contracting. □

Proof of Theorem 1.1. Assume ε be so small that the $\varepsilon L_1 < 1$ so that by Lemma 1.5, the Picard operator $\mathcal{P}$ is contracting. By Theorem 1.2 the fixed point of this operator (which exists by Theorem 1.3 and Lemma 1.5) is a *holomorphic* vector function $f\colon D_\varepsilon \to \mathbb{C}^n$ that satisfies the integral equation

$$f(t,v) = v + \int_{t_0}^{t} F(t, f(z,v))\,dz. \tag{1.7}$$

For each fixed x_0, the function $\varphi(t) = f(t, x_0)$ clearly satisfies both the initial condition (1.2) and the differential equation (1.1). By construction, it depends holomorphically on the initial condition x_0.

To prove holomorphic dependence on additional parameters, one can treat them as fictitious dependent variables. Assume that the vector function $F = F(t, x, y)$ depends holomorphically on additional parameters $y \in \mathbb{C}^m$, and consider the initial value problem (recall that the dot means the derivative $\frac{d}{dt}$)

$$\begin{cases} \dot{x} = F(t, x, y), & x(t_0) = x_0, \\ \dot{y} = 0, & y(t_0) = y_0. \end{cases} \tag{1.8}$$

Solution of this initial value problem is a function $f(t, x, y, x_0, y_0)$ holomorphically depending on all variables. □

Remark 1.6. For a differential equation with the right hand side $F(t, x)$ the *shifted solution* $x'(t) = x(t - y)$, $y \in \mathbb{C}^1$, satisfies the shifted equation $\dot{x}' = F(t-y, x')$ which analytically depends on the parameter y. By Theorem 1.1, this shows that solutions of the initial value problem depend holomorphically also on the t-component of the initial point $(t_0, x_0) \in U$.

1.4. Principal example: exponential formula for linear systems. The proof of the existence theorem is constructive: solution of a differential equation is the uniform limit of its *Picard approximations*, iterations of the Picard operator.

In the simplest case of a differential equation with *constant* (i.e., independent of t, x, y) right hand side $F = \text{const} \in \mathbb{C}^n$ the Picard approximations stabilize immediately: if $f_0(t, v) = v$, then $f_1(t, v) = f_2(t, v) = \cdots = v + tF$.

A *linear system with constant coefficients* is the system of equations

$$\dot{x} = Ax, \qquad x \in \mathbb{C}^n, \quad A \in \operatorname{Mat}(n, \mathbb{C}) \tag{1.9}$$

where $A = \|a_{ij}\|$ is a constant $(n \times n)$-matrix with complex entries independent of t. Reasoning by induction, one can see that the Picard approximations for solution of (1.9) which start with the constant initial term $f_0(t, v) = v$, have the form

$$f_k(t, v) = \left(E + tA + \tfrac{t^2}{2!}A^2 + \cdots + \tfrac{t^k}{k!}A^k\right) v. \tag{1.10}$$

Indeed,

$$\begin{aligned} \mathcal{P} f_k(t, v) &= v + \int_0^t A \cdot \left(E + sA + \cdots + \tfrac{s^k}{k!}A^k\right) v \, ds \\ &= Ev + \left(tA + \cdots + \tfrac{t^{k+1}}{(k+1)!}A^{k+1}\right) v = f_{k+1}(t, v). \end{aligned}$$

These formulas motivate the following fundamental object.

Definition 1.7 (matrix exponential)**.** For an arbitrary constant matrix $A \in \operatorname{Mat}(n, \mathbb{C})$ its *exponential* $\exp A$ is the sum of the matrix series

$$\exp A = E + A + \frac{1}{2!}A^2 + \cdots + \frac{1}{k!}A^k + \cdots . \tag{1.11}$$

Since $|A^k| \leqslant |A|^k$ and since the factorial series $\sum_{k\geqslant 0} r^k/k!$ converges absolutely for all values $r \in \mathbb{R}$, the matrix series (1.11) converges absolutely on the complex linear space $\operatorname{Mat}(n, \mathbb{C}) \simeq \mathbb{C}^{n^2}$ for any finite n.

Note that for any two *commuting* matrices A, B their exponents satisfy the *group identity*

$$\exp(A + B) = \exp A \cdot \exp B = \exp B \cdot \exp A. \tag{1.12}$$

This can be proved by substituting A, B instead of two scalars a, b in the formal identity obtained by expansion of the law $e^a e^b = e^{a+b}$ valid for all $a, b \in \mathbb{C}$.

The explicit formula (1.10) for Picard approximations for the linear system (1.9) immediately proves the following theorem.

Theorem 1.8. *Solution of the linear system $\dot{x} = Ax$, $A \in \operatorname{Mat}(n, \mathbb{C})$, with the initial value $x(0) = v$ is given by the matrix exponential,*

$$x(t) = (\exp tA)\, v, \qquad t \in \mathbb{C}, \quad v \in \mathbb{C}^n. \quad \square \tag{1.13}$$

Remark 1.9. Computation of the matrix exponential can be reduced to computation of a matrix polynomial of degree $\leqslant n-1$ and exponentials of eigenvalues of A. Indeed, assume that A has a Jordan normal form $A = \varLambda + N$, where $\varLambda = \operatorname{diag}\{\lambda_1, \dots, \lambda_n\}$ is the diagonal part and N an upper-triangular part *commuting* with $\varLambda$. Then $\exp \varLambda$ is a diagonal matrix with the exponentials of the eigenvalues of $\varLambda$ on the diagonal, $N^n = 0$ by nilpotency, and therefore

$$\begin{aligned} &\exp[t(\varLambda + N)] = \exp t\varLambda \cdot \exp tN \\ &= \begin{pmatrix} \exp t\lambda_1 & & \\ & \ddots & \\ & & \exp t\lambda_n \end{pmatrix} \cdot \left(E + tN + \frac{t^2}{2!}N^2 + \cdots + \frac{t^{n-1}}{(n-1)!}N^{n-1} \right). \end{aligned}$$

This provides a practical way of solving linear systems with constant coefficients: components of any solution in any basis are linear combinations of *quasipolynomials* $t^k \exp \lambda_j t$, $0 \leqslant k \leqslant n-1$ with complex coefficients.

Remark 1.10 (Liouville–Ostrogradskii formula)**.** From the group property and the multiplicativity of the determinant it follows immediately that the continuous function $f(t) = \det \exp tA$ satisfies the identity $f(t+s) = f(t)f(s)$. The only such functions are exponents, $f(t) = \exp ta$ with some

$a \in \mathbb{C}$. To compute the constant a, it is sufficient to consider the derivative at $t = 0$. By definition of f and a we have

$$\det(E + tA + O(t^2)) = 1 + ta + O(t^2).$$

On the other hand $\det(E + tA + O(t^2)) = 1 + t \operatorname{tr} A + O(t^2)$ as $t \to 0$. This can be proved by fully expanding the determinant. Thus we conclude that $a = \operatorname{tr} A$, i.e.,

$$\forall A \in \operatorname{Mat}(n, \mathbb{C}) \qquad \det \exp A = \exp \operatorname{tr} A. \tag{1.14}$$

1.5. Flow box theorem. Let $f(t, x_0)$ be the holomorphic vector function solving the initial value problem (1.2) for the differential equation (1.1).

Definition 1.11. The *flow map* for a differential equation (1.1) is the vector function of $n + 2$ complex variables (t_0, t_1, v) defined when $|t_0 - t_1|$ is sufficiently small and $(t_0, x) \in U$ by the formula

$$(t_0, t_1, v) \mapsto \Phi_{t_0}^{t_1}(v) = f(t_1, v), \tag{1.15}$$

where $f(t, v)$ is the fixed point of the Picard operator $\mathcal{P}$ (1.7) associated with the initial point t_0.

In other words, $\Phi_{t_0}^{t_1}(v)$ is the value $\varphi(t)$ which takes the solution of the initial value problem with the initial condition $\varphi(t_0) = v$, at the point t_1 sufficiently close to t_0.

Example 1.12. For a linear system (1.9) with constant coefficients, the flow map is linear:

$$\Phi_{t_0}^{t_1}(v) = [\exp(t_1 - t_0)A]\, v.$$

This map is defined for *all* values of t_0, t_1, v.

By Theorem 1.1, Φ is a holomorphic map. Since solution of the initial value problem is unique, it obviously must satisfy the functional equation

$$\Phi_{t_1}^{t_2}(\Phi_{t_0}^{t_1}(x)) = \Phi_{t_0}^{t_2}(x) \tag{1.16}$$

for all t_1, t_2 sufficiently close to t_0 and all x sufficiently close to x_0. Since for any x the vector function $t \mapsto \varphi_x(t) = \Phi_{t_0}^{t}(x)$ is a solution of (1.1), we have

$$\left.\frac{\partial}{\partial t}\right|_{t=t_0,\, x=x_0} \Phi_{t_0}^{t}(x) = -\left.\frac{\partial}{\partial t_0}\right|_{t=t_0,\, x=x_0} \Phi_{t_0}^{t}(x) = F(t_0, x_0).$$

From the integral equation (1.7) it follows that

$$\Phi_{t_0}^{t}(x_0) = x_0 + (t - t_0)F(t_0, x_0) + o(|t - t_0|), \tag{1.17}$$

and therefore the Jacobian matrix of Φ with respect to the x-variable is

$$\left(\frac{\partial \Phi_{t_0}^{t}(x)}{\partial x}\right)_{t=t_0,\, x=x_0} = E. \tag{1.18}$$

Differential equations can be transformed to each other by various transformations. The most important is the (bi)holomorphic equivalence, or holomorphic conjugacy.

Definition 1.13. Two differential equations, (1.1) and

$$\dot{x}' = F'(t', x'), \qquad (t', x') \in U', \tag{1.19}$$

are *conjugated* by the biholomorphism $H\colon U \to U'$ (the *conjugacy*), if H sends any integral trajectory of (1.1) into an integral trajectory of (1.19).

Two systems are *holomorphically equivalent* in their respective domains, if there exists a biholomorphic conjugacy between them.

Clearly, biholomorphically conjugate systems are indistinguishable in everything which concerns properties invariant by biholomorphisms. Finding a simple system biholomorphically equivalent to a given one, is therefore of paramount importance.

Theorem 1.14 (Flow box theorem)**.** *Any holomorphic differential equation* (1.1) *in a sufficiently small neighborhood of any point is biholomorphically conjugated by a suitable biholomorphic conjugacy $H\colon (t,x) \mapsto (t, h(t,x))$ preserving the independent variable t, to the trivial equation*

$$\dot{x}' = 0. \tag{1.20}$$

Proof of the Theorem. Consider the map $H'\colon \mathbb{C}^{n+1} \to \mathbb{C}^{n+1}$ which is defined near the point (t_0, x_0) using the flow map (1.15) for the equation (1.1),

$$H'\colon (t, x') \mapsto (t, \Phi^t_{t_0}(x')), \qquad (t, x') \in (\mathbb{C}^{n+1}, (t_0, x_0)).$$

By construction, it takes the lines $x' = \text{const}$ parallel to the t-axis, into integral trajectories of the equation (1.1), while preserving the value of t.

The Jacobian matrix $\partial H'(t,x')/\partial(t,x')$ of the map H' at the point (t_0, x_0) has by (1.18) the form $\left(\begin{smallmatrix} 1 & \\ * & E \end{smallmatrix}\right)$ and is therefore invertible.

Thus H' restricted on a sufficiently small neighborhood of the point (t_0, x_0), is a biholomorphic conjugacy between the trivial system (1.19), whose solutions are exactly the lines $x' = \text{const}$, and the given system (1.1). The inverse map also preserves t and conjugates (1.1) with (1.19). □

1.6. Vector fields and their equivalence. The above constructions become simpler (after small modification) in the *autonomous* case, when the application $x \mapsto F(x)$ can be considered as a *holomorphic vector field* on its domain $U \subseteq \mathbb{C}^n$. The space of vector fields holomorphic in a domain $U \subseteq \mathbb{C}^n$ will be usually denoted by $\mathcal{D}(U)$.

In the autonomous case translation of the independent variable preserves solutions of the equation

$$\dot{x} = F(x), \qquad F\colon U \to \mathbb{C}^n, \tag{1.21}$$

so the flow map $\Phi_{t_0}^{t_1}$ actually depends only on the difference $t = t_1 - t_0$ and hence will be denoted simply by $\Phi^t(\cdot) = \Phi_0^t(\cdot)$. The functional identity (1.16) takes the form

$$\Phi^t(\Phi^s(x)) = \Phi^{t+s}(x), \qquad t, s \in (\mathbb{C}, 0),\ x \in (\mathbb{C}^n, x_0), \tag{1.22}$$

which means that the maps $\{\Phi^t\}$ form a one-parametric *pseudogroup* of biholomorphisms. ("Pseudo" means that the composition in (1.22) is not always defined. The pseudogroup is a true group, $\Phi^t \circ \Phi^s = \Phi^{t+s}$, if the maps Φ^t are globally defined for all $t \in \mathbb{C}$).

For autonomous equations it is natural to consider biholomorphisms that are *time-independent*.

Definition 1.15. Two vector fields, $F(x)$ holomorphic in U and $F'(x')$ holomorphic in U', $U, U' \subseteq \mathbb{C}^n$, are *biholomorphically equivalent*, if there exists a biholomorphic map $H\colon U \to U'$ conjugating their respective flows,

$$H \circ \Phi^t = \Phi'^t \circ H \tag{1.23}$$

whenever both sides are defined. The biholomorphism H is said to be a *conjugacy* between F and F'.

A conjugacy H maps *phase* curves of the first field into phase curves of the second field; in the similar way the suspension

$$\operatorname{id} \times H\colon (\mathbb{C}, 0) \times U \to (\mathbb{C}, 0) \times U', \qquad (t, x) \mapsto (t, H(x)),$$

maps *integral* curves of the two fields into each other. Differentiating the identity (1.23) in t at $t = 0$, we conclude that the differential $dH(x)$ of a holomorphic conjugacy sends the vector $v = F(x)$ of the first field, attached to a point $x \in U$, to the vector $v' = F'(x')$ of the second field *at the appropriate point* $x' = H(x)$. In the coordinates this property takes the form of the identity

$$\left(\frac{\partial H(x)}{\partial x}\right) \cdot F(x) = F'(H(x)), \tag{1.24}$$

in which the Jacobian matrix $\left(\frac{\partial H}{\partial x}\right) = \left(\frac{\partial h_i}{\partial x_j}\right)$ is involved. The formula (1.24) is sometimes used as the alternative *definition* of the holomorphic equivalence. The third (algebraic, in some sense most natural) way to introduce this equivalence is explained in the next section.

1.7. Vector fields as derivations. It is sometimes convenient to define vector fields in a way independent of the coordinates. Each vector field $F = (F_1, \dots, F_n)$ in a domain $U \subset \mathbb{C}^n$ defines a *derivation* $\mathbf{F}$ of the $\mathbb{C}$-algebra $\mathcal{O}(U)$ of functions holomorphic in U, by the formula

$$\mathbf{F}f(x) = \sum_{j=1}^{n} F_j(x)\,\frac{\partial f}{\partial x_j}. \tag{1.25}$$

Thus we will often denote the vector fields with the components F_i as a differential operator, $\mathbf{F} = \sum F_j \frac{\partial}{\partial x_j}$.

Derivations can be defined in purely algebraic terms as $\mathbb{C}$-linear maps of the algebra $\mathcal{O}(U)$ satisfying the Leibnitz identity,

$$\mathbf{F}(fg) = f(\mathbf{F}g) + (\mathbf{F}f)g.$$

Indeed, any $\mathbb{C}$-linear operator with this property in any coordinate system $(x_1, \dots, x_n)$ defines n functions $F_j = \mathbf{F}x_j$ and (obviously) sends all constants to zero. Any analytic function f can be written $f(x) = f(a) + \sum_1^n h_j(x)\,(x_j - a_j)$ with $h_j(a) = \frac{\partial f}{\partial x_j}(a)$. Applying the Leibnitz rule, we conclude that $(\mathbf{F}f)(a) = \sum_j F_j h_j(a) + 0 \cdot \mathbf{F}h_j = \sum_j F_j \frac{\partial f}{\partial x_j}(a)$, as claimed.

A similar algebraic description can be given for holomorphic maps. With any holomorphic map $H \colon U \to U'$ between two domains $U, U' \subseteq \mathbb{C}^n$ one can associate the *pullback operator* $\mathbf{H} \colon \mathcal{O}(U') \to \mathcal{O}(U)$, acting on $f' \in \mathcal{O}(U')$ by composition, $(\mathbf{H}f')(x) = f'(H(x))$. This operator is a *homomorphism* of commutative $\mathbb{C}$-algebras, a $\mathbb{C}$-linear map respecting multiplication, $\mathbf{H}(f'g') = \mathbf{H}f' \cdot \mathbf{H}g'$ for any $f', g' \in \mathcal{O}(U')$. Conversely, any (reasonably continuous) homomorphism $\mathbf{H}$ between the two algebras is induced by a holomorphic map $H = (h_1, \dots, h_n)$ with $h_i = \mathbf{H}x_i$, where $x_i \in \mathcal{O}(U')$ are the coordinate functions (restricted on U'). The map H is a biholomorphism if and only if $\mathbf{H}$ is an isomorphism of $\mathbb{C}$-algebras.

In this language the action of biholomorphisms on vector fields can be described as a simple *conjugacy of operators*: two derivations $\mathbf{F}$ and $\mathbf{F}'$ of the algebras $\mathcal{O}(U)$ and $\mathcal{O}(U')$ respectively, are said to be conjugated by the biholomorphism $H \colon U \to U'$, if

$$\mathbf{F} \circ \mathbf{H} = \mathbf{H} \circ \mathbf{F}' \tag{1.26}$$

as two $\mathbb{C}$-linear operators from $\mathcal{O}(U')$ to $\mathcal{O}(U)$.

Another advantage of this invariant description is the possibility of defining the *commutator* of two vector fields naturally, as the commutator of the respective differential operators. One can immediately verify that $[\mathbf{F}, \mathbf{F}'] = \mathbf{F}\mathbf{F}' - \mathbf{F}'\mathbf{F}$ satisfies the Leibnitz identity as soon as $\mathbf{F}, \mathbf{F}'$ do, and hence corresponds to a vector field. In coordinates the commutator takes the form

$$[F, F'] = \left(\frac{\partial F'}{\partial x}\right) F - \left(\frac{\partial F}{\partial x}\right) F'. \tag{1.27}$$

However, in the future we will not make difference between the vector fields and derivations.

Example 1.16. If $\mathbf{F} = Ax$, $\mathbf{F}' = A'x$ are two *linear* vector fields, their commutator $[\mathbf{F}, \mathbf{F}']$ is again a linear vector filed with the linearization matrix $A'A - AA'$. It coincides (modulo the sign) with the usual matrix commutator $[A, A']$.

1.8. Rectification of vector fields. Singularities. A straightforward counterpart of the Flow box Theorem 1.14 for holomorphic vector fields holds only if the field is nonvanishing.

Definition 1.17. A point x is a *singular point* (*singularity*) of a holomorphic vector field F, if $F(x_0) = 0$. Otherwise the point is *nonsingular.*

Theorem 1.18 (Rectification theorem)**.** *A holomorphic vector field F is holomorphically equivalent to the constant vector field $F'(x') = (1, 0, \dots, 0)$ in a sufficiently small neighborhood of any nonsingular point.*

Proof. The flow Φ' of the constant vector field F' can be immediately computed: $(\Phi')^t(x') = x' + t \cdot (1, 0, \dots, 0)$. Consider any affine hyperplane $\Pi \subset U$ passing through x_0 and transversal to $F(x_0)$ and the hyperplane $\Pi' = \{x'_1 = 0\}$. Let $t = x'_1 \colon \mathbb{C}^n \to \mathbb{C}$ be the function equal to the first coordinate in $\mathbb{C}^n$, so that $(\Phi')^{-t}(x') \in \Pi'$. Let $h' \colon \Pi' \to \Pi$ be *any* biholomorphism (e.g., linear invertible map). Then the map

$$H' = \Phi^t \circ h \circ (\Phi')^{-t}, \qquad t = t(x'),$$

is a holomorphic map that sends any (parameterized) trajectory of F', passing through a point $x' \in \Pi'$, to the parameterized trajectory of F passing through $x = h(x')$. Being composition of holomorphic maps, H' is also holomorphic, and coincides with h' when restricted on Π'. It remains to notice that the differential of $dH'(x_0)$ the vector $(1, 0, \dots, 0)$ transversal to Π', to the vector $F(x_0)$ transversal to Π. This observation proves that H' is invertible in some sufficiently small neighborhood U of x_0, and the inverse map H conjugates F in U with F' in $H(U)$. □

1.9. One-parametric groups of holomorphisms. The Rectification theorem from §1 can be formulated in the language of germs as follows: *The germ of a holomorphic vector field at a non-singular point, is holomorphically equivalent to the germ of a nonzero constant vector field* at any point.

However, we will mostly be interested at germs of vector fields at *singular points*. The first result is existence of germs of the flow maps $\varPhi^t$ at the singular point, for all values of $t \in \mathbb{C}$.

Proposition 1.19. *Germs of the flow maps $\varPhi^t(\cdot)$ at a singular point of a holomorphic vector field, can be defined for all t and form a one-parametric subgroup of the group* $\operatorname{Diff}(\mathbb{C}^n, 0)$ *of germs of biholomorphisms: $\varPhi^t \circ \varPhi^s = \varPhi^{t+s}$ for any $t, s \in \mathbb{C}$.*

Proof. The existence of the flow maps $\varPhi^t$ for all sufficiently small $t \in (\mathbb{C}, 0)$, the possibility of their composition and validity of the group identity for such small t all follow from Theorem 1.1 and the fact that $\varPhi^t(x_0) = x_0$.

For an arbitrary large value of $t \in \mathbb{C}$ we may define $\varPhi^t$ as the composition of germs of the flow maps $\varPhi^{t_i}$, $i = 1, \dots, N$, taken in any order, where the complex numbers t_i are sufficiently small to satisfy conditions of Theorem 1.1 but added together give t. From the local group identity it follows that the definition does not depend on the particular choice of t_i and preserves the group property. □

Remark 1.20. Proposition 1.19 translates into the algebraic language as follows: for any derivation $\mathbf{F}\colon \mathcal{O}(\mathbb{C}^n, 0) \to \mathcal{O}(\mathbb{C}^n, 0)$ of the algebra of holomorphic germs there exist an one-parametric subgroup $\{\mathbf{H}^t \colon t \in \mathbb{C}\}$ of this algebra, such that $\frac{d}{dt}\big|_{t=0} \mathbf{H}^t = \mathbf{F}$.

2. Holomorphic foliations with singularities

By the Existence/Uniqueness Theorem 1.1, any open connected domain $U \subseteq \mathbb{C}^n$ with a holomorphic vector field F defined on it, can be represented as the disjoint union of connected phase curves passing through all points of U. The Rectification Theorem 1.18 provides a local model for the geometric object called *foliated space* of simply *foliation*.

2.1. Principal definitions.

Speaking informally, a foliation is a partition of the phase space into a continuum of connected sets called *leaves*, which locally look as the family of parallel affine subspaces.

Definition 2.1. The *standard holomorphic foliation* of dimension n (respectively, of codimension m) of a bidisk $B = \{(x, y) \in \mathbb{C}^n \times \mathbb{C}^m \colon |x| < 1,\ |y| < 1\}$ is the disjoint union

$$B = \bigsqcup_{|y|<1} L_y, \qquad L_y = \{|x| < 1\} \times \{y\} \subseteq B. \tag{2.1}$$

The connected holomorphic n-dimensional submanifolds L_y indexed by points of the m-dimensional manifold $\{|y| < 1\}$, are called *leaves* of the standard foliation.

Definition 2.2. A holomorphic foliation $\mathcal{F}$ of a domain $U \subset \mathbb{C}^{n+m}$ (or, more generally, a complex analytic manifold) is the partition (disjoint union) $U = \bigsqcup_\alpha L_\alpha$ into disjoint connected subsets L_α, called *leaves*, which locally is biholomorphically equivalent to the standard holomorphic foliation.

The latter phrase means that each point $a \in U$ admits an open neighborhood $B' \ni a$ and a biholomorphism $H\colon B' \to B$ of B' onto the standard bidisk B, which sends the *local leaves* $L_\alpha \cap B'$ to the leaves of the standard foliation,

$$\forall\alpha\ \exists y = y(\alpha): \qquad H(L_\alpha \cap B') = L_y. \tag{2.2}$$

Sometimes the connected components of the intersections $L_\alpha \cap B'$ are called the *local leaves* of the foliation $\mathcal{F}$ *restricted* on the open subset B'. Different local leaves can belong to one leaf of the initial (non-restricted, global) foliation.

Remark 2.3. The definition of foliation admits several flavors. In the weakest settings the standard foliations are families of parallel balls slicing the real cylinder in $\mathbb{R}^{n+m}$ (the formulas remain the same as in (2.1)), while the local equivalencies H are simply homeomorphisms or smooth maps of low or high differentiability (up to C^∞ or even real analytic).

Moreover, one can require *different* regularity of H along the leaves and in the transversal direction. We will not deal with such exotic cases.

Remark 2.4 (important)**.** The space of leaves of the standard foliation is naturally parameterized by points of a polydisk. In the global case the space of leaves may have very complicated structure even topologically, therefore for indexing the leaves arbitrary sets without any additional structure are used.

Definition 2.5. Two holomorphic foliations $\mathcal{F}$ and $\mathcal{F}'$ defined on the respective subsets U, U', are called *holomorphically equivalent* or *topologically equivalent*, if there exists a biholomorphism $H\colon U \to U'$ (respectively, a homeomorphism) which maps (necessarily biholomorphically or homeomorphically, depending on the context) the leaves of $\mathcal{F}$ to those of $\mathcal{F}'$: $H(L_\alpha) = L'_{\alpha'}$ for some indices α, α'.

The following result is an obvious reformulation of the Rectification theorem in the language of foliations.

Proposition 2.6. *For any holomorphic vector field F without singularities in a domain $U \subseteq \mathbb{C}^n$, partition of U into phase curves of F forms a holomorphic foliation $\mathcal{F}_F$ of (complex) dimension* 1 *and codimension* $n-1$. □

We say that the foliation $\mathcal{F}_F$ is generated by the vector field F. Speaking about foliations rather than about vector fields means that the parametrization of solutions by the (complex) time is to be ignored.

Proposition 2.7. *Two holomorphically equivalent vector fields $F \in \mathcal{D}(U)$ and $F' \in \mathcal{D}(U')$ generate two holomorphically equivalent one-dimensional foliations.*

Conversely, if the foliations $\mathcal{F}, \mathcal{F}'$ generated by two nonsingular vector fields, are holomorphically equivalent by a biholomorphism $H\colon U \to U'$, then there exists a nonvanishing holomorphic function $\rho \in \mathcal{O}(U)$ such that

$$\rho(x) \cdot \left(\frac{\partial H}{\partial x}\right) F(x) = F'(H(x)), \tag{2.3}$$

cf. with (1.24) *and Definition 1.15.*

Proof. The first assertion is obvious immediately. To prove the second, it is sufficient to show that two vector fields generating *the same* holomorphic foliation, differ by a nonvanishing holomorphic scalar factor ρ. This is obvious for the standard foliation: the first component must be nonzero while all other components identically zero. □

2.2. Digression: foliations and integrable distributions. For a given holomorphic foliation $\mathcal{F}$ of dimension n and codimension m, the tangent spaces to leaves at different points are n-dimensional complex spaces in an obvious sense analytically depending on the point.

Such geometric object is called *distribution*. To define formally subspaces analytically depending on parameters, one can choose between the language of holomorphic vector fields and that of holomorphic differential forms.

Definition 2.8. A (holomorphic nonsingular) n-dimensional distribution in a domain $U \subseteq \mathbb{C}^{n+m}$ is either

- a tuple of n holomorphic vector fields $F_1, \dots, F_n \in \mathcal{D}(U)$, linear independent at every point of U, or
- tuple of m holomorphic 1-forms $\omega_1, \dots, \omega_m \in \Lambda^1(U)$, linear independent at every point of U so that $\omega_1 \wedge \cdots \wedge \omega_m \in \Lambda^k(U)$ is nonvanishing.

Two tuples of the same type $\{F_j\}$ and $F_j'\}$ (resp., $\{\omega_i\}$ and $\{\omega_i'\}$ define the same distribution, if $F_j' = \sum_k c_{jk}(x) F_k$, resp., $\omega_i' = \sum_k c_{ik}'(x)\omega_k$) for some holomorphic functions $c_{jk}(x), c_{ik}'(x)$. The forms and the fields defining the same distribution must be dual to each other, $\omega_i \cdot F_j = 0$ for all i, j.

An one-dimensional distribution is usually called a *line field*.

Clearly, any holomorphic foliation defines the corresponding tangent distribution of the same dimension. The converse in general is not true unless $n = 1$.

A holomorphic n-dimensional distribution is called *integrable* in U, if it is tangent to leaves of a nonsingular holomorphic foliation in U.

Theorem 2.9 (Frobenius integrability criteria)**.** *A distribution defined by a tuple of holomorphic vector fields is integrable, if and only if the commutator of any two vector fields belongs to the same distribution, i.e., if*

$$[F_i, F_j] = \sum_{k=1}^{n} c_{ijk} \, F_k \qquad c_{ijk} \in \mathcal{O}(U). \tag{2.4}$$

A distribution defined by a tuple of holomorphic 1-forms is integrable, if and only if the ideal spanned by these forms in the exterior algebra $\Lambda^\bullet(U)$ over $\mathcal{O}(U)$, is closed by the exterior derivative, i.e., if

$$d\omega_i = \sum_{k=1}^{m} c'_{ik} \, \omega_k \wedge \eta_k, \qquad \eta_k \in \Lambda^1(U), \; c_{ik} \in \mathcal{O}(U). \tag{2.5}$$

We will not prove this theorem. Its proof can be derived from the local existence theorem for holomorphic vector fields in the same way as it is done, *mutatis mutandis*, in the C^∞-smooth case in [**War83**].

Remark 2.10. The Frobenius integrability condition trivially holds for $n = 1$. On the other hand, from the real point of view the holomorphic vector field F corresponds to a 2-dimensional distribution generated by *two* vector fields $F_1 = F$ and $F_2 = \sqrt{-1}F$ in $\mathbb{R}^{2n} \simeq \mathbb{C}^n$. The Frobenius integrability condition for this distribution reduces, as one can easily verify, to the Cauchy–Riemann identities for the components of the holomorphic vector field F.

Remark 2.11. In the (complex) 2-dimensional case when $U \subseteq \mathbb{C}^2$ that will be our principal object of studies later, the only nontrivial possibility is one-dimensional distribution that is automatically integrable. It can be defined either by one vector field $F \in \mathcal{D}(U)$ or by one Pfaffian form $\omega \in \Lambda^1(U)$. For many reasons the Pfaffian presentation is more convenient.

2.3. Holonomy. The notion of holonomy intends to be a replacement of the flow of the vector fields in the case when the natural parametrization of the solutions is absent.

Definition 2.12. A (parameterized) *cross-section* to a leaf L of a foliation $\mathcal{F}$ of codimension m on U at a point $a \in U$ is the map $\tau\colon (\mathbb{C}^m, 0) \to (U, a)$ transversal to L. Very often we identify the cross-section with the image of the map τ.

If $\mathcal{F}$ is a *standard* foliation and τ, τ' any two cross-sections (at different, in general) points a, a' of the leaf, say $L_0 = \{y = 0\}$, then any other leaf L_α

sufficiently close to L_0 intersects each cross-section exactly once. This defines in a unique way the holomorphic *correspondence map* $\Delta_{\tau,\tau'}\colon (\tau, a) \to (\tau', a')$: points with the same y-components are mapped into each other. In the charts on τ, τ' defined by the parameterizations, the correspondence map becomes the germ of a holomorphic map from $\operatorname{Diff}(\mathbb{C}^m, 0)$.

The correspondence maps obviously satisfy the identity

$$\Delta_{\tau,\tau''} = \Delta_{\tau',\tau''} \circ \Delta_{\tau,\tau'} \tag{2.6}$$

for any three cross-sections τ, τ', τ'' to the same leaf of the standard foliation.

Taking biholomorphic image of this construction, we arrive to the following conclusion. For any two cross-sections τ, τ' to two sufficiently close points on the same leaf, there exists a uniquely defined correspondence map $\Delta_{\tau,\tau'}$ between the cross-sections that satisfies the identity (2.6) for any third cross-section which is also sufficiently close.

Globalization of this construction associates the correspondence map not with just a pair of cross-sections to the same leaf, but rather with a *path* connecting the base points of these cross-sections. Let L be a leaf of a holomorphic foliation $\mathcal{F}$, τ, τ' two cross-sections cutting L at the points $a, a' \in L$ and $\gamma\colon [0,1] \to L$ an (oriented) path connecting $a = \gamma(0)$ with $a' = \gamma(1)$.

Since the segment $[0,1]$ and its image are compact, one can cover them by finitely many open sets U_j in such a way that in each set the foliation is locally trivial (biholomorphically equivalent to standard). One can insert between the cross-sections τ, τ' sufficiently many intermediate cross-sections τ_j, $j = 1, \dots, k$, $\tau_0 = \tau$, $\tau_k = \tau'$, such that every two consecutive cross-sections τ_j, τ_{j+1} belong to the same domain U_j (for this purpose one has to choose $\tau_j \subset U_{j-1} \cap U_j$. Let $\Delta_{\tau_j,\tau_{j+1}}$ be the corresponding local correspondence maps as defined earlier. The composition

$$\Delta_\gamma = \Delta_{\tau_{k-1},\tau_k} \circ \cdots \circ \Delta_{\tau_0,\tau_1}\colon (\tau, a) \to (\tau', a') \tag{2.7}$$

is a holomorphic map (more precisely, a germ) from $\operatorname{Diff}(\mathbb{C}^m, 0)$, also called the *correspondence map along the path* γ.

The identity (2.6) means that the correspondence map Δ_γ in fact does not depend on the choice of the intermediate cross-sections τ_j. In other words, Δ_γ depends on the *homotopy class* of the path γ (with fixed endpoints) rather than on the path itself.

Choosing another pair of cross-sections at the same endpoints (or another parametrization of the same cross-sections) results in composition of Δ_γ with two biholomorphisms from left and right, so using suitable charts, one can always bring any particular correspondence map Δ_γ to be the identity map. The situation changes completely if there is more than one homotopically distinct path connecting the same endpoints, or, what is the same, when one considers *closed paths*.

Let $a \in L$ be a point on the leaf L of a holomorphic foliation, $\gamma \in \pi_1(L, a)$ a (non-contractible) loop considered modulo the homotopic equivalence, and $\tau\colon (\mathbb{C}^m, 0) \to (U, a)$ a cross-section.

Definition 2.13. The *holonomy map* $\Delta_\gamma\colon (\tau, a) \to (\tau, a)$ is the correspondence map along the closed path $\gamma \in \pi_1(L, a)$.

The *holonomy group* of the foliation $\mathcal{F}$ along the leaf $L \in \mathcal{F}$ is the image of the fundamental group $\pi_1(L, a)$ in the group of holomorphic germs $\operatorname{Diff}(\tau, a)$.

The holonomy group is defined as a subgroup in $\operatorname{Diff}(\mathbb{C}^m, 0)$ modulo a simultaneous conjugacy of all holonomy maps, independently of the choice of the cross-section τ or even the base point $a \in L$. It is an obvious invariant of equivalence: for two (holomorphically or topologically) equivalent foliations $\mathcal{F}, \mathcal{F}'$ the holonomy groups of any two corresponding leaves $L \in \mathcal{F}, L' \in \mathcal{F}'$ are (respectively, holomorphically or topologically) conjugate.

However, for this invariant to be nontrivial, one has to find a leaf of the foliation which would be topologically nontrivial. Such leaves, rare among nonsingular holomorphic foliations, often can be easily found for *foliations with singularities*, or *singular foliations*.

2.4. Singular foliations. Starting from this moment, we consider only one-dimensional foliations unless explicitly stated otherwise.

A vector holomorphic vector field $F \in \mathcal{D}(U)$ defines a nonsingular holomorphic foliation on the complement to its singular locus $\varSigma = \varSigma_F = \{x \in U\colon F(x) = 0\}$. This singular locus can be an arbitrary analytic subset of U. However, very often the foliation can be extended from U on a bigger open subset eventually containing a part of $\varSigma$.

If $U \subset U'$ are two domains and $\mathcal{F}'$ a foliation on the larger domain, than $\mathcal{F}'$ can be restricted on U: by definition, this means the foliation whose leaves are *connected components* of the intersections $L'_\alpha \cap U$ for all leaves $L'_\alpha \in \mathcal{F}$.

Theorem 2.14. *Let U be a connected open domain in $\mathbb{C}^n$ and $0 \not\equiv F \in \mathcal{D}(U)$ a holomorphic vector field with the singular locus $\varSigma \subset U$.*

Then there exists an analytic subset $\varSigma' \subseteq \varSigma$ of complex codimension $\geqslant 2$ in U and the foliation $\mathcal{F}'$ of $U \smallsetminus \varSigma'$ whose restriction on $U \smallsetminus \varSigma$ coincides with the foliation generated by the initial vector field F.

Proof. The assertion needs the proof only when Σ is an analytic hypersurface (a complex analytic set of codimension 1).

Consider an arbitrary *smooth point* $a \in \Sigma$ of the singular locus Σ: nonsmooth points already form an analytic subset $\Sigma_1 \subset \Sigma$ of codimension $\geqslant 2$ in U. Locally near this point Σ can be described by one equation $\{f = 0\}$ with f holomorphic and $df(a) \neq 0$. Let $\nu \geqslant 1$ be the maximal power such that all components $F_1, \dots, F_n$ of the vector field F are divisible by f^ν. By construction, the vector field $f^{-\nu} F$ extends analytically on Σ near a and its singular locus is a *proper* analytic subset $\Sigma_2 \subset \Sigma$ (locally near a). Since the germ of Σ at a is smooth hence irreducible, such subset necessarily has codimension $\geqslant 2$ respective to the ambient space.

The union $\Sigma' = \Sigma_1 \cup \Sigma_2$ has codimension $\geqslant 2$ and in $U \smallsetminus \Sigma'$ the field locally represented as $f^{-\nu} F$ is nonsingular and thus defines a holomorphic foliation $\mathcal{F}'$ extending $\mathcal{F}$ as required. □

Remark 2.15. If U is two-dimensional, the holomorphic vector field F can be replaced by line field defined by a holomorphic 1-form $\omega \in \Lambda^1(U)$ with the singular locus Σ.

Theorem 2.14 means that when speaking about holomorphic foliations with singularities, generated by holomorphic vector fields, one can always assume that the singular locus has codimension $\geqslant 2$; in particular, singularities of holomorphic foliations on the plane (and more generally, on holomorphic surfaces) are *isolated points*. The inverse statement is also true, as was observed in [**Ily72b**].

Theorem 2.16 ([**Ily72b**]). *Assume that $\Sigma \subset U \subseteq \mathbb{C}^n$ is an analytic subset of codimension $\geqslant 2$ and $\mathcal{F}$ a holomorphic nonsingular 1-dimensional foliation of $U \smallsetminus \Sigma$.*

Then near each point $a \in \Sigma$ the foliation $\mathcal{F}$ is generated by a holomorphic vector field F with the singular locus Σ.

Proof. One can always assume that the local coordinates near a are chosen so that the line field tangent to leaves of $\mathcal{F}$, is not everywhere parallel to the first coordinate axis. Then this line field is spanned by the meromorphic vector field $G = (1, G_2, \dots, G_n)$, where G_j are *meromorphic* functions in $U \smallsetminus \Sigma$. By E. Levi's theorem, any meromorphic function can be meromorphically extended on analytic subsets of codimension $\geqslant 2$ [**GH78**, Chapter III, §2]. Therefore we may assume that G_j are in fact meromorphic in U. Decreasing if necessary the size of U, each G_j can be represented as the ratio $G_j = P_j/Q_j$, where $P_j, Q_j \in \mathcal{O}(U)$ are holomorphic in U and the representation is irreducible.

Let $\Sigma_j = \{P_j = Q_j = 0\}$, $j = 2, \dots, n$: by irreducibility, Σ_j is of codimension $\geqslant 2$, so $\bigcap_{j\geqslant 2} \Sigma_j$ is also of codimension $\geqslant 2$. Passing to another representation $G_j = F_j/F_1$ with the denominator $F_1 = Q_2 \cdots Q_n$, and the numerators $F_j = P_j Q_2 \cdots Q_n$, we conclude that the vector field $F = (F_1, \dots, F_n)$ is tangent to the foliation $\mathcal{F}$ in $U \smallsetminus \Sigma$.

It remains to show that the singular locus Σ' of the field F *coincides* with Σ locally in U. In one direction it is obvious: if Σ' is *smaller* than Σ, this means that $\mathcal{F}$ is analytically extended as a nonsingular holomorphic foliation to some parts of Σ, contrary to the assumption that Σ is the minimal singular locus. Assume that Σ' is *larger* than Σ, i.e., there exists a nonsingular point $b \in U \smallsetminus \Sigma$ at which F vanishes. Since the foliation $\mathcal{F}$ is biholomorphically equivalent to the standard foliation near b, in the suitable chart F is parallel to the first coordinate axis, so that singular points of F are zeros of its first component. But by construction Σ' is of codimension $\geqslant 2$ and hence cannot be the zero locus of any holomorphic function. The contradiction proves that $\Sigma' \cap U$ cannot be larger than $\Sigma \cap U$. □

Example 2.17. The vector field $\frac{\partial}{\partial x} + e^{1/x} \frac{\partial}{\partial y}$ is analytic outside the line $\Sigma = \{x = 0\}$ of codimension 1 on the plane and defines a holomorphic foliation in $\mathbb{C}^2 \smallsetminus \Sigma$. This foliation cannot be defined by a vector field holomorphically extendable on Σ, which shows that the condition on the codimension in Theorem 2.16 cannot be relaxed.

Together Theorems 2.14 and 2.16 motivate the following concise definition. Since we never consider holomorphic foliations of dimension other than 1, this is explicitly included in the definition.

Definition 2.18. A *singular holomorphic foliation* in a domain U (or a complex analytic manifold) is a holomorphic foliation $\mathcal{F}$ with complex one-dimensional leaves in the complement $U \smallsetminus \Sigma$ to an analytic subset Σ of codimension $\geqslant 2$, called the *singular locus* of $\mathcal{F}$.

The second part of Proposition 2.7 motivates the following important definitions.

Definition 2.19. Two holomorphic vector fields $F \in \mathcal{D}(U)$, $F' \in \mathcal{D}(U')$ are *holomorphically orbitally equivalent* if the singular foliations $\mathcal{F}, \mathcal{F}'$ they generate, are holomorphically equivalent, i.e., there exists a biholomorphism $H \colon U \to U'$ which maps the singular loci of F and F' into each other and is a biholomorphism of foliations outside these loci.

From Proposition 2.7 it follows that for two holomorphically orbitally equivalent fields there exists a holomorphic function ρ satisfying (2.3) and non-vanishing on Σ. Since Σ has codimension $\geqslant 2$, ρ must be nonvanishing *everywhere* on U.

Changing only one adjective in Definition 2.19 (requiring that H be merely a homeomorphism), we obtain the definition of *topologically* orbitally equivalent vector fields. This weaker equivalence cannot be translated into a formula similar to (2.3).

2.5. Complex separatrices. Foliations generated by linear systems. Foliations with singularities much more frequently have non-simply connected leaves which may carry a nontrivial holonomy group. Recall that a (singular) analytic curve $S \subset U$ is a complex analytic set of complex dimension 1 (at its smooth points). The *irreducible* germ of a complex analytic curve at any its point (smooth or singular) admits *uniformization* (local parametrization). As shown in [**Chi89**, §6], if S is an analytic curve in $S \subset (\mathbb{C}^n, 0)$ having singularity at the origin, then there exists a holomorphic *non-constant and injective* map

$$\gamma\colon (\mathbb{C}^1, 0) \to (\mathbb{C}^n, 0), \qquad t \mapsto \gamma(t) \in S.$$

Let $\mathcal{F}$ be a singular holomorphic foliation on an open domain U with the singular locus Σ.

Definition 2.20. A *complex separatrix* of a singular point $a \in \Sigma$ is a leaf $L \subset U \setminus \Sigma$ whose closure $\overline{L} \subset (U, a)$ is the germ of an analytic curve.

Since the leaves are by definition connected, the closure is irreducible (as a germ) at any its point, hence (by the above uniformization arguments) the complex separatrix is topologically a punctured disk near the singularity. The fundamental group of the separatrix is nontrivial (infinite cyclic), thus the holomorphic map generating the *local holonomy group* is an invariant of the singular foliation. Note that the leaves are naturally oriented by their complex structure, so the loop generating the local fundamental group is uniquely defined modulo free homotopy.

The rest of this section consists of examples, some of them very important for future applications.

Example 2.21. Consider first the singular foliation spanned by a *diagonal* linear system

$$\dot{x} = Ax, \qquad A = \operatorname{diag}\{\lambda_1, \dots, \lambda_n\}, \quad \lambda_j \neq 0, \tag{2.8}$$

This foliation has an isolated singularity (of codimension n) at the origin, and all coordinate axes are complex separatrices.

Consider the first coordinate axis $S_1 = \{x_2 = \cdots = x_n = 0\}$ and the separatrix $L_1 = S_1 \setminus \{0\}$. The loop $\gamma = \{|x_1| = 1\}$ parameterized counterclockwise is the canonical generator of L_1. Choose the affine hyperplane $\tau = \{x_1 = 1\} \subset \mathbb{C}^n$ as the cross-section to S_1 at the point $(1, 0, \dots, 0) \in S_1$.

A solution of the system (the parameterized leaf of the foliation) passing through the point $(1, b_2, \dots, b_n) \in \tau$ looks as follows,

$$\mathbb{C}^1 \ni t \mapsto x(t) = (\exp \lambda_1 t, b_2 \exp \lambda_2 t, \dots, b_n \exp \lambda_n t) \in \mathbb{C}^n.$$

The image of the straight line segment $[0, 2\pi i/\lambda_1] \subset \mathbb{C}$ on the t-plane coincides with the loop γ when $b = 0$ (i.e., on the separatrix S_1) and is uniformly close to this loop on all leaves near S_1. The endpoints $x(2\pi i/\lambda_1)$ all belong to τ and hence the holonomy map $M_1 \colon \mathbb{C}^{n-1} \to \mathbb{C}^{n-1}$ is linear diagonal,

$$b \mapsto M_1 b, \qquad M_1 = \operatorname{diag}\{2\pi i \lambda_j/\lambda_1\}_{j=2}^n. \tag{2.9}$$

The other holonomy maps M_k for the canonical loops on the separatrices S_k parallel to the kth axis, are obtained by obvious re-labelling of the indices.

However, it would be wrong to think that the linearity of the holonomy map is an automatic consequence of the linearity of the corresponding vector field.

Example 2.22. Let $n = 2$. Consider the vector field $F = (x + y)\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$ corresponding to a linear vector field with a nontrivial Jordan matrix. The corresponding singular foliation has only one complex separatrix, the punctured axis $S = \{y = 0\}$.

Consider the standard cross-section $\tau = \{x = 1\}$. Solutions of the differential equation with the initial condition (x_0, y_0) can be written explicitly,

$$x(t) = (x_0 + ty_0) \exp t, \quad y = y_0 \exp t.$$

Let $t(y_0)$ be another moment of complex time when the leaf close to the separatrix crosses again τ after continuation along a path close to the standard loop on the separatrix: by definition, this means that we consider the initial point with $x_0 = 1$ and $x(t(y_0)) \equiv 1$, i.e., $1 + t(y_0)y_0 = 1/\exp t(y_0)$. *If* the holonomy map is linear, then $y(t(y_0)) = \lambda y_0$ identically in y_0, i.e., $\exp t(y_0) = \lambda$ is a constant. Substituting this into the previous identity, we obtain $1 + t(y_0)y_0 = 1/\lambda$. This is impossible in the limit $y_0 \to 0$ unless $\lambda = 1$. On the other hand, $\lambda = 1$ is also impossible since $t(y_0) \not\equiv 0$.

Thus the holonomy map cannot be linear. The principal term of this map in a more general settings is computed in Proposition 7.15.

3. Formal flows and embedding theorem

The assumption on convergence of Taylor series for the right hand sides of differential equations and their respective solutions is a very serious restriction: if it holds, then one can use various geometric tools as described in §2. However, considerable information can be gained without the convergence assumption, on the level of *formal power* (*Taylor*) *series*. For natural reasons, the corresponding results have more algebraic flavor.

In this section we introduce the class of formal vector fields and formal morphisms (maps), and prove that the flow of any such formal field can be correctly defined as a formal automorphism. The correspondence "field $\mapsto$ flow" can be inverted for maps with unipotent linearization: as was shown by F. Takens in 1974, any such formal map can be embedded in a unique formal flow [**Tak01**].

The subsequent section §4 contains classification of formal vector fields by the natural action of formal changes of variables.

3.1. Formal vector fields and formal morphisms. For convenience, we will always assume that all Taylor series are centered at the origin.

Definition 3.1. A formal (Taylor) series at the origin in $\mathbb{C}^n$ is an expression

$$f = \sum_{\alpha} c_\alpha x^\alpha, \qquad \alpha \in \mathbb{Z}_+^n, \quad c_\alpha \in \mathbb{C}. \tag{3.1}$$

The minimal degree $|\alpha|$ corresponding to a nonzero coefficient c_α, is called the *order* of f.

The set of all formal series is denoted by $\mathbb{C}[[x]] = \mathbb{C}[[x_1, \dots, x_n]]$. It is a commutative *infinite-dimensional* algebra over $\mathbb{C}$ which contains as a proper subset the algebra of germs of holomorphic functions, isomorphic to the algebra $\mathbb{C}\{x_1, \dots, x_n\}$ of *converging* series.

Definition 3.2. The *canonical basis* of $\mathbb{C}[[x]]$ is the collection of all monomials x^α, $\alpha \in \mathbb{Z}_+^n$, ordered in the following way: (i) all monomials of lower degree $|\alpha|$ precede all monomials of higher degree, and (ii) all monomials of the same degree are ordered lexicographically. This order will be denoted `deglex`-order.

Since the series may diverge, evaluation of $f(x_0)$ at any point $x_0 \in \mathbb{C}^n$ other than $x_0 = 0$, makes no sense. However, without risk of confusion we will denote the free term of a series $f \in \mathbb{C}[[x]]$ by $f(0)$ and the coefficient c_α by $\frac{1}{\alpha!} \frac{\partial^\alpha f}{\partial x^\alpha}(0)$. Under these agreements the Taylor formula becomes a *definition* of the Taylor series $f = \sum_{\alpha \geqslant 0} \frac{1}{\alpha!} \frac{\partial^\alpha f}{\partial x^\alpha}(0)\, x^\alpha$.

All formal partial derivatives $\partial^\alpha f / \partial x^\alpha$ of a formal series f are well defined in the class $\mathbb{C}[[x]]$ as termwise derivatives.

The subset of $\mathbb{C}[[x]]$ which consists of formal series without the free term, is (as one can easily verify) a maximal ideal of the commutative ring $\mathbb{C}[[x]]$: it will be denoted by

$$\mathfrak{m} = \{f \in \mathbb{C}[[x]] \colon f(0) = 0\} = \left\{ \sum\nolimits_{|\alpha| \geqslant 1} c_\alpha x^\alpha \right\}.$$

For any finite $k \in \mathbb{N}$ the space of kth order jets can be described as the quotient

$$J^k(\mathbb{C}^n, 0) = \mathbb{C}[[x_1, \dots, x_n]]/\mathfrak{m}^{k+1},$$

As a quotient ring, the affine finite-dimensional $\mathbb{C}$-space $J^k(\mathbb{C}^n, 0)$ inherits the structure of a commutative $\mathbb{C}$-algebra.

Definition 3.3. The *truncation* of formal series to a finite order k is the canonical projection map $j^k \colon \mathbb{C}[[x]] \to J^k(\mathbb{C}^n, 0)$, $f \mapsto f \bmod \mathfrak{m}^{k+1}$.

The name comes from the natural identification of $J^k(\mathbb{C}^n, 0)$ with polynomials of degree $\leqslant k$ in the variables $x_1, \dots, x_n$. If $l > k$ is a higher order, then $\mathfrak{m}^{l+1} \subset \mathfrak{m}^{k+1}$ so that the truncation operator j^k naturally "descends" as the projection $J^l(\mathbb{C}^n, 0) \to J^k(\mathbb{C}^n, 0)$ which will also be denoted by j^k.

In other words, a formal Taylor series $f \in \mathbb{C}[[x]]$ uniquely defines the k-jet $j^k f$ of any finite order k so that $\mathbb{C}[[x_1, \dots, x_n]]$ is in a sense the limit of the jet spaces $J^k(\mathbb{C}^n, 0)$ as $k \to \infty$. We will sometimes refer to formal series as *infinite* jets and write $\mathbb{C}[[x_1, \dots, x_n]] = J^\infty(\mathbb{C}^n, 0)$.

The canonical monomial basis in $\mathbb{C}[[x]]$ projects into canonically `deglex`-ordered monomial bases in all jet spaces $J^k(\mathbb{C}^n, 0)$. Convergence in $\mathbb{C}[[x]]$ is defined via finite truncations.

Definition 3.4. A sequence $\{f_j\}_{j=1}^\infty \subset \mathbb{C}[[x]]$ is said to be convergent, if and only if all its truncations $j^k f_j$ converge in the respective finite-dimensional k-jet space $J^k(\mathbb{C}^n, 0)$ for all $k \geqslant 0$.

Remark 3.5 (important)**.** All formal algebraic constructions described above can be implemented over the field $\mathbb{R}$ rather than $\mathbb{C}$ as the ground field. Moreover, for future purposes we will need the algebra $\mathfrak{A}[[x]]$ of formal power series in the indeterminates $x = (x_1, \dots, x_n)$ with the coefficients belonging to more general $\mathbb{C}$- or $\mathbb{R}$-algebras $\mathfrak{A}$. The principal examples are the algebras $\mathfrak{A} = \mathbb{C}[\lambda_1, \dots, \lambda_m]$ of polynomials in auxiliary indeterminates or the algebra $\mathcal{A} = \mathcal{O}(U)$ of holomorphic functions of additional variables $\lambda_1, \dots, \lambda_m$.

After introducing the algebra of "formal functions" we can define formal vector fields and formal maps via their algebraic (functorial) properties as in §1.7.

With any *vector formal series* $F = (F_1, \dots, F_n)$ (n-tuple of elements from $\mathbb{C}[[x]]$) one can associate a derivation $\mathbf{F} = \sum_1^n F_j \partial/\partial x_j$ of the algebra $\mathbb{C}[[x]]$, a $\mathbb{C}$-linear application satisfying the Leibniz rule (cf. with (1.25)),

$$\mathbf{F} \colon \mathbb{C}[[x]] \to \mathbb{C}[[x]], \qquad \mathbf{F}(gh) = g\,(\mathbf{F}h) + h\,(\mathbf{F}g).$$

Conversely, any derivation $\mathbf{F}$ of $\mathbb{C}[[x]]$ is of the form $\mathbf{F} = \sum_1^n F_j \partial/\partial x_j$ with the components $F_j = \mathbf{F}x_j$. By a *formal vector fields*, we mean both realizations, $F \in \mathbb{C}[[x]]^n$ or $\mathbf{F} \in \operatorname{Der}(\mathbb{C}[[x]])$. The field F is said to have *singularity* (at the origin), if all these series are without free terms, $F_j(0) = 0$, $j = 1, \dots, n$.

The collection of formal vector fields will be denoted $\mathcal{D}[[\mathbb{C}^n, 0]]$. It is a $\mathbb{C}$-linear (infinite dimensional) space which possesses additional algebraic structures of the *module* over the ring $\mathbb{C}[[x]]$. The *commutator* (bracket) of formal fields is defined in the usual way as $[\mathbf{F}, \mathbf{G}] = \mathbf{FG} - \mathbf{GF}$.

In a parallel way, a vector formal series $H = (h_1, \dots, h_n) \in \mathbb{C}[[x]]^n$ can be identified with a *homomorphism* $\mathbf{H}$ of the algebra $\mathbb{C}[[x]]$ if $H(0) = 0$, i.e., $h_j \in \mathfrak{m}$. Under this assumption, for any formal series $f = \sum_\alpha c_\alpha x^\alpha \in \mathbb{C}[[x]]$ one can correctly define the *substitution*

$$\mathbf{H}f(x) = f(H(x)) = \sum_{\alpha \geqslant 0} c_\alpha h^\alpha = \sum_{\alpha \geqslant 0} c_\alpha h_1^{\alpha_1}(x) \cdots h_n^{\alpha_n}(x). \tag{3.2}$$

Indeed, any k-truncation of $f(H(x))$ is completely determined by the k-truncations of f and H.

The operator $\mathbf{H}$ defined by (3.2), is a *homomorphism* of the algebra, a $\mathbb{C}$-linear map respecting the multiplication,

$$H \colon \mathbb{C}[[x]] \to \mathbb{C}[[x]], \qquad \mathbf{H}(fg) = \mathbf{H}f \cdot \mathbf{H}f.$$

Conversely, any homomorphism preserving convergence in $\mathbb{C}[[x]]$ is of the form $f \mapsto f \circ H$ for an appropriate vector series $H \in \mathbb{C}[[x]]^n$ with the *components* $h_j = \mathbf{H}x_j \in \mathbb{C}[[x]]$. By a *formal map* we mean either H or $\mathbf{H}$, depending on the context. If $\mathbf{H}$ is an homomorphism, then it must map the maximal ideal $\mathfrak{m} \subset \mathbb{C}[[x]]$ into itself and hence $h_j(0) = 0$, $j = 1, \dots, n$, which can be abbreviated to $H(0) = 0$.

If $\mathbf{H}$ is invertible (an isomorphism of the algebra $\mathbb{C}[[x]]$), we say about a *formal isomorphism* of $\mathbb{C}^n$ at the origin. The collection of such isomorphisms forms a *group* denoted by $\operatorname{Diff}[[\mathbb{C}^n, 0]]$ with the operation of composition. The latter can be defined either via substitution of the series, or as the composition of the operators acting on $\mathbb{C}[[x]]$.

Since the maximal ideal $\mathfrak{m}$ is preserved by any formal map $\mathbf{H} \in \operatorname{Diff}[[\mathbb{C}^n, 0]]$ and any *singular* formal vector field $\mathbf{F} \in \mathcal{D}[[\mathbb{C}^n, 0]]$, $F(0) = 0$,

$$\mathbf{H}(\mathfrak{m}) = \mathfrak{m}, \qquad \mathbf{F}(\mathfrak{m}) \subseteq \mathfrak{m},$$

truncation of the series at the level of k-jets commutes with the action of $\mathbf{H}$ and $\mathbf{F}$, therefore defining correctly the isomorphism $j^k\mathbf{H} \colon J^k(\mathbb{C}^n, 0) \to J^k(\mathbb{C}^n, 0)$ and derivation $j^k\mathbf{F} \colon J^k(\mathbb{C}^n, 0) \to J^k(\mathbb{C}^n, 0)$ respectively, which can be identified with the k-jets of the formal map H and the formal vector field F.

3.2. Inverse function theorem. For future purposes we will need the formal inverse function theorem.

Theorem 3.6. *Let H be a formal map with the linearization matrix $A = \left(\frac{\partial H}{\partial x}\right)(0)$ which is nondegenerate. Then H is invertible in* $\operatorname{Diff}[[\mathbb{C}^n, 0]]$.

If $A = E$ is the identity matrix and $H = (h_1, \dots, h_n)$, $h_i(x) = x_i + v_i(x) \bmod \mathfrak{m}^{k+1}$, where v_i are homogeneous polynomials of degree $k \geqslant 2$, then the formal inverse map $H^{-1} = (h'_1, \dots, h'_n)$ has the components $h'_i(x) = x_i - v_i(x) \bmod \mathfrak{m}^{k+1}$.

Clearly, it the first assertion of the Theorem follows from the second assertion applied to the formal map $A^{-1}H$.

The following definition will play important role throughout this section.

Definition 3.7. A finite-dimensional linear operator $A\colon \mathbb{C}^n \to \mathbb{C}^n$ is *unipotent*, if $A - E$ is nilpotent, $(A - E)^n = 0$.

Lemma 3.8. *If H is a formal map with the identical linearization matrix, then $j^k\mathbf{H}$ is an unipotent map as an automorphism of any jet algebra $J^k(\mathbb{C}^n, 0)$.*

Proof. For any monomial x^α from the canonical basis, $\mathbf{H}x^\alpha = x^\alpha +$(higher order terms)$=$ $x^\alpha +$(linear combination of monomials of higher `deglex`-order). □

Proof of Theorem 3.6. Consider the homomorphism $\mathbf{H}$ of $\mathbb{C}[[x]]$ and denote $\mathbf{N} = \mathbf{H} - \mathbf{E}$ the formal "finite difference" operator ($\mathbf{E} = \mathrm{id}$ denotes the identical operator), $\mathbf{N}f(x) = f(H(x)) - f(x)$ (in the sense of the substitution of formal series). By Lemma 3.8, all finite truncations $j^k\mathbf{N}$ are nilpotent.

Define the operator H^{-1} as the series

$$\mathbf{H}^{-1} = \mathbf{E} - \mathbf{N} + \mathbf{N}^2 - \mathbf{N}^3 \pm \cdots . \tag{3.3}$$

This series converges (in fact, stabilizes) after truncation to any finite order because of the above nilpotency, hence by definition converges to an operator on $\mathbb{C}[[x]]$ satisfying the identities $\mathbf{H} \circ \mathbf{H}^{-1} = \mathbf{H}^{-1} \circ \mathbf{H} = \mathbf{E}$. It is an homomorphism of algebra(s), since for any $a, b \in \mathbb{C}[[x]]$ and their images $a' = \mathbf{H}a$, $b' = \mathbf{H}b$ which also can be chosen arbitrarily, we have $\mathbf{H}(ab) = a'b'$ and therefore

$$\mathbf{H}^{-1}(a'b') = \mathbf{H}^{-1}\mathbf{H}(ab) = ab = (\mathbf{H}^{-1}a')(\mathbf{H}^{-1}b').$$

Direct computation of the components of the inverse map yields

$$h'_i = \mathbf{H}^{-1}x_i = x_i - \mathbf{N}x_i + \cdots = x_i - (h_i(x) - x_i) + \cdots = x_i - v_i(x) + \cdots$$

as asserted by the Theorem. □

The above formal construction is nothing else than the recursive computation of the Taylor coefficients of the formal inverse map $H^{-1}(x)$. Note that stabilization of truncations of the series (3.3) means that computation of the terms of any finite degree k of the components h'_i of the inverse map is achieved in finite number of steps.

3.3. Integration and formal flow of formal vector fields. Consider an (autonomous) formal ordinary differential equation

$$\dot{x} = F(x), \qquad F = (F_1, \dots, F_n) \in \mathcal{D}[[\mathbb{C}^n, 0]] \simeq \mathbb{C}[[x]]^n \tag{3.4}$$

with a *formal* right hand side part F. Since evaluation of a formal series at any point other than the origin makes no sense, the "standard" definition of solutions can at best be applied to constructing a solution with the initial condition $x(0) = 0$. Yet in the most interesting case when $F(0) = 0$, this solution is trivial, $x(t) \equiv 0$.

The alternative, suggested by Remark 1.20, is to define a *one-parametric subgroup of formal automorphisms* $\{\mathbf{H}^t : t \in \mathbb{C}\} \subset \operatorname{Diff}[[\mathbb{C}^n, 0]]$ satisfying the condition

$$\mathbf{H}^t \circ \mathbf{H}^s = \mathbf{H}^{t+s} \qquad \forall t, s \in \mathbb{C}, \qquad \mathbf{H}^0 = \mathbf{E}.$$

The corresponding condition for formal maps takes the form $H^t(H^s(x)) = H^s(H^t(x)) = H^{t+s}(x)$.

This subgroup is said to be *holomorphic*, if all finite truncations $j^k\mathbf{H}^t$ depend holomorphically on t. For a holomorphic subgroup the derivative

$$\mathbf{F} = \left.\frac{d\mathbf{H}^t}{dt}\right|_{t=0} = \lim_{t\to 0} t^{-1}(\mathbf{H}^t - \mathbf{E}) \colon \mathbb{C}[[x]] \to \mathbb{C}[[x]]. \tag{3.5}$$

is a formal vector field:

$$\begin{aligned}
\mathbf{F}(fg) &= \lim_{t\to 0} t^{-1}(\mathbf{H}^t(fg) - fg) \\
&= \lim_{t\to 0} t^{-1}\bigl(\mathbf{H}^t f\,\mathbf{H}^t g + f\,\mathbf{H}^t g - f\,\mathbf{H}^t g - fg\bigr) \\
&= \lim_{t\to 0} \bigl(\mathbf{H}^t g \cdot t^{-1}(\mathbf{H}^t f - f)\bigr) + \lim_{t\to 0} \bigl(f \cdot t^{-1}(\mathbf{H}^t g - g)\bigr) \\
&= g\,\mathbf{F}f + f\,\mathbf{F}g.
\end{aligned}$$

Definition 3.9. A holomorphic one-parametric subgroup of formal maps $\{H^t\} \subseteq \operatorname{Diff}[[\mathbb{C}^n, 0]]$ is a *formal flow* of the formal vector field F, if

$$\mathbf{F} = \left.\frac{d\mathbf{H}^t}{dt}\right|_{t=0} \in \mathcal{D}[[\mathbb{C}^n, 0]]. \tag{3.6}$$

The above observation means that *any analytic one-parametric subgroup* of formal maps is always a formal flow of the formal field F (3.6), called the *generator* of the one-parametric subgroup. The following Theorem is a formal analog of Proposition 1.19, showing that, conversely, *any* formal vector field generates an holomorphic one-parametric subgroup of formal maps. Denote by $\mathbf{F}^k$ the iterated composition $\mathbf{F}\circ\cdots\circ\mathbf{F}\colon \mathbb{C}[[x]]\to\mathbb{C}[[x]]$ (k times) and consider the exponential series

$$\mathbf{H}^t=\exp t\mathbf{F}=\mathbf{E}+t\mathbf{F}+\frac{t^2}{2!}\mathbf{F}^2+\cdots+\frac{t^k}{k!}\mathbf{F}^k+\cdots. \tag{3.7}$$

Theorem 3.10. *Any singular formal vector field F admits a formal flow $\{H^t\}$. This flow is defined by the series* (3.7) *which always converges.*

Proof. We have to show that this series converges and its sum is an isomorphism of the algebra $\mathbb{C}[[x]]$ for any $t\in\mathbb{C}$. Then the identity (3.6) will follow automatically.

Convergence of the series (3.7) can be seen from the following argument. Let k be any finite order. Truncating the series (3.7), i.e., substituting $j^k\mathbf{F}$ instead of $\mathbf{F}$, we obtain a matrix formal power series. This series is always convergent: if $|j^k\mathbf{F}|=r<+\infty$, then it is majorized (in any norm $|\cdot|$) by the convergent scalar series $1+|t|r+|t|^2r^2/2!+\cdots=\exp|t|r<+\infty$ for any finite $t\in\mathbb{C}$, cf. with Definition 1.7. Denote its sum by $\exp j^k\mathbf{F}\colon J^k(\mathbb{C}^n,0)\to J^k(\mathbb{C}^n,0)$.

Truncations $\exp j^k\mathbf{F}$ for different orders k agree in common terms: if $l>k$, then $j^k(\exp t\, j^l\mathbf{F})=\exp t\, j^k\mathbf{F}$. This allows to define the sum of the series $\exp t\mathbf{F}$ as a linear operator $\mathbf{H}^t\colon\mathbb{C}[[x]]\to\mathbb{C}[[x]]$ via its finite truncations of all orders.

The group property $\mathbf{H}^{t+s}=\mathbf{H}^t\circ\mathbf{H}^s$ follows from the formal identity $\exp(t+s)=\exp t\cdot\exp s$, since $t\mathbf{F}$ and $s\mathbf{F}$ obviously commute. It remains to show that $\mathbf{H}^t$ is an algebra *homomorphism*, i.e., $\mathbf{H}^t(fg)=H^tf\,\mathbf{H}^tg$ for any two series $f,g\in\mathbb{C}[[x]]$.

Denote the multiplication operation in $\mathbb{C}[[x]]$ by the asterisk (not to be confused with composition of homomorphisms), we have for any $f,g\in\mathbb{C}[[x]]$ the iterated Leibnitz rule

$$\mathbf{F}^k(fg)=\sum_{p+q=k}\tfrac{k!}{p!q!}\,\mathbf{F}^pf\,\mathbf{F}^qg.$$

Substituting this identity into the exponential series, we have

$$\begin{aligned}\mathbf{H}^k(fg)&=\sum_k\sum_{p+q=k}\tfrac{t^k}{p!q!}\,\mathbf{F}^pf\,\mathbf{F}^qg\\&=\Big(\sum_p\tfrac{t^p}{p!}\,\mathbf{F}^pf\Big)\cdot\Big(\sum_q\tfrac{t^q}{q!}\,\mathbf{F}^qg\Big)=\mathbf{H}^tf\,\mathbf{H}^tg.\quad\square\end{aligned}$$

3.4. Embedding in the flow and matrix logarithms.

Definition 3.11. A holomorphic germ $H\in\operatorname{Diff}(\mathbb{C}^n,0)$ or a formal map $H\in\operatorname{Diff}[[\mathbb{C}^n,0]]$ is said to be *embeddable*, if there exists a holomorphic germ of a vector field F (resp., a formal vector field $F \in \mathcal{D}[[\mathbb{C}^n, 0]]$) such that H is a time one (resp., formal time one) flow map of F.

For a linear system $\dot{x} = Ax$ with constant coefficients, the flow consists of *linear* maps $x \mapsto (\exp tA)x$, see (1.11). Conversely, for a *linear* map $x \mapsto Mx$, $M \in \mathrm{GL}(n, \mathbb{C})$, it is natural to consider the embedding problem in the class of linear vector fields $F(x) = Ax$. Then the problem reduces to finding a *matrix logarithm*, a matrix solution of the equation

$$\exp A = M, \qquad A, M \in \mathrm{Mat}(n, \mathbb{C}). \tag{3.8}$$

Clearly, the necessary condition for solvability of this equation is nondegeneracy of M. It turns out to be also sufficient in the complex settings.

Lemma 3.12. *For any nondegenerate matrix* $M \in \mathrm{Mat}(n, \mathbb{C})$*,* $\det M \neq 0$*, there exists the matrix logarithm* $A = \ln M$*, a complex matrix satisfying the equation* (3.8)

Proof. We give two constructions of matrix logarithms for nondegenerate matrices.

First, for a scalar matrix $M = \lambda E$, $0 \neq \lambda \in \mathbb{C}$, the logarithm can be defined as $\ln M = \ln \lambda \cdot E$, for any choice of $\ln \lambda$. A matrix having only one (multiple) nonzero eigenvalue has the form $M = \lambda(E + N)$, where N is a nilpotent (upper-triangular) matrix, and its logarithm can be defined using the formal series for the scalar logarithm as follows,

$$\ln M = \ln(\lambda E) + \ln(E + N) = \ln \lambda \cdot E + N - \tfrac{1}{2}N^2 + \tfrac{1}{3}N^3 - \cdots \tag{3.9}$$

(the sum is finite). This formula gives a well-defined answer by virtue of the formal identity $\exp(x - \frac{x^2}{2} + \frac{x^3}{3} \pm \dots) = 1 + x$, since the matrices λE and N commute.

An arbitrary matrix M can be reduced to a block diagonal form with blocks having only one eigenvalue each. The block diagonal matrix formed by logarithms of individual blocks solves the problem of computing the matrix logarithm in the general case.

The second proof uses the integral representation: for any function $f(x)$ complex analytic in a domain $U \subset \mathbb{C}$ bounded by a simple curve ∂U and any matrix M with all eigenvalues in U, the value $f(M)$ can be computed by the contour integral

$$f(M) = \frac{1}{2\pi i} \oint_{\partial U} f(\lambda)(\lambda E - M)^{-1}\, d\lambda$$

[**Gan59**, Ch. V, §4]. In application to $f(x) = \ln x$ we have to choose a simple loop containing all eigenvalues of M inside but the origin $\lambda = 0$ outside. Then in the domain U one can choose a branch of complex logarithm $\ln \lambda$ and write the integral representation as above.

To prove that the integral representation gives the same answer as before, it is sufficient to verify it only for the diagonal matrices, when the inverse can be computed explicitly. The advantage of this formula is the possibility of bounding the norm $|\ln M|$ defined by the above integral, in terms of $|M|$ and $|M^{-1}|$. □

Remark 3.13. The matrix logarithm is *by no means unique*. In the first proof one has the freedom to choose branches of logarithms of eigenvalues arbitrarily and independently for different Jordan blocks. In the second proof the freedom to choose the domain U (i.e., the loop ∂U encircling all the eigenvalues of M but not the origin).

Remark 3.14. There is a natural obstruction for extracting the matrix logarithm in the class of *real* matrices. If $\exp A = M$ for some real matrix A, then M can be connected with the identity E by a path of nondegenerate matrices $\exp tA$, in particular, M should be orientation-preserving. If M is non-degenerate but orientation-reverting, it has no real matrix logarithm.

However, there are more subtle obstructions. Consider the real matrix $M = \begin{pmatrix} -1 & 1 \\ & -1 \end{pmatrix}$ with determinant 1. If $M = \exp A$, and A is real, then by (1.14) $\exp \operatorname{tr} A = 1$ so that necessarily $\operatorname{tr} A = 0$. The two eigenvalues cannot be simultaneously zero, since then the exponent will have the eigenvalues both equal to 1. Therefore the eigenvalues must be different, in which case the matrix A and hence its exponent M must be diagonalizable. The contradiction shows impossibility of solving the equation $\exp A = M$ in the class of real matrices.

3.5. Logarithms and derivations. Inspired by the construction of the matrix exponential, one can attempt to prove that for any formal map $H \in \operatorname{Diff}[[\mathbb{C}^n, 0]]$ there exists a formal vector field F whose formal flow coincides with H, as follows.

Consider an arbitrary finite order k and the k-jet $\mathbf{H}_k = j^k\mathbf{H}$ considered as an isomorphism of the finite-dimensional $\mathbb{C}$-algebra $\mathfrak{F}^k = J^k(\mathbb{C}^n, 0)$. By Lemma 3.12, there exists a linear map $\mathbf{F}_k \colon \mathfrak{F}^k \to \mathfrak{F}^k$ such that $\exp \mathbf{F}_k = \mathbf{H}_k$.

Assume that for some reasons

(i) jets of the logarithms $\mathbf{F}_k$ of different orders agree after truncation, i.e., $j^k\mathbf{F}_l = \mathbf{F}_k$ for $l > k$, and

(ii) each $\mathbf{F}_k$ is a *derivation* of the commutative algebra $\mathfrak{F}^k$, thus a k-jet of a vector field.

Then together these jets would define a derivation $\mathbf{F}$ of the algebra $\mathfrak{F} = \mathbb{C}[[x]]$.

The first objective can be achieved if $\mathbf{F}_k$ are truncations of some polynomial or infinite series. There is only one such candidate, the *logarithmic series* $\log \mathbf{H} \colon \mathbb{C}[[x]] \to \mathbb{C}[[x]]$, the formal series for $\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 \mp \cdots$ rewritten as

$$\log \mathbf{H} = (\mathbf{H} - \mathbf{E}) - \frac{1}{2}(\mathbf{H} - \mathbf{E})^2 + \frac{1}{3}(\mathbf{H} - \mathbf{E})^3 \mp \cdots \tag{3.10}$$

(cf. with (3.9)). To distinguish the "principal" branch of the logarithm given by the series (3.10) from the preimages by the exponential map $\exp^{-1}(\mathbf{H})$ introduced earlier, we use temporarily the notation $\log \mathbf{H}$ instead of $\ln \mathbf{H}$ until the end of this section.

The series for $\log \mathbf{H}$ does not converge everywhere even in the finite-dimensional case: the domain of convergence contains the open ball $|\mathbf{H} - \mathbf{E}| < 1$ and all unipotent finite-dimensional matrices, but most certainly *not* the matrix $-\mathbf{E}$. Besides, it is absolutely not clear why the formal logarithm of an isomorphism, even if it converges, must be a derivation: no simple arguments similar to used in the proof of Theorem 3.10, exist (sometimes this circumstance is overlooked).

Let $\mathfrak{F}$ be a finite-dimensional commutative $\mathbb{C}$-algebra and H an automorphism of $\mathfrak{F}$.

Theorem 3.15. *The series* (3.10) *converges for all unipotent automorphisms* $\mathbf{H}$ *of a finite dimensional algebra* $\mathfrak{F}$ *and its sum* $\mathbf{F} = \log \mathbf{H}$ *in this case is a derivation of this algebra.*

Proof using the Lie group tools. Consider the matrix Lie group $\mathfrak{T} \subset \mathrm{GL}(n, \mathbb{C})$ of upper-triangular matrices with units on the principal diagonal and the corresponding Lie algebra $\mathfrak{t} \subset \mathrm{Mat}(n, \mathbb{C})$ of *strictly* upper-triangular matrices.

The exponential series (3.7) and the matrix logarithm (3.10) restricted on $\mathfrak{t}$ and $\mathfrak{T}$ respectively, are *polynomial* maps defined everywhere. They are mutually inverse: for any $\mathbf{F} \in \mathfrak{t}$ and $\mathbf{H} \in \mathfrak{T}$ we have $\log \exp \mathbf{F} = \mathbf{F}$ and $\exp \log \mathbf{H} = \mathbf{H}$. This follows from the identities $\ln e^z = z$, $e^{\ln w} = w$ expanded in the series. In particular, exp is surjective.

For any Lie subalgebra $\mathfrak{g} \subseteq \mathfrak{t}$ and the corresponding Lie subgroup $\mathfrak{G} \subseteq \mathfrak{T}$ the exponential map $\exp \colon \mathfrak{g} \to \mathfrak{G}$ is the restriction of (3.7) on $\mathfrak{g}$.

By [**Var84**, Theorem 3.6.2], the exponential map remains surjective also on $\mathfrak{G}$, i.e., $\exp \mathfrak{g} = \mathfrak{G}$. We claim that in this case the logarithm maps $\mathfrak{G}$ *into* $\mathfrak{g}$. Indeed, if $\mathbf{H} \in \mathfrak{G}$ and $\mathbf{H} = \exp \mathbf{F}$ for some $\mathbf{F} \in \mathfrak{g}$, then $\log \mathbf{H} = \log \exp \mathbf{F} = \mathbf{F} \in \mathfrak{g}$.

The assertion of the Theorem arises if we take $\mathfrak{G} = \mathfrak{T} \cap \mathrm{Aut}(\mathfrak{F})$ to be the Lie subgroup of *triangular automorphisms* of $\mathfrak{F} \sim \mathbb{C}^n$ and $\mathfrak{g} = \mathfrak{t} \cap \mathrm{Der}(\mathfrak{F})$ of *triangular derivations* of the commutative algebra $\mathfrak{F}$. □

Remark 3.16. Surjectivity of the exponential map for a subgroup of the triangular group $\mathfrak{T}$ is a delicate fact that follows from the nilpotency of the Lie algebra $\mathfrak{t}$. Indeed, by the Campbell–Hausdorff formula, $\exp \mathbf{F} \cdot \exp \mathbf{G} = \exp \mathbf{K}$, where $\mathbf{K} = \mathbf{K}(\mathbf{F}, \mathbf{G})$ is a series which in the nilpotent case is a polynomial map $\mathfrak{t} \times \mathfrak{t} \to \mathfrak{t}$ defined everywhere. Thus the image $\exp \mathfrak{g}$ is a *Lie subgroup* in $\mathfrak{G} \subseteq \mathfrak{T}$ for *any* subalgebra $\mathfrak{g}$, containing a small neighborhood of the unit $\mathbf{E}$. It is well known that any such neighborhood generates (by the group operation) the whole connected component of the unit, so that $\exp \mathfrak{g}$ coincides with this component. If $\mathfrak{G}$ is simply connected, then $\exp \mathfrak{g} = \mathfrak{G}$ as asserted.

Without nilpotency the answer may be different: as follows from Remark 3.14, for two Lie algebras $\mathfrak{gl}(n, \mathbb{R}) \subset \mathfrak{gl}(n, \mathbb{C})$ and the respective Lie groups $\mathrm{GL}(n, \mathbb{R}) \subset \mathrm{GL}(n, \mathbb{C})$, the exponent is surjective on the ambient (bigger) group but *not* on the subgroup.

Remark 3.17. Using similar arguments, one can prove that for an arbitrary automorphism $\mathbf{H} \in \mathrm{Aut}(\mathfrak{F})$ *sufficiently close to the unit* $\mathbf{E}$, the logarithm $\log \mathbf{H}$ given by the series (3.10) is a derivation, $\log \mathbf{H} \in \mathrm{Der}(\mathfrak{F})$. This follows from the fact that log and exp are mutually inverse on sufficiently small neighborhoods of $\mathbf{E}$ and 0 respectively. However, the size of this neighborhood depends on $\mathfrak{F}$.

3.6. Embedding in the formal flow. Based on Theorem 3.15, one can prove the following general result obtained by F. Takens in 1974, see [**Tak01**].

Theorem 3.18. *Let* $H \in \mathrm{Diff}[[\mathbb{C}^n, 0]]$ *a formal map whose linearization matrix* $A = \frac{\partial H}{\partial x}(0)$ *is unipotent,* $(A - E)^n = 0$.

Then there exists a formal vector field $F \in \mathcal{D}[[\mathbb{C}^n, 0]]$ *whose linearization is a nilpotent matrix* N*, such that* H *is the formal time* 1 *map of* F*.*

Proof. As usual, we identify the formal map with an automorphism $\mathbf{H}$ of the algebra $\mathfrak{F} = \mathbb{C}[[x]]$ so that its finite k-jets $j^k\mathbf{H}$ become automorphisms of the finite dimensional algebras $\mathfrak{F}^k = J^k(\mathbb{C}^n, 0)$. Without loss of generality we may assume that the matrix A is upper-triangular so that the isomorphism $\mathbf{H}$ and all its truncations $j^k H$ in the canonical `deglex`-ordered basis becomes upper-triangular with units on the diagonal: the jets $j^k\mathbf{H}$ are finite-dimensional upper-triangular (unipotent) automorphisms of the algebras $\mathfrak{F}^k$.

Consider the infinite series (3.10) together with its finite-dimensional truncations obtained by applying the functor j^k to all terms. Each such truncation is a logarithmic series for $\log j^k\mathbf{H}$ which converges (actually, stabilizes after finitely many steps) and its sum is a derivation $j^k\mathbf{F}$ of $\mathfrak{F}^k$ by

Theorem 3.15. Clearly, different truncations agree on the lower order terms, thus $\log \mathbf{H}$ converges in the sense of Definition 3.4 to a derivation $\mathbf{F}$ of $\mathfrak{F}$. This derivation corresponds to the formal vector field F as required. □

4. Formal normal forms

In the same way as holomorphic maps act on holomorphic vector fields by conjugacy (1.24), formal maps act on formal vector fields. In this section we investigate the *formal normal forms* to which a formal vector field can be brought by a suitable formal isomorphism.

Definition 4.1. Two formal vector fields F, F' are *formally equivalent*, if there exists an invertible formal morphism H such that the identity (1.24) holds on the level of formal series.

The fields are formally equivalent if and only if the corresponding derivations $\mathbf{F}, \mathbf{F}'$ of the algebra $\mathbb{C}[[x]]$ are conjugated by a suitable isomorphism $\mathbf{H} \in \operatorname{Diff}[[\mathbb{C}^n, 0]]$: $\mathbf{H} \circ \mathbf{F}' = \mathbf{F} \circ \mathbf{H}$.

Obviously, two holomorphically equivalent (holomorphic) germs of vector fields are formally equivalent. The converse is in general not true, as the formal morphism may be divergent.

4.1. Formal classification theorem. Formal classification of formal vector fields is very much influenced by properties of its principal part, in particular, the linearization matrix $A = \left(\frac{\partial F}{\partial x}\right)(0)$ if the latter is nonzero.

We start with the most important example.

Definition 4.2. An ordered tuple of complex numbers $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{C}^n$ is called *resonant*, if there exist nonnegative integers $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{Z}_+^n$ such that the *resonance* identity occurs,

$$\lambda_j = \langle \alpha, \lambda \rangle, \qquad |\alpha| \geqslant 2, \tag{4.1}$$

where $\langle \alpha, \lambda \rangle = \alpha_1 \lambda_1 + \dots + \alpha_n \lambda_n$. The natural number $|\alpha|$ is the *order* of the resonance.

A square matrix is resonant, if the collection of its eigenvalues is resonant. A formal vector field $F = (F_1, \dots, F_n)$ at the origin is resonant if its linearization matrix $A = \left(\frac{\partial F}{\partial x}\right)(0)$ is resonant.

Though resonant tuples can be dense in some parts of $\mathbb{C}^n$ (see §5.1), their measure is zero.

Theorem 4.3 (Poincaré linearization theorem)**.** *A non-resonant formal vector field $F(x) = Ax + \cdots$ is formally equivalent to its linearization $F'(x) = Ax$.*

The proof of this theorem is given in the sections §4.2–§4.3. In fact, it is the simplest particular case of a more general statement valid for resonant formal vector fields that appears in §4.4.

4.2. Induction step: homological equation. The proof of Theorem 4.3 goes by induction. Assume that the formal vector field F is already partially normalized, and contains no terms of order less than some $m \geqslant 2$:

$$F(x) = Ax + V_m(x) + V_{m+1}(x) + \cdots ,$$

where $V_m, V_{m+1}, \ldots$ are arbitrary homogeneous vector fields of degrees $m, m+1$ etc.

We show that in the assumptions of the Poincaré theorem, the term V_m can be removed from the expansion of F, i.e., that F is formally equivalent to the formal field $F'(x) = Ax + V'_{m+1} + \cdots$. Moreover, the corresponding conjugacy can be in fact chosen polynomial of the form $H(x) = x + P_m(x)$, where P_m is a vector polynomial of degree m. The Jacobian matrix of such formal morphism is $E + \bigl(\frac{\partial P_m}{\partial x}\bigr)$.

The conjugacy H with these properties must satisfy the equation (1.24) on the formal level. Keeping only terms of order $\leqslant m$ from this equation and using dots to denote the rest, we obtain

$$\left(E + \frac{\partial P_m}{\partial x}\right)(Ax + V_m + \cdots) = A(x + P(x)) + V'_m(x + P_m(x)) + \cdots .$$

The homogeneous terms of order 1 on both sides coincide. The next nontrivial terms appear in the order m. Collecting them, we see that in in order meet the condition $V'_m = 0$, the homogeneous terms $P = P_m$ must satisfy the identity

$$[\mathbf{A}, P_m] = -V_m, \qquad \mathbf{A}(x) = Ax, \tag{4.2}$$

where $\mathbf{A} = \mathbf{A}(x) = Ax$ is the linear vector field, the principal part of F, and the homogeneous vector polynomials P_m and V_m are considered as vector fields on $\mathbb{C}^n$. The left hand side of (4.2) is the commutator, $[\mathbf{A}, P] = \bigl(\frac{\partial P}{\partial x}\bigr)\, Ax - AP(x)$.

Conversely, if the condition (4.2) is satisfied by P_m, the polynomial map $H(x) = x + P_m(x)$ conjugates $F = \mathbf{A} + V_m + \cdots$ with the (formal) vector field $F'(x) = \mathbf{A} + \cdots$ having no terms of degree m.

Definition 4.4. The identity (4.2), considered as an equation on the unknown homogeneous vector P, is called the *homological equation*.

4.3. Solvability of homological equation. Solvability of the homological equation depends on the properties of the operator of commutation with the linear vector field $\mathbf{A}$.

Let $\mathcal{D}_m$ be the linear space of all homogeneous vector fields of degree m. This linear space has the *standard monomial basis* consisting of the fields

$$F_{k\alpha} = x^\alpha \frac{\partial}{\partial x_k}, \qquad k = 1, \dots, n, \ |\alpha| = m. \tag{4.3}$$

We shall order elements of this basis lexicographically so that x_i precedes x_j if $i < j$, but $\frac{\partial}{\partial x_j}$ precedes $\frac{\partial}{\partial x_i}$. To that end, we assign to each formal variable $x_1, \dots, x_n$ pairwise different positive weights $w_1 > \cdots > w_n$ that are *rationally independent*. This assignment extends on all monomials and monomial vector fields if the symbol $\frac{\partial}{\partial x_j}$ is assigned the weight $-w_j$. Now the monomial vector fields can be arranged in the decreasing order of their weights: the independence condition guarantees that any two different monomials have different weights.

The operator

$$\operatorname{ad}_A \colon P \mapsto [\mathbf{A}, P], \qquad (\operatorname{ad}_A P)(x) = \left(\frac{\partial P}{\partial x}\right) \cdot Ax - AP(x), \tag{4.4}$$

preserves the space $\mathcal{D}_m$ for any $m \in \mathbb{N}$.

Lemma 4.5. *If A is nonresonant, then the operator ad_A is invertible. More precisely, if the coordinates $x_1, \dots, x_n$ are chosen such that A has the upper-triangular Jordan form, then ad_A is lower-triangular in the respective standard monomial basis ordered lexicographically.*

Proof. The assertion of the Lemma is completely transparent when A is a diagonal matrix $\varLambda = \operatorname{diag}\{\lambda_1, \dots, \lambda_n\}$. In this case each $F_{k\alpha} \in \mathcal{D}_m$ is an eigenvector for $\operatorname{ad}_\varLambda$ with the eigenvalue $\langle \lambda, \alpha \rangle - \lambda_k$. Indeed, by the Euler identity,

$$F_{k\alpha} = x^\alpha \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix}, \qquad \left(\frac{\partial F_{k\alpha}}{\partial x}\right) = x^\alpha \begin{pmatrix} & 0 & \\ & \vdots & \\ \frac{\alpha_1}{x_1} & \cdots & \frac{\alpha_n}{x_n} \\ & \vdots & \\ & 0 & \end{pmatrix},$$

so that in the diagonal case $\varLambda F_{k\alpha} = \lambda_k F_{k\alpha}$, and $\left(\frac{\partial F_{k\alpha}}{\partial x}\right) \varLambda x = \langle \lambda, \alpha \rangle F_{k\alpha}$. Being diagonal with nonzero eigenvalues, $\operatorname{ad}_\varLambda$ is invertible.

To prove the Lemma in the general case when A is in the upper-triangular Jordan form, we consider the weights, used when describing the order on the monomial vector fields.

The operator $\operatorname{ad}_\varLambda$ with the diagonal matrix $\varLambda$ preserves the weights, as can be seen directly from inspection of the above formulas. On the other hand, consider the monomial vector field $\mathbf{J}_j = x_j \frac{\partial}{\partial x_{j+1}}$ with the upper-diagonal constant matrix J_j. Its weight is strictly positive, and it can be immediately verified that the commutator $\operatorname{ad}_j = \operatorname{ad}_{J_j}$ increases the weights in the sense that for any monomial vector field $F_{k\alpha}$ the image $\operatorname{ad}_j F_{k\alpha}$ is a linear combination of monomial fields of weights strictly bigger than the weight of $F_{k\alpha}$.

It remains to notice that an arbitrary matrix A in the upper-triangular Jordan normal form is the sum of the diagonal part Λ and a linear combination of matrices $J_1, \dots, J_{n-1}$. The operator ad_A linearly depends on A, so ad_A is equal to $\operatorname{ad}_\Lambda$ modulo a linear combination of the weight-increasing operators ad_{J_j}. Therefore, if the monomial fields $F_{k\alpha}$ are ordered in the decreasing order of their weights, as in the standard basis, then the operator ad_A is lower-triangular with the diagonal part $\operatorname{ad}_\Lambda$. □

Proof of Theorem 4.3. Now we can prove the Poincaré linearization theorem. By Lemma 4.5, the operator ad_A is invertible and therefore the homological equation (4.2) is always solvable no matter what the term $V = V_m$ is. Repeating this process inductively, we can construct an infinite sequence of polynomial maps $H_1, H_2, \dots, H_m, \dots$ and the formal fields $F_1 = F$, $F_2, \dots, F_m, \dots$ such that $F_m = Ax +$ (terms of order m and more), while H_m conjugates F_m with F_{m+1}. Thus the composition $H^{(m)} = H_m \circ \cdots \circ H_1$ conjugates the initial field F_1 with the field F_{m+1} without nonlinear terms up to order m.

It remains to notice that by construction of H_{m+1} the composition $H^{(m+1)} = H_{m+1} \circ H^{(m)}$ has the same terms of order $\leqslant m$ as $H^{(m)}$ itself. Thus the limit

$$H = H^{(\infty)} = \lim_{m \to \infty} H^{(m)}$$

(the infinite composition) exists in the class of formal morphisms. By construction, $H_* F$ cannot contain any nonlinear terms and hence is linear, as required. □

Remark 4.6. The formal map linearizing a non-resonant formal vector field and tangent to the identity, is unique. Indeed, otherwise there would exist a *nontrivial* formal map $\operatorname{id} + h$ which conjugates the linear field with itself,

$$\left(\frac{\partial h}{\partial x}\right) Ax = Ah(x), \qquad \text{i.e.,} \qquad \operatorname{ad}_A h = 0.$$

But in the non-resonant case the commutator ad_A is injective, hence $h = 0$.

Thus the only formal maps conjugating a linear field with itself, are linear maps $x \mapsto Bx$, with the matrix B commuting with A, $[A, B] = 0$.

4.4. Resonant normal forms: Poincaré–Dulac paradigm. The inductive construction proving the Poincaré linearization theorem, can be used to *simplify* the series, i.e., to eliminate *some* of the Taylor terms, when resonances between eigenvalues occur.

In this *resonant* case the operator $\mathrm{ad}_A = [\mathbf{A}, \cdot]$ of commutation with the linear part may be no longer surjective and in general the condition $V'_m = 0$ meaning absence of linear terms after the transformation, cannot be achieved.

In this case one can choose in each linear space $\mathcal{D}_m$ a complementary (transversal) subspace $\mathcal{N}_m$ to the image of the operator ad_A, so that

$$\mathcal{D}_m = \mathcal{N}_m + \mathrm{ad}_A(\mathcal{D}_m) \tag{4.5}$$

(the sum not necessary should be direct).

Theorem 4.7 (Poincaré–Dulac paradigm)**.** *If the subspaces $\mathcal{N}_m \subset \mathcal{D}_m$ are transversal to the image of the commutator* ad_A *as in* (4.5)*, then any formal vector field $F(x) = Ax + \cdots$ with the linearization matrix A is formally conjugated to some formal vector field whose all nonlinear terms of degree m belong to the subspace $\mathcal{N}_m$.*

Proof. If the transformation $H_m(x) = x + P_m$ conjugates the field $F(x) = Ax + V_m(x) + \cdots$ with another field $F'(x) = Ax + V'_m(x) + \cdots$ on the level of terms of order m, then instead of the homological equation (4.2) in the case $V'_m \neq 0$, the correction term P_m must satisfy the equation

$$\mathrm{ad}_A\, P_m = V'_m - V_m. \tag{4.6}$$

If $\mathcal{N}_m$ satisfies (4.5), then (4.6) can be always solved with respect to P_m for any V_m provided that V'_m is suitably chosen from the subspace $\mathcal{N}_m$.

The transform H_m that does not affect the lower order terms and hence the process can be iterated for larger values of m exactly as in the non-resonant case. As a result, one can prove that any formal vector field F is formally equivalent to a formal field containing only terms belonging to the "complementary" parts $\mathcal{N}_m$ for all $m = 2, 3, \ldots$.

The rest of the proof of Theorem 4.7 is the same as that of the Poincaré–Dulac theorem. □

The choice of the transversal subspaces $\mathcal{N}_m$ depends very much on the linearization matrix A.

Example 4.8. Assume that the matrix $A = \Lambda = \mathrm{diag}\{\lambda_1, \ldots, \lambda_n\}$ is diagonal. In this case the operator ad_Λ was already shown to be diagonal, eventually with some zeros among the eigenvalues. For diagonal operators on finite-dimensional space the kernel and the image are complementary subspaces, so one may choose $\mathcal{N}_m = \ker \mathrm{ad}_L \subset \mathcal{D}_m$. The kernel of the diagonal operator ad_Λ can be immediately described.

Definition 4.9. A *resonant vector monomial* corresponding to the resonance $\lambda_k - \langle \lambda, \alpha \rangle = 0$, is the monomial vector field $F_{k\alpha}$, see (4.3).

The kernel $\ker \operatorname{ad}_\Lambda$ consists of linear combinations of resonant monomials. From the discussion above it follows immediately that a formal vector field with diagonal linear part Λx is formally equivalent to the vector field with the same linear part and only resonant monomials among the nonlinear terms.

Actually, the assumption on diagonalizability is redundant. The following example is one of the most popular formal classification results.

Theorem 4.10 (Poincaré–Dulac theorem)**.** *A formal vector field is formally equivalent to a vector field with the linear part in the Jordan normal form and only resonant monomials in the nonlinear part.*

Proof. Assume that the coordinates are already chosen so that the linearization matrix A is Jordan upper-triangular.

Choose the subspace $\mathcal{N}_m$ as the linear span of all resonant monomials, $\mathcal{N}_m = \{\mathbb{C} \cdot F_{k\alpha} \colon \langle \lambda, \alpha \rangle - \lambda_k = 0\}$.

By Lemma 4.5, the operator $L_m = \operatorname{ad}_A|_{\mathcal{D}_m}$ is upper triangular with the expressions $\langle \lambda, \alpha \rangle - \lambda_k = 0$ on the diagonal. By the choice of $\mathcal{N}_m$, whenever zero occurs on the diagonal of L, the corresponding basis vector was included in $\mathcal{N}_m$. This obviously means (4.5). The rest is the Poincaré–Dulac paradigm. □

4.5. Belitskii theorem. The choice of the "normal form" (i.e., the subspaces $\mathcal{N}_m$) in the Poincaré–Dulac theorem, is excessive in the sense that the *dimension* of these spaces (the number of parameters in the normal form) is not minimal. For example, if A is a nonzero nilpotent Jordan matrix, then *all* monomials are resonant in the sense of Definition 4.9, whereas the image of ad_A is clearly nontrivial. We describe now one possible *minimal* choice, introduced by G. Belitskii [**Bel79**, Ch. II, §7].

Consider the standard Hermitian structure on the space $\mathbb{C}^n$, so that the basis vectors $e_j = \frac{\partial}{\partial x_j}$ form an orthonormal basis.

For any natural $m \geqslant 1$ denote by $\mathcal{H}_m$ the complex linear space of all homogeneous polynomials of degree m. We introduce the *standard Hermitian structure* in $\mathcal{H}_m$ in such a way that the normalized monomials $\varphi_\alpha = \frac{1}{\sqrt{\alpha!}}\, x^\alpha$ form an orthonormal basis,

$$(\varphi_\alpha, \varphi_\beta) = \delta_{\alpha\beta}, \qquad \varphi_\alpha = \frac{1}{\sqrt{\alpha!}}\, x^\alpha, \quad \alpha, \beta \in \mathbb{Z}_+^n,\ |\alpha| = |\beta| = m. \qquad (4.7)$$

Here, as usual, $\alpha! = \alpha_1! \cdots \alpha_n!$ for $\alpha = (\alpha_1, \dots, \alpha_n)$, $0! = 1$ and $\delta_{\alpha\beta}$ is the standard Kronecker symbol.

Then the linear space $\mathcal{D}_m$ of homogeneous vector fields of degree m can be naturally identified with the tensor product $\mathcal{D}_m = \mathcal{H}_m \otimes_{\mathbb{C}} \mathbb{C}^n$ and inherits the standard Hermitian structure for which the monomials $\varphi_\alpha \otimes e_k = \frac{1}{\sqrt{\alpha!}} F_{\alpha k}$ form an orthonormal basis.

Given a matrix $A \in \operatorname{Mat}_n(\mathbb{C})$, denote by A^* the *adjoint matrix* obtained from A by transposition and complex conjugacy: $a^*_{ij} = \bar a_{ji}$. If $\mathbf{A}(x) = Ax$ is the corresponding linear vector field on $\mathbb{C}^n$ and, respectively, $\mathbf{A}^*(x) = A^*x$, then both act as linear (differential) operators $\mathbf{A} = \sum a_{ij} x_i \frac{\partial}{\partial x_j}$ and $\mathbf{A}^* = \sum \bar a_{ji} x_i \frac{\partial}{\partial x_j}$ on $\mathcal{H}_m$. Furthermore, the commutation operators $\operatorname{ad}_A = [\mathbf{A}, \cdot]$ and $\operatorname{ad}_{A^*} = [\mathbf{A}^*, \cdot]$ are linear operators on $\mathcal{D}_m$.

The following statement claims that the operators in each pair are mutually adjoint (dual to each other) with respect to the standard Hermitian structures on the respective spaces.

Lemma 4.11.

1. The derivation $\mathbf{A}^* \colon \mathcal{H}_m \to \mathcal{H}_m$ *is adjoint to the derivation* $\mathbf{A}$ (*with respect to the standard Hermitian structure*) *and vice versa.*

2. The commutator $\operatorname{ad}_{A^*} = [\mathbf{A}^*, \cdot] \colon \mathcal{D}_m \to \mathcal{D}_m$ *is adjoint to the commutator* $\operatorname{ad}_A = [\mathbf{A}, \cdot]$ (*with respect to the standard Hermitian structure*) *and vice versa.*

Proof. 1. The identity $(\mathbf{A}f, g) = (f, \mathbf{A}^* g)$ for any pair of polynomials $f, g \in \mathcal{H}_m$ "linearly" depends on the matrix A: if it holds for two matrices $A, A' \in \operatorname{Mat}_n(\mathbb{C})$, then it also holds for their combination $\lambda A + \lambda' A'$ with any two complex numbers $\lambda, \lambda' \in \mathbb{C}$.

Thus it is sufficient to verify the assertion for the monomial derivations $\mathbf{A} = x_i \frac{\partial}{\partial x_j}$ and $\mathbf{A}^* = x_j \frac{\partial}{\partial x_i}$.

If $i = j$, then $\mathbf{A} = \mathbf{A}^* = x_i \frac{\partial}{\partial x_i}$ is diagonal in the orthonormal basis $\{\varphi_\alpha\}$ with the real eigenvalues $\lambda_\alpha = \alpha_i = \alpha_j \in \mathbf{Z}_+$, and hence is self-adjoint.

Otherwise both $\mathbf{A}$ and $\mathbf{A}^*$ can be represented as permutations of the basic vectors composed with the diagonal operators. If β is the multiindex obtained from α by the operation

$$\beta_k = \begin{cases} \alpha_k, & k \ne i, j, \\ \alpha_i + 1, & k = i, \\ \alpha_j - 1, & k = j, \end{cases} \qquad \alpha_k = \begin{cases} \alpha_k, & k \ne i, j, \\ \beta_i - 1, & k = i, \\ \beta_j + 1, & k = j, \end{cases}$$

then $\beta!/\alpha! = (\alpha_i + 1)/\alpha_j = \beta_i/\alpha_j$ and

$$\mathbf{A}\varphi_\alpha = \frac{\alpha_j}{\sqrt{\alpha!}}\, x^\beta = \alpha_j \frac{\sqrt{\beta!}}{\sqrt{\alpha!}}\, \varphi_\beta = \alpha_j \frac{\sqrt{\beta_i}}{\sqrt{\alpha_j}}\, \varphi_\beta = \sqrt{\alpha_j \beta_i}\, \varphi_\beta.$$

Reciprocally, $\mathbf{A}^*\varphi_\beta = \beta_i\, x^\alpha/\sqrt{\beta!} = \cdots = \sqrt{\beta_i\alpha_j}\,\varphi_\alpha$. But since the vectors φ_α form an orthonormal basis,

$$(\mathbf{A}\varphi_\alpha, \varphi_\beta) = (\varphi_\alpha, \mathbf{A}^*\varphi_\beta) = \sqrt{\beta_i\alpha_j} \in \mathbb{R}$$

and all other matrix entries in the basis $\{\varphi_\alpha\}$ are zeros. Therefore the derivations $\mathbf{A}$ and $\mathbf{A}^*$ are mutually adjoint on $\mathcal{H}_m$.

2. Using the structure of the tensor product $\mathcal{D}_m = \mathcal{H}_m \otimes \mathbb{C}^n$, one can represent the commutators as follows,

$$\operatorname{ad}_A = \mathbf{A} \otimes E - \operatorname{id} \otimes A$$

Indeed, for any element φv, where $\varphi \in \mathcal{H}_m$ is a polynomial and $v \in \mathbb{C}^n$ a vector considered as a constant vector field on $\mathbb{C}^n$, by the Leibnitz rule

$$[\mathbf{A}, \varphi v] = (\mathbf{A}\varphi)v + \varphi[\mathbf{A}, v] = (\mathbf{A}\varphi)v - \varphi\, Av.$$

Obviously, because of the choice of the Hermitian structure on $\mathcal{H}_m \otimes \mathbb{C}^n$, the operator $\operatorname{id} \otimes A$ is adjoint to $\operatorname{id} \otimes A^*$ whereas the adjoint to $\mathbf{A} \otimes E$ is the tensor product of the adjoint to $\mathbf{A}$ by the identity. By the first statement of the Lemma, the former is equal to $\mathbf{A}^*$, so that the adjoint to $[\mathbf{A}, \cdot]$ is $\mathbf{A}^* \otimes E - \operatorname{id} \otimes A^*$ which coincides with $[\mathbf{A}^*, \cdot] = \operatorname{ad}_{A^*}$. □

Theorem 4.12 (G. Belitskii [**Bel79**], see also [**Dum93, Van89**])**.** *A formal vector field $F(x) = Ax + \cdots$ with the linearization matrix A is formally equivalent to the vector field $F'(x) = Ax + V_2(x)+$ whose nonlinear part commutes with the linear vector field $\mathbf{A}^*(x) = A^*x$:*

$$[F' - \mathbf{A}, \mathbf{A}^*] = 0. \tag{4.8}$$

If the vector field F is real (i.e., has only real Taylor coefficients, in particular, A is real), then both the formal normal form and the conjugating transformation can be chosen real.

Proof. The proof is based on the following well-known observation: if L is a linear endomorphism of a complex Hermitian or real Euclidean space H into itself, then the image of L and the kernel of its Hermitian (resp., Euclidean) adjoint L^* are orthogonal complements to each other:

$$(\operatorname{img} L)^\perp = \ker L^*.$$

It follows then that $\ker L^*$ is complementary to $\operatorname{img} L$ in H.

Indeed, $\xi \in (\operatorname{img} L)^\perp$ if and only if $(\xi, Lv) = 0$ for all $v \in H$, which means that any vector v is orthogonal to $L^*\xi$. This is possible if and only if $L^*\xi = 0$.

Applying this observation to the operator $L_m = \operatorname{ad}_A$ restricted on any space $\mathcal{D}_m$ and using Lemma 4.11, we see that the subspaces $\mathcal{N}_m = \ker \operatorname{ad}_{A^*}|_{\mathcal{D}_m}$ are orthogonal (hence complementary) to the image of L_m and therefore satisfy the assumption (4.5) of Theorem 4.7. Therefore all *nonlinear* terms $V_2, V_3, \dots$ can be chosen to commute with $\mathbf{A}^*(x) = A^*x$, which is in turn possible if and only if their formal sum, equal to $F - \mathbf{A}$, commutes with $\mathbf{A}^*$.

In the real case one has to replace the Hermitian spaces $\mathcal{H}_m$, $\mathbb{C}^n$ and $\mathcal{D}_m = \mathcal{H}_m \otimes_{\mathbb{C}} \mathbb{C}^n$ by their real (Euclidean) counterparts ${}^{\mathbb{R}}H_m$, $\mathbb{R}^n$ and ${}^{\mathbb{R}}\mathcal{D}_m = {}^{\mathbb{R}}\mathcal{H}_m \otimes_{\mathbb{R}} \mathbb{R}^n$. Then for any real matrix A the image of the commutator ad_A and the kernel of ad_{A^*}, where A^* is a transposed matrix, are orthogonal and hence complementary. Then the homological equation $\operatorname{ad}_A P_m = V_m' - V_m$ can be solved with respect to $P_m \in {}^{\mathbb{R}}\mathcal{D}_m$ and $V_m' \in \ker \operatorname{ad}_{A^*} \cap {}^{\mathbb{R}}\mathcal{D}_m$ when $V_m \in {}^{\mathbb{R}}\mathcal{D}_m$. The Poincaré–Dulac paradigm does the rest of the proof. $\square$

This general statement immediately implies a number of corollaries. For example, if A is diagonal matrix with the spectrum $\{\lambda_1, \dots, \lambda_n\}$, then A^* is also diagonal with the conjugate eigenvalues $\{\bar\lambda_1, \dots, \bar\lambda_n\}$. As was already noted, restriction of ad_{A^*} on $\mathcal{D}_m$ is diagonal with the eigenvalues $\langle \bar\lambda, \alpha \rangle - \bar\lambda_k = \overline{\langle \lambda, \alpha \rangle - \lambda_k}$. Its kernel consists of the same resonant monomials as defined previously, so in this case Theorem 4.12 yields the usual Poincaré–Dulac form.

Example 4.13. If $A = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right) = -A^*$ is the matrix of rotation on the *real* plane $\mathbb{R}^2$ with the coordinates (x, y), then $\ker \operatorname{ad}_{A^*} = \ker \operatorname{ad}_A$ and the *entire* formal normal form, including the linear part, commutes with the rotation vector field $\mathbf{A} = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}$. Any such rotationally symmetric real vector field must necessarily be of the form

add explanations

$$f(x^2 + y^2)\bigl(x\tfrac{\partial}{\partial x} + y\tfrac{\partial}{\partial y}\bigr) + g(x^2 + y^2)\bigl(x\tfrac{\partial}{\partial y} - y\tfrac{\partial}{\partial x}\bigr), \tag{4.9}$$

where $f(r), g(r) \in \mathbb{R}[[r]]$ are two real formal series in one variable. The linear part is of the prescribed form if $f(0) = 0$, $g(0) = 1$. The standard demonstration of this normal form requires preliminary diagonalization of A with subsequent transformations preserving complex conjugacy. Note that since g is formally invertible, the normal form (4.9) is formally orbitally equivalent to the formal vector field with $g \equiv 1$.

The same observation explains why the normal form is so often explicitly integrable.

Corollary 4.14. *Assume that the matrix $A \ne 0$ is* normal, *i.e., it commutes with the adjoint matrix A^*. Then the vector field can be formally transformed to a field which commutes with the* (*nontrivial*) *linear vector field* $\mathbf{A}^*$. $\square$

Indeed, in this case from (4.8) and $[\mathbf{A}, \mathbf{A}^*] = 0$ it follows that $[F, \mathbf{A}^*] = 0$. This observation allows to depress the dimension of the system, cf. with §4.10.

Remark 4.15. We wish to stress that *there is no distinguished Hermitian structure on* $\mathbb{C}^n$. One can choose this structure arbitrarily and only then the standard Hermitian structure appears on $\mathcal{H}_m$ and $\mathcal{D}_m$. Thus the assumption of this Corollary is not restrictive, in particular, it always holds whenever A is diagonalizable.

4.6. Parametric case. The Poincaré–Dulac method of normalization of any finite jet or the entire Taylor series, involves only the *polynomial* (*ring*) operations (additions, subtractions and multiplications) with the Taylor coefficients of the original field, *except for inversion of the operator* ad_A. This allows to construct formal normal forms depending on parameters.

Definition 4.16. A formal series $f \in \mathbb{C}[[x]]$ is said to depend analytically (resp., polynomially) on finitely many parameters $\lambda_1, \dots, \lambda_m$, if each coefficient of this series depends on the parameters analytically (resp., polynomially).

Theorem 4.17 (Formal normal form with parameters)**.**

1. *If the vector field* (*holomorphic or formal*) $F = F(\cdot, \lambda) = \mathbf{A}(\lambda) + F_2(\lambda) + \cdots$ *depends holomorphically on parameters* $\lambda \in (\mathbb{C}^m, 0)$, *then by a formal transformation one can bring the field to the formal normal form* F' *satisfying the condition*

$$[F' - \mathbf{A}, \mathbf{A}^*(0)] = 0, \tag{4.10}$$

where $\mathbf{A}(0)$ *is the linear vector field corresponding to* $\lambda = 0$, *and* $\mathbf{A}^*(0)$ *its adjoint linear field. Both the formal normal form* F' *and the transformation* H *reducing* F *to* F' *can be chosen analytically depending on the parameters* $\lambda \in (\mathbb{C}^m, 0)$ *in some* (*eventually, smaller*) *neighborhood of* $\lambda = 0$. *If* F *was real, then also* F' *and* H *can be chosen real.*

2. *If the linear part* $\mathbf{A}(\lambda) \equiv \mathbf{A}(0) \equiv \mathbf{A}$ *is constant* (*does not depend on* λ) *and the field itself depends holomorphically or polynomially on the parameters* $\lambda \in U$ (*resp.,* $\lambda \in \mathbb{C}^n$), *then both the normal form* (4.10) *and the corresponding normalizing transformation can be chosen holomorphically* (*resp., polynomially*) *depending on the parameters in exactly the same sense as* F *was.*

Proof. We start with a very general remark, basically, a geometrical rephrasing of the Implicit Function theorem.

If $L\colon X \to Y$ is a linear map of between vector spaces, which is *transversal* to a subspace $Z \subseteq Y$, then for any analytic or polynomial map $y\colon \lambda \mapsto y(\lambda)$, $\lambda \in U$ or $\lambda \in \mathbb{C}^n$, one can find two maps $x\colon \lambda \mapsto x(\lambda) \in X$ and $z\colon \lambda \mapsto z(\lambda) \in Z$, such that $Lx(\lambda) + z(\lambda) = y(\lambda)$. If in addition L also depends on λ and is transversal to Z for $\lambda = 0$, then the solutions still can be found, but only locally for the parameter values $\lambda \in (\mathbb{C}^m, 0)$ sufficiently close to the origin. In this case analyticity of $x(\lambda), z(\lambda)$ in the larger domain U or polynomiality in general may fail.

Both facts become obvious if the bases of the vector spaces X, Y are are suitably chosen as follows. The operator L maps the first several basis vectors of X into the first basis vectors of Y and vanishes on the rest, while Z is the span of the last basis vectors of Y. The rest is the transversality theorem.

This observation should be applied to the homological operator $L = \mathrm{ad}_A$ acting in the space $X = \mathcal{D}_m$, and the subspace $Y = \mathcal{N}_m$ of homogeneous vector fields commuting with $\mathbf{A}^*(0)$. Holomorphic (polynomial) solvability of the homological equation on each step guarantees the possibility of transforming the field to the normal form with the required properties. $\square$

Remark 4.18 (Warning)**.** The difference between constant and non-constant linearization matrices is rather essential in what concerns the size of the common domain of analyticity of all Taylor coefficients of the normal form and/or conjugating transformation.

Suppose that all coefficients of the analytic family $F(\lambda)$ of formal vector fields are defined and holomorphic in some *common* domain U (e.g., the field is analytic in $D \times U$, where D is a small polydisk).

If the linearization matrix of $F(\lambda)$ does not depend on the parameters, then by the second assertion of Theorem 4.17, one may remove from the expansion of F all terms that are nonresonant (i.e., the terms that do not commute with the linear field $\mathbf{A}^*$ which is independent of the parameters). All coefficients of all series (the normal form and the conjugacy) will be holomorphic in the maximal natural domain U.

All the way around, if the linearized field $\mathbf{A}(\lambda)$ depends on parameters, then by a formal transformation one can eliminate all terms that are resonant with respect to $\mathbf{A}(0)$. The coefficients of the normal form and the transformation will be still analytically depending on λ, but their domains should be expected to shrink as the degree of the corresponding terms grow.

Indeed, assume that the linear field $\mathbf{A}(0)$ is non-resonant. Then the formal normal form guaranteed by the first assertion of Theorem 4.17 is *linear*, $F' = \mathbf{A}(\lambda)$. Yet clearly for the values of the parameter λ arbitrarily close to $\lambda = 0$, the spectrum of the matrix $A(\lambda)$ can become resonant, hence it will be impossible to eliminate completely all terms of the corresponding order. The apparent contradiction is easily explained: the domain of analyticity of the coefficient of a high order cannot be so large as to include values of the parameter corresponding to resonances of that order. Note that if $A(0)$ is non-resonant, then the possible order of resonances occurring for $A(\lambda)$ necessarily grows to infinity as $\lambda \to 0$.

4.7. Cuspidal points. One important case when Theorem 4.12 is considerably stronger than the Poincaré–Dulac theorem 4.10 is that of vector fields with *nilpotent* linear parts. In this case *all* monomials will be resonant and no simplification possible.

For our purposes we need the 2-dimensional case when the linear part is the vector field $J = y\frac{\partial}{\partial x}$ (the linearization matrix is a nilpotent Jordan cell of size 2). From Theorem 4.12 we can immediately derive the following Corollary.

Theorem 4.19. *A vector field on the plane with the linear part $J = y\frac{\partial}{\partial x}$ is formally equivalent to the vector field*

$$y\frac{\partial}{\partial x} + B(x)\,(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}) + A(x)\frac{\partial}{\partial y}, \qquad A, B \in \mathbb{C}[[x]], \tag{4.11}$$

with the formal series $A, B \in \mathbb{C}[[x]]$ in one variable x starting with terms of order 2 and 1 respectively.

Proof. We need only to describe the kernel of the operator ad_{J^*}, where $J^* = x\frac{\partial}{\partial y}$ is the "adjoint" vector field. The kernel of the operator $\operatorname{ad}_{J^*} = [x\frac{\partial}{\partial y}, \cdot\,]$ restricted on $\mathcal{D}_m$ can be immediately computed. Indeed,

$$[x\frac{\partial}{\partial y}, u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y}] = xu_y\frac{\partial}{\partial x} + (xv_y - u)\frac{\partial}{\partial y},$$

and the commutator vanishes only if both u and hence v_y depend only on x. Since both u, v must be homogeneous of degree m, we conclude that

$$\ker[x\frac{\partial}{\partial y}, \cdot\,] = \beta(x^m\frac{\partial}{\partial x} + x^{m-1}y\frac{\partial}{\partial y}) + \alpha x^m\frac{\partial}{\partial y} = \beta x^m\,(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}) + \alpha x^m\frac{\partial}{\partial y}$$

for some two constants $\alpha = \alpha_m$ and $\beta = \beta_m$ which will be the coefficients of the respective series A, B. $\square$

The complementary subspaces $\mathcal{N}_m$ may be chosen in a different way, more convenient for some applications.

Theorem 4.20. *The planar formal vector field with the linear part $J = y\frac{\partial}{\partial x}$, is formally equivalent to the vector field*

$$y\frac{\partial}{\partial x} + [yb(x) + a(x)]\frac{\partial}{\partial y}, \tag{4.12}$$

where $a(x)$ and $b(x)$ are two formal series of orders 2 and 1 respectively.

Proof. We reduce this assertion directly to the general Poincaré–Dulac paradigm. The image of ad_J in $\mathcal{D}_m$ can be complemented by the 2-dimensional space $\mathcal{N}'_m$ of vector fields $(\alpha x^m + \beta x^{m-1}y)\frac{\partial}{\partial x}$, see [**Arn83**, §35 D]. Indeed, the condition $[y\frac{\partial}{\partial x}, f\frac{\partial}{\partial x} + g\frac{\partial}{\partial y}] = u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y}$ takes the form of the system of linear partial differential equations

$$yf_x - g = u, \qquad yg_x = v.$$

While it can be not solvable for some u, v, the system of equations

$$yf_x - g = u, \quad yg_x + \alpha x^m + \beta x^{m-1}y = v \tag{4.13}$$

can be always resolved for any pair of homogeneous polynomials $u, v \in \mathbb{C}[x, y]$ of degree m and the constants α, β. To see this, apply $y\frac{\partial}{\partial x}$ to the first equation:

$$y^2 f_{xx} = y u_x + v - \alpha x^m - \beta x^{m-1} y.$$

The equation $y^2 f_{xx} = w$ is uniquely solvable for any monomial w divisible by y^2. On the other hand, the constants α, β can be found to guarantee that the terms proportional to x^m and $x^{m-1}y$ in the right hand side of this equation vanish. This choice automatically guarantees solvability of the second equation in (4.13) as well. The constants found in this way, appear as coefficients of the respective series a, b. □

4.8. Formal classification of biholomorphisms. Besides formal vector fields, formal isomorphisms act also on themselves, namely, by conjugacy: if

$$G(x) = Mx + V_2(x) + \cdots, \qquad \det M \neq 0, \tag{4.14}$$

is the formal map of $(\mathbb{C}^n, 0)$ to itself, then a formal isomorphism $H(x)$ transforms G to $G' = H \circ G \circ H^{-1}$. In the same way as before, one may ask if all nonlinear terms $V_2, V_3, \dots$ can be removed from the expansion by applying a suitable formal conjugacy.

The strategy is the same: $H(x) = x + P(x)$ conjugates $G(x)$ as in (4.14) with $G'(x) = G(x) + R_m(x) + \cdots$, where R_m is a homogeneous vector field of order m, if and only if

$$G(x) + P_m(G(x)) = G(x + P_m(x)) + R_m(x + P_m(x)),$$

which after collection of terms of order m yields the identity

$$P(Mx) = MP(x) + R(x), \qquad P = P_m,\ R = R_m. \tag{4.15}$$

This is the multiplicative analog of the homological equation (4.2). The operator

$$S_M \colon \mathcal{D}_m \to \mathcal{D}_m, \qquad P(x) \mapsto MP(x) - P(Mx), \tag{4.16}$$

can be studied by the methods similar to the operator ad_A. If M is a diagonal matrix with the diagonal entries $\mu_1, \dots, \mu_n$, then all monomials $F_{k\alpha}$ of the standard basis in $\mathcal{D}_m$ are eigenvectors for S_M with the eigenvalues $\mu_j - \mu^\alpha = \mu_j - \mu_1^{\alpha_1} \cdots \mu_n^{\alpha_n}$.

Now all theorems concerning formal classification of formal holomorphisms can be obtained in exactly the same way as for the formal vector fields.

Definition 4.21. A *multiplicative* resonance between the complex numbers $\mu_1, \dots, \mu_n$ is any identity of the form

$$\mu_j - \mu^\alpha = 0, \qquad |\alpha| \geqslant 2,\ j = 1, \dots, n. \tag{4.17}$$

A nondegenerate matrix M and a formal holomorphism $G(x) = Mx + \cdots$ are non-resonant if there are no multiplicative resonances between the eigenvalues of M. A *multiplicative resonant monomial* corresponding to the resonance (4.17), is the vector whose jth component is x^α and all others are zeros.

Theorem 4.22 (Poincaré–Dulac theorem for formal automorphisms)**.** *Any invertible formal holomorphism is formally equivalent to a formal holomorphism whose linear part is in the Jordan normal form, and the nonlinear part contains only resonant monomials with complex coefficients. In particular, a nonresonant formal holomorphism is formally conjugated to the linear map $G'(x) = Mx$.* □

4.9. Generalization: classification of vector fields with zero linear parts. If the formal vector field F has zero linear part and starts with kth order terms, $F(x) = V_k(x) + V_{k+1}(x) + \cdots$, then application of the formal transformation $H(x) = x + P_2(x)$ conjugates F with the vector field $F'(x) = V_k + V'_{k+1} + \cdots$ with the same (nonlinear) principal part V_k, if

$$\left(\frac{\partial P_2}{\partial x}\right) V_k(x) + V_{k+1}(x) + \cdots = V_k(x + P_2(x)) + V'_{k+1}(x + P_2(x)) + \cdots$$

which after collecting the homogeneous terms of order $m + 1$ yields

$$[V_k, P_2] = V_{k+1} - V'_{k+1}.$$

If this equation is resolved for a suitably chosen V'_{k+1} (e.g., equal to zero if that is possible), one can pass to terms of order $k + 2$ by applying a transform of the form $H(x) = x + P_3(x)$ which does not affect the terms of order V_k and V_{k+1} and so on. As a result, one has to resolve in each order the homological equation

$$\operatorname{ad}_{V_k} P_m = V_{m+k-1} - V'_{m+k-1} \tag{4.18}$$

with respect to the homogeneous vector field P_m of degree m. As before, complete elimination of all non-principal terms of orders $k + 1$ and more, is possible if the operator ad_{V_k} is surjective, otherwise it will be necessary to introduce the "normal subspaces" $\mathcal{N}_{m+k-1} \subset \mathcal{D}_{m+k-1}$ complementary to the image $\operatorname{ad}_{V_k}(\mathcal{D}_m) \subseteq \mathcal{D}_{m+k-1}$ and choose the components V'_{m+k-1} of the formal normal form from these subspaces.

In contrast to the case $k = 1$ discussed earlier, the operator ad_{V_k} acts between *different* spaces, the dimension of the target space in general higher than that of the source space. Thus the number of parameters in the normal form will in general be infinite. A notable exception is the one-dimensional case $\dim x = 1$.

Theorem 4.23. *A nonzero formal vector field in $\mathbb{C}^1$ is formally equivalent to one of the vector fields of the form*

$$\bigl(x^k + ax^{2k-1}\bigr)\tfrac{\partial}{\partial x}, \qquad k \geqslant 2,\ a \in \mathbb{C}. \tag{4.19}$$

Proof. Any nonzero formal vector field on $\mathbb{C}^1$ starts with the term $a_k x^k \frac{\partial}{\partial x}$, $a_k \neq 0$. For $k > 1$ one can make a_k equal to 1 by a linear transformation $x \mapsto cx$.

In this case all spaces $\mathcal{D}_m$ are one-dimensional, and the commutator with the principal term $x^k \frac{\partial}{\partial x}$ can be immediately computed:

$$\bigl[x^k \tfrac{\partial}{\partial x}, x^m \tfrac{\partial}{\partial x}\bigr] = (k-m)x^{k+m-1}\tfrac{\partial}{\partial x}.$$

This operator is surjective for all $m \neq k$. □

Over the real case two signs, $\pm x^k + \cdots$ for even k. Collect all such results in §**??**?

Perhaps, expand?

We note in passing that a similar result holds for formal maps $x \mapsto x + x^k + \cdots$ of $\mathbb{C}^1$ into itself, tangent to the identity. Any such map is formally conjugated to the map $x \mapsto x + x^k + ax^{2k-1}$. The proof is completely similar to the proof of Theorem 4.23.

4.10. Formal normal forms of elementary singular points on the plane. In this section we summarize the (orbital) formal normal forms for all planar (i.e., for $n = 2$) vector fields with nonzero linear part. All these results are particular cases of the general results proved earlier. Everywhere below “singularity” means a formal vector field on $\mathbb{C}^2$ with a singular point at the origin and the eigenvalues $\lambda_1, \lambda_2 \in \mathbb{C}$.

Definition 4.24. A singularity of planar vector field is *elementary*, if at least one of its eigenvalues $\lambda_{1,2}$ is nonzero.

Definition 4.25. An elementary singularity is *resonant node*, if $\lambda_1 = r\lambda_2$, where $r \in \mathbb{N}$ or $1/r \in \mathbb{N}$. It is called a resonant saddle, if $m_1\lambda_1 + m_2\lambda_2 = 0$, $m_1, m_2 \in \mathbb{N}$. Finally, the singularity is a *saddle-node*, if exactly one eigenvalue is zero.

Proposition 4.26 (Formal normal forms of elementary singularities)**.**

1. *A nonresonant elementary singularity is formally linearizable.*
2. *A resonant node with $r \in \mathbb{N}$ is is formally equivalent to the field*

$$\begin{cases} \dot x_1 = \lambda_1 x_1 + a x_2^r, \\ \dot x_2 = \lambda_2 x_2. \end{cases} \tag{4.20}$$

(*this system is in fact linear if $r = 1$*).

3. A resonant saddle is either formally orbitally *linearizable, or equivalent to the field*

$$\begin{cases} \dot{x}_1 = \lambda_1 x_1, \\ \dot{x}_2 = \lambda_2 x_2 (1 + u^k + a_2 u^{2k}), \end{cases} \qquad u = x^m = x_1^{m_1} x_2^{m_2}, \tag{4.21}$$

where $a \in \mathbb{C}$, $2 \leqslant k \in \mathbb{N}$.

4. A saddle-node is either formally orbitally *linearizable* (*equivalent to the vector field* $x_1 \frac{\partial}{\partial x_1}$ *with a non-isolated singular point*) *or equivalent to the vector field*

$$\begin{cases} \dot{x}_1 = x_1, \\ \dot{x}_2 = x_2^k + a x_2^{2k}, \end{cases} \tag{4.22}$$

where $a \in \mathbb{C}$, $2 \leqslant k \in \mathbb{N}$.

Proof. The first two assertions literally coincide with the assertion of the Poincaré–Dulac theorem.

The fourth assertion is a combination of the Poincaré–Dulac theorem and Theorem 4.23. While the condition $\lambda_2 = 0$ is not a resonance, it implies infinitely many resonances $\lambda_j = \lambda_j + m$ for any $m \in \mathbb{N}$. Thus the field is formally (and time-preserving) equivalent to the field $x_1 f_1(x_2) \frac{\partial}{\partial x_1} + f_2(x_2) \frac{\partial}{\partial x_2}$ with $f_1(0) \neq 0$, $f_2(0) = 0$ (otherwise the singular point is not elementary degenerate). Dividing the vector filed by $f_1(x_2)$ (an orbital transform), one can achieve $f_1 \equiv 1$. It remains to make the formal change of the variable x_2 which puts the vector field $f_2(x_2) \frac{\partial}{\partial x_2}$ into the normal form (4.19).

The third assertion is proved similarly: the identity $\langle \lambda, m \rangle = 0$ itself is not a resonance, but its integer multiple can be added to either the identity $\lambda_1 = \lambda_1$ or $\lambda_2 = \lambda_2$, each time producing a resonance. Clearly, there are no other resonances and the Poincaré–Dulac normal form looks like $\lambda_1 x_1 f_1(u) \frac{\partial}{\partial x_1} + \lambda_2 x_2 f_2(u) \frac{\partial}{\partial x_2}$, $f_i(0) = 1$. Passing to an orbitally equivalent system, one can assume that $f_1 \equiv 1$.

The Poincaré–Dulac system admits the projection $\mathbb{C}^2 \ni x \mapsto u = x^m \in \mathbb{C}^1$. The projected system has the form

$$\dot{u} = u F(u), \qquad F(u) = f_2(u) - 1,$$

and by a suitable formal transformation $u \mapsto u' = u(1 + h(u))$ can be brought to the form (4.19), corresponding to $f_2(u) = 1 + u^{k-1} + a u^{2k-1}$. It remains to observe that any formal transformation of the variable u can be covered by the transformation $(x_1, x_2) \mapsto (x_1, x_2'(x_1, x_2))$,

$$x_2' = (u'/x_2'^{m_1})^{1/m_2} = x_2 [1 + h(x^m)]^{1/m_2} \in \mathbb{C}[[x]].$$

This transformation brings the field $\lambda_1 x_1 \frac{\partial}{\partial x_1} + \lambda_2 x_2 f_2(u) \frac{\partial}{\partial x_2}$ into the required form. □

Remark 4.27. The full (non-orbital) formal normal form contains more parameters. For instance, for the saddle-node the formal normal form is

$$\begin{cases} \dot{x}_1 = x_1(\lambda_1 + b_1 x_2 + \cdots + b_{k-1} x_2^{k-1}), \\ \dot{x}_2 = x_2^k + a x_2^{2k}, \qquad \lambda_1, b_1, \dots, b_{k-1}, a \in \mathbb{C}. \end{cases} \tag{4.23}$$

To prove this formula, we reduce the vector field to the form $x_1 f_1(x_2) \frac{\partial}{\partial x_1} + f_2(x_2) \frac{\partial}{\partial x_2}$ and then by a suitable change of the variable x_2 only put f_2 into the standard form as above. The function $f_1(x_2)$ can be further transformed by transformations of the form $(x_1, x_2) \mapsto (h(x_2)x_1, x_2)$, $h(0) \neq 0$, preserving the second component: one immediately verifies that such transformation results in replacing f_1 by

$$f_1' = f_1 + f_2 \cdot \frac{dh}{h\,dx_2}.$$

If f_2 begins with terms of order k, then the difference between f_1 and f_1' is necessarily $(k-1)$-flat (the logarithmic derivative $\frac{d}{dx_2} \ln h$ in the above formula is a formal series from $\mathbb{C}[[x_2]]$ since $h(0)$ is nonvanishing). On the other hand, if the difference $f_1 - f_1'$ is divisible by f_2, the quotient can be represented as the logarithmic derivative of a suitable series $h \in \mathbb{C}[[x_2]]$. Thus all terms of order k and above can be eliminated from f_1 by the formal transformation.

A similar result can be formulated for resonant saddles.

5. Holomorphic normal forms

5.1. Poincaré and Siegel domains. To linearize a given (say, nonresonant) vector field, on each step of the Poincaré–Dulac process one has to compute the inverse of the operator $\operatorname{ad}_A = [A, \cdot]$ on the spaces of homogeneous vector fields. To that end, one has to divide by the Taylor coefficients by the *denominators*, expressions of the form $\lambda_j - \langle \alpha, \lambda \rangle \in \mathbb{C}$ with $\alpha \in \mathbb{Z}_+^n$, $|\alpha| \geqslant 2$, that may a priori be *small.* These denominators associated with the spectrum λ of the linearization matrix A, behave differently as $|\alpha|$ grows to infinity, in the two different cases.

Definition 5.1. The *Poincaré domain* $\mathfrak{P} \subset \mathbb{C}^n$ is the collection of all tuples $\lambda = (\lambda_1, \dots, \lambda_n)$ such that the convex hull of the point set $\{\lambda_1, \dots, \lambda_n\} \subset \mathbb{C}$ does not contain the origin inside or on the boundary.

The Siegel domain $\mathfrak{S}$ is the complement to the Poincaré domain in $\mathbb{C}^n$.

The *strict Siegel domain* is the set of tuples for which the convex hull contains the origin strictly inside.

Sometimes we say about *tuples* or even *spectra* as being of Poincaré (resp., Siegel) type.

Proposition 5.2. *If λ is of Poincaré type, then only finitely many denominators $\lambda_j - \langle \alpha, \lambda \rangle$, $\alpha \in \mathbb{Z}_+^n$, $|\alpha| \geqslant 2$, may actually vanish.*

Moreover, nonzero *denominators are bounded away from the origin: the latter is an isolated point of the set of all denominators $\{\lambda_j - \langle \alpha, \lambda \rangle \,|\, j = 1, \dots, n,\ |\alpha| \geqslant 2\}$.*

On the contrary, if λ is of Siegel type, then either there are infinitely many vanishing denominators, or the origin $0 \in \mathbb{C}$ is their accumulation point.

Proof. If the convex hull of $\{\lambda_1, \dots, \lambda_n\} \subset \mathbb{C}$ does not contain the origin, by the convex separability theorem there exists a real linear functional $\ell \colon \mathbb{C}^2 \to \mathbb{R}$ such that $\ell(\lambda_j) \leqslant -r < 0$ for all λ_j, and hence $\ell(\langle \alpha, \lambda \rangle) \leqslant -r|\alpha|$. But then for any denominator we have

$$\ell(\lambda_j - \langle \alpha, \lambda \rangle) \geqslant \ell(\lambda_j) + |\alpha| r \to +\infty \qquad \text{as } |\alpha| \to \infty.$$

Since ℓ is bounded on any small neighborhood of the origin $0 \in \mathbb{C}$, the first two assertions are proved.

To prove the second assertion, notice that in the Siegel case there are either two or three numbers, whose linear combination with positive (real) coefficients is zero, depending on whether the origin lies on the boundary or in the interior of the convex hull.

In the second (more difficult) case, modulo re-enumeration of the eigenvalues and an (non-conformal) affine transformation of the real plane $\mathbb{R}^2 \simeq \mathbb{C}$, we may assume without loss of generality that $\lambda_1 = 1$, $\lambda_2 = +i$ and $-\lambda_3 \in \mathbb{R}_+^2 = \mathbb{R}_+ + i\mathbb{R}_+$. In this case all "fractional parts" $-\mathbb{N}\lambda_3 \bmod \mathbb{Z} + i\mathbb{Z}$ of natural multiples of $-\lambda_3$ either form a finite subset of the 2-torus $\mathbb{R}^2/\mathbb{Z}^2$ (in which case all points of this set correspond to infinitely many vanishing denominators), or are uniformly distributed along some 1-torus, or dense. In both latter cases the point $(0,0) \in \mathbb{R}^2/\mathbb{Z}^2$ is the accumulation point of the "fractional parts" which are affine images of the denominators. □

Remark 5.3. A similar statement, the claim that resonant tuples $\lambda \in \mathbb{C}^n$ are dense in the Siegel domain $\mathfrak{S}$ and not dense in the Poincaré domain $\mathfrak{P}$, can be found in [**Arn83**].

Corollary 5.4. *If the spectrum of the linearization matrix A of a formal vector field belongs to the Poincaré domain, then the resonant formal normal form for this field established in Theorem 4.10, is polynomial.* □

5.2. Holomorphic classification in the Poincaré domain. In the Poincaré domain, the difference between formal and holomorphic (convergent) equivalence disappears.

Theorem 5.5 (Poincaré normalization theorem)**.** *A holomorphic vector field with the linear part of Poincaré type is holomorphically equivalent to its polynomial Poincaré–Dulac formal normal form.*

In particular, if the field is non-resonant, then it can be linearized by a holomorphic transformation.

We prove this theorem first for vector fields with a diagonal nonresonant linear part $\Lambda = \operatorname{diag}\{\lambda_1, \dots, \lambda_n\}$. The resonant case will be addressed later in §5.3. The classical proof by Poincaré was achieved by the so called *majorant method*. In the modern language, it takes a more convenient form of the contracting map principle in an appropriate functional space, the *majorant space.*

Definition 5.6. The *majorant operator* is the nonlinear operator acting on formal series by replacing all Taylor coefficients by their absolute values,

$$\mathcal{M}\colon \sum_{\alpha\in\mathbb{Z}_n^+} c_\alpha\, z^\alpha \mapsto \sum_{\alpha\in\mathbb{Z}_n^+} |c_\alpha|\, z^\alpha.$$

The action of the majorant operator naturally extends on all formal objects (vector formal series, formal vector fields, formal transformations etc.)

Definition 5.7. The *majorant ρ-norm* is the functional on the space of formal power series $\mathbb{C}[[z_1, \dots, z_n]]$, defined as

$$[\![f]\!]_\rho = \sup_{|z|<\rho} |\mathcal{M}f(z)| = |\mathcal{M}f(\rho, \dots, \rho)| \leqslant +\infty. \tag{5.1}$$

For a formal vector function $F = (F_1, \dots, F_n)$ the majorant norm is

$$[\![F]\!]_\rho = [\![F_1]\!]_\rho + \cdots + [\![F_n]\!]_\rho\,. \tag{5.2}$$

The majorant space $\mathcal{B}_\rho$ is the subspace of formal (vector) functions from $\mathbb{C}[[x]]$ having finite majorant ρ-norm.

Proposition 5.8. *The space $\mathcal{B}_\rho$ with the majorant norm $[\![\cdot]\!]_\rho$ is complete.*

Proof. If $\rho = 1$, this is obvious: $\mathcal{B}_1$ is the space of infinite absolutely converging sequences $\{c_\alpha\}$, and hence is isomorphic to the standard Lebesgue space ℓ^1 which is complete. The general case of an arbitrary ρ follows from the fact that the correspondence $f(\rho x) \leftrightarrow f(x)$ is an isomorphism between $\mathcal{B}_\rho$ and $\mathcal{B}_1$. □

Remark 5.9. The space $\mathcal{B}_\rho$ is closely related but not coinciding with the space $\mathcal{A}_\rho = \mathcal{A}(D_\rho)$ of functions, holomorphic in the polydisk $D_\rho = \{|z| < \rho\}$, continuous on its closure and equipped with the usual sup-norm $\|f\|_\rho = \max_{|z|<\rho} |f(z)|$.

Clearly, $\mathcal{B}_\rho \subset \mathcal{A}_\rho$, since a series from $\mathcal{B}_\rho$ is absolutely convergent on the polydisk D_ρ. Conversely, if f is holomorphic in D_ρ and continuous on the boundary, then by the Cauchy estimates, the Taylor coefficients c_α of f satisfy the inequality

$$|c_\alpha| \leqslant \|f\|_\rho \cdot \rho^{-|\alpha|}, \quad \alpha \in \mathbb{Z}_+^n.$$

Though the series $[\![f]\!]_\rho = \sum |c_\alpha| \rho^{|\alpha|}$ may still diverge, any other norm $[\![f]\!]_{\rho'}$ with $\rho' < \rho$, will already be finite:

$$[\![f]\!]_{\rho'} \leqslant \|f\|_\rho \cdot \sum_{\alpha \in \mathbb{Z}_+^n} \delta^{|\alpha|} < C \|f\|_\rho, \qquad C = C(\delta, n), \quad \delta = \rho'/\rho < 1.$$

To construct a counterexample showing that indeed $\mathcal{A}_\rho \supsetneq \mathcal{B}_\rho$, consider a convergent but not absolutely convergent Fourier series $\sum_{k\in\mathbb{Z}} c_k e^{ikt}$ in one real variable t and let $f(z) = \sum c_k z^k$. Such series converges at all points of the boundary $|z| = 1$ and represents a function from $\mathcal{A}(D_1)$, but by construction its 1-norm is infinite. Details can be found in [**Edw79**, §10.6]

The important properties of the majorant spaces and norms concern operations on functions. We will use the notation $f \ll g$ for two vector series from $\mathbb{C}^n[[x]]$ with positive coefficients, if each coefficient of f is no greater than the corresponding coefficient of g. In a similar way will be used the notation $x \ll y$ for $x, y \in \mathbb{R}^n$. If $f \in \mathbb{R}^n[[x]]$ is a (vector) series with *nonnegative* coefficients, then it is monotonous: $f(x) \ll f(y)$ if $x \ll y$.

Lemma 5.10. *1. For any two series $f, g \in \mathbb{C}[[x]]$ and any ρ,*

$$[\![fg]\!]_\rho \leqslant [\![f]\!]_\rho \cdot [\![g]\!]_\rho, \tag{5.3}$$

provided that all norms are finite.

2. If $G \ll G'$, are two formal series from $\mathbb{R}^n[[x]]$ and F is a series with nonnegative coefficients, then $F \circ G \ll F \circ G'$.

3. If $F, G \in \mathbb{C}^n[[z_1, \dots, z_n]]$ are two formal vector series, $F(0) = G(0) = 0$, then for their composition we have

$$[\![F \circ G]\!]_\rho \leqslant [\![F]\!]_\sigma, \qquad \sigma = [\![G]\!]_\rho. \tag{5.4}$$

Proof. The first two statements are obvious: all Taylor coefficients of the product or composition are obtained from the coefficients of entering terms by operations of addition and multiplication only. In particular, $\mathcal{M}(fg) \ll \mathcal{M}f \cdot \mathcal{M}g$. Evaluating both parts at $\boldsymbol{\rho} = (\rho, \dots, \rho)$ proves the first statement.

Since all binomial coefficients are nonnegative (in fact, natural numbers), we have $\mathcal{M}(F \circ G) \ll (\mathcal{M}F) \circ (\mathcal{M}G)$. Evaluating at $\boldsymbol{\rho} = (\rho, \dots, \rho)$ yields $\mathcal{M}G(\boldsymbol{\rho}) = y \ll \boldsymbol{\sigma} = (\sigma, \dots, \sigma)$, where $\sigma = [\![G]\!]_\rho$. By monotonicity, $\|F \circ G\|_\rho = \big((\mathcal{M}F) \circ (\mathcal{M}G)\big)(\boldsymbol{\rho}) \ll \mathcal{M}F(y) \ll \mathcal{M}F(\boldsymbol{\sigma}) = [\![F]\!]_\sigma$. The last statement is proved. □

Lemma 5.11. *If* $\Lambda \in \operatorname{Mat}(n, \mathbb{C})$ *is a nonresonant diagonal matrix of Poincaré type, then the operator* $\operatorname{ad}_\Lambda$ *has a bounded inverse in the space of vector fields equipped with the majorant norm.*

Proof. The formal inverse operator $\operatorname{ad}_\Lambda^{-1}$ is diagonal,

$$\operatorname{ad}_\Lambda^{-1}\colon \sum_{k,\alpha} c_{k\alpha} x^\alpha \frac{\partial}{\partial x_k} \longmapsto \sum_{k,\alpha} \frac{c_{k\alpha}}{\lambda_k - \langle \alpha, \lambda\rangle}\, x^\alpha \frac{\partial}{\partial x_k}.$$

In the Poincaré domain the absolute values of all denominators are bounded from below by a positive constant $\varepsilon > 0$, therefore *any* majorant ρ-norm is increased by no more than ε^{-1}:

$$[\![\operatorname{ad}_\Lambda^{-1}]\!]_\rho \leqslant \Bigl(\inf_{j,\alpha} |\lambda_j - \langle \alpha, \lambda\rangle\, |\Bigr)^{-1} < +\infty.$$

This proves that $\operatorname{ad}_\Lambda$ has the bounded inverse. □

Remark 5.12. A diagonal operator of the form $\sum_\alpha c_\alpha z^\alpha \mapsto \sum \mu_\alpha c_\alpha x^\alpha$ with bounded entries, $\sup_\alpha |\mu_\alpha| < +\infty$, which is always bounded on the majorant space, may be *not bounded* on the holomorphic space $\mathcal{A}(D_\rho)$, see Remark 5.9. The "real" counterexample is even simpler: the operator which multiplies odd coefficients by -1, sends the series $1 - x^2 + x^4 - \cdots$, converging and bounded on $[-1, 1]$, into an unbounded function.

Let $F = (F_1, \dots, F_n)\colon \mathbb{C}^n \to \mathbb{C}^n$ be a holomorphic vector function defined in some polydisk near the origin. The *operator of argument shift* is the operator

$$S_F\colon h(x) \mapsto F(x + h(x)), \tag{5.5}$$

acting on formal vector series $h\colon \mathbb{C}^n \to \mathbb{C}^n$ without the free term, $h(0) = 0$. We want to show that S_F is in some sense strongly contracting. The formal description looks as follows.

Consider the family of Banach spaces $\mathcal{B}_\rho$ indexed by the real parameter $\rho \in (\mathbb{R}_+, 0)$ and decreasing in the sense that $\mathcal{B}_\rho$ is a subspace in $\mathcal{B}_{\rho'}$ for all $0 < \rho < \rho'$ (the embedding $\operatorname{id}_{\rho,\rho'}\colon \mathcal{B}_\rho \to \mathcal{B}_{\rho'}$ is continuous). Let S be an operator defined on all of these spaces for all sufficiently small values of ρ. Formally this means a family of operators $S_\rho\colon \mathcal{B}_\rho \to \mathcal{B}_\rho$ which commute with the "restriction operators" $\operatorname{id}_{\rho,\rho'}$ for any $\rho < \rho'$, but we will omit the subscript in the notation of $S_\rho = S$.

Definition 5.13. The operator S is *strongly contracting*, if

(1) $[\![S(0)]\!]_\rho = O(\rho^2)$ and
(2) S is Lipschitz on the ball $B_\rho = \{[\![h]\!]_\rho \leqslant \rho\} \subset \mathcal{B}_\rho$ of the majorant ρ-norm (with the same ρ), with the Lipschitz constant no greater than $O(\rho)$ as $\rho \to 0$.

Note that any strongly contracting operator takes the balls B_ρ strictly into themselves, since the center of the ball is shifted by $O(\rho^2)$ and the diameter of the image $S(B_\rho)$ does not exceed $2\rho\, O(\rho) = O(\rho^2)$.

Lemma 5.14. *Assume that the germ $F\colon (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ is holomorphic and its linearization is zero, $\left(\frac{\partial F}{\partial x}\right)(0) = 0$.*

Then the operator of argument shift (5.5) *is strongly contracting.*

Proof. To prove the Lemma, first notice that S_F takes $h = 0$ into $F(x)$; the latter function has ρ-norm $O(\rho^2)$ for all sufficiently small ρ, since F begins with quadratic terms.

Next we compute the Lipschitz constant for $S = S_F$ on $B_\rho \subset \mathcal{B}_\rho$. If $h, h' \in \mathbb{C}^n[[x_1, \dots, x_n]]$ are two vector formal series, then the formal series

$$g = Sh - Sh' = F(\mathrm{id} + h) - F(\mathrm{id} + h')$$

can be computed as follows:

$$g(x) = \int_0^1 \left(\frac{\partial F}{\partial x}\right)\bigl(x + \tau h(x) + (1-\tau)h'(x)\bigr) \cdot (h(x) - h'(x))\, d\tau.$$

By Lemma 5.10, since $\tau \in [0, 1]$, we have

$$[\![g]\!]_\rho \leqslant [\![\tfrac{\partial F}{\partial x}]\!]_\sigma \cdot [\![h - h']\!]_\rho, \qquad \sigma = [\![x + \tau h(x) + (1-\tau)h'(x)]\!]_\rho .$$

The norm σ is no greater than $[\![x]\!]_\rho + \max([\![h]\!]_\rho, [\![h']\!]_\rho) = (n+1)\rho$ if both h, h' are both from the ρ-ball B_ρ. On the other hand, if F is holomorphic vector function without free and linear terms, its Jacobian matrix is holomorphic without free order terms and hence its σ-norm is bounded by $C\sigma$ for all sufficiently small $\sigma > 0$. Collecting everything together, we see that S_F is Lipschitz on the ρ-ball B_ρ, with the Lipschitz constant (contraction rate) not exceeding $(n+1)C\rho$, so S_F is strongly contracting. □

Proof of Theorem 5.5 (non-resonant case). Now we can prove that a holomorphic vector field with diagonal non-resonant linearization matrix Λ of Poincaré type is holomorphically linearizable in a sufficiently small neighborhood of the origin.

The holomorphic transformation $H = \mathrm{id} + h$ conjugates the linear vector field Λx (the normal form) with the initial nonlinear field denoted by $\Lambda x + F(x)$, if and only if

$$\Lambda h(x) - \left(\tfrac{\partial h}{\partial x}\right) \Lambda x = F\bigl(x + h(x)\bigr),$$

i.e., in the operator form,

$$\mathrm{ad}_\Lambda\, h = S_F h, \qquad S_F\, h = F \circ (\mathrm{id} + h), \quad \mathrm{ad}_\Lambda = [\Lambda, \cdot\,]. \tag{5.6}$$

Let h be the fixed point of the operator $\operatorname{ad}_\Lambda^{-1} \circ S_F$ in some space $\mathcal{B}_\rho$ (assuming its existence), i.e., the holomorphic solution of the equation

$$h = (\operatorname{ad}_\Lambda^{-1} \circ S_F)h, \qquad h \in \mathcal{B}_\rho. \tag{5.7}$$

Applying to both parts the operator $\operatorname{ad}_\Lambda$, we conclude that h solves (5.6) and therefore $\operatorname{id} + h$ conjugates the linear field Λx with the nonlinear field $\Lambda x + F(x)$ in the polydisk $\{|x| < \rho\}$.

Consider this operator $\operatorname{ad}_\Lambda^{-1} \circ S_F$ in the space $\mathcal{B}_\rho$ with sufficiently small ρ. The operator $\operatorname{ad}_\Lambda^{-1}$ is bounded by Lemma 5.11; its norm is the the reciprocal to the smallest small divisor and is independent of ρ. On the other hand, the argument shift operator S_F is strongly contracting with the contraction rate (Lipschitz constant) going to zero with ρ as $O(\rho)$. Thus the composition will be contracting on the ρ-ball B_ρ in the ρ-majorant norm with the contraction rate $O(1) \cdot O(\rho) = O(\rho) \to 0$. By the contracting map principle, there exists a unique fixed point of the operator equation (5.7) in the space $\mathcal{B}_\rho$ which is therefore a holomorphic vector function. The corresponding map $H = (\operatorname{id} + h)^{-1}$ linearizes the holomorphic vector field. □

5.3. Resonant case: polynomial normal form. Modification of the previous construction allows to prove that a resonant holomorphic vector field in the Poincaré domain can be brought into a *polynomial* normal form.

Consider a holomorphic vector field $F(x) = Ax + V(x)$ with the linearization matrix A having eigenvalues in the Poincaré domain, and nonlinear part V of order $\geqslant 2$ (i.e., 1-flat) at the origin. Without loss of generality (passing, if necessary, to an orbitally equivalent field cF, $0 \neq c \in \mathbb{C}$, one may assume that the eigenvalues of A satisfy the condition

$$1 < \operatorname{Re} \lambda_j < r \qquad \forall j = 1, \dots, n \tag{5.8}$$

with some natural $r \in \mathbb{N}$.

Theorem 5.15 (A. M. Lyapunov, H. Dulac[1])**.** *If the eigenvalues of the linearization matrix A of a holomorphic vector fields $F(x) = Ax + V(x)$ satisfy the condition* (5.8) *with some integer $r \in \mathbb{N}$, then the holomorphic vector field $F(x)$ is locally holomorphically equivalent to any holomorphic vector field with the same r-jet.*

Proof. A holomorphic conjugacy $H = \operatorname{id} + h$ between the fields F and $F + g$ satisfies the functional equation $\left(\frac{\partial H}{\partial x}\right) F = (F + g) \circ H$ which can be expanded to

$$\left(\frac{\partial h}{\partial x}\right) Ax - Ah = (V \circ (\operatorname{id} + h) - V) + g \circ (\operatorname{id} + h) - \left(\frac{\partial h}{\partial x}\right) V. \tag{5.9}$$

[1] The proof of this theorem, given in [**Bru71**], is incomplete.

Consider the three operators,

$$T_V : h \mapsto V \circ (\mathrm{id} + h) - V, \quad S_g : h \mapsto g \circ (\mathrm{id} + h), \qquad \Psi : h \mapsto \left(\frac{\partial h}{\partial x}\right) V.$$

Using these three operators, the differential equation (5.9) can be written in the form

$$\mathrm{ad}_A h = Th + Sh + \Psi h, \tag{5.10}$$

where $T = T_V$, $S = S_g$ and, as before in (5.6), ad_A is the commutator with the *linear* field $\mathbf{A}(x) = Ax$. The key difference with the previous case is two-fold: first, because of the resonances, the operator ad_A is *not invertible* anymore, and second, since the field F is nonlinear, the additional operator Ψ occurs in the right hand side. Note that this operator is a derivation of h, thus is unbounded in *any* majorant norm $[\![\cdot]\!]_\rho$.

Let $\mathcal{B}_{m,\rho} = \{f : j^m f = 0\} \cap \mathcal{B}_\rho$ be a subspace of m-flat series in the Banach space $\mathcal{B}_\rho$, equipped with the same majorant norm $[\![\cdot]\!]_\rho$. Since V is 1-flat, all three operators T, S, Ψ map the subspace $\mathcal{B}_{m,\rho}$ into itself for any $m > 1$.

Moreover, by Lemma 5.14, the argument shift operator S is strongly contracting, regardless of the choice of m. The "finite difference" operator T_V differs from the argument shift, S_V by the constant operator $V = T(0)$ which does not affect the Lipschitz constant. Since $[\![V]\!]_\rho = O(\rho^2)$, the operator T is also strongly contracting.

The operator ad_A preserves the order of all monomial terms and hence also maps $\mathcal{B}_{m,\rho}$ into itself for all m, ρ, and is *invertible* on these spaces if m is sufficiently large. Indeed, if $|\alpha| > r + 1$, then by (5.8) $\operatorname{Re}(\langle \alpha, \lambda \rangle - \lambda_j) > 0$ and all denominators in the formula

$$\mathrm{ad}_A^{-1}\big|_{\mathcal{B}_{m,\rho}} : \sum_{|\alpha| \geqslant m} c_{k\alpha}\, x^\alpha \frac{\partial}{\partial x_j} \;\longmapsto\; \sum_{|\alpha| \geqslant m} \frac{c_{k\alpha}}{\langle \alpha, \lambda \rangle - \lambda_j}\, x^\alpha \frac{\partial}{\partial x_j} \tag{5.11}$$

are nonzero if $m \geqslant r + 1$, and the restriction of ad_A^{-1} on $\mathcal{B}_{m,\rho}$ is bounded. Moreover,

$$[\![\mathrm{ad}_A^{-1} h]\!]_\rho \leqslant O(1/m)\, [\![h]\!]_\rho\,. \tag{5.12}$$

uniformly over all $h \in \mathcal{B}_{m,\rho}$ of order $m \geqslant r + 1$.

Thus the two compositions, $\mathrm{ad}_A^{-1} \circ S$ and $\mathrm{ad}_A^{-1} \circ T$, are strongly contracting. To prove the Theorem, it remains to prove that the *linear* operator $\mathrm{ad}_A^{-1} \circ \Psi : \mathcal{B}_{m,\rho} \to \mathcal{B}_{m,\rho}$ is strongly contracting when m is larger than $r + 1$.

Consider the $[\![\cdot]\!]_\rho$-normalized vectors $h_{k\beta} = \rho^{-|\beta|} x^\beta \frac{\partial}{\partial x_k}$ for all $k = 1, \dots, m$ and all $|\beta| \geqslant m$ spanning the entire space $\mathcal{B}_{m,\rho}$. We prove that

$$[\![\mathrm{ad}_A^{-1} \Psi h_{k\beta}]\!]_\rho = O(\rho) \qquad \text{as } \rho \to 0 \tag{5.13}$$

uniformly over $|\beta| \geqslant m$ and all k. Since $\operatorname{ad}_A^{-1} \circ \Psi$ is linear, this would imply that $\operatorname{ad}_A^{-1} \circ \Psi$ is strongly contracting.

The direct computation yields

$$\Psi h_{k\beta} = \sum_{i=1}^{n} \rho^{-|\beta|} \frac{\beta_i}{x_i} x^\beta \, V_i \frac{\partial}{\partial x_k}.$$

Since V is 1-flat, $\llbracket V_i \rrbracket_\rho = O(\rho^2)$; substituting this into the definition of the majorant norm, we obtain

$$\llbracket \Psi h_{k\beta} \rrbracket_\rho \leqslant \sum_i \beta_i \, \rho^{-1} O(\rho^2) = \beta_i \, O(\rho),$$

where $O(\rho)$ is uniform over all β. Since the order of the products $\frac{x^\beta}{x_i} V_i$ is at least $|\beta| + 1$, by (5.12) we have

$$\llbracket \operatorname{ad}_A^{-1} \Psi h_{k\beta} \rrbracket_\rho \leqslant \frac{\beta_i}{|\beta|} \, O(\rho) = O(\rho)$$

uniformly over all β with $|\beta| \geqslant m \geqslant r + 1$. Thus the last remaining composition $\operatorname{ad}_A^{-1} \circ \Psi$ is also strongly contracting, which implies existence of a solution for the fixed point equation

$$h = \operatorname{ad}^{-1} \circ (T + S + \Psi) h$$

equivalent to (5.10), in a sufficiently small polydisk $\{|x| < \rho\}$. □

Now one can easily complete the proof of holomorphic normalization theorem in the Poincaré domain in the resonant case.

Proof of Theorem 5.5 (resonant case). By the Poincaré–Dulac normalization process, one can eliminate all nonresonant terms up to any finite order m by a polynomial transformation. By Theorem 5.15, m-flat holomorphic terms can be eliminated by a holomorphic transformation if m is large enough (depending on the spectrum of the linearization matrix). □

Remark 5.16. In the Poincaré domain one can prove even stronger claim: if a holomorphic vector field depends analytically on finitely many additional parameters $\lambda \in (\mathbb{C}^m, 0)$ and belongs to the Poincaré domain for $\lambda = 0$, then by a holomorphic change of variables holomorphically depending on parameters, the field can be brought to a polynomial normal form involving only resonant terms. In such form this assertion is formulated in [**Bru71**] (see the footnote on p. 54) The proof can be achieved by minor adjustment of the arguments used in the demonstration of Theorem 5.15.

5.4. Divergence dichotomy. Outside the Poincaré domain, even in the absence of resonances, the normalizing series may diverge for some nonlinearities. On the other hand, no matter how "bad" is the linearization and its eigenvalues, there are always nonlinear systems that can be linearized (e.g., linear systems in nonlinear coordinates). It turns out that in some sense, the convergence/divergence pattern is common for *most* nonlinearities.

Consider a *parametric* nonlinear system

$$\dot{x} = Ax + z\,f(x), \qquad x \in \mathbb{C}^n, \quad z \in \mathbb{C}, \tag{5.14}$$

holomorphic in some neighborhood of the origin with the *nonresonant* linearization matrix A and the nonlinear part linearly depending on the complex parameter z. For such system for each value of the parameter $z \in \mathbb{C}$ there is a *unique* (by Remark 4.6) formal series $H_z(x) = x + h_z(x) \in \operatorname{Diff}[[x,z]]$ linearizing (5.14). This series may converge for some values of z while diverging for the rest. It turns out that there is a dichotomy: either convergence occurs for all values of z without exception, or on the contrary the series H_z diverges for all z outside a small (or rather "short") exceptional set $K \Subset \mathbb{C}$.

The exceptional sets are small in the sense that their (electrostatic) *capacity* is zero. This condition, formally introduced below in §5.5, implies among other things, that its Lebesgue measure is zero.

Theorem 5.17 (Divergence dichotomy, Yu. Ilyashenko [**Ily79**], R. Perez Marco [**PM01**])**.** *For any non-resonant linear family* (5.14) *one has the following alternative*:

(1) *Either the linearizing series $H_z \in \operatorname{Diff}[[\mathbb{C}^n,0]]$ converges for all values of $z \in \mathbb{C}$ in a symmetric polydisk of positive radius decreasing as $O(|z|^{-1})$ as $z \to \infty$, or*

(2) *The linearizing series H_z diverges for all values of z except for a set $K_f \Subset \mathbb{C}$ of capacity zero.*

The proof is based on the following property of polynomials, which can be considered as a quantitative uniqueness theorem for polynomials. If K is a set of positive capacity and $p \in \mathbb{C}[z]$ a polynomial vanishing on K, then by definition p vanishes identically. One can expect that if p is small on K, then it is also uniformly small on *any* other compact subset, in particular, on all compact subsets of $\mathbb{C}$.

Theorem 5.18 (Bernstein inequality)**.** *If $K \Subset \mathbb{C}$ is a set of positive capacity, then for any polynomial $p \in \mathbb{C}[z]$ of degree $r \geqslant 0$,*

$$|p(z)| \leqslant \|p\|_K \, \exp(rG_K(z)), \tag{5.15}$$

where $\|p\|_K = \max_{z\in K} |p(z)|$ *is the supremum-norm of* p *and* $G_K(z)$ *is the non-negative Green function of the complement* $\mathbb{C} \smallsetminus K$ *with the source at infinity, see* (5.19).

We postpone the proof of this Theorem until §5.5.

Lemma 5.19. *Formal Taylor coefficients of the formal series linearizing the field* (5.14) *are polynomial in* z.

More precisely, every monomial x^α, $|\alpha| \geqslant 2$, *enters into the vector series* h_z *with the coefficient which is a polynomial of degree* $\leqslant |\alpha| - 1$ *in* z.

Proof. The equation determining $h = h_z$ is of the form

$$\left(\frac{\partial h_z}{\partial x}\right)(Ax + z\, f(x)) = Ah_z(x). \tag{5.16}$$

Collecting the terms of degree m in x, we obtain for the corresponding mth homogeneous (vector) components $h_z^{(m)}$, $f^{(l)}$, the recurrent identities

$$\left(\frac{\partial h_z^{(m)}}{\partial x}\right) Ax - Ah_z^{(m)} = -z \sum_{k+l=m,\, l\geqslant 2} \left(\frac{\partial h_z^{(k+1)}}{\partial x}\right) f^{(l)}.$$

From these identities it obviously follows by induction that each $h_z^{(m)}$ is a polynomial of degree $m-1$ in z for all $m \geqslant 1$ (recall that f does not depend on z). □

Proof of Theorem 5.17. Assume that the formal series $H_z(x) = x + h_z(x)$ linearizing the field $F_z(x) = Ax + z\, f(x)$ converges for values of z belonging to some set $K^* \subset \mathbb{C}$ of positive capacity.

Consider the subsets $K_{c\rho} \Subset \mathbb{C}$, $\rho > 0$, $c < +\infty$, defined by the condition

$$z \in K_{c\rho} \iff |h_z^{(m)}| \leqslant c\rho^{-m} \quad \forall m \in \mathbb{N}.$$

By this definition, $K^* = \bigcup K_{c\rho}$ (a series converges if and only if satisfies some Cauchy-type estimate). Each of the sets $K_{c\rho}$ obviously is a compact subset of $\mathbb{C}$, being an intersection of semialgebraic compact sets.

The compacts $K_{c\rho}$ are naturally nested. Passing to countable subcollection, one concludes that set K of positive capacity is a countable union of compacts $K_{c\rho}$. By Proposition 5.23 (2), one of these compacts must also be of positive capacity. Denote this compact by $K = K_{c\rho}$: by its definition,

$$|h_z^{(m)}| \leqslant c\rho^{-m}, \qquad \forall z \in K,\ \forall m \in \mathbb{N}.$$

Since the capacity of K is positive, Theorem 5.18 applies. By this Theorem and Lemma 5.19, the polynomial coefficients of the series h_z for *any* $z \in \mathbb{C}$ satisfy the inequalities

$$|h_z^{(m)}| \leqslant c\rho^{-m} \exp[(m-1)G_K(z)] \leqslant c(\rho/\exp G_K(z))^{-m}, \quad \forall z \in \mathbb{C},\ \forall m \in \mathbb{N}.$$

This means that the series h_z converges for any $z \in \mathbb{C}$ in the symmetric polydisk $\{|x| < \rho/\exp G_K(z)\}$. Together with the asymptotic growth rate $G_K(z) \sim \ln|z| + O(1)$ as $z \to \infty$, see (5.19), this proves the lower bound on the convergence radius of H_z. □

The dichotomy established in Theorem 5.17 may be instrumental in constructing "non-constructive" examples of diverging linearization series. Consider again the non-resonant case when the homological equation $\operatorname{ad}_A g = f$ is always formally solvable.

Theorem 5.20 ([**Ily79**])**.** *Assume that the formal solution $g \in \mathcal{D}[[\mathbb{C}^n, 0]]$ of the homological equation $\operatorname{ad}_A g = f$ is divergent.*

Then the series linearizing the vector field $F_z(x) = Ax + z\,f(x)$, diverges for most values of the parameter z, eventually except for a zero capacity set.

Proof. Assume the contrary, that the linearizing series H_z converges for a positive capacity set. By Theorem 5.17, it converges then for all values of z, in particular h_z is holomorphic in some small polydisk $\{|x| < \rho',\ |z| < \rho''\}$.

Differentiating (5.16) in z, we see that the derivative $g(x) = \frac{\partial h_z(x)}{\partial z}\big|_{z=0}$ is a converging solution of the equation $(\frac{\partial g}{\partial x})Ax - Ag = f$, contrary to the assumption of the Theorem. □

Remark 5.21. The divergence assumption appearing in Theorem 5.20 can be easily achieved. Assume that A is a diagonal matrix with the spectrum $\{\lambda_j\}_1^n$ such that the differences $|\lambda_j - \langle \lambda, \alpha\rangle\,|$ decrease faster than *any* geometric progression $\rho^{|\alpha|}$ for any nonzero ρ. Assume also that the Taylor coefficients of f are bounded *from below* by *some* geometric progression. Then the series $\operatorname{ad}_A^{-1} f$ diverges.

It remains to observe that a set of positive measure is necessarily of positive capacity (Proposition 5.23), hence divergence guaranteed in the assumptions of Theorem 5.20, occurs for almost all z in the measure-theoretic sense, as stated in [**Ily79**].

5.5. Digression: capacity and Bernstein inequality. The brief exposition below is based on [**PM01**] and the encyclopedic treatise [**Tsu59**].

Recall that the function $\ln|z-a|^{-1} = -\ln|z-a|$ is the electrostatic potential on the z-plane $\mathbb{C} \simeq \mathbb{R}^2$, created by a unit charge at the point $a \in \mathbb{C}$ and harmonic outside a. If μ is a nonnegative measure (charge distribution) on the compact $K \Subset \mathbb{C}$, then its potential is the function represented by the integral $u_\mu(z) = \int_K \ln|z-a|^{-1}\,d\mu(a)$ and the energy of this measure is

$$E_\mu(K) = \iint_{K\times K} \ln|z-w|^{-1}\,d\mu(z)\,d\mu(w).$$

This energy can be either infinite for all measures, or $E_\mu(K) < +\infty$ for some nonnegative measures. In the latter case one can show that among all nonnegative measures normalized by the condition $\mu(K) = 1$, the (finite) minimal energy $E^*(K) = \inf_{\mu(K)=1} E_\mu(K)$ is achieved by a unique *equilibrium distribution* μ_K. The corresponding potential $u_K(z)$ is called the *conductor potential* of K.

Definition 5.22. The (harmonic, electrostatic) *capacity* of the compact K is either zero (when $E_\mu = +\infty$ for any charge distribution on K) or $\exp(-E^*(K)) > 0$ otherwise,

$$\varkappa(K) = \begin{cases} 0, & \text{if } \forall\mu\ E_\mu(K) = +\infty, \\ \sup\limits_{\mu(K)=1,\,\mu\geqslant 0} \exp(-E_\mu(K)), & \text{otherwise} \end{cases} \tag{5.17}$$

Proposition 5.23. *Capacity of compact sets possesses the following properties*:

(1) *Countable union of zero capacity sets also has capacity zero,*

(2) $\varkappa(K) \geqslant \sqrt{\operatorname{mes}(K)/\pi e}$, *where* $\operatorname{mes}(K)$ *is the Lebesgue measure of* K, *in particular, if* K *is a set of positive measure, then* $\varkappa(K) > 0$,

(3) *If* K *is a Jordan curve of positive length, then* $\varkappa(K) > 0$.

Proof. All these assertions appear in [**Tsu59**] as Theorems III.8, III.10 and III.11 respectively. □

Proposition 5.24. *For compact sets of positive capacity, the conductor potential is harmonic outside* K, *and*

$$\begin{gathered} u_K \leqslant \varkappa^{-1}(K), \qquad u_K|_K = \varkappa^{-1}(K) \quad \textit{a.e.}, \\ u_K(z) = \ln|z| + O(|z|^{-1}) \quad \textit{as } z \to \infty. \end{gathered} \tag{5.18}$$

Proof. [**Tsu59**, Theorem III.12] □

As a corollary, we conclude that for sets of the positive capacity there exists the Green function

$$G_K(z) = \varkappa^{-1}(K) - u_K(z) = \ln|z| + \varkappa^{-1}(K) + o(1) \quad \text{as } z \to \infty, \tag{5.19}$$

nonnegative on $\mathbb{C} \setminus K$, vanishing on K and asymptotic to the fundamental solution of the Laplace equation with the source at infinity.

Proof of Theorem 5.18 (Bernstein inequality). Since the assertion is invariant by multiplication by scalars, it is sufficient to prove for monic polynomials only.

If $p(z) = z^r + \cdots$ is a monic polynomial of degree r, then the function

$$g(z) = \ln|p(z)| - \ln\|p\|_K - rG_K(z)$$

is negative near infinity since $g(z) = -\ln \|p\|_K - \varkappa^{-1}(K) + o(1)$ as $z \to \infty$ by (5.19), and has zero limit on K by (5.18). By construction this function is harmonic in $\mathbb{C} \setminus K$ outside isolated zeros of p where it tends to $-\infty$. By the maximum principle, the function g is nonnegative everywhere, which after passing to exponents proves the Theorem. □

Example 5.25. Assume that $K = [-1, 1]$ is the unit segment. Its complement is conformally mapped into the exterior of the unit disk $D = \{|w| < 1\}$ by the function $z = \frac{1}{2}(w + w^{-1})$, $w = z + \sqrt{z^2 - 1}$. The Green function G_D of the exterior is $\ln |w|$. Thus we obtain the explicit expression for G_K,

$$G_K = \ln \left| z + \sqrt{z^2 - 1} \right|,$$

wherefrom comes the classical form of the Bernstein inequality,

$$|p(z)| \leqslant \left| z + \sqrt{z^2 - 1} \right|^{\deg p} \max_{-1 \leqslant z \leqslant +1} |p(z)|. \tag{5.20}$$

5.6. Linearization in the Siegel domain: Siegel, Brjuno and Yoccoz theorems. Theorem 5.20 suggests that, at least when the nonzero *denominators* $\langle \alpha, \lambda \rangle - \lambda_j$ decrease *too fast* as $|\alpha| \to +\infty$ (say, faster than exponentially), the formal series linearizing a holomorphic vector field $F(x) = Ax + \cdots$, $A = \operatorname{diag}\{\lambda_1, \dots, \lambda_n\}$, may diverge. Thus in the natural way the question arises, what is the "admissible" decrease rate which still guarantees holomorphic linearizability.

In this section we briefly summarize the results known under the general name of KAM theory (after Kolmogorov, Arnold and Moser). The issue is very classical and proofs can be found in many excellent sources, e.g., [**CG93, Arn83**].

Definition 5.26. A tuple of complex numbers $\lambda \in \mathbb{C}^n$ from the Siegel domain $\mathfrak{S}$ is called *Diophantine* tuple, if the small denominators decay no faster than polynomially,

$$|\lambda_j - \langle \alpha, \lambda \rangle| \geqslant \varepsilon\, |\alpha|^{-N}, \tag{5.21}$$

for some positive $\varepsilon > 0$ and finite $N < \infty$.

Otherwise the tuple is called a *Liouville* tuple (vector, collection).

Liouville vectors are scarce: they form the set of measure zero in $\mathfrak{S}$ if $N > (n - 2)/2$, see [**Arn83**].

Theorem 5.27 (Siegel theorem). *If the the linearization matrix Λ of a holomorphic vector field is nonresonant of Siegel type and has Diophantine spectrum, then the field is holomorphically linearizable.*

This result can be further improved.

Definition 5.28. A non-resonant collection $\lambda \in \mathbb{C}^n$ is said to satisfy the *Brjuno condition*, if the small denominators decrease sub-exponentially,

$$|\lambda_j - \langle \alpha, \lambda \rangle|^{-1} \leqslant Ce^{|\alpha|^{1-\varepsilon}}, \qquad \text{as } |\alpha| \to \infty, \tag{5.22}$$

for some finite C and positive $\varepsilon > 0$.

Theorem 5.29 (Brjuno theorem)**.** *A holomorphic vector field with nonresonant linearization matrix of Siegel type satisfying the Brjuno condition, is holomorphically linearizable.*

However, at least in some cases the Brjuno condition is not only sufficient, but also necessary. An analog of Theorem 5.29 for holomorphic germs from $\mathrm{Diff}(\mathbb{C}^1, 0)$ claims the following. If the complex number $\lambda = \exp 2\pi i l$, $l \in \mathbb{R}$, satisfies the *multiplicative Brjuno condition*

$$|\lambda^k - 1|^{-1} < Ce^{k^{1-\varepsilon}}, \qquad C < +\infty,\ \varepsilon > 0, \tag{5.23}$$

then any holomorphic map $(\mathbb{C}, 0) \to (\mathbb{C}, 0)$, $z \mapsto \lambda z + z^2 + \cdots$, is holomorphically linearizable. The sufficient arithmetic condition (5.23) turns out to be also *necessary* in the following sense.

Theorem 5.30 (J.-C. Yoccoz [**Yoc88, Yoc95**])**.** *If the complex number* $\lambda = \exp 2\pi i l$, $l \in \mathbb{R}$, *violates the multiplicative Brjuno condition* (5.23), *then there exists a non-linearizable holomorphic germ* $(\mathbb{C}, 0) \to (\mathbb{C}, 0)$, $z \mapsto \lambda z + f(z)$, $f(z) = z^2 + \cdots$.

In fact, in the assumptions of this Theorem for almost all complex numbers $\mu \in \mathbb{C}$ the germ $\lambda z + \mu f(z)$ is non-linearizable. This assertion can be obtained using the Divergence Dichotomy (Theorem 5.17) as explained in [**PM01**].

Remark 5.31. The condition on the rate of convergence of small denominators can be reformulated in terms of the growth rate of coefficients of decomposition of the number $l \notin \mathbb{Q}$ into a continuous fraction.

Add reference on Żołądek–Stróżyna on holomorphic Takens form for cusps

6. Holomorphic invariant manifolds

In this section we show that under rather weak conditions one can eliminate enough nonresonant terms to ensure existence of *holomorphic invariant manifolds*. Recall that a holomorphic submanifold $W \subset (\mathbb{C}^n, 0)$ is invariant for a holomorphic vector field F, if the vector $F(x)$ is tangent to W at any point $x \in W$.

6.1. Invariant manifolds.

Definition 6.1. Two point sets $S^{\pm} \subset \mathbb{C}$ on the complex plane are said to be separated by a line through the origin (or simply *separated*), if there exists a real linear form $\ell\colon \mathbb{C} \to \mathbb{R}$ such that $\inf_{z\in S^+} \ell(z) > 0$, $\sup_{z\in S^-} \ell(z) < 0$.

Suppose that the spectrum $S \subset \mathbb{C}$ of linearization matrix A of a holomorphic vector field consists of two parts $S^{\pm} \subset \mathbb{C}$ separated by a line. In this case no eigenvalue from one part can be equal to a linear combination of eigenvalues from the other part with nonnegative coefficients,

$$\begin{gathered} \lambda_j^- - \sum \alpha_i \lambda_i^+ \neq 0, \qquad \lambda_i^+ - \sum \alpha_j \lambda_j^- \neq 0, \\ \lambda_i^+ \in S^+, \quad \lambda_j^- \in S^-, \quad \alpha_i, \alpha_j \in \mathbb{Z}_+, \end{gathered} \tag{6.1}$$

(we say that there are no *cross-resonances* between the two parts). Without loss of generality A can be assumed to be in the block diagonal form. By the Poincaré–Dulac theorem, there exists a formal transformation eliminating all nonresonant terms corresponding to the nonzero cross-combinations (6.1). The corresponding formal normal form has two invariant manifolds coinciding with the corresponding coordinate subspaces.

Moreover, all small denominators (6.1) are obviously bounded from below. Therefore one can expect that the corresponding transformation converge and the invariant manifolds will exist in the analytic category. This is indeed the case.

Theorem 6.2 (Hadamard–Perron theorem for holomorphic flows)**.** *Assume that the linearization operator of a holomorphic vector field $Ax + F(x)$ has a transversal pair of invariant subspaces $L^{\pm}$ such that the spectra of A restricted on these subspaces are separated from each other.*

Then the vector field has two holomorphic invariant manifolds $W^{\pm}$ tangent to the subspaces $L^{\pm}$.

However, the proof of this result is indirect. We start by formulating and proving a counterpart of Theorem 6.2 for biholomorphisms.

Definition 6.3. A holomorphic map $H\colon (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$, $x \mapsto Mx + h(x)$, $h(0) = \frac{\partial h}{\partial x}(0) = 0$, is said to be *hyperbolic* if no eigenvalue of the linearization matrix M has modulus 1.

The notion of invariant manifolds for biholomorphisms defined in neighborhoods of points that *do not* necessarily mapped into themselves, requires slight modification compared to the global situation.

Definition 6.4. A holomorphic submanifold W passing through a fixed point of a biholomorphism $H\colon (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ is *invariant*, if *the germ* of $H(W)$ at the origin coincides with *the germ* of W.

Theorem 6.5 (Hadamard–Perron theorem for biholomorphisms)**.** *A hyperbolic holomorphism in a sufficiently small neighborhood of the fixed point at the origin has two holomorphic invariant submanifolds W^+ and W^-.*

These manifolds pass through the origin, transversal to each other and are tangent to the corresponding invariant subspaces $L^\pm$ of the linearized map $x \mapsto \Lambda x$.

The dimensions of the invariant manifolds are necessarily equal to the dimension of the corresponding subspaces. The manifold W^+ is called *unstable manifold*, whereas W^- is referred to as the *stable manifold*, because the restriction of H on these manifolds is unstable and stable respectively.

Proof. The linearization matrix M of a holomorphic biholomorphism $H\colon (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ can be put into the block diagonal form. Choosing appropriate system of local holomorphic coordinates $(x, y) \in (\mathbb{C}^k, 0) \times (\mathbb{C}^l, 0)$, $k + l = n$, one can always assume that the map H has the form

$$H\colon \begin{pmatrix} x \\ y \end{pmatrix} \longmapsto \begin{pmatrix} Bx + g(x,y) \\ Cy + h(x,y) \end{pmatrix}, \qquad (x,y) \in (\mathbb{C}^k, 0) \times (\mathbb{C}^l, 0). \tag{6.2}$$

Here the square matrices B, C and the nonlinear terms g, h of order $\geqslant 2$ satisfy the conditions

$$\begin{aligned} |B| \leqslant \mu, \quad |C^{-1}| \leqslant \mu, \qquad \mu < 1,\\ |f(x,y)| + |g(x,y)| < |x|^2 + |y|^2, \qquad \text{for } |x| < 1,\ |y| < 1. \end{aligned} \tag{6.3}$$

with some *hyperbolicity parameter* $\mu < 1$.

It is sufficient to prove the existence of the *stable* manifold only; the unstable manifold for H is the stable manifold of the inverse map H^{-1} which is also hyperbolic.

The stable manifold W^+ tangent to $L^+ = \{(x, 0)\}$ is necessarily the graph of a holomorphic vector function $\varphi\colon \{|x| \leqslant \varepsilon\} \to \{|y| \leqslant \varepsilon\}$ defined in a sufficiently small polydisk, $\varphi(0) = 0$, $\frac{\partial \varphi}{\partial x}(0) = 0$. For this manifold to be invariant, the function φ must satisfy the functional equation

$$\varphi\bigl(Bx + g(x, \varphi(x))\bigr) = C\varphi(x) + h(x, \varphi(x)). \tag{6.4}$$

This equation can be transformed to the fixed point form as follows,

$$\varphi(x) = \mathcal{H}\varphi, \qquad \mathcal{H}\varphi = C^{-1}\varphi\bigl(Ax + g(x, \varphi(x))\bigr) - h(x, \varphi(x)). \tag{6.5}$$

All assertions of Theorem 6.5 follow from the contracting map principle and the following Lemma 6.7. □

Remark 6.6. The linearization of the operator $\mathcal{H}$ at the "point" $\varphi = 0$ is the linear operator

$$\varphi(x) \mapsto C^{-1}\varphi(Bx), \qquad |B|,\ |C^{-1}| \leqslant \mu < 1,$$

which is obviously contracting. Lemma 6.7 shows that nonlinear terms do not affect this property.

Lemma 6.7. *Under the assumptions* (6.3)*, the nonlinear operator* $\mathcal{H}$ *has the following properties*:

(1) $\mathcal{H}$ *is well defined for* φ *in the ball* $\mathcal{B}_\varepsilon = \{\varphi\colon \sup_{|x|<\varepsilon} |\varphi(x)| < \varepsilon\}$ *inside the space* $\mathcal{A}_\varepsilon$*, and takes this ball into itself,*

(2) *the subset* $\mathcal{B}^1_\varepsilon$ *of functions in* $\mathcal{B}_\varepsilon$ *with the Lipschitz constant* $\leqslant 1$*, is preserved by* $\mathcal{H}$*,*

(3) *the operator* $\mathcal{H}$ *is contracting on* $\mathcal{B}^1_\varepsilon$*,*

provided that the value $\varepsilon > 0$ *is sufficiently small.*

Proof. To prove the first assertion, note that $|Bx+g(x,\varphi(x))| < \mu|x|+|x|^2+|\varphi|^2 < \mu\varepsilon + 2\varepsilon^2 < \varepsilon$ for $|x| < \varepsilon$, if the latter parameter is sufficiently small. Thus the composition occurring in the definition of $\mathcal{H}$ makes perfect sense and $\mathcal{H}\varphi$ is well defined. For the same reason, $|\varphi|$ never exceeds $\mu\varepsilon + 2\varepsilon^2 < \varepsilon$ which means that $\mathcal{B}_\varepsilon$ is taken by $\mathcal{H}$ into itself.

The Jacobian matrix $J(x) = \frac{\partial\varphi}{\partial x}$ is transformed into $J' = C^{-1}J(\cdots)(B+\frac{\partial g}{\partial x}+\frac{\partial g}{\partial y}J)+(\frac{\partial h}{\partial x}+\frac{\partial h}{\partial y}J)$. Since the terms g, h are of order $\geqslant 2$, their derivatives vanish at the origin and therefore the Jacobian is no greater (in the sense of the matrix norm) than $(\mu^2 + O(\varepsilon))|J|$. As $\mu < 1$, this proves the assertion about the Lipschitz constant.

To prove the last assertion that $\mathcal{H}$ is contractive, notice that the operator $\varphi(x) \mapsto h(x, \varphi(x))$ is strongly contracting:

$$|h(x,\varphi_1(x)) - h(x,\varphi_2(x))| \leqslant \left|\frac{\partial h}{\partial y}\right| |\varphi_1(x) - \varphi_2(x)| \leqslant O(\varepsilon)\|\varphi_1 - \varphi_2\|_\varepsilon. \tag{6.6}$$

Consider the operator $\varphi \mapsto \mathcal{G}\varphi = \varphi(Bx + g(x,\varphi))$ and the difference of the values it takes on two functions $\varphi_1, \varphi_2 \in \mathcal{B}^1_\varepsilon$: by the triangle inequality,

$$\begin{aligned} |\mathcal{G}\varphi_1(x) - \mathcal{G}\varphi_2(x)| &= |\varphi_1(Bx + g_1(x)) - \varphi_2(Bx + g_2(x))| \\ &\leqslant |\varphi_1(Bx + g_2(x)) - \varphi_2(Bx + g_2(x))| \\ &\quad + |\varphi_1(Bx + g_1(x)) - \varphi_1(Bx + g_2(x))|, \end{aligned}$$

where we denoted $g_i(x) = g(x, \varphi_i(x))$ for brevity. The first term does not exceed $\|\varphi_1 - \varphi_2\|_\varepsilon$. Since the vector function $\varphi_1 \in \mathcal{B}^1_\varepsilon$ has Lipschitz constant 1, the second term does not exceed $|g_1(x) - g_2(x)| = |g(x,\varphi_1(x)) - g(x,\varphi_2(x))|$. Similarly to (6.6), this part is no greater than $O(\varepsilon)\|\varphi_1 - \varphi_2\|_\varepsilon$. Finally, we conclude that $\mathcal{G}$ is Lipschitz on $\mathcal{B}^1_\varepsilon$: $\|\mathcal{G}\varphi_1 - \mathcal{G}\varphi_2\|_\varepsilon \leqslant (1 + O(\varepsilon))\|\varphi_1 - \varphi_2\|$.

Adding all terms together for $\mathcal{H} = C^{-1}\mathcal{G} - h(x,\cdot)$, we conclude that if $\varphi_{1,2} \in \mathcal{B}^1_\varepsilon$, then

$$\|\mathcal{H}\varphi_1 - \mathcal{H}\varphi_2\|_\varepsilon \leqslant (\mu + O(\varepsilon))\,\|\varphi_1 - \varphi_2\|_\varepsilon.$$

Since $\mu < 1$, the operator $\mathcal{H}$ is contracting on the invariant subset $\mathcal{B}^1_\varepsilon$ of the complete metric space $\mathcal{B}_\varepsilon \subset \mathcal{A}_\varepsilon$. □

Remark 6.8. As typical for the proofs based on the contracting map principle, the germs of invariant manifolds are unique.

Now we can derive Theorem 6.2 from Theorem 6.5.

Proof. Passing if necessary to an orbitally equivalent field, one may assume that the linearization $A = \operatorname{diag}\{A_+, A_-\}$ is block diagonal with the spectra of the blocks are separated by the imaginary axis.

Consider the flow maps $\Phi^t\colon (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ for $t = 1/k$, $k = 1, 2, \dots$. Each of them is a biholomorphism with the linear part $x \mapsto \exp tAx$ whose eigenvalues are the corresponding exponentials $\{\exp t\lambda_i\colon \lambda_i \in S\}$ separated by the unit circle $\{|\lambda| = 1\}$. In the assumptions of the theorem, each flow map Φ^t is hyperbolic. By Theorem 6.5, the map Φ^t has a pair of invariant manifolds $W^\pm_t$, tangent to the corresponding invariant subspaces $L^\pm$ common for all $t \in \mathbb{R}$.

Apriori, the invariant subspaces $W^\pm_t$ do not have to coincide. However, $\left(\Phi^{1/k}\right)^k = \Phi^1$, therefore manifolds invariant for $\Phi^{1/k}$, are invariant also for Φ^1. Since the invariant manifolds for the latter map are unique, we conclude that all the maps $\Phi^{1/k}$ leave the pair $W^\pm = W^\pm_1$ invariant.

In other words, an analytic trajectory $x(t)$ of the vector field which begins on, say, W^-, $x(0) \in W^-$, remains on W^- for $t = 1/k$. Since isolated zeros of analytic functions cannot have accumulation points, $x(t)$ is on W^- for all (sufficiently small) values of $t \in (\mathbb{C}, 0)$. Then W^- is invariant for the vector field $Ax + F(x)$. The proof for W^+ is similar. □

Remark 6.9. Intersection of invariant manifolds is again an invariant manifold. This observation allows to construct small-dimensional invariant manifolds for holomorphic vector fields. For instance, if the linearization matrix Λ has a simple eigenvalue $\lambda_1 \neq 0$ such that $\lambda_1/\lambda_j \notin \mathbb{R}_+$ for all other eigenvalues λ_j, $j = 2, \dots, n$, then the vector field has a one-dimensional holomorphic invariant manifold (curve) tangent to the corresponding eigenvector.

The Hadamard–Perron theorem for holomorphic flows, as formulated above, is the nearest analog of the Hadamard–Perron theorem for smooth flows in $\mathbb{R}^n$. There are known stronger results in this direction, see [**Bib79**].

6.2. Holomorphic hyperbolic submanifolds for saddle-nodes. Consider a holomorphic vector field on the plane $(\mathbb{C}^2, 0)$ with the saddle-node at the origin. Recall that by Definition 4.25, this means that exactly one of the eigenvalues is zero, while the other eigenvalue must be nonzero. The null space (line) of the linearization operator is called the *central* direction.

The direction of eigenvector with the nonzero eigenvalue is referred to as *hyperbolic*.

The nonzero eigenvalue cannot be separated from the null one, thus the Hadamard–Perron theorem cannot be applied. However, the invariant manifold (smooth holomorphic curve) tangent to the eigenvector with nonzero eigenvalue, exists and is unique in this case as well. As before, we start with the case of biholomorphisms with one contracting eigenvalue $|\mu| < 1$ and the other eigenvalue equal to 1. For obvious reasons, such maps are called *saddle-node* biholomorphisms.

Any saddle-node biholomorphism $H\colon (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$ can be brought into the form

$$H\colon \begin{pmatrix} x \\ y \end{pmatrix} \longmapsto \begin{pmatrix} \mu x + g(x, y) \\ y + y^2 + h(x, y) \end{pmatrix}, \qquad \mu \in (0, 1) \subset \mathbb{R}, \tag{6.7}$$

with g, h holomorphic nonlinear terms of order $\geqslant 3$, by a suitable holomorphic choice of coordinates x, y. Indeed, this is the form in which only resonant quadratic terms are kept.

Theorem 6.10. *The biholomorphism* (6.7) *has a unique holomorphic invariant manifold* (*curve*) *tangent to the eigenvector* $(1, 0) \in \mathbb{C}^2$.

Proof. The operator

$$\varphi \longmapsto \mathcal{H}\varphi, \qquad \mathcal{H}\varphi(x) = \varphi\bigl(\mu x + g(x, \varphi(x))\bigr) - \varphi^2(x) - h(x, \varphi(x)), \tag{6.8}$$

corresponding to the functional equation $\varphi(\mu x + g(x, \varphi(x))) = \varphi(x) + \varphi^2(x) + h(x, \varphi(x))$ to be satisfied by the invariant manifold $W = \operatorname{graph} \varphi$, is no longer contracting: its linearization at $\varphi = 0$ is the operator $\varphi(x) \mapsto \varphi(\mu x)$ which keeps all constants fixed. To restore the contractivity, we have to restrict this operator on the subspace of functions vanishing at the origin, with the norm $\|\varphi\|' = \sup_{x \neq 0} \frac{|\varphi(x)|}{|x|}$. Technically it is more convenient to substitute $\varphi(x) = x\psi(x)$ into the functional equation and bring it back to the fixed point form. As a result, we obtain the equation

$$(\mu x + g(x, x\psi(x)) \cdot \psi\bigl(\mu x + g(x, x\psi(x))\bigr) = x\psi(x) + x^2\psi^2(x) + h(x, x\psi(x)),$$

which yields the nonlinear operator

$$\psi(x) \longmapsto \bigl(\mu + g'(x, \psi(x))\bigr) \cdot \psi\bigl(\mu x + g(x, x\psi(x))\bigr) - x\psi^2(x) - h'(x, \psi). \tag{6.9}$$

Here the holomorphic functions $g'(x, y) = g(x, xy)/x$, $h'(x, y) = h(x, xy)/x$ are of order $\geqslant 2$ at the origin.

The proof of Lemma 6.7 carries out almost literally for the operator (6.9), proving that it is contractible on the space of functions $\psi\colon \{|x| < \varepsilon\} \to \{|y| < \varepsilon\}$ with respect to the usual supremum-norm on sufficiently small discs. $\square$

Completely similar to derivation of Theorem 6.2 from Theorem 6.5 in the hyperbolic case, Theorem 6.10 implies the following result concerning holomorphic saddle-nodes.

Theorem 6.11. *A holomorphic vector field on the plane* $(\mathbb{C}^2, 0)$ *having a saddle-node at the origin, admits a unique holomorphic invariant curve passing through the singular point and tangent to the hyperbolic direction.* □

This curve is called the *hyperbolic invariant manifold*.

It is important to conclude this section by the explicit example showing that the other invariant manifold, the *central manifold* tangent to the central direction, may not exist in the analytic category. Note, however, that the formal invariant manifold always exists and is unique: this follows from the formal orbital classification of saddle-nodes (Proposition 4.26).

Example 6.12. The vector field

$$x^2 \tfrac{\partial}{\partial x} + (y - x)\tfrac{\partial}{\partial y} \tag{6.10}$$

has vertical hyperbolic direction $\mathbb{C} \cdot (0, 1)$ and the central direction $\mathbb{C} \cdot (1, 1)$. The central manifold, if it exists, must be represented as the graph of the function $y = \varphi(x)$, $\varphi(x) = x + \sum_{k\geqslant 2} c_k x^k$. However, this series diverges, as was noticed already by L. Euler. Indeed, the function φ must be solution to the differential equation

$$\frac{d\varphi}{dx} = \frac{\varphi(x) - x}{x^2}$$

which implies the recurrent formulas for the coefficients,

$$k\, c_k = c_{k+1}, \qquad k = 1, 2, \dots, \qquad c_1 = 1.$$

The factorial series with $c_k = (k-1)!$ has zero radius of convergence, hence no analytic central manifold exists.

However, sufficiently large "pieces" of the central manifold for the saddle-node can be shown to exist.

More about this in Non-Stokes part? or simply add reference.

7. Topological classification of holomorphic foliations

The famous Grobman–Hartman theorem [**Gro62, Har82**] asserts that any smooth vector field whose linearization matrix is hyperbolic (i.e., has no eigenvalues with zero real part), is topologically orbitally equivalent to its linearization. An elementary analysis shows that two hyperbolic linear vector fields are orbitally topologically conjugated if and only if they have the same number of eigenvalues to both sides of the imaginary axis.

This section describes complex counterparts of these results. From the real point of view a holomorphic 1-dimensional singular foliation on $(\mathbb{C}^n, 0)$ by phase curves of a holomorphic vector field is a 2-dimensional real analytic foliation on $(\mathbb{R}^{2n}, 0)$. If the singularity at the origin is in the Poincaré domain, this foliation induces a *nonsingular real* 1*-dimensional* foliation (*trace*) on all small $(2n-1)$-dimensional spheres $\mathbb{S}^{2n-1}_\varepsilon = \{|x_1|^2 + \cdots + |x_n|^2 = \varepsilon > 0\}$. Under the *complex* hyperbolicity-type conditions excluding resonances, the trace is generically structurally stable. Poincaré resonances manifest themselves via *bifurcations* of this trace foliation.

On the contrary, if the singularity is in the Siegel domain, the corresponding foliations exhibit *topological rigidity*: there are continuous invariants of topological classification.

7.1. Trace of the foliation on the small sphere. Consider the real sphere of radius $\varepsilon > 0$,

$$\mathbb{S}_r = \{r^2(x) = \varepsilon\} \subseteq \mathbb{C}^n, \qquad r^2(x) = |x|^2 = \sum_1^n x_i \bar{x}_i. \tag{7.1}$$

The differential of the (non-holomorphic) function $r^2 \colon \mathbb{C}^n \to \mathbb{R}$ is a complex-valued 1-form, $dr^2 = x\, d\bar{x} + \bar{x}\, dx$, which on the complex vector field $F(x) = (v_1(x), \ldots, v_n(x))$ takes the value

$$dr^2 \cdot v(x) = \sum_{i=1}^n x_i\, \bar{v}_i + \bar{x}_i\, v_i = 2 \operatorname{Re} \left(\sum x_i \bar{v}_i \right) \in \mathbb{R}.$$

If $F(x) = \Lambda x$ is a linear diagonal vector field with the eigenvalues $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$, then

$$dr^2 \cdot F = 2 \operatorname{Re} \sum \lambda_i |x_i|^2.$$

Proposition 7.1 (V. Arnold [**Arn69**])**.** *If the collection of eigenvalues belongs to the Poincaré domain, then all complex phase curves of the diagonal linear vector field Λx in $\mathbb{C}^n$ are transversal as 2-dimensional embedded surfaces, to all spheres $\mathbb{S}_\varepsilon$, $\varepsilon > 0$.*

Proof. The tangent space to any trajectory considered as a real 2-dimensional surface in $\mathbb{R}^{2n} = \mathbb{C}^n$, is spanned over $\mathbb{R}$ by the vectors $v(x) = \Lambda x$ and $v'(x) = i\Lambda x$. To prove the transversality, it is sufficient to verify that the 1-form dr^2 cannot vanish on both vectors simultaneously for $x \neq 0$.

If the spectrum belongs to the Poincaré domain, then without loss of generality we may assume that

$$\operatorname{Re} \lambda_i < 0, \qquad i = 1, \ldots, n. \tag{7.2}$$

Indeed, replacing the field Λx by the orbitally equivalent field $\alpha\Lambda x$, $|\alpha| = 1$, preserves all holomorphic phase curves but rotates the spectrum of Λ as a whole.

Under the assumption (7.2) the expression

$$dr^2 \cdot F = s(x) = \sum \lambda_i |x_i|^2 \in \mathbb{C} \tag{7.3}$$

is in the left half-plane, moreover,

$$\operatorname{Re} s(x) \leqslant \delta\, |x|^2 < 0, \qquad \delta > 0. \tag{7.4}$$

This implies the required transversality. □

Remark 7.2. Transversality is an open condition: sufficiently small perturbations of the vector field leave it transversal to the compact sphere.

In particular, if $F(x) = \Lambda x + w(x)$ is a nonlinear vector field, then the rescaling $x \mapsto \varepsilon x$ conjugates its restriction on the ε-sphere $\mathbb{S}^{2n-1}_\varepsilon$ with the restriction of the field $F_\varepsilon(x) = \Lambda x + \varepsilon^{-1} w(\varepsilon x)$ on the unit sphere $\mathbb{S}^{2n-1}_1$. But since the nonlinear part $w(x)$ is at least of second order, the field F_ε is ε-uniformly close on the unit sphere to the linear field $F_0(x) = \Lambda x$. Thus we conclude that the *nonlinear* vector field F is transversal to all sufficiently small spheres $\mathbb{S}^{2n-1}_\varepsilon$.

Definition 7.3. Let $\mathcal{F} = \{L_\alpha\}$ be a foliation on a manifold M. The *trace of the foliation* on a submanifold $N \subset M$ is the partition of N into connected components of intersection of the leaves L_α with N, $\mathcal{F}|_N = \{L_\alpha \cap N\}$.

In general, the trace of a foliation need not itself be foliation (the intersections $L_\alpha \cap N$ can be non-manifolds in generally). Even in the analytic context one cannot exclude appearance of singularities.

Formally the definition of trace coincides with the definition of restriction of a foliation on an open subset of the initial domain. However, we will use the term "restriction" when dealing with open subsets when the dimension of the leaves remains the same, while the term "trace" will indicate that the dimension drops down.

In these terms the transversality Proposition 7.1 implies the following result.

Corollary 7.4. *If $\mathcal{F}$ is a holomorphic singular foliation of $\mathbb{C}^n$ by phase curves of a vector field Λx in the Poincaré domain, then the trace of $\mathcal{F}$ on any sphere $\mathbb{S}^{2n-1}_\varepsilon$ is a smooth* (*actually, real analytic*) *nonsingular real 1-dimensional foliation $\mathcal{F}' = \mathcal{F}|_{\mathbb{S}_\varepsilon}$.*

Proof. By the implicit function, intersection of each leaf with the sphere is a smooth curve. □

In the Poncaré domain, the trace of the foliation on a (sufficiently small) sphere determines completely the foliation up to the topological equivalence.

Definition 7.5. A (topological) *cone* over a set $K \subset \mathbb{C}^n \smallsetminus \{0\}$ is the set $\complement K = \{rx\colon 0 \leqslant r \leqslant 1,\ x \in K\} \subseteq \mathbb{C}^n$. If $\mathcal{F}'$ is a foliation on the sphere $\mathbb{S}_1^{2n-1} \subset \mathbb{C}^n$, then the *cone over the foliation* $\complement\mathcal{F}'$ is the foliation of $\mathbb{C}^n \smallsetminus \{0\}$ whose leaves are the cones over the leaves of $\mathcal{F}'$.

Theorem 7.6. *If Λ is in the Poincaré domain, then the foliation $\mathcal{F}$ of $\mathbb{C}^n$ by phase curves of the field $F(x) = \Lambda x + w(x)$ is locally topologically equivalent to the cone over the trace $\mathcal{F}'_\varepsilon$ of $\mathcal{F}$ on any sufficiently small sphere $\mathbb{S}_\varepsilon^{2n-1}$, $\varepsilon > 0$.*

Proof. Under the normalizing assumption (7.2) the real flow of the vector field Λx, the one-parametric subgroup of linear maps $\{\Phi^t = \exp t\Lambda\colon t \in \mathbb{R}\}$ is locally contracting: orbits $\Phi^t(x)$, $x \in \mathbb{S}_1^{2n-1}$ of all points uniformly converge to the origin as $t \to +\infty$. This follows again from (7.4): if ε is so small that $|w(x)| < \frac{\delta}{2}|x|$ for $|x| < \varepsilon$, we have $|\Phi^t(x)| < \exp(-\delta t/4)\,|x|$ for all $t > 0$.

The real flow Φ^t is tangent to the foliation $\mathcal{F}$. Thus the map h of the small ε-ball $\{|x| \leqslant \varepsilon\}$ into itself,

$$h(rx) = \Phi^{-\ln r}(x), \qquad 0 < r \leqslant 1,\ x \in \mathbb{S}_\varepsilon^{2n-1}, \qquad h(0) = 0,$$

is a homeomorphism conjugating $\complement(\mathcal{F}|_{\mathbb{S}_\varepsilon})$ with $\mathcal{F}$. □

Remark 7.7. Theorem 7.6 implies, among other, that all foliations $\mathcal{F}'_\varepsilon$ are topologically equivalent to each other. Yet without the additional assumptions they may be non-equivalent to the foliation $\mathcal{F}'_0$ which is the trace of the linear foliation $\mathcal{F}_0$ on (any) sphere.

7.2. Structural stability of the trace foliation. Under an additional assumption of *complex hyperbolicity* we can completely describe the trace of the linear diagonal foliation and show that it is structurally stable: small perturbations remain topologically equivalent to it. The hyperbolicity-type condition that will be often required in this section, is less restrictive than its real counterpart. Add X-reference.

Definition 7.8. A holomorphic germ of a vector field $\dot{x} = Ax + \cdots$ in $(\mathbb{C}^n, 0)$ is *hyperbolic* (or *complex hyperbolic*, to distinguish this from the *real hyperbolicity*), if no two eigenvalues λ_i, λ_j of the linearization matrix A differ by a real factor,

$$\lambda_i/\lambda_j \notin \mathbb{R} \qquad \forall 1 \leqslant i \neq j \leqslant n. \tag{7.5}$$

In particular, A must be nondegenerate and diagonalizable.

Everywhere below in this section $\mathcal{F}$ is a singular foliation of $\mathbb{C}^n$ by phase curves of the complex hyperbolic vector field Λx with the eigenvalues

$\lambda_1, \dots, \lambda_n$ of the diagonal matrix Λ in the Poincaré domain and, if necessary, normalized by the condition (7.2). We denote by $\mathcal{F}'$ its restriction on $\mathbb{S}_1^{2n-1}$.

The first immediate consequence of complex hyperbolicity is the fact that the only non-simply-connected leaves of the foliation $\mathcal{F}$ by complex phase curves of a diagonal linear system, are the coordinate axes.

Proposition 7.9. *The only complex phase curves of a complex hyperbolic linear system $\dot{x} = Ax$ in $\mathbb{C}^n$ are the separatrices spanned by the eigenvectors of A.*

Proof. Without loss of generality we may assume that A is diagonal, $A = \Lambda = \operatorname{diag}\{\lambda_1, \dots, \lambda_n\}$. The map $t \mapsto \Phi^t a = (a_1 \exp t\lambda_1, \dots, a_n \exp t\lambda_n)$, $t \in \mathbb{C}$, parameterizes the phase curve passing through a point $a \in \mathbb{C}^n$. This parametrization is not injective, if $\exp t\lambda_j = 1$ for some t and all j corresponding to nonzero coordinates of the point a. If there is only one such coordinate, then the non-injectivity is indeed possible if $t = 0 \bmod T_j$, where T_j is the corresponding period. If a has at least two nonzero coordinates j and k, then the simultaneous occurrence $t = 0 \bmod T_j$ and $t = 0 \bmod T_k$ is impossible: it would mean that the ratio T_j/T_k is rational hence real. $\square$

Assume that in addition to the normalizing condition (7.2), the enumeration of the eigenvalues $\lambda_1, \dots, \lambda_n$ is chosen in the increasing order of their arguments,

$$\arg \lambda_1 < \arg \lambda_2 < \cdots < \arg \lambda_{n-1} < \arg \lambda_n \tag{7.6}$$

(this is possible since by the hyperbolicity assumption $\lambda_j/\lambda_k \notin \mathbb{R}$).

Since the coordinate axes are leaves of $\mathcal{F}$, the big circles circles $C_i = \{x_j = 0,\ j \neq i,\ |x_i| = 1\}$ are leaves of $\mathcal{F}'$. We show that all other leaves are bi-asymptotic to these circles.

Proposition 7.10. *If Λ is hyperbolic, then the limit set $\overline{L} \smallsetminus L$ of any leaf $L \in \mathcal{F}'$ different from C_j, is the union of two big circles $C_j \cup C_k$, $j \neq k$.*

Proof. Any leaf L_c passing through a point $c \in \mathbb{C}^n$, is naturally parameterized by $t \in \mathbb{C}$ as follows,

$$t \mapsto x(t) = \bigl(c_1 \exp(\lambda_1 t), \dots, c_n \exp(\lambda_n t)\bigr) \in \mathbb{C}^n. \tag{7.7}$$

The intersection $\gamma_c = L \cap \mathbb{S}_1^{2n-1}$ is defined by the equation

$$|c_1|^2 \exp 2\operatorname{Re}(\lambda_1 t) + \cdots + |c_n|^2 \exp 2\operatorname{Re}(\lambda_n t) = 1. \tag{7.8}$$

As follows from the transversality property, this is a smooth curve parameterized by a smooth curve $\widetilde{\gamma}_c$ on the t-plane, defined by the equation (7.8).

The curve $\widetilde{\gamma}_c$ *apriori* may have compact and non-compact components. But any compact component must bound a compact set in $\mathbb{C}$ so that the function $|x(t)|$ has critical points inside. Such critical points correspond to non-transversal intersections that are forbidden by Proposition 7.1.

Thus γ_c may consist of only non-compact components (eventually, several) along which $|t|$ tends to infinity. But as $|t| \to \infty$, the growth pattern of each exponential $\exp 2\operatorname{Re}\lambda_j t$ is determined by the angular behavior of t. In particular, since all exponentials in (7.8) should be bounded unless the corresponding coefficients c_j vanish, we have the necessary condition that all limit directions $\lim\{t/|t|\colon t \in \widetilde{\gamma}_c,\ |t| \to +\infty\}$ must be within the sector $S_c = \bigcap_{j\colon c_j \neq 0}\{\operatorname{Re}\lambda_j t \leqslant 0\}$. However, if t tends to infinity (asymptotically) *in the interior* of this sector, then all exponents will tend to zero in violation of (7.8). Thus, unless only one coefficient c_j is nonzero (and then γ_c is the cycle C_j), the curve $\widetilde{\gamma}_c$ must be bi-asymptotic to the two boundary rays of the sector S_c. This in turn means that the corresponding trajectory γ_c is bi-asymptotic to the two cycles $C_j \neq C_k$. One can immediately see that in fact j and k correspond to the eigenvalues λ_j and λ_k with the minimal and maximal argument respectively. □

Behavior of leaves near each cycle C_j is determined by the iterations of the corresponding holonomy map of the foliation $\mathcal{F}'$ which can be easily expressed in terms of the holonomy of the corresponding complex separatrix $\mathbb{C}e_j$, $e_j = (0, \dots, \underset{j}{1}, \dots, 0) \in \mathbb{C}^n$, of the initial holomorphic foliation $\mathcal{F}$.

Consider the circular leaf $C_j \subset \mathbb{S}_1^{2n-1}$ of the foliation $\mathcal{F}'$ with the orientation induced by the counterclockwise (positive) direction of going around the origin in the jth coordinate axis. Then for any (smooth) $(2n-2)$-dimensional cross-section $\tau_j'\colon (\mathbb{R}^{2n-2}, 0) \to (\mathbb{S}_1^{2n-1}, e_j)$ transversal to the trace foliation $\mathcal{F}'$ at the point $e_j \in C_j$, one can define the first return map $h_j = \Delta_{C_j}\colon (\tau_j', 0) \to (\tau_j', 0)$

Proposition 7.11. *The first return map* $h_j \in \operatorname{Diff}(\mathbb{R}^{2n-2}, 0)$ *of each cycle* C_j *is differentiably conjugate to the hyperbolic diagonal linear map* $\Lambda_j \in \operatorname{Diff}(\mathbb{C}^{n-1}, 0)$ *with the eigenvalues* $\{2\pi i\lambda_k/\lambda_j\}$, $k \neq j$.

Proof. Since the sphere $\mathbb{S}_1^{2n-1}$ is transversal to the foliation $\mathcal{F}$, any smooth (non-holomorphic) cross-section $\tau_j'\colon (\mathbb{R}^{2n-2}, 0) \to (\mathbb{S}_1^{n-1}, e_j)$ transversal to the trace foliation $\mathcal{F}'$ at the point $e_j \in C_j$ *inside* $\mathbb{S}_1^{2n-1}$, will be also transversal to the the complex separatrix of $\mathcal{F}$ lying on the jth coordinate axis.

The holonomy maps for the foliation $\mathcal{F}$ associated with the two cross-sections, τ_j' and the "standard" cross-section $\tau_j\colon (\mathbb{C}^{n-1}, 0) \to (\mathbb{C}^n, e_j)$, are smoothly conjugate, in fact, the conjugacy is real analytic as a germ between $(\mathbb{R}^{2n-2}, 0)$ and $(\mathbb{C}^{n-1}, 0)$. The holonomy for the "standard" cross-section can be immediately computed, since leaves of $\mathcal{F}$ are graphs of the vector functions $x_k = c_k\, x_j^{\lambda_k/\lambda_j}$, $c_k \in \mathbb{C}$, $k \neq j$. □

X-Ref to example of computation of the linear holonomy

Proposition 7.12. *The stable (unstable) manifold of the cycle C_j is the sphere $\mathbb{S}_1^{j-1} = \{x_{j+1} = \cdots = x_n = 0\} \cap \mathbb{S}_1^{2n-1}$ (resp., the sphere $\mathbb{S}_1^{n-j-1} = \{x_1 = \cdots = x_{j-1} = 0\} \cap \mathbb{S}_1^{2n-1}$).*

Proof. The corresponding complex coordinate planes $\mathbb{C}^{j-1}$ and $\mathbb{C}^{n-j-1}$ in $\mathbb{C}^n$ are invariant by the foliation $\mathcal{F}$ and the computations of the preceding proof show that the restriction of the first return map on the corresponding spheres (in intersection with the cross-section τ_j') has only eigenvalues $\exp 2\pi i\lambda_k/\lambda_j$. All these numbers are of modulus less than one (resp., greater than one). Since the stable (unstable) manifolds are uniquely defined, this proves the Proposition. □

The properties of the foliation $\mathcal{F}'$ established by these three propositions, imply its *structural stability*: any sufficiently close foliation is topologically equivalent to $\mathcal{F}'$.

Theorem 7.13 (J. Guckenheimer, 1972 [**Guc72**]). *Assume that the diagonal matrix Λ is complex hyperbolic and in the Poincaré domain.*

Then the holomorphic vector field $F(x) = \Lambda x + w(x)$ is topologically orbitally linearizable, i.e., the holomorphic singular foliation of $(\mathbb{C}^n, 0)$ by complex phase curves of the holomorphic vector field is locally topologically equivalent to the foliation defined by the linear vector field $F_0(x) = \Lambda x$.

Moreover, any sufficiently close vector field is locally topologically orbitally equivalent to F.

Proof. Consider the rescaling $F_\varepsilon(x) = \varepsilon^{-1}F(\varepsilon x)$, the corresponding foliation $\mathcal{F}_\varepsilon$ in the ball $\{|x| < 1\}$ and its trace $\mathcal{F}_\varepsilon'$ on the unit sphere $\mathbb{S}_1^{2n-1} = \varepsilon^{-1}\mathbb{S}_1^{2n-1}$.

By Theorem 7.6, both foliations $\mathcal{F}_\varepsilon$ and $\mathcal{F}_0$ are topological cones over their traces $\mathcal{F}_\varepsilon'$ and $\mathcal{F}_0'$. The assertion of the Theorem will follow from the topological equivalence of the latter two foliations on $\mathbb{S}_1^{2n-1}$.

By the Palis–Smale theorem [**PS70a**], a vector field on the compact manifold is structurally stable (i.e., its phase portrait is topologically orbitally equivalent to that of any sufficiently C^k-close vector field) if it meets the following *Morse–Smale conditions*:

(1) its singular points and limit cycles are hyperbolic (i.e., all eigenvalues of the linearization at any singular point have nonzero real parts, and all multiplicators of any limit cycle have modulus different from 1);

(2) its orbits can accumulate only to singular points or limit cycles;

(3) all stable and unstable invariant manifolds of singular points and limit cycles (which exist by the hyperbolicity assumption) intersect transversally.

All these conditions for the foliation $\mathcal{F}'_0$ are verified in Propositions 7.11, 7.10 and 7.12 respectively. Therefore the foliation $\mathcal{F}'_0$ is structurally stable and hence topologically equivalent to $\mathcal{F}'_\varepsilon$ for all small ε.

Returning to the initial nonlinear vector field $F = F_1$, we conclude that it is topologically orbitally equivalent to its linearization in all sufficiently small balls $\{|x| < \varepsilon\}$. □

Corollary 7.14. *Any two linear vector fields in $\mathbb{C}^n$ which are both in the Poincaré domain and hyperbolic, generate globally topologically equivalent singular foliations.*

Any two holomorphic vector fields in $(\mathbb{C}^n, 0)$ which are both in the Poincaré domain and hyperbolic, generate locally topologically equivalent singular holomorphic foliations.

Proof. Since topological equivalence is transitive, by Theorem 7.13 the second assertion of the Corollary follows from the first one.

To prove the assertion on linear systems, note that any two complex hyperbolic matrices in the Poincaré domain can be continuously deformed into each other within this class. Indeed, any such matrix can be first diagonalized and all its eigenvalues brought into the open left half-plane. Then all absolute values of these eigenvalues can be made equal to 1 without changing their arguments; this will not affect neither hyperbolicity nor the Poincaré property. Finally, the arguments of the eigenvalues can be assigned any positions, say, at equal angles between $\pi/2$ and $-\pi/2$. In this normal form the two diagonal matrices of the same size differ only by reordering of the coordinate axes. □

7.3. Resonances in the Poincaré domain. In the *linear* non-hyperbolic case the foliation traced by linear systems on the unit sphere, is still transversal but may have nontrivial recurrence. Indeed, in this case the first return map for one of the cycles will have a multiplicator $\exp 2\pi i\lambda_1/\lambda_2$ on the unit circle. Thus there will be a family of invariant 2-tori foliated by periodic or quasiperiodic orbits, depending on whether the ratio $\lambda_1/\lambda_2 \neq 1$ is rational or nor. Since both rational and irrational numbers are dense, two non-hyperbolic linear systems in the Poincaré domain can be arbitrarily close to each other but topologically non-equivalent.

If the system has two eigenvalues coincide, then typically the linearization matrix will have a nontrivial Jordan normal form with only one complex separatrix tangent to the corresponding 2-dimensional eigenspace. The same arguments as were used in the proof of Proposition 7.11, show that this separatrix leaves the trace in the form of a cycle on the sphere $\mathbb{S}^3$ whose first return map is conjugate to the complex holonomy of the separatrix.

Somewhat surprisingly and in contrast with the previously discussed diagonal cases, the holonomy map of this separatrix is *essentially nonlinear*: it cannot be linearized by a suitable choice of the cross-section (or, what is the same, a chart on it). The simple computation below shows that the holonomy has a fixed point of multiplicity exactly equal to 2 and thus a small perturbation will produce two close fixed points corresponding to two cycles of the trace foliation.

Occurrence of nonlinearities affects the situation in a similar way when (Poncaré) resonances occur, as was observed in [**Arn69**]. Consider the simplest case possible in the Poincaré domain in $\mathbb{C}^2$ and compute the holonomy map.

Eventually, all such computations will be collected at one place, say, in §**??**.

Proposition 7.15. *Consider the system in the formal normal form,*

$$\dot{x} = nx + a\,y^n, \quad \dot{y} = y, \qquad a \in \mathbb{C},\ n \geqslant 1. \tag{7.9}$$

Then the holonomy Δ of the unique separatrix $y = 0$, computed for the standard cross-section $\tau = \{x = 1\}$, is tangent to a rotation by the rational angle $2\pi/n$ and its nth iteration has an isolated fixed point of multiplicity $n + 1$ at the origin.

Proof. The system (7.9) is integrable: its general solution is $y(t) = C \exp t$, $x = (C' + aC^n t) \exp nt$, with arbitrary constants $C, C' \in \mathbb{C}$. The initial condition $(x(0), y(0)) = (1, s) \in \tau$ yields for the corresponding solution the formula

$$x(t) = (1 + as^n\, t) \exp nt, \qquad y(t) = s\, \exp t.$$

For $s = 0$ the x-component of the solution (separatrix) is $2\pi/n$-periodic. For small $s \in (\mathbb{C}, 0)$, the solution with this initial condition crosses again the section τ at the moments $t_k(s) = 2\pi ik/n + \delta_k(s)$, $\delta_k(s) = o(1)$, $k = 1, 2, \dots$, where $\delta_k(s)$ is the complex root of the equation

$$1 + as^n(2\pi ik/n + \delta_k(s)) = \exp\bigl(-n\delta_k(s)\bigr) = 1 - n\delta_k(s) + \cdots, \quad \lim_{s\to 0} \delta_k(s) = 0.$$

This equation can be resolved with respect to $\delta_k(s)$ defining the latter as an analytic function of s by the implicit function theorem. Computing the Taylor terms, we see immediately that

$$\delta_k(s) = -\frac{2\pi ika}{n^2}\, s^n + \cdots, \qquad t_k(s) = \frac{2\pi ik}{n} + \delta(s).$$

The iterated power of the holonomy map Δ^k is therefore

$$\Delta^k(s) = s\, \exp t_k(s) = \lambda^k\, s\, \exp \delta(s) = \lambda^k s(1 - kA\, s^n + \cdots),$$
$$\lambda = \exp \tfrac{2\pi i}{n}, \qquad A = \tfrac{2\pi ia}{n^2} \neq 0.$$

The nth iterated power of Δ is tangent to the identity and has an isolated fixed point of multiplicity exactly $n+1$. □

Corollary 7.16. *The resonant node corresponding to the resonance* $(1:n)$*,* $n \geqslant 2$*, can be analytically linearized if and only if it can be topologically linearized.*

Proof. Consider the trace of the foliation on the unit sphere. The first return map is a topological invariant of the foliation. For the nonlinear Jordan node (7.9) with $a \neq 0$ the holonomy map is nontrivial (its nth power has an isolated fixed point), whereas the holonomy map for the linear node is linear and its nth power identical. □

Note that in the real domain all nodes are topologically equivalent to each other.

Remark 7.17. Unlike the hyperbolic case in which the trace of the foliation on the unit sphere is structurally stable, the trace of the foliation corresponding to the system (7.9) changes its topological type by a small perturbation that affects the eigenvalues of the linear part (destroying thus the resonance).

Any small perturbation of the vector field preserves the transversality of the corresponding foliation to the unit sphere $\mathbb{S}^3_1$. Thus the holonomy map $\widetilde{\Delta}$ will be a small perturbation of the initial holonomy map Δ.

The map Δ has an isolated fixed point $x = 0$ which persists under small perturbations of the map if $n > 1$, since the multiplicator λ is different from 1 for such n.

Since the multiplicity of the fixed point $x = 0$ for Δ^n is $n+1$, any sufficiently close iterated power $\widetilde{\Delta}^n$ will have $n+1$ fixed points nearby. One of these points is fixed also for $\widetilde{\Delta}$ by the implicit function theorem. The remaining n points form a tuple of n*-periodic* points for $\widetilde{\Delta}$ that are positioned approximately at vertices of a regular n-gon and permuted by $\widetilde{\Delta}$ cyclically.

In terms of the traces of the foliations, this means that a vector field obtained by a sufficiently small perturbation of the non-linearizable resonant node, corresponds to a foliation on $\mathbb{S}^3_1$ that has two cycles close to each other and linked with the index $n \geqslant 2$.

The assertion remains true also for the Jordan node (linear or not) with the ratio of eigenvalues equal to 1.

7.4. Topological classification of linear complex flows in the Siegel domain. As opposite to the Poincaré case, the topological classification of holomorphic foliations generated by Siegel-type linear flows involves a number of continuous invariants. This means that in general an arbitrary small variation of the linear system results in a topologically different holomorphic foliation. This phenomenon is known as *rigidity*.

Consider a hyperbolic linear vector field $\dot{x} = Ax$ of Siegel type in $\mathbb{C}^n$, i.e., assume that the origin belongs to the convex hull of the eigenvalues $\lambda_1, \dots, \lambda_n$ of A, see §5.1. The complex hyperbolicity in the sense of Definition 7.8 implies that the matrix A can be assumed diagonal, $A = \Lambda = \operatorname{diag}\{\lambda_1, \dots, \lambda_n\}$, and the origin is necessarily *in the interior* of the convex hull $\operatorname{conv}\{\lambda_1, \dots, \lambda_n\} \subseteq \mathbb{C}$. In particular, hyperbolic Siegel systems exist only when $n \geqslant 3$.

Hyperbolicity means that the invariant axes (diagonalizing coordinates) of the linear vector field can be ordered to meet the following condition,

$$\begin{aligned} &\dot{x} = \Lambda x, \qquad x \in \mathbb{C}^n,\ \Lambda = \operatorname{diag}\{\lambda_1, \dots, \lambda_n\}, \quad n \geqslant 3, \\ &\operatorname{Im} \lambda_{j+1}/\lambda_j < 0, \quad j = 1, \dots, n, \qquad 0 \in \operatorname{conv}\{\lambda_1, \dots, \lambda_n\}. \end{aligned} \tag{7.10}$$

Here and everywhere below the enumeration of coordinates is cyclical modulo n, so that the assumption (7.10) includes the condition $\operatorname{Im} \lambda_n/\lambda_1 < 0$ as well. Denote by $\Phi^t = \exp t\Lambda \colon \mathbb{C}^n \to \mathbb{C}^n$ the *complex* flow of the linear system $\dot{x} = \Lambda x$ and $\mathcal{F}$ the (singular) holomorphic foliation by phase curves of this flow:

$$\mathcal{F} = \{L_x\}_{x \neq 0}, \qquad L_x = \{\Phi^t(x) \colon t \in \mathbb{C}\}.$$

Definition 7.18. The (complex) *periods* of the linear system (7.10) are the complex numbers

$$T_j = \frac{2\pi i}{\lambda_j} \in \mathbb{C}, \qquad j = 1, \dots, n.$$

For a hyperbolic system, the ratios of periods are never real.

Definition 7.19. Two sets of complex numbers $\mathbf{T} = (T_1, \dots, T_n)$, and $\mathbf{T}' = (T_1', \dots, T_n')$ are called *affine equivalent*, if after an eventual rearrangement, one of the following two equivalent conditions holds:

(1) The exists an $\mathbb{R}$-linear map $M \colon \mathbb{C} \to \mathbb{C}$ such that $MT_j = T_j'$ for all $j = 1, \dots, n$,

(2) The rank of the $(4 \times n)$-matrix $\mathfrak{T}$ whose columns are real 4-tuples $v_j = (\operatorname{Re} T_j, \operatorname{Im} T_j, \operatorname{Re} T_j', \operatorname{Im} T_j') \in \mathbb{R}^4$, is equal to 2.

The equivalence of the two conditions is immediate. If the rank of the matrix V is equal to 2 and the nonzero 2×2-minor occurs in the first two columns v_1, v_2, then any other column v_j, $j > 2$, can be represented as a real combination $\alpha v_1 + \beta v_2$, so that $T_j = \alpha T_1 + \beta T_2$ and $T_j' = \alpha T_1' + \beta T_2'$ with the same $\alpha, \beta \in \mathbb{R}$. If M is an $\mathbb{R}$-linear map taking T_1 and T_2 to T_1'

and T_2', then it will automatically map all other complex numbers (planar vectors) $T_3, \dots, T_n$ into $T_3', \dots, T_n'$ respectively: $MT_j = M(\alpha T_1 + \beta T_2) = \alpha T_1' + \beta T_2' = T_j'$.

Conversely, if the there exists a map M mapping T_j into T_j', then the last two rows of V are linear combinations of the first two rows, so that the rank of V is $\leqslant 2$. The equality occurs under the hyperbolicity assumption.

Theorem 7.20 (N. Ladis [**Lad77**], C. Camacho–N. H. Kuiper–J. Palis [**CKP76**], Yu. Ilyashenko [**Ily77**])**.**

Assume that the singular holomorphic foliations $\mathcal{F}, \mathcal{F}'$ generated by two hyperbolic linear systems of Siegel type are topologically equivalent.

Then the collections of the complex periods $\mathbf{T} = (T_1, \dots, T_n)$ and $\mathbf{T}' = (T_1', \dots, T_n')$ of the corresponding linear systems are affine equivalent: there exists an affine map $M \colon \mathbb{C} \to \mathbb{C}$ such that $MT_j = T_j'$ for all $j = 1, \dots, n$.

The proof of this Theorem begins §7.5 and occupies the rest of the section §7. The inverse statement is relatively easy.

Theorem 7.21. *If two collections of periods for two diagonal linear systems are affine equivalent, the corresponding holomorphic singular foliations on $\mathbb{C}^n$ are topologically equivalent.*

Proof. Without loss of generality we may assume that the $\mathbb{R}$-linear map $M \colon \mathbb{C} \to \mathbb{C}$ taking the collection $\{\lambda_1^{-1}, \dots, \lambda_n^{-1}\}$ into $\{\lambda_1'^{-1}, \dots, \lambda_n'^{-1}\}$, is orientation-preserving. Otherwise replace one of the foliations by its image by the total conjugacy $(x_1, \dots, x_n) \mapsto (\bar{x}_1, \dots, \bar{x}_n)$: the latter is generated by the linear system with the eigenvalues $\{\bar\lambda_1, \dots, \bar\lambda_n\}$ (note that the map $\lambda \mapsto \bar\lambda$ reverts the orientation).

Consider the map $h_\gamma \colon \mathbb{C} \to \mathbb{C}$, $x \mapsto x\,|x|^\gamma$, $\gamma \in \mathbb{C}$, extended as $h_\gamma(0) = 0$ at the origin. If $\operatorname{Re}\gamma > -1$, it is a homeomorphism of the complex plane into itself, since $\bigl||x|^\gamma\bigr| = |x|^{\operatorname{Re}\gamma}$ and therefore $|h_\gamma(x)| = |x|^{1+\operatorname{Re}\gamma}$.

We are looking for a diagonal homeomorphism H of the form $H(x) = \bigl(h_{\gamma_1}(x_1), \dots, h_{\gamma_n}(x_n)\bigr)$ which would conjugate two linear holomorphic foliations with the diagonal matrices $\varLambda = \operatorname{diag}\{\lambda_1, \dots, \lambda_n\}$ and $\varLambda' = \operatorname{diag}\{\lambda_1', \dots, \lambda_n'\}$ as follows,

$$H \circ \exp t\varLambda = \exp t'\varLambda' \circ H, \qquad t' = t(t), \tag{7.11}$$

where $t \mapsto t'(t)$ is a suitable $\mathbb{R}$-affine map, and $\gamma_1, \dots, \gamma_n$ are appropriate complex parameters with $\operatorname{Re}\gamma_j > -1$.

Since both H and $\varLambda, \varLambda'$ are diagonal maps, the condition (7.11) consists of n independent "scalar" conditions,

$$h_{\gamma_j}(z \exp t\lambda_j) = h_{\gamma_j}(z) \exp t'\lambda_j', \qquad j = 1, \dots, n, \tag{7.12}$$

which must hold identically for all $z \in \mathbb{C}$ and t; the affine map $t \mapsto t'$ must be the same for all j. Substituting the explicit formula for h_{γ_j}, we obtain after cancellation of $z\,|z|^{\gamma_j}$ the conditions

$$\exp[\lambda_j t + \gamma_j \operatorname{Re}(\lambda_j t)] = \exp t'\lambda'_j,$$

which is equivalent to the system of *linear* equations

$$\lambda_j'^{-1}[\lambda_j t + \gamma_j \operatorname{Re}(\lambda_j t)] = t', \qquad j = 1, \dots, n. \tag{7.13}$$

Notice that any $\mathbb{R}$-affine map has the form $t \mapsto t' = at + b\bar{t}$ with uniquely determined complex numbers $a, b \in \mathbb{C}$. This map is orientation-preserving if and only if $|a| > |b|$.

Since the map in the right hand side of all equations (7.13) is the same, after substituting $\operatorname{Re}(\lambda_j t) = \frac{1}{2}(\lambda_j t + \bar{\lambda}_j \bar{t})$ and equating the respective coefficients before t and $\bar{t}$, we obtain the system of equations

$$\tfrac{1}{2}\lambda_j'^{-1}\lambda_j(2 + \gamma_j) = a, \quad \tfrac{1}{2}\lambda_j'^{-1}\bar{\lambda}_j\gamma_j = b, \qquad j = 1, \dots, n.$$

The necessary and sufficient condition of solvability of these equations is obtained by rewriting them as follows,

$$2 + \gamma_j = 2a\lambda'_j\lambda_j^{-1}, \qquad \gamma_j = 2b\lambda'_j\bar{\lambda}_j^{-1}, \tag{7.14}$$

and subtracting one from the other:

$$1 = \lambda'_j(a\lambda_j^{-1} - b\bar{\lambda}_j^{-1}), \qquad j = 1, \dots, n.$$

Rewritten once again in the form

$$\lambda_j'^{-1} = a\lambda_j^{-1} - b\bar{\lambda}_j^{-1}, \qquad j = 1, \dots, n, \tag{7.15}$$

this solvability condition means affine equivalence of periods of the two systems in the sense of Definition 7.19: the inverse eigenvalues are simultaneously conjugated by the $\mathbb{R}$-affine map $M : w \mapsto aw - b\bar{w}$.

Conversely, if there exist complex a, b satisfying all identities (7.15), then one can resolve simultaneously all equations (7.14):

$$\gamma_j = 2b\frac{\lambda'_j}{\bar{\lambda}_j} = 2a\frac{\lambda'_j}{\lambda_j} - 2 = \lambda'_j(a\lambda_j^{-1} + b\bar{\lambda}_j^{-1}) - 1 = \frac{a\lambda_j^{-1} + b\bar{\lambda}_j^{-1}}{a\lambda_j^{-1} - b\bar{\lambda}_j^{-1}} - 1, \tag{7.16}$$

(the last transformation uses (7.15)). The corresponding conjugacy $H_\gamma = \big(h_{\gamma_1}(x_1), \dots, h_{\gamma_n}(x_n)\big)$ satisfies (7.11). It remains to verify that H is a homeomorphism, i.e., $\operatorname{Re}\gamma_j > -1$.

A direct computation yields

$$\operatorname{Re}\gamma_j + 1 = \operatorname{Re}\frac{a\lambda_j^{-1} + b\bar{\lambda}_j^{-1}}{a\lambda_j^{-1} - b\bar{\lambda}_j^{-1}} = \frac{|\lambda_j|^{-1}|^2(|a|^2 - |b|^2)}{|a\lambda_j^{-1} - b\bar{\lambda}_j^{-1}|^2}$$

(it is sufficient and easier to compute for $|\lambda_j| = 1$). This expression is positive if the $\mathbb{R}$-linear map M is orientation-preserving: then $|a| > |b|$, and hence $\operatorname{Re}\gamma_j > -1$ as required. □

Note that the sufficiency of affine equivalence of periods for topological equivalence of foliations is independent of whether the system is in the Siegel domain or not.

7.5. Complex transition time and topology of linear hyperbolic maps in $\mathbb{C}^2$. In this section we begin the proof of topological rigidity of linear systems in the Siegel domain (Theorem 7.20)

As follows from Proposition 7.9, all nontrivial (other than separatrices) solutions of the system (7.10) are simply connected. Therefore for each leaf $L \in \mathcal{F}$ of the foliation, other than one of the separatrices, the complex function

$$t(x, y) = t \iff \Phi^t(x) = y, \qquad x, y \in L. \tag{7.17}$$

is correctly defined on *pairs* of points of that leaf. We will refer $t(x, y)$ as the (complex) *transition time* from x to y. This function is holomorphic: indeed, $|\partial\Phi^t(x)/\partial t| \neq 0$ on the leaf, so the implicit function theorem applies.

The transition time satisfies the obvious *cocycle identity*: for any n points on *the same* leaf,

$$t(x_1, x_2) + \cdots + t(x_{n-1}, x_n) + t(x_n, x_1) = 0, \qquad x_1, \dots, x_n \in L.$$

The transition time depends continuously on the leaf unless it grows to infinity. More accurately, if x_m, y_m are two sequences of points on simply connected leaves L_m that converge to the limits $x = \lim x_m$, $y = \lim y_m$, then the transition times $t(x_m, y_m)$ converge to a finite limit *provided that x and y belong to the same simply connected leaf L*:

$$x, y \in L \neq S_j \implies \lim_{m\to\infty} t(x_m, y_m) = t(x, y).$$

Indeed, in this case there exists a curve $\gamma \subset L$ connecting x with y. Trivializing the foliation near this curve, we see that x_m can be connected by a close curve γ_m with y_m on L_m.

On the contrary, each separatrix S_j is a multiply connected domain and the flow Φ^t restricted on S_j, is T_j-periodic (whence the term "period").

Consider the case $n = 3$ and denote by τ_j the *standard cross-section* $\{x_j = 1\} \simeq \mathbb{C}^2$ to the separatrix $S_j = \mathbb{C}e_j$, $j = 1, 2, 3$, equipped with the coordinates (x_{j-1}, x_{j+1}) (recall that the enumeration of coordinates is cyclical). Denote by Δ_j the corresponding holonomy map: because of the periodicity,

$$\Delta_j = \Phi^{T_j}\big|_{\tau_j}, \qquad j = 1, 2, 3.$$

The operators Δ_j are linear diagonal with the eigenvalues $\exp 2\pi i \frac{\lambda_{j\pm1}}{\lambda_j}$. Given the assumption (7.10), we have

$$|\exp(2\pi i\lambda_{j-1}/\lambda_j)| < 1 < |\exp(2\pi i\lambda_{j+1}/\lambda_j)|.$$

Denote by $W_j^{\pm}$ the corresponding stable and unstable subspaces in τ_j: Δ_j is contracting on W_j^- and expanding on W_j^+ for all $j = 1, 2, 3$, see Figure 1.

This hyperbolic structure immediately implies the following lemma.

Lemma 7.22. *If $P = (1, 0, p) \in W_1^-$, $P' = (1, p', 0) \in W_1^+$ are two points, $pp' \neq 0$, then one can find two converging sequences of points $P_m \to P$ and $P'_m \to P'$ in the cross-section τ_1 such that $\Delta_1^m(P_m) = P'_m$.*

Proof. If μ and ν are the contracting and expanding eigenvalues of Δ_1, $|\mu| < 1 < |\nu|$, then the points

$$P_m = (1, \nu^{-m}p', p), \quad P'_m = (1, p', \mu^m p),$$

obviously meet all requirements. □

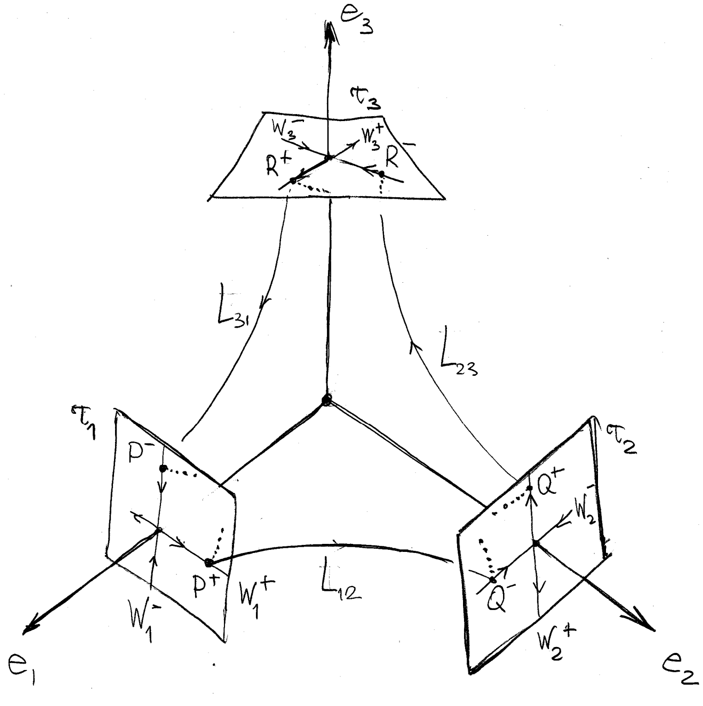

Figure 1. Demonstration of Theorem 7.20: construction of the sequences $P_m^{\pm}, Q_m^{\pm}, R_m^{\pm}$.

Before proceeding with the formal proof of this Theorem, we briefly discuss the differences which occur between the Poincaré and Siegel hyperbolic cases as seen on the trace left by a linear foliation $\mathcal{F}$ on the unit sphere $\mathbb{S}^{2n-1} \subset \mathbb{C}^n$. We will deal with the simplest case $n = 3$.

First, the transversality of $\mathcal{F}$ to $\mathbb{S}^5 = \{|x_1|^2 + |x_2|^2 + |x_3|^2 = 1\}$ holds no more: if $(\boldsymbol{\rho}, \mathbf{T}) = 0$ and $\boldsymbol{\rho} \in \mathbb{R}^3_+$, then on the 3-torus $\mathbb{T}^3 = \{|x_j| = \rho_j\}$ the leaves are tangent to the sphere. However, the coordinate axes (separatrices) are transversal to $\mathbb{S}^5$ and leave their traces on this sphere as the cycles $C_1, C_2, C_3 \subset \mathbb{S}^5$. These cycles are hyperbolic, and their corresponding invariant manifolds are 3-spheres $\mathbb{S}^{\pm}_j \simeq \mathbb{S}^3$, $\mathbb{S}^+_j = \mathbb{S}^5 \cap \{x_{j-1} = 0\}$: for each $j = 1, 2, 3$, $\mathbb{S}^-_j = \mathbb{S}^5 \cap \{x_{j+1} = 0\}$.

Here the similarity ends. First, the invariant manifolds do not intersect transversally. Quite contrary, $\mathbb{S}^+_j$ coincides with $\mathbb{S}^-_{j+1}$ and all trajectories inside this sphere are bi-asymptotic to C_j and C_{j+1}. Behavior of the trace foliation $\mathcal{F}$ on this sphere is of Poincaré type.

All other trajectories outside of the union of all invariant manifolds, are *closed*. Indeed, if $\lambda_1, \lambda_2, \lambda_3$ form a triangle, then at least one of the absolute values $|\exp \lambda_j t|$ tends to infinity as $|t| \to \infty$ along any ray. By (7.8), the trace of any leaf $L \in \mathcal{F}$ on $\mathbb{S}^5$ is compact (periodic).

Thus we see that the trace of the foliation has singularities and nontrivial recurrence on the sphere $\mathbb{S}^5$.

Topological description of the coordinate planes? To be required later...

7.6. Main construction. The proof of Theorem 7.20 for $n = 3$ is based on construction of a sequence of leaves L_m of the foliation $\mathcal{F}$ that accumulate to *all three complex separatrices* simultaneously as $m \to \infty$. It is the *relative portions of time* spent near each separatrix, which constitute the continuous invariant underlying Theorem 7.20. The traces of these leaves on the unit sphere $\mathbb{S}^5$ will be very long but closed curves, that "spend most of their length" near the separatrix cycles C_j.

Assume that $n = 3$ and the three eigenvalues $\lambda_1, \lambda_2, \lambda_3$ form a triangle on the complex plane, containing the origin in the interior. Then their respective periods $\mathbf{T} = (T_1, T_2, T_3)$ also form the triangle with the same property.

There exists a unique positive vector $\boldsymbol{\rho} = (\rho_1, \rho_2, \rho_3) \in \mathbb{R}^3_+$, such that

$$0 = (\boldsymbol{\rho}, \mathbf{T}) = \rho_1 T_1 + \rho_2 T_2 + \rho_3 T_3, \qquad |\boldsymbol{\rho}| = 1. \tag{7.18}$$

Approximating the positive numbers $\rho_i > 0$ (7.18) by rational numbers as in the proof of Proposition 5.2, we can construct a sequence of *natural* vectors $\mathbf{k}_m = (k_{1,m}, k_{2,m}, k_{3,m}) \in \mathbb{N}^3$ such that

$$(\mathbf{k}_m, \mathbf{T}) \to 0, \quad \frac{\mathbf{k}_m}{|\mathbf{k}_m|} \to \boldsymbol{\rho} \qquad \text{as } m \to \infty. \tag{7.19}$$

In the hyperbolic Siegel case $|\mathbf{k}_m| \to +\infty$ implies $k_{m,j} \to +\infty$ for all $j = 1, 2, 3$.

Choose two arbitrary points $P^{\pm} \in W_1^{\pm}$ on the invariant subspaces in the cross-section τ_1 and let $P_m^{\pm}$, $m = 1, 2, \dots$, be two sequences of points satisfying the condition

$$t(P_m^-, P_m^+) = k_{m,1}, \qquad \lim_{m\to\infty} P_m^{\pm} = P^{\pm}.$$

Existence of such sequence is asserted by Lemma 7.22.

The leaf $L_{12} \in \mathcal{F}$ passing through P^+ belongs to the invariant plane $x_3 = 0$ and intersects (transversally) the cross-section τ_2 at some point Q^- belonging to the Δ_2-invariant subspace W_2^-. By transversality arguments and continuity of the transition time along the leaf L_{12}, all nearby leaves L_m passing through P_m^+, cross τ_2 at some points Q_m^- that converge to Q^- so that the transition time between P_m^+ and Q_m^- has a limit as $m \to +\infty$, denoted by T_{12}:

$$\lim_{m\to\infty} t(P_m^+, Q_m^-) = t(P^+, Q^-) = T_{12}.$$

In the same way we can construct a sequence of points $R_m^+ \in \tau_3$ converging to $R^+ \in W_3^+$ such that P^-, R^+ belong to the same leaf of $\mathcal{F}$ denoted by L_{31}, and $t(R_m^+, P_m^-)$ has a limit,

$$\lim_{m\to\infty} t(R_m^+, P_m^-) = t(R^+, P^-) = T_{31}.$$

Now we construct two remaining sequences, $R_m^- \in \tau_3$ and $Q_m^+ \in \tau_2$, as follows,

$$R_m^+ = \Delta_3^{k_{m,3}}(R_m^-), \qquad Q_m^+ = \Delta_2^{k_{m,2}}(Q_m^-)$$

(more accurately, R_m^- should be defined starting from R_m^+ that were already constructed, iterating the inverse of the holonomy map, $R_m^- = \Delta_3^{-k_{m,3}}(R_m^+)$).

Unlike before, convergence of the sequences R_m^-, Q_m^+ to some limits R^-, Q^+ that belong to the respective subspaces W_3^-, W_2^+ requires separate proof. Computation of the following Lemma is a central step of the entire construction.

Lemma 7.23. *In the above settings, the sequences of points R_m^- and Q_m^+ converge,*

$$\lim_{m\to\infty} R_m^- = R^- \in W_3^-, \qquad \lim_{m\to\infty} Q_m^+ = Q^+ \in W_2^+.$$

The limit points Q^+ and R^- belong to the same leaf $L_{23} \in \mathcal{F}$, and the transition time $t(Q^+, R_-) = T_{23}$ satisfies the cocyclic identity

$$T_{12} + T_{23} + T_{31} = 0. \qquad (7.20)$$

Proof. The proof of convergence is nearly identical for the two sequences. By construction, $Q^+_m \in \tau_2$, so the second coordinate is identically 1 along this sequence. Next, since the first coordinate is contracting by iterations of Δ_2 and $k_{m,2} \to \infty$, from the definition $Q^+_m = \Delta^{k_{m,2}}(Q^-_m)$ it follows that the first coordinate of the points Q^+_m tends to zero. It remains to show only that the third coordinate has nonzero limit.

By construction of the points and taking into account the condition (7.19), we have

$$t(P^-_m, Q^+_m) = k_{m,1}T_1 + t(P^+_m, Q^-_m) + k_{m,2}T_2 = -k_{m,3}T_3 + T_{12} + o(1).$$

Since the third coordinate $x_3(t) = x_3(0)\exp\lambda_3 t$ is T_3-periodic along any solution $x(t) = (x_1(t), x_2(t), x_3(t))$, we conclude that the third coordinate tends to the nonzero limit equal to $[\exp\lambda_3 T_{12}]p$, where p is the third coordinate of the point $P^- = (1, 0, p)$.

The proof of the second limit is completely similar. For exactly the same reasons, the only coordinate whose convergence requires a proof, is the second coordinate x_2 that is T_2-periodic on leaves of $\mathcal{F}$. By construction, we have

$$t(P^+_m, R^-_m) = -k_{m,1}T_1 - T_{31} - k_{m,3}T_3 + o(1) = k_{m,2}T_2 - T_{31} + o(1),$$

and the limit exists: $x_2(R^-_m) \to [\exp(-\lambda_2 T_{31})]\, p'$, where p' is the second coordinate of the point $P^+ = (1, p', 0)$.

It remains to show that the points R^- and Q^+ belong to the same leaf of $\mathcal{F}$. This again follows from the same computation:

$$t(Q^+_m, R^-_m) = (\mathbf{k}_m, \mathbf{T}) - (T_{12} + T_{31}) + o(1).$$

By uniform continuity of the flow $\Phi^t(x)$ in x for all bounded values of t, the points R^- and Q^+ belong to the same leaf of $\mathcal{F}$. The identity (7.20) follows from (7.19). □

Remark 7.24. The construction depends on the initial choice of the two points $P^\pm$ as the parameters. A simple inspection shows that if these points are chosen sufficiently close to e_1, then the points $Q^\pm$ and $R^\pm$ will be arbitrarily close to e_2 and e_3 respectively.

7.7. Topological functoriality of the main construction and the proof of Theorem 7.20. Consider two complex hyperbolic linear flows of Siegel type in $\mathbb{C}^3$ and denote the corresponding holomorphic singular foliations by $\mathcal{F}$ and $\mathcal{F}'$ respectively. Let $\mathbf{T} = (T_1, T_2, T_3)$ and $\mathbf{T}' = (T'_1, T'_2, T'_3)$ be the corresponding periods.

Assume that $H\colon \mathbb{C}^3 \to \mathbb{C}^3$ is a homeomorphism conjugating the foliations. By Proposition 7.9 the complex separatrices are uniquely characterized by being not simply connected, hence H must map coordinate axes into coordinate axes. Without loss of generality we may assume that $H(e_j) = e_j$, where e_j, $j = 1, 2, 3$, are the three unit vectors in $\mathbb{C}^3$.

The construction described in §7.6 associates with the three positive real numbers $\boldsymbol{\rho} = (\rho_1, \rho_2, \rho_3)$ satisfying the condition (7.18), a sequence of leaves $L_m \in \mathcal{F}_m$ that accumulate to the union of three separatrices S_1, S_2, S_3 and the three "heteroclinic" leaves L_{12}, L_{23}, L_{31}. More precisely, each leaf L_m carries six points $P^\pm_m, Q^\pm_m, R^\pm_m$ each converging as $m \to \infty$ to the respective limits $P^\pm, Q^\pm, R^\pm$, in such a way that the transition times are as follows (see Figure 2),

$$\begin{aligned} t(P^-_m, P^+_m) &= k_{m,1}T_1, & t(P^+_m, Q^-_m) &= t(P^+, Q^-) + o(1),\\ t(Q^-_m, Q^+_m) &= k_{m,2}T_2, & t(Q^+_m, R^-_m) &= t(Q^+, R^-) + o(1),\\ t(R^-_m, R^+_m) &= k_{m,3}T_3, & t(R^+_m, P^-_m) &= t(R^+, P^-) + o(1). \end{aligned} \tag{7.21}$$

Denote by L'_m the images of the leaves $L'_m = H(L_m)$. Let τ'_j, $j = 1, 2, 3$ be three standard cross-sections to the separatrices S'_j of the second foliation $\mathcal{F}'$. (Note that τ'_j coincide with τ_j if we identify the phase spaces of the two foliations $\mathcal{F}, \mathcal{F}'$). The homeomorphism H in general does not map the cross-sections τ_j to τ'_j, but in any case the images $H(\tau'_j)$ are "topologically transversal" to the separatrices S'_j: each nearby local leaf of $\mathcal{F}'$ in a small neighborhood of e_j intersects $H(\tau_j)$ only once. This allows to define the local holonomy correspondences $h_j\colon (H(\tau_j), e_j) \to (\tau'_j, e_j)$ between the two cross-sections, at least in sufficiently small neighborhoods of the points e_j. They are local homeomorphisms.

Consider the following six points on the leaves L'_m,

$$\begin{aligned} \widetilde P^\pm_m &= h_1 \circ H(P^\pm_m) \in \tau'_1,\\ \widetilde Q^\pm_m &= h_2 \circ H(Q^\pm_m) \in \tau'_2,\\ \widetilde R^\pm_m &= h_3 \circ H(R^\pm_m) \in \tau'_3. \end{aligned} \tag{7.22}$$

All these sequences are converging, since $h_j \circ H\colon \tau_j \to \tau'_j$ are homeomorphisms and the preimages were converging by construction. Denote by $\widetilde P^\pm, \widetilde Q^\pm, \widetilde R^\pm$ their respective limits.

Let $t'(\cdot,\cdot)$ be the transition time function defined on pairs of points on the same leaf of the second foliation $\mathcal{F}'$ via the flow of the vector field $\dot x = \Lambda' x$ generating $\mathcal{F}'$.

Lemma 7.25.

$$\begin{aligned} t'(\widetilde P^-_m, \widetilde P^+_m) &= k_{m,1}T'_1, & t'(\widetilde P^+_m, \widetilde Q^-_m) &= t'(\widetilde P^+, \widetilde Q^-) + o(1),\\ t'(\widetilde Q^-_m, \widetilde Q^+_m) &= k_{m,2}T'_2, & t'(\widetilde Q^+_m, \widetilde R^-_m) &= t'(\widetilde Q^+, \widetilde R^-) + o(1),\\ t'(\widetilde R^-_m, \widetilde R^+_m) &= k_{m,3}T'_3, & t'(\widetilde R^+_m, \widetilde P^-_m) &= t'(\widetilde R^+, \widetilde P^-) + o(1). \end{aligned} \tag{7.23}$$

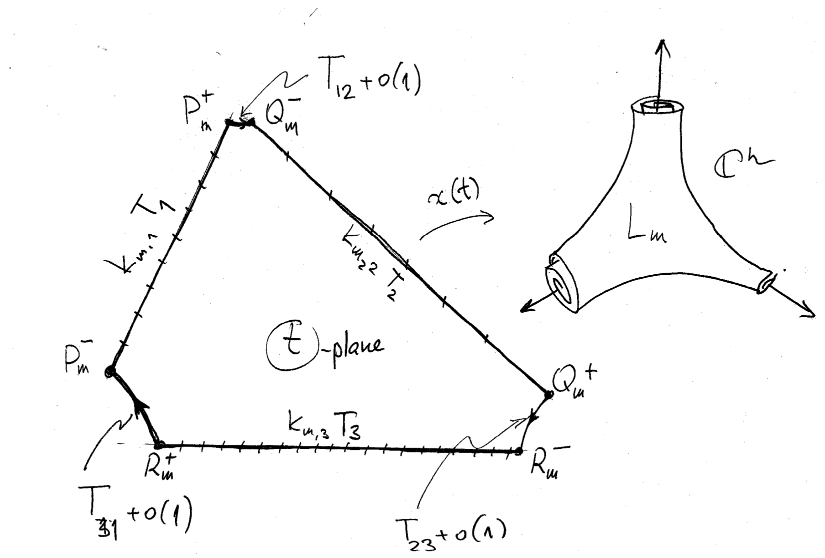

Figure 2. Demonstration of Theorem 7.20: topological functoriality

Proof. The three left equalities follow from the fact that $h_j \circ H$ conjugates the holonomy Δ_j of the foliation $\mathcal{F}$ on the cross-section τ_j, with the holonomy Δ'_j of the foliation $\mathcal{F}'$ on the cross-section τ'_j. To obtain P_m^+ from P_m^-, one has to iterate $k_{m,1}$ times the map Δ_1, therefore $\widetilde{P}_m^+ = (\Delta'_j)^{k_{m,2}}(\widetilde{P}_m^-)$. Since $t'(x, \Delta'_j(x)) = T'_1$, we conclude that $t'(\widetilde{P}_m^-, \widetilde{P}_m^+) = k_{m,1}T'_1$. The other three equalities are completely similar.

To prove the remaining three limits, we note that the limit points, say, $\widetilde{P}^+$ and $\widetilde{Q}^-$ belong to the same leaf $L'_{12} = H(L_{12})$, again by continuity of H. Therefore $t'(\widetilde{P}^+, \widetilde{Q}^-)$ is the finite limit of $t'(\widetilde{P}_m^+, \widetilde{Q}_m^-)$ as $m \to \infty$. The other two transition times $t'(\widetilde{Q}_m^+, \widetilde{R}_m^-)$, $t'(\widetilde{R}_m^+, \widetilde{P}_m^-)$ have finite limits in exactly the same way. □

Proof of Theorem 7.20 for $n = 3$. The cocycle identity

$$t'(\widetilde{P}_m^-, \widetilde{P}_m^+) + t'(\widetilde{P}_m^+, \widetilde{Q}_m^-) + t'(\widetilde{Q}_m^-, \widetilde{Q}_m^+) + t'(\widetilde{Q}_m^+, \widetilde{R}_m^-) + t'(\widetilde{R}_m^-, \widetilde{R}_m^+) + t'(\widetilde{R}_m^+, \widetilde{P}_m^-) = 0$$

together with (7.23) implies that

$$(\mathbf{k}_m, \mathbf{T}') = O(1), \qquad \text{as } m \to \infty.$$

Dividing this identity by $|\mathbf{k}_m| \to \infty$ yields in the limit

$$(\boldsymbol{\rho}, \mathbf{T}') = 0, \qquad \boldsymbol{\rho} = (\rho_1, \rho_2, \rho_3) \in \mathbb{R}_3^+.$$

In other words, the positive vector $\boldsymbol{\rho} \in \mathbb{R}^3_+$ satisfying the condition $(\boldsymbol{\rho}, \mathbf{T}) = 0$, satisfies also the condition $(\boldsymbol{\rho}, \mathbf{T}') = 0$.

Thus the system of four linear equations (over $\mathbb{R}$), equivalent to the two complex equalities,

$$(\boldsymbol{\rho}, \mathbf{T}) = 0, \quad (\boldsymbol{\rho}, \mathbf{T}') = 0, \tag{7.24}$$

has a nontrivial solution. This means that the rank of its coefficient matrix is 2. By Definition 7.19 (2), the two collections of periods $\mathbf{T}$ and $\mathbf{T}'$ are affine equivalent. □

Remark 7.26. The three-dimensional construction used in the above proof, in fact implies some multidimensional corollaries. Consider two linear hyperbolic Siegel-type systems in $\mathbb{C}^n$, $n > 3$, with the complex periods $\mathbf{T}$ and $\mathbf{T}'$ respectively, which are topologically orbitally equivalent (i.e., the corresponding foliations $\mathcal{F}$ and $\mathcal{F}'$ are topologically equivalent). By Proposition 7.9, without loss of generality (changing the enumeration of coordinates if necessary) we may assume that the conjugating homeomorphism H sends the complex separatrices S_j (the coordinate axes) to the separatrices S'_j for all $j = 1, \dots, n$.

Assume that the first three eigenvalues $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{C}$ of the first system already form a triangle containing the origin strictly inside. Then the respective triplets of periods (T_1, T_2, T_3) and (T'_1, T'_2, T'_3) are affine equivalent in the sense of Definition 7.19.

Indeed, the coordinate plane $\mathbb{C}^3$ spanned by the first three coordinates in $\mathbb{C}^n$, is invariant by the complex flow of the first system hence, the construction of the leaves $L_m \subset \mathbb{C}^3$ can be carried out without any changes. On the other hand, the three-dimensional proof of Theorem 7.20 does not use the fact that the images $L'_m = H(L_m)$ belong to any coordinate subspace invariant for the second system: the only fact required for the proof is accumulation of the leaves L'_m to the three complex separatrices S'_1, S'_2, S'_3 of the second system. The conclusion on affine equivalence of the respective periods obviously holds in this case.

One may be tempted to prove Theorem 7.20 for $n > 3$ by studying all 3-dimensional (invariant) coordinate planes the restriction on which is of Siegel type, based on the above Remark. However, the accurate proof goes along slightly different lines.

First we make some simple topological observations. It was already proved that the coordinate axes of a diagonal hyperbolic linear system are topologically functorial. On the other hand, *not every* (invariant) coordinate subspace is

If $\lambda_1, \dots, \lambda_n \in \mathbb{C}$ is a point set, its element is called a *corner point* if it can be separated from the rest of the set by a real line.

Lemma 7.27. *Assume that two diagonal hyperbolic linear systems in $\mathbb{C}^n$ are topologically equivalent by a homeomorphism preserving the coordinate axes (separatrices).*

If λ_n is a corner point of the spectrum of the first system, then H preserves the coordinate hyperplane $\mathbb{C}^{n-1} = \{x_n = 0\} \subset \mathbb{C}^n$.

Proof. The coordinate hyperplane $\{x_n = 0\}$ is distinguished by the following topological description: all leaves not belonging to this plane, accumulate to nonsingular points on the complex separatrix $S_n = \mathbb{C}e_n$. By our assumption on the enumeration of the coordinates, the separatrices S_n and S'_n are H-related, hence their "complementary" hyperplanes are also H-related. □

Proof of Theorem 7.20 for any $n > 3$. The proof goes by induction in n. The basis at $n = 3$ is already established.

Consider a hyperbolic Siegel-type linear system in $\mathbb{C}^{n+1}$ with the spectrum $\lambda_1, \dots, \lambda_{n+1}$ containing the origin strictly inside its convex hull. As before, we can assume without loss of generality that the system is diagonal, so any coordinate subspace of any (complex) dimension between 1 and n is invariant.

Assume that the enumeration of the axes is so chosen that 0 is inside the convex hull $\operatorname{conv}(\lambda_1, \dots, \lambda_n)$, while the last remaining eigenvalue λ_{n+1} is a corner point. Elementary geometric considerations show that this is always possible.

By Lemma 7.27, the invariant hyperplane $\{x_{n+1} = 0\} \subset \mathbb{C}^{n+1}$ is topologically invariant: any homeomorphism H between $\mathcal{F}$ and another such foliation $\mathcal{F}'$ defined by a diagonal hyperbolic linear system, necessarily conjugates the restrictions of these foliations on the respective hyperplanes $\{x_{n+1} = 0\}$ and $\{x'_{n+1} = 0\}$.

By the inductive assumption, the truncated collections of the periods $(T_1, \dots, T_n)$ and $(T'_1, \dots, T'_n)$ are affine equivalent: there exists an $\mathbb{R}$-linear map M of $\mathbb{C}$ into itself, taking one collection into the other.

To show that this map takes the last period T_{n+1} into T'_{n+1}, notice that for elementary reasons at least one of the triangles $\operatorname{conv}(\lambda_{n+1}, \lambda_j, \lambda_k)$, $1 \leqslant j \neq k \leqslant n$, also contains the origin in its interior (the union of these triangles contains the convex hull of all $n+1$ eigenvalues). By Remark 7.26, the triplets (T_{n+1}, T_j, T_k) and (T'_{n+1}, T'_j, T'_k) are affine equivalent by an $\mathbb{R}$-linear map $M' \colon \mathbb{C} \to \mathbb{C}$. But since $T_j/T_k \notin \mathbb{R}$, there exists *only one* $\mathbb{R}$-linear map $M = M'$ that takes (T_j, T_k) into (T'_j, T'_k), which therefore automatically maps the complete collection $\mathbf{T}$ into $\mathbf{T}'$. □

7.8. Further results: topological equivalence of linear Siegel-type foliations with Jordan blocks. If the matrix A of Siegel type is non-diagonalizable and "otherwise" hyperbolic (i.e., if the ratio of any two eigenvalues is non-real unless they are equal and occur in the same Jordan block), then the topological classification of the corresponding holomorphic foliations is even more rigid, as was discovered by L. Ortiz Bobadilla [**OB96**].

As before, the key result is low-dimensional. Consider the class of linear systems in $\mathbb{C}^4$ whose matrices have one (2×2)-block with the eigenvalue λ_1, and two other eigenvalues λ_2, λ_3 are such that the triangle $\lambda_1, \lambda_2, \lambda_3$ contains the origin in the interior.

Two foliations $\mathcal{F}$ and $\mathcal{F}'$ generated by systems of this class, are topologically equivalent if the two corresponding tuples of eigenvalues are proportional over $\mathbb{C}$, i.e., if

$$\boldsymbol{\lambda} = c\boldsymbol{\lambda}', \qquad \boldsymbol{\lambda} = (\lambda_1, \lambda_2, \lambda_3),\ \boldsymbol{\lambda}' = (\lambda_1', \lambda_2', \lambda_3'), \qquad 0 \neq c \in \mathbb{C} \tag{7.25}$$

The topological equivalence H in this case can be made *linear*, of the form $x \mapsto Cx$. Indeed, from the proportionality of the eigenvalues (7.25) and identical Jordan structure it follows that one can find a linear transformation such that the matrices A and $CA'C^{-1}$ would differ only by the scalar multiple c. But the leaves of two foliations $\mathcal{F}$ and $\mathcal{F}'$ defined by the *proportional* matrices, simply coincide.

It turns out that this is *the only* case when foliations of the considered class are topologically equivalent. In other words, the following result asserts the maximal topological rigidity of Siegel type foliations having Jordan blocks.

Theorem 7.28 (see [**OB96**]). *Two holomorphic foliations generated by Siegel-type linear vector fields in $\mathbb{C}^4$ having one Jordan block and hyperbolic otherwise, are topologically equivalent if and only if their eigenvalues are proportional over $\mathbb{C}$, in which case they are linear equivalent.*

Returning to the (truly) hyperbolic hence diagonalizable case, one may ask whether the study of holomorphic foliations generated by *nonlinear* vector fields, brings any new phenomena. In a surprising way, the answer is negative, as was established by M. Chaperon [**Cha86**] who proved the following complex analog of the Grobman–Hartman theorem.

Theorem 7.29 (M. Chaperon [**Cha86**]). *If the spectrum of a matrix A is hyperbolic and Siegel-type, then the singular holomorphic foliation by solutions of any nonlinear holomorphic vector field $\dot{x} = A(x) + \cdots$, is topologically linearizable (topologically equivalent to the foliation $\mathcal{F}'$ by solutions of the linearized field $\dot{x} = Ax$).*

The complete proofs of these results go beyond the scope of this book, though all the main tools required for the proof of, say, Theorem 7.28, were already described in this section.

8. Desingularization in the plane

Reasonably complete analysis of singular points of holomorphic vector fields using holomorphic normal forms and transformations by biholomorphisms, is possible under the assumption that the linear part is not very degenerate. The degenerate cases have to be treated by transformations that can alter the linear part. Such transformations, necessarily not holomorphically invertible, are known by the name *desingularization*, *resolution of singularities*, *sigma-process* or *blow-up*. Very roughly, the idea is to consider a holomorphic map $\pi\colon M \to (\mathbb{C}^2,0)$ of a holomorphic surface (2-dimensional manifold) M that squeezes (blows down) a sufficiently large set, usually a complex 1-dimensional curve $D \subset M$ to the single point $0 \in \mathbb{C}^2$, while being one-to-one between $M \smallsetminus D$ and $(\mathbb{C}^2,0) \smallsetminus \{0\}$. The second circumstance allows to pull back local objects (functions, curves, foliations, 1-forms *etc.*) from $(\mathbb{C}^2,0)$ to M and then extend them on D. These pull-backs are called desingularizations, or blow-up of the initial objects; sometimes M is itself called the blow-up of (the neighborhood of) the point $0 \in \mathbb{C}^2$.

Using desingularization one can ultimately simplify singularities of holomorphic foliations in dimension 2. The main result of this section, the fundamental Desingularization Theorem 8.14 asserts that by a suitable blow-up any singular holomorphic foliation in a neighborhood of a singular point can be resolved into a singular foliation defined in a neighborhood of a union $D = \bigcup_i D_i$ of one or more transversally intersecting holomorphic curves D_i and having only *elementary* singularities on D.

8.1. Polar blow-up. We start with a transcendental but geometrically more transparent construction in the real domain.

Definition 8.1. The *polar blow-down* is the map P of the real cylinder $C = \mathbb{R} \times S^1 \to \mathbb{R}^2$ onto the plane $\mathbb{R}^2$

$$P\colon (r,\varphi) \mapsto (r\cos\varphi, r\sin\varphi). \tag{8.1}$$

This map is a diffeomorphism between the open half-cylinder $C_+ = \{r > 0\} \subset C$ and the punctured plane $\mathbb{R}^2 \smallsetminus \{0\}$ and real analytic there. The image of the narrow band $C = (\mathbb{R},0) \times S^1$ (cylinder) is a double covering of the small neighborhood of the origin $\{|x| < \varepsilon\}$ except for the central equator $S = \{r = 0\} \subset C$, called also *exceptional divisor*. The latter is squeezed into one point, the origin $0 \in \mathbb{R}^2$.

The map P pulls back functions and differential 1-forms from $(\mathbb{R}^2, 0)$ on C (in non-invariant terms, passing to the polar coordinates and ignoring the inequality $r > 0$). However, the pullback $P^*\omega \in \Lambda^1(C)$ of any 1-form $\omega \in \Lambda^1(\mathbb{R}^2, 0)$ always has a non-isolated singularity on S. In the real analytic case one can always divide $P^*\omega$ by a suitable natural power r^ν so that the result $\widetilde{\omega} = r^{-\nu} P^*\omega$ remains still real analytic but has only isolated singularities on S.

Consider now the real analytic line field $\{\omega = 0\}$ and the corresponding foliation $\mathcal{F}$ of $(\mathbb{R}^2, 0) \smallsetminus \{0\}$ tangent to this field. As P is one-to-one outside the origin, $P^{-1}(\mathcal{F})$ is a foliation of $C \smallsetminus S$ tangent to the line field $\{P^*\omega = 0\}$. Since r is nonvanishing outside S, this foliation $\widetilde{\mathcal{F}}$ tangent to the line field $\{\widetilde{\omega} = 0\}$ provides the natural extension of $P^{-1}(\mathcal{F})$ onto most of S.

Definition 8.2. The line field $r^{-\nu} P^*\omega = 0$ with isolated singularities and the corresponding singular foliation $\widetilde{\mathcal{F}}$ on C are called the *trigonometric blow-up* of the line field $\omega = 0$ and the foliation corresponding $\mathcal{F}$ respectively.

Example 8.3.

(i) The form $dx = 0$ defining a nonsingular foliation, after trigonometric blow-up becomes $\cos\varphi\, dr - r \sin\varphi\, d\varphi$ and has two isolated singular points $(0,0)$ and $(0, \pi)$ on $\mathbb{R} \times S^1$. Both these points are nondegenerate saddles. The exceptional circle without these points is the leaf of the blow-up foliation.

(ii) The form $\omega = y\,dx - x\,dy$ defines the "radial" singular foliation on $\mathbb{R}^2$. The pullback $P^*\omega = -r^2\, d\varphi$, has a non-isolated singularity on $r = 0$, but after division the form $\widetilde{\omega} = r^{-2} P^*\omega = d\varphi$ defines the non-singular "parallel" foliation $\{\varphi = \text{const}\}$. All leaves of this foliation cross the exceptional circle S transversally.

(iii) The form $x\,dx + y\,dy = \frac{1}{2} d(x^2 + y^2)$ which defines foliation of $\mathbb{R}^2$ by the circles $x^2 + y^2 = \text{const}$, pulls back as the line field $r\,dr = 0$ which after division also becomes a non-singular form dr on C. The exceptional circle is a leaf of the blow-up foliation carrying no singular points.

The map P can be complexified and the above examples generalized. However, the complexification will also be a two-fold covering, which is not natural geometrically. Besides, using the trigonometric functions $\sin\varphi, \cos\varphi$ makes the corresponding formulas non-algebraic: preimage of the origin is the complex line $\mathbb{C}$ which is not compact.

There is an algebraic version of the map P, called the *sigma-process, monoidal transformation*, or simply the *blow-up* without the adjective trigonometric.

8.2. Algebraic blow-up (σ-process). It is not so easy to construct a holomorphic map $\sigma \colon C \to \mathbb{C}^2$ such that (i) the preimage of the origin is a compact irreducible holomorphic curve S and (ii) the map σ is one-to-one between $C \smallsetminus S$ and $\mathbb{C}^2 \smallsetminus \{0\}$.

Consider the *canonical map* from $\mathbb{C}^2 \smallsetminus \{0\}$ to the projective line $\mathbb{C}P^1$ that associates with each point $(x, y) \neq (0, 0)$ different from the origin, the line $\{(tx, ty) : t \in \mathbb{C}\}$ passing through this point. The graph of this map is a complex 2-dimensional surface in the complex 3-dimensional manifold (the Cartesian product) $\mathbb{C}^2 \times \mathbb{C}P^1$, which is not closed. To obtain the closure, one has to add the *exceptional curve* $S = \{0\} \times \mathbb{C}P^1 \subset \mathbb{C}^2 \times \mathbb{C}P^1$. The result is a non-singular surface $C \subseteq \mathbb{C}^2 \times \mathbb{C}P^1$ with the compact curve (Riemann sphere) $\mathbb{C}P^1 \simeq S \subseteq C$ on it. Projection $\mathbb{C}^2 \times \mathbb{C}P^1 \to \mathbb{C}^2$ on the first component, after restriction on C becomes a holomorphic map

$$\sigma : C \to \mathbb{C}^2, \qquad \sigma(S) = \{0\} \in \mathbb{C}^2,$$

which is by construction one-to-one between $C \smallsetminus S$ and $\mathbb{C}^2 \smallsetminus \{0\}$.

Definition 8.4. The map $\sigma : C \to \mathbb{C}^2$ is called the (standard) monoidal map. The curve $S \subset C$ is referred to as the (standard) *exceptional divisor*. The inverse map $\sigma^{-1} : \mathbb{C}^2 \smallsetminus \{0\} \to C$ is called the (standard) blow-up map, or simply the blow-up. Less frequently used term is *blow-down* for the map σ.

To see why C is a nonsingular manifold (and justify the assertions on the closure and smoothness), we produce a convenient ("standard") atlas on C. Let z, w be two affine charts on the Riemann sphere $\mathbb{C}P^1$, which take the line passing through the point $(x, y) \neq (0, 0)$ into the numbers $z = y/x$ and $w = x/y$ respectively: by construction, $w = 1/z$. These charts induce two affine charts in the respective domains V_1, V_2 on the Cartesian product $\mathbb{C}^2 \times \mathbb{C}P^1$. In these charts the graph of the canonical map is given by the equations

$$y - xz = 0, \qquad \text{resp.,} \qquad x - wy = 0, \qquad (x, y) \neq (0, 0).$$

The surfaces defined by these equations, clearly remain nonsingular after extension on the line $\{x = 0,\ y = 0\} \subseteq \mathbb{C}^3$. Moreover, the functions (x, z) in the chart V_1 and (y, w) in chart V_2 respectively, become two coordinate systems (charts) on C, defined in the two domains $U_i = C \cap V_i$, $i = 1, 2$. The transition map between these charts is a monomial transformation

$$y = zx, \quad w = 1/z, \qquad \text{and reciprocally,} \qquad x = wy, \quad z = 1/w. \tag{8.2}$$

It remains to observe that the map $\sigma : C \to \mathbb{C}^2$ in these charts is polynomial, hence holomorphic: $\sigma|_{U_i} = \sigma_i$, $i = 1, 2$, where

$$\sigma_1 : (x, z) \mapsto (x, xz), \qquad \text{resp.,} \qquad \sigma_2 : (y, w) \mapsto (yw, y). \tag{8.3}$$

The exceptional divisor S in the respective charts is given by the equations

$$S \cap U_1 = \{x = 0\}, \qquad \text{resp.,} \qquad S \cap U_2 = \{y = 0\}.$$

Remark 8.5. The formulas (8.2) and (8.3) are *real* algebraic, thus defining at the same time the real counterpart of the above construction. The real projective line $\mathbb{R}P^1$ is diffeomorphic to the circle $\mathbb{S}^1$, so the surface ${}^{\mathbb{R}}C$ is constructed as a submanifold of the cylinder $\mathbb{R}^2 \times \mathbb{S}^1$. This submanifold is homeomorphic to the Möbius band.

Picture or ref.?

Remark 8.6. Nontriviality of the construction becomes even more striking in the complex domain. Note that the exceptional divisor cannot be *globally* defined by a single equation $\{f = 0\}$ with a function f holomorphic on C near S. Indeed, if such function exists, it would uniquely define a function $f \circ \sigma^{-1}$ everywhere in $(\mathbb{C}^2, 0)$. This function is holomorphic and nonvanishing outside the origin and, since the point has codimension 2 in $\mathbb{C}^2$, f extends holomorphically at the origin. But the zero locus of a holomorphic function cannot have codimension 2—contradiction.

Moreover, similar arguments show that S is *exceptional* in the following sense: it cannot be deformed inside C, since in a sufficiently small neighborhood (C, S) there are simply no submanifolds other than S. Indeed, since S is compact, such manifold S' should necessarily also be compact, hence the image $\sigma(S')$ of any such manifold should be a compact subset of $(\mathbb{C}^2, 0)$. This is impossible unless this image is a point. Since σ is one-to-one outside the origin, the only remaining possibility is $\sigma(S') = \{0\}$, i.e., $S' = S$.

Remark 8.7. These properties of the map $\sigma\colon (C, S) \to (\mathbb{C}^2, 0)$ may seem to be caused by the artificial construction. However, one can prove that *any* holomorphic map $\sigma'\colon (C', S') \to (\mathbb{C}^2, 0)$ defined in a neighborhood of a compact holomorphic curve S', mapping it to a single point while being one-to-one on the complement, is necessarily equivalent to the standard monoidal map σ under the sole assumption that the curve S' is irreducible. (Without this condition σ' can be equivalent to a *composition* of several monoidal maps).

The above equivalence means that there exists a biholomorphic map $H\colon (C, S) \to (C', S')$ such that $\sigma = \sigma' \circ H$. In particular, the construction does not depend on the choice of the local coordinates (x, y) near the origin. The proof of these facts in the algebraic category can be found in [**Sha94**, Chapter IV, §3.4].

Proof: lift σ' to a map $\rho\colon (C', S') \to (C, S)$ and prove its regularity (requires some work). It must map S' to S and be non-constant hence one-to-one as well. A holomorphic map which is one-to-one, has a holomorphic inverse.

Using the local model provided by the standard monoidal transformation σ, we can construct a global map blowing up any finite set of points Σ on any two-dimensional complex manifold (surface) M.

Proposition 8.8. *There exists a holomorphic surface M' and holomorphic map $\pi\colon M' \to M$ such that the preimage of any point from Σ is a Riemann sphere $S_p = \pi^{-1}(p) \simeq \mathbb{C}P^1$ whereas π is one-to-one between $M' \smallsetminus \bigcup_{p \in \Sigma} S_p$ and $M \smallsetminus \Sigma$.*

The surface M' and the map π are unique modulo a biholomorphic isomorphism and the right equivalence respectively. As follows from Remark 8.7, the requirement that S_p are biholomorphically equivalent to the Riemann sphere, can be relaxed to a mere irreducibility.

The inverse map $\pi^{-1}: M \smallsetminus \Sigma \to M'$ is called the *simple blow-up* of the locus (finite point set) Σ. The map π itself is sometimes called a simple blow-down.

Proof. Construction of the map π from local monoidal transformations is tautological in the class of abstract manifolds. Consider an atlas of charts $\{U_\alpha\}$ on M including special charts U_p identifying neighborhoods of each point $p \in \Sigma$ with a neighborhood $(\mathbb{C}^2, 0)$ of the origin. Without loss of generality we can assume that all other charts do not intersect the locus Σ. The manifold M can be then described as the quotient space of the disjoint union, $M = \bigsqcup_\alpha U_\alpha / \sim$ by the equivalence relationship $\sim$ (images of the same points in different charts are identified). The manifold M' in these terms can be described as follows. Replace each special chart U_p by the neighborhood $U'_p = (C, S)_p$, different for different singular points p, and consider again the disjoint union $\bigsqcup_\alpha U'_\alpha$ with $U'_\alpha = U_\alpha$ when the chart does not intersect Σ. The equivalence relationship $\sim$ lifts to an equivalence relationship $\sim'$ on the new disjoint union (all non-singular points have unique preimages in U'_α). The quotient space $M' = \bigsqcup_\alpha U'_\alpha / \sim'$ by construction is a manifold. There are natural holomorphic maps $\pi: U'_\alpha \to U_\alpha$ which coincide with the monoidal map σ if the chart U_α was special, and identical otherwise. Clearly these maps agree with the equivalences $\sim, \sim'$ and hence define a holomorphic map $\pi: M' \to M$ with the required local properties. □

8.3. Blow-up of analytic curves and singular foliations. As any holomorphic map, the standard monoidal map $\sigma: (C, S) \to (\mathbb{C}^2, 0)$ carries holomorphic functions and forms (by pullback) and analytic subsets (by preimages) from $(\mathbb{C}^2, 0)$ to the surface C. However, these results usually are very degenerate on the exceptional divisor S.

The alternative is to carry the objects from the *punctured* plane $\mathbb{C}^2 \smallsetminus \{0\}$ to the *complement* $C \smallsetminus S$ of the exceptional divisor, and then *extend* them in one or another way on S. The result is called the *blow-up* (*desingularization*) of the initial object.

8.3.1. *Blow-up of analytic curves.* Recall that σ^{-1} is a well defined holomorphic map of $\mathbb{C}^2 \smallsetminus \{0\}$ to C.

Definition 8.9. The *blow-up* of an analytic curve $\gamma \subseteq (\mathbb{C}^2, 0)$ is the closure

$$\widetilde{\gamma} = \overline{\sigma^{-1}(\gamma \smallsetminus \{0\})} \tag{8.4}$$

in C of the preimage of the *punctured* curve $\gamma \smallsetminus \{0\}$.

Proposition 8.10. *The blow-up of any analytic curve is again an analytic curve in (C, S) intersecting the exceptional divisor only at isolated points.*

Proof. The equation of the blow-up in C is obtained by pulling back the equation of γ and cancelling out all terms vanishing identically on S. However, because of the special properties of S in C (see Remark 8.6), it can be done only locally.

Consider any holomorphic germ f defining γ and its pullback $f' = \sigma^* f = f \circ \sigma \in \mathcal{O}(C)$. For each point $a \in S$ the germ of f' in the local ring $\mathcal{O}(C, a)$ vanishes identically on S and can be divided by the maximal power g^ν, $\nu \geqslant 1$, of the function g which defines a local equation of $S = \{g = 0\}$ near a. After division we obtain the germ $\widetilde{f} = g^{-\nu} f$ with the following properties:

(1) outside S the loci $\sigma^{-1}(\gamma) = \{f' = 0\}$ and $\widetilde{\gamma} = \{\widetilde{f} = 0\}$ coincide, since g is invertible off S, and

(2) the locus $\widetilde{\gamma}$ is a closed analytic curve which intersects S only at the point a locally in near this point.

Thus the analytic curve $\widetilde{\gamma}$ is the closure of $\sigma^{-1}(\gamma) \smallsetminus S$. □

The blow-up can be described as the smallest analytic curve in C such that $\sigma(\widetilde{\gamma}) = \gamma$. Note that in general this curve can be non-connected.

8.3.2. *Blow-up of foliations.* Let $\mathcal{F}$ be a singular holomorphic foliation of $(\mathbb{C}^2, 0)$. By definition, this means that $\mathcal{F}$ is a non-singular holomorphic foliation of the punctured neighborhood $(\mathbb{C}^2, 0) \smallsetminus \{0\}$. Its preimage $\sigma^{-1}(\mathcal{F})$ is a nonsingular foliation of $C \smallsetminus S$.

Definition 8.11. The *blow-up of a singular foliation* $\mathcal{F}$ of $(\mathbb{C}^2, 0)$ is the singular holomorphic foliation $\widetilde{\mathcal{F}}$ of C extending the preimage foliation $\sigma^{-1}(\mathcal{F})$ of $C \smallsetminus S$.

The *existence* of such extension has to be proved: by definition, a singular holomorphic foliation may have only isolated singularities, so $\sigma^{-1}(\mathcal{F})$ has to be extended as a nonsingular foliation onto the most part of S.

Proposition 8.12. *There exists a finite set $\Sigma \subset S$ and a holomorphic foliation $\widetilde{\mathcal{F}}$ with the singular locus Σ, whose restriction on $C \smallsetminus S$ coincides with $\sigma^{-1}(\mathcal{F})$.*

Proof. By Theorem 2.16, the foliation $\mathcal{F}$ can be defined by a holomorphic line field (distribution) $\{\omega = 0\}$ with a suitable holomorphic 1-form $\omega \in \Lambda^1(\mathbb{C}^2, 0)$ having an isolated singular point at the origin (recall that we are dealing with the two-dimensional case).

The pullback $\sigma^*\omega \in \Lambda^1(C, S)$ is a holomorphic 1-form which vanishes identically on S and defines the foliation $\sigma^{-1}(\mathcal{F})$ outside S. By Theorem 2.14 and Remark 2.15, the foliation $\sigma^{-1}(\mathcal{F})$ can be extended on S except for a set Σ of codimension 2 which consists of finitely many points since S is compact. To do this, one should near each point $a \in S$ divide $\sigma^*\omega$ by the maximal power g^ν, $\nu \geqslant 1$, of the germ g locally defining S near a. As before, in the case of algebraic curves, the construction cannot be done globally because of special properties of S (Remark 8.6). □

Remark 8.13. One may have two apriori possibilities for the extended foliation $\widetilde{\mathcal{F}}$: either the exceptional divisor S, after deleting the singular locus $\Sigma \subset S$, is a single leaf of $\widetilde{\mathcal{F}}$, or different points of S belong to different leaves of the latter foliation, which cross S transversally at all points, eventually except for finitely many *tangency points*.

It will be shown that the first opportunity occurs generically whereas the second situation corresponds to certain degeneracy of the singularity which in such case will be called *dicritical*. The accurate description of this phenomena will be given later in Definition 8.16.

The previous arguments can be carried out *verbatim* for any holomorphic non-constant map $\pi\colon (M, D) \to (\mathbb{C}^2, 0)$ squeezing a holomorphic curve $D = \pi^{-1}(0)$ (eventually, singular or reducible) into the single point at the origin and one-to-one between $M \smallsetminus D$ and $(\mathbb{C}^2, 0) \smallsetminus \{0\}$. Any holomorphic foliation $\mathcal{F}$ on $(\mathbb{C}^2, 0)$ can be pulled back as a foliation $\pi^{-1}(\mathcal{F})$ on $M \smallsetminus D$ and then extended on D everywhere except for finitely many points. The resulting singular foliation on M will be denoted by $\pi^*\mathcal{F}$ and referred to as a *desingularization*, or *blow-up* of $\mathcal{F}$ by the map π.

8.4. Desingularization theorem. It turns out that singular points of *any* holomorphic foliation can be completely simplified by iterated blow-ups. The following result was first discovered by Ivar Bendixson [**Ben01**] in 1901 and improved and generalized by S. Lefschetz [**Lef56, Lef68**], A. F. Andreev [**And62, And65a, And65b**] and A. Seidenberg [**Sei68**]. A. van den Essen simplified the proof considerably [**vdE79**], see also [**MM80**]. In [**Dum77**] F. Dumortier obtained a generalization of this theorem for smooth rather than analytic foliations and showed that tangencies can also be eliminated. Recently O. Kleban in [**Kle95**] computed the number of iterates of simple blow-ups required to *desingularize* completely an isolated singularity of a holomorphic foliation.

Recall that a singularity of the foliation $\mathcal{F}$ defined by the Pfaffian equation $\omega = 0$, $\omega = f\,dx + g\,dy$ with the coefficients $f, g \in \mathcal{O}(\mathbb{C}^2, 0)$ without common factors, is *elementary* if the linearization matrix $A = \partial F(0,0)/\partial(x,y)$ of the dual vector field $F = -g\frac{\partial}{\partial x} + f\frac{\partial}{\partial y}$ has at least one nonzero eigenvalue.

Theorem 8.14 (I. Bendixson–A. Andreev–A. Seidenberg–S. Lefschetz; F. Dumortier)**.** *For any singularity of a holomorphic foliation $\mathcal{F}$ one can construct a holomorphic surface M with an analytic curve D on it and a non-constant holomorphic map $\pi\colon (M, D) \to (\mathbb{C}^2, 0)$, one-to-one between $M \smallsetminus D$ and $(\mathbb{C}^2, 0) \smallsetminus \{0\}$, such that the blow-up $\pi^*\mathcal{F}$ has only elementary singularities on D.*

More precisely, the map π resolving the singularity can be constructed as a composition of finitely many simple blow-downs.

The vanishing divisor $D = \pi^{-1}(0)$ is the union of finitely many projective planes intersecting transversally, $D = \bigcup D_j$, $D_j \simeq \mathbb{C}P^1$, $D_i \pitchfork D_j$.

In this section we give the *constructive* proof of this result, based on the idea of van den Essen [**vdE79, MM80**]. This idea is to introduce the *multiplicities* of isolated singularities of holomorphic foliations and monitor their decrease under blow-ups.

Detailed inspection of this algorithm yields the following estimate for the complexity of the desingularization map. It is formulated in terms of *multiplicity* of a singular point of holomorphic foliation, which will be introduced in §8.7–§8.9

Theorem 8.15. *Thee number of simple blow-ups required to resolve an isolated singularity of multiplicity μ, does not exceed $2\mu + 1$.*

A stronger result was proved by O. Kleban [**Kle95**]. One can not only achieve elementarity of all isolated singularities, but also eliminate all *tangency points* between the foliation $\pi^*\mathcal{F}$ and the vanishing divisor D, using the smaller number of simple blow-ups, no more than $\mu + 2$.

8.5. Blow-up in an affine chart: computations. In this section we compute the standard blow-up of an isolated singularity of the line field $\{\omega = 0\}$ and describe two essentially different possible results.

Let $\omega = f\,dx + g\,dy$ be a holomorphic 1-form having an isolated singularity of order n. By definition, this means that the Taylor expansion of the coefficients f, g of this form begin with homogeneous polynomials f_n, g_n of degree n and at least one of these two homogeneous polynomials does not vanish identically.

Consider the pullback $\sigma^*\omega$ on the complex Möbius band C. In the coordinates (x, z) in the chart U_1 the map $\sigma_1\colon (x, z) \mapsto (x, xz)$ pulls back the form ω to $\omega_1 = \sigma_1^*\omega$ which has the structure

$$\begin{aligned} \omega_1 &= [f(x, xz) + zg(x, xz)]\,dx + xg(x, xz)\,dz \\ &= x^{-1}[(\sigma_1^* h)\,dx + (\sigma_1^* g')\,dz], \qquad (8.5) \\ h = xf + yg,& \qquad g' = x^2 g, \qquad h, g' \in \mathcal{O}(\mathbb{C}^2, 0). \end{aligned}$$

Both coefficients of the form ω_1 are divisible at least by x^n. However, the second coefficient is in fact divisible even by x^{n+1}. On the other hand, the first coefficient can *accidentally* be also divisible by x^{n+1}, if the homogeneous polynomial $h_{n+1} = xf_n + yg_n$ vanishes identically.

In order to extend the foliation $\sigma_1^{-1}(\mathcal{F})$ on the line $S = \{x = 0\}$ in the chart U_1, we have to divide by the coefficients of the form (8.5) by the *maximal possible* power of x so that the result will be not identically zero on S. Thus we have two cases.

Definition 8.16. The singularity is called *non-dicritical*, if

$$\operatorname{ord}_0(xf + yg) = 1 + \operatorname{ord}_0 \omega = \min(\operatorname{ord}_0 f, \operatorname{ord}_0 g), \tag{8.6}$$

and *dicritical*, if

$$\operatorname{ord}_0(xf + yg) > 1 + \operatorname{ord}_0 \omega. \tag{8.7}$$

The *homogeneous* polynomial $h_{n+1} = xf_n + yg_n$ of degree $n+1$ will play an important role in computations pertinent to the dicritical case. It will be referred to as the *tangent form* for lack of a better name. The roots of h_{n+1} can be identified with the points of the projective line $\mathbb{C}P^1$ *globally* isomorphic to the exceptional divisor S.

8.5.1. *Non-dicritical case.* In this case the blow-up of $\mathcal{F}$ is given by the Pfaffian equation

$$\widetilde{\omega}_1 = 0, \qquad \widetilde{\omega}_1 = [h_{n+1}(1, z) + x(\cdots)]\, dx + x[g_n(1, z) + x(\cdots)]\, dz, \tag{8.8}$$

where f_n, g_n and $h_{n+1} = xf_n + yg_n$ are the homogeneous bivariate polynomials from $\mathbb{C}[x, y]$ as above and the dots denote some holomorphic functions of x and z.

The line $S = \{x = 0\}$ is integral for the line field $\widetilde{\omega}_1 = 0$. In the language introduced later, the exceptional divisor S in the non-dicritical case is a *separatrix* of the blow-up foliation.

The singular locus Σ consists of the isolated roots of the equation

$$\Sigma = \{x = 0,\, z = z_j\}, \qquad h_{n+1}(1, z_j) = 0. \tag{8.9}$$

Their number (counted with multiplicities) is equal to $\deg_z h_{n+1}(1, z)$ which can be *less* than $n+1$ if the homogeneous polynomial $h_{n+1}(x, y)$ is divisible by x. In the latter case the point $z = \infty \in \mathbb{C}P^1$ is singular and should be studied in the second affine chart U_2 on C. Globally the singular locus $\Sigma \subset \mathbb{C}P^1$ is defined by the tangent form h_{n+1} as the *projective* locus in the homogeneous coordinates $\{(x : y) \in \mathbb{C}P^1 \colon h_{n+1}(x, y) = 0\}$. There is a simple sufficient condition guaranteeing that a point $a \in \Sigma$ is elementary.

Proposition 8.17. *Each* simple (*non-multiple*) *linear factor* $ax + by$ *of the tangent form* $h_{n+1} = xf_n + yg_n$ *corresponds to an elementary singularity* $z = -a/b$ (*resp.*, $w = -b/a$) *of the blow-up foliation.*

Proof. In the assumptions of the Proposition, the singularity is obviously non-dicritical and without loss of generality we may assume that the factor is simply y, and $h_{n+1}(1,z) = zu(z)$ and $u(0) = 1$.

The vector spanning the same line field as the form (8.8), is

$$\dot{z} = z + ax + O(2), \qquad \dot{x} = -bx + O(2),$$

where a, b are some two complex numbers and $O(2)$ denote functions of order $\geqslant 2$. The linearization matrix $\left(\begin{smallmatrix} 1 & * \\ 0 & * \end{smallmatrix}\right)$ of this field has nonzero eigenvalue 1 for the eigenvector tangent to S. □

8.5.2. *Dicritical case.* In this case $h_{n+1} \equiv 0$ and the Pfaffian form defining the blow-up foliation in the affine chart U_1 has the structure

$$\widetilde{\omega}_1 = 0, \qquad \widetilde{\omega}_1 = [h_{n+2}(1,z) + x(\cdots)]\,dx + [g_n(1,z) + x(\cdots)]\,dz. \quad (8.10)$$

Outside the null set $T = \{g_n(1,z) = 0\} \subset S$ the form $\widetilde{\omega}_1$ is nonsingular and *transversal* to S, which means that the leaves of the blow-up foliation cross S transversally. Note that $g_n \not\equiv 0$: otherwise the condition $h_{n+1} \equiv 0$ would mean that $f_n \equiv 0$ in violation of the assumption that the order of ω is n.

The points of T may correspond to either *tangency points* if $h_{n+2}(1,z)$ does not vanish (and hence the point is nonsingular), or singularities if both $g_n(1,z)$ and $h_{n+2}(1,z)$ vanish simultaneously there.

8.6. Divisors. We proceed with demonstration of the desingularization theorems. To that end, we first introduce a convenient algebraic formalism for counting analytic subvarieties (points and analytic hypersurfaces) with integer *multiplicities* (positive or negative). While this formalism cannot be easily extended for subvarieties of intermediate dimensions, in two dimensions the theory is as complete as possible.

Xref to §13 where radicals are discussed.

The integer multiplicity can be easily attached to analytic subvarieties of codimension one (hypersurfaces) using the fact that the ring of germs admits unique irreducible factorization. This construction leads to the notion of a *divisor* introduced and discussed in this section. Multiplicity of zero-dimensional sets (isolated points) can be introduced in a different way via codimension of the respective ideals as explained in §8.7 as the *intersection multiplicity* of two analytic curves. Behavior of these multiplicities under blow-up is studied in §8.8–§8.9.

8.6.1. *Definitions.* A *divisor* on a complex manifold M is a finite union of irreducible analytic hypersurfaces (analytic subsets of codimension 1) with assigned integer multiplicities (coefficients). By this definition, each divisor D is a formal sum $\sum_\gamma k_\gamma \gamma$ where the summation is formally over *all* irreducible subvarieties of codimension 1, but only finitely many integer coefficients $k_\gamma \in \mathbb{Z}$ can be in fact nonzero. Divisors form an Abelian group denoted by $\operatorname{Div}(M)$ with the operation denoted additively, $\sum k_\gamma\,\gamma + \sum k'_\gamma\,\gamma = \sum (k_\gamma + k'_\gamma)\,\gamma$. The divisor is called *effective* if all k_γ are nonnegative; any divisor can be formally represented as a *difference* of two effective divisors. The *support* of a divisor is the union of all subvarieties entering into D with nonzero coefficients,

$$|D| = \bigcup_{k_\gamma \neq 0} \gamma \simeq \sum_{k_\gamma \neq 0} \gamma,$$

which can be alternatively though of as either the point set or an effective divisor with all k_γ being just 0 or 1.

If M is one-dimensional, divisors are finite point sets with integer multiplicities attached to each point. We will be interested here in the two-dimensional case when M is a holomorphic surface and the divisors are unions of irreducible curves counted with multiplicities.

8.6.2. *Divisors and meromorphic functions.* Each holomorphic function $f \in \mathcal{O}(M)$ defines an effective divisor D_f called the *divisor of zeros* of f as follows. The support $|D_f|$ is the zero locus $Z_f = \{f = 0\} \subseteq M$. To assign the multiplicity $k_\gamma \geqslant 0$ to an irreducible component $\gamma \subseteq Z_f$, take an arbitrary point $a \in Z_f$ and consider the irreducible factorization of the germ of f at this point, $f = \prod f_j^{\nu_j}$. By irreducibility, each function f_j either vanishes identically on γ, or not vanish at all outside the point a. We assign to γ the multiplicity

$$k_\gamma = \sum_{f_j|_\gamma \equiv 0} \nu_j \in \mathbb{N}.$$

This definition depends formally on the point $a \in \gamma$ but the answer is obviously locally constant as a varies along γ. Since γ is connected, the result does not depend on a, moreover, one can always choose a being a smooth point on γ.

For a meromorphic function $h = f/g$ the divisor D_h is defined as the formal difference,

$$D_{f/g} = D_f - D_g.$$

It obviously does not depend on the choice of the representation.

Conversely, any divisor can be associated with a meromorphic function, albeit only locally. Let $D = \sum k_\gamma\,\gamma$ be an divisor on M. Then M can be covered by a union of charts $\{U_\alpha\}$ so that in each chart U_α each hypersurface $\gamma \subseteq |D|$ is represented by a holomorphic equation $\{f_{\alpha,\gamma} = 0\}$ with the differential $df_{\alpha,\gamma}$ nonvanishing outside a set of codimension 2 on γ. The divisor D locally in U_α is defined by the meromorphic function $f_\alpha = \prod_\gamma f_{\alpha,\gamma}^{k_\gamma}$. The collection $\{f_\alpha\}$ is called a *holomorphic cochain* defining the divisor D.

Note that there can be divisors not defined by a single *global* equation on M.

Let $\pi\colon M' \to M$ be a non-constant holomorphic map between two connected manifolds of the same dimension and $D = \sum k_\gamma \gamma$ a divisor on M defined by the meromorphic cochain $\{f_\alpha\}$.

Definition 8.18. The *preimage* (pullback) $\pi^{-1}(D)$ of a divisor $D \in \operatorname{Div}(M)$ is the divisor on M' which in the charts $U'_\alpha = \pi^{-1}(U)$ is defined by the meromorphic cochain $f'_\alpha = \pi^* f_\alpha \in \mathcal{M}(U'_\alpha)$.

Since π^* is a ring homomorphism, taking preimages commutes with addition/subtraction of divisors: for any two divisors D, D' on M,

$$\pi^{-1}(D \pm D') = \pi^{-1}(D) \pm \sigma^{-1}(D').$$

In other words, $\pi^{-1}\colon \operatorname{Div}(M) \to \operatorname{Div}(M')$ is a homomorphism of Abelian groups.

8.7. Intersection multiplicity and intersection index. In this section we define the multiplicity of intersection of two divisors (curves) at an isolated point and the global intersection index between divisors. More details they can be found in [**vdE79, MM80, Chi89**], [**AGV85**, §5] and in the algebraic context in [**Sha94**, Chapter IV].

Let D_f, D_g be two *effective* divisors in $U \subset \mathbb{C}^2$ defined by two *holomorphic* germs $f, g \in \mathcal{O}(\mathbb{C}^2, a)$ at a point $a \in U$. We say that their intersection is *isolated* at a, if $|D_f| \cap |D_g| \cap (\mathbb{C}^2, a) = \{a\}$ (in the sense of germs of analytic sets). The intersection is isolated if and only if no irreducible component enters both divisors with positive coefficient, i.e., f, g have no common irreducible factors in the ring of germs $\mathcal{O}(\mathbb{C}^2, a)$.

Definition 8.19. The *multiplicity of intersection* $D_f \overset{a}{\cdot} D_g$ of two *effective* divisors D_f and D_g at a point $a \in U \subseteq \mathbb{C}^2$ is the codimension of the ideal $I = \langle f, g \rangle$ in the ring $\mathcal{O}(\mathbb{C}^2, a)$, i.e., dimension of the quotient *local algebra* $Q_{f,g} = \mathcal{O}(\mathbb{C}^2, a) / \langle f, g \rangle$:

$$D_f \overset{a}{\cdot} D_g = \dim_{\mathbb{C}} Q_{f,g}, \qquad Q_{f,g} = \mathcal{O}(\mathbb{C}^2, a) / \langle f, g \rangle . \tag{8.11}$$

By definition, the equality $\dim Q_{f,g} = \mu < +\infty$ means that there exist the germs $e_1, \dots, e_\mu$ which are a basis of the local algebra: any other germ $u \in \mathcal{O}(\mathbb{C}^2, a)$ admits the representation

$$u = \sum_1^\mu c_i\, e_i + af + bg, \qquad c_1, \dots, c_\mu \in \mathbb{C}, \quad a, b \in \mathcal{O}(\mathbb{C}^2, a), \tag{8.12}$$

and the constant coefficients c_i are defined uniquely.

According to this definition, the multiplicity of intersection depends only on the ideal $\langle f, g\rangle$ and is equal to zero if one of the divisors is empty (zero in the additive language).

Consider now the general case of divisors on an arbitrary surface M. Two divisors D, D' on M are said to have *isolated intersection*, if $|D| \cap |D'|$ is a finite point set.

Definition 8.20. The *intersection index* between two divisors D, D' with isolated intersection is the sum of intersection multiplicities at all points of M:

$$D \cdot D' = \sum_{a \in M} D \overset{a}{\cdot} D', \qquad \text{if } |D| \cap |D'| \text{ is a finite set.} \tag{8.13}$$

The summation in (8.13) is formally extended over all points in M, but only points from $|D| \cap |D'|$ may contribute nonzero terms.

This definition for the moment makes sense only for effective divisors with nonnegative coefficients. In a moment we will extend the definition of intersection multiplicity for *all* divisors with isolated intersection. Then Definition 8.20 will make sense without the nonnegativity assumption.

The intersection multiplicity as defined by (8.11), is a generalization for the number of geometrically different points of intersection between two curves in a generic position.

Theorem 8.21 ([**AGV85**]).

1. The intersection multiplicity at a point is finite if and only if the intersection is isolated at this point.

2. If D_f, D_g are two divisors in $(\mathbb{C}^2, 0)$ and $U \subseteq (\mathbb{C}^2, 0)$ is a neighborhood of the origin such that $|D_f| \cap |D_g| \cap U = \{0\}$, then for all sufficiently small $\varepsilon, \delta \in \mathbb{C}$,

$$D_{f-\varepsilon} \cdot D_{g-\delta} = D_f \overset{0}{\cdot} D_g \qquad \text{in } U. \tag{8.14}$$

Corollary 8.22. *If D_f, D_g have isolated intersection and $|D_f| \cap |D_g| \cap U = \{0\}$, then the multiplicity intersection $D_f \overset{0}{\cdot} D_g$ at the origin is equal to the number of geometrically different points in the locus $\{f = \varepsilon,\ g = \delta\} \cap U$, provided that ε, δ are sufficiently small and all these intersections are nondegenerate.*

Remark 8.23. By the Sard theorem, for every ε the assumptions of the Corollary hold for almost all (in the Lebesgue measure sense) small δ.

Proof of the Corollary. Elementary arguments show that $\langle f, g\rangle = \mathfrak{m} = \langle x, y\rangle$ if and only if both $df(0)$ and $dg(0)$ are nonzero and $df(0) \wedge dg(0) \neq 0$, which means that the curves are smooth and intersecting transversally. By the second assertion of Theorem 8.21, $D_f \cdot D_g = \sum_{a \in |D_f| \cap |D_g|} 1$. □

Another Corollary to Theorem 8.21 provides a convenient tool for computation of the intersection multiplicity.

The geometric description provided by the Corollary 8.22 can and *should* be used as a *definition* of μ instead of the algebraic definition. The former is convenient for all purposes except for proving that the multiplicity in fact depends on the *ideal* rather than two equations. But we do not need that anywhere except the self-consistency of the multiplicity of a form (independence of the choice of local coordinates) which can be immediately "geometrized". I plan to rewrite the section for the final draft.

Corollary 8.24. *Assume that an injective non-constant map* $\tau\colon(\mathbb{C}^1,0)\to(\mathbb{C}^2,0)$ parameterizes *the curve* $\{f=0\}\subseteq(\mathbb{C}^2,0)$, *i.e.,* $0\equiv f\circ\tau\in\mathcal{O}(\mathbb{C}^1,0)$.

Then the intersection of D_f *and another effective divisor* D_g *is isolated if and only if the germ* $g\circ\tau$ *is not identically zero, and the multiplicity* $D_f\overset{0}{\cdot}D_g$ *of this intersection is equal to the order* $\operatorname{ord}_0 f\circ\tau$.

Proof. By Corollary 8.22, $D_f\overset{0}{\cdot}D_g$ is the number of points in the set $\{f=0,\ g=\varepsilon\}$, where ε is a sufficiently small "generic" complex number. These points are τ-parameterized by the small roots of the holomorphic function of one variable $(g-\varepsilon)\circ\tau=g\circ\tau-\varepsilon$ which is a small perturbation of the function $g\circ\tau$. It remains to observe that a small perturbation of a germ of order ν in $\mathcal{O}(\mathbb{C}^1,0)$ is a function that has exactly ν roots in a sufficiently small neighborhood of the origin. □

Proposition 8.25. *For any three* effective *divisors* D,D',D'' *on any surface* M, *such that* $D\cap(|D'|\cup|D''|)$ *is a finite point set,*

$$D\cdot(D'+D'')=D\cdot D'+D\cdot D''. \tag{8.15}$$

Proof. Clearly, it is sufficient to prove this Proposition for an arbitrary point $a\in|D|\cap|D'+D''|$ in the intersection of the effective divisors $D=D_h$ and $D'+D''=D_{fg}=D_f+D_g$.

Assume for simplicity that $a=0$ and let h,f,g be three holomorphic germs defining the divisors D,D',D'' respectively in $(\mathbb{C}^2,0)$. The Proposition would follow from the identity

$$\dim Q_{h,fg}=\dim Q_{h,f}+\dim Q_{h,g} \tag{8.16}$$

provided that the germs h and fg have no common irreducible factors (note that f and g need not be mutually prime). It remains to prove (8.16).

Let $e_1,\dots,e_\mu$ be the basis of the local algebra $Q_{h,f}$ and $e'_1,\dots,e'_\nu$ the basis of $Q_{h,g}$. By their choice, any germ u can be represented as follows,

$$u=\sum_1^\mu c_ie_i+af+bh.$$

Furthermore, the germ a also admits a representation

$$a=\sum_1^\nu c'_je'_j+a'g+b'h.$$

Substituting it into the previous expansion, we conclude that $\mu+\nu$ germs $e_1,\dots,e_\mu,fe'_1,\dots,fe'_\nu$ form the basis of the quotient algebra $Q_{h,fg}$.

To show that they are linear independent, assume that some linear combination is in $\langle fg, h\rangle$, i.e., $\sum_1^\mu c_i e_i + f\sum_1^\nu c_j' e_j' = afg + bh$. Then $\sum_1^\mu c_i e_i = f(ag - \sum c_j' e_j') + bh \in \langle f, h\rangle$, which is possible only if $c_1 = \cdots = c_\mu = 0$, as e_i are linear independent over $\langle f, h\rangle$.

The germ f has no common irreducible factors with h (otherwise the multiplicity would be infinite), hence is not zero divisor modulo $\langle h\rangle$. Therefore the equality $f(ag - \sum_1^\nu c_j' e_j') = bh$ is possible if only if $\sum c_j' e_j' \in \langle g, h\rangle$, which is again possible only if $c_1' = \cdots = c_\nu' = 0$ for the same reasons as before. □

Using the geometric definition of deformations, the proof is much more obvious: one should consider the reducible deformation $(f-\varepsilon)(g-\delta)$ and shift h so that its values on the cross points of the locus are all nonzero.

Using Proposition 8.25, one can extend the intersection index to all divisors on M, including non-effective ones, by (bi)linearity and symmetry. Any divisor can be written as the difference $D' - D''$ of two effective divisors, and we *define* the intersection index between D and $D' - D''$ by the rule

$$D \cdot (D' - D'') = D \cdot D' - D \cdot D''. \tag{8.17}$$

The result of such extension is a bilinear (over $\mathbb{Z}$) symmetric form $\operatorname{Div}(M) \to \operatorname{Div}(M) \to \mathbb{Z}$, also called *intersection index*, defined on pairs of divisors with isolated intersection,

$$\begin{gathered} D, D' \longmapsto D \cdot D', \qquad \text{when } |D| \cap |D'| \text{ is finite set}, \\ D \cdot (D' \pm D'') = D \cdot D' \pm D \cdot D'', \qquad (D, D') = (D', D). \end{gathered} \tag{8.18}$$

8.8. Blow-up and intersection index. The intersection index is well-defined and invariant by *biholomorphisms*: if $\pi\colon M' \to M$ is a biholomorphism, then

$$\begin{gathered} \pi^{-1}(D) \cdot \pi^{-1}(D') = D \cdot D', \\ D, D' \in \operatorname{Div}(M), \quad \pi^{-1}(D), \pi^{-1}(D') \in \operatorname{Div}(M') \end{gathered} \tag{8.19}$$

for any two divisors D, D' on M with an isolated intersection. However, if σ is a *blow-up* then the preimage of the point $\{0\}$ is the exceptional divisor which therefore belongs to preimage of *any* divisor. Hence $\sigma^{-1}(D)$ and $\sigma^{-1}(D')$ necessarily have non-isolated intersection even if $|D| \cap |D'| = \{0\}$: this intersection always contains the exceptional divisor S with a positive multiplicity if D, D' were effective.

One can attempt to *extend* the intersection form on pairs of divisors $R, R' \in \operatorname{Div}(C)$ which have no *non-exceptional* common components, i.e., when

$$|R| \cap |R'| \subseteq S. \tag{8.20}$$

Formally any such extension can be achieved by setting *arbitrarily* the *self-intersection index* $S \cdot S$: then any other pair of divisors meeting the requirement (8.20) will be uniquely assigned the intersection index $R \cdot R'$ by bilinearity.

Yet there is a natural condition which makes only one extension natural. Theorem 8.21 can be interpreted as the *local continuity* of the intersection index: a small perturbation of divisors does not change the intersection index (while multiplicities of particular intersection points may of course change).

If the desired extension of the intersection index were to possess the same continuity, then preimages $\sigma^{-1}(D), \sigma^{-1}(D')$ of any two divisors $D, D' \in \operatorname{Div}(\mathbb{C}^2, 0)$ should intersect with the same multiplicity as the divisors themselves completely similarly to (8.19). Indeed, by a small perturbation one can always move the divisors D, D' off the origin while keeping their intersection index constant. But σ^{-1} is holomorphic off the origin, hence (8.19) applies.

Remark 8.26. Invariance of the intersection by σ, if postulated, would imply by bilinearity certain rather nontrivial intersection properties for the exceptional divisor S, the most unexpected of them the identity $S \cdot S = -1$ (cf. with Theorem 8.27). Note that the intersection multiplicity between any two analytic curves is always positive!

This identity *cannot* be obtained by small perturbation of S, since near S there are no other holomorphic curves (Remark 8.6).

Theorem 8.27. *The intersection form between divisors on C can be uniquely extended for pairs of divisors satisfying* (8.20) *as a symmetric bilinear form with the following properties,*

$$S \cdot S = -1, \tag{8.21}$$

$$\sigma^{-1}(D) \cdot S = \quad 0, \qquad \forall D \in \operatorname{Div}(\mathbb{C}^2, 0), \tag{8.22}$$

$$\sigma^{-1}(D) \cdot \sigma^{-1}(D') = D \cdot D', \qquad \forall D, D' \in \operatorname{Div}(\mathbb{C}^2, 0), \tag{8.23}$$

(*the last condition holds only for pairs of divisors $D, D' \in \operatorname{Div}(\mathbb{C}^2, 0)$ having isolated intersection*).

Proof. We need to prove that the rule (8.21) if adopted as an axiom and combined with bilinearity, would imply the identities (8.22) and (8.23) for arbitrary divisors $D, D' \in \operatorname{Div}(\mathbb{C}^2, 0)$. Because of the bilinearity and symmetry, it is sufficient to complete the proof when the divisor $D = D_f$ is an irreducible curve defined by an irreducible holomorphic germ $f \in \mathcal{O}(\mathbb{C}^2, 0)$ and "counted" with multiplicity 1.

Denote by $n = \operatorname{ord}_0 f$ the order of the holomorphic germ $f = f_n + f_{n+1} + \cdots$. Without loss of generality we may assume that the principal homogeneous part f_n is *not* divisible by x, so that $f_n(x, y) = cy^n + \cdots$, $c \neq 0$ (otherwise an affine change of coordinates should be first made). In the chart U_1 we have

$$\begin{aligned}\sigma_1^* f(x,z) &= x^n f_n(1,z) + x^{n+1}(1,z) + \cdots = x^n[f_n(1,z) + x f_{n+1} + \cdots] \\ &= x^n \widetilde{f}(x,z), \qquad \widetilde{f}(0,z) = f_n(1,z) \not\equiv 0,\end{aligned}$$

so that by definition of the preimage of divisors,

$$\sigma^{-1}(D_f) = nS + \widetilde{D}_f, \qquad \widetilde{D}_f = D_{\widetilde{f}}, \quad n = \operatorname{ord}_0 f. \tag{8.24}$$

As a curve, $|\widetilde{D}_f|$ is the blow-up of the curve $|D_f|$, since the function $\widetilde{f}$ does not vanish identically on S.

The intersection between $\widetilde{D}_f$ and S is isolated and consists of the roots of the polynomial $f_n(1,z)$ of degree exactly n. If $a = (0, a')$ is such a point, then the multiplicity of intersection $\widetilde{D}_f \overset{a}{\cdot} S$ at this point is equal to the multiplicity of the root of $f_n(1,z)$ at $z = a' \in \mathbb{C}$, since $\widetilde{f}(x,z) = f_n(1,z) \bmod \langle x\rangle$ and the quotient rings $\mathcal{O}(\mathbb{C}^2, a)/\langle x, \widetilde{f}\rangle$ and $\mathcal{O}(\mathbb{C}^1, a')/\langle f_n(1,\cdot)\rangle$ are naturally isomorphic. Adding the contributions of all points together, we obtain

$$\widetilde{D}_f \cdot S = \deg_z f_n(1,z) = \operatorname{ord} f = n. \tag{8.25}$$

Using the axiom (8.21), we obtain from (8.24) by linearity

$$\sigma^{-1}(D_f) \cdot S = (-1) \cdot n + \widetilde{D}_f \cdot S = -n + n = 0.$$

The proof of (8.22) is complete (in fact, we did not use the fact that D is irreducible).

To prove (8.23) we use the fact that an irreducible analytic curve $D = D_f$ can be parameterized in the following sense, see [**Chi89**]. There exists an *injective* holomorphic map $\tau\colon (\mathbb{C}^1, 0) \to (\mathbb{C}^2, 0)$, $t \mapsto \bigl(x(t), y(t)\bigr)$ such that $f \circ \tau \equiv 0$.

By Corollary 8.24, the intersection multiplicity $D_f \overset{0}{\cdot} D_g$ is equal to the multiplicity (order) $\operatorname{ord}_0 g \circ \tau$ of the root $t = 0$ of the composition $g \circ \tau$.

If $D_f = \gamma$ is an irreducible curve parameterized by τ, then the map $\widetilde{\tau}\colon t \mapsto \sigma^{-1} \circ \tau$, $t \neq 0$, parameterizes the points of $\sigma^{-1}(\gamma) \smallsetminus S$. It obviously extends holomorphically at the origin and becomes a map $\widetilde{\tau}\colon (\mathbb{C}^1, 0) \to C$ parameterizing the blow-up curve $\widetilde{D}_f = \widetilde{\gamma}$.

If $D' = D_g$ is an arbitrary divisor (reducible or not), then using Corollary 8.24 twice we obtain

$$\begin{aligned}D_g \cdot D_f &= \operatorname{ord}_0 g \circ \tau = \operatorname{ord}_0 g \circ \sigma \circ \sigma^{-1} \circ \tau = \operatorname{ord}_0(\sigma^* f) \circ \widetilde{\tau} \\ &= D_{\sigma^* g} \cdot \widetilde{D}_f = \sigma^{-1}(D_g) \cdot \widetilde{D}_f.\end{aligned}$$

Combining this with (8.24) and (8.22), we obtain

$$\begin{aligned}\sigma^{-1}(D_g)\cdot\sigma^{-1}(D_f) &= \sigma^{-1}(D_g)\cdot(nS+\widetilde{D}_f)\\ &= n\,\sigma^{-1}(D_g)\cdot S+\sigma^{-1}(D_g)\cdot\widetilde{D}_f\\ &= 0 + D_g\cdot D_f = D_g\cdot D_f.\end{aligned}$$

The proof of (8.23) is complete when D is irreducible. As was already mentioned, the proof in the general case follows from bilinearity of the intersection index. □

As a corollary to Theorem 8.27, we obtain a simple formula for the intersection index between *blow-up* of two analytic curves.

Corollary 8.28. *If $\gamma,\gamma'\subseteq(\mathbb{C}^2,0)$ are two holomorphic curves of orders m and m' at the origin respectively, and $\widetilde{\gamma},\widetilde{\gamma}'\subset(C,S)$ their blow-ups, then*

$$\gamma\cdot\gamma'=\widetilde{\gamma}\cdot\widetilde{\gamma}'-mm'. \tag{8.26}$$

Proof. By (8.24), on the level of divisors

$$\sigma^{-1}(\gamma)=mS+\widetilde{\gamma},\qquad \sigma^{-1}(\gamma')=m'S+\widetilde{\gamma}'.$$

Using bilinearity and the three rules (8.25), (8.21), (8.22) and (8.23), we achieve the proof. □

Remark 8.29. The formula $\widetilde{D}\cdot\widetilde{D}'=D\cdot D'-(\operatorname{ord}_0 D)(\operatorname{ord}_0 D')$ generalizing the assertion of the Corollary for arbitrary divisors, remains true in this broader context if the order of a divisor $D=\sum k_\gamma\,\gamma$ is defined to be $\operatorname{ord}_0 D=\sum k_\gamma\operatorname{ord}_0\gamma$ and the blow-up $\widetilde{D}$ is defined by the formula $\widetilde{D}=\sum k_\gamma\,\widetilde{\gamma}$.

Example 8.30. The calculus of intersection indices can replace direct computation of blow-ups.

For instance, if $\gamma=\{f=0\}\subset(\mathbb{C}^2,0)$ is smooth, then its order is 1 and by (8.24) the blow-up intersects the exceptional divisor with multiplicity 1. In particular, $\widetilde{\gamma}$ is smooth and *transversally* intersects S. The map σ restricted on $\widetilde{\gamma}$, is a biholomorphism between $\widetilde{\gamma}$ and γ. Clearly, the assertion can be globalized for any simple blow-up.

If γ,γ' are two smooth curves transversally intersecting at the origin, then by (8.26) their blow-ups intersect with multiplicity *zero*. This means that $\widetilde{\gamma}$ and $\widetilde{\gamma}'$ are disjoint. If γ,γ' are both smooth, then their intersection multiplicity decreases by 1 after blow-up. Since in the smooth case the intersection multiplicity is equal to the order of tangency between γ and γ' minus 1, *the order of tangency between smooth curves is also decreased by one by blow-up.*

8.9. Blow-up and multiplicity of singular foliations. Consider a singular holomorphic foliation $\mathcal{F}$ defined by the Pfaffian equation (line field) $\{\omega = 0\}$ near an isolated point at the origin. Denote by n the *order* of the form ω at the origin: by definition, it means that

$$\omega = f\,dx + g\,dy = (f_n + f_{n+1} + \cdots)\,dx + (g_n + g_{n+1} + \cdots)\,dy \qquad (8.27)$$

and the homogeneous polynomials f_n, g_n of lowest degree n do not vanish identically: $f_n\,dx + g_n\,dy \neq 0$. Without loss of generality we may assume that the origin is the isolated singularity of the form ω. In the language of divisors this means that the intersection of the divisors D_f and D_g is isolated.

Definition 8.31. The *multiplicity* $\mu_0(\omega)$ of the singular point of the form (8.27) at the origin is the intersection multiplicity $D_f \overset{0}{\bullet} D_g$ between the respective divisors.

The multiplicity $\mu_a(\mathcal{F})$ of a singular foliation $\mathcal{F}$ at a point a is the multiplicity of any holomorphic form ω tangent to $\mathcal{F}$ and having an isolated singular point at a.

By definition, multiplicities of *nonsingular* points are taken to be zero.

Proposition 8.32. *The multiplicity does not depend on the choice of local coordinates used for writing the coefficients of the form.*

Proof. By Theorem 8.21, the multiplicity is equal to the number of geometrically distinct singularities that appear by small perturbation of the form ω. Since this description is coordinate-free, so is the original definition.

An alternative argument is as follows: changing the coordinates results in replacing the coefficients (f, g) of the form by another tuple of functions (f', g') belonging to the same ideal $\langle f, g\rangle$. If the change of coordinates is invertible, the two ideals are equal and so are the local algebras. □

Our immediate goal is to compare the total multiplicity of all singularities of a foliation $\mathcal{F}$ and its blow-up $\widetilde{\mathcal{F}}$. Clearly, it is sufficient to consider the case when $\mathcal{F}$ has an isolated singularity on $(\mathbb{C}^2, 0)$ and the blow-up is the standard monoidal transformation σ.

The answer is different in the dicritical and non-dicritical cases. Consider the singular foliation $\mathcal{F}$ determined by 1-form $\omega = f\,dx + g\,dy$ of order n as in (8.27) and denote $\widetilde{\mathcal{F}}$ its blow-up as defined in Definition 8.11.

Theorem 8.33. *If $\mathcal{F}$ is a singular foliation on $(\mathbb{C}^2, 0)$ and $\widetilde{\mathcal{F}}$ its blow-up, then in all cases except the dicritical singularity of order* 1,

$$\sum_{a \in S} \mu_a(\widetilde{\mathcal{F}}) = \mu_0(\mathcal{F}) - k(k-2) + n. \qquad (8.28)$$

Here $n = \operatorname{ord}_0 \omega$, $m = \operatorname{ord}_0(xf + yg) \geqslant n+1$ (*with the equality occurring in the non-dicritical case*) *and*

$$k = \min(n+2, m) = \begin{cases} n+1, & \text{in the non-dicritical case,} \\ n+2, & \text{in the dicritical case.} \end{cases} \tag{8.29}$$

In the non-dicritical case the formula (8.28) implies

$$\sum_a \mu_a(\widetilde{\mathcal{F}}) = \mu_0(\mathcal{F}) - (n^2 - n - 1) = \begin{cases} \mu_0(\mathcal{F}) - 1, & \text{if } n = 2, \\ \mu_0(\mathcal{F}) + 1, & \text{if } n = 1. \end{cases} \tag{8.30}$$

In the dicritical case of order $n > 1$ the formula (8.28) yields

$$\sum_a \mu_a(\widetilde{\mathcal{F}}) = \mu_0(\mathcal{F}) - (n^2 + n). \tag{8.31}$$

In the dicritical case of order $n = 1$ we have $\mu_0(\mathcal{F}) = 1$ whereas the blow-up foliation $\widetilde{\mathcal{F}}$ is nonsingular, therefore

$$\sum_a \mu_a(\widetilde{\mathcal{F}}) = 0 = 1 - 1 = \mu_0(\mathcal{F}) - n^2. \tag{8.32}$$

Corollary 8.34. *If $n > 1$, then the total number of singularities of $\widetilde{\mathcal{F}}$ counted with their multiplicities, hence the multiplicity of every particular singularity, is strictly smaller than the multiplicity of the initial singularity,*

$$\sum_{a \in S} \mu_a(\widetilde{\mathcal{F}}) < \mu_0(\mathcal{F}). \quad \square \tag{8.33}$$

Proof of the Theorem 8.33. We start with a convenient choice of the affine chart to work in. Making an affine transformation if necessary, we will be able then to assume without loss of generality that this chart is the standard affine chart U_1 with the coordinates (x, z).

First, we can assume that the only point *not* covered by the affine chart, is nonsingular for the blow-up foliation $\widetilde{\mathcal{F}}$. In the non-dicritical case this is equivalent to assuming that the principal homogeneous part $h_{n+1} = xf_n + yg_n$ is not divisible by x.

Moreover, we can always assume in addition that the intersection of the divisors D_g and D_h is isolated: this happens if and only if g is *not divisible* by x.

The last assumption concerns the principal homogeneous part g_n of the coefficient g: we will assume that it is *not* divisible by x. Unlike the previous assumptions which can always be achieved by a suitable affine transformation, this last assumption can be achieved in all cases *except* for the dicritical case of order $n = 1$. In the latter case we always have $g_1(x, y) = x$ since the linear part of the corresponding vector field is a scalar matrix which remains scalar in any affine coordinates.

In the affine chart $U_1 \simeq \mathbb{C}^2$ with the coordinates (x, z) the pullback of the form ω was computed in (8.5). Technically it is more convenient to pull back the form $x\omega \in \Lambda^1(\mathbb{C}^2, 0)$: the fact that it has a non-isolated singularity does not matter, as the pullback will be in any case divided by a suitable power of x when extending on the exceptional divisor. The advantage is that the coefficients of the 1-*form* $\sigma_1^*(x\omega) = (\sigma_1^* h)\, dx + \sigma_1^*(x^2 g)\, dz$ are pullbacks of two *holomorphic germs* h and $g' = x^2 g$.

To extend the form $\sigma_1^*(x\omega)$ on the exceptional divisor $S = \{x = 0\}$, one has to divide the coefficients $\sigma_1^* h$ and $\sigma_1^* g'$ by the maximal positive power x^k of the function x which is the local (relative to the chart U_1) equation of the exceptional divisor. Depending on whether the initial singularity is dicritical or not, we have two possibilities for this maximal order k, given by (8.29). The intersection multiplicity between $x^{-k}\sigma_1^* h$ and $x^{-k}\sigma_1^* g'$ at any point on the line $x = 0$ will be then the multiplicity of the corresponding singularity of the blow-up foliation.

On the language of the divisors the total multiplicity of all singular points of $\widetilde{\mathcal{F}}$ on the exceptional divisor S reduces to computation of the intersection index between the divisors $\sigma^{-1}(D_h) - kS$ and $\sigma^{-1}(D_{x^2 g}) - kS = \sigma^{-1}(D_g) - (k-2)S$ in the open domain $U_1 \subset C$. However, by our assumption that the point not covered by U_1 is non-singular, we may extend the summation over all singular points on S using bilinearity and the rules established in Theorem 8.27:

$$\begin{aligned}\sum_a \mu_a(\widetilde{\mathcal{F}}) &= (\sigma^{-1}(D_h) - kS) \cdot (\sigma^{-1}(D_{x^2 g}) - kS) \\ &= (\sigma^{-1}(D_h) - kS) \cdot \bigl(\sigma^{-1}(D_g) - (k-2)S\bigr) \\ &= \sigma^{-1}(D_h) \cdot \sigma^{-1}(D_g) + k(k-2)\, S \cdot S \\ &= D_h \cdot D_g - k(k-2).\end{aligned} \tag{8.34}$$

It remains to compute the intersection index between two divisors in $(\mathbb{C}^2, 0)$ at the origin. Using the fact that it depends only on the ideal generated by these germs, we obtain

$$D_h \overset{0}{\cdot} D_g = D_{xf+yg} \overset{0}{\cdot} D_g = D_{xf} \overset{0}{\cdot} D_g = D_x \overset{0}{\cdot} D_g + D_f \overset{0}{\cdot} D_g.$$

The multiplicity of intersection $D_x \overset{0}{\cdot} D_g$ is equal to the order of the function $\operatorname{ord}_0 g(0, y)$. If g_n is not divisible by x, this order is equal to n, so that ultimately

$$D_h \overset{0}{\cdot} D_g = \mu_0(\mathcal{F}) + n, \qquad n = \operatorname{ord}_0 \mathcal{F}.$$

Putting everything together, we obtain the formula (8.28). $\square$

8.10. Desingularization of cuspidal points. Multiplicity of isolated singularities of order $n > 1$ goes down after blow-up (dicritical or not). To prove the desingularization theorems, we need to show that the only non-elementary points of order 1, the cuspidal points, can be desingularized in finitely many steps. Note that since the order of a cuspidal point is 1, the total multiplicity of all singularities which appear after blow-up (non-dicritical) goes up by 1 by (8.30). We will show that for cuspidal points the multiplicity decreases after *two* consecutive blow-ups if it was three or higher, whereas a cusp of multiplicity 2 after three blow-ups gets desingularized into elementary points.

Without loss of generality we may assume that the lower order terms of the form ω are brought to the normal form

$$\begin{gathered} \omega = y\,dy + [f(x) + yg(x)]\,dx, \qquad f, g \in \mathbb{C}[[x]], \\ \operatorname{ord}_0 f = \mu \geqslant 2, \qquad \operatorname{ord}_0 g > 0. \end{gathered} \tag{8.35}$$

(cf. with (4.12)). In fact, we need only terms of order 2 for the analysis below. The number $\mu \geqslant 2$ is the multiplicity of the singular point (8.35).

The tangent form $xf_1 + yg_1$ for (8.35) is equal to y^2. It is nonzero (hence the singularity is non-dicritical) and the only singular point after blow-up is the point $z = 0$ in the chart U_1, where the blow-up of ω takes the form

$$xz\,dz + (ax + bx^2 + cxz + z^2)\,dx + O(3), \tag{8.36}$$

where a, b are the leading coefficients of $f(x) = ax^2 + bx^3 + \cdots$ ($a \neq 0$ if and only if $\mu = 2$) and c the leading coefficient of $g(x) = cx + \cdots$. Here and below $O(k)$ means a holomorphic form of order $\geqslant k$.

Further arguments are different for *simple cusp* with $\mu = 0$ and *higher cusps* with $\mu > 2$.

8.10.1. *Simple cusp.* We show that after three consecutive blow-ups the simple cusp gives rise to three elementary singularities, of which two are non-degenerate and one a saddle-node of multiplicity 2.

If $\mu = 2$, then without loss of generality one may assume that $a = 1$. The order of the singularity (8.36) is again 1 so it is a simple cusp, its multiplicity by (8.30) is 3 and the tangent form is $x^2 \not\equiv 0$. After the *second* blow-up (substitution $x = uz$ and division by z) the cusp (8.36) is transformed into

$$uz\,dz + (u + z)(u\,dz + z\,du) + O(3). \tag{8.37}$$

having a unique singularity at $u = 0$. The order of this singularity is now 2 and multiplicity is equal to 4 again by (8.30).

The tangent form for (8.37), $uz^2 + 2uz(u + z) = uz(2u + 3z)$, is the product of three different (simple) linear factors which means that after the *third* blow-up the foliation will have three singular points of total multiplicity

4 (once again by (8.30)). This leaves only one combination of multiplicities $1, 1$ and 2 respectively. However, since all factors above are simple, all three singularities are elementary by Proposition 8.17. Desingularization of a simple cusp is complete.

8.10.2. *Higher cusp.* In this case already after the first blow-up the form (8.36) has order 2, multiplicity $\mu + 1$ by (8.30) and the tangent form $xz^2 + x(bx^2 + cxz + z^2) = x(bx^2 + cxz + 2z^2)$ which is divisible by x but not a power of x. In other words, after the *second* blow-up there will appear at least *two* distinct points (three if $c^2 \neq 8b$) of total multiplicity μ by (8.30). This means that each of these two points has multiplicity of the higher cusp goes down by 1 after *two* consecutive blow-ups.

Proof of Desingularization theorems 8.14 and 8.15. Now we can explicitly construct the sequence of blow-ups that would resolve completely an isolated singularity. In fact, the algorithm is very simple: starting from the initial singularity of a foliation $\mathcal{F} = \mathcal{F}_0$ at the origin $0 \in M_0 \simeq (\mathbb{C}^2, 0)$, one should construct simple blow-ups $\pi_k \colon M_k \to M_{k-1}$, $k = 1, 2, \dots,$ of all *non-elementary* singular points $\Sigma_{k-1} \subset M_{k-1}$ of the foliation $\mathcal{F}_{k-1}$ obtained on the previously constructed surface M_{k-1}.

The assertion on the vanishing divisor D (preimage of the origin) can be easily verified inductively. If $\gamma \subset M$ is a nonsingular curve biholomorphically equivalent to $\mathbb{C}P^1$ and $a \in \gamma$ a center of blow-up $\pi \colon M' \to M$, then by Example 8.30 the blow-up $\pi^*\gamma$ will be again a nonsingular curve $\widetilde{\gamma}$ biholomorphically equivalent to γ and therefore again equivalent to $\mathbb{C}P^1$ (note that the topology of embedding of $\widetilde{\gamma}$ in M' may change). If γ, γ' intersect transversally, then their blow-ups will be disjoint and both transversal to the exceptional divisor $\pi^{-1}(0) \subset M'$ created by π. Thus the assertion on the vanishing divisor reproduces itself inductively and holds at any moment.

To prove Theorem 8.15, it remains to estimate the number of simple blow-ups before the algorithm terminates, i.e., before all singularities become elementary. Note that all singularities appearing in the process, can be organized in a tree graph with branches connecting each singularity with its descendants appearing by the simple blow-up. Take the longest branch in this tree, $0 = a_0$, $a_1 \in \Sigma_1$, $a_2 \in \Sigma_2$ *etc.* We claim that, with the possible exception of the last three steps, the multiplicity of singularities a_i decreases at least by one every step or, at worst, every two steps. Denoting by μ_i the respective multiplicities, we already know that:

(1) if a_i is of order > 1 then $\mu_{i+1} < \mu_i$ by Corollary 8.34;
(2) if a_i is of order 1 and is neither elementary nor simple cusp, then $\mu_{i+2} < \mu_i$ by §8.10.2;

(3) if a_i is a simple cusp, then the branch terminates after three more steps by §8.10.1.

These inequalities constrain the length of the branch by $2(\mu-1)+3 = 2\mu+1$. The proof of Theorems 8.14 and 8.15 is complete. $\square$

8.11. Concluding remarks: elimination of resonant nodes and dicritical tangencies. Elementary singular points can be also to some extent simplified by blow-up. For instance, a nondegenerate singularity with the eigenvalues λ_1, λ_2, defined by the Pfaffian equation

$$x\,dy + \lambda y\,dx + \cdots = 0, \qquad \lambda = -\lambda_1/\lambda_2 \neq -1,$$

is “split” by the blow-up into two singularities which are both nondegenerate when $\lambda \neq -1$. The corresponding negative ratios of eigenvalues will be $\lambda+1$ and $(\lambda^{-1}+1)^{-1}$.

The case $\lambda = -1$ corresponds either to the dicritical node $x\,dy + y\,dx + \cdots = 0$ or to the Jordan node $(x+y)\,dy + y\,dx + \cdots = 0$. The former singularity *disappears* after blow-up, while the latter is produces an elementary singular point whose hyperbolic eigenspace is *transversal* to the exceptional divisor (the corresponding tangent form is y^2).

Combining these observations, one can make additional blow-ups on top of the desingularization achieved in Theorem 8.14 and *eliminate all resonant nodes with natural ratios of eigenvalues.* Indeed, such points correspond to negative natural values $\lambda = -n$ which can be increased by 1 in $n-1$ steps until the parameter λ reaches the threshold value $\lambda = -1$ (all other singularities appearing in the process will be resonant saddles with $\lambda = n/(n-1)$). On the next step the singularity either disappears or becomes a saddle-node.

In another development, one can refine the assertion of the Desingularization theorem 8.15 to *eliminate tangency points* between the foliation $\pi^*\mathcal{F}$ and the vanishing divisor D. We briefly outline here the required adjustments.

The tangency order between two smooth curves $\{f = 0\}$ and $\}g = 0\}$ is by definition the multiplicity of intersection $D_f \overset{a}{\cdot} D_g$ minus 1: if two curves intersect transversally, the tangency order is 0, for a true tangency it is always positive.

The *tangency order* between a foliation $\mathcal{F}$ defined by the Pfaffian equation $\omega = 0$ and a *smooth* analytic curve $\gamma = \{f = 0\}$ at a point a is defined only when γ is *not* a leaf or separatrix of $\mathcal{F}$.

If a is nonsingular for $\mathcal{F}$, then the tangency order $\tau_a(\mathcal{F}, \gamma)$ is by definition the tangency order between γ and the leaf of $\mathcal{F}$ passing through a. If γ is defined by the equation $\{f = 0\}$ locally near a, then one can easily verify that

$$\tau_a(\mathcal{F},\gamma) = D_{\omega\wedge df} \overset{a}{:} D_f, \tag{8.38}$$

where $D_{\omega\wedge df}$ is the divisor of zeros of the 2-form $\omega \wedge df = \rho(x,y)\,dx\wedge dy$ identified with its coefficient ρ, $D_{\omega\wedge df} = D_\rho$.

Indeed, if the tangency order is k then after choosing a suitable local coordinates one can assume that $\omega = dy$ (recall that a is non-singular) and $\gamma = \{f=0\}$, $f(x,y) = y - b(x)$, $\operatorname{ord}_0 b = k+1$. The expression in the right hand side of (8.38) will be then equal to the order of $\sigma(x,y) = db(x)/dx$ restricted on the smooth curve γ parameterized by x, i.e., to $k = \operatorname{ord}_0 b - 1$.

In the case when a is a singular point, one can use (8.38) as a *definition* of the tangency order. The important property of the tangency order thus defined, is the following one.

Proposition 8.35. *If a is a hyperbolic singular point of $\mathcal{F}$ which is not a resonant saddle, and L is a* separatrix *of the foliation $\mathcal{F}$ passing through it, then the order of tangency between L and any other smooth curve γ is by* 1 *greater than the order of tangency between $\mathcal{F}$ and γ,*

$$\gamma \overset{0}{:} L = \tau(\mathcal{F},\gamma) + 1.$$

Proof. We can assume that the local coordinates are chosen so that the separatrix L is a coordinate axis, $L = \{y=0\}$. Then $\omega = \lambda y(1+O(1))\,dx + (x + O(2))\,dy$, where $O(1), O(2)$ denote terms of order $\geqslant 1$ and $\geqslant 2$ respectively and λ is the negative ratio of eigenvalues.

A curve γ tangent to $\{y=0\}$ with multiplicity $k \geqslant 0$, is defined by the equation $y - b(x) = 0$, $\operatorname{ord}_0 b = k+1$. Direct computation of (8.38) yields

$$\tau_0(\mathcal{F},\gamma) = \operatorname{ord}_{x=0}[\lambda b(x)(1+O(1)) - b'(x)(x+O(2))] = k+1$$

if $\lambda \neq k+1$, i.e., if the singular point is not a saddle with the ratio of eigenvalues $-1:(k+1)$. □

Using the tangency order, one can combine the equalities (8.31) and (8.32) into a single identity valid for both $n>1$ and $n=1$. Assume that the origin is a *dicritical* singularity of a holomorphic foliation $\mathcal{F}$. Denote by Σ the singular locus of its blow-up $\widetilde{\mathcal{F}}$ and by T the collection of the tangency points between $\widetilde{\mathcal{F}}$ and the exceptional divisor.

Proposition 8.36. *If the singularity is dicritical of any order $n \geqslant 1$, then*

$$\sum_{a\in\Sigma} \mu_a(\widetilde{\mathcal{F}}) + \sum_{b\in T} \tau_b(\widetilde{\mathcal{F}}, S) = \mu_0(\mathcal{F}) - n^2. \tag{8.39}$$

Proof. When $n > 1$, the equality (8.39) follows from (8.31) and the observation that the order of tangency between $\widetilde{\mathcal{F}}$ given by the Pfaffian equation $x^{-n}[(\cdots)\,dx + g(x,xz)\,dz]$ and $S = \{x=0\}$ at any point is equal to the order of the root of the function $x^{-n}g(x, xz) = g_n(1, z) + \cdots$ restricted on S. The total multiplicity of all roots of $g_n(1, z)$ is equal to n, which proves (8.39) for $n > 1$. For $n = 1$ this formula is proved by direct inspection: there are neither singular no tangency points after blow-up, whereas the initial multiplicity $\mu_0(\mathcal{F})$ is equal to 1. □

Behavior of tangency points after blow-up can be easily controlled: by (8.26), the intersection multiplicity between two *smooth* analytic curves decreases by 1 after blow-up. Using this fact, one can conclude by elementary inductive arguments that in the formulation of the desingularization Theorem 8.14 one can further eliminate all tangency between the foliation $\pi^*\mathcal{F}$ and the dicritical components of the vanishing divisor $D = \pi^{-1}(0)$. Details can be found in [**Kle95**].

9. Complex separatrices of holomorphic line fields

Desingularization of singular points of holomorphic (line or vector) fields provides a powerful tool for their analysis. In this section we generalize the result on existence of holomorphic invariant curves from the hyperbolic or semihyperbolic context of §6.1 to arbitrary isolated *planar* singularities. Subsequent sections (Chapter II) deal with phase portraits of *real* analytic planar vector fields.

9.1. Invariant curves. Consider the germ of a holomorphic line field $\{\omega = 0\}$ on the complex plane, having an isolated singularity, $\omega \in \Lambda^1(\mathbb{C}^2, 0)$, $\mu_0(\omega) < +\infty$. Denote by $\mathcal{F}$ the corresponding singular foliation.

Definition 9.1. A (local) *complex separatrix* of the line field $\{\omega = 0\}$ is a non-constant germ of holomorphic curve $\gamma\colon (\mathbb{C}, 0) \to (\mathbb{C}^2, 0)$ tangent to the null spaces of the form ω: $0 \equiv \gamma^*\omega \in \Lambda^1(\mathbb{C}, 0)$.

A complex separatrix is a leaf ℓ of the foliation $\mathcal{F}$ whose closure $\ell \cup \{0\}$ is (the germ of) a holomorphic curve, $\{f = 0\} \subset (\mathbb{C}^2, 0)$.

For an elementary singular point, there always exists at least one *smooth* complex separatrix. More precisely, there are two smooth complex separatrices if the singular point is *not* a saddle-node or a resonant node, and one or two smooth separatrices in the latter cases. The question on existence of complex separatrices for more degenerate singular points was first discussed by C. Briot and J. Bouquet in 1856. However, the complete solution was achieved only in 1982 by C. Camacho and P. Sad [**CS82**].

Theorem 9.2 (C. Camacho–P. Sad, 1982)**.** *Every isolated singularity of a planar holomorphic vector field admits a complex separatrix.*

The idea of the proof is to blow up the singular point until it has only elementary singularities. Each such singularity has at least one complex separatrix. If this separatrix is *not contained* in the vanishing divisor D that blows down to one point, then the image of this separatrix will be a non-constant analytic curve and hence a complex separatrix. To prove the theorem, one has to show that at least one elementary singularity has an invariant curve (it will be always a hyperbolic invariant curve) transversal to D.

The most difficult combinatorial part of the original proof from [**CS82**] was simplified by J. Cano [**Can97**]. Both proofs use the geometric notion of of an *index* of a smooth separatrix.

9.2. Linearization along invariant manifolds and index of a complex separatrix. Assume that a smooth analytic curve S is a complex separatrix through an isolated singular point a of a foliation $\mathcal{F}$. We define the *index of the separatrix* S relative to the foliation $\mathcal{F}$ as follows.

Assume that S is given by the equation $\{y = 0\}$ in a suitable coordinate chart $M = (\mathbb{C}^2, 0)$ and consider any holomorphic 1-form $\omega = f\,dx + g\,dy \in \Lambda^1(\mathbb{C}^2, 0)$ tangent to $\mathcal{F}$ which has an isolated singular point at the origin. Invariance of S means that $f(x, 0) \equiv 0$, so that

$$f(x,y) = a(x)y + O(y^2), \qquad g(x,y) = b(x) + O(y),$$

with the holomorphic function $b(x)$ having an isolated root at $x = 0$. Linearization of the Pfaffian equation $\{\omega = 0\}$ on S (i.e., keeping only the first order terms in powers of y and dy) yields the equation

$$y\,a(x)\,dx + b(x)\,dy = 0, \tag{9.1}$$

which corresponds to the linear ordinary (nonautonomous) equation

$$\frac{dy}{dx} = r(x)\,y, \qquad r(x) = -\frac{a(x)}{b(x)}. \tag{9.2}$$

The point $x = 0$ is a pole of the meromorphic function $r(x)$. The meromorphic 1-form $\theta \in \Lambda^1(S, 0)$ on the curve S, defined as

$$\theta = -\frac{a(x)}{b(x)}\,dx, \tag{9.3}$$

is called the *linearization form* of ω along S; note, that this form *depends* on the choice of the local coordinates (x, y) used in the construction.

Definition 9.3. The *index* $i(a, S, \mathcal{F})$ of the smooth analytic invariant curve (separatrix) S passing through a singular point $a \in S$ of a singular foliation $\mathcal{F}$ is the residue $\operatorname{res}_0 \theta = \operatorname{res}_{x=0} r(x)$ of the linearization form (9.3) for the Pfaffian equation $\omega = 0$ along S.

In the notation below we will sometimes omit one or more arguments from the list $i(a, S, \mathcal{F})$, when they are unambiguously determined by the context.

To show that the index in fact does not depend neither on the coordinates used for the linearization, nor on the choice of ω (i.e., remains the same if ω is replaced by a multiple $u\omega$, $u \neq 0$), we re-expose *the same* construction in more invariant terms as follows.

Proposition 9.4. *Assume that M is a holomorphic 2-dimensional manifold and a smooth curve $S \subset M$ is given by the equation $\{h = 0\}$, where h is a holomorphic function on M with the differential dh not vanishing on S.*

Then any holomorphic 1-form ω tangent to S can be represented as

$$\omega = g(dh - h\,\theta), \tag{9.4}$$

where g is a holomorphic function and θ a meromorphic 1-form whose poles can be only at singular points of ω.

The restrictions of the function g and the form θ on S and the tangent bundle $TS = \bigcup_{a \in S} T_a S$ respectively, are uniquely defined by ω and h.

Proof. Since ω vanishes on vectors tangent to S, we have $\omega = g\,dh$ at all points of S (two forms with the same null space must be proportional). The holomorphic function $g\colon (S, 0) \to \mathbb{C}$, originally defined only on S, can be extended on the neighborhood of S in M; this extension (denoted again by g) is vanishing only at singular points of ω on S.

The difference $\omega - g\,dh$ is a 1-form vanishing identically at all points of S and hence divisible by h: $\omega - g\,dh = h\vartheta$, where ϑ is a holomorphic 1-form. Denote by θ the *meromorphic* 1-form $\theta = g^{-1}\vartheta$: this yields the representation (9.4).

The extension of g from S on M is non-unique, hence θ is non-unique. However, if $\omega = g'(dh - h\,\theta')$ is an alternative representation with a different choice of g', θ', then g and g' must coincide on S and hence their difference is divisible by h, $g - g' = uh$. From the the equality of two representations $g(dh - h\,\theta) = (g + uh)(dh - h\,\theta')$ of the same form ω it follows that $g(\theta' - \theta) = u(dh - h\theta')$. Both terms dh and $h\theta'$ in the right hand side vanish on vectors tangent to S, hence the restrictions of θ and θ' on TS coincide. $\square$

The restriction of the 1-form θ on S, the meromorphic 1-form $\theta \in \Lambda^1(S, 0)$, in the local coordinates coincides with the expression (9.3) obtained by the straightforward computation.

Corollary 9.5. *The linearization form θ is not changed when ω is replaced by a proportional form $u\omega$, $u|_S \neq 0$.*

If the function h is replaced by a proportional function $h' = uh$, $u|_S \neq 0$, then θ is replaced by the form

$$\theta' = \theta + u^{-1}\,du, \qquad u|_S \neq 0. \tag{9.5}$$

Consequently, the residue $\operatorname{res}_0 \theta$ *of the form* (9.4) *does not depend neither on the choice of ω nor on the choice of the holomorphic function h defining the local equation of S.*

Proof. To get an expansion (9.4) for $u\omega$, it is sufficient to multiply the corresponding expansion for ω by u and use the uniqueness. The second assertion is achieved by the substitution $h = vh'$, $v = u^{-1}$: we have $\omega = gv\big(dh' - h'(\theta - v^{-1}dv)\big) = g'(dh' - h'\theta')$ which by the uniqueness implies that

$$\theta' = \theta - v^{-1}dv = \theta + u^{-1}du.$$

Since both u, v are holomorphically invertible, the residue of the new form remains the same. □

9.3. Indices of separatrices of elementary singularities. Computation of the index of elementary singular points is an easy exercise: existence of nonsingular invariant curves tangent to nonzero eigenvalues follows from Theorem 6.2 (if the singular point is nondegenerate and not a resonant saddle) or Theorem 6.10 (for the saddle-nodes) respectively.

Proposition 9.6. *1. If the foliation $\mathcal{F}$ has a nondegenerate point different from the resonant node with the eigenvalues λ_1, λ_2, and S_1, S_2 the corresponding invariant curves, then*

$$i(0, S_1, \mathcal{F}) = \lambda_2/\lambda_1 = [i(0, S_2, \mathcal{F})]^{-1}. \tag{9.6}$$

2. Index of a hyperbolic invariant curve of a saddle-node is zero. □

Remark 9.7. If a saddle-node has a holomorphic center manifold, its index may well be nonzero: for the normal form $\omega = y\,dx - (x^n + ax^{2n-1})\,dy$ it is equal to

$$\operatorname{res}_{x=0} \frac{dx}{x^n + ax^{2n-1}} = \operatorname{res}_0[x^{-n}(1 - ax^{n-1} + \cdots)] = -a.$$

9.4. Total index along a smooth compact invariant curve. Consider a singular foliation $\mathcal{F}$ on a complex 2-dimensional surface M and assume that a smooth compact holomorphic curve S becomes a leaf of $\mathcal{F}$ after deleting from it the singular points $a_1, \ldots, a_n$ of the latter.

Theorem 9.8. *The sum of indices of S at all singular points* $\operatorname{sing} \mathcal{F} \cap S$ *is the same for all foliations $\mathcal{F}$ tangent to S:*

$$\sum_{a \in \operatorname{sing} \mathcal{F} \cap S} i(a, S, \mathcal{F}) = c(S). \tag{9.7}$$

Proof. Consider a covering of S by open neighborhoods U_α so that in each neighborhood $S \cap U_\alpha$ is defined by some local equation, $\{h_\alpha = 0\}$, $h_\alpha \in O(U_\alpha)$. On the overlapping $U_\alpha \cap U_\beta$ the equations differ by invertible factors, $h_\alpha = u_{\alpha\beta} h_\beta$, $u_{\beta\alpha} = u_{\alpha\beta}^{-1}$.

For each foliation $\mathcal{F}$ represented locally by the distribution $\{\omega_\alpha = 0\}$ in U_α, where ω_α is tangent to S in U_α, we construct a collection $\{\theta_\alpha\}$ of meromorphic 1-forms on $S \cap U_\alpha$. As follows from Corollary 9.5, the forms θ_α depend only on the foliation $\mathcal{F}$ and h_α and not on the forms ω_α.

For any two foliations $\mathcal{F}, \mathcal{F}'$ both tangent to S, we can thus construct *two collections* of the linearization forms, $\{\theta_\alpha\}$ and $\{\theta'_\alpha\}$, defined on the intersections of S with the corresponding domains U_α. Again by Corollary 9.5, we have on the overlapping $S \cap U_\alpha \cap U_\beta$ the identities

$$\theta_\beta = u_{\alpha\beta}^{-1} du_{\alpha\beta} + \theta_\alpha, \qquad \theta'_\beta = u_{\alpha\beta}^{-1} du_{\alpha\beta} + \theta'_\alpha.$$

But this means that the differences $\xi_\alpha = \theta_\alpha - \theta'_\alpha$ coincide on the overlapping, $\xi_\alpha = \xi_\beta$ on $S \cap U_\alpha \cap U_\beta$, that is, ξ is a *globally defined* meromorphic 1-form on a compact Riemann surface S. By the Cauchy theorem, the sum of residues of ξ is zero.

Let Σ be the union $(\operatorname{sing}\omega \cup \operatorname{sing}\omega') \cap S$. Then for any point $a \in \Sigma$ $\operatorname{res}_a \theta_\alpha - \operatorname{res}_a \theta'_\alpha = \operatorname{res}_a \xi$. Adding these equalities over all $a \in \Sigma$ proves that the sum of indices does not depend on the choice of the form. □

From this proof it follows immediately that in the case when S is defined by *one global* equation $\{h = 0\}$ on M, the total index of S at all singularities is zero for any foliation tangent to S.

Remark 9.9 (forward reference)**.** This elementary proof is a particular case of the general argument explained in full details in Chapter III (cf. with Proposition 25.10). In geometric terms introduced there, Corollary 9.5 means the following.

Any singular foliation $\mathcal{F}$ on a surface M, tangent to a smooth analytic curve $S \subset M$, induces a meromorphic connection on a certain holomorphic line bundle over S. This bundle, which depends only on the embedding of S in M, is the *normal bundle* whose fiber over any point $a \in S$ is the one-dimensional quotient space T_aM/T_aS. The corresponding holomorphic cocycle $\{u_{\alpha\beta}\}$, defining the bundle, is given by the fractions $u_{\alpha\beta} = h_\alpha/h_\beta$. The common number $c(S)$ is the degree of this bundle, equal to its (first) Chern class, a topological invariant of the embedding S in M.

9.5. Index and blow-up. Let S be an integral curve through a singular point a of a foliation $\mathcal{F} = \{\omega = 0\}$, and consider the blow-up σ of the point a. After the blow-up we obtain the singular foliation $\mathcal{F}'$. Denote by S'

the blow-up of the curve S and let $D = \sigma^{-1}(a) \simeq \mathbb{C}P^1$ be the exceptional divisor. Finally, denote $a' = S' \cap D$.

Lemma 9.10.
$$i(a', S', \mathcal{F}') = i(a, S, \mathcal{F}) - 1. \tag{9.8}$$

Proof. The Pfaffian equation $\omega = 0$ in suitable local coordinates takes the form
$$\frac{dy}{dx} = r(x)y + \cdots, \qquad x \in (\mathbb{C}, 0),$$
where the dots denote meromorphic terms divisible by y^2 and $r(x)$ is a meromorphic function whose residue is $i(0, D, \mathcal{F})$.

Blowing up means introducing the new variable $z = y/x$ linearly depending on y. Changing the variable in the above equation (i.e., applying a meromorphic gauge transform in the terminology of Chapter III) yields after linearization on $\{y = 0\}$ the differential equation
$$\frac{dz}{dx} = r'(x)z, \qquad r'(x) = r(x) - \tfrac{1}{x}.$$
Subtracting from the meromorphic function $r(x)$ the reciprocal $1/x$ decreases the residue by 1, as claimed. $\square$

Assume that the blow-up is non-dicritical. Then $\mathcal{F}'$ is tangent to the exceptional divisor D as well, which means that D is a complex separatrix through each singular point $b \in D$ of $\mathcal{F}'$.

Lemma 9.11. *If the blow-up is non-dicritical, then*
$$\sum_{b \in \operatorname{sing} \mathcal{F}' \cap D} i(b, D, \mathcal{F}') = -1. \tag{9.9}$$

Proof. This is an immediate corollary of Theorem 9.8. To compute the number $c(D)$ characterizing the embedding of the exceptional divisor after the blow-up, one can consider any foliation/form, for instance, $\omega = dy$. After blow-up and division by the local equation of D we obtain the form $\omega' = z\,dx + x\,dz$ which has a unique nondegenerate saddle with the ratio of eigenvalues -1 on D, so $c(D) = -1$. $\square$

9.6. Cano points. Recall that for two complex numbers a, b the notation $a \geqslant b$ means that $a - b \in \mathbb{R}_+$. We will also use the (obvious) negated notation $a \ngeqslant b$ meaning that $a - b \notin \mathbb{R}_+$.

Consider a divisor with normal crossings D on a complex 2-dimensional holomorphic manifold M, and a singular foliation $\mathcal{F}$ tangent to D. As before, this means that $D \smallsetminus \operatorname{sing} \mathcal{F}$ is the union of leaves of the foliation $\mathcal{F}$.

Definition 9.12. A singular middle point a on the divisor D is called the *Cano middle point* for the foliation $\mathcal{F}$, if

$$i(a, D) \ngeqslant 0. \tag{9.10}$$

A (singular) corner point $a \in D_+ \cap D_-$ on the intersection of two smooth components is called the *Cano corner point*, if

$$i(a, D_-) < 0, \tag{9.11}$$

$$i(a, D_+) \ngeqslant [i(a, D_-)]^{-1} \tag{9.12}$$

(note that the two curves play asymmetric roles).

A *Cano point* is a Cano middle point or a Cano corner point.

Proposition 9.13.

(1) *A Cano middle point which is elementary, must have an holomorphic separatrix passing through it and not contained in D;*

(2) *a Cano corner point cannot be elementary.*

Proof. Both assertions follow from Proposition 9.6.

1. If the Cano middle point is a saddle-node, then its hyperbolic invariant manifold (curve) cannot locally coincide with D, since in this case the index would be zero.

2. A nondegenerate elementary Cano point must have *two* hyperbolic invariant curves (complex separatrices). Indeed, as soon as the ratio of the two eigenvalues is not a positive real, this is asserted by the Hadamard–Perron theorem 6.2.

But two transversal separatrices of a *middle* point cannot simultaneously belong to the vanishing divisor.

3. A Cano corner point cannot have zero index along any smooth component, since then the other index $i(D_-)$ must be negative and the inequality $0 = i(D_+) \geqslant 1/i(D_-)$ means violation of the Cano property (9.12).

Thus a saddle-node cannot be a Cano corner point. Similarly, a nondegenerate singularity cannot be a Cano corner point since in this case $i(D_+) = 1/i(D_-)$ in contradiction with (9.12) even if both are negative and (9.11) holds. □

The principal property of the Cano points is their persistence under *non-dicritical* blow-up. Consider a singular foliation $\mathcal{F}$ tangent to a divisor D with normal crossings, and let $a \in D \cap \operatorname{sing} \mathcal{F}$ be a singular point, either corner or a middle point.

Lemma 9.14 (J. Cano [**Can97**]). *If $a \in D$ is a Cano point, then at least one of the singularities that appear by the non-dicritical blow-up of a on the blow-up of D, is again a Cano point.*

Proof. Denote by S the exceptional divisor and let the prime in the notations indicate the objects appearing by the blow-up.

1. Consider first the case when a is a middle Cano point. In this case D is a smooth curve and its blow-up D' intersects S transversally. Denote by $a' = S \cap D'$ the corner point. The singular locus for $\mathcal{F}'$ consists of a' and, eventually, several middle points $m_1, \dots, m_k$ on S.

Assume that all middle points are non-Cano. Then $i(m_j, S) \geqslant 0$ and by Lemma (9.11),

$$i(a', S) = -1 - \sum_j i(m_j, S) \leqslant -1.$$

If in addition the corner point a' is non-Cano, then by (9.12) necessarily $i(a', D') \geqslant 1/i(a', S) \geqslant -1$, since $i(a', S) < 0$ and (9.11) holds. By (9.8),

$$i(a, D) = 1 + i(a', D') \geqslant 1 + 1/i(a', S) \geqslant 1 - 1 = 0$$

and we have a contradiction with the assumption that a was a middle Cano point. Hence among $\operatorname{sing} \mathcal{F}' = \{a', m_1, \dots, m_k\}$ must be a Cano point.

2. Consider the case when $a \in D_- \cap D_+$ is a Cano corner point. After the blow-up it will produce *two* corner points $a'_\pm = D'_\pm \cap S$ on the intersection of S with the blow-ups $D'_\pm$ of the smooth components $D_\pm$, and eventually one or more middle points $m_1, \dot{;} m_k \in S$. Without loss of generality we assume that $I = i(a, D_-) < 0$. We again assume that all these singularities are non-Cano points and arrive to a contradiction.

First, note that $i(a'_-, D') = i(a, D_-) - 1 < 0$ and (9.11) holds. If a'_- is non-Cano, then by (9.12)

$$i(a'_-, S) \geqslant 1/i(a'_-, D'_-) = 1/[i(a, D_-) - 1] = 1/(I - 1).$$

If all middle points are non-Cano, their indices $i(m_j, S)$ are nonnegative and

$$i(a'_+, S) = -1 - i(a'_-, S) - \sum i(m_j, S) \leqslant -1 - 1/(I - 1) = I/(1 - I).$$

This last quantity is negative so (9.11) holds for a'_+. If the latter is non-Cano then (9.12) has to be violated so that $i(a'_+, D'_+) \geqslant 1/i(a'_+, S)$. Again by (9.8),

$$i(a, D_+) = 1 + i(a_+, D'_+) \geqslant 1 + 1/i(a'_+, S) \geqslant 1 + (1 - I)/I = 1/I.$$

As a result we conclude that $i(a, D_+) \geqslant 1/I = 1/i(a, D_-)$ in contradiction with the assumption that a was a corner Cano point. Thus $\operatorname{sing} \mathcal{F}' = \{a'_\pm, m_1, \dots, m_k\}$ must include at least one Cano point. □

9.7. Proof of the Camacho–Sad theorem. Consider a singular foliation $\mathcal{F}_0$ at an isolated singular point. By Theorem 8.14, there exists a map $\pi\colon (M,D) \to (\mathbb{C}^2, 0)$ resolving all singularities of $\mathcal{F}$. Expanding π as a composition of simple blow-ups, we obtain a chain of holomorphic 2-dimensional surfaces M_k carrying singular foliations $\mathcal{F}_k$ and simple blow-down maps $\pi_k\colon M_{k+1} \to M_k$ such that the preimage of the origin by any composition $\pi_k \circ \cdots \circ \pi_1$ is a vanishing divisor with normal crossings only, and the foliation $\mathcal{F}_n$ has only elementary singularities on D_n.

If one of the blow-ups π_k was dicritical, there were infinitely many leaves of $\mathcal{F}_k$ transversal to D_k, which after blow-down produce complex separatrices. Thus we may assume that all blow-ups π_k are non-dicritical.

We claim that in this case at least one singularity of each $\mathcal{F}_k$ is a Cano point. Indeed, $D_1 = \pi^{-1}(0) \simeq \mathbb{C}P^1$ is smooth, so $\mathcal{F}_1$ has no corner points. One of the singularities from $\operatorname{sing} \mathcal{F}_1$ must be Cano middle point: otherwise the sum of their indices will be a nonnegative real number in contradiction with Lemma 9.11.

By Lemma 9.14, π_2-preimage of the Cano middle point p_1 on D_1 must contain a Cano point $p_2 \in D_2$, either corner or middle. For the same reason the preimage $\pi_3^{-1}(p_2)$ must contain a Cano point $p_3 \in D_3$ etc., until we find a Cano point $p_n \in D_n$. By the assumption on the resolution, $\mathcal{F}_n$ has only elementary points.

By Proposition 9.13, an elementary Cano point has a complex separatrix not contained in D_n. Its blow-down is the complex separatrix of the initial singularity. $\square$

Chapter 2

Singular points of planar analytic vector fields

In this Chapter we apply the analytic tools developed earlier in Chapter I, to the study of singular points of planar real analytic vector fields. Whenever explicitly stated otherwise, an isolated singularity is assumed to be at the origin $0 \in \mathbb{R}^2$.

10. Singularities of planar vector fields with characteristic trajectories

A real analytic vector field F on the plane or, more generally, a real analytic 2-dimensional manifold U (*surface*) defines a *real analytic* foliation $\mathcal{F}_F$ by real analytic curves on the complement to the zero locus $\Sigma_F = \{F = 0\}$. The leaves of this foliation are naturally *oriented* by the field F.

Everywhere in this section we will assume that the singularities of $\mathcal{F}$ are *complex isolated*: the locus Σ_F even *after complexification* consists of isolated points. The vector field $F = (x^2 + y^2)\frac{\partial}{\partial x}$ illustrates the possibility when the "visible part" of the zero locus on $\mathbb{R}^2$ is an isolated point while after complexification the singularity on $\mathbb{C}^2$ is non-isolated.

10.1. First steps of topological classification: Poincaré types and saddle-nodes. Two vector fields F and F' defined on two surfaces U and U' respectively, are *topologically equivalent* if there exists an orientation-preserving homeomorphism $H : U \to U'$ mapping Σ_F to $\Sigma_{F'}$ and the leaves of $\mathcal{F}$ to the leaves of $\mathcal{F}'$ respecting the orientations.

One of the principal problems of the local theory of analytic differential equations on the plane is to construct topological classification of isolated singularities of planar analytic vector fields, corresponding to $U = (\mathbb{R}^2, 0)$, $\Sigma = \{0\}$. The topological equivalence class is sometimes referred to as the *phase portrait* of a given singularity.

The initial steps of this classification were implemented by H. Poincaré who obtained a complete topological classification of nondegenerate *linear* planar vector fields (a degenerate singularity cannot be linear and isolated simultaneously). Poincaré introduced the following topological types of phase portraits (in parenthesis we indicate a simple representative):

(1) saddle $x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}$,

(2) node, $\pm(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y})$, stable or unstable depending on the choice of the sign (respectively, minus or plus);

(3) center $x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}$.

All other types of linear phase portraits, e.g., foci or Jordanian nodes, turn out to be topologically equivalent to the above types.

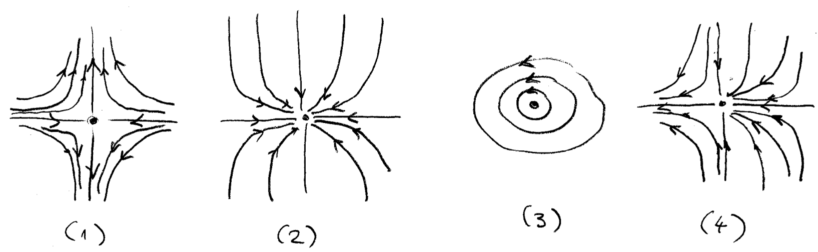

Figure 1. Poincaré types of phase portraits

For *nonlinear nondegenerate* singularities no new types arise: except for one case (center), any analytic (and even smooth) germ of vector field is topologically equivalent to its linear part. This follows from the Grobman–Hartman topological linearization theorem for hyperbolic singularities [**Gro62, Har82**], already mentioned in §7. A vector field whose linearization is a center, may be center or focus: we shall explore this issue in depth in §11.4 below.

For *elementary* (degenerate) singularities there is only one new topological type,

(4) saddle-node $x^2\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$.

Altogether there is a finite (in fact, very short) list of topologically different phase portraits of elementary singularities.

Any isolated singularity can be resolved into elementary ones by Theorem 8.14. Blowing down the corresponding two-dimensional surfaces with foliations on them, one can obtain description of phase portraits of degenerate singularities in terms of *sectors*. As explained in [**ALGM73**], in many cases a small punctured neighborhood of a singular point can be represented as the union of sectors bounded by phase curves of the vector field, with the standard foliations of three types (hyperbolic, parabolic and elliptic sectors, see Figure 2.

Since Theorem 8.14 is constructive, one may expect that there exists an efficient algorithm for determination of the topological type of isolated singularities, based on the desingularization process.

This is indeed the case under an additional assumption of existence of a *characteristic orbit*.

10.2. Cycles, monodromic singularities and characteristic orbits. All general constructions with foliations from Chapter I can be implemented also for real analytic foliations by real curves. In particular, if L is a non-simply-connected leaf of such foliation and τ a cross-section to L, then every

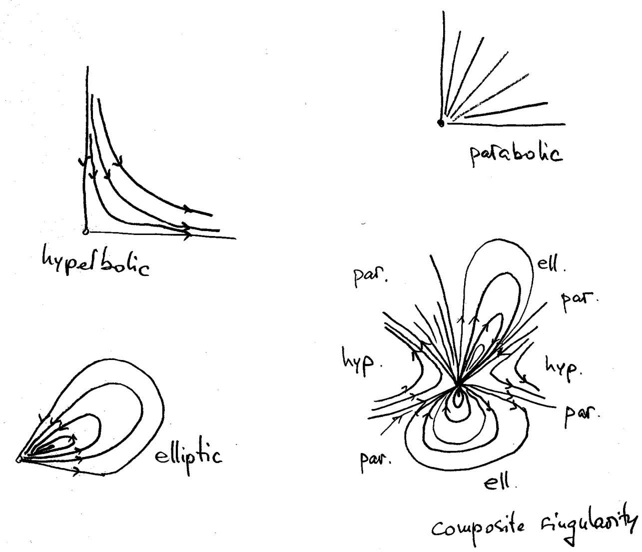

Figure 2. Hyperbolic, parabolic and elliptic sectors of a degenerate singular point.

non-contractible loop on L can be associated with an analytic homeomorphism of τ.

However, in the real analytic case we discuss, the only possibility for a one-dimensional leaf L to be non-simply-connected is when L is itself a periodic trajectory of the vector field. Its fundamental group is cyclical generated by L considered as a loop with the natural orientation. The corresponding holonomy map, usually referred to as *Poincaré return map* or *monodromy*, analytic by the general theorems of Chapter I, can either be identical or have an isolated fixed point at the intersection $\tau \cap L$.

Definition 10.1. A *limit cycle* of a planar vector field is an *isolated* nontrivial periodic phase curve .

In the language of foliations the limit cycle is a compact leaf which has no other compact leaves nearby. In case when the return map is identical, the compact leaf is called the *identical cycle.*

Remark 10.2. For smooth vector fields one may have a third possibility when the return map has a non-isolated fixed point while being nonidentical. This would correspond to a periodic phase curve to which an infinite number of isolated periodic curves accumulates in the sense of Hausdorff distance. For analytic vector fields such pathology is impossible.

An isolated singular point by definition is *not* a leaf of the foliation. However, sometimes one can define a (first) return map around this point.

Definition 10.3. A *cross-section* to a vector field F *at a singular point* $0 \in \mathbb{R}^2$ is a (parameterized non-constant) analytic curve $\tau\colon (\mathbb{R}^1, 0)$ restricted on the positive semiaxis $(\mathbb{R}^1_+, 0)$, such that $\tau(0) = 0$ and at all other points the field F is transversal to τ.

By analyticity, any analytic "semi-curve" τ with $\tau(0) = 0$, e.g., a line segment, is either a reparameterized phase curve of the vector field, or becomes a cross-section after restriction on a sufficiently small sub-semi-interval $(\mathbb{R}^1_+, 0)$. It can be crossed by phase curves from one side only.

Definition 10.4. A singular point of an analytic vector field F is *monodromic*, if there exists a cross-section at this point with the following property: all phase curves passing through points of τ sufficiently close to the singularity, intersect τ at least once again after continuation forward.

The map $P\colon (\tau, 0) \to (\tau, 0)$ taking a point $x \in \tau$ into the point of the first intersection of the phase curve with τ, is called the *Poincaré map* or *return map*.

Remark 10.5. The property of being monodromic is *not* invariant by topological equivalence: a focus is monodromic, while a node (topologically equivalent to it) is not.

Phase trajectories of a monodromic singularity are spiralling around the singular point. An opposite type of behavior is as follows. Assume that the origin $0 \in \mathbb{R}^2$ is an isolated singularity.

Definition 10.6. A phase curve $(x(t), y(t))$ of an analytic vector field is called a *characteristic orbit* (curve) of a singular point, if it tends to the origin (in the forward or backward time) with a certain limit tangent,

$$\lim_{t\to\pm\infty} \bigl(x^2(t) + y^2(t)\bigr) = 0, \qquad \lim_{t\to\pm\infty} \frac{y(t)}{x(t)} = c \in \mathbb{R} \cup \{\infty\}.$$

Remark 10.7. Definition of a characteristic orbit is similar to the definition of a separatrix. The difference is two-fold: the characteristic orbit may be not an analytic curve (e.g., all phase curves of a node are characteristic), but it must be real (i.e., two complex separatrices of a center are not characteristic curves).

Elementary geometric considerations immediately show that existence of a characteristic orbit which tends to the singular point in a "radial" direction, is incompatible with existence of "spiralling" orbits typical for a monodromic singularity, and vice versa. However, these considerations alone cannot exclude some rather pathological behavior of phase curves of a planar vector field.

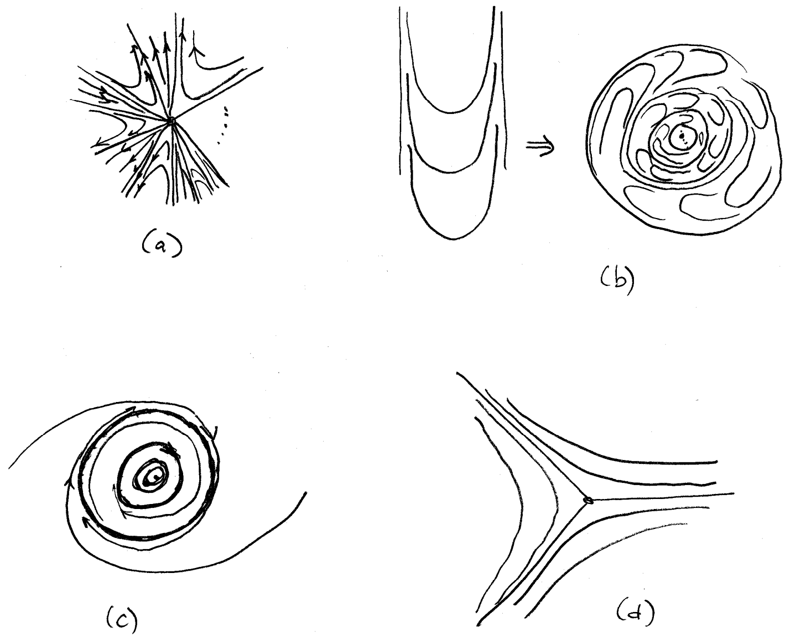

Figure 3. "Pathological" behavior of C^∞-smooth vector fields: (a) infinitely many sectors, (b) non-monodromic singularity without characteristic orbit, (c) accumulating limit cycles, (d) non-orientable foliation.

10.3. "Three nightmares". In this section we describe three examples of isolated singularities of planar vector fields, which can be constructed in the class of C^∞-smooth planar vector fields. *None* of these examples can exist in the real analytic category, yet the proof of such impossibility varies from simple geometric arguments to very deep analytic study.

Example 10.8 (Infinitely many sectors)**.** The singular point schematically pictured on Figure 3(a), has infinitely many alternating hyperbolic and parabolic sectors. Similar examples can be designed with elliptic sectors.

Example 10.9 (Non-monodromic singularity without characteristic orbits)**.** Consider a function of one real variable, defined on the interval $(-1,1)$ and having infinite limits at the endpoints. Shifting the graph of this function in the vertical direction, one can construct a foliation without singular points on the infinite strip $[-1,1]\times\mathbb{R}$ tangent to the two border lines of the strip which are themselves the leaves. Rolling this strip (say, by the exponential map of the plane $\mathbb{R}^2 \simeq \mathbb{C}^1$), a foliation on the annulus $\{1 \leqslant |z| \leqslant 2\}$ can be constructed. Finally, assembling countably many homothetic copies of such annulus, we obtain a foliation shown on Figure 3(b). This foliation is neither monodromic (it simply admits no cross-section) nor does it have characteristic orbits. In §10.4 it will be shown that this is impossible for analytic foliations.

Example 10.10. It was already remarked that infinitely limit cycles cannot accumulate to a compact (nontrivial) leaf of a real analytic foliation. Clearly, for non-analytic (C^∞-smooth) foliations this prohibition does not hold. In a similar way, one can easily construct a C^∞-smooth vector filed with infinitely many limit cycles accumulating to an isolated singular point, see Figure 3(c). It is very difficult to prove that such accumulation is impossible for analytic vector fields (the so called *Nonaccumulation theorem*, see [**Ily91, Ily02**] and §**??**).

We conclude this section by an example of a foliation with real analytic leaves and the only singular point at the origin, that cannot be generated by a real analytic vector field.

Example 10.11. Consider the foliation $\mathcal{F}$ by the level curves $\operatorname{Im} z^{3/2} = \text{const}$ of the plane $\mathbb{R}^2 \simeq \mathbb{C}^1$. This foliation is nonsingular outside the origin and the leaves are all real analytic, see Figure 3(4).

However, the real foliation $\mathcal{F}$ cannot be complexified: there does not exist a complex singular foliation $\mathcal{F}^{\mathbb{C}}$ of $\mathbb{C}^2$ whose trace on the real plane $\mathbb{R}^2 \subset \mathbb{C}^2$ is $\mathcal{F}$. Indeed, by Theorem 2.16, such foliation would be generated by a holomorphic vector field which takes real (vector) values at real points. However, the foliation $\mathcal{F}$ is non-orientable and hence cannot be generated by any real analytic vector field.

10.4. Principal alternative: characteristic or monodromic? We already noted that existence of characteristic orbits is incompatible with existence of the Poincaré return map. For analytic vector fields, this is a genuine alternative.

Theorem 10.12. *An isolated singular point of a real analytic vector field is either monodromic or has a characteristic trajectory.*

The assertion of the Theorem follows from the following geometrically rather obvious observation. For brevity we say that a trajectory $x(t)$ of a vector field *lands* on a set S, if $\lim_{t\to\infty} x(t) \in S$. Note that no trajectory can *land* on a cross-section, only *cross* it in finite time. By the implicit function theorem, if a trajectory crosses a cross-section, then all close trajectories also cross it.

Let S be a separatrix of a smooth planar vector field F, denote T_1, T_2 two cross-sections to S at two different points $a_1, a_2 \in S$ and let B be an arbitrary smooth curve intersecting both $T_{1,2}$ but disjoint with S.

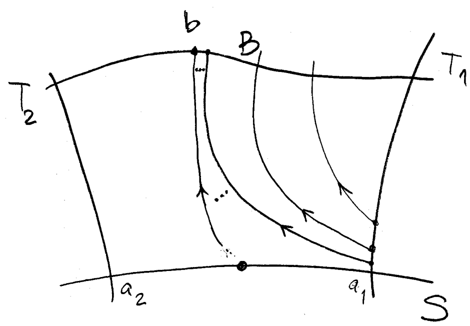

Figure 4. Correspondence map near separatrix

Lemma 10.13. *Either there is a trajectory of the field F that lands somewhere on S in forward or reverse time, or the correspondence map is well-defined as a germ $P\colon (T_1, a_1) \to (T_2, a_2)$.*

Proof of the Lemma. Without loss of generality we assume that T_1 is the entrance cross-section for the curvilinear rectangle $R = T_1ST_2B$, see Figure 4. Assume the contrary, that there are infinitely many points on T_1, accumulating to a_1, such that the trajectories l_α starting at these points never cross T_2, while no trajectory lands on S.

Then the only possibility is that these trajectories leave R through the the side B and the exit points necessarily accumulate to a point $b \in B$.

The trajectory l_b passing through b in the *reverse* time cannot cross neither T_1 nor T_2. Indeed, the former case is impossible since then all l_α in the reverse time should cross T_1 near $b' = l_b \cap T_1$ which, by construction of l_α, is possible only if $b' = a_1$. But this is impossible since the trajectory crossing T_1 at a_1, remains on the separatrix S both in the direct and reverse time.

The latter case is also impossible. Indeed, if l_b crosses T_2, then all points near b in the inverse time move out from R across T_2, whereas at least some of them should move out across T_1.

The only remaining possibility would be for l_b to land on S somewhere between a_1 and a_2, but this contradicts to our assumption. □

Proof of Theorem 10.12. Consider the polar blow-up of the vector field. By definition, it is a real analytic vector field on the cylinder $(\mathbb{R}, 0) \times S^1$, having only finitely many isolated singular points on the circle $\{0\} \times S^1$.

If the field was dicritical, then there are infinitely many (real analytic) leaves of the foliation, transversally crossing S. All of them correspond to characteristic trajectories, since the property of being characteristic means that after blow-up the trajectory lands on some point in S.

In the non-dicritical case S is a separatrix of the resulting real analytic foliation. Choose two nonsingular points $a_1, a_2 \in S$ and let T_i be arbitrary arcs transversal to S at a_i. This transversality means that T_i are cross-sections of the field near a_i. Without loss of generality we may assume that T_i are blow-ups of suitable analytic curves $\tau_i \colon (\mathbb{R}^1, 0) \to (\mathbb{R}^2, 0)$, e.g., two transversal lines.

By Lemma 10.13, if the field has no characteristic orbit, then there exist two correspondence maps $P_1 \colon (T_1, a_1) \to (T_2, a_2)$ and $P_2 \colon (T_2, a_2) \to (T_1, a_1)$. Their composition $P_2 \circ P_1$ after blowing down to $(\mathbb{R}^2, 0)$ is the monodromy of the singular point, associated with the cross-section τ_1 (restricted on the positive semiaxis). □

10.5. Algorithm of decision for the principal alternative. The above proof of Theorem 10.12 does not allow for an efficient decision process to decide between characteristic and monodromic case. In this section we describe an "algebraic" algorithm which works for all (complex) isolated singularities and gives an answer in finite time. The algorithm is based on the desingularization.

Consider the (real) complete desingularization of the isolated singularity, as described in §8. By definition, this means a holomorphic singular foliation $\mathcal{F}^{\mathbb{C}}$ defined in a complex neighborhood $U^{\mathbb{C}}$ of the exceptional divisor $D^{\mathbb{C}}$ which is a finite union of normally crossing Riemann spheres (projective lines). By construction, the foliation $\mathcal{F}^{\mathbb{C}}$ has only elementary singularities on $D^{\mathbb{C}}$.

Since the initial singularity was real analytic and all blow-up formulas have only real coefficients, the "*real part*" U of $U^{\mathbb{C}}$ is well-defined real analytic 2-dimensional surface (eventually, non-orientable) which is a neighborhood of the real part D of the exceptional divisor: the latter is the union of normally crossing circles D_i (real equators $\mathbb{R}P^1 \simeq \mathbb{S}^1$ of the corresponding spheres $\mathbb{C}P^1 \simeq \mathbb{S}^2$). The intersections $D_i \cap D_j$ of different components are referred as the *corner points*; they are always singular for the blow-up foliation $\mathcal{F}'$.

Proposition 10.14. *An isolated singularity does not have a characteristic orbit if and only if its complete desingularization does not involve dicritical blow-ups and the only elementary singularities that appear at the end, are topological saddles at the corner points of D.*

Proof. Any phase curve which after desingularization tends to some *point* on D in forward or backward time, blows down to a characteristic orbit.

If the desingularization map represented as the composition of simple blow-ups, involves a dicritical blow-up, then there will immediately be uncountably many phase curves crossing D transversally. All of them would blow down to characteristic orbits.

Thus the only case to be considered is the composition of non-dicritical simple blow-down maps, when separate pieces D_i of the vanishing divisor D are all invariant curves (leaves) of the blow-up foliation $\mathcal{F}'$. This invariance implies that no phase curve can tend to D outside the singular locus of $\mathcal{F}'$. In other words, existence of characteristic curves can be verified by inspection of possible types of elementary singularities in different position with respect to D.

Note first that no centers or foci are allowed if the desingularization is non-dicritical (none of them has real separatrices). Thus the only *admissible types* are saddles (degenerate or not), nodes and saddle-nodes. Each admits finite or infinite number of integral curves that land at the singularity. This number differentiates between saddles and other admissible types as follows.

(1) In the saddle case there are two analytic invariant curves tangent to two transversal invariant curves (Theorem 6.2). These curves carry *four* different phase curves which land at the singular point.

(2) In the nodal or saddle-nodal cases there are infinitely (uncountably) many *different* phase curves which land at the singular point.

For singularities at smooth points of D_i ("middle points" in the terminology of §9) in the worst case only two of the phase curves that land at the singularity, may be part of D, therefore occurrence of a middle singular point of any admissible type always implies existence of a characteristic orbit for the initial singularity.

The same argument excludes the possibility of saddle-nodes or nodes at the corner points: out of infinitely many orbits landing at such singularities, at worst 4 can belong to D.

The only remaining possibility is a saddle occurring at a corner point. It can have have both its invariant manifolds on D. Such singularity does not imply existence of a characteristic curve. □

Remark 10.15. If the foliation obtained by complete desingularization of an isolated singularity is tangent to the exceptional divisor and all singularities are corner saddles, then the singularity *is monodromic.* The return map can be constructed as a composition of the correspondence maps for individual hyperbolic sectors of the corner saddles. This observation constitutes an independent proof of Theorem 10.12.

10.6. Algebraicity of the decision. Inspection of the above algorithm for decision of the characteristic/monodromic alternative suggests that it is "effective" and "algebraic". This means, in particular, that:

(1) the result is achieved in finitely many steps, their number being determined by the multiplicity μ of the singular point;

(2) on each step calculations and tests involve only finitely many Taylor coefficients;

(3) both the calculations and the tests are algebraic (polynomial).

In this and the next section we give the formal definitions of algebraic decidability. The (subtle) difference between the constructions of this section and that in §11 is the explicit reference to the parameter μ, the multiplicity of the singular point, which determines, among other things, the maximal order of Taylor coefficients involved in the decision algorithm.

We start with describing "computable" subsets in affine finite-dimensional spaces. Without going into deep discussion on the nature of computability, we postulate the class of semialgebraic sets as the only reasonable class of subsets of $\mathbb{R}^n$ or $\mathbb{C}^n$, which are finitely presented. For any such set, one can imagine an "algorithm" involving only algebraic computations and sign tests, that in a finite number of steps allows to decide, whether a given input (point) belongs to the set or not.

Definition 10.16. A *real semialgebraic* set in $\mathbb{R}^n$ is any set defined by finitely many polynomial equalities and inequalities of the form $p(x) = 0$, $p(x) < 0$ or $p(x) \leqslant 0$, where $p \in \mathbb{R}[x_1, \dots, x_n]$.

Semialgebraic sets form a Boolean algebra (their finite unions and intersections are obviously semialgebraic). What is more important, the class of semialgebraic sets is closed by taking complements and affine projections (and, more generally, polynomial maps). This is the famous Tarski–Seidenberg theorem [**vdD88**]. Semialgebraic spaces are *decidable* in the sense that any such set is defined by a finite formula involving polynomial (in)equalities over $\mathbb{R}[x_1, \dots, x_n, y_1, \dots, y_m]$ involving "auxiliary" variables $y_1, \dots, y_m$, the logical operations "and", "or", "not", and the *quantifiers* $\forall y_i, \exists y_j$ which tie down the auxiliary variables. The Tarski–Seidenberg theorem asserts that all quantifiers can be effectively eliminated, meaning that the decision process fully constructive.

Consider a subset M in the space, say, of germs of complex analytic vector fields at the origin on the plane $\mathcal{D} = \mathcal{D}(\mathbb{C}^2, 0)$. Note that for any finite order n the space $J^n = J^n\mathcal{D}(\mathbb{C}^2, 0)$ of n-jets of such vector fields is a finite-dimensional complex affine space. Usually the the set M is defined by some *properties* of the vector fields (e.g., multiplicity, order, existence of analytic separatrix *etc.*).

Definition 10.17. A jet $g \in J^n$ of order n is said to be *sufficient for the set M* (or for the corresponding property), if all germs having the given jet, either belong to M or to its complement $\mathcal{D} \smallsetminus M$:

$$(j^n)^{-1}(g) \subseteq M \quad \text{or} \quad (j^n)^{-1}(g) \subseteq \mathcal{D} \smallsetminus M.$$

Definition 10.18. The set M is said to be *algebraically decidable at the level of n-jets*, if there exists a semialgebraic subset $M^{(n)} \subseteq J^n\mathcal{D}(\mathbb{C}^2, 0)$ such that $F \in M$ if and only if $j^n F \in M^{(n)}$.

In other words, the set (or the respective property) is algebraically decidable at the level of n-jets, if all such jets are sufficient. This is a relatively rare opportunity, as we will see in §11: in most cases when M is described by its topological or analytic properties, there always are some jets that are insufficient to guarantee whether their representatives belong to M or don't.

Clearly, similar definitions can be constructed for other classes of objects (e.g., germs of real analytic vector fields, germs of functions or differential forms *etc.*).

10.7. Decidability of multiplicity. The first example of the property algebraically decidable at the level of finite order jets, is that of having explicitly bounded multiplicity.

Theorem 10.19. *For any finite μ the set M_μ of holomorphic vector fields having multiplicity $\leqslant \mu$ at the origin, is algebraically decidable at the level of n-jets with $n = \mu$.*

Proof. First we show that if F is a germ of multiplicity $\leqslant \mu$, then its μ-jet is sufficient in the sense that any germ F' with the same μ-jet also has the same multiplicity. To prove that, we use the definition of the multiplicity as the dimension of the quotient local algebra, $\mu = \dim_{\mathbb{C}} \mathcal{O}_0 / \langle F_1, F_2 \rangle$, where $F_{1,2}$ are the coordinate functions of the germ F of the vector field.

Indeed, by [**AGV85**, Lemma 1, §5.5], any power $x^a y^b$ of order $a + b \geqslant \mu+1$ belongs to the ideal of any finite codimension μ. Thus any analytic germ of the form $F_i' = F_i + o((|x| + |y|)^\mu)$, $i = 1, 2$, belongs to the ideal $\langle F_1, F_2 \rangle$ and hence $\langle F_1', F_2' \rangle = \langle F_1, F_2 \rangle$. Clearly, the arguments are symmetric and all germs with the same μ-jet generate the same ideals and hence the same multiplicity.

Thus we can define the set $M_\mu^{(\mu)}$ as the set of polynomial vector fields of degree μ, having a singularity of multiplicity $\leqslant \mu$ at the origin. Regardless of the local coordinates, if the Taylor polynomial (truncation) of F belongs to $M_\mu^{(\mu)}$, then the corresponding μ-jet is sufficient for M_μ.

It remains to prove that $M_\mu^{(\mu)}$ is semialgebraic in the space of μ-jets $J^\mu\mathcal{D}(\mathbb{C}^2, 0)$. Consider the affine space $\mathcal{D}_\mu \simeq \mathbb{C}^N$, $N = N(\mu)$, of polynomial vector fields of degree μ. By Corollary 8.22, the polynomial formula

$$\forall \varepsilon > 0\, \exists y \in \mathbb{C}^2,\, x_1 \neq \cdots \neq x_\mu \neq x_{\mu+1} : |x_i|,\, |y| < \varepsilon,\, F(x_i) = y$$

defines a subset in $\mathcal{D}_\mu$ whose elements are germs of vector fields having multiplicity $\geqslant \mu + 1$ (or infinite multiplicity) at the origin, i.e., the complement to $M_\mu^{(\mu)}$. Eliminating the quantifiers by the Tarski–Seidenberg theorem, we see that $M_\mu^{(\mu)}$ is semialgebraic. □

Remark 10.20. If a certain set (property) M is algebraically decidable at the level of n-jets, then for trivial reasons it is algebraically decidable at the level of any higher order jets.

10.8. Algebraic decidability of the principal alternative. We prove now that among all singularities of bounded multiplicity $\leqslant \mu$, those having characteristic trajectory are algebraically decidable.

Theorem 10.21. *For each multiplicity $\mu \in \mathbb{N}$ there exists a finite order $n = n(\mu) \in \mathbb{N}$ and two disjoint semialgebraic subsets $C^{(n)}, M^{(n)} \subseteq J^n(\mathcal{D}(\mathbb{R}^2, 0))$ in the space of n-jets of planar vector fields, such that a field F of multiplicity μ at the origin has a characteristic orbit (resp., is monodromic) if and only if its jet $j^n F$ belongs to $C^{(n)}$ (resp., $M^{(n)}$).*

In other words, Theorem 10.21 means that the combined properties "characteristic orbit & multiplicity $\leqslant \mu$" and "monodromic singularity & multiplicity $\leqslant \mu$" are *algebraically decidable* at the level of n-jets for some $n = n(\mu)$. The complement to $C^{(n)} \cup M^{(n)}$ consists of jets of singularities of multiplicity greater than μ.

Sketch of the proof of Theorem 10.21. By Theorem 10.19 and Remark 10.20, in all sufficiently high order jets there exists semialgebraic subsets guaranteeing that the corresponding singularities have multiplicity $\leqslant \mu$. By the Desingularization Theorem 8.14, any such singularity can be completely resolved into elementary singularities in a bounded (in terms of μ) number of *steps* (consecutive simple blow-ups).

As follows from Proposition 10.14, to decide between characteristic and monodromic cases, it is sufficient to identify ("recognize") the location and topological types of these elementary singularities which appear after complete desingularization.

Non-degenerate singularities (saddles and nodes) can be recognized looking at their 1-jets; the criteria (inequalities for the discriminants of characteristic polynomials of degree 2) are obviously semialgebraic in the elements of the linearization matrices.

Degenerate isolated elementary singularities (of finite multiplicity μ) can be either saddles or saddle-nodes. To decide between these two types, one has to know the jet of order μ, as will be shown in §11.3. The test condition is polynomial.

Finally, the decision on whether a given non-elementary singularity has a dicritical blow-up or not, depends on the terms of lower order (and is obviously expressed by an algebraic condition involving these terms). Since the order of a singularity cannot exceed its multiplicity (as follows from [**AGV85**, Lemma 1, §5.5] already cited in the proof of Theorem 10.19), we arrive at the following conclusion: *existence of a characteristic orbit can be expressed as a semialgebraic condition on the jets of order* $\leqslant \mu + 1$ *at all singularities that appear in the process of complete desingularization.*

Inspection of the process shows that the multiplicities and hence orders of all intermediate singularities do not exceed $\mu + 1$, while the number of steps in the desingularization process is also bounded in terms of μ (does not exceed $2\mu + 1$ by Theorem 8.15 and even $\mu + 2$ by [**Kle95**]). Thus all information sufficient to determine uniquely the desingularization process and the topological types of elementary singularities that appear after this construction terminates, is contained in a sufficiently high order jet of the initial singularity. The order $n = n(\mu)$ of this jet should be so large as to determine uniquely $(\mu + 1)$-jets at all intermediate singularities.

Consider an isolated singularity of order ν (hence of multiplicity $\leqslant \nu$) and its blow-up. The corresponding transformation of the Pfaffian equation involves change of variables from (x, y) to (x, z), $z = y/x$, and division by an appropriate power of x, more precisely, by x^ν in the non-dicritical case and by $x^{\nu+1}$ in the dicritical case respectively. This construction implies that jets of order $k+\nu$ (respectively, $k+\nu+1$) at the initial point determine uniquely the jet of order k at any singularity that appears on the exceptional divisor after blow-up. Clearly, the formulas describing the transformation on the level of jets, are (real) algebraic.

Iterating these arguments, one obtains an upper bound for the order $n(\mu)$ of the initial jet that encodes all $(\mu + 1)$-jets on all $\mu + 2$ steps of the desingularization process. In other words, all representatives of n-jets of vector fields of multiplicity μ have the same desingularization schemes and the same jets of order $\mu + 1$ at all elementary singular points of multiplicity $\leqslant \mu + 1$ that appear after complete desingularization.

Based on this information and a computable algorithm of detecting topological types of elementary singularities which will be discussed in more details in §11, one can use Proposition 10.14 to write down explicitly the semialgebraic conditions necessary and sufficient for existence of a characteristic orbit. □

10.9. Topologically sufficient jets. Theorem 10.21 asserts that the principal alternative of monodromic *vs.* characteristic singular points, is algebraically decidable. After working out some additional details, this result may be further improved: assuming existence of the characteristic orbit, the entire *topological type* of the singularity is determined by its sufficiently high order jet.

Definition 10.22. An m-jet of a planar vector field is called *topologically sufficient*, if any two real analytic vector fields extending this jet, are topologically equivalent to each other.

Theorem 10.23 (O. Kleban [**Kle95**]). *For an isolated singularity of planar vector field of multiplicity μ, its $2\mu+2$-jet is topologically sufficient.*

Idea of the proof. As follows from Theorem 8.14 and 8.15, the "scheme of desingularization" (the number and choice of centers of subsequent blow-ups until all singular points become elementary) is completely determined by the jet of some order $n = n(\mu)$ depending only on the multiplicity μ of the initial non-elementary (real) analytic singularity.

It is rather clear that for two singularities of vector fields to be topologically equivalent, it is *not sufficient* to have the same desingularization schemes with topologically equivalent elementary singularities. Indeed, what is important is not only the topological types of singularities, but also their position relative to the vanishing divisor D.

In particular, if some of the blow-ups were dicritical, the blow-up foliation will be non-transversal to D. Such *tangency points* may be of different topological types. Besides, the singular points that can appear by a dicritical blow-up, may produce different sectors depending on the relative position of the vanishing divisor and the invariant manifolds of these points.

However, as was mentioned in §8.11, using *additional blow-ups*, one can resolve such singularities and remove all nontrivial tangencies, see [**Kle95, Dum77**]. For instance, a point where nonsingular phase curves have quadratic tangency with the exceptional divisor D at a smooth point of the latter, by two blow-up can be transformed to two *singular* points (nondegenerate saddle) in such a way that the tangency disappears, see Figure 5(a). One can show that using such additional blow-ups, one can assure that the foliation has only singular points of the types shown on Figure 5(b), in addition to the elementary singularities at the middle and corner points which can appear by non-dicritical blow-ups. Computations similar to those proving Theorem 8.15 show that to reach this goal, it is sufficient to do no more than $\mu+2$ blow-ups, the result being determined by $2\mu+2$-jet of the initial singular point. The core of the argument is the inequality (8.39)

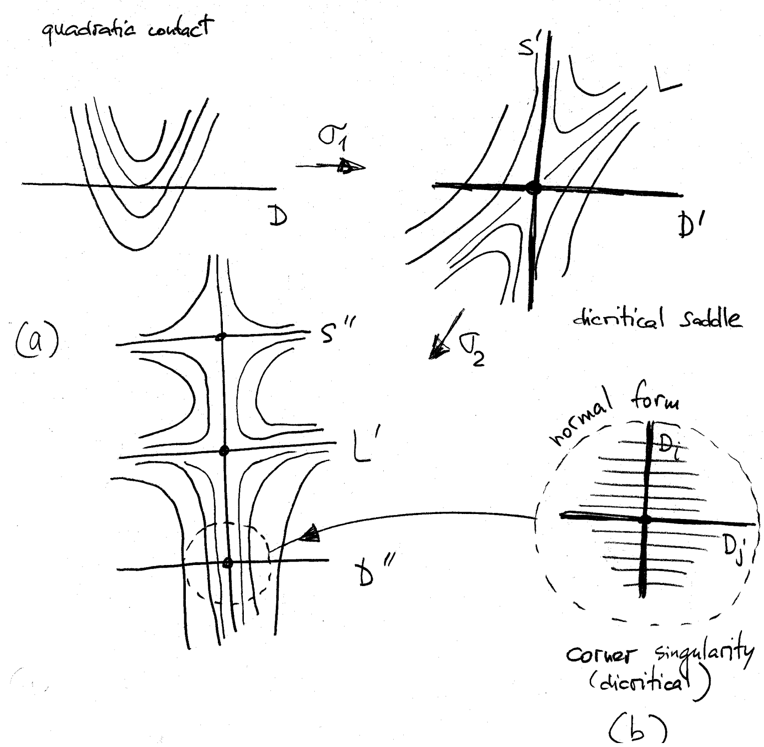

Figure 5. Resolution of points of contact: (a) quadratic tangency, (b) additional type of corner dicritical singularity.

which provides an upper bound for the total order of tangency between the foliation and the vanishing divisor after a dicritical blow-up.

Now one can refer to a (rather intuitively obvious) result from [**Dum77**]: two germs of smooth vector fields with characteristic orbits, which have topologically equivalent elementary singularities and no tangency with the exceptional divisor, are topologically equivalent. This completes the proof; the details can be found in [**Dum77, Kle95**]. □

10.10. Concluding remarks. Together with the previously established results, Theorem 10.23 proves that for any finite value of μ, the space $J^n = J^n\mathcal{D}(\mathbb{R}^2, 0)$ of n-jets of planar vector fields for $n \geqslant 2\mu + 2$ has the following structure of the disjoint union:

$$J^n = C \sqcup M \sqcup Z, \qquad C = \bigcup_{\alpha=1}^{N} C_\alpha.$$

Here C is the subset of jets sufficient to guarantee existence of the characteristic orbit whose different components C_α correspond to topologically different germs of vector fields, M consists of jets sufficient to guarantee that all their representatives are monodromic, and Z is the collection of jets whose representatives have multiplicity $\geqslant \mu + 1$. All three sets are semialgebraic and their defining equation depend only on μ as soon as $n \geqslant 2\mu + 1$.

In its turn, all jets from C are topologically sufficient and guarantee that their representatives belong to one out of N different topological types (their number N depends on μ). Though it seems to be not rigorously proved anywhere, there is no doubt that the respective components are semialgebraic in J^n.

This means that the topological classification of singularities of any finite multiplicity is algebraically decidable under an additional semialgebraic assumption that there exists a characteristic trajectory and the multiplicity μ of the singularity is an apriori known parameter.

On the complementary set one can have three possibilities: a monodromic isolated singularity can be a focus, a center or an accumulation point of infinitely many limit cycles. The latter case is forbidden by the Nonaccumulation theorem, see §10.3, so that one can discuss the *center–focus alternative*. In the next section we will introduce a class of generalized elliptic singularities for which the Nonaccumulation theorem holds true for trivial reasons and show that in general the center–focus alternative is *not* algebraically solvable.

11. Algebraic decidability of local problems. Center–focus alternative

Decidability of the principal alternative (characteristic *vs.* monodromic case) and the topological classification was discussed in §10 in the presence of the additional integer parameter, the multiplicity μ of the singular point. For any finite value of this parameter the problems turned out to be algebraically solvable, but it is not clear if one can omit explicitly mention of μ in the formulation, considering *all* finite values of μ. Besides purely logical reasons (one may be reluctant to self-impose apriori restrictions), there are situations when multiplicity is irrelevant for the topological classification. One such example is exactly the center-focus alternative.

The general notion of *algebraic decidability* was introduced by V. Arnold in [**Arn70a, Arn70b**], see also [**Arn83**, §37], where he proved that (i) the Lyapunov stability for singularities in dimension $n \geqslant 3$ and (ii) topological classification of holomorphic singular foliations in $(\mathbb{C}^2, 0)$ are algebraically undecidable.

We discuss decidability of the topological classification for *real* planar singularities. The principal result of this section concerns decidability of classification for degenerate elementary singularities and undecidability in general for for monodromic singularities.

11.1. Decidability in the jet spaces. The space of germs of real analytic vector fields $\mathcal{D}(\mathbb{R}^2, 0)$ (or, what is the same in the planar case, the space of germs of real analytic 1-forms $\Lambda^1(\mathbb{R}^2, 0)$) is infinite-dimensional and thus the decidability of subsets of this space cannot be defined in terms of semialgebraic sets. Yet this infinite-dimensional space is naturally endowed with infinitely many projections j^k associating with each germ its k-jet at the singular point. The jets of any finite order form a finite-dimensional space with the natural affine stricture. Thus one can define decidable sets of germs in terms of decidability of their jet projections.

Consider a subset M in a space of all analytic germs $\mathcal{G}$, for example, in the space of germs of 1-forms $\mathcal{G} = \Lambda^1(\mathbb{R}^2, 0) = \Lambda^1$. By $J^k(\mathcal{G})$ we will denote the finite-dimensional space of k-jets of germs from $\mathcal{G}$.

Definition 11.1. A set $M \subset \mathcal{G}$ is *algebraically decidable to codimension* $r \in \mathbb{N}$, if for some jet order k there exist two disjoint semialgebraic subspaces $S_k^{\pm} \subseteq J^k(\mathcal{G})$ such that:

(1) any germ whose k-jet belongs to S_k^+, necessarily belongs to M;
(2) any germ whose k-jet belongs to S_k^-, necessarily belongs to the complement $\mathcal{G} \smallsetminus M$;
(3) the complement $N_k = J^k(\mathcal{G}) \smallsetminus (S_k^+ \cup S_k^-)$, automatically semialgebraic, has codimension $\geqslant r$ in $J^k(\mathcal{G})$.

Jets from the subsets $S_k^{\pm}$ are referred to as *sufficient* jets, while the complementary set N_k consists of *neutral* jets.

Usually the terminology of sufficient or neutral jets applies to sets of germs defined by certain *properties* (e.g., topological type of the phase portrait for 1-forms, extremum type for functions *etc.*).

Formally, algebraic decidability means that the set M in the infinite-dimensional space can be approximated from two sides by "cylindrical" semialgebraic subspaces,

$$\mathcal{S}_k^+ \subseteq M \subseteq \mathcal{G} \smallsetminus \mathcal{S}_k^-, \qquad \mathcal{S}_k^{\pm} = (j^k)^{-1}(S_k^{\pm}),$$

so that the "accuracy" of this approximation, $\mathcal{N}_k = \mathcal{G} \smallsetminus (\mathcal{S}_k^+ \cup \mathcal{S}_k^-)$, has a well-defined codimension that is at least r.

Remark 11.2. The order k of the jets is not as important as the codimension r. More precisely, it is sufficient to guarantee that at least one such order exists. Then in any higher order jet space $J^l(\mathcal{G})$, $l > k$, one can immediately construct the partition into three semialgebraic sets $S_l^{\pm}, N_l$ with the required properties, taking $S_l^{\pm}$ as all l-jets whose truncation to order k lies in $S_k^{\pm}$ respectively.

Then the codimension of N_l inside $J^l(\mathcal{G})$ will be the same as the codimension of N_k, that is, at least r.

However, one may hope that using higher order jet space one can approximate M with *more* accuracy.

Definition 11.3. A subset M of the space of germs is algebraically decidable to *infinite codimension* (or simply *decidable*), if it is algebraically decidable to any finite codimension r.

According to this definition, there exists an infinite sequence of two-sided semialgebraic cylindrical approximations for M,

$$\cdots \subseteq \mathcal{S}_k^+ \subseteq \mathcal{S}_{k+1}^+ \subseteq \cdots \subseteq M \subseteq \cdots \subseteq (\mathcal{G} \smallsetminus \mathcal{S}_{k+1}^-) \subseteq (\mathcal{G} \smallsetminus \mathcal{S}_{k+1}^-) \subseteq \cdots$$

such that, unless the stabilization occurs and $\mathcal{N}_k = \varnothing$ for some k, the codimension of the decreasing differences $\mathcal{N}_k = \mathcal{G} \smallsetminus (\mathcal{S}_k^+ \cup \mathcal{S}_k^-)$ grows to infinity:

$$\mathcal{G} \supseteq \mathcal{N}_1 \supseteq \cdots \supseteq \mathcal{N}_k \supseteq \mathcal{N}_{k+1} \supseteq \cdots, \qquad \operatorname{codim}_{\mathcal{G}} \mathcal{N}_k \to +\infty.$$

The intersection $\mathcal{N}_\infty = \bigcap_{k \geqslant 0} \mathcal{N}_k$, eventually empty even if all $\mathcal{N}_k$ are nonzero, may still be nontrivial, since the space of germs $\mathcal{G}$ is infinite-dimensional.

Definition 11.4. The subset $M \subseteq \mathcal{G}$ is *ultimately* (algebraically) decidable, if the intersection $\mathcal{N}_\infty$ entirely belongs either to M or to its complement.

Remark 11.5. Speaking in terms of algorithms, a set of germs $M \subseteq \mathcal{G}$ is decidable (i.e., algebraically decidable to infinite codimension), if there exists an algorithm that allows for any given germ $g \in \mathcal{G}$ to verify whether it belongs to M or not. This algorithm must be algebraic, meaning that it tests conditions expressed by polynomial equalities and inequalities on Taylor coefficients. On each step either the decision is made, whether $g \in M$ or $g \notin M$, or the computations should be continued involving higher order Taylor coefficients. The algorithm should terminate for almost all germs except for an eventual set of infinite codimension. The set is *ultimately* decidable, if all germs on which the algorithm never stops, belong to M or its complement simultaneously.

Remark 11.6. The definition of decidability admits possible variations. Clearly, the constructions remain the same for any other types of germs, e.g., vector fields in $(\mathbb{R}^n, 0)$, as well as for the holomorphic objects, e.g., holomorphic diffeomorphisms $\operatorname{Diff}(\mathbb{C}, 0)$. In the latter case the jet spaces are complex and one has to explain what means semialgebraicity in the complex space $\mathbb{C}^k$. By definition, it means quasialgebraicity in its *realification* $\mathbb{R}^{2k}$.

Remark 11.7. One more variation appears when instead of just two sets M and $\mathcal{G} \smallsetminus M$, there is given a partition of the total space of germs into finitely many sets $M_1, \dots, M_m$, $m \geqslant 2$, pairwise disjoint (as is typical for classification problems with several possible normal forms). The decision problem in this context is algebraically solvable, if there can be constructed pairwise disjoint semialgebraic subsets $S_k^t \in J^k(\mathcal{G})$, $t = 1, \dots, m$, $k = 1, 2, \dots$, which exhaust J^k in the sense that the complement $N_k = J^k(\mathcal{G}) \smallsetminus \bigcup_t S_k^t$ of neutral ("undecided") jets has codimension growing to infinity together with k. The decidability is *ultimate*, if the intersection $\mathcal{N}_\infty = \bigcap_{k \geqslant 0} (j^k)^{-1}(N_k)$ belongs to *only one* of sets $M_1, \dots, M_m$ (classification types).

The classification problems are seldom decidable in the whole set of germs. However, some parts of the respective subsets (and sometimes large parts) can be.

Let $\mathcal{B} \subset \mathcal{G}$ be a semialgebraic subset in the space of germs. By definition, this means that for some l there is a semialgebraic subset $B_l \subset J^k(\mathcal{G})$ such that $\mathcal{B} = (j^l)^{-1}(B_l)$.

Definition 11.8. A subset M is decidable (resp., ultimately decidable) *relative* to a semialgebraic set $\mathcal{B}$, if the corresponding sufficient sets $S_k^\pm$ are semialgebraic in the intersection with $B_k = \{j^k g : j^l g \in B_l\}$ for all $k \geqslant l$.

When speaking about classification problems or alternatives, discussing relative decidability means that from the outset the problem is *restricted* only on a subclass of germs already defined by some semialgebraic conditions on their l-jets. In this case the relative (ultimate or not) decidability means that the property is determined by algebraic conditions imposed on the higher order jets. Sometimes we say about decidability of an alternative *for the specific class*. For example, the center-focus alternative is undecidable in general, but decidable (and even ultimately decidable) for germs with nondegenerate linear part, see §11.4.

Having introduced this formal language, we will immediately switch back to informal description.

11.2. First examples of decidability. The most obvious example of an algebraically decidable problem is the problem of determining the type of local extremum for functions of one real variable.

For this problem any nonzero jet is sufficient: if $f(x) = ax^k + \cdots$, $a \neq 0$, then the point $x = 0$ is a local maximum, minimum of monotonicity point, depending on the sign of a and parity of k. Zero jets form the neutral subset, and knowledge of further Taylor coefficients is required.

The classification is *ultimately* decidable for real analytic germs: if all Taylor coefficients are zero, then $f \equiv 0$ and such germs constitute a separate topological type. On the contrary, for infinitely smooth germs the ultimate decidability fails: there exist flat functions with zero Taylor series and any type of local extremum.

We will discuss now two easy but less artificial examples, both concerning relative decidability.

Example 11.9. In the space $\mathcal{G} = \operatorname{Diff}(\mathbb{R}, 0)$ the Lyapunov stability and asymptotic stability is ultimately algebraically decidable for germs tangent to identity, $g(x) = x + O(x^2)$.

Indeed, any jet $g(x) = x + a_2x^2 + \cdots a_kx^k + \cdots$ with $a_2 = \cdots = a_{k-1} = 0$ and $a_k \neq 0$ is sufficient. The germs whose any jet is neutral (not sufficient), are only identical, $g(x) \equiv x$. This germ is Lyapunov stable, but not asymptotically stable.

As in the case of extrema of functions, the ultimate decidability fails for infinitely smooth diffeomorphisms.

Example 11.10. The same question for the class of *flip germs* tangent to the involution, $g(x) = -x + O(x^2)$.

The square $g^2 = g \circ g \in \operatorname{Diff}(\mathbb{R}, 0)$ belongs to the class considered in the previous example. The coefficients b_j of $g \circ g = x + b_2x^2 + b_3x^3 + \cdots$ are polynomially depending on the coefficients a_j of the jet $g = -x + a_2x^2 + a_3x^3 + \cdots$:

$$b_2 = a_2 - a_2 = 0, \quad a_3 = -2(a_2 + a_3), \quad \ldots \tag{11.1}$$

Algebraicity of the stability conditions on $h = g \circ g$ implies the ultimate decidability of the stability (both asymptotic and Lyapunov) for the flip germs.

Example 11.11. Using arguments similar to those from the proof of Theorem 10.19, one can prove that the property of having an isolated singularity is ultimately algebraically decidable.

Example 11.12. Consider the *periodicity alternative*: for a given $m \in \mathbb{N}$ determine, whether $g \in \operatorname{Diff}(\mathbb{C}^n, 0)$ is periodic of period $\leqslant m$.

The same arguments as before show that the periodicity alternative is ultimately decidable for any finite m.

Example 11.13 (Warning)**.** The periodicity alternative *without specifying the period* is not algebraically decidable. Indeed, if it were, then restricted on linear maps $g\colon \mathbb{C} \to \mathbb{C}$, $z \mapsto \nu z$, it should distinguish periodic maps from aperiodic by a semialgebraic test. Yet clearly the set $2\pi i\mathbb{Q}$ of values ν corresponding to periodic germs is not semialgebraic (a dense subset of the circle). This example shows why, say, an alternative that is algebraically decidable for singularities of any finite multiplicity μ (cf. with §10.8), can cease to be decidable without restricting μ.

11.3. Topological classification of degenerate elementary singularities on the plane. In this section we discuss algebraic decidability of topological classification of isolated *degenerate elementary* singularities.

An isolated degenerate elementary singular point of a real analytic vector field on the real plane $(\mathbb{R}^2, 0)$ may be of three topological types: saddle-node, topological node or topological saddle, represented by the three standard models as described in §10.1. We show that this classification is algebraically decidable to infinite codimension and, moreover, ultimately decidable. This classification problem constitutes perhaps the simplest nontrivial example of algebraic decidability.

To fit the formal settings, we consider only the subspace $\mathcal{B}_{\text{elem}} = \mathcal{B}$ of germs of holomorphic 1-forms having one zero and one nonzero eigenvalue of the linearization: on the level of 1-jets this subspace is determined by the semialgebraic conditions $\det A = 0$, $\operatorname{tr} A \neq 0$ on the linearization matrix A of the corresponding vector field. Without loss of generality we may assume that A is already reduced to the diagonal form, so that

$$\mathcal{B} = \{\omega \colon j^1\omega = y\,dx\} \subset \Lambda^1(\mathbb{R}^2, 0).$$

The three subsets of $\mathcal{B}$, corresponding to different topological types, will be denoted M_S (saddles), M_N (nodes), M_{SN} (saddle-nodes). However, for the sake of completeness one has to introduce the fourth class $M_I \subseteq \mathcal{B}$ of germs having a non-isolated singularity (such germs become nonsingular after division by a non-invertible function $y + \cdots$). Clearly, then

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$$\mathcal{B} = M_S \sqcup M_N \sqcup M_{SN} \sqcup M_I. \tag{11.2}$$

Theorem 11.14. *The problem of topological classifications of degenerate elementary singular points of analytic vector fields on the real plane is ultimately algebraically decidable.*

Formally the assertion of the Theorem means that the partition (11.2) is ultimately decidable in the sense explained in Remark 11.7. The proof occupies the sections §11.3.1 till §11.3.3.

The decision is very easy for germs in the formal normal form: if

$$\omega = (\pm x^k + a\,x^{2k-1})\,dy + y\,dx, \qquad k \geqslant 2,\ a \in \mathbb{R}, \tag{11.3}$$

then $\omega \in M_{SN}$ if k is even and is *not* in M_{SN} if k is odd (in this case it can be either saddle or node, depending on the sign $\pm$ before x^k). Any singularity in the normal form (11.3) is always isolated, hence cannot be in M_I.

To derive decidability from this observation, one has to show that the normal form is determined by some *semialgebraic* conditions on the jet of an appropriate order and make sure that if all these conditions fail to detect one of the three "isolated" types, the germ is actually of the fourth type (non-isolated).

In principle it is possible to infer the required semialgebraicity by inspection of the Poincaré–Dulac algorithm of transformation to the normal form. However, we suggest a circumventive approach based on constructing jets of *first integrals*. The full power of this approach will be revealed later in §12.

11.3.1. *Topological sufficiency in the normal form.* Denote by $N_k \subseteq J^k = J^k(\Lambda^1)$ the collection of k-jets of 1-forms $y\,dx + \cdots \in \mathcal{B}$ orbitally equivalent to the linear jet $y\,dx$: in suitable coordinates, any germ ω with $j^k\omega \in N_k$, takes the form

$$\omega \in N_k \iff \omega = f(x,y)(y\,dx + \omega'), \qquad \omega' \in \mathfrak{m}^{k+1}\Lambda^1, \quad f(0,0) \neq 0. \tag{11.4}$$

Let $S_k = \mathcal{B} \smallsetminus N_k$ be the complement. We claim that all jets from this complement are topologically sufficient.

Lemma 11.15. *The jets from the set S_k are topologically sufficient. More precisely, germs with the k-jet in $S_k = \mathcal{B}\smallsetminus N_k$ have one of the three "isolated" topological types,*

$$(j^k)^{-1}(S_k) \subseteq M_S \sqcup M_N \sqcup M_{SN}.$$

Proof. If $j^k\omega \notin N_k$, then the k-jet of ω by a polynomial (i.e., jet) orbital transformation can be brought to the form

$$\omega = (\pm x^l + a\,x^{2l-1})\,dy + y\,dx, \qquad 2 \leqslant l \leqslant k, \quad a = 0 \text{ if } 2l-1 > k,$$

similar to (11.3), eventually with a smaller value of the order $l \leqslant k$ and truncated at the level k if necessary.

We claim that ω is a saddle-node, saddle or node depending on the parity of l and the sign (as explained above) regardless of the terms of order $> k$ that may occur after. Clearly, it is sufficient to consider the case $l = k$ only.

By the *center manifold theorem* [**Kel67**], there exists an invariant curve C tangent to the axis $y = 0$ (in general, this center manifold is only finitely smooth, but in the planar case one can prove its C^∞-smoothness, see [**Ily85**]). One can immediately verify that C is tangent to the axis with order $k+1$ at least: $C = \{y = o(x^k)\}$.

Consider a planar vector field $F = (\pm x^k + \cdots)\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}$ dual to the form ω. The k-jet of its restriction on the center manifold C is determined by k and the sign: if x is chosen as the local coordinate on C, then $F|_C =$

$(\pm x^k + o(x^k))\frac{\partial}{\partial x}$; this field is topologically equivalent to the field $\pm x^k \frac{\partial}{\partial x}$ by an orientation-preserving homeomorphism of the x-axis.

By the *Pugh–Shub–Shoshitaishvili reduction principle* [**PS70b, Šoš72, Šoš75**], see also [**Tak71**], any vector field is topologically orbitally equivalent to its linearization along the center manifold. In our case this means that the germ F is topologically orbitally equivalent to the vector field

$$F' = -y\frac{\partial}{\partial y} \pm x^k \frac{\partial}{\partial x}.$$

Topological classification of these fields is obvious. □

Remark 11.16. The description of the jet sets S_k and N_k becomes completely transparent: within the class $\mathcal{B}$ of degenerate elementary singularities, the set S_k corresponds to jets of germs having multiplicity $\leqslant k$, while N_k is the collection of jets of germs with multiplicity $> k$ at the origin,

$$\begin{aligned} S_k &= \{j^k\omega\colon j^1\omega = y\,dx,\ \mu_0(\omega) \leqslant k\}, \\ N_k &= \{j^k\omega\colon j^1\omega = y\,dx,\ \mu_0(\omega) > k\}. \end{aligned} \tag{11.5}$$

Thus Lemma 11.15 can be reformulated as follows.

Corollary 11.17. *The k-jet of a germ $\omega \in \mathcal{B}$ is topologically sufficient, if and only if its multiplicity $\mu_0(\omega)$ is no greater than k.* □

11.3.2. *First integrals and semialgebraicity of S_k.* Now we can explain why the sets S_k, N_k are semialgebraic.

Definition 11.18. A degenerate elementary jet $j^k\omega \in J^k(\Lambda^1)$ with $j^1\omega = y\,dx$ is *integrable*, if there exists a k-jet $j^k u \in J^k(\mathbb{R}^2, 0)$ of a function u such that $j^1 u = x$ and the k-jet of wedge product $\omega \wedge du$ is zero.

We claim that the neutral in the sense of (11.4) (i.e., orbitally linearizable) jets and only them are integrable.

Lemma 11.19.

$$N_k = \{j^k\omega \in J^k(\Lambda^1)\colon \exists j^k u \in J^k(\mathbb{R}^2, 0),\ j^1u = x,\ j^k(\omega \wedge du) = 0\}. \tag{11.6}$$

Proof. Both the assumption and the assertion of the Lemma are invariant by jet orbital equivalence tangent to the identity. Indeed, N_k is already defined in the invariant terms. The equation $j^k(\omega \wedge du) = 0$ is also independent of the choice of local coordinates x, y, so its solvability is also an invariant fact.

Thus it is sufficient to verify the Lemma only for jets already having the normalized form. Clearly, if $j^k\omega = y\,dx$, then the jet $j^k u = x$ *is* the first integral.

Conversely, if $j^k\omega \notin N_k$, then $j^l\omega = y\,dx \pm x^l\,dy$ for some $l \leqslant k$. In this case for any $j^k u = x + j^k u'$, $j^1 u' = 0$, we would have

$$\omega \wedge du = (y\,dx \pm x^l\,dy) \wedge (dx + du') = \pm x^l\,dx \wedge dy + y\,dx \wedge du' \bmod \mathfrak{m}^{l+1}\Lambda^2.$$

But the l-jet of the restriction of the coefficient of this 2-form on $y = 0$ is $\pm x^l \neq 0$, so the whole l-jet is nonzero. Thus a jet $j^k\omega \notin N_k$ cannot be integrable. □

The invariant form provided by Lemma 11.19, immediately allows to prove semialgebraicity of the neutral sets N_k, *without referring to the normalizing chart.*

Lemma 11.20. *The sets $N_k \subset J^k(\Lambda^1)$ are algebraic. Their codimension in $J^k(\Lambda^1)$ grows to infinity together with k.*

Proof. Consider the Taylor polynomial representing k-jet from J^k, writing it as the sum of homogeneous components

$$\omega = y\,dx + \omega_2 + \cdots + \omega_k, \qquad \deg \omega_j = j.$$

Its first integral, if it exists, can be found in the form

$$u = x + u_2 + \cdots + u_k, \qquad u_k, \qquad \deg u_j = j.$$

Substituting these two expansions into the integrability condition and equating homogeneous terms of the wedge product, we obtain a system of equations

$$\begin{aligned} y\,dx \wedge du_2 &= dx \wedge \omega_2, \\ y\,dx \wedge du_3 &= dx \wedge \omega_3 + du_2 \wedge \omega_2, \\ &\vdots \\ y\,dx \wedge du_k &= dx \wedge \omega_k + du_2 \wedge \omega_{k-1} + \cdots + du_{k-1} \wedge \omega_2. \end{aligned} \tag{11.7}$$

This is a *linear* system with respect to the homogeneous components $u_2, \dots, u_k$. The coefficient matrix of this system contains linear combinations of coefficients of the homogeneous components of the forms $\omega_2, \dots, \omega_k$.

The well-known criterion of solvability of systems of linear nonhomogeneous equations is that the rank of the matrix of its coefficient should be equal to the rank of the extended matrix obtained by adjoining the column of the free terms. This rank condition is *polynomial* with respect to the entries of the matrices.

To see why the codimension of the set N_k in $J^k(\Lambda^1)$ is growing with k, it is enough to observe that with each new line in (11.7), coefficients of the new form ω_k enter for the first time in a nontrivial way in the free terms column. □

Remark 11.21. The *semi*algebraicity of the set N_k can be seen without any computations. Indeed, it is a projection on the second component of the *algebraic* subset of the Cartesian product, defined by bilinear equations as follows,

$$(j^k u, j^k \omega) \in J^k(\mathbb{R}^2, 0) \times J^k(\Lambda^1), \quad j^1\omega = y\,dx,\ j^1 u = x,\ j^k(\omega \wedge du) = 0.$$

By the Tarski–Seidenberg principle, the projection is semialgebraic.

Remark 11.22. Partition of the sufficient sets $S_k \subset J^k(\Lambda^1)$ into subsets S_k^S, S_k^N, S_k^{SN} corresponding to the topological classes M_S, M_N, M_{SN}, can also be done in terms of first integrals, though the description is more technically involved.

Yet if we are interested only in establishing the *semialgebraicity* of these sets, then it can be derived from the algebraicity of N_k. Indeed, consider the algebraic subsets

$$B_{k+1} = (j^k)^{-1}(N_k) \subseteq J^{k+1}(\Lambda^1)$$

of $(k+1)$-jets whose k-truncation is neutral. The set N_{k+1} is algebraic in B_k by Lemma 11.20 the connected components of the semialgebraic complement $B_{k+1} \smallsetminus N_{k+1}$ belong to only one of the three classes. But a connected component of a semialgebraic set is itself semialgebraic. The sets S_{k+1}^t are obtained by attaching some of these components to the respective preimages $(j^k)^{-1}(S_k^t)$ for all $t = S, N, SN$.

11.3.3. *Ultimate decidability.* In topological classification of degenerate elementary singularities, ultimate decidability is an easy fact.

Proposition 11.23.

$$\bigcap_{k \geqslant 2} (j^k)^{-1}(N_k) = M_I.$$

Proof. By Corollary 11.17, the set $(j^k)^{-1}(N_k)$ consists of all analytic germs of multiplicity $> k$. Thus the intersection $\mathcal{N}_\infty = \bigcup (j^k)^{-1}(N_k)$ cannot include any germ of finite multiplicity, i.e., all germs from $\mathcal{N}_\infty$ are non-isolated. □

Remark 11.24. We can formulate now the algorithm for decision on the topological type of an elementary singularity $\omega = y\,dx + \omega_2 + \omega_3 + \cdots$ of finite multiplicity μ.

One has to resolve recursively the (infinite) system of linear equations (11.7), determining the homogeneous components $u_2, u_3, \dots$ of the first integral. If at some moment k this is impossible (the corresponding linear equation is not solvable), then the singularity is of one of the 3 types (saddle, node or saddle-node). To decide between them, one has to compute the

k-jet of the central manifold and compute the sign of the leading term of restriction on it.

The algorithm stops no later after μ steps, where $\mu = \mu_0(\omega)$ is the multiplicity of ω.

11.4. Generalized elliptic points and center–focus alternative. Ultimate decidability of degenerate elementary singular points is in a sense a model problem serving to illustrate the concepts and use of some important tools. On the contrary, the problem of distinction between center and focus traditionally, since the times of Poincaré, is one of the most challenging in the qualitative theory of ordinary differential equations on the plane. We discuss this problem (in terms of algebraic decidability) for *generalized elliptic* singularities. By definition, generalized ellipticity means that the principal homogeneous terms guarantee nonexistence of characteristic trajectories, so that generalized elliptic singularities are always monodromic and the only two possible topological types for them are center and focus. In this section we show that the center–focus alternative for generalized elliptic singularities is *ultimately algebraically decidable* if the principal homogeneous part is fixed. Yet if the principal part is variable, the boundary between stable and unstable foci is non-algebraic, as will be shown in §11.7. This undecidability was first conjectured by A. Brjuno and proved in [**Ily72a**]. We give a simplified proof.

Consider a monodromic singular point on the real plane. As was mentioned in Example 10.10, this point may be focus, center or (apriori, at least while Nonaccumulation Theorem is not proved in full generality) an accumulation point for limit cycles. We introduce a class of singularities (*generalized elliptic points*), for which infinite accumulation is impossible and study decidability of the center–focus alternative for such singularities.

Everywhere in this section we use the Pfaffian forms. Consider the real singular foliation $\omega = 0$ defined by the real analytic Pfaffian form whose expansion into homogeneous components begins with terms of order n,

$$\begin{gathered}\omega = \omega_n + \omega_{n+1} + \cdots, \quad \omega_k = p_k(x,y)\,dx + q_k(x,y)\,dy, \quad n \geqslant 1, \\ p_k, q_k \in \mathbb{R}[x,y], \quad \deg p_k = \deg q_k = k, \quad k = n, n+1, \dots. \end{gathered} \tag{11.8}$$

Definition 11.25. The singular point is called *generalized elliptic*, if the *real* homogeneous polynomial $h_{n+1} = yp_n + xq_n \in \mathbb{R}[x,y]$ is nonvanishing except at the origin,

$$h_{n+1}(x,y) = xp_n(x,y) + yq_n(x,y) \neq 0 \qquad \text{for} \quad (x,y) \in \mathbb{R}^2 \smallsetminus (0,0). \tag{11.9}$$

This definition is in fact invariant by real analytic transformations, as we shall see in a moment.

Example 11.26. If two eigenvalues $a \pm ib$ of the linearization of the real analytic vector field on the plane, are non-real, $b \neq 0$, then the singularity is generalized elliptic.

Indeed, after a suitable linear transformation the linearization matrix can be brought to the form $\left(\begin{smallmatrix} a & b \\ -b & a \end{smallmatrix}\right)$ with real $a, b \in \mathbb{R}$. The corresponding dual form ω is $(ax + by)\,dy + (bx - ay)\,dx$, and the polynomial $h_2 = x(bx - ay) + y(ax + by) = b(x^2 + y^2)$ is nonvanishing outside the origin.

The (non-universal) term *generalized* elliptic is motivated by the following universally accepted definition.

Definition 11.27. The singular point of a planar vector field is *elliptic*, if the eigenvalues of its linearization are non-real complex conjugate (in particular, nonzero).

By this definition, a *linear* elliptic singularity is a center if the two eigenvalues are imaginary (with zero real part) and a focus otherwise.

Consider the complexification of a singularity (11.8) and its subsequent blow-up. By definition, this is a singular holomorphic foliation $\widetilde{\mathcal{F}}$ defined in a small complex neighborhood of the exceptional divisor $S = \mathbb{C}P^1$ in a complex 2-dimensional surface C. This surface is covered by the two charts, (x, z), $z = y/x$, and (y, w), $w = x/y$ respectively. In the chart (x, z) the blow-up foliation is defined by the Pfaffian form

$$\begin{aligned} \omega = \bigl(h_{n+1}(1, z) + x h_{n+2}(1, z) + x^2 h_{n+3}(1, z) + \cdots\bigr)\, dx + \\ + x\left(q_n(1, z) + x q_{n+1}(1, z) + x^2 q_{n+2}(1, z) + \cdots\right)\, dz, \end{aligned} \tag{11.10}$$

Here $h_{k+1} = x p_k + y q_k$ are homogeneous polynomials of degree $k + 1$ in two variables, see §8.5.1.

The singular points on the exceptional divisor are roots of the polynomial $p_n(1, z) + z q_n(1, z) = x^{-(n+1)} h_{n+1}(x, xz)$. For a generalized elliptic singularity this polynomial is not identically zero, hence the blow-up is always non-dicritical and Definition 11.25 guarantees that there are no singular points on the real line $\mathbb{R} \subset S$ in the chart (x, z). For similar reasons the point $z = \infty$ (mapped as $w = 0$ in the second chart) is also non-singular.

Thus we obtain an invariant description of generalized elliptic singularities.

Corollary 11.28 (invariant definition of generalized elliptic singularities)**.** *A real analytic singularity is generalized elliptic, if and only if it is non-dicritical (in the sense of Definition 8.16) and after the blow-up has no singularities on the real projective line $\mathbb{R}P^1 \subset \mathbb{C}P^1$ of the exceptional divisor.* □

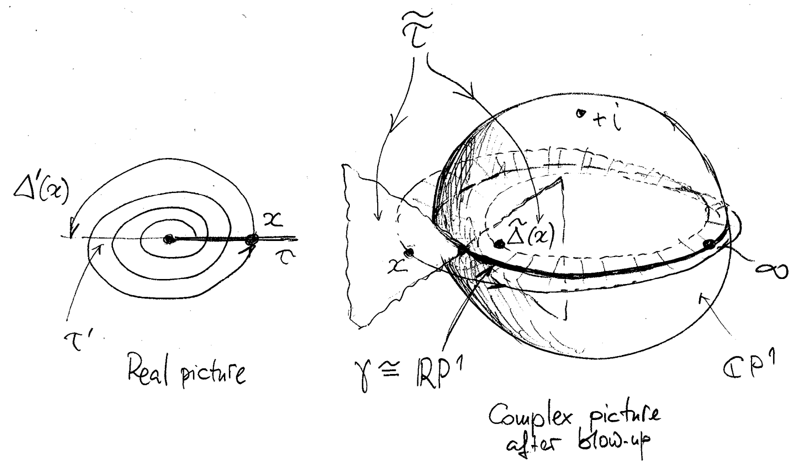

Figure 6. Real equator and its complexification

Elliptic singularity whose linearization matrix A is normalized to $A = \left(\begin{smallmatrix} a & b \\ -b & a \end{smallmatrix}\right)$, after blow-up has two singular points at $z = \pm\sqrt{-1}$.

The real projective line $\mathbb{R}P^1$ is a closed loop on the Riemann sphere $\mathbb{S}^2 \simeq \mathbb{C}P^1$, which is "visible" as the real line $\mathbb{R}$ in the affine chart $\mathbb{C}^1$. Thus the holonomy map $\Delta_{\mathbb{R}}$ along this loop is well defined, e.g., for the cross-section $\tau = \{z = 0\}$ with the coordinate x as a local chart on it. As the form ω was *real* analytic, the blow-up is a well-defined real singular foliation on the Möbius band which is the neighborhood of its central circle. The holonomy map is therefore real analytic.

Note, however, that this loop *does not* belong entirely to any of the two canonical charts: to compute the holonomy, one has to "continue" across infinity $z = \infty$, that is, pass to the other chart.

Still this problem can be easily avoided after complexification: if the singularity is generalized elliptic, the holonomy can be computed in the chart (x, z) as the result of analytic continuation along the semi-circular loop $[-R, R] \cup \{|z| = R, \operatorname{Im} z > 0\}$ homotopic to $\mathbb{R}P^1$.

The holonomy operator $\Delta_{\mathbb{R}}$ is visible on the real plane $(\mathbb{R}^2, 0)$ before the blow-up: the cross-section τ blows down as the x-axis on the (x, y)-plane. By construction, $(\Delta_{\mathbb{R}}(x), 0)$ is the first point of intersection with the x-axis of a solution starting at $(x, 0)$, after continuation counter-clockwise. The standard monodromy is the *square* $\Delta_{\mathbb{R}} \circ \Delta_{\mathbb{R}}$ of the holonomy.

Definition 11.29. The holonomy map $\Delta_{\mathbb{R}}$ (as well as its complexification) is called the *semi-monodromy* of a generalized elliptic singular point.

The complex description of the semi-monodromy immediately allows to prove analyticity of it and the full monodromy, and hence the non-accumulation result.

Theorem 11.30. *The semi-monodromy of a generalized elliptic singular point is real analytic on* $(\mathbb{R}, 0)$*, including the origin.*

If the Pfaffian form or the vector field depends analytically on additional parameters, the semi-monodromy depends analytically on these parameters as far as the singularity remains generalized elliptic. □

Corollary 11.31. *Limit cycles cannot accumulate to a generalized elliptic point.* □

11.5. Relative decidability of center–focus alternative for generalized elliptic singularities. Corollary 11.31 means that decision between center and focus is the true alternative for generalized elliptic points (no third possibility exists). It is equivalent to the *periodicity alternative* (see Example 11.12) for the semi-monodromy, namely, testing whether $\Delta_{\mathbb{R}}$ is of period 2. The latter alternative is ultimately algebraically decidable *in terms of the coefficients of the map* $\Delta_{\mathbb{R}}$. Thus decidability of the center-focus alternative is reduced to *algebraic computability* of the Taylor coefficients of $\Delta_{\mathbb{R}}$ via the Taylor coefficients of the form ω.

The Pfaffian equation $\omega = 0$ which after blow-up takes the form (11.10) in the chart (x, z) can be rewritten as a convergent expansion

$$dx = x\theta_1 + x^2\theta_2 + x^3\theta_3 + \cdots, \tag{11.11}$$

where θ_i are *rational* 1-forms,

$$\theta_i = R_i(z)\,dz, \qquad i = 1, 2, \dots,$$

all holomorphic (nonsingular) outside the polar locus

$$\Sigma = \{z \in \mathbb{C} \colon h_{n+1}(1, z) = 0\}.$$

This expansion can be obtained by division of both parts of (11.10) by the holomorphic function $\sum_{j\geqslant 0} x^j\, h_{n+1+j}(1, z)$ non-vanishing on the line $\{x = 0\} \smallsetminus \Sigma$. In particular,

$$\theta_1 = -\frac{q_n(1, z)\,dz}{h_{n+1}(1, z)}. \tag{11.12}$$

The equation (11.11) can be rewritten the other chart (y, w) of the blow-up, using the change of variables $z = 1/w$, $x = yw$. After this change we obtain the equation

$$dy = y\vartheta_1 + y^2\vartheta_2 + \cdots, \qquad \vartheta_1 = \theta_1 - \frac{dw}{w}, \quad \vartheta_k = w^{k-1}\theta_k,\ k \geqslant 2. \tag{11.13}$$

The nontrivial formula for transition from θ_1 to ϑ_1 is the consequence of the fact that the complex Möbius band C on which the blow-up is defined, *is not* the product $\mathbb{C}P^1 \times (\mathbb{C}, 0)$. The *linearization form* θ_1 should rather be considered as a meromorphic connection on the nontrivial line bundle (cf. with remark 9.9 and especially §25.7).

Remark 11.32. Conversely, the holomorphic (convergent) Pfaffian equation (11.11) is always the blow-up of an appropriate equation $\omega = 0$ with a holomorphic form ω having an isolated singularity at the origin, *provided that* the point at infinity $z = \infty$ is a nonsingular or at worst a finite order pole for *all* forms ϑ_k (meaning that $\sup_k \operatorname{ord}_{w=0} \vartheta_k < +\infty$).

In particular, assume that $\Sigma \subset \mathbb{C}$ is a finite set (necessarily symmetric with respect to the involution $z \mapsto \bar{z}$), disjoint with the real axis $\Sigma \cap \mathbb{R} = \varnothing$, and θ_k are rational forms whose singularities always belong to Σ. Then the equation (11.11) corresponds to a generalized elliptic singularity, if the point $w = 0$ is *nonsingular* for all forms ϑ_k, i.e., when

$$\theta_1 + \frac{dz}{z}, \ z^{-1}\theta_2, \ \dots, \ z^{-k}\theta_k, \ \dots \text{ are holomorphic at } z = \infty \tag{11.14}$$

as 1-forms on $\mathbb{C}P^1$ at the point $z = \infty$ (recall that this holomorphy for $\theta = R(z)\,dz$ means that $a(z) = O(z^{-2})$). In this case the identities (11.14) imply that

$$\sum_{\Sigma} \operatorname{res} \theta_1 = -1, \qquad \sum_{\Sigma} \operatorname{res} \theta_i = 0, \quad i = 2, 3, \dots, \tag{11.15}$$

where the summation is extended on all finite singularities of the forms θ_i.

As we will be interested in the dependence on Taylor coefficients, let us make the following obvious observation.

Lemma 11.33. *Assume that the blow-up of the real analytic form* $\omega = \omega_n + \omega_{n+1} + \cdots$ *is non-dicritical. Then*:

(1) *The coefficients of the rational forms θ_k depend rationally on the coefficients of the initial form ω.*

(2) *The form θ_k* does not *depend on the coefficients of the homogeneous components of order $n + k$ and higher.*

(3) *If the principal homogeneous part ω_n is fixed, the first form θ_1 is uniquely determined and all other forms θ_k, $k \geqslant 2$, depend* linearly *on the remaining coefficients of higher order terms $\omega_{n+1}, \omega_{n+2}, \dots$ of the form ω.*

Proof. Everything follows immediately from (11.10) and computation of the reciprocal

$$\frac{1}{h_{n+1}(z) + xh_{n+2}(z) + \cdots} = \frac{1}{h_{n+1}(z)} \left(1 - x \cdot \frac{h_{n+2}(z)}{h_{n+1}(z)} + \cdots\right)$$

on any compact set $K \times (\mathbb{C}, 0)$, $K \Subset \mathbb{C} \smallsetminus \Sigma$. □

Remark 11.34. It would be wrong to assume that the principal homogeneous part ω_n is determined by the linearization form θ_1 only. In particular, the form θ_1 may be nonsingular at some points of Σ (when p_n and q_n have common factor), whereas some of the higher forms θ_k, $k \geqslant 2$, may have poles and therefore necessary contribute to ω_n. The reason is, of course, the fact that blowing down is given by the change of coordinates $z = y/x$ that is only rational and not polynomial.

Now we are in a position to prove *relative* decidability of the center–focus alternative for generalized elliptic singularities with fixed principal part. Denote by $\mathcal{B}(\omega_n) = (j^n)^{-1}(\omega_n) = \{\omega = \omega_n + \omega_{n+1} + \cdots\} \subseteq \Lambda^1(\mathbb{R}^2, 0)$ the space of all holomorphic forms with the fixed principal homogeneous part ω_n.

Theorem 11.35 (see [**Ily72a**]). *If ω_n is generalized elliptic, then the center-focus alternative is ultimately decidable within the class $\mathcal{B}(\omega_n)$.*

Proof. We show that in the assumptions of the Theorem, the coefficients $a_k = a_k(\omega) = a_j(\omega_{n+1}, a_{n+2}, \dots)$ of the semi-monodromy map $\Delta_{\mathbb{R}}(x) = a_1x + a_2x^2 + \cdots$ are *quasi*homogeneous polynomials in the Taylor coefficients of $\omega - \omega_n = \omega_{n+1} + \omega_{n+2} + \cdots$. When written as an argument, each ω_k is identified with the linear space of all its coefficients.

By Lemma 11.33, each coefficient a_k depends only on the components $\omega_n, \dots, \omega_{n+k-1}$ and this dependence is real analytic.

Consider an arbitrary real number $0 \neq \mu \in \mathbb{R}$ and the linear transformation $D_\mu = (x, y) \mapsto (\mu x, \mu y)$. This transformation acts diagonally on 1-forms: the coefficients of $D_\mu^* \omega_k$ are multiplied by μ^{k+1} by homogeneity so that the form

$$\mu^{-(n+1)}\, D_\mu^* \omega = \omega_n + \mu\omega_{n+1} + \mu^2\omega_{n+2} + \cdots$$

again belongs to $\mathcal{B}(\omega_n)$.

On the other hand, D_μ changes the chart on the x-axis and hence transforms the semi-monodromy map $\Delta_{\mathbb{R}}$ into

$$\mu^{-1}\Delta_{\mathbb{R}}(\mu x) = a_1x + \mu a_2x^2 + \mu^2 a_3 x + \cdots .$$

Since the coefficients of the semi-monodromy are uniquely defined, we conclude that

$$a_k(\mu\omega_{n+1}, \mu^2\omega_{n+2}, \dots, \mu^{k-1}\omega_{n+k-1}) = \mu^{k-1}\, a_k(\omega_{n+1}, \omega_{n+2}, \dots, \omega_{n+k-1}).$$

In other words, each a_k is a quasihomogeneous real analytic function of its arguments. Such function is necessarily a quasihomogeneous polynomial.

The ultimate algebraic decidability of the center-focus alternative now follows immediately from Example 11.12. Indeed, since a_j are polynomial functions on $\mathcal{B}(\omega_n)$, vanishing of any finite number of coefficients of $\Delta_{\mathbb{R}} \circ \Delta_{\mathbb{R}}$ is an algebraic condition on a finite jet of ω. If *all* nonlinear coefficients of $\Delta_{\mathbb{R}} \circ \Delta_{\mathbb{R}}$ vanish, then the singularity is a center. □

Remark 11.36. This proof works under less restrictive assumption that only the singular points of the blow-up form are fixed. Thus decidability holds on larger semialgebraic subsets of $\Lambda^1(\mathbb{R}^2, 0)$ defined by prescribing positions of the singular points from Σ *outside* the real axis.

11.6. Decidability to codimension 1. As follows from Remark 11.36, coefficients of the semi-monodromy map $\Delta_{\mathbb{R}}$ depend polynomially on the Taylor coefficients as far as the singular locus Σ remains constant. It turns out that dependence of the coefficients on the *location* of points in Σ is *non-algebraic*. This implies undecidability of the center-focus alternative in some rather low codimension.

To compute the coefficients of the semi-monodromy map, we will integrate the equation (11.11) in the form $x = X(z, u)$ subject to the initial condition $X(0, u) = u$. Expanding this solution in the series $X(z, u) = \sum_{k\geqslant 1} u^k X_k(z)$ and substituting this expansion into (11.11), we obtain a triangular (infinite) system of the differential equations with the initial conditions

$$\begin{aligned} dX_1 &= X_1\theta_1, & X_1(0) &= 1, \\ dX_2 &= X_2\theta_1 + X_1^2\theta_2, & X_2(0) &= 0, \\ dX_3 &= X_3\theta_1 + 2X_1X_2\theta_2 + X_1^3\theta_3, & X_3(0) &= 0, \\ &\vdots & &\vdots \end{aligned} \tag{11.16}$$

This system can be recursively solved in quadratures, since on each step the equation for X_k is linear nonhomogeneous with known nonhomogeneity.

The coefficients of the semi-monodromy map are obtained as the result of analytic continuation of solutions of the system (11.16) along the loop $\mathbb{R}P^1$ (i.e., along the real line across infinity),

$$\Delta_{\mathbb{R}}(x) = \sum_{k\geqslant 1} a_k x^k, \qquad a_k = (\varDelta_{\mathbb{R}P^1} X_k)(0) \in \mathbb{R}, \qquad k = 1, 2, \dots, \tag{11.17}$$

where by $\varDelta_{\mathbb{R}P^1}$ is denoted the operator of analytic continuation of the function $X_k(\cdot)$ along $\mathbb{R}P^1$, not to be confused with the *map* $\Delta_{\mathbb{R}}$.

Analyzing this system, we immediately see that the first coefficient $a_1(\omega)$ is non-algebraically depending on (the Taylor coefficients of) θ_1. Yet despite this non-algebraicity, the *neutrality condition*$a_1(\omega) - 1 = 0$ is algebraically decidable.

Theorem 11.37.

1. *The multiplicator* $a_1 = a_1(\omega_n)$ *of the semi-monodromy map* $\Delta_{\mathbb{R}}$ *of a generalized elliptic singular point is equal to* -1*, if and only if*

$$\operatorname{Im} \sum_{\operatorname{Re} z>0} \operatorname{res}_z \theta_1 = -\frac{1}{2}. \tag{11.18}$$

2. *The center-focus alternative for generalized elliptic singularities is algebraically decidable to codimension* 1*.*

Proof. The first equation of (11.16) can be immediately integrated, yielding for the solution $X_1(z)$ and the multiplicator a_1 of its continuation along $\mathbb{R}P^1$ the *transcendental* expressions

$$X_1(z) = \exp \int_1^z \theta_1, \qquad a_1 = \exp \oint_{\mathbb{R}P^1} \theta_1.$$

The *neutrality condition* $a_1 = -1$ holds if and only if $\oint_{\mathbb{R}P^1} \theta_1 = \pi i(2m+1)$, $m \in \mathbb{Z}$, i.e.,

$$\sum_{\operatorname{Re} z>0} \operatorname{res}_z \theta_1 = \frac{1}{2} + m, \qquad m = 0, \pm 1, \pm 2, \dots. \tag{11.19}$$

This equality is not yet an algebraic condition, since it is the union of *infinitely many* conditions for different values of $m \in \mathbb{Z}$. However, since ω is *real* on the real axis, its singular locus $\varSigma$ is symmetric by the reflection $z \mapsto \bar{z}$, and the residues at symmetric points are complex conjugate. The total of *all* residues of θ_1 on the whole plane $\mathbb{C}$ is -1 by (11.15). Therefore, the real part of the expression in the left hand side of (11.19) is $-\frac{1}{2}$, which is compatible with the right hand side only when $m = -1$, proving thus (11.18).

The second assertion of the Theorem immediately follows from the first one, since (11.18) is an algebraic condition on the form θ_1. □

Remark 11.38. The algorithm of computation of the semi-monodromy and monodromy maps for generalized elliptic points, provides also a tool for *definition* of the (semi-)monodromy for *formal vector fields* or *formal* Pfaffian forms. Indeed, consider a formal Pfaffian form $\omega =$ as in (11.8) but *without* assuming that the series converges. The condition (11.9) makes sense since it involves only the lowest order homogeneous terms ω_n of ω.

The blow-up of this form is well defined and gives a Pfaffian equation (11.11) with the forms θ_i still rational, but the series in the powers of x in the right hand side only formal.

It remains to notice now that the infinite triangular system of Pfaffian equations (11.16) remains exactly the same (no changes are required) and solving any finite number of equations from this system determines uniquely the finite jet of the holonomy $\Delta_{\mathbb{R}}$ of the initial formal singularity. Thus the map $\Delta_{\mathbb{R}}$ gets consistently defined, at least for the specific choice of the cross-section $\tau = \{z = 0\}$. Choosing any other cross-section $\{z = \varphi(x)\}$, even formal so that $\varphi \in \mathbb{C}[[x]]$, may change $\Delta_{\mathbb{R}}$ by the formal conjugacy: the arguments remain the same.

Finally, we remark that if the homogeneous forms $\omega_n, \omega_{n+1}, \dots$ depend analytically on any additional parameters $\lambda_1, \dots, \lambda_m$, then the coefficients of the formal holonomy (semi-monodromy) will depend analytically on λ as far as the form remains generalized elliptic, that is, the roots of the homogeneous polynomial h_{n+1} in (11.9) remain off the real axis.

11.7. Undecidability of the of the general center-focus alternative. Inspection of the next nontrivial equation in (11.16) already reveals non-algebraicity of the second nontrivial condition $a_3(\omega) = 0$. Since $a_2(\omega)$ does not affect the sufficiency of the square $\Delta_{\mathbb{R}} \circ \Delta_{\mathbb{R}}$ (as follows from the first condition in (11.1)), this non-algebraicity would mean that the unrestricted center-focus alternative is undecidable to codimension 2.

To prove the non-algebraicity of the condition $a_3(\omega) = 0$, we construct a polynomial family of 1-forms, from the very beginning in the chart (x, z), as follows.

$$\begin{gathered} dx = x\theta_1 + x^3\theta_3, \qquad \theta_1 = \left(\frac{A}{z^2+2} - \frac{A+1}{z^2+1}\right) z\,dz, \\ \theta_3 = \mu dz + \frac{zdz}{z^2+\lambda^2}, \qquad \lambda, \mu \in \mathbb{R}, \quad \lambda \neq 0. \end{gathered} \tag{11.20}$$

Here $A \in \mathbb{R} \smallsetminus \mathbb{Z}$ is any fixed *non-integer* number. The Pfaffian equation (11.20) can be blown down to a polynomial form $\omega = 0$ in $\mathbb{C}^2$ by Remark 11.32.

The conditions (11.14) for this system are obviously verified, meaning that in the semialgebraic domain $\lambda \neq 0$ the equation (11.20) is generalized elliptic. The total residue of the form θ_1 at the singular points $i, i\sqrt{2}$ in the upper half-plane is exactly $-\frac{1}{2}$, so the condition (11.18) is automatically verified for all values of λ, μ.

Obviously, the second coefficient $a_2 = a_2(\lambda, \mu)$ of $\Delta_{\mathbb{R}}$ is zero, since the term $x^2\theta_2$ is absent in (11.20). The third coefficient, $a_3 = a_3(\lambda, \mu)$ is a real analytic function of λ, μ in the domain $\lambda \neq 0$ where (11.20) is generalized elliptic.

Theorem 11.39. *The second integrability condition* $a_3(\lambda, \mu) = 0$ *for the family* (11.20) *defines a non-algebraic real curve on the plane of parameters* $\{\lambda > 0,\ \mu \in \mathbb{R}\}$.

The complement of this curve $\{a_3(\lambda, \mu) = 0\}$ consists of sufficient jets (foci), thus Theorem 11.39 indeed proves undecidability of the center-focus problem.

Proof of Theorem 11.39. The first equation $dX_1 = X_1\theta_1$ together with the initial condition $X_1(0) = 1$ yields the unique solution

$$X_1(z) = \frac{1}{\sqrt{z^2+1}} \left(\frac{z^2+2}{z^2+1}\right)^{A/2}, \qquad z \in \mathbb{R}. \tag{11.21}$$

This solution has two branches and the monodromy multiplicator for going around $\mathbb{R}P^1$ is -1, exactly as expected. The *square* of this function is the function

$$F(z) = \frac{1}{z^2+1} \left(\frac{z^2+2}{z^2+1}\right)^{A} \tag{11.22}$$

ramified over the four points $\pm i, \pm i\sqrt{2}$ but admitting a single-valued meromorphic branch over the neighborhood of the loop $\mathbb{R}P^1$, *positive* on the loop $\mathbb{R}P^1$ itself.

The third equation in (11.16), which becomes

$$dX_3 = X_3\theta_1 + X_1^3\theta_3,$$

can be solved by the variation of constants, i.e., the substitution $X_3 = f(z)X_1$. Since $X_1^2 = F$, the function $f(z)$ must satisfy the equation

$$df(z) = F(z) \cdot \theta_3, \qquad z \in (\mathbb{C}P^1, \mathbb{R}P^1).$$

The periodicity condition $a_3 = 0$ is equivalent to the condition

$$\oint_{\mathbb{R}P_1} F(z)\,\theta_3 = 0. \tag{11.23}$$

Note that, despite the multivaluedness of the function $F(z)$, the integral is well-defined using the uniquely defined real branch of this function.

In our example (11.20), the form θ_3 consists of two terms depending on two parameters. The integral

$$K_1(\mu) = \oint_{\mathbb{R}P_1} \mu F(z)\,dz = \mu c,$$

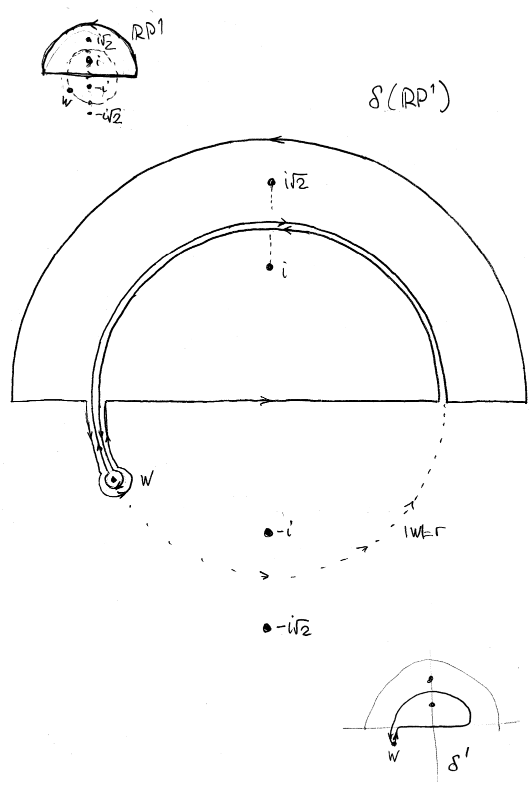

Figure 7. Monodromy of the integral $I(w)$

where c is a *nonzero* constant. Indeed, according to our choice of branches, the function $F(z)$ is positive on the real line $\mathbb{R}$. The contribution from the semi-circle $\{|z| = R,\ \operatorname{Im} z > 0\}$ vanishes as $R \to \infty$, so c is *real positive.*

We will show now that the other integral,

$$K_2(\lambda) = \oint_{\mathbb{R}P^1} F(z)\,\frac{z\,dz}{z^2+\lambda^2} = \frac{1}{2}\cdot\oint_{\mathbb{R}P^1} F(z)\left(\frac{dz}{z-i\lambda}+\frac{dz}{z+i\lambda}\right) \qquad (11.24)$$

where $F(z)$ is the transcendental function defined in (11.22), is a non-algebraic function of the parameter $\lambda \neq 0$. To do this, consider the auxiliary Cauchy-type integral

$$I(w) = \oint_{\mathbb{R}P^1} \frac{F(z)\,dz}{z-w}, \qquad w \in \mathbb{C} \smallsetminus \{\pm i,\ \pm i\sqrt{2}\} \qquad (11.25)$$

and its symmetrization $J(w) = \frac{1}{2}(I(w) + I(-w))$, so that $K_2(\lambda) = J(i\lambda)$. The integrals $I(w)$ and $J(w)$ are holomorphic functions of w, eventually ramified over the singular locus $\Sigma = \{\pm i, \pm i\sqrt{2}\}$. As will be shown, this ramification is indeed nontrivial and of logarithmic type.

Consider the result $\delta I(w)$ of analytic continuation of $I(w)$ when w goes counterclockwise around the circle $\{|w| = r\}$, $1 < r < \sqrt{2}$ (the operator of analytic continuation in the w-plane, denoted by δ, should not be confused with previously considered operators of continuation in the z-plane).

To that end, we need to deform the loop $\mathbb{R}P^1$ continuously with w so that it remains disjoint from the locus $\Sigma \cup \{w\} = \{\pm i, \pm i\sqrt{2}, w\}$. Together with the deformation of the loop, we have to choose the branch of F along it that is continuous.

The result of such continuation, the loop denoted $\delta(\mathbb{R}P^1)$, is shown on Figure 7. It is homologous to the initial loop $\mathbb{R}P^1$ and the two small circular loops around the point w, oriented in the opposite senses. However, on the Riemann surface of the function F these two small loops lie on two *different* sheets. More precisely, the arc connecting one such cycle to the other along $\delta(\mathbb{R}P^1)$, is close to the loop γ in the z-plane, beginning and ending at the point w and going counterclockwise around the point $+i$ as shown on Fig. 7. Denote the operator of analytic continuation along this loop by δ'.

Expressing the integrals over these small loops in terms of the residue $\operatorname{res}_w (z-w)^{-1} F(w)\,dz = F(w)$ (recall that F is multivalued), we conclude finally that

$$\begin{aligned} \delta I(w) &= I(w) + 2\pi i F(w) - 2\pi i \delta' F(w) \\ &= I(w) + 2\pi i F(w) B, \qquad B = 1 - \exp[2\pi i (A-1)] \neq 0, \end{aligned}$$

where δ' is the operator of analytic continuation along the loop described above, see Figure 7. The inequality $B \neq 0$ follows from our assumption that $A \notin \mathbb{Z}$.

The monodromy of the symmetrization $J(w) = \frac{1}{2}(I(w) + I(-w))$ is

$$\delta J(w) = J(w) + \pi i B[F(w) + F(-w)] = J(w) + 2\pi i B \cdot F(w),$$

since F, as a function of z^2 only, is even in the domain $\mathbb{C} \smallsetminus [\pm i, \pm i\sqrt{2}]$ with two symmetric slits. The function $J(w)$ has therefore logarithmic branching along the circle $|z| = r$, and cannot be algebraic, for example, because it has infinitely many different branches. This proves that $K_2(\lambda) = J(i\lambda)$ is transcendental (and by construction, it is real for $\lambda \in \mathbb{R} \smallsetminus \{0\}$).

The curve $a_3(\lambda, \mu) = 0$ is defined by the equation

$$K_1(\mu) + K_2(\lambda) = c\mu + K_2(\lambda) = 0, \qquad c > 0,$$

and hence is non-algebraic as the graph of a transcendental function $-K_2(\lambda) = -J(i\lambda)$. □

Remark 11.40. The observed undecidability of the center-focus alternative is relatively "mild": at least, the neutral jets are defined by analytic (though non-algebraic) equations.

For other local problems the situation can be much more grave, and the structure of neutral sets can be very pathological. The first obvious example that comes to mind is the *integrability alternative*. Consider the Pfaffian form

$$\omega = a\,\frac{dx}{x} + b\frac{dy}{y} + c\frac{d(x+y)}{x+y}, \qquad a,b,c>0.$$

This form is "integrable" in the Darboux sense: $\omega = du/u$, therefore $\omega \wedge d(u^r) = 0$ for any power r, where $u(x,y) = x^a y^b (x+y)^c$. However, u^r is not analytic at the origin for any r unless the ratio $(ra, rb, rc) \in \mathbb{Z}^3_+$. In the latter case u^r is a polynomial.

Thus the problem of detecting integrability reduces to testing whether the ratios of the parameters $a : b : c$ are rational or not. Clearly, this cannot be done by analytic functions on the coordinates of the 2-jet of ω reduced to the polynomial form. More precisely, in the set of all holomorphic 1-forms $\Lambda^1(\mathbb{C}^2, 0)$ there is a 3-dimensional linear subspace which intersects the subset of integrable forms by a subset that is dense in some open parts of $\mathbb{C}^3$. In other words, the integrability alternative is not analytically decidable.

In [**Ily76**] it is shown that the *stability alternative* for germs of vector fields in $\mathbb{R}^5$ is not analytically decidable.

A similar situation occurs when studying dynamics of iterations of polynomial (in fact, quadratic!) maps of the plane $\mathbb{C}^1$ into itself of the form $z \mapsto z^2 + c$. In the space of parameters $c \in \mathbb{C}$ the *Mandelbrot set* corresponding to two very different dynamical patterns, is known to have a very complicated *fractal*, or *self-similar* structure, see [**CG93**].

12. Holonomy and first integrals

A useful tool in studying the center-focus alternative is first integrals. In the first part of this section we show that for *elliptic* points existence of a formal first integral is equivalent its centrality (the Poincaré–Lyapunov theorem). The proof is based on the paper [**Mou82**].

The second part of the section is devoted to complexification of this theorem for arbitrary isolated singularities of holomorphic foliations on $(\mathbb{C}^2, 0)$. We show that simple topology of the holomorphic foliation is necessary and sufficient for its analytic integrability. Exposition of the second part is based on the papers [**MM80, EISV93**].

12.1. Integrability and decidability. Integrability (complete or on the level of finite jets at the origin) of a vector field F means existence of a nontrivial function u (resp., a jet $j^k u$ at the point $0 \in \mathbb{R}^2$), called the *first integral*, whose derivative Fu vanishes identically or on the level of jets respectively. In terms of the Pfaffian form ω the condition $Fu = 0$ takes the form $\omega \wedge du = 0$. The function u must be nontrivial: the precise form of this condition is related to the principal part of the form ω. In the particular case of elliptic singularities the natural definition looks as follows.

Definition 12.1. (The k-jet of) an elliptic singularity is *integrable*, if there exists (the k-jet of) a function $u = u(x, y)$ such that

$$j^2 u \text{ is } \textit{nondegenerate} \text{ quadratic form and} \qquad \omega \wedge du = 0 \tag{12.1}$$

(respectively, $j^k(\omega \wedge du) = 0$). The function (resp., k-jet) is called the *first integral* of ω (resp., of the jet $j^k\omega$).

The linear part ω_1 of an integrable elliptic singularity can be reduced to the canonical form $\omega_1 = \frac{1}{2} d(x^2+y^2)$. Indeed, one can choose the coordinates so that the quadratic part u_2 of the series (jet) u is the sum of squares. The integrability condition $\omega_1 \wedge du_2 = 0$ means that $\omega_1 = ax\,dx + by\,dy$, $a, b \in \mathbb{R}$. If the form is elliptic, then necessarily $ab \neq 0$ and by a diagonal linear change of variables one may ensure that an integrable elliptic form (jet) has the expansion

$$\omega = \omega_1 + \omega_2 + \cdots, \qquad \omega_1 = x\,dx + y\,dy = \tfrac{1}{2}\, d(x^2 + y^2), \tag{12.2}$$

and the corresponding first integral u begins with the terms

$$u = u_2 + u_3 + \cdots, \qquad u_2(x, y) = \tfrac{1}{2}(x^2 + y^2). \tag{12.3}$$

Everywhere below in this section we consider only elliptic forms meeting the assumption (12.2) and their integrals normalized as in (12.3).

Proposition 12.2. *The following three conditions on the k-jet of an elliptic singularity are equivalent*:

1. (1) *the jet is neutral with respect to the center-focus alternative,*
2. (2) *the jet is integrable,*
3. (3) *the jet is orbitally linearizable, i.e., there exists a k-jet of a plane holomorphism bringing $j^k\omega$ to its linear part ω_1 modulo an invertible scalar factor.*

Proof. 1. The assertion is absolutely transparent if the k-jet is normalized,

$$\begin{gathered} \omega = \omega_1 + \Big(\sum c_{2j}\, r^{2j}\Big)(x\,dy - y\,dx), \qquad 1 \leqslant j, \quad 2j+1 \leqslant k, \\ c_{2j} \in \mathbb{R}, \qquad \omega_1 = \tfrac{1}{2} dr^2, \qquad r^2 = x^2 + y^2. \end{gathered} \tag{12.4}$$

Indeed, if not all coefficients c_{2j} are zeros, the singularity is a focus (stable or unstable). This follows from the fact that $u_2(x, y) = x^2 + y^2$ is a Lyapunov function (its derivative has constant sign). On the other hand, if all c_{2j} are zeros, u_2 is the jet of a first integral.

2. By a suitable orbital transformation, the jet of any order can be normalized to the form (12.4). Being invariant by orbital transformations, the Proposition holds therefore for all elliptic germs (jets). □

The following Proposition shows that integrability of an elliptic jet is an algebraic condition. This gives an alternative proof of decidability of the center-focus alternative for elliptic germs.

Proposition 12.3. *Integrable elliptic jets constitute a semialgebraic subset of* $J^k(\Lambda^1)$ *for all* k.

Proof. The jet $u_2 + u_3 + \cdots + u_k$, where u_j is a homogeneous polynomial of degree $j \leqslant k$, will be a first integral for the jet $\omega_1 + \cdots + \omega_k$, if

$$\begin{aligned} -\omega_1 \wedge du_2 &= 0, \\ -\omega_1 \wedge du_3 &= \omega_2 \wedge du_2, \\ &\vdots \\ -\omega_1 \wedge du_k &= \omega_2 \wedge du_{k-1} + \cdots + \omega_{k-1} \wedge du_2, \end{aligned} \tag{12.5}$$

cf. with (11.7). The first condition is automatically satisfied, since $\omega_1 = 2du_2$.

The same arguments as were used in the proof of Lemma 11.20 when discussing solvability of (11.7), can be literally used for the system (12.5). Namely, this system determines an algebraic subvariety in the space $J^k(\mathbb{R}^2, 0) \times J^k(\Lambda^1)$. By the Tarski–Seidenberg theorem, the projection of this subvariety on the ω-component is semialgebraic. □

Definition 12.4. An elliptic germ is *formally integrable*, if there exists a formal series $u \in \mathbb{R}[[x, y]]$ (not necessarily converging), such that $\omega \wedge du = 0$ as a formal 2-form.

Proposition 12.5. *Formally integrable elliptic germs are centers.*

Proof. For any finite order k the k-jet of a formally integrable elliptic germ is neutral and hence the monodromy map $\Delta_\mathbb{R} \circ \Delta_\mathbb{R}$ differs from identity by terms of order $o(x^k)$. Thus the order of contact between $\Delta_\mathbb{R} \circ \Delta_\mathbb{R}$ and the identity must be infinite. Since $\Delta_\mathbb{R}$ is real analytic, this means that $\Delta_\mathbb{R} \circ \Delta_\mathbb{R} = \text{id}$, and we have a center. □

12.2. Analytic first integrals and Poincaré–Lyapunov theorem. Out of the three conditions,

(AI) existence of non-identical analytic first integral,

(FI) existence of nonzero formal first integral,

(C) center (identical return map),

the first obviously implies the second and the third, regardless of whether the monodromic singularity is elliptic or not.

The implication (FI) $\implies$ (C) is asserted by Proposition 12.5. We will discuss now the remaining implication (C) $\implies$ (AI) showing that for *elliptic* singularities, all three conditions are equivalent. This is the famous Poincaré–Lyapunov theorem, proved by Poincaré for polynomial differential equations and by Lyapunov in the analytic category. The modern proof given below, is based on [**Mou82**].

Theorem 12.6 (Poincaré–Lyapunov)**.** *A real analytic elliptic singularity which is a center, admits a real analytic first integral with the nondegenerate quadratic part.*

The stress in the assertion of this theorem is on analyticity of the first integral. Indeed, existence of a first integral that is simply continuous at the origin $x = y = 0$ and real analytic outside, is obvious. Indeed, take the cross-section $\tau = \{y = 0,\, x > 0\}$ and the function x^2 on it, and extend this function on the entire neighborhood of the origin as constant along the trajectories of the vector field. Since all trajectories are closed, this extension is unambiguous and real analytic outside the origin where its continuity is obvious. Applying this construction in the coordinates linearizing any finite order jet (they exist by Proposition 12.2), we can in fact guarantee smoothness of the constructed first integral to any finite order and even its C^∞-smoothness (by the Borel theorem).

Remark 12.7. Note that the isolated point where the analyticity break may eventually occur, is a *small* set of codimension 2. Thus, if all objects were defined in $(\mathbb{C}^2, 0)$ rather than in $(\mathbb{R}^2, 0)$, the analyticity would follow automatically unlike in the real context where no removable singularity theorems are available. In other words, the natural way to prove analyticity is to complexify the situation.

The proof of Theorem 12.6 is based on complexification with subsequent blow-up. For an elliptic singularity the result looks especially simple, cf. with §11.4.

Consider the singular foliation $\widetilde{\mathcal{F}}$ on the complex Möbius band C in a neighborhood of the exceptional divisor $\mathbb{C}P^1$ that appears by blow-up of the complex analytic singular foliation $\mathcal{F}$ in $(\mathbb{C}^2, 0)$ defined by the Pfaffian equation $\omega = 0$. The only two singular points of $\widetilde{\mathcal{F}}$ on $\mathbb{C}P^1$ are at the points $z = \pm i$, both of them saddles with the ratio of eigenvalues equal to $-\frac{1}{2}$. By the Hadamard–Perron Theorem 6.2, each saddle has two holomorphic invariant curves. One of them is the common complex separatrix $\mathbb{C}P^1$ of both singular points, the other is a holomorphic curve W_+ (resp., W_-) transversal to $\mathbb{C}P^1$ at $+i$ (resp., $-i$).

The real line corresponds to the loop $\mathbb{R}P^1 \subset \mathbb{C}P^1$ on the exceptional leaf $L = \mathbb{C}P^1 \smallsetminus \{\pm i\}$. The fundamental group of L is cyclic generated by the loop $\mathbb{R}P^1$, therefore the holonomy group is generated by the single germ of the semi-monodromy $\Delta_{\mathbb{R}}$ which will be denoted by H for brevity:

$$H = \Delta_{\mathbb{R}}|_\tau, \qquad \tau = \{z = 0\}, \qquad H(x) = -x + \cdots .$$

The Poincaré–Lyapunov theorem asserts that if H is 2-periodic and all leaves of the real elliptic foliation $\mathcal{F} = \{\omega = 0\}$ on $(\mathbb{R}^2, 0)$ are closed, then there exists a real analytic function u constant along the the integral curves.

Proof of Theorem 12.6. Assume that the singularity is a center. Then the semi-monodromy map $H = \Delta_{\mathbb{R}} \colon (\tau, 0) \to (\tau, 0)$ must be 2-periodic (an *involution*),

$$H \colon (\tau, 0) \to (\tau, 0), \qquad x \mapsto -x + \cdots, \qquad H \circ H = \mathrm{id}. \tag{12.6}$$

We prove Theorem 12.6 very much like the "quasi-proof" in the real context (see Remark 12.7), by the following steps:

(1) construction of a holomorphic function $u \colon (\tau, 0) \to (\mathbb{C}, 0)$ that would be H-invariant and starts with the quadratic term, $u(x) = x^2 + \cdots$,

(2) extending this function along leaves of the foliation $\widetilde{\mathcal{F}}$ onto an open neighborhood of the exceptional leaf L,

(3) extension in the full neighborhood of $\mathbb{C}P^1$,

(4) blowing down the obtained integral of the foliation $\widetilde{\mathcal{F}}$, back onto $(\mathbb{C}^2, 0)$ to a holomorphic integral of the foliation $\mathcal{F}$.

1. Construction on the cross-section. Let $u_0 \colon (\tau, 0) \to (\mathbb{C}, 0)$, $u_0(x) = x^2 + \cdots$, be an arbitrary function on the cross-section, having a Morse critical point at the origin and real on $\tau \cap \mathbb{R}$. Define the function $u \colon (\tau, 0) \to (\mathbb{C}, 0)$ as

$$u = \tfrac{1}{2}(u_0 + u_0 \circ H).$$

Since H is an involution, $H \circ H = \mathrm{id}$, we have $u \circ H = u$, i.e., u is H-invariant. The function u is real on the real part of τ, since H maps this part into itself. Since linearization $x \mapsto -x$ of H preserves the quadratic function x^2, the quadratic parts of u and u_0 are the same.

2. Continuation along leaves. For any compact subset $K \Subset L$ containing the point $z = 0$, there exists a small neighborhood $U = U(K)$ such that the integral u can be extended as a holomorphic function on U constant along the leaves of the foliation $\widetilde{\mathcal{F}}|_U$ and real on the real Möbius band $\operatorname{Im} z = 0$, $\operatorname{Im} x = 0$.

Indeed, for any point $z \in L$ consider the transversal τ_z with the chart $x|_{\tau_z} \in (\mathbb{C}^1, 0)$ and an arbitrary curve $\gamma_{z0} = \gamma_{0z}^{-1}$ connecting z with 0 in

$L = \mathbb{C}P^1 \smallsetminus \{\pm i\}$. Since the points of L are nonsingular, the germ of the holonomy correspondence $\Delta_{0z} = \Delta_{z0}^{-1} \colon (\tau, 0) \to (\tau_z, 0)$ is well defined and hence one can define the function $u|_{\tau_z}$ as

$$u|_{\tau_z} = (u|_\tau) \circ \Delta_{0z}^{-1} = (u|_\tau) \circ \Delta_{z0}.$$

This germ is locally holomorphically depending on $z \in L$ and therefore can be considered as a holomorphic function defined in some neighborhood of the point $(z, 0)$.

The definition of u does not depend on the choice of the curve γ_{z0} connecting z with 0. Indeed, any other path will differ by the multiple of the loop $\gamma = \mathbb{R}P^1$ modulo homotopy with fixed endpoints, since the fundamental group of L is cyclic and generated by γ. But since $u|_\tau$ was H-invariant, the function u obtained by continuation along leaves, is well defined in $U(K)$ for any compact set $K \Subset L$.

Since foliation is real, the function constructed this way will be real for real points (x, z).

3. Extension onto neighborhoods of singularities. To extend the function u onto the full neighborhood of the Riemann sphere, it remains to treat neighborhoods of the two remaining singular points $(\pm i, 0)$, each of them independently.

Consider a base point z_0 sufficiently close to $z = +i$ and a cross-section $\tau_0 = \{z = z_0, |x| < \varepsilon\}$ at z_0, transversal to $\overline{L}$ (with the chart x). On the previous step we constructed a holomorphic function $u|_{\tau_0}$ invariant by the holonomy operator $\Delta_0 \colon (\tau_0, 0) \to (\tau_0, 0)$ for a small loop around $z = +i$ (since this loop is freely homotopic to $\mathbb{R}P^1$, the holonomy Δ_0 is 2-periodic). By construction, the function $u|_{\tau_0}$ has a double root at $x = 0$.

Give the definition of saturation when foliations are first introduced!

Consider the *saturation* of the cross-section τ_0 by leaves of the foliation $\widetilde{\mathcal{F}}$ in a small bidisk D around $(i, 0)$. By definition, it is the set of points (z, x) that can be connected with some point $(z_0, x_0) \in \tau_0$ by a continuous curve which entirely belongs to some leaf while remains inside the bidisk D.

By Lemma 12.25 from §12.7 below, this saturation contains a (smaller) bidisk D' centered at the singular point, with the exception of the second complex separatrix W_+. This Lemma asserts that the leaf of the restricted foliation $\widetilde{\mathcal{F}}|_D$ passing through an arbitrary point $(z, x) \in D \smallsetminus W_+$ sufficiently close to the singular point $(i, 0)$, intersects the cross-section τ_0 at two points belonging to one Δ_0-orbit, and these points tend to the base point $(z_0, 0)$ if (z, x) tends to $(i, 0)$.

This topological description ensures that the function $u|_{\tau_0}$ can be uniquely holomorphically extended along leaves of $\widetilde{\mathcal{F}}$ to the slit bidisk $D \smallsetminus W_+$ as a holomorphic function u' *bounded* near W_+ (actually, having zero limit on W_+). By the removable singularity theorem, u' can be

Xref to: Removable bounded singularities—introduction!

further extended on the bidisk D' as the local first integral of $\widetilde{\mathcal{F}}$ near $(i, 0)$.

Thus we have constructed the holomorphic first integral u of the blown-up foliation $\widetilde{\mathcal{F}}$ in a small neighborhood of the exceptional divisor $\overline{L} = \mathbb{C}P^1 \subset C$ which has the second order root on any cross-section to this divisor. By construction, this integral is real on the real part of C.

4. Blowing down the analytic first integral. The complex blow-up map $\pi\colon (C, \mathbb{C}P^1) \to (\mathbb{C}^2, 0)$ is one-to-one outside the exceptional divisor (resp., the origin). Thus the first integral u on $C \smallsetminus \mathbb{C}P^1$ can be blown down to a function $u'' = u \circ \pi^{-1}$ defined and holomorphic on the punctured neighborhood $(\mathbb{C}^2, 0) \smallsetminus \{0\}$.

Again by the removable singularity theorem, u'' extends as a holomorphic function on $(\mathbb{C}^2, 0)$. Its restriction on the real plane $(\mathbb{R}^2, 0)$ is real and has a quadratic root on any real line passing through the origin. By construction, it is the first integral of the initial real foliation with an elliptic singularity. This completes the proof of the Poincaré–Lyapunov theorem. □

Example 12.8 (Center without analytic first integral)**.** The above proof clarifies the role played by the assumption on the linear part in the Poincaré–Lyapunov theorem 12.6. One can easily construct examples of real analytic (generalized) elliptic singularities which are centers but do not admit real analytic first integrals.

Let $\theta = \theta_1$ be a real rational meromorphic 1-form on $\mathbb{C}P_1$ without real poles, satisfying the condition (11.14). Consider the corresponding Pfaffian equation (11.11) with $\theta_2 = \theta_2 = \cdots = 0$. By Remark 11.32, this equation can be blown down to a generalized elliptic singularity.

Being linear in x, the equation (11.11) is integrable, and all holonomy maps are linear in the natural chart x. By (11.15) and the symmetry of θ by the involution $z \mapsto \bar{z}$, the total residue of all singularities in each half-sphere $\pm \operatorname{Im} z > 0$ on $\mathbb{C}P^1$ is $-\frac{1}{2} \pm ic$, $c \in \mathbb{R}$. If $c = 0$, the holonomy of the real (projective) line is 2-periodic, so the first return map of the real singularity would be a center.

On the other hand, if there is more than one pole of θ, the above constraint $c = 0$ is compatible with the fact that the corresponding residues are *not negative rational numbers*. This means that the holonomy operators for small loops around these singularities cannot be periodic.

Clearly, this is impossible for an integrable singularity: in the latter case the holonomy operator should swap branches of the analytic integral and hence necessarily were periodic.

Remark 12.9. Another "counterexample" to the Poincaré–Lyapunov theorem was suggested in [**Mou82**]. The real polynomial 1-form

$$\omega = x^3\,dx + y^3\,dy - \tfrac{1}{2}x^2y^2\,dx \tag{12.7}$$

defines a real analytic singular foliation on $(\mathbb{R}^2, 0)$. The singular point at the origin is the center, being symmetric by the mirror symmetry (involution) $(x, y) \mapsto (-x, y)$.

The principal part (3-jet) of ω is integrable: $j^3\omega = \frac{1}{4}d(x^4+y^4)$. However, by direct inspection one can show that there is no 5-jet of the form $u = x^4 + y^4 + \cdots$ such that $j^5(\omega \wedge du) = 0$.

The explanation is that after blow-up the form has 4 singularities on the exceptional divisor at the points $z = \sqrt[4]{-1}$ in the chart $z = y/x$. The monodromy around these singularities are tangent to linear 4-periodic but not 2-period maps. However, these observations alone do not allow to conclude that there is no first integral with *non-isolated* critical point at the origin, beginning with terms of degree greater than 4.

* * *

The Poincaré–Lyapunov Theorem 12.6 relates certain topological simplicity of a real analytic foliation with its integrability which is an analytic property. In the remaining part of this section we will describe generalizations of this Theorem for arbitrary singular holomorphic foliations on $(\mathbb{C}^2, 0)$ having an isolated singularity.

12.3. Finitely generated subgroups of germs. The simplest context in which such generalization is possible, is that of finitely generated subgroups of holomorphic germs inside the group $\operatorname{Diff}(\mathbb{C}, 0)$. The results obtained for this problem, serve as the principal tool of investigation of holomorphic singular foliations in $(\mathbb{C}^2, 0)$.

We consider groups generated by finitely many holomorphic germs $g_1, \dots, g_n \in \operatorname{Diff}(\mathbb{C}, 0)$ with the operation "composition". For elements of this group we will use the exponential notation

$$\forall g \in \operatorname{Diff}(\mathbb{C}, 0) \qquad g^k = \underbrace{h \circ \cdots \circ h}_{k \text{ times}}, \quad g^{-k} = (g^{-1})^k.$$

Theorem 12.10 (Bochner linearization theorem)**.** *Any* finite *subgroup* $G \subseteq \operatorname{Diff}(\mathbb{C}, 0)$ *can be simultaneously linearized*: *there exists a biholomorphism* $h \in \operatorname{Diff}(\mathbb{C}, 0)$ *such that all germs* $h \circ g \circ h^{-1}$ *are linear,*

$$\forall g \in G \qquad h \circ g \circ h^{-1}(x) = \nu_g x, \qquad \nu_g = \tfrac{dg}{dx}(0) \in \mathbb{C}^*. \tag{12.8}$$

Proof. Define the germ $h \in \operatorname{Diff}(\mathbb{C},0)$ by the formula $h = \sum_{g\in G} \nu_g^{-1} g$. The germ h has the linear part $x \mapsto nx + \cdots$, $n = |G|$ and is therefore invertible.

By the chain rule, the correspondence $G \to \mathbb{C}^*$, $g \mapsto \nu_g = \frac{dg}{dx}(0)$, is a group homomorphism, $\nu_f \nu_g = \nu_{f\circ g}$. Therefore for any germ $f \in G$,

$$h \circ f = \sum_{g\in G} \nu_g^{-1}(g \circ f) = \nu_f \sum_{g\in G} (\nu_g \nu_f)^{-1}(g \circ f) = \nu_f \sum_{g'\in G} \nu_{g'}\, g' = \nu_f h,$$

which means that h conjugates f with the multiplication by ν_f. □

Remark 12.11. The image of the group homomorphism $g \mapsto \nu_g$ is a subgroup in the *commutative* group $(\mathbb{C}^*, \times)$ of nonzero complex numbers with the operation of multiplication. Its only finite subgroups are rotations generated by primitive roots of unity of degree n.

Theorem 12.10 means that for *finite* groups this correspondence is injective (an isomorphism) and therefore finite subgroup of $\operatorname{Diff}(\mathbb{C},0)$ is cyclic, $G = \{g^{\mathbb{Z}}\}$ generated by a single germ $g \in G$ of finite order equal to the order of the group. To some extent this argument can be inverted.

Theorem 12.12. *A finitely generated subgroup of germs $G \subset \operatorname{Diff}(\mathbb{C},0)$ whose all elements have finite order, is necessarily finite, commutative and cyclic.*

The proof of based on the following lemma.

Lemma 12.13. *Commutator of any two germs, if nontrivial, necessarily has infinite order.*

Proof of the Lemma. The commutator h of any two elements $h = f \circ g \circ f^{-1} \circ g^{-1}$ is tangent to the identity: $h(x) = x + \cdots$. This is an immediate consequence of the fact that the correspondence $g \mapsto \nu_g \in \mathbb{C}^*$ is a homomorphism.

If $h \neq \mathrm{id}$, i.e., $h(x) = x + cx^m + \cdots$, $m < \infty$, $c \neq 0$, then $(h \circ h)(x) = x + 2c\,x^m + \cdots$ and, more generally, $h^k(x) = x + kcx^m + \cdots$ which means that $h^k \neq \mathrm{id}$ for $k \neq 0$ and hence the cyclic subgroup $\{h^{\mathbb{Z}}\}$ is infinite. □

Proof of Theorem 12.12. By Lemma 12.13, the subgroup of $\operatorname{Diff}(\mathbb{C},0)$ with all elements of finite order, must be commutative. But the commutative group generated by finitely many elements of finite orders, is itself finite. The rest follows from Theorem 12.10 and Remark 12.11. □

12.4. Integrable germs. Let $u \in \mathcal{O}_0$ be a nonzero germ of analytic function, $u(x) = cx^m + \cdots$, $c \neq 0$.

Definition 12.14. A *symmetry group* of an analytic germ u is the subgroup $S_u = \{g \in \operatorname{Diff}(\mathbb{C},0) \colon u \circ g = u\}$ of holomorphisms preserving u.

Alternatively, we say that an analytic germ u is the *first integral* of a group $G \subseteq \operatorname{Diff}(\mathbb{C},0)$, if $G \subseteq S_u$. The group G is said then to be *integrable*.

If G is cyclic and generated by a holomorphism g, then we say that u is a *first integral* of g. The germ g is *integrable* if it admits a nontrivial holomorphic first integral.

Proposition 12.15. *An holomorphism is periodic if and only if it is integrable.*

More precisely, $h \in \operatorname{Diff}(\mathbb{C},0)$ admits a first integral $u = cx^m + \cdots$, $c \neq 0$, if and only if $h^k = \mathrm{id}$, where k divides m.

Proof. A periodic holomorphism h is linearizable by Theorem 12.10 and any linear map $x \mapsto \nu x$, $\nu^k = 1$, has the first integrals $u(x) = x^m$ for all m divisible by k (the case $m = 0$ is trivial and has to be excluded).

Conversely, if h is integrable and $u = x^m + \cdots$ is the integral, then every level set $M_c = \{u(x) = c\} \subseteq (\mathbb{C},0)$ in a sufficiently small neighborhood of 0 consists of exactly m points that are mapped into each other by h. By the Lagrange theorem, $h|_{M_c}$ is of period $k = k(c)$ that divides m. Let k be the minimal value such that the set of k-periodic points is infinite. Then the kth iterate of h is identity by the uniqueness theorem. □

12.5. Orbits of (pseudo)groups of biholomorphisms. If a group G acts on a set X, the action denoted by $x \mapsto g(x)$, then the G-orbit of a point $x \in X$ for this action is the subset $\{g(x)\colon g \in G\} \subseteq X$.

A germ $g \in \operatorname{Diff}(\mathbb{C},0)$ "acts" on an unspecified small neighborhood $(\mathbb{C},0)$ of the origin, but the value $g(x)$ makes no sense unless $x = 0$. One can replace the germ g by its representative $g \in \mathcal{O}(U)$ defined in some open neighborhood $U \ni 0$, but in general $g(U) \not\subseteq U$ so that the action in the rigorous sense is not defined. Thus we need a special definition for orbits of subgroups of $\operatorname{Diff}(\mathbb{C},0)$.

Definition 12.16. Let $G \Subset \operatorname{Diff}(\mathbb{C},0)$ be a subgroup generated by finitely many germs $g_1,\dots,g_n$ and $U \ni 0$ an open subset in which representatives of all $2n$ germs $g_i^{\pm1}$ are defined.

For any point $x \in U$ the *orbit* $G(x|U)$ of the point x in U is the *maximal* set with the following property. Any two points x', x'' from this set can be connected by a finite chain of points $x_1 = x', x_2, \dots, x_{k-1}, x_k = x''$, all belonging to U, such that any two consecutive points in this chain are obtained from each other by the application of some generator g_i or its inverse g_i^{-1}.

This somewhat technical definition becomes completely transparent when $G = \{g^{\mathbb{Z}}\}$ is a cyclic group generated by a single germ g. Choose an open domain U in which a representative of g and its inverse g^{-1} are both defined. In this case the G-orbit $G(x|U)$ is the bi-directional sequence of iterates $x_{k\pm1} = g^{\pm1}(x_k)$, $k = 0, \pm1, \pm2, \dots$ defined until all iterates remain in U. The orbit may be finite, infinite or bi-infinite. In the cyclic case $G = \{g^{\mathbb{Z}}\}$ we will sometimes speak about g-orbit rather than G-orbit.

Periodicity of a germ g (meaning that $g^n = \mathrm{id}$) means that all infinite g-orbits are periodic. The inverse statement is less obvious.

Lemma 12.17. *If the germ $g \in \operatorname{Diff}(\mathbb{C}, 0)$ is aperiodic, i.e., if the cyclic group $G = \{g^{\mathbb{Z}}\}$ is infinite, then for any small open domain $U \ni 0$ there are uncountably many infinite aperiodic orbits $G(x|U)$.*

Proof. Consider an arbitrary circular disk $D_\rho = \{|x| < \rho\}$ and its boundary circle $K_\rho = \partial D_\rho$, $\rho > 0$.

1. We prove that there are uncountably many points on D_ρ with infinite g-orbits in D_ρ (either in the past or in the future). To that end, we will show that on each circle K_r, $r \leqslant \rho$, there is at least one point with an infinite orbit in D_ρ. Since the number of different circles which can intersect any given orbit is at most countable, this will prove that the number of infinite orbits in uncountable.

Define two extended-integer-valued functions,

$$\overline{\nu}(x) = \#G(x|\overline{D}_r), \qquad \underline{\nu}(x) = \#G(x|D_r),$$

the length of the orbit $G(x)$ inside the *closed* (respectively, *open*) disks $\overline{D}_r$ (resp., D_r). The values can be either finite or infinite. Since g is continuous, these two functions are semicontinuous in two opposite senses:

- if $\overline{\nu}(x) < +\infty$, then for all y sufficiently close to x, $\overline{\nu}(y) \leqslant \overline{\nu}(x)$,
- if $\underline{\nu}(x) < +\infty$, then for all y sufficiently close to x, $\underline{\nu}(y) \geqslant \underline{\nu}(x)$.

Assume that all points on the circumference K_r have finite orbits, i.e., the function $\overline{\nu}$ takes only finite values on K_r. Since K_r is compact, this means that $\overline{\nu}$ is bounded from above on K_r:

$$K_r \subseteq \{\overline{\nu} \leqslant N\} = \text{a relatively open subset in } \overline{D}_r.$$

On the other hand, since $g(0) = 0$, we have $\underline{\nu}(0) = +\infty$ so that the relatively open subset $\{\underline{\nu} > N\} \subset D_r$ is nonempty. Since the disk is connected and the two open sets $\{\overline{\nu} \leqslant N\}$ and $\{\underline{\nu} > N\}$ are disjoint, there is a point x_0 not in their union. This means that the orbits of x_0 in D_r and $\overline{D}_r$ are of *different* (finite) lengths, which is possible only if the orbit intersects the boundary K_r. But this contradicts our construction since the length of this orbit is greater than $N = \sup \overline{\nu}|_{K_r}$.

2. To complete the proof of the Lemma, note that the set of points with infinite orbits is the union of periodic points and the infinite aperiodic

orbits. For each finite N, the N-periodic points inside $\overline{D}_r$, roots of the equation $g^N(x) - x = 0$, form a finite subset of D_r. Indeed, otherwise by the uniqueness theorem, the germ g^N should be identity. The union of these finite sets is at most countable. Therefore the complement, the union of infinite aperiodic orbits in D_r, is uncountable. □

Thus we have a clear alternative.

Theorem 12.18. *Any finitely generated group $G \Subset \operatorname{Diff}(\mathbb{C}, 0)$ is integrable, or has uncountably many infinite aperiodic orbits.*

Proof. If G includes an aperiodic germ g, then this germ has uncountably many aperiodic orbit by Lemma 12.17. Conversely, if all elements of G are of finite order, then by Theorem 12.12 the group is finite, cyclic hence linearizable. Its integrability follows from Proposition 12.15. □

12.6. Singular foliations on $(\mathbb{C}^2, 0)$. A singular foliation $\mathcal{F} = \{\omega = 0\}$ on $(\mathbb{C}^2, 0)$ is said to be *integrable*, if there exists a nonzero holomorphic function (germ) u such that $\omega \wedge du = 0$.

Every leaf of an integrable foliation entirely belongs to a level curve $\{u = \text{const}\}$. This implies, among other, that the vanishing holonomy group of the integrable foliation is an integrable subgroup of the group $\operatorname{Diff}(\mathbb{C}, 0)$.

Topology of integrable foliations is necessarily simple.

Definition 12.19. A singular foliation $\mathcal{F}$ on $(\mathbb{C}^2, 0)$ is simple, if

(1) all leaves are relatively closed in $(\mathbb{C}^2, 0) \smallsetminus \{0\}$, and
(2) at most countably many leaves contain the isolated singular point $\{0\}$ in their closure.

Obviously, any integrable foliation is simple in the sense of this Definition. Moreover, the number of leaves adjacent to the singular point, is finite.

The inverse result generalizes both the Poincaré–Lyapunov theorem and Theorem 12.18.

Theorem 12.20 (J.-F. Mattei and R. Moussu [**MM80**])**.** *A simple singular holomorphic foliation is always integrable.*

The proof of this theorem, more precisely, its reduction to Theorem 12.18, is based on the following purely topological Lemma 12.22. This lemma asserts that simple foliations behave saddle-like from the topological point of view: almost all of their leaves “exit” any small neighborhood of the singularity. To formulate this property precisely, we use the construction of saturation.

Definition 12.21. The *saturation* of a subset $K \subset U$ by leaves of the (singular) foliation $\mathcal{F}$ defined in an open set U, is the union of all leaves of $\mathcal{F}|_U$ intersecting the subset K.

Avoid redundancy! Move the definition to the intro section? Xref?

Consider *any* complex separatrix L of the foliation $\mathcal{F}$ (such separatrix always exists by Theorem 9.2) and a cross-section τ to it at a nonsingular point $0 \neq a \in L$.

Lemma 12.22 (Saturation Lemma)**.** *Assume that a holomorphic singular foliation $\mathcal{F}$ is simple and has an isolated singularity at the origin.*

Then for any open neighborhood $V \subseteq \tau$ of the point $a = \tau \cap L$ on any cross-section τ to an arbitrary complex separatrix L, its saturation by the leaves of $\mathcal{F}$ contains an open neighborhood U of the singular point, eventually after deleting an analytic curve $S \subseteq U$.

Moreover, if S is non-empty and a variable point $b \in U \smallsetminus S$ tends to $S \smallsetminus \{0\}$, then the leaf $\mathcal{F}_b$ passing through b, crosses τ by a point set $\mathcal{F}_b \cap \tau$ such that $\operatorname{dist}(\mathcal{F}_n \cap \tau, a) \to 0$.

Proof of Theorem 12.20 modulo Lemma 12.22. Consider an arbitrary separatrix L of a holomorphic simple singular foliation and choose a cross-section $(\tau, a) \simeq (\mathbb{C}, 0)$ to it. Let $G \subset \operatorname{Diff}(\mathbb{C}, 0)$ be the holonomy group of $\mathcal{F}$, associated with this choice.

The group G cannot have uncountably many infinite aperiodic orbits. Indeed, otherwise there should be uncountably many leaves of $\mathcal{F}$ that intersect τ by an infinite point set, hence are either non-closed in $U \smallsetminus L$ or contain L hence $\overline{L} = L \cup \{0\}$ in their closure.

By Theorem 12.18, the group G is integrable, i.e., there exists a function $u\colon (\tau, a) \to (\mathbb{C}, 0)$ which is holomorphic and G-invariant. Because of this invariance, it can be extended on the saturation U' of (τ, a) by leaves of $\mathcal{F}$ as a first integral of $\mathcal{F}$, holomorphic on this saturation.

It remains to invoke Lemma 12.22, the saturation U' contains some open set of the form $U \smallsetminus S$, where U is an open neighborhood of the origin and S some analytic curve. Moreover, the extension of u on U' has the zero limit on S. This is sufficient to guarantee that u extends from $U \smallsetminus S$ on the whole of S while remaining holomorphic first integral. This proves Theorem 12.20. □

12.7. Simple elementary foliations. In this section we begin the proof of Lemma 12.22. The proof is based on desingularization theorem and goes by induction in the number of blow-ups required for making all singularities elementary.

As a base of induction, we verify case by case the assertion of the Lemma for elementary singularities. First, we eliminate the saddle-node case.

Lemma 12.23. *A foliation with an isolated degenerate elementary singularity is never simple.*

Proof. without loss of generality one may assume that the hyperbolic invariant curve coincides with the x-axis. Since the singularity is isolated, there exists a finite natural number m such that the corresponding differential equation takes the form

$$\frac{dy}{dx} = \frac{y^m(1+\cdots)}{x+\cdots}, \qquad |x|<1,\ |y|<1.$$

Consider the cross-section $\tau = \{x = 1, |y| < \delta\}$ and the corresponding cyclic holonomy group. Its generator can be obtained by integration of the equation over the loop around the origin in the x-plane. Parameterizing the loop as $x = \exp is$, $s \in [0, 2\pi]$, we obtain an ordinary differential equation $\frac{dy}{ds} = y^m F(s, y)$ with a complex-valued function F which is 2π-periodic in s and holomorphic in y. Integrating this equation on the interval $s \in [0, 2\pi]$, we obtain the generator germ of the holonomy group

$$g(y) = y + 2\pi y^m + \cdots$$

of infinite order (aperiodic, non-linearizable). This implies that there exists a leaf of the foliation that accumulates to the separatrix $y = 0$, i.e., is not closed. In other words, a foliation defined by a saddle-node is never simple in the sense of Definition 12.19. □

The complex topology of a nondegenerate elementary singularity depends on the hyperbolicity ratio λ. Consider a singular foliation $\mathcal{F}$ defined in $(\mathbb{C}^2, 0)$ by the holomorphic line field

$$\frac{dy}{dx} = \frac{\lambda y + \beta x + \cdots}{x+\cdots}, \qquad 0 \neq \lambda \in \mathbb{C}, \qquad \beta \in \{0,1\} \tag{12.9}$$

(the dots denote nonlinear terms). The hyperbolicity ratio λ is a nonzero complex number. The coefficient $\beta \in \{0, 1\}$ can be made zero unless $\lambda = 1$.

For real foliations, the cases $\lambda < 0$ (saddle) and $\lambda > 0$ (node) differ by the way how the (real) phase curves approach the origin on the plane $(\mathbb{R}^2, 0)$. In the nodal case all trajectories adhere to the singular point at the origin (i.e., contain the origin in their closure). On the other hand, in the saddle case the vector field has two separatrices and the union of all trajectories starting at a cross-section to any of these separatrices (saturation of the cross-section), contains an open neighborhood of the origin with only the second separatrix deleted.

This description survives complexification, though the complex "saddles" and "nodes" are not mutually exclusive.

Lemma 12.24 (Complex "nodal" case)**.** *If $\lambda \notin \mathbb{R}_+$, then all leaves of the foliation* (12.9) *contain the origin in their closure.*

Proof. The case when the ratio of eigenvalues is not a negative number or zero, corresponds to a planar vector field in the Poincaré domain. By Proposition 7.1, all leaves of the corresponding foliation intersect transversally all small spheres $\{|x|^2+|y|^2=\varepsilon\}$ and hence contain the origin in their closure. □

In the saddle case $\operatorname{Re}\lambda<0$ the Hadamard–Perron Theorem 6.2 always applies, and therefore the foliation $\mathcal{F}$ has two holomorphic smooth complex separatrices that can be normalized to become coordinate axes. The differential equation (12.9) defining the foliation $\mathcal{F}$ in these coordinates will takes the form

$$\frac{dy}{dx}=\frac{y}{x}(\lambda+a(x,y)), \qquad a(0,0)=0. \tag{12.10}$$

Rescaling the variables if necessary, we assume that the equation (12.10) is defined in the bidisk $\{|x|<1,\ |y|<1\}$ and the holomorphic term $a(x,y)$ is bounded in this bidisk,

$$|a|<\tfrac{1}{2}|\lambda|.$$

Lemma 12.25 (Complex saddle case)**.** *If $\operatorname{Re}\lambda<0$, then saturation of the cross-section $\tau=\{x=1,\ |y|<\delta\}$ by leaves of the foliation* (12.9) *contains a sufficiently small open punctured bidisk $\{0<|x|<\varepsilon,\ |y|<\varepsilon\}$ (with the deleted separatrix $x=0$).*

If the point (x,y) rends to the separatrix $x=0$, then the leaf of the foliation passing through this point, intersects τ by a point set, at least one point of which tends to $(1,0)$.

Proof. We need to prove that solutions of this differential equation with an arbitrary initial condition (x_0,y_0), close enough to the origin, $0<|x_0|<\varepsilon$, $|y_0|<\varepsilon$, can be continued without quitting the bidisk $\{|x|<1,\ |y|<1\}$ along an appropriate (real) curve $x=x(t)$ in the x-axis (plane) that ends at $x=1$, so that the continuation will satisfy the condition $|y(1)|<\delta$.

1. For a point over the positive real semiaxis, i.e., with $x_0\in\mathbb{R}_+$, we continue solutions over the *real* interval $[x_0,1]$, see Figure 8. Along this interval,

$$\frac{d|y|^2}{dx}=\frac{\bar y y}{x}(\lambda+a(x,y))+\frac{y\bar y}{x}(\overline{\lambda+a(x,y)})=2\frac{|y|^2}{x}\operatorname{Re}(\lambda+a(x,y))<0$$

This means that $|y(x)|$ decreases along the real semiaxes as x increases from $x_0>0$ to 1 and therefore $|y(1)|<|y_0|$. Moreover, the same computation in fact shows that $|y(1)|^2<|y_0|^2|x_0|^{\operatorname{Re}\lambda}$, so that as $x_0\to 0^+$, the value $y_{x_0}(1)$ of the corresponding solution tends to zero.

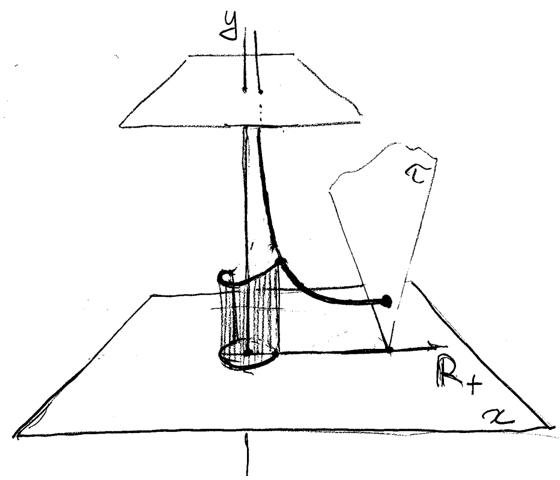

Figure 8. Saturation of a cross-section τ near a saddle singular point of a holomorphic foliation.

2. For all other initial points (x_0, y_0) with $x_0 \notin \mathbb{R}_+$, we first continue the solution over any circular arc $|x| = |x_0|$ until x becomes real positive at some point $x_1 \in \mathbb{R}_+$. This continuation is not unique (hence the solutions are in general multivalued, as excepted), but the growth of $|y(x)|$ along this arc is well bounded. Indeed, parameterizing the arc as $x = x_0 \exp is$, $s \in [0, 2\pi]$, $dx = ix\,ds$, we derive from (12.10) the ordinary equation

$$\frac{dy}{ds} = -iy(\lambda + a(x,y)), \qquad s \in [0, \pi],$$

with the real time, which implies the bound

$$|y(x_1)| \leqslant (\exp 4\pi |\operatorname{Im} \lambda|) \cdot |y_0|.$$

Continuing this solution further over the segment $[x_1, 1]$, we achieve the proof of the Lemma in the general case. □

12.8. Proof of the Saturation lemma. Assume that the Saturation lemma 12.22 is proved for all holomorphic foliations with an isolated singularity, that require less than N blow-ups for full desingularization. The case $N = 1$ (the base of induction) was proved in §12.7.

Let L be a complex separatrix for a foliation $\mathcal{F}$ and $\tau \pitchfork L$ a cross-section to it. Assume that $\mathcal{F}$ requires N blow-ups for full desingularization. After blow-up $\mathcal{F}$ lifts to a holomorphic singular foliation $\mathcal{F}'$ defined near an exceptional divisor $S_0 = \mathbb{C}P^1$ and having finitely many singular points $a_1, \dots, a_n \in S_0$, each of them requiring less than N blow-ups.

Obviously, $\mathcal{F}'$ must be simple near each singularity a_i and the blow-up must be non-dicritical (otherwise one can immediately can find uncountably many leaves of $\mathcal{F}$ with forbidden closures).

The separatrix L becomes a separatrix of one of the singularities, say, a_1, while τ becomes a cross-section τ_1. The exceptional divisor after deleting the singular points becomes another separatrix L_0, common for all singularities. Choose a cross-section τ_0 to L_0 at a nonsingular point $a_0 \notin \{a_1, \dots, a_n\}$.

By the inductive assumption, saturation of both τ_0 and τ by leaves of $\mathcal{F}'$ restricted on a sufficiently small neighborhood U_1 of a_1, contains a (smaller) neighborhood without a complex analytic curve S_1, i.e., the saturation of τ contains some small (punctured) neighborhood of the origin on τ_0.

For obvious reasons (rectification of foliation near a nonsingular point), saturation of τ_0 by leaves of $\mathcal{F}'$ contains a small neighborhood of any compact subset $K \Subset L_0$, in particular, open neighborhoods of the origin on small cross-sections $\tau_2, \dots, \tau_n$ to L_0 near the singular points $a_2, \dots, a_n$.

Finally, again by the induction assumption, saturation of any small neighborhood on τ_i contains open neighborhoods of the points a_i, $i = 2, \dots, n$, eventually after deleting some analytic curves $S_2, \dots, S_n$.

As a result, we conclude that saturation of τ_1 contains a small neighborhood of the exceptional divisor $S_0 \simeq \mathbb{C}P^1$, eventually without the union S of n analytic curves $S_1, \dots, S_n$. The assertion on leaves passing close to this union, is also obvious.

After blowing down, we obtain the proof of the Saturation lemma 12.22 for the initial foliation, completing thus the inductive step. □

12.9. Survey of further results. Here we briefly mention some of the results that link integrability with properties of the holonomy group, and also mention some generalizations.

12.9.1. *Formal and true integrability.* Existence of a first integral is difficult to establish. One can look for a solution as the formal series $u = u_m + u_{m+1} + \cdots$ and write a triangular system of linear equations similar to (11.7) and (12.5) for the homogeneous terms u_k.

The formal series $u \in \mathbb{C}[[x, y]]$ such that $\omega \wedge du = 0$, if it can be found in such way, does not necessarily have to converge. Indeed, if there exists at least one convergent solution $u(x, y)$, then among *different* solutions of this formal system there are always divergent solutions of the form $g(u(x, y))$, where g is a *divergent* series in one variable.

However, existence of at least one nonzero formal solution implies existence of holomorphic first integrals. For elliptic singular points it was proved in Proposition 12.5. The general result, also due to Mattei and Moussu, holds under the only assumption that the singularity is isolated, see [**MM80**]. Its proof can be also achieved using desingularization.

Recall that any formal series or holomorphic germ can be factorized as a product of powers of irreducible series (resp., germs), $u = f_1^{d_1} \cdots f_k^{d_k}$, $d_j \geqslant 1$. We say that u is not a power, if $\gcd(d_1, \dots, d_k) = 1$.

Theorem 12.26 (J.-F. Mattei and R. Moussu, [**MM80**]). *Assume that the holomorphic foliation $\mathcal{F} = \{\omega = 0\}$ in $(\mathbb{C}^2, 0)$ has a formal first integral $u \in \mathbb{C}[[x, y]]$. Then there exists a holomorphic first integral $0 \neq v \in \mathcal{O}(\mathbb{C}^2, 0)$.*

If v is not *a power, then any other formal (resp., holomorphic) first integral is of the form $g(v)$, where g is a formal (resp., holomorphic) germ in one variable.* □

In fact, both Theorems 12.20 and 12.26 are particular 2-dimensional cases of more general results concerning holomorphic singular foliations in $(\mathbb{C}^n, 0)$. We will not discuss these generalizations.

Two survey sections will be added later

12.9.2. *Liouville and Darboux integrability.* Besides holomorphic integrals, *polynomial* vector fields may possess more general kind of first integral, the so called Darboux integrals...

12.9.3. *Reversibility.* Another reason for existence of real center is symmetry (reversibility). ... Discussion...

Yet it would be wrong to conclude that there are no other reasons for centrality. In [**BCLN96**] an example of a real analytic foliation is constructed, that absolutely “asymmetric” and does not admit even Liouvillean integral.

13. Zeros of analytic functions depending on parameters and small amplitude limit cycles

This section, somewhat aside from the mainstream, deals with analytic local multiparametric families (deformations) of functions of one variable (real or complex). If a function has an isolated root of multiplicity $\mu < \infty$, then by the Weierstrass preparation theorem any its deformation has no more than μ zeros nearby (exactly μ in the complex analytic settings). We describe an object, called *Bautin ideal*, that determines the bound for the number of isolated zeros in the case when deformations of an *identically zero* function are considered.

This subject is traditionally linked to the problem of describing bifurcations of limit cycles from an elliptic center. The problem was studied first by Poincaré and H. Hopf and later by A. Andronov and L. Pontryagin. In the least degenerate case it is customarily referred to as the *Andronov–Hopf bifurcation*. N. Bautin formulated the problem in full generality, including cases of infinite degeneracy (centers), and gave a complete solution for quadratic vector fields in 1939, see [**Bau54**]. We give in §13.8 the modern exposition of this work, based on [**Żoł94**].

13.1. Poincaré–Andronov–Hopf–Takens bifurcation: small limit cycles bifurcating from elliptic points. Consider a real analytic *local family* of planar vector fields $F_\lambda = F(x,y;\lambda)$ defined in a small neighborhood $(\mathbb{R}^2, 0)$ of the origin on the real plane and depending analytically on a number of real parameters $\lambda = (\lambda_1, \dots, \lambda_n) \in (\mathbb{R}^n, 0)$. Suppose that this family is *elliptic*, i.e., for all (sufficiently small) values of the parameters the eigenvalues of the linearization matrix $A(\lambda)$ are nonzero complex conjugate numbers.

This assumption immediately implies that the singular point itself depends analytically on the parameters (by the implicit function theorem). Moreover, the local coordinates (x,y) can be chosen that linear part $\mathbf{A}$ of F has the form

$$\mathbf{A} = \alpha(\lambda)\mathbf{E} + \beta(\lambda)\mathbf{I}, \qquad \mathbf{E} = x\tfrac{\partial}{\partial x} + y\tfrac{\partial}{\partial y}, \quad \mathbf{I} = y\tfrac{\partial}{\partial x} - x\tfrac{\partial}{\partial y}, \tag{13.1}$$

with real analytic coefficients (germs) $\alpha(\lambda)$ and $\beta(\lambda)$ before the radial (Euler) vector field $\mathbf{E}$ and the rotation field $\mathbf{I}$. The ellipticity assumption means that the real analytic function $\beta(\lambda)$ is non-vanishing.

The monodromy (first return) map $P(\cdot,\lambda)$ for any elliptic family is real analytic and depends analytically on the parameters by Theorem 11.30. Denote by $f(x,\lambda)$ the displacement function $f = P - \mathrm{id}$ for some choice of a cross-section, say, the semiaxis $\tau_+ = \{y=0,\ x>0\}$, and an analytic chart x on this cross-section. By definition, sufficiently small limit cycles of the field F_λ intersect τ_+ at *isolated* zeros of f.

The number of small limit cycles born by small perturbations from a singular point, is usually referred to as the *cyclicity* of this singular point relative to the family $F = \{F_\lambda\}$.

This cyclicity can be relatively easily determined (or rather majorized) if the field $F(\cdot,0)$ is *not a center*. In this case the displacement function $f(\cdot,0)$ is different from the identical zero and hence there exists a finite natural number μ such that $f(x,0) = cx^\mu + O(x^{\mu+1})$ with some $c \neq 0$.

In this case there exist $\varepsilon > 0$ and $\delta > 0$ such that for all $|\lambda| < \varepsilon$ the function $f(\cdot,\lambda)$ has no more than μ roots in the interval $(0,\delta)$, necessarily isolated. In fact, in the analytic case we are dealing with, the number of zeros of the complexified function is bounded by the same constant μ in the small complex *disk* $\{|x| < \delta\} \subseteq (\mathbb{C}^1, 0)$.

The proof is standard. The function $f(x,0) = cx^\mu(1+o(1))$ is nonvanishing along a sufficiently small circle $\{|x| = \delta\}$ and its *variation of argument* (index) along this circle is equal to $2\pi\mu$. By continuity in the parameters, the variation of argument of $f(\cdot,\lambda)$ along $\{|x|=\delta\}$ remains the same for all $|\lambda| < \varepsilon$ if $\varepsilon > 0$ is sufficiently small. By the argument principle, the number of complex roots of $f(\cdot,\lambda)$ in the disk $\{|x| < \delta\}$ is equal to μ.

The bound for cyclicity established by this simple argument, does not depend on the family, only on the field $F(\cdot, 0)$. On the other hand, these arguments break almost completely if the field $F(\cdot, 0)$ is integrable (center). In this case the bound necessarily depends on the family.

This section describes the algebraic procedure that allows to produce an upper bound for the cyclicity of an elliptic family of real analytic planar vector fields.

Remark 13.1. In fact, for elliptic vector fields the order μ of the displacement function must be always odd and no more than $(\mu - 1)/2$ small limit cycles can be generated near an elliptic singular point by any analytic perturbation. The reason is that the origin is always a zero of the displacement function, while every small limit cycle crosses twice any analytic curve through the origin. To restore the uniformity, in the elliptic case one has to consider only positive values of the real parameters. The issue is addressed in more details later in §13.5.

13.2. Bautin ideal and generating functions. The initial steps of the construction exposed below, refer to *semi*formal series, i.e., *formal* series in one independent variable x, whose coefficients depend *analytically* (or even polynomially) on several real or complex parameters $\lambda_1, \dots, \lambda_n$.

Let $\mathfrak{A}$ be a Noetherian ring of functions. The most important are the particular cases when $\mathfrak{A}$ is:

(1) the rings of germs $\mathcal{O}(\mathbb{C}^m, 0)$ or $\mathcal{O}(\mathbb{R}^n, 0)$, complex or real analytic respectively,
(2) the ring $\mathcal{O}(U)$ of analytic functions in a domain $U \subseteq \mathbb{R}^n$, or $U \subseteq \mathbb{C}^n$,
(3) the ring of polynomials in m variables $\lambda_1, \dots, \lambda_m$ (again, real or complex).

In this section we refer to the variables $\lambda_1, \dots, \lambda_m$ as the *parameters* and U the *parameter space*. Using anyone of these rings, we can construct the ring $\mathfrak{A}[[x, y, \dots]]$ of *semiformal series*, formal in the variables $x, y, \dots$ with coefficients analytically depending on the parameters $\lambda_1, \dots, \lambda_m$. Of course, it contains as a subring the ring of analytic functions or germs defined in an appropriate domains.

With any sequence of functions

$$a_0(\lambda), a_1(\lambda), \dots, a_k(\lambda), \dots, \qquad a_k \in \mathfrak{A}, \tag{13.2}$$

we can associate a growing chain of ideals

$$\begin{gathered} B_0 \subseteq B_1 \subseteq \cdots \subseteq B_k \subseteq \cdots \subseteq (1) = \mathfrak{A}, \\ B_k = \langle a_0, a_1, \dots, a_k \rangle . \end{gathered} \tag{13.3}$$

Since the ring $\mathfrak{A}$ is Noetherian, the chain (13.3) stabilizes at some moment, $B_\nu = B_{\nu+1} = \cdots$.

With the sequence (13.2) we will associate the *generating function*, the *semi*formal series in one variable

$$a(x, \lambda) = \sum_{k \geqslant 0} a_k(\lambda)\, x^k \in \mathfrak{A}[[x]]. \tag{13.4}$$

Conversely, with any formal or converging series $a(x, \lambda)$ of the form (13.4) we can associate the sequence of its coefficients (13.2), the ascending chain of ideals (13.3), denoted by $B_k(a)$, and the ideal

$$B(a) = \lim_{k \to \infty} B_k(a) = B_\nu(a). \tag{13.5}$$

Definition 13.2. The ideal $B(a)$ is called the *Bautin ideal* of the semiformal series $a(x, \lambda)$. The chain of ideals (13.3) will be referred to as the *Bautin chain* and denoted $\mathfrak{B}(a)$. The stabilization moment ν is the *Bautin index*.

We stress that the enumeration of ideals in the Bautin chain begins with B_0 which, however, may be zero ideal. For application to *real analytic* problems instead of the Bautin index we will use another number, the Bautin depth that is by one less the number of *nonzero different* ideals in the chain (13.3).

Definition 13.3. The *Bautin depth* of the chain (13.3) is the number of instances in which the inclusion is *strict* and *nontrivial*,

$$\mu = \#\{k \in \mathbb{N} \colon 0 \neq B_{k-1} \neq B_k\} \geqslant 0.$$

Obviously, $\mu \leqslant \nu$, with the equality possible only if $0 \neq B_0 \subsetneq \cdots \subsetneq B_\nu = B_{\nu+1} = \cdots$.

For two Bautin chains of ideals $\mathfrak{B} = \{B_k\}$ and $\mathfrak{B}' = \{B'_k\}$ in the same ring $\mathfrak{A}[[x]]$ we will write $\mathfrak{B} = \mathfrak{B}'$ if all ideals in the two chains coincide, and $\mathfrak{B} \subseteq \mathfrak{B}'$ when $B_k \subseteq B'_k$ for all $k = 0, 1, 2, \ldots$.

Remark 13.4 (terminological)**.** The term "Bautin ideal" is rather standard and widely used [**Rou98, Yom99**], whereas the combination "Bautin chain" is not. Speaking formally, the Bautin chain $\mathfrak{B}(a)$ defines a *filtration* on the Bautin ideal $B(a)$. In order to be consistent with the accepted terminology, we will speak mostly about Bautin ideals, while always bearing in mind that they possess the additional structure induced by this filtration. We will use the notation $\mathfrak{B}(a)$ for the Bautin ideal in order to stress the fact that it is considered together with the filtration, whereas $B(a)$ usually denotes the unfiltered ideal.

On the contrary, the term "Bautin depth" seems to be new. The reason why the Bautin depth is introduced, is closely related to the so called *fewnomials theory* developed by A. Khovanskii [**Kho91**]. Its usefulness will be clear from Example 13.9.

Recall that the *radical* of an ideal $B \subseteq \mathfrak{A}$ is

$$\sqrt{B} = \{f \in \mathfrak{A}\colon f^k \in B \text{ for some } k \in \mathbb{N}\}. \tag{13.6}$$

Obviously, $B \subseteq \sqrt{B}$. The ideal is *radical* (adjective), is $B = \sqrt{B}$.

For *polynomial* ideals in $\mathfrak{A} = \mathbb{C}[\lambda_1, \dots, \lambda_n]$ over the algebraically closed field, the radical consists of all polynomials vanishing on the *complex null locus* $X_B = \{\lambda \in \mathbb{C}^n \colon f(\lambda) = 0 \, \forall f \in B\}$ of the ideal $B \subseteq \mathbb{C}[\lambda_1, \dots, \lambda_n]$. This is the famous *Hilbert Nullstellensatz*. Thus the radical polynomial ideals over $\mathbb{C}$ are in one-to-one correspondence with their null loci: any radical ideal can be characterized as the *biggest* ideal with the same null locus.

The *null locus* X_B (real or complex) of the Bautin ideal B corresponds to the parameter values when the series $a(\cdot, \lambda)$ vanishes identically.

The Bautin ideal (and more generally, the Bautin chain) describes parametric deformations of the identically zero functions (series). In a similar way, we can introduce ideals describing deformations of "maximally degenerate" objects of other types, that can be translated into univariate series. Besides "obvious" candidates, like semiformal families of vector fields on the line that can vanish identically or semiformal families of maps that can contain periodic series, the Bautin ideal can be associated with families of elliptic vector fields that can exhibit formal centers for some values of the parameters.

To introduce the formal definitions, together with semiformal "functions" from the ring $\mathfrak{A}\,[[x, y, \dots]] = \mathfrak{A} \otimes \mathbb{C}[[x, y, \dots]]$ we consider families of other types of formal objects depending analytically on the parameters. In particular, we will be interested in the following classes.

(1) Families of formal morphisms $\mathfrak{A} \otimes \operatorname{Diff}[[\mathbb{C}^n, 0]]$ (we will be only interested in the one-dimensional case $n = 1$),

(2) Families of formal vector fields $\mathfrak{A} \otimes \mathcal{D}[[\mathbb{C}^n, 0]]$, for $n = 1, 2$,

(3) Real analytic counterparts of all of the above, with the ground field $\mathbb{C}$ replaced by $\mathbb{R}$.

Every time the tensor product $\otimes$ is considered over the appropriate ground field, $\mathbb{R}$ or $\mathbb{C}$ respectively. As a common term, we will refer to these objects as *semiformal* families (of maps, fields, forms *etc.*). The prefix *semi-* indicates that the coefficients of the formal series are *analytic* functions of the parameters λ. Sometimes we will write these families as collections, $\{f_\lambda\}_{\lambda \in U}$, $\{F_\lambda\}_{\lambda \in U}$ rather than in the full form $f(\cdot, \lambda)$, $F(\cdot, \lambda)$ *etc.*

13.3. Basics of formal theory. We begin by pointing out several almost obvious properties of the Bautin ideals of "univariate" objects. These properties reflect simple combinatorics of coefficients of product and composition of formal series in one independent variable. All of them become trivial if instead of ideals their null loci were involved, see Remark 13.13.

Proposition 13.5. *If $f, g \in \mathfrak{A}[[x]]$, then $\mathfrak{B}(fg) \subseteq \mathfrak{B}(f)$. If g is invertible in $\mathfrak{A}[[x]]$, then $\mathfrak{B}(fg) = \mathfrak{B}(f)$.*

Proof. Denote by $f_k, g_k \in \mathfrak{A}$ the Taylor coefficients of f and g respectively, and by f'_k the coefficients of their product fg. Then, obviously,

$$f'_k = g_0 f_k \mod \langle f_0, f_1, \dots, f_{k-1} \rangle ,$$

which means that $B_k(fg) \subseteq B_k(f)$ for all $k = 0, 1, \dots$. The first assertion is thus proved; the second assertion follows from the fact that g is invertible in $\mathfrak{A}[[x]]$ if and only if the principal (free) Taylor coefficient $g(0)$ is invertible in $\mathfrak{A}$. □

The Bautin ideal is in fact independent of the choice of the coordinate x, or, in algebraic terms, of the generator of the ring $\mathfrak{A}[[x]]$. This ring can be generated by any element $y = c_1 x + c_2 x^2 + \cdots \in \mathfrak{A}[[x]]$ (semiformal series), provided that the leading coefficient $c_1 = c_1(\lambda) \in \mathfrak{A}$ is *invertible*: $c_1^{-1} \in \mathfrak{A}$. Indeed, $\mathfrak{A}[[y]] \subseteq \mathfrak{A}[[x]]$ regardless of the choice of y. If c_1 is invertible, then by the formal inverse function theorem x can be expanded as a formal series in powers of y, so that the equality $\mathfrak{A}[[y]] = \mathfrak{A}[[x]]$ holds.

Recall (see §3.1) that the operator $f \mapsto f \circ y$ is an endomorphism of the ring $\mathfrak{A}[[x]]$ over $\mathfrak{A}$ (i.e., additive, multiplicative and identical on $\mathfrak{A}$). Conversely, any such endomorphism $H \colon \mathfrak{A}[[x]] \to \mathfrak{A}[[x]]$ is induced by a composition $f \mapsto f \circ y$ with $y = Hx$. If y is a generator of $\mathfrak{A}[[x]]$, then H is an invertible endomorphism (automorphism) of $\mathfrak{A}[[x]]$ and vice versa.

Proposition 13.6. *If $y = \sum_1^\infty c_k x^k$ is a generator of the ring $\mathfrak{A}[[x]]$, then for any $f \in \mathfrak{A}[[x]]$ the Bautin ideals of f and $f \circ y$ coincide.*

Proof. Denote the Taylor coefficients of f and $f' = f \circ y$ by f_k and f'_k respectively, and let $y = c_1 x + c_2 x^2 + \cdots$, $c_j \in \mathfrak{A}$. Expanding $f' = \sum f_k y^k$, we obtain the formula

$$f'_k = c_1^k f_k \mod \langle f_0, f_1, \dots, f_{k-1} \rangle .$$

The series y is a generator if and only if c_1 is invertible in $\mathfrak{A}$. This immediately means that $B_k(f') = B_k(f)$. □

A semiformal family of vector fields $F \in \mathfrak{A} \otimes \mathcal{D}[[\mathbb{C}^1, 0]]$ on the line, having a singularity at the origin, can be identified with a derivation $g \mapsto Fg$ of the algebra $\mathfrak{A}[[x]]$ over the ring $\mathfrak{A}$ (i.e., $Fc = 0$ for any $c \in \mathfrak{A}$) and preserving the maximal ideal $\mathfrak{m} = \mathfrak{A} \otimes \langle x \rangle$. For any (semi)formal series $y \in \mathfrak{A}[[x]]$, we have $Fy = \frac{\partial y}{\partial x} \cdot Fx$. By Proposition 13.5, $\mathfrak{B}(Fy) \subseteq \mathfrak{B}(Fx)$ with the equality occurring when y is a generator of $\mathfrak{A}[[x]]$. This motivates the following definition.

Definition 13.7. The Bautin ideal of the semiformal family of vector fields F is the Bautin ideal of the semiformal series Fy for any generator $y \in \mathfrak{A}[[x]]$ of the semiformal ring.

By this definition, $\mathfrak{B}(Fg) \subseteq \mathfrak{B}(F)$ for any series g. In coordinates, the Bautin ideal of the semiformal family of vector fields $F = f(x,\lambda)\frac{\partial}{\partial x}$ is the Bautin ideal of the coefficient (series) f. If $g = \sum_{k\geqslant 0} g_k\, x^k$, $F = \sum_{k\geqslant 1} f_k\, x^k\, \frac{\partial}{\partial x}$ and $Fg = g' = \sum_{k\geqslant 0} g'_k\, x^k$, then

$$g'_0 = 0, \quad g'_k = k a_1 g_k \bmod \langle g_0, \dots, g_{k-1} \rangle, \qquad k = 1, 2, \dots. \tag{13.7}$$

Remark 13.8. Note that since a formal derivation F must have zero "free terms", the Bautin chain $\mathfrak{B}(F)$ always starts with the zero ideal $B_0(F) = 0$.

In the same way as the Bautin ideal of a formal series, the Bautin chain of a semiformal field is invariant by automorphisms: if $F' = GFG^{-1}$, where $G\colon f \mapsto f \circ y$ is an automorphism of $\mathfrak{A}[[x]]$ induced by the change of the independent variable $x \mapsto y$, then the Bautin ideals of F and F' coincide.

Unlike power series transformations, *fractional* transformations of the independent variable may change the Bautin chain (i.e., the filtration on the Bautin ideal) without changing its limit (the ideal itself).

Example 13.9. Consider a semiformal vector field $F = f(z,\lambda)\frac{\partial}{\partial z}$ with $f(z,\lambda) = a_1(\lambda)\, z + a_2(\lambda)\, z^2 + a_3(\lambda)\, z^3 + \cdots$. The substitution $z = x^2$ brings this vector field to the field $f'(x,\lambda)\frac{\partial}{\partial x}$ with $f'(x,\lambda) = \frac{1}{2}x^{-1} f(x^2,\lambda) = \frac{1}{2}[a_1(\lambda)\, x + a_2(\lambda)\, x^3 + a_3(\lambda)\, x^5 + \cdots]$.

The Bautin chain $\mathfrak{B}'$ for the transformed vector field is obtained by "shearing transformation" of the chain $\mathfrak{B}$:

$$B'_1 = B'_2 = B_1, \quad B'_3 = B'_4 = B_2, \quad \dots \quad B'_{2k-1} = B'_{2k} = B_k.$$

Clearly, this transformation does not affect the Bautin ideal as the limit of the Bautin chain, and changes the Bautin index. Yet the Bautin depth remains the same.

Now consider an endomorphism $H \in \mathfrak{A} \otimes \operatorname{Diff}[[\mathbb{C}^1, 0]]$ of the algebra $\mathfrak{A}[[x]]$ identical on $\mathfrak{A}$. This endomorphism can be identified with a family of formal maps of the complex line into itself.

Definition 13.10. The (filtered) Bautin ideal of an endomorphism H is the Bautin ideal of the difference $H - \mathrm{id}$, i.e., the (filtered) Bautin ideal of the series $Hy - y$ for an arbitrary generator $y \in \mathfrak{A}[[x]]$.

Again, in less abstract terms this means that for a semiformal map $H\colon x \mapsto h(x,\lambda)$ the Bautin ideal $\mathfrak{B}(H)$ is the Bautin ideal of the *displacement function* $h(x,\lambda)-x$. As before, this definition in fact does not depend on the choice of the generator (i.e., the local coordinate).

Remark 13.11. More generally, let $d \geqslant 2$ be a natural number. Then one can introduce the dth *iterated Bautin ideal* of semiformal families of maps as the Bautin (filtered) ideal of the displacement of the dth *iterated* power $H^{[d]} = \underbrace{H \circ \cdots \circ H}_{d \text{ times}}$ of the formal map H, $\mathfrak{B}^{[d]}(H) = \mathfrak{B}(H^{[d]})$. This iterated ideal describes analytic perturbations of *periodic* formal maps.

The main (though still very simple) result of this section compares the Bautin ideals of a (semi)formal vector field F and that of its (semi)formal flow $\varPhi = \exp F$. Denote by $\varPhi^t = \exp tF$ the formal flow of F (the time t map).

Proposition 13.12. *The Bautin ideals $\mathfrak{B}(F)$ and $\mathfrak{B}(\varPhi^1)$ of a vector field F and its time 1 map respectively, coincide.*

More generally, if $\tau(x) = t_0 + t_1 x + t_2 x^2 + \cdots \in \mathfrak{A}[[x]]$ is a formal series with the coefficients $t_i = t_i(\lambda) \in \mathfrak{A}$, and the leading term t_0 is invertible in $\mathfrak{A}$, then the Bautin ideal $\mathfrak{B}(\varPhi_\tau)$ of the map $\varPhi_\tau\colon x \mapsto \varPhi_\tau(x) = \varPhi^{\tau(x)}(x)$, does not depend on the series $\tau(\cdot)$ and coincides with $\mathfrak{B}(F)$.

Proof. We first write the exponential sum with the independent formal variable t, according to which the flow $\varPhi^t$, $t \in \mathbb{C}$, is the automorphism H^t of $\mathfrak{A}[[x]]$ over $\mathfrak{A}$ given by the formal series

$$H^t = \operatorname{id} + tF + \frac{t^2}{2!}\,F^2 + \cdots + \frac{t^k}{k!}\,F^k + \cdots . \tag{13.8}$$

The "matrix" of the operator F in the basis $1, x, x^2, x^3, \ldots$ of $\mathbb{C}[[x]]$ is the bi-infinite matrix

$$M_F = \begin{pmatrix} 0 & & & & & \\ & a_1 & & & & \\ & a_2 & 2a_1 & & & \\ & a_3 & 2a_2 & 3a_1 & & \\ & a_4 & 2a_3 & 3a_2 & 4a_1 & \\ & \vdots & & & & \ddots \end{pmatrix}$$

The proposition can be proved by inspection of the structure of the powers F^k and hence of the entire sum (13.8). Looking at the first coefficient, we see that $F^k x = a_1^k\, x + O(x^2)$, so that $H^t x = x + \sum_{k\geqslant 1} \frac{t^k}{k!}\, a_1^k\, x + O(x^2)$, and therefore the first Bautin ideal $B_1(\varPhi^t)$ is the ideal

$$B_1(\varPhi^t) = \langle \exp(ta_1) - 1 \rangle = \langle ta_1(1 + \cdots) \rangle = \langle ta_1 \rangle .$$

that coincides with $\langle a_1\rangle = B_1(F)$ if instead of t an invertible series τ is substituted.

Assume by induction that the equalities $B_i(\Phi^t) = B_i(F) = \langle a_1,\dots,a_i\rangle$ are proved for all $i = 1,2,\dots,k-1$. To prove that $B_k(\Phi^t) = B_k(F)$, note that modulo the ideal $\langle a_1,\dots,a_{k-1}\rangle\,[[x]] \subseteq \mathfrak{A}[[x]]$, the derivation F coincides with the derivation $[a_k\,x^k + O(x^{k+1})]\frac{\partial}{\partial x}$. Substituting it into the exponential series, we obtain

$$H^t x = x + t\,a_k\,x^k + O(x^{k+1}) + \frac{t^2}{2!}O(x^{2k-1}) + \cdots \mod \langle a_1,\dots,a_{k-1}\rangle\,.$$

Thus if the free term t_0 of the series $t = \sum_{j\geqslant 0} t_j x^j$ is invertible,

$$B_k(\Phi^t) = \langle t_0 a_k\rangle \bmod B_{k-1}(\Phi^t) = \langle a_k\rangle \bmod \langle a_1,\dots,a_{k-1}\rangle = B_k(F).$$

By induction, the coincidence of the ideals is proved. □

Remark 13.13. We conclude this section by stressing again that all the properties listed above, reflect the simple combinatorics of coefficients behind formulas for multiplication, composition and differentiation of formal Taylor series in one variable. The parallel statements involving null loci rather than the ideals, are completely obvious. For instance, the statement parallel to Proposition 13.12, would mean that the flow map of a vector field on the line is identity if and only if the field itself vanishes identically.

13.4. Bautin ideal of a convergent series. It was already noted (see Proposition 13.6), that formal changes of variables leaving the origin fixed, preserve the Bautin ideals of various "one-dimensional" objects.

For *convergent* (analytic) families of functions the *translation* (shift) of the variable x also keeps the Bautin ideal.

Theorem 13.14. *Assume that the series $\sum_{k\geqslant 0} a_k(\lambda)\,x^k$ is convergent in some small neighborhood of the origin $(x,\lambda) \in (\mathbb{C}^1,0)\times(\mathbb{C}^n,0)$.*

Then for any analytic germ $t\colon (\mathbb{C}^n,0)\to(\mathbb{C},0)$ the non-filtered Bautin ideal of the shifted function $S_t(a)$, $S_t(a)(x,\lambda) = a(x+t(\lambda),\lambda)$ does not depend on the germ t,

$$B(S_t(a)) = \lim_k B_k(S_t(a)) = \lim_k B_k(a) = B(a).$$

In other words, the ideal

$$B(a;y) = \Bigl\langle a(y,\lambda), \frac{\partial a}{\partial x}(y,\lambda), \frac{\partial^2 a}{\partial x^2}(y,\lambda),\dots,\frac{\partial^k a}{\partial x^k}(y,\lambda),\dots\Bigr\rangle \subseteq \mathfrak{A} \qquad (13.9)$$

generated by the derivatives at a variable point $y \in (\mathbb{C}^1,0)$, is independent of this point as far as it remains in the domain of analyticity of a.

Remark 13.15 (important)**.** The Bautin chains (*filtrations*) induced on the limit ideal, are not preserved by the shift. In other words, the ideal $B(a; y)$ depends on the point y if considered as a filtered ideal.

The following Corollary restores the complete invariance of the Bautin ideals by *arbitrary* analytic changes of variables.

Corollary 13.16. *If* $H\colon (x, \lambda) \to (h(x, \lambda), \lambda)$ *is the germ of an analytic change of variables depending on parameters,* $H(0, 0) = (0, 0)$*, then the analytic families* a *and* $a \circ H$ *have the same Bautin ideal.*

Proof of the Corollary. An arbitrary family H can be represented as a composition of a translation (shift) $(x, \lambda) \mapsto (x+t(\lambda), \lambda)$, and a holomorphic transformation preserving the origin. The germ $t(\lambda)$ is holomorphic and $t(0) = 0$. $\square$

The proof of Theorem 13.14 is based on a rather nontrivial fact, the *closedness* of analytic ideals, which in turn is a consequence of the fact that division by an analytic ideal is a bounded operation.

Let $I \subseteq \mathcal{O}(C^n, 0)$ be an ideal generated by the germs of analytic functions $a_1(\lambda), \dots, a_n(\lambda)$. Denote by $D \simeq (\mathbb{C}^n, 0)$ a small polydisk D centered at the origin, on which all germs a_k extend as holomorphic functions. Recall that $\|f\|_D = \sup_{\lambda \in D} |f(\lambda)|$ denotes the norm on the space of holomorphic functions $\mathcal{O}(D)$.

Theorem 13.17 (Division theorem for germs [**Her63**])**.** *For any polydisk* $D' \Subset D$ *there exist a constant* K *depending, in general, on* D'*, such that any holomorphic function* $f \in \mathcal{O}(D')$ *whose germ at the origin belongs to* I*, admits expansion* $f = \sum_1^m h_i a_i$ *with* h_i *also holomorphic in* D' *and*

$$\|h_i\|_{D'} \leqslant K\|f\|_{D'}.$$

This theorem implies that ideals in the ring of germs are *closed*.

Corollary 13.18 (closedness of ideals)**.** *If a sequence of functions* $\{f_i\}_{i=1}^{\infty}$ *is defined in a common open neighborhood of the origin, converges uniformly on a smaller set, and their germs at the origin belong to an arbitrary ideal* $I \subseteq \mathcal{O}(\mathbb{C}^n, 0)$*, then the germ of the limit function also lies in this ideal.* $\square$

Remark 13.19. Formulation of Theorem 13.17 is somewhat technical because of the interplay between germs and representing them holomorphic functions: the ring of germs cannot be equipped by a single norm with respect to which the ideals are closed.

There exists a parallel assertion for polynomials that is free of this drawback. For a (multivariate) polynomial $p = \sum c_\alpha \lambda^\alpha \in \mathbb{C}[\lambda]$ denote by $|p|$ the sum of absolute values of all its coefficients, $|p| = \sum_\alpha |c_\alpha|$. The correspondence $p \mapsto |p|$ is a multiplicative norm on the algebra of the complex polynomials, $|p+q| \leqslant |p| + |q|$, $|pq| = |p|\,|q|$.

Consider an arbitrary *polynomial* ideal $I = \langle a_1, \dots, a_m \rangle \subset \mathbb{C}[\lambda_1, \dots, \lambda_n]$. By definition of the basis, any other polynomial $q \in I$ from this ideal can be expanded as $q = \sum_1^m h_i a_i$ with some polynomial coefficients $h_1, \dots, h_m \in \mathbb{C}[\lambda]$. This expansion is by no means unique, however, it is well-posed in the following precise sense.

Theorem 13.20 (Hironaka division theorem for polynomial ideals)**.** *For some (hence, for any) basis $a_1, \dots, a_m$ of an arbitrary polynomial ideal $I \subseteq \mathbb{C}[\lambda_1, \dots, \lambda_n]$ there exist two finite constants K_1, K_2, depending in general on the choice of the basis, such that any member $q \in I$ admits expansion $q = \sum_1^m h_i a_i$ with*

$$\deg h_i \leqslant \deg q + K_1, \qquad |h_i| \leqslant K_2^{\deg q}\,|q|.$$

This result can be proved by thorough inspection of the division algorithm involving Gröbner bases of ideals [**CLO97**]. In this form the result appears in [**Yom99**].

Proof of Theorem 13.14. Consider a series $\sum a_k(\lambda)\,x^k$ converging to a function $a(x,\lambda)$ holomorphic in some polydisk $U \times D \subseteq (\mathbb{C}^{n+1}, 0)$. Consider first the case when $t \in \mathbb{C}$ is an independent variable parameter. The coefficients $a_{k,t} \in \mathfrak{A}$ of the expansion of $S_t(a)(x,\lambda) = a(t+x,\lambda)$ with the center t i.e., the derivatives of $a(\cdot,\lambda)$ at the point t, coincide (modulo the factorial coefficients) with the derivatives of the shifted function at t. In particular,

$$a_{0,t}(\lambda) = a(t,\lambda) = \sum_0^\infty a_k(\lambda)\,t^k.$$

This series converges if $|t|$ is sufficiently small and its kth partial sums belongs to $B_k(a) \subseteq B(a)$. By Corollary 13.18, the limit belongs to $B(a)$. Differentiating this converging series termwise in t proves that the kth partial sum for $k!\,a_{j,t}(\lambda) = \partial^k a(t,\lambda)/\partial t^k$ belongs to $B_{k-j}(a) \subseteq B(a)$ for all $j = 1, 2, \dots$. Thus the ideal generated by $a_{j,t}$ belongs to $B(a)$,

$$B(S_t(a)) = \langle a_{0,t}, a_{1,t}, \dots, a_{j,t}, \dots, \rangle \subseteq B(a).$$

The inclusion remains valid after substitution of a holomorphic germ $t = t(\lambda)$ instead of the formal parameter t. The arguments being symmetric (reversible), we conclude that the two ideals in fact coincide. □

Another very important corollary of the closedness of the ideals is the possibility of grouping their terms. Consider a convergent series $a(x,\lambda) = \sum a_k(\lambda)x^k$ and its filtered Bautin ideal $\mathfrak{B}(a)$ in the ring $\mathfrak{A} = \mathcal{O}(\mathbb{C}^n, 0)$.

Lemma 13.21. *If the Bautin depth of the Bautin ideal* $\mathfrak{B}(a)$ *is equal to* μ, *then the germ* a *can be represented as the finite sum*

$$a(x,\lambda) = \sum_{j=0}^{\mu} a_{k_j}(\lambda)\, x^{k_j} h_j(x,\lambda),$$
$$0 \leqslant k_0 < k_1 < \cdots < k_\mu, \qquad h_j(0,0) = 1, \qquad j = 0,1,\dots,\mu. \tag{13.10}$$

Here k_j *are the instances where the strict inclusions in the chain* (13.3) *occur,* $I_{k_j-1} \neq I_{k_j}$.

Proof. If the series a converges, then $\|a_k\|_U \leqslant Cr^{-k}$ for some positive constants $0 < r, C < +\infty$.

By definition of the Bautin depth, the coefficients $a_{k_0}, a_{k_1}, \dots, a_{k_\mu}$ generate the limit Bautin ideal $B(a)$. Therefore all other coefficients can be expressed as combinations

$$a_k = \sum_{j\colon k_j \leqslant k} h_{kj} a_j, \qquad h_{kj} \in \mathfrak{A}, \quad k = 0,1,\dots. \tag{13.11}$$

By Theorem 13.17, the representation can be chosen so that $\|h_{kj}\|_U \leqslant C'\, r^{-k}$ with another constant C'. But this means that the series

$$h'_j(x,\lambda) = \sum_{k\geqslant 0} h_{kj}(\lambda)\, x^k = x^{k_j} h_j(x,\lambda),$$
$$h_{kj}(x,\lambda) = 1, \qquad j = 0,1,\dots,\mu,$$

is convergent and begins with the term x^{k_j}. Multiplying the identities (13.11) by x^k and rearranging the terms of the converging series, we obtain the required representation. □

13.5. Bautin index and cyclicity. Let $f = f(x,\lambda) \in \mathcal{O}(\mathbb{C}^{n+1},0)$ be a holomorphic (or real analytic) germ represented by a function holomorphic in a small polydisk $D \times U$. This function can be considered as an analytic *local family* of functions in $\mathfrak{A} \otimes \mathcal{O}(D)$, $\mathfrak{A} = \mathcal{O}(U)$.

Definition 13.22. The *complex cyclicity* (sometimes referred to as *local valency*) of the complex analytic local family of functions $f(x,\lambda)$ is the smallest integer number $\mu \in \mathbb{N}$ such that the number of isolated zeros of the function $f(\cdot,\lambda)$ in a sufficiently small polydisk $\{|x| < \delta, |\lambda| < \varepsilon\}$ does not exceed μ,

$$\exists \varepsilon > 0, \delta > 0 \quad \forall |\lambda| < \varepsilon, \quad \#\{x\colon |x| < \delta,\ f(x,\lambda) = 0\} \leqslant \mu. \tag{13.12}$$

Here and below by $\#M$ we will denote the number of *isolated* points in a real or complex analytic set $M \subseteq U$.

Remark 13.23 (terminological)**.** The term *cyclicity* is related to bifurcations of limit cycles, as explained in §13.1. Assume that L is a limit cycle of a planar real analytic vector field analytically depending on parameters $\lambda_1, \dots, \lambda_n$ varying near the origin in $\mathbb{R}^n$. Let $f(x, \lambda)$ be the displacement function for the first return (real holonomy) map associated with any choice of the cross-section to L. Then cyclicity of the germ f is equal to the maximal number of limit cycles that can be observed in a small annulus around L for any sufficiently small values of the parameters.

Somewhat ironically, for the elliptic families (the first and the best studied bifurcation problem, see §13.6 below) cyclicity μ of the displacement function for the first return map around the *singular point* is always odd and the number of limit cycles is $(\mu - 1)/2$, see Remark 13.1.

Definition 13.24. If f is a *real analytic* local family of functions, then its *real cyclicity* is defined as the maximal number of *positive* isolated roots of $f(\cdot, \lambda)$ in $(\mathbb{R}^1_+, 0)$ uniform over all small values of the parameters $\lambda \in (\mathbb{R}^n, 0)$.

The formal definition with quantifiers coincides with (13.12) except that instead of the disk $\{|x| < \delta\}$ one has to take the real interval $\{0 < x < \delta\}$.

By definition, cyclicity is defined for a family, i.e., for a deformation, though if $f_0 = f(\cdot, 0)$ is not identically zero, it can be majorized uniformly over all analytic families containing f_0, as shown in §13.1.

Theorem 13.25.

1. *If f is a real analytic germ and the associated Bautin ideal $\mathfrak{B}(f) \subseteq \mathcal{O}(\mathbb{R}^n, 0)$ has the depth μ, then the real cyclicity of the family on the real semiaxis is $\leqslant \mu$.*

2. *If $f(x, \lambda) = \sum_0^\infty a_k(\lambda)\, x^k$ is an holomorphic germ and the associated Bautin ideal $\mathfrak{B}(f) \subseteq \mathcal{O}(\mathbb{C}^n, 0)$ has index ν, then the complex cyclicity of the family is $\leqslant \nu$.*

Proof. The real assertion is proved by the classical derivation-division process which is one of ingredients of the much broader *fewnomials theory* [**Kho91**]. The complex counterpart is treated using the Cartan inequality and the perturbation technique following [**Yak00**].

1. By Lemma 13.21, the germ f can be represented as the finite sum $f(x, \lambda) = \sum_0^\mu a_j(\mu)\, x^{k_j} h_j(x, \lambda)$, see (13.10), with $k_0 < k_1 < \cdots < k_\mu$.

The neighborhood $U = (\mathbb{R}^n, 0)$ of the origin in the parameter space can be represented as the union of the domains where the jth coefficient a_j is not too small compared to the other coefficients a_i, $i \neq j$,

$$U = Z \cup U_0 \cup \cdots \cup U_\mu, \qquad Z = \{\lambda \colon a_0 = \cdots = a_\mu = 0\},$$

$$U_j = \{\lambda \colon 2(\mu+1)\,|a_j| > \sum_{i \neq j} |a_i|\}, \qquad j = 0, \dots, \mu.$$

For $\lambda \in Z$ there is nothing to prove since $f(x,\lambda) \equiv 0$ there. It remains to show that $f(x,\lambda)$ has no more than μ zeros in some interval $(0,\varepsilon)$ uniformly over λ restricted to each U_j.

Consider the following *derivation-division process*. The sum involving $\mu+1$ terms $f(x,\lambda) = f_0(x,\lambda) = \sum_{j \geqslant 0} a_j(\lambda)\, x^{k_j} h_j(x,\lambda)$ is divided by the function $x^{k_0} h_0(x,\lambda)$ and then the derivative in x is taken. This division leaves the sum real analytic since the exponents k_j increase and $h_0(0,0) \neq 0$. As a result, the first term disappears completely and the remainder $f_1(x,\lambda)$ has the same structure, $f_1(x,\lambda) = \sum_{j \geqslant 1} a_j(\lambda) x^{k_j - k_0} h_{j1}(x,\lambda)$, but with different exponents $k_j - k_0 > 0$ and some analytic invertible coefficients, $h_{j1}(0,0) \neq 0$.

After j such “division+derivation” steps we arrive at the function

$$f_j(x,\lambda) = a_j(\lambda)\, x^{k_j - k_{j-1}} + \sum_{i > j} a_i(\lambda)\, x^{k_i - k_{j-1}} h_{ij}(x,\lambda)$$

This function is nonvanishing for all values of $\lambda \in U_j$ on a sufficiently small real interval $(0,\varepsilon)$. Indeed, the exponents $k_i - k_{j-1}$ are all bigger than $k_j - k_{j-1}$ because of the same monotonicity, and the ratios $|a_i(\lambda)|/|a_j(\lambda)|$ do not exceed $\frac{1}{2(\mu+1)}$ by construction of U_j. Thus the first term in f_j dominates on a sufficiently small interval $(0,\varepsilon)$ the rest of the sum, therefore f_j has the same sign as $a_j(\lambda) \neq 0$ in U_j.

It remains to notice that each step “division+derivation” may decrease the number of isolated zeros on $(0,\varepsilon)$ at most by 1:

$$\#\{x \in (0,\varepsilon) \colon f_j(x,\lambda) = 0\} \geqslant \#\{x \in (0,\varepsilon) \colon f_{j-1}(x,\lambda) = 0\}$$

for any $j = 1, 2, \dots, \mu$. Indeed, multiplication by any power of x does not affect the number of roots on any positive interval, while derivation can decrease the number of roots by 1 at worst. This follows from the Rolle lemma, since (i) between any two *distinct* roots of f there must be at least one root of the derivative, and (ii) the *multiplicity* of a multiple root decreases after derivation exactly by 1.

Since $f_j(x,\lambda)$ is nonvanishing on $(0,\varepsilon)$ for $\lambda \in U_j$, the function $f = f_0$ has no more than j isolated zeros there. On the union $\bigcup_0^\mu U_j$ the function f has no more than μ real roots. The statement on real zeros is proved.

2. To prove the assertion on complex zeros, we use the same representation $f(x,\lambda) = \sum_0^\mu a_j(\lambda) x^{k_j}\, h_j(x,\lambda)$, see (13.10), which should be further prepared as follows. Let $D = \{|x| < \varepsilon\} \subset \mathbb{C}$ be a small disk on which the functions h_j are explicitly bounded, say, by 2 uniformly over λ. Restricting the parameters on the domain U_j and dividing the function f by a_j there, we obtain

$$a_j^{-1}(\lambda)\, f(x,\lambda) = p_j(x,\lambda) + x^{k_j+1} q_j(x,\lambda), \tag{13.13}$$

where p_j are monic polynomials of degree k_j, while the remainders q_j are explicitly bounded,

$$\begin{aligned} p_j(x,\lambda) &= x^{k_j} + \sum_{k<k_j} b_{kj}(\lambda)\, x^k, \qquad b_j \in \mathcal{O}(U_j), \\ |q_j(x,\lambda)| &\leqslant C = 4(\mu+1), \qquad (x,\lambda) \in D \times U_j. \end{aligned} \tag{13.14}$$

The rest of the proof goes independently for each domain U_j. We show that a function (13.13) constrained by the inequality (13.14) may have at most k_j complex zeros in a disk of radius

$$r_0 = \frac{1}{2}\big((8e)^{k_j}(C+1)\big)^{-1} \geqslant \frac{1}{2}\big((8e)^{\nu}(C+1)\big)^{-1}. \tag{13.15}$$

This will prove the theorem since $k_0 < \cdots < k_\mu = \nu$. To simplify the notation, we omit explicit dependence on λ.

Let r be a positive number between 0 and ε to be chosen later. As the polynomial p_j is monic, by Cartan inequality [**Lev80**] there exists a finite number of *exceptional disks* with the sum of their diameters less than r such that *outside* their union p_j admits the *lower* bound $|p_j(x)| \geqslant (r/4e)^{k_j}$, where $e \approx 2.71828\ldots$ is the Euler number.

Consider the annulus $\{r \leqslant |x| \leqslant 2r\}$ foliated by concentric circumferences $\{|x| = \rho\}$, $r \leqslant \rho \leqslant 2r$. As the sum of diameters of the exceptional disks is less than r, at least one such circumference is disjoint with their union and hence p_j is bounded from below on it by $(r/4e)^{k_j}$.

On the other hand, on any such circumference the term $x^{k_j+1}q_j(x)$ admits an explicit upper bound using (13.14):

$$|x^{k_j+1} q_j(x)|_{|x|=\rho} \leqslant C\frac{\rho^{k_j+1}}{1-\rho} \leqslant C\frac{(2r)^{k_j+1}}{1-2r}.$$

The *domination inequality* $(r/4e)^{k_j} > (2r)^{k_j+1}C/(1-2r)$ ensures that the Rouché theorem applies to the circumference $\{|x| = \rho\}$ and guarantees that the number of roots of p_j and $a_j^{-1}f$ (the former being at most k_j) in the disk $\{|x| \leqslant r\}$ coincide. Resolving the domination inequality with respect to r gives $r < \frac{1}{2}\big((8e)^{k_j}(C+1)\big)^{-1}$. □

Remark 13.26. The proof of Theorem 13.25 is constructive in the sense that, knowing the parameters K characterizing the ideal in Theorem 13.17, one can produce explicitly the lower bound for the size of the interval or disk containing no more than the asserted number of roots (in the complex case this was done explicitly).

The simple bound of this type asserted by Theorem 13.25, is not the best known one. In [**RY97**] N. Roytwarf and Y. Yomdin considered the general problem of uniform localization of zeros of an analytic family of functions with the specified Bautin ideal and explicit constraints on the growth of Taylor coefficients, the so called *Bernstein classes*. Using a dual description of the Bernstein classes in terms of the growth rate of the functions represented by the series, they obtain a lower bound for the radius of the disk in which at most ν zeros can occur. This bound was achieved in the form $r_0 = \bigl(8^\nu \max(C,2)\bigr)^{-1}$ (in the equivalent settings). These results are generalized in [**FY97**] for A_0-series with polynomial coefficients in $\mathfrak{A} = \mathbb{C}[\lambda_1, \dots, \lambda_n]$ of degree growing at most linearly and the norms at most exponentially.

Yet somewhat surprisingly, the best result can be obtained by properly "complexifying" the derivation-division process, based on the complex analog of the Rolle lemma [**KY96**]. On this way one can prove that the number of small complex isolated roots in the family (13.13)–(13.14) does not exceed ν in the disk of radius $r_s = \frac{1}{2}(1-s^{-1})\bigl(s^{\nu+1}C+1\bigr)^{-1}$ for any value of $s > 1$. All details can be found in [**Yak00**].

The assertion of Theorem 13.25 can be improved in another direction. An *integral closure* of an ideal $I \subset \mathfrak{A}$ is the collection of all roots $y \in \mathfrak{A}$ of all equations of the form $y^n + q_1 y^{n-1} + \dots + q_{n-1}y + q_n = 0$ with the coefficients q_k belonging to the kth powers of I, $q_k \in I^k$. If $B = \langle a_0, a_1, \dots, a_n, \dots\rangle$ is the filtered Bautin ideal, its *reduced Bautin index* is defined in [**HRT99**] as the minimal number $r \in \mathbb{N}$ such that the integral closure of $\langle a_0, \dots, a_r\rangle$ coincides with B. Obviously, the reduced Bautin index does not exceed its (usual) Bautin index. In [**HRT99**] an analog (also constructive) of the second assertion of Theorem 13.25 is proved for the reduced Bautin index rather than ν.

Theorem 13.25 is a general tool linking cyclicity of roots in analytic families of functions of one variable (real or complex) with the depth (or Bautin index) of the corresponding Bautin chain of ideals generated by the coefficients. In the next sections it will be applied to bifurcations of limit cycles in analytic vector fields on the plane.

13.6. Elliptic vector fields on the plane: Bautin and Dulac ideals. Consider a real analytic family of vector fields on the plane,

$$\begin{gathered} F = \mathbf{A} + \text{nonlinear terms}, \qquad \mathbf{A} = \alpha(\lambda)\mathbf{E} + \beta(\lambda)\mathbf{I}, \\ \mathbf{E} = x\tfrac{\partial}{\partial x} + y\tfrac{\partial}{\partial y}, \qquad \mathbf{I} = y\tfrac{\partial}{\partial x} - x\tfrac{\partial}{\partial y}, \\ \mathfrak{A} = \mathcal{O}(\mathbb{R}^n, 0), \quad \alpha, \beta \in \mathfrak{A}, \quad F \in \mathfrak{A} \otimes \mathcal{D}(\mathbb{R}^2, 0), \end{gathered} \tag{13.16}$$

with the linear part $\mathbf{A}$ normalized as in (13.1) and elliptic, i.e., $\beta(\lambda) \neq 0$. By Theorem 4.17, there exists a semiformal transformation bringing the family (13.16) to the rotationally invariant normal form (4.9), which after division by the nonvanishing formal series can be further transformed to

$$F' = f(r^2)\mathbf{E} + \mathbf{I}, \qquad f(u) = \sum_1^\infty f_k(\lambda)\, u^k \in \mathfrak{A}[[u]]. \tag{13.17}$$

There are *two* univariate series with coefficients analytically depending on parameters, naturally related to the family F. One series is the first return map $P \in \mathfrak{A} \otimes \operatorname{Diff}(\mathbb{R}^1, 0)$ that is always convergent. The other series is the coefficient $f(u) \in \mathfrak{A}[[u]]$ occurring the orbital formal normal form (13.17), which apriori can diverge and is not uniquely defined. Each series generates a growing chain of ideals in $\mathfrak{A}$, the Bautin ideal $\mathfrak{B}(F)$ and another filtered ideal, the *Dulac ideal* $\mathfrak{D}(F)$ which will be later introduced in invariant terms (Definition 13.30).

Vanishing of all coefficients of the return map P means that the field F exhibits a center for the corresponding values of the parameters. Vanishing of all coefficients of the normal form (13.17) means that the field is formally orbitally linearizable and hence admits a formal first integral. By Proposition 12.5, the two properties are equivalent for elliptic vector fields, which means that the respective zero loci of the two (unfiltered) ideals $\mathfrak{B}(F)$ and $\mathfrak{D}(F)$ coincide.

This observation suggests a conjecture that the ideals generated by coefficients of these two series, should also coincide. This can be considered as a parametric generalization of Proposition 12.5 and the Poincaré–Lyapunov Theorem 12.6.

This conjecture turns out, broadly speaking, true. However, in order to make its formulation precise, one has to overcome several technical obstacles arising since the normal form can be divergent. Besides, we will give an alternative construction for the Dulac ideal that will be invariant by formal transformations.

13.6.1. *Formal first return map for semiformal families.* Consider instead of a real analytic family (13.16), the *semiformal* family of real elliptic vector fields on the plane

$$
\begin{gathered}
F = \alpha(\lambda)\mathbf{E} + \beta(\lambda)\mathbf{I} + \text{nonlinear terms}, \\
\mathfrak{A} = \mathcal{O}(\mathbb{R}^n, 0), \qquad \alpha, \beta \in \mathfrak{A}, \quad \beta \neq 0, \quad F \in \mathfrak{A} \otimes \mathcal{D}[[\mathbb{R}^2, 0]].
\end{gathered}
\tag{13.18}
$$

We need to *define* the first return map $P(x, \lambda)$ for such family on the level of semiformal real maps. Consider the formal flow $\varPhi^t$ as a formal map from $\operatorname{Diff}[[\mathbb{R}^2, 0]]$ with coefficients being entire functions of t, defined as in §3.3. Assume that there exists the semiformal series $t(x, \lambda) \in \mathfrak{A}[[x]]$ which together with the P satisfies the formal identity

$$
\varPhi^{t(x,\lambda)}(x, 0) = (P(x, \lambda), 0), \qquad t(0, \lambda) = \frac{2\pi}{\beta(\lambda)}. \tag{13.19}
$$

Solvability of this equation can be established by inspection of the coefficients. In the convergent case the series $t(x, \lambda)$ is the *return time* between two subsequent intersections of the x-axis and $P(x, \lambda)$ is the true first return map. In the formal case the series P may be used as the *definition* of the formal return map.

An alternative (easier) way to define P is to consider an arbitrary C^∞-*smooth* family $\widetilde{F}$ of vector fields extending the formal family F (e.g., extending its coefficients). Since the family is elliptic, after passing to the polar coordinates (r, φ) on the plane $(\mathbb{R}^2, 0)$, the associated differential equation on the cylinder $\mathbb{S}^1 \times (\mathbb{R}^1, 0)$ has the form

$$
\dot{\varphi} = \beta + O(r), \qquad \dot{r} = r(1 + O(r)),
$$

in particular, this field has no singular points on the equator. Moreover, any solution starting on the cross-section $\tau_+ = \{\varphi = 0\}$ at the point $(x, 0)$, $x > 0$, again intersects τ_+ after some time $\widetilde{t}(x, \lambda) = 2\pi/\beta(\lambda) + O(x)$ at the point $(\widetilde{P}(x, \lambda), 0)$, where $\widetilde{P}(\cdot, \lambda)$ is the first return map of the extended smooth field and $\widetilde{t}(x, \lambda)$ the return time, both C^∞-smooth since the flow is transversal to τ_+. By construction, the pair $\widetilde{P}(x, \lambda), \widetilde{t}(x, \lambda)$ and the flow $\widetilde{\varPhi}^t$ of the smooth field $\widetilde{F}$ satisfy the equation $\widetilde{\varPhi}^{\widetilde{t}(x,\lambda)}(x, 0) = (\widetilde{P}(x, \lambda), 0)$ having the same meaning as (13.19). Taking their Taylor (semiformal) series, we obtain a semiformal map $P \in \mathfrak{A} \otimes \operatorname{Diff}[[\mathbb{R}^1, 0]]$ which automatically satisfies (13.19) and can be used as the definition of the first return map for the semiformal field F.

Definition 13.27. The Bautin ideal $\mathfrak{B}(F)$ (with the corresponding filtration) of a semiformal elliptic family of vector fields $F \in \mathfrak{A} \otimes \mathcal{D}[[\mathbb{R}^2, 0]]$, is the Bautin ideal $\mathfrak{B}(P)$ of its semiformal first return map P as defined by (13.19).

13.6.2. *Quotient equation.* In this section we give invariant definition of the second ideal associated with a semiformal elliptic family (13.18).

Definition 13.28. The *quotient equation* for a semiformal elliptic family F of real planar vector fields is the equation $Fu = g(u)$, or, with more details,

$$\begin{aligned} &Fu(x,y) = g(u(x,y)), \qquad u \in \mathfrak{A}[[x,y]], \quad g \in \mathfrak{A}[[z]] \\ &u(x,y) = (x^2+y^2) + \cdots, \qquad g(z) = b_1(\lambda)z + \cdots. \end{aligned} \tag{13.20}$$

The quotient equation contains the result of the Lie derivation Fu in the left hand side and has to be solved with respect to the unknown semiformal families of functions u, g. The semiformal series $u = u_2 + u_3 + \cdots$ must have the fixed (independent of the parameters) 2-jet $u_2 = x^2 + y^2$ and the semiformal series $g = g_1 z + g_2 z^2 + \cdots$ in one variable should be without the free term. In [**Arn69**] the quotient equation is introduced under the name *cocycle*, but this term is too overburdened and will be never used in such sense.

Solution of the quotient equation, if it exists, is a semiformal map $u\colon (\mathbb{R}^2,0) \to (\mathbb{R}^1,0)$ "projecting" the elliptic family F onto the one-dimensional semiformal vector field

$$G = g(z,\lambda)\tfrac{\partial}{\partial z}, \qquad G \in \mathfrak{A} \otimes \mathcal{D}[[\mathbb{R}^1,0]]. \tag{13.21}$$

This means that for any real value of t the formal flows $\Phi_F^t \in \mathfrak{A} \otimes \operatorname{Diff}[[\mathbb{R}^2,0]]$ and $\Phi_G^t \in \mathfrak{A} \otimes \operatorname{Diff}[[\mathbb{R}^1,0]]$ of the fields F and G respectively, are linked by u as a formal map: on the level of semiformal series,

$$u \circ \Phi_F^t = \Phi_G^t \circ u, \qquad \forall t \in \mathbb{R}. \tag{13.22}$$

Existence of such map (i.e., solvability of the quotient equation) is closely related to the *integrability* of the (semi)formal normal form of an elliptic vector field on the plane. Note that most certainly there is no uniqueness of solution of the quotient equation (13.20): if (u,g) is one such solution and $z \mapsto w(z,\lambda) = z + c_2(\lambda)z^2 + \cdots$ is any semiformal family of maps of one variable into itself with the identical linear part, then clearly the composition $w \circ u$ will be the first component of another solution (the second component is obtained by changing the variable $z \mapsto w(z)$ in the formal vector field $G = g\frac{\partial}{\partial z}$).

Recall that the ring of coefficients $\mathfrak{A} = \mathcal{O}(\mathbb{R}^n,0)$ is the ring of germs of real analytic functions.

Lemma 13.29. *For any semiformal elliptic family $F \in \mathfrak{A} \otimes \mathcal{D}[[\mathbb{R}^2,0]]$ the quotient equation is always solvable in the class of real semiformal power series.*

Proof. Solvability of the equation (13.20) is invariant by a semiformal conjugacy of vector fields. If $H \in \mathfrak{A} \otimes \operatorname{Diff}[[\mathbb{R}^2, 0]]$ is a semiformal invertible transformation (formal change of the variables x, y analytically depending on parameters), then the quotient equations for two semiformal families F and F' conjugated by H, are both solvable or not solvable simultaneously: the respective solutions (u, g) and (u', g') have the common series $g' = g$ and the conjugate series $u' = u \circ H$.

Thus one can assume without loss of generality that the elliptic semiformal family already is in the normal form given by the first assertion of Theorem 4.17:

$$F' = a(u)\mathbf{E} + b(u)\mathbf{I}, \qquad a, b \in \mathfrak{A}[[z]], \quad u = u(x,y) = x^2 + y^2, \qquad (13.23)$$

cf. with (4.9). For the field F' in the normal for (13.23) the functions $u(x, y) = x^2 + y^2$ and $g(z) = 2z\,a(z)$ give a solution of (13.20), since

$$\mathbf{E}u = 2u \quad \text{(Euler identity)}, \qquad \mathbf{I}u = 0 \quad \text{(symmetry)}.$$

Because of the invariance by conjugacy, any quotient system for an elliptic semiformal family is solvable. □

Besides referring to the normal form, solvability of the quotient equation can be established by direct arguments that give at the same time the practical algorithm for its solution, see §13.7.1.

13.6.3. *Dulac ideal.*

Definition 13.30. The (filtered) *Dulac ideal* $\mathfrak{D}_F \subseteq \mathfrak{A}$ of the semiformal elliptic family F is the (filtered) Bautin ideal $\mathfrak{B}_G$ of the semiformal family of vector fields $G = g(z, \lambda)\frac{\partial}{\partial z}$ on the real line $(\mathbb{R}^1, 0)$, where (u, g) is any solution of the quotient equation (13.20).

As follows from the computation proving Lemma 13.29, the Dulac ideal is generated by the coefficients of the formal normal form of the elliptic family. The conjectured relationship between the two semiformal series, discussed in the beginning of §13.6, can be now formulated as follows.

Theorem 13.31. *The Dulac ideal $\mathfrak{D}(F) = \{D_k\}$ and the Bautin ideal $\mathfrak{B}(F) = \{B_k\}$ of any semiformal elliptic family F are related as follows,*

$$D_1 = D_2 = B_1,\ D_3 = D_4 = B_2,\ \dots\ D_{2k-1} = D_{2k} = B_k,\ \dots \qquad (13.24)$$

In particular,

$$D(F) = \lim D_k(F) = \lim B_k(F) = B(F).$$

In other words, the Dulac and Bautin ideals coincide as unfiltered ideals in $\mathfrak{A}$, and have the same depth as filtered ideals.

Remark 13.32. As the Bautin ideal (of the return map) is defined without any reference to the quotient equation, this Theorem implies, among other things, that the Dulac ideal is independent of the particular solution (u, g) of the quotient equation.

The proof of this Theorem is based on representation of the first return map P of an elliptic family in terms of the flow of another semiformal field, using the identity (13.22). Let

$$G'(w,\lambda) = \frac{1}{2w} g(w^2,\lambda)\frac{\partial}{\partial w}, \qquad G' \in \mathfrak{A} \otimes \mathcal{D}[[\mathbb{R}^1, 0]], \tag{13.25}$$

be the semiformal vector field obtained by substitution $u = w^2$ from the field G defined in (13.21). Recall that $g(0, \lambda) = 0$ so that the coefficient of G' is again a semiformal series without the free term, cf. with Example 13.9.

Proposition 13.33. *There exists a semiformal series* $t'(w, \lambda) =\in \mathfrak{A}[[w]]$ *with an invertible free term* $2\pi/\beta(\lambda)$ *such that the semiformal first return map* P *is* (*semi-*)*formally conjugate to the flow map*

$$P'(w) = \Phi_{G'}^{t'(w,\lambda)}(w), \tag{13.26}$$

where $\Phi_{G'}$ *is the flow of the formal field* (13.25).

Proof of the Proposition. This assertion is almost obvious. Indeed, let $t(x, \lambda) \in \mathfrak{A}[[x]]$ be the first return time for the cross-section $\tau_+ = \{y = 0,\ x > 0\}$. Then the identities (13.19), (13.22) together mean that the formal map $\Phi_F^{t(x,\lambda)}$ maps τ_+ into itself, its restriction on τ_+ coincides with the first return map P and $u \circ P = \Phi_G^{t(x,\lambda)}(u)$. In other words, if u were a formal coordinate on τ_+ (a generator of the ring $\mathfrak{A}[[\tau_+, 0]]$), then P would be the flow map as required.

However, u is not a formal coordinate function on τ_+, so $t(x, \lambda)$ in general is *not* a formal series in powers of u (restricted on τ). Yet the *square root* of u, the series $w = \sqrt{u(x, 0)} \in \mathfrak{A}[[x]]$, already is an invertible formal series, so $t(x, \lambda)$ can be re-expanded as $t'(w, \lambda)$, the free term remaining the same. The vector field G' is obtained from G by passing to the square root of the phase variable, and in the new formal coordinate w the first return map $P(w)$ coincides with the flow map $\Phi_{G'}^{t'(w,\lambda)}(w)$. □

Proof of Theorem 13.31. By Proposition 13.33, the formal first return map of the semiformal elliptic family F is represented via the flow map of the vector field G' obtained from the quotient vector field G by the substitution $u = w^2$. The Bautin ideal of the map $\Phi_{G'}^{t'(\cdot,\lambda)}$ is equal to the Bautin ideal of the semiformal family G' by Proposition 13.12. The relationship between the Bautin ideals of the families G and G' was established in Example 13.9. □

As a corollary, we obtain a description of cyclicity of elliptic singular points.

Theorem 13.34. *Cyclicity of the singular point in a real analytic family of elliptic planar vector fields is equal to the depth of the Dulac ideal* $\mathfrak{D}$ *constructed using any formal solution of the quotient equation* (13.20).

Proof. The depth of the Dulac and Bautin ideals for a given elliptic family coincides by Theorem 13.31. Cyclicity of the singular point (the maximal number of small limit cycles occurring near this point) is equal to the maximal number of fixed points of the monodromy map for the positive semiaxis $\tau_+ = \{y = 0,\ x > 0\}$, i.e., cyclicity of the corresponding displacement. In turn, the latter cyclicity is equal to the depth of the Bautin ideal by the first assertion of Theorem 13.25. ☐

13.7. Universal polynomial families, cyclicity and localized Hilbert problem. Consider the *universal family* of elliptic polynomial vector fields of a given degree d,

$$F = \alpha\mathbf{E} + \beta\mathbf{I} + \sum_{2\leqslant i+j\leqslant d} \lambda'_{ij}x^iy^j\,\tfrac{\partial}{\partial x} + \lambda''_{ij}x^iy^j\,\tfrac{\partial}{\partial y}. \tag{13.27}$$

parameterized by the real parameters

$$\alpha \in \mathbb{R}^1, \quad \beta \in \mathbb{R}^1 \smallsetminus \{0\}, \quad \lambda = \{\lambda'_{ij}, \lambda''_{ij}\} \in \mathbb{R}^n, \qquad n = n(d).$$

Cyclicity of the origin in the family (13.27) is closely related to the Hilbert 16th problem about the number and location of limit cycles of a polynomial vector field of degree d, see §**??**. Knowing this cyclicity would answer the question about the maximal number of *small* limit cycles near the origin, at least for vector fields close to linear centers. As follows from Theorem 13.34, this cyclicity is equal to the depth of the Dulac (or Bautin) filtered ideal. The Dulac and Bautin ideals for the universal family (13.27) apriori belong to the ring $\mathcal{O}(\mathbb{R}^{n+2}, 0)$ of real analytic germs of functions of $n + 2$ variables α, β, λ.

However, for reasons similar to those guaranteeing algebraic decidability of the center–focus alternative, one can reduce the question about cyclicity of the family (13.27) to the depth of some *polynomial* filtered ideal, the Dulac ideal of an auxiliary family with fixed linear part $\mathbf{I} = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}$ (pure rotation),

$$F' = \mathbf{I} + \sum_{2\leqslant i+j\leqslant d} \lambda'_{ij}x^iy^j\,\tfrac{\partial}{\partial x} + \lambda''_{ij}x^iy^j\,\tfrac{\partial}{\partial y}. \tag{13.28}$$

Denote by $\mathfrak{D} = \{D_k\}$ and $\mathfrak{D}' = \{D'_k\}$ the Dulac chain of ideals for the corresponding families (13.27) and (13.28),

$$\mathfrak{D} = \{D_k\},\ D_k \subseteq \mathcal{O}(\mathbb{R}^{n+2}, 0), \qquad \mathfrak{D}' = \{D'_k\},\ D'_k \subseteq \mathcal{O}(\mathbb{R}^n, 0).$$

Denote the depths of these chains by μ and μ' respectively.

Proposition 13.35. *The auxiliary chain of ideals $\mathfrak{D}'$ is generated by polynomials in λ and $D'_1 = 0$.*

The depths of the two chains differ by 1,

$$\mu = \mu' + 1.$$

Proof. We prove that generators of the chain $\mathfrak{D}$ may be chosen so that $D_k = \langle \alpha, a_2, \dots, a_k \rangle = \langle \alpha, D'_k \rangle$, where $a_k = a_k(\lambda)$ are polynomials in the variables $\lambda \in \mathbb{R}^n$ only, generating the Dulac chain for the reduced equation (13.28).

Indeed, resolving the quotient equation for this equation,

$$\alpha\,\mathbf{E}u + \beta\,\mathbf{I}u + \cdots = g_1(\lambda)\,u + \cdots, \qquad u = x^2 + y^2 + \cdots,$$

we immediately obtain that $g_1(\lambda) = 2\alpha$ so that $D_1 = \langle \alpha \rangle$, $D'_1 = \{0\}$.

Computation of the higher ideals in the Dulac chain can be done modulo the ideal $\langle \alpha \rangle$, i.e., their generators are sufficient to compute only on the zero locus $\{\alpha = 0\}$, a hyperplane in the space of the parameters $\{\alpha, \beta, \lambda\}$.

Moreover, since multiplication of F by a nonzero constant does not change the Dulac ideal (all its generators will be multiplied by this constant), without loss of generality one can compute the ideal of the *reduced* elliptic family (13.28) with the fixed (independent of the parameters) linear part $\mathbf{I}$.

It was already observed in the proof of Theorem 11.35, that the coefficients of the first return map polynomially depend on the nonlinear coefficients of the algebraic vector field. These polynomials generate the Bautin chain $\mathfrak{B}'$ for the family F' which coincides with the Dulac chain $\mathfrak{D}'$ modulo "shearing transformation" (13.24) as described in Theorem 13.31. □

Both the depth and the Bautin index of a chain of ideals generated by polynomials, do not depend on whether the ideals are considered in the ring $\mathbb{R}[\lambda]$ or in the larger ring $\mathcal{O}(\mathbb{R}^n, 0)$. Thus the transcendental problem on the number of small limit cycles that can appear near an elliptic singular point of a polynomial vector field of degree d, is reduced to a completely algebraic problem of determination of the depth of a growing chain of *polynomial* ideals $D'_i \subseteq \mathbb{R}[\lambda]$.

Computing any finite number of ideals in the Dulac chain $\mathfrak{D}$ is theoretically feasible and can be relegated to one of many existing symbolic computation programs. Yet computation of the Bautin index (or depth) of the chain is the problem beyond the reach of any computer algebra system, even if we ignore the practical limitations on memory and time. Indeed, after observing that the chain $\mathfrak{D}$ stops growing at some moment μ, one has to *prove* that all *infinitely many* remaining coefficients of, say, the series $g(u,\lambda)$, belong to the ideal generated by the first μ of them.

13.7.1. *Practical computation of the Dulac chain.* The most important advantage of working with Dulac chain (ideal) rather than with the Bautin ideal, is practical: computation of $\mathfrak{D}$ does not require solving differential equations which is a necessary step when computing the first return map (cf. with §11.5).

Denote by $\mathbf{A}$ the linear part of $F = \mathbf{A} + F_2 + F_3 + \cdots$. Without loss of generality we may assume that $\mathbf{A} = \mathbf{I}$ does not depend on the parameters, replacing the family F by the family F' as in (13.28). For the same reason we look for the expansion for g beginning with the quadratic term $g(u) = g_2 u^2 + \cdots$.

Assume that all terms $u_2, u_3, \dots$ of degree $< k$ of the function u are already known and the coefficients $g_2, \dots, g_r$ of the series g are selected in some way, where r is the integer part $r = \lfloor (k-1)/2 \rfloor$.

If we substitute the expansion for u into the quotient equation $Fu = g(u)$ and compare the homogeneous terms of degree k, then we will obtain the identity

$$\mathbf{A}u_k + \sum_{j=2}^{k-2} F_j u_{k-j} = \sum_{j=2}^{r} g_j v_j + \begin{cases} g_{k/2}(u_2)^{k/2}, & k \text{ even}, \\ 0, & k \text{ odd}. \end{cases}$$

Here v_k are some homogeneous polynomials, all of degree k, obtained by products of the previously found homogeneous components $u_2, \dots, u_{k-1}$ with coefficients $g_2, \dots, g_r$.

To determine u_k, we have thus to resolve the equation

$$\mathbf{A}u_k = V_k + \begin{cases} g_{k/2}(x^2+y^2)^{k/2}, & k \text{ even}, \\ 0, & k \text{ odd}. \end{cases} \tag{13.29}$$

with some homogeneous polynomial V_k, choosing the coefficient $g_{k/2}$ appropriately when k is even.

The Lie derivative operator $\mathbf{A} = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}$ restricted on the finite-dimensional space of homogeneous polynomials of degree k, is diagonalizable (conjugate to $\frac{i}{2}(z\frac{\partial}{\partial z} - w\frac{\partial}{\partial w})$ with $z = x+iy$, $w = x-iy$) with the eigenvalues $j-(k-j)$, $j = 0, 1, \dots, k$. If k is odd, then $\mathbf{A}$ is invertible and the equation (13.29) is solvable for any expression V_k. If k is even, the nontrivial kernel of $\mathbf{A}$ consists of the polynomial $(x^2+y^2)^{k/2}$ and is transversal to the image of $\mathbf{A}$, therefore one can choose $g_{k/2}$ in such a way that the equation (13.29) will be again solvable with respect to u_k. Clearly, both g_k and the coefficients of the solution depend polynomially on the parameters (coefficients before the nonlinear terms of F).

This construction is very much parallel to the one described in §4.5, but in the particular case of elliptic families it is considerably simpler.

13.7.2. *Dulac ideal and Poincaré–Lyapunov constants.* The quotient equation (13.20) is not a unique way to associate a semiformal series with an elliptic family. For instance, in [**Sch93**] and in some other sources the following equation appears,

$$Fv = b(r^2), \qquad b(r^2) = \sum_{j\geqslant 1} b_j(\lambda)\, r^{2j}, \tag{13.30}$$

where $r^2 = (x^2 + y^2)$ and $v \in \mathfrak{A}[[x, y]]$. This equation also always admits a formal solution (v, b). The coefficients $b_j \in \mathfrak{A}$ are called *Poincaré–Lyapunov constants*, *Lyapunov values*, *focal values* etc. In the standard way one can associate with any solution of (13.30) a growing chain of ideals

$$\langle b_1\rangle \subseteq \langle b_1, b_2\rangle \subseteq \langle b_1, b_2, b_3\rangle \subseteq \cdots \tag{13.31}$$

The definition of Poincaré–Lyapunov constants and the chain of ideals (13.31) is not intrinsically invariant (unlike the definition of the Dulac ideal). Nevertheless, the common zero locus of the first k polynomials, $\{b_1 = \cdots = b_k = 0\} \subseteq (\mathbb{R}^n, 0)$ corresponds to parameter values for which the elliptic field admits a jet of order $2k$ of the first integral. The same condition in terms of the Dulac ideal translates as vanishing of the first k coefficients of the vector fields G.

Moreover, computation of the Dulac coefficient g_k and the Poincaré–Lyapunov constant b_k as polynomials in the parameters on the zero locus of $g_1, \dots, g_{k-1}$ or $b_1, \dots, b_{k-1}$ respectively, lead to the same equation (13.29), which proves that, at least as far as the Dulac ideals remain radical, the chains of ideals $\mathfrak{D}$ and (13.31) coincide.

13.8. Quadratic vector fields. The only universal polynomial family for which the depth of the Dulac ideal was computed, is the family of *quadratic vector fields* corresponding to $d = 2$. In this and the next section we prove the following famous theorem.

Theorem 13.36 (N. Bautin [**Bau39, Bau54**]). *Cyclicity of an elliptic singular point in the family of quadratic vector fields on the plane is equal to* 3.

Bautin theorem generated the conjecture that the number of *all* limit cycles of a planar quadratic vector field can be at most 3. This conjecture was believed to be true until in 1980 Shi Songling discovered an example of a quadratic vector field in which 3 small limit cycles coexist with one “large” limit cycle away from the elliptic singularity [**Shi80**].

The core of Theorem 13.36 constitutes the following purely algebraic fact. Consider the family F' of quadratic vector fields with the fixed linear part, which we write as a system of differential equations

$$\begin{aligned} \dot{x} &= \quad y + \lambda_1 x^2 + \lambda_2 xy + \lambda_3 y^2, \\ \dot{y} &= -x + \lambda_4 x^2 + \lambda_5 xy + \lambda_6 y^2. \end{aligned} \tag{13.32}$$

Theorem 13.37. *The Dulac chain of ideals $\mathfrak{D}'$ for the family* (13.32) *of quadratic vector fields with the rotation linear part* $\mathbf{I}$*, has depth* 2*,*

$$0 \neq D_2' \subsetneq D_3' \subsetneq D_4' = D_5' = D_6' = \cdots . \tag{13.33}$$

The proof of Theorem 13.37 occupies the rest of the rest of §13.8 and the whole section §13.9.

Reduction of Theorem 13.36 to Theorem 13.37. Let $\mathfrak{D}$ be the Dulac chain for the universal polynomial family (13.27) of degree $d = 2$. Assuming Theorem 13.37 proved, the depth of the Dulac chain $\mathcal{D}$ is equal to 3 by Proposition 13.35. By Theorem 13.31, the depth of the corresponding Bautin chain is also 3. By the first assertion of the fundamental Theorem 13.25, the *real* cyclicity of the displacement function (equal to cyclicity of the elliptic point) is 3. This completes the proof of Theorem 13.36. ☐

It was already noted on several occasions that many assertions concerning Bautin ideals admit counterparts concerning the respective zero loci in the space of the parameters, and almost always these assertions are much simpler. The Bautin theorem is not an exception: its proof is based on a no less remarkable theorem proved by H. Dulac in 1908.

Together with the chain of *real* polynomial ideals $\mathfrak{D}' \subseteq \mathbb{R}[\lambda]$ consider the chain of their *complexified* zero loci

$$\begin{gathered} \mathbb{C}^n \supseteq X_2 \supseteq X_3 \supseteq X_4 \supseteq \cdots \supseteq X_k \supseteq \cdots , \\ X_k = \{\lambda \in \mathbb{C}^n : p(\lambda) = 0 \, \forall p \in D_k'\}. \end{gathered} \tag{13.34}$$

The limit $X = \lim_{k\to\infty} X_k$ of the chain (13.34) consists of the complex values of the parameters λ for which the *complex* vector field is formally integrable, i.e., there exists a formal solution $u = (x^2 + y^2) + \cdots$ of the quotient equation $F'u \equiv 0$ corresponding to $g \equiv 0$. By Proposition 12.5, in this case there exists another, convergent formal integral.

Theorem 13.38 (H. Dulac [**Dul08**])**.** *The complex variety $X_4 \subseteq \mathbb{C}^6$ corresponds to integrable quadratic systems.*

In other words, the chain of complex algebraic varieties (13.34) stabilizes on the 4th term, $X_4 = X_5 = \cdots = X$.

The chain of ideals (13.33), starting from the term D'_4, in principle may exhibit nontrivial growth, but only in such a way that the zero loci of all subsequent ideals $D'_4, D'_5, \dots$ remain constant. This is, however, impossible, because the following Theorem asserts that D'_4 is *the biggest* ideal with the null locus X_4, so that further growth of the Dulac chain $\mathfrak{D}'$ is impossible.

Theorem 13.39 (H. Żołądek [**Żoł94**]). *The ideal D'_4 from the Dulac chain* (13.33) *is radical: any polynomial $p \in \mathbb{C}[\lambda]$ vanishing on X_4, belongs to D'_4.*

Theorem 13.37 obviously follows from Theorems 13.38 and 13.39, whose complete proofs are postponed until §13.9. Here we outline the general structure of these proofs in a brief historical discourse. From the outset it should be stressed that heavy computations cannot be avoided, though almost all of them can be now done by computers.

The first step is to compute the initial segment of the Dulac chain. On the level of null loci this computation was done by Dulac in [**Dul08**]. To minimize the number of independent parameters, Dulac used rotation of the coordinates (x, y) on the real plane to reduce the vector field to the so called Kapteyn form involving only 5 parameters $\widetilde{\lambda}_2, \dots, \widetilde{\lambda}_6$ (different from the initial parameters $\lambda_1, \dots, \lambda_6$),

$$\begin{aligned} \dot{x} &= -y - \widetilde{\lambda}_3 x^2 + (2\widetilde{\lambda}_2 + \widetilde{\lambda}_5)\, xy + \widetilde{\lambda}_6\, y^2, \\ \dot{y} &= \quad x + \widetilde{\lambda}_2 x^2 + (2\widetilde{\lambda}_3 + \widetilde{\lambda}_4)\, xy - \widetilde{\lambda}_2\, y^2. \end{aligned} \tag{13.35}$$

For this family Dulac derived the polynomial conditions over $\mathbb{R}[\widetilde{\lambda}]$ necessary for existence of a 7-jet of a first integral $u = (x^2 + y^2) + \cdots$, and discovered that under these conditions the vector field is integrable.

Bautin used the computations of Dulac to compute (by hand!) the coefficients of the return map and discovered that the ideal $\widetilde{D}_7 = \langle \widetilde{a}_3, \widetilde{a}_5, \widetilde{a}_7 \rangle \subseteq \mathbb{R}[\widetilde{\lambda}]$ is *not radical*. The main lemma of the paper [**Bau54**], proved by lengthy calculations (partially explained in [**Yak95**]), claims that all higher coefficients of the return map in fact belong to $\widetilde{D}_7$.

This circumstance remained completely mysterious until H. Żołądek in 1994 realized that both non-radicality of the ideal $\widetilde{D}_7$ in the Dulac chain and the fact that this chain stabilizes *despite* this non-radicality, are aberrations caused by the Kapteyn form, since transformation of the general equation (13.32) to the form (13.35) is singular (discontinuous). When written with respect to the original parameters λ, the respective (Dulac or Bautin) ideals $D'_4 = B_7$ become radical. Żołądek himself in [**Żoł94**] gave an elementary (though long and technical) proof of this radicality with respect to the ring of polynomials equivariant by a natural circle action (see Remark 13.41 below) and noted in passing that the equivariance is irrelevant and the fact remains true in the full ring $\mathbb{C}[\lambda]$, though the proof of this is "much more complicated" [**Żoł94**, Remark 1, p. 236].

However, unlike the claim on effective termination of the infinite chain of ideals which amounts to the *infinite* number of equalities between individual ideals in the chain, the claim on radicality of a single ideal admits verification in finite time. Moreover, algorithms for computing the radical of a polynomial ideal given by its generators, as well as the coincidence test for two such ideals are well developed and efficient computer algebra systems exist for implementing them. Proving Theorem 13.39 can be completely delegated to computer in the same way as computation of the initial coefficients of formal integrals, normal forms *etc.* This observation in some sense "downgrades" Theorem 13.39 to the level of a polynomial identity which for the moment cannot be proved by any method other than direct tedious computation. Below we give a five-line script for `CoCoA` (Commutative Computer Algebra, [**CNR00**]), which computes the radical $\sqrt{D'_4}$ and verifies that it coincides with D'_4.

Unlike Theorem 13.39, Dulac Theorem 13.38 is a claim that requires human intervention and ingenuity (together with unavoidable computations).

13.9. Demonstration of Dulac and Żołądek theorems. It was another observation of H. Żołądek that using the "complex notation" greatly simplifies computations. If we identify a point (x, y) on the real plane $\mathbb{R}^2$ with the complex number $z = x + iy \in \mathbb{C}$, then any quadratic vector field with the linear part $\mathbf{I}$ can be written as

$$\dot z = iz + A\,z^2 + B\,z\bar z + C\,\bar z^2, \qquad A, B, C \in \mathbb{C}, \tag{13.36}$$

with *complex* coefficients A, B, C. This observation can be explained by the fact that after complexification (allowing the coefficients λ to take complex values) the linear part can be diagonalized by passing to the coordinates $z = x + iy$, $w = x - iy$. The complex quadratic vector field acquires then the form

$$\begin{aligned} \dot z &= iz + Az^2 + Bzw + Cw^2, \\ \dot w &= -iw + C'z^2 + B'zw + A'w^2, \end{aligned} \qquad A, \dots, C' \in \mathbb{C}. \tag{13.37}$$

The real vector fields (with real values of the parameters λ) correspond to systems of the form (13.37) with the complex parameters $A, \dots, C'$ meeting the conditions

$$A' = \bar A, \quad B' = \bar B, \quad C' = \bar C \tag{13.38}$$

(the bar denotes the complex conjugation), after restriction on the real subspace $\mathbb{R}^2 \simeq \{w = \bar z\} \subseteq \mathbb{C}^2$. Clearly, solving the quotient system (13.20) when the vector field F has diagonal linear part, is much easier, see §13.7.1.

13.9.1. *Solution of the quotient equation.* The first several steps of formal solution of the quotient equation for the equation (13.37) yield the following results for coefficients the series $g(u) = g_1u + g_2u^2 + \cdots$,

$$\begin{aligned} g_1 &= 0, \\ g_2 &= c_2\,(AB - A'B'), \\ g_3 &= c_3\,[(2A+B')(A-2B')CB' - (2A'+B)(A'-2B)C'B], \\ g_4 &= c_4\,(BB' - CC')[(2A+B')B'^2C - (2A'+B)B^2C'], \end{aligned} \tag{13.39}$$

where $c_i \neq 0$ are *nonzero* constants, $i = 2, 3, 4$. Under the "reality" assumptions (13.38) these conditions take the form

$$\begin{aligned} g_1 &= 0, \\ g_2 &= c_2\ \operatorname{Im}(AB), \\ g_3 &= c_3\ \operatorname{Im}[(2A+\bar B)(A-2\bar B)\bar B C], \\ g_4 &= c_4\ \operatorname{Im}[(|B|^2 - |C|^2)(2A+\bar B)\bar B^2 C], \end{aligned} \tag{13.40}$$

as they appear in [**Żoł94**]. Clearly, cancellation of the nonzero constants does not change the chains of ideals, so from now on we will omit them.

In §13.7.1 we explained how the computations of the polynomials $g_{2,3,4}$ should be organized; the algorithm described there, can be easily made into a code for `Mathematica`.

Remark 13.40. Computation of the coefficients of the first return map is considerably more resource-consuming than that of the quotient equation. Bautin in [**Bau54**] reveals no details, only the ultimate results. This computation was reproduced using computers, see [**FLLL89**], confirming Bautin's formulas modulo an inessential error in the numeric coefficient c_4. Żołądek in [**Żoł94**] double-checked part of the results using perturbations technique. All existing methods corroborate the formulas (13.40).

13.9.2. *The Dulac variety.* The variety $X_4 = \{g_2 = g_3 = g_4 = 0\} \subseteq \mathbb{C}^6$ is reducible and consists of 4 components (their names will be later explained by the different mechanisms of integrability),

$$\begin{aligned} &V_\triangle = \{B = B' = 0\}, && \text{(Darbouxian)} \\ &V_H = \{2A + B' = 2A' + B = 0\}, && \text{(Hamiltonian)} \\ &V_\ominus = \{AB - A'B' = B'^3C - B^3C' = 0\}, && \text{(symmetric)} \\ &V_\between = \{A - 2B' = A' - 2B = BB' - CC' = 0\} && \text{(meromorphic)} \end{aligned} \tag{13.41}$$

Indeed, the locus $B = B' = 0$ of codimension 2 satisfies all equations (13.39) and gives the component of X_4 denoted by $V_\triangle$. Outside $V_\triangle$ the equation $g_2 = 0$ yields $A/B' = A'/B$; denoting this common value by R, we transform the remaining equations $g_3 = 0$, $g_4 = 0$ respectively to

$$(2R+1)(R-2)(B'^3C - B^3C') = 0,$$
$$(BB' - CC')(2R+1)(B'^3C - B^3C') = 0.$$

Two more components are given by the equations $2R+1 = 0$ which (together with $g_2 = 0$) corresponds to the locus V_H, and the equation $B'^3C - B^3C = 0$ that defines $V_\ominus$. Outside all these components of codimension 2 the last remaining component is defined by the equations $R = 2$, $BB' - CC' = 0$ which gives us $V_{\between}$.

Proof of Dulac Theorem 13.38. We begin the proof by noting that the linear part of normal form (13.37) is invariant by diagonal transformations $(z, w) \mapsto (\gamma z, \gamma' w)$, $\gamma, \gamma' \in \mathbb{C} \smallsetminus \{0\}$, in particular, by the transformations $(z, w) \mapsto (\gamma z, \gamma^{-1} w)$. These transformations, however, change the coefficients $A, \dots, C'$ of the field as follows,

$$\begin{gathered}(z, w) \mapsto (\gamma z, \gamma^{-1} w),\\ (A, B, C, A', B', C') \mapsto (\gamma A, \gamma^{-1} B, \gamma^{-3} C, \gamma^3 A', \gamma B', \gamma^{-1} C').\end{gathered} \tag{13.42}$$

These formulas define an action of $\mathbb{C} \smallsetminus \{0\}$ on the space of the coefficients; all components of the loci (13.41) are invariant by this action.

Remark 13.41. Though the action of $(\mathbb{C} \smallsetminus \{0\})^2$ or $\mathbb{C} \smallsetminus \{0\}$ does not preserve the subset of real systems (13.38), the restriction of this action on the circle $\mathbb{S}^1 = \{|\gamma| = 1,\ \gamma' = \gamma^{-1} = \bar{\gamma}\}$, corresponding to the rigid rotation of the real plane $z \mapsto \gamma z$, induces the circle action on the space of real quadratic vector fields with an elliptic singular point at the origin. It is this circle action that was used by Żołądek in [**Żoł94**] to simplify the proof of radicality.

We prove Theorem 13.38 by proving separately that each of the four components (13.41) corresponds to integrable systems.

1. V_H: *Hamiltonian case*. The divergence of the vector field (13.37) is

$$i + 2Az + Bw + (-i) + B'z + 2A'w = z(2A + B') + w(2A' + B)$$

and vanishes identically along the component V_H. The corresponding Hamiltonian is a cubic polynomial $\frac{1}{2}zw + \cdots$.

When establishing integrability of vector fields for the three remaining components of the locus (13.41), we will first establish it for a particular combination of parameters in the corresponding component and then show that by a suitable action (13.42) any other point on this component can be brought to this particular form.

2. $V_\ominus$: *Symmetric*, or *reversible case*. The component $V_\ominus$ parameterizes systems whose phase portrait is symmetric by a line passing through the origin.

Indeed, if

$$A' = -A, \quad B' = -B, \quad C' = -C, \tag{13.43}$$

then the vector field (13.37) is *anti*invariant by the symmetry $\sigma\colon (z,w) \mapsto (w,z)$: this symmetry preserves the field modulo the constant factor -1, $\sigma_* F = -F$. Therefore the complex holomorphic foliation $\mathcal{F}$ is symmetric (σ sends leaves into leaves). We claim that this symmetry implies integrability.

Indeed, denote by $\Delta_{\mathbb{R}}$ the holonomy (semi-monodromy) map of $\mathcal{F}$ after blow-up, corresponding to the symmetric cross-section $\tau = \{z + w = 0\}$, see Definition 11.29. The symmetry σ changes the orientation of the loop (equator) $\mathbb{R} \subset \mathbb{C}P^1$ on the exceptional divisor $\mathbb{C}P^1$, on the other hand, it does not change the intersection points between the leaves and the cross-section. Therefore

$$\Delta_{\mathbb{R}}^{-1} = \Delta_{\sigma(\mathbb{R})} = \Delta_{\mathbb{R}},$$

which means that $\Delta_{\mathbb{R}}$ is 2-periodic, $\Delta_{\mathbb{R}}^2 = \mathrm{id}$, and the field is a center.

Now we claim that any other combination of parameters on $V_\ominus$ can be brought to the special form (13.43) by a suitable action (13.42). Indeed, the equations of $V_\ominus$ can be reduced to the form

$$A/A' = B'/B, \qquad (B'/B)^3 = C'/C. \tag{13.44}$$

By a suitable choice of γ one can make the ratio A/A' equal to -1. The equations (13.44) imply then that the other two ratios B'/B and C'/C are automatically equal to -1, i.e., the conditions (13.43) are achieved. Thus any combination of the parameters on $V_\ominus$ corresponds to a field having a symmetry axis and hence integrable.

Darbouxian cases. In both the two remaining cases the vector field has several (real algebraic) invariant curves $p_i(z,w) = 0$. Starting from the functions p_i one can construct Darbouxian integrals of the form $\Phi = \prod p_i^{\alpha_i}$ with suitable (in general, non-integer or even non-real) exponents $\alpha_i \in \mathbb{C}$.

3. $V_\triangle$: *Darbouxian triangle*. The component $V_\triangle$ defined by the condition $B = B' = 0$, corresponds to vector fields having (generically) three invariant lines. To see them, note that the straight line $\{w - z = \alpha\}$, $\alpha \in \mathbb{C}$ is invariant by the field (13.37) with $B = 0$, if and only if

$$C' + A' = C + A, \quad 2\alpha(C - A') + 2i = 0, \quad \alpha^2(C - A') + i\alpha = 0 \tag{13.45}$$

(the result $iz + Az^2 + Cw^2 + iz - C'z^2 - A'w^2$ of the differentiation of $z - w$ after restriction on the line $w - z = \alpha$ must vanish identically).

This system (13.45) admits solution α only when

$$C' + A' = C + A, \tag{13.46}$$

moreover, if $C \neq A'$ (i.e., generically), this solution indeed exists. For an arbitrary combination A, C, A', C' the condition (13.46) can be achieved by a suitable diagonal action (13.42): one should resolve the equation

$$\gamma^{-3}C + \gamma A = \gamma^{3}C' + \gamma^{-1}A' \tag{13.47}$$

with respect to $\gamma \in \mathbb{C} \smallsetminus \{0\}$. This equation of degree 6, cubic with respect to γ^2, generically has three pairs of roots differing by a sign in each pair; each pair of roots corresponds to an invariant line.

Thus we conclude that for the parameter values in the component $V_\triangle$, the vector field F has (generically) three invariant straight lines $p_i = 0$, $i = 1, 2, 3$, two of them eventually conjugate. The invariance means that the derivatives Fp_i are divisible by p_i in the ring of polynomials in z, w. Denote by q_i the corresponding *cofactors*, the polynomials such that

$$Fp_i = q_i p_i, \qquad i = 1, 2, 3, \quad \deg q_i = 1.$$

Clearly, $q_i(0, 0) = 0$. Since any three homogeneous linear forms on $\mathbb{C}^2$ are linear dependent, there exist three nonzero complex numbers $\alpha_1, \alpha_2, \alpha_3$ such that $\sum \alpha_i q_i = 0$.

Now the direct computation shows that the function $\varPhi = \prod_1^3 p_i^{\alpha_i}$ is the Darbouxian first integral:

$$F\varPhi = \varPhi \cdot \sum_1^3 \frac{Fp_i^{\alpha_i}}{p_i^{\alpha_i}} = \varPhi \cdot \sum_1^3 \alpha_i q_i = 0.$$

Since $p_i(0, 0) \neq 0$, every branch of $\varPhi$ is analytic at the singular point. Thus the component $V_\triangle$ corresponds to the Darbouxian integrable vector fields having an invariant triangle $p_1 p_2 p_3 = 0$.

Thus a generic vector field corresponding to the component $V_\triangle$ is a center. Yet since being center is a closed property, the entire component $V_\triangle$ consists of centers.

4. $V_{\between}$: *Meromorphic integrable systems.* In the last remaining case when the parameters belong to the component $V_{\between}$, we show that one can find a meromorphic (rational) first integral as a ratio of two degree 6 polynomials, both nonzero at the singular point.

By a suitable action $(z, w) \mapsto (\gamma z, \gamma' w)$ multiplying B by γ and B' by γ', the vector field can be brought to the form with $B = B' = 1$. The remaining equations of $V_{\between}$ imply then that

$$B = B' = 1, \qquad A = A' = 2, \qquad CC' = 1, \tag{13.48}$$

so that the vector field has the form

$$\begin{aligned} \dot z &= \quad iz + 2z^2 + zw + Cw^2, \\ \dot w &= -iw + (1/C)\, z^2 + zw + 2w^2. \end{aligned} \tag{13.49}$$

We show that this vector field has two invariant curves, a quadric $\{p_2(z,w) = 0\}$ and a cubic $\{p_3(z,w) = 0\}$, with the corresponding cofactors *coinciding* modulo the rational coefficient,

$$Fp_2 = 2(z+w)p_2, \quad Fp_3 = 3(z+w)p_3. \tag{13.50}$$

Consequently, the *rational* first integral of the field F has the form $\Phi = p_2^3 p_3^{-2}$. The polynomials $p_2(z,w) = \sum_{i+j\leqslant 3} (P_2)_{ij}\, z^{i-1} w^{j-1}$ and $p_3(z,w) = \sum_{i+j\leqslant 4} (P_3)_{ij}\, z^{i-1} w^{j-1}$ have the following coefficient matrices,

$$P_2 = \begin{pmatrix} -1 & -2\,i & C \\ 2\,i & -2 & \\ \frac{1}{C} & & \end{pmatrix}, \quad P_3 = \begin{pmatrix} \frac{2\,i}{1+C} & \frac{-6}{1+C} & -3\,i & C \\ \frac{6}{1+C} & \frac{3\,i\,(1+C)}{C} & -3 & \\ \frac{-3\,i}{C} & \frac{3}{C} & & \\ -\frac{1}{C^2} & & & \end{pmatrix}$$

and the fact that they satisfy the condition (13.50), can be verified by a direct (though tedious) computation. Actually, they were found by `Mathematica` [**Wol96**] as solutions of (13.50) using indeterminate coefficients method.

Thus all four components (13.41) correspond to nonlinear centers, which completes the proof of Theorem 13.38. □

"Proof" of Żołądek Theorem 13.39. To prove this result, we need to show that the ideal generated in the polynomial ring in 6 variables $\mathbb{C}[A,B,C,A',B',C']$ by the three polynomials g_2, g_3, g_4 from (13.39), is radical. Checking radicality is a task that is well algorithmized. The computer system `CoCoA` includes both computation of the complex radical and the coincidence test for two ideals defined by their generators, as the standard functions, see [**CNR00**].

The code checking radicality, is given on Figure 9. Due to the technical constraints (independent variables should be denoted by lowercase letters) we denoted by `a,b,c,x,y,z` the variables A, B, C, A', B', C' respectively. The first line instructs to use the ring of characteristic zero in the six indeterminates, then `D` is defined as the ideal generated by the polynomials `G2,G3,G4` encoding respectively g_2, g_3, g_4. Finally, the last line is the logical command checking equality between the ideal `D` and its radical `Radical(D)`. After 2 sec. of computations on a laptop, the program prints `TRUE`. □

```
Use R::=Q[axbycz];

  G2:=ab-xy;
  G3:=(2a+y)(a-2y)cy-(2x+b)(x-2b)zb;
  G4:=(by-cz)((2a+y)y^2c-(2x+b)b^2z);

D:=Ideal(G2,G3,G4);
D=Radical(D);
```

Figure 9. The `CoCoA` code verifying radicality of the Dulac ideal.

13.9.3. *Concluding remarks.* We conclude the proof of Bautin theorem by two technical remarks.

Remark 13.42. The "complex notation" (i.e., writing the quadratic vector field so that its linear part is diagonal) simplifies computations not only for humans, but also for computers. An attempt to compute the radical of the Dulac ideal D_4' written for the real system (13.32) fails miserably, apparently because the corresponding polynomials g_i have too many monomial terms for the standard algorithms to cope with (recall that we are dealing with polynomials of degree 6 in 6 independent variables!).

Remark 13.43. One may recycle information already stored in the equations of the four Dulac loci (13.41) to simplify computation of the radical $\sqrt{D_4'}$. Indeed, this radical is the *intersection* of the four ideals $J_\triangle, J_H, J_\ominus$ and $J_\between$ in $\mathbb{C}[A,\dots,C']$ which consist of polynomials vanishing on the respective components.

However, one has to bear in mind that while the three ideals,

$$\begin{aligned} J_\triangle &= \langle B, B' \rangle, \\ J_H &= \langle 2A+B', 2A'+B \rangle, \\ J_\between &= \langle A-2B', A'-2B, BB'-CC' \rangle, \end{aligned}$$

are all radical, the polynomial equations defining $J_\ominus$ in (13.41), span a *non*-radical ideal,

$$\begin{aligned} J_\ominus &= \sqrt{\langle AB-A'B', B'^3C-B^3C' \rangle} \\ &= \langle AB-A'B', B'^3C-B^3C', AB'^2C-A'B^2C', A^2B'C-A'^2BC' \rangle. \end{aligned}$$

In any case, computing intersection of ideals (i.e., computing a basis for the intersection) is in general a tedious task which amounts to computing resultants and elimination of variables. On top of that one should solve the membership problem, checking that all elements of the constructed basis again belong to D_4'. To double-check the above described `CoCoA`-proof of Theorem 13.39, these computations were also implemented (by another `CoCoA` script) and gave the same answer, thus further reducing the chances of computer- or human-generated errors.

Chapter 3

Linear systems: local and global theory

20. General facts about linear systems

We start with discussing analytic (nonsingular) systems of linear ordinary differential equations on Riemann surfaces and later introduce the class of systems with singularities.

20.1. Linear differential equations: vector, matrix, Pfaffian. Let $T \subset \mathbb{C}$ be an open domain with the chart t on it, and $A(t) = \|a_{ij}(t)\|_{i,j=1}^n$ a holomorphic matrix function on T.

Definition 20.1. A system of n linear ordinary differential equations with holomorphic matrix of coefficients $A(t) \in \operatorname{Mat}(n, \mathcal{O}(T))$ on T (a linear system on T, in short) is the vector differential equation

$$\dot{x}(t) = A(t)x(t), \qquad t \in T \subset \mathbb{C}, \quad x = (x_1 \dots, x_n)^{\mathrm{T}} \in \mathbb{C}^n. \tag{20.1}$$

Solutions of the system are holomorphic vector-functions $x(\cdot)\colon T \to \mathbb{C}^n$.

The linear system (20.1) can be also identified with a vector field in $T \times \mathbb{C}^n$ with the components

$$\frac{\partial}{\partial t} + \sum_{i,j} a_{ij}(t)x_j \frac{\partial}{\partial x_i} \tag{20.2}$$

in the coordinates $(t, x_1, \dots, x_n)$. This form will be especially convenient when dealing with singularities of linear systems.

It is possible to define linear systems avoiding explicit reference to the chart t (which is important when considering linear systems defined globally on Riemann surfaces). Let T be a Riemann surface. Consider n^2 holomorphic 1-forms $\omega_{ij} \in \Lambda^1(T)$, arranged in the form of the *Pfaffian matrix* Ω,

$$\Omega = \begin{pmatrix} \omega_{11} & \cdots & \omega_{1n} \\ \vdots & \ddots & \vdots \\ \omega_{n1} & \cdots & \omega_{nn} \end{pmatrix} \in \operatorname{Mat}(n, \Lambda^1(T)).$$

A column vector function $x = (x_1, \dots, x_n)^{\mathrm{T}}$, $x_i = x_i(t) \in \mathcal{O}(T)$, is said to be a solution of the linear system defined by the Pfaffian matrix Ω as above, if $dx_i = \sum_{j=1}^n \omega_i j x_j$, or, in the matrix form,

$$dx = \Omega x, \qquad dx = (dx_1, \dots, dx_n)^{\mathrm{T}}. \tag{20.3}$$

Clearly, this definition is equivalent to (20.1), if the Pfaffian matrix in the local chart t has the form $\Omega = A(t)\,dt$, i.e., $\omega_{ij} = a_{ij}(t)\,dt$.

Any number k of (column) vector solutions of (20.1) or (20.3) can be arranged into a rectangular holomorphic $n \times k$-matrix function $X = X(t)$ satisfying the corresponding *matrix* equations, differential or Pfaffian. The case of *square* matrices with $k = n$ is especially important: when mentioning a linear system, we will most often deal with the matrix equations, either ordinary or Pfaffian, of the form

$$\begin{array}{ccc} \dot X(t) = A(t)X(t), & & dX = \Omega X, \\ A(t) \in \operatorname{Mat}(n, \mathcal{O}(T)), & \text{or} & \Omega \in \operatorname{Mat}(n, \Lambda^1(T)), \\ \text{(ordinary)} & & \text{(Pfaffian)}, \end{array} \tag{20.4}$$

$$X = X(t) \in \operatorname{Mat}(n, \mathcal{O}(t)), \quad \Omega = A(t)\,dt, \quad t \in T$$

Such matrices represent n-tuples of vector solutions of (20.1) or (20.3).

20.2. Fundamental solutions. Notice that the coefficients matrix $A(t)$ (resp., the Pfaffian matrix Ω) can be restored from a holomorphic matrix solution $X(t)$ at each point of nondegeneracy of the latter, where $X^{-1}(t)$ is defined and holomorphic:

$$A(t) = \dot X(t) \cdot X^{-1}(t), \qquad \text{resp.,} \quad \Omega = dX \cdot X^{-1}.$$

This observation motivates the following definition.

Definition 20.2. A *fundamental matrix solution* of the system (20.4) is a matrix solution $X(t)$ of (20.4) that is everywhere nondegenerate:

$$X \in \operatorname{GL}(n, \mathcal{O}(T)), \qquad \det X(t) \neq 0 \quad \forall t \in T.$$

The mere existence of a single fundamental matrix solution allows to describe completely the structure of all solutions of a linear system (20.1) and the associated matrix equation (20.4).

Theorem 20.3 (structure of solutions of linear system)**.** *Let T be a connected domain in which some fundamental matrix solution of a linear system* (20.4) *exists. Then*:

1. *For any point $t_0 \in T$ and any initial value $X_0 \in \operatorname{Mat}(n, \mathbb{C})$ there exists a unique matrix solution $X(t)$ of the matrix equation* (20.4) *satisfying the initial condition $X(t_0) = X_0$. This solution is fundamental if* $\det X_0 \neq 0$.

2. *For any point $t_0 \in T$ and any initial value $x_0 \in \mathbb{C}^n$ there exists a unique vector solution $x(t)$ of the linear system* (20.1) *satisfying the initial condition $x(t_0) = x_0$.*

3. *All vector solutions of the linear system* (20.1) *in T form a linear space of dimension n over $\mathbb{C}$ in the infinite-dimensional space of analytic vector-functions in T. For any point $t_0 \in T$ the map $x(\cdot) \mapsto x(t_0)$ is an isomorphism between this space and $\mathbb{C}^n$.*

Proof. For any solution $X = X(t)$ of (20.4) and any constant matrix $C \in \operatorname{Mat}(n, \mathbb{C})$ the product $Y(t) = X(t)C$ will again be a matrix solution, since $d(XC) = dX \cdot C = \Omega XC$. Conversely, if $X(t)$ is a fundamental solution in a *connected* domain T, then any other solution $Y(t)$ can be obtained from

$X(t)$ by multiplication by a constant matrix C from the right. Indeed, the (matrix) differential of the matrix ratio $X^{-1}Y$ of two solutions is zero,

$$d(X^{-1}Y) = -X^{-1}dX \cdot X^{-1}Y + X^{-1}dY = -X^{-1}\Omega Y + X^{-1}\Omega Y = 0.$$

Therefore this matrix ratio is a constant matrix C. For two different solutions satisfying identical initial conditions, $C = E$. Hence these two solutions coincide identically.

Given an arbitrary fundamental matrix solution $Y(t)$, one can multiply it from the right by the constant matrix $C = Y(t_0)^{-1}X_0$ to construct a matrix solution satisfying the initial condition required in the theorem: its determinant $\det X(t) = \text{const} \cdot \det Y(t)$ will be nonzero everywhere together with $\det Y(t)$.

In the same way differentiation of the vector function $Y^{-1}(t)x(t)$, where $x(t) = (x_1(t), \dots, x_n(t))^{\mathrm{T}}$ is an arbitrary vector solution of (20.1), shows that it is locally constant. Therefore any vector solution the system (20.1) can be obtained as a linear combination of columns of a fundamental matrix solution, and the initial value problem is solved by the vector function

$$x(t) = Y(t)c, \qquad c = Y^{-1}(t_0)x_0 \in \mathbb{C}^n.$$

To prove the last assertion, note that the map $x(\cdot) \mapsto x(t_0)$ is a linear map that is injective and surjective, hence an isomorphism. □

This structural theorem in turn motivates the following invariant definition.

Definition 20.4. A *fundamental system of solutions* of the linear system (20.1) is a basis over $\mathbb{C}$ in the linear space of solutions.

Columns of any fundamental matrix solutions form a fundamental system of vector solutions of (20.1) and conversely, any fundamental system of vector solutions arranged in the form of a square matrix, form a fundamental matrix solution of (20.4).

20.3. Gauge equivalence. When considering linear systems, it is natural to restrict the class of admissible transformations to those preserving the linearity. Any such transformation of $T \times \mathbb{C}^n$ has the form

$$(t, x) \mapsto (t, H(t)x), \qquad H \in \mathrm{GL}(n, \mathcal{O}(t)),$$

with an analytic matrix $H(t)$ which is holomorphically invertible (i.e., the inverse matrix $H^{-1}(t)$ exists and is also holomorphic). This transformation acts on linear systems (20.4) in an obvious way: if $X(t)$ is a fundamental matrix solution of (20.4), then the matrix function $X'(t) = H(t)X(t)$ is a fundamental matrix solution for another system with the Pfaffian matrix

$\Omega' = dX' \cdot (X')^{-1}$. Direct computation of the differential yields the following definition.

Definition 20.5. Two linear systems

$$dX = \Omega X \quad \text{and} \quad dX' = \Omega' X', \qquad \Omega, \Omega' \in \mathrm{Mat}(n, \Lambda^1(T)),$$

are said to be *gauge equivalent*, *linear equivalent*, or simply *conjugate*, if

$$\Omega' = dH \cdot H^{-1} + H\Omega H^{-1} \tag{20.5}$$

for some analytically invertible matrix function $H = H(t) \in \mathrm{GL}(n, \mathcal{O}(T))$. This matrix is called the *conjugacy matrix*.

20.4. Existence and uniqueness of solutions. The local existence theorem for linear systems is a corollary to the general local existence theorem for arbitrary analytic systems. Consider a holomorphic linear system (20.4) on the Riemann surface T.

Lemma 20.6. *For any point $t_0 \in T$ there exists a fundamental matrix solution $X(t)$ defined in some small neighborhood U of t_0.*

Proof. The linear vector-function $(t, x) \mapsto A(t)x$ is holomorphic everywhere on $T \times \mathbb{C}^n$. By the local existence theorem, for any initial condition $(t_0, x_0) \in T \times \mathbb{C}^n$ there exists a holomorphic vector solution $x(t)$, defined on some neighborhood of t_0, meeting the condition $x(t_0) = x_0$. Choose n solutions satisfying n linear independent initial conditions at t_0, arranged as columns of a square matrix $X(t)$ and considered on their common domain. By construction, $\det X(t_0) \neq 0$, hence the holomorphic matrix $X(t)$ is holomorphically invertible in some neighborhood U of the point t_0. □

In contrast to the local nature of the general existence and uniqueness theorem for analytic differential equations, the analogous theorems in the linear case are global.

Theorem 20.7 (global existence theorem)**.** *A linear system on a Riemann surface T admits a fundamental solution in any simply connected subdomain $U \subset X$.*

Proof. By Lemma 20.6, local solutions exist near each point of T. We show that they can be glued together over any simply connected domain U.

Choose a base point $t_0 \in U$ and let $X_0(t)$ be a local fundamental matrix solution at this point. We extend it to an arbitrary point $t_1 \in U$.

Since U is connected, there exists a compact piecewise smooth curve (path) γ connecting t_0 with t_1. Being compact, γ can be covered by a union of finitely many charts $U_0, U_1, \ldots, U_m$ carrying the respective local fundamental matrix solutions $X_0(t), \dots, X_m(t)$, such that all intersections $U_{ij} = U_i \cap U_j$ are connected, and $U_{ij} \neq \varnothing$ if and only if $|i-j| \leqslant 1$.

By the uniqueness theorem, on U_{ij} we have $X_i(t)C_{ij} = X_j(t)$ for some constant nondegenerate matrix $C_{ij} \in \mathrm{GL}(n, \mathbb{C})$ (by definition, $C_{ij} = C_{ji}^{-1}$). Assume that the constant matrices $B_i \in \mathrm{Mat}(n, \mathbb{C})$ satisfy the identities

$$C_{ij} = B_i B_j^{-1}, \qquad |i-j| \leqslant 1,\ i \neq j. \tag{20.6}$$

Then the modified fundamental solutions

$$X_i'(t) = X_i(t) B_i, \qquad i = 1, \dots, m,$$

agree on the intersections:

$$X_j'(t) = X_j(t)B_j = X_i(t)C_{ij}B_j = X_i(t)B_i = X_i', \qquad t \in U_{ij} = U_i \cap U_j.$$

Thus the question on existence of a holomorphic solution $X(t)$ along γ is reduced to solution of a linear (algebraic) system of matrix equations (20.6). In our assumptions the solution can be explicitly constructed:

$$B_0 = E, \quad B_{i+1} = C_{i+1,i} B_i, \qquad i = 0, 1, \dots, m-1.$$

This completes the proof of existence of analytic continuation of solutions along paths.

The result of such continuation will not be affected by sufficiently small variations of γ with frozen endpoints. By continuity, this result will be the same when γ is replaced by any homotopic curve with the same endpoints t_0, t_1. Finally, for a simply connected domain U all paths connecting two points, are homotopic. This proves coincidence of all continuations along all paths.

Now one can define the global solution $X(t)$ in U as the result of the above continuation along any path γ connecting t_0 with t. □

As a corollary to Theorems 20.3 and 20.7, we conclude that on a simply connected Riemann surface T, for any initial condition $(t_0, X_0) \in (T \times \mathrm{Mat}(n, \mathbb{C}))$ there exists a unique solution $X(t) \in \mathrm{Mat}(n, \mathcal{O}(T))$ of the matrix system (20.4) satisfying this condition. The solution is automatically fundamental if $\det X_0 \neq 0$.

Remark 20.8. An alternative proof of the fact that any solution of a linear system can be continued along any path, can be achieved by purely real arguments. We start with a general a priori growth rate bound characteristic for linear systems.

Lemma 20.9 (Gronwall inequality). *Let $A(\cdot)$ be a continuous matrix function on the real interval $t \in [t_0, t_1] \subset \mathbb{R}$ of explicitly bounded norm,*

$$\forall t \in [t_0, t_1] \qquad A(t) \in \mathrm{Mat}(n, \mathbb{C}), \qquad \|A(t)\| \leqslant C.$$

Then any solution $x(t)$ of the linear system (20.1) *satisfies the inequality*

$$\|x(t)\| \leqslant \|x(t_0)\| \exp(C\,|t-t_0|)$$

Proof. By the limit triangle inequality, $\frac{d}{dt}\|x(t)\| \leqslant \|\frac{d}{dt}x(t)\|$, therefore

$$\tfrac{d}{dt}\|x(t)\| \leqslant \|A(t)\|\,\|x(t)\| \leqslant C\,\|x(t)\|.$$

Therefore the logarithmic derivative $\frac{d}{dt}\ln\|x(t)\|$ is bounded by C everywhere on $[t_0,t_1]$, so that its growth between t_0 and an arbitrary t is no greater than $C\,|t-t_0|$. This immediately implies the inequality for the norm $\|x(t)\|$ itself. □

By the Gronwall inequality, any solution with the initial condition $x_0 \in \mathbb{R}^n$ cannot leave the compact set $K = [t_0,t_1]\times\{\|x\| \leqslant R'\} \subset \mathbb{R}^{1+n}$, $R' = \|x_0\|\,\exp(R\,|t_1-t_0|)$, except for the right section $K\cap\{t_1\}\times\mathbb{R}^n$. On the other hand, by the fundamental continuation theorem for real ordinary differential equations [**Arn92**], any solution beginning in any compact K can be continued until it reaches the boundary of K. Together with the above argument, this implies that solutions of linear systems on real intervals are always globally defined.

One can use this real theorem to continue solutions along arbitrary parameterized curves in T. It remains to prove that these restricted solutions are in fact holomorphic on T and prove (in the same way as before) that the results are independent of the choice of the curves in case the domain is simply connected.

Remark 20.10 (variation of constants)**.** Solution of nonhomogeneous systems can be reduced to that of homogeneous systems using the method of *variation of constants*. If $X(t)$ is a fundamental matrix solution of the linear system $\dot X = A(t)X$, then solution the nonhomogeneous system $\dot Y = A(t)Y + B(t)$, where $B(t)$ is a known matrix function, is given by the formulas

$$Y(t) = X(t)C(t), \qquad \dot C(t) = X^{-1}(t)B(t), \tag{20.7}$$

where solutions of the second equation can be found by immediate integration, $C = \int x^{-1}B\,dt$.

20.5. Matrix exponent. In the particular case of *linear systems with constant coefficients* the fundamental matrix solution can be explicitly found either in the form of a converging matrix series, or in the closed form involving exponents.

Most contents of the subsection moved to Chapter I, §3.3

Consider the system

$$\dot x = Ax, \qquad A = \text{const} \in \operatorname{Mat}(n,\mathbb{C}). \tag{20.8}$$

Solution of this system, normalized by the initial condition $X(0) = E$, is called the *matrix exponent* and denoted $\exp tA$. One can easily prove that this solution is given by the sum of an everywhere absolutely converging matrix series

$$X(t) = \exp tA = E + tA + \frac{t^2}{2!}A^2 + \cdots + \frac{t^k}{k!}A^k + \cdots,$$

obtained by substituting the powers of A into the Taylor series for $\exp at = 1 + at + \cdots$. Indeed, from commutativity of powers of A and the property

$$\exp at_1 \cdot \exp at_2 = \exp a(t_1 + t_2), \qquad \forall t_1, t_2 \in \mathbb{C},$$

of the scalar exponent it follows that $X(t_1 + t_2) = X(t_1)X(t_2)$ for any $t_1, t_2 \in \mathbb{C}$. This property immediately implies that $\dot{X}(t) = AX(t)$, since $\dot{X}(0) = A$.

The infinite series defining the matrix exponent can be reduced to computation of a matrix polynomial. Assume that $A = \varLambda + N$ is in the Jordan normal form, being a sum of a diagonal matrix $\varLambda = \operatorname{diag}\{\lambda_1, \dots, \lambda_n\}$ and the nilpotent part N commuting with $\varLambda$, then

$$\exp tA = \exp t\varLambda \cdot \exp tN = \operatorname{diag}\{e^{\lambda_1}, \dots, e^{\lambda_n}\} \cdot P(t),$$

where $P(t) = \exp tN$ is a matrix polynomial of degree $\leqslant n-1$ obtained by truncating the exponential series.

An arbitrary matrix can be brought to the Jordan normal form by a suitable conjugacy $A \mapsto CAC^{-1}$. On the other hand, by definition of the matrix exponent as a solution to the linear system (20.8), $\exp t(CAC^{-1}) = C(\exp tA)C^{-1}$. This proves that all entries of the exponent $\exp tA$ are *quasipolynomials*, linear combinations of monomials $t^{k_i-1}\exp \lambda_i t$, with k_i ranging between 1 and the maximal size of the Jordan block for the corresponding eigenvalue λ_i.

20.6. Monodromy and holonomy. The condition of Theorem 20.7 that the domain U is simply connected, cannot be omitted. Solutions of a linear system on the Riemann surface T may become multivalued if T is not simply connected. The best one can get from Theorems 20.3 and 20.7 is that global solutions exist on the universal covering space $\widetilde{T}$ of T.

Consider a linear system (20.4) on a multiply connected Riemann surface T. Fix a *base point* $t_0 \in T$. Then the *germs* of solutions at the point t_0 also form an n-dimensional linear space, isomorphic to the space of solutions of this system in any simply connected domain $U \subset T$ containing the point t_0. Let $\gamma \in \pi_1(T, t_0)$ be a *closed loop* beginning and ending at t_0.

Definition 20.11. The *monodromy operator* Δ_γ corresponding to the loop γ is the linear automorphism of the linear n-space of germs of solutions at the point t_0, obtained by continuation along γ.

The monodromy operator depends only on the homotopy class of the loop γ in the fundamental group $\pi_1(T, t_0)$. The correspondence between (homotopy classes) of closed loops and the monodromy operators is a linear n-dimensional *anti*representation of the fundamental group:

$$\Delta_{\gamma_1 \cdot \gamma_2} = \Delta_{\gamma_2} \Delta_{\gamma_1}, \qquad \forall \gamma_1, \gamma_2 \in \pi_1(T, t_0).$$

where $\gamma_1 \cdot \gamma_2$ is the composite loop circumscribing first γ_1 and then γ_2. The image, a subgroup of the linear group $\mathrm{GL}(n, \mathbb{C})$, is called the *monodromy group* of the linear system (20.1). (The difference between a representation and an antirepresentation is not essential: the monodromy can be treated as a representation if the fundamental group is considered with another operation $\gamma_1 * \gamma_2 = \gamma_2 \cdot \gamma_1$.)

Choosing a basis in the space of solutions in a neighborhood of the base point is equivalent to choosing a (germ of) fundamental matrix solution $X(t)$. After analytic continuation along the loop γ we obtain *another* germ of fundamental matrix solution $X'(t)$. Expanding columns of $X'(t)$ as linear combinations of columns of $X(t)$, we can represent the result of this analytic continuation in the matrix form,

$$\Delta_\gamma X(t) = X(t) M_\gamma, \qquad \gamma \in \pi_1(T, t_0), \tag{20.9}$$

where $M_\gamma \in \mathrm{GL}(n, \mathbb{C})$ is a constant matrix. The correspondence

$$\pi_1(T, t_0) \to \mathrm{Mat}(n, \mathbb{C}), \qquad \gamma \mapsto M_\gamma,$$

is a matrix antirepresentation of the fundamental group:

$$M_{\gamma_1 \cdot \gamma_2} = M_{\gamma_2} M_{\gamma_1}.$$

It is the matrix group generated by the matrices M_γ that is sometimes referred to as the *monodromy group* of a linear system.

The monodromy matrices *do not* depend on the choice of the base point t_0 in the sense that if *the same* fundamental solution $X(t)$ is used for defining the monodromy, then the matrices will be the same for all base points sufficiently closed to t_0. On the contrary, the monodromy matrices *do* depend on the choice of the basis in the space of solutions (the fundamental solution $X(t)$ to be continued along different loops). For another fundamental solution $X(t)C$ with a nondegenerate constant matrix C, the monodromy matrix M_γ relative to this basis will be replaced by $C^{-1} M_\gamma C$ with the same C for all loops. On the other hand, the definition of monodromy is apriori not linked to any specific point, as soon as the fundamental solution is already chosen.

Cf. with the new section §2.3: much of the contents is to be moved there.

Another way to represent the monodromy group by matrices is to identify solutions with the vector values they take at the base point (cf. with the last assertion of Theorem 20.3). Consider an arbitrary nonsingular point $t_0 \in T$ and a closed loop $\gamma \in \pi_1(T, t_0)$.

Definition 20.12. The *holonomy map* $F_\gamma = F_{\gamma,t_0}$ is the linear map of the linear space $\{t_0\} \times \mathbb{C}^n$ into itself obtained by continuation of solutions with a specified initial condition over a closed loop γ.

By construction, for the composition loop $\gamma_1 \cdot \gamma_2 \in \pi_1(T, t_0)$,

$$F_{\gamma_1 \cdot \gamma_2, t_0} = F_{\gamma_2, t_0} \circ F_{\gamma_1, t_0},$$

which means that the correspondence $\gamma \mapsto F_{\gamma,t_0}$ is again an *anti*representation of the fundamental group. In the standard basis in $\{t_0\} \times \mathbb{C}^n$ the holonomy operators become matrices that can be expressed through a fundamental matrix solution $X(t)$ as follows,

$$(\Delta_\gamma X)(t_0) = F_{\gamma,t_0} \cdot X(t_0), \qquad F_{\gamma,t_0} \in \operatorname{Mat}(n, \mathbb{C}).$$

From this definition it is clear that the holonomy operators, unlike the monodromy matrices, *do not* depend on the choice of the fundamental solution, but *do* depend on the choice of the transversal. Choosing a different chart on the transversal (or a different transversal through another base point) results in a simultaneous conjugacy of all these matrices.

As follows from the definitions, the monodromy matrices M_γ *numerically coincide* with the holonomy matrices F_γ for the standard choice of coordinates on $\mathbb{C}^n$ and a special choice of the fundamental solution $X(t)$, normalized by the condition $X(t_0) = E$.

Clearly, *gauge equivalent systems have isomorphic monodromy and holonomy groups*. The corresponding matrix representations are equivalent in the sense that all matrices $\{M_\gamma\}$ of one antirepresentation are simultaneously conjugated to corresponding matrices $\{M'_\gamma\}$ of the other (anti)representation by means of the same nondegenerate matrix $C \in \operatorname{GL}(n, \mathbb{C})$, depending on the choice of the fundamental solutions. The same obviously holds for the holonomy (anti)representation.

20.7. Systems with isolated singularities. The notions of holonomy and monodromy are nontrivial only in the case of multiply connected Riemann surfaces. Such surfaces naturally appear after deleting singular points from compact Riemann surfaces if the coefficients of a linear system are only meromorphic. By far the most important particular case (and the only one considered in this book) is that of systems with *meromorphic coefficients on the Riemann sphere* $\mathbb{C}P^1$. As is well-known, all meromorphic functions and forms on the Riemann sphere are *rational* (a 1-form is rational if in the affine chart t its coefficient is a rational function).

Definition 20.13. A *linear system with singularities* on the Riemann sphere is a system of the form (20.4) with *meromorphic* Pfaffian matrix Ω. The *singular locus* of such a system is the polar locus of the Pfaffian form Ω.

The notion of gauge equivalence of nonsingular linear systems can be accordingly modified.

Definition 20.14. Two linear systems on the Riemann sphere are *globally meromorphically equivalent*, if they have the same singular locus $\Sigma \subset \mathbb{C}P^1$ and are gauge equivalent in the sense (20.5) on the complement $\mathbb{C}P^1 \smallsetminus \Sigma$ with a *rational* conjugacy matrix $H(t)$, holomorphic and holomorphically invertible outside Σ.

Definition 20.15. The monodromy group of a linear system with singularities on the Riemann sphere is the monodromy group of its restriction on $\mathbb{C}P^1 \smallsetminus \Sigma$.

As before, the monodromy group is a subgroup of $\mathrm{GL}(n, \mathbb{C})$ defined up to a simultaneous conjugacy of all monodromy matrices, and it is the same for two globally meromorphically equivalent systems.

20.8. Euler system. A linear system with constant coefficients, $\Omega = A\,dt$, has no singularities on $\mathbb{C}$ but when considered on $\mathbb{C}P^1$, it has a pole of second order at infinity: in the chart $z = 1/t$, $\Omega = -Az^{-2}dz$.

There are no global holomorphic forms on $\mathbb{C}P^1$: the difference between the number of poles and the number of isolated zeros for any meromorphic 1-form on $\mathbb{C}P^1$ is always equal to 2 if counted with multiplicities. Therefore at least two poles must always occur.

A simplest nontrivial example of a linear system on $\mathbb{C}P^1$ having the minimal number of *simple* poles, is that of an *Euler system*,

$$dX = \Omega X, \qquad \Omega = A\,t^{-1}dt, \qquad A \in \mathrm{Mat}(n, \mathbb{C}), \tag{20.10}$$

defined by a single constant matrix A called the *residue matrix*. The singular locus of the system (20.10) consists of two points $\{0, \infty\}$. Actually, any linear system on $\mathbb{C}P^1$ with two simple poles takes the form (20.10) after a conformal transformation of the sphere, bringing the two singular points to 0 and ∞ respectively.

The Euler system can be immediately integrated. Consider the logarithmic chart $z = \ln t$ on the universal covering $\mathbb{C}$ of $\mathbb{C}P^1 \smallsetminus \Sigma$. In this chart (20.10) becomes a system with constant coefficients $\Omega = A\,dz$, whose fundamental solution is given by the matrix exponent. In the initial chart the exponential solution takes the form

$$X(t) = t^A = \exp(A \ln t), \qquad t \neq 0$$

which is indeed ramified over Σ.

The fundamental group of $\mathbb{C}P^1 \smallsetminus \Sigma = \mathbb{C} \smallsetminus \{0\}$ is cyclic generated by the loop $s \mapsto \exp 2\pi i s$, $s \in [0, 1]$, around the origin. The monodromy matrix of the Euler equation, corresponding to the above constructed fundamental solution, can be easily computed:

$$M = \exp(2\pi i A) \tag{20.11}$$

(going around the origin corresponds to choosing a different branch of the logarithm, shifted by $2\pi i$ from the initial one).

After the monodromy matrix of the Euler system is explicitly computed, we can show that *any* nondegenerate matrix M can in fact be realized as the monodromy of an appropriate Euler system. This follows from existence of matrix logarithms for nondegenerate matrices over $\mathbb{C}$ (Lemma 3.12) one could simply put

$$A = \frac{1}{2\pi i} \ln M.$$

21. Local theory of regular singular points

The global theory of systems with singularities is based on an extensive study of the local objects, singular points or simply singularities.

21.1. Holomorphic and meromorphic local equivalence.

Definition 21.1. A *singularity* of a linear system is the germ of a meromorphic Pfaffian form Ω (or, what is essentially the same, of a meromorphic matrix function $A(t)$) at an isolated pole of the latter. The order of this pole diminished by 1 is called the *Poincaré rank* of the singularity.

For convenience we shall always assume in this section that the singularity occurs at the origin $t_0 = 0$. The fundamental group of a punctured neighborhood of the origin is cyclic generated by a single small loop γ. Being unique, indication of the loop can be omitted in the notation: the corresponding monodromy operator will be denoted by Δ and the associated monodromy matrix by M, so that $\Delta X(t) = X(t)M$.

The notion of gauge equivalence can be localized in several possible ways.

Definition 21.2. Two singularities Ω, Ω' are (locally) *holomorphically equivalent*, if there exists a holomorphic invertible germ of a conjugacy matrix $H(t)$ such that the identity (20.5) holds locally in a neighborhood of the singular point.

These two singularities are (locally) *meromorphically equivalent*, if the germ of the conjugacy matrix $H(t)$ can be chosen only meromorphic and not identically degenerate (any such meromorphic matrix is automatically meromorphically invertible).

Example 21.3. If $t_0 = 0$ is a nonsingular point for a Pfaffian matrix Ω, then the latter is holomorphically equivalent to the trivial (identically zero) form $\Omega' \equiv 0$: it is sufficient to take $H(t) = X^{-1}(t)$, where $X(t)$ is a fundamental matrix solution of (20.4).

Example 21.4. Holomorphically equivalent systems must have the same Poincaré rank. However, meromorphic equivalence can create (resp., eliminate) poles or otherwise change their Poincaré rank. For an instance, the meromorphic transformation $H(t) = \operatorname{diag}\{t^{d_1}, \dots, t^{d_n}\}$, where $d_i \in \mathbb{Z}$ are integer numbers, conjugates the trivial system with $\Omega' = 0$ with the system having a simple pole at the origin,

$$\Omega = dH \cdot H^{-1} + H\Omega' H^{-1} = D\, t^{-1}\, dt, \qquad D = \operatorname{diag}\{d_1, \dots, d_n\}.$$

21.2. Regular singularities. A pole of an analytic function $f(t)$ can be described as an isolated singular point at which the absolute value $|f(t)|$ grows at most polynomially in $|t|^{-1}$ (assuming the singular point at the origin). This moderate growth condition ensures numerous important properties, the most important of them being finiteness of the number of Laurent terms for f. A parallel notion can be defined for singularities of linear systems, but a special care has to be exercised because of the multivaluedness of their solutions.

Definition 21.5. A vector or matrix function $X(t)$, eventually ramified at the origin, is said to be of *moderate growth* there if its norm grows at most polynomially in $|t|^{-1}$ as t tends to the origin along any ray,

$$\|X(t)\| \leqslant C|t|^{-d}, \qquad \text{as } |t| \to 0^+,\ \operatorname{Arg} t = \phi = \text{const}, \tag{21.1}$$

for some finite d and C (which a priori may depend on ϕ).

Definition 21.6. A singular point t_0 of a linear system is called *regular*, if any fundamental matrix solution $X(t)$ of the system has moderate growth at this point.

The moderate growth condition may be postulated not for rays but for all sectors with opening less than 2π: the result will be the same.

Remark 21.7. This terminology is counter-intuitive, since "regular" does not mean "nonsingular". However, it is too firmly established to replace the adjective "regular" by "tame" or "moderate" which would be less confusing.

Lemma 21.8. *For a regular singularity, the inverse $X^{-1}(t)$ of any fundamental solution also grows moderately.*

Proof. From the monodromy property, the determinant $h(t) = \det X(t)$ of any solution, is ramified over the origin:

$$\Delta h(t) = \mu h(t), \qquad \mu = \det M \in \mathbb{C}^*.$$

The function $t^{-\lambda}h(t)$, $\lambda = (2\pi i)^{-1}\ln\mu$, is therefore single-valued and growing no faster than polynomially as $t \to 0$. Hence it must have a zero of some finite order k so that

$$|\det X(t)| \geqslant C|t|^{k+1} \qquad \text{as } |t| \to 0,\ \operatorname{Arg} t = \text{const},\ C \neq 0.$$

Now the formula expressing the inverse X^{-1} as $(\det X)^{-1}$ times the adjugate matrix formed by all $(n-1)\times(n-1)$-minors of $X(t)$, shows that $\|X^{-1}(t)\|$ also grows moderately. □

Lemma 21.9. *If the homogeneous linear system* (20.4) *is regular at the origin and $b(t)$ a vector function of moderate growth at $t = 0$, then solutions of the nonhomogeneous system $\dot{x} = A(t)x + b(t)$ also have moderate growth.*

Proof. This follows from the explicit formula (20.7). □

Meromorphic classification of regular singularities is very simple.

Theorem 21.10 (meromorphic classification of regular singularities)**.** *Any regular singularity is meromorphically equivalent to a suitable Euler system.*

Proof. Let M be the monodromy matrix for a fundamental solution $X(t)$ of a regular singularity. As was already observed, for any nondegenerate matrix $M \in \mathrm{GL}(n, \mathbb{C})$ there always exists an Euler system (20.10) with $A = \frac{1}{2\pi i}\ln M$ such that its fundamental solution $Y(t) = t^A$ has the monodromy equal to M.

The matrix quotient $H(t) = X(t)\,t^{-A}$ is single-valued in $(\mathbb{C}, 0)$, since $\Delta H(t) = X(t)M \cdot \exp(-2\pi i A)\,t^{-A} = X(t)\,t^{-A} = H(t)$. As both $X(t)$ and $Y^{-1}(t) = t^{-A}$ grow at most polynomially along rays, so does $H(t)$. Considered as a conjugacy matrix, $H(t)$ realizes a local meromorphic equivalence $X(t) = H(t)Y(t)$ between the Euler system and the initial singularity. □

Remark 21.11. The theorem proves in fact that two regular singularities are meromorphically equivalent if and only if their monodromies are conjugate. In particular, two Euler systems with residues A, A' are meromorphically equivalent if and only if $\exp 2\pi i A = \exp 2\pi i A'$; explicit formulas for the matrix exponent allow to translate this condition into the terms involving exponentials of eigenvalues and the structure of Jordan blocks of the residues.

For arbitrary regular singularities the classification problem reduces to computation of the monodromy matrices. We note in passing that the problem of detecting regularity is rather nontrivial in general (see §21.3).

Corollary 21.12. *Any solution of a linear system exhibiting regular singularity at the origin, can be represented as*

$$X(t) = H(t)\,t^A, \qquad H \in \mathrm{GL}(n, \mathcal{M}_0),\ A \in \mathrm{Mat}(n, \mathbb{C}). \quad \square \tag{21.2}$$

21.3. Fuchsian singularities. The problem of detecting regular singularities is in general very difficult: in particular, Example 21.4 shows that no necessary condition of regularity can be given in terms of the Poincaré rank. However,there exists a simple *sufficient condition* of regularity.

Definition 21.13. A singularity is called *Fuchsian*, if its Pfaffian matrix has a simple pole, i.e., if its Poincaré rank is equal to zero,

$$\Omega = (A_0 + A_1 t + \cdots)\, t^{-1} dt, \qquad A_0, A_1, \dots \in \mathrm{Mat}(n, \mathbb{C}).$$

The matrix coefficient A_0 before the term t^{-1} is called the *residue* of the Fuchsian singularity.

Remark 21.14. The residue matrix of a germ of a meromorphic 1-form Ω at the origin can be defined as the Cauchy matrix integral

$$\operatorname{res}_0 \Omega = \frac{1}{2\pi i} \oint_\gamma \Omega, \qquad \operatorname{res}_0 \Omega \in \mathrm{Mat}(n, \mathbb{C})$$

along a small positive loop γ around the origin. By the Cauchy integral formula, this coincides with the previous definition, but the integral is independent of the choice of the chart.

Theorem 21.15 (L. Sauvage (1886), see [**Har82**])**.** *Any Fuchsian singularity is regular.*

Proof. We start with the following observation. If $\Omega = A(t)\, dt$ is a Pfaffian form whose coefficients matrix is bounded in a *convex* domain $U \subset \mathbb{C}$, $\|A(t)\| \leqslant C$, then for any two points $t_0, t_1 \in U$ and any fundamental matrix solution $X(t)$ its growth between these points is explicitly bounded: by the Gronwall inequality (Lemma 20.9) applied to the restriction of Ω on the real segment $[t_0, t_1]$,

$$\|X(t_1)\| \leqslant \|X(t_0)\| \exp(C\,|t_1 - t_0|).$$

For a Fuchsian singularity $\Omega = A(t)\, t^{-1}\, dt$ with a holomorphic hence bounded matrix function $A(t)$ the ratio $A(t)/t$ is unbounded, but in the logarithmic chart $z = \ln t$ the Pfaffian matrix $\Omega = A(\exp z)\, dz$ has bounded coefficients in some shifted left half-plane. By the above observation, its solution $X(z)$ grows at most exponentially as $\operatorname{Re} z$ tends to $-\infty$ along any horizontal line $\operatorname{Im} z = \text{const}$, which corresponds to the polynomial growth in the initial chart t. □

Passing to the logarithmic chart allows to see how the residue matrix can be described through the limit of holonomy operators. The same loop γ_0 generating the monodromy group (the loop going full turn counterclockwise around the origin) can actually be “translated” to any base point, providing thus for the natural identification between the fundamental groups $\pi_1(\mathbb{C} \smallsetminus 0, t_i)$, for any $t_1 \neq t_2$. (This should not be necessarily the case were the fundamental group non-commutative). As a result, one can define the family of holonomy operators $\{F_{\gamma_0,t}\colon \{t\}\times\mathbb{C}^n \to \{t\}\times\mathbb{C}^n,\ t\in(\mathbb{C},0)\}$ for the same loop but different fibers $\{t\}\times\mathbb{C}^n$. These operators depend analytically on $t\neq 0$ are all nondegenerate.

Proposition 21.16. *For a system exhibiting a Fuchsian singularity at the origin with the residue matrix A_0, there is a uniform limit $F_0=\lim_{t\to 0} F_{\gamma,t}$. It satisfies the equality*

$$F_0 = \exp 2\pi i A_0. \tag{21.3}$$

Proof. In the logarithmic chart $z=\ln t$, the system becomes $2\pi i$-periodic with the coefficients matrix $A(z)=A_0+A_1\exp z+A_2\exp 2z+\cdots$ which exponentially fast converges to the constant matrix A_0 as $\operatorname{Re} z$ tends to $-\infty$. The holonomy operator F_t in this chart corresponds to the time-$2\pi i$-shift, the value $X(z+2\pi i)$ of a solution of the system with the initial condition $X(z)=E$, $z=\ln t$ (any branch can be chosen because of the periodicity) of the system). By the theorem on continuous dependence on the right hand side, the limit as $\operatorname{Re} z\to -\infty$ exists and can be computed using the limit equation $dX/dz = A_0 X$, which yields the formula (21.3). □

Remark 21.17. While all holonomy operators $F_{\gamma,t}$ for $t\neq 0$ are conjugate to each other in the group $\mathrm{GL}(n,\mathbb{C})$, the limit holonomy F_0 may well have a different Jordan structure. The question is discussed in details below, see Corollary 21.27.

In the next several subsections we establish a polynomial integrable normal form for the local holomorphic classification of Fuchsian systems and prove its integrability, computing explicitly the fundamental solution and the monodromy.

21.4. Formal classification of Fuchsian singularities. Resonances. The first step in the local holomorphic classification of Fuchsian singularities consists in studying *formal equivalence*. Two singularities are said being formally equivalent, if there exists a *formal gauge transformation* defined by a formal series

$$\widehat{H}(t) = H_0 + tH_1 + t^2H_2 + \cdots, \qquad H_i\in \mathrm{Mat}(n,\mathbb{C}),\ \det H_0\neq 0,$$

conjugating the corresponding systems in the sense that the identity (20.5) holds on the level of formal power series. No assumption on convergence of the series is made.

As was observed by V. I. Arnold, the formal classification of Fuchsian singularities of linear systems can be reduced to the formal classification of nonlinear vector fields. Indeed, consider the vector field (20.2) associated with the system of differential equations

$$\dot{x} = t^{-1}(A_0 + tA_1 + t^2 A_2 + \cdots)x,$$

at the Fuchsian singular point $t = 0$. This vector field in $(\mathbb{C}, 0) \times \mathbb{C}^n$ is *orbitally* equivalent to the analytic vector field in $(\mathbb{C}^{1+n}, 0)$, corresponding to the system of nonlinear ordinary differential equations

$$\begin{aligned} \dot{x} &= A_0 x + tA_1 x + \cdots, \\ \dot{t} &= t, \end{aligned} \tag{21.4}$$

having an isolated singular point at the origin $(0, 0)$ and the linearization matrix that is block diagonal with two blocks, A_0 of size $n \times n$ and the 1×1-block consisting of the single entry 1. For simplicity we will assume that the residue matrix A_0 is in the Jordan normal form; its eigenvalues are denoted $\lambda_1, \ldots, \lambda_n$.

By the Poincaré–Dulac theorem, after an appropriate formal transformation one can remove from the system (21.4) all nonresonant terms. Characteristically for a system *linear* in the variables $x = (x_1, \ldots, x_n)$, only part of the resonances between eigenvalues $\{\lambda_1, \ldots, \lambda_n, 1\}$ (those corresponding to terms of higher order in t but linear in x) do matter.

Definition 21.18. The residue matrix A_0 of a Fuchsian singularity is called *nonresonant*, if the difference between any two eigenvalues in non-natural,

$$\lambda_i - \lambda_j \notin \mathbb{N}.$$

Otherwise any identity of the form

$$\lambda_i = \lambda_j + k, \qquad k \in \mathbb{N}, \tag{21.5}$$

is called *resonance*. The *resonant monomial* corresponding to the resonance (21.5), is the vector monomial

$$a_{ij,k}\, t^k x_j \frac{\partial}{\partial x_i}.$$

A formal matrix function

$$A(t) = t^{-1} \sum_{j=0}^{\infty} t^k A_k, \qquad A_k \in \operatorname{Mat}(n, \mathbb{C})$$

is called a (preliminary) *resonant normal form*, if the residue matrix A_0 is in the Jordan normal form with the eigenvalues $\lambda_1, \ldots, \lambda_n$ and the (i, j)th matrix entry $a_{ij,k}$ of the matrix coefficient A_k is nonzero only when the resonance (21.5) occurs.

In other words, a linear system (20.4) is in the formal normal form, if its coefficient matrix $A(t) = t^{-1}B(t)$, where B is a matrix polynomial, has the following structure,

$$\dot{x} = t^{-1}B(t)\,x, \qquad B(t) = \left\| \sum_k b_{ij,k} t^k \right\|_{i,j=1}^{n}, \\ \mathbb{C} \ni b_{ij,k} \neq 0 \implies \lambda_i - \lambda_j = k \in \mathbb{Z}_+. \tag{21.6}$$

Note that any other integer combination of the eigenvalues, either of the form $1 = \sum k_j \lambda_j$ or of the form $\lambda_i = k_0 + \sum_j k_j \lambda_j$, with $\sum_1^n k_j > 1$, is not considered as resonance, since the corresponding nonlinear in x resonant terms do not occur in linear systems of the form (21.4).

In a certain sense the formal theory of Fuchsian singularities is parallel to the formal classification of nonlinear vector fields in the Poincaré domain. In particular, the resonant normal form is always polynomial (since only finitely many integer differences between eigenvalues are possible).

Theorem 21.19 (Poincaré–Dulac theorem for Fuchsian singularities)**.** *A Fuchsian singularity is formally equivalent to a system in the resonant normal form.*

In particular, a Fuchsian system with a nonresonant residue matrix is formally equivalent to an Euler system.

Proof. By the standard Poincaré–Dulac scheme, all nonresonant monomials can be eliminated from the system (21.4). As for the resonant monomials, only those corresponding to resonances of the form (21.5) are linear in x and hence could occur in (21.4).

The only remaining question is why the resulting formal transformation will be linear in x_i and preserving the t-coordinate identically. This can be seen by inspection of the Poincaré–Dulac method: the normalizing map is constructed as an infinite composition of polynomial maps, each preserving the t-coordinate and linear in the x-coordinates, since only monomials of such form may need to be eliminated on each step. □

An instructive exercise is to reproduce the same arguments in the linear settings from the very beginning. To remove nonresonant terms of order $k-1$ from the Fuchsian system whose matrix $A(t) = t^{-1}\sum_{j\geqslant 0} t^j A_j$ has all lower order terms already normalized, consider a gauge equivalence with the conjugacy matrix $H(t) = E + t^k H_k$, whose inverse is $H^{-1}(t) = E - t^k H_k + \cdots$. The transformed system will have the terms of order $(k-1)$ as follows,

$$\begin{aligned} A'(t) &= kt^{k-1}H_k + t^{-1}(E + t^k H_k)A(t)(E - t^k H_k + \cdots) \\ &= A(t) + t^{k-1}(kH_k + H_k A_0 - A_0 H_k) + \cdots . \end{aligned}$$

This computation shows that all matrix coefficients $A'_0, \dots, A'_{k-1}$ of $A'(t)$ will remain the same as the matrix coefficients of $A(t)$, while the last matrix coefficient A'_k can be modified by subtracting (or adding) any matrix B representable as $kH + [H, A_0]$ for some $H \in \operatorname{Mat}(n, \mathbb{C})$. The image of the linear operator $k + [\cdot, A_0]$ involving the commutator with a Jordan normal form, was already computed: it is a linear subspace in the space of square matrices $\{b_{ij}\}$, defined by the linear equations $b_{ij} = 0$ for all pairs i, j such that $\lambda_i - \lambda_j = k$. In other words, all nonresonant matrices belong to the image, hence A'_k can be brought into the resonant normal form and the process continues further by induction in k.

back ref.: lemma on image of commutator with a Jordan normal form

21.5. Holomorphic classification of Fuchsian singularities. Convergence of formal normalizing transformations for arbitrary nonlinear vector fields can be a rather delicate issue. However, for Fuchsian systems the answer is very simple.

Theorem 21.20 (holomorphic classification of Fuchsian singularities)**.** *Any formal gauge transformation conjugating two Fuchsian singularities, always converges.*

In particular, any Fuchsian singularity is locally holomorphically equivalent to a polynomial Fuchsian system in the resonant normal form (21.6)*. A nonresonant Fuchsian system is holomorphically equivalent to an Euler system.*

The proof of this result can be obtained by several arguments. First, one can modify the proof of the Poincaré normalization theorem to show that the series converges: this is possible since all nonzero "small denominators" $\lambda_i - \lambda_j - k$ are bounded away from zero, exactly like in the Poincaré domain.

back ref.: to be inserted later

An alternative proof requires the following lemma concerning convergence of formally meromorphic solutions of Fuchsian systems. By definition, a formally meromorphic solution of a linear system (20.1) is a formal vector Laurent series

$$x(t) = \sum_{t=-d}^{+\infty} t^k x_k, \qquad x_{-d}, \dots, x_0, x_1, \dots \in \mathbb{C}^n, \tag{21.7}$$

satisfying formally the equation (20.1).

Lemma 21.21. *Any formal meromorphic solution of a regular system is convergent and hence truly meromorphic.*

Proof. The property of having only convergent formally meromorphic solutions, is obviously invariant by (truly) meromorphic equivalence of linear systems. As any regular system is meromorphically equivalent to an Euler system, the assertion of the Lemma is sufficient to prove only in this particular case.

For an Euler system $t\dot{x} = Ax$, $A \in \operatorname{Mat}(n, \mathbb{C})$, any formal solution (21.7) after substitution gives an infinite number of conditions

$$kx_k = Ax_k, \qquad k = -d, \dots, 0, 1, \dots.$$

Each of these conditions means that the vector coefficient x_k must be either zero or an eigenvector of A with the eigenvalue $k \in \mathbb{Z}$. But as soon as $|k|$ exceeds the spectral radius of A, the second possibility becomes impossible and hence all formal meromorphic solutions of the Euler system must be Laurent (vector) polynomials, thus converging. □

Proof of Theorem 21.20. Let $H(t)$ be a formal matrix Taylor series conjugating two Fuchsian singularities $\Omega_i = A_i(t)\, t^{-1} dt$, $i = 1, 2$. By (20.5), it means that

$$t^{-1} A_2 = \dot{H} \cdot H^{-1} + t^{-1} H A_1 H^{-1},$$

implying the "matrix differential equation" for the matrix function $H(t)$,

$$t\dot{H} = A_1 H - H A_2.$$

This is *not* the equation in the form (20.4) with respect to the unknown matrix function H, since both left and right matrix multiplication occurs in the right hand side of this equation. However, it is still a *system* of n^2 linear ordinary differential equations with respect to all n^2 *entries* of the matrix H. The coefficients of this large $(n^2 \times n^2)$-system are picked from among the entries of $t^{-1} A_i(t)$ and hence exhibit at most a simple pole at the origin.

All this means that $H(t)$ is a formal vector solution to a Fuchsian system of order n^2. By Lemma 21.21, it converges. □

21.6. Integrability of the normal form. Results of the previous section show that when studying Fuchsian singularities up to holomorphic equivalence, it is sufficient to deal only with resonant normal forms. We show that these normal forms are *integrable* and compute explicitly some of their characteristics. The integrability is obvious in the nonresonant case. We show how the normal form can be integrated in the presence of resonances. The key fact is that all resonant eigenvalues and hence the corresponding resonant monomials can be ordered so that the system becomes upper-triangular. Integrability of upper-triangular systems is a relatively simple observation. We incorporate this ordering in the definition of the normal form, slightly strengthening the previous definition.

In the resonant case some of the eigenvalues $\lambda_1, ..., \lambda_n$ of the residue matrix A_0 are related by the partial order. Two complex numbers can be compared if their difference is integer; this allows to write

$$\lambda_i \geqslant \lambda_j \quad \Longleftrightarrow \quad \lambda_i = \lambda_j + k, \qquad k \in \mathbb{Z}_+ \tag{21.8}$$

(the numbers λ_i, λ_j can still be complex). This partial order defines partition of the spectrum into disjoint *resonant groups* so that all eigenvalues in the same group are mutually comparable (i.e., either $\lambda_i \geqslant \lambda_j$, or $\lambda_i \leqslant \lambda_j$, or both inequalities hold which means that $\lambda_i = \lambda_j$). The direct sum of root subspaces of A_0 corresponding to the same resonant group of eigenvalues, is called the *resonant subspace* of A_0.

Definition 21.22. We say that a constant matrix $A_0 \in \operatorname{Mat}(n, \mathbb{C})$ in the Jordan normal form with the spectrum $\lambda_1, \dots, \lambda_n$ is in the *Levelt form*, if the coordinates are ordered in such a way that:

(1) A_0 has an *upper* triangular Jordan form,
(2) the eigenvalues are nonincreasing in the sense of the partial order (21.8):
$$\lambda_i > \lambda_j \implies i < j.$$

Definition 21.23. A Fuchsian singularity $\Omega = (A_0 + tA_1 + \cdots)\, t^{-1} dt$ is said to be in the *Poincaré–Dulac–Levelt resonant normal form*, or simply in the resonant normal form, if it is in the resonant normal form in the sense of Definition 21.18 (that is, only resonant monomials may occur) and its residue A_0 is in the Levelt form.

This definition is a particular case of the Poincaré–Dulac normal form as introduced earlier in §21.4 (in case we need to distinguish between the two definitions, the previous definition was introduced as *preliminary* resonant normal form).

Clearly, any matrix A_0 can be brought into a Levelt form by a suitable linear transformation (permutation of coordinates of a Jordan basis). This observation together with Theorems 21.19 and 21.20 asserts that any Fuchsian system can be brought into the Poincaré–Dulac–Levelt resonant normal form by a local holomorphic transformation.

The key property of the Poincaré–Dulac–Levelt form is its triangularity.

Lemma 21.24. *A Fuchsian system in the Poincaré–Dulac–Levelt resonant normal form has an upper triangular coefficient matrix $A(t)$. Moreover, in the expansion*

$$A(t) = t^{-1}(A_0 + tA_1 + \cdots + t^d A_d)$$

the matrix coefficients A_k satisfy the condition

$$t^{\Lambda} A_k\, t^{-\Lambda} = t^k A_k, \qquad \Lambda = \operatorname{diag}\{\lambda_1, \dots, \lambda_n\}, \tag{21.9}$$

where Λ is the diagonal part of A_0.

Proof. The term of degree $k \geqslant 1$ can appear on the (i,j)th place in the expansion of $A(t)$ only in case of the resonance $\lambda_i - \lambda_j = k$, that is, when $\lambda_i > \lambda_j$. According to the ordering of eigenvalues in the Levelt form, this may happen only when $i < j$, which means that the polynomial matrix $A(t)$ is in fact upper-triangular: $a_{ij}(t) \equiv 0$ for all $i > j$.

Moreover, all terms of degree k arranged into the matrix A_k of the above decomposition, correspond to the same value k of the differences $\lambda_i - \lambda_j$. Note that the conjugation by the matrix t^{Λ} multiplies the (i,j)th entry $a_{ij}(t)$ of the matrix A_k by $t^{\lambda_i - \lambda_j}$. This proves the assertion (21.9) concerning the matrices A_k with $k > 0$. Note that the Jordan matrix A_0 commutes with its diagonal part Λ, so the condition (21.9) holds also for $k = 0$. □

Corollary 21.25. *A Fuchsian singularity can be brought to the upper triangular polynomial form by a local holomorphic transformation.* □

A system in the normal form can be immediately explicitly integrated. By Lemma 21.24, one can always write

$$tA(t) = A_0 + tA_1 + \cdots + t^d A_d, \qquad \Lambda = \operatorname{diag}\{\lambda_1, \dots, \lambda_n\},$$

where the constant residue A_0 is in the Levelt form, and all matrix coefficients A_k are strictly upper-triangular. The degree d is the maximal integer difference between the eigenvalues (zero in the nonresonant case).

Consider the matrix $A(1) = tA(t)|_{t=1}$. The difference $I = A(1) - \Lambda$ is called a *characteristic matrix* of the corresponding Poincaré–Dulac–Levelt normal form, definition,

$$I = (A_0 - \Lambda) + A_1 + \cdots + A_d.$$

By construction I is a strictly upper-triangular and hence nilpotent matrix involving contributions from both off-diagonal terms of the Jordan form of the residue A_0 and also from the higher order terms of $A(t)$. Notice that in general Λ and I *do not commute.*

Lemma 21.26. *The system in the Poincaré–Dulac–Levelt normal form has a fundamental matrix solution*

$$X(t) = t^{\Lambda} t^{I}.$$

Proof. Direct computation yields

$$\begin{aligned} t\dot{X}X^{-1} &= \Lambda + t^{\Lambda} I t^{-\Lambda} \\ &= t^{\Lambda}(\Lambda + A_0 - \Lambda + A_1 + \cdots + A_d)t^{-\Lambda} \\ &= (\Lambda + A_0 - \Lambda) + tA_1 + \cdots + t^d A_d \\ &= A(t) \end{aligned}$$

by virtue of Lemma 21.24. □

If the matrices t^I and t^Λ were commuting, the monodromy of the system would be equal to the product $\exp(2\pi i\Lambda)\exp(2\pi iI)$ (in any order). It turns out that the formula still holds even in the noncommuting case.

Corollary 21.27. *The monodromy matrix of the Poincaré–Levelt normal form is the product of two commuting matrices,*

$$\exp(2\pi i\Lambda)\exp(2\pi iI) = \exp(2\pi iI)\exp(2\pi i\Lambda).$$

Proof. The exponent of the diagonal term

$$\exp(2\pi i\Lambda) = \operatorname{diag}\{\exp 2\pi i\lambda_1, \dots, \exp 2\pi i\lambda_n\}$$

is a *scalar* matrix on each resonant subspace of A, because all eigenvalues corresponding to this subspace have integer differences. Hence on each resonant subspace $\exp(2\pi i\Lambda)$ commutes with I, t^I and $\exp(2\pi iI)$, so that for the monodromy operator Δ around the singularity,

$$\Delta X(t) = t^\Lambda \exp(2\pi i\Lambda)\, t^I \exp(2\pi iI) = X(t)M, \quad M = \exp(2\pi i\Lambda)\exp(2\pi iI).$$

This completes the computation. □

Different Poincaré–Dulac–Levelt normal forms may still be holomorphically equivalent to each other. The problem of *complete* holomorphic classification, including recognition of pairwise nonequivalent normal forms, was only very recently reduced to a purely algebraic problem of classification of upper-triangular matrices by the Heisenberg group.

Recall that the characteristic matrix for a system in the Poincaé–Dulac–Levelt normal form is the difference $I = A(1) - \Lambda$, where Λ is the diagonal part of the residue.

Theorem 21.28 (Complete holomorphic classification of Fuchsian singularities, V. Kleptsyn and B. Rabinovich (1995))**.** *Two different systems in the Poincaré–Dulac–Levelt normal form with diagonal residue matrices are holomorphically equivalent if and only if their characteristic matrices are conjugated by a unipotent upper triangular matrix.*

A similar result holds for an arbitrary residue matrix. The proof of this result, though relatively simple, goes beyond the scope of this book.

22. Analytic and rational matrix functions. Matrix factorization theorems

In this section we collect necessary results on factorization of holomorphic matrix functions (both local and global). These results admit a natural interpretation in terms of analytic vector bundles. Basic notions of the corresponding geometric theory are briefly recalled below in §25.

22.1. Matrix cocycles, equivalence, solvability. Let $\mathfrak{U} = \{U_i\}_{i=1}^m$ be a finite open covering of the Riemann surface T. Throughout this section we will always assume that the domains U_i are connected and simply connected, and their finite unions and intersections are bounded by finitely many smooth arcs. When T is an open disk, the complex line $\mathbb{C}$ or even the Riemann sphere $\mathbb{C}P^1 = \mathbb{C} \cup \{\infty\}$, we identify U_i with subdomains of $\mathbb{C}$ and refer to them as *charts* equipped with the corresponding complex coordinate function $t \in \mathbb{C}$.

For a *closed* subset $K \subset T$ denote by $\mathrm{GL}(n, \mathcal{O}(K))$ the space of matrix functions of size $n \times n$, holomorphic on some neighborhood of K in T and holomorphically invertible there.

Definition 22.1. A *holomorphic matrix cochain* inscribed in the covering $\mathfrak{U}$ is a collection $\mathcal{H} = \{H_i(t)\}$ of matrix functions $H_i \in \mathrm{GL}(n, \mathcal{O}(\overline{U}_i))$, holomorphic and holomorphically invertible in the closure of the respective domains U_i.

Definition 22.2. A holomorphic *matrix cocycle* inscribed in the covering $\mathfrak{U}$ is a collection of holomorphic matrix functions $\mathcal{F} = \{F_{ij}(t)\}$, $F_{ij} \in \mathrm{GL}(n, \mathcal{O}(\overline{U}_{ij}))$, defined on the closure of all nonempty pairwise intersections $U_{ij} = U_i \cap U_j$, holomorphically invertible there and satisfying the *cocyclic identities*

$$\begin{aligned} F_{ij}(t)F_{ji}(t) &= E, \qquad & t \in U_{ij} = U_i \cap U_j, \\ F_{ij}(t)F_{jk}(t)F_{ki}(t) &= E, \qquad & t \in U_{ijk} = U_i \cap U_j \cap U_k. \end{aligned} \tag{22.1}$$

Definition 22.3. Two cocycles $\mathcal{F} = \{F_{ij}(t)\}$, $\mathcal{F}' = \{F'_{ij}(t)\}$ inscribed in the same covering are *holomorphically equivalent*, if there exists a matrix cochain $\mathcal{H} = \{H_i(t)\}$, also inscribed in the same covering, which is *conjugating* them in the following sense: on any nonempty intersection $U_{ij} \neq \varnothing$

$$F_{ij}(t)H_j(t) = H_i(t)F'_{ij}(t). \tag{22.2}$$

If $\mathcal{H} = \{H_i\}$ conjugates $\mathcal{F}$ with $\mathcal{F}'$, then the cochain $\mathcal{H}^{-1} = \{H_i^{-1}\}$ conjugates $\mathcal{F}'$ with $\mathcal{F}$, so that the holomorphic equivalence is symmetric (and obviously reflexive and transitive).

Definition 22.4. A cocycle $\mathcal{F}$ is called *holomorphically solvable*, if it is holomorphically equivalent to the *trivial cocycle* $\mathcal{E}$ having all matrices identical, $F_{ij}(t) \equiv E$.

Resolving the cocycle $\mathcal{F}$ means constructing the *matrix factorization*,

$$F_{ij}(t) = H_i(t)H_j^{-1}(t), \qquad t \in U_{ij} = U_i \cap U_j, \tag{22.3}$$

with holomorphic invertible matrix factors H_i, H_j defined in domains U_i, U_j *larger* than the domain of their ratio F_{ij}.

The problem of resolving and classification of cocycles naturally appears in many situations when global objects (equations, solutions *etc.*) are constructed by piecing together local objects. A typical example is the proof of the global existence theorem (Theorem 20.7), where the matrix cocycle $\{C_{ij}\}$ of *locally constant* matrix functions inscribed in the linearly ordered covering $\mathfrak{U}$, was solved also in the class of locally constant matrices. However, the main application of the results of this section is the Riemann–Hilbert problem discussed in §23.

Remark 22.5. The case $n = 1$ is also nontrivial though considerably simpler than the general case with $n > 1$, because of its commutativity.

In parallel with holomorphic cochains, cocycles and equivalence, their relaxed *meromorphic* counterparts can be defined. Actually, meromorphic cocycles will never be used, though meromorphic equivalence of holomorphic cocycles is an important tool.

Definition 22.6. A *meromorphic* matrix cochain inscribed in a covering $\mathfrak{U} = \{U_i\}$, is a collection of meromorphic matrix functions $H_i(t)$, each of them defined in some neighborhood of the closure of the respective domain U_i (to ensure meromorphy also on the boundary) and not identically degenerate, $\det H_i(t) \not\equiv 0$.

Two holomorphic matrix cocycles $\mathcal{F}, \mathcal{F}'$ inscribed in the same covering, are *meromorphically equivalent* if there exists a meromorphic cochain $\mathcal{H} = \{H_i(t)\}$ conjugating them in the sense (22.2).

22.2. Meromorphic solvability of cocycles. Cartan and Birkhoff–Grothendieck cocycles. The principal reason for introducing meromorphic equivalence of matrix cocycles is the following very general and difficult fundamental theorem. It is valid for any covering on any Riemann surface, and generalizes the theorem on existence of meromorphic functions on an arbitrary Riemann surface, see §25.

Theorem 22.7. *Any holomorphic matrix cocycle on a Riemann surface is meromorphically solvable* (*meromorphically equivalent to the trivial cocycle*).

We will neither need nor prove this result in full generality. Instead, we formulate two particular cases of Theorem 22.7 for two simplest types of cocycles inscribed in coverings with only two charts. This will be sufficient for all our purposes; a complete proof can be found in [**For91**].

In the simplest case the covering consists of only two charts $U_0, U_1 \subseteq \mathbb{C}P^1$ with a nonempty intersection. A cocycle inscribed in such covering consists of a single matrix function $F = F_{01}(t) = F_{10}^{-1}(t)$ holomorphic and holomorphically invertible in the intersection $U_{01} = U_0 \cap U_1$ up to the boundary.

Definition 22.8. A covering of a domain $T = U_0 \cup U_1 \subset \mathbb{C}P^1$ with two charts U_0, U_1 is called *Cartan covering*, if these two charts have connected and simply connected intersection $U_{01} = U_0 \cap U_1$.

A *Cartan cocycle* is a holomorphic matrix cocycle inscribed in a Cartan covering.

Remark 22.9. Note if the entire Riemann sphere $\mathbb{C}P^1$ is covered by two open disks, then their intersection must be an annulus that is not simply connected. Thus without loss of generality we may assume that in the definition of the Cartan cocycle, $T = U_0 \cup U_1 \subseteq \mathbb{C}$, that us, the point $t = \infty$ *does not* belong to T.

Since the Riemann sphere $\mathbb{C}P^1$ cannot be covered by two charts with a connected simply connected intersection, we need to consider topologically different cases. Consider the covering of $\mathbb{C}P^1$ by two circular disks, $\mathbb{C}P^1 = U_0 \cup U_1$,

$$U_0 = \{|t| < r_0\}, \quad U_1 = \{|t| > r_1\}, \qquad r_1 < 1 < r_0, \tag{22.4}$$

their intersection being the open *annulus*

$$U_{01} = U = \{r_1 < |t| < r_0\}.$$

Definition 22.10. A *Birkhoff–Grothendieck cocycle* is a holomorphic cocycle inscribed in the covering (22.4). The covering itself is called a *Birkhoff–Grothendieck covering*.

In other words, a Birkhoff–Grothendieck cocycle consists of a single holomorphically invertible matrix function $F(t) = F_{01} = F_{10}^{-1} \in \mathrm{GL}(n, \mathcal{O}(\overline{U}_{01}))$ on the closed annulus.

The two particular cases of Theorem 22.7 that will be proved in the Appendix to this section, concern cocycles inscribed in the Cartan and Birkhoff–Grothendieck coverings. Both assertions belong to the realm of analytic matrix functions theory.

Theorem 22.11. *Any Cartan cocycle is meromorphically solvable.*

Theorem 22.12. *Any Birkhoff–Grothendieck cocycle is meromorphically solvable.*

In the remaining part of §22 we will derive from these theorems several results on *holomorphic* solvability and equivalence of cocycles. All are obtained by elementary row and column operations with matrix functions. Theorems 22.11 and 22.12 themselves are proved in the appendix to this section.

Recall that an elementary operation on rows of a matrix is one of the following three:

(1) transposition of two rows of a matrix,

(2) adding to one of the rows a linear combination of other rows,

(3) multiplication of a row by a nonzero scalar.

Each elementary operation can be achieved by the left multiplication of the matrix by an appropriate *elementary matrix*. Except for the third type, the determinant of the corresponding elementary matrix is 1. Three parallel elementary operations on columns of a matrix can be achieved by an appropriate right multiplication.

In an obvious way, these elementary operations can be generalized for meromorphic matrix functions: transformations of the second type consist in adding to a row of a matrix function a linear combination of other rows with meromorphic coefficients. Transformations of the third type consist of multiplication of a row by a nonzero meromorphic function. Elementary operations on columns of meromorphic matrix functions are also self-explanatory.

22.3. Cartan lemma.

Theorem 22.13 (Cartan factorization lemma)**.** *Any Cartan cocycle is holomorphically solvable.*

In other words, if the intersection $U = U_0 \cap U_1$ is connected and simply connected, then any matrix function $F = F_{01} \in \mathrm{GL}(n, \mathcal{O}(\overline{U}))$, holomorphic and holomorphically invertible on the closure $\overline{U}$, can be factorized as

$$F_{01}(t) = H_0(t)H_1^{-1}(t),$$

with $H_i(t)$ holomorphic and holomorphically invertible on $\overline{U}_i$, $i = 0, 1$.

Proof. By Theorem 22.11, any Cartan cocycle can be resolved by a meromorphic cochain: there exist two meromorphic functions $M_0(t), M_1(t)$ such that $F(t)M_1(t) = M_0(t)$. The proof of Theorem 22.13 consists in a series of modifications transforming this meromorphic cochain to a holomorphic cochain.

First, the meromorphic cochain can be modified so that all matrix functions $M_i(t)$ become holomorphic in the corresponding domains $U_i \subseteq \mathbb{C}$. To that end, all functions $M_i(t)$ should be multiplied by a suitable scalar power $(t - t_k)^{\nu_k}$, $\nu_k \in \mathbb{N}$, for each finite pole t_k of order ν_k. Clearly, the determinants of the holomorphic matrices $H_i(t)$ obtained by such multiplication, remain not identically vanishing, though they still may have isolated zeros of finite order.

In order to get rid of these zeros, we will further multiply H_i simultaneously by rational matrix functions from the right (this operation obviously will preserve the identity $FH_1 = H_0$). If t_* is an isolated root of, say, $\det H_1(t)$, then one of the columns of the matrix $H_1(t_*)$ is a linear combination of other columns, so that after the right multiplication by an appropriate constant matrix C one of the columns of $H_1(t_*)$ becomes zero. Then all entries from this column of the matrix *function* $H_1(t)C$ have the common factor $(t - t_*)$. After the right multiplication by the rational matrix function $R(t) = \operatorname{diag}\{1, \dots, (t - t_*)^{-1}, \dots, 1\}$, the modified matrix function $H_1(t)CR(t) = H_1'(t)$ remains holomorphic at t_*, and so apparently is $H_0'(t) = F(t)H_1'(t) = H_0(t)CR(t)$.

The total number of zeros of $\det H_i'(t)$, counted with multiplicities in $\mathbb{C}$, will decrease by 1 compared to that of $\det H_i(t)$. After a finite number of such steps we will get rid of all zeros of the determinant. The resulting cochain will resolve the cocycle, since by Remark 22.9, both U_0 and U_1 belong to the finite part $\mathbb{C}$. □

22.4. Global solvability of cocycles on the plane. The same proof actually would apply to *any* cocycle inscribed in any covering $\mathfrak{U}$ with $T = \bigcup_i U_i \subset \mathbb{C}$, provided it is meromorphically solvable as asserted by Theorem 22.7. However, since this theorem is not proved here in full generality, we derive solvability of certain types of cocycles directly from the Cartan factorization lemma (Theorem 22.13).

For an arbitrary covering with more than two domains, the "pairwise solvability" established by the Cartan lemma, does not in general guarantee the global holomorphic solvability. Obstructions may arise because of the global topology of the Riemann surface T. However, there can be formulated a simple sufficient condition on the covering, guaranteeing that any cocycle inscribed in this covering is solvable.

Theorem 22.14. *Suppose that a finite covering $\mathfrak{U} = \{U_i\}$, $i = 1, \dots, m$, satisfies the following topological triviality condition: for any k between 1 and $m - 1$, the intersection*

$$(U_1 \cup \dots \cup U_k) \cap U_{k+1}, \qquad k = 1, 2, \dots, m-1, \tag{22.5}$$

is connected and simply connected.

Then any matrix cocycle $\mathcal{F}$ inscribed in this covering, is solvable.

Example 22.15. The condition (22.5) is not very artificial. For example, if a circular disk $\{|t| < 1\}$ is subdivided by a number of rays into sectors $S_i = \{\alpha_i \leqslant \operatorname{Arg} t \leqslant \alpha_{i+1}\}$, $S_m = \{\alpha_{m-1} \leqslant \operatorname{Arg} t \leqslant \alpha_1 + 2\pi\}$ with $0 \leqslant \alpha_1 < \dots < \alpha_{m-1} < 2\pi$, then their (convex) ε-neighborhoods U_i, $i = 1, \dots, m$, form a covering of the disk satisfying (22.5).

Another example is that of a rectangle $\{0 < \operatorname{Re} t < k,\ 0 < \operatorname{Im} t < \ell\}$ covered by convex ε-neighborhoods of the closed square cells $\{i \leqslant \operatorname{Re} t \leqslant$

$i+1,\ j \leqslant \operatorname{Im} t \leqslant j+1\}$. In this case the condition (22.5) holds if the cells are ordered lexicographically.

Proof of the Theorem. When $m = 2$, the theorem coincides with the Cartan lemma. The general case follows from the following inductive construction reducing the number of charts in the covering.

Consider the first two sets U_1, U_2 and the matrix $F_{12} \in \mathcal{F}$ on their intersection U_{12}. By the Cartan lemma, F_{12} can be factorized,

$$H_2' = F_{21} H_1' \tag{22.6}$$

for some two holomorphically invertible matrices H_1', H_2' defined in U_1, U_2 respectively.

We construct a new covering $\mathfrak{U}' = \{U_0, U_3, \dots, U_m\}$ replacing U_1 and U_2 by their union $U_0 = U_1 \cup U_2$. On this covering we define the matrix cocycle $\mathcal{F}' = \{F_{ij}'\}$ as follows: for $i, j \geqslant 3$ we leave $F_{ij}' = F_{ij}$, whereas for $i = 0$ the function $F_{j0} = F_{0j}^{-1}$ are defined on $U_{0j} = (U_1 \cup U_2) \cap U_j = U_{1j} \cup U_{2j}$ as follows,

$$F_{j0}' = \begin{cases} F_{j1} H_1' & \text{on} \quad U_{1j}, \\ F_{j2} H_2' & \text{on} \quad U_{2j}, \end{cases} \qquad F_{0j}' = (F_{j0}')^{-1}, \qquad j = 3, \dots, m.$$

This definition is self-consistent, since on the triple intersection $U_{12j} = U_{1j} \cap U_{2j}$ the two expressions for F_{j0}' coincide,

$$F_{j1} H_1' = F_{j2} F_{21} H_1' = F_{j2} H_2',$$

by the cocyclic identity (22.1) and the characteristic property (22.6).

Assume that $\mathcal{H}' = \{H_0', H_3', \dots, H_m'\}$ is a solution of the cocycle $\mathcal{F}'$. Then the cochain $\mathcal{H} = \{H_i\}$ with

$$H_1 = H_1' H_0', \quad H_2 = H_2' H_0', \qquad H_j = H_j', \quad j = 3, \dots, m,$$

is a solution of $\mathcal{F}$. Indeed, for all $j \geqslant 3$

$$F_{j1} H_1 = F_{j1} H_1' H_0' = F_{j0}' H_0' = H_j' = H_j,$$

and the same for $F_{j2} H_2$. To prove the remaining identity $F_{21} H_1 = H_2$, both sides of (22.6) should be multiplied by H_0' from the right.

Thus the problem of resolving of the initial cocycle $\mathcal{F}$ is reduced to resolving the auxiliary cocycle $\mathcal{F}'$ inscribed in the covering with $m-1$ charts $U_0, U_3, \dots, U_m$. Clearly, the property (22.5) remains valid if the first two domains U_1, U_2 are replaced by their union. This allows to prove the theorem by induction in the number of charts m. □

22.5. Birkhoff–Grothendieck theorem. Unlike Cartan cocycles, meromorphic solvability of Birkhoff–Grothendieck cocycles does not imply their holomorphic solvability. The obstruction can be represented as a tuple of n integer numbers.

The scalar case with $n = 1$ can be completely studied by elementary methods. Let $d \in \mathbb{Z}$ be an integer number.

Definition 22.16. The *standard* one-dimensional Birkhoff–Grothendieck cocycle $\mathcal{F}^d$ is defined by the function $f(t) = t^d \in \mathrm{GL}(1, \mathcal{O}(\overline{U}_{01}))$ in the annulus.

Proposition 22.17. *Every one-dimensional holomorphic Birkhoff–Grothendieck cocycle is holomorphically equivalent to one of the standard cocycles $\mathcal{F}^d$. Cocycles with different values of d are not holomorphically equivalent to each other.*

Proof. Denote by d the integer number equal to the increment of argument of $F(t)$ along the positively (counterclockwise) oriented mid-circle $\{|t| = 1\}$, divided by 2π.

A cocycle $f(t)$ with $d = 0$ is solvable. Indeed, in this case one can choose an analytic branch of the logarithm $g(t) = \ln f(t)$ holomorphic in the annulus U. The Laurent series for g, converging in U, can be split into two parts,

$$g(t) = g_1(t) - g_0(t),$$

where g_0 contains only terms with nonnegative degrees, while g_1 is the sum of all terms having negative degrees in t. As a consequence, the functions $g_i(t)$ are holomorphic in the respective domains U_i. The cochain $\{h_0, h_1\}$, $h_i = \exp g_i$, solves the cocycle f:

$$h_0(t)h_1^{-1}(t) = \exp\bigl(g_0(t) - g_1(t)\bigr) = \exp \ln f(t) = f(t).$$

An arbitrary cocycle $h'(t)$ inscribed in the Birkhoff–Grothendieck covering, can be represented as $h'(t) = t^d f(t)$ with an appropriate $d \in \mathbb{Z}$ and f having a well defined logarithm in U as above. Factorization of $F(t)$ yields the factorization of $f'(t)$,

$$f'(t) = t^d f(t) = h_0(t) t^d(t) h_1^{-1}(t)$$

which means that the cocycle $f'(t)$ is equivalent to t^d in the sense (22.2).

Two cocycles t^d and $t^{d'}$ with $d \neq d'$ cannot be equivalent, since in this case one would have

$$t^{d'} = h_0(t)\, t^d h_1^{-1}(t)$$

with two nonvanishing holomorphic functions h_0 and h_1 in U_0 and U_1 respectively. The variation of argument of both $h_0(t)$ and $h_1(t)$ along the mid-circle $\{|t| = 1\}$ of U must be zero by the argument principle (h_i has neither zeros, nor poles inside U_i). But then the variation of argument of $t^{d-d'}$ along this circle must also be zero, which is possible only if $d = d'$. □

This proposition completes the study of one-dimensional cocycles. The general multidimensional case admits a similar answer. Denote by D an ordered collection of n integer numbers considered as a diagonal integer matrix $D = \operatorname{diag}\{d_1, \dots, d_n\}$.

Definition 22.18. The *standard* multidimensional Birkhoff–Grothendieck cocycle $\mathcal{F}^D$ is the Birkhoff–Grothendieck cocycle of the form

$$F(t) = t^D, \qquad D = \operatorname{diag}\{d_1, \dots, d_n\}, \quad d_i \in \mathbb{Z}. \tag{22.7}$$

The integer numbers $d_1, \dots, d_n$ are called *partial indices*.

Theorem 22.19 (Birkhoff–Grothendieck theorem)**.** *Every holomorphic Birkhoff–Grothendieck cocycle is holomorphically equivalent to a standard cocycle $\mathcal{F}^D$ of the form* (22.7) *with an appropriate diagonal integer matrix $D = \operatorname{diag}(d_1, \dots, d_n)$.*

The collection $D = \{d_1, \dots, d_n\}$ of partial indices is a complete invariant of classification: *two cocycles of the form* (22.7) *are equivalent if and only if their collections of partial indices coincide modulo a permutation.*

In other words, any matrix function $F(t)$ in the annulus $U = U_{01}$ can be factored as

$$F(t) = H_0(t)\, t^D H_1(t),$$

where the matrix functions $H_0(t)$ and $H_1(t)$ are holomorphic and invertible in the disks $U_0 = \{|t| < r_0\}$ and $U_1 = \{|t| > r_1\} \subset \mathbb{C}P^1$ respectively.

Remark 22.20. Applying the Birkhoff–Grothendieck theorem to the inverse matrix $F^{-1}(t)$ and then inverting the result, one can construct the factorization of $F(t)$ with the inverted order of terms,

$$F(t) = H_1'(t)\, t^D H_0'(t), \tag{22.8}$$

with holomorphic invertible factors $H_i' \in \operatorname{GL}(n, \mathcal{O}(U_i))$.

22.6. “Small” cocycles on $\mathbb{C}P^1$. Sauvage lemma. An important particular (or rather limit) case of Theorem 22.19 can be achieved by elementary arguments.

Any *germ* of a matrix function $F(t)$, holomorphic and invertible at a *punctured* neighborhood of a point a on the Riemann sphere, can be considered as a cocycle inscribed in the covering of $\mathbb{C}P^1$ by two connected simply connected charts, one “large” $\mathbb{C}P^1 \smallsetminus \{a\} \simeq \mathbb{C}$ and one ”small”, an arbitrarily small neighborhood $(\mathbb{C}P^1, a)$ of the point. Their intersection will indeed be a small punctured neighborhood of a. Since all constructions are conformally invariant, we can assume without loss of generality that $a = \infty \in \mathbb{C}P^1$. Then

$$U_0 = \mathbb{C} \subset \mathbb{C}P^1, \quad U_1 = (\mathbb{C}P^1, \infty), \quad U = U_{01} = (\mathbb{C}P^1, \infty) \smallsetminus \{\infty\}. \tag{22.9}$$

The regularity condition imposed in the definition of holomorphic cocycle, needs to be modified (since holomorphy F, F^{-1} on the closure means in this particular case that the singularity $t = a$ is removable and $U_0 = \mathbb{C}P^1$). We will assume that $F(t)$ is only *meromorphic* at $a = \infty$, having a pole of finite order there.

The limit case of the Birkhoff–Grothendieck theorem for the covering (22.9), is known as the *Sauvage lemma*. Unlike the general case requiring reference to the fundamental Theorem 22.7, the Sauvage lemma can be proved by explicit construction of the conjugacy.

Lemma 22.21 (Sauvage lemma)**.** *The germ of a meromorphic matrix function $F(t)$ at $(\mathbb{C}P^1, \infty)$ considered as a cocycle on the covering* (22.9) *of the Riemann sphere, is holomorphically equivalent to a standard cocycle $\mathcal{F}^D$ with an appropriate diagonal integer matrix D.*

Note that though the cocycle $\mathcal{F} = \{F(t)\}$ is “small” ($F(t)$ is a meromorphic germ having a representative defined in an arbitrarily small punctured neighborhood of a point), the factorization problem is intrinsically global. Indeed, the conjugating cochain is inscribed into the covering (22.9) of the whole sphere $\mathbb{C}P^1$.

22.7. Monopoles. The cochain $\mathcal{H} = \{H_0, H_1\}$ realizing equivalence between a given meromorphic cocycle and its normal form t^D in the limit case considered in Lemma 22.21, is also rather particular. The matrix function $H_1(t)$ is reduced to a holomorphic invertible *germ* at $t = \infty$. On the contrary, the function $H_0(t)$ must be a matrix function, holomorphic and holomorphically invertible everywhere in $\mathbb{C}$ and having a finite order pole at infinity. By Liouville theorem, $H_0(t)$ is a matrix polynomial with the constant (nonzero) determinant. In more invariant terms, $H_0(t)$ is a particular case of the following class of matrix functions that will play an important role later.

Definition 22.22. A *monopole* is a rational matrix function on the Riemann sphere, holomorphic and holomorphically invertible everywhere except for one point.

If the singular point $a \in \mathbb{C}P^1$ of a monopole has to be explicitly mentioned, we will say about the monopole at a. The following simple observation serves as an important example of monopoles.

Lemma 22.23. *If* $D = \operatorname{diag}\{d_1, \dots, d_n\}$ *is a diagonal matrix with non-increasing integer entries* $d_1 \geqslant \cdots \geqslant d_n$ *and* $\Gamma(t)$ *a constant or polynomial* upper-triangular *matrix function, then the conjugated matrix* $t^D \Gamma(t)\, t^{-D}$ *will be again an upper-triangular matrix polynomial.*

Proof. After the conjugacy, every nonzero (i,j)th entry of $\Gamma(t)$ will be multiplied by $t^{d_i - d_j}$ which is a Taylor monomial for all $i \leqslant j$. □

In particular, if C is a constant upper-triangular matrix and D as above, then $t^D C\, t^{-D}$ is a monopole, since its determinant is a nonzero constant. Lemma 22.23 has a twin statement concerning *lower triangular* matrices conjugated by powers of diagonal matrices with *nondecreasing* entries.

Remark 22.24. All monopoles at a form a *monopole* group that is a proper subgroup of the group of meromorphic germs at a. This group acts on meromorphic germs of matrix functions by left multiplications and on singularities of linear systems by gauge transformations. In both cases we will say about *monopole equivalence* of the corresponding objects.

Monopoles are important as conjugacy matrices of global gauge transformations for linear systems on the sphere, which are holomorphic equivalences at all singular points but one. The global theory of regular systems on the Riemann sphere is largely a question of local classification of regular singularities by the monopole group. Notice that neither the monopole group nor the group of holomorphic invertible germs are subgroups of each other.

22.8. Proof of Sauvage lemma and Birkhoff–Grothendieck theorem. In terms of monopoles, the Sauvage lemma asserts that any meromorphic germ $F(t)$ at $(\mathbb{C}P^1, \infty)$ can be factored as

$$F(t) = \Gamma(t)\, t^D H(t), \tag{22.10}$$

where $H(t)$ is a holomorphic invertible germ at $(\mathbb{C}P^1, \infty)$, and $\Gamma(t)$ a monopole with the pole at $t = \infty$. As before, the order of terms in this factorization can be reversed.

Proof of Lemma 22.21.

1. If a *holomorphic* germ $H(t)$ at $(\mathbb{C}P^1, \infty)$ is degenerate at $t = \infty$, then there exists a *constant upper-triangular* matrix C and a holomorphic germ $H'(t)$ such that

$$CH(t) = t^{D'} H'(t), \qquad D' = \operatorname{diag}\{0, \dots, -1, \dots, 0\}. \tag{22.11}$$

Indeed, if $\det H(\infty) = 0$, then the rows of the *constant* matrix $M = H(\infty)$ must be linear dependent, in particular, some row of M must be equal to a linear combination of the subsequent (relatively lower) rows. In other words, there exists an *upper*-triangular constant matrix C with determinant

1, such that $CM = CH(\infty)$ has a zero row. But then this row of the matrix function $CH(t)$ is divisible t^{-1}, so that the matrix $H'(t) = t^{-D'}CH(t)$ is holomorphic at $t = \infty$.

Clearly, the order of zero of $\det H'(t)$ is strictly inferior (by one less) than the order of zero of $\det H(t)$:

$$\operatorname{ord}_\infty \det H'(t) = \operatorname{ord}_\infty \det H(t) - 1. \tag{22.12}$$

2. If D is an integer diagonal matrix $D = \operatorname{diag}\{d_1, \dots, d_n\}$ with non-increasing entries $d_1 \geqslant \cdots \geqslant d_n$, and $H(t)$ is holomorphic and degenerate at infinity, then the product $t^D H(t)$ is monopole equivalent to $t^{D+D'}H'(t)$ with D' and $H'(t)$ as above.

Indeed, by Step 1, there exists a constant upper-triangular matrix C such that $CH(t) = t^{D'}H'(t)$ with holomorphic $H'(t)$ satisfying (22.12). Consider the conjugacy of C by t^D, $\varGamma(t) = t^D C\, t^{-D}$. By Lemma 22.23, $\varGamma(t)$ is an upper-triangular monopole. Since D and D' commute,

$$\varGamma(t)\, t^D H(t) = t^D C\, t^{-D} \cdot t^D H = t^D CH = t^D t^{D'} H' = t^{D+D'} H'.$$

3. For an arbitrary diagonal matrix D one can find a *constant* permutation matrix $P \in \operatorname{GL}(n, \mathbb{C})$ (particular case of monopole) such that the diagonal entries of $D' = Pt^D P^{-1}$ will be monotonous as required on Step 2. This shows that the condition on the order of the diagonal entries d_i, imposed on Step 2, can be always achieved by a suitable monopole equivalence (left multiplication by P):

$$P\, t^D H = P\, t^D P^{-1} \cdot PH = t^{D'} H',$$

with a holomorphic H' degenerate at infinity together with H.

4. The proof of Sauvage lemma follows by simple induction. Any meromorphic germ $F(t)$ can be represented as $t^{D_1}H_1(t)$ with $H_1(t)$ holomorphic at infinity: it is sufficient to multiply $F(t)$ by a suitable (scalar) power of t. Since $\det F(t) \not\equiv 0$, the multiplicity of the root of $\det H_1(t)$ at $t = \infty$ is finite. The inductive application of the construction described above, allows to construct a sequence of monopole transformations reducing $F(t)$ to the form of a product of two terms, $t^{D_k}H_k(t)$ as above (diagonal and holomorphic at infinity respectively), with strictly decreasing orders of the roots $\operatorname{ord}_\infty \det H_k(t)$. After finitely many steps the holomorphic term $H_m(t)$ becomes nondegenerate at infinity, and the Sauvage lemma is proved. □

Proof of the Birkhoff–Grothendieck theorem.

The Birkhoff–Grothendieck theorem is an easy corollary to the results already obtained. As follows from Theorem 22.12, any Birkhoff–Grothendieck cocycle $\mathcal{F} = \{F_{01}(t)\}$ is meromorphically solvable. The procedure used in the proof of Cartan theorem, allows to modify the corresponding meromorphic cochain $\mathcal{H} = \{H_0, H_1\}$ so that:

(1) H_0 is holomorphic and holomorphically invertible everywhere in U_0, and

(2) H_1 is holomorphic and holomorphically invertible in $U_1 \smallsetminus \{\infty\}$.

The only remaining obstruction is an eventual pole of the meromorphic functions $H_1(t)$ or $H_1^{-1}(t)$ at $t = \infty$.

By the Sauvage lemma, $H_1^{-1}(t)$ can be represented as $\Gamma(t)\, t^D \widetilde{H}(t)$ with a polynomial and polynomially invertible (monopole) $\Gamma(t)$ and holomorphically invertible germ $\widetilde{H}(t)$ at $t = \infty$. The germ $\widetilde{H}(t) = t^{-D}\Gamma^{-1}H_1^{-1}$ actually extends on the entire domain U_1 as a holomorphically invertible matrix function, since all terms in the latter equality are holomorphically invertible in $U_1 \smallsetminus \{\infty\}$. Substituting this into the identity $F_{01}(t) = H_0(t)H_1^{-1}(t)$, we get

$$F_{01}(t) = H_0(t)\Gamma(t)\, t^D \widetilde{H}(t).$$

The holomorphic cochain $\{H_0\Gamma, \widetilde{H}^{-1}\}$ conjugates the initial Birkhoff–Grothendieck cocycle $\mathcal{F}$ with $\mathcal{F}^D$. □

22.9. Lemma on matrix permutation. We will need one more result on matrix factorization, which differs from Birkhoff–Grothendieck or Sauvage factorizations by reordering of terms.

By the Sauvage lemma, any meromorphic not identically degenerate matrix germ at $t = \infty$ is monopole equivalent to the product $t^D H(t)$, where $H(t)$ is holomorphic and invertible in a full neighborhood of infinity. The following result shows that the terms in this representation can be permuted.

Lemma 22.25. *Any matrix germ at $t = \infty$ of the form $F(t) = t^D H(t)$ with a holomorphically invertible factor $H(t)$ is monopole equivalent to a germ of the form $H'(t)\, t^{D'}$ with $H'(t)$ also holomorphic and invertible and D' a diagonal matrix with the same diagonal entries d_i, eventually in a permuted order.*

In other words, for any $D = \operatorname{diag}\{d_1, \dots, d_n\}$, $d_i \in \mathbb{Z}$, and any holomorphically invertible germ $H(t) \in \mathrm{GL}(n, \mathcal{O}_\infty)$ there exists a matrix function $\Gamma(t)$ which is a matrix polynomial in t with determinant 1, and $H'(t) \in \mathrm{GL}(n, \mathcal{O}_\infty)$ such that

$$\Gamma(t)\, t^D H(t) = H'(t)\, t^{D'}, \tag{22.13}$$

where D' is a diagonal matrix whose entries are obtained by permutation of the numbers d_i.

Proof of Lemma 22.25. We start by proving the Lemma in a simple particular case, and then reduce the general case to the former one by a series of suitable gauge transformations.

1. Consider first the case when the (constant) matrix $H(\infty)$ has all nonzero principal (upper-left) minors, while the diagonal matrix D is of the form $\begin{pmatrix} 0 & \\ & \nu E \end{pmatrix} = \operatorname{diag}\{0,\dots,0,\nu,\dots,\nu\}$, $\nu > 0$. This means that D is block diagonal with only two distinct eigenvalues and they are arranged in the ascending order. We show that in this case the meromorphic conjugated germ $R(t) = t^D H(t)\, t^{-D}$ is monopole equivalent to a holomorphic germ $H'(t)$ that is automatically nondegenerate at infinity. This is a particular case of the Lemma, when $D' = D$.

More precisely, we will show that in this case the monopole transformation can be chosen lower triangular with the block structure $\begin{pmatrix} E & 0 \\ * & E \end{pmatrix}$, so that the upper left blocks of $H'(t)$ and $H(t)$ are the same. Denoting the appropriate blocks of $H(t)$ as follows yields

$$H(t) = \begin{pmatrix} M(t) & N(t) \\ P(t) & Q(t) \end{pmatrix}, \qquad R(t) = t^D H(t)\, t^{-D} = \begin{pmatrix} M(t) & t^{-\nu} N(t) \\ t^{\nu} P(t) & Q(t) \end{pmatrix}.$$

The upper left block $M(t)$ is nondegenerate by assumption. The only elements that may have poles at infinity, are these of the lower right block $t^\nu P$. We show how these poles can be removed by lower triangular monopole transformations.

The principal Laurent part of the matrix $t^\nu P(t)$ can be expanded as

$$t^\nu P(t) = t^\nu P_\nu + t^{\nu-1} P_{\nu-1} + \cdots + t\, P_1 + P_0,$$

with constant rectangular matrices P_i. Linear combinations of rows of the nondegenerate matrix $M(0)$ generate any row of the appropriate length, in particular, any row of the constant matrix P_ν. Subtracting these combinations with the rational factor t^ν allows to eliminate from $t^\nu P(t)$ all terms with poles of order ν at infinity. Being an elementary row operation, this corresponds to the left multiplication by an appropriate lower triangular monopole matrix $\Gamma^\nu(t)$, polynomial in t and with determinant 1. Since elements of the upper right block of $R(t)$ were all divisible by $t^{-\nu}$, the lower right block of $R(t)$ will remain holomorphic after multiplication by $\Gamma^\nu(t)$. Iterating this step, by suitable left multiplications one can eliminate consecutively all terms with poles of order $\nu - 1$, $\nu - 2$ and so on until the constant terms will be eliminated. The overall product $\Gamma^0(t)\Gamma^1(t)\cdots\Gamma^\nu(t)$ of all monopoles used in the process, will again be a monopole at infinity (polynomial in t), also lower triangular. This completes the proof in the particular case when the matrix D has only two distinct eigenvalues $0 < \nu$ ordered in the ascending (nondecreasing) order.

2. Any diagonal matrix D with ascending integer eigenvalues $d_1 \leqslant \cdots \leqslant d_n$ can be represented as a sum of several matrices of the type considered above. More precisely, we can always represent such D as the sum

$$D = D_0 + D_1 + \cdots + D_m, \qquad m \leqslant n - 1, \tag{22.14}$$

so that D_0 is scalar (diagonal with a single eigenvalue) and each D_i with $i > 1$ is block diagonal with two eigenvalues 0 and $\nu_i > 0$ arranged in the ascending order. To see this, consider the monotonous integer function $i \mapsto d_i$, $i \in \{1, \dots, n\}$. This function can be represented as a sum of $m - 1$ "step functions" (nonincreasing integer functions assuming only two values, one of them zero) plus a constant term. Indeed, the first difference $i \mapsto d_{i+1} - d_i$ is a nonnegative integer function which can be represented as the sum of $\leqslant m - 1$ "delta-functions" taking a positive nonzero value only once. Taking "primitives" of these "delta-functions" (the sums restoring integer functions from their differences) and adding the "constant of integration" proves the claim: each step function can be considered as a diagonal matrix D_i with one zero and one positive eigenvalue.

Since the powers t^{D_i} commute between themselves, the terms in the representation (22.14) can be arranged so that the matrices with biggest-size upper-left (zero) block come last.

3. Splitting (22.14) permits to prove the assertion of the Lemma for every product $t^D H(t)$ where the diagonal matrix is ascending (its eigenvalues nondecreasing) and $H(t)$ having nonzero principal minors. In this case one can also choose $D = D'$. Indeed, in the representation

$$t^{D_0} t^{D_1} \cdots t^{D_m} H(t)$$

the term t^{D_m} can be permuted with $H(t)$ if the appropriate monopole $\varGamma(t)$ is inserted between $t^{D_{m-1}}$ and t^{D_m}, as shown on Step 1. To do this, the whole product must be multiplied from the left by the matrix function

$$\varGamma'(t) = t^{D_0 + \cdots + D_{m-1}} \varGamma(t)\, t^{-(D_0 + \cdots + D_{m-1})}.$$

But since both D and all matrices D_i were ascending and $\varGamma(t)$ lower triangular, the matrix $\varGamma'(t)$ will again be a monopole by Lemma 22.23 (more precisely, by its lower-diagonal "twin"). By construction,

$$\varGamma'(t)\, t^D H(t) = t^{D_0 + \cdots + D_{m-1}} H'(t)\, t^{D_m},$$

and the upper-left corner of $H'(t)$ will coincide with that of $H(t)$. The process can be clearly continued by induction, since on the next step one may require nondegeneracy of only smaller or same size upper-left minors of $H(t)$, thus preserving inductively the assumptions required on Step 1. After m permutations all terms t^{D_i} will appear to the right from the holomorphically invertible term, while the scalar term t^{D_0} commutes with everything.

4. For an arbitrary nondegenerate $H(\infty)$, the required condition on principal minors can always be achieved by a suitable permutation of columns, that is, multiplying $t^D H$ from the right by a suitable constant permutation matrix P. By Step 3, $t^D H(t)P$ is monopole equivalent to $H'(t)\,t^D$ for any *ascending* matrix D. But then $t^D H(t)$ is monopole equivalent to $H'(t)P^{-1}\cdot P\,t^D P^{-1} = H''(t)t^{D'}$, where $D' = PDP^{-1}$ is a diagonal matrix with entries obtained by the permutation of entries of D.

5. The last remaining assumption that D is ascending, can also be removed by a suitable permutation of rows. Indeed, if P is a permutation matrix such that the entries of $D' = PDP^{-1}$ are ascending, then $t^D H$ is monopole equivalent to $t^{D'}H'$ with H' holomorphically invertible at infinity:

$$P\cdot t^D H = P\,t^D P^{-1}\cdot PH = t^{D'}H'.$$

By Step 4, $t^{D'}H'$ is monopole equivalent to $H''\,t^{D''}$ as required.

This proves Lemma 22.25 in full generality. □

Together with the Birkhoff–Grothendieck factorization theorem, Lemma 22.25 implies the following corollary[1]. Let $\mathfrak{U} = \{U_0, U_1\}$ be a covering as in the Birkhoff–Grothendieck theorem.

Corollary 22.26. *Any holomorphically invertible matrix function $F(t)$ in the annulus $\{r_1 < |t| < r_0\}$ can be factored out as*

$$F(t) = H_0(t)H_1(t)\,t^D, \tag{22.15}$$

with the terms $H_i(t)$ holomorphically invertible in U_i, $i = 0, 1$, and an integer diagonal matrix D.

In particular, any nonzero meromorphic germ of a matrix function $F(t)$ at the infinity admits factorization

$$F(t) = \Gamma(t)H(t)\,t^D,$$

with a monopole $\Gamma(t)$ and a holomorphically invertible germ $H(t)$ at infinity.

Proof. By the Birkhoff–Grothendieck theorem,

$$H_0^{-1}(t)F(t) = t^D H_1^{-1}(t)$$

with $H_i(t)$ holomorphically invertible in U_i. By Lemma 22.25, for a suitable monopole $\Gamma(t)$,

$$\Gamma(t)\,t^D H_1^{-1}(t) = H(t)\,t^{D'},$$

where $H(t)$ is a priori only a holomorphic germ (invertible) at infinity. However, since all other terms of this identity are defined and holomorphically invertible in $U_1 \smallsetminus \{\infty\}$, $H(t)$ also can be extended as a holomorphically

[1]It is this form that is sometimes called the Birkhoff factorization or Birkhoff normal form, see [**FM98**].

invertible matrix function everywhere in U_1. Substituting this into the Birkhoff–Grothendieck factorization, we prove the corollary. □

Remark 22.27 (nonuniqueness of the Birkhoff form)**.** The representation (22.15) is *not unique* in any sense, including the non-uniqueness of eigenvalues of the diagonal matrices D, D'. For example, the matrix function $H(t) = \left(\begin{smallmatrix} 1 & 0 \\ t^{-1} & 1 \end{smallmatrix}\right)$, holomorphically invertible near $t = \infty$, besides the trivial representation $H(t) = H(t)\, t^D$ with $D = 0$, can be represented as

$$\begin{pmatrix} 1 & 0 \\ t^{-1} & 1 \end{pmatrix} = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & t^{-1} \end{pmatrix} \begin{pmatrix} t^{-1} & \\ & t \end{pmatrix} = \Gamma(t) H'(t)\, t^{D'}.$$

This means that $H(t)\, t^D$, $D = 0$, is monopole equivalent to $H'(t)\, t^{D'}$ with $D' = \operatorname{diag}\{\pm 1\}$.

In application to the local theory of linear systems treated in §21, this means that the eigenvalues of a Fuchsian singular point are not invariant not only by meromorphic, but also by the monopole classification.

Corollary 22.26 is an important example of monopole gauge classification of regular non-Fuchsian singularities. Recall that a singular point of a linear system is an *apparent singularity*, if the fundamental matrix solution is a meromorphic germ, i.e., if the monodromy around this point is trivial (identical). The assertion of the Corollary means that any apparent singularity of a linear system on the Riemann sphere can be made Fuchsian by a suitable rational transformation holomorphic at all other singular points.

Indeed, if $F(t) = \Gamma(t) H(t)\, t^D$ is a meromorphic germ of solution at $t = \infty$, then in general $\Omega = dF \cdot F^{-1}$ has a pole of order greater than 1, but the form Ω is monopole gauge equivalent to the form

$$\Omega' = dH \cdot H^{-1} + HDH^{-1}\, t^{-1} dt,$$

which is Fuchsian at infinity.

Appendix: meromorphic solvability of cocycles

Theorems 22.11 and 22.12 are proved in two steps. On the first (difficult) step we show that any cocycle on two charts (either Cartan or Birkhoff–Grothendieck) which is *sufficiently close* to the identical cocycle $\mathcal{E}$, is in fact *holomorphically* equivalent to the latter, i.e., holomorphically solvable. The second (easy) step is to show that *any* cocycle is meromorphically equivalent to a cocycle arbitrarily close to $\mathcal{E}$.

22.10. Solvability of near identical cocycles.

Lemma 22.28. *For both types of coverings, in the Cartan case as well as in the Birkhoff–Grothendieck case, there exists $\delta > 0$ depending on the geometry* *of the domains U_0, U_1, such that any cocycle satisfying $\|F(t) - E\| < \delta$ for any $t \in \overline{U}_{01}$, is holomorphically solvable.*

The idea is to consider the solvability condition

$$H_0(t)H_1^{-1}(t) = F_{01}(t), \qquad H_k \in \mathrm{GL}(n, \mathcal{O}(\overline{U}_k)), \tag{22.16}$$

as a *nonlinear* functional equation on the unknown matrix functions H_0, H_1, and solve it using the Newton method. More precisely, we linearize the equation (22.16) at $F_{10} = H_1 = H_0 \equiv E$, show that the linearized *homological equation* is solvable, and then construct solution of the nonlinear equation as a limit of rapidly converging iterations, characteristic for the Newton method of solving nonlinear equations. The scheme is very much similar to the method of the proof of Poincaré–Dulac theorem from Part 1. On the other hand, this method (especially for the Cartan cocycles) is in a nutshell the core of KAM theory.

Back reference to Poincaré–Dulac theorem

22.11. Homological equation and its solution. The linearized equation can be obtained by substituting $F_{01}(t) = E + B(t)$, $H_k(t) = E + A_k(t)$ into (22.16) and keeping only terms of the first order in A_k, B. The resulting equation will be

$$A_0(t) - A_1(t) = B(t). \tag{22.17}$$

In order to discuss its solvability for $B(t)$ sufficiently close to zero, we introduce convenient Banach spaces and describe the solution in terms of linear operators.

Let $U \Subset \mathbb{C}$ be a connected simply connected domain bounded by finitely many smooth arcs. Denote by $\mathbf{B}(U)$ the Banach space of functions holomorphic in U, continuous in the closure $\overline{U}$, and equipped with the norm

$$\|H(\cdot)\| = \max_{t \in \overline{U}} \|H(t)\|$$

(any usual matrix norm can be chosen in the right hand side of this pointwise definition).

Assuming that $B(t)$ is holomorphic on the *closure* of U_{01}, one can easily write an integral operator resolving the homological equation (22.17) in matrix functions $A_0(t)$, $A_1(t)$ holomorphic in the *open* domains U_0, U_1. The boundary γ of U_{01} consists of two (piecewise smooth) arcs: the arc γ_0 belonging to the boundary of U_0 and an arc γ_1 on the boundary of U_1. In the Cartan case these two arcs have common endpoints, in the Birkhoff–Grothendieck case γ_k are disjoint concentric circles.

In both cases, by the Cauchy residue formula applied to $B(t)$ in U_{01}

$$B(t) = \frac{1}{2\pi i} \oint\limits_{\gamma_0 \cup \gamma_1} \frac{B(z)}{z - t}\, dz = A_0(t) - A_1(t),$$

where $A_0(t), A_1(t)$ are the integrals over the arcs γ_0 and $-\gamma_1$ respectively. The function $A_1(t)$ is holomorphic everywhere except for the points of the arc γ_1 that is a part of the boundary of U_1, hence $A_1(t)$ is holomorphic in the interior U_1. In a similar way $A_0(t)$ is holomorphic in U_0.

Thus the two Cauchy integral operators $\mathbf{L}_0, \mathbf{L}_1$, defined on the Banach space $\mathbf{B}(U_{01})$ by the formulas

$$\mathbf{L}_k \colon B(t) \mapsto \frac{(-1)^k}{2\pi i} \int_{\gamma_k} \frac{B(z)\,dz}{z-t},$$

give a solution of the equation (22.17) in the open domains U_0, U_1,

$$B(t) = \mathbf{L}_0(B(t)) - \mathbf{L}_1(B(t)). \tag{22.18}$$

In general, the Cauchy integral gives a function with singularities on the boundary, so that the operators $\mathbf{L}_k$ *do not extend* to bounded linear operators $\mathbf{B}(U_{01}) \to \mathbf{B}(U_k)$. However, this extension is possible in the Birkhoff–Grothendieck case which is simpler in this respect and will be treated first. On the qualitative level this happens because on each of the two disjoint boundary circles γ_0 and γ_1, two out of the three terms in the identity (22.18) are holomorphic, hence the third is also holomorphic and thus both A_1 and A_0 belong to the Banach spaces $\mathbf{B}(U_k)$. The quantitative statement is almost as easy as the qualitative one.

Lemma 22.29. *In the Birkhoff–Grothendieck case when the intersection U_{01} is the annulus of the conformal width $r = (r_0 - r_1)/r_0 > 0$, the operators $\mathbf{L}_k$ extend as bounded operators*

$$\mathbf{L}_k \colon \mathbf{B}(U_{01}) \to \mathbf{B}(U_k),$$

with bounded operator norms,

$$\|\mathbf{L}_k\| \leqslant 1 + 2r^{-1} < +\infty.$$

Proof. Denote $A_k = \mathbf{L}_k(B)$, $k = 0, 1$, and assume that $\|B(\cdot)\| = 1$ in the space $\mathbf{B}(U_{01})$. For any point in the annulus U_{01} the distance from t to one of the boundary circles is at least $(r_0 - r_1)/2$ (recall that $r_0 > r_1$). The corresponding Cauchy integral will give the matrix $A_k(t)$ whose norm *at this point* is no greater than $2r_0/(r_0 - r_1) = 2r^{-1}$, but since $A_1(t) - A_0(t) = B(t)$, the other matrix function at the same point has the norm not exceeding $1 + \|A_k(t)\| \leqslant 1 + 2r^{-1}$. □

In the Cartan case where the two parts of the boundary of U_{01} are not disjoint, the Cauchy operators are unbounded and the integrals $\mathbf{L}_k(B)$ in general admit no holomorphic extension on the closure $\overline{U}_k$. The best quantitative bound one can get in this case is a bound for the operator norm of $\mathbf{L}_k$ restricted on *smaller* domains. For simplicity we will assume that the smooth arcs bounding U_k, intersect transversally. This transversality implies all regularity conditions imposed on the covering, will remain valid also if the charts U_k are replaced by their sufficiently small ε-neighborhoods.

Let $\varepsilon > 0$ be a sufficiently small positive number and

$$U_k^\varepsilon = \{t \in \mathbb{C}\colon \operatorname{dist}(t, U_k) < \varepsilon\}, \quad k = 0, 1, \qquad U_{01}^\varepsilon = U_0^\varepsilon \cap U_1^\varepsilon,$$

denote ε-neighborhoods of the charts U_k and the intersection of these neighborhoods respectively (note that U_{01}^ε is *not* the ε-neighborhood of the intersection U_{01}, though the difference is not essential). In the same way as before, the boundary $\gamma^\varepsilon = \partial U_{01}^\varepsilon$ can be subdivided into two arcs,

$$\gamma^\varepsilon = \gamma_0^\varepsilon \cup \gamma_1^\varepsilon, \qquad \gamma_k^\varepsilon \subset \partial U_k^\varepsilon.$$

The Cauchy formula applied to the loop γ^ε, yields two integrals that are holomorphic in U_k^ε and can be restricted on any pair of smaller domains. For our purposes it is sufficient to take $U_k^{\varepsilon/2}$, introducing two integral operators

$$\mathbf{L}_k^\varepsilon\colon \mathbf{B}(U_{01}^\varepsilon) \to \mathbf{B}(U_k^{\varepsilon/2}), \quad \mathbf{L}_k^\varepsilon(B(t)) = \frac{(-1)^k}{2\pi i} \int_{\gamma_k^\varepsilon} \frac{B(z)\,dz}{z-t}\bigg|_{U_k^{\varepsilon/2}}, \quad k = 0, 1.$$

The following result obviously follows from the estimates of the Cauchy kernel.

Lemma 22.30. *The integral operators $\mathbf{L}_k^\varepsilon$ have bounded norms for all sufficiently small $\varepsilon > 0$. More precisely, there exist a constant c depending only on the geometry of the domains U_k, such that*

$$\|\mathbf{L}_k^\varepsilon\| \leqslant c\varepsilon^{-1}. \quad \square$$

Back ref. — Poincaré domain

Thus from the point of view of solvability of the homological equation (22.17), the Birkhoff–Grothendieck case is similar to the Poincaré domain, whereas the Cartan case is more like the Siegel domain.

22.12. Holomorphic solvability of near identical Birkhoff–Grothendieck cocycles. Consider a cocycle $F(t) = E + B(t)$ sufficiently close to identity so that the bound $\|B(\cdot)\|$ in the appropriate space $\mathbf{B}(U_{01})$ is no greater than some sufficiently small δ. This cocycle is equivalent to the cocycle $E + B'(t)$ defined by the equation

$$(E + B')(E + \mathbf{L}_0(B)) = (E + \mathbf{L}_1(B))(E + B),$$

provided that $(1 + 2r^{-1})\delta < \frac{1}{2}$ so that the matrix functions $E + A_k(t)$, $A_k = \mathbf{L}_k(B)$, $i = 0, 1$, forming the cochain, are invertible and their inverses have norms bounded by 2. By construction, $B = A_0 - A_1$.

The norm of the term $B' \in \mathbf{B}(U_{01})$ can be easily estimated, knowing the norm of the resolvents $\mathbf{L}_k$:

$$B' = A_1 B (E + A_0)^{-1},$$

hence $\|B'\| \leqslant 2c\delta^2$, where $c = 1 + 2r^{-1}$ is a bound for the operator norms of $\mathbf{L}_k$.

Thus we see that if δ is smaller than some δ_0, than a cocycle δ-close to the trivial cocycle $\mathcal{E}$, is equivalent to the cocycle that is $c\delta^2$-close to $\mathcal{E}$, and the conjugating cochain is $c\delta$-close to the trivial cochain consisting of identity functions. Iteration of this step yields a sequence of pairwise equivalent cocycles $\mathcal{F}_j = \{E + B_j(t)\}$, $j = 1, 2, \dots$, inscribed in the same covering, and very fast (super-exponentially) converging to the trivial one,

$$\|B_j\| \leqslant (c\delta)^{2^j} \quad \text{in} \quad \mathbf{B}(U_{01}), \qquad j = 1, 2, \dots.$$

By transitivity, the initial cocycle is equivalent to all cocycles $E + B_j(t)$, and the corresponding conjugating cochains converge very fast to a holomorphic matrix cochain conjugating the initial cocycle to the trivial limit $\mathcal{E} = \{E + \lim B_j\}$. This proves Lemma 22.28 in the Birkhoff–Grothendieck case.

Remark 22.31. Strictly speaking, this proof guarantees that the conjugating cochain is only continuous on the boundary. However, the initial covering could be slightly enlarged since by the regularity assumption the initial cocycle could be extended on a larger annulus. The above argument proves existence of the conjugating cochain inscribed in this larger covering, that will be automatically holomorphic on the closure of each initial chart.

22.13. Holomorphic solvability of near identical Cartan cocycles. To treat the Cartan case, consider the iteration step on which a cocycle $E + B(t)$ with $B(t) \in \mathbf{B}(U_{01}^{\varepsilon})$ is replaced by an equivalent cocycle $B'(t)$ on the *smaller* domain, $B'(t) \in \mathbf{B}\big(U_{01}^{\varepsilon/2}\big)$, found from the condition

$$(E + B')(E + \mathbf{L}_0^{\varepsilon}(B)) = (E + \mathbf{L}_1^{\varepsilon}(B))(E + B).$$

The same arguments as before, prove that in this case

$$\|B'\| \leqslant c\varepsilon^{-1}\|B\|^2,$$

where the matrix norms in the two sides of this inequality refer to two different Banach spaces $\mathbf{B}(U_{01}^{\varepsilon/2})$ and $\mathbf{B}\big(U_{01}^{\varepsilon}\big)$ respectively.

In order to iterate this construction, we have first to find $\varepsilon_0 > 0$ such that $F(t)$ extends on $U_{01}^{\varepsilon_0}$. By the regularity assumption on the domains and the cocycle, this is possible. Then we will consider a shrinking system of coverings $\mathfrak{U}_j = \{U_0^{\varepsilon_j}, U_1^{\varepsilon_j}\}$, $\varepsilon_j = \varepsilon_0/2^j > 0$, $j = 0, 1, 2 \dots$. The operators $\mathbf{L}_0^{\varepsilon_j}, \mathbf{L}_1^{\varepsilon_j}$ define the sequence of cocycles $\mathcal{F}_j = \{E + B_j(t)\}$,

$$E + B_{k+1} = (E + \mathbf{L}_0^{\varepsilon_j}(B))(E + B_j)(E + \mathbf{L}_1^{\varepsilon_j}(B))^{-1},$$

inscribed in these coverings. The cocycles $\mathcal{F}_j$ are by construction pairwise conjugated to each other in the sense that the conjugating cochain is defined on the smaller of the two coverings. The gap between the $U_i^{\varepsilon_{j-1}}$ and $U_i^{\varepsilon_j}$ in this sequence will be $\varepsilon_0 \cdot 2^{-j}$, and therefore the sequence of norms $\delta_j = \|B_j\|$ with respect to the corresponding Banach spaces, satisfies the recurrent inequalities

$$\delta_{j+1} \leqslant c \cdot 2^j \delta_j^2, \qquad c < +\infty,\ j = 0, 1, 2, \dots.$$

This sequence also decays super-exponentially fast, though not so fast as in the Birkhoff–Grothendieck case: to see this, notice that the negative of binary logarithms $r_j = -\log_2 \delta_j$ satisfy the inequalities

$$r_{j+1} \geqslant 2r_j - j + c', \qquad c' < +\infty,$$

and obviously grow exponentially in j as $j \to \infty$. As a result, we can conclude that on the intersection

$$\overline{U}_{01} = \bigcap_{j \geqslant 0} U_{01}^{\varepsilon_j},$$

the cocycles $\mathcal{F}_j = \{E + B_j(t)\}$ converge to the trivial cocycle $\mathcal{E}$. Since the operators $\mathbf{L}_i^{\varepsilon_j}$ are bounded, the cochains $\mathcal{H}_j$ conjugating $\mathcal{F}_j$ with $\mathcal{F}_0$, converge uniformly on the closure $\overline{U}_i$ to holomorphic invertible functions. The proof of Lemma 22.28 is complete. □

Remark 22.32. As before, the sequence of iterations in fact converges in the space $\mathbf{B}(U_{01})$, and the limit $\lim \mathcal{H}_j$ is only continuous on the boundary. However, holomorphy on the closure can be achieved by initial arbitrarily small enlarging of the domain of the cocycle.

22.14. Meromorphic solvability of arbitrary cocycles. To prove Theorems 22.11 and 22.12, it is sufficient to approximate any matrix cocycle $\mathcal{F} = \{F(t)\}$ on U_{01} by a polynomial (in case of Cartan cocycles when the intersection is simply connected) or at worst by a rational matrix function (for Birkhoff–Grothendieck cocycles, when U_{01} is an annulus). In both cases we can find a rational matrix function $R(t)$ without poles or degeneracy points on the closure $\overline{U}_{01}$ such that $\|R^{-1}(t)F(t) - E\| < \delta$ with a positive δ small enough to guarantee that Lemma 22.28 will be applicable. Then the cocycle $R^{-1}(t)F(t)$ is solvable and factors as $H_0(t)H_1^{-1}(t)$ with an appropriate holomorphic cochain $\mathcal{H} = \{H_0, H_1\}$.

But then the initial cocycle itself admits meromorphic factorization,

$$F(t) = R(t)H_0(t) \cdot H_1^{-1}(t). \quad \square$$

22.15. Variations. Theorems on holomorphic solvability of cocycles may be formulated and proved under various additional constraints. One such variation concerns cocycles on punctured neighborhood of the origin, subject to specific asymptotic behavior.

Consider covering $\mathfrak{U}$ of a punctured disk $\{0 < |t| < 1\}$ by open sectors U_j bounded by rays and an *asymptotically trivial* holomorphic cocycle $\mathcal{F} =$

$\{F_{ij}(t)\}$. By definition, this means that for any matrix function $F_{ij} \in \mathcal{F}$ the difference $F_{ij}(t) - E$ is flat at the origin, $t^{-N}\|F_{ij}(t) - E\|$ tends to zero as $t \to 0$, $t \in U_{ij}$, for any finite N.

Theorem 22.33 (Birkhoff, 1913; Y. Sibuya [**Sib90**])**.** *Any asymptotically trivial cocycle is solvable by a holomorphic cochain $\mathcal{H} = \{H_j\}$ bounded together with its inverse.*

This theorem can be proved similarly to the Birkhoff–Grothendieck theorem in §22.12. The key step is again the bounded solvability of the homological equation, the linearization of the cocycle identity.

More precisely, assume that the sectors $U_0, \dots, U_{m-1}$ forming the covering $\mathfrak{U}$ are chosen so that only the pairwise intersection $U_{j,j+1}$ are non-empty for $j = 0, 1, \dots \bmod m$. Assume that in each intersection a holomorphic matrix function $B_j(t)$ is defined, which is flat as $t \to 0$. We claim that in this case there exists a collection of functions $A_j(t) \in \mathcal{O}(U_j)$, $j = 0, 1, \dots, m-1$, holomorphic and bounded in the respective sectors U_j, such that on the intersections

$$B_j = A_{j+1} - A_j, \qquad j = 0, 1, \dots, m-1 \bmod m. \tag{22.19}$$

These equations, a variation on the theme of the equation (22.17), are also solved by the integral Cauchy-type operator.

It is convenient to pass to the inverse chart $z = 1/t$ and consider the sectors U_j with the vertex at infinity and their pairwise intersections $U_{j,j+1}$. Choose a system of rays $R_j \subset U_{j,j+1}$ and assume that the neighborhood of infinity $\{|z| > r\}$ in which the covering is considered, is so small (i.e., r is so large) that the rays $R_j \cap \{|z| > r\}$ are at least 2-distant from each other. The collection $\{B_j(z)\}_{j=0}^{m-1}$ defines a holomorphic matrix function $B(z)$ on the union $U_{01} \cup U_{12} \cup \cdots \cup U_{m-1,0}$ containing the union of the rays $R = R_0 \cup \cdots \cup R_{m-1}$,

$$B(z)|_{R_j} = B_j(z), \qquad j = 0, \dots, m-1.$$

Consider the Cauchy-type integral operator

$$\mathbf{L}\colon B(\cdot) \mapsto A(\cdot), \qquad A(z) = \frac{1}{2\pi i} \int\limits_R \frac{B(\zeta)}{\zeta - z}\, d\zeta. \tag{22.20}$$

It defines the matrix function holomorphic on the complement $\{|z| > r\} \smallsetminus R$. One can show that the operator $\mathbf{L}$ is bounded in the following sense: if $\sup \|B(z)\|_R \leqslant 1$ for $|z| > r$ and $\int_R \|B(z)\|\, d|z| \leqslant 1$, then $\|A(z)\| \leqslant 1 + \pi$ for $|z| > r + 1$.

Indeed, if the distance from the point z to R is greater than 1, then the inequality $\|A(z)\| \leqslant 1$ follows directly from the definition. If there is (at most one) ray R_j whose distance from z is less than 1, then the path of integration has to be slightly changed, "pushed away" from z. One has to replace the chord $R_j \cap \{\zeta \colon |\zeta - z| \leqslant 1\}$ of the unit disk centered at z by the smaller of the two arcs supported by this chord. This replacement transforms R_j into another path R_j' which is 1-distant from z and differs from R_j (in the homological sense) by the compact closed arc γ, the boundary of the circular segment. The integral $\oint_\gamma (\zeta - z)^{-1} B(\zeta)\, d\zeta$ is zero since B is holomorphic inside, so the change of path of integration does not affect the value of the integral. On the other hand, the integral along the new path is at most 1+integral over the arc, the latter being no more than π in the sense of the norm.

The same argument actually shows that the function $A(z)$ defined by (22.20) can be analytically extended from each sector bounded by two consecutive rays R_j, R_{j+1} to a larger domain (sectorial, if necessary) as a holomorphic matrix function $A_j(z)$, $j = 0, 1, \dots$. By the Plemelj–Sokhotski formula, the jump of the value (the difference between the limits from two sides) of the integral (22.20) along the ray R_j, equal to the difference $A_{j+1} - A_j$, is exactly B_j.

Based on the boundedness of the operator $\mathbf{L}$, one can prove solvability of the asymptotically trivial cocycle exactly as in §22.12. Details can be found in [**Sib90**].

23. The Riemann–Hilbert problem: positive results

The Riemann–Hilbert problem, also known as Hilbert Twenty-First problem, requires to construct a linear system with the prescribed monodromy group and positions of all singularities:

> ... This problem is as follows: *To show that there always exists a linear differential equation of the Fuchsian class, with given singular points and monodromy group.* The problem requires the production of n functions of the variable z, regular throughout the complex z plane except at the given singular points; at these points the functions may become infinite of only finite order, and when z describes circuits about these points the functions shall undergo the prescribed linear substitutions (*D. Hilbert* [**Hil00**]).

This formulation is somewhat confusing, since the clarification given in the text after it, describes only the regularity condition, while the main formulation was about Fuchsian systems. One can think of *three* different accurate formulations, when a given monodromy group is required to be realized by:

(i) a Fuchsian linear nth order differential equation,

(ii) a linear system having only regular singularities, or

(iii) a Fuchsian system on the whole Riemann sphere $\mathbb{C}P^1$.

In each case it is required that the equation (resp., the system) be nonsingular outside the preassigned points.

The negative answer in the first problem was known already by A. Poincaré: the reason is that the dimension of the space of all Fuchsian equations having m prescribed singular points on $\mathbb{C}P^1$, is strictly smaller than the dimension of all admissible monodromy data, except for the case of second order equations with three singular points studied by Riemann.

Only recently it became clear that there is a substantial difference between the formulations (ii) and (iii). J. Plemelj [**Ple64**] gave a solution of problem (ii) while claiming solution of the strongest problem (iii). The gap was discovered by Yu. Ilyashenko [**AI88**] and A. Treibich [**Tre83**] in the earlier eighties. The positive part of Plemelj theorem is described below. Later it was proved independently by A. Bolibruch [**Bol92**] and V. Kostov [**Kos92**] that an irreducible monodromy group can be always realized by a Fuchsian system. In this section we explain a remarkably simple proof of the Bolibruch–Kostov theorem which was communicated to us by A. Bolibruch.

However, for a reducible monodromy group the answer to problem (iii) may be negative. The counterexample, also due to Bolibruch, is described in §24.

23.1. Solution of the Riemann–Hilbert problem for an open disk and the affine plane. The local version of the Riemann–Hilbert problem is very simple: any nondegenerate matrix M can be realized as the monodromy matrix of a Fuchsian singularity (as was already noticed, it is sufficient to take the Euler system with the residue $A = \frac{1}{2\pi i} \ln M$). It is important to stress that this local solution is by no means unique: one can always replace A by its conjugate or add to it an integer multiple of the identity matrix E.

Remark 23.1. The freedom to choose the matrix logarithm is different for different matrices. For an instance, when M is a diagonal matrix, $M = \operatorname{diag}\{\lambda_1, \dots, \lambda_n\}$, then for $\ln M = \operatorname{diag}\{\ln \lambda_1, \dots, \ln \lambda_n\}$ one can choose the values of logarithms $\ln \lambda_j$ for each entry λ_j independently from all other entries, so for a scalar matrix the logarithm may well be non-scalar. On the other hand, for matrices having only one Jordan block of the maximal size n, the freedom of choice of the logarithm is reduced to the above transformations, as described in Remark 24.3.

Solution of the Riemann–Hilbert problem for the open disk $U \subset \mathbb{C}$ can be constructed by patching together any collection of local solutions.

In order to specify the monodromy antirepresentation of the fundamental group of a multiply connected Riemann surface, one has to specify the choice of loops generating this group.

Let $U = \{|t| < 1\}$ be the open unit disk, $\Sigma = \{t_1, \dots, t_m\}$ a finite set of points and $M_1, \dots, M_m \in \mathrm{GL}(n, \mathbb{C})$ an arbitrary collection of invertible matrices; together with Σ it will be referred to as the *monodromy data*. Choose a base point $t_0 \in U \smallsetminus \Sigma$ in such a way that the rectilinear segments $[t_0, t_i]$, $i = 1, \dots, m$ are disjoint except for the common endpoint t_0, and enumeration is chosen so that arguments $\arg(t_i - t_0)$, $i = 1, \dots, m$, are increasing between 0 and 2π. Consider the loops $\gamma_i \in \pi_1(U \smallsetminus \Sigma, t_0)$ corresponding to going from t_0 to t_i along the segment $[t_0, t_1]$, encircling t_i along a small circular path in the counterclockwise (positive) direction and returning again to t_0 along the same segment.

The loops $\gamma_1, \dots, \gamma_m$ generate freely the fundamental group, thus there is a unique antirepresentation $M \colon \pi_1(U \smallsetminus \Sigma, t_0) \to \mathrm{GL}(n, \mathbb{C})$, $\gamma \mapsto M_\gamma$, such that $M_{\gamma_i} = M_i$, $i = 1, \dots, m$. We say that a linear system Ω with only Fuchsian singularities in U *realizes* the monodromy data $(\Sigma, \{M_i\})$, if the monodromy group of Ω coincides with the above described group generated by the matrices M_i. The (multivalued) solution $X(t)$ ramified over Σ with the monodromy factors *equal* to the matrices M_i, will be referred to as the *privileged* (matrix) solution realizing the prescribed monodromy.

Theorem 23.2. *Any monodromy data in the unit disk U can be realized by a Fuchsian system.*

Proof. Consider partition of the disk U by m rays all meeting at the point t_0, such that each open sector S_j between two adjacent rays contains only one singular point t_j. Let U_j be a convex sufficiently small neighborhood of the closure $\overline{S}_j$.

In each domain U_j there exists a multivalued matrix function $X_j(t)$ obtained as continuation of the germ $X_j(t) = (t - t_j)^{A_j}$, where $A_j = \frac{1}{2\pi i} \ln M_j$. Fix any *privileged branch* of each $X_j(t)$ at t_0, considering X_j as the full analytic continuation on U_j of the germ on the privileged branch. By construction, the loop γ_j entirely belongs to U_j and

$$\Delta_{\gamma_j} X_j = X_j M_j, \quad j = 1, \dots, m.$$

Let $\Omega_j = dX_j \cdot X_j^{-1}$ be the corresponding Pfaffian matrices: by construction, each Ω_j is holomorphic in $U_j \smallsetminus \{t_j\}$ and has a simple pole at t_j. The collection $\{\Omega_j\}_{j=1}^m$ disagrees on the intersections of domains U_j, but this can be corrected by a suitable gauge transformation.

Let $F_{ij}(t)$ be a collection of analytic matrices defined on the intersections $U_{ij} = U_i \cap U_j$ as

$$F_{ij}(t) = X_i(t)X_j^{-1}(t), \qquad t \in U_{ij}. \tag{23.1}$$

This matrix ratio is defined in U_{ij} unambiguously as the result of analytic continuation of the germ of privileged branch, and forms a matrix cocycle $\mathcal{F} = \{F_{ij}(t)\}$ inscribed in the covering $\mathfrak{U}$. Indeed, on the triple intersections $U_{ijk} = U_i \cap U_j \cap U_k$ the cocyclic identities hold,

$$F_{ij}F_{jk} = X_iX_j^{-1}X_jX_k^{-1} = X_iX_k^{-1} = F_{ik}, \qquad F_{ij}F_{ji} = E.$$

As was observed in Example 22.15, the covering $\mathfrak{U} = \{U_j\}$ meets the condition (22.5). Therefore by Theorem 22.14, the matrix cocycle is solvable, and there exist holomorphic invertible matrix functions H_i defined in U_i and satisfying

$$F_{ij}(t) = H_i(t)H_j^{-1}(t), \qquad t \in U_{ij}, \tag{23.2}$$

on all intersections. Then (23.1), (23.2) imply that

$$X_i(t)X_j^{-1}(t) = H_i(t)H_j(t), \qquad t \in U_{ij},$$

which means that the privileged branches of the functions $X_i' = H_i^{-1}X_i$ coincide on the pairwise intersections,

$$H_i^{-1}(t)X_i(t) = H_j(t)^{-1}X_j(t), \qquad t \in U_{ij}.$$

In other words, the gauge transforms

$$\Omega_j' = d(H_j^{-1}) \cdot H_j + H_j^{-1}\Omega_j H_j, \qquad j = 1, \dots, m,$$

coincide on all intersections and together define a Pfaffian matrix form Ω' on the union $U = \bigcup_{j=1}^m U_j$ with simple (Fuchsian) singularities only at the points of $\varSigma$. The common germ $X'(t) = H_i^{-1}X_i$ at $t_0 \in \bigcap_{i=1}^m U_i$ after complete analytic continuation along each loop γ_i extends as the privileged solution of the Riemann–Hilbert problem for the disk: by construction, $X'(t)$ acquires the preassigned monodromy matrix factor M_i. □

To formulate the Riemann–Hilbert problem on the Riemann sphere, one has to take into account the fact that the loops around singular points are related by a single relation. Assume that the singular locus $\varSigma$ consists of $m+1$ distinct points, and choose the affine chart t on $\mathbb{C}P^1$ so that the last point is at infinity, $t_{m+1} = \infty$.

Assume that the base point t_0 and the loops γ_i, $i = 1, \dots, m$ around all other (finite) singular points are chosen as described above. Construct the loop γ_{m+1} encircling $t_{m+1} = \infty$ as follows. Choose a (real) ray through the point t_0 so that $\arg(t_m - t_0) < \arg(t - t_0) < \arg(t_1 - t_0) + 2\pi$ along this ray (it goes to infinity "between" t_1 and t_m). Then the loop γ_{m+1} goes from t_0 along this ray close enough to infinity, then makes a full *clockwise* turn along a (sufficiently large) circle centered at t_0 containing all other singularities and returns back to t_0 along the same ray.

The loops $\gamma_1, \dots, \gamma_m, \gamma_{m+1}$ satisfy the identity $\gamma_1 \cdot \gamma_2 \cdots \gamma_m \cdot \gamma_{m+1} = \mathrm{id}$ which implies that the corresponding monodromy matrices M_i must satisfy the identity $M_1 \cdots M_m M_{m+1} = E$.

Theorem 23.3 (Röhrl–Plemelj theorem [**Ple64, For91**]). *Any matrix group with $m+1$ generators $M_1, \dots, M_m, M_{m+1}$ satisfying the identity*

$$M_1 \cdots M_m M_{m+1} = E$$

can be realized as the monodromy group of a regular system on the Riemann sphere $\mathbb{C}P^1$ having all singularities Fuchsian with at most one exception.

Proof. After a suitable conformal automorphism of $\mathbb{C}P^1$ one may assume the last singular point being at infinity and all other singularities inside the disk of radius $\frac{1}{2}$ around the origin.

By Theorem 23.2, one can construct a meromorphic Pfaffian matrix Ω_0 having only simple poles in the unit disk $U_0 = \{|t| < 1\}$ and realizing the monodromy data concerning all finite singularities. To prove Theorem 23.3, one has to extend the form Ω_0 on the sphere so that it would have a regular singularity at infinity.

Let $M = M_1 \cdots M_m$ be the monodromy operator corresponding to going around the point at infinity: by construction, this matrix factor is acquired by a solution $X_0(t)$ of the linear system $dX_0 = \Omega_0 X_0$ after analytic continuation along the unit circle γ (in the positive direction). Consider restriction of the matrix function $X_1(t) = t^M$ on the $U_1 = \{|t| > \frac{1}{2}\}$, the exterior of the disk containing all finite singularities. This matrix is a fundamental matrix solution of the Euler system $dX_1 = \Omega_1 X_1$ with $\Omega_1 = At^{-1}\,dt$, $A = \frac{1}{2\pi i} \ln M$.

The monodromy factor for the solution $X_1(t)$ along γ is the same as for X_0, therefore their matrix ratio $F_{01}(t)$,

$$F_{01}(t) = X_0(t) X_1^{-1}(t), \qquad t \in U_{01} = \{\tfrac{1}{2} < |t| < 1\}$$

is single-valued holomorphically invertible matrix function in the annulus, in other words, a Birkhoff–Grothendieck cocycle inscribed in the covering $\mathfrak{U} = \{U_0, U_1\}$.

By the Birkhoff–Grothendieck theorem, this cocycle admits factorization: there exist $H_0(t)$ holomorphically invertible in U_0, H_1 holomorphic and holomorphically invertible in U_1 except for $t = \infty$ (where it has an isolated pole) so that on the intersection U_{01}

$$X_0 X_1^{-1} = H_0 H_1^{-1}.$$

This means that the two matrix functions $X_i' = H_i^{-1} X_i$ coincide on the intersection, as well as the two gauge transforms

$$\Omega_i' = dX_i' \cdot (X_i')^{-1} = d(H_i^{-1})H_i + H_i^{-1}\Omega_i H_i, \qquad i = 0, 1.$$

Together Ω_0', Ω_1' define a Pfaffian matrix Ω' on $\mathbb{C}P^1$. This form is holomorphically equivalent to Ω_0 in U_0, hence has only simple poles there and the same monodromy around all finite singularities.

As for the point at infinity, the gauge transformation matrix $H_1^{-1}(t)$ conjugating the Fuchsian singularity Ω_1 with Ω_1', is only meromorphic at $t = \infty$, since in general the matrix D is nonzero. However, the singularity at $t = \infty$ remains regular. □

23.2. Plemelj theorem. In the previous section the problem of constructing a linear system with the preassigned monodromy group was solved in the class of regular systems having all singular points Fuchsian with at most one exception. In this section we show that the last remaining singularity can sometimes be made Fuchsian by an appropriate gauge transformation with a monopole rational matrix.

Assume that the regular non-Fuchsian point is at infinity $t = \infty$. By (21.2), the fundamental solution constructed in §23.1 in a small neighborhood $(\mathbb{C}P^1, \infty)$ can be represented as

$$X(t) = H(t)\, t^A, \qquad H(\cdot) \in \mathrm{GL}(n, \mathcal{M}_\infty), \quad A \in \mathrm{Mat}(n, \mathbb{C}), \tag{23.3}$$

with a meromorphic matrix germ $H(t)$ and a constant matrix A that is a (normalized) logarithm of the corresponding monodromy matrix $M = M_{m+1}$.

Since the monodromy group is defined modulo a simultaneous conjugacy of all monodromy matrices, without loss of generality one may assume that both M and A are upper triangular. More generally, if M is diagonalizable, one may assume that both M and A are already diagonal.

Theorem 23.4 (Plemelj)**.** *If one of the monodromy matrices is diagonalizable, then the monodromy group can be realized by a Fuchsian system.*

Proof. Consider the fundamental solution $X(t)$ constructed in §23.1, assuming that the non-Fuchsian singularity is at infinity and the corresponding monodromy $M_{m+1} = M$ and its logarithm $A = \frac{1}{2\pi i} \ln M$ are diagonal. Let $H(t)$ be the meromorphic factor from the representation (23.3).

By Corollary 22.26,

$$H(t) = \Gamma(t) H'(t)\, t^D, \qquad D = \mathrm{diag}\{d_1, \dots, d_n\},$$

with a monopole $\Gamma(t)$ and $H'(t)$ holomorphically invertible at $t = \infty$.

After the gauge transformation $X \mapsto X' = \Gamma^{-1}X$ the new fundamental solution will have the local representation

$$X'(t) = H'(t)\, t^D t^A = H'(t)\, t^{D+A},$$

since two diagonal matrices D and A always commute with each other. This means that after this gauge transformation the singular point $t = \infty$ became Fuchsian (holomorphically equivalent to t^{D+A}, since H' is invertible). As the Fuchsian nature of all other points was not affected by the gauge transformation, the system is globally Fuchsian. □

Since the identical transformation is obviously diagonal, the Plemelj theorem implies that any monodromy group can be solved by a Fuchsian system having singularities at all preassigned positions and at most one more *apparent singularity* (a singular point where all solutions remain meromorphic) at any other point on the sphere.

Remark 23.5. In his book [**Ple64**] Plemelj formulated this theorem without assuming that the monodromy matrix is diagonal. Clearly, without this assumption the terms t^A and t^D cannot be permuted. This is the gap that was discovered by Ilyashenko and Treibich.

23.3. Bolibruch–Kostov theorem: construction of a Fuchsian system with an irreducible monodromy group. A considerably more elaborated construction allows to prove that the last remaining regular non-Fuchsian point occurring in Theorem 23.3 can be made Fuchsian under the *global* assumption that the monodromy group is irreducible, that is, the monodromy operators M_i have no nontrivial common invariant subspace.

Theorem 23.6 (Bolibruch–Kostov theorem)**.** *Any irreducible monodromy group can be realized by a Fuchsian system on* $\mathbb{C}P^1$*.*

By the Röhrl–Plemelj theorem 23.3, we can assume that the monodromy data is realized by a regular system with only one non-Fuchsian singular point, all other m points being already Fuchsian. Following Bolibruch, we show that the global irreducibility condition implies a local restriction on the analytic type of the only non-Fuchsian point. This information will be then used to construct a monopole equivalence putting the last singular point into the Fuchsian form.

We will assume this time that the non-Fuchsian singularity is at the origin (the point at infinity may be regular or Fuchsian singular, this is unimportant).

Lemma 23.7. *Suppose that a regular system* Ω *on the Riemann sphere has* $m \geqslant 1$ *Fuchsian points and a non-Fuchsian point at the origin. Assume* *that locally near this point the fundamental solution of the system admits representation*

$$X(t) = t^N Y(t), \qquad N = \operatorname{diag}\{\nu_1, \dots, \nu_n\}, \quad \nu_i \in \mathbb{Z},$$

where $Y(t)$ *has a Fuchsian singularity (so that* $dY \cdot Y^{-1}$ *has a first order pole at the origin) and* ν_i *some integer numbers.*

If the global monodromy group of the system is irreducible, then the difference between the numbers ν_i *is explicitly bounded,*

$$|\nu_i - \nu_j| \leqslant (m-1)^2, \qquad \forall i, j = 1, \dots, n. \tag{23.4}$$

Proof. The Pfaffian matrix of the system locally near the origin has the form

$$\Omega = N\, t^{-1} dt + t^N \Omega'\, t^{-N},$$

where $\Omega' = dY \cdot Y^{-1}$ has a first order pole at the origin. Without loss of generality, we may assume that the entries of the integer diagonal matrix N are arranged in the nonincreasing order,

$$\nu_1 \geqslant \cdots \geqslant \nu_n$$

(one can always permute the rows by a global constant gauge transformation that preserves the irreducibility). If $\nu_k - \nu_{k+1} > m - 1$ for some k between 1 and $n-1$, then all entries in some upper right corner of the matrix Ω will have zero of order $> m-2$ at the origin. More precisely, if $i \leqslant k$ and $j \geqslant k+1$, then the (i, j)th matrix element of the Pfaffian matrix Ω is obtained by multiplying the corresponding element ω'_{ij} of Ω' by t^d, $d = \nu_i - \nu_j \geqslant \nu_k - \nu_{k+1} > m - 1$. Since Ω' is Fuchsian, its entries have at most first order pole, thus the order of zero of all ω_{ij} with $i \leqslant k$ and $j \geqslant k+1$ will be greater than $m-2$.

On the other hand, since the form Ω is globally defined on the whole sphere, its entries are rational 1-forms. By assumptions, these forms have at most simple poles at no more than m other points of $\mathbb{C}P^1$. Thus the order of zero at the origin cannot be greater than $m-2$, unless the form is identically zero (the difference between the total number of poles and zeros for any rational form is always equal to 2). This necessarily implies that $\omega_{ij} \equiv 0$ for all combinations of i, j such that $i \leqslant k$ and $j \geqslant k+1$.

But the simultaneous occurrence of a corner of identical zeros as was described above, in the (rational, i.e., globally defined) Pfaffian matrix Ω means that the coordinate subspace $\{x_1 = \cdots = x_k = 0\}$ is invariant by the system, hence by all monodromy operators, contrary to the irreducibility assumption.

Thus for the case when the diagonal entries ν_i are arranged in the nonincreasing order, the difference between any two *consecutive* numbers cannot be greater than $m-1$. Hence the difference between *any* two ν_i is no greater than $(m-1)^2$ in the absolute value, and this assertion is already independent on the order of these numbers. □

Proof of Theorem 23.6. Consider a linear system on the Riemann sphere, having m Fuchsian singularities outside the origin and a regular non-Fuchsian singular point at the origin. By the local meromorphic classification theorem, solution of the system near the origin can be represented as

$$X(t) = M(t)\, t^A, \qquad A \in \operatorname{Mat}(n, \mathbb{C}), \quad M(t) \in GL(n, \mathcal{M}_0).$$

Without loss of generality we may assume that A is upper triangular.

Let D be an integer diagonal matrix with *very fast decreasing* entries $d_1 > \cdots > d_n$. For our purposes it would be sufficient to assume that

$$d_k - d_{k+1} > (m-1)^2, \qquad k = 1, \dots, m-1. \tag{23.5}$$

Inserting the trivial term $E = t^{-D} t^D$ between the terms of the representation above, we can apply the Sauvage lemma to $M'(t) = M(t)\, t^{-D}$ and then permute the terms applying Lemma 22.25. Using the symbol $\sim$ for the monopole gauge equivalence at the origin, we have

$$M(t)\, t^{-D} = M'(t) \sim t^N H(t) \sim H'(t)\, t^{N'},$$

with a diagonal integer matrix $N = \operatorname{diag}\{\nu_1, \dots, \nu_n\}$, its permutation N' and holomorphically invertible germs $H(t), H'(t)$ at the origin. Thus for one and the same system we have two different but monopole gauge equivalent local representations,

$$X(t) = M'(t)\, t^D t^A \sim t^N \cdot H(t)\, t^D t^A \tag{23.6}$$

$$\sim H'(t)\, t^{N'} t^D t^A$$

$$= H'(t) \cdot t^{D+N'} t^A. \tag{23.7}$$

Note that in the first form (23.6), the singularity $Y'(t) = t^D t^A$ is Fuchsian, since A is upper-triangular and D has decreasing eigenvalues: the corresponding Pfaffian form is $(D + t^D A t^{-D})\, t^{-1} dt$ and one may apply Lemma 22.23. Since $H(t)$ is holomorphically invertible, $Y(t) = H(t)\, t^D t^A$ is also Fuchsian.

Any monopole gauge equivalence preserves irreducibility of the global monodromy group. Lemma 23.7 and the representation (23.6) imply that the entries ν_i are not very different from each other, $|\nu_i - \nu_j| \leqslant (m-1)^2$. Since N' is a diagonal matrix obtained by permutation of diagonal entries of N, the same inequality is valid also for the elements ν_i' of N'. Note that though the construction depends on the choice of D, the bounds on the differences $|\nu_i' - \nu_j'|$ are uniform.

Now we use the fact that the sequence d_i was decreasing fast: the diagonal matrix $D' = D + N'$ also has nonincreasing integer entries. Indeed,

$$d'_k - d'_{k+1} = (d_k - d_{k+1}) + (\nu'_k - \nu'_{k+1}) > (m-1)^2 - (m-1)^2 = 0$$

by (23.5) and (23.4).

But then again by Lemma 22.23 the product $t^{D'}t^A$ will be Fuchsian, and its multiplication by a holomorphically invertible germ $H'(t)$ cannot change this fact. The equality (23.7) means that the initial system is gauge monopole equivalent to a system having a Fuchsian singularity at the origin. This proves the theorem, since all other singularities remain Fuchsian. □

24. Negative answer for the Riemann–Hilbert problem in the reducible case

By Bolibruch–Kostov theorem, any irreducible matrix group can be realized as a monodromy of a Fuchsian system on the Riemann sphere. In this section we explain why certain *reducible* matrix groups cannot be realized by Fuchsian systems. Plemelj theorem (Theorem 23.4) indicates that such counterexamples are possible only when all monodromy matrices have nontrivial Jordan form.

24.1. Systems of the class B.

Definition 24.1. A linear operator $M\colon \mathbb{C}^n \to \mathbb{C}^n$ is said to be *of class* B, if its Jordan normal form consists of a single block of maximal size.

From this definition, it follows that an operator of class B has a unique eigenvalue ν and for any $k \leqslant n$ the power $(M - \nu E)^k$ has the rank *exactly* equal to $n - k$. Invariant subspaces of operators of class B can be easily described.

Lemma 24.2. *For any $k \leqslant n$ an operator of class B has a unique k-dimensional invariant subspace. In a basis in which M has an upper triangular matrix, this subspace is spanned by the first k vectors.*

Proof. Without loss of generality assume that the unique eigenvalue of M is zero, $\nu = 0$, that is, M is nilpotent.

If V is an invariant subspace of dimension $k \leqslant n$ for M, then the restriction of M on V must also be nilpotent, more precisely, $M^k|_V = 0$. But for a nilpotent operator of class B the rank of M^k is exactly $n - k$, which means that $\dim \operatorname{Ker} M^k = k$, and hence V must coincide with $\operatorname{Ker} M^k$, being thus uniquely defined.

It remains to notice that for an upper-triangular nilpotent matrix M, $\operatorname{Ker} M^k$ consists of the first k basic vectors. □

Remark 24.3. It is important to notice that operators of class B admit in a sense unique matrix logarithm. More precisely, any two matrix logarithms A, A' of the same operator of class B differ by an integer multiple of the identity matrix modulo conjugacy:

$$\exp A = \exp A' \text{ is of class } B \implies A - CA'C^{-1} = 2\pi i k E$$

for a suitable integer number $k \in \mathbb{Z}$ and an invertible conjugacy matrix $C \in \mathrm{GL}(n, \mathbb{C})$. Moreover, the spectrum of either logarithm consists of a single number of maximal multiplicity n.

To see this, consider the Jordan basis for A. If A has more than one block, then its exponent will also be block diagonal, resulting in more invariant subspaces than allowed by definition of the class B. In particular, the spectra of both A and A' must be singletons (consist of single complex numbers). Denote them by λ and λ' respectively: they must differ by $2\pi i k$, $k \in \mathbb{Z}$. The differences $N = A - \lambda E$ and $N' = A' - \lambda' E$ are both nilpotent and hence are obviously conjugated by an invertible matrix. If A is in the Jordan form, then C must be upper-triangular.

This observation means that the freedom in constructing local solutions X_j in the proof of the Röhrl–Plemelj theorem is very limited: if chosen among solutions of Euler systems, they are defined uniquely modulo transformations $X(t) \rightsquigarrow t^k C X(t)$.

Definition 24.4. The matrix group generated by the invertible matrices $M_1, \ldots, M_m \in \mathrm{GL}(n, \mathbb{C})$ with the single restriction $M_1 \cdots M_m = E$, is called the *group of class B*, if:

(1) each M_j is of class B with the eigenvalue $\nu_j \neq 0$, and
(2) the group generated by $M_1, \ldots, M_m$ is reducible, i.e., the operators $M_1, \ldots, M_m$ have a common nontrivial invariant subspace.

A Fuchsian system (24.2) on the Riemann sphere is called a *system of class B*, if its monodromy is a group of class B.

Reducibility of a matrix group means that there exists a common invariant subspace for all matrices from this group. Choosing a suitable basis in the linear space, one can reduce all matrices generating the matrix group of class B to a block upper-triangular form with a zero lower left corner,

$$M_j = \begin{pmatrix} M_j' & * \\ 0 & * \end{pmatrix}, \qquad j = 1, \ldots, m, \qquad M_j' \in \mathrm{GL}(k, \mathbb{C}). \tag{24.1}$$

The square $k \times k$-matrices M_j' are nondegenerate and correspond to the restriction of the matrix group on the invariant k-dimensional subspace.

24.2. Residues of systems of class B and their eigenvalues. Consider a Fuchsian system on the Riemann sphere with m singular points, for simplicity all being in the finite part $\mathbb{C} \subset \mathbb{C}P^1$:

$$\dot{x} = A(t)x, \qquad A(t) = \sum_{j=1}^{m} \frac{A_j}{t - t_j}, \quad \sum_j A_j = 0. \tag{24.2}$$

Eigenvalues of the local monodromy operators M_j are exponentials of eigenvalues of the respective residues A_j. The fact that each M_j has only one eigenvalue ν_j, means in general only that the eigenvalues of A_j are all within one resonant group, i.e., all of them differ by integer numbers. However, for systems of class B this cannot happen: *all eigenvalues of each residue must coincide.*

Theorem 24.5. *For a system of class B, the spectrum of each residue matrix A_j consists of only one eigenvalue λ_j.*

This is the key assertion whose proof we postpone until §24.7. Later, in §25 we give a geometric explanation of this result, stressing its global nature.

Note that the assertion of Theorem 24.5 *does not follow* from the observation made in Remark 24.3. The uniqueness of eigenvalues of the residue, asserted there, concerns only Euler systems. Lemma 21.26 easily allows to construct a system in the Poincaré–Dulac–Levelt normal form with different eigenvalues of the residue matrix and a prescribed monodromy of class B.

Corollary 24.6. *For a system of class B, the product of all eigenvalues ν_j of all monodromy operators, must be equal to* 1.

Proof of the Corollary. Because of the uniqueness of each eigenvalue, $\lambda_j = \frac{1}{n} \operatorname{tr} A_j$. Since $\sum_{j=1}^m \operatorname{tr} A_j = \operatorname{tr} \sum_{j=1}^m A_j = 0$, we have $\sum_j \lambda_j = 0$. But $\nu_j = \exp 2\pi i \lambda_j$, hence $\prod_j \nu_j = 1$. □

As an immediate conclusion, we obtain a necessary condition for a matrix group to be the monodromy group of a Fuchsian system.

Corollary 24.7. *A matrix group of the class B can be realized as the monodromy group of a Fuchsian system on the unit sphere, only if the product of all eigenvalues of the matrices M_j is* 1. □

Note that for systems of the class B the product $\nu_1 \cdots \nu_m$ is always a root of unity, since the product of all *determinants* $\det M_j = \nu_j^n$ is equal to $1 = \det E$.

24.3. Monodromy group that cannot be realized by a Fuchsian system. The following statement can be verified by the straightforward computations.

Lemma 24.8. *The three matrices* M_1, M_2, M_3,

$$\begin{pmatrix} 1 & 1 & & \\ & 1 & 1 & \\ & & 1 & 1 \\ & & & 1 \end{pmatrix}, \quad \begin{pmatrix} 3 & 1 & 1 & -1 \\ -4 & -1 & 1 & 2 \\ & & 3 & 1 \\ & & -4 & -1 \end{pmatrix}, \quad \begin{pmatrix} -1 & & 2 & -1 \\ 4 & -1 & & 1 \\ & & -1 & \\ & & 4 & -1 \end{pmatrix} \tag{24.3}$$

generate the matrix group of class B*. At the same time, their eigenvalues*

$$\nu_1 = \nu_2 = 1, \qquad \nu_3 = -1$$

do not meet the product condition from Corollary 24.6. □

As an immediate corollary, we obtain the following impossibility theorem.

Theorem 24.9 (Bolibruch counterexample)**.** *The matrix group generated by the three matrices* (24.3)*, cannot be realized as the monodromy group of a Fuchsian system with three singular points* t_1, t_2, t_3 *on the Riemann sphere in such a way that the operator* M_j *corresponds to a positive circuit around* t_j*.* □

The rest of this section is devoted to the proof of Theorem 24.5. We first give it in elementary terms and later in §25 describe the geometric construction behind these arguments.

24.4. Determinant exponents. We introduce an invariant of holomorphic classification of regular singularities. Consider a linear system having a regular singular point at the origin, and denote by $X(t)$ an arbitrary fundamental matrix solution of this system.

Lemma 24.10. *The determinant* $h(t) = \det X(t)$ *can be represented as*

$$\det X(t) = t^\alpha u(t), \qquad \alpha \in \mathbb{C}, \quad u(0) \neq 0,$$

with some complex number α *and a holomorphic invertible germ* u*.*

The number α *does not depend on the choice of the fundamental solution* $X(t)$ *and is the same for two holomorphically equivalent singularities.*

For a Fuchsian singular point with the residue matrix $A \in \operatorname{Mat}(n, \mathbb{C})$

$$\alpha = \operatorname{tr} A.$$

Definition 24.11. The number α will be called the *determinant exponent* of the regular singularity.

Proof of the Lemma. All assertions follow from the Liouville–Ostrogradskii formula: if $\dot X(t) = A(t)X(t)$ and $h(t) = \det X(t)$, then

$$\dot h(t) = a(t)\, h(t), \qquad a(t) = \operatorname{tr} A(t).$$

The exponent α is the residue of the meromorphic function $a(t) = \frac{\alpha}{t} + \cdots$ at the origin, which proves the assertion for Fuchsian singularities.

If $X'(t) = H(t)X(t)$ with $\det H(t) \neq 0$ or $X'(t) = X(t)C$, $\det C \neq 0$, then the determinant changes by a holomorphic invertible factor and hence the exponent α remains the same. $\square$

Since the sum of residues of the rational function $a(t) = \operatorname{tr} A(t)$ on the Riemann sphere is zero, we have immediately the following Corollary.

Corollary 24.12. *For any regular system on the Riemann sphere $\mathbb{C}P^1$, the sum of determinant exponents of all singular points is zero.* $\square$

24.5. Systems with reducible monodromy. Subsolutions. Consider a linear system (24.2) with reducible monodromy. By definition, this means that in the linear n-dimensional space of its (vector) solutions, there is a k-dimensional subspace invariant by all monodromy transformations. This subspace is spanned by some k vector solutions. By the isomorphism established in Theorem 20.3 (assertion 3), the values of these solutions are linear independent at any nonsingular point $t \notin \Sigma$.

Arranged in the form of a rectangular $n \times k$-matrix $X'(t)$, they satisfy the identities

$$\Delta_\gamma X'(t) = X'(t) M'_\gamma, \tag{24.4}$$

where M'_γ are nondegenerate $k \times k$-matrices (restrictions of the reducible monodromy matrices on the invariant subspace).

Since the rank of $X'(t)$ is k for any $t \notin \Sigma$, one of its $k \times k$-minors is not identically zero; without loss of generality we assume that this minor consists of the first k rows, writing

$$X'(t) = \begin{pmatrix} Y(t) \\ * \end{pmatrix}$$

where $Y(t)$ is a square $k \times k$-matrix function not identically degenerate. Its monodromy properties follow from (24.4):

$$\Delta_\gamma Y(t) = Y(t) M'_\gamma.$$

The matrix 1-form $\Omega' = dY \cdot Y^{-1} = B(t)\,dt$ is univalent, $\Delta_\gamma \Omega = \Omega$, hence rational because all singularities of $Y(t)$ are regular.

This means that $Y(t)$ must satisfy a system of linear ordinary differential equations with rational coefficients

$$\dot{Y}(t) = B(t) Y(t), \tag{24.5}$$

with an appropriate rational matrix function $B(t)$. We will refer to (24.5) as the invariant *subsystem* of (24.2), calling $Y(t)$ a *subsolution*. The construction is not canonical: first, one can choose a different basis in the k-subspace (resulting in another subsolution of the same system (24.5)). Besides, one can choose the rows containing a nonzero minor of X' in a different way. The system (24.5) will be replaced then by an equivalent system, in general the equivalence being only meromorphic.

Singular points of the subsystem (24.5) can be of two kinds,

(1) *true*, occurring at the same places where the singularities $t_1, \dots, t_m$ of the initial system (24.2) were, and

(2) *apparent*, occurring at the points where the minor $Y(t)$ degenerates while remaining holomorphic.

24.6. Determinant exponents of a subsolution. The following key result asserts certain inequalities on the determinant exponents of subsolutions of a system of class B. In this section we write $\alpha \geqslant \beta$ for two *complex* numbers $\alpha, \beta \in \mathbb{C}$ if their difference is a nonnegative *real* number.

Lemma 24.13. *For a Fuchsian system of class B, the determinant exponent α_* of a k-dimensional subsolution $Y(t)$ at a singular point t_* satisfies the following inequalities*:

(1) *if $t_* \notin \Sigma$ is an apparent singularity of the subsolution $Y(t)$, then $\alpha_* \geqslant 1$;*

(2) *if $t_* = t_j$ is a true singularity of the subsolution $Y(t)$, then*

$$\alpha_* \geqslant \alpha'_j,$$

where α'_j is the sum of k biggest eigenvalues of the residue matrix A_j.

Proof. The first assertion is obvious, since $\det Y(t)$ is holomorphic at an apparent singular points, and the determinant exponent is simply its order of zero, a natural number. We will prove the second assertion in two steps.

1. Assume first that the initial Fuchsian system is in the Poincaré–Dulac–Levelt normal form: its coefficient matrix is upper-triangular and the eigenvalues ordered in the nonincreasing order. Notice that for a singularity of the class B *all* eigenvalues belong to the same resonant group, hence any two eigenvalues must by comparable in the sense of the order $\geqslant$. This observation makes the expression "biggest eigenvalues" occurring in the formulation of the lemma, unambiguous.

One k-dimensional locally invariant subsolution of this system can be immediately constructed. Since $A(t)$ is upper triangular, the subspace spanned by any first k coordinate axes, is invariant. The upper left $k \times k$-block $A'(t)$ of the upper triangular matrix $A(t)$ is the coefficient matrix for the restriction of the initial system on this invariant subspace.

In coordinates this means that the rectangular $n \times k$-matrix

$$X'(t) = \begin{pmatrix} Y'(t) \\ 0 \end{pmatrix}$$

will satisfy the equation $\dot{X}'(t) = A(t)X'(t)$ provided that its upper part $Y(t)$ satisfies the system

$$\dot{Y}'(t) = A'(t)Y'(t).$$

The determinant exponent of the subsolution $Y(t)$ is the trace of the residue of $A'(t)$ which is the sum of k first eigenvalues of the residue of $A(t)$ at t_*. But the first are the biggest, so the assertion is proved for this particular subsolution of a system in the normal form.

2. Let t_* be an arbitrary singularity of class B, not necessarily in the normal form, and $Y(t)$ a k-dimensional subsolution. By definition, this means that there exists a rectangular $n \times k$-matrix solution $X''(t) = \begin{pmatrix} Y'(t) \\ * \end{pmatrix}$ of the system, invariant by the local monodromy operator.

By the holomorphic classification theorem (specifically, by the Corollary 21.25), there exists a holomorphic local gauge transformation conjugating the system with an upper-triangular Poincaré–Dulac–Levelt normal form. Denote by $H(t)$ the matrix of the *inverse* transformation.

Let $X'(t)$ be a rectangular invariant subsolution for the normal form, constructed on Step 1. The rectangular $n \times k$-matrix $H(t)X'(t)$ is a locally invariant rectangular solution of the initial system near t_*, so that, in particular, the linear span of the columns is invariant by the local monodromy operator that is by assumption an operator of the class B. But by Lemma 24.2, such subspace must be unique, hence $H(t)X'(t)$ must coincide with $X''(t)$, eventually modulo a constant invertible right $k \times k$-matrix factor C.

Writing $H(t)$ in the block form, we obtain

$$X''(t) = \begin{pmatrix} Y(t) \\ * \end{pmatrix} = H(t)X'(t)C = \begin{pmatrix} H_{11}(t) & H_{12}(t) \\ H_{21}(t) & H_{22}(t) \end{pmatrix} \begin{pmatrix} Y'(t) \\ 0 \end{pmatrix} C$$

with holomorphic blocks $H_{ij}(t)$. Hence

$$\det Y(t) = \det H_{11}(t) \det Y'(t) \det C = (t - t_j)^{\alpha_j} u(t),$$
$$\alpha_j = r + \alpha'_j, \qquad u(t_j) \neq 0,$$

where α'_j is the determinant exponent of the subsolution $Y'(t)$ computed on Step 1 and r a nonnegative integer, the order of zero of $\det H_{11}(t)$ at t_j. This proves that the determinant exponent α_j is greater or equal to α'_j. □

Corollary 24.14.

$$\alpha_j \geqslant \tfrac{k}{n} \operatorname{tr} A_j, \tag{24.6}$$

and the inequality is strict unless all eigenvalues of A_j coincide between themselves.

Proof. This follows from Lemma 24.13 and the inequality $\alpha'_j \geqslant \frac{k}{n} \operatorname{tr} A_j$ which is obvious. Indeed, the right hand side of it is k times the *average* eigenvalue of A_j, while the left hand side is the sum of k *biggest* eigenvalues. Clearly, this latter inequality is *strict* if there are unequal eigenvalues. □

24.7. Proof of Theorem 24.5. Theorem 24.5 follows immediately from the above inequalities. Indeed, consider a Fuchsian system of class B and any its subsolution $Y(t)$ corresponding to the invariant subspace of the monodromy. By Corollary 24.12, the sum σ of *all* determinant exponents for this subsolution is zero. On the other hand, let $\nu \geqslant 0$ be the number of apparent singularities for A'. Then by Corollary 24.14,

$$0 = \sigma \geqslant \nu + \sum_j \alpha'_j \geqslant \nu + \tfrac{k}{n} \operatorname{tr} A_j \geqslant \nu + \tfrac{k}{n} \sum_j \operatorname{tr} A_j = \nu \geqslant 0,$$

where the summation is extended over all true singular points of the subsystem. This is possible only if $\nu = 0$ and for all $j = 1, \ldots, m$,

$$\alpha'_j = \tfrac{k}{n} \operatorname{tr} A_j.$$

In other words, for any Fuchsian system of class B the corresponding system (24.5) for any subsolution has no apparent singular points. Moreover, the sum of k biggest eigenvalues of each residue matrix for a system of class B is equal to k times the average eigenvalue. The latter equality in turn is possible only if all eigenvalues of each residue coincide. □

25. Riemann–Hilbert problem on holomorphic vector bundles

Many constructions of this chapter admit a natural geometric interpretation in the language of *holomorphic vector bundles* over Riemann surfaces. The subject is fairly canonical: its excellent treatment can be found in numerous textbooks, among them [**GH78, For91**] and very recently in [**Bol00**]. In this short section we simply recall the basic vocabulary of the language and supply geometric "translations" for constructions from the preceding sections, focusing on explanation of Theorem 24.5. The first part of this section contains the proof of the fact that the sum of traces of residues of a meromorphic connection on a holomorphic vector bundle is equal to the degree of this bundle, an integer number that is always nonnegative for subbundles of a trivial bundle. The second half explains why Fuchsian connections with singularities of class B only, cannot have invariant subbundles unless each residue has a single eigenvalue.

25.1. Holomorphic vector bundles. A holomorphic n-dimensional vector bundle over a Riemann surface T is a holomorphic map of constant rank ("projection") $\pi\colon S \to T$ of an analytic manifold of dimension $n+1$, the *total space* of the bundle, onto the Riemann surface T (the *base*), which is *locally trivial* in the following sense.

Every point $t \in T$ admits a neighborhood $U \subset T$ and a biholomorphic map (*local trivialization*) $\varPhi_U\colon \pi^{-1}(U) \to U \times \mathbb{C}^n$ between the preimage $\pi^{-1}(U) \subset S$ and the Cartesian product $U \times \mathbb{C}^n$. The linear structure these trivializations induce on the preimages $S_t = \pi^{-1}(t)$, called *fibers*, must be coherent: if $U, V \subset T$ are two intersecting open sets with the respective trivializations $\varPhi_U, \varPhi_V$, then the *transition map* $\varPhi_V \circ \varPhi_U^{-1}\colon U \times \mathbb{C}^n \mapsto V \times \mathbb{C}^n$ between them must be linear in the second component,

$$\varPhi_V \circ \varPhi_U^{-1}(t,x) = (t, F(t)x), \qquad F = F_{VU} \in \mathrm{GL}(n, \mathcal{O}(U \cap V)),\ x \in \mathbb{C}^n.$$

Here $F = F_{UV}$ is a holomorphic holomorphically invertible $n \times n$-matrix function, called the *transition matrix*. The trivializations and the respective transition maps play the same role in the definition of vector bundles, as the charts and the transition maps play in the definition of smooth manifolds.

Speaking informally, a holomorphic bundle is a union of linear spaces (fibers) parameterized by points of the Riemann surface in a locally trivial way. The *trivial bundle* (*standard cylinder*) $T \times \mathbb{C}^n$ is the simplest example of a holomorphic bundle (in this case all transition matrices are identical). The main source of bundles is geometry: the set of all (complex) vectors tangent to a complex manifold at different points, is the *tangent bundle* (it can be defined over manifolds of any dimension and in various categories,—smooth, real analytic, complex analytic). In a similar way, the *cotangent* bundle, whose sections are holomorphic 1-forms, can be defined (see Example 25.3 below).

Most linear algebraic definitions and constructions can be extended for the bundles. Thus, a *subbundle* $S' \subset S$ is an analytic submanifold of the total space, such that the intersection $S'_t = S' \cap S_t$ with any fiber S_t is a linear subspace of the latter. The sum of two subbundles $S', S'' \subset S$ is the bundle whose fibers are the sums $S'_t + S''_t$ for all $t \in T$. A *direct sum* of two bundles S', S'' is the bundle whose fibers are direct sums of fibers of the initial bundles. The dimension n of this new bundle is equal to the sum of dimensions $n' + n''$, and the transition matrices are block diagonal,

$$F_{VU} = \begin{pmatrix} F'_{VU} & \\ & F''_{VU} \end{pmatrix} \in \mathrm{GL}(n'+n'', \mathcal{O}(U \cap V)).$$

For our purposes we will need the *determinant* of an n-dimensional bundle S introduced in §25.8.

A *bundle map* between two bundles $\pi\colon S \to T$ and $\pi'\colon S' \to T'$ over two (in general, different) Riemann surfaces T, T', is a holomorphic map which sends fibers to fibers and is linear after restriction on each fiber. Such map is called an *equivalence* between bundles, if it is holomorphically invertible (the inverse map will be automatically a bundle map). A bundle over T is *trivial*, if it is equivalent to the standard cylinder $T \times \mathbb{C}^n$.

By definition, a bundle map B induces a holomorphic map b between the bases, $b\colon T \to T'$ (points of the fiber over each $t \in T$ are mapped to those in the fiber over $t' = b(t)$). The map B is said to be *fibered* over b. If B is a bundle equivalence, then b is necessarily a biholomorphic isomorphism between T and T'.

After choosing any two trivializations, $\varPhi_U\colon \pi^{-1}(U) \to U \times \mathbb{C}^n$ for the bundle S and $\varPhi'_{U'}\colon (\pi')^{-1}(U') \to U' \times \mathbb{C}^{n'}$ for S' respectively, the bundle map $B\colon S \to S'$ is represented by a holomorphic map $\varPhi_{U'} \circ B \circ \varPhi_U^{-1}\colon U \times \mathbb{C}^n \to U' \times \mathbb{C}^{n'}$. This map renders commutative the diagram

$$\begin{array}{ccc} (t,x) & \xrightarrow{\varPhi_U^{-1} B \varPhi_{U'}} & (t',x') \\ {\scriptstyle\pi}\big\downarrow & & \big\downarrow{\scriptstyle\pi'} \\ t & \xrightarrow{b(\cdot)} & t' \end{array} \qquad \begin{gathered} (t,x) \in U \times \mathbb{C}^n, \quad (t',x') \in U' \times \mathbb{C}^{n'}, \\ t' = b(t), \quad x' = B_{U'U}(t)\,x, \end{gathered} \tag{25.1}$$

where $B_{U'U}(t) \in \operatorname{Mat}_{n\times n'}(\mathcal{O}(U))$ is a matrix function considered as a linear map from $\mathbb{C}^n$ to $\mathbb{C}^{n'}$ analytically depending on t.

We will be mostly interested in bundle maps between bundles over the same base T. Moreover, for our purposes it will be sufficient to consider only maps fibered over the identity map of T, i.e., sending each fiber S_t into S'_t over the same point t over the base.

25.2. Sections. A *holomorphic section* of the holomorphic bundle $\pi\colon S \to T$ is a holomorphic map $s\colon T \to S$ which satisfies the condition $\pi(s(t)) \equiv t$. In other words, sections can be described as *fiber-valued functions* on T which choose one vector from each fiber S_t holomorphically depending on the point of t the base, determining the fiber.

In each trivialization $\varPhi_U$ a section is represented by a holomorphic vector-function $x_U\colon U \to \mathbb{C}^n$ so that $\varPhi_U(s(t)) = (t, x_U(t)) \in U \times \mathbb{C}^n$. Over a nonempty intersection $U \cap V$, two coordinate representations $x_U(t)$ and $x_V(t)$ of the same section must be related by the transition matrix,

$$x_V(t) = F_{VU}(t)\,x_U(t), \qquad t \in U \cap V, \quad x_U(t), x_V(t) \in \mathbb{C}^n. \tag{25.2}$$

A section is *nonzero* if $x_U \not\equiv 0$, and *nonvanishing*, if x_U is a nonvanishing vector-function on its domain U. Both properties are invariant by the transition maps.

Not all bundles admit globally defined holomorphic sections. A *local* section may be defined over a proper subset $U \subsetneq T$.

Instead of attempting to define meromorphic maps of T to S in invariant terms, we can use the local representations to introduce the notion of *meromorphic* sections of a holomorphic bundle as a collection of *meromorphic* vector-functions $\{x_U(t)\}$ associated to each local trivialization, and satisfying the transition conditions (25.2) on the intersections of any two trivializing charts.

The linear structure on each fiber induces the structure of a module on the set of holomorphic (resp., meromorphic) sections of a holomorphic bundle over the ring of holomorphic (resp., meromorphic) functions on the base. This means that sections can be added between themselves and multiplied by (scalar) functions. In particular, one can say about linear (in)dependence of sections. The module of holomorphic sections will be denoted by $\Gamma^0(S)$ or simply $\Gamma(S)$.

In a similar way, one can introduce the notions of *fiber-valued holomorphic or meromorphic* 1*-forms* on T: any such form is a linear functional on the tangent spaces at different points t of the base, which takes values in the respective fiber S_t. In coordinates (after trivialization) such forms are represented by collection of holomorphic (resp., meromorphic) vector-valued 1-forms $\omega_U \in \Lambda^1(U) \otimes \mathbb{C}^n$ associated with each trivialization Φ_U, satisfying the transition condition $\omega_{U'} = F_{U'U} \cdot \omega_U$ on the intersections (recall that all transition maps between trivializations are fibered over the identity map of T). For a given vector bundle S, we denote by $\Gamma^1(S)$ the module of holomorphic S-valued 1-forms on T over the ring $\mathcal{O}(T)$. Their meromorphic counterparts, meromorphic fiber-valued k-forms on T, $k = 0, 1$, will be denoted by $\mathcal{M}^k(S)$. They are modules over the field $\mathcal{M}(T)$ of meromorphic functions on T.

A point $t_0 \in T$ is a pole for a meromorphic section $s \in \mathcal{M}^0(S)$, if it is a pole for any coordinate representation $x_U(\cdot)$ of s. Since the transformations (25.2) preserve the *order of pole* of meromorphic vector functions, this order is well-defined for singular points of meromorphic sections. A singular point is *simple*, if this order is equal to 1. For a simple pole, the notion of the *residue* $\operatorname{res}_{t_0} s$ is well defined as an element of the fiber S_{t_0}.

When discussing connections, we also will use *operator-valued* functions on T, assigning to each point $t \in T$ a linear endomorphism of the respective fiber in a fashion holomorphically depending on t. In coordinates such objects are represented by collection of holomorphic matrix-functions $\{A_U(t)\}$ meeting the condition $A_{U'} = F_{U'U} A_U F_{UU'}$ on the intersections. The notion of *operator-valued* 1*-forms* on T is obtained by obvious modifications. The same refers to *meromorphic* counterparts of the holomorphic prototypes.

To conclude this brief synopsis, we mention the fundamental fact: a holomorphic n-dimensional vector bundle is trivial if and only if this bundle admits n holomorphic sections linear independent everywhere. In one direction it is obvious. To prove the 'if' part, consider the map $B\colon T \times \mathbb{C}^n \to S$, $(t, x) \mapsto (t, x_1\, s_1(t) + \cdots + x_n\, s_n(t))$, where $x = (x_1, \ldots, x_n) \in \mathbb{C}^n$ and $s_1, \ldots, s_n \in \Gamma^0(S)$ are the sections. This bundle map is holomorphic and obviously invertible.

25.3. Connections on holomorphic bundles. A *holomorphic connection* on a holomorphic vector bundle S over a Riemann surface T is a geometric object corresponding to a system of linear differential equations with the independent variable ranging over T and the dependent variables ranging over the fibers of the bundle.

More formally, a holomorphic connection on a bundle $\pi\colon S \to T$ is a differential operator $\nabla\colon \Gamma^0(S) \to \Gamma^1(S)$, which is $\mathbb{C}$-linear and satisfies the Leibniz rule

$$\nabla(fs) = df \otimes s + f\nabla s, \qquad s \in \Gamma^0(S), \quad f \in \mathcal{O}(T) \tag{25.3}$$

for any holomorphic section $s \in \Gamma^0(S)$ and any holomorphic function $f \in \mathcal{O}(T)$. The value of ∇s on a holomorphic (or meromorphic) vector field v on T is denoted by $\nabla_v s$ and is a holomorphic (resp., meromorphic) section of the same bundle; the value of $(\nabla_v s)(t)$ over a point $t \in T$ depends linearly on the vector $v(t)$ at this point. The Leibniz rule means that

$$\left(\nabla_v(fs)\right)(t) = g(t) \cdot s(t) + f(t) \cdot (\nabla_v s)(t),$$

where $g = df \cdot v = \partial f/\partial v$ is the Lie derivative of f along v, whose value at t is $g(t) = df(t) \cdot v(t) \in \mathbb{C}$.

Given any two connections ∇, ∇' on the same bundle, their difference $\Theta = \nabla - \nabla'$ is a differential operator of zero order. This means that $\Theta\colon \Gamma^0(S) \to \Gamma^1(S)$ is an operator-valued holomorphic 1-form on T in the sense described in §25.2. Indeed, from (25.3) it immediately follows that

$$\Theta(fs) = f \cdot \Theta s, \qquad s \in \Gamma^0(S), \quad f \in \mathcal{O}(T),$$

that is, the value of Θs of the section s at any point $t_0 \in T$ depends only on the value $s(t_0) \in S_{t_0}$ at this point.

On the standard cylinder $T \times \mathbb{C}^n$ there always exists the 'standard' connection

$$\nabla' s = ds, \qquad s\colon T \to \mathbb{C}^n,$$

where d is applied as an exterior derivative to each component of the section s considered as a holomorphic vector function from T to $\mathbb{C}^n$. By the above observation, any other connection ∇ acting on sections of the trivial bundle can always be written as

$$\nabla s = ds - \Omega s, \qquad \Omega \in \operatorname{Mat}(n, \Lambda^1(T)),\ s\colon T \to \mathbb{C}^n,$$

where Ω is a $n \times n$-matrix valued 1-form, called the *connection form*.[2]

Definition 25.1. A section s is called *horizontal*, if $\nabla s = 0$.

A section of the trivial bundle is horizontal for a connection with the form Ω, if and only if it satisfies the system of linear ordinary differential equations $ds = \Omega s$, coinciding with (20.3).

If ∇ is a holomorphic connection on a nontrivial bundle S, then each trivializing chart $\Phi_U\colon S \supseteq \pi^{-1}U \to U \times \mathbb{C}^n$ transforms ∇ into a connection on the trivial bundle $U \times \mathbb{C}^n$ and uniquely associates with ∇ the corresponding connection form Ω_U. On the intersection $U \cap V$ of two trivializations, the respective connection forms $\Omega = \Omega_U$ and $\Omega' = \Omega_V$ are gauge equivalent: if $F = F_{VU} = F_{UV}^{-1}$ is the transition matrix between the two trivializations, then

$$\Omega' = dF \cdot F^{-1} + F\Omega F^{-1}, \qquad F = F_{VU} = F_{UV}^{-1},\ \Omega = \Omega_U,\ \Omega' = \Omega_V. \tag{25.4}$$

Horizontal sections uniquely determine the connection: if $s_1, \dots, s_n \in \Gamma^0(S)$ are n sections of an n-dimensional vector bundle, that are linearly independent at each point, then there is a unique connection ∇ for which all these sections are horizontal. In each trivializing chart, if the sections are represented by columns of a holomorphic nowhere degenerating $n \times n$-matrix function $X_U(t)$, then the connection matrix Ω_U must satisfy the identity

$$\Omega_U = dX_U \cdot X_U^{-1}. \tag{25.5}$$

This defines the matrix connection forms uniquely, and they obviously satisfy the condition (25.4).

Remark 25.2. Existence of *local* horizontal sections is a consequence of the fact that the base T is one-dimensional: after choosing an arbitrary trivialization, this follows from the local existence theorem for linear systems (Lemma 20.6). Similarly to the usual linear systems, local horizontal sections may not extend globally if the base is multiply connected.

Add reference to [**Del70**]/modify the text

For connections on bundles over multidimensional manifolds, there is an obstruction to existence of horizontal connections, even locally. This obstruction is called the *curvature* of the connection. To distinguish this general case, connections that locally admit horizontal sections through any point on a fiber, are called *flat* connections.

[2] Very often the connection form differs from our definition by the sign, e.g., see [**Del70**].

25.4. Meromorphic connections. Singular points. Residues. The definition of meromorphic connection differs from that of a holomorphic connection by obvious modifications only. A meromorphic connection is a differential operator $\nabla\colon \mathcal{M}^0(S) \to \mathcal{M}^1(S)$, taking meromorphic sections of S to meromorphic fiber-valued 1-forms and satisfying the axiom (25.3). In coordinates (i.e., after choosing a trivialization Φ_U over an open set $U \subset T$), a meromorphic connection is completely determined by a matrix connection form Ω_U with meromorphic entries. The connection forms associated with different trivialization, are related by the same gauge equivalence (25.4). Thus any property of a linear system that is invariant by gauge transformations, admits generalization for meromorphic connections.

A point $t_0 \in T$ is singular for a connection, if it is singular for the connection form Ω in some (hence in any) trivializing chart containing the fiber S_{t_0}. A singular point is *Fuchsian* (sometimes referred to as a *logarithmic singularity*), if all connection forms have a first order pole at this point (this definition is specific for connections over one-dimensional base).

Let Ω, Ω' be two connection forms in two trivializations over the same Fuchsian singular point t_0, related by the transition identity (25.4). Since the term $dF \cdot F^{-1}$ is holomorphic, the *matrix residues* $A = \operatorname{res}_{t_0} \Omega$, $A' = \operatorname{res}_{t_0} \Omega'$ are related by the identity

$$A' = CAC^{-1}, \qquad C = F(t_0) \in \mathrm{GL}(n, \mathbb{C}).$$

This means that the *residue of the connection* $\operatorname{res}_{t_0} \nabla$ makes an invariant sense as a well defined linear map of the fiber $S_{t_0} = \pi^{-1}(t_0)$ into itself. This linear map is related to the limit holonomy operator F_{t_0}, the linear automorphism of the fiber S_{t_0}, exactly as in Proposition 21.16.

The determinant, trace and the spectrum of a residue $\operatorname{res}_{t_0} \nabla$ at a given Fuchsian singular point t_0 are well defined complex numbers (resp., a collection of complex numbers), since these notions are invariant by gauge transformations.

25.5. Cocycles and holomorphic vector bundles. Holomorphic vector bundles can be constructing by patching together cylinders (trivial bundles over open subsets of the base), using holomorphic matrix cocycles for the patching. This construction is similar to constructing a manifold from an atlas of charts and transition maps between them.

Let $\mathfrak{U} = \{U_i\}$ be an open covering of the Riemann surface $T = \bigcup_i U_i$ and $\mathcal{F} = \{F_{ij}\}$ a holomorphic matrix cocycle inscribed in this covering. Consider the disjoint union $\bigsqcup_i U_i \times \mathbb{C}^n$ of the cylinders $U_i \times \mathbb{C}^n$ over U_i, and let S be the quotient space obtained by the following identification. Two points,

$(t,x) \in U_i \times \mathbb{C}^n$ and $(t',x') \in U_j \times \mathbb{C}^n$ are identified, if and only if

$$t = t' \in U_i \cap U_j, \qquad x = F_{ij}(t)x' \iff x' = F_{ji}(t)x. \tag{25.6}$$

The cocycle identities (22.1) ensure that this identification is a consistent transitive equivalence relationship on the disjoint union $\bigsqcup U_i \times \mathbb{C}^n$. Hence the quotient space S can be equipped with the structure of an analytic manifold with the cylinders $U_i \times \mathbb{C}^n$ playing the role of coordinate charts.

The canonical projections $(t,v) \mapsto t$ of $U_i \times \mathbb{C}^n$ on the first component together define a holomorphic *projection*

$$\pi \colon S \to T. \tag{25.7}$$

The local triviality of the constructed map π is tautological: for any point $t \in T$ one can choose any of the domains U_i containing t as the trivializing chart. Two such charts are related by a transformation (25.6) linear in x, x'.

Example 25.3. Consider an open covering $\{U_i\}$ of a Riemann surface T and assume that a nonvanishing holomorphic 1-form ω_i is defined in each U_i. Since any two 1-forms on T are proportional, on each intersection U_{ij} the holomorphic invertible functions f_{ij} appear, so that $\omega_i = f_{ij}\omega_j$.

The one-dimensional cocycle $\{f_{ij}\}$ corresponds to the cotangent bundle over T. Indeed, any section of this bundle, represented by a collection of holomorphic (scalar) functions $\{x_i(\cdot)\}$ satisfying the identities $x_i = f_{ij}x_j$ on the intersections U_{ij}, corresponds to a globally defined holomorphic 1-form ω equal to $x_i\omega_i$ in U_i, and vice versa.

Consider two bundles S, S' constructed from two cocycles $\mathcal{F}, \mathcal{F}'$ inscribed in the *same* covering but eventually of different dimensions. Any bundle map B from S to S' fibered over the identity map of the base, corresponds to a collection of holomorphic matrix functions $\mathcal{B} = \{B_i\}$ (of appropriate dimensions) which satisfy the identity

$$F_{ij}B_j = B_iF'_{ij} \qquad \text{on} \quad U_i \cap U_j.$$

If B is invertible (being thus a holomorphic equivalence) then the matrices B_i must be square and holomorphically invertible. This coincides with the definition of holomorphic equivalence of matrix cocycles: *two vector bundles built from cocycles inscribed in the same covering are equivalent if and only if the respective cocycles are equivalent.* In the same way *solvability of a cocycle means holomorphic triviality of the corresponding bundle.*

Theorems of §22 on solvability and equivalence of cocycles can be interpreted as theorems on holomorphic classification of vector bundles over the disk and the Riemann sphere. Thus, from Theorem 22.14 one can derive that any holomorphic vector bundle over the unit disk is trivial. This is the particular case of a more general claim.

Theorem 25.4 ([**For91**]). *Any holomorphic vector bundle over a noncompact Riemann surface is trivial.*

Among compact Riemann surfaces, the most important is the Riemann sphere $\mathbb{C}P^1 \simeq \mathbb{C} \cup \{\infty\}$. It can be covered by two charts (circular disks), e.g., $U_0 = \{|t| < 2\}$ and $U_1 = \{|t| > 1\} \cup \{\infty\}$. Over each disk the bundle is trivial by Theorem 25.4. Therefore any bundle over the entire sphere can be built from the two cylinders $U_i \times \mathbb{C}^n$, $i = 0, 1$, using an appropriate transition (gluing), represented by a Birkhoff–Grothendieck cocycle. By the Birkhoff–Grothendieck theorem (Theorem 22.19), classification of holomorphic bundles over the Riemann sphere reduces to classification of *standard bundles* corresponding to the standard cocycles with the transition function $F(t) = t^D$, $d = \operatorname{diag}\{d_1, \dots, d_n\}$, $d_i \in \mathbb{Z}$. Note that since the transition matrix of a standard Birkhoff–Grothendieck cocycle is diagonal, all coordinate axes are preserved when gluing the bundle. In other words, we have the following result.

Theorem 25.5. *Any holomorphic vector bundle over the Riemann sphere splits as a direct sum of n one-dimensional subbundles.*

Each standard one-dimensional subbundle has the transition function $f(t) = t^{d_i}$ for an appropriate integer number d_i, $i = 1, \dots, n$ (some of these numbers may coincide). In the next section we describe this number as the *degree* of a bundle.

One can show that the collection of the integer numbers $\{d_1, \dots, d_n\}$ (called *partial indices*), defined up to a permutation, is indeed a complete invariant of holomorphic equivalence of vector bundles over the Riemann sphere $\mathbb{C}P^1$: if two bundles are equivalent then their collections of partial indices must be the same [**GK60**].

25.6. Line bundles. One-dimensional bundles (corresponding to $n = 1$), referred to as *line bundles*, are especially important because of the commutativity of 1×1-matrices.

Let $s \in \mathcal{M}^0(S)$ be a meromorphic section of a line bundle, represented by meromorphic functions $x_U(\cdot)$ in respective trivializing charts.

Definition 25.6. The *order* of s at a point $t_0 \in T$ is an integer number $\operatorname{ord}_{t_0} s$ equal to the order of zero or the negative order of pole of any local representation of s. If s has neither zero nor pole at t_0, then $\operatorname{ord}_{t_0} s = 0$.

From (25.2) it follows that the order is well defined, since the transition matrix F_{UV} in the one-dimensional case is a holomorphic nonvanishing function. The *degree* of a holomorphic or meromorphic section is the total order of all points,

$$\deg s = \sum_{t\in T} \operatorname{ord}_t s, \qquad 0 \not\equiv s \in \mathcal{M}^0(S)$$

(this sum is in fact finite if T is compact). Degree of a *holomorphic* section s is always nonnegative, since $\operatorname{ord}_t s \geqslant 0$ everywhere on T.

Any two nonzero sections of a line bundle differ by a meromorphic factor: if $s, s' \in \mathcal{M}^0(S)$, then the ratio $f = s/s'$ does not depend on the trivialization and hence is a globally defined meromorphic *function* on T. Obviously, $\operatorname{ord}_t f = \operatorname{ord}_t s - \operatorname{ord}_t s'$ for any point $t \in T$. Since the total order $\sum_{t\in T} \operatorname{ord}_t f$ is zero for any meromorphic function $f \not\equiv 0$ on a compact Riemann surface, we obtain the following result.

Proposition 25.7. *All meromorphic sections of any line bundle over a compact Riemann surface T have the same degree.* □

The common degree of all meromorphic sections is called the *degree of the line bundle*. For trivial bundles there exist constant nonzero sections, hence degree of a trivial bundle is always zero. Degree of the tangent and cotangent bundles over $\mathbb{C}P^1$ are equal to $+2$ and -2 respectively: to see this, it is sufficient to compute the order of zero (resp., pole) of the "constant" vector field $\partial/\partial t$ (resp., 1-form dt) in the chart $z = 1/t$, at $z = 0$.

The degree is non-increasing by holomorphic bundle maps.

Proposition 25.8. *If $B\colon S \to S'$ is a holomorphic bundle map between two line bundles over the same base T, fibered over a holomorphically invertible base map b, then $\deg S \leqslant \deg S'$.*

Note that though the base map is assumed to be invertible, the bundle map B itself is not.

Proof. For any pair of trivializations $\Phi_U, \Phi_{U'}$, the map $\Phi_{U'} B \Phi_U^{-1}$ is represented as $(t, x) \mapsto (b(t), a(t)x)$ with a holomorphic factor $a(t)$ which may have zeros but not poles. For any meromorphic section $s \in \mathcal{M}^0(S)$ and its image $s' = Bs \in \mathcal{M}^1(S')$, this implies that $\operatorname{ord}_{b(t)} s' = \operatorname{ord}_t s + \operatorname{ord}_t a$. Since $\operatorname{ord}_t a \geqslant 0$, we conclude that

$$\operatorname{ord}_{b(t)} s' \geqslant \operatorname{ord}_t s, \qquad \forall t \in T.$$

Adding together these inequalities over all $t \in T$, we complete the proof. □

Holomorphically equivalent bundles obviously have the same degrees. For line bundles over the Riemann sphere the converse is also true.

Proposition 25.9. *Two line bundles of the same degree over $\mathbb{C}P^1$, are holomorphically equivalent.*

Proof. Let s, s' be any two meromorphic sections of the respective line bundles S, S' over $\mathbb{C}P^1$. Consider a meromorphic (rational) function $a(t)$ on $\mathbb{C}P^1$, which satisfies the condition

$$\operatorname{ord}_t a = \operatorname{ord}_t s - \operatorname{ord}_t s', \qquad \forall t \in \mathbb{C}P^1.$$

Since $\deg s = \deg s'$ by assumption, $\sum_{t \in \mathbb{C}P^1} \operatorname{ord}_t a = 0$, and such function can be explicitly constructed. Assuming that $t = \infty$ is nonsingular for both s and s', one may choose a as the product

$$a(z) = \prod_{t \in \mathbb{C}P^1} (z - t)^{\operatorname{ord}_t s - \operatorname{ord}_t s'}$$

(this product is in fact finite). The section $s'' = as'$ will have the same order as s at all points on the sphere. For each point t outside zeros and poles of s, there exists a unique invertible linear map that takes each fiber S_t into the fiber S'_t while mapping $s(t)$ to $s''(t)$. Since the two sections have the same orders at each point, this map extends analytically to the exceptional locus of zeros and poles and this extension remains holomorphically invertible there, defining thus a bundle equivalence. □

25.7. Connections on line bundles. A meromorphic connection on a line bundle is locally represented by *scalar* connection forms ω_U depending on the trivializations Φ_U. However, for any two connections their difference is a globally defined meromorphic 1-form on T. Indeed, this difference is a meromorphic 1-form with values in the space of linear maps from $\mathbb{C}^1$ to $\mathbb{C}^1$, which is isomorphic to $\mathbb{C}$ itself. Accordingly, residues of a meromorphic connection on a line bundle are well defined complex numbers.

Proposition 25.10. *The sum of residues of any meromorphic connection on a line bundle over a compact Riemann surface, is the same for all connections and depends only on the bundle.*

Proof. If ∇, ∇' are two meromorphic connections with Fuchsian (simple) singularities only, and $\Theta = \nabla - \nabla' \in \Lambda^1(T)$ is their difference, then by linearity $\operatorname{res}_{t_0} \nabla - \operatorname{res}_{t_0} \nabla' = \operatorname{res}_{t_0} \Theta$ for any singular point $t_0 \in T$.

By the Cauchy residue theorem, the sum of residues of any meromorphic 1-form on any compact Riemann surface is zero. Thus

$$\sum_{t_0} \operatorname{res}_{t_0} \nabla - \sum_{t_0} \operatorname{res}_{t_0} \nabla' = \sum_{t_0} \operatorname{res}_{t_0} \Theta = 0$$

for any two meromorphic connections ∇, ∇' on the same bundle. □

This sum can be immediately computed.

Proposition 25.11. *The sum of residues of any meromorphic connection on a line bundle over a compact Riemann surface, is equal to the degree of this bundle.*

Proof. To prove this assertion, it is sufficient to compute the total of all residues for any single meromorphic connection. Let $s \in \mathcal{M}^0(S)$ be an arbitrary meromorphic section. Consider the connection ∇ for which s is horizontal. The connection forms of this connection are the logarithmic derivatives of the local representations of s:

$$\omega = dx \cdot x^{-1}, \qquad \omega = \omega_U \in \Lambda^1(U),\ x = x_U \in \mathcal{M}(U).$$

The residue of ∇ at any singular point $t_0 \in T$ is equal to $\operatorname{ord}_{t_0} s$, as follows immediately from the local representation: if $x(t) = (t - t_0)^r\, h(t)$, $h(t_0) \neq 0$, then $\omega = r(t - t_0)^{-1}\, dt + dh/h$. The asserted claim follows now from Proposition 25.7. □

25.8. Determinant bundle. Any holomorphic vector bundle is in a canonical way related to a line bundle over the same base.

Definition 25.12. The *determinant* of a holomorphic vector bundle $\pi\colon S \to T$ is the line bundle, denoted by $\det S$, whose fibers are wedge powers $S_t \wedge \cdots \wedge S_t$ (n times).

The transition matrices of $\det S$ are the determinants (considered as 1×1-matrix functions),

$$f_{VU} = \det F_{VU}, \qquad f_{VU} \in \operatorname{GL}(1, \mathcal{O}(U \cap V)). \tag{25.8}$$

Any holomorphic bundle map B between holomorphic bundles S, S' of the same dimension, descends as a holomorphic bundle map, denoted by $\det B$, between their determinants $\det S$ and $\det S'$. In local trivializing coordinates (25.1), the determinant map $\det B$ corresponds to multiplication by the holomorphic function $\det B_{U'U}$.

Definition 25.13. Degree of a holomorphic vector bundle S over a compact Riemann surface is by definition the degree of its determinant, the line bundle $\det S$.

Degree can be only increased by holomorphic bundle maps.

Proposition 25.14. *If $B : S \to S'$ is a holomorphic bundle map between two vector bundles of the same dimension, that is not identically degenerate and fibered over an invertible map of the bases $T \to T'$, then* $\deg S' \geqslant \deg S$.

Proof. A holomorphic map $B\colon S \to S'$ induces the holomorphic determinant map $\det B\colon \det S \to \det S'$. By the definition of degree and Proposition 25.8, $\deg S = \deg \det S \geqslant \deg \det S' = \deg S'$. □

As a corollary, we obtain the following result.

Lemma 25.15. *A subbundle of a trivial bundle has a nonpositive degree. If this degree is zero, the subbundle itself is trivial.*

Proof. Let $S' = T \times \mathbb{C}^{n'}$ be the trivial bundle and S a holomorphic subbundle of dimension $n < n'$. One can always find a trivial n-dimensional subbundle $S'' = T \times \mathbb{C}^n$ and a projection $p\colon \mathbb{C}^{n'} \to \mathbb{C}^n$ such that the corresponding bundle map $B\colon S' \to S''$, $(t,x) \mapsto (t, p(x))$, restricted on $S \subset S'$, will be not identically degenerate bundle map from S to S'', fibered over the identity. By Proposition 25.14, $\deg S \leqslant \deg S'' = 0$, with equality possible only if the two bundles are equivalent and the determinant $\det B$ being invertible bundle map. Therefore the map B itself is invertible and realizes a holomorphic equivalence between S' and the trivial (sub)bundle S''. □

25.9. Trace of a meromorphic connection. A meromorphic connection ∇ on any holomorphic vector bundle $\pi\colon S \to T$ induces a meromorphic connection on the determinant bundle $\det S$, called the trace of ∇ and denoted by $\operatorname{tr} \nabla$.

Definition 25.16. The *trace* of a meromorphic connection ∇ on a holomorphic vector bundle S is the unique connection on the determinant bundle $\det S$ such that the wedge product of any n horizontal local sections of S is a horizontal local section of $\det S$.

To compute the connection form for the trace $\operatorname{tr} \nabla$ near a nonsingular point of ∇, choose a trivialization and assume that n horizontal sections correspond to columns of a holomorphic $n \times n$-matrix function $X(t)$. Then the connection form for ∇ will be $\Omega = dX \cdot X^{-1}$, see (25.5). The corresponding section of the determinant bundle is given by $f(t) = \det X(t)$, and the only connection for which it is horizontal, is given by the 1-form $\omega = df \cdot f^{-1}$. By the Liouville–Ostrogradskii formula,

$$\omega = f^{-1}\, df = \operatorname{tr}(dX \cdot X^{-1}) = \operatorname{tr} \Omega, \qquad f = \det X,$$

which explains the term "trace": the connection form ω for $\operatorname{tr} \nabla$ is the trace of the matrix connection form Ω for the initial connection ∇.

For any choice of trivialization the connection form $\omega = \operatorname{tr} \Omega$ extends meromorphically to the singular locus of the connection, defining therefore the trace $\operatorname{tr} \nabla$ globally as a meromorphic connection. By linearity, the residue of the trace at any singular point is the trace of the corresponding residue of the connection:

$$\operatorname{res}_{t_0} \operatorname{tr} \nabla = \operatorname{tr} \operatorname{res}_{t_0} \nabla, \qquad \forall t_0 \in T.$$

This, together with Proposition 25.10, proves the following principal result.

Lemma 25.17. *The sum of traces of residues of any meromorphic connection on the holomorphic bundle over a compact Riemann surface is equal to the degree of this bundle and does not depend on the connection.* □

25.10. Monodromy and holonomy of a meromorphic connection. In a way almost completely similar to that for linear systems, a meromorphic connection ∇ on a holomorphic vector bundle $\pi\colon S \to T$ with singularities on a finite locus $\Sigma \subset T$ may have monodromy. For any point $t_0 \notin \Sigma$ and any initial value $s_0 \in S_{t_0} = \pi^{-1}(t_0)$, there exists a unique local horizontal section s passing through s_0. This section can be uniquely continued as a horizontal section over any path $\gamma \subset T$ starting at t_0 and avoiding Σ. All horizontal sections over a simply connected domain $U \subset T \smallsetminus \Sigma$ form a linear space, isomorphic to the fiber S_{t_0} if $t_0 \in U$, by the isomorphism $s(\cdot) \mapsto s(t_0)$. Analytic continuation over closed loops γ beginning and ending at t_0, yields linear automorphisms M_γ of this linear space, called the monodromy transformations. (If they are interpreted as linear automorphisms of a fixed fiber S_{t_0} using the above isomorphism, then more frequently the term "holonomy transformations" is used). In any case, the correspondence $\gamma \mapsto M_\gamma$ is an antirepresentation of the fundamental group $\pi_1(T \smallsetminus \Sigma, t_0)$. Choosing a different fiber S_{t_1} results in an equivalent antirepresentation (simultaneous conjugacy of all operators M_γ by the same constant invertible matrix).

The *Riemann–Hilbert problem for holomorphic bundles* is formulated as follows. Given a finite set $\Sigma \subset T$ and a linear n-dimensional antirepresentation of the fundamental group $\pi_1(T \smallsetminus \Sigma, t_0)$, $t_0 \notin \Sigma$, one has to construct a holomorphic vector bundle $\pi\colon S \to T$ of a *prescribed type* and a meromorphic connection ∇ on this bundle, having only logarithmic singularities on Σ, such that the monodromy of this connection is equivalent to the given antirepresentation. The classical Riemann–Hilbert problem arises when the base is the Riemann sphere $\mathbb{C}P^1$ and the bundle is required to be trivial. In such case the connection can be identified with a single globally defined meromorphic matrix connection 1-form Ω and horizontal sections with solutions of the linear system $dx = \Omega x$ on $T \times \mathbb{C}^n$.

Prescribing the holomorphic type of the bundle becomes the central point of this formulation. Indeed, the following general result is essentially a tautology, being valid for any Riemann surface (compact or not) and any dimension n.

Theorem 25.18 (H. Röhrl, 1957, see [**For91**]). *Any linear antirepresentation of the fundamental group $\pi_1(T \smallsetminus \Sigma, t_0)$, can be realized as the monodromy of a meromorphic connection ∇ on some holomorphic bundle $\pi\colon S \to T$ having only Fuchsian singularities on Σ.*

Proof. We give a brief sketch of the proof, referring to the book [**For91**] for technical details.

The first step is to construct a holomorphic bundle and a nonsingular connection on it with the preassigned monodromy, over the set $T' = T \smallsetminus \Sigma$ of nonsingular points. The construction is similar to the standard suspension used to construct a flow with the preassigned Poincaré map. On the second step the bundle is extended to singular points where the connection exhibits a Fuchsian singularity.

Back ref.—suspension of a map to a flow.

Consider the universal covering space $p\colon \widetilde{T} \to T'$ and a covering of T' by connected simply connected charts U_i such that $p^{-1}(U_i) \simeq U_i \times G$, where G is the group of covering transformations of $\widetilde{T}$, isomorphic to the fundamental group $\pi_1(T', \cdot)$. Each preimage $\widetilde{U}_i = U_i \times G$ is the disjoint union of copies of the chart U_i, and one can define a matrix function X_i on $\widetilde{U}_i$ by letting $X_i|_{U_i \times e} = E$, $X_i|_{U_i \times \gamma} = M_\gamma$, where e is the unit of the group G, γ is considered as the covering transformation corresponding to a loop γ beginning and ending in U_i, and M_γ the respective preassigned monodromy factor. Since $\widetilde{U}_i$ is a *disjoint* union (not connected), the functions X_i are holomorphic (being locally constant) and nondegenerate.

Over the intersections $\widetilde{U}_{ij} = \widetilde{U}_i \cap \widetilde{U}_j$ on the universal cover, the matrix ratios $F_{ij} = X_i X_j^{-1}$ are invariant by the covering transformations, hence can be considered as nondegenerate locally constant matrix functions on $U_i \cap U_j \subset T'$. Clearly, they satisfy the cocycle identity and can be used to construct a holomorphic bundle over T' by the patchwork procedure described in §25.5. The collection of *zero connection forms* induces a holomorphic connection on this bundle. Columns of the matrix functions X_i induce horizontal sections of this bundle, and by construction it has the prescribed monodromy.

The second step in the proof of the theorem is to "seal the gaps" around the deleted singularities. This is a *local* problem of extending a holomorphic bundle S' over a punctured neighborhood of an isolated singular point, to a bundle over the full neighborhood. Consider a holomorphic bundle over $U' = \{0 < |t| < 1\}$ and a collection of n linear independent locally horizontal sections represented by a multivalued matrix function $X'(t)$ which acquires the monodromy factor M after going around 0. Choose *any* matrix logarithm A of M and consider the multivalued matrix function $X(t) = t^A$ acquiring the same matrix factor. The corresponding connection form $\Omega = dX \cdot X^{-1} = A\,dt/t$ has a Fuchsian singularity. The matrix quotient $X'(t)X^{-1}(t)$ is holomorphic invertible in U' and hence determines a cocycle that can be used to glue the cylinders $U' \times \mathbb{C}^n$ with the holomorphic connection form $\Omega' = dX' \cdot (X')^{-1}$ on it and $U \times \mathbb{C}^n$, where $U = U' \cup \{0\}$ is the disk, with the connection form Ω having an isolated Fuchsian singularity.

The result will be a bundle extending the bundle S' to the isolated singular point. By construction, the meromorphic connection with the connection form $\Omega' = dX' \cdot (X')^{-1}$ outside the singular locus and $\Omega = dX \cdot X^{-1}$ near the singular point, possesses the required monodromy group. □

As was already remarked, all holomorphic vector bundles over noncompact Riemann surfaces are trivial. This implies solvability of the classical Riemann–Hilbert problem for the open disk $T = \{|t| < 1\}$ and the affine plane $T = \mathbb{C}$. Bundles over the Riemann sphere are completely classified and may be nontrivial, but the problem of recognizing the holomorphic type of the bundle constructed in the proof of Theorem 23.3, is transcendental for $n > 1$. Moreover, the bundle obtained after sealing the gaps, depends essentially on the choice of the matrix logarithms (this choice is independent at each singular point). Thus there is *no canonical bundle* associated with a given monodromy group, which makes the Riemann–Hilbert problem even more difficult.

25.11. Reducible representations and invariant subbundles. Consider a meromorphic connection ∇ on a bundle S. A nontrivial subbundle $S' \subset S$ (different from S and $T \times \{0\} \subset S$) is invariant by ∇ if any local horizontal section passing through a point $s_0 \in S'$, $\pi(s_0) \notin \Sigma$, remains in S'.

If $s_1, \dots, s_k \in \Gamma^0(S)|_U$ represent horizontal local sections spanning S' over $U \subset T$, then any holomorphic section $s' \in \Gamma^0(S')|_U$ of S' over U can be represented as $s' f_1 s_1 + \cdots + f_k s_k$, where $f_1, \dots, f_k$ are holomorphic functions in U. Applying the Leibnitz rule (25.3) ana taking into account that $\nabla s_i = 0$, we conclude that $\nabla s' = df_1 \cdot s_1 + \cdots + df_k \cdot s_k \in \Lambda^1(U) \otimes \Gamma^0(S')$, that is, ∇ induces a meromorphic connection on S'. It is referred to as the *restriction* of ∇ on the invariant subbundle S'.

Note that the condition of invariance is local, while the requirement that S' is a globally defined subbundle, is global.

Over simply connected subsets of $T' = T \smallsetminus \Sigma$, invariant subbundles are abundant. Indeed, any linear subspace $L_{t_0} \subset S_{t_0}$ of any dimension k, $0 < k < n$, can be saturated by horizontal sections defined everywhere over $U \subset T'$ provided U is open, connected and simply connected. The union of these sections is a k-dimensional subbundle in $S|_U$.

For a multiply connected subset of the base (in particular, for the whole regular locus T') the answer depends on the monodromy. The following proposition is almost tautological.

Proposition 25.19. *A meromorphic connection with a singular locus Σ admits an invariant subbundle over $T' = T \smallsetminus \Sigma$ if and only if the monodromy* *of ∇ is reducible, i.e., when all monodromy operators have a common nontrivial invariant subspace.*

Proof. It is more convenient to use the holonomy operators for making this statement obvious. Let $L_{t_0} \subset S_{t_0}$ be a linear subset invariant by all holonomy operators. Define the fibers $L_t \subset S_t$ for all $t \in T'$ as the set of endpoints of all horizontal sections passing through L_{t_0}, continued along *any* path γ connecting T_0 with t. By the invariance assumption, L_t as a linear space does not depend on the homotopy class of γ (though the result of each horizontal continuation depends). Clearly, L_t depends on t analytically so that their union $S' = \bigcup_{t \in T'} L_t$ is an analytic manifold.

The inverse statement is obvious: if S' is a subbundle, then $L_{t_0} = S' \cap S_{t_0}$ is invariant by all monodromy operators by definition. □

Thus the only obstruction for existence of invariant subbundles, besides irreducibility of the monodromy, may occur only when attempting to extend the invariant subbundle over T' to the singular fibers. However, such extension is always feasible for regular singularities.

Proposition 25.20. *Any holomorphic subbundle invariant by a meromorphic connection, admits an analytic extension to an isolated regular singularity of this connection.*

Remark 25.21. We will need this assertion only for Fuchsian singularities of class B, where it can be derived from the uniqueness (see the proof of Lemma 25.22).

Proof of the Proposition. The assertion is local, hence can be proved in a trivializing chart around the singular point that can without loss of generality be assumed at the origin, $\Sigma = \{0\}$.

First consider the case when the monodromy of the regular singularity at $t = 0$ is trivial and the subbundle is generated by one or several meromorphic sections $s_1(t), \dots, s_k(t)$ (vector functions), holomorphic and linear independent outside the origin.

We prove by induction that there exists a holomorphic invertible gauge transformation $F(t) \in \mathrm{GL}(n, \mathcal{O}_0)$ and a meromorphic invertible matrix $R(t) = \{r_{ij}(t)\} \in \mathrm{GL}(n, \mathcal{M}_0)$, such that the transformed vector functions

$$F(t)s'_i(t), \qquad i = 1, \dots, k, \quad s'_i(t) = \sum_{j=1} r_{ij}(t)s_j(t),$$

are constant vector functions, either identically zero or the coordinate vectors $(0, \dots, 1, \dots, 0) \in \mathbb{C}^n$. The bundle map F transforms the subbundle spanned by the sections s_i, into the constant subbundle which is obviously holomorphic at the origin.

One meromorphic vector function $s_1 \colon (\mathbb{C}, 0) \to \mathbb{C}^n$, unless identically zero, after multiplication by an appropriate power $r_{11}(t) = t^{\nu_1}$, $\nu_1 \in \mathbb{Z}$, can be made holomorphic and nonvanishing at $t = 0$. There exists holomorphic invertible transformation $F(t)$ taking the result into the first basic vector $(1, 0, \dots, 0)$. This case serves as a base for induction.

To prove the inductive step, we may assume that out of any number of $k+1$ vector functions $s_1, \dots, s_k, s_{k+1}$, the first k are already constant basic vectors as asserted. Without loss of generality we assume that they are all linear independent (over $\mathbb{C}$) and coincide with the first k basic vectors.

Subtracting from $s_{k+1} = (x_1(t), \dots, x_k(t), x_{k+1}(t), \dots, x_n(t))$, $x_i(t) \in \mathcal{M}_0$, the first k sections $s_1, \dots, s_k$ with the meromorphic coefficients $r_{k+1,i} = x_i$, $i = 1, \dots, k$, we may assume that $x_1 = \cdots = x_k \equiv 0$.

As before, multiplying by an appropriate power $r_{k+1,k+1}(t) = t^{\nu_{k+1}}$, we may assume that the "tail" $(x_{k+1}(t), \dots, x_n(t)) \in \mathbb{C}^{n-k}$ is holomorphic at the origin and nonvanishing. The holomorphic invertible transformation of $\mathbb{C}^{n-k}$ into itself, sending s_{k+1} into the (constant) basic vector $(1, 0 \dots, 0) \in \mathbb{C}^{n-k}$, after being extended by the identical transformation of the subspace $\mathbb{C}^k \subset \mathbb{C}^n$ normalizes the collection $s_1, \dots, s_k, s_{k+1}$ as required. The induction is complete.

Analytically the assertion just proved means that any meromorphic (in general, rectangular) $n \times k$-matrix matrix germ $Y(t)$, $k \leqslant n$, can be represented as the product $Y(t) = F(t)CR(t)$ with holomorphically invertible left factor $F \in \operatorname{GL}(n, \mathcal{O})$, a meromorphic invertible right factor $R \in \operatorname{GL}(k, \mathcal{M}_0)$ and a constant rectangular matrix C.

The general case of singularities with a nontrivial monodromy, is only slightly more difficult. A subbundle of dimension k is spanned by k vector solutions, linear independent everywhere outside the origin. Arranged in the form of a rectangular $n \times k$-matrix $X(t)$, they satisfy the condition $\Delta X(t) = X(t)M$, where M is an invertible $k \times k$-matrix, since the subspace spanned by these solutions is invariant. Thus $X(t) = Y(t)\, t^A$ for an appropriate constant matrix A, and $Y(t)$ a rectangular matrix germ of rank k meromorphic at the origin. By construction, the columns of $Y(t)$ span the same subspace at every nonsingular point. By the first part of the proof, it extends analytically to the singular point. □

25.12. Connections of class B. A singular point $t_* \in \Sigma$ of a meromorphic connection is of class B, if the monodromy operator M for a small loop around this singularity is of the class B in the sense of Definition 24.4, that is, has only one eigenvalue and a single maximal size Jordan block.

The local analysis carried out in §24.1 shows that already the cyclic subgroup $\{M^k : k \in \mathbb{Z}\}$ generated by the operator M has very few invariant subspaces, more precisely, only one in which dimension between 1 and $n-1$. While such subspaces indeed exist for the cyclic subgroup, they may be non-invariant by other monodromy operators.

Each of the subbundles invariant by M, can be analytically extended to the singular point in a unique way. While the extensibility follows from the general claim (Proposition 25.20), the condition B implies both existence and uniqueness of this extension.

Recall that two complex numbers can be compared by the relation $\geqslant$, if their difference is a nonnegative real (in particular, integer) number.

Lemma 25.22. *Let a meromorphic connection ∇ on a holomorphic n-dimensional bundle S have an invariant k-dimensional subbundle S'. Denote by ∇' the restriction of ∇ on S'.*

If t_0 is a singular point of class B for ∇, then

$$\operatorname{tr}\operatorname{res}_{t_0}\nabla' \geqslant \tfrac{k}{n}\operatorname{tr}\operatorname{res}_{t_0}\nabla.$$

The equality is possible if and only if all n eigenvalues of $\operatorname{res}_{t_0}\nabla$ coincide.

Proof. This assertion is local and invariant. Hence it is sufficient to prove it for a trivial bundle S over a neighborhood of just one singular point t_0, assuming that the connection matrix 1-form $\Omega = t^{-1}A(t)\,dt$ of ∇ has the upper-triangular Poincaré–Dulac–Levelt normal form in the sense of Definition 21.23 (see §21.6). Recall that in this form the diagonal entries of the residue matrix $A = A(0)$ differ by integer numbers and are ordered in the *nonincreasing* order.

The (trivial constant) subbundle S'' spanned by the first k coordinate axes, is invariant by ∇ because Ω is triangular. Being unique by Lemma 24.2, S'' must coincide with the given invariant k-dimensional bundle S' over $t \neq t_0$ in the chosen trivializing chart.

While $\operatorname{tr}\operatorname{res}_0\Omega$ is equal to the sum of *all* eigenvalues of A, the trace of its restriction on $S' = S''$ is equal to the sum of the *first* k eigenvalues. Since the *largest* eigenvalues come first, the mean eigenvalue $\frac{1}{n}\operatorname{tr}A$ of the residue A is less or equal than the mean eigenvalue $\frac{1}{k}\operatorname{tr}A'$ of its upper left $k\times k$-block A'. The equality is possible if the maximal and the mean eigenvalues coincide, i.e., when they are all equal to each other. This proves both assertions of the Lemma. □

As a corollary, we conclude with the following geometric generalization of Theorem 24.5, valid for connections on bundles over any compact Riemann surface.

Theorem 25.23. *If a meromorphic connection on a trivial bundle over a compact Riemann surface T has only singularities of class B and admits* *a nontrivial invariant subbundle, then the spectrum of each residue* $\operatorname{res}_{t_i}\nabla$, $t_i \in \Sigma$, *consists of a single number. The invariant subbundle in this case must also be trivial.*

Proof. Let S be the trivial bundle, and S' the invariant subbundle. For each singularity $t_i \in \Sigma$, we have $\operatorname{tr}\operatorname{res}_{t_i}\nabla' \geqslant c \operatorname{tr}\operatorname{res}_{t_i}\nabla$, with $c = k/n > 0$. Adding these inequalities together and noting that the degree of the trivial bundle is zero, we have

$$\deg S' = \sum_{t_i\in\Sigma} \operatorname{tr}\operatorname{res}_{t_i}\nabla' \geqslant c \sum_{t_i\in\Sigma} \operatorname{tr}\operatorname{res}_{t_i}\nabla = c\,\deg S = 0,$$

with the equality possible only if the spectra of all residues are singletons.

On the other hand, by Lemma 25.15,

$$\deg S' \leqslant \deg S$$

with the equality possible only if S' is also trivial.

Combination of these opposite inequalities proves the Theorem. □

The Bolibruch impossibility theorem (Theorem 24.9) becomes completely transparent now. If the group generated by three matrices (24.3) were realized as a monodromy group of a meromorphic connection with logarithmic (Fuchsian) singularities on the *trivial* bundle, then by Theorem 25.23 the corresponding residues must have the singleton spectra consisting of the numbers $\lambda_{1,2} = \frac{1}{2\pi i}\ln 1$ and $\lambda_3 = \frac{1}{2\pi i}\ln(-1)$. The choice of the logarithm in each case is not known, but since all 4 eigenvalues of each residue coincide, we have

$$\operatorname{tr} A_1 = \operatorname{tr} A_2 = 0 \bmod 4\mathbb{Z}, \qquad \operatorname{tr} A_3 = 2 \bmod 4\mathbb{Z}.$$

In such situation the equality $\operatorname{tr} A_1 + \operatorname{tr} A_2 + \operatorname{tr} A_3 = 0$, necessary for the bundle to be trivial, is impossible.

26. Linear nth order differential equations

Linear high order differential equation can be reduced to a rather special class of *companion* linear systems which are naturally defined on the *jet bundle*, in general nontrivial. For companion systems the difference between regular and Fuchsian singularities disappears. Additional feature is the structure of (noncommutative) algebra on the set of linear differential operators, which implies the possibility of *factorization* of operators. The latter circumstance plays an important role when studying *roots of solutions* of linear ordinary differential equations.

26.1. High order differential operators. Jet extensions, companion system. In the beginning we assume that $T \subset \mathbb{C}$ is an open domain of the complex plane with the fixed chart t (the independent variable) on it. The corresponding derivation $\frac{d}{dt} \colon \mathcal{O}(T) \to \mathcal{O}(T)$ will be denoted by T.

Definition 26.1. A linear nth order differential operator with holomorphic coefficients $a_0(t), \dots, a_n(t) \in \mathcal{O}(T)$, $n \geqslant 0$, is the $\mathbb{C}$-linear operator $L \colon \mathcal{O}(T) \to \mathcal{O}(T)$,

$$L = a_0 D^n + a_1 D^{n-1} + \cdots + a_{n-1} D + a_n, \qquad D = \frac{d}{dt}, \quad a_0 \not\equiv 0. \tag{26.1}$$

The operator L is called *monic*, if $a_0 \equiv 1$. The operator $a_0 D^n$ is called the *leading term* of L. A linear nth order homogeneous differential equation has the form

$$Lf = 0. \tag{26.2}$$

We will also allow meromorphic coefficients, denoting by $\mathcal{D}^n(T)$ the collection of all linear differential operators of order n with coefficients meromorphic in T, and $\mathcal{D} = \bigcup_{n \geqslant 0} \mathcal{D}^n(T)$ the graded linear space of all differential operators. However, when studying homogeneous equations, one can always assume that the coefficients are holomorphic, multiplying, if necessary, all coefficients of L by an appropriate holomorphic common denominator. Conversely, it is possible to deal with only monic operators but having in general meromorphic coefficients.

In this section it will be convenient to enumerate coordinates of the complex space $\mathbb{C}^{n+1} = \{(x_0, \dots, x_n)\}$ starting from x_0. With any holomorphic function $f \in \mathcal{O}(T)$ and any order $n \in \mathbb{N}$ one can associate a holomorphic vector-function $\mathbf{j}^n f \colon T \mapsto \mathbb{C}^{n+1}$, called *$n$-jet extension* of f,

$$\mathbf{j}^n f \colon t \mapsto (x_0(t), \dots, x_n(t)), \qquad x_j(t) = D^j f(t), \tag{26.3}$$

the collection of all derivatives of f up to order n (recall that the chart t on T is assumed fixed). Clearly, the components of the jet extension $x(\cdot) = \mathbf{j}^n f$ of *any function* satisfy the differential identities

$$D x_j = x_{j+1}, \qquad j = 0, 1, , \dots, n-1. \tag{26.4}$$

On the other hand, any differential equation of the form (26.1)–(26.2) is simply a linear identity between the components of the n-jet extension,

$$a_0 x_n + a_1 x_{n-1} + \cdots + a_{n-1} x_1 + a_n x_0 = 0 \tag{26.5}$$

The following reduction is obvious.

Proposition 26.2. *A holomorphic function $f \in \mathcal{O}(T)$ is a solution of the n-th order differential equation* (26.2) *with the operator L as in* (26.1)*, if its*

$(n-1)$-jet extension $x(\cdot) = \mathbf{j}^{n-1} f$ satisfies the linear system

$$\dot{x} = A(t)x, \qquad A(t) = \begin{pmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ & & & 0 & 1 \\ -\frac{a_n}{a_0} & -\frac{a_{n-1}}{a_0} & \cdots & -\frac{a_2}{a_0} & -\frac{a_1}{a_0} \end{pmatrix}, \tag{26.6}$$

called the companion system *for the linear equation* (26.1)–(26.2).

Conversely, any holomorphic solution $x(t)$ of the system (26.6) *is the jet extension, $x = \mathbf{j}^{n-1} f$ of the function $f(t) = x_0(t) \in \mathcal{O}(T)$ which in turn satisfies the equation* (26.1)–(26.2). □

Remark 26.3. The Pfaffian form of the companion system can be described as follows. Consider the cylinder $T \times \mathbb{C}^{n+1}$ and n differential forms

$$\theta_k = dx_k - \omega x_{k-1}, \qquad \omega = dt \in \Lambda^1(T \times \mathbb{C}^{n+1}),\ k = 1, \dots, n. \tag{26.7}$$

The ordinary differential equations (26.4) correspond to the Pfaffian equations $\theta_1 = \cdots = \theta_{k-1} = 0$. The last equation of the system (26.6) is obtained by eliminating the variable x_n from the Pfaffian equation $\theta_n = 0$, using the linear identity (26.5). The result will be the system (20.3) on $T \times \mathbb{C}^n$ with the Pfaffian matrix $\Omega = A(t)\omega$ with $A(t)$ as above and $\omega = dt$.

After this reduction almost all general notions and results on linear systems can be reformulated for high order equations. Thus, a point t_0 is *nonsingular*, if all the ratios $a_i(t)/a_0(t)$, $i = 1, \dots, n$, are holomorphic at t_0; otherwise (if at least one of the ratios has a pole) the point t_0 is singular. In a simply connected domain T free of singular points, the linear equation (26.2) of order n has n-dimensional $\mathbb{C}$-linear space of solutions. If T is multiply connected, a nontrivial monodromy group in general arises. A *fundamental system of solutions* is any basis $f_1, \dots, f_n$ in this linear space.

Definition 26.4. A singular point $t_0 \in T$ for a linear equation (26.1)–(26.2) is *regular*, if it is regular for the companion system (26.6), i.e., if all derivatives of order $\leqslant n-1$ of any solution f grow no faster than (negative) powers of $|t - t_0|$ as $t \to t_0$ in sectors with the vertex at t_0.

However, the definition of a Fuchsian singularity for high order equations (companion systems) has its own specifics and will be discussed later, in §26.4.

26.2. Wronskian. Restoring a linear system from its solutions. In the same way as any holomorphic invertible matrix function is a fundamental (matrix) solution of an appropriate linear system (20.4), any n-dimensional linear subspace in the space of analytic functions is a solution space for an appropriate linear nth order equation. The difference is that the equation in general has singularities.

Definition 26.5. The *Wronskian*, or Wronski determinant, of n functions is the determinant

$$W(f_1,\dots,f_n)=\det\begin{pmatrix} f_1 & f_2 & \cdots & f_n \\ Df_1 & Df_2 & \cdots & Df_n \\ \vdots & \vdots & \ddots & \vdots \\ D^{n-1}f_1 & D^{n-1}f_2 & \cdots & D^{n-1}f_n \end{pmatrix}. \tag{26.8}$$

The Wronskian is a holomorphic (resp., meromorphic) function of t if all functions $f_1,\dots,f_n$ were holomorphic (resp., meromorphic). It depends multi-linearly (over $\mathbb{C}$) and antisymmetrically on the functions f_j. In particular, it vanishes *identically* if the functions f_j are linear dependent over $\mathbb{C}$. If $f_1,\dots,f_n$ are solutions of a linear equation (26.2), then $W(f_1,\dots,f_n)$ is the determinant of the matrix solution $X(t)$ of the associated companion system (26.6). The link between Wronskians and linear equations is very intimate.

Proposition 26.6. *If $f_1,\dots,f_n\in\mathcal{M}(T)$ are meromorphic functions such that their Wronskian $W(f_1,\dots,f_n)$ is not identically zero, then the operator*

$$L=W(f_1,\dots,f_n,\bullet),\qquad Lf=W(f_1,\dots,f_n,f), \tag{26.9}$$

is a differential operator of order n, vanishing on all functions $f_1,\dots,f_n$. It can be expanded as

$$L=a_0D^n+a_1D^{n-1}+\cdots+a_{n-1}D+a_n,\qquad a_0=W(f_1,\dots,f_n). \tag{26.10}$$

Proof. To see that L is the differential operator, it is sufficient to expand the $(n+1)\times(n+1)$-determinant $W(f_1,\dots,f_n,f)$ in the elements of the last column. Since the Wronskian vanishes when two of the functions coincide, each f_j belongs to the null space of L. □

It was already noticed that the Wronskian of linearly dependent functions vanishes identically. The inverse in general is not true if the functions f_j are only smooth (and t a real variable), but for analytic functions identical vanishing of the Wronskian implies the linear dependence of the functions.

Proposition 26.7. *If $f_1,\dots,f_n$ are meromorphic functions such that*

$$W(f_1,\dots,f_n)\not\equiv 0,\qquad W(f_1,\dots,f_n,f_{n+1})\equiv 0,$$

then f_{n+1} is a linear combination of $f_1,\dots,f_n$ over $\mathbb{C}$.

Consequently, if the Wronskian of several meromorphic functions is identically zero, then these functions are linear dependent over $\mathbb{C}$.

Proof. The functions $f_1, \dots, f_n$ should be linear independent since their Wronskian is not identically zero, and clearly satisfy the equation $Lf = 0$, where $L = W(f_1, \dots, f_n, \bullet)$ is the linear operator (26.9) of order n. Hence any other solution of this equation is a linear combination of f_j, at least over a simply connected open domain U containing no singular points of the equation.

By assumption, f_{n+1} is such a solution, hence $f_{n+1} = \sum_1^n c_j f_j$, $c_j \in \mathbb{C}$ is a linear combination of $f_1, \dots, f_n$ over U. By analyticity, the identity remains is true over the entire domain T.

To prove the corollary, one has to apply the previous claim to the first occurrence of identical zero in the sequence of Wronskians $w_1, \dots, w_n$, $w_k = W(f_1, \dots, f_k)$. This sequence begins with $w_1 = f_1$ and terminates by the Wronskian w_n which is identically zero by assumption. □

Combining these two Propositions, we conclude that the expression (26.9) gives the general form for the nth order linear differential operator vanishing on n prescribed linear independent functions, holomorphic or meromorphic. This operator is clearly unique modulo proportionality (i.e., simultaneous multiplication of all coefficients a_j by a nonzero meromorphic function). Indeed, assuming that there are two operators with the same solution space and the same leading coefficients, their difference would be a linear operator of order $\leqslant n - 1$ with n independent solutions.

Two remarks must be made. First, even if all functions f_j are holomorphic, the operator L may well have singular points at the (isolated) roots of the Wronskian $W(f_1, \dots, f_n)$.

Second, the formula (26.9) makes sense even for multivalued (ramified) functions f_j provided that their Wronskian is not vanishing identically (this condition makes sense even for multivalued functions). The coefficients a_j of the operator L restored by (26.9), in general will be only multivalued. However, in one important situation their ratios a_j/a_0 are single-valued.

Theorem 26.8 (Riemann theorem)**.** *Assume that $f_1, \dots, f_n$ are multivalued functions ramified over a finite locus $\Sigma \subset T$ such that:*

(1) *$f_1, \dots, f_n$ are linear independent over $\mathbb{C}$,*

(2) *the linear space spanned by the branches of the functions f_j is invariant by the monodromy, i.e., for any closed loop $\gamma \in \pi_1(T \smallsetminus \Sigma, \bullet)$*

$$\Delta_\gamma(f_1, \dots, f_n) = (f_1, \dots, f_n) \cdot M_\gamma, \qquad M_\gamma \in \mathrm{GL}(n, \mathbb{C}), \tag{26.11}$$

(3) *the functions f_j grow moderately at each ramification point t_i.*

Then the ratios of the coefficients a_j/a_0 *of the differential operator* (26.9) *are meromorphic in* T *and hence the unique monic operator of order* n *annulled by the functions* $f_1, \dots, f_n$*, given by the formula*

$$L = \frac{1}{W(f_1, \dots, f_n)}\, W(f_1, \dots, f_n, \bullet) \tag{26.12}$$

has meromorphic coefficients and only regular singular points.

Proof. The coefficients a_j given by the formula (26.10), are certain $n \times n$-minors of the matrix X formed by n-jet extensions (columns) of the functions f_j. After analytic continuation along γ all minors are multiplied by the same determinant $\det M_\gamma$, so that their ratios are single-valued on $T \smallsetminus \Sigma$. These ratios may have at worst poles of finite order at isolated roots of the principal Wronskian a_0.

To prove the Theorem, it is sufficient to show that the ramification points $t_j \in \Sigma$ are also at worst poles for the ratios a_j/a_0. Consider the row vector function

$$h(t) = (h_1(t), \dots, h_n(t) = (f_1(t), \dots, f_n(t))\, (t - t_j)^{-A_j},$$

where $A_j = \frac{1}{2\pi i} \ln M_j$ is any logarithm of the monodromy matrix M_j corresponding to a small loop around t_j.

From our assumptions it follows that near each singular point t_j, the function $h(t)$ is single-valued near t_j. Since both f_j and the entries of the matrix function $(t - t_j)^{-A_j}$ grow moderately, $h(t)$ also has moderate growth at t_j, being thus meromorphic. Differentiation of the reciprocal formula $f(t) = h(t)\,(t - t_j)^{A_j}$ shows that any order derivatives of the functions $f_1, \dots, f_n$ also grow moderately in sectors near t_j. Thus each coefficient $a_0, \dots, a_n$ grows moderately at any point $t_j \in \Sigma$.

It remains to prove that the reciprocal $1/a_0$ grows moderately. Again by the monodromy argument,

$$a_0(t) = h_0(t)\,(t - t_j)t^{\alpha_j}, \qquad \alpha_j = \frac{1}{2\pi i} \ln \det M_j,$$

where h_0 is a single-valued hence meromorphic function. The assumption that $a_0 \not\equiv 0$ guarantees that $h_0 \not\equiv 0$ and hence the reciprocal $1/a_0 = (1/h_0)\,(t - t_j)^{-\alpha_j}$ grows moderately.

Thus all ratios a_k/a_0 have moderate growth near t_j. Being single-valued, they have at worst poles of finite order at all points of Σ. □

Remark 26.9. The above proof shows that for *monodromic tuples* of moderately growing functions (satisfying the condition (26.11)) their derivatives of all orders also grow moderately. Thus Definition 26.4 of regular singular points can be formally relaxed: a point t_0 is regular for a linear equation $Lf = 0$ with meromorphic coefficients, if any solution of this equation grows moderately at t_0 (then the moderate growth of derivatives will follow automatically).

26.3. Algebra of differential operators. Factorization. Application of an nth order differential operator to a meromorphic function is again a meromorphic function. This allows to introduce the structure of a *noncommutative algebra* with the operation of composition in the $\mathbb{C}$-linear space $\mathcal{D}(T)$ of all differential operators of all finite orders. Obviously, if L, L' are two differential operators, then for their compositions LL' and $L'L$ we have

$$\operatorname{ord} LL' = \operatorname{ord} L'L = \operatorname{ord} L + \operatorname{ord} L'.$$

The representation (26.1) can be considered now as a (noncommutative) polynomial expansion in $\mathcal{D}(T)$ in powers of the derivation $D \in \mathcal{D}^1(T)$ with all coefficients occurring *to the left* of all powers $D, D^2, \dots, D^n$. The only units of $\mathcal{D}(T)$ are zero order operators corresponding to multiplication by a nonzero meromorphic function.

Despite non-commutativity, the algebra $\mathcal{D}(T)$ admits division with remainder, very much like division of univariate polynomials.

Lemma 26.10. *If $L \in \mathcal{D}^n(T)$ and $Q \in \mathcal{D}^k(T)$ are two differential operators of orders $n \geqslant k$, then there exist two linear ordinary differential operators P (the incomplete ratio) and R (the remainder), such that*

$$L = PQ + R, \qquad \operatorname{ord} P = \operatorname{ord} L - \operatorname{ord} Q, \quad \operatorname{ord} R < \operatorname{ord} Q. \tag{26.13}$$

Proof. The operators P, Q can constructed by the following algorithm which is a modification of the division algorithm for polynomials in one variable. If the operators L, Q are expanded in powers of $D = \frac{d}{dt}$ as follows,

$$\begin{aligned} L &= a_0 D^n + a_1 D^{n-1} + \cdots + a_n, \\ Q &= b_0 D^k + b_1 D^{k-1} + \cdots + b_k, \end{aligned} \tag{26.14}$$

then the leading term of the operator $D^{n-k}Q$ is $b_0 D^n$ and hence the operator $L_1 = L - P_0 Q$, where $P_0 = (a_0/b_0)D^{n-k}$, has the order $\leqslant n-1$. Repeating this step, we construct P_1 so that $L_2 = L_1 - P_1 Q$ is of the order strictly inferior to that of L_1, *etc.*

After at most $n-k$ steps we will be left with an operator of order strictly less than k, which is designated to be the residue R. The "partial incomplete ratios" $P_0, P_1, \dots$ add together to form the operator $P = P_0 + P_1 + \cdots$. $\square$

Remark 26.11. If all coefficients a_i, b_j of the operators L and Q in (26.14) are holomorphic at a given point $t_0 \in T$, and the leading coefficient b_0 of the divisor Q is nonvanishing, then both the remainder and the incomplete ratio will be obtained as expansions in powers of D with holomorphic coefficients. This can be seen by direct inspection of the algorithm.

Definition 26.12. An operator $L \in \mathcal{D}^n(T)$ is *divisible* by $Q \in \mathcal{D}^k(T)$, if $L = PQ$ with $P \in \mathcal{D}^{n-k}(T)$. An operator L is *reducible*, if it is divisible by an operator $Q \in \mathcal{D}^k(T)$ with $0 < k < n$. Otherwise L is called *irreducible*.

Divisibility can be easily described in terms of common solutions of the homogeneous equations $Lf = 0$ and $Qf = 0$.

Proposition 26.13. *An operator L is divisible by another operator Q, if and only if any solution of $Qf = 0$ is also solution of $Lf = 0$.*

Proof. The "if" part is obvious. To prove divisibility, consider a fundamental system $f_1, \dots, f_k$ of solutions of the equation $Qf = 0$ and divide L by Q with remainder R, $L = PQ + R$, as in Lemma 26.10. Being in the null space for L and Q by assumption, $f_1, \dots, f_k$ also belong to the null space of PQ and hence to the null space of R. Since $\operatorname{ord} R < k$, this is possible only when $R = 0$. □

The possibility of division with remainder allows to depress the order of differential equation when some of its solutions are known. Indeed, if $0 \not\equiv f_1 \in \mathcal{M}(T)$ is a known meromorphic solution of the equation $Lf = 0$, then L can be divided out as $L = L'Q$, where $Q = W(f_1, \bullet) = f_1 D - (Df_1)$ is the first order operator vanishing on f. Solving the equation $L'Qf = 0$ is reduced now to solving the homogeneous equation $L'f' = 0$ of order $n-1$ and subsequently solving the nonhomogeneous equation $Qf = f'$ of first order.

If all n solutions $f_1, \dots, f_n$ of the homogeneous nth order equation $Lf = 0$ are known, this procedure allows to construct *complete factorization* of L as a composition of n first order operators in any subdomain $U \subseteq T$ where these solutions are meromorphic (recall that in general they can be multivalued in the whole domain T). To simplify the expressions, denote by

$$\begin{gathered} w_k = W(f_1, \dots, f_k) \in \mathcal{M}(U), \qquad k = 1, \dots, n, \\ w_{-1} = w_0 = 1, \qquad w_{n+1} = w_n. \end{gathered} \tag{26.15}$$

the Wronskians of the first k functions from the ordered tuple $f_1, \dots, f_n$ (the functions w_{-1}, w_0 and w_{n+1} are introduced for convenience).

Theorem 26.14. *If $f_1, \dots, f_n \in \mathcal{M}(U)$ are linear independent solutions of the equation $Lf = 0$ with a monic operator $L = D^n + \cdots$, then L is a composition of n derivations D interspersed with $n+1$ multiplications $b_0, \dots, b_n \in \mathcal{D}^0(U)$,*

$$\begin{gathered} L = b_n \, D \, b_{n-1} \, D \, b_{n-2} \cdots b_2 \, D \, b_1 \, D \, b_0, \\ b_k = \frac{w_k^2}{w_{k-1} w_{k+1}}, \quad k = 0, 1, \dots, n, \end{gathered} \tag{26.16}$$

where $w_{-1}, w_0, \dots, w_n, w_{n+1}$ are the Wronskians (26.15).

Proof. Consider the monic differential operators L_k of order $k = 0, 1, \dots, n$,

$$L_0 = \mathrm{id}, \quad L_k = w_k^{-1}(t) \cdot W(f_1, \dots, f_k, \bullet), \qquad k = 1, \dots, n.$$

We claim that these operators satisfy the operator identity

$$D \, \frac{w_{k-1}}{w_k} \, L_{k-1} = \frac{w_{k-1}}{w_k} \, L_k, \qquad k = 1, \dots, n. \tag{26.17}$$

Indeed, both parts are differential operators of the same order k with the same leading terms $(w_{k-1}/w_k)\, D^k$. The null spaces of both operators also coincide with the linear span of $f_1, \dots, f_k$ and hence with each other. Indeed, the functions $f_1, \dots, f_{k-1}$ obviously belong to the null space of both parts. On the last function f_k the operator L_k vanishes by definition, whereas $L_{k-1} f_k = w_k / w_{k-1}$, so the left hand side of (26.17) also vanishes. Being both monic and having the same null space, the operators occurring in the two sides of (26.17), must coincide.

The identity (26.17) can be rewritten as

$$L_k = \frac{w_k}{w_{k-1}} \, D \, \frac{w_{k-1}}{w_k} L_{k-1}, \qquad k = 1, \dots, n.$$

Applying it recursively to the monic operator $L = L_n$ which is what we are interested in by Proposition 26.6, we obtain its decomposition into n terms

$$L_n = \left(\frac{w_n}{w_{n-1}} \, D \, \frac{w_{n-1}}{w_n} \right) \cdots \left(\frac{w_2}{w_1} \, D \, \frac{w_1}{w_2} \right) \cdot \left(\frac{w_1}{w_0} \, D \, \frac{w_0}{w_1} \right) \cdot L_0,$$

which coincides with (26.16). □

The advantage of complete factorization becomes clear when solving homogeneous or non-homogeneous equations. Denote by D^{-1} *any* "primitive" operator $D^{-1} f = \int f \, dt$ (defined modulo a constant). Then solution of the equation $Lf = g$ for L factored as in (26.16), is given by the symbolic formula

$$f = b_0^{-1} \, D^{-1} \, b_1^{-1} \, D^{-1} \cdots D^{-1} \, b_{n-1}^{-1} \, D^{-1} \, b_n^{-1} g. \tag{26.18}$$

In other words, *knowing a fundamental system of solutions of a homogeneous differential equation allows to solve any nonhomogeneous equation by taking n quadratures.* This may be a convenient alternative to reducing the equation to the companion system and using the method of variation of constants.

In general, solutions of linear equations are ramified at singular points hence the formal factorization (26.16) has in general multivalued coefficients, being thus *not* a factorization in the algebra $\mathcal{D}(T)$. Reducibility of operators in $\mathcal{D}(T)$ is closely related to reducibility of their monodromy group.

Theorem 26.15. *A linear operator $L \in \mathcal{D}(T)$ having only regular singularities in T, is reducible in the algebra $\mathcal{D}(T)$ if and only if its monodromy group is reducible (i.e., has a nontrivial invariant subspace). More precisely, L is divisible from right by any operator Q, defined modulo a left unit, whose solution space is the invariant subspace of solutions of $L = 0$.*

Proof. Assume that $L = PQ$ and $f_1, \dots, f_k$ is a fundamental system of solutions for $Qf = 0$. Then these functions also solve the equation $Lf = 0$ and span an invariant subspace of the monodromy group which is therefore reducible. Conversely, assume (without loss of generality) that an invariant subspace of the monodromy group for $Lf = 0$ is generated by the first k functions $f_1, \dots, f_n$ of some fundamental system of solutions. Then by the Riemann Theorem 26.8, there exists an operator $Q \in \mathcal{D}(T)$ of order k, annulled by these first functions. By Proposition 26.13, L is divisible by Q and hence reducible in $\mathcal{D}(T)$. □

Factorization of operators is compatible with regularity. For brevity we say that a differential operator $L \in \mathcal{D}(T)$ is *regular* in $U \subset T$, if it has only regular singular points there.

Lemma 26.16. *Composition of two regular operators is regular. Conversely, if a regular operator is reducible in $\mathcal{D}(T)$, then both factors are also regular.*

Proof. If $L = PQ$, then any solution of the equation $Lf = 0$ is solution of the non-homogeneous equation $Qf = g$, where g is some solution of the lower order equation $Pg = 0$. For any singular point $t_0 \in T$, the function g grows moderately at t_0 since P is regular. Since Q is also regular at this point, by Lemma 21.9 we conclude that f also grows moderately at t_0. This proves regularity of PQ.

Conversely, if $L = PQ$ is regular, then any function from the null space of Q grows moderately at any singular point t_0 regardless of regularity of P. To prove regularity of P, choose any solution g of the equation $Pg = 0$. Let as before f be any solution of $Qf = g$: by construction, f grows moderately as a solution of $Lf = 0$ and can be represented as

$$f(t) = (h_1, \dots, h_n)\,(t - t_0)^A (c_1, \dots, c_n)^\intercal,$$

where the row vector function $(h_1, \dots, h_n)$ is meromorphic at t_0, the column vector $(c_1, \dots, c_n)^\intercal$ has constant entries and A is any logarithm of the monodromy matrix around t_0. Any such function admits any number of derivations and multiplications by meromorphic functions while retaining the moderate growth at t_0. Therefore application of any operator $Q \in \mathcal{D}(T)$ proves that $g = Qf$ grows moderately at t_0, so that P is regular. □

As an immediate application of this result, we have the local theorem on complete factorization.

Theorem 26.17. *Any differential operator $L \in \mathcal{D}(T)$ having a regular singularity at a point $t_0 \in T$, admits complete factorization in a small neighborhood $U = (\mathbb{C}, t_0)$ of this point,*

$$L = P_n P_{n-1} \cdots P_1, \qquad P_i \in \mathcal{D}(U), \quad \operatorname{ord} P_i = 1, \tag{26.19}$$

with first order factors P_i having meromorphic coefficients in U and regular singularity at t_0. The leading terms of $P_1, \dots, P_{n-1}$ can be prescribed arbitrarily.

Proof. The monodromy group of any operator in a punctured neighborhood U of an (isolated) singular point is cyclic and hence always admits a one-dimensional invariant subspace. By Theorem 26.15, $L = L_0$ is divisible from the right by a first order operator $P_1 \in \mathcal{D}(U)$ whose leading term can be prescribed arbitrary. By Lemma 26.16, both P_1 and its left cofactor L_1 are regular at t_0. Thus the process can be continued by induction until the complete factorization is achieved. □

26.4. Fuchsian singularities of nth order equation. The choice of the "standard" derivation $D = \frac{d}{dt}$ when expanding differential operators as in (26.1), is rather arbitrary and linked only to the choice of the chart t on the domain T. From the algebraic point of view, *any* nonzero meromorphic derivation $D' \colon \mathcal{M}(t) \to \mathcal{M}(T)$, can be used to write the (noncommutative) polynomial expansions. Since T is one-dimensional, such derivation necessary is of the form $D' = rD$, $r \in \mathcal{M}(T)$, $r \not\equiv 0$.

To pass from D to another derivation $D' = r(t)D$ differing by a meromorphic factor r, it is sufficient to iterate the Leibnitz rule. By induction we obtain explicit formulas for the iterated derivations,

$$\begin{pmatrix} 1 \\ D' \\ D'^2 \\ \vdots \\ D'^n \end{pmatrix} = \begin{pmatrix} 1 & & & & \\ & r & & & \\ & \vdots & r^2 & & \\ & \vdots & & \ddots & \\ & \dots & \dots & \dots & r^n \end{pmatrix} \cdot \begin{pmatrix} 1 \\ D \\ D^2 \\ \vdots \\ D^n \end{pmatrix} \tag{26.20}$$

(the first line and column are added for future convenience). The coefficients $c_{kj}(t)$, $j, k = 0, \dots, n$ of the transformation matrix are obtained recurrently by applying the derivation D' to both parts of the identity $D'^k = \sum_{j \leqslant k} c_{kj}(t) D^j$ obtained on the previous step:

$$\begin{gathered} c_{k+1,j} = D' c_{kj} + r c_{k,j-1} = r\bigl(D c_{kj} + c_{k,j-1}\bigr), \\ j = 0, 1, \dots, k+1, \qquad c_{00} \equiv 1, \; c_{11} = r. \end{gathered} \tag{26.21}$$

They are in general only meromorphic, however, if D is a *holomorphic derivation* (i.e., preserve holomorphy of functions to which they are applied) and $r(t)$ is holomorphic, then all coefficients c_{kj} are holomorphic. Reciprocally, if $r(t)$ is holomorphically invertible, then powers of D can be expressed as combinations of powers of D' with holomorphic coefficients c'_{kj}. Substituting the formulas (26.20) into expansions, one can easily pass from one base derivation to another.

Passing to a different derivation D' is equivalent to choosing a different independent "time" variable, at least if D' is holomorphic and nonvanishing in T.

If the coefficient matrix $A(t)$ of the linear system $Dx = A(t)x$ has a simple pole, then this system can be written as $D'x = A'(t)x$, where $D' = (t - t_0)D$ and $A'(t)$ a *holomorphic* matrix function. This suggests using the "alternative" holomorphic derivation

$$D' \colon \mathcal{M}(U) \to \mathcal{M}(U), \qquad U = (\mathbb{C}, t_0), \qquad D' = (t - t_0)\tfrac{d}{dt}, \tag{26.22}$$

instead of the standard holomorphic derivation $D = \frac{d}{dt}$ and motivates the following definition.

Definition 26.18. A differential operator $L \in \mathcal{D}(T)$ is *Fuchsian* at a singular point t_0 if, after being expanded in powers of the derivation $D' = (t - t_0)\,\frac{d}{dt}$,

$$L = a'_0 D'^n + a'_1 D'^{n-1} + \cdots + a'_{n-1} D' + a'_n, \tag{26.23}$$

with meromorphic coefficients $a'_0(t), \dots, a'_n(t) \in \mathcal{M}(T)$, it has all ratios a'_k/a'_0 holomorphic at t_0.

Reciprocally, a singular point t_0 of a differential operator L, resp., a homogeneous linear differential equation $Lf = 0$, is the *Fuchsian singularity*, if L is Fuchsian at t_0.

Remark 26.19. One can conclude by easy inductive arguments that in the particular case $D' = (t - t_0)D$, $D = \frac{d}{dt}$, the formulas (26.20) take the form

$$D'^k = (t - t_0)^k D^k + \sum_{j=0}^{k-1} \beta_{jk}\,(t - t_0)^j D^j, \qquad \beta_{jk} \in \mathbb{C}.$$

This means that after returning to the "initial" derivation $D = \frac{d}{dt}$ and division by $(t - t_0)^n$, any Fuchsian operator $L = (D')^n + \sum_{j=1}^n a'_j(t) D'^{n-j}$ with the leading term D'^n and the coefficients a'_j holomorphic at t_0, will be re-expanded as

$$L = D^n + \sum_{j=1}^n a_j D^{n-j}, \qquad a_j(t) \text{ has a pole of order } \leqslant j \text{ at } t_0. \tag{26.24}$$

This is the standard definition of the Fuchsian singularity [**Inc44, Har82**].

The "alternative" representation (26.23) of the linear operator L can be reduced to a companion system using the "alternative" jet extension

$$t \mapsto x'(t) = (f(t), D'f(t), \dots, D'^{m} f(t)). \tag{26.25}$$

The result will be an "alternative" companion system

$$D'x' = A'(t)x', \qquad A'(t) = \begin{pmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ & & & 0 & 1 \\ -\frac{a'_n}{a'_0} & -\frac{a'_{n-1}}{a'_0} & \cdots & -\frac{a'_2}{a'_0} & -\frac{a'_1}{a'_0} \end{pmatrix}, \tag{26.26}$$

of the form with the matrix function $A'(t)$ holomorphic if and only if the point t_0 is Fuchsian. The "alternative" Pfaffian form $dx' = \Omega' x'$ can be derived in the same way as in Remark 26.3, using the forms

$$\theta'_k = dx'_k - \omega' x'_{k-1}, \qquad \omega' = (t-t_0)^{-1} dt \in \mathcal{M}^1(T),\ k = 1, \dots, n, \tag{26.27}$$

which are now only meromorphic on $T \times \mathbb{C}^{n+1}$.

The "alternative" companion system $D'x' = A'(t)x'$ (26.26) is meromorphically gauge equivalent to the initial companion system $Dx = A(t)x$ (26.6). The gauge transformation (and its inverse) is defined by the formulas (26.20) which express D'-derivatives via D-derivatives.

26.5. Characteristic exponents. The residue of the system (26.26) at t_0 is the matrix $A'(t_0) \in \operatorname{Mat}(n, \mathbb{C})$. The eigenvalues of the residue matrix, called *characteristic exponents* of the equation, can be easily computed: they are roots $\lambda_1, \dots, \lambda_n$ of the *characteristic equation*

$$\begin{gathered} \alpha'_0 \lambda^n + \alpha'_1 \lambda^{n-1} + \cdots + \alpha'_1 \lambda + \alpha'_n = 0, \\ \alpha'_j = a'_j(t_0) \in \mathbb{C},\ j = 0, \dots, n. \end{gathered} \tag{26.28}$$

If the operator is expanded as in (26.24), then the characteristic exponents are roots of the equation

$$\begin{aligned} \lambda(\lambda-1)\cdots(\lambda-n+1) &+ \alpha_1 \lambda(\lambda-1)\cdots(\lambda-n+2) + \cdots \\ &+ \alpha_2 \lambda(\lambda-1) + \alpha_1 \lambda + \alpha_n = 0, \qquad \alpha_j = \lim_{t \to t_0} t^j a_j(t) \in \mathbb{C}. \end{aligned} \tag{26.29}$$

The characteristic exponents describe with the growth exponents of solutions: near the Fuchsian singular point t_0, there exists a fundamental system of solutions of the form

$$f_k(t) = (t-t_0)^{\lambda_k}(1 + o(1)), \qquad k = 1, \dots, n.$$

Indeed, at least in the non-resonant case when no two characteristic roots differ by an integer number, the companion system (26.26) is holomorphically gauge equivalent to the diagonal Euler system and hence admits n distinct solutions of the form $x(t) = (t-t_0)^{\lambda_k}(v_k + o(1))$, $0 \neq v_k \in \mathbb{C}^n$. Substituting each such solution to the companion system, we see that v_k is the eigenvector of the residue matrix $A'(t_0)$ with the eigenvalue λ_k. Because the residue also has a companion form, the first component of each v_k must be nonvanishing.

26.6. Regular singularities are Fuchsian. Since meromorphic gauge equivalence preserves regularity and (26.26) is obviously regular (in fact, Fuchsian), we see immediately that *any Fuchsian singular point of a linear differential equation is always regular.* In a somewhat surprising development and unlike in the case of general linear systems, the inverse statement is true.

Theorem 26.20 (L. Fuchs, 1868)**.** *Any regular singularity of a linear ordinary differential equation with meromorphic coefficients, is Fuchsian.*

Proof. 1°. For equations of the first order the assertion of the Theorem is verified by a straightforward computation. Consider the equation $L'f = 0$, $L = D' + a_1'(t)$. If it has a regular singularity at t_0, we can represent its solution as $f(t) = (t-t_0)^{\lambda}h(t)$ with an appropriate complex $\lambda \in \mathbb{C}$ and some meromorphic function $h(t)$. Changing λ by a suitable integer number, we can assume in addition that h is holomorphic and holomorphically invertible at t_0. Substituting this representation for f into the equation $D'f + a_1'f = 0$, we obtain the formula $-a_1'(t) = D'f/f = \lambda + (D'h/h)$. Since h is holomorphically invertible and $D' = (t-t_0)\frac{d}{dt}$ holomorphic, we conclude that a_1' is holomorphic at t_0 and hence $L = D' + a_1'$ is Fuchsian.

2°. The case of an arbitrary order follows from the factorization Theorem 26.17. By this Theorem, any regular operator L can be factored as $L = a_0' P_n \cdots P_1$ with each P_i being a first order operator regular at t_0. Since the leading terms of P_i can be chosen arbitrary, we assume that

$$P_i = (t-t_0)D + a_i' = D' + a_i', \qquad i = 1, \dots, n.$$

By Step 1°, each P_i is Fuchsian, that is, the free terms $a_1', \dots, a_n'$ are necessarily holomorphic at t_0. But then the composition $P_n \cdots P_1$ begins with the leading term $(D')^n$ and has all holomorphic coefficients after the complete expansion. In other words, L differs from a Fuchsian operator by a meromorphic factor a_0' and hence is also Fuchsian. □

26.7. Jet bundles and invariant constructions. The notion of a linear nth order differential equation can be defined in invariant terms without referring specifically to any coordinate. Any such equation corresponds to a meromorphic connection on a codimension 1 holomorphic subbundle of the n-jet bundle $J^n(T)$. We recall briefly the construction of the latter bundle; more details can be found in [**AVL91**].

Two holomorphic functions on T are said to be n-equivalent at a point $p \in T$, if their difference vanishes with order $n+1$ at p (the order of vanishing is defined independently of any choice of local coordinate). The n-jet at a point p is the equivalence class with respect to this n-equivalence. If $f \in \mathcal{O}(T)$ is a holomorphic function, its n-jet extension is the map associating with each point p the n-jet of f at $p \in T$.

The n-jet space $J^n(T)$ is the union of all jets at all points of T; it is equipped with the natural projection $\pi\colon J^n(T) \to T$. Moreover, since any n-jet uniquely determines the jets of all order inferior to n, there are canonical maps (projections) of $J^n(T)$ to all $J^k(T)$ with $k < n$. Clearly, $J^0(T) \simeq T \times \mathbb{C}$. We will need the projection $\rho\colon J^n(T) \to J^{n-1}(T)$.

The structure of a holomorphic vector bundle on the jet space can be defined by the following construction. Let $\mathfrak{U} = \{U_i\}$ be an open covering of T by simply connected domains, and $\omega_i \in \Lambda^1(U)$ a collection of *holomorphic nonvanishing* 1-forms. Each such form ω_i defines a local chart t_i such that $dt_i = \omega_i$ and the corresponding vector field (derivation) $D_i = \frac{d}{dt_i}$.

As soon as the local coordinate is chosen, n-jets can be identified with tuples of the derivatives $x = (x_0, \dots, x_n) \in \mathbb{C}^{n+1}$ (including the value of the function as the zero order derivative), $x_k = D_i^k f$, $k = 0, 1, \dots, n$. This identification serves as the local trivializing map $\pi^{-1}(U_i) \to U_i \times \mathbb{C}^{n+1}$.

On the overlapping $U_{ij} = U_i \cap U_j$ of two domains where two forms ω_i and ω_j are defined, the transition map appears,

$$U_{ij} \times \mathbb{C}^{n+1} \to U_{ij} \times \mathbb{C}^{n+1}, \qquad (p, x) \mapsto (p, x'), \quad x' = C_{ij}(p)x. \tag{26.30}$$

The matrix function C_{ij} describes how higher derivatives of the same function are to be recomputed. This computation is explained in (26.20)–(26.21): the matrix C_{ij} of size $(n+1) \times (n+1)$ is obtained if in (26.20) one replaces r by the ratio

$$r_{ij} = \frac{\omega_i}{\omega_j} = \frac{D_j}{D_i} \in \mathcal{M}(U_{ij}). \tag{26.31}$$

If all these ratios are holomorphic and invertible (and this is the case we are discussing now), the matrix functions C_{ij} are holomorphically invertible. The collection $\{C_{ij}\}$ forms a holomorphic matrix cocycle corresponding to the bundle $J^n(T)$. For our purpose it is important to remark that, because of the triangularity,

$$\det C_{ij} = 1 \cdot r_{ij} \cdot r_{ij}^2 \cdots r_{ij}^n. \tag{26.32}$$

In each trivializing chart, passing to $(n-1)$-jets is truncation ("forgetting the last derivative") and the kernel of this projection is the one-dimensional subbundle in $J^n(T)$. We refer to the direction of the last coordinate axis x_n as *vertical*. A subspace of the fiber $\pi^{-1}(p)$ in $J^n(T)$ is vertical, if it contains the vertical axis.

A linear nth order equation in each trivialization is a linear (homogeneous) identity between the derivatives, i.e., in the invariant terms, a holomorphic subbundle $\mathcal{A} \subset J^n(T)$. The fibers of this subbundle cannot be everywhere vertical: otherwise $\mathcal{A}$ will be a ρ-preimage of a subbundle in $J^{n-1}(T)$, that is, an equation of order $n-1$ or even less. Thus the set of points $p \in T$, for which the fiber $\mathcal{A}_p$ is vertical, is a discrete subset $\Sigma \subset T$. It corresponds to singular points of the equation.

To describe in invariant terms the differential equation associated with such subbundle, we use the natural additional structure on the jet spaces, the *Cartan distribution*. This is a distribution of 2-dimensional planes on $J^n(T)$, which in each trivializing chart $U_i \times \mathbb{C}^{n+1}$ is given by $n-1$ holomorphic differential forms $\theta_1, \dots, \theta_{n-1} \in \Lambda^1(U_i \times \mathbb{C}^{n+1})$, $\theta_k = dx_{k-1} - \omega_i x_k$. One can easily verify that the distribution $\{\theta_1 = \cdots = \theta_{n-1} = 0\}$ is mapped by the transition map (26.30) into the distribution $\{\theta'_1 = \cdots = \theta'_{n-1} = 0\}$ of $U_j \times \mathbb{C}^{n+1}$, where $\theta'_k = dx'_{k-1} - \omega_j x'_k$.

The characteristic property of the Cartan distribution is obvious: a holomorphic section of the jet bundle is the jet extension of a holomorphic function if and only if it is tangent to the Cartan distribution. The 2-planes of the latter can be described as the closure of the union of tangent lines to jet extensions of all holomorphic functions. Since the notion of the jet extension of a function is defined without reference to any trivializing chart, this gives an invariant description of the Cartan distribution, see [**AVL91**].

Restriction of the Cartan distribution on the holomorphic subbundle $\mathcal{A} \subset J^n(T)$, defines a meromorphic connection $\nabla_{\mathcal{A}}$ on $\mathcal{A}$, holomorphic on the union $\pi^{-1}(T \smallsetminus \Sigma)$ of non-vertical fibers of $\mathcal{A}$. On this open set the intersection of the 2-planes of the Cartan distribution with the tangent hyperplanes to $\mathcal{A}$ as a submanifold in $J^n(T)$, is a line field tangent to $\mathcal{A}$ and hence necessary non-vertical; moreover, this line field projects nicely on T. Thus any integral trajectory of this line field is by construction a jet extension of a holomorphic function, entirely belonging to $\mathcal{A}$. This function is thus the solution of the differential equation.

The line field on the holomorphic bundle $\pi\colon \mathcal{A} \to T$ can be viewed as the field of horizontal spaces of an abstract connection $\nabla_{\mathcal{A}}$ defined outside the union of vertical fibers of $\mathcal{A}$. On that open part of $\mathcal{A}$, the projection $\rho\colon J^n(T) \to J^{n-1}(T)$ is an isomorphism, so the line field can be projected on fibers of the junior jet bundle $J^{n-1}(T)$. In the trivializing chart $U_i \times \mathbb{C}^n$

on $J^{n-1}(T)$, the result will be a Pfaffian system in the companion form with meromorphic coefficients, as explained in Remark 26.3.

However, it should be remarked that the projection ρ restricted on the subbundle $\mathcal{A}$, is by no means a holomorphic isomorphism of vector bundles $\mathcal{A}$ and $J^{n-1}(T)$. Thus the companion connection on $J^{n-1}(T)$ has no intrinsic meaning (unlike the connection on $\mathcal{A}$ that was defined invariantly).

26.8. Globally Fuchsian equations. For Fuchsian equations the above construction can be refined. Let $\Sigma \subset T$ be a finite point set, $\mathfrak{U} = \{U_i\}$ the open covering of T and $\{\omega_i'\}$ collection of 1-forms that are this time assumed having *simple poles* at the points of Σ, remaining holomorphic and nonvanishing outside Σ. The corresponding derivations D_i' will be holomorphic and having simple "zeros" (hyperbolic singular points in the language of vector fields) on Σ.

Since on the pairwise intersections U_{ij} the ratios $r_{ij}' = \omega_i'/\omega_j'$ are again holomorphically invertible, the the matrix functions $\{C_{ij}'\}$ built from these ratios using the same formulas (26.20) with $r = r_{ij}'$, form another holomorphic matrix cocycle. The corresponding holomorphic vector bundle, denoted by $J^n(T,\Sigma)$, will be referred to as the *twisted* n-jet bundle. Together with $J^n(T,\Sigma)$ one has at the same time all junior bundles, in particular, $J^{n-1}(T,\Sigma)$ and the corresponding projection $\rho'\colon J^n(T,\Sigma) \mapsto J^{n-1}(T,\Sigma)$.

By construction, the twisted jet bundle $J^n(T,\Sigma)$ is meromorphically equivalent to the standard jet bundle $J^n(T) = J^n(T,\varnothing)$. Over each domain U_i, this bundle map is represented by the meromorphic matrix function F_i which recomputes powers of the derivation D_i^k as combinations of powers of $D_i'^k$ in terms of the ratio $r_i = \omega_i/\omega_i'$, in general only meromorphic. (The same argument shows also that the construction of the bundle does not depend on the choice of the forms ω_i': for any other choice the cocycle will be holomorphically equivalent.) The meromorphic map F naturally conjugates ρ with ρ'.

Any holomorphic subbundle $\mathcal{A} \subset J^n(T)$ is mapped by the above meromorphic map into a holomorphic subbundle $\mathcal{A}' \subset J^n(T,\Sigma)$. Consider a singular point $p \in T$, that is, the point such that the corresponding fiber $\mathcal{A}_p$ is vertical. If this point is Fuchsian and belongs to Σ, then by Definition 26.18, the fiber $\mathcal{A}_p'$ of the second subbundle is *non-vertical*.

Thus linear equations having only Fuchsian singularities on the finite point set Σ, can be defined as *holomorphic subbundles of the twisted n-jet bundle $J^n(T,\Sigma)$, whose fibers are never vertical* with respect to the projection $\rho'\colon J^n(T,\Sigma) \mapsto J^{n-1}(T,\Sigma)$. This definition immediately implies that, as a holomorphic bundle over T, any Fuchsian equation is holomorphically equivalent to the *junior* twisted jet bundle $J^{n-1}(T,\Sigma)$.

The twisted jet bundles carry the *twisted Cartan distribution*, obtained as the preimage by F of the standard Cartan distribution on $J^n(T)$. However, this distribution is now only *meromorphic*. In each local trivialization $U_i \times \mathbb{C}^{n+1}$ it is given by the forms $\theta'_k = dx_{k-1} - \omega'_i x_k$, meromorphic with simple poles on $\varSigma$. Outside $\varSigma$ the restriction of the twisted Cartan distribution on $\mathcal{A}'$ defines a line field nicely projecting onto the base T and hence a meromorphic connection $\nabla_{\mathcal{A}'}$ with only simple poles on $\mathcal{A}'$. Since $\mathcal{A}'$ is holomorphically equivalent to $J^{n-1}(T, \varSigma)$, the connection $\nabla_{\mathcal{A}'}$ can be considered as a meromorphic connection on the latter bundle. This explains why *Fuchsian* equations correspond to naturally defined Fuchsian connections on $J^{n-1}(T, \varSigma)$. The residues of these connections have the companion form, with the characteristic exponents at the eigenvalues.

The topology of the bundle $J^{n-1}(T, \varSigma)$ depends on the number of points in the singular locus $\varSigma$. This explains the following result (which is especially useful when T is the Riemann sphere $\mathbb{C}P^1$).

Theorem 26.21. *The sum of characteristic exponents of a Fuchsian equation of order n on the compact Riemann surface T of Euler characteristics χ with m singular points, is equal to $(m - \chi)n(n-1)/2$.*

Recall that the Euler characteristics $\chi(T)$ of a compact Riemann surface T is the total order of poles minus total order of zeros of any meromorphic differential form ω on T,

$$\chi = -\sum_{p \in T} \operatorname{ord}_p \omega, \qquad \omega \in \mathcal{M}^1(T).$$

The Euler characteristics of the Riemann sphere $\mathbb{C}P^1$ is equal to 2.

We will also need a general result on the degree of line bundles.

Lemma 26.22. *If $\{a_{ij}\}$ and $\{b_{ij}\}$ are two 1-dimensional holomorphic cocycles corresponding to the line bundles of degrees A and B respectively, then the cocycle $\{a_{ij}b_{ij}\}$ corresponds to the line bundle of degree $A + B$.*

Proof of the Lemma. Let $\{f_i\}$ and $\{g_i\}$ be any two meromorphic cochains representing meromorphic sections of the former line bundles. Then $\{f_i g_i\}$ is the section of the latter bundle. Its degree is equal to the algebraic sum of zeros and poles of any section,

$$\sum_{p \in T} \operatorname{ord}_p(f_i g_i) = \sum_p \operatorname{ord}_p f_i + \sum_p \operatorname{ord}_p g_i.$$

Reference to Proposition 25.7 proves the Lemma. □

Proof of the Theorem. The sum of the traces of all residues of a meromorphic connection on the vector bundle is, by Lemma 25.17, equal to the degree of this bundle, by definition equal to the degree of the determinant bundle.

In other words, one has, starting from the holomorphic (scalar) cocycle $r_{ij} = D_j/D_i = \omega_i/\omega_j$, compute the degree of the cocycle $h_{ij} = \det C_{ij}$ corresponding to the *junior* twisted bundle $J^{n-1}(T, \Sigma)$. Since the $(n \times n)$-matrices C_{ij} are all lower-triangular with the powers $1, r_{ij}, \dots, r_{ij}^{n-1}$ on the diagonal, we immediately have

$$h_{ij} = 1 \cdot r_{ij} \cdot r_{ij}^2 \cdots r_{ij}^{n-1} = r_{ij}^{n(n-1)/2}.$$

By Lemma 26.22, the degree of the line bundle with the cocycle $\{r_{ij}^{n(n-1)/2}\}$ is $n(n-1)/2$ times the degree $\deg \mathcal{R}_\Sigma$ of the bundle $\mathcal{R}_\Sigma$ with the cocycle $\{r_{ij}\}$. To compute $\deg \mathcal{R}_\Sigma$, we construct some meromorphic section of this bundle. For that purpose, take any meromorphic 1-form $\omega \in \mathcal{M}^1(T)$ and let $f_i \in \mathcal{M}(U_i)$ be the value of ω on the vector fields D_i, $f_i = \omega \cdot D_i$. Then the ratios $f_i/f_j = D_i/D_j$ will be equal to r_{ij} as required, so the cochain $\{f_i\}$ indeed represents a section of the bundle.

The degree of the bundle $\mathcal{R}_\Sigma$ is equal to the algebraic sum of zeros and poles of the functions f_i, that is,

$$\sum_{p \in T} \operatorname{ord}_p f_i = \sum_p \operatorname{ord}_p D_i + \sum_p \operatorname{ord}_p df = \sum_{p \in \Sigma} 1 + \sum_{p \in T} \operatorname{ord}_p df = m - \chi(T).$$

Returning to the bundle $J^{n-1}(T, \Sigma)$, we obtain for its degree the expression $(m - \chi)n(n-1)/2$, as asserted. □

27. Irregular singularities and the Stokes phenomenon

Unlike Fuchsian singularities which admit simple formal normal form by means of a transformation that is always convergent, the irregular singularities have the formal classification considerably more involved and the normalizing transformations as a rule diverge.

27.1. One-dimensional irregular singular points. The one-dimensional (scalar) case admits complete investigation. Consider the equation

$$t^m \dot{x} = a(t)x, \qquad m \geqslant 2, \quad a(t) = \lambda + a_1 t + a_2 t^2 + \cdots \in \mathcal{O}_0. \tag{27.1}$$

Its solution is given by the explicit formula

$$x(t) = \exp \int \frac{a(t)}{t^m}\, dt = \exp[-t^{1-m}\lambda(1 + o(1))]. \tag{27.2}$$

Consider the set on the complex plane $\mathbb{C}$, described by the condition

$$\operatorname{Re}(\lambda / t^{m-1}) = 0. \tag{27.3}$$

It consists of $2(m-1)$ rays from the origin, dividing the neighborhood $(\mathbb{C}, 0)$ into sectors of equal opening $\pi/(m-1)$.

For any sector of the form $\alpha < \operatorname{Arg} t < \beta$ not containing any of the *exceptional rays* (27.3) inside or on the boundary, the real part of the function $b(t) = \int t^{-m} a(t)\, dt$ tends either to plus infinity ("growth" sectors), or to minus infinity ("fall" sectors). Accordingly, the solution $x(t)$ grows exponentially fast in the growth sectors and is *flat* at $t = 0$ (i.e., decreases faster than any finite power $|t|^N$, $N \in \mathbb{N}$) in the fall sectors. Thus we indeed see that as $m \geqslant 2$, the "system" (27.1) has an irregular singularity at the origin.

Holomorphic classification of one-dimensional systems is very simple. Clearly, the order m is invariant.

Proposition 27.1. *Two meromorphic "one-dimensional systems" (equations) of the form* (27.1) *with the coefficients $a(t)$ and $a'(t)$ are holomorphically gauge equivalent if and only $a(t) - a'(t)$ is m-flat at the origin. In particular, any such equation is equivalent to a unique polynomial equation*

$$t^m \dot{x} = p(t), \qquad p \in \mathbb{C}[t], \quad \deg p \leqslant m-1, \quad p(0) = \lambda. \tag{27.4}$$

Proof. Any conjugacy $x \mapsto h(t)x$ between these equations must satisfy the condition $\dot{h}/h = (a - a')/t^m$ so h is holomorphic and invertible at the origin if and only if the right hand side is holomorphic at the origin. □

Remark 27.2. The same assertion (with the same proof) holds for *formal* equations with respect to *formal equivalence*, i.e., when both a, a' and the conjugacy h are in the class $\mathbb{C}[[t]]$ of formal Taylor series.

27.2. Birkhoff normal form. The possibility of reducing a general (matrix) linear system of any dimension near a non-Fuchsian singular point to a polynomial normal form depends on the monodromy M of the singular point at the origin.

Consider a linear system of the form

$$t^m \dot{X} = A(t)X, \qquad A(t) \in \operatorname{Mat}(n, \mathcal{O}_0), \quad A(0) = A_0, \tag{27.5}$$

with the *leading matrix* coefficient $A_0 \in \operatorname{Mat}(n, \mathbb{C})$. The integer number $m-1$ is the Poincaré rank of the singularity.

Theorem 27.3 (Birkhoff, 1913)**.** *If the monodromy operator M of a system* (27.5) *is diagonal(izable), then this system is holomorphically gauge equivalent to a polynomial system*

$$t^m \dot{X} = A'_0 + tA'_1 + t^2 A_2 + \cdots + t^{m-1} A'_{m-1}, \qquad A'_i \in \operatorname{Mat}(n, \mathbb{C}).$$

Proof. Let A be a diagonal matrix logarithm satisfying the condition $\exp 2\pi i A = M$. Then any fundamental matrix solution has the form

$X(t) = F(t)\,t^A$, where F is a matrix function single-valued and holomorphically invertible in the punctured neighborhood of the origin but eventually having an essential singularity there. By Corollary 22.26, the function $F(t)$ can be represented as $F(t) = H_0(t)H_1(t)\,t^D$, with an integer diagonal matrix D, the matrix germ H_0 holomorphically invertible in a neighborhood $(\mathbb{C}, 0)$ of the origin and the matrix functions $H_1^{\pm 1}(t)$ holomorphic on $\mathbb{C}P^1 \smallsetminus \{0\}$ (i.e., both H_1 and H_1^{-1} are entire functions of $1/t$). Since A and D commute, the solution X can be represented as $X(t) = H_0 \cdot H_1\,t^{D'}$, $D' = D + A$.

After the holomorphic at the origin gauge transform $X \mapsto X' = H_0^{-1}X$, the logarithmic derivative

$$\Omega' = dX' \cdot (X')^{-1} = dH_1 \cdot H_1^{-1} + t^{-1}\,H_1 D' H_1^{-1}$$

can be extended on the whole Riemann sphere $\mathbb{C}P^1$ with a simple pole at infinity and no other singularities except for $t = 0$.

The origin $t = 0$ is a pole of order m for Ω'. Indeed, it was a pole of order m for $\Omega = dX \cdot X^{-1}$; since Ω' and Ω are locally holomorphically conjugate at the origin by construction, this assertion is valid also for Ω'.

Thus the matrix coefficient $A'(t)$ of $\Omega' = A'\,dt$ must be a matrix polynomial of degree m in t^{-1} without the free term (so that Ω' has at most a simple pole at infinity), exactly as was asserted. □

If the monodromy is not diagonalizable, then the assertion is in general false [**Gan59**]. However, if the system is not holomorphically (or meromorphically, which is the same in this case) reducible, i.e., if one cannot put the matrix function into a block upper-triangular form, then the condition on the monodromy can be dropped [**Bol94**].

The Birkhoff normal form is simple but not very convenient, since it cannot in general be integrated. Besides, it is inefficient: the matrix coefficients A_i' cannot be computed.

27.3. Resonances. Formal diagonalization. The first step in the "genuine" classification of general irregular singularities is the formal classification similar to that described in §21.4 for Fuchsian systems with $m = 1$. Exactly like there, the linear system

$$t^m \dot{x} = A(t)x, \qquad A(t) \in \operatorname{Mat}(n, \mathcal{O}_0), \tag{27.6}$$

associated with the matrix equation (27.5), can be reduced to a holomorphic vector field in $(\mathbb{C}^{n+1}, 0)$ corresponding to the "nonlinear" system of differential equations

$$\begin{cases} \dot{x} = A_0 x + tA_1 x + \cdots, & x \in (\mathbb{C}^n, 0), \\ \dot{t} = t^m, & t \in (\mathbb{C}, 0). \end{cases} \tag{27.7}$$

The spectrum of linearization of the system (27.7) at the singular point $(0,0)$ consists of zero $\lambda_0 = 0$ (since $m \geqslant 2$) and the eigenvalues $\lambda_1, \dots, \lambda_n \in \mathbb{C}$ of the leading matrix $A_0 \in \operatorname{Mat}(n, \mathbb{C})$ (repetitions allowed).

Applying the Poincaré–Dulac technique to the nonlinear system (27.7), we can eliminate from its Taylor expansion all nonresonant terms. Exactly as was the case with Fuchsian systems in §21.4, only occurrence of *cross-resonances* $\lambda_i = \lambda_j + k\lambda_0$ corresponding to the vector-monomials $t^k x_j \frac{\partial}{\partial x_i}$ will matter. As $\lambda_0 = 0$, this motivates the following definition.

Definition 27.4. The system (27.5) is said to be *non-resonant* at the origin, if all eigenvalues $\lambda_1, \dots, \lambda_n$ of the leading matrix A_0 are pairwise different.

Theorem 27.5. *A non-Fuchsian system* (27.5) *at a non-resonant singular point $t = 0$ is formally gauge equivalent to a diagonal polynomial system of degree m,*

$$\begin{aligned} t^m \dot{x} &= \varLambda(t)x, \qquad \varLambda(t) = \operatorname{diag}\{p_1(t), \dots, p_n(t)\}, \\ p_i &\in \mathbb{C}[t], \quad \deg p_i = m, \qquad \varLambda(0) = \operatorname{diag}\{\lambda_1, \dots, \lambda_n\}. \end{aligned} \tag{27.8}$$

Proof. The same (literally) arguments that proved Theorem 21.19 in §21.4, prove also that only resonant monomials of the form $c_k\, x_k \frac{\partial}{\partial x_k}$ should be kept in the expansion (27.7), all others being removable. Elimination of the resonant monomials of degree $k \geqslant m$ can be achieved by Proposition (27.1) and the remark after it. □

As follows from the analysis of the scalar case in §27.1, a system in the formal normal form (27.8) is integrable: there are diagonal matrix polynomial $B(t^{-1}) = B_0 t^{1-m} + B_1 t^{2-m} + \cdots + B_{m-2} t^{-1}$ and a constant diagonal matrix C, such that a fundamental matrix solution of (27.5) has the form $X(t) = t^C \exp B(t^{-1})$.

27.4. Formal simplification in the resonant case. The direct proof of the formal diagonalization Theorem 27.5 looks as follows. The formal gauge transformation $X \mapsto X' = HX$ defined by a formal matrix series

$$H = E + \sum_{k>0} t^k H_k \in \operatorname{GL}(n, \mathbb{C}[[t]])$$

conjugates two systems (formal or convergent)

$$\begin{aligned} t^m \dot{X} &= A(t)X, & & & t^m \dot{X}' &= A'(t)X', \\ A(t) &= A_0 + \sum_{k>0} t^k A_k, & &\text{and} & A'(t) &= A_0 + \sum_{k>0} t^k A'_k, \end{aligned}$$

with the same principal part $A(0) = A'(0) = A_0$, if and only if H is a formal solution to the following matrix differential equation,

$$t^m \dot{H} = A'(t)H - HA(t). \tag{27.9}$$

Termwise, this equation is equivalent to the sequence of matrix equations involving the coefficients A_k, A'_k of the expansions for $A(t)$ and $A'(t)$ respectively,

$$0 = (A'_0 H_k - H_k A_0) + (A'_k - A_k) + \\ + \sum_{i,j>0,\ i+j<k} (A'_i H_j - H_i A_j) - \begin{cases} kH_{k+1-m}, & k \geqslant m-1, \\ 0, & k < m-1. \end{cases} \qquad (27.10)$$

These equations can be rewritten in the form

$$[A_0, H_k] + A'_k = \text{matrix polynomial in } \{A'_j, H_j,\ 0 < j < k\}.$$

Denote by $L_0 \subset \operatorname{Mat}(n, \mathbb{C})$ the linear subspace of commutators $[A_0, B]$, $B \in \operatorname{Mat}(n, \mathbb{C})$ and let $L_1 \subset \operatorname{Mat}(n, \mathbb{C})$ be any complementary subspace. Then the matrix equations (27.10) can be recursively solved with respect to $H_k \in \operatorname{Mat}(n, \mathbb{C})$ and $A'_k \in L_1$ for all $k = 1, 2, \ldots$ starting from $H_0 = E$, $A'_0 = A_0$.

If A_0 is nonresonant, it can be diagonalized, $A_0 = \operatorname{diag}\{\lambda_1, \ldots, \lambda_n\}$, and the entries of the commutator $[A_0, B]$, $B = \|b_{ij}\|$, will have the form $b_{ij}(\lambda^i - \lambda_j)$. In this case L_0 consists of all matrices with zero diagonal elements. The subspace L_1 of the diagonal matrices is complementary to L_0,

$$[A_0, \operatorname{Mat}] + \operatorname{diag}\{\mathbb{C}, \ldots, \mathbb{C}\} = \operatorname{Mat}, \qquad \operatorname{Mat} = \operatorname{Mat}(n, \mathbb{C}),$$

which proves that a formal solution $H(t), A'(t)$ for (27.9) exists with a diagonal matrix series $A' = \sum t^k A'_k$. Slightly more generally, if A_0 is block diagonal with each block having only one eigenvalue different for different blocks, then the complementary subspace can be chosen as matrices having the same block diagonal structure. This proves the following generalization of Theorem 27.5.

Theorem 27.6. *By a formal gauge transformation one can reduce an irregular system to the block-diagonal form with each block having the leading matrix with a single eigenvalue.* □

Example 27.7. Assume that the leading matrix A_0 is a single Jordan block of size n with the eigenvalue λ_0. Then the subspace L_1 can be chosen consisting of matrices with only the last row nonzero. As a result, we see that by a formal gauge transformation the system can be reduced to the companion form modulo a scalar matrix,

Back reference. See [Dop. Glavy, p. 219], Lemma on commutators.

$$A(t) = \lambda_0 E + \begin{pmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ & & & 0 & 1 \\ a_n(t) & a_{n-1}(t) & \cdots & a_2(t) & a_1(t) \end{pmatrix}. \qquad (27.11)$$

with formal series $a_i \in \mathbb{C}[[t]]$. The eigenvalues of the matrix $A(t)$ are the roots $\lambda_1(t), \dots, \lambda_n(t)$ of the polynomial equation

$$\lambda^n = a_1(t)\lambda^{n-1} + \cdots + a_{n-1}(t)\lambda + a_n(t),$$

shifted by λ_0. Since $\lambda_1(0) = \cdots = \lambda_n(0) = 0$ by assumption, we see that the formal series $a_i \in \mathbb{C}[[t]]$ are all without the free terms.

Remark 27.8. If $f(t) = \exp(m\lambda_0/t^{m-1})$ is a solution of the equation $\dot f = -\lambda_0 t^{-m} f$, then the gauge transformation $X \mapsto f(t)X$ brings the system (27.11) to the true companion form (without the diagonal term $\lambda_0 E$). Being scalar, this transformation commutes with any other gauge equivalence, formal or convergent.

27.5. Shearing transformation. Ramified formal normal form. Further simplification of the system is possible only if we extend the class of formal gauge transformations, allowing for *ramified formal transformations* which are formal series in fractional powers of t. It was E. Fabry who realized (1885) the necessity of passing to fractional powers.

Example 27.9 (continuation of Example 27.7)**.** Consider again the case of a system whose leading matrix is a maximal size Jordan block. By Remark 27.8, without loss of generality we may assume that $\lambda_0 = 0$. Assume that $r \in \mathbb{Q}$ is a *positive* rational number, and consider the gauge transformation (cf. with Example 21.4)

$$H(t) = \operatorname{diag}\left\{1, t^{-r}, t^{-2r}, \dots, t^{(1-n)r}\right\}. \tag{27.12}$$

This transformation takes the system (27.5) with the matrix $A(t)$ as in (27.11), into that with the matrix

$$\begin{pmatrix} 0 & t^r & & & \\ & 0 & t^r & & \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ & & & 0 & t^r \\ t^{(1-n)r}a_n & t^{(2-n)r}a_{n-1} & \cdots & t^{-r}a_2 & a_1 \end{pmatrix} - t^{m-1}R,$$

where $R = \operatorname{diag}\{0, r, 2r, \dots, (n-1)r\}$ is the diagonal matrix. The orders of zeros $\nu_k \in \mathbb{N}$ of the formal series $a_k(t)$ were all positive, since $a_k(0) = 0$. Choose r so that the orders of all terms $a'_k(t) = t^{-kr}a_k(t)$ are still nonnegative but the smallest of them is zero, $r = \min_k \nu_k/k$. The denominator of r is no greater than n.

After the conjugacy by H the matrix of the system will take the form

$$\dot X = [t^{-m+r}A'(t) + t^{-1}R]X, \qquad r > 0, \tag{27.13}$$

where $A'(t)$ is a companion matrix similar to (27.11) but with the entries $a'_k(t) \in \mathbb{C}[[t^{1/q}]]$, $k = 1, \dots, n$, being now formal series in *fractional* powers of

t (and without the diagonal term λ_0). The leading (matrix) coefficient $A'(0)$ of $A'(t)$ is the companion matrix with the complex numbers $a'_n(0),\dots,a'_1(0)$ in the last row. By the choice of r, *not all of them are simultaneously zero*, yet their sum is zero, since $\operatorname{tr} A'(0) = a'_1(0) = a_1(0) = 0$. Therefore if after the shearing transformation the system remains non-Fuchsian (i.e., if $r < m-1$), at least some of the leading eigenvalues must be nonzero.

Somewhat more elaborate computations allow to prove similar statement also in the case when the leading matrix coefficient A_0 has several Jordan blocks with the common eigenvalue.

Notice now that the construction described in §27.4, applies without any changes to the *ramified* formal series in fractional powers of t (i.e., when the indices i, j, k range over an arithmetic progression with rational non-integer difference). Applying Theorem 27.5 in these extended settings, we see that the system (27.13) can be now formally split into two subsystems.

Iteration of these two steps (splitting the system and subsequent shearing transformation) sufficiently many times, one can prove the following result.

Theorem 27.10 (Hukuhara (1942), Turritin (1955), Levelt (1975))**.** *By a suitable formal ramified gauge transformation an irregular singularity can be reduced to the diagonal form*

$$A(t) = t^{-r_1}P_1 + t^{-r_2}P_2 + \cdots + t^{-r_k}P_k + t^{-1}C,$$

where $r_1 > r_2 > \cdots > r_k > 1$ *are rational numbers with the denominators not exceeding* $n!$ *and* $P_1,\dots,P_k \in \operatorname{Mat}(n,\mathbb{C})$ *are diagonal matrices commuting with* C*.*

We will not give the proof in full details, see [**Var96**] and references therein. Instead, we focus on the more transparent nonresonant case and study the problems of *holomorphic* rather than formal classification.

27.6. Holomorphic sectorial normalization.

Definition 27.11 (cf. with (27.3))**.** A *separating ray* corresponding to a pair of complex numbers $\lambda \neq \lambda' \in \mathbb{C}$ is any of the $2(m-1)$ rays defined by the condition

$$\operatorname{Re}[(\lambda-\lambda')/t^{m-1}] = 0. \tag{27.14}$$

The following property is characteristic for separation rays, being an immediate consequence of the explicit formula (27.2). Consider two solutions $x(t), x'(t)$ of two scalar systems (27.1) with the same order m and the holomorphic coefficients $a(t), a'(t)$. Denote $\lambda = a(0)$, $\lambda' = a'(0)$.

Back reference: Recall that a function $f\colon(\mathbb{R}_+,0)\to\mathbb{C}$ is *flat* at $t=0$ if $|f(t)|$ decreases faster than any finite power of t as $t\to 0^+$. The function is *vertical*, if $1/f$ is flat.

Proposition 27.12. *If* $R = \rho\cdot\mathbb{R}_+$, $|\rho| = 1$, *is* not *a separating ray for the pair* λ, λ', *then out of the two reciprocal ratios* $x(t)/x'(t)$ *and* $x'(t)/x(t)$ *one* *after restriction on R is flat and the other is vertical, depending on whether $(\lambda - \lambda')/\rho^{m-1}$ is respectively negative or positive.* □

Everywhere here and below we always assume that any sector is bounded by two straight rays coming from the vertex (a finite point, mostly the origin, or infinity); the angle between these rays is the *opening* of the sector. If $\widehat{H} \in \mathrm{GL}(n, \mathbb{C}[[t]])$ is a formal power series, we say that a holomorphic matrix function $H \in \mathrm{GL}(n, \mathcal{O}(S))$ *extends* this series, if $\widehat{H}$ is asymptotic for H in S.

Theorem 27.13 (Sibuya's sectorial normalization theorem, 1962)**.** *Assume that the leading matrix A_0 of the linear system* (27.5) *is non-resonant* (*i.e., has pairwise different eigenvalues*) *and $S \subset (\mathbb{C}, 0)$ is an arbitrary sector not containing two separating rays for any pair of the eigenvalues.*

Then any formal conjugacy $\widehat{H}(t) \in \mathrm{GL}(n, \mathbb{C}[[t]])$ conjugating (27.5) *with its polynomial diagonal normal form* (27.8)*, can be extended to a holomorphic conjugacy $H_S(t) \in \mathrm{GL}(n, \mathcal{O}(S))$ between these systems in S.*

This theorem, published in [**Sib62, Was87**] will be proved in the Appendix to this section, see §27.10 below.

27.7. Sectorial automorphisms and Stokes matrices. Consider a linear system (27.5) and a sector $S \subset (\mathbb{C}, 0)$.

Definition 27.14. A holomorphic invertible matrix $H(t) \in \mathrm{GL}(n, \mathcal{O}(S))$ is called a sectorial automorphism of the system (27.5), if

(1) H conjugates the system with itself,

$$t^m \dot{H}(t) \cdot H^{-1}(t) = A(t)H(t) - H(t)A(t), \qquad t \in S,$$

(2) the asymptotic series for $H(t)$ is identical, i.e., $H(t) - E$ is flat.

Assume that $X(t)$ is any fundamental matrix solution of the system (27.5) in S, and $H(t)$ is a sectorial automorphism of this system. Then $H(t)X(t)$ is another solution of this system, therefore

$$H(t)X(t) = X(t)C, \qquad C \in \mathrm{GL}(n, \mathbb{C}), \tag{27.15}$$

for an appropriate *constant* invertible matrix C representing a linear automorphism of the space of solutions of the system.

Definition 27.15. The matrix C is called the *Stokes matrix* of the sectorial automorphism H with respect to the given solution X.

Since the diagonal formal normal form (27.8) is integrable, sectorial automorphisms in this particular case can be easily described.

Consider a nonresonant system in the diagonal formal normal form (27.8) with the (pairwise different) eigenvalues of the leading matrix denoted by $\lambda_1, \dots, \lambda_n$. Without loss of generality we may assume that the real parts $\operatorname{Re}\lambda_i$ are also all different (if not, the t-plane can be first rotated by an arbitrary small angle) and the enumeration of the coordinates is chosen so that these real part are increasing, $\operatorname{Re}\lambda_1 < \cdots < \operatorname{Re}\lambda_n$.

We fix a diagonal fundamental solution $W(t) = \operatorname{diag}\{w_1(t), \dots, w_n(t)\}$ for (27.8).

Lemma 27.16. *Suppose that neither of the two rays bounding a sector S is separating for the system* (27.8) *in the formal normal form with the eigenvalues ordered so that* $\operatorname{Re}\lambda_1 < \cdots < \operatorname{Re}\lambda_n$.

Then the Stokes matrix C of any sectorial automorphism with respect to the diagonal solution $W(t)$ possesses the following properties:

(1) *For any pair i, j of indices, one of the matrix elements c_{ij}, c_{ji} must be zero, in particular,*
(2) *if $S \supset \mathbb{R}_+$, then $C - E$ is an upper-triangular matrix.*
(3) *If S contains a separating ray for the pair $\lambda_i \neq \lambda_j$ then both $c_{ij} = c_{ji} = 0$, in particular,*
(4) *if S contains one separating ray for each pair of eigenvalues, then necessarily $C = E$.*

Proof. All assertions immediately follow from inspection of the asymptotic behavior of the sectorial automorphism

$$H(t) = W(t)CW^{-1}(t) = \|h_{ij}(t)\|, \qquad h_{ij}(t) = c_{ij}\, w_i(t)/w_j(t),$$

and the observation in Proposition 27.12.

Indeed, if the ratio $w_i(t)/w_j(t)$ along some ray in S is vertical, the corresponding coefficient c_{ij} must necessarily be zero. This proves the first two assertions.

To prove the remaining assertions, note that the two reciprocal ratios w_i/w_j and w_j/w_i have reciprocal asymptotical behavior along any two rays sufficiently close but separated by the separating ray for the eigenvalues λ_i and λ_j. By the preceding arguments, in this case both c_{ij} and c_{ji} must be absent. □

Proposition 27.17 (rigidity)**.** *If a sector S has opening bigger than $\pi/(m-1)$, then the sectorial normalization H_S described in Theorem 27.13, is unique.*

Proof. If there were two sectorial normalizations H, H' with the same asymptotic series $\widehat{H}$, then their matrix ratio $H'H^{-1}$ must be a sectorial automorphism of the formal normal form (27.8). Since all separating rays for the same pair of eigenvalues are separated by the angle $\pi/(m-1)$, the sector S must contain at least one such ray. By the last assertion of Lemma 27.16, the corresponding Stokes matrix must be identity, which means that the ratio itself is identity. □

27.8. Stokes phenomenon. Holomorphic classification of irregular singularities. Consider a linear system (27.5) of Poincaré rank $m-1$ at the nonresonant non-Fuchsian singular point $t=0$, and let (27.8) be its formal normal form.

As before, we can assume without loss of generality that the leading matrix has eigenvalues ordered so that

$$\operatorname{Re}\lambda_1 < \cdots < \operatorname{Re}\lambda_n, \tag{27.16}$$

which means that neither the positive semiaxis $\mathbb{R}_+$ nor its rotated copies $\rho^k\,\mathbb{R}_+$, $k=1,\dots,2(m-1)$, where $\rho = \exp\frac{\pi i}{m-1}$, are separating rays for any two eigenvalues $\lambda_i \ne \lambda_j$.

Consider the covering of the punctured neighborhood $(\mathbb{C},0)\smallsetminus\{0\}$ by $2(m-1)$ rotated congruent sectors $S_k = \{(k-1)\operatorname{Arg}\rho - \delta < \operatorname{Arg} t < k\operatorname{Arg}\rho + \delta\}$, $k=1,\dots,2(m-1)$, of opening $\pi/(m-1)+2\delta$. Here the positive δ can be chosen so small that each sector S_k contains *exactly one separating ray* for each pair of eigenvalues $\lambda_i \ne \lambda_j$. This collection of sectors will be referred to as the *convenient covering*.

By Theorem 27.13, over each sector S_k there exists a holomorphic gauge conjugacy $H_k(t) \in \operatorname{GL}(n,\mathcal{O}(S_k))$ between the initial system (27.5) and its formal normal form (27.8). This conjugacy is unique by Proposition 27.17. The collection $\{H_k\}$ of these sectorial normalizing maps will be referred to a *normalizing cochain* inscribed in the convenient covering $\{S_k\}$.

Since all maps forming the normalizing cochain have the same common asymptotic series, the matrix ratios $F_{ij} = H_i H_j^{-1} = F_{ji}^{-1}$ defined on the nonempty intersections $S_i \cap S_j$, are sectorial automorphisms of the formal normal form (27.8). Clearly, the intersections $S_i \cap S_j$ are non-void if and only if $j = i+1$ cyclically modulo $2(m-1)$; they are thin sectors around the rotated copies $\rho^j\,\mathbb{R}_+$ of the real axis.

Definition 27.18. Let $\{H_i\}$ be a uniquely defined normalizing cochain inscribed in the convenient covering. The *Stokes collection* of a linear system at a nonresonant irregular singular point is the collection of Stokes matrices $\{C_j\}$, $j=1,\dots,2(m-1)$ of the sectorial automorphisms $F_{ij} = H_i H_j^{-1}$,

$i + 1 = j$, corresponding to a diagonal solution $W(t)$ of the formal normal form.

By Proposition 27.17, the Stokes collection is uniquely defined, as soon as the diagonal fundamental solution $W(t)$ is fixed.

Proposition 27.19. *The matrices C_j from the Stokes collection are unipotent. Moreover, under the normalizing assumption* (27.16) *they are simultaneously upper-triangular.*

Proof. This follows from the second assertion of Lemma 27.16. □

Remark 27.20. Note that the diagonal formal normal form of a non-Fuchsian system is uniquely defined, while its diagonal solution $W(t)$ is defined only modulo constant diagonal gauge transformations. Thus the Stokes matrices for a given formal formal normal form $\Lambda(t)$ are also defined only modulo a simultaneous conjugacy

$$C_j \mapsto DC_jD^{-1}, \qquad D = \operatorname{diag}\{\alpha_1, \dots, \alpha_n\}, \qquad \forall j = 1, \dots, 2(m-1).$$

However, we will always assume that a diagonal fundamental solution is fixed for each given formal normal form (27.8).

Theorem 27.21 (classification theorem for irregular singularities)**.** *A linear system is holomorphically gauge equivalent to its formal normal form at a nonresonant irregular singular point, if and only if the Stokes collection is trivial, $C_1 = \cdots = C_{2m-2} = E$.*

More generally, two such linear systems with a common formal normal form are holomorphically gauge equivalent if and only if their Stokes collections coincide.

Proof. If the system is holomorphically gauge equivalent to its formal normal form and $H(t)$ is the corresponding holomorphic matrix function yielding the equivalence, then the restrictions $H_j = H|_{S_j}$ of H on the sectors of the convenient covering, form the (unique) normalizing cochain and trivially coincide on the intersections. Hence the corresponding Stokes matrices C_j are all identical.

Conversely, if all Stokes matrices $C_j = E$ are trivial, the sectorial normalizing maps coincide on the intersections of the sectors and hence together constitute a map H holomorphically invertible in the punctured neighborhood $(\mathbb{C}, 0) \smallsetminus 0$. Since this map admits a formally invertible asymptotic series $\widehat{H}$, it has a removable singularity at the origin and hence extends as a holomorphic conjugacy between the system and its formal normal form.

More generally, consider two systems with the same formal normal form and the uniquely defined normalizing cochains $\{H_j\}$ and $\{H'_j\}$ respectively.

If G is a holomorphic conjugacy between these systems, then the cochain $\{H_j G\}$ will also be normalizing for the second system. By the uniqueness (Proposition 27.17), $H'_j = H_j G$ and hence $H'_i(H'_j)^{-1} = H_i H_j^{-1}$ for all meaningful i, j, that is, the Stokes operators C'_j and C_j coincide. This argument works also in the inverse direction: if all Stokes operators coincide, then the "ratios" $G_j = H'_j H_j^{-1}$ coincide on the non-void intersections and hence together define a function G holomorphically invertible outside the origin. This function extends to the origin for the same reasons as before: it has an asymptotic series equal to the ratio $\widehat{H}'\widehat{H}^{-1}$ of the formal normalizing series of the two systems. □

27.9. Realization theorem. Proposition 27.19 describes the necessary property of Stokes operators. It turns out that this is a unique requirement.

Theorem 27.22 (Birkhoff, 1909)**.** *Any collection of unipotent upper triangular matrices $\{C_i\}$ can be realized as a Stokes collection of a non-resonant irregular singularity with a preassigned formal normal form* (27.8) *normalized by the condition* (27.16)*.*

Proof. The proof reproduces with only minor repetitions the proof of Theorem 23.2 modulo the solvability result for cocycles of special form. Consider the convenient covering S_j and the collection of holomorphic invertible matrix functions

$$F_{ij}(t) = W(t) C_j W^{-1}(t), \qquad j = 1, \dots, 2(m-1),\ j - i = 1,$$

defined in the corresponding nonempty intersections $S_{ij} = S_i \cap S_j$, where $W(t)$ is a diagonal fundamental solution of the formal normal form. Since C_j are upper-triangular and the eigenvalues λ_j are arranged to satisfy (27.16), the differences $F_{ij}(t) - E$ are flat in the thin sectors S_{ij} and define an asymptotically trivial cocycle in the sense §22.15.

By Theorem 22.33 which is a refinement of the Cartan theorem, the asymptotically identical cocycle $\mathcal{F} = \{F_{ij}\}$ is solvable by a bounded cochain $\mathcal{H} = \{H_j\}$. This means that the sectorial solutions $X_j(t) = H_j^{-1}(t) W(t) = X_i(t) C_j$, for $i + 1 = j$, satisfy linear systems with the coefficient matrices

$$A_j(t) = t^m \tfrac{d}{dt}(H_j^{-1}) H_j + H_j^{-1}(t) \Lambda(t) H_j(t)$$

coinciding on the intersections, $A_i(t) = A_j(t)$ for $t \in S_i \cap S_j$. The resulting matrix function $A(t)$, defined in the punctured neighborhood of the origin, is bounded hence holomorphic and by construction the system $t^m \dot{X} = A(t) X$ is holomorphically equivalent to the formal normal form $t^m \dot{W} = \Lambda(t) W$.

Clearly, the Stokes collection of the constructed system coincide with the prescribed data $\{C_j\}$. □

As a corollary we conclude that there exist non-Fuchsian systems for which the formal diagonalizing series diverge. Moreover, in some sense this divergence is characteristic for the *majority* of non-Fuchsian singularities: Theorems 27.21 and 27.22 imply that classes of holomorphic gauge equivalence are parameterized by $(m-1)n(n-1)$ complex parameters (entries of the Stokes collections).

Appendix: demonstration of Sibuya theorem

In this section we prove the Sectorial Normalization Theorem 27.13. This theorem can be reduced to an analytic claim asserting existence of flat solutions for a non-homogeneous system of linear equations in a sector.

Throughout this Appendix we fix a non-resonant linear system (27.5), its diagonal formal normal form (27.8) with $\Lambda(0) = \operatorname{diag}\{\lambda_1, \dots, \lambda_n\}$, $\lambda_i \neq \lambda_j$, and a formal transformation $\widehat{H} \in \mathrm{GL}(n, \mathbb{C}[[t]])$ conjugating the two. Given a sector S, we can speak then about sectorial conjugacy (or conjugacies) extending $\widehat{H}$ in this sector.

27.10. Extension on sectors without separation rays. First we show that the problem of constructing sectorial normalization for the sector described in the Sibuya theorem can be reduced to that for smaller sectors.

Lemma 27.23. *Assume that S_0 and S_1 are two overlapping sectors in which sectorial conjugacies H_0 and H_1 exist. If S_1 contains no separating rays inside or on the boundary, then the conjugacy H_0 can be extended on the union $S_0 \cup S_1$.*

Proof. Without loss of generality we may assume that the intersection $S_{01} = S_0 \cap S_1$ contains the positive semiaxis $\mathbb{R}_+$ and the eigenvalues of the leading matrix are arranged as in (27.16). Then the Stokes matrix C for this pair must be upper-triangular, and on the intersection S_{01} we have

$$H_0(t) = H_1(t) W(t) C W^{-1}(t), \tag{27.17}$$

where $W(t)$ is a fixed diagonal solution of the formal normal form. But since S contains no separation rays, the difference $E - W(t)CW^{-1}(t)$ remains flat not only on $S_{01} \subset S_1$, but also on the entire section S_1. In other words, the right hand side of (27.17) extends the same series $\widehat{H}$ and, being defined also in $S_1 \smallsetminus S_0$, extends H_0 onto this complement while remaining a sectorial conjugacy with the same asymptotic series. □

As a corollary, we conclude that *it would be sufficient to prove the Sectorial normalization theorem for an arbitrary sector with opening less than* $\pi/(m-1)$. Indeed, since separating rays for the same pair of eigenvalues are equidistributed with the angle $\pi/(m-1)$, any sector with at most one such ray for each pair contains a subsector of opening strictly less than $\pi/(m-1)$ containing these rays, and eventually one or two flaps free from the separating rays from the sides.

From now on we will always assume that the sector S is *acute*, meaning that its opening is less than $\pi/(m-1)$.

27.11. Homotopy method: the construction. We show first how the problem of constructing a sectorial conjugacy between a linear system (27.5) and its formal normal form, can be reduced to construction of a flat solution of an auxiliary linear system in an acute sector.

This description of the homotopy method may be expanded or replaced by a back reference if the method is used somewhere else.

By the Borel–Ritt theorem [**Was87**, §9.2], in any sector S there exists an analytic matrix function $F(t)$ whose asymptotic series in S is the prescribed normalizing series $\widehat{H}$. Conjugating the system (27.5) by F, we obtain a new system of the form $t^m\dot{X} = A'(t)X$ with the matrix $A'(t)$ holomorphic in S and having the same asymptotic series at the origin as the Taylor series $\Lambda(t)$ of the formal normal form $t^m\dot{X} = \Lambda(t)X$. Thus to construct the sectorial conjugacy between the system and its initial normal form, it is sufficient to remove by a suitable sectorial gauge transformation the *flat* non-diagonal part $B(t)$ from the system

$$\begin{gathered}\dot{X} = (\Lambda(t) + B(t))X, \qquad B(t) = \|b_{ij}(t)\|,\\ b_{ij} \in \mathcal{O}(S), \quad b_{ii} \equiv 0, \quad t^{-N}b_{ij}(t) \to 0 \text{ in } S \text{ for any } N.\end{gathered} \tag{27.18}$$

The diagonal entries of B can be assumed absent by Proposition 27.1.

The conjugacy between (27.18) and (27.8) can be constructed now as a shift along trajectories of an auxiliary vector field. Let $\varepsilon \in \mathbb{C}$ be an auxiliary variable and consider the holomorphic vector field V in the space $S\times\mathbb{C}^n\times\mathbb{C}$ corresponding to the system of equations

$$\dot{t} = 1, \qquad \dot{x} = t^{-m}(\Lambda(t) + \varepsilon B(t))x, \quad \dot{\varepsilon} = 0$$

(cf. with (20.2) with an additional coordinate ε). Suppose that there exists another vector field Q defined in the same domain, defined by a system of equations

$$\dot{t} = 0, \quad \dot{x} = H(t,\varepsilon)x, \quad \dot{\varepsilon} = 1$$

with a matrix function $H(t,\varepsilon)$ flat with respect to $t \in S$ for all values of ε. The flow map of Q carries the hyperplanes $\varepsilon = \text{const}$ to themselves. If Q commutes with V, then this flow map will conjugate the restrictions of the vector field V on these (invariant for V) hyperplanes. In particular, the flow of Q will conjugate the systems (27.8) and (27.18) corresponding to the values of $\varepsilon = 0$ and $\varepsilon = 1$ respectively. Since H is flat, the flow of Q differs from the identity (i.e., the translation along the ε-axis) by a flat term.

Thus the problem of constructing the sectorial gauge transformation removing the off-diagonal terms from the system (27.18) is reduced to constructing the field Q, that is, the flat matrix function $H(t,\varepsilon)$ holomorphically depending on ε as a parameter.

The condition $[V,Q]=0$ translates into the identity

$$\dot H = t^{-m}[H, \Lambda + \varepsilon B] + B, \qquad H = H(t,\varepsilon) \in \operatorname{Mat}(n, \mathcal{O}(S\times\mathbb{C})), \quad (27.19)$$

with the (usual) matrix commutator in $\mathbb{C}^n$ in the right side. This identity, sometimes referred to as the *homological equation*, can be considered as a system of n^2 first order linear ordinary differential equations on the components of the matrix function H. Moreover, since the matrix B has identically zero diagonal, only *off-diagonal* entries can be considered so that ultimately the solution of (27.19) will be constructed also with identical zeros on the diagonal.

Denote by $y=(y_1,\dots,y_k)\in\mathbb{C}^k$, $k=n(n-1)/2$, the collection of all off-diagonal entries of the matrix H. The system (27.19) takes then the form

$$t^m \dot y(t) = [D + G(t,\varepsilon)y](t) + g(t), \qquad t\in S,\ \varepsilon\in\mathbb{C}, \qquad (27.20)$$

where D is a diagonal matrix corresponding to the commutator with the leading term $\Lambda_0 = \operatorname{diag}\{\lambda_1,\dots,\lambda_n\}$ of the formal normal form $\Lambda(t)$. Since the system was assumed nonresonant, all eigenvalues of D are nonzero,

$$D = \operatorname{diag}\{\mu_1,\dots,\mu_k\}, \qquad \mu_i \neq 0,\ i=1,\dots,k,\ k=n(n-1)/2. \quad (27.21)$$

The term $G(t,\varepsilon)$ corresponds to the commutator with the non-leading terms and hence tends to zero as $t\to 0$ uniformly in ε, and the non-homogeneity $g(t)$, (accidentally) independent of ε, consists of the off-diagonal terms of the matrix $B(t)$ and is flat at the origin.

It is convenient to simplify the system further to reduce the Poincaré rank to the minimum and place the singular point at infinity so that the leading part would be a system with constant coefficients easy for explicit integration.

Changing the independent variable from $t\in S\subset(\mathbb{C},0)$ to $z=1/t^{m-1}\in(\mathbb{C},\infty)$ transforms the 1-form $t^{-m}\,dt$ to $(1-m)\,dz$. This transformation brings the system (27.20) to the form $dy/dz = (1-m)[D+G(z^{1/(1-m)},\varepsilon)] + (1-m)g(z^{1/(1-m)})$ defined in a sector S' with the vertex at infinity and the opening strictly less than π, i.e., acute in the conventional sense of this word. Rotating the z-plane if necessary, we can always assume that $S'=\{|z|>r,\ |\operatorname{Arg} z|<\pi-\delta\}$, where $\delta>0$ is a small positive parameter.

Returning to the previous notations, we can rewrite the system (27.20) with respect to the new variable z as follows,

$$
\begin{gathered}
\tfrac{d}{dz}y = [D + G(z,\varepsilon)]y + g(z), \qquad y \in \mathbb{C}^k,\\
z \in S' = \{|z| > r,\ |\operatorname{Arg} z| < \pi - \delta\},\\
\left.\begin{aligned} G(z,\varepsilon) &= o(1) \quad \text{uniformly over } |\varepsilon| < 2,\\ g(z) &= o(z^{-N}) \quad \text{for any } N \in \mathbb{N}, \end{aligned}\right\} \text{as } z \to \infty \text{ in } S',\\
D = \operatorname{diag}\{\mu_1, \dots, \mu_k\}, \quad \mu_i \neq 0.
\end{gathered} \tag{27.22}
$$

By constructions of the homotopy method, existence of a sectorial conjugacy between a linear system (27.5) and its formal normal form (27.8) is reduced to the existence of a flat solution to the linear non-homogeneous system (27.22).

Theorem 27.24. *The system* (27.22) *admits a unique flat solution in the acute sector S'.*

Back reference

The rest of the Appendix is the proof of this Theorem achieved by application of the contraction mapping principle. From now on we treat ε as a *parameter* noting in passing that all results are valid *uniformly over all values of this parameter*, say, in the disk $|\varepsilon| < 2$. To simplify the notation, we omit ε everywhere.

Remark 27.25. Consider the matrix equation (27.9) with the matrices $A(t)$ and $A'(t)$ having the same diagonal terms and the common diagonal leading matrix $A_0 = \operatorname{diag}\{\lambda_1, \dots, \lambda_n\}$ in the non-resonant case $\lambda_i - \lambda_j \neq 0$. The corresponding "nonlinear" system of differential equations

$$\dot t = t^m, \qquad \dot y = D(t)y, \qquad t \in \mathbb{C}^1,\ y \in \mathbb{C}^k,$$

for the off-diagonal components of the conjugacy H, has a one-dimensional central direction (the t-axis). Theorem 27.24 asserts existence of a sufficiently large sector-like piece of the analytic center manifold for this system. The Stokes phenomenon describes obstructions for existence of analytic center manifold in the entire neighborhood of the singular point.

27.12. Core example. Consider first the one-dimensional particular case of the system (27.22),

$$\tfrac{d}{dz}y = \mu y + g(z), \qquad 0 \neq \mu \in \mathbb{C}, \quad y \in \mathbb{C}^1, \quad z \in S'. \tag{27.23}$$

with a flat non-homogeneity $g(z)\mathcal{O}(S')$ and the absent term $G \equiv 0$. We are looking for a solution flat in the acute sector S'.

Solution of this system is given by the explicit formula obtained by variation of constants method (see Remark 20.10): for an arbitrary choice of the base point $b \in S'$,

$$y(z) = e^{\mu z}\left(y(b) + \int_b^z e^{-\mu\zeta} g(\zeta)\,d\zeta\right) = e^{\mu z} y(b) + \int_b^z e^{\mu(z-\zeta)} g(\zeta)\,d\zeta. \quad (27.24)$$

The upper limit of integration is the variable point z. The lower limit $b \in S'$ and the respective boundary condition $y(b)$ have to be chosen so that the solution (27.24) would be flat in S'.

Two cases have to be treated separately, depending on the relative position of $0 \neq \mu \in \mathbb{C}$ and S', namely,

(1) $\operatorname{Re} \mu a > 0$ for some $a \in S'$, that is, the solution of the homogeneous equation is unbounded in S'; this happens when S' overlaps with some growth sector (in the sense of §27.1), and

(2) $\operatorname{Re} \mu z < 0$ for all $z \in S'$, that is, the solution of the homogeneous equation decays exponentially fast in S' (i.e., when S' belongs to a fall sector).

The intermediate case when $\operatorname{Re} \mu z = 0$ along one of the boundary rays of S', will not be discussed, as we will not need it. Abusing the language, we will refer to the sector of the first type as a growth sector as well.

In the growth sector we chose the base point at infinity, $b = +\infty \cdot a$. More precisely, consider the ray $R_z = z + \mathbb{R}_+ a = \{\zeta = z + sa \colon s \in \mathbb{R}_+\}$ (with the orientation inherited from $\mathbb{R}_+$) and the integral operator $\mathbf{S}_+ \colon f \mapsto \mathbf{S}_+ f$,

$$\begin{aligned} \mathbf{S}_+ f(z) &= -\int_{R_z} e^{\mu(z-\zeta)} f(\zeta)\,d\zeta \\ &= -a \cdot \int_0^{+\infty} e^{-s\cdot\mu a} f(z+sa)\,ds, \qquad s \in \mathbb{R}_+. \end{aligned} \quad (27.25)$$

This integral converges since both the function $e^{-s\mu a}$ and $f(z+sa)$ decrease very fast as $s \to +\infty$.

In the sector of fall we choose the base point $b = r$ on the "exterior circumference" of the sector S', and fix the initial condition $y(b) = 0$. Then the solution $y(\cdot)$ is given by the integral operator $\mathbf{S}_-$ along the segment $[r, z] = -[z, r] = \{z - sa \colon 0 \leqslant s \leqslant |z-r|\}$, where $a = a(z) = (z-r)/|z-r|$,

$$\begin{aligned} \mathbf{S}_- f(z) &= -\int_{[z,r]} e^{\mu(z-\zeta)} f(\zeta)\,d\zeta \\ &= -a \cdot \int_0^{|z-r|} e^{s\cdot\mu a} f(z-sa)\,ds, \quad a(z) = \frac{z-r}{|z-r|}. \end{aligned} \quad (27.26)$$

There is no question of convergence, since the segment is always finite.

Definition 27.26. Given an acute sector S' and a nonzero complex number μ such that $\operatorname{Re} \mu z \neq 0$ on the boundary of S', we denote by $\mathbf{S} = \mathbf{S}_{\mu,S'}$ the appropriate integral operator,

$$\mathbf{S}_{\mu,S'} = \begin{cases} \mathbf{S}_+, & \text{if } \operatorname{Re}\mu a > 0 \text{ for some } a \in S', \\ \mathbf{S}_-, & \text{if } \operatorname{Re}\mu z/|z| \leqslant \delta_0 < 0 \text{ for all } z \in S'. \end{cases} \tag{27.27}$$

Lemma 27.27. *The operator $\mathbf{S}_{\mu,S'}$ is bounded as a linear operator acting on the subspace $\mathcal{O}(S';0)$ of bounded functions from $\mathcal{O}(S')$ equipped with the sup-norm $\|f\| = \sup_{S'} |f(z)|$.*

Moreover, it remains bounded when considered as an operator on the space $\mathcal{O}(S';N)$ of functions decreasing as fast as $O(|z|^{-N})$ equipped with the norm

$$\|f\|_N = \|f\|_{S';N} = \sup_{z\in S'} |z|^N\, |f(z)|. \tag{27.28}$$

Proof. We fix the sector S' and treat separately the two possibilities of S' being the sector of growth or fall, depending on the choice of μ. First we consider the case $N = 0$ corresponding to the usual sup-norm.

If S' is the sector of growth and $\|f\| = 1$, that is, $|f(z)| \leqslant 1$, then $|\mathbf{S}_+ f(z)| \leqslant |a| \int_0^\infty e^{-cs}\, ds = |a|/c$, $c = \operatorname{Re}\mu a > 0$.

If S' is the sector of fall, then $|\mathbf{S}_- f(z)| \leqslant |a| \int_0^{|z-r|} e^{cs}\, ds \leqslant 1/|c|$, where $c = c(z) = \operatorname{Re}\mu a(z)$. If z belongs to the translate $r + S'$ of the sector S', then $a(z) = (z-r)/|z-r|$ of modulus 1 belongs to S', hence by the second assumption (27.27) we have $|c(z)| \geqslant \delta_0 > 0$ bounded from from below. This proves that $\mathbf{S}_- f$ is bounded in $r + S'$.

Moreover, one can replace S' by another sector $S'' \supset S'$ of slightly bigger opening but still a fall sector; the above arguments would prove then that $\mathbf{S}_- f$ is bounded in $r + S''$. It remains to notice that the difference $S' \smallsetminus (r + S'')$ is bounded of diameter depending only on S', S'' and r, so the integral (27.26) is bounded also there. Thus we have proved the boundedness of $\mathbf{S}_-$ with respect to the usual sup-norm $\|\cdot\|_0$ on S'.

To prove the boundedness with respect to the "weighted sup-norms" $\|\cdot\|_N$, assume that $\|f\|_N \leqslant 1$, i.e., $|f(z)| \leqslant |z|^{-N}$, and consider again both possibilities for S'.

Let S' be a sector of growth. Since S' is acute and $z, a \in S'$, we have $|z+sa| \geqslant c'\,|z|$ for some constant $c' > 0$ depending only on S' and all $s \in \mathbb{R}_+$, by obvious geometric considerations. Substituting this inequality into the integral (27.25), we majorize $\mathbf{S}_+ f$ in S' by $|c'z|^{-N} \cdot /|c|$. This proves the boundedness of $\mathbf{S}_+$.

To see why $\mathbf{S}_-$ is bounded in $r + S''$ with respect to this norm (where S'' is chosen as in the case $N = 0$), we split the segment of integration $[r, z]$ in (27.26) into two equal parts. On the initial part $\zeta \in [r, \frac{1}{2}(r+z)]$ the exponential factor $e^{\mu(z-\zeta)}$ is exponentially small, since $|z - \zeta| \geqslant \frac{1}{2}|z|$. On the distant part $\zeta \in [\frac{1}{2}(z+r), z]$ we have the inequality $|\zeta| \geqslant \frac{1}{2}|z|$ and hence by our assumption on f, $|f(\zeta)| \leqslant 2^{-N}|z|^{-N}$, so that the full integral $\mathbf{S}_- f(z)$ is bounded by $2^{-N}|z|^{-N}/|c(z)|$. Exactly as in the case $N = 0$, this implies that $\mathbf{S}_-$ is bounded in the $\|\cdot\|_N$-norm. □

Remark 27.28. In all these constructions the bound for the norm $\|\mathbf{S}_\pm\|_{S';N}$ may depend on N and the opening of the sector S' but does not depend on the "radius" r of the sector. This can be verified independently by the rescaling arguments.

27.13. Integral equation and demonstration of Theorem 27.24. If instead of the simple equation (27.23) we would have a slightly more general form

$$\tfrac{d}{dz}y = [\mu + G(z)]y + g(z), \tag{27.29}$$

then the method of variation of constants, instead of giving an explicit solution, would reduce (27.29) to an integral equation.

After the substitution $y(z) = e^{\mu z}y'(z)$ (27.29) is transformed to the equation $\frac{d}{dz}y'(z) = e^{-\mu z}[G(z)y(z) + g(z)]$ which after taking primitive and multiplication by $e^{\mu z}$ yields

$$y(z) = e^{\mu z}y(b) + \int_b^z e^{\mu(z-\zeta)}[G(z)y(z) + g(z)]\,dz.$$

Again the base point b can be chosen freely, and this freedom can be again used to ensure the flatness of solutions. As before, we conclude that

$$y = \mathbf{S}[Gy + g], \qquad \mathbf{S} = \mathbf{S}_{\mu,S'}, \tag{27.30}$$

if it exists, satisfies the differential equation (27.29).

A multidimensional generalization of this example for the k-dimensional system (27.22) is straightforward. Denote by $\mathbf{S}$ the diagonal integral operator defined on vector-functions bounded in the sector S', as follows:

$$\mathbf{S}(y_1, \dots, y_k) = (\mathbf{S}_1 y_1, \dots, \mathbf{S}_k y_k), \qquad \mathbf{S}_i = \mathbf{S}_{\mu_i,S'},\ i = 1, \dots, k. \tag{27.31}$$

This operator, a Cartesian product of integral operators of the form (27.27), depends on the eigenvalues of the diagonal matrix $D = \operatorname{diag}\{\mu_1, \dots, \mu_k\}$, with the path of integration being in general different for each component.

In complete analogy with (27.30), solution of the system (27.22) can be constructed by solving the integral equation

$$y = \mathbf{S}[Gy + g], \qquad \mathbf{S} = \operatorname{diag}\{\mathbf{S}_1, \dots, \mathbf{S}_k\}. \tag{27.32}$$

The diagonal integral operator $\mathbf{S}$ is bounded by Lemma 27.27, if the boundary rays of S' are not exceptional for any μ_i, that is, not separating for the initial system (27.5). We show that the composition occurring in the right hand side (27.32) is a contraction, if the sector $S' = \{|z| > r,\ |\operatorname{Arg} z| < \pi - \delta\}$ is sufficiently small, i.e., r is sufficiently large.

Proposition 27.29. *In the assumptions of Theorem 27.24 the operator*

$$y \mapsto \mathbf{G}y = Gy + g$$

is Lipschitz in the sense of any norm $\|\cdot\|_{S';N}$ on the space of vector-functions holomorphic in $S'_r = S' \cap \{|z| > r\}$,

$$\|\mathbf{G}y - \mathbf{G}y'\|_{S'_r;N} < \rho\,\|y - y'\|_{S'_r;N}, \qquad \rho = \rho(r) > 0.$$

The Lipschitz constant $\rho(r)$ tends to zero as $r \to +\infty$.

Proof. The Lipschitz constant $\rho = \rho(r)$, actually independent of N, can be chosen as $\rho(r) = \sup_z\{\|G(z)\|\colon\ z \in S'_r\}$. By assumption, $G(z)$ tends to zero as $z \to \infty$ in S', hence $\rho(r) \to 0^+$ as $r \to +\infty$. □

Proof of Theorem 27.24. Our goal already has been reduced to showing that the integral equation (27.32) admits a solution flat in the sector S'. Without loss of generality we may assume that the rays bounding S' are not exceptional (otherwise one can increase slightly the opening while keeping the sector acute).

Let $N \geqslant 0$ be an arbitrary order of decay. As soon as r is sufficiently large, $r \geqslant r(N)$, the Lipschitz constant $\rho(r)$ of the operator $\mathbf{G}$ becomes smaller than the bound for the norm of the operator $\mathbf{S}$ with respect to any given N (recall that $\|\mathbf{S}\|_N$ does not depend on r, see Remark 27.28). In the corresponding $S'_r = S' \cap \{|z| > r(N)\}$ the composition $\mathbf{S} \cdot \mathbf{G}$ will be contracting in the $\|\cdot\|_N$-norm. Hence the fixed point-type integral equation (27.32) possesses a *unique* solution, a vector function with each component belonging to the space $\mathcal{O}(S'_N, N)$. Any such solution can in fact be extended to a function holomorphic in the entire sector S' by virtue of the differential equation (27.22) non-singular in S'. By the uniqueness, any two such extensions necessarily coincide with each other on the intersection of their domains. Together they yield a vector function $y(z)$ holomorphic in S' and decreasing faster than $|z|^{-N}$ for any N as $|z| \to \infty$. In other words, the constructed solution $y(z)$ is flat as required. □

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28. To Do List

Chapter I. Siegel–Brjuno–Yoccoz survey. ??? Existence of a piece of central manifold by contracting map principle. Linearization of foliation and meromorphic connection.

Chapter III: Shorten the first section. Made more geometric the part on bundles, perhaps, elaborate more. Add the section on how bounds and connections appear by linearization of foliations.