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Chapter 1
Introduction

For convenience we turn off the spell checker by typing

```
Off@General::spell1D
```

à 1.1 Preliminary Remarks

à 1.2 Symbolic Computation

The *Mathematica* programming language allows not only for interactive manipulations of a wide range of objects, but it also encourages the automation of complicated computations. *Mathematica* actually works by using systems of rewrite rules, and the user can freely create and use his or her own systems of rules. This distinguishes it from traditional programming languages, which normally have no such features. We have applied rule-based and functional programming techniques in *Mathematica* throughout this book. Hence, we briefly cover the basic ingredients of rule-based programming in Sections 1.2.1--1.2.5 and some functional programming features in Sections 1.2.6--1.2.8.

For further information on *Mathematica*, consult the standard *Mathematica* reference: *The Mathematica Book*, Stephen Wolfram, Third Edition, Wolfram Media and Cambridge University Press, 1996. Full descriptions of the functions in the standard *Mathematica* packages are available in *Mathematica 3.0 Standard Add-On Packages manual*, Wolfram Media and Cambridge University Press, 1996. Alternatively, simply click on the Help menu to view the Help Browser.

In the Text cells of this book, we use **bold-magenta** to indicate **user-defined functions** or **names** and **bold-black** to indicate *Mathematica*'s **built-in functions**.

Ÿ 1.2.1 Global Rules

Global rules are applied whenever the appropriate left-hand side is encountered or matched. The advantage of using global rules is that one does not need to apply explicitly a rule to get things done. However, it is almost impossible to prevent such global rules from being applied to a certain expression. There are two types of user-defined global rewrite rules: those using **=** and those using **:=**. The distinction between them lies in when the right-hand side is evaluated. Moreover, each of these rules has two forms, depending on where the rule is stored, resulting in four rules indicated by **=** , **^=** , **:=** , and **^:=**.

Ÿ 1.2.1.1 = rules

We can think of =, an assignment statement in analogy with traditional procedural programming languages, as a (global) rewrite rule. The characteristic property of using the rule **=** is that the right-hand side is evaluated immediately when the assignment is made, and all subsequent matched left-hand sides are replaced by the evaluated right-hand side. For example, let us consider the rule

```
a = 2;
```

The output is suppressed by ending the input with a semicolon. From now on, whenever **a** is encountered in any expression, *Mathematica* will replace it by its value 2.

```
expr1 = a^2 + 3 - a + Tan@aD
```

```
5 + Tan@2D
```

Let us try to evaluate **expr1** for another value,

```
a = 3;
```

```
expr1
```

```
5 + Tan@2D
```

The new value of **a** is not recognized in **expr1**.

Let us try to clear the definition of **a** by using **Clear** or

```
a =.
```

Then, let us evaluate **expr1**,

```
expr1
```

```
5 + Tan@2D
```

Again, **expr1** is not changed accordingly.

To be able to evaluate **expr1** for different values of **a**, we need to either define **expr1** before the assignment of **a** or use "**:=**", instead of "**=**". The precise meaning of **:=** is taken up in the next section. Here, we illustrate the former choice.

```
Clear@aD; expr1 = a^2 + 3 - a + Tan@aD
```

```
3 - a + a^2 + Tan@aD
```

```
a = 2; expr1
```

```
5 + Tan@2D
```

```
a = 3; expr1
```

```
9 + Tan@3D
```

In traditional programming languages, the left-hand side of an assignment statement is required to be a simple identifier (i.e., a symbol). In computer algebra, the left-hand side can be arbitrarily complicated. For example,

```
look@here + 9D = Expand@H1 + xL^2 + yD
```

$1 + 2x + x^2 + y$

Note that the output of an **=** expression is the evaluated form of the right-hand side. *Mathematica* regards the left-hand side as a pattern. Whenever *Mathematica* finds *something* that matches this pattern, it replaces the *something* by the evaluated right-hand side.

```
look@there + look@here + 5 + 2^2DD
```

look@1 + there + 2 x + x^2 + yD

In this evaluation, the pattern **look@here + 5 + 2^2D** simplifies to **look@here + 9D**, which is replaced by $1 + 2x + x^2 + y$. The resulting expression does not match any pattern involving **look** and so it is left in unevaluated form. This rule is stored with **look**.

```
? look
```

```
Global`look

look@9 + hereD = 1 + 2*x + x^2 + y
```

Again, we see the evaluated form on the right-hand side.

There are some problems associated with left-hand sides that are not symbols. For example, suppose we try to make the following rule:

```
x + y = z
```

```
Set::write : Tag Plus in x+y is Protected.

z
```

We get an error message stating that **Plus** is protected, which means new rules cannot be added for **Plus**. Every time *Mathematica* encounters **Plus**, it searches through the rules for **Plus** to see if anything applies. If we add a new rule for **Plus**, then that rule would have to be examined at every subsequent addition. When a rule of the form $x + y = z$ is given, *Mathematica* interprets it as a rule of the form **Plus**@x, yD = z. Rules have to be stored somewhere and the default rule is that for the head of the left-hand side. One way of adding a new rule for **Plus** is to unprotect **Plus**, make the rule, and then reprotect it.

```
Unprotect@PlusD;
x + y = z;
Protect@PlusD;
```

Now whenever *Mathematica* sees $x + y$, it rewrites it as z.

```
x + s + y + t
```

```
s + t + z
```

Alternatively, we can use **UpValues** to associate the rule with the unprotected argument of the left-hand side. (Definitions that attach a value to the head of the left-hand side are called **DownValues** of the head.) For example,

```
s + t ^= u
```

```
u
```

Note the caret ^ before the = sign. This rule is associated with the symbol *s* or *t*.

```
? s
```

```
Global`s

s •: s + t = u
```

```
t + s + t
```

```
t + u
```

A given symbol can have both up and down values. Let's give **s** a down value in addition to the up value it already has.

```
s@x_D := Exp@I a xD;
```

Then looking at **s** shows both kinds of values:

```
? s
```

```
Global`s

s •: s + t = u

s@x_D := Exp@I * a * xD
```

Finally, we can access the up and down values individually.

```
8UpValues@sD, DownValues@sD<
```

```
88HoldPattern@s + tD ¦ u<, 8HoldPattern@s@x_DD ¦ Exp@I a xD<<
```

Now let us try naively define the sine function using an = rule according to

```
m@xD = Sin@xD;
```

It works properly for the symbol *x* but not for anything else.

```
8m@xD, m@yD, m@1D<
```

```
8Sin@xD, m@yD, m@1D<
```

This is where the special symbol _ comes in. The form x_ means a pattern named *x*. We can show the internal representation of any expression, such as x_ , by either using

```
FullForm@x_D
```

```
Pattern@x, Blank@DD
```

or using

```
x_ •• FullForm
```

```
Pattern@x, Blank@DD
```

An underscore _ in a pattern matches anything, so it is a dummy variable or a "wild card". If it appears on the left-hand side of "=" rule with a name, like *x*, then the left-hand side is rewritten as the right-hand side with *x* replaced by the desired variable. Let us use this to redefine **m**; that is,

```
m@x_D = Sin@xD;
```

Now **m** works for any argument:

```
8m@xD, m@yD, m@1D, m@s - tD<
```

```
8Sin@xD, Sin@yD, Sin@1D, Sin@s - tD<
```

Thus, one can use = rules to define functions.

Ÿ 1.2.1.2 := rules

Rules using := do not evaluate the right-hand side immediately but instead leave it unevaluated until the function is actually used. They can be used with simple left-hand sides or with left-hand sides containing patterns. As an example, let us define and compare two rules using = or :=.

```
Clear@xD;
lhs1 = Expand@Hx + 1L^2D;
lhs2 := Expand@Hx + 1L^2D;
```

If these are evaluated, they give the same result.

```
8lhs1, lhs2<
```

```
81 + 2 x + x^2, 1 + 2 x + x^2<
```

If we now give a value to *x*, then **lhs1** and **lhs2** will use this value in different ways.

```
x = p + q;
```

```
8lhs1, lhs2<
```

```
81 + 2 Hp + qL + Hp + qL^2, 1 + 2 p + p^2 + 2 q + 2 p q + q^2<
```

If the left-hand side of a := rule contains a pattern, then on a subsequent occurrence of the left-hand side with actual arguments, the formal (dummy) arguments (or names of patterns) on the right-hand side are replaced by the actual arguments and then the right-hand side is evaluated. Thus, each time the left-hand side of such a rule matches something, it is replaced by a new evaluation of the right-hand side. To see the difference, we again set up the two rules

```
f@k_D = Expand@k^2D
```

k^2

```
g@k_D := Expand@k^2D
```

We note that the right-hand side of the rule for **f** is evaluated immediately when it is defined, whereas the right-hand side of **g** is kept unevaluated. Now, try out these two definitions on the same value.

```
8f@p + qD, g@p + qD<
```

8Hp + qL^2, $p^2 + 2\,p\,q + q^2$<

Because the right-hand side of the rule for **f** was evaluated immediately when it was defined, there is nothing to expand, and k is replaced with $p+q$ without expanding the result. On the other hand, the right-hand side of the rule for **g** was not expanded when defined and it retains the whole expression **Expand**[k^2]. When the two functions are subsequently used, f@$p + q$D is just replaced by H$p + q$L^2, while g@$p + q$D is replaced by **Expand**@H$p + q$L^2D , which evaluates to $p^2 + 2\,p\,q + q^2$. The internal representation of such a definition has the following form:

```
FullForm@g@k_D := pD
```

```
Null
```

To see the actual head of an expression before it is evaluated, we wrap the expression with **Hold** and then request its **FullForm**

```
FullForm@Hold@g@k_D := pDD
```

```
Hold@SetDelayed@g@Pattern@k, Blank@DDD, pDD
```

Thus, the symbol := is the infix form of **SetDelayed**. This makes dramatically clear the distinction between the evaluation when the rule is given and the evaluation when the rule is used.

When the head of the left-hand side is protected, we can use **UpValues** to associate the rule with the unprotected argument of the left-hand side. For example, when we try to differentiate a user-defined definite integral with constant bounds, we obtain

```
Clear@int, f, xD
```

```
D@int@f@xD, 8x, 0, 1<D, xD
```

int$^{H0,81,0,0<L}$@f@xD, 8x, 0, 1<D + f£@xD int$^{H1,80,0,0<L}$@f@xD, 8x, 0, 1<D

which is not what we intended. We can either unprotect **D** and add a rule or use ^:= to assign the definition with **int**; that is,

```
D@int@x_, a_D, y_D ^:= int@D@x, yD, aD
```

Now we check to see if *Mathematica* gives us the answer we want.

```
D@int@f@xD, 8x, 0, 1<D, xD
```

```
int@f¢@xD, 8x, 0, 1<D
```

As a second example, we define **Cos**@*n* pD = H- 1L^n and **Sin**@*n* pD = 0 for integer *n*:

```
Unprotect@Cos, SinD;
Cos@n_ pD := H-1L^n •; IntegerQ@nD
Sin@n_ pD := 0 •; IntegerQ@nD
Protect@Cos, SinD;
```

If we declare m to be an integer (here **UpValues** is used),

```
IntegerQ@mD ^= True;
```

then

```
8Cos@m pD, Sin@m pD<
```

```
8H-1L^m, 0<
```

Ÿ 1.2.2 Local Rules

Local rewrite rules (*lhs*->*rhs* or *lhs*:>*rhs*) are useful for making substitutions without making the definitions permanent. These rules are applied to an expression using the operation •. (means **ReplaceAll**) or ••. (means **ReplaceRepeated** in *Mathematica*).

Ÿ 1.2.2.1 -> rules

We can use **Solve** to obtain the solutions of some algebraic equations. For example,

```
eq1 = u^2 - 3 u + 2 == 0;
sol1 = Solve@eq1, uD
```

```
88u ® 1<, 8u ® 2<<
```

We can check the result by substituting **sol1** into **eq1** and obtain

```
eq1 •. sol1
```

```
8True, True<
```

The output of **Solve** is a list of lists of local rules. Thus, *u* -> 1 is a local rule, which is the analog of the global rule *u* = 1. The rule is applied to an expression by using /. , so the expression **eq1**/.**sol1** means "use the rewrite rules *u* -> 1 and *u* -> 2 in

eq1". The result of this is 80 == 0, 0 == 0<, which is then evaluated as {**True, True**}. The usual form of the right-hand side of /. is a list of local rules for some of the symbols that appear on the left-hand side.

```
u v w •. 8u -> 3, w -> 5<
```

```
15 v
```

If the right-hand side is a list of lists of local rules, then the result is a list of modified expressions, one for each substitution in the list.

```
u v w •. 88u -> 3, w -> 5<, 8v -> 2, w -> 3<<
```

```
815 v, 6 u<
```

Local replacement rules also work on expressions which have a **Hold** wrapped around them. For example, without a **Hold**, ¶$_x$ Cos@*x*D immediately evaluates to - Sin@*x*D.

```
¶x Cos@xD
```

```
-Sin@xD
```

However, the rewrite rule goes under the **Hold** to make the replacement.

```
Hold@¶x Cos@xDD •. Cos -> Sin
```

```
Hold@¶x Sin@xDD
```

Local rules with "->" share with "=" rules the property that their right-hand sides are evaluated at the time they are defined.

Ÿ 1.2.2.2 :> rules

The local analog of a ":=" rule is a ":>" rule; i.e., a local rule that evaluates its right-hand side only when it is used. For example,

```
expr1 = Exp@I Hw2 - w1L T0D;
```

```
Clear@aD
```

```
expr1 •. Exp@a_D -> Exp@Expand@aDD
```

$E^{I\, T_0\, H-w_1+w_2L}$

```
8expr1 •. Exp@a_D :> Exp@Expand@aDD, expr1 •• ExpandAll<
```

$8E^{-I\, T_0\, w_1+I\, T_0\, w_2},\ E^{-I\, T_0\, w_1+I\, T_0\, w_2}<$

where the immediate rule (*lhs*->*rhs*) fails to do the expansion. The left-hand sides of the local rules involve patterns rather than just symbols. The difference is that the immediate rule for **Exp**@a_D replaces it by the evaluation of **Exp@Expand@***a*DD which equals **Exp@***a*D. Hence, when this is used with *a* equal to IHw_2 - w_1L T_0, we get the result $E^{I\ T_0\ \mathrm{H}-w_1+w_2\mathrm{L}}$. On the other hand, the delayed rule (*lhs*:>*rhs*) for **Exp**@a_D replaces it by the unevaluated **Exp@Expand@***a*DD which, when used with *a* equal to IHw_2 - w_1L T_0, gives $E^{-I\ T_0\ w_1+I\ T_0\ w_2}$. Alternatively, for this simple example, we can use **ExpandAll** to obtain the same result.

We can check how these expressions are represented internally.

```
FullForm@Hold@p •. q -> rDD
```

```
Hold@ReplaceAll@p, Rule@q, rDDD
```

```
FullForm@Hold@p •. q :> rDD
```

```
Hold@ReplaceAll@p, RuleDelayed@q, rDDD
```

Thus, /. is the infix form of **ReplaceAll**, the arrow -> is the infix form of **Rule**, and the arrow :> is the infix form of **RuleDelayed**, corresponding to **Set** and **SetDelayed** for = and :=.

Ÿ 1.2.2.3 •. and ••.

There is another form of •. given by ••. which applies a local rule repeatedly until the expression no longer changes. Internally, ••. is represented by

```
p ••. q -> r •• FullForm •• HoldForm
```

```
ReplaceRepeated@p, Rule@q, rDD
```

An example of the difference between •. and ••. follows. This example uses a list of rules rather than just a single rule. When a list of rules is applied to a single expression, then the rule for each symbol is tried from the left until a match is found. In the following example, the right-hand side of the •. expression consists of a list of two rules for the same symbol, **fac**. This list is searched from the left until a pattern is found that matches the left-hand side of the •. expression. In the first case using •., as soon as a match is found, the evaluation is finished. In the second case using ••., the rules are tried repeatedly from the left on the output of the previous evaluation until no matches are found.

```
fac@5D •. 8fac@1D -> 1, fac@i_D -> i fac@i - 1D<
```

```
5 fac@4D
```

```
fac@5D ••. 8fac@1D -> 1, fac@i_D -> i fac@i - 1D<
```

```
120
```

In the first case, the left-hand side of the rule fac@1D -> 1 does not match anything in fac@5D, but fac@i_D -> *i* fac@*i* - 1D does with i_ equal to 5, so the output is 5 fac@4D. In the second case, the left-hand side of the rule fac@i_D -> *i* fac@*i* - 1D continues to match a part of the existing expression until 120 fac@1D is obtained. Then the left-hand side of the rule fac@1D -> 1 matches, leading to 120 * 1, which simplifies to 120 where neither rule matches, so the output is 120.

If such rules are given globally, then the order in which they are given does not matter since *Mathematica* will put the more specific rule, fac@1D = 1, first. However, in a list of local rules, applied with •••., we are completely responsible for the ordering. Thus, the following gives the wrong answer:

```
fac@5D •••. 8fac@i_D -> i fac@i - 1D, fac@1D -> 1<
```

```
0
```

If several local rules are given for different symbols, then these rules are applied simultaneously. For example,

```
8u, v, w< •. 8u -> v, v -> w, w -> s<
```

```
8v, w, s<
```

If the substitutions are carried out sequentially, then the results are quite different.

```
8u, v, w< •. 8u -> v< •. 8v -> w< •. 8w -> s<
```

```
8s, s, s<
```

In particular, this means that variables can be interchanged without introducing an intermediate temporary variable.

```
8u, v< •. 8u -> v, v -> u<
```

```
8v, u<
```

Ÿ 1.2.3 Pattern Matching

Mathematica is a language based on pattern matching. Patterns refer to the structure of expressions. The symbol _ by itself, without any symbol on the left, can be used to describe a pattern. (Recall that the **FullForm** of _ is **Blank**@D). For instance, the expression _^_ matches anything of the form *x*^ *y*, where *x* and *y* are any (dummy) expressions.

```
Clear@a, xD
```

```
g1@_, _^_D := u;
8g1@a, a^aD, g1@a, b^cD<
```

```
8u, u<
```

A pattern of the form x_ is matched by any expression, which is then used for evaluating the right-hand side.

```
g2@x_^y_, z_D := u@x, y, v@zDD;
8g2@a^a, aD, g2@a^b, cD<
```

```
8u@a, a, v@aDD, u@a, b, v@cDD<
```

```
g3@x_^y_, x_D := u@y, v@xDD;
8g3@a^a, aD, g3@a^b, cD, g3@a^b, aD<
```

```
8u@a, v@aDD, g3@a^b, cD, u@b, v@aDD<
```

A pattern of the form x_h is matched by any expression whose head is h.

```
g4@x_^y_Integer, z_D := u@x, y, v@zDD;
8g4@a^a, aD, g4@a^3, arg2D<
```

```
8g4@a^a, aD, u@a, 3, v@arg2DD<
```

If we give a rule for an expression involving two separate arguments with underscores, then we are constructing a function of two variables. Such a function only works if it is given exactly two arguments.

```
g5@x_, y_D := x + y;
8g5@aD, g5@a, bD, g5@a, b, cD<
```

```
8g5@aD, a + b, g5@a, b, cD<
```

Besides rule-schemes using a single underscore_, there are rule-schemes using double and triple underscores. A double underscore, __, is matched by any number of arguments, excluding zero arguments, separated by commas, while a triple underscore, ___, is matched by any number of arguments, including zero arguments. The form x__ means a sequence of one or more arguments, named x, and x__Head means a sequence of one or more arguments, named x, with the head of each argument is Head. Similarly, the form x___ means zero or more arguments, named x, and x___Head means zero or more arguments, named x, with the head of each argument is Head.

```
g6@x__D := Length@8x<D;
8g6@D, g6@aD, g6@a, bD, g6@a, b, cD<
```

```
8g6@D, 1, 2, 3<
```

```
g7@x___D := Length@8x<D;
8g7@D, g7@aD, g7@a, bD, g7@a, b, cD<
```

```
80, 1, 2, 3<
```

Default values and double and triple underscores are important techniques, which give optional number of arguments to functions; that is, a variable number of arguments can be given to such functions. Alternatively, there is another rule in which a specific argument can be optional.

```
g8@x_^y_, z_D := u@x, y, zD;
8g8@a^b, cD, g8@a, cD<
```

```
8u@a, b, cD, g8@a, cD<
```

Because *a* = *a*^1, one might think that **g8**@*a*, *c*D should match the pattern **g8**@*a*^1, *c*D, which would mean that it should be rewritten as *u*@*a*, 1, *c*D, but of course *Mathematica* cannot guess that this is what we intend. However, there is a provision to take care of such default values. To assign a default value *v* to a pattern, one can write _ : *v*. So, the effect we wanted to achieve is given by the form

```
g9@x_^Hy_: 1L, z_D := u@x, y, zD;
8g9@a^b, cD, g9@a, cD<
```

```
8u@a, b, cD, u@a, 1, cD<
```

In this case, the default value 1 for the exponent is the natural and obvious choice, and *Mathematica* knows this. It has standard built-in default values for a number of such positions. The notation _. tells *Mathematica* to use the built-in default value. Note the almost invisible period after the underscore. Thus, the effect we wanted at the beginning is given by a tiny modification of the original form.

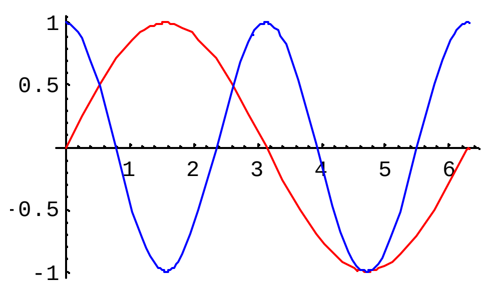
```
g10@x_^y_., z_D := u@x, y, zD;
8g10@a^b, cD, g10@a, cD<
```

```
8u@a, b, cD, u@a, 1, cD<
```

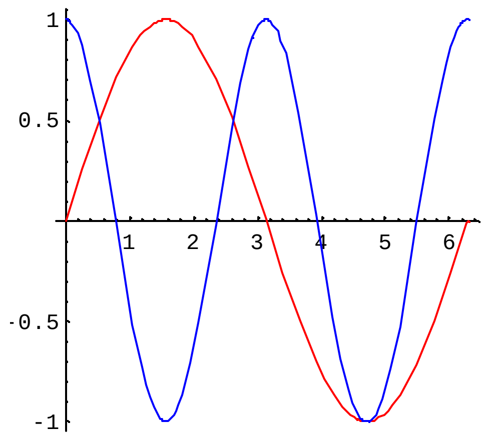
There is still another way that optional arguments occur in *Mathematica*. Some functions, such as **Plot**, can take named optional arguments such as AspectRatio -> 1. By incorporating such functions into our own definitions, we can use these named optional arguments too. Consider an example of a plotting function.

```
plotwithSin@func_, var_, opts___D := Plot@8Sin@varD, func@varD<,
   8var, 0, 2 Pi<, opts, PlotStyle -> 8RGBColor@1, 0, 0D, RGBColor@0, 0, 1D<D;
```

```
plotwithSin@Cos@2 #D &, xD;
```



```
plotwithSin@Cos@2 #D &, x, AspectRatio -> 1D;
```



In an expression of the form g[x_^n_Integer, z_], the patterns are named with the symbols *x*, *n*, and *z*, but there is no name for the whole compound pattern x_^n_Integer. There is a way to give names to such compound patterns so that they can be referred to directly on the right-hand side. The syntax consists of a name followed by a colon followed by the compound pattern.

```
g11@expr : x_^y_Real, z_D := z expr;
8g11@Ha + b - 2L^0.5, 2D, g11@Ha + b - 2L^3, 2D<
```

```
82 H-2 + a + bL^0.5, g11@H-2 + a + bL^3, 2D<
```

Ÿ 1.2.4 Using Patterns in Rules

Patterns play an important role in both global and local rules.

Ÿ 1.2.4.1 Patterns in Global Rules

For example, it is very easy to give rewrite rules for differentiating polynomials of one variable.

```
diff@x_^n_., x_D := n x^Hn - 1L;
diff@a_ + b_, x_D := diff@a, xD + diff@b, xD;
```

Notice the default value for *n* in the first rule. Try it out on some typical functions.

```
8diff@y, yD, diff@z^2.1, zD, diff@u^2 + v^3, uD<
```

```
81, 2.1 z^1.1, 2 u + diff@v^3, uD<
```

But notice that **diff** does not know what to do with a constant times *x*, or just a constant for that matter, and we have no obvious way as yet to teach it what to do.

```
8diff@3, xD, diff@3 x, xD<
```

```
8diff@3, xD, diff@3 x, xD<
```

We could try the following: first, give a rule for products.

```
diff@a_ b_, xD := a diff@b, xD + b diff@a, xD
```

Using this for *a x* gives

```
diff@a x, xD
```

```
a + x diff@a, xD
```

The program does not know that *a* is supposed to be a constant, so we have to tell it that explicitly, with a last rule.

```
diff@a, xD = 0;
```

Then it gives the "correct" answer.

```
diff@a x, xD
```

```
a
```

However, this is not very satisfactory. We would like some general way to say that *a* is not a function of *x*. Section 1.2.5 below will continue this discussion.

Ÿ 1.2.4.2 Patterns in Local Rules

We will encounter this type of application quite often in the following chapters. In the following example, we show how to change the appearance of a matrix. Using **Array** to make a matrix with indexed entries, we then use a local rule to display the indices as subscripts.

```
Array@x, 83, 5<D •. x@i__D -> xi •• MatrixForm
```

$$\begin{pmatrix} x_{1,1} & x_{1,2} & x_{1,3} & x_{1,4} & x_{1,5} \\ x_{2,1} & x_{2,2} & x_{2,3} & x_{2,4} & x_{2,5} \\ x_{3,1} & x_{3,2} & x_{3,3} & x_{3,4} & x_{3,5} \end{pmatrix}$$

where the two consecutive underscores next to *i* on the left-hand side of the rule represent one or more arguments.

We can also use pattern matching to replace the coefficients of various terms any way we want. For example,

```
Clear@fD
```

```
eq1 = u²@tD + u@tD f¢@x0D + 1/2 u@tD^2 f²@x0D + 1/6 u@tD^3 f^H3L@x0D == 0;
```

```
eq2 = eq1 •. _. u@tD^ b_. -> kb u@tD^ b
```

$$k_1\, u@tD + k_2\, u@tD^2 + k_3\, u@tD^3 + u''@tD == 0$$

Sometimes we may obtain, in a bifurcation analysis, a data set, which includes a set of bifurcation points x_i and the corresponding control parameters *cpSearch* and *cpFix*, such as 88x_1, x_2, x_3, x_4, *cpSearch*<, *cpFix*<. If we want to switch these two parameters and then use the same data format as our initial input to a program, we have at least two choices. First, we take apart the data list and then assemble them:

```
data1 = 88x1, x2, x3, x4, cp1<, cp2<;
```

```
8Append@Drop@data1@@1DD, -1D, data1@@2DDD, data1@@1, -1DD<
```

```
88x1, x2, x3, x4, cp2<, cp1<
```

Second, we simply apply a local rule using patterns to the data list as a whole:

```
data1 •. 88a__, b_<, c_< -> 88a, c<, b<
```

```
88x1, x2, x3, x4, cp2<, cp1<
```

Ÿ 1.2.5 Restricting Pattern Matching with Predicates

So far, all of the rules we have considered have been "context free" rewrite rules. Whenever the pattern is matched, the rewriting is carried out. There can be a restriction on the head of the matching expression included in the pattern. However, there are also conditional rewrite rules, which are only applied when some condition is satisfied. First, we have to discuss predicates (functions that return the value **True** or **False**) in *Mathematica* because all of the conditions will always be expressed in terms of them. All built-in predicates that are defined for all expressions end with **Q**. It is easy to display all of them.

```
? *Q
```

```
ArgumentCountQ        LinkConnectedQ      OrderedQ
AtomQ                 LinkReadyQ          PartitionsQ
DigitQ                ListQ               PolynomialQ
EllipticNomeQ         LowerCaseQ          PrimeQ
EvenQ                 MachineNumberQ      SameQ
ExactNumberQ          MatchLocalNameQ     StringMatchQ
FreeQ                 MatchQ              StringQ
HypergeometricPFQ     MatrixQ             SyntaxQ
InexactNumberQ        MemberQ             TrueQ
IntegerQ              NameQ               UnsameQ
IntervalMemberQ       NumberQ             UpperCaseQ
InverseEllipticNomeQ  NumericQ            ValueQ
LegendreQ             OddQ                VectorQ
LetterQ               OptionQ
```

Specifically requesting information on our function gives the usage message.

```
? OddQ
```

```
OddQ@exprD gives True if expr is an odd integer, and False otherwise.
```

Predicates are used to control pattern matching. In general, they are applied using •; , which is the infix form of **Condition**.

```
FullForm@Hold@x •; yDD
```

```
Hold@Condition@x, yDD
```

However, the position of the predicate in an expression allows it to be used in different ways. If the predicate is placed at the end of a global rule definition, it will restrict the application of the rule. For instance, we can use predicates to extend our definition of differentiation in Section 1.2.4.1 to deal with arbitrary polynomials in a very natural way by adding a single conditional rule.

```
Clear@diffD
```

```
diff@x_^n_., x_D := n x^Hn - 1L;
diff@a_ + b_, x_D := diff@a, xD + diff@b, xD;
diff@a_ b_, x_D := a diff@b, xD + b diff@a, xD;
diff@a_, x_D := 0 •; FreeQ@a, xD
```

Now constants and products are handled properly

```
diff@a@xD + b x^2 + c, xD
```

```
2 b x + diff@a@xD, xD
```

Using **rule•;Predicate** restricts the rule to those situations in which the predicate evaluates to **True**; that is, to those expressions belonging to the type given by the predicate. An unrestricted rule is the same as a conditional rule where the predicate always equals **True**.

If a predicate is immediately applied after a pattern, it will restrict the pattern matching rather than the rule application. For instance,

```
fac1@1D = 1; fac1@n_D := n fac1@n - 1D
```

This will work perfectly well if positive integers are given as arguments.

```
fac1@5D
```

```
120
```

However, if another argument is given, then it will fail badly.

```
8fac1@-3D, fac1@endlessD<;
```

```
$RecursionLimit::reclim : Recursion depth of 256 exceeded.

$RecursionLimit::reclim : Recursion depth of 256 exceeded.

$RecursionLimit::reclim : Recursion depth of 256 exceeded.

General::stop : Further output of $RecursionLimit::reclim will be suppressed during this calculation.
```

A very large output is suppressed. What happens, of course, in these cases is that the value 1 is never encountered as an argument, so the function keeps calling itself recursively until the built-in recursion limit is reached. This bad behavior can be corrected by using a conditional rule.

```
fac2@1D = 1; fac2@n_D := n fac2@n - 1D •; n > 1;
```

```
8fac2@5D, fac2@-2D, fac2@somethingD<
```

```
8120, fac2@-2D, fac2@somethingD<
```

One can also use the form _?**Predicate**, which restricts the pattern to something for which the predicate evaluates to **True**.

```
fac3@1D = 1; fac3@n_?PositiveD := n fac3@n - 1D;
fac4@1D = 1; fac4@n_ •; Positive@nDD := n fac4@n - 1D;
```

```
8fac3@5D, fac3@-2D, fac4@5D, fac4@-2D<
```

```
8120, fac3@-2D, 120, fac4@-2D<
```

In n_?**Positive**, **Positive** is a pure function, while in the form using •; , the condition is the value of the predicate for the name of the pattern. In either case, **Positive** or **Positive**[n] is a positive test in the sense that the pattern is matched and the rule applied only if the test succeeds. Now try **fac4** on a real number and see what happens.

```
fac4@3.6D
```

```
8.9856 fac4@-0.4D
```

The rule is applied until the negative value -0.4 is reached, where the condition fails so **fac4**[-0.4] is returned in unevaluated form. Of course, the real problem is that we only intend the function to apply to integers. But this additional restriction can easily be added.

```
fac5@1D = 1; fac5@n_Integer ?PositiveD := n fac5@n - 1D;
```

```
8fac5@5D, fac5@3.6D, fac5@-2D<
```

```
8120, fac5@3.6D, fac5@-2D<
```

The predicate that appears after •; or ? can also be a user defined expression.

```
test@x_Integer?HH# > 2L &LD := x + 1
```

```
8test@1D, test@2D, test@3D, test@4D, test@5D<
```

```
8test@1D, test@2D, 4, 5, 6<
```

The form **? Predicate** can only be used after single slots, but the form **•; Predicate** can be used after any pattern, simple or compound. For instance,

```
f1@a_, b_D •; EvenQ@a + bD := a ^ b;
f1@a_, b_D •; OddQ@a + bD := a ^H- bL;
```

```
8f1@1, 3D, f1@2, 3D, f1@3, 3D<
```

```
     1
91, ---, 27=
     8
```

Predicates also play an important role in manipulating lists.

```
Select@Range@-5, 5D, PositiveD
```

```
81, 2, 3, 4, 5<
```

```
Cases@Range@-5, 5D, _?PositiveD
```

```
81, 2, 3, 4, 5<
```

```
Cases@Range@-5, 5D, a_ •; a > 0D
```

```
81, 2, 3, 4, 5<
```

There is also another form of **Cases** in which an operation is applied to the entries that are selected.

```
Cases@Range@-5, 5D, Ha_ •; a > 0L :> Sqrt@aDD
```

```
91, •!!2, •!!3, 2, •!!5=
```

The following example was the 1992 *Mathematica* programming competition question. The problem is to write a function called **maxima** that starts with a list of numbers and constructs the sublist of the numbers bigger than all previous ones from the given list. For instance, **maxima**[{3, 6, 4, 2, 8, 7, 9}] should return {3, 6, 8, 9}. The winning entry used a pattern with a condition in a local rule.

```
maxima@list_ListD := list ••. 8a___, x_, y_, b___< •; y <= x -> 8a, x, b<
```

```
maxima@83, 6, 4, 2, 8, 7, 9<D
```

```
83, 6, 8, 9<
```

Ÿ 1.2.6 Pure Functions

Given an expression *expr* involving a variable *x*, we can think of *expr* as describing a function with *x* being the argument. If we need to refer to this function by name, we can either define a rule for **f** in the form *f*@x_D := *expr* or use the object **Function**[***x, expr***]. Either **f** or **Function**[*x*, *expr*] is a name for this function, and we can use the two interchangeably. To apply them to an argument, we write **f**[*arg*] in the familiar way or in the so-called pure function form **Function**[*x*, *expr*][*arg*].

```
f@x_D := 1 + x^2
g = Function@x, 1 + x^2D;
```

```
8f@2D, g@2D<
```

```
85, 5<
```

The name of the formal argument in a pure function does not matter. **Function**[*x*, *x^2*] is the same function as **Function**[*y*, *y^2*]. This fact is easy to see if you apply such a function to an argument, the result is the same in both cases.

```
8Function@x, x^2D@aD, Function@y, y^2D@aD, Sin@aD<
```

```
8a^2, a^2, Sin@aD<
```

Note that **Sin** and all built-in functions, with no arguments, are pure functions. Because the names of the variables in a pure function do not matter, *Mathematica* denotes these variables with symbols. The symbols #1, #2, ... are used for the first, second, ... variable. The internal form of #i is **Slot**[i]. If we use these forms, then we do not give the arguments of **Function**[ ] in order to declare the names of the variables. Thus, instead of **Function**[*x*, *x^2*] we simply write **Function**[#1^2]. A convenient abbreviation for #1 is #, so we can simplify our example a bit more and write **Function**[#^2]. Finally, there is a postfix operator **&** for **Function**. That is, body**&** is the same as **Function**[body]. Using this operator, we arrive at the shortest form #^2 **&** that our example can take. We have used this form quite frequently in this book. Let us see the internal form of this short form of a pure function.

```
FullForm@#^2 &D
```

```
Function@Power@Slot@1D, 2DD
```

The operator **&** has a very low priority, just above assignment. Therefore, *x* **=** body **&** is understood as *x* **= Function**[body], but *x* **->** body **&** is interpreted as **Function**[*x* **->** body]. If we want the right-hand side of the rule to be the pure function, we use *x* **->** (body **&**). Beware of *x* **->** (body) **&**. Another case where the low priority of pure functions requires the use of parentheses is in predicates for patterns: x_?body**&** is not the same as x_?(body**&**). The latter is usually correct.

```
1 + # + #^2 &@3D
```

```
13
```

The whole expression to the left of **&** is part of the body of the pure function. No parentheses are necessary. In other situations, parentheses are necessary.

```
h@aD •. h -> H#^2 &L
```

```
a^2
```

Parentheses around the pure function are needed here; otherwise, the whole rule would be considered a part of the body of the function.

Pure functions written in the form with # and & are very concise and useful, especially, in functional operations, such as **Map**, **Apply**, **Thread**, and iteration functions, such as **FoldList**. For example,

```
eq1 = ¶t,t u@x, tD • a + u@x, tD == F@xD • a
```

```
           uH0,2L@x, tD     F@xD
u@x, tD + ƒƒƒƒƒƒƒƒƒƒƒƒƒƒƒƒ == ƒƒƒƒƒƒ
               a             a
```

```
Clear@lhs2D
```

```
eq2 = lhs2 == rhs2;
eq3 = lhs3 == rhs3;
```

We can multiply both sides of **eq1** by *a* and use **Thread** and obtain

```
eq1 •• Thread@a #, EqualD & •• ExpandAll
```

```
a u@x, tD + u^H0,2L@x, tD == F@xD
```

Moreover, we can add **eq2** to **eq3** by

```
eq2 + eq3 •• Thread@#, EqualD &
```

```
lhs2 + lhs3 == rhs2 + rhs3
```

Ÿ 1.2.7 Map and Apply

In *Mathematica*, two important commands that take functions as arguments are **Map**[ ] and **Apply**[ ]. The various versions of **Map** act on the arguments of an expression, while **Apply** acts only on its head.

Ÿ 1.2.7.1 Mapping Functions onto Expressions

The operation **Map**[*f*, *list*] or in infix form *f* •ž *list* applies the function *f* to each element of the list *list*. The second argument of **Map**[ ] need not be a list, however. Any expression of the form $h[e_1, e_2, ..., e_n]$ will do. The result of the mapping is the expression *h*@ *f*@e_1D, *f*@e_2D, ..., *f*@e_nDD.

```
Clear@fD
```

```
8Map@f, a + b + cD, #^2 & •ž 81, 2, 3<<
```

```
8f@aD + f@bD + f@cD, 81, 4, 9<<
```

Map[*f*, *expr*, *levelspec*] has an optional third argument that specifies the levels at which to map. The default level is {1}; that is, at the elements of *expr*. In a matrix, the entries are at level 2 (there are two levels of lists). If we want to map a function at these entries, we can use **Map**[*f*, *matrix*, {2}].

```
expr1 = 88a1,1, a1,2<, 8a2,1, a2,2<<;
```

```
Map@entry, expr1, 82<D •• MatrixForm
```

```
¡ entry@a1,1D  entry@a1,2D y
k entry@a2,1D  entry@a2,2D {
```

This is a very powerful facility and is one of our main tools in manipulating expressions. The purpose of the operations based on **Map** is to make it possible to treat lists as wholes. For instance, a really poor way to square the entries in a list is as follows:

```
list1 = 8a, b, c<;
Table@list1@@iDD^2, 8i, Length@list1D<D
```

$8a^2, b^2, c^2<$

The term **list1[[i]]** tears apart the original list by extracting its parts one at a time, ^2 squares each part, and then **Table** reassembles the parts into a new list. **Map**, however, allows us to treat mathematical structures as wholes so that we do not tear them apart and rebuild them again.

Map[ ] always maps the function at all elements of the given levels. The command **MapAt**[*f*, *expr*, *poslist*] allows you to apply a function at selective positions.

```
MapAt@G, a b + c d + e f, 82, 1<D
```

```
a b + e f + d G@cD
```

```
expr2 = a + b • a + c Exp@a + 1D
```

$a + \frac{b}{a} + c\,E^{1+a}$

If we want to map a function *f* at all occurrences of *a*, we first find all the positions of *a* in our expression.

```
Position@expr2, aD
```

```
881<, 82, 1, 1<, 83, 2, 2, 2<<
```

This list of positions is in the right form for **MapAt** and can be used directly.

```
MapAt@f, expr2, %D
```

$c\,E^{1+f@aD} + \frac{b}{f@aD} + f@aD$

This particular example could have been done more easily with a replacement rule.

```
expr2 •. a -> f@aD
```

$c\,E^{1+f@aD} + \frac{b}{f@aD} + f@aD$

In some applications, the function to be performed on an element e_i may depend on its position i. The operation **Map-Indexed**[*f*, {e_1, e_2, ..., e_n}] behaves essentially like **Map**[ ], but it passes the position of each element as a second argument to *f*. The resulting expression is

```
MapIndexed@1 + #^2 &, 8a, b, c, d<D
```

$81 + a^2, 1 + b^2, 1 + c^2, 1 + d^2<$

```
MapIndexed@1 + #^2 &, 88a, b<, 8c, d<<, 82<D
```

$881 + a^2, 1 + b^2<, 81 + c^2, 1 + d^2<<$

Therefore, the function that is mapped must be a function of two arguments. The second argument is a position (note the list braces) that can be used to modify the operation performed on the first argument.

```
MapIndexed@#1^First@#2D &, 8a, b, c, d, e<D
```

8a, b^2, c^3, d^4, e^5<

Ÿ 1.2.7.2 Apply

The rule **Apply**[*h*, *expr*] or *h* žž *expr* replaces the head of *expr* by *h*.

```
a + b + c •• FullForm
```

```
Plus@a, b, cD
```

Replacing the head **Plus** by **Times** gives the product of the three terms in the sum above.

```
8Apply@Times, a + b + cD, Times žž Ha + b + cL<
```

```
8a b c, a b c<
```

With a third argument, **Apply**[*h*, *expr*, *levelspec*] replaces heads in the parts of *expr* described by *levelspec* by *h*.

```
Apply@b, 88a@1, 1D, a@1, 2D<, 8a@2, 1D, a@2, 2D<<, 82<D
```

```
88b@1, 1D, b@1, 2D<, 8b@2, 1D, b@2, 2D<<
```

This simple definition finds the average of a list of numbers

```
average@list1_ListD := Apply@Plus, list1D • Length@list1D
average@8a, b, c<D
```

$\frac{1}{3}$ Ha + b + cL

Note that computing the average of a list in this way requires no do-loops or knowledge of the length of the list. **Apply** is frequently used if one first wants to prepare a number of ingredients and then apply some operation to them. The ingredients can be held in a list until they are ready and then the head of the list is changed to the appropriate operation by using **Apply**.

To generate more concise notations for the outputs created in the following chapters, we can introduce **displayRule** to meet our needs. For example,

$$\text{output} = u_2^{H2,0,0L}@T_0,\ T_1,\ T_2D + w^2\, u_2@T_0,\ T_1,\ T_2D == 2\, u_1^{H1,1,0L}@T_0,\ T_1,\ T_2D - d\, u_1@T_0,\ T_1,\ T_2D^2;$$

$$\text{displayRule} = 9\text{Derivative@a\_\_D@}u_{i}\text{\_D@\_\_D} :> \text{SequenceFormATimes žž MapIndexedAD}^{\#1}_{\#2@@1DD-1}\ \&,\ 8a<E,\ u_iE,\ u_{i}\text{\_@\_\_D} \to u_i=;$$

```
output •. displayRule
```

$D_0^2 u_2 + w^2\, u_2 == 2\, HD_0\, D_1 u_1 L - d\, u_1^2$

Ÿ 1.2.8 Nest and Fold

Nest and its related operations **NestList** and **FixedPoint** apply a function to its argument many times. **Nest**[*function*, *x*, *n*] applies *function* to *x* and repeats the application *n* times

```
NestAH1 + #L^2 &, x, 3E
```

```
J1 + H1 + H1 + xL^2L^2N^2
```

```
NestListAH1 + #L^2 &, x, 3E
```

```
9x, H1 + xL^2, H1 + H1 + xL^2L^2, J1 + H1 + H1 + xL^2L^2N^2=
```

Here is an example producing a simple continued fraction.

```
Nest@1 • H1 + #L &, a, 3D
```

$$\frac{1}{1+\frac{1}{1+\frac{1}{1+a}}}$$

An operation that is closely related to **Nest** is **FixedPoint**, which nests its operation until there is no change. For instance, everyone is familiar with what happens if the **Cos** key on a pocket calculator is pushed repeatedly. In principle, **FixedPoint** is what happens if it is pushed forever.

```
8Nest@Cos, 0.3, 5D, Nest@Cos, 0.3, 10D, FixedPoint@Cos, 0.3D<
```

```
80.784436, 0.732698, 0.739085<
```

Actually, **FixedPoint** stops after machine accuracy is achieved. We can use the option **SameTest** to get some control of **FixedPoint**.

```
FixedPoint@Cos, 0.3, SameTest -> HAbs@#1 - #2D < 10^-5 &LD
```

```
0.739089
```

The second pair of functions, **Fold** and **FoldList**, do something similar to **Nest** and **NestList**, but for functions of two variables.

```
Fold@f, u, 8x, y, z<D
```

```
f@f@f@u, xD, yD, zD
```

```
FoldList@f, u, 8x, y, z<D
```

```
8u, f@u, xD, f@f@u, xD, yD, f@f@f@u, xD, yD, zD<
```

Chapter 2
The Duffing Equation

```
Off@General::spell1D
```

To symbolize some notations, we need to load the package

```
Needs@"Utilities`Notation`"D
```

à 2.1 The Duffing Equation

The free oscillations of many conservative systems having a single degree of freedom are governed by an equation of the form

```
eq21a = x² @tD + f@x@tDD == 0;
H* An equation is set and then assigned to eq21a *L
```

where f is a nonlinear function of x@tD. Here, x^{2}@tD is the acceleration of the system, whereas f@x@tDD is the restoring force. Let $x = x_0$ be an equilibrium position of the system. Then x_0 is a constant and hence f@x_0D = 0. In order for *Mathematica* to interpret x_0 as a constant, we need to symbolize it. Otherwise, if we replace x with an expression g@tD, *Mathematica* will replace x_0 with g@tD sub 0 and not treat it as a constant. For example, replacing x with Sin@tD in the expression $x^2 + x_0$ yields

```
x^2 + x_0 •. x -> Sin@tD
```

```
Sin@tD^2 + HSin@tDL_0
```

Clearly, x_0 is treated as a variable x with subscript 0, which is wrong. Therefore, we need to symbolize x_0 as (enter and select x_0 followed by clicking **Symbolize**@ŸD from NotationPalette)

```
Symbolize@x_0D;
H* Note that symbol form is also required in certain built-in functions *L
```

Then, *Mathematica* treats the symbolized expression x_0 as a constant. Consequently, replacing x with Sin@tD in the expression $x^2 + x_0$ yields

```
x^2 + x_0 •. x -> Sin@tD
```

```
x_0 + Sin@tD^2
```

which is correct.

Next, we assume that f is an analytic function of x at $x = x_0$ and expand it in a Taylor series around x_0 as

```
fexp = Series@f@x@tDD, 8x@tD, x0, 3<D
```

```
f@x0D + f¢@x0D Hx@tD - x0L + 1/2 f²@x0D Hx@tD - x0L^2 + 1/6 f^H3L@x0D Hx@tD - x0L^3 + O@x@tD - x0D^4
```

where terms up to cubic are retained. Using **Normal** to truncate the higher-order terms from **fexp**, we have

```
fpoly = fexp •• Normal
```

```
f@x0D + H-x0 + x@tDL f¢@x0D + 1/2 H-x0 + x@tDL^2 f²@x0D + 1/6 H-x0 + x@tDL^3 f^H3L@x0D
```

Substituting **fpoly** for $f@x@tDD$ and using the equilibrium condition $f@x_0D = 0$, we obtain

```
temp = eq21a •. 8f@x@tDD -> fpoly< •. f@x0D -> 0
```

```
H-x0 + x@tDL f¢@x0D + 1/2 H-x0 + x@tDL^2 f²@x0D + x²@tD + 1/6 H-x0 + x@tDL^3 f^H3L@x0D == 0
```

Now, we introduce the transformation $x@tD ® x_0 + u@tD$ in **temp** to obtain the final form of the equation that will be used in the subsequent analysis. To accomplish this, we need to express this transformation in a pure function form as $x ® Hx_0 + u@\#D \&L$; otherwise, *Mathematica* will not transform the derivatives of $x@tD$ in terms of the derivatives of $u@tD$ unless we explicitly define the rules for those derivatives; that is, it will not replace $x^{¢}@tD$ with $u^{¢}@tD$ and $x^{²}@tD$ with $u^{²}@tD$ by simply substituting $x@tD ® x_0 + u@tD$. Moreover, it will not replace $x@pD$ with $x_0 + u@pD$ as desired. To see the difference between these two transformations, we compare the following two statements:

```
8x@tD, x@pD, x¢@tD, x²@tD< •. x@tD -> x0 + u@tD
```

```
8x0 + u@tD, x@pD, x¢@tD, x²@tD<
```

```
8x@tD, x@pD, x@a, bD, y@tD, x¢@tD, x²@tD< •. x -> Hx0 + u@#D &L
```

```
8x0 + u@tD, x0 + u@pD, x0 + u@aD, y@tD, u¢@tD, u²@tD<
```

We note that the pure function form can be used for any expression with the head x, especially, for the derivatives. Consequently, we introduce this transformation in pure function form into **temp** and obtain

```
eq21b = temp •. x -> Hx0 + u@#D &L
```

```
u@tD f¢@x0D + 1/2 u@tD^2 f²@x0D + u²@tD + 1/6 u@tD^3 f^H3L@x0D == 0
```

To simplify the notation in **eq21b**, we denote the coefficient of $u@tD^b$ by k_b. The result is

```
eq21c = eq21b •. _. u@tD^ b_. -> kb u@tD^ b
```

```
k1 u@tD + k2 u@tD^2 + k3 u@tD^3 + u²@tD == 0
```

We relate the k_b to the derivatives of the function $f@xD$ by using the built-in function **Cases** as

```
CasesAeq21b, a_. u@tD^b_· -> 8k_b -> a<, 82<E •• Flatten
```

```
9k1 ® f¢@x0D, k2 ® 1/2 f²@x0D, k3 ® 1/6 f^H3L@x0D=
```

Most of this chapter is devoted to the special case of cubic nonlinearities. Thus, we set the coefficient of the quadratic term in **eq21c** equal to zero and obtain

```
eq21d = eq21c •. k2 -> 0
```

```
k1 u@tD + k3 u@tD^3 + u²@tD == 0
```

which is called the Duffing equation. We note that k_1 and k_3 may be positive or negative.

It is a good practice to nondimensionalize the governing equations before treating them with perturbation methods. To this end, we nondimensionalize u and t using a characteristic length U and a characteristic time T of the motion and denote the nondimensional variables by an asterisk. In nondimensional form, **eq21d** becomes

```
eq21e = eq21d •. u -> HU u*@# • TD &L •. t -> T t*
```

```
U k1 u*@t*D + U^3 k3 u*@t*D^3 + U Hu*L²@t*D / T^2 == 0
```

Next, we multiply the left-hand side of **eq21e** (using **eq21e**[[1]]) by $T^2 \cdot U$ so that the coefficient of u^{*2}@t*D is unity. The result is

```
eq21f = Ieq21e@@1DD T^2 • U •• ExpandM == 0
```

```
T^2 k1 u*@t*D + T^2 U^2 k3 u*@t*D^3 + Hu*L²@t*D == 0
```

We choose T so that the linear natural frequency of the system is unity; that is, we let $k_1 T^2 = 1$, for ease of notation, we let e $= k_3 T^2 U^2$, drop the asterisk, and rewrite **eq21f** as

```
DuffingEq = eq21f •. 9k1 T^2 -> 1, k3 T^2 U^2 -> e= •. anything_* -> anything
```

```
u@tD + e u@tD^3 + u²@tD == 0
```

We note that e is a nondimensional quantity, which is a measure of the strength of the nonlinearity.

à 2.2 Straightforward Expansion

To solve the **DuffingEq**, we augment it with the initial conditions

```
ic = 8u@0D == u0, u¢@0D == v0<;
```

The solution u of the initial-value problem (IVP), consisting of the **DuffingEq** and associated initial conditions **ic**, is a function of the independent variable t and the parameter e. Next, we determine an approximation of the IVP for weak nonlinearity; that is, for small but finite e.

First, we set e equal to zero in the **DuffingEq**, augment the result with the initial conditions by using the *Mathematica* command **Join**, and obtain the following linear IVP:

```
linearIVP = Join@8DuffingEq •. e -> 0<, icD
```

```
8u@tD + u''@tD == 0, u@0D == u0, u'@0D == v0<
```

The solution of this linear IVP can be obtained by using the *Mathematica* command **DSolve**. The result is

```
linearSol = DSolve@linearIVP, u@tD, tD@@1DD
```

```
8u@tD ® u0 Cos@tD + v0 Sin@tD<
```

When e is small but different from zero, the solution of the nonlinear initial-value problem is no longer given by **linearSol**, but deviates from it. We try a correction in the form of a power series in e; that is, we expand *u*@*t*D in a power series in e as

```
uExpRule@maxOrder_D := u -> ISumAe^i u_i@#D, 8i, 0, maxOrder<E &M
```

where u_0@*t*D is the solution of the linear problem. For example, to first order, we have

```
u@tD •. uExpRule@1D
```

$u_0@tD + e\, u_1@tD$

and to third order, we have

```
u@tD •. uExpRule@3D
```

$u_0@tD + e\, u_1@tD + e^2\, u_2@tD + e^3\, u_3@tD$

and to fifth order, we have

```
u@tD •. uExpRule@5D
```

$u_0@tD + e\, u_1@tD + e^2\, u_2@tD + e^3\, u_3@tD + e^4\, u_4@tD + e^5\, u_5@tD$

Next, we restrict our discussion to first order; that is, we keep two terms in the power series. Thus, we apply the rule **uExp-Rule**[1] to the **DuffingEq**, expand the result, and obtain

```
eq22atemp = DuffingEq •. uExpRule@1D •• ExpandAll
```

$u_0@tD + e\, u_0@tD^3 + e\, u_1@tD + 3\, e^2\, u_0@tD^2\, u_1@tD + 3\, e^3\, u_0@tD\, u_1@tD^2 + e^4\, u_1@tD^3 + u_0''@tD + e\, u_1''@tD == 0$

Since we kept only terms up to *O*HeL in the expansion of *u*@*t*D, we need to neglect terms of order higher than e in **eq22atemp** for consistency. To accomplish this, we use the rule $e^{n•;n>k}$ ® 0 to discard terms with order higher than e^k and obtain

```
eq22a = eq22atemp •. e^(n_•;n>1) -> 0
```

$u_0@tD + e\, u_0@tD^3 + e\, u_1@tD + u_0''@tD + e\, u_1''@tD == 0$

Setting the coefficients of like powers of e in **eq22a** equal to zero, we obtain the following hierarchy of linear equations:

```
eqEps = CoefficientList@eq22a@@1DD, eD == 0 •• Thread
```

```
8u0@tD + u0''@tD == 0, u0@tD^3 + u1@tD + u1''@tD == 0<
```

which needs to be solved in succession. That is, we first solve the first equation in **eqEps** to obtain u_0@*t*D and then substitute the result into the second equation in **eqEps** to obtain a linear nonhomogeneous equation, which can be solved for u_1@*t*D.

The general solution of the zeroth-order problem (**eqEps**[[1]]) can be expressed in Cartesian form as

```
DSolve@eqEps@@1DD, u0@tD, tD@@1DD
```

```
8u0@tD ® C@2D Cos@tD - C@1D Sin@tD<
```

Alternatively, we can express the solution of the zeroth-order problem in the polar form u_0@*t*D ® a_0 Cos@*t* + b_0D by adding two appropriate initial conditions to **eqEps**[[1]] as

```
sol0 =
 DSolve@8eqEps@@1DD, u0@0D == a0 Cos@b0D, u0'@0D == -a0 Sin@b0D<, u0@tD, tD@@1DD •• Simplify
```

```
8u0@tD ® Cos@t + b0D a0<
```

where a_0 and b_0 are arbitrary constants. Because **eqEps**[[1]] represents a linear undamped oscillator, one can write down its general solution in either Cartesian or polar form without using the *Mathematica* command **DSolve**.

Substituting for u_0@*t*D in the first-order equation (**eqEps**[[2]]) and determining the general solution of the resulting equation for u_1@*t*D, we obtain

```
sol1temp = DSolve@eqEps@@2DD •. sol0, u1@tD, tD@@1DD •• Simplify
```

```
9u1@tD ®
  1/32 H32 HC@2D Cos@tD - C@1D Sin@tDL + H-6 Cos@t + b0D + Cos@3 Ht + b0LD - 12 t Sin@t + b0DL a0^3L=
```

We note that the homogeneous part of **sol1temp** consists of the terms involving the *C*@*i*D (the usual case) and Cos@*t* + b_0D (obtained from **DSolve** in this case). Hence, we can replace them by a_1 Cos@*t* + b_1D and rewrite **sol1temp** as

```
sol1 = u1@tD ® Hsol1temp@@1, 2DD •. 8C@_D -> 0, Cos@t + _D -> 0<L + a1 Cos@t + b1D
```

```
u1@tD ® 1/32 HCos@3 Ht + b0LD - 12 t Sin@t + b0DL a0^3 + Cos@t + b1D a1
```

Combining the solutions of the zeroth- and first-order problems, we obtain the general solution of the **DuffingEq**, to first order, as

```
sol = u@tD == Hu@tD •. uExpRule@1D •. sol0 •. sol1L
```

```
u@tD == Cos@t + b0D a0 + e (1/32 HCos@3 Ht + b0LD - 12 t Sin@t + b0DL a0^3 + Cos@t + b1D a1)
```

where a_0, a_1, b_0, and b_1 are arbitrary constants.

We started with a second-order equation that can support two initial conditions but it appears that we ended up with four arbitrary constants. It turns out that only two of the constants a_0, a_1, β_0, and β_1 are arbitrary and that the two initial conditions in **ic** are sufficient to determine all of them. To see this, we first change the **Equal** form in **ic** to the **Rule** form as

```
icRule = ic /. Equal -> Rule
```

```
{u[0] → u0, u'[0] → v0}
```

Then, we impose the initial conditions **icRule** by evaluating **sol** and its derivative at $t = 0$ according to

```
eq22b = {sol, D[sol, t]} /. t -> 0 /. icRule
```

$$\left\{u0 == \text{Cos}[\beta_0]\, a_0 + \epsilon \left(\frac{1}{32}\text{Cos}[3\beta_0]\, a_0^3 + \text{Cos}[\beta_1]\, a_1\right),\right.$$
$$\left. v0 == -\text{Sin}[\beta_0]\, a_0 + \epsilon\left(\frac{1}{32}(-12\,\text{Sin}[\beta_0] - 3\,\text{Sin}[3\beta_0])\, a_0^3 - \text{Sin}[\beta_1]\, a_1\right)\right\}$$

We equate the coefficient of each power of ϵ in **eq22b** and obtain four algebraic equations for the constants a_0, a_1, β_0, and β_1. The result is

```
eq22c = Table[Map[Coefficient[#, ε, i] &, eq22b, {2}], {i, 0, 1}]
```

$$\{\{u0 == \text{Cos}[\beta_0]\, a_0,\ v0 == -\text{Sin}[\beta_0]\, a_0\},$$
$$\{0 == \frac{1}{32}\text{Cos}[3\beta_0]\, a_0^3 + \text{Cos}[\beta_1]\, a_1,\ 0 == \frac{1}{32}(-12\,\text{Sin}[\beta_0] - 3\,\text{Sin}[3\beta_0])\, a_0^3 - \text{Sin}[\beta_1]\, a_1\}\}$$

Thus, once a_0 and β_0 are known from **eq22c**[[1]], a_1 and β_1 can be calculated from **eq22c**[[2]].

Alternatively, instead of including the homogeneous solution at each order and consider the arbitrary constants to be independent of ϵ as done above, one may disregard the homogeneous solution at all orders except the first and consider the arbitrary constants to depend on ϵ in imposing the initial conditions. With the latter approach, the solution to first order can be rewritten as

```
sol /. {a0 -> a, β0 -> β, a1 -> 0} // ExpandAll
```

$$u[t] == a\,\text{Cos}[t + \beta] + \frac{1}{32} a^3\, \epsilon\, \text{Cos}[3t + 3\beta] - \frac{3}{8} a^3\, t\, \epsilon\, \text{Sin}[t + \beta]$$

We note that the correction term $\frac{1}{32} a^3 \epsilon \cos(3t + 3\beta) - \frac{3}{8} a^3 t \epsilon \sin(t + \beta)$ is small compared to the principal term $a \cos(t + \beta)$, as it is supposed to be, only when ϵt is small compared with unity. When ϵt is $O(1)$, the term that is supposed to be a small correction becomes the same order as the principal term. Moreover, when $\epsilon t > O(1)$, the "small-correction" term becomes larger than the principal term. Hence, the above obtained straightforward expansion is valid only for times such that $\epsilon t < O(1)$; that is, $t < O(\epsilon^{-1})$. Consequently, we say such expansions are nonuniform or breakdown for long times and we call them pedestrian or naive expansions. The reason for the breakdown of the above expansion is the presence of the term $t \sin(t + \beta)$, a product of algebraic and circular terms. Such terms are called mixed-secular terms. In subsequent sections, we implement four methods that avoid secular terms, and hence yield uniform expansions.

à 2.3 The Lindstedt-Poincaré Technique

The breakdown in the straightforward expansion is due to its failure to account for the nonlinear dependence of the frequency of the system on the nonlinearity. A number of techniques that yield uniformly valid expansions have been developed. Four of these techniques are discussed in this chapter. We start with the Lindstedt-Poincaré technique in this section.

To account for the dependence of the frequency w of the system on the nonlinearity, we explicitly exhibit w in the governing differential equation. To this end, we introduce the transformation t = w*t*, where w is a constant that depends on e, and obtain

```
DuffingEq = u'' @tD + u@tD + e u@tD^3 == 0;
eq23a = DuffingEq •. u -> Hu@w #D &L •. t -> t • w
```

```
u@tD + e u@tD^3 + w^2 u'' @tD == 0
```

To determine a uniform expansion of the solution of **eq23a**, we expand both *u*@t; eD and w@eD in powers of e using **uExpRule[k]** and

```
omgRule@maxOrder_D := w -> w0 + SumAe^i wi, 8i, maxOrder<E
```

where w_0 is the linear natural frequency of the system. In the present case, the linear natural frequency was normalized to unity. Hence, we set $w_0 = 1$. The corrections to the linear frequency are determined in the course of the analysis by requiring the expansion of *u* to be uniform for all t. Next, we show how one can determine w_1.

Substituting the expansions of *u* and w into **eq23a**, using the fact that $w_0 = 1$, setting **maxOrder** = 1, expanding the result, and discarding terms of order higher than e, we obtain

```
eq23b = ExpandAll@eq23a •. uExpRule@1D •. omgRule@1DD •. w0 -> 1 •. e^n_•;n>1 -> 0
```

```
u0@tD + e u0@tD^3 + e u1@tD + u0''@tD + 2 e w1 u0''@tD + e u1''@tD == 0
```

Equating coefficients of like powers of e on both sides of **eq23b** yields

```
eqEps = CoefficientList@eq23b@@1DD, eD == 0 •• Thread
```

```
8u0@tD + u0''@tD == 0, u0@tD^3 + u1@tD + 2 w1 u0''@tD + u1''@tD == 0<
```

The general solution of **eqEps[[1]]**, the zeroth-order equation, can be expressed in polar form as

```
sol0 = DSolve@8eqEps@@1DD, u0@0D == a Cos@bD, u0'@0D == -a Sin@bD<, u0, tD@@1DD
```

```
8u0 ® Ha Cos@bD Cos@#1D - a Sin@bD Sin@#1D &L<
```

where *a* and b are constants. We have expressed u_0 as a pure function so that we can evaluate its derivatives in the higher-order problems.

Substituting for u_0@tD in **eqEps[[2]]**, the first-order equation, yields

```
order1Eq = eqEps@@2DD •. sol0 •• Simplify
```

```
a^3 Cos@b + tD^3 - 2 a Cos@b + tD w1 + u1@tD + u1''@tD == 0
```

whose particular solution is

```
u1pSol = DSolve@order1Eq, u1@tD, tD@@1DD •. C@_D -> 0 •• Simplify
```

```
9u1@tD ®
  1/32 a Ha^2 H-6 Cos@b + tD + Cos@3 Hb + tLD - 12 t Sin@b + tDL + 16 HCos@b + tD + 2 t Sin@b + tDL w1L=
```

It is clear that **u1pSol** contains a mixed-secular term, which makes the expansion nonuniform. In contrast with the straightforward expansion, where the secular term cannot be annihilated unless *u* is trivial, in this case, we can choose the parameter w_1 to eliminate the secular term according to

```
omg1Rule = Solve@Coefficient@u1pSol@@1, 2DD, Sin@b + tDD == 0, w1D@@1DD
```

```
:w1 ® 3 a^2/8 >
```

Substituting for w_1 into **u1pSol**, we have

```
sol1 = u1pSol •. omg1Rule •• Simplify
```

```
9u1@tD ® 1/32 a^3 Cos@3 Hb + tLD=
```

We note that, to determine the condition for the elimination of the secular term from u_1, we do not need to determine the particular solution first as done above. Instead, we only need to inspect **order1Eq** and choose w_1 so that the coefficient of Cos@b + tD, which produces secular terms in u_1, is equal to zero. To this end, we calculate the nonhomogeneous part of **order1Eq** and simplify the result using trigonometric identities as

```
expr23a = order1Eq@@1DD •. u1 -> H0 &L •• TrigReduce
```

```
1/4 H3 a^3 Cos@b + tD + a^3 Cos@3 b + 3 tD - 8 a Cos@b + tD w1L
```

Next, we choose w_1 to annihilate the term in **expr23a** that produces secular terms, namely Cos@b + tD, and obtain

```
Solve@Coefficient@expr23a, Cos@b + tDD == 0, w1D@@1DD
```

```
:w1 ® 3 a^2/8 >
```

Substituting **sol0** and **sol1** into **uExpRule[1]** yields

```
u@tD = Hu@tD •. uExpRule@1D •. sol0 •. sol1 •• SimplifyL
```

```
a Cos@b + tD + 1/32 a^3 e Cos@3 Hb + tLD
```

Replacing t with ωt, substituting for ω, using the fact that $\omega_0 = 1$, and expanding the arguments of the trigonometric functions, we obtain

```
solLP =
 u[t] == (u[t] /. t -> ω t /. omgRule[1] /. omg1Rule /. ω0 -> 1 /. f_[arg_] :> f[arg // Expand])
```

$$u[t] == a \operatorname{Cos}\left[t + \beta + \frac{3}{8} a^2 t \epsilon\right] + \frac{1}{32} a^3 \epsilon \operatorname{Cos}\left[3 t + 3 \beta + \frac{9}{8} a^2 t \epsilon\right]$$

Clearly, the expansion **solLP** is free of secular terms and the correction term $\frac{1}{32} a^3 \epsilon \operatorname{Cos}[3t + 3\beta + \frac{9}{8} \epsilon a^2 t]$ is small compared with the principal term $a \operatorname{Cos}[t + \beta + \frac{3}{8} \epsilon a^2 t]$ for all t and hence it is uniformly valid.

2.4 The Method of Multiple Scales

We note from **solLP** that the functional dependence of u on t and ϵ is not disjoint. In fact, u depends on the combination ϵt as well as on the individual t and ϵ. Carrying out the expansion to higher order, we find that u depends on the combinations $\epsilon t, \epsilon^2 t, \epsilon^3 t, \ldots$ as well as on the individual t and ϵ. Hence, $u = u(\epsilon t, \epsilon^2 t, \epsilon^3 t, \ldots; \epsilon)$. For small ϵ, $T_n = \epsilon^n t$, for $n = 1, 2, 3, \ldots$, represent different time scales. For example, if $\epsilon = \frac{1}{60}$, variations on the time scales T_0, T_1, and T_2 can be observed, respectively, on the second, minute, and hour arms of a watch. Thus, T_0 represents a fast time scale, T_1 represents a slower time scale, T_2 represents an even slower time scale, and so on. Since the dependence of u on t and ϵ occurs on different time scales, we imagine that we have a watch and attempt to observe the behavior of u using the different scales of the watch. Therefore, instead of determining u as a function of t, we determine u as a function of $T_0, T_1, T_2, \ldots$. To this end, we change the independent variable t in the governing equation from t to $T_0, T_1, T_2, \ldots$, which are symbolized by

```
Symbolize[T0]; Symbolize[T1]; Symbolize[T2];

timeScales = {T0, T1, T2};
```

We define the derivative operators as

```
dt[1][expr_] := Sum[ε^i D[expr, timeScales[[i + 1]]], {i, 0, maxOrder}];
dt[2][expr_] := (dt[1][dt[1][expr]] // Expand) /. ε^(i_ /; i > maxOrder) -> 0;
```

and treat the time scales T_0, T_1, and T_2 as independent variables.

Next, we use the method of multiple scales to determine an approximate solution for the **DuffingEq**. In Section 2.4.1, we treat the **DuffingEq** in its second-order form; in Section 2.4.2, we transform it first into a system of two real-valued equations; and in Section 2.4.3, we transform it first into a single complex-valued equation.

2.4.1 Second-Order Real-Valued System

```
DuffingEq = u''[t] + u[t] + ε u[t]^3 == 0;
```

To determine a first-order uniform expansion of the solution of the Duffing equation, we use the two time scales T_0 and T_1 and hence let

```
maxOrder = 1;
```

Expressing the time derivative in **DuffingEq** in terms of the two time scales T_0 and T_1, we have

```
eq241a = DuffingEq •. 8u@tD -> u@T0, T1D, Derivative@n_D@uD@tD -> dt@nD@u@T0, T1DD<
```

```
u@T0, T1D + e u@T0, T1D^3 + 2 e u^H1,1L@T0, T1D + u^H2,0L@T0, T1D == 0
```

Consequently, we have replaced the original ordinary-differential equation by a partial-differential equation, and it appears that, as a result, we have complicated the problem. This is partially true, but experience with this method has shown that the disadvantages of introducing this complication are far outweighed by the advantages. Not only does this method provide a uniform expansion, but it also provides all the various nonlinear resonance phenomena, as we shall see in subsequent chapters.

We seek a uniform second-order expansion of the solution of **eq241a** in the form

```
solRule = u -> ISumAe^i ui@#1, #2D, 8i, 0, maxOrder<E &M;
```

Substituting this expansion into **eq241a**, expanding the result, and discarding terms of order higher than e, we obtain

```
eq241b = Heq241a •. solRule •• ExpandAllL •. e^n_•;n>maxOrder -> 0
```

```
u0@T0, T1D + e u0@T0, T1D^3 + e u1@T0, T1D + 2 e u0^H1,1L@T0, T1D + u0^H2,0L@T0, T1D + e u1^H2,0L@T0, T1D == 0
```

Equating coefficients of like powers of e in **eq241b** yields

```
eqEps = CoefficientList@eq241b@@1DD, eD == 0 •• Thread
```

```
9u0@T0, T1D + u0^H2,0L@T0, T1D == 0, u0@T0, T1D^3 + u1@T0, T1D + 2 u0^H1,1L@T0, T1D + u1^H2,0L@T0, T1D == 0=
```

The general solution of **eqEps**[[1]] can be written in a pure function form as

```
sol0 = u0 -> Ha@#2D Cos@#1 + b@#2DD &L;
```

where #1 stands for the scale T_0 and #2 stands for the scale T_1. The functional dependence of a and b on T_1 is not known at this level of approximation; it is determined at subsequent levels of approximation by eliminating the secular terms.

Substituting **sol0** into **eqEps**[[2]] and moving the nonhomogeneous parts to the right-hand side of the resulting equation yields

```
eq241c = HeqEps@@1, 1DD •. u0 -> u1L - # & •ž Reverse@eqEps@@2DD •. sol0 •• SimplifyD
```

```
u1@T0, T1D + u1^H2,0L@T0, T1D ==
 - a@T1D^3 Cos@T0 + b@T1DD^3 + 2 Sin@T0 + b@T1DD a¢@T1D + 2 a@T1D Cos@T0 + b@T1DD b¢@T1D
```

Next, we expand the right-hand side of **eq241c** in a Fourier series using trigonometric identities and obtain

```
eq241c@@2DD = eq241c@@2DD •• TrigFactor •• Expand
```

```
- 3/4 a@T1D^3 Cos@T0 + b@T1DD - 1/4 a@T1D^3 Cos@3 T0 + 3 b@T1DD +
 2 Sin@T0 + b@T1DD a¢@T1D + 2 a@T1D Cos@T0 + b@T1DD b¢@T1D
```

To produce a uniform expansion, we eliminate the terms that produce secular terms from **eq241c**[[2]]; that is, we choose a_1¢@T_1D and b¢@T_1D to annihilate each of the coefficients of Sin@T_0 + b@T_1DD and Cos@T_0 + b@T_1DD. The result is

```
eq241d = Coefficient@eq241c@@2DD, #D == 0 & •ž 8Sin@T0 + b@T1DD, Cos@T0 + b@T1DD<
```

```
92 a¢@T1D == 0, - 3/4 a@T1D^3 + 2 a@T1D b¢@T1D == 0=
```

Solving **eq241d** for a_1¢@T_1D and b¢@T_1D when a@T_1D ¹ 0 yields

```
SCond = Solve@eq241d, 8a¢@T1D, b¢@T1D<D@@1DD
```

```
9a¢@T1D ® 0, b¢@T1D ® 3/8 a@T1D^2=
```

The solution of **SCond**[[1]] is $a = a_0$ = constant. Then, the solution of **SCond**[[2]] is

```
betaRule = DSolve@eq241d@@2DD •. a@T1D -> a0, b@T1D, T1D@@1DD •. C@1D -> b0
```

```
9b@T1D ® 3/8 T1 a0^2 + b0=
```

With the above solvability conditions, **eq241c** becomes

```
eq241e = eq241c •. SCond
```

```
u1@T0, T1D + u1^H2,0L@T0, T1D == - 1/4 a@T1D^3 Cos@3 T0 + 3 b@T1DD
```

The particular solution of **eq241e** can be obtained as

```
sol1 = DSolve@eq241e, u1@T0, T1D, 8T0, T1<D@@1DD •. C@_D -> H0 &L •• Simplify
```

```
9u1@T0, T1D ® 1/32 a@T1D^3 Cos@3 HT0 + b@T1DLD=
```

```
eq241f = u@T0, T1D == Hu@T0, T1D •. solRule •. sol0 •. sol1 •. a@T1D -> a0 •. betaRuleL
```

```
u@T0, T1D == CosAT0 + 3/8 T1 a0^2 + b0E a0 + 1/32 e CosA3 jT0 + 3/8 T1 a0^2 + b0zE a0^3
                                                        k                       {
```

where a_0 and b$_0$ are constants to within the order of the error indicated. In terms of the original variable t, **eq241f** can be expressed as

```
u@tD == eq241f@@2DD •. 8T0 -> t, T1 -> e t<
```

```
u@tD == CosAt + 3/8 t e a0^2 + b0E a0 + 1/32 e CosA3 jt + 3/8 t e a0^2 + b0zE a0^3
                                                   k                      {
```

in agreement with the expansion obtained by using the Lindstedt-Poincaré technique.

In the higher-order approximations, we include the scales $T_0, T_1, \ldots, T_N$ but we do not include the term $O[\epsilon^N]$ in an N th-order expansion.

Before closing this section, we present an alternate representation of the solutions of the perturbation equations. Instead of the real-valued form **sol0**, we represent the solution of **eqEps**[[1]] in a complex-valued form; that is,

```
sol0 = u0 -> IA@#2D Exp@I #1D + Ā@#2D Exp@- I #1D &M;
```

where $\bar{A}$ is the complex conjugate of A and

```
ruleA = 9A -> (1/2 a@#D Exp@I b@#DD &), Ā -> (1/2 a@#D Exp@- I b@#DD &)=;
```

Substituting **sol0** into **eqEps**[[2]] and moving the nonhomogeneous terms to the right-hand side of the resulting equation, we have

```
eq241g = u1@T0, T1D + u1^H2,0L@T0, T1D - # & •ž HeqEps@@2DD •. sol0L •• ExpandAll •• Reverse
```

```
u1@T0, T1D + u1^H2,0L@T0, T1D == - E^3 I T0 A@T1D^3 - 3 E^I T0 A@T1D^2 Ā@T1D -
   3 E^-I T0 A@T1D Ā@T1D^2 - E^-3 I T0 Ā@T1D^3 - 2 I E^I T0 A'@T1D + 2 I E^-I T0 Ā'@T1D
```

We note that the terms proportional to $\exp[I T_0]$ and $\exp[-I T_0]$ produce secular terms in the particular solution of u_1. Thus, to produce a uniform expansion, we set the coefficient of each of these functions equal to zero; that is,

```
eq241h = Coefficient@eq241g@@2DD, #D == 0 & •ž 8Exp@I T0D, Exp@- I T0D<
```

```
: - 3 A@T1D^2 Ā@T1D - 2 I A'@T1D == 0, - 3 A@T1D Ā@T1D^2 + 2 I Ā'@T1D == 0>
```

These two equations are not independent because they are complex conjugates of each other. Hence, if one of them is satisfied, the other is automatically satisfied.

To analyze the solutions of **eq241h**[[1]], we multiply it by $\exp[-I\,\beta[T_1]]$ and replace A with its polar form **ruleA**. The result is

```
eq241i = Expand@eq241h@@1, 1DD Exp@- I b@T1DD •. ruleAD == 0
```

```
- 3/8 a@T1D^3 - I a'@T1D + a@T1D b'@T1D == 0
```

Next, we separate the real and imaginary parts of **eq241i**. To accomplish this, we define the following rule:

```
realRule = 8Re@s_D -> s, Im@s_D -> 0<;
```

Using the **realRule**, we find that the imaginary part of **eq241i** is

```
ampEq = Im@eq241i@@1DDD == 0 •. realRule
```

```
- a'@T1D == 0
```

which governs the modulation of the amplitude. Similarly, the real part of **eq241i** is

```
phaseEq = Re@eq241i@@1DDD == 0 •. realRule
```

```
- 3/8 a@T1D^3 + a@T1D b¢@T1D == 0
```

which governs the modulation of the phase. These modulation equations are in agreement with those obtained above by expressing the solution in real form.

Comparing the complex-valued and the real-valued representations, we find it more convenient to use the complex-valued form. Therefore, the complex-valued form is used in the remainder of this book.

Ÿ 2.4.2 First-Order Real-Valued System

We let $v@tD = u¢@tD$ and transform the **DuffingEq** into the following set of two first-order equations:

```
eq242a = 9u¢@tD == v@tD, v¢@tD + u@tD == -e u@tD^3=;
```

We seek a first-order uniform expansion of the solution of **eq242a** in terms of the two time scales T_0 and T_1 in the form

```
solRule = 9u -> ISumAe^j uj@#1, #2D, 8j, 0, 1<E &M, v -> ISumAe^j vj@#1, #2D, 8j, 0, 1<E &M=;
```

Substituting **solRule** into **eq242a**, transforming the derivative with respect to t in terms of the derivatives with respect to T_0 and T_1, and discarding terms of order higher than e, we obtain

```
eq242b = Heq242a •. 8u_¢@tD -> dt@1D@u@T0, T1DD, u_@tD -> u@T0, T1D< •. solRule •• ExpandAllL •.
   e^n_•;n>1 -> 0
```

```
9e u0^H0,1L@T0, T1D + u0^H1,0L@T0, T1D + e u1^H1,0L@T0, T1D == v0@T0, T1D + e v1@T0, T1D,
  u0@T0, T1D + e u1@T0, T1D + e v0^H0,1L@T0, T1D + v0^H1,0L@T0, T1D + e v1^H1,0L@T0, T1D == -e u0@T0, T1D^3=
```

Equating coefficients of like powers of e, we have

```
eqEps = Thread@CoefficientList@Subtract žž #, eD == 0D & •ž eq242b •• Transpose
```

```
99-v0@T0, T1D + u0^H1,0L@T0, T1D == 0, u0@T0, T1D + v0^H1,0L@T0, T1D == 0=,
  9-v1@T0, T1D + u0^H0,1L@T0, T1D + u1^H1,0L@T0, T1D == 0,
   u0@T0, T1D^3 + u1@T0, T1D + v0^H0,1L@T0, T1D + v1^H1,0L@T0, T1D == 0==
```

Ÿ Zeroth-Order Problem: Linear System

The zeroth-order problem is given by the linear system

```
linearSys = #@@1DD & •ž eqEps@@1DD
```

```
9-v0@T0, T1D + u0^H1,0L@T0, T1D, u0@T0, T1D + v0^H1,0L@T0, T1D=
```

To determine the solution of this linear system, we seek a solution proportional to Exp@$I\, T_0$D as

```
coefList = E^(-I T0) linearSys /. {u0 -> (P E^(I #) &), v0 -> (Q E^(I #) &)} // Expand
```

```
8I P - Q, P + I Q<
```

Next, we determine the coefficient matrix as

```
coefMat = Outer@Coefficient, coefList, 8P, Q<D
```

```
88I, -1<, 81, I<<
```

To determine the eigenvalues and eigenvectors of **coefMat**, we define the following conjugate rule and Hermitian matrix:

```
conjugateRule = 9A -> Ā, Ā -> A, Complex@0, n_D -> Complex@0, -nD=;

hermitian@mat_? MatrixQD := mat /. conjugateRule // Transpose
```

Then the left and right eigenvectors of **coefMat** are given by

```
leftVec = 81, c1< /. Solve@Hhermitian@coefMatD.81, c1<L@@1DD == 0, c1D@@1DD
```

```
81, I<
```

```
rightVec = 81, c1< /. Solve@HcoefMat.81, c1<L@@1DD == 0, c1D@@1DD
```

```
81, I<
```

To express the solution of the zeroth-order problem, we introduce the following basic function:

```
basicH = A@T1D E^(I T0);
```

In terms of this function and the right eigenvector, the solution of the zeroth-order problem can be expressed in terms of

```
sol0Form = rightVec basicH
```

```
8E^(I T0) A@T1D, I E^(I T0) A@T1D<
```

and its complex conjugate as

```
sol0 =
 8u0 -> Function@8T0, T1<, sol0Form@@1DD + Hsol0Form@@1DD /. conjugateRuleL // EvaluateD,
  v0 -> Function@8T0, T1<, sol0Form@@2DD + Hsol0Form@@2DD /. conjugateRuleL // EvaluateD<
```

```
8u0 ® Function@8T0, T1<, E^(I T0) A@T1D + E^(-I T0) Ā@T1DD,
 v0 ® Function@8T0, T1<, I E^(I T0) A@T1D - I E^(-I T0) Ā@T1DD<
```

We have expressed the solution of the zeroth-order problem in function form so that its partial derivatives can be readily evaluated.

Ÿ First-Order Problem

Substituting the zeroth-order solution **sol0** into the first-order problem **eqEps**[[2]] and moving the nonhomogeneous terms to the right-hand sides of the resulting equations, we obtain

```
order1Eq = HlinearSys •. u_0 -> u1L ==
   HlinearSys •. u_0 -> u1L - HSubtract žž # & •ž eqEps@@2DD •. sol0 •• ExpandL •• Thread
```

```
:-v1@T0, T1D + u1^H1,0L@T0, T1D == -E^(I T0) A¢@T1D - E^(-I T0) Ā¢@T1D,
 u1@T0, T1D + v1^H1,0L@T0, T1D == -E^(3 I T0) A@T1D^3 - 3 E^(I T0) A@T1D^2 Ā@T1D -
   3 E^(-I T0) A@T1D Ā@T1D^2 - E^(-3 I T0) Ā@T1D^3 - I E^(I T0) A¢@T1D + I E^(-I T0) Ā¢@T1D>
```

Eliminating the terms that produce secular terms in **order1Eq** demands that their right-hand sides are orthogonal to every solution of the adjoint homogeneous problem. To determine these conditions, we first calculate the coefficient vector of Exp@$I\,T_0$D as

```
STerms = Coefficient@#@@2DD, Exp@I T0DD & •ž order1Eq
```

```
8-A¢@T1D, -3 A@T1D^2 Ā@T1D - I A¢@T1D<
```

Then, we demand that this vector is orthogonal to the left eigenvector and obtain the solvability condition

```
SCond = Expand@Conjugate@leftVecD.STermsD == 0
```

```
3 I A@T1D^2 Ā@T1D - 2 A¢@T1D == 0
```

which is in agreement with that obtained by treating the second-order form of the **DuffingEq**. Repeating the procedure for the coefficient vector of Exp@$-I\,T_0$D, we obtain the complexconjugate of **SCond**.

Ÿ 2.4.3 First-Order Complex-Valued System

In this section, we determine a first-order uniform expansion of the solution of the **DuffingEq** by transforming it first into a single first-order complex-valued equation. To this end, we introduce the transformation

```
transfRule = 9u@tD -> z@tD + z̄@tD, u¢@tD -> I Hz@tD - z̄@tDM=;
```

whose inverse is

```
zetaRule = SolveAtransfRule •. Rule -> Equal, 9z@tD, z̄@tD=E@@1DD
```

```
9z@tD ® 1/2 Hu@tD - I u¢@tDL, z̄@tD ® 1/2 Hu@tD + I u¢@tDL=
```

It follows from the **DuffingEq** that the acceleration is related to the displacement according to

```
acceleration = Solve@DuffingEq, u''@tDD@@1DD
```

```
8u''@tD ® -u@tD - e u@tD^3<
```

Differentiating z@tD, **zetaRule**[[1]], with respect to *t* and substituting for *u*@tD, *u*¢@tD, and *u*''@tD using the **acceleration** and **transfRule**, we obtain

```
eq243a = D@zetaRule@@1DD, tD •. acceleration •. transfRule •. Rule -> Equal •• ExpandAll
```

$$z'[t] == I\,z[t] + \frac{1}{2} I\,e\,z[t]^3 + \frac{3}{2} I\,e\,z[t]^2\,\bar{z}[t] + \frac{3}{2} I\,e\,z[t]\,\bar{z}[t]^2 + \frac{1}{2} I\,e\,\bar{z}[t]^3$$

Next, we use the method of multiple scales to determine a first-order uniform expansion of the solution of **eq243a** in the form

```
solRule = 9z -> ISumAe^j z_j@#1, #2D, 8j, 0, 1<E &M, ż -> ISumAe^j ż_j@#1, #2D, 8j, 0, 1<E &M=;
```

Substituting this expansion into **eq243a**, expanding the result, and discarding terms of order higher than e, we have

```
eq243b =
  Ieq243a •. 9z@tD -> z@T0, T1D, ż@tD -> ż@T0, T1D, z¢@tD -> dt@1D@z@T0, T1DD= •. solRule ••
    ExpandAllM •. e^n_•;n>1 -> 0
```

$$e\,z_0^{(0,1)}[T_0, T_1] + z_0^{(1,0)}[T_0, T_1] + e\,z_1^{(1,0)}[T_0, T_1] == I\,z_0[T_0, T_1] + \frac{1}{2} I\,e\,z_0[T_0, T_1]^3 + I\,e\,z_1[T_0, T_1] + \frac{3}{2} I\,e\,z_0[T_0, T_1]^2\,\bar{z}_0[T_0, T_1] + \frac{3}{2} I\,e\,z_0[T_0, T_1]\,\bar{z}_0[T_0, T_1]^2 + \frac{1}{2} I\,e\,\bar{z}_0[T_0, T_1]^3$$

Equating coefficients of like power of e in **eq243b** yields

```
eqEps = CoefficientList@Subtract žž eq243b, eD == 0 •• Thread
```

$$\{-I\,z_0[T_0, T_1] + z_0^{(1,0)}[T_0, T_1] == 0,$$
$$-\frac{1}{2} I\,z_0[T_0, T_1]^3 - I\,z_1[T_0, T_1] - \frac{3}{2} I\,z_0[T_0, T_1]^2\,\bar{z}_0[T_0, T_1] - \frac{3}{2} I\,z_0[T_0, T_1]\,\bar{z}_0[T_0, T_1]^2 - \frac{1}{2} I\,\bar{z}_0[T_0, T_1]^3 + z_0^{(0,1)}[T_0, T_1] + z_1^{(1,0)}[T_0, T_1] == 0\}$$

The solution of the zeroth-order problem, **eqEps**[[1]], can be expressed as

```
sol0Form = DSolve@eqEps@@1DD, z0@T0, T1D, 8T0, T1<D@@1DD •. C@1D -> A
```

$$\{z_0[T_0, T_1] \to E^{I\,T_0}\,A[T_1]\}$$

which we express in function form as

```
sol0 = 8z0 -> Function@8T0, T1<, sol0Form@@1, 2DD •• EvaluateD<
```

$$\{z_0 \to \text{Function}[\{T_0, T_1\}, E^{I\,T_0}\,A[T_1]]\}$$

To evaluate the complex conjugates of *A* and z, we define the conjugate rule

```
conjugateRule = 9A -> Ȧ, Ȧ -> A, z -> ż, ż -> z, Complex@0, n_D -> Complex@0, -nD=;
```

Substituting **sol0** into the first-order problem, **eqEps**[[2]], using **conjugateRule**, and moving the nonhomogeneous terms to the right-hand side of the resulting equation, we obtain

```
order1Eq = HeqEps@@1, 1DD •. z0 -> z1L ==
  HeqEps@@1, 1DD •. z0 -> z1L - HeqEps@@2, 1DD •. sol0 •. Hsol0 •. conjugateRuleLL
```

$$-I\, z_1[T_0, T_1] + z_1^{(1,0)}[T_0, T_1] == \frac{1}{2} I E^{3 I T_0} A[T_1]^3 + \frac{3}{2} I E^{I T_0} A[T_1]^2 \dot{A}[T_1] + \frac{3}{2} I E^{-I T_0} A[T_1] \dot{A}[T_1]^2 + \frac{1}{2} I E^{-3 I T_0} \dot{A}[T_1]^3 - E^{I T_0} A'[T_1]$$

Elimintaing the terms that lead to secular terms from **order1Eq** yields the solvability condition

```
SCond = CoefficientAorder1Eq@@2DD, E^(I T0)E == 0
```

$$\frac{3}{2} I A[T_1]^2 \dot{A}[T_1] - A'[T_1] == 0$$

in agreement with that obtained by treating the second-order form of the **DuffingEq** as well as that obtained by transforming it first into two first-order real-valued equations.

à 2.5 Variation of Parameters

In the next section, we use the method of averaging to determine a first-order uniform expansion of the solution of the Duffing equation. To this end, we use the method of variation of parameters to transform it into a system of two first-order equations. The Duffing equation is repeated here; that is,

```
DuffingEq = u''@tD + u@tD + e u@tD^3 == 0;
```

When $\epsilon = 0$, the solution of this equation can be written as

```
usolEq = u@tD == a Cos@t + bD;
```

where *a* and β are constants, which are sometimes referred to as parameters. It follows from **usolEq** that

```
cond01 = D@usolEq, tD
```

$$u'[t] == -a \sin[t + \beta]$$

When $\epsilon \neq 0$, we assume that the solution of **DuffingEq** is still given by **usolEq** but with time-varying *a* and β; that is,

```
tdepRule = 8a -> a@tD, b -> b@tD<;
```

and hence

```
usolEqt = usolEq •. tdepRule
```

$$u[t] == a[t] \cos[t + \beta[t]]$$

In other words, we consider **usolEq** as a transformation from *u*@*t*D to *a*@*t*D and b@*t*D. This is why this approach is called the method of variation of parameters. Using this view, we note that we have two equations, namely **DuffingEq** and **usolEq**, for the three unknowns *u*@*t*D, *a*@*t*D, and b@*t*D. Hence, we have the freedom of imposing a third condition (third equation). This condition is arbitrary except that it must be independent of **DuffingEq** and **usolEq**. This arbitrariness can be used to advantage, namely to produce a simple and convenient transformation. Out of all possible conditions, we choose to impose the condition **cond01**, thereby assuming that *u*@*t*D as well as *u*¢@*t*D have the same form as in the linear case. This condition leads to a convenient transformation; it leads to a set of first-order rather than second-order equations for *a*@*t*D and b@*t*D.

Differentiating **usolEq** with respect to *t* and recalling that *a* and b are functions of *t*, we have

```
cond02 = D@usolEqt, tD
```

```
u¢@tD == Cos@t + b@tDD a¢@tD - a@tD Sin@t + b@tDD H1 + b¢@tDL
```

Comparing **cond02** with **cond01**, we conclude that

```
cond1 = Expand@cond02@@2DD - Hcond01@@2DD •. tdepRuleLD == 0
```

```
Cos@t + b@tDD a¢@tD - a@tD Sin@t + b@tDD b¢@tD == 0
```

Differentiating **cond01** with respect to *t*, we obtain

```
cond03 = D@cond01 •. tdepRule, tD
```

```
u²@tD == -Sin@t + b@tDD a¢@tD - a@tD Cos@t + b@tDD H1 + b¢@tDL
```

Substituting for *u*@*t*D and *u*²@*t*D from **usolEqt** and **cond03** into **DuffingEq**, we have

```
cond2 = DuffingEq •. H8usolEqt, cond03< •. Equal -> RuleL •• ExpandAll
```

```
e a@tD^3 Cos@t + b@tDD^3 - Sin@t + b@tDD a¢@tD - a@tD Cos@t + b@tDD b¢@tD == 0
```

Solving **cond1** and **cond2** for *a*¢@*t*D and b¢@*t*D, we obtain the desired two first-order equations

```
transformedEq = HSolve@8cond1, cond2<, 8a¢@tD, b¢@tD<D@@1DD •• SimplifyL •. Rule -> Equal
```

```
8a¢@tD == e a@tD^3 Cos@t + b@tDD^3 Sin@t + b@tDD, b¢@tD == e a@tD^2 Cos@t + b@tDD^4<
```

if *a* T 0. Thus, the original second-order **DuffingEq** for *u*@*t*D has been replaced by two first-order equations for *a*@*t*D and b@*t*D. We emphasize that no approximations have been made in deriving **transformedEq**. Comparing **transformedEq** with **DuffingEq**, we find that the transformed equations are more nonlinear than the original equation. Then, the question arises what is the value of this transformation? The answer depends on the value of e. If e is small, the major parts of *a* and b vary more slowly with *t* than *u*. This fact can be used to advantage analytically and numerically. The analytical advantage is utilized in the method of averaging, as discussed in the next section. Numerically, it is advantageous to solve the transformed equations instead of the original equation because a large step size can be used in the integration. This is the reason why astronomers numerically solve the variational equations rather than the original equations. Usually, astronomers and celestial mechanicians refer to this approach as the special method of perturbations.

2.6 The Method of Averaging

In this section, we determine a first-order approximation to the transformed equations obtained in the preceding section. To this end, we rewrite **transformedEq** as

```
ampEq = TrigReduce /@ transformedEq[[1]]
```

$$a'[t] == \frac{1}{8}\left(2\,\epsilon\, a[t]^3\, \mathrm{Sin}[2\,t + 2\,\beta[t]] + \epsilon\, a[t]^3\, \mathrm{Sin}[4\,t + 4\,\beta[t]]\right)$$

```
phaseEq = TrigReduce /@ transformedEq[[2]]
```

$$\beta'[t] == \frac{1}{8}\left(3\,\epsilon\, a[t]^2 + 4\,\epsilon\, a[t]^2\, \mathrm{Cos}[2\,t + 2\,\beta[t]] + \epsilon\, a[t]^2\, \mathrm{Cos}[4\,t + 4\,\beta[t]]\right)$$

The major parts of a and β are slowly varying functions of time if ϵ is small. Hence, they change very little during the time interval π (the period of the circular functions) and, to the first approximation, they can be considered constant in the interval $[0, \pi]$. Hence, we replace $a[t]$ and $\beta[t]$ on the right-hand sides of **ampEq** and **phaseEq** with time-independent a and β and obtain

```
ampEq[[2]] = ampEq[[2]] /. (Reverse /@ tdepRule)
```

$$\frac{1}{8}\left(2\,a^3\,\epsilon\, \mathrm{Sin}[2\,t + 2\,\beta] + a^3\,\epsilon\, \mathrm{Sin}[4\,t + 4\,\beta]\right)$$

```
phaseEq[[2]] = phaseEq[[2]] /. (Reverse /@ tdepRule)
```

$$\frac{1}{8}\left(3\,a^2\,\epsilon + 4\,a^2\,\epsilon\, \mathrm{Cos}[2\,t + 2\,\beta] + a^2\,\epsilon\, \mathrm{Cos}[4\,t + 4\,\beta]\right)$$

We average the right-hand sides of **ampEq** and **phaseEq** over the interval $[0, \pi]$ and obtain

```
list26a = 1/π Integrate[#, {t, 0, π}] & /@ {ampEq[[2]], phaseEq[[2]]} // Expand
```

```
Integrate::gener : Unable to check convergence

Integrate::gener : Unable to check convergence
```

$$\left\{0, \frac{3\,a^2\,\epsilon}{8}\right\}$$

It follows from the above result that

```
averagingEq = {a'[t], β'[t]} == list26a // Thread
```

$$\left\{a'[t] == 0,\ \beta'[t] == \frac{3\,a^2\,\epsilon}{8}\right\}$$

This averaging method is usually referred to as the Krylov-Bogoliubov or van der Pol technique.

Solving the averaged equations yields

```
rule26a = DSolve@averagingEq, 8a@tD, b@tD<, tD@@1DD •. 8C@1D -> a0, C@2D -> b0<
```

```
9a@tD ® a0, b@tD ® 1/8 H3 a^2 t e + 8 b0L=
```

Substituting for *a*@*t*D and b@*t*D into **usolEq**, we obtain, to the first approximation, that

```
usolEq •. tdepRule •. rule26a •. Cos@arg_D :> Cos@Expand@argDD
```

```
u@tD == CosAt + 3/8 a^2 t e + b0E a0
```

in agreement with the solutions obtained by using the Lindstedt-Poincaré technique and the method of multiple scales.

Before closing this section, we note that one can arrive at the final results in **ampEq** and **phaseEq** without going through the averaging process. The right-hand sides of **ampEq** and **phaseEq** are the sum of two groups of terms — a group that is a linear combination of fast varying terms and a group that is a linear combination of slowly varying terms. Then, to the first approximation, *a*¢@*t*D in **ampEq** is equal to the slowly varying group on its right-hand side, which is zero. And, to the first approximation, b¢@*t*D in **phaseEq** is equal to the slowly varying group on its right-hand side, which is $\frac{3}{8} e a^2$.

Chapter 3
Systems with Quadratic and Cubic Nonlinearities

```
Off@General::spell1D
```

à 3.1 Nondimensional Equation of Motion

We consider the free oscillations of a particle of mass m under the action of gravity and restrained by a nonlinear spring. The equation of motion is

```
eq31a = m D@x*@t*D, 8t*, 2<D + f@x*@t*DD == m g
```

```
f@x*@t*DD + m Hx*L''@t*D == g m
```

where g is the gravitational acceleration and $f@x^*D$ is the restraining force of the spring. We assume that $f@x^*D$ is an odd cubic function of x^*; that is,

```
eq31b = eq31a •. f@a_D -> k1 a + k3 a^3
```

```
k1 x*@t*D + k3 x*@t*D^3 + m Hx*L''@t*D == g m
```

The equilibrium positions $x^* = x_s^*$ = constant can be obtained by dropping the acceleration term. The result is

```
eq31c = eq31b •. x* -> Hxs* &L
```

```
k1 HxsL* + k3 HHxsL*L^3 == g m
```

In this chapter, we investigate small oscillations about one of the equilibrium positions. To this end, we shift this equilibrium position to the origin by using the transformation

```
xRule = 8x* -> Hxs* + u*@#D &L<;
```

Substituting this transformation into **eq31b** and using the equilibrium condition **eq31c**, we obtain

```
eq31d = Expand@Subtract žž eq31b •. xRuleD == 0 •. Heq31c •. Equal -> RuleL
```

```
k1 u*@t*D + 3 k3 HHxsL*L^2 u*@t*D + 3 k3 HxsL* u*@t*D^2 + k3 u*@t*D^3 + m Hu*L''@t*D == 0
```

As before, we introduce the following dimensionless quantities:

```
dimenRule = 8u* -> Hxs* u@# • TD &L, t* -> T t<;
```

into **eq31d** and obtain

```
eq31e = Ieq31d@@1DD T^2 • Hm x_s^* L •. dimenRule •• ExpandM == 0
```

$$\frac{T^2 k_1 u@tD}{m} + \frac{3 T^2 k_3 HHx_s L^* L^2 u@tD}{m} + \frac{3 T^2 k_3 HHx_s L^* L^2 u@tD^2}{m} + \frac{T^2 k_3 HHx_s L^* L^2 u@tD^3}{m} + u^{\prime\prime}@tD == 0$$

We rewrite **eq31e** as

```
eq31f = HCollect@eq31e@@1DD, u@tDD •. 8_ u@tD^n_. -> a_n u@tD^n<L == 0
```

$$a_1 u@tD + a_2 u@tD^2 + a_3 u@tD^3 + u^{\prime\prime}@tD == 0$$

where the a_j are given by

```
alphas = Ia_# -> CoefficientAeq31e@@1DD, u@tD^#EM & •ž 81, 2, 3<
```

$$:a_1 ® \frac{T^2 k_1}{m} + \frac{3 T^2 k_3 HHx_s L^* L^2}{m},\ a_2 ® \frac{3 T^2 k_3 HHx_s L^* L^2}{m},\ a_3 ® \frac{T^2 k_3 HHx_s L^* L^2}{m}>$$

We choose T so that $a_1 = 1$ and hence the natural frequency is unity; that is,

```
TRule = Solve@a_1 == 1 •. alphas, TD@@2DD
```

$$:T ® \frac{\sqrt{m}}{\sqrt{k_1 + 3 k_3 HHx_s L^* L^2}}>$$

Consequently, a_2 and a_3 can be rewritten as

```
alphas = alphas •. TRule •• Rest
```

$$:a_2 ® \frac{3 k_3 HHx_s L^* L^2}{k_1 + 3 k_3 HHx_s L^* L^2},\ a_3 ® \frac{k_3 HHx_s L^* L^2}{k_1 + 3 k_3 HHx_s L^* L^2}>$$

With a_1 being unity, **eq31f** becomes

```
eq31g = eq31f •. a_1 -> 1
```

$$u@tD + a_2 u@tD^2 + a_3 u@tD^3 + u^{\prime\prime}@tD == 0$$

In contrast with the **DuffingEq**, **eq31g** contains a quadratic as well as a cubic term.

In the next section, we determine a second-order straightforward expansion to the solutions of **eq31g** for small but finite amplitudes. In Section 3.3, we determine a uniform second-order expansion by using the Lindstedt-Poincaré technique. In Section 3.4, we determine a uniform second-order expansion by using the method of multiple scales. In Section 3.5, we show that the first approximation obtained with the method of averaging yields an incomplete solution. In Section 3.6, we introduce the generalized method of averaging and obtain a uniform second-order expansion. Finally, in Sections 3.7 and 3.8, we introduce the Krylov-Bogoliubov-Mitropolsky technique and the method of normal forms, respectively.

à 3.2 Straightforward Expansion

To carry out a straightforward expansion for small but finite amplitudes for **eq31g**, we need to introduce a small parameter e as a bookkeeping parameter. In terms of this parameter, we seek a third-order expansion in the form

```
uExpRule@maxOrder_D := u -> ISumAe^i u_i@#D, 8i, 1, maxOrder<E &M
```

where maxOrder is the order of the expansion sought. In what follows, we let maxOrder = 3 for an expansion of order three. Clearly, e is a measure of the amplitude of oscillation and can be set equal to unity in the final solution if the amplitude is taken to be small, as described below. Substituting this expansion into **eq31g**, expanding the result for small e, and discarding terms of order higher than e^3, we have

```
eq32a = Heq31g •. uExpRule@3D •• ExpandAllL •. e^n_•;n>3 -> 0
```

```
e u1@tD + e^2 a2 u1@tD^2 + e^3 a3 u1@tD^3 + e^2 u2@tD +
  2 e^3 a2 u1@tD u2@tD + e^3 u3@tD + e u1''@tD + e^2 u2''@tD + e^3 u3''@tD == 0
```

Equating coefficients of like powers of e in **eq32a** yields

```
eqEps = TableACoefficientAeq32a@@1DD, e^iE == 0, 8i, 3<E
```

```
8u1@tD + u1''@tD == 0, a2 u1@tD^2 + u2@tD + u2''@tD == 0, a3 u1@tD^3 + 2 a2 u1@tD u2@tD + u3@tD + u3''@tD == 0<
```

which can be solved sequentially for u_1, u_2, and u_3.

The general solution of the first-order equation, **eqEps[[1]]**, can be expressed as

```
sol1 = DSolve@8eqEps@@1DD, u1@0D == a Cos@bD, u1'@0D == -a Sin@bD<, u1@tD, tD@@1DD •• Simplify
```

```
8u1@tD ® a Cos@t + bD<
```

where a and b are constants. Substituting **sol1**, u_1@tD, into the second-order equation, **eqEps[[2]]**, and solving the resulting equation for u_2@tD, we obtain

```
sol2 = DSolve@eqEps@@2DD •. sol1, u2@tD, tD@@1DD •. C@_D -> 0 •• Simplify
```

```
9u2@tD ® 1/6 a^2 H-3 + Cos@2 Ht + bLDL a2=
```

As before, we do not include the solution of the homogeneous problem for u_2@tD.

Substituting **sol1** and **sol2** into the third-order equation, **eqEps[[3]]**, and solving the resulting equation u_3@tD, we have

```
sol3 = DSolve@eqEps@@3DD •. sol1 •. sol2, u3@tD, tD@@1DD •. C@_D -> 0 •• Simplify
```

```
9u3@tD ® 1/96 a^3 H2 H10 Cos@t + bD + Cos@3 Ht + bLD + 20 t Sin@t + bDL a2^2 +
     3 H-6 Cos@t + bD + Cos@3 Ht + bLD - 12 t Sin@t + bDL a3L=
```

Simply letting *C*@_D ® 0 did not remove all of the solutions of the homogeneous equation in this case. Therefore, we clear it one more time and obtain

```
sol3 = sol3 •. Cos@t + _D -> 0 •• ExpandAll
```

```
9u3@tD ®
  1/48 a^3 Cos@3 t + 3 bD a2^2 + 5/12 a^3 t Sin@t + bD a2^2 + 1/32 a^3 Cos@3 t + 3 bD a3 - 3/8 a^3 t Sin@t + bD a3=
```

Combining the first-, second-, and third-order solutions, we obtain, to the third approximation, that

```
sol = u@tD == Hu@tD •. uExpRule@3D •. Flatten@8sol1, sol2, sol3<DL
```

```
u@tD == a e Cos@t + bD + 1/6 a^2 e^2 H-3 + Cos@2 Ht + bLDL a2 +
  e^3 (1/48 a^3 Cos@3 t + 3 bD a2^2 + 5/12 a^3 t Sin@t + bD a2^2 + 1/32 a^3 Cos@3 t + 3 bD a3 - 3/8 a^3 t Sin@t + bD a3)
```

We note that the dependence of *u* on e and *a* is in the combination e*a*. Thus, one can set e = 1 and consider *a* as the perturbation parameter.

The straightforward expansion **sol** breaks down for *t* ³ *O*He^{-1} *a*$^{-1}$L because the third-order term is the same order or larger than the second-order term owing to the presence of the mixed-secular term. Next, we use the Lindstedt-Poincaré technique to determine a third-order uniform solution.

à 3.3 The Lindstedt-Poincaré Technique

To apply the Lindstedt-Poincaré technique to **eq31g**, we introduce the transformation t = w*t* and rewrite it as

```
eq33a = eq31g •. u -> Hu@w #D &L •. t -> t • w
```

```
u@tD + a2 u@tD^2 + a3 u@tD^3 + w^2 u''@tD == 0
```

To determine a uniform expansion of order maxOrder of the solution of **eq33a**, we expand both w and *u*@tD in powers of e as

```
omgRule@maxOrder_D := w -> 1 + SumAe^i wi, 8i, maxOrder<E
uExpRule@maxOrder_D := u -> ISumAe^i ui@#D, 8i, maxOrder + 1<E &M
```

As discussed in the preceding chapter, the first term in the expansion of w is taken to be the linear natural frequency of the system, which is unity in this case. We note that the order of the expansion for *u* is larger than that of w and the uniform expansion required. As discussed below, we eliminate the terms that lead to secular terms from the equation at order max-Order+1 but do not include in the solution of the resulting equation in the expansion.

For a uniform second order expansion, we let

```
maxOrder = 2;
```

Substituting the expansions of *u* and w into **eq33a**, expanding the result for small e, and discarding terms of order higher than e^3, we obtain

```
eq33b = Heq33a •. 8uExpRule@maxOrderD, omgRule@maxOrderD< •• ExpandAllL •. e^(n_•;n>maxOrder+1) -> 0
```

```
e u1@tD + e^2 a2 u1@tD^2 + e^3 a3 u1@tD^3 + e^2 u2@tD + 2 e^3 a2 u1@tD u2@tD + e^3 u3@tD + e u1''@tD +
   2 e^2 w1 u1''@tD + e^3 w1^2 u1''@tD + 2 e^3 w2 u1''@tD + e^2 u2''@tD + 2 e^3 w1 u2''@tD + e^3 u3''@tD == 0
```

```
eqEps = TableACoefficientAeq33b@@1DD, e^iE == 0, 8i, 3<E
```

```
8u1@tD + u1''@tD == 0, a2 u1@tD^2 + u2@tD + 2 w1 u1''@tD + u2''@tD == 0,
 a3 u1@tD^3 + 2 a2 u1@tD u2@tD + u3@tD + w1^2 u1''@tD + 2 w2 u1''@tD + 2 w1 u2''@tD + u3''@tD == 0<
```

The general solution of the first-order equation, **eqEps[[1]]**, can be expressed as

```
sol1 = DSolve@8eqEps@@1DD, u1@0D == a Cos@bD, u1'@0D == -a Sin@bD<, u1, tD@@1DD
```

```
8u1 ® Ha Cos@bD Cos@#1D - a Sin@bD Sin@#1D &L<
```

where *a* and b are constants. Instead of using u_1@tD, we use u_1 in the second argument to **DSolve** so that **sol1** can be used in the terms involved with derivatives.

Substituting **sol1**, u_1@tD, into the second-order equation, **eqEps[[2]]**, moving the nonhomogeneous terms to the right-hand side of the resulting equation, we obtain

```
eq33c = u2@tD + u2''@tD - # & •ž Reverse@eqEps@@2DD •. sol1 •• SimplifyD
```

```
u2@tD + u2''@tD == -a^2 Cos@b + tD^2 a2 + 2 a Cos@b + tD w1
```

Expanding the right-hand side of **eq33c** in a Fourier series using trigonometric identities yields

```
eq33c@@2DD = eq33c@@2DD •• TrigReduce
```

```
1/2 H-a^2 a2 - a^2 Cos@2 b + 2 tD a2 + 4 a Cos@b + tD w1L
```

Eliminating the terms, Cos@b + tD and Sin@b + tD, demands that $w_1 = 0$. Then, the particular solution of **eq33c** can be expressed as

```
sol2 = DSolve@eq33c •. w1 -> 0, u2@tD, tD@@1DD •. C@_D -> 0 •• Simplify
```

```
9u2@tD ® 1/6 a^2 H-3 + Cos@2 Hb + tLDL a2=
```

or in a pure function format as

```
sol21 = u2 -> Function@t, sol2@@1, 2DD •• EvaluateD
```

```
u2 ® FunctionAt, 1/6 a^2 H-3 + Cos@2 Hb + tLDL a2E
```

Substituting **sol1** and **sol21** into the third-order equation, **eqEps[[3]]**, and using the fact that $w_1 = 0$, we obtain

```
eq33d = u3[t] + u3''[t] - # & /@ Reverse[eqEps[[3]] /. ω1 -> 0 /. sol1 /. sol21 // Simplify]
```

$$u_3[\tau] + u_3''[\tau] == -\frac{1}{3}\alpha^3 \mathrm{Cos}[\beta+\tau](-3+\mathrm{Cos}[2(\beta+\tau)])\alpha_2^2 - \alpha^3\mathrm{Cos}[\beta+\tau]^3\alpha_3 + 2\alpha\,\mathrm{Cos}[\beta+\tau]\,\omega_2$$

Expanding the right-hand side of **eq33d** in a Fourier series using trigonometric identities, we have

```
eq33d[[2]] = eq33d[[2]] // TrigReduce
```

$$\frac{1}{12}(10\alpha^3\mathrm{Cos}[\beta+\tau]\alpha_2^2 - 2\alpha^3\mathrm{Cos}[3\beta+3\tau]\alpha_2^2 - 9\alpha^3\mathrm{Cos}[\beta+\tau]\alpha_3 - 3\alpha^3\mathrm{Cos}[3\beta+3\tau]\alpha_3 + 24\alpha\,\mathrm{Cos}[\beta+\tau]\,\omega_2)$$

Eliminating the terms that lead to secular terms from **eq33d[[2]]** demands that

```
omg2Rule = Solve[Coefficient[eq33d[[2]], Cos[τ + β]] == 0, ω2][[1]] // ExpandAll
```

$$\left\{\omega_2 \to -\frac{5}{12}\alpha^2\alpha_2^2 + \frac{3\alpha^2\alpha_3}{8}\right\}$$

As discussed above, for a second-order uniform expansion, we do not need to solve for $u_3[\tau]$. Combining the first- and second-order solutions, we obtain, to the second approximation, that

```
u[t] == (u[t] /. uExpRule[2] /. sol1 /. sol21 // Simplify // Expand)
```

$$u[t] == \alpha\,\epsilon\,\mathrm{Cos}[\beta+\tau] - \frac{1}{2}\alpha^2\epsilon^2\alpha_2 + \frac{1}{6}\alpha^2\epsilon^2\mathrm{Cos}[2(\beta+\tau)]\alpha_2 + \epsilon^3 u_3[\tau]$$

where

```
τ == ω t /. omgRule[2] /. omg2Rule /. ω1 -> 0
```

$$\tau == t\left(1+\epsilon^2\left(-\frac{5}{12}\alpha^2\alpha_2^2 + \frac{3\alpha^2\alpha_3}{8}\right)\right)$$

The above expansion is uniform to second order because secular terms do not appear in it and the correction term (the term proportional to ϵ^2) is small compared with the first term.

Returning to **sol** in the previous section, we note that the first secular term appears at $O(\epsilon^3)$. Consequently, we could have concluded that $\omega_1 = 0$ before carrying out the expansion because the term $\epsilon\omega_1$ in **omgRule[2]** creates secular terms at $O(\epsilon^2)$ and not at $O(\epsilon^3)$, as needed to eliminate the secular term from **sol**.

■ 3.4 The Method of Multiple Scales

We use the method of multiple scales to attack directly **eq31g** in Section 3.4.1, the corresponding first-order real-valued equations of **eq31g** in Section 3.4.2, and the corresponding first-order complex-valued equation of **eq31g** in Section 3.4.3. To obtain a second-order uniform expansion by using the method of multiple scales, we need the three time scales $T_0 = t$, $T_1 = \epsilon t$, and $T_2 = \epsilon^2 t$, which we symbolize by

```
Needs["Utilities`Notation`"]
```

```
Symbolize@T0D; Symbolize@T1D; Symbolize@T2D;
```

and list as

```
timeScales = 8T0, T1, T2<;
```

In terms of the time scales T_n, the time derivatives become

```
dt@1D@expr_D := SumAe^i D@expr, timeScales@@i + 1DDD, 8i, 0, maxOrder<E;
dt@2D@expr_D := Hdt@1D@dt@1D@exprDD •• ExpandL •. e^(i_•;i>maxOrder) -> 0;
```

For a uniform second-order expansion, we let

```
maxOrder = 2;
```

To represent some of the expressions in a more concise way, we introduce the following display rule:

```
displayRule =
  9Derivative@a__DAu_i_E@__D :> SequenceFormATimes žž MapIndexedAD#1#2@@1DD-1 &, 8a<E, uiE,
   Derivative@a__D@AD@__D :> SequenceFormATimes žž MapIndexedAD#1#2@@1DD &, 8a<E, AE,
   Derivative@a__D@ȦD@__D :> SequenceFormATimes žž MapIndexedAD#1#2@@1DD &, 8a<E, ȦE,
   u_i_@__D -> ui, A@__D -> A, Ȧ@__D -> Ȧ=;
```

3.4.1 Second-Order Real-Valued System

Using the derivative rule, we transform **eq31g** into the partial-differential equation

```
eq341a = eq31g •. 8u@tD -> u@T0, T1, T2D, Derivative@n_D@uD@tD -> dt@nD@u@T0, T1, T2DD<
```

$$u[T_0, T_1, T_2] + a_2\, u[T_0, T_1, T_2]^2 + a_3\, u[T_0, T_1, T_2]^3 + e^2\, u^{(0,2,0)}[T_0, T_1, T_2] + 2\, e^2\, u^{(1,0,1)}[T_0, T_1, T_2] + 2\, e\, u^{(1,1,0)}[T_0, T_1, T_2] + u^{(2,0,0)}[T_0, T_1, T_2] == 0$$

Again, to determine a uniform expansion of order maxOrder, we expand u to order maxOrder+1, eliminate the secular terms from the maxOrder+1 equation, but do not include its solution in the final approximate solution. Hence, we seek a uniform expansion of the solution of **eq341a** in the form

```
solRule = u -> ISumAe^i ui@#1, #2, #3D, 8i, maxOrder + 1<E &M;
```

Substituting this expansion into **eq341a**, expanding the result for small e, and discarding terms of order higher than e^3, we have

```
eq341b = Heq341a •. solRule •• ExpandAllL •. e^(n_•;n>maxOrder+1) -> 0
```

$$e\, u_1[T_0, T_1, T_2] + e^2\, a_2\, u_1[T_0, T_1, T_2]^2 + e^3\, a_3\, u_1[T_0, T_1, T_2]^3 + e^2\, u_2[T_0, T_1, T_2] + 2\, e^3\, a_2\, u_1[T_0, T_1, T_2]\, u_2[T_0, T_1, T_2] + e^3\, u_3[T_0, T_1, T_2] + e^3\, u_1^{(0,2,0)}[T_0, T_1, T_2] + 2\, e^3\, u_1^{(1,0,1)}[T_0, T_1, T_2] + 2\, e^2\, u_1^{(1,1,0)}[T_0, T_1, T_2] + 2\, e^3\, u_2^{(1,1,0)}[T_0, T_1, T_2] + e\, u_1^{(2,0,0)}[T_0, T_1, T_2] + e^2\, u_2^{(2,0,0)}[T_0, T_1, T_2] + e^3\, u_3^{(2,0,0)}[T_0, T_1, T_2] == 0$$

Equating coefficients of like powers of e in **eq341b**, we obtain

```
eqEps = CoefficientList@eq341b@@1DD, eD == 0 •• Thread •• Rest
```

```
9u1@T0, T1, T2D + u1^H2,0,0L@T0, T1, T2D == 0,
 a2 u1@T0, T1, T2D^2 + u2@T0, T1, T2D + 2 u1^H1,1,0L@T0, T1, T2D + u2^H2,0,0L@T0, T1, T2D == 0,
 a3 u1@T0, T1, T2D^3 + 2 a2 u1@T0, T1, T2D u2@T0, T1, T2D + u3@T0, T1, T2D + u1^H0,2,0L@T0, T1, T2D +
   2 u1^H1,0,1L@T0, T1, T2D + 2 u2^H1,1,0L@T0, T1, T2D + u3^H2,0,0L@T0, T1, T2D == 0=
```

To place the linear operator on one side and the rest of the terms on the other, we define

```
eqOrder@i_D := HeqEps@@1, 1DD •. u1 -> uiL - # & •ž eqEps@@iDD •• Reverse
```

Using **eqOrder[i]** and **displayRule**, we rewrite **eqEps** in a concise way as

```
Array@eqOrder, 3D •. displayRule •• TableForm
```

$D_0^2 u_1 + u_1 == 0$

$D_0^2 u_2 + u_2 == -2\,(D_0\,D_1 u_1) - u_1^2\,a_2$

$D_0^2 u_3 + u_3 == -2\,(D_0\,D_1 u_2) - D_1^2 u_1 - 2\,(D_0\,D_2 u_1) - 2\,u_1\,u_2\,a_2 - u_1^3\,a_3$

where $D_n\,u_i = \partial u_i[T_0, T_1, T_2]/\partial T_n$.

The general solution of the first-order equation, **eqOrder[1]**, can be expressed as

```
sol1 = 9u1 -> IA@#2, #3D Exp@I #1D + Ā@#2, #3D Exp@-I #1D &M=;
```

where $\bar{A}$ is the complex conjugate of A defined by the conjugate rule

```
conjugateRule = 9A -> Ā, Ā -> A, Complex@0, n_D -> Complex@0, -nD=;
```

Then, the second-order equation, **eqOrder[2]**, becomes

```
eq341c = eqOrder@2D •. sol1 •• ExpandAll;
eq341c •. displayRule
```

$D_0^2 u_2 + u_2 == -2\,I\,E^{I\,T_0}\,(D_1 A) + 2\,I\,E^{-I\,T_0}\,(D_1 \bar{A}) - A^2\,E^{2\,I\,T_0}\,a_2 - 2\,A\,\bar{A}\,a_2 - E^{-2\,I\,T_0}\,\bar{A}^2\,a_2$

Eliminating the term $E^{I\,T_0}$ that produces secular terms in u_2 from the right-hand side of **eq341c** demands that

```
SCond1 = Coefficient@eq341c@@2DD, Exp@I T0DD == 0
```

$-2\,I\,A^{(1,0)}[T_1, T_2] == 0$

or

```
SCond1Rule = SolveASCond1, A^H1,0L@T1, T2DE@@1DD
H* either copy and paste A^H1,0L@T1,T2D from SCond1 or
  use Derivative@1,0D@AD@T1,T2D as the second argument to Solve *L
```

```
8A^H1,0L@T1, T2D ® 0<
```

Eliminating the term $E^{-I\,T_0}$ that produces secular terms in u_2 from the right-hand side of **eq341c** yields the complex conjugate of **SCond1Rule**; that is,

```
ccSCond1Rule = SCond1Rule •. conjugateRule
```

```
:Ā^H1,0L@T1, T2D ® 0>
```

It follows from **SCond1** that $A = A@T_2D$. Substituting the solvability conditions into **eq341c**, we have

```
eq341d = eq341c •. SCond1Rule •. ccSCond1Rule
```

```
u2@T0, T1, T2D + u2^H2,0,0L@T0, T1, T2D ==
 - E^2 I T0 A@T1, T2D^2 a2 - 2 A@T1, T2D a2 Ā@T1, T2D - E^-2 I T0 a2 Ā@T1, T2D^2
```

The particular solution of **eq341d** can be obtained by using **DSolve**; the result is

```
u2Sol =
 DSolve@eq341d, u2@T0, T1, T2D, timeScalesD@@1DD •. C@_D -> H0 &L •• TrigToExp •• ExpandAll
```

```
9u2@T0, T1, T2D ® 1/3 E^2 I T0 A@T1, T2D^2 a2 - 2 A@T1, T2D a2 Ā@T1, T2D + 1/3 E^-2 I T0 a2 Ā@T1, T2D^2=
```

whose right-hand side can be used directly to express u_2 in a pure function form as

```
sol2 = 8u2 -> Function@8T0, T1, T2<, u2Sol@@1, 2DD •• EvaluateD<
```

```
9u2 ®
  FunctionA8T0, T1, T2<, 1/3 E^2 I T0 A@T1, T2D^2 a2 - 2 A@T1, T2D a2 Ā@T1, T2D + 1/3 E^-2 I T0 a2 Ā@T1, T2D^2E=
```

Substituting the first- and second-order solutions, **sol1** and **sol2**, into the third-order equation, **eqOrder[3]**, we obtain

```
eq341e = eqOrder@3D •. sol1 •. sol2 •• ExpandAll;
eq341e •. displayRule
```

```
D0^2 u3 + u3 == - E^I T0 HD1^2 AL - E^-I T0 HD1^2 ĀL - 2 I E^I T0 HD2 AL + 2 I E^-I T0 HD2 ĀL -
  8/3 I A E^2 I T0 HD1 AL a2 + 8/3 I E^-2 I T0 Ā HD1 ĀL a2 - 2/3 A^3 E^3 I T0 a2^2 + 10/3 A^2 E^I T0 Ā a2^2 +
  10/3 A E^-I T0 Ā^2 a2^2 - 2/3 E^-3 I T0 Ā^3 a2^2 - A^3 E^3 I T0 a3 - 3 A^2 E^I T0 Ā a3 - 3 A E^-I T0 Ā^2 a3 - E^-3 I T0 Ā^3 a3
```

Eliminating the terms that produce secular terms in u_3 from the right-hand side of **eq341e** demands that

```
SCond2 = Coefficient@eq341e@@2DD, Exp@I T0DD == 0
```

$\frac{10}{3} A@T_1, T_2D^2 a_2^2 \dot{A}@T_1, T_2D - 3 A@T_1, T_2D^2 a_3 \dot{A}@T_1, T_2D - 2 I A^{H0,1L}@T_1, T_2D - A^{H2,0L}@T_1, T_2D == 0$

Using **SCond1Rule** and the fact that $A = A@T_2D$, we can rewrite **SCond2** as

```
SCond = SCond2 •. D@SCond1Rule, T1D •. 9A -> HA@#2D &L, Ȧ -> IȦ@#2D &M=
```

$\frac{10}{3} A@T_2D^2 a_2^2 \dot{A}@T_2D - 3 A@T_2D^2 a_3 \dot{A}@T_2D - 2 I A^{¢}@T_2D == 0$

Expressing A in the polar form

```
ruleA = 9A -> (1/2 a@#D Exp@I b@#DD &), Ȧ -> (1/2 a@#D Exp@-I b@#DD &)=;
```

where a and b are real and using the rule

```
realRule = 8Re@s_D -> s, Im@s_D -> 0<;
```

we write **SCond** as

```
eq341f = SCond@@1DD Exp@-I b@T2DD •. ruleA •• Expand
```

$\frac{5}{12} a@T_2D^3 a_2^2 - \frac{3}{8} a@T_2D^3 a_3 - I a^{¢}@T_2D + a@T_2D b^{¢}@T_2D$

Separating the real and imaginary parts in **eq341f**, we obtain the modulation equations

```
ampEq = Im@eq341fD == 0 •. realRule
```

$- a^{¢}@T_2D == 0$

```
phaseEq = Re@eq341fD == 0 •. realRule
```

$\frac{5}{12} a@T_2D^3 a_2^2 - \frac{3}{8} a@T_2D^3 a_3 + a@T_2D b^{¢}@T_2D == 0$

The solution of **ampEq** is $a = a_0$ = const. Then, if $a_0 \neq 0$, the solution of **phaseEq** is

```
betaRule = DSolve@phaseEq •. a@T2D -> a0, b@T2D, T2D@@1DD •. C@1D -> b0 •• ExpandAll
```

$9b@T_2D \to -\frac{5}{12} T_2 a_0^2 a_2^2 + \frac{3}{8} T_2 a_0^2 a_3 + b_0=$

where b_0 is a constant. Substituting **ruleA** into **sol1** and **sol2** and recalling that $T_0 = t$ and $T_2 = \epsilon^2 t$, we obtain, to the second approximation, that

```
u@tD == IIu@T0, T1, T2D •. solRule •. sol1 •. sol2 •. e^3 -> 0 •. 9A@__D -> A@T2D, Ȧ@__D -> Ȧ@T2D= •.
        ruleA •• ExpToTrigM •. 8a@T2D -> a0< •. betaRule •.
    9T0 -> t, T2 -> e^2 t= •. Cos@arg_D :> Cos@Collect@arg, tDDM
```

$$u[t] == \epsilon \, \mathrm{Cos}\left[t\left(1 - \frac{5}{12}\epsilon^2 a_0^2 \alpha_2^2 + \frac{3}{8}\epsilon^2 a_0^2 \alpha_3\right) + \beta_0\right] a_0 -$$
$$\frac{1}{2}\epsilon^2 a_0^2 \alpha_2 + \frac{1}{6}\epsilon^2 \mathrm{Cos}\left[t\left(2 - \frac{5}{6}\epsilon^2 a_0^2 \alpha_2^2 + \frac{3}{4}\epsilon^2 a_0^2 \alpha_3\right) + 2\beta_0\right] a_0^2 \alpha_2$$

which is in full agreement with that obtained by using the Lindstedt-Poincaré technique.

Ÿ 3.4.2 First-Order Real-Valued System

In this section, we first transform **eq31g** into a system of two real-valued first-order equations using the transformation $v[t] = u'[t]$ and obtain

```
eq342a = 8u'@tD == v@tD, eq31g •. u''@tD -> v'@tD<
```

$$\{u'[t] == v[t],\ u[t] + \alpha_2 u[t]^2 + \alpha_3 u[t]^3 + v'[t] == 0\}$$

To determine a second-order uniform expansion of the solution of **eq342a** using the method of multiple scales, we first introduce

```
multiScales = 8u@tD -> u@T0, T1, T2D, v@tD -> v@T0, T1, T2D,
  u'@tD -> dt@1D@u@T0, T1, T2DD, v'@tD -> dt@1D@v@T0, T1, T2DD<;
```

to transform the derivative with respect to t in terms of the derivatives with respect to the three time scales T_0, T_1, and T_2 and then expand u and v in the form

```
solRule = 9u -> ISumAe^j uj@#1, #2, #3D, 8j, 3<E &M, v -> ISumAe^j vj@#1, #2, #3D, 8j, 3<E &M=;
```

Substituting **multiScales** and **solRule** into **eq342a**, expanding the result for small ϵ, and discarding terms of order higher than ϵ^3, we obtain

```
eq342b = Heq342a •. multiScales •. solRule •• ExpandAllL •. e^n_•;n>3 -> 0;
```

Equating coefficients of like powers of ϵ in **eq342b** yields

```
eqEps = Thread@CoefficientList@Subtract žž #, eD == 0D & •ž eq342b •• Transpose •• Rest;
```

To place the linear operator on one side and the rest of the terms on the other, we define

```
eqOrder@i_D := H#@@1DD & •ž eqEps@@1DD •. u_1 -> uiL ==
   H#@@1DD & •ž eqEps@@1DD •. u_1 -> uiL - H#@@1DD & •ž eqEps@@iDDL •• Thread
```

Using **eqOrder[i]** and **displayRule**, we rewrite **eqEps** in a concise way as

```
eqOrder@1D •. displayRule •• TableForm
eqOrder@2D •. displayRule •• TableForm
eqOrder@3D •. displayRule •• TableForm
```

$D_0u_1 - v_1 == 0$
$D_0v_1 + u_1 == 0$

$D_0u_2 - v_2 == -HD_1u_1L$
$D_0v_2 + u_2 == -HD_1v_1L - u_1^2\, a_2$

$D_0u_3 - v_3 == -HD_1u_2L - D_2u_1$
$D_0v_3 + u_3 == -HD_1v_2L - D_2v_1 - 2\, u_1\, u_2\, a_2 - u_1^3\, a_3$

Ÿ First-Order Equations: Linear System

To determine the solution of the first-order equations, **eqOrder[1]**, we list their left-hand sides and obtain

```
linearSys = #@@1DD & •ž eqOrder@1D
```

$9 - v_1@T_0,\ T_1,\ T_2D + u_1^{H1,0,0L}@T_0,\ T_1,\ T_2D,\ u_1@T_0,\ T_1,\ T_2D + v_1^{H1,0,0L}@T_0,\ T_1,\ T_2D=$

Next, we seek a solution of **linearSys** in the form $u_1 = P E^{I\, T_0}$ and $v_1 = Q E^{I\, T_0}$ and obtain

```
coefList = E^(-I T0) linearSys •. 9u1 -> IP E^(I #) &M, v1 -> IQ E^(I #) &M= •• Expand
```

8I P - Q, P + I Q<

The coefficient matrix of **coefList** is

```
coefMat = Outer@Coefficient, coefList, 8P, Q<D
```

88I, -1<, 81, I<<

and its adjoint is defined by

```
hermitian@mat_? MatrixQD := mat •. conjugateRule •• Transpose
```

Hence, the right and left eigenvectors of **coefMat** are

```
rightVec = 81, c1< •. Solve@HcoefMat.81, c1<L@@1DD == 0, c1D@@1DD
```

81, I<

```
leftVec = 81, c1< •. Solve@Hhermitian@coefMatD.81, c1<L@@1DD == 0, c1D@@1DD
```

81, I<

Then the solution of the first-order equations can be expressed as

```
sol1Form = rightVec A@T1, T2D E^(I T0)
```

```
8E^(I T0) A@T1, T2D, I E^(I T0) A@T1, T2D<
```

and hence u_1 and v_1 can be expressed in a pure function form as

```
order1Sol =
 8u1 -> Function@8T0, T1, T2<, sol1Form@@1DD + Hsol1Form@@1DD •. conjugateRuleL •• EvaluateD,
  v1 -> Function@8T0, T1, T2<, sol1Form@@2DD + Hsol1Form@@2DD •. conjugateRuleL •• EvaluateD<
```

```
8u1 ® Function@8T0, T1, T2<, E^(I T0) A@T1, T2D + E^(-I T0) Ȧ@T1, T2DD,
 v1 ® Function@8T0, T1, T2<, I E^(I T0) A@T1, T2D - I E^(-I T0) Ȧ@T1, T2DD<
```

Ÿ Second-Order Equations

Substituting the first-order solution, **order1Sol**, into the second-order equations, **eqOrder[2]**, we have

```
order2Eq = eqOrder@2D •. order1Sol •• ExpandAll;
order2Eq •. displayRule
```

```
:D0u2 - v2 == -E^(I T0) HD1AL - E^(-I T0) HD1ȦL,
 D0v2 + u2 == -I E^(I T0) HD1AL + I E^(-I T0) HD1ȦL - A^2 E^(2 I T0) a2 - 2 A Ȧ a2 - E^(-2 I T0) Ȧ^2 a2>
```

To eliminate the terms that produce secular terms in u_2 and v_2, we determine the vector proportional to $E^{I\,T_0}$ in the right-hand sides of **order2Eq** and obtain

```
STerms1 = Coefficient@#@@2DD, Exp@I T0DD & •ž order2Eq
```

```
8-A^(H1,0L)@T1, T2D, -I A^(H1,0L)@T1, T2D<
```

Then, the condition for the elimination of the terms that produce secular terms, solvability condition, demands that **STerms1** be orthogonal to the adjoint, left eigenvector. The result is

```
SCond1 = SolveAConjugate@leftVecD.STerms1 == 0, A^(H1,0L)@T1, T2DE@@1DD
```

```
8A^(H1,0L)@T1, T2D ® 0<
```

whose complex conjugate is

```
ccSCond1 = SCond1 •. conjugateRule
```

```
:Ȧ^(H1,0L)@T1, T2D ® 0>
```

With this solvability condition, **order2Eq** becomes

```
order2Eqm = order2Eq •. SCond1 •. ccSCond1;
order2Eqm •. displayRule
```

```
:D0u2 - v2 == 0, D0v2 + u2 == -A^2 E^(2 I T0) a2 - 2 A Ȧ a2 - E^(-2 I T0) Ȧ^2 a2>
```

whose particular solution can be expressed as

```
DSolve@order2Eqm, 8u2@T0, T1, T2D, v2@T0, T1, T2D<, timeScalesD
```

```
DSolve::pde : Partial differential equation may not have
   a general solution. Try loading Calculus`DSolveIntegrals` to find special solutions.
```

```
DSolveB:-v2@T0, T1, T2D + u2^H1,0,0L@T0, T1, T2D == 0, u2@T0, T1, T2D + v2^H1,0,0L@T0, T1, T2D ==
   -E^(2 I T0) A@T1, T2D^2 a2 - 2 A@T1, T2D a2 Ȧ@T1, T2D - E^(-2 I T0) a2 Ȧ@T1, T2D^2>,
 8u2@T0, T1, T2D, v2@T0, T1, T2D<, 8T0, T1, T2<F
```

Here, directly applying **DSolve** to solve for **order2Eqm** does not work. Since the differential operators on the left-hand sides of **order2Eqm** are essentially ordinary-differential operators in terms of T_0, we can first transform **order2Eqm** into an ordinary-differential-equation form and then use **DSolve** to obtain the particular solution as

```
sol2pForm =
 DSolve@order2Eqm •. 8u2 -> Hu2@#1D &L, v2 -> Hv2@#1D &L<, 8u2@T0D, v2@T0D<, T0D@@1DD •.
   C@_D -> 0 •• TrigToExp •• ExpandAll
```

```
9u2@T0D ® 1/3 E^(2 I T0) A@T1, T2D^2 a2 - 2 A@T1, T2D a2 Ȧ@T1, T2D + 1/3 E^(-2 I T0) a2 Ȧ@T1, T2D^2,
 v2@T0D ® 2/3 I E^(2 I T0) A@T1, T2D^2 a2 - 2/3 I E^(-2 I T0) a2 Ȧ@T1, T2D^2=
```

We can then express the solution of the second-order equations in a pure function form as

```
order2Sol = 8u2 -> Function@8T0, T1, T2<, sol2pForm@@1, 2DD •• EvaluateD,
  v2 -> Function@8T0, T1, T2<, sol2pForm@@2, 2DD •• EvaluateD<
```

```
9u2 ®
  FunctionA8T0, T1, T2<, 1/3 E^(2 I T0) A@T1, T2D^2 a2 - 2 A@T1, T2D a2 Ȧ@T1, T2D + 1/3 E^(-2 I T0) a2 Ȧ@T1, T2D^2E,
 v2 ® FunctionA8T0, T1, T2<, 2/3 I E^(2 I T0) A@T1, T2D^2 a2 - 2/3 I E^(-2 I T0) a2 Ȧ@T1, T2D^2E=
```

Ÿ Third-Order Equations

Substituting the first- and second-order solutions into the third-order equations, **eqOrder[3]**, yields

```
order3Eq = eqOrder@3D •. order1Sol •. order2Sol •• ExpandAll;
order3Eq •. displayRule
```

$$9D_0u_3 - v_3 == -E^{I\,T_0}\,HD_2AL - E^{-I\,T_0}\,HD_2\dot{A}L - \frac{2}{3}\,A\,E^{2\,I\,T_0}\,HD_1AL\,a_2 + 2\,\dot{A}\,HD_1AL\,a_2 + 2\,A\,HD_1\dot{A}L\,a_2 - \frac{2}{3}\,E^{-2\,I\,T_0}\,\dot{A}\,HD_1\dot{A}L\,a_2,$$

$$D_0v_3 + u_3 == -I\,E^{I\,T_0}\,HD_2AL + I\,E^{-I\,T_0}\,HD_2\dot{A}L - \frac{4}{3}\,I\,A\,E^{2\,I\,T_0}\,HD_1AL\,a_2 + \frac{4}{3}\,I\,E^{-2\,I\,T_0}\,\dot{A}\,HD_1\dot{A}L\,a_2 - \frac{2}{3}\,A^3\,E^{3\,I\,T_0}\,a_2^2 + \frac{10}{3}\,A^2\,E^{I\,T_0}\,\dot{A}\,a_2^2 + \frac{10}{3}\,A\,E^{-I\,T_0}\,\dot{A}^2\,a_2^2 - \frac{2}{3}\,E^{-3\,I\,T_0}\,\dot{A}^3\,a_2^2 - A^3\,E^{3\,I\,T_0}\,a_3 - 3\,A^2\,E^{I\,T_0}\,\dot{A}\,a_3 - 3\,A\,E^{-I\,T_0}\,\dot{A}^2\,a_3 - E^{-3\,I\,T_0}\,\dot{A}^3\,a_3=$$

To eliminate the terms that lead to secular terms from **order3Eq**, we calculate the vector proportional to $E^{I\,T_0}$ in their right-hand sides and obtain

```
STerms2 = Coefficient@#@@2DD, Exp@I T0DD & •ž order3Eq
```

$$9-A^{H0,1L}@T_1,\ T_2D,\ \frac{10}{3}\,A@T_1,\ T_2D^2\,a_2^2\,\dot{A}@T_1,\ T_2D - 3\,A@T_1,\ T_2D^2\,a_3\,\dot{A}@T_1,\ T_2D - I\,A^{H0,1L}@T_1,\ T_2D=$$

Hence, the solvability condition at this order is

```
SCond2 = Conjugate@leftVecD.STerms2 == 0 •• ExpandAll
```

$$-\frac{10}{3}\,I\,A@T_1,\ T_2D^2\,a_2^2\,\dot{A}@T_1,\ T_2D + 3\,I\,A@T_1,\ T_2D^2\,a_3\,\dot{A}@T_1,\ T_2D - 2\,A^{H0,1L}@T_1,\ T_2D == 0$$

This solvability condition is the same as that obtained by attacking the second-order form of the governing equation.

Ÿ 3.4.3 First-Order Complex-Valued System

In this section, we first transform **eq31g** into a single first-order complex-valued equation using the transformation

```
transfRule = 9u@tD -> z@tD + ż@tD, u¢@tD -> I Iz@tD - ż@tDM=;
```

whose inverse is given by

```
zetaRule = SolveAtransfRule •. Rule -> Equal, 9z@tD, ż@tD=E@@1DD
```

$$9z@tD \circledR \frac{1}{2}\,Hu@tD - I\,u^{¢}@tDL,\ \dot{z}@tD \circledR \frac{1}{2}\,Hu@tD + I\,u^{¢}@tDL=$$

where $\bar{z}$ is the complex conjugate of z as defined by the complex conjugate rule

```
conjugateRule = 9A -> Ȧ, Ȧ -> A, z -> ż, ż -> z, Complex@0, n_D -> Complex@0, -nD=;
```

It follows from **eq31g** that the acceleration is given by

```
acceleration = Solve@eq31g, u''@tDD@@1DD
```

```
8u''@tD ® -u@tD - a2 u@tD^2 - a3 u@tD^3<
```

Differentiating **zetaRule[[1]]** once with respect to *t* and using **acceleration** and **transfRule**, we obtain the following first-order complex-valued equation:

```
eq343a = D@zetaRule@@1DD, tD •. acceleration •. transfRule •. Rule -> Equal •• ExpandAll
```

$$z'[t] == I\,z[t] + \frac{1}{2} I\,a_2\,z[t]^2 + \frac{1}{2} I\,a_3\,z[t]^3 + I\,a_2\,z[t]\,\bar{z}[t] + \frac{3}{2} I\,a_3\,z[t]^2\,\bar{z}[t] + \frac{1}{2} I\,a_2\,\bar{z}[t]^2 + \frac{3}{2} I\,a_3\,z[t]\,\bar{z}[t]^2 + \frac{1}{2} I\,a_3\,\bar{z}[t]^3$$

To determine a second-order uniform expansion of the solution of **eq343a** using the method of multiple scales, we first introduce

```
multiScales = 9z@tD -> z@T0, T1, T2D, ż@tD -> ż@T0, T1, T2D, z'@tD -> dt@1D@z@T0, T1, T2DD=;
```

and then expand z and z̈ in the form

```
solRule = 9z -> ISumAe^j zj@#1, #2, #3D, 8j, 3<E &M, ż -> ISumAe^j żj@#1, #2, #3D, 8j, 3<E &M=;
```

Substituting **multiScales** and **solRule** into **eq343a**, expanding the result of small e, and discarding terms of order higher than e^3, we obtain

```
eq343b = Heq343a •. multiScales •. solRule •• ExpandAllL •. e^n_•;n>3 -> 0;
```

Equating coefficients of like powers of e yields

```
eqEps = CoefficientList@Subtract žž eq343b, eD == 0 •• Thread •• Rest;
```

To place the linear operator on one side and the rest of the terms on the other, we define

```
eqOrder@i_D := HeqEps@@1, 1DD •. z1 -> ziL - # & •ž eqEps@@iDD •• Reverse
```

Using **eqOrder[i]** and **displayRule**, we rewrite **eqEps** in a concise way as

```
Array@eqOrder, 3D •. displayRule •• TableForm
```

$$D_0 z_1 - I\,z_1 == 0$$

$$D_0 z_2 - I\,z_2 == -(D_1 z_1) + \frac{1}{2} I\,a_2\,z_1^2 + I\,a_2\,z_1\,\dot{z}_1 + \frac{1}{2} I\,a_2\,\dot{z}_1^2$$

$$D_0 z_3 - I\,z_3 == -(D_1 z_2) - D_2 z_1 + \frac{1}{2} I\,a_3\,z_1^3 + I\,a_2\,z_1\,z_2 + \frac{3}{2} I\,a_3\,z_1^2\,\dot{z}_1 + I\,a_2\,z_2\,\dot{z}_1 + \frac{3}{2} I\,a_3\,z_1\,\dot{z}_1^2 + \frac{1}{2} I\,a_3\,\dot{z}_1^3 + I\,a$$

The general solution of the first-order equation, **eqOrder[1]**, can be expressed in a pure function form as

```
sol1 = DSolve@eqOrder@1D, z1, timeScalesD@@1DD •. C@1D -> A
```

```
8z1 ® HE^(I #1) A@#2, #3D &L<
```

whose complex conjugate is

```
ccsol1 = sol1 •. conjugateRule
```

```
8ż1 ® HEComplex@0,-1D #1 Ȧ@#2, #3D &L<
```

Substituting the first-order solution into the second-order equation, **eqOrder[2]**, we have

```
order2Eq = eqOrder@2D •. sol1 •. ccsol1;
order2Eq •. displayRule
```

$$D_0 z_2 - I\, z_2 == -E^{I T_0}\, HD_1 AL + \frac{1}{2}\, I\, A^2\, E^{2 I T_0}\, a_2 + I\, A\, \dot{A}\, a_2 + \frac{1}{2}\, I\, E^{-2 I T_0}\, \dot{A}^2\, a_2$$

Eliminating the terms that lead to seculr terms in z_2 from **order2Eq** yields

```
SCond1 = CoefficientAorder2Eq@@2DD, E^(I T0) E == 0
```

$$-A^{H1,0L}@T_1,\ T_2D == 0$$

or

```
SCond1Rule = SolveASCond1, A^(H1,0L)@T1, T2DE@@1DD
```

$$8A^{H1,0L}@T_1,\ T_2D ® 0<$$

With this solvability condition, the particular solution of **order2Eq** can be expressed in a pure function form as

```
sol2 = DSolve@order2Eq •. SCond1Rule, z2, timeScalesD@@1DD •. C@_D -> H0 &L
```

$$9z_2 ® \Big(E^{I \#1}\, H0\ \&L@\#2,\ \#3D + \frac{1}{6}\, E^{-2 I \#1}\, a_2\, J3\, E^{4 I \#1}\, A@\#2,\ \#3D^2 - 6\, E^{2 I \#1}\, A@\#2,\ \#3D\, \dot{A}@\#2,\ \#3D - \dot{A}@\#2,\ \#3D^2 N\ \&\Big)=$$

whose complex conjugate is

```
ccsol2 = sol2 •. conjugateRule;
```

Substituting the first- and second-order solutions in the third-order equation, **eqOrder[3]**, we obtain

```
order3Eq = eqOrder@3D •. sol1 •. ccsol1 •. sol2 •. ccsol2 •• ExpandAll;
order3Eq •. displayRule
```

$$D_0 z_3 - I\, z_3 == -E^{I T_0}\, HD_2 AL - A\, E^{2 I T_0}\, HD_1 AL\, a_2 + \dot{A}\, HD_1 AL\, a_2 + A\, HD_1 \dot{A}L\, a_2 + \frac{1}{3}\, E^{-2 I T_0}\, \dot{A}\, HD_1 \dot{A}L\, a_2 + \frac{1}{3}\, I\, A^3\, E^{3 I T_0}\, a_2^2 - \frac{5}{3}\, I\, A^2\, E^{I T_0}\, \dot{A}\, a_2^2 - \frac{5}{3}\, I\, A\, E^{-I T_0}\, \dot{A}^2\, a_2^2 + \frac{1}{3}\, I\, E^{-3 I T_0}\, \dot{A}^3\, a_2^2 + \frac{1}{2}\, I\, A^3\, E^{3 I T_0}\, a_3 + \frac{3}{2}\, I\, A^2\, E^{I T_0}\, \dot{A}\, a_3 + \frac{3}{2}\, I\, A\, E^{-I T_0}\, \dot{A}^2\, a_3 + \frac{1}{2}\, I\, E^{-3 I T_0}\, \dot{A}^3\, a_3$$

Eliminating the terms that produce secular terms in z_3 from **order3Eq** demands the following condition

```
SCond2 = CoefficientAorder3Eq@@2DD, E^ITθ E == 0;
SCond2 •. displayRule
```

```
-HD2AL - 5/3 I A^2 Ȧ a2^2 + 3/2 I A^2 Ȧ a3 == 0
```

which is in full agreement with that obtained by attacking directly **eq31g**.

à 3.5 The Method of Averaging

As before, we use the method of variation of parameters to transform the dependent variable *u* into *a* and b according to

```
eq35a = u@tD == e a Cos@t + bD;
```

subject to the constraint

```
eq35b = D@eq35a, tD
```

```
u¢@tD == -a e Sin@t + bD
```

where *a* and b are functions of *t*

```
tdependentRule = 8a -> a@tD, b -> b@tD<;
```

and e is a small dimensionless parameter that is a measure of the amplitude of oscillation. Differentiating **eq35a** with respect to *t* yields

```
eq35c = D@eq35a •. tdependentRule, tD
```

```
u¢@tD == e Cos@t + b@tDD a¢@tD - e a@tD Sin@t + b@tDD H1 + b¢@tDL
```

Comparing **eq35b** and **eq35c**, we conclude that

```
cond35a = Expand@eq35c@@2DD - Heq35b@@2DD •. tdependentRuleLD == 0
```

```
e Cos@t + b@tDD a¢@tD - e a@tD Sin@t + b@tDD b¢@tD == 0
```

Differentiating **eq35b** once with respect to *t* yields

```
eq35d = D@eq35b •. tdependentRule, tD
```

```
u²@tD == -e Sin@t + b@tDD a¢@tD - e a@tD Cos@t + b@tDD H1 + b¢@tDL
```

Substituting **eq35a** and **eq35d** into **eq31g**, we have

```
cond35b = eq31g •. H8eq35a •. tdependentRule, eq35d< •. Equal -> RuleL •• ExpandAll
```

```
e^2 a@tD^2 Cos@t + b@tDD^2 a2 + e^3 a@tD^3 Cos@t + b@tDD^3 a3 -
   e Sin@t + b@tDD a¢@tD - e a@tD Cos@t + b@tDD b¢@tD == 0
```

Solving **cond35a** and **cond35b** for $a'(t)$ and $\beta'(t)$, we have

```
transformedEq = Solve@8cond35a, cond35b<, 8a¢@tD, b¢@tD<D@@1DD •. Rule -> Equal •• Simplify
```

$$\{a'(t) == \epsilon\, a(t)^2 \operatorname{Cos}[t+\beta(t)]^2 \operatorname{Sin}[t+\beta(t)] (\alpha_2 + \epsilon\, a(t) \operatorname{Cos}[t+\beta(t)]\, \alpha_3),$$
$$\beta'(t) == \epsilon\, a(t) \operatorname{Cos}[t+\beta(t)]^3 (\alpha_2 + \epsilon\, a(t) \operatorname{Cos}[t+\beta(t)]\, \alpha_3)\}$$

where a is assumed to be different from zero in arriving at **transformedEq[[2]]**. Using trigonometric identities, we expand the right-hand sides of **transformedEq** in Fourier series and obtain

```
ampEq1 = TrigReduce •ž transformedEq@@1DD
```

$$a'(t) == \frac{1}{8} (2\,\epsilon\, a(t)^2 \operatorname{Sin}[t+\beta(t)]\, \alpha_2 + 2\,\epsilon\, a(t)^2 \operatorname{Sin}[3\,t+3\,\beta(t)]\, \alpha_2 + 2\,\epsilon^2\, a(t)^3 \operatorname{Sin}[2\,t+2\,\beta(t)]\, \alpha_3 + \epsilon^2\, a(t)^3 \operatorname{Sin}[4\,t+4\,\beta(t)]\, \alpha_3)$$

```
phaseEq1 = TrigReduce •ž transformedEq@@2DD
```

$$\beta'(t) == \frac{1}{8} (6\,\epsilon\, a(t) \operatorname{Cos}[t+\beta(t)]\, \alpha_2 + 2\,\epsilon\, a(t) \operatorname{Cos}[3\,t+3\,\beta(t)]\, \alpha_2 + 3\,\epsilon^2\, a(t)^2\, \alpha_3 + 4\,\epsilon^2\, a(t)^2 \operatorname{Cos}[2\,t+2\,\beta(t)]\, \alpha_3 + \epsilon^2\, a(t)^2 \operatorname{Cos}[4\,t+4\,\beta(t)]\, \alpha_3)$$

Since ϵ is small, $a'(t)$ and $\beta'(t)$ are slowly varying functions of t. Then, according to the method of averaging, we can neglect the fast-varying terms in **ampEq1** and **phaseEq1** and obtain the following averaged equations:

```
ampAvgEq = ampEq1 •. Sin@_. t + _D -> 0
```

$$a'(t) == 0$$

```
phaseAvgEq = phaseEq1 •. Cos@_. t + _D -> 0
```

$$\beta'(t) == \frac{3}{8}\,\epsilon^2\, a(t)^2\, \alpha_3$$

Whereas **ampAvgEq** agrees with **ampEq**, **phaseAvgEq** does not agree with **phaseEq** obtained in the preceding section by using the method of multiple scales. There is a term $\frac{5}{12}\,\epsilon^2 a^2 \alpha_2^2$ missing from **phaseAvgEq**. Following the details of the solution in the preceding section, one finds that this term is the result of the interaction of the first- and second-order approximations. This interaction was not taken into account in arriving at **ampAvgEq** and **phaseAvgEq**. To include the effect of this interaction, we need to carry out the solutions of **ampEq1** and **phaseEq1** to higher order. This is accomplished by using the generalized method of averaging, which is discussed next, or its variant the Krylov-Bogoliubov-Mitropolsky technique, which is discussed in Section 3.7.

3.6 The Generalized Method of Averaging

To apply this method, we introduce the variable $\phi = t + \beta$ and rewrite **ampEq1** and **phaseEq1** as

```
ampEq2 = ampEq1 •. Sin@n_. t + a_D :> Sin@Expand@n Hf@tD - b@tDL + aDD
```

```
a¢@tD == 1/8 H2 e a@tD^2 Sin@f@tDD a2 +
    2 e a@tD^2 Sin@3 f@tDD a2 + 2 e^2 a@tD^3 Sin@2 f@tDD a3 + e^2 a@tD^3 Sin@4 f@tDD a3L
```

```
phaseEq2 = f¢@tD == 1 + phaseEq1@@2DD •. Cos@n_. t + a_D :> Cos@Expand@n Hf@tD - b@tDL + aDD
```

```
f¢@tD == 1 + 1/8 H6 e a@tD Cos@f@tDD a2 + 2 e a@tD Cos@3 f@tDD a2 +
    3 e^2 a@tD^2 a3 + 4 e^2 a@tD^2 Cos@2 f@tDD a3 + e^2 a@tD^2 Cos@4 f@tDD a3L
```

We seek approximate solutions to **ampEq2** and **phaseEq2** in the form

```
asol = a@tD == a0@tD + SumAe^i ai@a0@tD, f0@tDD, 8i, 2<E
```

```
a@tD == a0@tD + e a1@a0@tD, f0@tDD + e^2 a2@a0@tD, f0@tDD
```

```
phisol = f@tD == f0@tD + SumAe^i fi@a0@tD, f0@tDD, 8i, 2<E
```

```
f@tD == f0@tD + e f1@a0@tD, f0@tDD + e^2 f2@a0@tD, f0@tDD
```

where a_0@tD and f_0@tD are expanded in power series in e as

```
a0Eq = a0¢@tD == SumAe^i Ai@a0@tDD, 8i, 2<E
```

```
a0¢@tD == e A1@a0@tDD + e^2 A2@a0@tDD
```

```
phi0Eq = f0¢@tD == 1 + SumAe^i Fi@a0@tDD, 8i, 2<E
```

```
f0¢@tD == 1 + e F1@a0@tDD + e^2 F2@a0@tDD
```

The functions a_1, a_2, …, a_n and f_1, f_2, …, f_n are fast varying functions of f_0, while it follows from **a0Eq** and **phi0Eq** that a_0, and hence the A_n and F_n are slowly varying functions of t.

To the second approximation, we differentiate a@tD and f@tD once with respect to t, use **a0Eq** and **phi0Eq**, expand the result for small e, discard terms of order higher than e^2, and obtain

```
aEq1 = HD@asol, tD •. H8a0Eq, phi0Eq< •. Equal -> RuleL •• ExpandAllL •. e^(n_•;n>2) -> 0
```

```
a¢@tD == e A1@a0@tDD + e^2 A2@a0@tDD + e a1^H0,1L@a0@tD, f0@tDD +
    e^2 F1@a0@tDD a1^H0,1L@a0@tD, f0@tDD + e^2 a2^H0,1L@a0@tD, f0@tDD + e^2 A1@a0@tDD a1^H1,0L@a0@tD, f0@tDD
```

```
phiEq1 = HD@phisol, tD •. H8a0Eq, phi0Eq< •. Equal -> RuleL •• ExpandAllL •. e^(n_•;n>2) -> 0
```

```
f¢@tD == 1 + e F1@a0@tDD + e^2 F2@a0@tDD + e f1^H0,1L@a0@tD, f0@tDD +
    e^2 F1@a0@tDD f1^H0,1L@a0@tD, f0@tDD + e^2 f2^H0,1L@a0@tD, f0@tDD + e^2 A1@a0@tDD f1^H1,0L@a0@tD, f0@tDD
```

Next, we substitute **asol** and **phisol** into **ampEq2** and **phaseEq2**, expand the right-hand sides for small e keeping terms up to OHe^{2}L, and rewrite their right-hand sides as

```
ampEqrhs = Series@ampEq2@@2DD •. H8asol, phisol< •. Equal -> RuleL, 8e, 0, 2<D •• Normal
```

```
1
ŒŒŒ e H2 Sin@f0@tDD a2 a0@tD^2 + 2 Sin@3 f0@tDD a2 a0@tD^2L +
8
 1
 ŒŒŒ e^2 H2 Sin@2 f0@tDD a3 a0@tD^3 + Sin@4 f0@tDD a3 a0@tD^3 +
 8
    2 a2 H2 Sin@f0@tDD a0@tD a1@a0@tD, f0@tDD + Cos@f0@tDD a0@tD^2 f1@a0@tD, f0@tDDL +
    2 a2 H2 Sin@3 f0@tDD a0@tD a1@a0@tD, f0@tDD + 3 Cos@3 f0@tDD a0@tD^2 f1@a0@tD, f0@tDDLL
```

```
phaseEqrhs = Series@phaseEq2@@2DD •. H8asol, phisol< •. Equal -> RuleL, 8e, 0, 2<D •• Normal
```

```
     1
1 + ŒŒŒ e H6 Cos@f0@tDD a2 a0@tD + 2 Cos@3 f0@tDD a2 a0@tDL +
     8
 1
 ŒŒŒ e^2 H3 a3 a0@tD^2 + 4 Cos@2 f0@tDD a3 a0@tD^2 + Cos@4 f0@tDD a3 a0@tD^2 +
 8
    6 a2 HCos@f0@tDD a1@a0@tD, f0@tDD - Sin@f0@tDD a0@tD f1@a0@tD, f0@tDDL +
    2 a2 HCos@3 f0@tDD a1@a0@tD, f0@tDD - 3 Sin@3 f0@tDD a0@tD f1@a0@tD, f0@tDDLL
```

Equating coefficients of like powers of e from **aEq1**, **ampEqrhs**, **phiEq1**, and **phaseEqrhs**, we obtain

```
ampEq3 = CoefficientList@#, eD & •ž HaEq1@@2DD == ampEqrhsL •• Thread •• Rest
```

```
                                                         1
9A1@a0@tDD + a1^H0,1L@a0@tD, f0@tDD == ŒŒŒ H2 Sin@f0@tDD a2 a0@tD^2 + 2 Sin@3 f0@tDD a2 a0@tD^2L,
                                                         8
 A2@a0@tDD + F1@a0@tDD a1^H0,1L@a0@tD, f0@tDD + a2^H0,1L@a0@tD, f0@tDD +
                                                 1
   A1@a0@tDD a1^H1,0L@a0@tD, f0@tDD == ŒŒŒ H2 Sin@2 f0@tDD a3 a0@tD^3 + Sin@4 f0@tDD a3 a0@tD^3 +
                                                 8
     2 a2 H2 Sin@f0@tDD a0@tD a1@a0@tD, f0@tDD + Cos@f0@tDD a0@tD^2 f1@a0@tD, f0@tDDL +
     2 a2 H2 Sin@3 f0@tDD a0@tD a1@a0@tD, f0@tDD + 3 Cos@3 f0@tDD a0@tD^2 f1@a0@tD, f0@tDDL=
```

```
phaseEq3 = CoefficientList@#, eD & •ž HphiEq1@@2DD == phaseEqrhsL •• Thread •• Rest
```

```
                                                         1
9F1@a0@tDD + f1^H0,1L@a0@tD, f0@tDD == ŒŒŒ H6 Cos@f0@tDD a2 a0@tD + 2 Cos@3 f0@tDD a2 a0@tDL,
                                                         8
 F2@a0@tDD + F1@a0@tDD f1^H0,1L@a0@tD, f0@tDD +
   f2^H0,1L@a0@tD, f0@tDD + A1@a0@tDD f1^H1,0L@a0@tD, f0@tDD ==
   1
  ŒŒŒ H3 a3 a0@tD^2 + 4 Cos@2 f0@tDD a3 a0@tD^2 + Cos@4 f0@tDD a3 a0@tD^2 +
   8
     6 a2 HCos@f0@tDD a1@a0@tD, f0@tDD - Sin@f0@tDD a0@tD f1@a0@tD, f0@tDDL +
     2 a2 HCos@3 f0@tDD a1@a0@tD, f0@tDD - 3 Sin@3 f0@tDD a0@tD f1@a0@tD, f0@tDDL=
```

Next, we use the method of separation of variables to separate fast and slowly varying terms in the first-order equations, **ampEq3[[1]]** and **phaseEq3[[1]]**. The slowly varying parts yield

```
cond36a = 8A1@a0@tDD -> 0, F1@a0@tDD -> 0<;
```

whereas the fast varying parts yield

```
order1Eq = 8ampEq3@@1DD, phaseEq3@@1DD< •. cond36a
```

```
9a_1^H0,1L@a_0@tD, f_0@tDD == 1/8 H2 Sin@f_0@tDD a_2 a_0@tD^2 + 2 Sin@3 f_0@tDD a_2 a_0@tD^2L,
 f_1^H0,1L@a_0@tD, f_0@tDD == 1/8 H6 Cos@f_0@tDD a_2 a_0@tD + 2 Cos@3 f_0@tDD a_2 a_0@tDL=
```

To determine the solutions of **order1Eq** using **DSolve**, we transform them from partial-differential to ordinary-differential equations, replace f_0@tD with *s*, and obtain

```
order1EqTransf = order1Eq •. 8a_1 -> Ha_1@#2D &L, f_1 -> Hf_1@#2D &L< •. f_@tD -> f •. f_0 -> s
```

```
9a_1'@sD == 1/8 H2 Sin@sD a_0^2 a_2 + 2 Sin@3 sD a_0^2 a_2L, f_1'@sD == 1/8 H6 Cos@sD a_0 a_2 + 2 Cos@3 sD a_0 a_2L=
```

Instead of replacing f_0@tD, we could have symbolized it. Using **DSolve,** we find that a_1 and f_1 are given by

```
rule36@1D = DSolve@order1EqTransf@@1DD, a_1@sD, sD@@1DD •. C@_D -> 0 •.
   8s -> f_0@tD, a_0 -> a_0@tD< •. a_1@arg_D -> a_1@a_0@tD, argD
```

```
9a_1@a_0@tD, f_0@tDD ® 1/12 H-3 Cos@f_0@tDD a_2 a_0@tD^2 - Cos@3 f_0@tDD a_2 a_0@tD^2L=
```

```
rule36@2D = DSolve@order1EqTransf@@2DD, f_1@sD, sD@@1DD •. C@_D -> 0 •.
   8s -> f_0@tD, a_0 -> a_0@tD< •. f_1@arg_D -> f_1@a_0@tD, argD
```

```
9f_1@a_0@tD, f_0@tDD ® 1/12 H9 Sin@f_0@tDD a_2 a_0@tD + Sin@3 f_0@tDD a_2 a_0@tDL=
```

which can be combined into

```
cond36b = Join@rule36@1D, rule36@2DD;
```

Substituting the slow- and fast- varying components of the first-order solution, **cond36a** and **cond36b**, into the second-order equations, we have

```
order2Eq = {ampEq3[[2]], phaseEq3[[2]]} /. cond36a /. cond36b
```

$$\Big\{A_2[a_0[t]] + a_2^{(0,1)}[a_0[t], \phi_0[t]] == \frac{1}{8}\Big(2\,\mathrm{Sin}[2\phi_0[t]]\,\alpha_3 a_0[t]^3 + \mathrm{Sin}[4\phi_0[t]]\,\alpha_3 a_0[t]^3 + 2\alpha_2\Big(\frac{1}{12}\mathrm{Cos}[\phi_0[t]]\,a_0[t]^2\,(9\,\mathrm{Sin}[\phi_0[t]]\,\alpha_2 a_0[t] + \mathrm{Sin}[3\phi_0[t]]\,\alpha_2 a_0[t]) + \frac{1}{6}\mathrm{Sin}[\phi_0[t]]\,a_0[t]\,(-3\,\mathrm{Cos}[\phi_0[t]]\,\alpha_2 a_0[t]^2 - \mathrm{Cos}[3\phi_0[t]]\,\alpha_2 a_0[t]^2)\Big) + 2\alpha_2\Big(\frac{1}{4}\mathrm{Cos}[3\phi_0[t]]\,a_0[t]^2\,(9\,\mathrm{Sin}[\phi_0[t]]\,\alpha_2 a_0[t] + \mathrm{Sin}[3\phi_0[t]]\,\alpha_2 a_0[t]) + \frac{1}{6}\mathrm{Sin}[3\phi_0[t]]\,a_0[t]\,(-3\,\mathrm{Cos}[\phi_0[t]]\,\alpha_2 a_0[t]^2 - \mathrm{Cos}[3\phi_0[t]]\,\alpha_2 a_0[t]^2)\Big)\Big),$$

$$\Phi_2[a_0[t]] + \phi_2^{(0,1)}[a_0[t], \phi_0[t]] == \frac{1}{8}\Big(3\alpha_3 a_0[t]^2 + 4\,\mathrm{Cos}[2\phi_0[t]]\,\alpha_3 a_0[t]^2 + \mathrm{Cos}[4\phi_0[t]]\,\alpha_3 a_0[t]^2 + 6\alpha_2\Big(-\frac{1}{12}\mathrm{Sin}[\phi_0[t]]\,a_0[t]\,(9\,\mathrm{Sin}[\phi_0[t]]\,\alpha_2 a_0[t] + \mathrm{Sin}[3\phi_0[t]]\,\alpha_2 a_0[t]) + \frac{1}{12}\mathrm{Cos}[\phi_0[t]]\,(-3\,\mathrm{Cos}[\phi_0[t]]\,\alpha_2 a_0[t]^2 - \mathrm{Cos}[3\phi_0[t]]\,\alpha_2 a_0[t]^2)\Big) + 2\alpha_2\Big(-\frac{1}{4}\mathrm{Sin}[3\phi_0[t]]\,a_0[t]\,(9\,\mathrm{Sin}[\phi_0[t]]\,\alpha_2 a_0[t] + \mathrm{Sin}[3\phi_0[t]]\,\alpha_2 a_0[t]) + \frac{1}{12}\mathrm{Cos}[3\phi_0[t]]\,(-3\,\mathrm{Cos}[\phi_0[t]]\,\alpha_2 a_0[t]^2 - \mathrm{Cos}[3\phi_0[t]]\,\alpha_2 a_0[t]^2)\Big)\Big)\Big\}$$

Next, we use trigonometric identities to expand the right-hand sides of the second-order equations in Fourier series and obtain

```
order2Eqrhs = (#[[2]] // TrigReduce // Expand) & /@ order2Eq
```

$$\Big\{-\frac{9}{32}\mathrm{Sin}[2\phi_0[t]]\,\alpha_2^2 a_0[t]^3 + \frac{5}{24}\mathrm{Sin}[4\phi_0[t]]\,\alpha_2^2 a_0[t]^3 + \frac{1}{96}\mathrm{Sin}[6\phi_0[t]]\,\alpha_2^2 a_0[t]^3 + \frac{1}{4}\mathrm{Sin}[2\phi_0[t]]\,\alpha_3 a_0[t]^3 + \frac{1}{8}\mathrm{Sin}[4\phi_0[t]]\,\alpha_3 a_0[t]^3,$$
$$-\frac{5}{12}\alpha_2^2 a_0[t]^2 - \frac{3}{16}\mathrm{Cos}[2\phi_0[t]]\,\alpha_2^2 a_0[t]^2 + \frac{1}{4}\mathrm{Cos}[4\phi_0[t]]\,\alpha_2^2 a_0[t]^2 + \frac{1}{48}\mathrm{Cos}[6\phi_0[t]]\,\alpha_2^2 a_0[t]^2 + \frac{3}{8}\alpha_3 a_0[t]^2 + \frac{1}{2}\mathrm{Cos}[2\phi_0[t]]\,\alpha_3 a_0[t]^2 + \frac{1}{8}\mathrm{Cos}[4\phi_0[t]]\,\alpha_3 a_0[t]^2\Big\}$$

Since we are seeking an expansion valid to $O(\epsilon^2)$, we do not need to solve for a_2 and ϕ_2. All we need to do is to investigate the above expressions to determine the slowly varying parts and determine A_2 and Φ_2. The result is

```
cond36c = {A2[a0[t]], Φ2[a0[t]]} -> (order2Eqrhs /. {Sin[_] -> 0, Cos[_] -> 0}) // Thread
```

$$\Big\{A_2[a_0[t]] \to 0,\ \Phi_2[a_0[t]] \to -\frac{5}{12}\alpha_2^2 a_0[t]^2 + \frac{3}{8}\alpha_3 a_0[t]^2\Big\}$$

Substituting the fast-varying components of the first-order solution, **cond36b**, into the expansions for a and ϕ and discarding terms of order higher than ϵ, we obtain, to the second approximation, that

```
asolF = asol /. cond36b /. ε^2 -> 0
```

$$a[t] == a_0[t] + \frac{1}{12}\epsilon\,(-3\,\mathrm{Cos}[\phi_0[t]]\,\alpha_2 a_0[t]^2 - \mathrm{Cos}[3\phi_0[t]]\,\alpha_2 a_0[t]^2)$$

```
phisolF = phisol /. cond36b /. ε^2 -> 0
```

$$\phi[t] == \frac{1}{12}\,\epsilon\,(9\,\mathrm{Sin}[\phi_0[t]]\,\alpha_2\,a_0[t] + \mathrm{Sin}[3\,\phi_0[t]]\,\alpha_2\,a_0[t]) + \phi_0[t]$$

Substituting the solw-varying components of the first- and second-order solutions, **cond36a** and **cond36c**, into **a0Eq** and **phi0Eq** yields

```
a0Eq1 = a0Eq /. cond36a /. cond36c
```

$$a_0'[t] == 0$$

```
phi0Eq1 = phi0Eq /. cond36a /. cond36c
```

$$\phi_0'[t] == 1 + \epsilon^2\left(-\frac{5}{12}\,\alpha_2^2\,a_0[t]^2 + \frac{3}{8}\,\alpha_3\,a_0[t]^2\right)$$

It follows from the above equations that a_0 = constant and

```
phi0Rule = DSolve[phi0Eq1 /. a0[t] -> a0, φ0[t], t][[1]] /. C[1] -> β0
```

$$\left\{\phi_0[t] \to t\left(1 - \frac{5}{12}\,\epsilon^2\,a_0^2\,\alpha_2^2 + \frac{3}{8}\,\epsilon^2\,a_0^2\,\alpha_3\right) + \beta_0\right\}$$

where β_0 is a constant. Substituting these expansions into **eq35a**, we obtain the following second-order expansion:

```
eq36a = u[t] == ε a[t] Cos[φ[t]] /. {asolF, phisolF} /. Equal -> Rule
```

$$u[t] == \epsilon\,\mathrm{Cos}\left[\frac{1}{12}\,\epsilon\,(9\,\mathrm{Sin}[\phi_0[t]]\,\alpha_2\,a_0[t] + \mathrm{Sin}[3\,\phi_0[t]]\,\alpha_2\,a_0[t]) + \phi_0[t]\right]\left(a_0[t] + \frac{1}{12}\,\epsilon\,(-3\,\mathrm{Cos}[\phi_0[t]]\,\alpha_2\,a_0[t]^2 - \mathrm{Cos}[3\,\phi_0[t]]\,\alpha_2\,a_0[t]^2)\right)$$

To compare the present solution with those obtained by using the method of multiple scales and the Lindstedt-Poincaré technique, we expand the circular functions in **eq36a** for small ϵ about ϕ_0 and obtain

```
u[t] == (Series[eq36a[[2]], {ε, 0, 2}] // Normal // TrigReduce // Expand)
```

$$u[t] == \epsilon\,\mathrm{Cos}[\phi_0[t]]\,a_0[t] - \frac{1}{2}\,\epsilon^2\,\alpha_2\,a_0[t]^2 + \frac{1}{6}\,\epsilon^2\,\mathrm{Cos}[2\,\phi_0[t]]\,\alpha_2\,a_0[t]^2$$

This expansion and **phi0Rule** are in full agreement with those obtained by using the method of multiple scales.

3.7 The Krylov-Bogoliubov-Mitropolsky Technique

In this section, we describe a variant of the generalized method of averaging, namely the Krylov-Bogoliubov-Mitropolsky technique. It is often referred to as the asymptotic method. When the nonlinear terms are neglected, the solution of **eq31g** is

```
linearSol = u == ε a Cos[t + β];
```

where *a* and b are constants and e is a small dimensionless parameter that is a measure of the amplitude. When the nonlinear terms are included, we consider **linearSol** to be the first term in an approximate solution of **eq31g** but with slowly varying rather than constant *a* and b. Moreover, we introduce the fast scale f = *t* + b and use *a* to represent the slow variations. Thus, we seek a second-order uniform expansion of the solution of **eq31g** in the form

```
uSol = u@a@tD, f@tDD == e a@tD Cos@f@tDD + SumAe^i u_i@a@tD, f@tDD, 8i, 2, 3<E
```

```
u@a@tD, f@tDD == e a@tD Cos@f@tDD + e^2 u_2@a@tD, f@tDD + e^3 u_3@a@tD, f@tDD
```

Again, terms of order e^3 have been included in the expansion, but these terms will not be included in the final result. Since *a* and b are slowly varying functions of *t*, we express them in power series of e in terms of *a* as

```
apEq = a '@tD == SumAe^i A_i@a@tDD, 8i, 2<E
```

```
a¢@tD == e A_1@a@tDD + e^2 A_2@a@tDD
```

```
phipEq = f '@tD == 1 + SumAe^i F_i@a@tDD, 8i, 2<E
```

```
f¢@tD == 1 + e F_1@a@tDD + e^2 F_2@a@tDD
```

In what follows, we need *a*ˆ@*t*D and fˆ@*t*D. To this end, we differentiate **apEq** and **phipEq** with respect to *t*, use these equations to express *a*¢@*t*D and f¢@*t*D in terms of the A_i@*a*@*t*DD and F_i@*a*@*t*DD, discard terms of order higher than e^2, and obtain

```
appEq = HD@apEq, tD ê. HapEq ê. Equal -> RuleL êê ExpandAllL ê. e^(n_ê;n>2) -> 0
```

```
aˆ@tD == e^2 A_1@a@tDD A¢_1@a@tDD
```

```
phippEq = HD@phipEq, tD ê. HapEq ê. Equal -> RuleL êê ExpandAllL ê. e^(n_ê;n>2) -> 0
```

```
fˆ@tD == e^2 A_1@a@tDD F¢_1@a@tDD
```

Thus, this method can be viewed as a multiple scales procedure with *a* and f being the scales.

Substituting the assumed expansion for *u*@*t*D, expressed in function form, and the expansions for the derivatives of *a*@*t*D and f@*t*D into **eq31g**, expanding the result for small e, and keeping terms up to order e^3, we have

```
eq37a = Heq31g ê. u -> Function@t, uSol@@2DD êê EvaluateD ê.
    H8apEq, phipEq, appEq, phippEq< ê. Equal -> RuleL êê ExpandAllL ê. e^(n_ê;n>3) -> 0
```

```
e^2 a@tD^2 Cos@f@tDD^2 a_2 + e^3 a@tD^3 Cos@f@tDD^3 a_3 - 2 e^2 Sin@f@tDD A_1@a@tDD -
  2 e^3 Sin@f@tDD A_2@a@tDD + e^2 u_2@a@tD, f@tDD + 2 e^3 a@tD Cos@f@tDD a_2 u_2@a@tD, f@tDD +
  e^3 u_3@a@tD, f@tDD - 2 e^2 a@tD Cos@f@tDD F_1@a@tDD - 2 e^3 Sin@f@tDD A_1@a@tDD F_1@a@tDD -
  e^3 a@tD Cos@f@tDD F_1@a@tDD^2 - 2 e^3 a@tD Cos@f@tDD F_2@a@tDD + e^3 Cos@f@tDD A_1@a@tDD A¢_1@a@tDD -
  e^3 a@tD Sin@f@tDD A_1@a@tDD F¢_1@a@tDD + e^2 u_2^(H0,2L)@a@tD, f@tDD +
  2 e^3 F_1@a@tDD u_2^(H0,2L)@a@tD, f@tDD + e^3 u_3^(H0,2L)@a@tD, f@tDD + 2 e^3 A_1@a@tDD u_2^(H1,1L)@a@tD, f@tDD == 0
```

Equating coefficients of like powers of e in **eq37a** yields

```
eqEps = TableACoefficientAeq37a@@1DD, e^i E == 0, 8i, 2, 3<E •. f_@tD -> f
```

```
9a^2 Cos@fD^2 a2 - 2 Sin@fD A1@aD + u2@a, fD - 2 a Cos@fD F1@aD + u2^H0,2L@a, fD == 0,
 a^3 Cos@fD^3 a3 - 2 Sin@fD A2@aD + 2 a Cos@fD a2 u2@a, fD + u3@a, fD -
   2 Sin@fD A1@aD F1@aD - a Cos@fD F1@aD^2 - 2 a Cos@fD F2@aD + Cos@fD A1@aD A1¢@aD -
   a Sin@fD A1@aD F1¢@aD + 2 F1@aD u2^H0,2L@a, fD + u3^H0,2L@a, fD + 2 A1@aD u2^H1,1L@a, fD == 0=
```

Next, we use trigonometric identities to expand the nonhomogeneous terms in **eqEps[[1]]** in a Fourier series and obtain

```
order2Eq = TrigReduce •ž eqEps@@1DD
```

```
1/2 Ia^2 a2 + a^2 Cos@2 fD a2 - 4 Sin@fD A1@aD + 2 u2@a, fD - 4 a Cos@fD F1@aD + 2 u2^H0,2L@a, fDM == 0
```

Eliminating the terms that produce secular terms in u_2 from **order2Eq** demands that

```
rule37a =
 Solve@Coefficient@order2Eq@@1DD, 8Cos@fD, Sin@fD<D == 0 •• Thread, 8F1@aD, A1@aD<D@@1DD
```

```
8F1@aD ® 0, A1@aD ® 0<
```

Then, the solution of the resulting **order2Eq** can be written as

```
rule37b = DSolve@order2Eq •. rule37a, u2@a, fD, 8a, f<D@@1DD •. C@_D -> H0 &L •• Simplify
```

```
9u2@a, fD ® 1/6 a^2 H-3 + Cos@2 fDL a2=
```

which can be expressed in function form as

```
rule37c = 8u2 -> Function@8a, f<, rule37b@@1, 2DD •• EvaluateD<
```

```
9u2 ® FunctionA8a, f<, 1/6 a^2 H-3 + Cos@2 fDL a2E=
```

Substituting **rule37a** and **rule37c** into **eqEps[[2]]**, we obtain

```
eq37b = eqEps@@2DD •. rule37a •. rule37c
```

```
1/3 a^3 Cos@fD H-3 + Cos@2 fDL a2^2 + a^3 Cos@fD^3 a3 -
  2 Sin@fD A2@aD + u3@a, fD - 2 a Cos@fD F2@aD + u3^H0,2L@a, fD == 0
```

Eliminating the terms that produce secular terms in u_3 from **eq37b** demands that

```
eq37c = Coefficient@eq37b@@1DD •• TrigReduce, 8Cos@fD, Sin@fD<D == 0 •• Thread
```

```
:- 5/6 a^3 a2^2 + 3 a^3 a3/4 - 2 a F2@aD == 0, -2 A2@aD == 0>
```

or

```
rule37d = Solve@eq37c, 8A2@aD, F2@aD<D@@1DD
```

$\{A_2[a] \to 0,\ F_2[a] \to -\frac{1}{24}\, a^2 (10\, a_2^2 - 9\, a_3)\}$

Substituting **rule37a** and **rule37d** into **apEq** and **phipEq** yields

```
apEq /. (rule37a ~Join~ rule37d /. a -> a[t])
```

$a'[t] == 0$

```
phipEq /. (rule37a ~Join~ rule37d /. a -> a[t]) // ExpandAll
```

$\phi'[t] == 1 - \frac{5}{12}\, \epsilon^2\, a[t]^2\, a_2^2 + \frac{3}{8}\, \epsilon^2\, a[t]^2\, a_3$

which are in full agreement with those obtained by using the method of multiple scales and the generalized method of averaging.

3.8 The Method of Normal Forms

In this section, we use the method of normal forms to determine a second-order uniform expansion of the solution of **eq31g**. To this end, we start with the corresponding first-order complex-valued equation

$$eq343a = \zeta'[t] == I\,\zeta[t] + \frac{1}{2}\, I\, \alpha_2\, \zeta[t]^2 + \frac{1}{2}\, I\, \alpha_3\, \zeta[t]^3 + I\, \alpha_2\, \zeta[t]\, \bar{\zeta}[t] + \frac{3}{2}\, I\, \alpha_3\, \zeta[t]^2\, \bar{\zeta}[t] + \frac{1}{2}\, I\, \alpha_2\, \bar{\zeta}[t]^2 + \frac{3}{2}\, I\, \alpha_3\, \zeta[t]\, \bar{\zeta}[t]^2 + \frac{1}{2}\, I\, \alpha_3\, \bar{\zeta}[t]^3;$$

According to the method of normal forms, we introduce the near-identity transformation

```
basicTerms = {η[t], η̄[t]};

zetaRule = {
ζ -> Function[t, ε η[t] + Sum[ε^j h_j @@ basicTerms, {j, 2, 3}] // Evaluate],
ζ̄ -> Function[t, ε η̄[t] + Sum[ε^j h̄_j @@ basicTerms, {j, 2, 3}] // Evaluate]}
```

$\{\zeta \to \text{Function}[t,\ \epsilon\, \eta[t] + \epsilon^2\, h_2[\eta[t], \bar{\eta}[t]] + \epsilon^3\, h_3[\eta[t], \bar{\eta}[t]]],$
$\bar{\zeta} \to \text{Function}[t,\ \epsilon\, \bar{\eta}[t] + \epsilon^2\, \bar{h}_2[\eta[t], \bar{\eta}[t]] + \epsilon^3\, \bar{h}_3[\eta[t], \bar{\eta}[t]]]\}$

that results in the simplest possible equation

```
etaRule = {η'[t] -> I η[t] + Sum[ε^j g_j[t], {j, 2}]}
```

$\{\eta'[t] \to I\, \eta[t] + \epsilon\, g_1[t] + \epsilon^2\, g_2[t]\}$

where the overbar denotes the complex conjugate

```
conjugateRule = {h -> h̄, h̄ -> h, g -> ḡ, ḡ -> g, Complex[0, n_] -> Complex[0, -n]};
```

Substituting the expansion for z, the **zetaRule**, into **eq343a**, using **etaRule**, expanding the result for small e, and discarding terms of order higher than e^3, we have

```
eq38a =
  Heq343a •. zetaRule •. etaRule •. HetaRule •. conjugateRuleL •• ExpandAllL •. e^(n_•;n>3) -> 0;
```

Equating coefficients of like powers of e in **eq38a** yields

```
eqEps = TableACoefficientASubtract žž eq38a, e^i E == 0, 8i, 2, 3<E
```

```
9- 1/2 I a2 h@tD^2 - I a2 h@tD ḣ@tD - 1/2 I a2 ḣ@tD^2 + g1@tD -
   I h2@h@tD, ḣ@tDD - I ḣ@tD h2^(H0,1L)@h@tD, ḣ@tDD + I h@tD h2^(H1,0L)@h@tD, ḣ@tDD == 0,
 - 1/2 I a3 h@tD^3 - 3/2 I a3 h@tD^2 ḣ@tD - 3/2 I a3 h@tD ḣ@tD^2 - 1/2 I a3 ḣ@tD^3 + g2@tD -
   I a2 h@tD h2@h@tD, ḣ@tDD - I a2 ḣ@tD h2@h@tD, ḣ@tDD - I h3@h@tD, ḣ@tDD -
   I a2 h@tD ḣ2@h@tD, ḣ@tDD - I a2 ḣ@tD ḣ2@h@tD, ḣ@tDD + ġ1@tD h2^(H0,1L)@h@tD, ḣ@tDD -
   I ḣ@tD h3^(H0,1L)@h@tD, ḣ@tDD + g1@tD h2^(H1,0L)@h@tD, ḣ@tDD + I h@tD h3^(H1,0L)@h@tD, ḣ@tDD == 0=
```

Ÿ Second-Order Solution

We choose the h_i to eliminate as many terms from **eqEps**, thereby reducing them into their simplest possible form. It turns out that we can eliminate all nonresonance terms. To determine the resonance terms in **eqEps[[1]]**, we note that all of the possible forms of its nonhomogeneous terms are

```
possibleQTerms = Outer@Times, basicTerms, basicTermsD •• Flatten •• Union
```

```
9h@tD^2, h@tD ḣ@tD, ḣ@tD^2=
```

It follows from the linear parts of the **etaRule** that

```
form = 9h@tD -> E^(I t), ḣ@tD -> E^(-I t)=;
```

Hence, the possible resonance terms are given by

```
ResonantQTerm = IE^(-I t) possibleQTerms •. form •. E^(_ t) -> 0M possibleQTerms •• Union •• Rest
```

```
8<
```

Consequently, there are no resonance terms in the second-order problem and the nonresonance terms can be defined as

```
NRQT = Complement@possibleQTerms, ResonantQTermD
```

```
9h@tD^2, h@tD ḣ@tD, ḣ@tD^2=
```

We associate with each of them a coefficient according to

```
coeffsQ = Table@Gj, 8j, Length@NRQTD<D
```

```
8G1, G2, G3<
```

Therefore, h_2 and its complex conjugate have the form

```
hFormQ = 9h2 -> HEvaluate@coeffsQ.NRQT •. Thread@basicTerms -> 8#1, #2<DD &L,
  h̄2 -> HEvaluate@coeffsQ.NRQT •. conjugateRule •. Thread@basicTerms -> 8#1, #2<DD &L=
```

```
8h2 ® H#1^2 G1 + #1 #2 G2 + #2^2 G3 &L, h̄2 ® H#2^2 G1 + #1 #2 G2 + #1^2 G3 &L<
```

Substituting for h_2 in **eqEps[[1]]**, equating the coefficients of the possible nonresonance terms to zero, and solving the resulting equations for the G_i, we obtain

```
coeffsQRule = Solve@Coefficient@eqEps@@1, 1DD •. hFormQ, NRQTD == 0 •• Thread, coeffsQD@@1DD
```

$$\left\{G_1 \to \frac{a_2}{2},\ G_2 \to -a_2,\ G_3 \to -\frac{a_2}{6}\right\}$$

We choose g_1 to eliminate the resonance terms in **eqEps[[1,1]]** according to

```
gRuleQ = g1@tD -> - Coefficient@eqEps@@1, 1DD, ResonantQTermD.ResonantQTerm
```

```
g1@tD ® 0
```

In this case, there are on resonance terms and hence $g_1 = 0$.

Ÿ Third-Order Equations

Substituting the second-order results into the third-order equation yields

```
order3expr =
  eqEps@@2, 1DD •. hFormQ •. coeffsQRule •. gRuleQ •. HgRuleQ •. conjugateRuleL •• Expand
```

$$-\frac{1}{3} I a_2^2 h[t]^3 - \frac{1}{2} I a_3 h[t]^3 + \frac{5}{3} I a_2^2 h[t]^2 \bar{h}[t] - \frac{3}{2} I a_3 h[t]^2 \bar{h}[t] +$$
$$\frac{5}{3} I a_2^2 h[t] \bar{h}[t]^2 - \frac{3}{2} I a_3 h[t] \bar{h}[t]^2 - \frac{1}{3} I a_2^2 \bar{h}[t]^3 - \frac{1}{2} I a_3 \bar{h}[t]^3 + g_2[t] -$$
$$I h_3[h[t], \bar{h}[t]] - I \bar{h}[t] h_3^{(0,1)}[h[t], \bar{h}[t]] + I h[t] h_3^{(1,0)}[h[t], \bar{h}[t]]$$

The nonhomogeneous terms in **order3expr** are proportional to

```
possibleCTerms = Outer@Times, possibleQTerms, basicTermsD •• Flatten •• Union
```

$$\{h[t]^3,\ h[t]^2 \bar{h}[t],\ h[t] \bar{h}[t]^2,\ \bar{h}[t]^3\}$$

Next, we determine the resonance terms according to

```
ResonantCTerm = I E^(-I t) possibleCTerms /. form /. E^(-t) -> 0M possibleCTerms // Union // Rest
```

```
{h[t]^2 h̄[t]}
```

Then, the nonresonance terms are the complement of these resonance terms; that is,

```
NRCT = Complement[possibleCTerms, ResonantCTerm]
```

```
{h[t]^3, h[t] h̄[t]^2, h̄[t]^3}
```

Again, we associate coefficients with these terms as follows:

```
coeffsC = Table[Λ_j, {j, Length[NRCT]}]
```

$\{\Lambda_1, \Lambda_2, \Lambda_3\}$

Consequently, h_3 and its complex conjugate have the forms

```
hFormC = {h_3 -> (Evaluate[coeffsC.NRCT /. Thread[basicTerms -> {#1, #2}]] &),
  h̄_3 -> (Evaluate[coeffsC.NRCT /. conjugateRule /. Thread[basicTerms -> {#1, #2}]] &)}
```

$\{h_3 \to (\#1^3 \Lambda_1 + \#1\, \#2^2 \Lambda_2 + \#2^3 \Lambda_3 \&),\ \bar{h}_3 \to (\#2^3 \Lambda_1 + \#1^2\, \#2\, \Lambda_2 + \#1^3 \Lambda_3 \&)\}$

Substituting for h_3 into **order3expr**, equating the coefficient of each possible nonresonance term to zero, and solving the resulting algebraic equations for the Λ_i, we obtain

```
coeffsCRule = Solve[Coefficient[order3expr /. hFormC, NRCT] == 0 // Thread, coeffsC][[1]]
```

$\{\Lambda_1 \to \frac{1}{12}(2\, a_2^2 + 3\, a_3),\ \Lambda_2 \to \frac{1}{12}(10\, a_2^2 - 9\, a_3),\ \Lambda_3 \to \frac{1}{24}(-2\, a_2^2 - 3\, a_3)\}$

Choosing g_2 to eliminate the resonance terms from **order3expr** yields

```
gRuleC = g_2[t] -> -Coefficient[order3expr, ResonantCTerm] ResonantCTerm
```

$g_2[t] \to -\left(\frac{5\, I\, a_2^2}{3} - \frac{3\, I\, a_3}{2}\right) h[t]^2\, \bar{h}[t]$

Combining **etaRule**, **gRuleQ**, and **gRuleC** and letting $h = A[t]E^{I\,t}$, we obtain the modulation equation

```
moduEq = 2 I E^(-I t) (h'[t] - (h'[t] /. etaRule /. gRuleQ /. gRuleC)) == 0 /.
   {h -> (A[#] E^(I #) &), h̄ -> (Ā[#] E^(-I #) &)} // ExpandAll
```

$-\frac{10}{3} \epsilon^2 A[t]^2 a_2^2 \bar{A}[t] + 3\, \epsilon^2 A[t]^2 a_3 \bar{A}[t] + 2\, I\, A'[t] == 0$

which is in full agreement with that obtained by using the other techniques.

Chapter 4
Forced Oscillations of the Duffing Equation

```
Off@General::spell1D
```

In contrast with the preceding chapters, which deal with free oscillations, this chapter and the next two deal with forced oscillations. We consider

```
FDuffingEq = u²@tD + w² u@tD + 2 m̀ u¢@tD + d u@tD² + a u@tD³ == F Cos@W tD;
```

where m̀ is a positive constant. In this chapter, we determine second-order solutions to **FDuffingEq** beginning with the straightforward expansion in the next section. We investigate this straightforward expansion and determine under what conditions it breaks down. This leads to the so-called resonance values of W. In Section 4.2, we use the method of multiple scales to determine second-order uniform expansions of the solutions of **FDuffingEq** for all resonance cases, including the effect of light viscous damping. In Sections 4.3 and 4.4, we use the generalized method of averaging and the method of normal forms, respectively, to determine second-order uniform expansions for the case of subharmonic resonance of order one-half; that is, W » 2w.

à 4.1 Straightforward Expansion

We seek a straightforward expansion for the solutions of **FDuffingEq** in the form

```
uSol = u -> ISumAe^i u_i@#D, 8i, 3<E &M;
```

Substituting **uSol** into **FDuffingEq** , letting F = e f and m̀ = e m, expanding the result for small e, and discarding terms of order higher than e^3, we have

```
eq41a = HFDuffingEq •. uSol •. 8F -> e f, m̀ -> e m< •• ExpandAllL •. e^(n_•;n>3) -> 0
```

```
e w² u1@tD + d e² u1@tD² + a e³ u1@tD³ + e² w² u2@tD + 2 d e³ u1@tD u2@tD +
  e³ w² u3@tD + 2 e² m u1¢@tD + 2 e³ m u2¢@tD + e u1²@tD + e² u2²@tD + e³ u3²@tD == f e Cos@t WD
```

Equating coefficients of like powers of e on both sides yields

```
eqEpsa = CoefficientList@Subtract žž eq41a, eD == 0 •• Thread •• Rest
```

```
8-f Cos@t WD + w² u1@tD + u1²@tD == 0, d u1@tD² + w² u2@tD + 2 m u1¢@tD + u2²@tD == 0,
 a u1@tD³ + 2 d u1@tD u2@tD + w² u3@tD + 2 m u2¢@tD + u3²@tD == 0<
```

The particular solution of the first-order equation, linear and undamped problem, can be expressed as

```
sol1p = DSolve@eqEpsa@@1DD, u1@tD, tD@@1DD •. C@_D -> 0 •• TrigToExp •• Simplify
```

```
:u1@tD ® - E^-I t W H1 + E^2 I t W L f / (2 H-w^2 + W^2 L) >
```

We note that **sol1p** consists of a term whose denominator is very small when W » w. Such a term is called a small-divisor term. Moreover, *u*@*t*D tends to infinity as W ® w, and the excitation is referred to as a resonance excitation. Because the small-divisor term appears in the first-order problem, we speak of a primary or main resonance. When the small-divisor terms appear in the higher-order problems, we speak of secondary resonances. In the case of primary resonance, the scaling *F* ® e *f* is not valid. Physically, as soon as the motion becomes large, the damping and nonlinearity are activated to counter the effect of the resonance. Consequently, to obtain a uniform expansion in this case, we rescale *F* and m, as discussed in Section 4.2, so that the influence of the damping and nonlinearity balances the influence of the primary resonance.

When W is away from w, we add the homogeneous solution to **sol1p** and obtain the general solution of the first-order problem as

```
fRule = 9f -> 2 L IW^2 - w^2M=;

sol1 = u1 -> FunctionAt, A E^I w t + Ā E^-I w t + sol1p@@1, 2DD •. fRule •• Expand •• EvaluateE
```

```
u1 ® Function@t, A E^I t w - E^-I t W L - E^I t W L + E^-I t w ĀD
```

where *A* is a complex-valued constant. Substituting **sol1** into the second-order equation, **eqEpsa[[2]]**, yields

```
order2Eqa = u2''@tD + w^2 u2@tD - # & •ž eqEpsa@@2DD •. sol1 •• ExpandAll •• Reverse
```

```
w^2 u2@tD + u2''@tD ==
 -A^2 E^2 I t w d + 2 A E^I t w-I t W d L + 2 A E^I t w+I t W d L - 2 d L^2 - E^-2 I t W d L^2 - E^2 I t W d L^2 - 2 I A E^I t w m w -
  2 I E^-I t W L m W + 2 I E^I t W L m W - 2 A d Ā + 2 E^-I t w-I t W d L Ā + 2 E^-I t w+I t W d L Ā + 2 I E^-I t w m w Ā - E^-2 I t w d Ā^2
```

Solving for the particular solution of **order2Eqa**, we obtain

```
sol2p = Hu2@tD •. DSolve@order2Eqa@@1DD == #, u2@tD, tD@@1DD •. C@_D -> 0 •• TrigToExp ••
      SimplifyL & •ž order2Eqa@@2DD
```

```
A^2 E^2 I t w d/(3 w^2) - 2 d L^2/w^2 - A E^I t w m HI + 2 t wL/(2 w) + 2 A E^I t Hw-WL d L/(H2 w - WL W) -
 2 A E^I t Hw+WL d L/(W H2 w + WL) + 2 I E^I t W L m W/(w^2 - W^2) + 2 I E^-I t W L m W/(-w^2 + W^2) + E^-2 I t W d L^2/(-w^2 + 4 W^2) + E^2 I t W d L^2/(-w^2 + 4 W^2) -
 2 A d Ā/w^2 - E^-I t w m H-I + 2 t wL Ā/(2 w) + 2 E^-I t Hw-WL d L Ā/(H2 w - WL W) - 2 E^-I t Hw+WL d L Ā/(W H2 w + WL) + E^-2 I t w d Ā^2/(3 w^2)
```

Clearly, **sol2p** breaks down because it contains secular terms proportional to the damping coefficient m. Moreover, **sol2p** breaks down when W » 2w, W » w•2, or W » 0 because they produce small-divisor terms and hence secondary resonances. As discussed in Section 4.2, the case W » 2w is called subharmonic resonance of order one-half and the case W » w•2 is called superharmonic resonance of order two.

When Ω is away from ω, 2ω, $\omega/2$, or 0, **sol2p** is valid only if we rescale the damping term so that it first appears in the third-order equation; that is, $\mu \to \epsilon^2 \mu$. Using this scaling, we rewrite **eq41a** as

```
eq41b = eq41a /. μ -> ε μ /. ε^4 -> 0
```

$$\epsilon\,\omega^2 u_1[t] + \delta\,\epsilon^2 u_1[t]^2 + \alpha\,\epsilon^3 u_1[t]^3 + \epsilon^2 \omega^2 u_2[t] + 2\,\delta\,\epsilon^3 u_1[t]\,u_2[t] + \epsilon^3 \omega^2 u_3[t] + 2\,\epsilon^3 \mu\, u_1'[t] + \epsilon\, u_1''[t] + \epsilon^2 u_2''[t] + \epsilon^3 u_3''[t] == f\,\epsilon \operatorname{Cos}[t\,\Omega]$$

```
eqEpsb = CoefficientList[Subtract @@ eq41b, ε] == 0 // Thread // Rest
```

$$\{-f \operatorname{Cos}[t\,\Omega] + \omega^2 u_1[t] + u_1''[t] == 0,\ \delta\, u_1[t]^2 + \omega^2 u_2[t] + u_2''[t] == 0,\ \alpha\, u_1[t]^3 + 2\,\delta\, u_1[t]\,u_2[t] + \omega^2 u_3[t] + 2\,\mu\, u_1'[t] + u_3''[t] == 0\}$$

Comparing **eqEpsb** to **eqEpsa**, we immediately obtain its solution from **sol2p** as

```
sol2pNew = sol2p /. μ -> 0
```

$$\frac{A^2 e^{2 i t \omega}\delta}{3\omega^2} - \frac{2\delta\Lambda^2}{\omega^2} + \frac{2 A e^{i t(\omega-\Omega)}\delta\Lambda}{(2\omega-\Omega)\Omega} - \frac{2 A e^{i t(\omega+\Omega)}\delta\Lambda}{\Omega(2\omega+\Omega)} + \frac{e^{-2 i t\Omega}\delta\Lambda^2}{-\omega^2+4\Omega^2} + \frac{e^{2 i t\Omega}\delta\Lambda^2}{-\omega^2+4\Omega^2} - \frac{2 A\delta\bar{A}}{\omega^2} + \frac{2 e^{-i t(\omega-\Omega)}\delta\Lambda\bar{A}}{(2\omega-\Omega)\Omega} - \frac{2 e^{-i t(\omega+\Omega)}\delta\Lambda\bar{A}}{\Omega(2\omega+\Omega)} + \frac{e^{-2 i t\omega}\delta\bar{A}^2}{3\omega^2}$$

Hence, the general solution of **eqEpsb[[2]]** can be written as

```
sol2New = u2 -> Function[t, B E^(I ω t) + B̄ E^(-I ω t) + sol2pNew // Evaluate]
```

$$u_2 \to \operatorname{Function}\Big[t,\ B e^{i t\omega} + \frac{A^2 e^{2 i t \omega}\delta}{3\omega^2} - \frac{2\delta\Lambda^2}{\omega^2} + \frac{2 A e^{i t(\omega-\Omega)}\delta\Lambda}{(2\omega-\Omega)\Omega} - \frac{2 A e^{i t(\omega+\Omega)}\delta\Lambda}{\Omega(2\omega+\Omega)} + \frac{e^{-2 i t\Omega}\delta\Lambda^2}{-\omega^2+4\Omega^2} + \frac{e^{2 i t\Omega}\delta\Lambda^2}{-\omega^2+4\Omega^2} - \frac{2 A\delta\bar{A}}{\omega^2} + \frac{2 e^{-i t(\omega-\Omega)}\delta\Lambda\bar{A}}{(2\omega-\Omega)\Omega} - \frac{2 e^{-i t(\omega+\Omega)}\delta\Lambda\bar{A}}{\Omega(2\omega+\Omega)} + \frac{e^{-2 i t\omega}\delta\bar{A}^2}{3\omega^2} + e^{-i t\omega}\bar{B}\Big]$$

where *B* is a complex-valued constant.

Substituting **sol1** and **sol2New** into **eqEpsb[[3]]** yields

```
order3Eqb = u3 ²@tD + w² u3@tD - # & •ž eqEpsb@@3DD •. sol1 •. sol2New •• ExpandAll •• Reverse
```

$$
\begin{aligned}
& w^2 u_3@tD + u_3''@tD == -A^3 E^{3Itw} a - 2ABE^{2Itw} d + 3A^2 E^{2Itw-ItW} a L + 3A^2 E^{2Itw+ItW} a L + \\
& 2BE^{Itw-ItW} d L + 2BE^{Itw+ItW} d L - 6AE^{Itw} a L^2 - 3AE^{Itw-2ItW} a L^2 - 3AE^{Itw+2ItW} a L^2 + \\
& 3E^{-ItW} a L^3 + 3E^{ItW} a L^3 + E^{-3ItW} a L^3 + E^{3ItW} a L^3 - \frac{2A^3 E^{3Itw} d^2}{3w^2} + \frac{2A^2 E^{2Itw-ItW} d^2 L}{3w^2} + \\
& \frac{2A^2 E^{2Itw+ItW} d^2 L}{3w^2} + \frac{4AE^{Itw} d^2 L^2}{w^2} - \frac{4E^{-ItW} d^2 L^3}{w^2} - \frac{4E^{ItW} d^2 L^3}{w^2} - 2IAE^{Itw} m w - \\
& 2IE^{-ItW} L m W + 2IE^{ItW} L m W - \frac{4A^2 E^{2Itw-ItW} d^2 L}{2wW - W^2} + \frac{4AE^{Itw} d^2 L^2}{2wW - W^2} + \frac{4AE^{Itw-2ItW} d^2 L^2}{2wW - W^2} + \\
& \frac{4A^2 E^{2Itw+ItW} d^2 L}{2wW + W^2} - \frac{4AE^{Itw} d^2 L^2}{2wW + W^2} - \frac{4AE^{Itw+2ItW} d^2 L^2}{2wW + W^2} - \frac{2AE^{Itw+2ItW} d^2 L^2}{-w^2 + 4W^2} + \frac{2E^{ItW} d^2 L^3}{-w^2 + 4W^2} + \\
& \frac{2E^{3ItW} d^2 L^3}{-w^2 + 4W^2} - \frac{2AE^{Itw} d^2 L^2}{-E^{2ItW} w^2 + 4E^{2ItW} W^2} + \frac{2E^{-ItW} d^2 L^3}{-E^{2ItW} w^2 + 4E^{2ItW} W^2} + \frac{2E^{ItW} d^2 L^3}{-E^{2ItW} w^2 + 4E^{2ItW} W^2} - \\
& 3A^2 E^{Itw} a \dot{A} - 2Bd\dot{A} + 6AE^{-ItW} a L \dot{A} + 6AE^{ItW} a L \dot{A} - 6E^{-Itw} a L^2 \dot{A} - 3E^{-Itw-2ItW} a L^2 \dot{A} - \\
& 3E^{-Itw+2ItW} a L^2 \dot{A} + \frac{10A^2 E^{Itw} d^2 \dot{A}}{3w^2} - \frac{4AE^{-ItW} d^2 L \dot{A}}{w^2} - \frac{4AE^{ItW} d^2 L \dot{A}}{w^2} + \frac{4E^{-Itw} d^2 L^2 \dot{A}}{w^2} + \\
& 2IE^{-Itw} m w \dot{A} - \frac{4AE^{-ItW} d^2 L \dot{A}}{2wW - W^2} + \frac{4AE^{ItW} d^2 L \dot{A}}{2wW + W^2} - \frac{2E^{-Itw+2ItW} d^2 L^2 \dot{A}}{-w^2 + 4W^2} - \frac{2E^{-Itw} d^2 L^2 \dot{A}}{-E^{2ItW} w^2 + 4E^{2ItW} W^2} - \\
& \frac{4AE^{Itw} d^2 L \dot{A}}{2E^{Itw-ItW} wW - E^{Itw-ItW} W^2} + \frac{4E^{-ItW} d^2 L^2 \dot{A}}{2E^{Itw-ItW} wW - E^{Itw-ItW} W^2} + \frac{4E^{ItW} d^2 L^2 \dot{A}}{2E^{Itw-ItW} wW - E^{Itw-ItW} W^2} + \\
& \frac{4AE^{Itw} d^2 L \dot{A}}{2E^{Itw+ItW} wW + E^{Itw+ItW} W^2} - \frac{4E^{-ItW} d^2 L^2 \dot{A}}{2E^{Itw+ItW} wW + E^{Itw+ItW} W^2} - \frac{4E^{ItW} d^2 L^2 \dot{A}}{2E^{Itw+ItW} wW + E^{Itw+ItW} W^2} - \\
& 3AE^{-Itw} a \dot{A}^2 + 3E^{-2Itw-ItW} a L \dot{A}^2 + 3E^{-2Itw+ItW} a L \dot{A}^2 + \frac{10AE^{-Itw} d^2 \dot{A}^2}{3w^2} + \frac{2E^{-2Itw-ItW} d^2 L \dot{A}^2}{3w^2} + \\
& \frac{2E^{-2Itw+ItW} d^2 L \dot{A}^2}{3w^2} - \frac{4E^{-Itw} d^2 L \dot{A}^2}{2E^{Itw-ItW} wW - E^{Itw-ItW} W^2} + \frac{4E^{-Itw} d^2 L \dot{A}^2}{2E^{Itw+ItW} wW + E^{Itw+ItW} W^2} - \\
& E^{-3Itw} a \dot{A}^3 - \frac{2E^{-3Itw} d^2 \dot{A}^3}{3w^2} - 2Ad\dot{B} + 2E^{-Itw-ItW} d L \dot{B} + 2E^{-Itw+ItW} d L \dot{B} - 2E^{-2Itw} d \dot{A} \dot{B}
\end{aligned}
$$

The particular solution of **order3Eqb** is given by

```
Hsol3p = Hu3@tD •. DSolve@order3Eqb@@1DD == #, u3@tD, tD@@1DD •. C@_D -> 0 •• TrigToExp ••
    SimplifyL & •ž order3Eqb@@2DDL •• Timing
```

:64.803 Second,

$$\frac{A^3 E^{3 I t w} d^2}{12 w^4} + \frac{A^3 E^{3 I t w} a}{8 w^2} + \frac{2 A B E^{2 I t w} d}{3 w^2} + \frac{A E^{I t w} d^2 L^2 H1 - 2 I t wL}{w^4} +$$

$$\frac{3 I A E^{I t w} a L^2 HI + 2 t wL}{2 w^2} - \frac{A E^{I t w} m HI + 2 t wL}{2 w} - \frac{3 A E^{I t Hw-2 WL} a L^2}{4 Hw - WL W} + \frac{2 B E^{I t Hw-WL} d L}{H2 w - WL W} +$$

$$\frac{A E^{I t w} d^2 L^2 H1 - 2 I t wL}{w^2 H2 w - WL W} + \frac{3 A E^{I t Hw+2 WL} a L^2}{4 W Hw + WL} - \frac{2 B E^{I t Hw+WL} d L}{W H2 w + WL} + \frac{I A E^{I t w} d^2 L^2 HI + 2 t wL}{w^2 W H2 w + WL} +$$

$$\frac{A E^{I t Hw+2 WL} d^2 L^2}{W^2 Hw + WL H2 w + WL} - \frac{3 A^2 E^{I t H2 w+WL} a L}{Hw + WL H3 w + WL} - \frac{2 A^2 E^{I t H2 w+WL} d^2 L}{3 w^2 Hw + WL H3 w + WL} - \frac{4 A^2 E^{I t H2 w+WL} d^2 L}{W Hw + WL H2 w + WL H3 w + WL} +$$

$$\frac{E^{-3 I t W} a L^3}{w^2 - 9 W^2} + \frac{E^{3 I t W} a L^3}{w^2 - 9 W^2} + \frac{A E^{I t Hw-2 WL} d^2 L^2}{2 Hw - WL W Hw^2 - 4 W^2 L} - \frac{A E^{I t Hw+2 WL} d^2 L^2}{2 W Hw + WL Hw^2 - 4 W^2 L} + \frac{3 E^{-I t W} a L^3}{w^2 - W^2} +$$

$$\frac{3 E^{I t W} a L^3}{w^2 - W^2} + \frac{2 I E^{I t W} L m W}{w^2 - W^2} + \frac{4 E^{-I t W} d^2 L^3}{w^2 H- w^2 + W^2 L} + \frac{4 E^{I t W} d^2 L^3}{w^2 H- w^2 + W^2 L} + \frac{2 I E^{-I t W} L m W}{- w^2 + W^2} - \frac{3 A^2 E^{I t H2 w-WL} a L}{3 w^2 - 4 w W + W^2} -$$

$$\frac{2 A^2 E^{I t H2 w-WL} d^2 L}{3 w^2 H3 w^2 - 4 w W + W^2 L} + \frac{A E^{I t Hw-2WL} d^2 L^2}{W^2 H2 w^2 - 3 w W + W^2 L} - \frac{4 A^2 E^{I t H2 w-WL} d^2 L}{W H-6 w^3 + 11 w^2 W - 6 w W^2 + W^3 L} -$$

$$\frac{2 E^{-I t W} d^2 L^3}{w^4 - 5 w^2 W^2 + 4 W^4} - \frac{2 E^{I t W} d^2 L^3}{w^4 - 5 w^2 W^2 + 4 W^4} - \frac{2 E^{-3 I t W} d^2 L^3}{w^4 - 13 w^2 W^2 + 36 W^4} - \frac{2 E^{3 I t W} d^2 L^3}{w^4 - 13 w^2 W^2 + 36 W^4} - \frac{2 B d \dot A}{w^2} +$$

$$\frac{5 A^2 E^{I t w} d^2 H1 - 2 I t wL \dot A}{6 w^4} + \frac{E^{-I t w} d^2 L^2 H1 + 2 I t wL \dot A}{w^4} - \frac{3 I E^{-I t w} a L^2 H-I + 2 t wL \dot A}{2 w^2} -$$

$$\frac{E^{-I t w} m H-I + 2 t wL \dot A}{2 w} + \frac{3 I A^2 E^{I t w} a HI + 2 t wL \dot A}{4 w^2} + \frac{E^{-I t w} d^2 L^2 H1 + 2 I t wL \dot A}{w^2 H2 w - WL W} +$$

$$\frac{3 E^{-I t Hw+2 WL} a L^2 \dot A}{4 W Hw + WL} - \frac{4 A E^{-I t W} d^2 L \dot A}{W H-2 w + WL H-w + WL Hw + WL} - \frac{4 A E^{I t W} d^2 L \dot A}{W H-2 w + WL H-w + WL Hw + WL} -$$

$$\frac{I E^{-I t w} d^2 L^2 H-I + 2 t wL \dot A}{w^2 W H2 w + WL} + \frac{E^{-I t Hw+2 WL} d^2 L^2 \dot A}{W^2 Hw + WL H2 w + WL} - \frac{4 A E^{-I t W} d^2 L \dot A}{W H-w + WL Hw + WL H2 w + WL} -$$

$$\frac{4 A E^{I t W} d^2 L \dot A}{W H-w + WL Hw + WL H2 w + WL} + \frac{E^{-I t Hw-2 WL} d^2 L^2 \dot A}{2 Hw - WL W Hw^2 - 4 W^2 L} - \frac{E^{-I t Hw+2 WL} d^2 L^2 \dot A}{2 W Hw + WL Hw^2 - 4 W^2 L} +$$

$$\frac{6 A E^{-I t W} a L \dot A}{w^2 - W^2} + \frac{6 A E^{I t W} a L \dot A}{w^2 - W^2} + \frac{4 A E^{-I t W} d^2 L \dot A}{w^2 H- w^2 + W^2 L} + \frac{4 A E^{I t W} d^2 L \dot A}{w^2 H- w^2 + W^2 L} + \frac{E^{-I t Hw-2 WL} d^2 L^2 \dot A}{W^2 H2 w^2 - 3 w W + W^2 L} +$$

$$\frac{3 E^{-I t w+2 I t W} a L^2 \dot A}{-4 w W + 4 W^2} + \frac{5 A E^{-I t w} d^2 H1 + 2 I t wL \dot A^2}{6 w^4} - \frac{3 I A E^{-I t w} a H-I + 2 t wL \dot A^2}{4 w^2} -$$

$$\frac{3 E^{-I t H2 w+WL} a L \dot A^2}{Hw + WL H3 w + WL} - \frac{2 E^{-I t H2 w+WL} d^2 L \dot A^2}{3 w^2 Hw + WL H3 w + WL} - \frac{4 E^{-I t H2 w+WL} d^2 L \dot A^2}{W Hw + WL H2 w + WL H3 w + WL} -$$

$$\frac{3 E^{-2 I t w+I t W} a L \dot A^2}{3 w^2 - 4 w W + W^2} - \frac{2 E^{-2 I t w+I t W} d^2 L \dot A^2}{3 w^2 H3 w^2 - 4 w W + W^2 L} - \frac{4 E^{-2 I t w+I t W} d^2 L \dot A^2}{W H-6 w^3 + 11 w^2 W - 6 w W^2 + W^3 L} +$$

$$\frac{E^{-3 I t w} d^2 \dot A^3}{12 w^4} + \frac{E^{-3 I t w} a \dot A^3}{8 w^2} - \frac{2 A d \dot B}{w^2} + \frac{2 E^{-I t Hw-WL} d L \dot B}{H2 w - WL W} - \frac{2 E^{-I t Hw+WL} d L \dot B}{W H2 w + WL} + \frac{2 E^{-2 I t w} d \dot A \dot B}{3 w^2} >$$

Clearly, **sol3p** breaks down because it contains secular terms and small-divisor terms when W » w, W » 0, W » 2w, W » w•2, W » 3w, and W » w•3. As shown in the next section, the small-divisor terms arising from W » 3w produce a subharmonic reosonance of order one-third and those arising from W » w•3 produce a superharmonic resonance of order three. Carrying out the expansion to higher order, one finds that other resonances may occur.

We note that the resonances that occur depend on the order of the nonlinearity. Quadratic nonlinearities produce (a) subharmonic resonances of order one-half and superharmonic resonances of order two at second order and (b) subharmonic resonances of order one-third and superharmonic resonances of order three at third order. On the other hand, cubic nonlinearities produce subharmonic resonances of order one-third and superharmonic resonances of order three at third order. For a given system and order of approximation, the resonances produced can be easily identified by carrying out a straightforward expansion as done above.

In the next three sections, we use the method of multiple scales, generalized method of averaging, and method of normal forms to determine second-order uniform expansions for the solutions of **FDuffingEq** that do not contain secular or small-divisor terms.

à 4.2 The Method of Multiple Scales

Ÿ 4.2.1 Preliminaries

For a uniform second-order expansion, we need three time scales, which we symbolize and list as follows:

```
Needs@"Utilities`Notation`"D
```

```
Symbolize@T0D; Symbolize@T1D; Symbolize@T2D;
```

```
timeScales = 8T0, T1, T2<;
```

The maximum order of the expansion is related, in general, to the number of time scales by

```
maxOrder = Length@timeScalesD - 1;
```

In what follows, we need the complex conjugates of A, z, and G, which we define by

```
conjugateRule = 9Complex@0, n_D -> Complex@0, -nD, A -> Ȧ, Ȧ -> A, z -> ż, ż -> z, G -> Ġ, Ġ -> G=;
```

In terms of the time scales T_n, the time derivatives become

```
dt@1D@expr_D := SumAe^i D@expr, timeScales@@i + 1DDD, 8i, 0, maxOrder<E;
dt@2D@expr_D := Hdt@1D@dt@1D@exprDD •• ExpandL •. e^(i_•;i>maxOrder) -> 0;
```

To represent some of the expressions in a more concise way, we introduce the following display rule:

```
displayRule =
  9Derivative@a__DAz_i_E@__D :> SequenceFormATimes žž MapIndexedAD#1#2@@1DD-1 &, 8a<E, ziE,
   Derivative@a__D@AD@__D :> SequenceFormATimes žž MapIndexedAD#1#2@@1DD &, 8a<E, AE,
   Derivative@a__D@ȦD@__D :> SequenceFormATimes žž MapIndexedAD#1#2@@1DD &, 8a<E, ȦE,
   z_i_@__D -> zi, A@__D -> A, Ȧ@__D -> Ȧ=;
```

Using the time scales T_0, T_1, and T_2, we transform **FDuffingEq** from an ordinary-differential equation into a partial-differential equation according to the rule

```
multiScalesRule =
  8u@tD -> u@T0, T1, T2D, Derivative@n_D@uD@tD :> dt@nD@u@T0, T1, T2DD, t -> T0<;
```

We seek a second-order uniform expansion of the solution of **FDuffingEq** in the form

```
solRule = u -> ISumAe^i ui@#1, #2, #3D, 8i, maxOrder + 1<E &M;
```

For a uniform expansion, we need to eliminate the secular and small-divisor terms. To accomplish this, we need to distinguish between primary and secondary resonances. They are treated separately beginning with the case of primary resonance.

Ÿ 4.2.2 Primary Resonance

We scale the damping $\grave{m}$ and forcing F so that the damping term and resonance terms appear at the same order at which the secular and / or small-divisor terms generated by the nonlinearities appear. In this case, they appear at order e^3, and hence we scale the forcing and damping as

```
scaling = 9F -> e^3 f, m̀ -> e^2 m=;
```

Substituting the **solRule** into **FDuffingEq**, using the **scaling** and **multiScalesRule**, expanding the result for small e, and discarding terms of order higher than e^3, we obtain

```
eq422a =
  HFDuffingEq •. scaling •. multiScalesRule •. solRule •• ExpandAllL •. e^(n_•;n>maxOrder+1) -> 0;
eq422a •. displayRule
```

$$2\,e^3\,m\,HD_0u_1L + e\,HD_0^2u_1L + e^2\,HD_0^2u_2L + e^3\,HD_0^2u_3L + 2\,e^2\,HD_0\,D_1u_1L + 2\,e^3\,HD_0\,D_1u_2L + e^3\,HD_1^2u_1L + 2\,e^3\,HD_0\,D_2u_1L + e\,w^2\,u_1 + d\,e^2\,u_1^2 + a\,e^3\,u_1^3 + e^2\,w^2\,u_2 + 2\,d\,e^3\,u_1\,u_2 + e^3\,w^2\,u_3 == f\,e^3\,Cos@T_0\,WD$$

Equating coefficients of like powers of e in **eq422a**, we obtain

```
eqEps = CoefficientList@Subtract žž eq422a, eD == 0 •• Thread •• Rest •• TrigToExp;
```

To place the linear operator on one side and the nonhomogeneous terms on the other side, we define

```
eqOrder@i_D := HeqEps@@1, 1DD •. u1 -> ui L - # & •ž eqEps@@iDD •• Reverse
```

Using **eqOrder[i]** and the **displayRule**, we rewrite **eqEps** in a concise way as

```
Array@eqOrder, 3D •. displayRule
```

$9D_0^2u_1 + w^2 u_1 == 0,\ D_0^2u_2 + w^2 u_2 == -2\,HD_0\,D_1u_1L - d\,u_1^2,$
$D_0^2u_3 + w^2 u_3 == \frac{1}{2}\,HE^{-I\,T_0\,W} + E^{I\,T_0\,W}L\,f - 2\,m\,HD_0u_1L - 2\,HD_0\,D_1u_2L - D_1^2u_1 - 2\,HD_0\,D_2u_1L - a\,u_1^3 - 2\,d\,u_1\,u_2=$

The general solution of **eqOrder[1]** can be expressed as

```
sol1 = u1 -> FunctionA8T0, T1, T2<, A@T1, T2D Exp@I w T0D + Ā@T1, T2D Exp@-I w T0DE;
```

Using **sol1**, we rewrite **eqOrder[2]** as

```
order2Eq = eqOrder@2D •. sol1 •• ExpandAll;
order2Eq •. displayRule
```

$D_0^2u_2 + w^2 u_2 == -A^2 E^{2\,I\,T_0\,w}\,d - 2\,A\,d\,\bar{A} - E^{-2\,I\,T_0\,w}\,d\,\bar{A}^2 - 2\,I\,E^{I\,T_0\,w}\,w\,HD_1AL + 2\,I\,E^{-I\,T_0\,w}\,w\,HD_1\bar{A}L$

Eliminating the terms that produce secular terms in **order2Eq** demands that

```
SCond1 = Coefficient@order2Eq@@2DD, Exp@I w T0DD == 0
```

$-2\,I\,w\,A^{H1,0L}@T_1,\ T_2D == 0$

Solving **SCond1** for $A^{H1,0L}@T_1,\ T_2D$ yields

```
SCond1Rule = SolveASCond1, A^H1,0L@T1, T2DE@@1DD
```

$8A^{H1,0L}@T_1,\ T_2D \circledR 0<$

Hence, $A = A@T_2D$.

Substituting the solvability condition into **order2Eq**, we have

```
order2Eqm = order2Eq •. SCond1Rule •. HSCond1Rule •. conjugateRuleL;
order2Eqm •. displayRule
```

$D_0^2u_2 + w^2 u_2 == -A^2 E^{2\,I\,T_0\,w}\,d - 2\,A\,d\,\bar{A} - E^{-2\,I\,T_0\,w}\,d\,\bar{A}^2$

The particular solution of **order2Eqm** can be obtained by using **DSolve**; the result is

```
sol2p = DSolve@order2Eqm, u2 žž timeScales, timeScalesD@@1DD •. C@_D -> H0 &L •• TrigToExp ••
   ExpandAll;
sol2p •. displayRule
```

$:u_2 \circledR \frac{A^2 E^{2\,I\,T_0\,w}\,d}{3\,w^2} - \frac{2\,A\,d\,\bar{A}}{w^2} + \frac{E^{-2\,I\,T_0\,w}\,d\,\bar{A}^2}{3\,w^2}>$

Next, we use **sol2p[[1,2]]** directly to express u_2 in a pure function form as

```
sol2 = u2 -> Function@8T0, T1, T2<, sol2p@@1, 2DD •• EvaluateD;
```

Substituting the first- and second-order solutions, **sol1** and **sol2**, into the third-order equation, **eqOrder[3]**, we obtain

```
order3Eq = eqOrder[[3]] /. sol1 /. sol2 // ExpandAll;
order3Eq /. displayRule
```

$$D_0^2u_3 + \omega^2 u_3 == \frac{1}{2} E^{-I T_0 \Omega} f + \frac{1}{2} E^{I T_0 \Omega} f - A^3 E^{3 I T_0 \omega} \alpha - \frac{2 A^3 E^{3 I T_0 \omega} \delta^2}{3 \omega^2} - 2 I A E^{I T_0 \omega} \mu \omega - 3 A^2 E^{I T_0 \omega} \alpha \bar{A} + \frac{10 A^2 E^{I T_0 \omega} \delta^2 \bar{A}}{3 \omega^2} + 2 I E^{-I T_0 \omega} \mu \omega \bar{A} - 3 A E^{-I T_0 \omega} \alpha \bar{A}^2 + \frac{10 A E^{-I T_0 \omega} \delta^2 \bar{A}^2}{3 \omega^2} - E^{-3 I T_0 \omega} \alpha \bar{A}^3 - \frac{2 E^{-3 I T_0 \omega} \delta^2 \bar{A}^3}{3 \omega^2} - \frac{8 I A E^{2 I T_0 \omega} \delta (D_1 A)}{3 \omega} + \frac{8 I E^{-2 I T_0 \omega} \delta \bar{A} (D_1 \bar{A})}{3 \omega} - E^{I T_0 \omega} (D_1^2 A) - E^{-I T_0 \omega} (D_1^2 \bar{A}) - 2 I E^{I T_0 \omega} \omega (D_2 A) + 2 I E^{-I T_0 \omega} \omega (D_2 \bar{A})$$

To express the nearness of Ω to ω, we introduce a detuning parameter $\sigma = O(1)$ defined by

```
OmgRule = {Ω -> ω + ε^2 σ};
```

Using the **OmgRule**, we convert the small-divisor term arising from the excitation into a secular term. Then, eliminating the secular terms from u_3 demands that

```
expRule1 = Exp[a_] :> Exp[Expand[a /. OmgRule] /. ε^2 T0 -> T2];

SCond2 = Coefficient[order3Eq[[2]] /. expRule1, Exp[I ω T0]] == 0;
SCond2 /. displayRule
```

$$\frac{1}{2} E^{I T_2 \sigma} f - 2 I A \mu \omega - 3 A^2 \alpha \bar{A} + \frac{10 A^2 \delta^2 \bar{A}}{3 \omega^2} - D_1^2 A - 2 I \omega (D_2 A) == 0$$

Using the fact that $A = A[T_2]$ from **SCond1Rule**, we rewrite **SCond2** as

```
SCond = SCond2 /. D[SCond1Rule, T1] /. {A -> (A[#2] &), Ā -> (Ā[#2] &)}
```

$$\frac{1}{2} E^{I T_2 \sigma} f - 2 I \mu \omega A[T_2] - 3 \alpha A[T_2]^2 \bar{A}[T_2] + \frac{10 \delta^2 A[T_2]^2 \bar{A}[T_2]}{3 \omega^2} - 2 I \omega A'[T_2] == 0$$

4.2.3 Secondary Resonances Due to Cubic Nonlinearities

In this case Ω is away from ω, 2ω, and $\omega/2$ and small-divisor terms first appear at $O(\epsilon^3)$. We scale the damping $\hat{\mu}$ and forcing F so that the damping term and resonance terms appear at the same order as the cubic nonlinearity according to

```
scaling = {F -> ε f, μ̂ -> ε^2 μ};
```

```
eq423a =
  HFDuffingEq •. scaling •. multiScalesRule •. solRule •• ExpandAllL •. e^(n_•;n>maxOrder+1) -> 0;
eq423a •. displayRule
```

2 e^3 m HD_0u_1L + e HD_0^2u_1L + e^2 HD_0^2u_2L + e^3 HD_0^2u_3L + 2 e^2 HD_0 D_1u_1L + 2 e^3 HD_0 D_1u_2L + e^3 HD_1^2u_1L +
  2 e^3 HD_0 D_2u_1L + e w^2 u_1 + d e^2 u_1^2 + a e^3 u_1^3 + e^2 w^2 u_2 + 2 d e^3 u_1 u_2 + e^3 w^2 u_3 == f e Cos@T_0 WD

Equating coefficients of like powers of e, we obtain

```
eqEps = CoefficientList@Subtract žž eq423a, eD == 0 •• Thread •• Rest •• TrigToExp;
```

To place the linear operator on one side and the nonhomogeneous terms on the other side, we define

```
eqOrder@i_D := HeqEps@@1, 1DD •. f -> 0 •. u_1 -> u_iL - # & •ž eqEps@@iDD •• Reverse
```

Using **eqOrder[i]** and **displayRule**, we rewrite **eqEps** in a concise way as

```
Array@eqOrder, 3D •. displayRule •• TableForm
```

D_0^2u_1 + w^2 u_1 == 1/2 HE^(-I T_0 W) + E^(I T_0 W)L f
D_0^2u_2 + w^2 u_2 == -2 HD_0 D_1u_1L - d u_1^2
D_0^2u_3 + w^2 u_3 == -2 m HD_0u_1L - 2 HD_0 D_1u_2L - D_1^2u_1 - 2 HD_0 D_2u_1L - a u_1^3 - 2 d u_1 u_2

The particular solution of **eqOrder[1]** can be expressed as

```
sol1p = DSolve@eqOrder@1D, u_1 žž timeScales, timeScalesD@@1DD •. C@_D -> H0 &L •• Simplify
```

:u_1@T_0, T_1, T_2D ® - (E^(-I T_0 W) H1 + E^(2 I T_0 W)L f) / (2 H-w^2 + W^2L) >

The general solution of **eqOrder[1]** can be written as

```
fRule = 9f ® -2 L I-w^2 + W^2M=;
```

```
sol1 = u_1 -> FunctionA8T_0, T_1, T_2<,
   A@T_1, T_2D Exp@I w T_0D + Ȧ@T_1, T_2D Exp@-I w T_0D + sol1p@@1, 2DD •. fRule •• Expand •• EvaluateE
```

u_1 ® Function@8T_0, T_1, T_2<, E^(-I T_0 W) L + E^(I T_0 W) L + E^(I T_0 w) A@T_1, T_2D + E^(-I T_0 w) Ȧ@T_1, T_2DD

Substituting **sol1** into **eqOrder[2]** yields

```
order2Eq = eqOrder@2D •. sol1 •• ExpandAll;
order2Eq •. displayRule
```

D_0^2u_2 + w^2 u_2 == -A^2 E^(2 I T_0 w) d - 2 A E^(I T_0 w-I T_0 W) d L - 2 A E^(I T_0 w+I T_0 W) d L - 2 d L^2 - E^(-2 I T_0 W) d L^2 - E^(2 I T_0 W) d L^2 -
  2 A d Ȧ - 2 E^(-I T_0 w-I T_0 W) d L Ȧ - 2 E^(-I T_0 w+I T_0 W) d L Ȧ - E^(-2 I T_0 w) d Ȧ^2 - 2 I E^(I T_0 w) w HD_1AL + 2 I E^(-I T_0 w) w HD_1ȦL

The Case Ω ≈ 3ω

To express the nearness of Ω to 3ω, we introduce a detuning parameter $\sigma = O(1)$ defined by

```
OmgRule = {Ω -> 3 ω + ε^2 σ};

expRule1 = {Exp[arg_] :> Exp[Expand[arg /. OmgRule] /. ε^2 T0 -> T2]};
```

Eliminating the secular terms from u_2 demands that

```
SCond1 = Coefficient[order2Eq[[2]] /. expRule1, Exp[I ω T0]] == 0
```

$$-2 I \omega A^{(1,0)}[T_1, T_2] == 0$$

```
SCond1Rule = Solve[SCond1, A^(1,0)[T1, T2]][[1]]
```

$$\{A^{(1,0)}[T_1, T_2] \to 0\}$$

whose complex conjugate is

```
ccSCond1Rule = SCond1Rule /. conjugateRule
```

$$\{\bar{A}^{(1,0)}[T_1, T_2] \to 0\}$$

Substituting the solvability condition, **SCond1Rule**, and its complex conjugate, into **order2Eq**, we have

```
(order2Eqm = order2Eq /. SCond1Rule /. ccSCond1Rule) /. displayRule
```

$$D_0^2 u_2 + \omega^2 u_2 == -A^2 E^{2 I T_0 \omega} \delta - 2 A E^{I T_0 \omega - I T_0 \Omega} \delta \Lambda - 2 A E^{I T_0 \omega + I T_0 \Omega} \delta \Lambda - 2 \delta \Lambda^2 - E^{-2 I T_0 \Omega} \delta \Lambda^2 - E^{2 I T_0 \Omega} \delta \Lambda^2 - 2 A \delta \bar{A} - 2 E^{-I T_0 \omega - I T_0 \Omega} \delta \Lambda \bar{A} - 2 E^{-I T_0 \omega + I T_0 \Omega} \delta \Lambda \bar{A} - E^{-2 I T_0 \omega} \delta \bar{A}^2$$

In order to efficiently use **DSolve** to determine the particular solution of **order2Eqm**, we first transform the partial-differential equation to an ordinary-differential equation. Then we solve for the particular solution of the resulting equation and obtain

```
sol2p =
  (u2[T0] /. DSolve[(order2Eqm[[1]] /. u2 -> (u2[#1] &)) == #, u2[T0], T0][[1]] /. C[_] -> 0 //
      TrigToExp // Expand) & /@ order2Eqm[[2]];
sol2p /. displayRule
```

$$\frac{A^2 E^{2 I T_0 \omega} \delta}{3 \omega^2} - \frac{2 \delta \Lambda^2}{\omega^2} - \frac{2 A E^{I T_0 (\omega - \Omega)} \delta \Lambda}{(2 \omega - \Omega) \Omega} + \frac{2 A E^{I T_0 (\omega + \Omega)} \delta \Lambda}{\Omega (2 \omega + \Omega)} + \frac{E^{-2 I T_0 \Omega} \delta \Lambda^2}{(-\omega + 2 \Omega)(\omega + 2 \Omega)} + \frac{E^{2 I T_0 \Omega} \delta \Lambda^2}{(-\omega + 2 \Omega)(\omega + 2 \Omega)} - \frac{2 A \delta \bar{A}}{\omega^2} - \frac{2 E^{-I T_0 (\omega - \Omega)} \delta \Lambda \bar{A}}{(2 \omega - \Omega) \Omega} + \frac{2 E^{-I T_0 (\omega + \Omega)} \delta \Lambda \bar{A}}{\Omega (2 \omega + \Omega)} + \frac{E^{-2 I T_0 \omega} \delta \bar{A}^2}{3 \omega^2}$$

which can be used directly to express u_2 in a pure function form as

```
sol2 = u2 -> Function[{T0, T1, T2}, sol2p // Evaluate];
```

Substituting **sol1** and **sol2** into the third-order equation, **eqOrder[3]**, we obtain

```
order3Eq = eqOrder@3D •. sol1 •. sol2 •• ExpandAll;
```

Eliminating the secular terms from u_3 demands that

```
SCond2 =
  Coefficient@order3Eq@@2DD •. expRule1 •. W -> 3 w, Exp@I w T0DD == 0 •. D@SCond1Rule, T1D;
SCond2 •. displayRule
```

$$-6\,A\,a\,L^2+\frac{12\,A\,d^2\,L^2}{5\,w^2}-2\,I\,A\,m\,w-3\,A^2\,a\,\dot{A}+\frac{10\,A^2\,d^2\,\dot{A}}{3\,w^2}-3\,E^{I\,T_2\,s}\,a\,L\,\dot{A}^2-\frac{2\,E^{I\,T_2\,s}\,d^2\,L\,\dot{A}^2}{w^2}-2\,I\,w\,HD_2AL==0$$

Ÿ The Case $W \approx \frac{1}{3} w$

To express the nearness of W to $\frac{1}{3}$ w, we introduce a detuning parameter $s = O\text{H}1\text{L}$ defined by

```
OmgRule = 9W -> 1/3 Iw + e^2 sM=;
```

```
expRule1 = 9Exp@arg_D :> ExpAExpand@arg •. OmgRuleD •. e^2 T0 -> T2E=;
```

Eliminating the secular terms from u_2 demands that

```
SCond1 = Coefficient@order2Eq@@2DD •. expRule1, Exp@I w T0DD == 0
```

```
-2 I w A^H1,0L@T1, T2D == 0
```

or

```
SCond1Rule = SolveASCond1, A^H1,0L@T1, T2DE@@1DD
```

```
8A^H1,0L@T1, T2D ® 0<
```

whose complex conjugate is

```
ccSCond1Rule = SCond1Rule •. conjugateRule;
```

Substituting the solvability condition, **SCond1Rule**, and its complex conjugate, into **order2Eq**, we have

```
order2Eqm = order2Eq •. SCond1Rule •. ccSCond1Rule;
order2Eqm •. displayRule
```

$$D_0^2u_2+w^2\,u_2==-A^2\,E^{2\,I\,T_0\,w}\,d-2\,A\,E^{I\,T_0\,w-I\,T_0\,W}\,d\,L-2\,A\,E^{I\,T_0\,w+I\,T_0\,W}\,d\,L-2\,d\,L^2-$$
$$E^{-2\,I\,T_0\,W}\,d\,L^2-E^{2\,I\,T_0\,W}\,d\,L^2-2\,A\,d\,\dot{A}-2\,E^{-I\,T_0\,w-I\,T_0\,W}\,d\,L\,\dot{A}-2\,E^{-I\,T_0\,w+I\,T_0\,W}\,d\,L\,\dot{A}-E^{-2\,I\,T_0\,w}\,d\,\dot{A}^2$$

Again, we first transform the partial-differential equation into an ordinary-differential equation and then use **DSolve** to obtain the particular solution of **order2Eqm** as

```
Hsol2p =
  Hu2@T0D •. DSolve@Horder2Eqm@@1DD •. u2 -> Hu2@#1D &LL == #, u2@T0D, T0D@@1DD •. C@_D -> 0 ••
     TrigToExpL & •ž order2Eqm@@2DDL •. displayRule
```

$$\frac{A^2 E^{2 I T_0 w}\, d}{3 w^2} - \frac{2 d L^2}{w^2} - \frac{2 A E^{I T_0 (w-W)} d L}{(2 w - W) W} + \frac{2 A E^{I T_0 (w+W)} d L}{W (2 w + W)} + \frac{E^{-2 I T_0 W} d L^2}{(-w + 2 W)(w + 2 W)} +$$
$$\frac{E^{2 I T_0 W} d L^2}{(-w + 2 W)(w + 2 W)} - \frac{2 A d \bar{A}}{w^2} - \frac{2 E^{-I T_0 (w-W)} d L \bar{A}}{(2 w - W) W} + \frac{2 E^{-I T_0 (w+W)} d L \bar{A}}{W (2 w + W)} + \frac{E^{-2 I T_0 w} d \bar{A}^2}{3 w^2}$$

```
sol2 = u2 -> Function@8T0, T1, T2<, sol2p •• EvaluateD;
```

Substituting **sol1** and **sol2** into the third-order equation, **eqOrder[3]**, we obtain

```
order3Eq = eqOrder@3D •. sol1 •. sol2 •• ExpandAll;
```

Converting the terms that produce small-divisor terms into terms that produce secular terms and then eliminating the secular terms from u_3 demands that

```
SCond2 =
  CoefficientAorder3Eq@@2DD •. expRule1 •. W -> 1/3 w, Exp@I w T0DE == 0 •. D@SCond1Rule, T1D;
SCond2 •. displayRule
```

$$-6 A a L^2 - E^{I T_2 s} a L^3 + \frac{212 A d^2 L^2}{35 w^2} + \frac{18 E^{I T_2 s} d^2 L^3}{5 w^2} - 2 I A m w - 3 A^2 a \bar{A} + \frac{10 A^2 d^2 \bar{A}}{3 w^2} - 2 I w (D_2 A) == 0$$

Ÿ 4.2.4 Secondary Resonances Due to Quadratic Nonlinearities

In this case W is away from w and small-divisor terms first appear at $O(e^2)$. We scale the damping $\grave{m}$ and forcing F so that the damping term and resonance terms appear at the same order as the quadratic nonlinearity; that is,

```
scaling = 8F -> e f, m̀ -> e m<;
```

```
eq424a =
  HFDuffingEq •. scaling •. multiScalesRule •. solRule •• ExpandAllL •. e^(n_•;n>maxOrder+1) -> 0;
eq424a •. displayRule
```

$$2 e^2 m (D_0 u_1) + 2 e^3 m (D_0 u_2) + e (D_0^2 u_1) + e^2 (D_0^2 u_2) + e^3 (D_0^2 u_3) +$$
$$2 e^3 m (D_1 u_1) + 2 e^2 (D_0 D_1 u_1) + 2 e^3 (D_0 D_1 u_2) + e^3 (D_1^2 u_1) + 2 e^3 (D_0 D_2 u_1) +$$
$$e w^2 u_1 + d e^2 u_1^2 + a e^3 u_1^3 + e^2 w^2 u_2 + 2 d e^3 u_1 u_2 + e^3 w^2 u_3 == f e \,\mathrm{Cos}[T_0 W]$$

Equating coefficients of like powers of e, we obtain

```
eqEps = CoefficientList@Subtract žž eq424a, eD == 0 •• Thread •• Rest •• TrigToExp;
```

To place the linear operator on one side and the nonhomogeneous terms on the other side, we define

```
eqOrder@i_D := HeqEps@@1, 1DD •. f -> 0 •. u1 -> uiL - # & •ž eqEps@@iDD •• Reverse
```

Using **eqOrder[i]** and **displayRule**, we rewrite **eqEps** in a concise way as

```
Array[eqOrder, 3] /. displayRule // TableForm
```

$$D_0^2u_1 + \omega^2 u_1 == \frac{1}{2}\left(E^{-I T_0 \Omega} + E^{I T_0 \Omega}\right) f$$
$$D_0^2u_2 + \omega^2 u_2 == -2\,\mu\,(D_0u_1) - 2\,(D_0\,D_1u_1) - \delta\, u_1^2$$
$$D_0^2u_3 + \omega^2 u_3 == -2\,\mu\,(D_0u_2) - 2\,\mu\,(D_1u_1) - 2\,(D_0\,D_1u_2) - D_1^2u_1 - 2\,(D_0\,D_2u_1) - \alpha\, u_1^3 - 2\,\delta\, u_1\, u_2$$

The particular solution of **eqOrder[1]** can be expressed as

```
sol1p = DSolve[eqOrder[1], u1 @@ timeScales, timeScales[[1]]] /. C[_] -> (0 &) // Simplify
```

$$\left\{u_1[T_0, T_1, T_2] \to -\frac{E^{-I T_0 \Omega}\left(1 + E^{2 I T_0 \Omega}\right) f}{2\left(-\omega^2 + \Omega^2\right)}\right\}$$

```
fRule = f -> -2 Λ (-ω^2 + Ω^2);
```

The general solution of **eqOrder[1]** can be written in pur function form as

```
sol1 = u1 -> Function[{T0, T1, T2},
   A[T1, T2] Exp[I ω T0] + Ā[T1, T2] Exp[-I ω T0] + sol1p[[1, 2]] /. fRule // Expand // Evaluate]
```

$$u_1 \to \text{Function}[\{T_0, T_1, T_2\}, E^{-I T_0 \Omega}\Lambda + E^{I T_0 \Omega}\Lambda + E^{I T_0 \omega} A[T_1, T_2] + E^{-I T_0 \omega}\bar{A}[T_1, T_2]]$$

Substituting **sol1** into **eqOrder[2]** yields

```
order2Eq = eqOrder[2] /. sol1 // ExpandAll;
order2Eq /. displayRule
```

$$D_0^2u_2 + \omega^2 u_2 == -A^2 E^{2 I T_0 \omega}\delta - 2 A E^{I T_0 \omega - I T_0 \Omega}\delta\Lambda - 2 A E^{I T_0 \omega + I T_0 \Omega}\delta\Lambda - 2\delta\Lambda^2 - E^{-2 I T_0 \Omega}\delta\Lambda^2 - E^{2 I T_0 \Omega}\delta\Lambda^2 - 2 I A E^{I T_0 \omega}\mu\omega + 2 I E^{-I T_0 \Omega}\Lambda\mu\Omega - 2 I E^{I T_0 \Omega}\Lambda\mu\Omega - 2 A\delta\bar{A} - 2 E^{-I T_0 \omega - I T_0 \Omega}\delta\Lambda\bar{A} - 2 E^{-I T_0 \omega + I T_0 \Omega}\delta\Lambda\bar{A} + 2 I E^{-I T_0 \omega}\mu\omega\bar{A} - E^{-2 I T_0 \omega}\delta\bar{A}^2 - 2 I E^{I T_0 \omega}\omega(D_1A) + 2 I E^{-I T_0 \omega}\omega(D_1\bar{A})$$

• The Case $\Omega \approx 2\omega$

To express the nearness of Ω to 2ω, we introduce a detuning parameter $\sigma = O(1)$ defined by

```
OmgRule = {Ω -> 2 ω + ε σ};
```

```
sigRule = Solve[OmgRule /. Rule -> Equal, σ][[1]]
```

$$\left\{\sigma \to -\frac{2\omega - \Omega}{\epsilon}\right\}$$

```
expRule2 = {Exp[arg_] :> Exp[Expand[arg /. OmgRule] /. ε T0 -> T1]};
```

Converting the terms that produce small-divisor terms into terms that produce secular terms and then eliminating the secular terms from u_2 demands that

```
SCond1 = Coefficient@order2Eq@@2DD •. expRule2, Exp@I w T0DD == 0;
SCond1 •. displayRule
```

$$-2\,I\,A\,m\,w - 2\,E^{I\,T_1\,s}\,d\,L\,\dot{A} - 2\,I\,w\,HD_1AL == 0$$

or

```
SCond1Rule = ISolveASCond1, A^H1,0L@T1, T2DE@@1DD •• ExpandAllM;
SCond1Rule •. displayRule
```

$$:D_1A \,®\, -A\,m + \frac{I\,E^{I\,T_1\,s}\,d\,L\,\dot{A}}{w}>$$

Substituting **sigRule** into **SCond1Rule** yields

```
SCond1Rulem = SCond1Rule •. Exp@a_D :> Exp@a •. sigRule •. T1 -> e T0 •• ExpandD;
SCond1Rulem •. displayRule
```

$$:D_1A \,®\, -A\,m + \frac{I\,E^{-2\,I\,T_0\,w+I\,T_0\,W}\,d\,L\,\dot{A}}{w}>$$

whose complex conjugate is

```
ccSCond1Rulem = SCond1Rulem •. conjugateRule;
```

Substituting these conditions into **order2Eq** yields

```
order2Eqm = order2Eq •. SCond1Rulem •. ccSCond1Rulem •• ExpandAll;
order2Eqm •. displayRule
```

$$D_0^2u_2 + w^2\,u_2 == -A^2\,E^{2\,I\,T_0\,w}\,d - 2\,A\,E^{I\,T_0\,w+I\,T_0\,W}\,d\,L - 2\,d\,L^2 - E^{-2\,I\,T_0\,W}\,d\,L^2 - E^{2\,I\,T_0\,W}\,d\,L^2 + 2\,I\,E^{-I\,T_0\,W}\,L\,m\,W - 2\,I\,E^{I\,T_0\,W}\,L\,m\,W - 2\,A\,d\,\dot{A} - 2\,E^{-I\,T_0\,w-I\,T_0\,W}\,d\,L\,\dot{A} - E^{-2\,I\,T_0\,w}\,d\,\dot{A}^2$$

Transforming **order2Eqm** into an ordinary-differential equation, we can obtain the particular solution as

```
sol2p =
  Hu2@T0D •. DSolve@Horder2Eqm@@1DD •. u2 -> Hu2@#1D &LL == #, u2@T0D, T0D@@1DD •. C@_D -> 0 ••
      TrigToExpL & •ž order2Eqm@@2DD;
sol2p •. displayRule
```

$$\frac{A^2\,E^{2\,I\,T_0\,w}\,d}{3\,w^2} - \frac{2\,d\,L^2}{w^2} - \frac{2\,I\,E^{-I\,T_0\,W}\,L\,m\,W}{H-w+WL\,Hw+WL} + \frac{2\,I\,E^{I\,T_0\,W}\,L\,m\,W}{H-w+WL\,Hw+WL} + \frac{2\,A\,E^{I\,T_0\,Hw+WL}\,d\,L}{W\,H2\,w+WL} + \frac{E^{-2\,I\,T_0\,W}\,d\,L^2}{H-w+2\,WL\,Hw+2\,WL} + \frac{E^{2\,I\,T_0\,W}\,d\,L^2}{H-w+2\,WL\,Hw+2\,WL} - \frac{2\,A\,d\,\dot{A}}{w^2} + \frac{2\,E^{-I\,T_0\,Hw+WL}\,d\,L\,\dot{A}}{W\,H2\,w+WL} + \frac{E^{-2\,I\,T_0\,w}\,d\,\dot{A}^2}{3\,w^2}$$

```
sol2 = u2 -> Function@8T0, T1, T2<, sol2p •• EvaluateD;
```

Then, using **sol1** and **sol2** in **eqOrder[3]**, we have

```
order3Eq = eqOrder@3D •. sol1 •. sol2 •• ExpandAll;
```

Converting the terms that produce small-divisor terms into terms that produce secular terms and then eliminating the secular terms from u_3 demands that

```
SCond2 =
  Coefficient[order3Eq[[2]] /. expRule2 /. Ω -> 2 ω, Exp[I ω T0]] == 0 /. D[SCond1Rule, T1] /.
    SCond1Rule /. (SCond1Rule /. conjugateRule) // ExpandAll;
SCond2 /. displayRule
```

$$-6\,A\,\alpha\,\Lambda^2+A\,\mu^2+\frac{5\,A\,\delta^2\,\Lambda^2}{2\,\omega^2}-3\,A^2\,\alpha\,\bar{A}+\frac{10\,A^2\,\delta^2\,\bar{A}}{3\,\omega^2}-\frac{8\,I\,E^{I\,T_1\,\sigma}\,\delta\,\Lambda\,\mu\,\bar{A}}{3\,\omega}+\frac{E^{I\,T_1\,\sigma}\,\delta\,\Lambda\,\sigma\,\bar{A}}{\omega}-2\,I\,\omega\,(D_2A)==0$$

or

```
SCond2Rule = Solve[SCond2, A^(0,1)[T1, T2]][[1]] // ExpandAll;
SCond2Rule /. displayRule
```

$$\left\{D_2A\to-\frac{5\,I\,A\,\delta^2\,\Lambda^2}{4\,\omega^3}+\frac{3\,I\,A\,\alpha\,\Lambda^2}{\omega}-\frac{I\,A\,\mu^2}{2\,\omega}-\frac{5\,I\,A^2\,\delta^2\,\bar{A}}{3\,\omega^3}-\frac{4\,E^{I\,T_1\,\sigma}\,\delta\,\Lambda\,\mu\,\bar{A}}{3\,\omega^2}-\frac{I\,E^{I\,T_1\,\sigma}\,\delta\,\Lambda\,\sigma\,\bar{A}}{2\,\omega^2}+\frac{3\,I\,A^2\,\alpha\,\bar{A}}{2\,\omega}\right\}$$

The two partial-differential equations, **SCond1Rule** and **SCond2Rule**, can be reconstituted to obtain an ordinary-differential equation governing A. The result is

```
(2 I ω A' == (2 I ω dt[1][A[T1, T2]] /. SCond1Rule /. SCond2Rule // Expand // Collect[#, ε] &)) /.
  displayRule
```

$$2\,I\,\omega\,A'==\epsilon\,(-2\,I\,A\,\mu\,\omega-2\,E^{I\,T_1\,\sigma}\,\delta\,\Lambda\,\bar{A})+\epsilon^2\left(-6\,A\,\alpha\,\Lambda^2+A\,\mu^2+\frac{5\,A\,\delta^2\,\Lambda^2}{2\,\omega^2}-3\,A^2\,\alpha\,\bar{A}+\frac{10\,A^2\,\delta^2\,\bar{A}}{3\,\omega^2}-\frac{8\,I\,E^{I\,T_1\,\sigma}\,\delta\,\Lambda\,\mu\,\bar{A}}{3\,\omega}+\frac{E^{I\,T_1\,\sigma}\,\delta\,\Lambda\,\sigma\,\bar{A}}{\omega}\right)$$

4.2.5 First-Order Real-Valued System

As in Section 4.2.4, we determine a second-order uniform expansion of the solution of the **FDuffingEq** for the case of subharmonic resonance of order one-half. However, instead of treating the second-order form of this equation, we treat its corresponding first-order form; that is, we first transform it into a system of two real-valued first-order equations. To this end, we introduce the transformation

```
vRule = u'[t] -> v[t];
```

In order that the influence of the nonlinearity and damping balance the subharmonic resonance, we scale the damping coefficient $\grave{\mu}$ and forcing amplitude F as

```
scaling = {F -> ε f, μ̀ -> ε μ};
```

Using the **scaling** and **vRule**, we transform the **FDuffingEq** into the following system of two real-valued first-order equations:

```
eq425a = {vRule /. Rule -> Equal, FDuffingEq /. scaling /. vRule /. D[vRule, t]}
```

$$\{u'[t]==v[t],\ \omega^2\,u[t]+\delta\,u[t]^2+\alpha\,u[t]^3+2\,\epsilon\,\mu\,v[t]+v'[t]==f\,\epsilon\,\mathrm{Cos}[t\,\Omega]\}$$

To determine a second-order uniform expansion of the solution of **eq425a** using the method of multiple scales, we first transform these equations from ordinary-differential equations into partial-differential equations in terms of the three time scales T_0, T_1, and T_2 according to

```
multiScales =
 8u@tD -> u@T0, T1, T2D, v@tD -> v@T0, T1, T2D, u_¢@tD -> dt@1D@u@T0, T1, T2DD, t -> T0<;
```

Next, we expand u and v in the form

```
solRule = 9u -> ISumAe^j uj@#1, #2, #3D, 8j, 3<E &M, v -> ISumAe^j vj@#1, #2, #3D, 8j, 3<E &M=;
```

Substituting **multiScales** and **solRule** into **eq425a**, expanding the result for small e, and discarding terms of order higher than e^3, we obtain

```
eq425b = Heq425a •. multiScales •. solRule •• ExpandAllL •. e^(n_•;n>3) -> 0;
```

Equating coefficients of like powers of e in **eq425b** yields

```
eqEps = Thread@CoefficientList@Subtract žž #, eD == 0D & •ž eq425b •• Transpose •• Rest ••
   TrigToExp;
```

To place the linear operator on one side and the nonhomogeneous terms on the other side, we define

```
linearSys = #@@1DD & •ž eqEps@@1DD •. f -> 0;
linearSys •. displayRule
```

```
8D0u1 - v1, D0v1 + w^2 u1<
```

```
eqOrder@i_D :=
 HlinearSys •. u_1 -> uiL == HlinearSys •. u_1 -> uiL - H#@@1DD & •ž eqEps@@iDDL •• Thread
```

Using **eqOrder[i]** and the **displayRule**, we rewrite **eqEps** in a concise way as

```
eqOrder@1D •. displayRule •• TableForm
eqOrder@2D •. displayRule •• TableForm
eqOrder@3D •. displayRule •• TableForm
```

```
D0u1 - v1 == 0
D0v1 + w^2 u1 == 1/2 HE^(-I T0 W) + E^(I T0 W)L f

D0u2 - v2 == -HD1u1L
D0v2 + w^2 u2 == -HD1v1L - d u1^2 - 2 m v1

D0u3 - v3 == -HD1u2L - D2u1
D0v3 + w^2 u3 == -HD1v2L - D2v1 - a u1^3 - 2 d u1 u2 - 2 m v2
```

Ÿ First-Order Equations: Linear System

We seek a solution of the **linearSys** in the form $u_1 = P E^{I\,w\,T_0}$ and $v_1 = Q E^{I\,w\,T_0}$ and obtain

```
coefList = E^(-I w T0) linearSys /. {u1 -> (P E^(I w #) &), v1 -> (Q E^(I w #) &)} // Expand
```

```
{-Q + I P w, I Q w + P w^2}
```

The coefficient matrix of **coefList** is

```
coefMat = Outer[Coefficient, coefList, {P, Q}]
```

```
{{I w, -1}, {w^2, I w}}
```

whose adjoint is defined by

```
hermitian[mat_?MatrixQ] := mat /. conjugateRule // Transpose
```

Hence, the right and left eigenvectors of **coefMat** are

```
rightVec = {1, c1} /. Solve[(coefMat.{1, c1})[[1]] == 0, c1][[1]]
```

```
{1, I w}
```

```
leftVec = {c1, 1} /. Solve[(hermitian[coefMat].{c1, 1})[[1]] == 0, c1][[1]]
```

```
{-I w, 1}
```

The complex conjugate of **leftVec** is

```
ccleftVec = leftVec /. conjugateRule
```

```
{I w, 1}
```

Thefore, the homogeneous solution of **eqOrder[1]** is given by

```
sol1h = # + (# /. conjugateRule) & /@ (rightVec A[T1, T2] E^(I w T0))
```

$$\{E^{I T_0 w} A[T_1, T_2] + E^{-I T_0 w} \bar{A}[T_1, T_2],\ I E^{I T_0 w} w A[T_1, T_2] - I E^{-I T_0 w} w \bar{A}[T_1, T_2]\}$$

Transforming **eqOrder[1]** into ordinary-differential equations and using **DSolve**, we obtain the particular solution of **eqOrder[1]** as

```
sol1p = DSolve[eqOrder[1] /. {u1 -> (u1[#1] &), v1 -> (v1[#1] &)}, {u1[T0], v1[T0]}, T0][[1]] /.
    C[_] -> 0 // TrigToExp // Simplify
```

$$\left\{u_1[T_0] \to -\frac{E^{-I T_0 W}(1 + E^{2 I T_0 W}) f}{2(-w^2 + W^2)},\ v_1[T_0] \to -\frac{I E^{-I T_0 W}(-1 + E^{2 I T_0 W}) f W}{2(-w^2 + W^2)}\right\}$$

Then, the general solution of the first-order equations can be expressed in pure function form as

```
fRule = f -> -2 (-w^2 + W^2) L;
```

```
sol1 =
 8u1 -> Function@8T0, T1, T2<, sol1h@@1DD + Hu1@T0D •. sol1p •. fRule •• ExpandL •• EvaluateD,
  v1 -> Function@8T0, T1, T2<, sol1h@@2DD + Hv1@T0D •. sol1p •. fRule •• ExpandL •• EvaluateD<
```

```
8u1 ® Function@8T0, T1, T2<, E^(-I T0 W) L + E^(I T0 W) L + E^(I T0 w) A@T1, T2D + E^(-I T0 w) Ȧ@T1, T2DD,
 v1 ® Function@8T0, T1, T2<, -I E^(-I T0 W) L W + I E^(I T0 W) L W + I E^(I T0 w) w A@T1, T2D - I E^(-I T0 w) w Ȧ@T1, T2DD<
```

Ÿ Second-Order Equations

Substituting the first-order solution, **sol1**, into the second-order equations, **eqOrder[2]**, we have

```
order2Eq = eqOrder@2D •. sol1 •• ExpandAll;
order2Eq •. displayRule
```

```
:D0u2 - v2 == -E^(I T0 w) HD1AL - E^(-I T0 w) HD1ȦL,
 D0v2 + w^2 u2 == -A^2 E^(2 I T0 w) d - 2 A E^(I T0 w-I T0 W) d L - 2 A E^(I T0 w+I T0 W) d L - 2 d L^2 - E^(-2 I T0 W) d L^2 -
   E^(2 I T0 W) d L^2 - 2 I A E^(I T0 w) m w + 2 I E^(-I T0 W) L m W - 2 I E^(I T0 W) L m W - 2 A d Ȧ - 2 E^(-I T0 w-I T0 W) d L Ȧ -
   2 E^(-I T0 w+I T0 W) d L Ȧ + 2 I E^(-I T0 w) m w Ȧ - E^(-2 I T0 w) d Ȧ^2 - I E^(I T0 w) w HD1AL + I E^(-I T0 w) w HD1ȦL>
```

To express quantitatively the nearness of the subharmonic resonance of order one-half, we introduce the detuning parameter s defined by

```
OmgRule = 8W -> 2 w + e s<;
```

To convert the terms that produce small-divisor terms into terms that produce secular terms, we use the rule

```
expRule1 = Exp@arg_D :> Exp@Expand@arg •. OmgRuleD •. e T0 -> T1D;
```

To eliminate the terms that produce secular terms in u_2 and v_2 (i.e., the solvability condition of **order2Eq**), we first determine the vector proportional to $E^{I\,w\,T_0}$ in the right-hand sides of **order2Eq** and obtain

```
STerms1 = Coefficient@#@@2DD •. expRule1, Exp@I w T0DD & •ž order2Eq
```

```
8-A^(H1,0L)@T1, T2D, -2 I m w A@T1, T2D - 2 E^(I T1 s) d L Ȧ@T1, T2D - I w A^(H1,0L)@T1, T2D<
```

Then, the solvability condition of **order2Eq** demands that **STerms1** be orthogonal to the solution of the adjoint; that is, **ccleftVec**. The result is

```
SCond1 = SolveAccleftVec.STerms1 == 0, A^(H1,0L)@T1, T2DE@@1DD
```

```
:A^(H1,0L)@T1, T2D ® - (m w A@T1, T2D - I E^(I T1 s) d L Ȧ@T1, T2D) / w >
```

whose complex conjugate is

```
ccSCond1 = SCond1 •. conjugateRule
```

```
:Ȧ^(H1,0L)@T1, T2D ® - (I E^(-I T1 s) d L A@T1, T2D + m w Ȧ@T1, T2D) / w >
```

Next, we use the solvability condition to eliminate $D_1\, A$ and its complex conjugate from **order2Eq** and then find their particular solution. To simplify the resulting equations using *Mathematica*, we need to replace T_1 with e T_0. To this end, we express the detuning parameter s in terms of w and W as

```
sigRule = Solve@OmgRule •. Rule -> Equal, sD@@1DD
```

```
      2 w - W
9s ® - ─────── =
         e
```

and define the rule

```
expRule2 = Exp@a_D :> Exp@a •. T1 -> e T0 •. sigRule •• ExpandD;
```

This rule enables us to rewrite **SCond1** and **ccSCond1** as

```
SCond1f = Join@SCond1, ccSCond1D •. expRule2;
SCond1f •. displayRule
```

```
           A m w - I E^(-2 I T0 w+I T0 W) d L Ȧ           I A E^(2 I T0 w-I T0 W) d L + m w Ȧ
:D1A ® - ───────────────────────────────────── , D1Ȧ ® - ──────────────────────────────────── >
                          w                                              w
```

With this form of the solvability condition, **order2Eq** becomes

```
order2Eqm = order2Eq •. SCond1f •• ExpandAll;
order2Eqm •. displayRule
```

```
                               I A E^(I T0 w-I T0 W) d L                          I E^(-I T0 w+I T0 W) d L Ȧ
:D0u2 - v2 == A E^(I T0 w) m + ───────────────────────── + E^(-I T0 w) m Ȧ - ─────────────────────────── ,
                                          w                                              w
 D0v2 + w^2 u2 == -A^2 E^(2 I T0 w) d - A E^(I T0 w-I T0 W) d L - 2 A E^(I T0 w+I T0 W) d L - 2 d L^2 -
   E^(-2 I T0 W) d L^2 - E^(2 I T0 W) d L^2 - I A E^(I T0 w) m w + 2 I E^(-I T0 W) L m W - 2 I E^(I T0 W) L m W -
   2 A d Ȧ - 2 E^(-I T0 w-I T0 W) d L Ȧ - E^(-I T0 w+I T0 W) d L Ȧ + I E^(-I T0 w) m w Ȧ - E^(-2 I T0 w) d Ȧ^2>
```

Next, we use the method of undetermined coefficients to determine the particular solution of **order2Eqm**. To accomplish this, we first identify the form of their nonhomogeneous terms, which we will refer to as possible terms, and then seek the solution as a linear combination of them. Substituting this solution into **order2Eqm** and equating the coefficient of each possible term to zero, we obtain a set of pairs of algebraic equations that need to be solved for the unknown coefficients. The solutions of these pairs are unique except for the coefficients of $E^{I\, w\, T_0}$ and $E^{-I\, w\, T_0}$. To determine a unique solution corresponding to these terms, we use two different approaches.

Approach 1:

In the first approach, we replace w with another symbol, say w_0, find the solution of the system of algebraic equations, and then take the limit as w_0 tends to w. To accomplish this, we define

```
collectForm = 9E^(I T0 w) A@T1, T2D, E^(-I T0 w) Ȧ@T1, T2D, E^(I T0 W) L, E^(-I T0 W) L=;
```

Then, the form of the possible terms on the right-hand sides of **order2Eqm** can be determined as

```
possibleTerms =
 HcollectForm •. w -> w0L ~Join~ HOuter@Times, collectForm, collectFormD •• Flatten •• UnionL
```

$$:E^{I T_0 w_0} A@T_1, T_2D, E^{-I T_0 w_0} \dot{A}@T_1, T_2D, E^{I T_0 W} L, E^{-I T_0 W} L, L^2, E^{-2 I T_0 W} L^2,$$
$$E^{2 I T_0 W} L^2, E^{I T_0 w - I T_0 W} L A@T_1, T_2D, E^{I T_0 w + I T_0 W} L A@T_1, T_2D, E^{2 I T_0 w} A@T_1, T_2D^2,$$
$$E^{-I T_0 w - I T_0 W} L \dot{A}@T_1, T_2D, E^{-I T_0 w + I T_0 W} L \dot{A}@T_1, T_2D, A@T_1, T_2D \dot{A}@T_1, T_2D, E^{-2 I T_0 w} \dot{A}@T_1, T_2D^2>$$

Next, we associate with each possible term an undetermined coefficient in u_2 by

```
symbolList1 = Table@Gi, 8i, Length@possibleTermsD<D
```

$8G_1, G_2, G_3, G_4, G_5, G_6, G_7, G_8, G_9, G_{10}, G_{11}, G_{12}, G_{13}, G_{14}<$

and an undetermined coefficient in v_2 by

```
symbolList2 = symbolList1 •. G -> L
```

$8L_1, L_2, L_3, L_4, L_5, L_6, L_7, L_8, L_9, L_{10}, L_{11}, L_{12}, L_{13}, L_{14}<$

Then, we seek a particular solution of **order2Eqm** in the form

```
sol2Rule = 8u2 -> Function@8T0, T1, T2<, symbolList1.possibleTerms •• EvaluateD,
   v2 -> Function@8T0, T1, T2<, symbolList2.possibleTerms •• EvaluateD<;
```

Substituting **sol2Rule** into **order2Eqm** and replacing $E^{\pm I\, w\, T_0}$ on the right-hand sides of **order2Eqm** by $E^{\pm I\, w_0\, T_0}$, we have

```
order2Eqf = Subtract žž # & •ž order2Eqm •. sol2Rule •.
    8Exp@I w T0D -> Exp@I w0 T0D, Exp@-I w T0D -> Exp@-I w0 T0D< •• Expand;
```

Applying directly the function **Coefficient** to collect the coefficients of all **possibleTerms** could result in extra terms that are functions of T_0, which we eliminate. The result is

```
algEqs = Flatten@Coefficient@order2Eqf, #D & •ž possibleTermsD == 0 •. Exp@_ T0D -> 0 •• Thread
```

$$9 - m - L_1 + I G_1 w_0 == 0,\ I m w + w^2 G_1 + I L_1 w_0 == 0,\ -m - L_2 - I G_2 w_0 == 0,$$
$$-I m w + w^2 G_2 - I L_2 w_0 == 0,\ I W G_3 - L_3 == 0,\ 2 I m W + w^2 G_3 + I W L_3 == 0,\ -I W G_4 - L_4 == 0,$$
$$-2 I m W + w^2 G_4 - I W L_4 == 0,\ -L_5 == 0,\ 2 d + w^2 G_5 == 0,\ -2 I W G_6 - L_6 == 0,$$
$$d + w^2 G_6 - 2 I W L_6 == 0,\ 2 I W G_7 - L_7 == 0,\ d + w^2 G_7 + 2 I W L_7 == 0,\ -\frac{I d}{w} + I w G_8 - I W G_8 - L_8 == 0,$$
$$d + w^2 G_8 + I w L_8 - I W L_8 == 0,\ I w G_9 + I W G_9 - L_9 == 0,\ 2 d + w^2 G_9 + I w L_9 + I W L_9 == 0,$$
$$2 I w G_{10} - L_{10} == 0,\ d + w^2 G_{10} + 2 I w L_{10} == 0,\ -I w G_{11} - I W G_{11} - L_{11} == 0,$$
$$2 d + w^2 G_{11} - I w L_{11} - I W L_{11} == 0,\ \frac{I d}{w} - I w G_{12} + I W G_{12} - L_{12} == 0,\ d + w^2 G_{12} - I w L_{12} + I W L_{12} == 0,$$
$$-L_{13} == 0,\ 2 d + w^2 G_{13} == 0,\ -2 I w G_{14} - L_{14} == 0,\ d + w^2 G_{14} - 2 I w L_{14} == 0=$$

Using the function **Solve**, we directly solve the system of algebraic equations **algEqs** for these undetermined coefficients and obtain

```
var = Join@symbolList1, symbolList2D;
```

```
coefs = Solve@algEqs, varD@@1DD;
```

Substituting **coefs** into **sol2Rule** and changing w_0 back to w, we obtain the solution of the second-order equations in pure function form as

```
sol2a = sol2Rule •. Function@8T0, T1, T2<, b_D :>
    Function@8T0, T1, T2< •• Evaluate, b •. coefs •. w0 -> w •• EvaluateD;
sol2a •. displayRule
```

$$:u_2 \text{ ® FunctionB8}T_0, T_1, T_2\text{<}, -\frac{2 d L^2}{w^2} - \frac{E^{-2 I T_0 W} d L^2}{w^2 - 4 W^2} - \frac{E^{2 I T_0 W} d L^2}{w^2 - 4 W^2} + \frac{2 I E^{-I T_0 W} L m W}{w^2 - W^2} - \frac{2 I E^{I T_0 W} L m W}{w^2 - W^2} - \frac{I E^{I T_0 w} m A}{2 w} - \frac{E^{I T_0 w - I T_0 W} d L A}{w W} + \frac{2 E^{I T_0 w + I T_0 W} d L A}{W H2 w + WL} + \frac{E^{2 I T_0 w} d A^2}{3 w^2} + \frac{I E^{-I T_0 w} m \dot{A}}{2 w} - \frac{E^{-I T_0 w + I T_0 W} d L \dot{A}}{w W} + \frac{2 E^{-I T_0 w - I T_0 W} d L \dot{A}}{W H2 w + WL} - \frac{2 d A \dot{A}}{w^2} + \frac{E^{-2 I T_0 w} d \dot{A}^2}{3 w^2} F,$$

$$v_2 \text{ ® FunctionB8}T_0, T_1, T_2\text{<}, \frac{2 I E^{-2 I T_0 W} d L^2 W}{w^2 - 4 W^2} - \frac{2 I E^{2 I T_0 W} d L^2 W}{w^2 - 4 W^2} + \frac{2 E^{-I T_0 W} L m W^2}{w^2 - W^2} + \frac{2 E^{I T_0 W} L m W^2}{w^2 - W^2} - \frac{1}{2} E^{I T_0 w} m A - \frac{I E^{I T_0 w - I T_0 W} d L A}{W} + \frac{2 I E^{I T_0 w + I T_0 W} d L Hw + WL A}{W H2 w + WL} + \frac{2 I E^{2 I T_0 w} d A^2}{3 w} - \frac{1}{2} E^{-I T_0 w} m \dot{A} + \frac{I E^{-I T_0 w + I T_0 W} d L \dot{A}}{W} - \frac{2 I E^{-I T_0 w - I T_0 W} d L Hw + WL \dot{A}}{W H2 w + WL} - \frac{2 I E^{-2 I T_0 w} d \dot{A}^2}{3 w} F\text{>}$$

Approach 2:

As discussed above, solutions of the algebraic equations resulting from applying the method of undetermined coefficients are unique except those corresponding to $E^{I w T_0}$ and $E^{-I w T_0}$. To render them unique, we demand that they be orthogonal to solutions of the corresponding adjoint problems; that is, we damand that the coefficient vector of $E^{I w T_0}$ be $8u_2, v_2<$ be orthogonal to **ccleftVec**. To accomplish this, we first determine the form of all possible terms on the right-hand sides of **order2Eqm** and then identify those among them corresponding to the resonance terms and those corresponding to nonresonance terms.

```
possibleTerms =
 collectForm ~ Join ~ HOuter@Times, collectForm, collectFormD •• Flatten •• UnionL
```

$$:E^{I T_0 w} A@T_1, T_2D, E^{-I T_0 w} \dot{A}@T_1, T_2D, E^{I T_0 W} L, E^{-I T_0 W} L, L^2, E^{-2 I T_0 W} L^2, E^{2 I T_0 W} L^2, E^{I T_0 w - I T_0 W} L A@T_1, T_2D, E^{I T_0 w + I T_0 W} L A@T_1, T_2D, E^{2 I T_0 w} A@T_1, T_2D^2, E^{-I T_0 w - I T_0 W} L \dot{A}@T_1, T_2D, E^{-I T_0 w + I T_0 W} L \dot{A}@T_1, T_2D, A@T_1, T_2D \dot{A}@T_1, T_2D, E^{-2 I T_0 w} \dot{A}@T_1, T_2D^2\text{>}$$

Next, we identify the resonance terms among them by

```
RT = I# •. 8a_ •; a =!= 0 -> 1< & •ž IE^(-I w T0) possibleTerms •. expRule1 •. Exp@_ T0 + _.D -> 0MM
    possibleTerms •• Union •• Rest
```

$$8E^{I T_0 w} A@T_1, T_2D, E^{-I T_0 w + I T_0 W} L \dot{A}@T_1, T_2D\text{<}$$

Hence, the nonresonance part of **possibleTerms** is the complement of **RT**; that is,

```
NRT = Complement@possibleTerms, Join@RT, RT •. conjugateRuleDD
```

```
:E^(-I T0 W) L, E^(I T0 W) L, L^2, E^(-2 I T0 W) L^2, E^(2 I T0 W) L^2, E^(I T0 w+I T0 W) L A@T1, T2D,
 E^(2 I T0 w) A@T1, T2D^2, E^(-I T0 w-I T0 W) L Ȧ@T1, T2D, A@T1, T2D Ȧ@T1, T2D, E^(-2 I T0 w) Ȧ@T1, T2D^2>
```

Next, we associate with each possible resonance term an undetermined coefficient by using the rule

```
RTsymbolList = Table@Gj, 8j, Length@RTD<D
```

```
8G1, G2<
```

and we associate with each possible nonresonance term an undetermined coefficient by using the rule

```
NRTsymbolList@i_D = Table@Li,j, 8j, Length@NRTD<D
```

```
8Li,1, Li,2, Li,3, Li,4, Li,5, Li,6, Li,7, Li,8, Li,9, Li,10<
```

Hence, the coefficients of **RT** in the nonhomogeneous terms in **order2Eqm** are

```
coefRT = Coefficient@#@@2DD, RTD & •ž order2Eqm •. Exp@_ T0 + _.D -> 0
```

```
99m, - I d/w =, 8- I m w, - d<=
```

It follows from

```
ccleftVec.coefRT === 80, 0<
```

```
True
```

that **coefRT** is orthogonal to the solution of the adjoint, **ccleftVec**, as it should.

Imposing the condition that $\{u_2, v_2\}$ is orthogonal to **ccleftVec**, we seek u_2 and v_2 in pure function form as

```
sol2Form = 8
     u2 -> Function@8T0, T1, T2<, RTsymbolList.RT +
    HRTsymbolList.RT •. conjugateRuleL + NRTsymbolList@1D.NRT •• EvaluateD, v2 ->
  Function@8T0, T1, T2<, -I w RTsymbolList.RT + H-I w RTsymbolList.RT •. conjugateRuleL +
    NRTsymbolList@2D.NRT •• EvaluateD<;
```

Substituting **sol2Form** into **order2Eqm** and collecting coefficients of **NRT**, we obtain the following set of algebraic equations:

```
algEqs1 = HCoefficient@Subtract žž #, NRTD & •ž Horder2Eqm •. sol2FormL •. Exp@_ T0D -> 0 ••
    FlattenL == 0 •• Thread
```

$$8- I\, W\, L_{1,1} - L_{2,1} == 0,\ I\, W\, L_{1,2} - L_{2,2} == 0,\ -L_{2,3} == 0,\ -2\, I\, W\, L_{1,4} - L_{2,4} == 0,$$
$$2\, I\, W\, L_{1,5} - L_{2,5} == 0,\ I\, w\, L_{1,6} + I\, W\, L_{1,6} - L_{2,6} == 0,\ 2\, I\, w\, L_{1,7} - L_{2,7} == 0,$$
$$-I\, w\, L_{1,8} - I\, W\, L_{1,8} - L_{2,8} == 0,\ -L_{2,9} == 0,\ -2\, I\, w\, L_{1,10} - L_{2,10} == 0,\ -2\, I\, m\, W + w^2\, L_{1,1} - I\, W\, L_{2,1} == 0,$$
$$2\, I\, m\, W + w^2\, L_{1,2} + I\, W\, L_{2,2} == 0,\ 2\, d + w^2\, L_{1,3} == 0,\ d + w^2\, L_{1,4} - 2\, I\, W\, L_{2,4} == 0,$$
$$d + w^2\, L_{1,5} + 2\, I\, W\, L_{2,5} == 0,\ 2\, d + w^2\, L_{1,6} + I\, w\, L_{2,6} + I\, W\, L_{2,6} == 0,\ d + w^2\, L_{1,7} + 2\, I\, w\, L_{2,7} == 0,$$
$$2\, d + w^2\, L_{1,8} - I\, w\, L_{2,8} - I\, W\, L_{2,8} == 0,\ 2\, d + w^2\, L_{1,9} == 0,\ d + w^2\, L_{1,10} - 2\, I\, w\, L_{2,10} == 0<$$

Solving **algEqs1** for the undetermined coefficients **NRTsymbolList** yields

```
coef1 = Solve@algEqs1, Array@NRTsymbolList, 2D •• FlattenD@@1DD
```

$$:L_{2,9} ® 0,\ L_{1,3} ® -\frac{2\, d}{w^2},\ L_{1,9} ® -\frac{2\, d}{w^2},\ L_{2,3} ® 0,\ L_{1,1} ® \frac{2\, I\, m\, W}{w^2 - W^2},\ L_{1,4} ® -\frac{d}{w^2 - 4\, W^2},$$
$$L_{1,2} ® -\frac{2\, I\, m\, W}{w^2 - W^2},\ L_{1,5} ® -\frac{d}{w^2 - 4\, W^2},\ L_{2,1} ® \frac{2\, m\, W^2}{w^2 - W^2},\ L_{1,7} ® \frac{d}{3\, w^2},\ L_{2,2} ® \frac{2\, m\, W^2}{w^2 - W^2},$$
$$L_{2,5} ® -\frac{2\, I\, d\, W}{w^2 - 4\, W^2},\ L_{1,10} ® \frac{d}{3\, w^2},\ L_{2,10} ® -\frac{2\, I\, d}{3\, w},\ L_{2,4} ® \frac{2\, I\, d\, W}{w^2 - 4\, W^2},\ L_{2,7} ® \frac{2\, I\, d}{3\, w},$$
$$L_{2,6} ® \frac{2\, I\, d\, H w + W L}{W\, H 2\, w + W L},\ L_{1,6} ® \frac{2\, d}{W\, H 2\, w + W L},\ L_{1,8} ® \frac{2\, d}{2\, w\, W + W^2},\ L_{2,8} ® -\frac{2\, I\, d\, H w + W L}{2\, w\, W + W^2}>$$

Substituting **sol2Form** into either of the two **order2Eqm** and collecting the coefficients of **RT**, we obtain the following set of algebraic equations:

```
algEqs2 =
  Coefficient@Subtract žž order2Eqm@@1DD •. sol2Form, RTD == 0 •. Exp@_ T0D -> 0 •• Thread
```

$$9- m + 2\, I\, w\, G_1 == 0,\ \frac{I\, d}{w} + I\, W\, G_2 == 0=$$

Solving **algEqs2** for the undetermined coefficients **RTsymbolList** yields

```
coef2a = Solve@algEqs2, RTsymbolListD@@1DD
```

$$9G_1 ® -\frac{I\, m}{2\, w},\ G_2 ® -\frac{d}{w\, W}=$$

```
coef2 = Join@coef2a, coef2a •. conjugateRuleD
```

$$9G_1 ® -\frac{I\, m}{2\, w},\ G_2 ® -\frac{d}{w\, W},\ \dot{G}_1 ® \frac{I\, m}{2\, w},\ \dot{G}_2 ® -\frac{d}{w\, W}=$$

Substituting **coef1** and **coef2** into **sol2Form**, we obtain the solution of the second-order equations in pure function form as

```
sol2b = sol2Form /. Function[{T0, T1, T2}, b_] :>
    Function[{T0, T1, T2} // Evaluate, b /. coef1 /. coef2 // Expand // Evaluate];
sol2b /. displayRule
```

$$\Big\{u_2 \to \text{Function}\Big[\{T_0, T_1, T_2\}, -\frac{2\,\delta\,\Lambda^2}{\omega^2} - \frac{e^{-2 i T_0 \Omega}\,\delta\,\Lambda^2}{\omega^2 - 4\,\Omega^2} - \frac{e^{2 i T_0 \Omega}\,\delta\,\Lambda^2}{\omega^2 - 4\,\Omega^2} + \frac{2\, i\, e^{-i T_0 \Omega}\,\Lambda\,\mu\,\Omega}{\omega^2 - \Omega^2} -$$
$$\frac{2\, i\, e^{i T_0 \Omega}\,\Lambda\,\mu\,\Omega}{\omega^2 - \Omega^2} - \frac{i\, e^{i T_0 \omega}\,\mu\, A}{2\,\omega} - \frac{e^{i T_0 \omega - i T_0 \Omega}\,\delta\,\Lambda\, A}{\omega\,\Omega} + \frac{2\, e^{i T_0 \omega + i T_0 \Omega}\,\delta\,\Lambda\, A}{\Omega\,(2\,\omega + \Omega)} + \frac{e^{2 i T_0 \omega}\,\delta\, A^2}{3\,\omega^2} +$$
$$\frac{i\, e^{-i T_0 \omega}\,\mu\,\bar{A}}{2\,\omega} - \frac{e^{-i T_0 \omega + i T_0 \Omega}\,\delta\,\Lambda\,\bar{A}}{\omega\,\Omega} + \frac{2\, e^{-i T_0 \omega - i T_0 \Omega}\,\delta\,\Lambda\,\bar{A}}{2\,\omega\,\Omega + \Omega^2} - \frac{2\,\delta\, A\,\bar{A}}{\omega^2} + \frac{e^{-2 i T_0 \omega}\,\delta\,\bar{A}^2}{3\,\omega^2}\Big],$$
$$v_2 \to \text{Function}\Big[\{T_0, T_1, T_2\}, \frac{2\, i\, e^{-2 i T_0 \Omega}\,\delta\,\Lambda^2\,\Omega}{\omega^2 - 4\,\Omega^2} - \frac{2\, i\, e^{2 i T_0 \Omega}\,\delta\,\Lambda^2\,\Omega}{\omega^2 - 4\,\Omega^2} + \frac{2\, e^{-i T_0 \Omega}\,\Lambda\,\mu\,\Omega^2}{\omega^2 - \Omega^2} +$$
$$\frac{2\, e^{i T_0 \Omega}\,\Lambda\,\mu\,\Omega^2}{\omega^2 - \Omega^2} - \frac{1}{2}\, e^{i T_0 \omega}\,\mu\, A - \frac{i\, e^{i T_0 \omega - i T_0 \Omega}\,\delta\,\Lambda\, A}{\Omega} + \frac{2\, i\, e^{i T_0 \omega + i T_0 \Omega}\,\delta\,\Lambda\, A}{2\,\omega + \Omega} +$$
$$\frac{2\, i\, e^{i T_0 \omega + i T_0 \Omega}\,\delta\,\Lambda\,\omega\, A}{\Omega\,(2\,\omega + \Omega)} + \frac{2\, i\, e^{2 i T_0 \omega}\,\delta\, A^2}{3\,\omega} - \frac{1}{2}\, e^{-i T_0 \omega}\,\mu\,\bar{A} + \frac{i\, e^{-i T_0 \omega + i T_0 \Omega}\,\delta\,\Lambda\,\bar{A}}{\Omega} -$$
$$\frac{2\, i\, e^{-i T_0 \omega - i T_0 \Omega}\,\delta\,\Lambda\,\omega\,\bar{A}}{2\,\omega\,\Omega + \Omega^2} - \frac{2\, i\, e^{-i T_0 \omega - i T_0 \Omega}\,\delta\,\Lambda\,\Omega\,\bar{A}}{2\,\omega\,\Omega + \Omega^2} - \frac{2\, i\, e^{-2 i T_0 \omega}\,\delta\,\bar{A}^2}{3\,\omega}\Big]\Big\}$$

These two approaches generate the same results because

```
Subtract @@ (u2[T0, T1, T2] /. {sol2a, sol2b}) // Simplify
```

0

```
Subtract @@ (v2[T0, T1, T2] /. {sol2a, sol2b}) // Simplify
```

0

The second approach is more convenient to use for a general system of equations. Hence, we will adopt this approach in the remainder of this book.

■ Third-Order Equations

Substituting the first- and second-order solutions into the third-order equations, **eqOrder[3]**, yields

```
order3Eq = eqOrder[3] /. sol1 /. sol2b // ExpandAll;
```

Using the **expRule1** to convert the terms that produce small-divisor terms into terms that produce secular terms in the right-hand sides of **order3Eq** and collecting the terms that could produce secular terms, we have

```
STerms2 = Coefficient[#[[2]] /. expRule1 /. Ω -> 2 ω, Exp[I ω T0]] & /@ order3Eq;
STerms2 /. displayRule
```

$$\left\{\frac{I\,\mu\,(D_1A)}{2\,\omega}+\frac{E^{I T_1 \sigma}\,\delta\,\Lambda\,(D_1\bar{A})}{2\,\omega^2}-D_2A,\ -6\,A\,\alpha\,\Lambda^2+A\,\mu^2+\frac{9\,A\,\delta^2\,\Lambda^2}{2\,\omega^2}-3\,A^2\,\alpha\,\bar{A}+\frac{10\,A^2\,\delta^2\,\bar{A}}{3\,\omega^2}-\frac{14\,I\,E^{I T_1 \sigma}\,\delta\,\Lambda\,\mu\,\bar{A}}{3\,\omega}+\frac{1}{2}\,\mu\,(D_1A)-\frac{I\,E^{I T_1 \sigma}\,\delta\,\Lambda\,(D_1\bar{A})}{2\,\omega}-I\,\omega\,(D_2A)\right\}$$

Then, the solvability condition demands that **STerms2** be orthogonal to the solution of the adjoint **ccleftVec**. The result is

```
SCond2 = Solve[ccleftVec.STerms2 == 0, A^(0,1)[T1, T2]][[1]] // ExpandAll;
SCond2 /. displayRule
```

$$\left\{D_2A\to-\frac{9\,I\,A\,\delta^2\,\Lambda^2}{4\,\omega^3}+\frac{3\,I\,A\,\alpha\,\Lambda^2}{\omega}-\frac{I\,A\,\mu^2}{2\,\omega}-\frac{5\,I\,A^2\,\delta^2\,\bar{A}}{3\,\omega^3}-\frac{7\,E^{I T_1 \sigma}\,\delta\,\Lambda\,\mu\,\bar{A}}{3\,\omega^2}+\frac{3\,I\,A^2\,\alpha\,\bar{A}}{2\,\omega}\right\}$$

The two partial-differential equations, **SCond1** and **SCond2**, can be reconstituted to obtain an ordinary-differential equation governing *A*. The result is

```
(2 I ω A' == (2 I ω dt[1][A[T1, T2]] /. SCond1 /. SCond2 // Expand // Collect[#, ε] &)) /.
  displayRule
```

$$2\,I\,\omega\,A'==$$

$$\epsilon\left(-2\,I\,A\,\mu\,\omega-2\,E^{I T_1 \sigma}\,\delta\,\Lambda\,\bar{A}\right)+\epsilon^2\left(-6\,A\,\alpha\,\Lambda^2+A\,\mu^2+\frac{9\,A\,\delta^2\,\Lambda^2}{2\,\omega^2}-3\,A^2\,\alpha\,\bar{A}+\frac{10\,A^2\,\delta^2\,\bar{A}}{3\,\omega^2}-\frac{14\,I\,E^{I T_1 \sigma}\,\delta\,\Lambda\,\mu\,\bar{A}}{3\,\omega}\right)$$

4.2.6 First-Order Complex-Valued System

In this section, we first transform **FDuffingEq** into a single first-order complex-valued equation using the transformation

```
transformRule = {u -> (ζ + ζ̄), u' -> I ω (ζ - ζ̄)};

gRule = γ -> -2 ε μ u' - δ u^2 - α u^3 + ε f Cos[Ω t];
```

then **FDuffingEq** becomes

```
eq426a = ζ' == I ω ζ - I/(2 ω) γ /. gRule /. transformRule // ExpandAll
```

$$\zeta'==-\epsilon\,\zeta\,\mu+\frac{I\,\delta\,\zeta^2}{2\,\omega}+\frac{I\,\alpha\,\zeta^3}{2\,\omega}+I\,\zeta\,\omega-$$

$$\frac{I\,f\,\epsilon\,\mathrm{Cos}[t\,\Omega]}{2\,\omega}+\epsilon\,\mu\,\bar{\zeta}+\frac{I\,\delta\,\zeta\,\bar{\zeta}}{\omega}+\frac{3\,I\,\alpha\,\zeta^2\,\bar{\zeta}}{2\,\omega}+\frac{I\,\delta\,\bar{\zeta}^2}{2\,\omega}+\frac{3\,I\,\alpha\,\zeta\,\bar{\zeta}^2}{2\,\omega}+\frac{I\,\alpha\,\bar{\zeta}^3}{2\,\omega}$$

To determine a second-order uniform expansion of the solution of **eq426a** using the method of multiple scales, we first introduce

```
multiScales = {ζ -> ζ[T0, T1, T2], ζ̄ -> ζ̄[T0, T1, T2], ζ' -> dt[1][ζ[T0, T1, T2]], t -> T0};
```

and then expand z and $\ddot{z}$ in the form

```
solRule =
  9z -> ISumAe^j zj@#1, #2, #3D, 8j, 1, 3<E &M, ż -> ISumAe^j żj@#1, #2, #3D, 8j, 1, 3<E &M=;
```

Substituting **multiScales** and **solRule** into **eq426a**, expanding the result for small e, and discarding terms of order higher than e^3, we obtain

```
eq426b = Heq426a •. multiScales •. solRule •• ExpandAllL •. e^n_•;n>3 -> 0;
```

Equating coefficients of like powers of e yields

```
eqEps = CoefficientList@Subtract žž eq426b, eD == 0 •• Thread •• Rest •• TrigToExp;
```

To place the linear operator on one side and the nonhomogeneous terms on the other side, we define

```
eqOrder@i_D := HeqEps@@1, 1DD •. f -> 0 •. z1 -> ziL - # & •ž eqEps@@iDD •• Reverse
```

Using **eqOrder[i]** and **displayRule**, we rewrite **eqEps** in a concise way as

```
Array@eqOrder, 3D •. displayRule •• TableForm
```

$D_0 z_1 - I\, w\, z_1 == -\frac{I\,(E^{-I T_0 W} + E^{I T_0 W})\, f}{4 w}$

$D_0 z_2 - I\, w\, z_2 == -(D_1 z_1) - m\, z_1 + \frac{I\, d\, z_1^2}{2 w} + m\, \dot{z}_1 + \frac{I\, d\, z_1 \dot{z}_1}{w} + \frac{I\, d\, \dot{z}_1^2}{2 w}$

$D_0 z_3 - I\, w\, z_3 == -(D_1 z_2) - D_2 z_1 + \frac{I\, a\, z_1^3}{2 w} - m\, z_2 + \frac{I\, d\, z_1 z_2}{w} + \frac{3 I\, a\, z_1^2 \dot{z}_1}{2 w} + \frac{I\, d\, z_2 \dot{z}_1}{w} + \frac{3 I\, a\, z_1 \dot{z}_1^2}{2 w} + \frac{I\, a\, \dot{z}_1^3}{2 w} + m\, \dot{z}_2 + \frac{I\, d\, z_1 \dot{z}_2}{w}$

The general solution of the first-order equation, **eqOrder[1]**, can be expressed in pure function form as

```
sol1Form =
 HDSolve@eqOrder@1D, z1@T0, T1, T2D, timeScalesD@@1, 1, 2DD •. C@1D -> A •• ExpandL •.
   Exp@a_D :> Exp@a •• ExpandD ••. c1_ Exp@x_D + c2_ Exp@x_D :> Factor@c1 + c2D Exp@xD
```

$\frac{E^{I T_0 W} f}{4 w (w - W)} + \frac{E^{-I T_0 W} f}{4 w (w + W)} + E^{I T_0 w} A[T_1, T_2]$

```
sol1 = 9z1 -> Function@8T0, T1, T2<, sol1Form •• EvaluateD,
   ż1 -> Function@8T0, T1, T2<, sol1Form •. conjugateRule •• EvaluateD=
```

$\{z_1 \to \text{Function}[\{T_0, T_1, T_2\}, \frac{E^{I T_0 W} f}{4 w (w - W)} + \frac{E^{-I T_0 W} f}{4 w (w + W)} + E^{I T_0 w} A[T_1, T_2]],$

$\dot{z}_1 \to \text{Function}[\{T_0, T_1, T_2\}, \frac{E^{-I T_0 W} f}{4 w (w - W)} + \frac{E^{I T_0 W} f}{4 w (w + W)} + E^{-I T_0 w} \dot{A}[T_1, T_2]]\}$

Substituting the first-order solution into the second-order equation, **eqOrder[2]**, we have

```
order2Eq = Expand •ž HeqOrder@2D •. sol1L;
order2Eq •. displayRule
```

$$D_0 z_2 - I w z_2 == -A E^{I T_0 w} m + \frac{I A^2 E^{2 I T_0 w} d}{2 w} + \frac{I f^2 d}{16 w^3 (w - W)^2} + \frac{I E^{-2 I T_0 W} f^2 d}{32 w^3 (w - W)^2} + \frac{I E^{2 I T_0 W} f^2 d}{32 w^3 (w - W)^2} + \frac{I A E^{I T_0 w - I T_0 W} f d}{4 w^2 (w - W)} +$$
$$\frac{I A E^{I T_0 w + I T_0 W} f d}{4 w^2 (w - W)} + \frac{E^{-I T_0 W} f m}{4 w (w - W)} - \frac{E^{I T_0 W} f m}{4 w (w - W)} + \frac{I f^2 d}{16 w^3 (w + W)^2} + \frac{I E^{-2 I T_0 W} f^2 d}{32 w^3 (w + W)^2} + \frac{I E^{2 I T_0 W} f^2 d}{32 w^3 (w + W)^2} +$$
$$\frac{I A E^{I T_0 w - I T_0 W} f d}{4 w^2 (w + W)} + \frac{I A E^{I T_0 w + I T_0 W} f d}{4 w^2 (w + W)} - \frac{E^{-I T_0 W} f m}{4 w (w + W)} + \frac{E^{I T_0 W} f m}{4 w (w + W)} + \frac{I f^2 d}{8 w^3 (w - W)(w + W)} +$$
$$\frac{I E^{-2 I T_0 W} f^2 d}{16 w^3 (w - W)(w + W)} + \frac{I E^{2 I T_0 W} f^2 d}{16 w^3 (w - W)(w + W)} + E^{-I T_0 w} m \dot{A} + \frac{I A d \dot{A}}{w} + \frac{I E^{-I T_0 w - I T_0 W} f d \dot{A}}{4 w^2 (w - W)} +$$
$$\frac{I E^{-I T_0 w + I T_0 W} f d \dot{A}}{4 w^2 (w - W)} + \frac{I E^{-I T_0 w - I T_0 W} f d \dot{A}}{4 w^2 (w + W)} + \frac{I E^{-I T_0 w + I T_0 W} f d \dot{A}}{4 w^2 (w + W)} + \frac{I E^{-2 I T_0 w} d \dot{A}^2}{2 w} - E^{I T_0 w} (D_1 A)$$

Eliminating the terms that lead to secular terms in z_2 from **order2Eq** yields

```
SCond1 = Coefficient@order2Eq@@2DD •. expRule1, Exp@I w T0DD == 0
```

$$-m A[T_1, T_2] + \frac{I E^{I T_1 s} f d \dot{A}[T_1, T_2]}{4 w^2 (w - W)} + \frac{I E^{I T_1 s} f d \dot{A}[T_1, T_2]}{4 w^2 (w + W)} - A^{(1,0)}[T_1, T_2] == 0$$

or

```
SCond1Rule =
  ISolveASCond1, A^(1,0)@T1, T2DE@@1DD •• ExpandAllM •. c1_ m + c2_ m :> Factor@c1 + c2D m
```

$$\left\{A^{(1,0)}[T_1, T_2] \to -m A[T_1, T_2] + \frac{I E^{I T_1 s} f d \dot{A}[T_1, T_2]}{2 w^3 - 2 w W^2}\right\}$$

With this solvability condition, we rewrite **order2Eq** as

```
order2Eqm = Expand •ž
   Horder2Eq •. SCond1Rule •. Exp@arg_D :> Exp@arg •. T1 -> e T0 •. sigRule •• ExpandDL;
```

There are at least two approaches we can use to solve for the particular solution of **order2Eqm**. In the first approach, we transform **order2Eqm** into an ordinary-differential equation and then use **DSolve** and **Map** to determine the following particular solution:

```
Hsol2a =
  Hz2@T0D /. DSolve@Horder2Eqm@@1DD /. z2 -> Hz2@#1D &LL == #, z2@T0D, T0D@@1DD /. C@_D ->
      0L & /@ order2Eqm@@2DD /. Exp@a_D :> Exp@Expand@aDD // ExpandL // Timing
```

$$:6.629\,\text{Second},\ -\frac{f^2\,d}{16\,w^4\,Hw-WL^2}-\frac{I\,E^{I\,T_0\,W}\,f\,m}{4\,w\,Hw-WL^2}-\frac{E^{2\,I\,T_0\,W}\,f^2\,d}{32\,w^3\,Hw-2\,WL\,Hw-WL^2}-\frac{f^2\,d}{16\,w^4\,Hw+WL^2}-$$

$$\frac{I\,E^{-I\,T_0\,W}\,f\,m}{4\,w\,Hw+WL^2}-\frac{E^{2\,I\,T_0\,W}\,f^2\,d}{32\,w^3\,Hw-2\,WL\,Hw+WL^2}-\frac{f^2\,d}{8\,w^4\,Hw-WL\,Hw+WL}+\frac{I\,E^{-I\,T_0\,W}\,f\,m}{4\,w\,Hw-WL\,Hw+WL}+$$

$$\frac{I\,E^{I\,T_0\,W}\,f\,m}{4\,w\,Hw-WL\,Hw+WL}-\frac{E^{2\,I\,T_0\,W}\,f^2\,d}{16\,w^3\,Hw-2\,WL\,Hw-WL\,Hw+WL}-\frac{E^{-2\,I\,T_0\,W}\,f^2\,d}{32\,w^3\,Hw-WL^2\,Hw+2\,WL}-$$

$$\frac{E^{-2\,I\,T_0\,W}\,f^2\,d}{32\,w^3\,Hw+WL^2\,Hw+2\,WL}-\frac{E^{-2\,I\,T_0\,W}\,f^2\,d}{16\,w^3\,Hw-WL\,Hw+WL\,Hw+2\,WL}-\frac{E^{I\,T_0\,w-I\,T_0\,W}\,f\,d\,A@T_1,\,T_2D}{4\,w^2\,Hw-WL\,W}+$$

$$\frac{E^{I\,T_0\,w+I\,T_0\,W}\,f\,d\,A@T_1,\,T_2D}{4\,w^2\,Hw-WL\,W}-\frac{E^{I\,T_0\,w-I\,T_0\,W}\,f\,d\,A@T_1,\,T_2D}{4\,w^2\,W\,Hw+WL}+\frac{E^{I\,T_0\,w+I\,T_0\,W}\,f\,d\,A@T_1,\,T_2D}{4\,w^2\,W\,Hw+WL}+$$

$$\frac{E^{2\,I\,T_0\,w}\,d\,A@T_1,\,T_2D^2}{2\,w^2}+\frac{I\,E^{-I\,T_0\,w}\,m\,\dot{A}@T_1,\,T_2D}{2\,w}-\frac{E^{-I\,T_0\,w+I\,T_0\,W}\,f\,d\,\dot{A}@T_1,\,T_2D}{4\,w^2\,Hw-WL\,H2\,w-WL}-$$

$$\frac{E^{-I\,T_0\,w+I\,T_0\,W}\,f\,d\,\dot{A}@T_1,\,T_2D}{4\,w^2\,H2\,w-WL\,Hw+WL}-\frac{E^{-I\,T_0\,w-I\,T_0\,W}\,f\,d\,\dot{A}@T_1,\,T_2D}{4\,w^2\,Hw-WL\,H2\,w+WL}-\frac{E^{-I\,T_0\,w-I\,T_0\,W}\,f\,d\,\dot{A}@T_1,\,T_2D}{4\,w^2\,Hw+WL\,H2\,w+WL}+$$

$$\frac{E^{-I\,T_0\,w+I\,T_0\,W}\,f\,d\,\dot{A}@T_1,\,T_2D}{2\,w\,H2\,w-WL\,Hw^2-W^2L}-\frac{d\,A@T_1,\,T_2D\,\dot{A}@T_1,\,T_2D}{w^2}-\frac{E^{-2\,I\,T_0\,w}\,d\,\dot{A}@T_1,\,T_2D^2}{6\,w^2}>$$

In the second approach, we multiply **order2Eqm** by an integrating factor and then use the function **Integrate.** To reduce the computation time, we define first the rule

```
intRule = int@a_ + b_, arg_D -> int@a, argD + int@b, argD;
```

Then, using this rule and an integrating factor, we find a particular solution of **order2Eqm** as

```
Hsol2b = HHint@order2Eqm@@2DD * Exp@-I T0 wD •• Expand, T0D ••. intRule •. int -> IntegrateL *
    Exp@I T0 wD •• ExpandL •. Exp@a_D :> Exp@Expand@aDDL •• Timing
```

$$
:0.841\ \text{Second},\ -\frac{f^2 d}{16 w^4 Hw - WL^2} - \frac{I E^{I T_0 W} f m}{4 w Hw - WL^2} - \frac{E^{2 I T_0 W} f^2 d}{32 w^3 Hw - 2 WL Hw - WL^2} - \frac{f^2 d}{16 w^4 Hw + WL^2} -
$$

$$
\frac{I E^{-I T_0 W} f m}{4 w Hw + WL^2} - \frac{E^{2 I T_0 W} f^2 d}{32 w^3 Hw - 2 WL Hw + WL^2} - \frac{f^2 d}{8 w^4 Hw - WL Hw + WL} + \frac{I E^{-I T_0 W} f m}{4 w Hw - WL Hw + WL} +
$$

$$
\frac{I E^{I T_0 W} f m}{4 w Hw - WL Hw + WL} - \frac{E^{2 I T_0 W} f^2 d}{16 w^3 Hw - 2 WL Hw - WL Hw + WL} - \frac{E^{-2 I T_0 W} f^2 d}{32 w^3 Hw - WL^2 Hw + 2 WL} -
$$

$$
\frac{E^{-2 I T_0 W} f^2 d}{32 w^3 Hw + WL^2 Hw + 2 WL} - \frac{E^{-2 I T_0 W} f^2 d}{16 w^3 Hw - WL Hw + WL Hw + 2 WL} - \frac{E^{I T_0 w - I T_0 W} f d A@T_1, T_2D}{4 w^2 Hw - WL W} +
$$

$$
\frac{E^{I T_0 w + I T_0 W} f d A@T_1, T_2D}{4 w^2 Hw - WL W} - \frac{E^{I T_0 w - I T_0 W} f d A@T_1, T_2D}{4 w^2 W Hw + WL} + \frac{E^{I T_0 w + I T_0 W} f d A@T_1, T_2D}{4 w^2 W Hw + WL} +
$$

$$
\frac{E^{2 I T_0 w} d A@T_1, T_2D^2}{2 w^2} + \frac{I E^{-I T_0 w} m \dot{A}@T_1, T_2D}{2 w} - \frac{E^{-I T_0 w + I T_0 W} f d \dot{A}@T_1, T_2D}{4 w^2 Hw - WL H2 w - WL} -
$$

$$
\frac{E^{-I T_0 w + I T_0 W} f d \dot{A}@T_1, T_2D}{4 w^2 H2 w - WL Hw + WL} - \frac{E^{-I T_0 w - I T_0 W} f d \dot{A}@T_1, T_2D}{4 w^2 Hw - WL H2 w + WL} - \frac{E^{-I T_0 w - I T_0 W} f d \dot{A}@T_1, T_2D}{4 w^2 Hw + WL H2 w + WL} +
$$

$$
\frac{E^{-I T_0 w + I T_0 W} f d \dot{A}@T_1, T_2D}{2 w H2 w - WL Hw^2 - W^2 L} - \frac{d A@T_1, T_2D \dot{A}@T_1, T_2D}{w^2} - \frac{E^{-2 I T_0 w} d \dot{A}@T_1, T_2D^2}{6 w^2} >
$$

The CPU time is decreased by a factor of

```
%%%@@1DD • %@@1DD
```

```
7.88228
```

```
sol2a === sol2b
```

```
True
```

Then, the particular solution of **order2Eqm** can be expressed in pure function form as

```
sol2 = 9z2 -> Function@8T0, T1, T2<, sol2b •• EvaluateD,
   ż2 -> Function@8T0, T1, T2<, sol2b •. conjugateRule •• EvaluateD=;
```

Substituting the first- and second-order solutions into the third-order equation, **eqOrder[3]**, we obtain

```
order3Eq = Expand •ž HeqOrder@3D •. sol1 •. sol2L;
```

Eliminating the terms that produce secular terms in z_3 from **order3Eq** yields

```
SCond2Rule =
 ISolveACoefficient@order3Eq@@2DD •. expRule1, Exp@I w T0DD == 0, A^H0,1L@T1, T2DE@@1DD •.
     SCond1Rule •. HSCond1Rule •. conjugateRuleL •• ExpandAllM ••.
  9c1_ m + c2_ m :> Factor@c1 + c2D m, c1_ f^2 d^2 + c2_ f^2 d^2 :> Factor@c1 + c2D f^2 d^2,
   c1_ f^2 a + c2_ f^2 a :> Factor@c1 + c2D f^2 a=;
SCond2Rule •. displayRule
```

$$:D_2A ® -\frac{I A m^2}{2 w} + \frac{3 I A f^2 a}{4 w Hw - WL^2 Hw + WL^2} - \frac{I A f^2 d^2 H5 w + 2 WL}{4 w^3 Hw - WL^2 Hw + WL^2 H2 w + WL} - \frac{5 I A^2 d^2 \dot{A}}{3 w^3} + \frac{3 I A^2 a \dot{A}}{2 w} - \frac{E^{I T_1 s} f d m H2 w^3 + w^2 W - 6 w W^2 - W^3 L \dot{A}}{4 w^2 Hw - WL^2 W Hw + WL^2} >$$

or

```
fRule = 9f -> 2 Iw^2 - W^2M L=;

SCond2Rule •. fRule •. W -> 2 w •. displayRule
```

$$:D_2A ® -\frac{9 I A d^2 L^2}{4 w^3} + \frac{3 I A a L^2}{w} - \frac{I A m^2}{2 w} - \frac{5 I A^2 d^2 \dot{A}}{3 w^3} - \frac{7 E^{I T_1 s} d L m \dot{A}}{3 w^2} + \frac{3 I A^2 a \dot{A}}{2 w} >$$

which is in agreement with that obtained in the preceding section.

Ÿ 4.2.7 The Function MMS1

Collecting the steps described in Section 4.2.5 for a system of two real-valued first-order equations, we can build a function named **MMS1** (**M**ethod of **M**ultiple **S**cales for **1** DOF system) specifically for **FDuffingEq**. A more general function (a Package) can be similarly created by considering as arguments the governing equation, symbols for the dependent variable, independent variable, excitation amplitudes and frequencies, and all other related quantities which allow the program to identify their respective meanings. We then use **MMS1** to solve for different resonance cases.

```
MMS1@scaling_List, ResonanceCond : 8__Equal<D :=
 ModuleA8<,
       vRule = u¢@tD -> v@tD;
  OmgRule = Solve@ResonanceCond, WD@@1DD;
       eqa = 8vRule •. Rule -> Equal, FDuffingEq •. scaling •. vRule •. D@vRule, tD<;
       multiScales =
  8u@tD -> u@T0, T1, T2D, v@tD -> v@T0, T1, T2D, u_¢@tD -> dt@1D@u@T0, T1, T2DD, t -> T0<;
       solRule = 9u -> ISumAe^j uj@#1, #2, #3D, 8j, 3<E &M,
   v -> ISumAe^j vj@#1, #2, #3D, 8j, 3<E &M=;
       eqb = Heqa •. multiScales •. solRule •• TrigToExp •• ExpandAllL •. e^n_•;n>3 -> 0;
       eqEps =
  Thread@CoefficientList@Subtract žž #, eD == 0D & •ž eqb •• Transpose •• Rest;

    H* First-Order Problem *L
```

```
linearSys = #@@1DD & •ž eqEps@@1DD •. f -> 0;
eqOrder@i_D :=
HlinearSys •. u_1 -> u_i L == HlinearSys •. u_1 -> u_i L - H#@@1DD & •ž eqEps@@iDDL •• Thread;
coefList = E^-I w T0 linearSys •. 9u1 -> IP E^I w # &M, v1 -> IQ E^I w # &M= •• Expand;
coefMat = Outer@Coefficient, coefList, 8P, Q<D;
hermitian@mat_?MatrixQD := mat •. conjugateRule •• Transpose;
rightVec = 81, c1< •. Solve@HcoefMat.81, c1<L@@1DD == 0, c1D@@1DD;
leftVec = 8c1, 1< •. Solve@Hhermitian@coefMatD.8c1, 1<L@@1DD == 0, c1D@@1DD;
ccleftVec = leftVec •. conjugateRule;
sol1h = # + H# •. conjugateRuleL & •ž IrightVec A@T1, T2D E^I w T0 M;
sol1p =
DSolve@eqOrder@1D •. 8u1 -> Hu1@#1D &L, v1 -> Hv1@#1D &L<, 8u1@T0D, v1@T0D<, T0D@@1DD •.
   C@_D -> 0 •• TrigToExp •• Simplify;
fRule = f -> -2 I-w^2 + W^2M L;
sol1 =
8u1 -> Function@8T0, T1, T2<, sol1h@@1DD + Hu1@T0D •. sol1p •. fRule •• ExpandL •• EvaluateD,
 v1 -> Function@8T0, T1, T2<,
   sol1h@@2DD + Hv1@T0D •. sol1p •. fRule •• ExpandL •• EvaluateD<;

  H* Second-Order Problem *L
order2Eq = eqOrder@2D •. sol1 •• ExpandAll;
expRule1 =
Exp@arg_D :> Exp@Expand@arg •. OmgRuleD •. e^n_. T0 :> timeScales@@n + 1DDD;
STerms1 = Coefficient@#@@2DD •. expRule1, Exp@I w T0DD & •ž order2Eq;
SCond1 = SolveAccleftVec.STerms1 == 0, A^H1,0L@T1, T2DE@@1DD;
ccSCond1 = SCond1 •. conjugateRule;
sigRule = Solve@OmgRule •. Rule -> Equal, sD@@1DD;
expRule2 = Exp@a_D :> ExpAa •. 9T1 -> e T0, T2 -> e^2 T0= •. sigRule •• ExpandE;
SCond1f = Join@SCond1, ccSCond1D •. expRule2;
order2Eqm = order2Eq •. SCond1f •• ExpandAll;
collectForm = JoinA9E^I T0 w A@T1, T2D, E^-I T0 w Ā@T1, T2D=,
 IfAHF • f •. scalingL === e, 9E^I T0 W L, E^-I T0 W L=, 8<EE;
possibleTerms = collectForm ~Join~
 HOuter@Times, collectForm, collectFormD •• Flatten •• UnionL;
RT = I# •. 8a_ •; a =!= 0 -> 1< & •ž IE^-I w T0 possibleTerms •. expRule1 •.
   Exp@_ T0 + _.D -> 0MM possibleTerms •• Union •• Rest;
NRT = Complement@possibleTerms, Join@RT, RT •. conjugateRuleDD;
RTsymbolList = Table@Gj, 8j, Length@RTD<D;
NRTsymbolList@i_D = Table@Li,j, 8j, Length@NRTD<D;
sol2Form = 8
   u2 -> Function@8T0, T1, T2<,
  RTsymbolList.RT + HRTsymbolList.RT •. conjugateRuleL + NRTsymbolList@1D.NRT ••
   EvaluateD, v2 -> Function@8T0, T1, T2<, -I w RTsymbolList.RT +
```

```
H-I w RTsymbolList.RT •. conjugateRuleL + NRTsymbolList@2D.NRT •• EvaluateD<;
algEqs1 = HCoefficient@Subtract žž #, NRTD & •ž Horder2Eqm •. sol2FormL •.
 Exp@_ T0D -> 0 •• FlattenL == 0 •• Thread;
coef1 = Solve@algEqs1, Array@NRTsymbolList, 2D •• FlattenD@@1DD;
algEqs2 =
Coefficient@Subtract žž order2Eqm@@1DD •. sol2Form, RTD == 0 •. Exp@_ T0D -> 0 •• Thread;
coef2a = Solve@algEqs2, RTsymbolListD@@1DD;
coef2 = Join@coef2a, coef2a •. conjugateRuleD;
sol2 = sol2Form •. Function@8T0, T1, T2<, b_D :>
 Function@8T0, T1, T2< •• Evaluate, b •. coef1 •. coef2 •• Expand •• EvaluateD;

H* Third-Order Problem *L
order3Eq = eqOrder@3D •. sol1 •. sol2 •• ExpandAll;
STerms2 =
Coefficient@#@@2DD •. expRule1 •. HOmgRule •. e -> 0L, Exp@I w T0DD & •ž order3Eq;
SCond2 = SolveAccleftVec.STerms2 == 0, A^H0,1L@T1, T2DE@@1DD •• ExpandAll;

H* Reconstitution *L
moduEq = 2 I w A¢ == H2 I w dt@1D@ A@T1, T2DD •. SCond1 •. SCond2 •• Collect@#, eD &L;
Print@"The second-order approximate solution:"D;
Print@
u@tD == Hu žž timeScales •. solRule •. e^3 -> 0 •. sol1 •. sol2 •. displayRuleLD;
IfAHF • f •. scalingL === e, PrintA"where\n L==f•H2Hw^2 -W^2LL"E
E;
Print@"\nThe modulation equations:"D;
Print@moduEq •. displayRuleD
E
```

Ÿ Primary Resonance: W » w

```
scaling1 = 9F -> e^3 f, m` -> e^2 m=;

ResonanceCond1 = 9W == w + e^2 s=;
```

```
MMS1[scaling1, ResonanceCond1] // Timing
```

The second-order approximate solution:

$$u[t] == e\left(A E^{I T_0 w} + E^{-I T_0 w}\bar{A}\right) + e^2\left(\frac{A^2 E^{2 I T_0 w} d}{3 w^2} - \frac{2 A d \bar{A}}{w^2} + \frac{E^{-2 I T_0 w} d \bar{A}^2}{3 w^2}\right)$$

The modulation equations:

$$2 I w A' == 2 I e^2 w\left(-A m - \frac{I E^{I T_2 s} f}{4 w} - \frac{5 I A^2 d^2 \bar{A}}{3 w^3} + \frac{3 I A^2 a \bar{A}}{2 w}\right)$$

{81.513 Second, Null}

Ÿ Subharmonic Resonance of Order 1/2: W » 2w

```
scaling2 = {F -> e f, m̀ -> e m};

ResonanceCond2 = {W == 2 w + e s};

MMS1[scaling2, ResonanceCond2] // Timing
```

The second-order approximate solution:

$$u[t] ==$$

$$e\left(A E^{I T_0 w} + E^{-I T_0 W} L + E^{I T_0 W} L + E^{-I T_0 w}\bar{A}\right) + e^2\left(\frac{A^2 E^{2 I T_0 w} d}{3 w^2} - \frac{2 d L^2}{w^2} - \frac{I A E^{I T_0 w} m}{2 w} - \frac{A E^{I T_0 w - I T_0 W} d L}{w W} + \frac{2 A E^{I T_0 w + I T_0 W} d L}{W(2 w + W)} - \frac{E^{-2 I T_0 W} d L^2}{w^2 - 4 W^2} - \frac{E^{2 I T_0 W} d L^2}{w^2 - 4 W^2} + \frac{2 I E^{-I T_0 W} L m W}{w^2 - W^2} - \frac{2 I E^{I T_0 W} L m W}{w^2 - W^2} - \frac{2 A d \bar{A}}{w^2} + \frac{I E^{-I T_0 w} m \bar{A}}{2 w} - \frac{E^{-I T_0 w + I T_0 W} d L \bar{A}}{w W} + \frac{2 E^{-I T_0 w - I T_0 W} d L \bar{A}}{2 w W + W^2} + \frac{E^{-2 I T_0 w} d \bar{A}^2}{3 w^2}\right)$$

where

$$L == f/(2(w^2 - W^2))$$

The modulation equations:

$$2 I w A' == -2 I e\left(A m w - I E^{I T_1 s} d L \bar{A}\right) +$$

$$2 I e^2 w\left(-\frac{9 I A d^2 L^2}{4 w^3} + \frac{3 I A a L^2}{w} - \frac{I A m^2}{2 w} - \frac{5 I A^2 d^2 \bar{A}}{3 w^3} - \frac{7 E^{I T_1 s} d L m \bar{A}}{3 w^2} + \frac{3 I A^2 a \bar{A}}{2 w}\right)$$

{88.562 Second, Null}

Ÿ Subharmonic Resonance of Order 1/3: W » 3w

```
scaling3 = {F -> e f, m̀ -> e^2 m};

ResonanceCond3 = {W == 3 w + e^2 s};
```

MMS1@scaling3, ResonanceCond3D •• Timing

The second-order approximate solution:

$$u@tD == e\ lA\ E^{I\,T_0\,w} + E^{-I\,T_0\,W} L + E^{I\,T_0\,W} L + E^{-I\,T_0\,w}\,\dot{A}M +$$
$$e^2 \left| \frac{A^2 E^{2\,I\,T_0\,w}\,d}{3\,w^2} - \frac{2\,d\,L^2}{w^2} - \frac{2\,A\,E^{I\,T_0\,w-I\,T_0\,W}\,d\,L}{H2\,w - WL\,W} + \frac{2\,A\,E^{I\,T_0\,w+I\,T_0\,W}\,d\,L}{W\,H2\,w + WL} - \frac{E^{-2\,I\,T_0\,W}\,d\,L^2}{w^2 - 4\,W^2} - \right.$$
$$\left. \frac{E^{2\,I\,T_0\,W}\,d\,L^2}{w^2 - 4\,W^2} - \frac{2\,A\,d\,\dot{A}}{w^2} - \frac{2\,E^{-I\,T_0\,w+I\,T_0\,W}\,d\,L\,\dot{A}}{H2\,w - WL\,W} + \frac{2\,E^{-I\,T_0\,w-I\,T_0\,W}\,d\,L\,\dot{A}}{W\,H2\,w + WL} + \frac{E^{-2\,I\,T_0\,w}\,d\,\dot{A}^2}{3\,w^2} \right|$$

where

$L == f \bullet H2Hw^2 - W^2 LL$

The modulation equations:

$$2\,I\,w\,A^{¢} ==$$
$$2\,I\,e^2\,w \left| -A\,m - \frac{6\,I\,A\,d^2\,L^2}{5\,w^3} + \frac{3\,I\,A\,a\,L^2}{w} - \frac{5\,I\,A^2\,d^2\,\dot{A}}{3\,w^3} + \frac{3\,I\,A^2\,a\,\dot{A}}{2\,w} + \frac{I\,E^{I\,T_2\,s}\,d^2\,L\,\dot{A}^2}{w^3} + \frac{3\,I\,E^{I\,T_2\,s}\,a\,L\,\dot{A}^2}{2\,w} \right|$$

87.03 Second, Null<

Ÿ Superharmonic Resonance of Order 2: 2W » w

scaling4 = 8F -> e f, m̀ -> e m<;

ResonanceCond4 = 82 W == w + e s<;

MMS1@scaling4, ResonanceCond4D •• Timing

The second-order approximate solution:

$$u@tD ==$$
$$e\ lA\ E^{I\,T_0\,w} + E^{-I\,T_0\,W} L + E^{I\,T_0\,W} L + E^{-I\,T_0\,w}\,\dot{A}M + e^2 \left| \frac{A^2 E^{2\,I\,T_0\,w}\,d}{3\,w^2} - \frac{2\,d\,L^2}{w^2} - \frac{I\,A\,E^{I\,T_0\,w}\,m}{2\,w} + \frac{2\,A\,E^{I\,T_0\,w+I\,T_0\,W}\,d\,L}{W\,H2\,w + WL} - \right.$$
$$\frac{E^{-2\,I\,T_0\,W}\,d\,L^2}{2\,w\,Hw + 2\,WL} - \frac{E^{2\,I\,T_0\,W}\,d\,L^2}{2\,w\,Hw + 2\,WL} + \frac{2\,I\,E^{-I\,T_0\,W}\,L\,m\,W}{w^2 - W^2} - \frac{2\,I\,E^{I\,T_0\,W}\,L\,m\,W}{w^2 - W^2} - \frac{2\,A\,E^{I\,T_0\,w-I\,T_0\,W}\,d\,L}{2\,w\,W - W^2} -$$
$$\left. \frac{2\,A\,d\,\dot{A}}{w^2} + \frac{I\,E^{-I\,T_0\,w}\,m\,\dot{A}}{2\,w} + \frac{2\,E^{-I\,T_0\,w-I\,T_0\,W}\,d\,L\,\dot{A}}{W\,H2\,w + WL} - \frac{2\,E^{-I\,T_0\,w+I\,T_0\,W}\,d\,L\,\dot{A}}{2\,w\,W - W^2} + \frac{E^{-2\,I\,T_0\,w}\,d\,\dot{A}^2}{3\,w^2} \right|$$

where

$L == f \bullet H2Hw^2 - W^2 LL$

The modulation equations:

$$2\,I\,w\,A^{¢} == -I\,e\,H-I\,E^{I\,T_1\,s}\,d\,L^2 + 2\,A\,m\,wL +$$
$$2\,I\,e^2\,w \left| -\frac{46\,I\,A\,d^2\,L^2}{15\,w^3} + \frac{13\,E^{I\,T_1\,s}\,d\,L^2\,m}{12\,w^2} + \frac{3\,I\,A\,a\,L^2}{w} - \frac{I\,A\,m^2}{2\,w} - \frac{5\,I\,A^2\,d^2\,\dot{A}}{3\,w^3} + \frac{3\,I\,A^2\,a\,\dot{A}}{2\,w} \right|$$

88.993 Second, Null<

Superharmonic Resonance of Order 3: $3\Omega \approx \omega$

```
scaling5 = {F -> ε f, μ̂ -> ε^2 μ};
```

```
ResonanceCond5 = {3 Ω == ω + ε^2 σ};
```

```
MMS1[scaling5, ResonanceCond5] // Timing
```

The second-order approximate solution:

$$u[t] == \epsilon\left(A E^{I T_0 \omega} + E^{-I T_0 \Omega}\Lambda + E^{I T_0 \Omega}\Lambda + E^{-I T_0 \omega}\bar{A}\right) + \epsilon^2\left(\frac{A^2 E^{2 I T_0 \omega}\delta}{3\omega^2} - \frac{2\delta\Lambda^2}{\omega^2} - \frac{2 A E^{I T_0 \omega - I T_0 \Omega}\delta\Lambda}{(2\omega-\Omega)\Omega} + \frac{2 A E^{I T_0 \omega + I T_0 \Omega}\delta\Lambda}{\Omega(2\omega+\Omega)} - \frac{E^{-2 I T_0 \Omega}\delta\Lambda^2}{\omega^2 - 4\Omega^2} - \frac{E^{2 I T_0 \Omega}\delta\Lambda^2}{\omega^2 - 4\Omega^2} - \frac{2 A\delta\bar{A}}{\omega^2} - \frac{2 E^{-I T_0 \omega + I T_0 \Omega}\delta\Lambda\bar{A}}{(2\omega-\Omega)\Omega} + \frac{2 E^{-I T_0 \omega - I T_0 \Omega}\delta\Lambda\bar{A}}{\Omega(2\omega+\Omega)} + \frac{E^{-2 I T_0 \omega}\delta\bar{A}^2}{3\omega^2}\right)$$

where

$$\Lambda == f/(2(\omega^2 - \Omega^2))$$

The modulation equations:

$$2 I \omega A' ==$$

$$2 I \epsilon^2 \omega\left(-A\mu - \frac{106 I A \delta^2\Lambda^2}{35\omega^3} - \frac{9 I E^{I T_2 \sigma}\delta^2\Lambda^3}{5\omega^3} + \frac{3 I A \alpha\Lambda^2}{\omega} + \frac{I E^{I T_2 \sigma}\alpha\Lambda^3}{2\omega} - \frac{5 I A^2\delta^2\bar{A}}{3\omega^3} + \frac{3 I A^2\alpha\bar{A}}{2\omega}\right)$$

```
{87.08 Second, Null}
```

Slowly-Modulated Load: $\Omega \approx 0$

```
scaling6 = {F -> ε f, μ̂ -> ε μ};
```

```
ResonanceCond6 = {Ω == ε σ};
```

```
MMS1[scaling6, ResonanceCond6D] // Timing
```

The second-order approximate solution:

u[t] ==

$$\epsilon\left(A E^{I T_0 \omega} + E^{-I T_0 \Omega}\Lambda + E^{I T_0 \Omega}\Lambda + E^{-I T_0 \omega}\bar{A}\right) + \epsilon^2\left(\frac{A^2 E^{2 I T_0 \omega}\delta}{3\omega^2} - \frac{2\delta\Lambda^2}{\omega^2} - \frac{I A E^{I T_0 \omega}\mu}{2\omega} - \frac{A E^{I T_0 \omega - I T_0 \Omega}\delta\Lambda}{\omega(2\omega-\Omega)} - \frac{A E^{I T_0 \omega + I T_0 \Omega}\delta\Lambda}{\omega(2\omega+\Omega)} - \frac{E^{-2 I T_0 \Omega}\delta\Lambda^2}{\omega^2 - 4\Omega^2} - \frac{E^{2 I T_0 \Omega}\delta\Lambda^2}{\omega^2 - 4\Omega^2} + \frac{2 I E^{-I T_0 \Omega}\Lambda\mu\Omega}{\omega^2-\Omega^2} - \frac{2 I E^{I T_0 \Omega}\Lambda\mu\Omega}{\omega^2-\Omega^2} - \frac{2A\delta\bar{A}}{\omega^2} + \frac{I E^{-I T_0 \omega}\mu\bar{A}}{2\omega} - \frac{E^{-I T_0 \omega + I T_0 \Omega}\delta\Lambda\bar{A}}{\omega(2\omega-\Omega)} - \frac{E^{-I T_0 \omega - I T_0 \Omega}\delta\Lambda\bar{A}}{\omega(2\omega+\Omega)} + \frac{E^{-2 I T_0 \omega}\delta\bar{A}^2}{3\omega^2}\right)$$

where

$\Lambda == f/(2(\omega^2 - \Omega^2))$

The modulation equations:

$$2 I \omega A' == -2 I A E^{-I T_1 \sigma}\epsilon\left(-I\delta\Lambda - I E^{2 I T_1 \sigma}\delta\Lambda + E^{I T_1 \sigma}\mu\omega\Lambda + 2 I \epsilon^2\omega\left(-\frac{3 I A\delta^2\Lambda^2}{\omega^3} - \frac{3 I A E^{-2 I T_1 \sigma}\delta^2\Lambda^2}{2\omega^3} - \frac{3 I A E^{2 I T_1 \sigma}\delta^2\Lambda^2}{2\omega^3} + \frac{3 I A\alpha\Lambda^2}{\omega} + \frac{3 I A E^{-2 I T_1 \sigma}\alpha\Lambda^2}{2\omega} + \frac{3 I A E^{2 I T_1 \sigma}\alpha\Lambda^2}{2\omega} - \frac{I A\mu^2}{2\omega} - \frac{5 I A^2\delta^2\bar{A}}{3\omega^3} + \frac{3 I A^2\alpha\bar{A}}{2\omega}\right)\right.$$

{88.002 Second, Null}

4.3 The Generalized Method of Averaging

To apply either the method of averaging or the generalized method of averaging to **FDuffingEq**, we need first to use the method of variation of parameters to transform it into a system of two first-order equations governing the amplitude and phase. To this end, we introduce the following transformation:

```
states = {a[t], φ[t]};

transformRule = {u -> a[t] Cos[φ[t]] + Λ Cos[Ψ[t]], u' -> -ω a[t] Sin[φ[t]] - Ω Λ Sin[Ψ[t]]};

gRule = g -> -2 ε μ u' - ε δ u^2 - ε^2 α u^3;
```

After some algebraic manipulations (Nayfeh, 1973, 1981), we transform **FDuffingEq** into two first-order equations as

```
eq43a =
  (D[states, t] == {-g/ω Sin[φ[t]], ω - g/(a[t] ω) Cos[φ[t]]} /. gRule /. transformRule // Expand //
      TrigReduce // Expand) // Thread;
```

We seek approximate solutions to **eq43a** in the form

```
basicTerms = {η[t], ϕ[t], Ψ[t]};
```

```
solRule = 9a -> I EvaluateAh@tD + SumAe^j a_j ê. basicTerms, 8j, 2<E ê. t -> #E &M,
  f -> I EvaluateAj@tD + SumAe^j f_j ê. basicTerms, 8j, 2<E ê. t -> #E &M=
```

```
8a Ø Hh@#1D + e a_1@h@#1D, j@#1D, Y@#1DD + e^2 a_2@h@#1D, j@#1D, Y@#1DD &L,
 f Ø Hj@#1D + e f_1@h@#1D, j@#1D, Y@#1DD + e^2 f_2@h@#1D, j@#1D, Y@#1DD &L<
```

where h@tD and j@tD are expanded in power series in e as

```
basicDRule = D@basicTerms@@81, 2<DD, tD ->
  9SumAe^i A_i ê. basicTerms, 8i, 2<E, w + SumAe^i F_i ê. basicTerms, 8i, 2<E= êê Thread
```

```
8h'@tD Ø e A_1@h@tD, j@tD, Y@tDD + e^2 A_2@h@tD, j@tD, Y@tDD,
 j'@tD Ø w + e F_1@h@tD, j@tD, Y@tDD + e^2 F_2@h@tD, j@tD, Y@tDD<
```

The functions a_1, a_2, f_1, and f_2 are fast varying functions of j, while it follows from **basicDRule** that h, and hence the A_n and F_n are slowly varying functions of *t*.

To the second approximation, we differentiate *a*@*t*D and f@*t*D once with respect to *t*, use **solRule** and **basicDRule**, expand the result for small e, discard terms of order higher than e^2, and obtain

```
eq43bLHS =
  CoefficientListAExpand@#@@1DD ê. solRule ê. basicDRuleD ê. e^n_ê;n>2 -> 0, eE & êž eq43a;
```

Next, we substitute **solRule** into the right-hand sides of **eq43a**, expand the result for small e, keep terms up to *O*He^2L, and rewrite their right-hand sides as

```
eq43bRHS =
  CoefficientList@Series@#@@2DD ê. solRule, 8e, 0, 2<D êê Normal êê Expand, eD & êž eq43a;
```

Equating coefficients of like powers of e in **eq43bLHS** and **eq43bRHS**, we obtain

```
eqEps = MapThread@Equal, 8eq43bLHS, eq43bRHS<, 2D êê Transpose êê Rest;
```

Next, we use the method of separation of variables to separate fast and slowly varying terms in the first-order equations, **eqEps[[1]]**. We first introduce the rules

```
SVT@j_D := 8A_j ê. basicTerms, F_j ê. basicTerms<
solVar@j_D := 8a_j, f_j<

psiRule = 8j@tD -> w t + b@tD, Y@tD -> W t<;

betaRule = Solve@psiRule ê. Rule -> Equal, 8b@tD, W<D@@1DD
```

```
9b@tD Ø - t w + j@tD, W Ø Y@tD/t =
```

For the case of subharmonic resonance of *O*I1/2M, we define

```
OmgRule = 8W -> 2 w + e s<;
```

```
sigRule = Solve@OmgRule •. Rule -> Equal, sD@@1DD
```

```
        2 w - W
9s ® - ‑‑‑‑‑‑‑‑‑ =
           e
```

```
expRule1 = f_@a_D :> f@Expand@a •. psiRule •. OmgRuleD •. e t -> t1D;

expRule2 = f_@a_D :> f@Expand@a •. t1 -> e t •. sigRule •. betaRuleDD;
```

Using these rules, we find that the slowly varying parts of **eqEps[[1]]** are given by

```
SVT1Rule =
 Table@Solve@eqEps@@1, iDD •. Thread@solVar@1D -> H0 &LD •. expRule1 •. f_@_ t + _.D -> 0 •.
     expRule2, SVT@1D@@iDDD @@1DD, 8i, 2<D •• Flatten
```

```
                           2 m w h@tD - d L Sin@2 j@tD - Y@tDD h@tD
9A1@h@tD, j@tD, Y@tDD ® - ‑‑‑‑‑‑‑‑‑‑‑‑‑‑‑‑‑‑‑‑‑‑‑‑‑‑‑‑‑‑‑‑‑‑‑‑‑‑‑‑‑ ,
                                             2 w
                         d L Cos@2 j@tD - Y@tDD
 F1@h@tD, j@tD, Y@tDD ® ‑‑‑‑‑‑‑‑‑‑‑‑‑‑‑‑‑‑‑‑‑‑‑ =
                                 2 w
```

whereas the fast varying parts are given by

```
FVT1 =
 Table@Subtract žž eqEps@@1, iDD •. Thread@solVar@1D -> H0 &LD, 8i, 2<D •. SVT1Rule •• Expand
```

```
 L m W Cos@j@tD - Y@tDD   L m W Cos@j@tD + Y@tDD   d L^2 Sin@j@tDD
:‑‑‑‑‑‑‑‑‑‑‑‑‑‑‑‑‑‑‑‑‑‑ - ‑‑‑‑‑‑‑‑‑‑‑‑‑‑‑‑‑‑‑‑‑‑ - ‑‑‑‑‑‑‑‑‑‑‑‑‑‑‑ -
          w                        w                    2 w

  d L^2 Sin@j@tD - 2 Y@tDD   d L^2 Sin@j@tD + 2 Y@tDD
  ‑‑‑‑‑‑‑‑‑‑‑‑‑‑‑‑‑‑‑‑‑‑‑‑ - ‑‑‑‑‑‑‑‑‑‑‑‑‑‑‑‑‑‑‑‑‑‑‑‑ - m Cos@2 j@tDD h@tD -
            4 w                        4 w

  d L Sin@2 j@tD + Y@tDD h@tD   d Sin@j@tDD h@tD^2   d Sin@3 j@tDD h@tD^2
  ‑‑‑‑‑‑‑‑‑‑‑‑‑‑‑‑‑‑‑‑‑‑‑‑‑‑‑ - ‑‑‑‑‑‑‑‑‑‑‑‑‑‑‑‑‑‑ - ‑‑‑‑‑‑‑‑‑‑‑‑‑‑‑‑‑‑‑‑ ,
              2 w                      4 w                  4 w

  d L Cos@Y@tDD   d L Cos@2 j@tD + Y@tDD                  d L^2 Cos@j@tDD
- ‑‑‑‑‑‑‑‑‑‑‑‑‑ - ‑‑‑‑‑‑‑‑‑‑‑‑‑‑‑‑‑‑‑‑‑‑ + m Sin@2 j@tDD - ‑‑‑‑‑‑‑‑‑‑‑‑‑‑‑ -
        w                  2 w                               2 w h@tD

  d L^2 Cos@j@tD - 2 Y@tDD   d L^2 Cos@j@tD + 2 Y@tDD   L m W Sin@j@tD - Y@tDD
  ‑‑‑‑‑‑‑‑‑‑‑‑‑‑‑‑‑‑‑‑‑‑‑‑ - ‑‑‑‑‑‑‑‑‑‑‑‑‑‑‑‑‑‑‑‑‑‑‑‑ - ‑‑‑‑‑‑‑‑‑‑‑‑‑‑‑‑‑‑‑‑‑‑ +
         4 w h@tD                   4 w h@tD                   w h@tD

  L m W Sin@j@tD + Y@tDD   3 d Cos@j@tDD h@tD   d Cos@3 j@tDD h@tD
  ‑‑‑‑‑‑‑‑‑‑‑‑‑‑‑‑‑‑‑‑‑‑ - ‑‑‑‑‑‑‑‑‑‑‑‑‑‑‑‑‑‑ - ‑‑‑‑‑‑‑‑‑‑‑‑‑‑‑‑‑‑ >
         w h@tD                   4 w                  4 w
```

To determine a particular solution corresponding to these fast varying terms, we use the method of undetermined coefficients. To accomplish this, we first determine the possible forms of the terms in **FVT1** as follows:

```
FVT1Forms =
 HCases@#, HCos È SinL@a_D -> 8Cos@aD, Sin@aD<, InfinityD •• Flatten •• UnionL & •ž FVT1
```

```
88Cos@j@tDD, Cos@2 j@tDD, Cos@3 j@tDD, Cos@j@tD - 2 Y@tDD,
  Cos@j@tD - Y@tDD, Cos@j@tD + Y@tDD, Cos@2 j@tD + Y@tDD, Cos@j@tD + 2 Y@tDD,
  Sin@j@tDD, Sin@2 j@tDD, Sin@3 j@tDD, Sin@j@tD - 2 Y@tDD, Sin@j@tD - Y@tDD,
  Sin@j@tD + Y@tDD, Sin@2 j@tD + Y@tDD, Sin@j@tD + 2 Y@tDD<,
 8Cos@j@tDD, Cos@2 j@tDD, Cos@3 j@tDD, Cos@j@tD - 2 Y@tDD, Cos@j@tD - Y@tDD,
  Cos@Y@tDD, Cos@j@tD + Y@tDD, Cos@2 j@tD + Y@tDD, Cos@j@tD + 2 Y@tDD,
  Sin@j@tDD, Sin@2 j@tDD, Sin@3 j@tDD, Sin@j@tD - 2 Y@tDD, Sin@j@tD - Y@tDD,
  Sin@Y@tDD, Sin@j@tD + Y@tDD, Sin@2 j@tD + Y@tDD, Sin@j@tD + 2 Y@tDD<<
```

Using the principle of superposition, we seek a particular solution corresponding to the fast-varying terms as a linear combination of these possible forms:

```
sol1Form = MapIndexed@Hcoeffs1@#2@@1DDD = Array@c, Length@#1DDL.#1 &, FVT1FormsD
```

```
8c@1D Cos@j@tDD + c@2D Cos@2 j@tDD + c@3D Cos@3 j@tDD +
  c@4D Cos@j@tD - 2 Y@tDD + c@5D Cos@j@tD - Y@tDD + c@6D Cos@j@tD + Y@tDD +
  c@7D Cos@2 j@tD + Y@tDD + c@8D Cos@j@tD + 2 Y@tDD + c@9D Sin@j@tDD + c@10D Sin@2 j@tDD +
  c@11D Sin@3 j@tDD + c@12D Sin@j@tD - 2 Y@tDD + c@13D Sin@j@tD - Y@tDD +
  c@14D Sin@j@tD + Y@tDD + c@15D Sin@2 j@tD + Y@tDD + c@16D Sin@j@tD + 2 Y@tDD,
 c@1D Cos@j@tDD + c@2D Cos@2 j@tDD + c@3D Cos@3 j@tDD + c@4D Cos@j@tD - 2 Y@tDD +
  c@5D Cos@j@tD - Y@tDD + c@6D Cos@Y@tDD + c@7D Cos@j@tD + Y@tDD + c@8D Cos@2 j@tD + Y@tDD +
  c@9D Cos@j@tD + 2 Y@tDD + c@10D Sin@j@tDD + c@11D Sin@2 j@tDD + c@12D Sin@3 j@tDD +
  c@13D Sin@j@tD - 2 Y@tDD + c@14D Sin@j@tD - Y@tDD + c@15D Sin@Y@tDD +
  c@16D Sin@j@tD + Y@tDD + c@17D Sin@2 j@tD + Y@tDD + c@18D Sin@j@tD + 2 Y@tDD<
```

Substituting **sol1Form** into **eqEps[[1]]**, using **SVT1Rule**, collecting the coefficients of **FVT1Forms**, solving the resulting algebraic equations for the undetermined coefficients, and then substituting the result back into **sol1Form**, we obtain the solution

```
sol1rhs = Table@sol1Form@@iDD •.
    Solve@Coefficient@Subtract žž eqEps@@1, iDD •. SVT1Rule •. solVar@1D@@iDD -> HEvaluate@
            sol1Form@@iDD •. Thread@basicTerms -> 8#1, #2, #3<DD &L, FVT1Forms@@iDDD ==
        0 •• Thread, coeffs1@iDD@@1DD •. D@psiRule, tD •• Expand, 8i, 2<D
```

```
:- d L^2 Cos@j@tDD/(2 w^2) - d L^2 Cos@j@tD - 2 Y@tDD/(4 w Hw - 2 WL) - d L^2 Cos@j@tD + 2 Y@tDD/(4 w Hw + 2 WL) -
  L m W Sin@j@tD - Y@tDD/(w Hw - WL) + L m W Sin@j@tD + Y@tDD/(w Hw + WL) - d L Cos@2 j@tD + Y@tDD h@tD/(2 w H2 w + WL) +
  m Sin@2 j@tDD h@tD/(2 w) - d Cos@j@tDD h@tD^2/(4 w^2) - d Cos@3 j@tDD h@tD^2/(12 w^2),
 m Cos@2 j@tDD/(2 w) + d L Sin@Y@tDD/(w W) + d L Sin@2 j@tD + Y@tDD/(2 w H2 w + WL) - L m W Cos@j@tD - Y@tDD/(w Hw - WL h@tD) +
  L m W Cos@j@tD + Y@tDD/(w Hw + WL h@tD) + d L^2 Sin@j@tDD/(2 w^2 h@tD) + d L^2 Sin@j@tD - 2 Y@tDD/(4 w Hw - 2 WL h@tD) +
  d L^2 Sin@j@tD + 2 Y@tDD/(4 w Hw + 2 WL h@tD) + 3 d Sin@j@tDD h@tD/(4 w^2) + d Sin@3 j@tDD h@tD/(12 w^2)>
```

which can be expressed in pure function form as

```
sol1Rule = Table@solVar@1D@@iDD ->
   HEvaluate@sol1rhs@@iDD •. Thread@basicTerms -> 8#1, #2, #3<DD &L, 8i, 2<D
```

```
:a1 ® i- d L^2 Cos@#2D/(2 w^2) - d L^2 Cos@#2 - 2 #3D/(4 w Hw - 2 WL) -
      k
    d L^2 Cos@#2 + 2 #3D/(4 w Hw + 2 WL) - L m W Sin@#2 - #3D/(w Hw - WL) + L m W Sin@#2 + #3D/(w Hw + WL) -
    d L Cos@2 #2 + #3D #1/(2 w H2 w + WL) + m Sin@2 #2D #1/(2 w) - d Cos@#2D #1^2/(4 w^2) - d Cos@3 #2D #1^2/(12 w^2) &y,
                                                                                                                  z
                                                                                                                  {
 f1 ® i m Cos@2 #2D/(2 w) + d L Sin@#3D/(w W) + d L Sin@2 #2 + #3D/(2 w H2 w + WL) - L m W Cos@#2 - #3D/(w Hw - WL #1) + L m W Cos@#2 + #3D/(w Hw + WL #1) +
      k
    d L^2 Sin@#2D/(2 w^2 #1) + d L^2 Sin@#2 - 2 #3D/(4 w Hw - 2 WL #1) + d L^2 Sin@#2 + 2 #3D/(4 w Hw + 2 WL #1) + 3 d Sin@#2D #1/(4 w^2) + d Sin@3 #2D #1/(12 w^2) &y>
                                                                                                                  z
                                                                                                                  {
```

Substituting the slow- and fast-varying components of the first-order solution, **SVT1Rule** and **sol1Rule**, into the second-order equations, we have

```
Horder2Expr = Table@TrigReduce •ž
    HSubtract žž eqEps@@2, iDD •. SVT1Rule •. sol1Rule •• ExpandL, 8i, 2<D;L •• Timing
```

```
812.36 Second, Null<
```

Since we are seeking an expansion valid up to OHe^{2}L, we do not need to solve for a_2 and f_2. All we need to do is to investigate the above expressions to determine the slowly varying parts and determine A_2 and F_2. The result is

```
SVT2Rule =
 Table@Solve@order2Expr@@iDD == 0 •. Thread@solVar@2D -> H0 &LD •. expRule1 •. f_@_ t + _.D ->
       0 •. expRule2 •. HOmgRule •. e -> 0L, SVT@2D@@iDDD, 8i, 2<D •• Flatten •• ExpandAll
```

```
:A2@h@tD, j@tD, Y@tDD ® - (7 d L m Cos@2 j@tD - Y@tDD h@tD)/(6 w^2), F2@h@tD, j@tD, Y@tDD ®
  - (9 d^2 L^2)/(16 w^3) + (3 a L^2)/(4 w) - m^2/(2 w) + (7 d L m Sin@2 j@tD - Y@tDD)/(6 w^2) - (5 d^2 h@tD^2)/(12 w^3) + (3 a h@tD^2)/(8 w)>
```

Hence, to the second approximation, we find that

```
u@tD ==
 HTrigReduce •ž HNormal@Series@u •. transformRule •. solRule, 8e, 0, 1<DD •. sol1Rule ••
     ExpandL •• Collect@#, eD &L
```

```
u@tD ==
 L Cos@Y@tDD + Cos@j@tDD h@tD + e (- (d L^2)/(2 w^2) - (d L^2 Cos@2 Y@tDD)/(4 w Hw - 2 WL) - (d L^2 Cos@2 Y@tDD)/(4 w Hw + 2 WL) + (L m W Sin@Y@tDD)/(w Hw - WL) +
    (L m W Sin@Y@tDD)/(w Hw + WL) - (d L Cos@j@tD - Y@tDD h@tD)/(2 w W) + (d L Cos@j@tD + Y@tDD h@tD)/(2 w W) -
    (d L Cos@j@tD + Y@tDD h@tD)/(2 w H2 w + WL) + (m Sin@j@tDD h@tD)/(2 w) - (d h@tD^2)/(2 w^2) + (d Cos@2 j@tDD h@tD^2)/(6 w^2))
```

where the equations governing the amplitude and phase are

```
8ampEqs, phaseEqs< = HbasicDRule •. SVT1Rule •. SVT2Rule •• ExpandAllL •.
   f_@a_D :> f@Collect@a •. psiRule, tDD •. Rule -> Equal
```

```
:h¢@tD == - e m h@tD - (7 d e^2 L m Cos@t H2 w - WL + 2 b@tDD h@tD)/(6 w^2) + (d e L Sin@t H2 w - WL + 2 b@tDD h@tD)/(2 w),
 j¢@tD == - (9 d^2 e^2 L^2)/(16 w^3) + (3 a e^2 L^2)/(4 w) - (e^2 m^2)/(2 w) + w + (d e L Cos@t H2 w - WL + 2 b@tDD)/(2 w) +
   (7 d e^2 L m Sin@t H2 w - WL + 2 b@tDD)/(6 w^2) - (5 d^2 e^2 h@tD^2)/(12 w^3) + (3 a e^2 h@tD^2)/(8 w)>
```

à 4.4 The Method of Normal Forms

In this section, we use the method of normal forms to determine a second-order uniform expansion of the solution of **FDuffingEq**. To this end, we start with the corresponding first-order complex-valued equation, which was derived in Section 4.2.6 and a summary of the derivation is given below:

```
transformRule = 9u@tD -> I z@tD + z̄@tDM, u¢@tD -> I w I z@tD - z̄@tDM, f Cos@W tD -> 1/2 Hz@tD + z̄@tDL=;
```

```
gRule = g -> - 2 e m u¢@tD - e d u@tD^2 - e^2 a u@tD^3 + f Cos@W tD;
```

```
eq44a = 8z¢@tD == I w z@tD - I g • H2 wL •. gRule •. transformRule •• ExpandAll, z¢@tD == I W z@tD<
```

```
:z¢@tD ==

   I z@tD                     I d e z@tD^2   I a e^2 z@tD^3   I ż@tD                I d e z@tD ż@tD
 - €€€€€€ - e m z@tD + I w z@tD + €€€€€€€€€€€ + €€€€€€€€€€€€€ - €€€€€€ + e m ż@tD + €€€€€€€€€€€€€€ +
   4 w                          2 w            2 w            4 w                       w

  3 I a e^2 z@tD^2 ż@tD   I d e ż@tD^2   3 I a e^2 z@tD ż@tD^2   I a e^2 ż@tD^3
  €€€€€€€€€€€€€€€€€€€€ + €€€€€€€€€€€ + €€€€€€€€€€€€€€€€€€€€ + €€€€€€€€€€€€€ , z¢@tD == I W z@tD>
          2 w               2 w               2 w                  2 w
```

According to the method of normal forms, we introduce the near-identity transformation

```
basicTerms = 8h@tD, ḣ@tD, z@tD, ż@tD<;
```

```
zetaRule = 9z -> FunctionAt, h@tD + SumAe^j hj@Sequence žž basicTermsD, 8j, 0, 2<E •• EvaluateE,
  ż -> FunctionAt, ḣ@tD + SumAe^j ḣj@Sequence žž basicTermsD, 8j, 0, 2<E •• EvaluateE=
```

```
8z ® Function@t, h@tD + h0@h@tD, ḣ@tD, z@tD, ż@tDD +
    e h1@h@tD, ḣ@tD, z@tD, ż@tDD + e^2 h2@h@tD, ḣ@tD, z@tD, ż@tDDD,
 ż ® Function@t, ḣ@tD + ḣ0@h@tD, ḣ@tD, z@tD, ż@tDD +
    e ḣ1@h@tD, ḣ@tD, z@tD, ż@tDD + e^2 ḣ2@h@tD, ḣ@tD, z@tD, ż@tDDD<
```

that results in the simplest possible equation

```
etaRule = 9h¢@tD -> I w h@tD + SumAe^j gj@tD, 8j, 2<E, z¢@tD -> I W z@tD=
```

```
8h¢@tD ® I w h@tD + e g1@tD + e^2 g2@tD, z¢@tD ® I W z@tD<
```

where the overbar denotes the complex conjugate

```
conjugateRule = 8h -> ḣ, ḣ -> h, g -> ġ, ġ -> g, z -> ż, ż -> z, Complex@0, n_D -> Complex@0, -nD<;
```

Substituting the expansion for z, the **zetaRule**, into **eq44a**, using the **etaRule**, expanding the result for small e, and discarding terms of order higher than e^2, we have

```
eq44b = Heq44a@@1DD •. zetaRule •. etaRule •. HetaRule •. conjugateRuleL •• ExpandAllL •.
    e^(n_•;n>2) -> 0;
```

Equating coefficients of like powers of e in **eq44b** yields

```
eqEps = CoefficientList@Subtract žž eq44b, eD == 0 •• Thread;
```

Ÿ First-Order Solution

Using the method of undetermined coefficients, we assume

```
coeffsL = Table@Di, 8i, Length@basicTermsD<D
```

```
8D1, D2, D3, D4<
```

```
hFormL = 9h0 -> HEvaluate@coeffsL.basicTerms •. Thread@basicTerms -> 8#1, #2, #3, #4<DD &L,
  ḣ0 -> HEvaluate@coeffsL.basicTerms •. conjugateRule •.
      Thread@basicTerms -> 8#1, #2, #3, #4<DD &L=;
```

Substituting the **hFormL** into **eqEps[[1]]**, collecting the coefficients of the **basicTerms**, and solving for the **coeffsL**, we obtain

```
coeffsLRule =
 Solve@Coefficient@eqEps@@1, 1DD •. hFormL, basicTermsD == 0 •• Thread, coeffsLD@@1DD
```

```
Solve::svars : Equations may not give solutions for all "solve" variables.
```

$:D_2 ® 0,\ D_3 ® \frac{1}{4\ w\ Hw - WL},\ D_4 ® \frac{1}{4\ w\ Hw + WL}>$

Substituting the **coeffsLRule** into the **hFormL** yields

```
hSolL = h0 žž basicTerms •. hFormL •. coeffsLRule •. Di_ -> 0
```

$\frac{z@tD}{4\ w\ Hw - WL} + \frac{\dot{z}@tD}{4\ w\ Hw + WL}$

Then, we write the first-order solution in pure function form as

```
hRuleL = 9h0 -> HEvaluate@hSolL •. Thread@basicTerms -> 8#1, #2, #3, #4<DD &L,
  ḣ0 -> HEvaluate@hSolL •. conjugateRule •. Thread@basicTerms -> 8#1, #2, #3, #4<DD &L=
```

$:h_0 ® \left(\frac{\#3}{4\ w\ Hw - WL} + \frac{\#4}{4\ w\ Hw + WL}\ \&\right),\ \dot{h}_0 ® \left(\frac{\#3}{4\ w\ Hw + WL} + \frac{\#4}{4\ w\ Hw - WL}\ \&\right)>$

Ÿ Second-Order Solution

Substituting the **hRuleL** into the left-hand side of the second-order equation, **eqEps[[2]]**, yields

```
order2expr = eqEps@@2, 1DD •. hRuleL •• Expand
```

$$\frac{m\ z@tD}{4\ w\ Hw - WL} - \frac{m\ z@tD}{4\ w\ Hw + WL} - \frac{I\ d\ z@tD^2}{32\ w^3\ Hw - WL^2} - \frac{I\ d\ z@tD^2}{32\ w^3\ Hw + WL^2} - \frac{I\ d\ z@tD^2}{16\ w^3\ Hw - WL\ Hw + WL} + m\ h@tD -$$

$$\frac{I\ d\ z@tD\ h@tD}{4\ w^2\ Hw - WL} - \frac{I\ d\ z@tD\ h@tD}{4\ w^2\ Hw + WL} - \frac{I\ d\ h@tD^2}{2\ w} - \frac{m\ \dot{z}@tD}{4\ w\ Hw - WL} + \frac{m\ \dot{z}@tD}{4\ w\ Hw + WL} - \frac{I\ d\ z@tD\ \dot{z}@tD}{16\ w^3\ Hw - WL^2} -$$

$$\frac{I\ d\ z@tD\ \dot{z}@tD}{16\ w^3\ Hw + WL^2} - \frac{I\ d\ z@tD\ \dot{z}@tD}{8\ w^3\ Hw - WL\ Hw + WL} - \frac{I\ d\ h@tD\ \dot{z}@tD}{4\ w^2\ Hw - WL} - \frac{I\ d\ h@tD\ \dot{z}@tD}{4\ w^2\ Hw + WL} - \frac{I\ d\ \dot{z}@tD^2}{32\ w^3\ Hw - WL^2} -$$

$$\frac{I\ d\ \dot{z}@tD^2}{32\ w^3\ Hw + WL^2} - \frac{I\ d\ \dot{z}@tD^2}{16\ w^3\ Hw - WL\ Hw + WL} - m\ \dot{h}@tD - \frac{I\ d\ z@tD\ \dot{h}@tD}{4\ w^2\ Hw - WL} - \frac{I\ d\ z@tD\ \dot{h}@tD}{4\ w^2\ Hw + WL} - \frac{I\ d\ h@tD\ \dot{h}@tD}{w} -$$

$$\frac{I\ d\ \dot{z}@tD\ \dot{h}@tD}{4\ w^2\ Hw - WL} - \frac{I\ d\ \dot{z}@tD\ \dot{h}@tD}{4\ w^2\ Hw + WL} - \frac{I\ d\ \dot{h}@tD^2}{2\ w} + g_1@tD - I\ w\ h_1@h@tD,\ \dot{h}@tD,\ z@tD,\ \dot{z}@tDD -$$

$$I\ W\ \dot{z}@tD\ h_1^{H0,0,0,1L}@h@tD,\ \dot{h}@tD,\ z@tD,\ \dot{z}@tDD + I\ W\ z@tD\ h_1^{H0,0,1,0L}@h@tD,\ \dot{h}@tD,\ z@tD,\ \dot{z}@tDD -$$

$$I\ w\ \dot{h}@tD\ h_1^{H0,1,0,0L}@h@tD,\ \dot{h}@tD,\ z@tD,\ \dot{z}@tDD + I\ w\ h@tD\ h_1^{H1,0,0,0L}@h@tD,\ \dot{h}@tD,\ z@tD,\ \dot{z}@tDD$$

We choose h_1 to eliminate as many terms from **order2expr**, thereby reducing it to its simplest possible form. It turns out that we can eliminate all nonresonance terms. To determine the resonance terms in **order2expr**, we first determine all of the possible forms of its nonhomogeneous terms:

```
possibleQTerms =
 basicTerms~Join~HOuter@Times, basicTerms, basicTermsD •• Flatten •• UnionL
```

```
9h@tD, ḣ@tD, z@tD, ż@tD, z@tD^2, z@tD h@tD, h@tD^2,
 z@tD ż@tD, h@tD ż@tD, ż@tD^2, z@tD ḣ@tD, h@tD ḣ@tD, ż@tD ḣ@tD, ḣ@tD^2=
```

It follows from the linear parts of the **etaRule** that

```
form = 9h@tD -> E^(I w t), ḣ@tD -> E^(-I w t), z@tD -> E^(I W t), ż@tD -> E^(-I W t)=;
```

Hence, the possible resonance terms are given by

```
ResonantQTerm =
 IE^(-I w t) possibleQTerms •. form •. W -> 2 w •. E^(–t) -> 0M possibleQTerms •• Union •• Rest
```

```
8h@tD, z@tD ḣ@tD<
```

We choose g_1 to eliminate the resonance terms in **order2expr** according to

```
gRuleQ =
 g1@tD -> -HCoefficient@order2expr •. h1 -> H0 &L, ResonantQTermD •. Thread@basicTerms -> 0DL.
    ResonantQTerm
```

```
                         j        I d                  I d          y
g1@tD ® -m h@tD - j- ¢¢¢¢¢¢¢¢¢¢¢¢¢¢¢¢¢¢¢¢¢¢¢¢ - ¢¢¢¢¢¢¢¢¢¢¢¢¢¢¢¢¢¢¢¢¢¢¢¢ z z@tD ḣ@tD
                         k   4 w^2 Hw - WL       4 w^2 Hw + WL   {
```

The nonresonance terms are the complement of the resonance terms; that is,

```
NRQT = Complement@possibleQTerms, ResonantQTermD;
```

We associate with each of them an undetermined coefficient according to

```
coeffsQ = Table@Gj, 8j, Length@NRQTD<D
```

```
8G1, G2, G3, G4, G5, G6, G7, G8, G9, G10, G11, G12<
```

Therefore, h_1 has the form

```
hFormQ = 8h1 -> HEvaluate@coeffsQ.NRQT •. Thread@basicTerms -> 8#1, #2, #3, #4<DD &L<
```

```
8h1 ® H#3 G1 + #3^2 G2 + #1 #3 G3 + #1^2 G4 + #4 G5 +
      #3 #4 G6 + #1 #4 G7 + #4^2 G8 + #2 G9 + #1 #2 G10 + #2 #4 G11 + #2^2 G12 &L<
```

Substituting for h_1 in **order2expr**, equating the coefficients of the possible nonresonance terms to zero, and solving the resulting equations for the G_i, we obtain

```
coeffsQRule = Solve@Coefficient@order2expr •. gRuleQ •. hFormQ •• Expand, NRQTD == 0 •.
    Thread@basicTerms -> 0D •• Thread, coeffsQD@@1DD
```

9G3 ® - d/(2 w W H-w + WL Hw + WL), G4 ® d/(2 w^2), G6 ® - d/(4 w^2 Hw - WL^2 Hw + WL^2), G7 ® d/(2 w W H-w + WL Hw + WL),
G9 ® I m/(2 w), G10 ® - d/w^2, G12 ® - d/(6 w^2), G1 ® - I m W/(2 w Hw - WL^2 Hw + WL), G2 ® - d/(8 w Hw - 2 WL Hw - WL^2 Hw + WL^2),
G5 ® I m W/(2 w Hw - WL Hw + WL^2), G11 ® - d/(2 w Hw - WL Hw + WL H2 w + WL), G8 ® - d/(8 w Hw - WL^2 Hw + WL^2 Hw + 2 WL) =

Substituting the **coeffsQRule** into the **hFormQ** yields

```
hSolQ = h1 žž basicTerms •. hFormQ •. coeffsQRule
```

- I m W z@tD/(2 w Hw - WL^2 Hw + WL) - d z@tD^2/(8 w Hw - 2 WL Hw - WL^2 Hw + WL^2) - d z@tD h@tD/(2 w W H-w + WL Hw + WL) +
d h@tD^2/(2 w^2) + I m W ż@tD/(2 w Hw - WL Hw + WL^2) - d z@tD ż@tD/(4 w^2 Hw - WL^2 Hw + WL^2) + d h@tD ż@tD/(2 w W H-w + WL Hw + WL) -
d ż@tD^2/(8 w Hw - WL^2 Hw + WL^2 Hw + 2 WL) + I m ḣ@tD/(2 w) - d h@tD ḣ@tD/w^2 - d ż@tD ḣ@tD/(2 w Hw - WL Hw + WL H2 w + WL) - d ḣ@tD^2/(6 w^2)

Hence, we can write the second-order solution in pure function form as

```
hRuleQ = 9h1 -> HEvaluate@hSolQ •. Thread@basicTerms -> 8#1, #2, #3, #4<DD &L,
   ḣ1 -> HEvaluate@hSolQ •. conjugateRule •. Thread@basicTerms -> 8#1, #2, #3, #4<DD &L=;
```

Ÿ Third-Order Equations

Substituting the **hRuleL**, **hRuleQ**, and **gRuleQ** into the left-hand side of the third-order equation, **eqEps[[3]]**, yields

```
order3expr =
  eqEps@@3, 1DD •. hRuleL •. hRuleQ •. gRuleQ •. HgRuleQ •. conjugateRuleL •• Expand;
```

The nonhomogeneous terms in **order3expr** are proportional to

```
possibleCTerms = Outer@Times, possibleQTerms, basicTermsD •• Flatten •• Union
```

9z@tD^2, z@tD^3, z@tD h@tD, z@tD^2 h@tD, h@tD^2, z@tD h@tD^2, h@tD^3, z@tD ż@tD, z@tD^2 ż@tD,
h@tD ż@tD, z@tD h@tD ż@tD, h@tD^2 ż@tD, ż@tD^2, z@tD ż@tD^2, h@tD ż@tD^2, ż@tD^3, z@tD ḣ@tD,
z@tD^2 ḣ@tD, h@tD ḣ@tD, z@tD h@tD ḣ@tD, h@tD^2 ḣ@tD, ż@tD ḣ@tD, z@tD ż@tD ḣ@tD,
h@tD ż@tD ḣ@tD, ż@tD^2 ḣ@tD, ḣ@tD^2, z@tD ḣ@tD^2, h@tD ḣ@tD^2, ż@tD ḣ@tD^2, ḣ@tD^3=

```
possibleTerms = Join@possibleQTerms, possibleCTermsD •• Union
```

```
9z@tD, z@tD^2, z@tD^3, h@tD, z@tD h@tD, z@tD^2 h@tD, h@tD^2, z@tD h@tD^2, h@tD^3, ż@tD, z@tD ż@tD,
 z@tD^2 ż@tD, h@tD ż@tD, z@tD h@tD ż@tD, h@tD^2 ż@tD, ż@tD^2, z@tD ż@tD^2, h@tD ż@tD^2,
 ż@tD^3, ḣ@tD, z@tD ḣ@tD, z@tD^2 ḣ@tD, h@tD ḣ@tD, z@tD h@tD ḣ@tD, h@tD^2 ḣ@tD, ż@tD ḣ@tD,
 z@tD ż@tD ḣ@tD, h@tD ż@tD ḣ@tD, ż@tD^2 ḣ@tD, ḣ@tD^2, z@tD ḣ@tD^2, h@tD ḣ@tD^2, ż@tD ḣ@tD^2, ḣ@tD^3=
```

Next, we determine the resonance terms according to

```
ResonantCTerm =
 IE^-Iwt possibleTerms •. form •. W -> 2 w •. E^-t -> 0M possibleTerms •• Union •• Rest
```

```
8h@tD, z@tD h@tD ż@tD, z@tD ḣ@tD, h@tD^2 ḣ@tD<
```

Choosing g_2 to eliminate the resonance terms from **order3expr** yields

```
gRuleC = g2@tD ->
  - Factor@Coefficient@order3expr •. h2 -> H0 &L, ResonantCTermD •. Thread@basicTerms -> 0DD.
    ResonantCTerm
```

```
g2@tD ® - (I m^2 h@tD)/(2 w) + (I H- 5 d^2 w + 6 a w^3 - 2 d^2 W + 3 a w^2 WL z@tD h@tD ż@tD)/(4 w^3 Hw - WL^2 Hw + WL^2 H2 w + WL) -
  (d m H2 w^3 + w^2 W - 6 w W^2 - W^3L z@tD ḣ@tD)/(4 w^2 Hw - WL^2 W Hw + WL^2) - (I H10 d^2 - 9 a w^2L h@tD^2 ḣ@tD)/(6 w^3)
```

The nonresonance terms are the complement of the resonance terms; that is,

```
NRCT = Complement@possibleTerms, ResonantCTermD;
```

Again, we associate an undetermined coefficient with each of these terms as follows:

```
coeffsC = Table@Lj, 8j, Length@NRCTD<D
```

```
8L1, L2, L3, L4, L5, L6, L7, L8, L9, L10, L11, L12, L13, L14, L15,
 L16, L17, L18, L19, L20, L21, L22, L23, L24, L25, L26, L27, L28, L29, L30<
```

Consequently, h_2 has the form

```
hFormC = 8h2 -> HEvaluate@coeffsC.NRCT •. Thread@basicTerms -> 8#1, #2, #3, #4<DD &L<;
```

Substituting for h_2 into **order3expr**, equating the coefficient of each possible nonresonance term to zero, and solving the resulting algebraic equations for the L_i, we obtain

```
coeffsCRule = Solve@Coefficient@order3expr •. gRuleC •. hFormC •• Expand, NRCTD == 0 •.
     Thread@basicTerms -> 0D •• Thread, coeffsCD@@1DD •• ExpandAll;
```

Substituting the **coeffsCRule** into the **hFormC** yields

```
hSolC = h2 ěě basicTerms •. hFormC •. coeffsCRule;
```

Combining the **etaRule**, **gRuleQ**, and **gRuleC** and letting

```
solForm = {η -> (A[#] E^(I ω #) &), η̄ -> (Ā[#] E^(-I ω #) &), z -> (f E^(I Ω #) &), z̄ -> (f E^(-I Ω #) &)};
```

we obtain the modulation equation

```
moduEq = (2 I ω E^(-I ω t) (η'[t] - η'[t] /. etaRule /. gRuleQ /. gRuleCLL /. solForm // Expand //
    Collect[#, ε] &) == 0
```

$$\epsilon\left(2 I \mu \omega A[t]+\frac{E^{-2 I t \omega+I t \Omega} f \delta \bar{A}[t]}{2 \omega(\omega-\Omega)}+\frac{E^{-2 I t \omega+I t \Omega} f \delta \bar{A}[t]}{2 \omega(\omega+\Omega)}\right)+$$
$$\epsilon^2\left(-\mu^2 A[t]-\frac{5 f^2 \delta^2 A[t]}{2 \omega(\omega-\Omega)^2(\omega+\Omega)^2(2 \omega+\Omega)}+\frac{3 f^2 \alpha \omega A[t]}{(\omega-\Omega)^2(\omega+\Omega)^2(2 \omega+\Omega)}+\right.$$
$$\frac{3 f^2 \alpha \Omega A[t]}{2(\omega-\Omega)^2(\omega+\Omega)^2(2 \omega+\Omega)}-\frac{f^2 \delta^2 \Omega A[t]}{\omega^2(\omega-\Omega)^2(\omega+\Omega)^2(2 \omega+\Omega)}+$$
$$\frac{I E^{-2 I t \omega+I t \Omega} f \delta \mu \omega \bar{A}[t]}{2(\omega-\Omega)^2(\omega+\Omega)^2}+\frac{I E^{-2 I t \omega+I t \Omega} f \delta \mu \omega^2 \bar{A}[t]}{(\omega-\Omega)^2 \Omega(\omega+\Omega)^2}-\frac{3 I E^{-2 I t \omega+I t \Omega} f \delta \mu \Omega \bar{A}[t]}{(\omega-\Omega)^2(\omega+\Omega)^2}-$$
$$\left.\frac{I E^{-2 I t \omega+I t \Omega} f \delta \mu \Omega^2 \bar{A}[t]}{2 \omega(\omega-\Omega)^2(\omega+\Omega)^2}+3 \alpha A[t]^2 \bar{A}[t]-\frac{10 \delta^2 A[t]^2 \bar{A}[t]}{3 \omega^2}\right)+2 I \omega A'[t]==0$$

or

```
moduEq /. f -> 2 Λ (Ω^2 - ω^2) /. Ω -> 2 ω
```

$$\epsilon\left(2 I \mu \omega A[t]-2 \delta \Lambda \bar{A}[t]\right)+$$
$$\epsilon^2\left(6 \alpha \Lambda^2 A[t]-\mu^2 A[t]-\frac{9 \delta^2 \Lambda^2 A[t]}{2 \omega^2}-\frac{14 I \delta \Lambda \mu \bar{A}[t]}{3 \omega}+3 \alpha A[t]^2 \bar{A}[t]-\frac{10 \delta^2 A[t]^2 \bar{A}[t]}{3 \omega^2}\right)+$$
$$2 I \omega A'[t]==0$$

which is in agreement with those obtained by using the method of multiple scales and the generalized method of averaging.

Chapter 5
Higher-Order Approximations for Systems with Internal Resonances

```
Off@General::spell1D
```

à 5.1 Euler-Lagrange Equations

In this chapter, we use different methods to determine approximate solutions of nonlinear systems possessing internal resonances to orders higher than the order at which the influence of the internal resonance first appears. To describe the methods with minimum algebra, we consider the free oscillations of a two-degree-of-freedom conservative system possessing a two-to-one internal resonance and having simple quadratic nonlinearities. In particular, we consider a system governed by the **Lagrangian**

```
Lagrangian = 1/2 u1¢@tD^2 + 1/2 u2¢@tD^2 - 1/2 w1^2 u1@tD^2 - 1/2 w2^2 u2@tD^2 + d u1@tD^2 u2@tD;
```

where w_2 » $2w_1$. Writing down the Euler-Lagrange equations corresponding to the **Lagrangian**, we obtain the following second-order equations of motion:

```
EOM1 = Table@D@D@Lagrangian, ui¢@tDD, tD - D@Lagrangian, ui@tDD == 0, 8i, 2<D
```

```
8w1^2 u1@tD - 2 d u1@tD u2@tD + u1²@tD == 0, -d u1@tD^2 + w2^2 u2@tD + u2²@tD == 0<
```

Approximate solutions of general two-degree-of-freedom system with quadratic and cubic nonlinearities having a two-to-one internal resonance can be obtained by replacing the **Lagrangian** with the Lagrangian corresponding to the general system.

As discussed in the next section, treating this second-order form, **EOM1**, of the governing equations by using the method of multiple scales may lead to results that violate the conservative nature of the system under consideration. To determine an approximate solution that preserves the conservative nature of the system, we treat a first-order form of the governing equations. To transform **EOM1** into a system of first-order equations, we form a modified **Lagrangian** by introducing two more states v_1@tD and v_2@tD such that

```
velRule = 8u1¢@tD -> v1@tD, u2¢@tD -> v2@tD<;
```

Then, the **Lagrangian** becomes

```
Lag1 = Lagrangian •. velRule
```

```
- 1/2 w1^2 u1@tD^2 + d u1@tD^2 u2@tD - 1/2 w2^2 u2@tD^2 + 1/2 v1@tD^2 + 1/2 v2@tD^2
```

Substituting **Lag1** into Euler-Lagrange equations, we obtain the following first-order equations

```
eq51a = Table@D@D@Lag1, vi@tDD, tD - D@Lag1, ui@tDD == 0, 8i, 2<D
```

```
8w1^2 u1@tD - 2 d u1@tD u2@tD + v1¢@tD == 0, -d u1@tD^2 + w2^2 u2@tD + v2¢@tD == 0<
```

Using the **velRule** and **eq51a**, we obtain the following first-order equations of motion:

```
EOM2 = 8velRule •. Rule -> Equal, eq51a< •• Transpose •• Flatten
```

```
8u1¢@tD == v1@tD, w1^2 u1@tD - 2 d u1@tD u2@tD + v1¢@tD == 0,
 u2¢@tD == v2@tD, -d u1@tD^2 + w2^2 u2@tD + v2¢@tD == 0<
```

The corresponding linear system of **EOM2** can be defined as $x = A x$, where x is the state vector; that is,

```
states = Table@8ui@tD, vi@tD<, 8i, 2<D •• Flatten
```

```
8u1@tD, v1@tD, u2@tD, v2@tD<
```

Using these states, we find that the coefficient matrix A of the linear part of **EOM2** is given by

```
matrixA = Outer@D, #@@2DD & •ž Solve@EOM2, D@states, tDD@@1DD •. d -> 0, statesD
```

```
880, 1, 0, 0<, 8-w1^2, 0, 0, 0<, 80, 0, 0, 1<, 80, 0, -w2^2, 0<<
```

whose adjoint is defined by

```
conjugateRule = 9A -> Ȧ, Ȧ -> A, Complex@0, n_D -> Complex@0, -nD=;
```

```
hermitian@mat_? MatrixQD := mat •. conjugateRule •• Transpose
```

The eigenvalues and eigenvectors of **matrixA** can be obtained as

```
matrixA •• Eigensystem
```

```
:8-I w1, I w1, -I w2, I w2<, ::I/w1, 1, 0, 0>, :-I/w1, 1, 0, 0>, :0, 0, I/w2, 1>, :0, 0, -I/w2, 1>>>
```

Hence, the right eigenvectors of **matrixA** corresponding to the eigenvalues $I\,w_1$ and $I\,w_2$, respectively, are

```
rightVec = Eigenvectors@matrixAD@@82, 4<DD
```

```
::-I/w1, 1, 0, 0>, :0, 0, -I/w2, 1>>
```

Moreover, the left eigenvectors of **matrixA** corresponding to the eigenvalues $I\,w_1$ and $I\,w_2$, respectively, are

```
leftVec = Eigenvectors@hermitian@matrixADD@@81, 3<DD
```

```
88-I w1, 1, 0, 0<, 80, 0, -I w2, 1<<
```

whose complex conjugate is

```
ccleftVec = leftVec •. conjugateRule
```

```
88I w1, 1, 0, 0<, 80, 0, I w2, 1<<
```

à 5.2 Method of Multiple Scales

We use the method of multiple scales to obtain a second-order uniform expansion of the solution of **EOM1** in Section 5.2.1, **EOM2** in Section 5.2.2, and the corresponding first-order complex-valued equations of **EOM1** in Section 5.2.3. Again, we need the three time scales $T_0 = t$, $T_1 = \epsilon t$, and $T_2 = \epsilon^2 t$, which we symbolize by

```
Needs@"Utilities`Notation`"D
```

```
Symbolize@T0D; Symbolize@T1D; Symbolize@T2D;
```

```
timeScales = 8T0, T1, T2<;
```

In terms of the new time scales, we can express the time derivatives as

```
dt@1D@expr_D := SumAe^i D@expr, timeScales@@i + 1DDD, 8i, 0, 2<E;
dt@2D@expr_D := Hdt@1D@dt@1D@exprDD •• ExpandL •. e^(i_•;i>3) -> 0;
```

To represent some of the expressions in a more concise way, we introduce the following display rule:

```
displayRule =
  9Derivative@a__DAu_(i_,j_)E@__D :> SequenceFormATimes žž MapIndexedAD^#1_(#2@@1DD-1) &, 8a<E, u_(i,j)E,
   Derivative@a__DAA_(i_)E@__D :> SequenceFormATimes žž MapIndexedAD^#1_(#2@@1DD) &, 8a<E, A_iE,
   u_(i_,j_)@__D -> u_(i,j), A_(i_)@__D -> A_i=;
```

Ÿ 5.2.1 Second-Order Real-Valued System

Using method of multiple scales, we assume that the solution of **EOM1** can be expressed in the form

```
solRule = u_(i_) -> ISumAe^j u_(i,j)@#1, #2, #3D, 8j, 3<E &M;
```

```
multiScales = 8u_(i_)@tD -> u_i@T0, T1, T2D, Derivative@n_D@u_(i_)D@tD :> dt@nD@u_i@T0, T1, T2DD<;
```

Substituting **multiScales** and **solRule** into **EOM1**, expanding the result for small ϵ, and discarding terms of order higher than ϵ^3, we obtain

```
eq521a = HEOM1 •. multiScales •. solRule •• ExpandAllL •. e^(n_•;n>3) -> 0;
eq521a •. displayRule
```

$$8 e\, HD_0^2 u_{1,1}L + e^2\, HD_0^2 u_{1,2}L + e^3\, HD_0^2 u_{1,3}L + 2\, e^2\, HD_0\, D_1 u_{1,1}L + 2\, e^3\, HD_0\, D_1 u_{1,2}L + e^3\, HD_1^2 u_{1,1}L + 2\, e^3\, HD_0\, D_2 u_{1,1}L + e\, w_1^2\, u_{1,1} + e^2\, w_1^2\, u_{1,2} + e^3\, w_1^2\, u_{1,3} - 2\, d\, e^2\, u_{1,1}\, u_{2,1} - 2\, d\, e^3\, u_{1,2}\, u_{2,1} - 2\, d\, e^3\, u_{1,1}\, u_{2,2} == 0,$$
$$e\, HD_0^2 u_{2,1}L + e^2\, HD_0^2 u_{2,2}L + e^3\, HD_0^2 u_{2,3}L + 2\, e^2\, HD_0\, D_1 u_{2,1}L + 2\, e^3\, HD_0\, D_1 u_{2,2}L + e^3\, HD_1^2 u_{2,1}L + 2\, e^3\, HD_0\, D_2 u_{2,1}L - d\, e^2\, u_{1,1}^2 - 2\, d\, e^3\, u_{1,1}\, u_{1,2} + e\, w_2^2\, u_{2,1} + e^2\, w_2^2\, u_{2,2} + e^3\, w_2^2\, u_{2,3} == 0<$$

Equating coefficients of like powers of e, we obtain

```
eqEps = Rest@Thread@CoefficientList@Subtract žž #, eD == 0DD & •ž eq521a •• Transpose;
```

To place the linear operators on one side and the nonhomogeneous terms on the other side, we define

```
eqOrder@i_D := I#@@1DD & •ž eqEps@@1DD •. u_{k_,1} -> u_{k,i}M ==
   I#@@1DD & •ž eqEps@@1DD •. u_{k_,1} -> u_{k,i}M - H#@@1DD & •ž eqEps@@iDDL •• Thread
```

Using **eqOrder[i]** and **displayRule**, we rewrite **eqEps** in a concise way as

```
eqOrder@1D •. displayRule •• TableForm
eqOrder@2D •. displayRule •• TableForm
eqOrder@3D •. displayRule •• TableForm
```

$D_0^2 u_{1,1} + w_1^2\, u_{1,1} == 0$

$D_0^2 u_{2,1} + w_2^2\, u_{2,1} == 0$

$D_0^2 u_{1,2} + w_1^2\, u_{1,2} == -2\, HD_0\, D_1 u_{1,1}L + 2\, d\, u_{1,1}\, u_{2,1}$

$D_0^2 u_{2,2} + w_2^2\, u_{2,2} == -2\, HD_0\, D_1 u_{2,1}L + d\, u_{1,1}^2$

$D_0^2 u_{1,3} + w_1^2\, u_{1,3} == -2\, HD_0\, D_1 u_{1,2}L - D_1^2 u_{1,1} - 2\, HD_0\, D_2 u_{1,1}L + 2\, d\, u_{1,2}\, u_{2,1} + 2\, d\, u_{1,1}\, u_{2,2}$

$D_0^2 u_{2,3} + w_2^2\, u_{2,3} == -2\, HD_0\, D_1 u_{2,2}L - D_1^2 u_{2,1} - 2\, HD_0\, D_2 u_{2,1}L + 2\, d\, u_{1,1}\, u_{1,2}$

Ÿ First-Order Equations: Linear System

To obtain the solution of **eqOrder[1]** by using **DSolve**, we transform **eqOrder[1]** into a set of ordinary-differential equations as

```
order1Eq = eqOrder@1D •. u_{i_,j_} -> Hu_{i,j}@#1D &L
```

$8 w_1^2\, u_{1,1}@T_0D + u_{1,1}''@T_0D == 0,\; w_2^2\, u_{2,1}@T_0D + u_{2,1}''@T_0D == 0<$

The particular solution of **order1Eq** can be expressed as

```
sol1p = DSolve@order1Eq, 8u_{1,1}@T_0D, u_{2,1}@T_0D<, T_0D@@1DD •. C@_D -> 0
```

$8u_{1,1}@T_0D \to 0,\; u_{2,1}@T_0D \to 0<$

Hence, we write the first-order solution in function form as

```
sol1 = TableAu_{i,1} -> FunctionA8T_0, T_1, T_2<,
    A_i@T_1, T_2D Exp@I w_i T_0D + Ā_i@T_1, T_2D Exp@- I w_i T_0D + u_{i,1}@T_0D •. sol1p •• EvaluateE, 8i, 2<E
```

```
8u_{1,1} ® Function@8T_0, T_1, T_2<, E^{I T_0 w_1} A_1@T_1, T_2D + E^{-I T_0 w_1} Ā_1@T_1, T_2DD,
 u_{2,1} ® Function@8T_0, T_1, T_2<, E^{I T_0 w_2} A_2@T_1, T_2D + E^{-I T_0 w_2} Ā_2@T_1, T_2DD<
```

Ÿ Second-Order Equations

Substituting **sol1** into **eqOrder[2]** yields

```
order2Eq = eqOrder@2D •. sol1 •• ExpandAll;
order2Eq •. displayRule
```

```
:D_0^2 u_{1,2} + w_1^2 u_{1,2} == 2 E^{I T_0 w_1+I T_0 w_2} d A_1 A_2 - 2 I E^{I T_0 w_1} HD_1A_1L w_1 +
    2 I E^{-I T_0 w_1} HD_1Ā_1L w_1 + 2 E^{-I T_0 w_1+I T_0 w_2} d A_2 Ā_1 + 2 E^{I T_0 w_1-I T_0 w_2} d A_1 Ā_2 + 2 E^{-I T_0 w_1-I T_0 w_2} d Ā_1 Ā_2,
 D_0^2 u_{2,2} + w_2^2 u_{2,2} == E^{2 I T_0 w_1} d A_1^2 - 2 I E^{I T_0 w_2} HD_1A_2L w_2 + 2 I E^{-I T_0 w_2} HD_1Ā_2L w_2 + 2 d A_1 Ā_1 + E^{-2 I T_0 w_1} d Ā_1^2>
```

To describe quantitatively the nearness of the two-to-one internal resonance $w_2 » 2 w_1$, we introduce the detuning parameter s defined by

```
ResonanceCond = 8w_2 == 2 w_1 + e s<;
```

In eliminating secular terms, we need to express w_2 in terms of w_1 in some of the equations and w_1 in terms of w_2 in the other equations. To accomplish this, we let

```
omgList = Table@w_i, 8i, 2<D;
```

```
omgRule = Solve@ResonanceCond, Drop@omgList, 8#<DD@@1DD & •ž 81, 2<
```

```
98w_2 ® e s + 2 w_1<, 9w_1 ® 1/2 H- e s + w_2L==
```

To convert small-divisor terms into secular terms, we define the rule

```
expRule1@j_D := Exp@arg_D :> Exp@Expand@arg •. omgRule@@jDDD •. e T_0 -> T_1D
```

Substituting **expRule1[i]** into the right-hand sides of **order2Eq** and collecting the coefficients of $E^{I w_i T_0}$, we obtain the solvability conditions as

```
SCond1 = Coefficient@order2Eq@@#, 2DD •. expRule1@#D, Exp@I w_# T_0DD == 0 & •ž 81, 2<
```

```
92 E^{I T_1 s} d A_2@T_1, T_2D Ā_1@T_1, T_2D - 2 I w_1 A_1^{H1,0L}@T_1, T_2D == 0,
 E^{-I T_1 s} d A_1@T_1, T_2D^2 - 2 I w_2 A_2^{H1,0L}@T_1, T_2D == 0=
```

Because the system is conservative, the modulation equations given by **SCond1** must be derivable from a Lagrangian. These equations are the Euler-Lagrange equations corresponding to the Lagrangian

```
Lag = E^(I T1 σ) d A2[T1, T2] A1'[T1, T2]^2 + E^(-I T1 σ) d A1[T1, T2]^2 A2'[T1, T2] -
  I w1 A1'[T1, T2] A1^(1,0)[T1, T2] - I w2 A2'[T1, T2] A2^(1,0)[T1, T2] +
   I w1 A1[T1, T2] A1'^(1,0)[T1, T2] + I w2 A2[T1, T2] A2'^(1,0)[T1, T2];
```

because

```
alSCond1 = Table[-D[D[Lag, Ak'^(1,0)[T1, T2]], T1] + D[Lag, Ak'[T1, T2]] == 0, {k, 2}]
```

```
{2 E^(I T1 σ) d A2[T1, T2] A1'[T1, T2] - 2 I w1 A1^(1,0)[T1, T2] == 0,
 E^(-I T1 σ) d A1[T1, T2]^2 - 2 I w2 A2^(1,0)[T1, T2] == 0}
```

Equations **alSCond1** are identical to equations **SCond1** as shown below

```
alSCond1 === SCond1
```

```
True
```

Alternatively, we can determine the modulation equations by using the time-averaged Lagrangian. To this end, we first define the rule

```
expRule2 = Exp[a_] :> Exp[Expand[a /. (ResonanceCond /. Equal -> Rule) /. ε T0 -> T1]];
```

Substituting **multiScales**, **solRule**, **sol1**, and **expRule2** into the **Lagrangian** and selecting the slow-varying terms, we obtain the following time-averaged Lagrangian:

```
TAL1 =
  Select[(Lagrangian /. multiScales /. solRule // Expand) /. ε^(n_ /; n>3) -> 0 /. sol1 // Expand] /.
    expRule2, FreeQ[#, T0] &] /. ε -> 1;
TAL1 /. displayRule
```

```
I (D1A1') A1 w1 + I (D1A2') A2 w2 - I (D1A1) w1 A1' + E^(I T1 σ) d A2 A1'^2 + E^(-I T1 σ) d A1^2 A2' - I (D1A2) w2 A2'
```

One can easily verify that **TAL1** is identical to **Lag**. In fact,

```
TAL1 == Lag
```

```
True
```

Next, we rewrite **SCond1** as

```
SCond1Rule1 = Solve[SCond1, {A1^(1,0)[T1, T2], A2^(1,0)[T1, T2]}][[1]]
```

```
{A1^(1,0)[T1, T2] -> -(I E^(I T1 σ) d A2[T1, T2] A1'[T1, T2])/w1, A2^(1,0)[T1, T2] -> -(I E^(-I T1 σ) d A1[T1, T2]^2)/(2 w2)}
```

whose complex conjugate is

```
ccSCond1Rule1 = SCond1Rule1 /. conjugateRule;
```

To solve for the particular solution of **order2Eq**, we express the detuning parameter σ in terms of the ω_i; that is,

```
sigRule = Solve@ResonanceCond, sD@@1DD
```

$$\left\{\sigma \to -\frac{2\,\omega_1 - \omega_2}{\epsilon}\right\}$$

Moreover, we express the scale T_1 in terms of the scale T_0 by using the rule

```
expRule3 = Exp@a_D :> Exp@a •. T1 -> e T0 •. sigRule •• ExpandD;
```

With the solvability conditions and the **expRule3**, **order2Eq** becomes

```
order2Eqm = order2Eq •. SCond1Rule1 •. ccSCond1Rule1 •. expRule3;
order2Eqm •. displayRule
```

$$\{D_0^2 u_{1,2} + \omega_1^2 u_{1,2} == 2\,E^{I T_0 \omega_1 + I T_0 \omega_2}\,\delta\,A_1 A_2 + 2\,E^{-I T_0 \omega_1 - I T_0 \omega_2}\,\delta\,\bar{A}_1 \bar{A}_2,\ D_0^2 u_{2,2} + \omega_2^2 u_{2,2} == 2\,\delta\,A_1 \bar{A}_1\}$$

Again, to use **DSolve**, we first transform **order2Eqm** into a set of ordinary-differential equations and then obtain the particular solution of **order2Eqm** as

```
sol2p = DSolve@order2Eqm •. ui_,j_ -> Hui,j@#1D &L, 8u1,2@T0D, u2,2@T0D<, T0D@@1DD •. C@_D -> 0 ••
  Simplify
```

$$\Big\{u_{1,2}[T_0] \to -\frac{1}{\omega_2\,(2\,\omega_1 + \omega_2)}\,\big(2\,E^{-I T_0 (\omega_1+\omega_2)}\,\delta\,(E^{2 I T_0 (\omega_1+\omega_2)} A_1[T_1, T_2] A_2[T_1, T_2] + \bar{A}_1[T_1, T_2] \bar{A}_2[T_1, T_2])\big),$$
$$u_{2,2}[T_0] \to \frac{2\,\delta\,A_1[T_1, T_2]\,\bar{A}_1[T_1, T_2]}{\omega_2^2}\Big\}$$

The second-order solution can be expressed in pure function form as

```
sol2 = TableAui,2 -> FunctionA8T0, T1, T2<, Bi@T1, T2D Exp@I wi T0D +
        B̄i@T1, T2D Exp@-I wi T0D + sol2p@@i, 2DD •• Expand •• EvaluateE, 8i, 2<E;
sol2 •. displayRule
```

$$\Big\{u_{1,2} \to \text{Function}\Big[\{T_0, T_1, T_2\},\ -\frac{2\,E^{I T_0 (\omega_1+\omega_2)}\,\delta\,A_1 A_2}{\omega_2\,(2\,\omega_1 + \omega_2)} + E^{I T_0 \omega_1} B_1 - \frac{2\,E^{-I T_0 (\omega_1+\omega_2)}\,\delta\,\bar{A}_1 \bar{A}_2}{\omega_2\,(2\,\omega_1 + \omega_2)} + E^{-I T_0 \omega_1}\,\bar{B}_1\Big],$$
$$u_{2,2} \to \text{Function}\Big[\{T_0, T_1, T_2\},\ E^{I T_0 \omega_2} B_2 + \frac{2\,\delta\,A_1 \bar{A}_1}{\omega_2^2} + E^{-I T_0 \omega_2}\,\bar{B}_2\Big]\Big\}$$

As shown below, the obtained modulation equations will violate the conservative nature of system unless we include an appropriate part of the homogeneous solutions of the second-order problem. This is the reason why we included the homogeneous solutions with the undetermined functions $B_i[t]$. They will be determined so as to preserve the conservative nature of the system.

Ÿ Third-Order Equations

Substituting **sol1** and **sol2** into **eqOrder[3]** yields

```
order3Eq = eqOrder@3D •. sol1 •. sol2 •• ExpandAll;
```

Substituting **expRule1[i]** into the right-hand sides of **order3Eq** and collecting the coefficients of $E^{I w_i T_0}$, we obtain the solvability conditions as

```
SCond2 = Coefficient@order3Eq@@#, 2DD •. expRule1@#D, Exp@I w# T0DD == 0 & •ž 81, 2<;
SCond2 •. displayRule
```

$$:-HD_1^2A_1L - 2\,I\,HD_1B_1L\,w_1 - 2\,I\,HD_2A_1L\,w_1 + 2\,E^{I\,T_1\,s}\,d\,B_2\,\dot{A}_1 + \frac{4\,d^2\,A_1^2\,\dot{A}_1}{w_2^2} - \frac{4\,d^2\,A_1\,A_2\,\dot{A}_2}{2\,w_1\,w_2 + w_2^2} + 2\,E^{I\,T_1\,s}\,d\,A_2\,\dot{B}_1 == 0,$$

$$-HD_1^2A_2L + 2\,E^{-I\,T_1\,s}\,d\,A_1\,B_1 - 2\,I\,HD_1B_2L\,w_2 - 2\,I\,HD_2A_2L\,w_2 - \frac{4\,d^2\,A_1\,A_2\,\dot{A}_1}{2\,w_1\,w_2 + w_2^2} == 0>$$

Using the solvability conditions at second order, **SCond1Rule1**, we obtain

```
SCond1Rule2 = D@SCond1Rule1, T1D •. SCond1Rule1 •. ccSCond1Rule1;
SCond1Rule2 •. displayRule
```

$$:D_1^2A_1 \text{®} \frac{E^{I\,T_1\,s}\,d\,s\,A_2\,\dot{A}_1}{w_1} - \frac{d^2\,A_1^2\,\dot{A}_1}{2\,w_1\,w_2} + \frac{d^2\,A_1\,A_2\,\dot{A}_2}{w_1^2},\; D_1^2A_2 \text{®} -\frac{E^{-I\,T_1\,s}\,d\,s\,A_1^2}{2\,w_2} - \frac{d^2\,A_1\,A_2\,\dot{A}_1}{w_1\,w_2}>$$

Hence, we rewrite the solvability conditions at third order as

```
SCond2Rule1 = SolveASCond2, 9A1^H0,1L@T1, T2D, A2^H0,1L@T1, T2D=E@@1DD •. SCond1Rule2 •• ExpandAll;
SCond2Rule1 •. displayRule
```

$$:D_2A_1 \text{®} -HD_1B_1L + \frac{I\,E^{I\,T_1\,s}\,d\,s\,A_2\,\dot{A}_1}{2\,w_1^2} - \frac{I\,E^{I\,T_1\,s}\,d\,B_2\,\dot{A}_1}{w_1} -$$

$$\frac{2\,I\,d^2\,A_1^2\,\dot{A}_1}{w_1\,w_2^2} - \frac{I\,d^2\,A_1^2\,\dot{A}_1}{4\,w_1^2\,w_2} + \frac{I\,d^2\,A_1\,A_2\,\dot{A}_2}{2\,w_1^3} + \frac{2\,I\,d^2\,A_1\,A_2\,\dot{A}_2}{w_1\,H2\,w_1\,w_2 + w_2^2L} - \frac{I\,E^{I\,T_1\,s}\,d\,A_2\,\dot{B}_1}{w_1},$$

$$D_2A_2 \text{®} -HD_1B_2L - \frac{I\,E^{-I\,T_1\,s}\,d\,s\,A_1^2}{4\,w_2^2} - \frac{I\,E^{-I\,T_1\,s}\,d\,A_1\,B_1}{w_2} - \frac{I\,d^2\,A_1\,A_2\,\dot{A}_1}{2\,w_1\,w_2^2} + \frac{2\,I\,d^2\,A_1\,A_2\,\dot{A}_1}{w_2\,H2\,w_1\,w_2 + w_2^2L}>$$

Ÿ Reconstitution

Using **SCond1Rule1** and **SCond2Rule1**, we reconstitute the modulation equations as

```
moduEq = Table@2 I wk Ak¢, 8k, 2<D ==
    HTable@2 I wk dt@1D@ Ak@T1, T2DD, 8k, 2<D •. SCond1Rule1 •. SCond2Rule1 •• Expand ••
      Collect@#, eD &L •• Thread;
moduEq •. displayRule
```

$$\Big\{2\,I\,w_1\,A_1' == 2\,E^{I T_1 s}\,d\,e\,A_2\,\dot{A}_1 + e^2\Big(-2\,I\,(D_1B_1)\,w_1 + 2\,E^{I T_1 s}\,d\,B_2\,\dot{A}_1 - \frac{E^{I T_1 s}\,d\,s\,A_2\,\dot{A}_1}{w_1} + \frac{4\,d^2\,A_1^2\,\dot{A}_1}{w_2^2} + \frac{d^2\,A_1^2\,\dot{A}_1}{2\,w_1\,w_2} - \frac{d^2\,A_1\,A_2\,\dot{A}_2}{w_1^2} - \frac{4\,d^2\,A_1\,A_2\,\dot{A}_2}{2\,w_1\,w_2 + w_2^2} + 2\,E^{I T_1 s}\,d\,A_2\,\dot{B}_1\Big),\ 2\,I\,w_2\,A_2' == E^{-I T_1 s}\,d\,e\,A_1^2 + e^2\Big(2\,E^{-I T_1 s}\,d\,A_1\,B_1 + \frac{E^{-I T_1 s}\,d\,s\,A_1^2}{2\,w_2} - 2\,I\,(D_1B_2)\,w_2 + \frac{d^2\,A_1\,A_2\,\dot{A}_1}{w_1\,w_2} - \frac{4\,d^2\,A_1\,A_2\,\dot{A}_1}{2\,w_1\,w_2 + w_2^2}\Big)\Big\}$$

Without including the B_i in the second-order solution, the following two coefficients:

```
CoefficientAmoduEq@@1, 2DD, A1@T1, T2D A2@T1, T2D Ȧ2@T1, T2DE •. w2 -> 2 w1 •• Simplify
```

$$-\frac{3\,d^2\,e^2}{2\,w_1^2}$$

```
CoefficientAmoduEq@@2, 2DD, A1@T1, T2D A2@T1, T2D Ȧ1@T1, T2DE •. w2 -> 2 w1 •• Simplify
```

```
0
```

are not the same, which violates the conservative nature of the system in that the modulation equations must be derivable from a Lagrangian. Using the flexibility given by the homogeneous parts of the solutions of the second-order problem, we choose the functions B_i@tD so that these coefficients are the same. To this end, we assume that

```
BForm = 9B1 -> FunctionA8T1, T2<, c1 E^(I T1 s) Ȧ1@T1, T2D A2@T1, T2D E,
   Ḃ1 -> FunctionA8T1, T2<, c1 E^(-I T1 s) A1@T1, T2D Ȧ2@T1, T2D E,
   B2 -> FunctionA8T1, T2<, c2 E^(-I T1 s) A1@T1, T2D^2 E=;
```

where c_1 and c_2 are undetermined constants. Substituting **BForm** and the solvability conditions into the right-hand sides of **moduEq** yields

```
moduEqMod = HExpand@# •. BForm •. SCond1Rule1 •. ccSCond1Rule1D & •ž moduEqL •• ExpandAll;
moduEqMod •. displayRule
```

$$\Big\{2\,I\,w_1\,A_1' == 2\,E^{I T_1 s}\,d\,e\,A_2\,\dot{A}_1 + 2\,d\,e^2\,A_1^2\,c_2\,\dot{A}_1 - \frac{E^{I T_1 s}\,d\,e^2\,s\,A_2\,\dot{A}_1}{w_1} + 2\,E^{I T_1 s}\,e^2\,s\,A_2\,c_1\,w_1\,\dot{A}_1 + \frac{4\,d^2\,e^2\,A_1^2\,\dot{A}_1}{w_2^2} + \frac{d^2\,e^2\,A_1^2\,\dot{A}_1}{2\,w_1\,w_2} - \frac{d\,e^2\,A_1^2\,c_1\,w_1\,\dot{A}_1}{w_2} + 4\,d\,e^2\,A_1\,A_2\,c_1\,\dot{A}_2 - \frac{d^2\,e^2\,A_1\,A_2\,\dot{A}_2}{w_1^2} - \frac{4\,d^2\,e^2\,A_1\,A_2\,\dot{A}_2}{2\,w_1\,w_2 + w_2^2},$$

$$2\,I\,w_2\,A_2' == E^{-I T_1 s}\,d\,e\,A_1^2 + \frac{E^{-I T_1 s}\,d\,e^2\,s\,A_1^2}{2\,w_2} - 2\,E^{-I T_1 s}\,e^2\,s\,A_1^2\,c_2\,w_2 + 2\,d\,e^2\,A_1\,A_2\,c_1\,\dot{A}_1 + \frac{d^2\,e^2\,A_1\,A_2\,\dot{A}_1}{w_1\,w_2} - \frac{4\,d\,e^2\,A_1\,A_2\,c_2\,w_2\,\dot{A}_1}{w_1} - \frac{4\,d^2\,e^2\,A_1\,A_2\,\dot{A}_1}{2\,w_1\,w_2 + w_2^2}\Big\}$$

In order that the **moduEqMod** be derivable from a Lagrangian, the following condition must be satisfied:

```
CoefficientAmoduEqMod@@1, 2DD, A1@T1, T2D A2@T1, T2D Ȧ2@T1, T2DE -
    CoefficientAmoduEqMod@@2, 2DD, A1@T1, T2D A2@T1, T2D Ȧ1@T1, T2DE == 0 •. e -> 1 •. w2 -> 2 w1
```

$$2\, d\, c_1 + 8\, d\, c_2 - \frac{3\, d^2}{2\, w_1^2} == 0$$

This provides a compatibility condition. However, one extra constant is floating !!!

To obtain a consistent second-order uniform expansion, we apply the method of multiple scales to the system of four first-order real-valued equations **EOM2** in Section 5.2.2 and to a system of two first-order complex-valued equations in Section 5.2.3.

Ÿ 5.2.2 First-Order Real-Valued System

As shown in the preceding section, treating the second-order form of the equations governing a system may lead to inconsistent results unless appropriate parts of the homogeneous solutions of the perturbation equations are included. Moreover, the consistent expansions might contain arbitrary constants that need to be chosen judiciously. In this section, we show that treatment of an equivalent set of first-order real-valued equations, namely **EOM2**, leads to consistent results without floating constants.

We seek a second-order uniform expansion of the solution of **EOM2** in the form

```
solRule =
  9ui_ -> ISumAej ui,j@#1, #2, #3D, 8j, 3<E &M, vi_ -> ISumAej vi,j@#1, #2, #3D, 8j, 3<E &M=;
```

Expressing the time derivative in **EOM2** in terms of the time scales T_0, T_1, and T_2, we have

```
multiScales = 9u_i_@tD -> ui@T0, T1, T2D, Derivative@n_DAu_i_E@tD :> dt@nD@ui@T0, T1, T2DD=;
```

Substituting **multiScales** and **solRule** into **EOM2**, expanding the result for small e, and discarding terms of order higher than e^3, we obtain

```
eq522a = HEOM2 •. multiScales •. solRule •• ExpandAllL •. en_•;n>3 -> 0;
```

Equating coefficients of like powers of e, we obtain

```
eqEps = Rest@Thread@CoefficientList@Subtract žž #, eD == 0DD & •ž eq522a •• Transpose;
```

To place the linear operators on one side and the nonhomogeneous terms on the other side, we define

```
eqOrder@i_D := I#@@1DD & •ž eqEps@@1DD •. u_k_,1 -> uk,iM ==
   I#@@1DD & •ž eqEps@@1DD •. u_k_,1 -> uk,iM - H#@@1DD & •ž eqEps@@iDDL •• Thread
```

Using **eqOrder[i]** and **displayRule**, we rewrite **eqEps** in a concise way as

```
eqOrder@1D •. displayRule •• TableForm
eqOrder@2D •. displayRule •• TableForm
eqOrder@3D •. displayRule •• TableForm
```

$D_0u_{1,1} - v_{1,1} == 0$
$D_0v_{1,1} + w_1^2 u_{1,1} == 0$
$D_0u_{2,1} - v_{2,1} == 0$
$D_0v_{2,1} + w_2^2 u_{2,1} == 0$

$D_0u_{1,2} - v_{1,2} == -HD_1u_{1,1}L$
$D_0v_{1,2} + w_1^2 u_{1,2} == -HD_1v_{1,1}L + 2\, d\, u_{1,1} u_{2,1}$
$D_0u_{2,2} - v_{2,2} == -HD_1u_{2,1}L$
$D_0v_{2,2} + w_2^2 u_{2,2} == -HD_1v_{2,1}L + d\, u_{1,1}^2$

$D_0u_{1,3} - v_{1,3} == -HD_1u_{1,2}L - D_2u_{1,1}$
$D_0v_{1,3} + w_1^2 u_{1,3} == -HD_1v_{1,2}L - D_2v_{1,1} + 2\, d\, u_{1,2} u_{2,1} + 2\, d\, u_{1,1} u_{2,2}$
$D_0u_{2,3} - v_{2,3} == -HD_1u_{2,2}L - D_2u_{2,1}$
$D_0v_{2,3} + w_2^2 u_{2,3} == -HD_1v_{2,2}L - D_2v_{2,1} + 2\, d\, u_{1,1} u_{1,2}$

Ÿ First-Order Equations

The homogeneous solution of **eqOrder[1]** can be obtained by using

```
eigenForm = #@@2DD Exp@#@@1DD T0D & •ž Transpose@Eigensystem@matrixADD
```

$::\frac{I\, E^{-I\, T_0\, w_1}}{w_1},\ E^{-I\, T_0\, w_1},\ 0,\ 0>,\ :-\frac{I\, E^{I\, T_0\, w_1}}{w_1},\ E^{I\, T_0\, w_1},\ 0,\ 0>,$
$:0,\ 0,\ \frac{I\, E^{-I\, T_0\, w_2}}{w_2},\ E^{-I\, T_0\, w_2}>,\ :0,\ 0,\ -\frac{I\, E^{I\, T_0\, w_2}}{w_2},\ E^{I\, T_0\, w_2}>>$

```
sol1hForm = FlattenATableAI wi 9-Ȧi@T1, T2D, Ai@T1, T2D=, 8i, 2<EE.eigenForm
```

$8E^{I\, T_0\, w_1} A_1@T_1,\ T_2D + E^{-I\, T_0\, w_1} \dot{A}_1@T_1,\ T_2D,\ I\, E^{I\, T_0\, w_1} w_1 A_1@T_1,\ T_2D - I\, E^{-I\, T_0\, w_1} w_1 \dot{A}_1@T_1,\ T_2D,$
$E^{I\, T_0\, w_2} A_2@T_1,\ T_2D + E^{-I\, T_0\, w_2} \dot{A}_2@T_1,\ T_2D,\ I\, E^{I\, T_0\, w_2} w_2 A_2@T_1,\ T_2D - I\, E^{-I\, T_0\, w_2} w_2 \dot{A}_2@T_1,\ T_2D<$

Since the first-order equations are homogeneous, the solution of **eqOrder[1]** can be expressed as

```
states1 = 8u1,1, v1,1, u2,1, v2,1<;
```

```
sol1 = Table@states1@@iDD -> Function@8T0, T1, T2<, sol1hForm@@iDD •• EvaluateD, 8i, 4<D
```

$8u_{1,1} ® Function@8T_0,\ T_1,\ T_2<,\ E^{I\, T_0\, w_1} A_1@T_1,\ T_2D + E^{-I\, T_0\, w_1} \dot{A}_1@T_1,\ T_2DD,$
$v_{1,1} ® Function@8T_0,\ T_1,\ T_2<,\ I\, E^{I\, T_0\, w_1} w_1 A_1@T_1,\ T_2D - I\, E^{-I\, T_0\, w_1} w_1 \dot{A}_1@T_1,\ T_2DD,$
$u_{2,1} ® Function@8T_0,\ T_1,\ T_2<,\ E^{I\, T_0\, w_2} A_2@T_1,\ T_2D + E^{-I\, T_0\, w_2} \dot{A}_2@T_1,\ T_2DD,$
$v_{2,1} ® Function@8T_0,\ T_1,\ T_2<,\ I\, E^{I\, T_0\, w_2} w_2 A_2@T_1,\ T_2D - I\, E^{-I\, T_0\, w_2} w_2 \dot{A}_2@T_1,\ T_2DD<$

Ÿ Second-Order Equations

Substituting the first-order solution into the second-order equations, **eqOrder[2]**, yields

```
order2Eq = eqOrder@2D •. sol1 •• ExpandAll;
order2Eq •. displayRule
```

$$\{D_0u_{1,2} - v_{1,2} == -E^{I T_0 \omega_1}(D_1A_1) - E^{-I T_0 \omega_1}(D_1\bar{A}_1),$$
$$D_0v_{1,2} + \omega_1^2 u_{1,2} == 2E^{I T_0 \omega_1 + I T_0 \omega_2}\delta A_1 A_2 - I E^{I T_0 \omega_1}(D_1A_1)\omega_1 + I E^{-I T_0 \omega_1}(D_1\bar{A}_1)\omega_1 + 2E^{-I T_0 \omega_1 + I T_0 \omega_2}\delta A_2 \bar{A}_1 + 2E^{I T_0 \omega_1 - I T_0 \omega_2}\delta A_1 \bar{A}_2 + 2E^{-I T_0 \omega_1 - I T_0 \omega_2}\delta \bar{A}_1 \bar{A}_2,\ D_0u_{2,2} - v_{2,2} == -E^{I T_0 \omega_2}(D_1A_2) - E^{-I T_0 \omega_2}(D_1\bar{A}_2),$$
$$D_0v_{2,2} + \omega_2^2 u_{2,2} == E^{2 I T_0 \omega_1}\delta A_1^2 - I E^{I T_0 \omega_2}(D_1A_2)\omega_2 + I E^{-I T_0 \omega_2}(D_1\bar{A}_2)\omega_2 + 2\delta A_1 \bar{A}_1 + E^{-2 I T_0 \omega_1}\delta \bar{A}_1^2\}$$

Next, we substitute the **expRule1[i]** into the right-hand sides of **order2Eq** to transform the small-divisor terms into secular terms, collect the terms responsible for secular terms, and obtain

```
ST11 = Coefficient@#@@2DD •. expRule1@1D, Exp@I w1 T0DD & •ž order2Eq;
ST11 •. displayRule
```

$$\{-(D_1A_1),\ -I(D_1A_1)\omega_1 + 2E^{I T_1 \sigma}\delta A_2 \bar{A}_1,\ 0,\ 0\}$$

```
ST12 = Coefficient@#@@2DD •. expRule1@2D, Exp@I w2 T0DD & •ž order2Eq;
ST12 •. displayRule
```

$$\{0,\ 0,\ -(D_1A_2),\ E^{-I T_1 \sigma}\delta A_1^2 - I(D_1A_2)\omega_2\}$$

The solvability conditions of the second-order equations demands that **ST11** and **ST12** be orthogonal to every solution of the adjoint homogeneous problems, namely the components of **ccleftVec**. Imposing these conditions, we obtain

```
SCond1 = 8ccleftVec@@1DD.ST11 == 0, ccleftVec@@2DD.ST12 == 0<;
SCond1 •. displayRule
```

$$\{-2I(D_1A_1)\omega_1 + 2E^{I T_1 \sigma}\delta A_2 \bar{A}_1 == 0,\ E^{-I T_1 \sigma}\delta A_1^2 - 2I(D_1A_2)\omega_2 == 0\}$$

or

```
SCond1Rule1 = SolveASCond1, 9A1H1,0L@T1, T2D, A2H1,0L@T1, T2D=E@@1DD;
SCond1Rule1 •. displayRule
```

$$\left\{D_1A_1 \to -\frac{I E^{I T_1 \sigma}\delta A_2 \bar{A}_1}{\omega_1},\ D_1A_2 \to -\frac{I E^{-I T_1 \sigma}\delta A_1^2}{2\omega_2}\right\}$$

whose complex conjugate is

```
ccSCond1Rule1 = SCond1Rule1 •. conjugateRule;
```

Using these solvability conditions and **expRule3**, we rewrite **order2Eq** as

```
order2Eqm = order2Eq •. SCond1Rule1 •. ccSCond1Rule1 •. expRule3;
order2Eqm •. displayRule
```

$$\Big\{D_0u_{1,2}-v_{1,2}==\frac{I\,E^{-I T_0 w_1+I T_0 w_2}\,d\,A_2\,\dot{A}_1}{w_1}-\frac{I\,E^{I T_0 w_1-I T_0 w_2}\,d\,A_1\,\dot{A}_2}{w_1},\ D_0v_{1,2}+w_1^2\,u_{1,2}==$$
$$2\,E^{I T_0 w_1+I T_0 w_2}\,d\,A_1\,A_2+E^{-I T_0 w_1+I T_0 w_2}\,d\,A_2\,\dot{A}_1+E^{I T_0 w_1-I T_0 w_2}\,d\,A_1\,\dot{A}_2+2\,E^{-I T_0 w_1-I T_0 w_2}\,d\,\dot{A}_1\,\dot{A}_2,$$
$$D_0u_{2,2}-v_{2,2}==\frac{I\,E^{2 I T_0 w_1}\,d\,A_1^2}{2\,w_2}-\frac{I\,E^{-2 I T_0 w_1}\,d\,\dot{A}_1^2}{2\,w_2},$$
$$D_0v_{2,2}+w_2^2\,u_{2,2}==\frac{1}{2}\,E^{2 I T_0 w_1}\,d\,A_1^2+2\,d\,A_1\,\dot{A}_1+\frac{1}{2}\,E^{-2 I T_0 w_1}\,d\,\dot{A}_1^2\Big\}$$

Transforming **order2Eqm** into a set of ordinary-differential equations and using **DSolve**, we obtain the particular solutions as

```
Isol2p = DSolveAorder2Eqm •. u_i_,2 -> Hui,2@#1D &L,
      8u1,2@T0D, v1,2@T0D, u2,2@T0D, v2,2@T0D<, T0E@@1DD •. C@_D -> 0 •• Simplify;M •• Timing
```

```
813.139 Second, Null<
```

Hence, the second-order solutions of **order2Eqm** can be expressed as

```
states2 = states1 •. u_i_,1 -> ui,2
```

$$\{u_{1,2},\ v_{1,2},\ u_{2,2},\ v_{2,2}\}$$

```
sol2 = Table@states2@@iDD -> Function@8T0, T1, T2<,
     Expand@sol2p@@i, 2DDD •. Exp@a_D :> Exp@a •• ExpandD •• EvaluateD, 8i, 4<D;
sol2 •. displayRule
```

$$\Big\{u_{1,2}\to\text{Function}\Big[\{T_0,T_1,T_2\},\ -\frac{2\,E^{I T_0 w_1+I T_0 w_2}\,d\,A_1\,A_2}{w_2\,(2\,w_1+w_2)}+\frac{E^{-I T_0 w_1+I T_0 w_2}\,d\,A_2\,\dot{A}_1}{w_1\,(2\,w_1+w_2)}+$$
$$\frac{2\,E^{-I T_0 w_1+I T_0 w_2}\,d\,A_2\,\dot{A}_1}{w_2\,(2\,w_1+w_2)}+\frac{E^{I T_0 w_1-I T_0 w_2}\,d\,A_1\,\dot{A}_2}{w_1\,(2\,w_1+w_2)}+\frac{2\,E^{I T_0 w_1-I T_0 w_2}\,d\,A_1\,\dot{A}_2}{w_2\,(2\,w_1+w_2)}-\frac{2\,E^{-I T_0 w_1-I T_0 w_2}\,d\,\dot{A}_1\,\dot{A}_2}{w_2\,(2\,w_1+w_2)}\Big],$$
$$v_{1,2}\to\text{Function}\Big[\{T_0,T_1,T_2\},\ -\frac{2\,I\,E^{I T_0 w_1+I T_0 w_2}\,d\,A_1\,A_2}{2\,w_1+w_2}-\frac{2\,I\,E^{I T_0 w_1+I T_0 w_2}\,d\,w_1\,A_1\,A_2}{w_2\,(2\,w_1+w_2)}-$$
$$\frac{I\,E^{-I T_0 w_1+I T_0 w_2}\,d\,A_2\,\dot{A}_1}{2\,w_1+w_2}-\frac{2\,I\,E^{-I T_0 w_1+I T_0 w_2}\,d\,w_1\,A_2\,\dot{A}_1}{w_2\,(2\,w_1+w_2)}+\frac{I\,E^{I T_0 w_1-I T_0 w_2}\,d\,A_1\,\dot{A}_2}{2\,w_1+w_2}+$$
$$\frac{2\,I\,E^{I T_0 w_1-I T_0 w_2}\,d\,w_1\,A_1\,\dot{A}_2}{w_2\,(2\,w_1+w_2)}+\frac{2\,I\,E^{-I T_0 w_1-I T_0 w_2}\,d\,\dot{A}_1\,\dot{A}_2}{2\,w_1+w_2}+\frac{2\,I\,E^{-I T_0 w_1-I T_0 w_2}\,d\,w_1\,\dot{A}_1\,\dot{A}_2}{w_2\,(2\,w_1+w_2)}\Big],$$
$$u_{2,2}\to\text{Function}\Big[\{T_0,T_1,T_2\},\ \frac{E^{2 I T_0 w_1}\,d\,A_1^2}{2\,w_2\,(2\,w_1+w_2)}+\frac{4\,d\,w_1\,A_1\,\dot{A}_1}{w_2^2\,(2\,w_1+w_2)}+\frac{2\,d\,A_1\,\dot{A}_1}{w_2\,(2\,w_1+w_2)}+\frac{E^{-2 I T_0 w_1}\,d\,\dot{A}_1^2}{2\,w_2\,(2\,w_1+w_2)}\Big],$$
$$v_{2,2}\to\text{Function}\Big[\{T_0,T_1,T_2\},\ -\frac{I\,E^{2 I T_0 w_1}\,d\,A_1^2}{2\,(2\,w_1+w_2)}+\frac{I\,E^{-2 I T_0 w_1}\,d\,\dot{A}_1^2}{2\,(2\,w_1+w_2)}\Big]\Big\}$$

Ÿ Third-Order Equations

Substituting the first- and second-order solutions into the right-hand sides of the third-order equations, **eqOrder[3]**, yields

```
order3Eqrhs = Expand@#@@2DD •. sol1 •. sol2D & •ž eqOrder@3D;
```

To eliminate the terms that lead to secular terms from **order3Eqrhs**, we use **expRule1[i]** to transform the terms that convert small-divisor terms into secular terms, calculate the vectors proportional to the $E^{I\, w_i\, T_0}$, and obtain

```
ST21 = Coefficient@order3Eqrhs •. expRule1@1D, Exp@I w1 T0DD;
ST21 •. displayRule
```

$$\Big\{-(D_2A_1) - \frac{E^{I T_1 s}\, d\, (D_1\dot{A}_1)\, A_2}{w_1(2w_1+w_2)} - \frac{2E^{I T_1 s}\, d\, (D_1\dot{A}_1)\, A_2}{w_2(2w_1+w_2)} - \frac{E^{I T_1 s}\, d\, (D_1A_2)\, \dot{A}_1}{w_1(2w_1+w_2)} - \frac{2E^{I T_1 s}\, d\, (D_1A_2)\, \dot{A}_1}{w_2(2w_1+w_2)},$$
$$-I(D_2A_1)w_1 + \frac{I E^{I T_1 s}\, d\, (D_1\dot{A}_1)\, A_2}{2w_1+w_2} + \frac{2 I E^{I T_1 s}\, d\, (D_1\dot{A}_1)\, A_2\, w_1}{w_2(2w_1+w_2)} + \frac{I E^{I T_1 s}\, d\, (D_1A_2)\, \dot{A}_1}{2w_1+w_2} +$$
$$\frac{8 d^2 A_1^2 w_1 \dot{A}_1}{w_2^2(2w_1+w_2)} + \frac{5 d^2 A_1^2 \dot{A}_1}{w_2(2w_1+w_2)} + \frac{2 I E^{I T_1 s}\, d\, (D_1A_2)\, w_1 \dot{A}_1}{w_2(2w_1+w_2)} + \frac{2 d^2 A_1 A_2 \dot{A}_2}{w_1(2w_1+w_2)},\ 0,\ 0\Big\}$$

```
ST22 = Coefficient@order3Eqrhs •. expRule1@2D, Exp@I w2 T0DD;
ST22 •. displayRule
```

$$\Big\{0,\ 0,\ -(D_2A_2) - \frac{E^{-I T_1 s}\, d\, (D_1A_1)\, A_1}{w_2(2w_1+w_2)},\ -I(D_2A_2)w_2 + \frac{I E^{-I T_1 s}\, d\, (D_1A_1)\, A_1}{2w_1+w_2} + \frac{2 d^2 A_1 A_2 \dot{A}_1}{w_1(2w_1+w_2)}\Big\}$$

Hence, demanding that **ST21** and **ST22** be orthogonal to the components of **ccleftVec** yields the solvability conditions

```
SCond2 = 8Expand@ccleftVec@@1DD.ST21D == 0, Expand@ccleftVec@@2DD.ST22D == 0<;
SCond2 •. displayRule
```

$$\Big\{-2I(D_2A_1)w_1 + \frac{8 d^2 A_1^2 w_1 \dot{A}_1}{w_2^2(2w_1+w_2)} + \frac{5 d^2 A_1^2 \dot{A}_1}{w_2(2w_1+w_2)} + \frac{2 d^2 A_1 A_2 \dot{A}_2}{w_1(2w_1+w_2)} == 0,$$
$$-2I(D_2A_2)w_2 + \frac{2 d^2 A_1 A_2 \dot{A}_1}{w_1(2w_1+w_2)} == 0\Big\}$$

or

```
SCond2Rule1 = SolveASCond2, 9A1^H0,1L@T1, T2D, A2^H0,1L@T1, T2D=E@@1DD;
SCond2Rule1 •. displayRule
```

$$\Big\{D_2A_1 \to -\frac{I(8 d^2 A_1^2 w_1^2 \dot{A}_1 + 5 d^2 A_1^2 w_1 w_2 \dot{A}_1 + 2 d^2 A_1 A_2 w_2^2 \dot{A}_2)}{2 w_1^2 w_2^2 (2w_1+w_2)},\ D_2A_2 \to -\frac{I d^2 A_1 A_2 \dot{A}_1}{w_1 w_2 (2w_1+w_2)}\Big\}$$

Ÿ Reconstitution

We reconstitute **SCond1Rule1** and **SCond2Rule1** and obtain

```
moduEq = Table@2 I wk Ak¢, 8k, 2<D ==
    HTable@2 I wk dt@1D@ Ak@T1, T2DD, 8k, 2<D •. SCond1Rule1 •. SCond2Rule1 ••
      ExpandL •• Thread;
moduEq •. displayRule
```

$$\left\{2\,I\,\omega_1 A_1' == 2\,E^{I T_1 \sigma}\,\delta\,\epsilon\,A_2\,\bar{A}_1 + \frac{8\,\delta^2\,\epsilon^2\,A_1^2\,\omega_1\,\bar{A}_1}{\omega_2^2\,(2\,\omega_1+\omega_2)} + \frac{5\,\delta^2\,\epsilon^2\,A_1^2\,\bar{A}_1}{\omega_2\,(2\,\omega_1+\omega_2)} + \frac{2\,\delta^2\,\epsilon^2\,A_1\,A_2\,\bar{A}_2}{\omega_1\,(2\,\omega_1+\omega_2)},\right.$$
$$\left. 2\,I\,\omega_2 A_2' == E^{-I T_1 \sigma}\,\delta\,\epsilon\,A_1^2 + \frac{2\,\delta^2\,\epsilon^2\,A_1\,A_2\,\bar{A}_1}{\omega_1\,(2\,\omega_1+\omega_2)}\right\}$$

These equations are derivable from a Lagrangian because

```
CoefficientAmoduEq@@1, 2DD, A1@T1, T2D A2@T1, T2D Ā2@T1, T2DE -
    CoefficientAmoduEq@@2, 2DD, A1@T1, T2D A2@T1, T2D Ā1@T1, T2DE == 0 •. e -> 1 •. w2 -> 2 w1
```

```
True
```

Therefore, treatment of the first-order version of the governing equations yields consistent results without floating constants.

5.2.3 First-Order Complex-Valued System

In this section, we show that treatment of a first-order complex-valued version of the governing equations yields consistent results without floating constants. We start by transforming the second-order form **EOM1** of the governing equations into a system of first-order complex-valued equations by introducing the transformation

```
transfRule = 9uk_@tD -> zk@tD + z̄k@tD, uk_¢@tD -> I wk Izk@tD - z̄k@tDM=;
```

Then, we define the dependent variables

```
depVar = 8u1@tD, u1¢@tD, u2@tD, u2¢@tD<;
```

```
nmodes = 9z1@tD, z̄1@tD, z2@tD, z̄2@tD=;
```

Substituting **transfRule** into **depVar** and solving for the **nmodes**, we obtain

```
zetaRule = Solve@depVar == HdepVar •. transfRuleL •• Thread, nmodesD@@1DD
```

$$\left\{z_1[t] \to -\frac{-\omega_1\,u_1[t] + I\,u_1'[t]}{2\,\omega_1},\ \bar{z}_1[t] \to -\frac{-\omega_1\,u_1[t] - I\,u_1'[t]}{2\,\omega_1},\right.$$
$$\left. z_2[t] \to -\frac{-\omega_2\,u_2[t] + I\,u_2'[t]}{2\,\omega_2},\ \bar{z}_2[t] \to -\frac{-\omega_2\,u_2[t] - I\,u_2'[t]}{2\,\omega_2}\right\}$$

It follows from **EOM1** that the acceleration is given by

```
acceleration = Solve@EOM1, 8u1''@tD, u2''@tD<D@@1DD
```

$$\{u_1''[t] \to -\omega_1^2\,u_1[t] + 2\,\delta\,u_1[t]\,u_2[t],\ u_2''[t] \to \delta\,u_1[t]^2 - \omega_2^2\,u_2[t]\}$$

Then, differentiating the dependent variables in **zetaRule** and using **transfRule** and **acceleration**, we transform the system of two second-order equations **EOM1** into the following two first-order complex-valued equations:

```
EOM3 = z#¢@tD == HD@z#@tD •. zetaRule, tD •. acceleration •. transfRule •• ExpandL & •ž 81, 2<
```

$$\left\{z_1'[t] == I\,w_1\,z_1[t] - \frac{I\,d\,z_1[t]\,z_2[t]}{w_1} - \frac{I\,d\,z_2[t]\,\dot{z}_1[t]}{w_1} - \frac{I\,d\,z_1[t]\,\dot{z}_2[t]}{w_1} - \frac{I\,d\,\dot{z}_1[t]\,\dot{z}_2[t]}{w_1},\right.$$
$$\left. z_2'[t] == -\frac{I\,d\,z_1[t]^2}{2\,w_2} + I\,w_2\,z_2[t] - \frac{I\,d\,z_1[t]\,\dot{z}_1[t]}{w_2} - \frac{I\,d\,\dot{z}_1[t]^2}{2\,w_2}\right\}$$

To determine a second-order uniform expansion of the solution of **EOM3** using the method of multiple scales, we expand z and $\ddot{z}$ in the form

```
solRule =
  9z_i_ -> ISumAe^j z_i,j@#1, #2, #3D, 8j, 3<E &M, ż_i_ -> ISumAe^j ż_i,j@#1, #2, #3D, 8j, 3<E &M=;

eq523a = HEOM3 •. multiScales •. solRule •• ExpandAllL •. e^n_•;n>3 -> 0;
```

Equating coefficients of like powers of ϵ, we obtain

```
eqEps = Rest@Thread@CoefficientList@Subtract žž #, eD == 0DD & •ž eq523a •• Transpose;
```

To place the linear operator on one side and the nonhomogeneous terms on the other side, we define

```
eqOrder@i_D := I#@@1DD & •ž eqEps@@1DD •. u_k_,1 -> u_k,iM ==
   I#@@1DD & •ž eqEps@@1DD •. u_k_,1 -> u_k,iM - H#@@1DD & •ž eqEps@@iDDL •• Thread
```

Using **eqOrder[i]** and the **displayRule**, we rewrite **eqEps** in a concise way as

```
eqOrder@1D •. displayRule •• TableForm
eqOrder@2D •. displayRule •• TableForm
eqOrder@3D •. displayRule •• TableForm
```

$$D_0 z_{1,1} - I\,w_1\,z_{1,1} == 0$$
$$D_0 z_{2,1} - I\,w_2\,z_{2,1} == 0$$

$$D_0 z_{1,2} - I\,w_1\,z_{1,2} == -(D_1 z_{1,1}) - \frac{I\,d\,z_{1,1}\,z_{2,1}}{w_1} - \frac{I\,d\,z_{2,1}\,\dot{z}_{1,1}}{w_1} - \frac{I\,d\,z_{1,1}\,\dot{z}_{2,1}}{w_1} - \frac{I\,d\,\dot{z}_{1,1}\,\dot{z}_{2,1}}{w_1}$$
$$D_0 z_{2,2} - I\,w_2\,z_{2,2} == -(D_1 z_{2,1}) - \frac{I\,d\,z_{1,1}^2}{2\,w_2} - \frac{I\,d\,z_{1,1}\,\dot{z}_{1,1}}{w_2} - \frac{I\,d\,\dot{z}_{1,1}^2}{2\,w_2}$$

$$D_0 z_{1,3} - I\,w_1\,z_{1,3} == -(D_1 z_{1,2}) - D_2 z_{1,1} - \frac{I\,d\,z_{1,2}\,z_{2,1}}{w_1} - \frac{I\,d\,z_{1,1}\,z_{2,2}}{w_1} - \frac{I\,d\,z_{2,2}\,\dot{z}_{1,1}}{w_1} - \frac{I\,d\,z_{2,1}\,\dot{z}_{1,2}}{w_1} - \frac{I\,d\,z_{1,2}\,\dot{z}_{2,1}}{w_1} - \frac{I\,d\,z_{1,2}\,\dot{z}}{w_1}$$
$$D_0 z_{2,3} - I\,w_2\,z_{2,3} == -(D_1 z_{2,2}) - D_2 z_{2,1} - \frac{I\,d\,z_{1,1}\,z_{1,2}}{w_2} - \frac{I\,d\,z_{1,2}\,\dot{z}_{1,1}}{w_2} - \frac{I\,d\,z_{1,1}\,\dot{z}_{1,2}}{w_2} - \frac{I\,d\,\dot{z}_{1,1}\,\dot{z}_{1,2}}{w_2}$$

Ÿ First-Order Equations

Since **eqOrder[1]** are homogeneous, we can write their solutions as

```
sol1 = Table[{z_{i,1} -> Function[{T_0, T_1, T_2}, A_i[T_1, T_2] Exp[I ω_i T_0] // Evaluate],
    z̄_{i,1} -> Function[{T_0, T_1, T_2}, Ā_i[T_1, T_2] Exp[-I ω_i T_0] // Evaluate]}, {i, 2}] // Flatten
```

$\{z_{1,1} \to \text{Function}[\{T_0, T_1, T_2\}, e^{i T_0 \omega_1} A_1[T_1, T_2]],$
$\bar{z}_{1,1} \to \text{Function}[\{T_0, T_1, T_2\}, e^{-i T_0 \omega_1} \bar{A}_1[T_1, T_2]],$
$z_{2,1} \to \text{Function}[\{T_0, T_1, T_2\}, e^{i T_0 \omega_2} A_2[T_1, T_2]],$
$\bar{z}_{2,1} \to \text{Function}[\{T_0, T_1, T_2\}, e^{-i T_0 \omega_2} \bar{A}_2[T_1, T_2]]\}$

Second-Order Equations

Substituting the first-order solution into the second-order equations, **eqOrder[2]**, we have

```
order2Eq = eqOrder[2] /. sol1;
order2Eq /. displayRule
```

$$\Big\{D_0 z_{1,2} - i\,\omega_1 z_{1,2} == -e^{i T_0 \omega_1} (D_1 A_1) - \frac{i\,e^{i T_0 \omega_1 + i T_0 \omega_2}\,\delta A_1 A_2}{\omega_1} - \frac{i\,e^{-i T_0 \omega_1 + i T_0 \omega_2}\,\delta A_2 \bar{A}_1}{\omega_1} - \frac{i\,e^{i T_0 \omega_1 - i T_0 \omega_2}\,\delta A_1 \bar{A}_2}{\omega_1} - \frac{i\,e^{-i T_0 \omega_1 - i T_0 \omega_2}\,\delta \bar{A}_1 \bar{A}_2}{\omega_1},$$
$$D_0 z_{2,2} - i\,\omega_2 z_{2,2} == -e^{i T_0 \omega_2} (D_1 A_2) - \frac{i\,e^{2 i T_0 \omega_1}\,\delta A_1^2}{2\,\omega_2} - \frac{i\,\delta A_1 \bar{A}_1}{\omega_2} - \frac{i\,e^{-2 i T_0 \omega_1}\,\delta \bar{A}_1^2}{2\,\omega_2}\Big\}$$

Converting the terms that produce small-divisor terms by using the **expRule1** and then eliminating the terms that lead to secular terms from **order2Eq** yields

```
SCond1 = Coefficient[order2Eq[[#, 2]] /. expRule1[#], Exp[I ω_# T_0]] == 0 & /@ {1, 2};
SCond1 /. displayRule
```

$$\Big\{-(D_1 A_1) - \frac{i\,e^{i T_1 \sigma}\,\delta A_2 \bar{A}_1}{\omega_1} == 0,\ -(D_1 A_2) - \frac{i\,e^{-i T_1 \sigma}\,\delta A_1^2}{2\,\omega_2} == 0\Big\}$$

or

```
SCond1Rule1 = Solve[SCond1, {A_1^(1,0)[T_1, T_2], A_2^(1,0)[T_1, T_2]}][[1]];
SCond1Rule1 /. displayRule
```

$$\Big\{D_1 A_1 \to -\frac{i\,e^{i T_1 \sigma}\,\delta A_2 \bar{A}_1}{\omega_1},\ D_1 A_2 \to -\frac{i\,e^{-i T_1 \sigma}\,\delta A_1^2}{2\,\omega_2}\Big\}$$

Substituting **SCond1Rule1** and **expRule3** into **order2Eq** yields

```
order2Eqm = order2Eq •. SCond1Rule1 •. expRule3;
order2Eqm •. displayRule
```

$$\Big\{D_0 z_{1,2} - I\, w_1 z_{1,2} == -\frac{I\, E^{I T_0 w_1 + I T_0 w_2}\, d\, A_1 A_2}{w_1} - \frac{I\, E^{I T_0 w_1 - I T_0 w_2}\, d\, A_1 \dot{A}_2}{w_1} - \frac{I\, E^{-I T_0 w_1 - I T_0 w_2}\, d\, \dot{A}_1 \dot{A}_2}{w_1},$$

$$D_0 z_{2,2} - I\, w_2 z_{2,2} == -\frac{I\, d\, A_1 \dot{A}_1}{w_2} - \frac{I\, E^{-2 I T_0 w_1}\, d\, \dot{A}_1^2}{2 w_2}\Big\}$$

We transform **order2Eqm** into a set of ordinary-differential equations and obtain their particular solutions as

```
sol2p = DSolve@order2Eqm •. zi_,2 -> Hzi,2@#1D &L, 8z1,2@T0D, z2,2@T0D<, T0D@@1DD •. C@_D -> 0;
```

```
expRule4 = Exp@a_D :> Exp@Expand@aDD;
```

We write the second-order solution in function form as

```
sol2 = TableA9zi,2 -> Function@8T0, T1, T2<, Hsol2p@@i, 2DD •• ExpandL •. expRule4 •• EvaluateD,
    żi,2 -> Function@8T0, T1, T2<, Hsol2p@@i, 2DD •• ExpandL •. expRule4 •. conjugateRule ••
      EvaluateD=, 8i, 2<E •• Flatten;
sol2 •. displayRule
```

$$\Big\{z_{1,2} \to \mathrm{Function}\Big[\{T_0, T_1, T_2\}, -\frac{E^{I T_0 w_1 + I T_0 w_2}\, d\, A_1 A_2}{w_1 (2 w_1 + w_2)} - \frac{2 E^{I T_0 w_1 + I T_0 w_2}\, d\, A_1 A_2}{w_2 (2 w_1 + w_2)} + \frac{E^{I T_0 w_1 - I T_0 w_2}\, d\, A_1 \dot{A}_2}{w_1 (2 w_1 + w_2)} + \frac{2 E^{I T_0 w_1 - I T_0 w_2}\, d\, A_1 \dot{A}_2}{w_2 (2 w_1 + w_2)} + \frac{E^{-I T_0 w_1 - I T_0 w_2}\, d\, \dot{A}_1 \dot{A}_2}{w_1 (2 w_1 + w_2)}\Big],$$

$$\dot{z}_{1,2} \to \mathrm{Function}\Big[\{T_0, T_1, T_2\}, \frac{E^{I T_0 w_1 + I T_0 w_2}\, d\, A_1 A_2}{w_1 (2 w_1 + w_2)} + \frac{E^{-I T_0 w_1 + I T_0 w_2}\, d\, A_2 \dot{A}_1}{w_1 (2 w_1 + w_2)} + \frac{2 E^{-I T_0 w_1 + I T_0 w_2}\, d\, A_2 \dot{A}_1}{w_2 (2 w_1 + w_2)} - \frac{E^{-I T_0 w_1 - I T_0 w_2}\, d\, \dot{A}_1 \dot{A}_2}{w_1 (2 w_1 + w_2)} - \frac{2 E^{-I T_0 w_1 - I T_0 w_2}\, d\, \dot{A}_1 \dot{A}_2}{w_2 (2 w_1 + w_2)}\Big],$$

$$z_{2,2} \to \mathrm{Function}\Big[\{T_0, T_1, T_2\}, \frac{2\, d\, w_1 A_1 \dot{A}_1}{w_2^2 (2 w_1 + w_2)} + \frac{d\, A_1 \dot{A}_1}{w_2 (2 w_1 + w_2)} + \frac{E^{-2 I T_0 w_1}\, d\, \dot{A}_1^2}{2 w_2 (2 w_1 + w_2)}\Big],$$

$$\dot{z}_{2,2} \to \mathrm{Function}\Big[\{T_0, T_1, T_2\}, \frac{E^{2 I T_0 w_1}\, d\, A_1^2}{2 w_2 (2 w_1 + w_2)} + \frac{2\, d\, w_1 A_1 \dot{A}_1}{w_2^2 (2 w_1 + w_2)} + \frac{d\, A_1 \dot{A}_1}{w_2 (2 w_1 + w_2)}\Big]\Big\}$$

Ÿ Third-Order Equations

Substituting the first- and second-order solutions into the right-hand sides of the third-order equations, **eqOrder[3]**, we obtain

```
order3Eqrhs = Expand@#@@2DD •. sol1 •. sol2D & •ž eqOrder@3D;
```

Converting the terms that produce small-divisor terms and eliminating the terms that produce secular terms in $z_{i,3}$ from **order3Eqrhs** demands the following conditions:

```
SCond2 = Coefficient@order3Eqrhs@@#DD •. expRule1@#D, Exp@I w# T0DD == 0 & •ž 81, 2<;
SCond2 •. displayRule
```

$$\left\{-(D_2A_1) - \frac{4 i \delta^2 A_1^2 \bar{A}_1}{\omega_2^2 (2\omega_1+\omega_2)} - \frac{5 i \delta^2 A_1^2 \bar{A}_1}{2\omega_1\omega_2(2\omega_1+\omega_2)} - \frac{i \delta^2 A_1 A_2 \bar{A}_2}{\omega_1^2(2\omega_1+\omega_2)} == 0,\ -(D_2A_2) - \frac{i\delta^2 A_1 A_2 \bar{A}_1}{\omega_1\omega_2(2\omega_1+\omega_2)} == 0\right\}$$

or

```
SCond2Rule1 = Solve[SCond2, {A1^(0,1)[T1, T2], A2^(0,1)[T1, T2]}][[1]];
SCond2Rule1 •. displayRule
```

$$\left\{D_2A_1 \to -\frac{i(8\delta^2 A_1^2\omega_1^2\bar{A}_1 + 5\delta^2 A_1^2\omega_1\omega_2\bar{A}_1 + 2\delta^2 A_1 A_2\omega_2^2\bar{A}_2)}{2\omega_1^2\omega_2^2(2\omega_1+\omega_2)},\ D_2A_2 \to -\frac{i\delta^2 A_1 A_2\bar{A}_1}{\omega_1\omega_2(2\omega_1+\omega_2)}\right\}$$

Reconstitution

We reconstitute **SCond1Rule1** and **SCond2Rule1** and obtain

```
moduEq = Table[2 I wk Ak', {k, 2}] ==
   (Table[2 I wk dt[1][ Ak[T1, T2]], {k, 2}] •. SCond1Rule1 •. SCond2Rule1 ••
     Expand) •• Thread;
moduEq •. displayRule
```

$$\left\{2 i \omega_1 A_1' == 2 e^{i T_1 \sigma} \delta\, \epsilon\, A_2 \bar{A}_1 + \frac{8\delta^2\epsilon^2 A_1^2\omega_1\bar{A}_1}{\omega_2^2(2\omega_1+\omega_2)} + \frac{5\delta^2\epsilon^2 A_1^2\bar{A}_1}{\omega_2(2\omega_1+\omega_2)} + \frac{2\delta^2\epsilon^2 A_1 A_2\bar{A}_2}{\omega_1(2\omega_1+\omega_2)},\right.$$

$$\left.2 i \omega_2 A_2' == e^{-i T_1\sigma}\delta\,\epsilon\, A_1^2 + \frac{2\delta^2\epsilon^2 A_1 A_2\bar{A}_1}{\omega_1(2\omega_1+\omega_2)}\right\}$$

5.3 Method of Normal Forms

To apply the method of normal forms, we find it convenient to start with the first-order complex-valued form **EOM3**

$$\text{EOM3} = \left\{z_1'[t] == i\omega_1 z_1[t] - \frac{i\delta z_1[t] z_2[t]}{\omega_1} - \frac{i\delta z_2[t]\bar{z}_1[t]}{\omega_1} - \frac{i\delta z_1[t]\bar{z}_2[t]}{\omega_1} - \frac{i\delta \bar{z}_1[t]\bar{z}_2[t]}{\omega_1},\right.$$

$$\left. z_2'[t] == -\frac{i\delta z_1[t]^2}{2\omega_2} + i\omega_2 z_2[t] - \frac{i\delta z_1[t]\bar{z}_1[t]}{\omega_2} - \frac{i\delta\bar{z}_1[t]^2}{2\omega_2}\right\};$$

of the governing equations. Then, according to the method of normal forms, we introduce the near-identity transformation

```
basicTerms = Table[{hk[t], h̄k[t]}, {k, 2}] •• Flatten
```

$\{h_1[t], \bar{h}_1[t], h_2[t], \bar{h}_2[t]\}$

```
zetaRule =
 9zi_ -> FunctionAt, e hi@tD + SumAe^(j+1) hi,j@Sequence žž basicTermsD, 8j, 2<E •• EvaluateE,
  z̄i_ -> FunctionAt, e h̄i@tD + SumAe^(j+1) h̄i,j@Sequence žž basicTermsD, 8j, 2<E •• EvaluateE=
```

```
8zi_ ® Function@t,
    e hi@tD + e^2 hi,1@h1@tD, h̄1@tD, h2@tD, h̄2@tDD + e^3 hi,2@h1@tD, h̄1@tD, h2@tD, h̄2@tDDD, z̄i_ ®
  Function@t, e h̄i@tD + e^2 h̄i,1@h1@tD, h̄1@tD, h2@tD, h̄2@tDD + e^3 h̄i,2@h1@tD, h̄1@tD, h2@tD, h̄2@tDDD<
```

that results in the simplest possible equations

```
etaRule = hi_'@tD -> I wi hi@tD + SumAe^j gi,j@tD, 8j, 2<E
```

```
h'i_@tD ® I wi hi@tD + e gi,1@tD + e^2 gi,2@tD
```

where the overbar denotes the complex conjugate

```
conjugateRule = 8h -> h̄, h̄ -> h, g -> ḡ, ḡ -> g, Complex@0, n_D -> Complex@0, -nD<;
```

Substituting the expansion for the z_i, the **zetaRule**, into **EOM3**, using the **etaRule**, expanding the result for small e, and discarding terms of order higher than e^3, we have

```
eq53a =
  HEOM3 •. zetaRule •. etaRule •. HetaRule •. conjugateRuleL •• ExpandAllL •. e^(n_•;n>3) -> 0;
```

Ÿ Second-Order Equations

Collecting the coefficients of e^2 in **eq53a** yields

```
order2expr = CoefficientASubtract žž #, e^2E & •ž eq53a;
```

To determine the resonance terms in **order2expr**, we note that all of the possible forms of its nonhomogeneous terms are

```
possibleQTerms = Outer@Times, basicTerms, basicTermsD •• Flatten •• Union
```

```
9h1@tD^2, h1@tD h2@tD, h2@tD^2, h1@tD h̄1@tD,
 h2@tD h̄1@tD, h̄1@tD^2, h1@tD h̄2@tD, h2@tD h̄2@tD, h̄1@tD h̄2@tD, h̄2@tD^2=
```

It follows from the linear parts of the **etaRule** that to the first approximation

```
form = 9hi_@tD -> E^(I wi t), h̄i_@tD -> E^(-I wi t)=;
```

Hence, the possible resonance terms are given by

```
ResonantQTerm@1D =
 IE^(-I w1 t) possibleQTerms •. form •. w2 -> 2 w1 •. E^(_ t) -> 0M possibleQTerms •• Union •• Rest
```

```
8h2@tD h̄1@tD<
```

```
ResonantQTerm@2D =
 jE^-I w2 t possibleQTerms •. form •. w1 -> 1/2 w2 •. E^-t -> 0y possibleQTerms •• Union •• Rest
 k                                                        {
```

```
8h1@tD^2<
```

Then, the nonresonance terms and their associated coefficients can be defined as

```
Do@NRQT@iD = Complement@possibleQTerms, ResonantQTerm@iDD;
 coeffsQ@iD = Table@Gi,j, 8j, Length@NRQT@iDD<D, 8i, 2<D
```

Thus, the $h_{i,1}$ and their complex conjugates have the form

```
hFormQ =
  TableA9hi,1 -> HEvaluate@coeffsQ@iD.NRQT@iD •. Thread@basicTerms -> 8#1, #2, #3, #4<DD &L,
    h̄i,1 -> HEvaluate@coeffsQ@iD.NRQT@iD •. conjugateRule •.
        Thread@basicTerms -> 8#1, #2, #3, #4<DD &L=, 8i, 2<E •• Flatten;
```

Substituting for the $h_{i,1}$ in **order2expr**, equating the coefficients of the possible nonresonance terms to zero, and solving the resulting algebraic equations for the $G_{i,\,j}$, we obtain

```
coeffsQRule = Table@
   Solve@Coefficient@order2expr@@iDD •. hFormQ, NRQT@iDD == 0 •• Thread, coeffsQ@iDD@@1DD,
   8i, 2<D •• Flatten
```

$\{G_{1,1}\to 0,\ G_{1,2}\to -\frac{d}{w_1 w_2},\ G_{1,4}\to 0,\ G_{1,5}\to 0,\ G_{1,6}\to \frac{d}{w_1 w_2},\ G_{1,7}\to 0,$
$G_{1,3}\to 0,\ G_{1,8}\to \frac{d}{w_1 (2 w_1 + w_2)},\ G_{1,9}\to 0,\ G_{2,1}\to 0,\ G_{2,2}\to 0,\ G_{2,3}\to \frac{d}{w_2^2},$
$G_{2,4}\to 0,\ G_{2,7}\to 0,\ G_{2,9}\to 0,\ G_{2,5}\to \frac{d}{2 w_2 (2 w_1 + w_2)},\ G_{2,6}\to 0,\ G_{2,8}\to 0\}$

Using these coefficients, we rewrite **hFormQ** as

```
hSolQ = Table@hi,1@Sequence žž basicTermsD ->
   Hhi,1@Sequence žž basicTermsD •. hFormQ •. coeffsQRuleL, 8i, 2<D
```

$\{h_{1,1}[h_1[t], \bar{h}_1[t], h_2[t], \bar{h}_2[t]] \to -\frac{d\, h_1[t]\, h_2[t]}{w_1 w_2} + \frac{d\, h_1[t]\, \bar{h}_2[t]}{w_1 w_2} + \frac{d\, \bar{h}_1[t]\, h_2[t]}{w_1 (2 w_1 + w_2)},$
$h_{2,1}[h_1[t], \bar{h}_1[t], h_2[t], \bar{h}_2[t]] \to \frac{d\, h_1[t]\, \bar{h}_1[t]}{w_2^2} + \frac{d\, \bar{h}_1[t]^2}{2 w_2 (2 w_1 + w_2)}\}$

or in pure function form as

```
hRuleQ = TableA9hi,1 -> HEvaluate@hSolQ@@i, 2DD •. Thread@basicTerms -> 8#1, #2, #3, #4<DD &L,
    h̄i,1 -> HEvaluate@hSolQ@@i, 2DD •. conjugateRule •.
        Thread@basicTerms -> 8#1, #2, #3, #4<DD &L=, 8i, 2<E •• Flatten;
```

We choose the $g_{i,1}$ to eliminate the resonance terms in **order2expr** according to

```
gRuleQ = Table@
  gi,1@tD -> - Coefficient@order2expr@@iDD, ResonantQTerm@iDD.ResonantQTerm@iD, 8i, 2<D
```

```
          I d h2@tD h̄1@tD                 I d h1@tD^2
:g1,1@tD ® - ──────────────── , g2,1@tD ® - ──────────── >
              w1                              2 w2
```

Ÿ Third-Order Equations

Collecting the coefficients of ϵ^3 in **eq53a** and using **hRuleQ** and **gRuleQ**, we have

```
order3expr = MapACoefficientASubtract žž #, e^3E &, eq53aE •. hRuleQ •. gRuleQ •.
    HgRuleQ •. conjugateRuleL •• Expand;
```

The nonhomogeneous terms in **order3expr** are proportional to

```
possibleCTerms = Outer@Times, possibleQTerms, basicTermsD •• Flatten •• Union
```

```
9h1@tD^3, h1@tD^2 h2@tD, h1@tD h2@tD^2, h2@tD^3, h1@tD^2 h̄1@tD,
 h1@tD h2@tD h̄1@tD, h2@tD^2 h̄1@tD, h1@tD h̄1@tD^2, h2@tD h̄1@tD^2, h̄1@tD^3,
 h1@tD^2 h̄2@tD, h1@tD h2@tD h̄2@tD, h2@tD^2 h̄2@tD, h1@tD h̄1@tD h̄2@tD,
 h2@tD h̄1@tD h̄2@tD, h̄1@tD^2 h̄2@tD, h1@tD h̄2@tD^2, h2@tD h̄2@tD^2, h̄1@tD h̄2@tD^2, h̄2@tD^3=
```

Next, we determine the resonance terms according to

```
ResonantCTerm@1D =
 IE^(-I w1 t) possibleCTerms •. form •. w2 -> 2 w1 •. E^(-t) -> 0M possibleCTerms •• Union •• Rest
```

```
8h1@tD^2 h̄1@tD, h1@tD h2@tD h̄2@tD<
```

```
ResonantCTerm@2D =
 jE^(-I w2 t) possibleCTerms •. form •. w1 -> 1/2 w2 •. E^(-t) -> 0y possibleCTerms •• Union •• Rest
 k                                                              {
```

```
8h1@tD h2@tD h̄1@tD, h2@tD^2 h̄2@tD<
```

Then, the nonresonance terms and their associated coefficients are given by

```
Do@NRCT@iD = Complement@possibleCTerms, ResonantCTerm@iDD;
 coeffsC@iD = Table@Li,j, 8j, Length@NRCT@iDD<D, 8i, 2<D
```

Consequently, the $h_{i,2}$ and their complex conjugates have the forms

```
hFormC =
  TableA9hi,2 -> HEvaluate@coeffsC@iD.NRCT@iD •. Thread@basicTerms -> 8#1, #2, #3, #4<DD &L,
     h̄i,2 -> HEvaluate@coeffsC@iD.NRCT@iD •. conjugateRule •.
          Thread@basicTerms -> 8#1, #2, #3, #4<DD &L=, 8i, 2<E •• Flatten;
```

Substituting for the $h_{i,2}$ in **order3expr**, equating the coefficient of each possible nonresonance term to zero, and solving the resulting algebraic equations for the $\Gamma_{i,j}$, we obtain

```
coeffsCRule = Table@
  Solve@Coefficient@order3expr@@iDD •. hFormC, NRCT@iDD == 0 •• Thread, coeffsC@iDD@@1DD,
  8i, 2<D •• Flatten
```

$$\{\Gamma_{1,1}\to-\frac{\delta^2\omega_1+\delta^2\omega_2}{2\omega_1^2\omega_2^2(2\omega_1+\omega_2)},\ \Gamma_{1,3}\to\frac{\delta^2}{\omega_1\omega_2^2(2\omega_1+\omega_2)},\ \Gamma_{1,7}\to-\frac{-5\delta^2\omega_1-3\delta^2\omega_2}{2\omega_1^2\omega_2^2(2\omega_1+\omega_2)},$$

$$\Gamma_{1,9}\to\frac{\delta^2}{4\omega_1^2\omega_2(2\omega_1+\omega_2)},\ \Gamma_{1,13}\to-\frac{\delta^2}{\omega_1^2\omega_2(2\omega_1+\omega_2)},\ \Gamma_{1,15}\to-\frac{-\delta^2\omega_1-\delta^2\omega_2}{\omega_1^2\omega_2^2(2\omega_1+\omega_2)},$$

$$\Gamma_{1,2}\to0,\ \Gamma_{1,16}\to0,\ \Gamma_{1,4}\to0,\ \Gamma_{1,17}\to-\frac{\delta^2}{\omega_1\omega_2(\omega_1+\omega_2)(2\omega_1+\omega_2)},\ \Gamma_{1,5}\to0,$$

$$\Gamma_{1,18}\to0,\ \Gamma_{1,10}\to0,\ \Gamma_{1,8}\to0,\ \Gamma_{1,6}\to\frac{\delta^2}{\omega_1^2(\omega_1-\omega_2)\omega_2},\ \Gamma_{1,11}\to0,\ \Gamma_{1,14}\to0,\ \Gamma_{1,12}\to0,$$

$$\Gamma_{2,2}\to\frac{\delta^2}{\omega_1\omega_2^2(2\omega_1+\omega_2)},\ \Gamma_{2,4}\to0,\ \Gamma_{2,8}\to0,\ \Gamma_{2,12}\to\frac{\delta^2}{\omega_1\omega_2^2(2\omega_1+\omega_2)},\ \Gamma_{2,16}\to0,\ \Gamma_{2,18}\to0,$$

$$\Gamma_{2,1}\to0,\ \Gamma_{2,3}\to0,\ \Gamma_{2,5}\to0,\ \Gamma_{2,6}\to0,\ \Gamma_{2,7}\to0,\ \Gamma_{2,9}\to0,\ \Gamma_{2,10}\to-\frac{\delta^2}{\omega_1(\omega_1-\omega_2)\omega_2^2},$$

$$\Gamma_{2,11}\to0,\ \Gamma_{2,17}\to0,\ \Gamma_{2,13}\to0,\ \Gamma_{2,14}\to-\frac{\delta^2}{\omega_2^2(\omega_1+\omega_2)(2\omega_1+\omega_2)},\ \Gamma_{2,15}\to0\}$$

Using these coefficients, we rewrite **hFormC** as

```
hSolC = Table@h_{i,2}@Sequence žž basicTermsD ->
  Hh_{i,2}@Sequence žž basicTermsD •. hFormC •. coeffsCRuleL, 8i, 2<D
```

$$\{h_{1,2}[\eta_1[t],\bar\eta_1[t],\eta_2[t],\bar\eta_2[t]]\to-\frac{(\delta^2\omega_1+\delta^2\omega_2)\eta_1[t]^3}{2\omega_1^2\omega_2^2(2\omega_1+\omega_2)}+\frac{\delta^2\eta_1[t]\eta_2[t]^2}{\omega_1\omega_2^2(2\omega_1+\omega_2)}+\frac{\delta^2\eta_2[t]^2\bar\eta_1[t]}{\omega_1^2(\omega_1-\omega_2)\omega_2}-$$

$$\frac{(-5\delta^2\omega_1-3\delta^2\omega_2)\eta_1[t]\bar\eta_1[t]^2}{2\omega_1^2\omega_2^2(2\omega_1+\omega_2)}+\frac{\delta^2\bar\eta_1[t]^3}{4\omega_1^2\omega_2(2\omega_1+\omega_2)}-\frac{\delta^2\eta_2[t]\bar\eta_1[t]\bar\eta_2[t]}{\omega_1^2\omega_2(2\omega_1+\omega_2)}-$$

$$\frac{(-\delta^2\omega_1-\delta^2\omega_2)\eta_1[t]\bar\eta_2[t]^2}{\omega_1^2\omega_2^2(2\omega_1+\omega_2)}-\frac{\delta^2\bar\eta_1[t]\bar\eta_2[t]^2}{\omega_1\omega_2(\omega_1+\omega_2)(2\omega_1+\omega_2)},\ h_{2,2}[\eta_1[t],\bar\eta_1[t],\eta_2[t],\bar\eta_2[t]]\to$$

$$\frac{\delta^2\eta_1[t]^2\eta_2[t]}{\omega_1\omega_2^2(2\omega_1+\omega_2)}-\frac{\delta^2\eta_1[t]^2\bar\eta_2[t]}{\omega_1(\omega_1-\omega_2)\omega_2^2}+\frac{\delta^2\eta_1[t]\bar\eta_1[t]\bar\eta_2[t]}{\omega_1\omega_2^2(2\omega_1+\omega_2)}-\frac{\delta^2\bar\eta_1[t]^2\bar\eta_2[t]}{\omega_2^2(\omega_1+\omega_2)(2\omega_1+\omega_2)}\}$$

or in pure function form as

```
hRuleC = TableA9h_{i,2} -> HEvaluate@hSolC@@i, 2DD •. Thread@basicTerms -> 8#1, #2, #3, #4<DD &L,
   h̄_{i,2} -> HEvaluate@hSolC@@i, 2DD •. conjugateRule •.
      Thread@basicTerms -> 8#1, #2, #3, #4<DD &L=, 8i, 2<E •• Flatten;
```

Choosing the $g_{i,2}$ to eliminate the resonance terms from **order3expr** yields

```
gRuleC = Table@
  gi,2@tD -> - Coefficient@order3expr@@iDD, ResonantCTerm@iDD.ResonantCTerm@iD, 8i, 2<D
```

$$:g_{1,2}@tD \circledR -\left(\frac{2\,I\,d^2}{w_1\,w_2^2} + \frac{I\,d^2}{2\,w_1\,w_2\,H2\,w_1 + w_2L} \right) h_1@tD^2\,\dot{h}_1@tD - \frac{I\,d^2\,h_1@tD\,h_2@tD\,\dot{h}_2@tD}{w_1^2\,H2\,w_1 + w_2L},$$

$$g_{2,2}@tD \circledR -\frac{I\,d^2\,h_1@tD\,h_2@tD\,\dot{h}_1@tD}{w_1\,w_2\,H2\,w_1 + w_2L}>$$

Combining **etaRule**, **gRuleQ**, and **gRuleC** and letting $h_i = A_i@tD\,E^{I\,w_i\,t}$, we obtain the modulation equation

```
moduEq = TableA2 I wk E^(-I wk t) Hhk¢@tD - Hhk¢@tD •. etaRule •. gRuleQ •. gRuleCLL == 0 •.
    9hi_ -> IAi@#D E^(I wi #) &M, ḣi_ -> IȦi@#D E^(-I wi #) &M= •.
   Exp@a_D :> ExpAa •. w2 -> 2 w1 + e^2 s •• ExpandE •• ExpandAll, 8k, 2<E
```

$$:-\frac{4\,d^2\,e^2\,A_1@tD^2\,\dot{A}_1@tD}{w_2^2} - \frac{2\,d^2\,e^2\,w_1\,A_1@tD^2\,\dot{A}_1@tD}{4\,w_1^2\,w_2 + 2\,w_1\,w_2^2} -$$

$$2\,E^{I\,t\,e^2\,s}\,d\,e\,A_2@tD\,\dot{A}_1@tD - \frac{2\,d^2\,e^2\,w_1\,A_1@tD\,A_2@tD\,\dot{A}_2@tD}{2\,w_1^3 + w_1^2\,w_2} + 2\,I\,w_1\,A_1^{¢}@tD == 0,$$

$$-E^{-I\,t\,e^2\,s}\,d\,e\,A_1@tD^2 - \frac{2\,d^2\,e^2\,w_2\,A_1@tD\,A_2@tD\,\dot{A}_1@tD}{2\,w_1^2\,w_2 + w_1\,w_2^2} + 2\,I\,w_2\,A_2^{¢}@tD == 0>$$

Using **zetaRule**, **hRuleQ**, and **hRuleC**, we obtain the following second-order uniform expansion of the solution of **EOM3**:

```
solution = TableAuk@tD == CollectAzk@tD + żk@tD •. zetaRule •. hRuleQ •. hRuleC, eE, 8k, 2<E
```

```
:u1@tD == e Hh1@tD + ḣ1@tDL + e^2 j - (d h1@tD h2@tD)/(w1 w2) + (d h1@tD h2@tD)/(w1 H2 w1 + w2L) +
    (d h2@tD ḣ1@tD)/(w1 w2) + (d h1@tD ḣ2@tD)/(w1 w2) - (d ḣ1@tD ḣ2@tD)/(w1 w2) + (d ḣ1@tD ḣ2@tD)/(w1 H2 w1 + w2L) z +
  e^3 j (d^2 h1@tD^3)/(4 w1^2 w2 H2 w1 + w2L) - (Hd^2 w1 + d^2 w2L h1@tD^3)/(2 w1^2 w2^2 H2 w1 + w2L) + (d^2 h1@tD h2@tD^2)/(w1 w2^2 H2 w1 + w2L) - (d^2 h1@tD h2@tD^2)/(w1 w2 Hw1 + w2L H2 w1 + w2L) -
    (H-5 d^2 w1 - 3 d^2 w2L h1@tD^2 ḣ1@tD)/(2 w1^2 w2^2 H2 w1 + w2L) + (d^2 h2@tD^2 ḣ1@tD)/(w1^2 Hw1 - w2L w2) - (H-d^2 w1 - d^2 w2L h2@tD^2 ḣ1@tD)/(w1^2 w2^2 H2 w1 + w2L) -
    (H-5 d^2 w1 - 3 d^2 w2L h1@tD ḣ1@tD^2)/(2 w1^2 w2^2 H2 w1 + w2L) + (d^2 ḣ1@tD^3)/(4 w1^2 w2 H2 w1 + w2L) - (Hd^2 w1 + d^2 w2L ḣ1@tD^3)/(2 w1^2 w2^2 H2 w1 + w2L) -
    (d^2 h1@tD h2@tD ḣ2@tD)/(w1^2 w2 H2 w1 + w2L) - (d^2 h2@tD ḣ1@tD ḣ2@tD)/(w1^2 w2 H2 w1 + w2L) + (d^2 h1@tD ḣ2@tD^2)/(w1^2 Hw1 - w2L w2) -
    (H-d^2 w1 - d^2 w2L h1@tD ḣ2@tD^2)/(w1^2 w2^2 H2 w1 + w2L) + (d^2 ḣ1@tD ḣ2@tD^2)/(w1 w2^2 H2 w1 + w2L) - (d^2 ḣ1@tD ḣ2@tD^2)/(w1 w2 Hw1 + w2L H2 w1 + w2L) z,
 u2@tD == e^2 j (d h1@tD^2)/(2 w2 H2 w1 + w2L) + (2 d h1@tD ḣ1@tD)/(w2^2) + (d ḣ1@tD^2)/(2 w2 H2 w1 + w2L) z + e Hh2@tD + ḣ2@tDL +
  e^3 j (d^2 h1@tD^2 h2@tD)/(w1 w2^2 H2 w1 + w2L) - (d^2 h1@tD^2 h2@tD)/(w2^2 Hw1 + w2L H2 w1 + w2L) + (d^2 h1@tD h2@tD ḣ1@tD)/(w1 w2^2 H2 w1 + w2L) - (d^2 h2@tD ḣ1@tD^2)/(w1 Hw1 - w2L w2^2) -
    (d^2 h1@tD^2 ḣ2@tD)/(w1 Hw1 - w2L w2^2) + (d^2 h1@tD ḣ1@tD ḣ2@tD)/(w1 w2^2 H2 w1 + w2L) + (d^2 ḣ1@tD^2 ḣ2@tD)/(w1 w2^2 H2 w1 + w2L) - (d^2 ḣ1@tD^2 ḣ2@tD)/(w2^2 Hw1 + w2L H2 w1 + w2L) z>
```

à 5.4 Generalized Method of Averaging

We define the nonlinear part of **EOM1**, a transformation rule, and the states vector as

```
NLTerms = 9f1 -> 2 e d u1@tD u2@tD, f2 -> e d u1@tD^2=;
transformRule = ui_ -> Hai@#D Sin@fi@#DD &L;
states = 8a1@tD, a2@tD, f1@tD, f2@tD<;
```

Substituting the above rules into the four first-order equations obtained by using the method of variation of parameters, we have

```
eq54a = D@states, tD ==
   j9 f1/w1 Cos@f1@tDD, f2/w2 Cos@f2@tDD, w1 - f1/(a1@tD w1) Sin@f1@tDD, w2 - f2/(a2@tD w2) Sin@f2@tDD= •.
   k
      NLTerms •. transformRule •• TrigReduce •• Expand¥ •• Thread
                                                     {
```

```
:a¢1@tD == d e Cos@2 f1@tD - f2@tDD a1@tD a2@tD/(2 w1) - d e Cos@2 f1@tD + f2@tDD a1@tD a2@tD/(2 w1), a¢2@tD ==
 - d e Cos@2 f1@tD - f2@tDD a1@tD^2/(4 w2) + d e Cos@f2@tDD a1@tD^2/(2 w2) - d e Cos@2 f1@tD + f2@tDD a1@tD^2/(4 w2),
 f¢1@tD == w1 - d e Sin@2 f1@tD - f2@tDD a2@tD/(2 w1) - d e Sin@f2@tDD a2@tD/w1 +
   d e Sin@2 f1@tD + f2@tDD a2@tD/(2 w1), f¢2@tD ==
  w2 - d e Sin@2 f1@tD - f2@tDD a1@tD^2/(4 w2 a2@tD) - d e Sin@f2@tDD a1@tD^2/(2 w2 a2@tD) + d e Sin@2 f1@tD + f2@tDD a1@tD^2/(4 w2 a2@tD)>
```

We seek a second-order approximate solution of **eq54a** in the form

```
basicTerms = 8h1@tD, h2@tD, j1@tD, j2@tD<;
```

```
solRule = 9 ai_ -> I EvaluateAhi@tD + SumAe^j ai,j@Sequence žž basicTermsD, 8j, 2<E •. t -> #E &M,
   fi_ -> I EvaluateAji@tD + SumAe^j fi,j@Sequence žž basicTermsD, 8j, 2<E •. t -> #E &M=;
```

where the h_i@tD and j_i@tD are expanded in power series in e as

```
basicDRule =
  D@basicTerms, tD -> 9SumAe^i A1,i@h1@tD, h2@tDD, 8i, 2<E, SumAe^i A2,i@h1@tD, h2@tDD, 8i, 2<E,
     w1 + SumAe^i F1,i@h1@tD, h2@tDD, 8i, 2<E, w2 + SumAe^i F2,i@h1@tD, h2@tDD, 8i, 2<E= •• Thread
```

```
8h¢1@tD ® e A1,1@h1@tD, h2@tDD + e^2 A1,2@h1@tD, h2@tDD,
 h¢2@tD ® e A2,1@h1@tD, h2@tDD + e^2 A2,2@h1@tD, h2@tDD,
 j¢1@tD ® w1 + e F1,1@h1@tD, h2@tDD + e^2 F1,2@h1@tD, h2@tDD,
 j¢2@tD ® w2 + e F2,1@h1@tD, h2@tDD + e^2 F2,2@h1@tD, h2@tDD<
```

The functions $a_{i,\ j}$ and $f_{i,\ j}$ are fast varying functions of the j_i, while it follows from **basicDRule** that the h_i, and hence the $A_{i,\ j}$ and $F_{i,\ j}$, are slowly varying functions of *t*.

To the second approximation, we differentiate the a_i@tD and f_i@tD once with respect to *t*, use **solRule** and **basicDRule**, expand the result for small e, discard terms of order higher than e^2, and obtain

```
eq54bLHS =
  CoefficientListAExpand@#@@1DD •. solRule •. basicDRuleD •. e^n_•;n>2 -> 0, eE & •ž eq54a;
```

Next, we substitute **solRule** into the right-hand sides of **eq54a**, expand the result for small e, keep terms up to *O*He^2L, and rewrite their right-hand sides as

```
eq54bRHS = CoefficientList@Series@#@@2DD •. solRule, 8e, 0, 2<D •• Normal, eD & •ž eq54a;
```

Equating coefficients of like powers of e in **eq54bLHS** and **eq54bRHS**, we obtain

```
eqEps = MapThread@Equal, 8eq54bLHS, eq54bRHS<, 2D •• Transpose •• Rest;
```

To express the nearness of the two-to-one internal resonance, we introduce the detuning parameter s defined according to

```
omgRule = 8w2 -> 2 w1 + e s<;
```

Hence, s can be related to the w_i by

```
sigRule = Solve@omgRule •. Rule -> Equal, sD@@1DD
```

9s ® - (2 w1 - w2)/e =

To use the method of averaging, we define the following rules:

```
psiRule = 8j1@tD -> w1 t + b1, j2@tD -> w2 t + b2<;
```

```
betaRule = Solve@psiRule •. Rule -> Equal, 8b1, b2<D@@1DD
```

8b1 ® -t w1 + j1@tD, b2 ® -t w2 + j2@tD<

```
expRule1 = f_@a_D :> f@Expand@a •. psiRule •. omgRuleD •. e t -> t1D;
```

```
expRule2 = f_@a_D :> f@Expand@a •. t1 -> e t •. sigRule •. betaRuleDD;
```

Next, we separate the fast and slowly varying terms in the first-order equations, **eqEps[[1]]**. To this end, we define a slow state vector and a fast state vector according to

```
SVT@j_D := 8A1,j@h1@tD, h2@tDD, A2,j@h1@tD, h2@tDD, F1,j@h1@tD, h2@tDD, F2,j@h1@tD, h2@tDD<
solVar@j_D := 8a1,j, a2,j, f1,j, f2,j<
```

Then, the slowly varying parts of **eqEps[[1]]** are given by

```
SVT1Rule =
 Table@Solve@eqEps@@1, iDD •. Thread@solVar@1D -> H0 &LD •. expRule1 •. f_@_ t + _.D -> 0 •.
      expRule2 •. f_@a_D :> f@Expand@aDD, SVT@1D@@iDDD @@1DD, 8i, 4<D •• Flatten
```

:A1,1@h1@tD, h2@tDD ® (d Cos@2 j1@tD - j2@tDD h1@tD h2@tD)/(2 w1),

A2,1@h1@tD, h2@tDD ® - (d Cos@2 j1@tD - j2@tDD h1@tD^2)/(4 w2),

F1,1@h1@tD, h2@tDD ® - (d Sin@2 j1@tD - j2@tDD h2@tD)/(2 w1),

F2,1@h1@tD, h2@tDD ® - (d Sin@2 j1@tD - j2@tDD h1@tD^2)/(4 w2 h2@tD)>

whereas the fast varying parts are given by

```
FVT1 = Table@Subtract žž eqEps@@1, iDD •. Thread@solVar@1D -> H0 &LD, 8i, 4<D •. SVT1Rule
```

$$:\frac{d\,Cos@2\,j_1@tD + j_2@tDD\,h_1@tD\,h_2@tD}{2\,w_1},\ -\frac{d\,Cos@j_2@tDD\,h_1@tD^2}{2\,w_2}+\frac{d\,Cos@2\,j_1@tD + j_2@tDD\,h_1@tD^2}{4\,w_2},$$

$$\frac{d\,Sin@j_2@tDD\,h_2@tD}{w_1}-\frac{d\,Sin@2\,j_1@tD + j_2@tDD\,h_2@tD}{2\,w_1},$$

$$\frac{d\,Sin@j_2@tDD\,h_1@tD^2}{2\,w_2\,h_2@tD}-\frac{d\,Sin@2\,j_1@tD + j_2@tDD\,h_1@tD^2}{4\,w_2\,h_2@tD}>$$

To determine the solution corresponding to these fast varying terms, we use the method of undetermined coefficients. To accomplish this, we first identify the possible forms of the nonhomogeneous terms. The result is

```
FVT1Forms = Flatten@Cases@#, HCos È SinL@a_D -> 8Cos@aD, Sin@aD<, InfinityDD & •ž FVT1
```

```
88Cos@2 j1@tD + j2@tDD, Sin@2 j1@tD + j2@tDD<,
 8Cos@j2@tDD, Sin@j2@tDD, Cos@2 j1@tD + j2@tDD, Sin@2 j1@tD + j2@tDD<,
 8Cos@j2@tDD, Sin@j2@tDD, Cos@2 j1@tD + j2@tDD, Sin@2 j1@tD + j2@tDD<,
 8Cos@j2@tDD, Sin@j2@tDD, Cos@2 j1@tD + j2@tDD, Sin@2 j1@tD + j2@tDD<<
```

The fast-varying component of the first-order solution can be expressed as a linear combination of these forms; that is,

```
sol1Form = MapIndexed@Hcoeffs1@#2@@1DDD = Array@c, Length@#1DDL.#1 &, FVT1FormsD
```

```
8c@1D Cos@2 j1@tD + j2@tDD + c@2D Sin@2 j1@tD + j2@tDD,
 c@1D Cos@j2@tDD + c@3D Cos@2 j1@tD + j2@tDD + c@2D Sin@j2@tDD + c@4D Sin@2 j1@tD + j2@tDD,
 c@1D Cos@j2@tDD + c@3D Cos@2 j1@tD + j2@tDD + c@2D Sin@j2@tDD + c@4D Sin@2 j1@tD + j2@tDD,
 c@1D Cos@j2@tDD + c@3D Cos@2 j1@tD + j2@tDD + c@2D Sin@j2@tDD + c@4D Sin@2 j1@tD + j2@tDD<
```

Substituting **sol1Form** into **eqEps[[1]]**, using **SVT1Rule**, collecting the coefficients of **FVT1Forms**, solving the resulting algebraic equation for the undetermined coefficients, and then substituting the result back into **sol1Form**, we obtain

```
sol1rhs = Table@sol1Form@@iDD •.
   Solve@Coefficient@Subtract žž eqEps@@1, iDD •. SVT1Rule •. solVar@1D@@iDD ->
        HEvaluate@sol1Form@@iDD •. Thread@basicTerms -> 8#1, #2, #3, #4<DD &L,
      FVT1Forms@@iDDD == 0 •• Thread, coeffs1@iDD@@1DD, 8i, 4<D
```

$$:-\frac{d\,Sin@2\,j_1@tD + j_2@tDD\,h_1@tD\,h_2@tD}{2\,w_1\,H2\,w_1 + w_2L},\ \frac{d\,Sin@j_2@tDD\,h_1@tD^2}{2\,w_2^2}-\frac{d\,Sin@2\,j_1@tD + j_2@tDD\,h_1@tD^2}{4\,w_2\,H2\,w_1 + w_2L},$$

$$\frac{d\,Cos@j_2@tDD\,h_2@tD}{w_1\,w_2}-\frac{d\,Cos@2\,j_1@tD + j_2@tDD\,h_2@tD}{2\,w_1\,H2\,w_1 + w_2L},$$

$$\frac{d\,Cos@j_2@tDD\,h_1@tD^2}{2\,w_2^2\,h_2@tD}-\frac{d\,Cos@2\,j_1@tD + j_2@tDD\,h_1@tD^2}{4\,w_2\,H2\,w_1 + w_2L\,h_2@tD}>$$

or in a pure function form as

```
sol1Rule = Table@solVar@1D@@iDD ->
   HEvaluate@sol1rhs@@iDD •. Thread@basicTerms -> 8#1, #2, #3, #4<DD &L, 8i, 4<D
```

$$:a_{1,1} \circledR \left(-\frac{d\,Sin@2\,\#3+\#4D\,\#1\,\#2}{2\,w_1\,H2\,w_1+w_2L}\,\& \right),\; a_{2,1} \circledR \left(\frac{d\,Sin@\#4D\,\#1^2}{2\,w_2^2} - \frac{d\,Sin@2\,\#3+\#4D\,\#1^2}{4\,w_2\,H2\,w_1+w_2L}\,\& \right),$$

$$f_{1,1} \circledR \left(\frac{d\,Cos@\#4D\,\#2}{w_1\,w_2} - \frac{d\,Cos@2\,\#3+\#4D\,\#2}{2\,w_1\,H2\,w_1+w_2L}\,\& \right),\; f_{2,1} \circledR \left(\frac{d\,Cos@\#4D\,\#1^2}{2\,\#2\,w_2^2} - \frac{d\,Cos@2\,\#3+\#4D\,\#1^2}{4\,\#2\,w_2\,H2\,w_1+w_2L}\,\& \right)>$$

Substituting the slow- and fast-varying components of the first-order solution, **SVT1Rule** and **sol1Rule**, into the second-order equations, we have

```
Horder2Expr = Table@Subtract žž eqEps@@2, iDD •. SVT1Rule •. sol1Rule •• Expand •• TrigReduce,
     8i, 4<D;L •• Timing
```

```
85.748 Second, Null<
```

Since we are seeking an expansion valid up to $O\text{He}^2\text{L}$, we do not need to solve for the $a_{i,2}$ and $f_{i,2}$. All we need to do is to investigate the above expressions to determine the slowly varying parts and hence determine the $A_{i,2}$ and $F_{i,2}$. The result is

```
SVT2Rule = Table@
   Solve@order2Expr@@iDD == 0 •. Thread@solVar@2D -> H0 &LD •. expRule1 •. f_@_ t + _.D -> 0 •.
      expRule2 •. f_@a_D :> f@Expand@aDD, SVT@2D@@iDDD, 8i, 4<D •• Flatten
```

$$:A_{1,2}@h_1@tD,\ h_2@tDD \circledR 0,\ A_{2,2}@h_1@tD,\ h_2@tDD \circledR 0,$$

$$F_{1,2}@h_1@tD,\ h_2@tDD \circledR -\frac{8\,d^2\,w_1^2\,h_1@tD^2 + 5\,d^2\,w_1\,w_2\,h_1@tD^2 + 2\,d^2\,w_2^2\,h_2@tD^2}{8\,w_1^2\,w_2^2\,H2\,w_1+w_2L},$$

$$F_{2,2}@h_1@tD,\ h_2@tDD \circledR -\frac{d^2\,h_1@tD^2}{4\,w_1\,w_2\,H2\,w_1+w_2L}>$$

Hence, we find that, to the second approximation, the solution is given by

```
eq54c =
  8u1@tD, u2@tD< == HNormal@Series@8u1@tD, u2@tD< •. transformRule •. solRule, 8e, 0, 1<DD •.
      sol1Rule •• TrigReduce •• ExpandL •• Thread
```

$$:u_1@tD == \frac{2\,Sin@j_1@tDD\,w_1\,h_1@tD}{2\,w_1+w_2} + \frac{Sin@j_1@tDD\,w_2\,h_1@tD}{2\,w_1+w_2} + \frac{d\,e\,Cos@j_1@tD - j_2@tDD\,h_1@tD\,h_2@tD}{2\,w_1\,H2\,w_1+w_2L} +$$

$$\frac{d\,e\,Cos@j_1@tD - j_2@tDD\,h_1@tD\,h_2@tD}{w_2\,H2\,w_1+w_2L} + \frac{d\,e\,Cos@j_1@tD + j_2@tDD\,h_1@tD\,h_2@tD}{w_2\,H2\,w_1+w_2L},$$

$$u_2@tD == \frac{d\,e\,w_1\,h_1@tD^2}{w_2^2\,H2\,w_1+w_2L} + \frac{d\,e\,h_1@tD^2}{2\,w_2\,H2\,w_1+w_2L} - \frac{d\,e\,Cos@2\,j_1@tDD\,h_1@tD^2}{4\,w_2\,H2\,w_1+w_2L} +$$

$$\frac{2\,Sin@j_2@tDD\,w_1\,h_2@tD}{2\,w_1+w_2} + \frac{Sin@j_2@tDD\,w_2\,h_2@tD}{2\,w_1+w_2}>$$

where the amplitudes and phases are given by

```
ampEqs = basicDRule@@81, 2<DD •. SVT1Rule •. SVT2Rule •. psiRule •.
   f_@a_D :> f@Collect@a, tDD •. Rule -> Equal
```

```
: h¢1@tD == (d e Cos@2 b1 - b2 - t H-2 w1 + w2LD h1@tD h2@tD) / (2 w1) ,
  h¢2@tD == - (d e Cos@2 b1 - b2 - t H-2 w1 + w2LD h1@tD^2) / (4 w2) >
```

```
phaseEqs = basicDRule@@83, 4<DD •. SVT1Rule •. SVT2Rule •. psiRule •.
   f_@a_D :> f@Collect@a, tDD •. Rule -> Equal
```

```
: j¢1@tD ==
   w1 - (d e Sin@2 b1 - b2 - t H-2 w1 + w2LD h2@tD) / (2 w1) - (e^2 H8 d^2 w1^2 h1@tD^2 + 5 d^2 w1 w2 h1@tD^2 + 2 d^2 w2^2 h2@tD^2L) / (8 w1^2 w2^2 H2 w1 + w2L) ,
  j¢2@tD == w2 - (d^2 e^2 h1@tD^2) / (4 w1 w2 H2 w1 + w2L) - (d e Sin@2 b1 - b2 - t H-2 w1 + w2LD h1@tD^2) / (4 w2 h2@tD) >
```

Chapter 6
Forced Oscillators of Systems with Finite Degrees of Freedom

In this chapter, we discuss nonlinear systems having finite degrees of freedom. The discussion is limited to weakly nonlinear systems, and approximate solutions are obtained by using the method of multiple scales. In the case of strongly nonlinear systems, perturbation methods can be used in cases for which a basic exact nonlinear solution exists. For the other cases, recourse is often made to numerical methods and / or geometrical methods to obtain a qualitative description of the behavior of the system, including its stability.

In contrast with a single-degree-of-freedom system, which has only a single linear natural frequency and a single mode of oscillation, an n-degree-of-freedom system has n linear natural frequencies and n corresponding modes. Let us denote these frequencies by ω_1, ω_2, …, ω_n and assume that all of them are real and different from zero. An important case occurs whenever two or more of these frequencies are commensurate or nearly commensurate. Examples of near-commensurability are

$$\{\omega_2 \approx 2\,\omega_1,\ \omega_3 \approx \omega_2 \pm \omega_1,\ \omega_2 \approx 3\,\omega_1,\ \omega_3 \approx 2\,\omega_2 \pm \omega_1,\ \omega_4 \approx \omega_3 \pm \omega_2 \pm \omega_1\}$$

Depending on the order of the nonlinearity in the system, these commensurate relationships of frequencies can cause the corresponding modes to be strongly coupled, and an internal or autoparametric resonance is said to exist. For example, if the system has quadratic nonlinearities, then to first order an internal resonance may exist if $\omega_m \approx 2\omega_k$ or $\omega_q \approx \omega_p \pm \omega_m$. For a system with cubic nonlinearities, to first order an internal resonance may exist if $\omega_m \approx 3\omega_k$ or $\omega_q \approx \omega_p \pm \omega_m \pm \omega_k$ or $\omega_m \approx \omega_k$ or $\omega_q \approx 2\omega_p \pm \omega_m$. When an internal resonance exists in a free undamped system, energy imparted initially to one of the modes involved in the internal resonance will be continuously exchanged among the modes involved in that internal resonance. If damping is present in the system, then the energy will decay with time as it is being continuously exchanged.

In a conservative nongyroscopic system, if the linear motion is oscillatory, then the nonlinear motion is bounded. For a conservative gyroscopic multidegree-of-freedom system, the nonlinear motion may be unbounded and hence unstable if an internal resonance exists.

If an external harmonic excitation of frequency Ω acts on a multidegree-of-freedom system, then in addition to all of the primary and secondary resonances ($p\,\Omega \approx q\,\omega_m$, with p and q being integers) of a single-degree-of-freedom system, there might exist other resonance combinations of the frequencies in the form

$$p\,\Omega == a_1\,\omega_1 + a_2\,\omega_2 + \cdots + a_N\,\omega_N$$

where p and the a_n are positive or negative integers such that

$$|p| + \sum_{n=1}^{N} |a_n| == M$$

where M is the order of the nonlinearity plus one and N is the number of degrees of freedom. The type of combination resonance which might exist in a system depends on the order of its nonlinearity. For a system having quadratic nonlinearities, to first order the combination resonances that might exist involve two frequencies in addition to Ω; that is, $\Omega \approx \omega_m + \omega_k$

or W » w_m - w_k. The first of these is called a summed combination resonance or a combination resonance of the additive type, whereas the second is called a difference combination resonance or a combination resonance of the difference type. These types of combination resonances were predicted theoretically by Malkin (1956) and found experimentally by Yamamoto (1957, 1960). For a system having cubic nonlinearities, to first order the combination resonances that might exist involve either two or three of the natural frequencies in addition to W; that is,

```
8W » wp ± wm ± wk, W » 2 wp ± wm, W » wp ± 2 wm, 2 W » wm ± wk<
```

à Preliminaries

```
Off@General::spell1D
```

```
Needs@"Utilities`Notation`"D
```

To use the method of multiple scales, we introduce different time scales, symbolize them as

```
Symbolize@T0D; Symbolize@T1D; Symbolize@T2D;
```

and form a list of them as follows:

```
timeScales = 8T0, T1, T2<;
```

In terms of the time scales T_n, the time derivatives become

```
dt@1D@expr_D := SumAe^i D@expr, timeScales@@i + 1DDD, 8i, 0, maxOrder<E;
dt@2D@expr_D := Hdt@1D@dt@1D@exprDD •• ExpandL •. e^(i_•;i>maxOrder) -> 0;
```

In the course of the analysis, we need the complex conjugates of *A* and G. We define them using the following rule:

```
conjugateRule = 9A -> Ȧ, Ȧ -> A, G -> Ġ, Ġ -> G, Complex@0, n_D -> Complex@0, -nD=;
```

To represent some of the expressions in a more concise way, we introduce the following display rule:

```
displayRule =
  9Derivative@a__DAu_i_,j_E@__D :> SequenceFormATimes žž MapIndexedAD^#1_#2@@1DD-1 &, 8a<E, ui,jE,
   Derivative@a__DAA_i_E@__D :> SequenceFormATimes žž MapIndexedAD^#1_#2@@1DD &, 8a<E, AiE,
   u_i_,j_@__D :> ui,j, A_i_@__D :> Ai=;
```

à 6.1 Externally Excited Linearly Uncoupled Systems

Ÿ System of Equations

We consider the response of the following two-degree-of-freedom system with quadratic and cubic geometric nonlinearities to an external (additive) excitation:

```
eq61a = 9
u1''@tD + w1^2 u1@tD + 2 m1 u1¢@tD + D@V@u1@tD, u2@tDD, u1@tDD == F1 Cos@W t + t1D,
u2''@tD + w2^2 u2@tD + 2 m2 u2¢@tD + D@V@u1@tD, u2@tDD, u2@tDD == F2 Cos@W t + t2D=
```

```
8w1^2 u1@tD + 2 m1 u1¢@tD + u1''@tD + V^H1,0L@u1@tD, u2@tDD == Cos@t W + t1D F1,
 w2^2 u2@tD + 2 m2 u2¢@tD + u2''@tD + V^H0,1L@u1@tD, u2@tDD == Cos@t W + t2D F2<
```

where the potential function V possesses cubic and quartic nonlinearities. The most general such potential function can be expressed as follows:

```
basicModes = 8u1@tD, u2@tD<;
```

```
cubicTerms = Nest@Outer@Times, basicModes, #D &, basicModes, 2D •• Flatten •• Union
```

```
8u1@tD^3, u1@tD^2 u2@tD, u1@tD u2@tD^2, u2@tD^3<
```

```
quarticTerms = Outer@Times, basicModes, cubicTermsD •• Flatten •• Union
```

```
8u1@tD^4, u1@tD^3 u2@tD, u1@tD^2 u2@tD^2, u1@tD u2@tD^3, u2@tD^4<
```

```
potential =
 V -> HEvaluate@Sum@di cubicTerms@@iDD, 8i, Length@cubicTermsD<D + Sum@ai quarticTerms@@iDD,
       8i, Length@quarticTermsD<D •. Thread@basicModes -> 8#1, #2<DD &L
```

```
V ® H#1^4 a1 + #1^3 #2 a2 + #1^2 #2^2 a3 + #1 #2^3 a4 + #2^4 a5 + #1^3 d1 + #1^2 #2 d2 + #1 #2^2 d3 + #2^3 d4 &L
```

Approximate solutions of such a system can be obtained using a number of techniques, including the method of normal forms, the method of multiple scales, the Krylov-Bogoliubov-Mitropolsky technique, and the method of averaging. With these methods, one determines a set of ordinary-differential equations (modulation equations) that govern the time evolution of the amplitudes and phases of the modes participating in the response. If these equations are time-dependent, one usually transforms them into a set of autonomous equations. In this chapter, we describe how the method of multiple scales can be used to determine uniform expansions of this system, including the modulation equations.

Secondary resonances of nonlinear systems include subharmonic, superharmonic, and combination resonances. One or more of these resonances might be activated in the presence or absence of internal resonances. When a combination resonance is activated in the presence of an internal resonance, fractional harmonics might be excited.

For a consistent expansion, we first transform **eq61a** into a system of four first-order equations. To this end, we introduce the two states v_1@tD and v_2@tD defined by

```
vel = 8u1¢@tD -> v1@tD, u2¢@tD -> v2@tD<;
```

Substituting the velocity and acceleration terms, using **vel** and **potential**, into **eq61a** and combining the result with **vel**, we transform **eq61a** into the following set of four first-order equations:

```
eq61b = 8vel •. Rule -> Equal, eq61a •. D@vel, tD •. vel •. potential< •• Transpose •• Flatten
```

8u¢1@tD == v1@tD, w21 u1@tD + 3 d1 u1@tD2 + 4 a1 u1@tD3 + 2 d2 u1@tD u2@tD + 3 a2 u1@tD2 u2@tD +
d3 u2@tD2 + 2 a3 u1@tD u2@tD2 + a4 u2@tD3 + 2 m1 v1@tD + v¢1@tD == Cos@t W + t1D F1,
u¢2@tD == v2@tD, d2 u1@tD2 + a2 u1@tD3 + w22 u2@tD + 2 d3 u1@tD u2@tD + 2 a3 u1@tD2 u2@tD +
3 d4 u2@tD2 + 3 a4 u1@tD u2@tD2 + 4 a5 u2@tD3 + 2 m2 v2@tD + v¢2@tD == Cos@t W + t2D F2<

We seek a uniform second-order expansion of the solution of **eq61b** in the form

```
solRule =
  9ui_ -> ISumAej ui,j@#1, #2, #3D, 8j, 3<E &M, vi_ -> ISumAej vi,j@#1, #2, #3D, 8j, 3<E &M=;
```

and set

```
maxOrder = 2;
```

The damping and forcing terms in **eq61b** need to be scaled so that they balance the influence of the nonlinearity. The scaling of the forcing depends on the type of resonance. Next, we treat different resonance cases in the following sections.

Ÿ 6.1.1 W » w2 and w2 » 2 w1

In this case, we have a combination of a primary resonance of the second mode, a subharmonic resonance of order one-half of the first mode, and a two-to-one internal resonance. In order to bring the effects of damping, forcing, and nonlinearity at the same order, we let

```
scaling = 9mn_ -> e mn, F1 -> e f1, F2 -> e2 f2=;
```

To describe quantitatively the nearness of the resonances, we introduce the two detunings s1 and s2 defined by

```
ResonanceCond = 8w2 == 2 w1 + e s1, W == w2 + e s2<;
```

Then, we define the following lists:

```
omgList = 8w1, w2<;
```

```
omgRule = Solve@ResonanceCond, 8W, #< •• FlattenD@@1DD & •ž omgList •• Reverse
```

98W ® e s1 + e s2 + 2 w1, w2 ® e s1 + 2 w1<, 9W ® e s2 + w2, w1 ® $\frac{1}{2}$ H- e s1 + w2L==

Using the time scales T_0, T_1, and T_2, we express the time derivative as

```
multiScales = 8ui_@tD -> ui žž timeScales, vi_@tD -> vi žž timeScales,
   Derivative@1D@u_D@tD -> dt@1D@u žž timeScalesD, t -> T0<;
```

Substituting the **scaling**, **multiScales**, and **solRule** into **eq61b**, expanding the result for small e, and discarding terms of order higher than e3, we obtain

```
eq611a = (eq61b /. scaling /. multiScales /. solRule // TrigToExp // ExpandAll) /. ε^(n_ /; n>3) -> 0;
```

Equating coefficients of like powers of ε, we obtain

```
eqEps = Rest[Thread[CoefficientList[Subtract @@ #, ε] == 0]] & /@ eq611a // Transpose;
```

To place the linear operator on one side and the nonhomogeneous terms on the other side, we define

```
eqOrder[i_] := ((#[[1]] & /@ eqEps[[1]] /. f_s_ -> 0 /. u_{k_,1} -> u_{k,i}) ==
    (#[[1]] & /@ eqEps[[1]] /. f_s_ -> 0 /. u_{k_,1} -> u_{k,i}) - (#[[1]] & /@ eqEps[[i]])) // Thread
```

Using the **eqOrder[i]** and **displayRule**, we rewrite **eqEps** in a concise way as

```
eqOrder[1] /. displayRule
eqOrder[2] /. displayRule
eqOrder[3] /. displayRule
```

$$\{D_0u_{1,1} - v_{1,1} == 0,\ D_0v_{1,1} + \omega_1^2 u_{1,1} == \frac{1}{2} E^{-I T_0 \Omega - I \tau_1} f_1 + \frac{1}{2} E^{I T_0 \Omega + I \tau_1} f_1,$$
$$D_0u_{2,1} - v_{2,1} == 0,\ D_0v_{2,1} + \omega_2^2 u_{2,1} == 0\}$$

$$\{D_0u_{1,2} - v_{1,2} == -(D_1u_{1,1}),\ D_0v_{1,2} + \omega_1^2 u_{1,2} == -(D_1v_{1,1}) - 3 \delta_1 u_{1,1}^2 - 2 \delta_2 u_{1,1} u_{2,1} - \delta_3 u_{2,1}^2 - 2 \mu_1 v_{1,1},$$
$$D_0u_{2,2} - v_{2,2} == -(D_1u_{2,1}),\ D_0v_{2,2} + \omega_2^2 u_{2,2} ==$$
$$-(D_1v_{2,1}) + \frac{1}{2} E^{-I T_0 \Omega - I \tau_2} f_2 + \frac{1}{2} E^{I T_0 \Omega + I \tau_2} f_2 - \delta_2 u_{1,1}^2 - 2 \delta_3 u_{1,1} u_{2,1} - 3 \delta_4 u_{2,1}^2 - 2 \mu_2 v_{2,1}\}$$

$$\{D_0u_{1,3} - v_{1,3} == -(D_1u_{1,2}) - D_2u_{1,1},$$
$$D_0v_{1,3} + \omega_1^2 u_{1,3} == -(D_1v_{1,2}) - D_2v_{1,1} - 4 \alpha_1 u_{1,1}^3 - 6 \delta_1 u_{1,1} u_{1,2} - 3 \alpha_2 u_{1,1}^2 u_{2,1} -$$
$$2 \delta_2 u_{1,2} u_{2,1} - 2 \alpha_3 u_{1,1} u_{2,1}^2 - \alpha_4 u_{2,1}^3 - 2 \delta_2 u_{1,1} u_{2,2} - 2 \delta_3 u_{2,1} u_{2,2} - 2 \mu_1 v_{1,2},$$
$$D_0u_{2,3} - v_{2,3} == -(D_1u_{2,2}) - D_2u_{2,1},\ D_0v_{2,3} + \omega_2^2 u_{2,3} == -(D_1v_{2,2}) - D_2v_{2,1} - \alpha_2 u_{1,1}^3 - 2 \delta_2 u_{1,1} u_{1,2} -$$
$$2 \alpha_3 u_{1,1}^2 u_{2,1} - 2 \delta_3 u_{1,2} u_{2,1} - 3 \alpha_4 u_{1,1} u_{2,1}^2 - 4 \alpha_5 u_{2,1}^3 - 2 \delta_3 u_{1,1} u_{2,2} - 6 \delta_4 u_{2,1} u_{2,2} - 2 \mu_2 v_{2,2}\}$$

■ First-Order Problem: Linear System

The first-order problem, **eqOrder[1]**, consists of two sets of uncoupled linear nonhomogeneous differential equations. Hence, the general solution of each set can be obtained by using the principle of superposition as the sum of any particular solution and the solution of the homogeneous equations. To determine the general solution of the homogeneous sets, we rewrite the homogeneous part of **eqOrder[1]** as

```
linearSys = #[[1]] & /@ eqOrder[1];
linearSys /. displayRule
```

$$\{D_0u_{1,1} - v_{1,1},\ D_0v_{1,1} + \omega_1^2 u_{1,1},\ D_0u_{2,1} - v_{2,1},\ D_0v_{2,1} + \omega_2^2 u_{2,1}\}$$

Next, we seek a solution of the **linearSys** in the form $u_{i,1} = P E^{I\,\omega_i\,T_0}$ and $v_{i,1} = Q E^{I\,\omega_i\,T_0}$ and obtain

```
expr1 =
 linearSys •. 9ui_,1 -> IP E^(I wi #1) &M, vi_,1 -> IQ E^(I wi #1) &M= •. Exp@a_D -> 1 •• Partition@#, 2D &
```

```
88-Q + I P w1, I Q w1 + P w1^2<, 8-Q + I P w2, I Q w2 + P w2^2<<
```

The coefficient matrices of **expr1** are

```
coefMat = Outer@D, #, 8P, Q<D & •ž expr1
```

```
888I w1, -1<, 8w1^2, I w1<<, 88I w2, -1<, 8w2^2, I w2<<<
```

and their adjoints are defined by

```
hermitian@mat_? MatrixQD := mat •. conjugateRule •• Transpose
```

Hence, the right and left eigenvectors of **coefMat** are

```
rightVec = # • #@@1DD & •ž HNullSpace@#D@@1DD & •ž coefMatL
```

```
881, I w1<, 81, I w2<<
```

```
leftVec = NullSpace@hermitian@#DD@@1DD & •ž coefMat
```

```
88-I w1, 1<, 8-I w2, 1<<
```

whose complex conjugates are

```
ccleftVec = leftVec •. conjugateRule
```

```
88I w1, 1<, 8I w2, 1<<
```

To determine particular solutions of the partial-differential equations **eqOrder[1]** using the *Mathematica* function **DSolve**, we first transform them into a set of ordinary-differential equations as

```
order1Eq = eqOrder@1D •. u_i_,1 -> Hui,1@#1D &L
```

```
9-v1,1@T0D + u¢1,1@T0D == 0, w1^2 u1,1@T0D + v¢1,1@T0D == 1/2 E^(-I T0 W-I t1) f1 + 1/2 E^(I T0 W+I t1) f1,
 -v2,1@T0D + u¢2,1@T0D == 0, w2^2 u2,1@T0D + v¢2,1@T0D == 0=
```

Then, the particular solutions of **order1Eq** are

```
sol1p =
 DSolve@order1Eq, 8u1,1@T0D, v1,1@T0D, u2,1@T0D, v2,1@T0D<, T0D@@1DD •. C@_D -> 0 •• Simplify
```

```
:u1,1@T0D ® - E^(-I HT0 W+t1L) H1 + E^(2 I HT0 W+t1L)L f1 / (2 HW^2 - w1^2L),
 v1,1@T0D ® - I E^(-I HT0 W+t1L) H-1 + E^(2 I HT0 W+t1L)L W f1 / (2 HW^2 - w1^2L), u2,1@T0D ® 0, v2,1@T0D ® 0>
```

To simplify some of the expressions, we let

```
fRule = 9fi_ -> 2 Li Iwi2 - W2M=;
```

and express $u_{i,1}$ and $v_{i,1}$ in pure function form as

```
sol1u = TableAui,1 -> FunctionA8T0, T1, T2<, Ai@T1, T2D Exp@I wi T0D + Ȧi@T1, T2D Exp@-I wi T0D +
      Hui,1@T0D •. sol1p •. fRule •• Simplify •• ExpandL •• EvaluateE, 8i, 2<E
```

$8u_{1,1}$ ® Function@8T_0, T_1, T_2<, $E^{-I T_0 W - I t_1} L_1 + E^{I T_0 W + I t_1} L_1 + E^{I T_0 w_1} A_1$@$T_1$, T_2D + $E^{-I T_0 w_1} \dot{A}_1$@T_1, T_2DD,
$u_{2,1}$ ® Function@8T_0, T_1, T_2<, $E^{I T_0 w_2} A_2$@T_1, T_2D + $E^{-I T_0 w_2} \dot{A}_2$@T_1, T_2DD<

```
sol1v =
 Table@vi,1 -> Function@8T0, T1, T2<, D@ui,1 žž timeScales, T0D •. sol1u •• EvaluateD, 8i, 2<D
```

$8v_{1,1}$ ® Function@8T_0, T_1, T_2<,
$-I E^{-I T_0 W - I t_1} W L_1 + I E^{I T_0 W + I t_1} W L_1 + I E^{I T_0 w_1} w_1 A_1$@$T_1$, T_2D $- I E^{-I T_0 w_1} w_1 \dot{A}_1$@$T_1$, T_2DD,
$v_{2,1}$ ® Function@8T_0, T_1, T_2<, $I E^{I T_0 w_2} w_2 A_2$@T_1, T_2D $- I E^{-I T_0 w_2} w_2 \dot{A}_2$@$T_1$, T_2DD<

Then, we combine them using the *Mathematica* function **Join** as

```
sol1 = Join@sol1u, sol1vD;
```

Ÿ Second-Order Problem

Substituting the first-order solution **sol1** into the second-order problem, **eqOrder[2]**, yields

```
order2Eq = eqOrder@2D •. sol1 •• ExpandAll;
order2Eq •. displayRule
```

$9D_0 u_{1,2} - v_{1,2} == -E^{I T_0 w_1} HD_1 A_1 L - E^{-I T_0 w_1} HD_1 \dot{A}_1 L,$
$D_0 v_{1,2} + w_1^2 u_{1,2} == -3 E^{2 I T_0 w_1} A_1^2 d_1 - 2 E^{I T_0 w_1 + I T_0 w_2} A_1 A_2 d_2 - E^{2 I T_0 w_2} A_2^2 d_3 -$
$6 E^{-I T_0 W - I t_1 + I T_0 w_1} A_1 d_1 L_1 - 6 E^{I T_0 W + I t_1 + I T_0 w_1} A_1 d_1 L_1 - 2 E^{-I T_0 W - I t_1 + I T_0 w_2} A_2 d_2 L_1 -$
$2 E^{I T_0 W + I t_1 + I T_0 w_2} A_2 d_2 L_1 - 6 d_1 L_1^2 - 3 E^{-2 I T_0 W - 2 I t_1} d_1 L_1^2 - 3 E^{2 I T_0 W + 2 I t_1} d_1 L_1^2 +$
$2 I E^{-I T_0 W - I t_1} W L_1 m_1 - 2 I E^{I T_0 W + I t_1} W L_1 m_1 - I E^{I T_0 w_1} HD_1 A_1 L w_1 + I E^{-I T_0 w_1} HD_1 \dot{A}_1 L w_1 -$
$2 I E^{I T_0 w_1} A_1 m_1 w_1 - 6 A_1 d_1 \dot{A}_1 - 2 E^{-I T_0 w_1 + I T_0 w_2} A_2 d_2 \dot{A}_1 - 6 E^{-I T_0 W - I t_1 - I T_0 w_1} d_1 L_1 \dot{A}_1 -$
$6 E^{I T_0 W + I t_1 - I T_0 w_1} d_1 L_1 \dot{A}_1 + 2 I E^{-I T_0 w_1} m_1 w_1 \dot{A}_1 - 3 E^{-2 I T_0 w_1} d_1 \dot{A}_1^2 - 2 E^{I T_0 w_1 - I T_0 w_2} A_1 d_2 \dot{A}_2 - 2 A_2 d_3 \dot{A}_2 -$
$2 E^{-I T_0 W - I t_1 - I T_0 w_2} d_2 L_1 \dot{A}_2 - 2 E^{I T_0 W + I t_1 - I T_0 w_2} d_2 L_1 \dot{A}_2 - 2 E^{-I T_0 w_1 - I T_0 w_2} d_2 \dot{A}_1 \dot{A}_2 - E^{-2 I T_0 w_2} d_3 \dot{A}_2^2,$
$D_0 u_{2,2} - v_{2,2} == -E^{I T_0 w_2} HD_1 A_2 L - E^{-I T_0 w_2} HD_1 \dot{A}_2 L,$
$D_0 v_{2,2} + w_2^2 u_{2,2} == \frac{1}{2} E^{-I T_0 W - I t_2} f_2 + \frac{1}{2} E^{I T_0 W + I t_2} f_2 - E^{2 I T_0 w_1} A_1^2 d_2 - 2 E^{I T_0 w_1 + I T_0 w_2} A_1 A_2 d_3 -$
$3 E^{2 I T_0 w_2} A_2^2 d_4 - 2 E^{-I T_0 W - I t_1 + I T_0 w_1} A_1 d_2 L_1 - 2 E^{I T_0 W + I t_1 + I T_0 w_1} A_1 d_2 L_1 -$
$2 E^{-I T_0 W - I t_1 + I T_0 w_2} A_2 d_3 L_1 - 2 E^{I T_0 W + I t_1 + I T_0 w_2} A_2 d_3 L_1 - 2 d_2 L_1^2 - E^{-2 I T_0 W - 2 I t_1} d_2 L_1^2 -$
$E^{2 I T_0 W + 2 I t_1} d_2 L_1^2 - I E^{I T_0 w_2} HD_1 A_2 L w_2 + I E^{-I T_0 w_2} HD_1 \dot{A}_2 L w_2 - 2 I E^{I T_0 w_2} A_2 m_2 w_2 -$
$2 A_1 d_2 \dot{A}_1 - 2 E^{-I T_0 w_1 + I T_0 w_2} A_2 d_3 \dot{A}_1 - 2 E^{-I T_0 W - I t_1 - I T_0 w_1} d_2 L_1 \dot{A}_1 - 2 E^{I T_0 W + I t_1 - I T_0 w_1} d_2 L_1 \dot{A}_1 -$
$E^{-2 I T_0 w_1} d_2 \dot{A}_1^2 - 2 E^{I T_0 w_1 - I T_0 w_2} A_1 d_3 \dot{A}_2 - 6 A_2 d_4 \dot{A}_2 - 2 E^{-I T_0 W - I t_1 - I T_0 w_2} d_3 L_1 \dot{A}_2 -$
$2 E^{I T_0 W + I t_1 - I T_0 w_2} d_3 L_1 \dot{A}_2 + 2 I E^{-I T_0 w_2} m_2 w_2 \dot{A}_2 - 2 E^{-I T_0 w_1 - I T_0 w_2} d_3 \dot{A}_1 \dot{A}_2 - 3 E^{-2 I T_0 w_2} d_4 \dot{A}_2^2 =$

Any particular solution of **order2Eq** contains secular terms and small-divisor terms, making the expansion nonuniform. For a uniform expansion, we choose $D_1 A_1$ and $D_1 A_2$ to eliminate these terms. To accomplish this, we first convert the small-divisor terms into secular terms using the rule

```
expRule1@i_D := Exp@a_D :> Exp@Expand@a •. omgRule@@iDDD •. e T0 -> T1D
```

To eliminate the terms that produce secular terms (i.e., determine the solvability conditions) from **order2Eq**, we collect the terms proportional to $E^{I\,w_1 T_0}$ and $E^{I\,w_2 T_0}$ and obtain

```
ST11 = Coefficient@order2Eq@@#, 2DD •. expRule1@1D, Exp@I w1 T0DD & •ž 81, 2<;
ST11 •. displayRule
```

```
8-HD1A1L, -I HD1A1L w1 - 2 I A1 m1 w1 - 2 E^(I T1 s1) A2 d2 Ā1 - 6 E^(I T1 s1+I T1 s2+I t1) d1 L1 Ā1<
```

```
ST12 = Coefficient@order2Eq@@#, 2DD •. expRule1@2D, Exp@I w2 T0DD & •ž 83, 4<;
ST12 •. displayRule
```

```
9-HD1A2L, 1/2 E^(I T1 s2+I t2) f2 - E^(-I T1 s1) A1^2 d2 - I HD1A2L w2 - 2 I A2 m2 w2=
```

Then, the solvability conditions, conditions for the elimination of the terms that produce secular terms, demand that **ST11** and **ST12** be orthogonal to every solution of the corresponding adjoint problem, namely, the components of **ccleftVec**. The result is

```
SCond1 = 8ccleftVec@@1DD.ST11 == 0, ccleftVec@@2DD.ST12 == 0<;
SCond1 •. displayRule
```

```
9-2 I HD1A1L w1 - 2 I A1 m1 w1 - 2 E^(I T1 s1) A2 d2 Ā1 - 6 E^(I T1 s1+I T1 s2+I t1) d1 L1 Ā1 == 0,
 1/2 E^(I T1 s2+I t2) f2 - E^(-I T1 s1) A1^2 d2 - 2 I HD1A2L w2 - 2 I A2 m2 w2 == 0=
```

Solving **SCond1** for $D_1 A_1$ and $D_1 A_2$ yields

```
SCond1Rule1 = SolveASCond1, 9A1^H1,0L@T1, T2D, A2^H1,0L@T1, T2D=E@@1DD •• ExpandAll;
SCond1Rule1 •. displayRule
```

```
:D1A1 ® -A1 m1 + (I E^(I T1 s1) A2 d2 Ā1)/w1 + (3 I E^(I T1 s1+I T1 s2+I t1) d1 L1 Ā1)/w1 ,
 D1A2 ® -A2 m2 - (I E^(I T1 s2+I t2) f2)/(4 w2) + (I E^(-I T1 s1) A1^2 d2)/(2 w2) >
```

With these conditions, **order2Eq** are solvable. However, their solutions are not unique. To render them unique, we demand that they be orthogonal to solutions of the corresponding adjoint problems; that is, we damand that $\{u_{1,2}, v_{1,2}\}$ be orthogonal to **ccleftVec[[1]]** and $\{u_{2,2}, v_{2,2}\}$ be orthogonal to **ccleftVec[[2]]**.

Next, we use the solvability conditions to eliminate $D_1 A_1$ and $D_1 A_2$ from **order2Eq**. To simplify the resulting equations using *Mathematica*, we replace the scale T_1 in **SCond1Rule1** with ϵT_0. To this end, we express the detuning parameters σ_i in terms of the ω_i and Ω as

```
sigRule = Solve@ResonanceCond, 8s1, s2<D@@1DD
```

$$9s_1 ® -\frac{2\,w_1 - w_2}{e},\ s_2 ® -\frac{-W + w_2}{e} =$$

and replace T_1 with e T_0 using the rule

```
expRule2 = Exp@a_D :> Exp@a •. T1 -> e T0 •. sigRule •• ExpandD;
```

Using the **expRule2**, we rewrite **SCond1Rule1** as

```
SCond1Rule2 = SCond1Rule1 •. expRule2;
SCond1Rule2 •. displayRule
```

$$:D_1A_1 ® -A_1\,m_1 + \frac{I\,E^{-2\,I\,T_0\,w_1 + I\,T_0\,w_2}\,A_2\,d_2\,\dot{A}_1}{w_1} + \frac{3\,I\,E^{I\,T_0\,W + I\,t_1 - 2\,I\,T_0\,w_1}\,d_1\,L_1\,\dot{A}_1}{w_1},$$

$$D_1A_2 ® -A_2\,m_2 - \frac{I\,E^{I\,T_0\,W + I\,t_2 - I\,T_0\,w_2}\,f_2}{4\,w_2} + \frac{I\,E^{2\,I\,T_0\,w_1 - I\,T_0\,w_2}\,A_1^2\,d_2}{2\,w_2} >$$

whose complex conjugate is

```
ccSCond1Rule2 = SCond1Rule2 •. conjugateRule;
```

Substituting this form for the solvability conditions into **order2Eq** yields

```
order2Eqm = order2Eq •. SCond1Rule2 •. ccSCond1Rule2 •• ExpandAll;
order2Eqm •. displayRule
```

$$9D_0 u_{1,2} - v_{1,2} == E^{I T_0 w_1} A_1 m_1 + \frac{3 I E^{-I T_0 W - I t_1 + I T_0 w_1} A_1 d_1 L_1}{w_1} + E^{-I T_0 w_1} m_1 \dot{A}_1 - \frac{I E^{-I T_0 w_1 + I T_0 w_2} A_2 d_2 \dot{A}_1}{w_1} - \frac{3 I E^{I T_0 W + I t_1 - I T_0 w_1} d_1 L_1 \dot{A}_1}{w_1} + \frac{I E^{I T_0 w_1 - I T_0 w_2} A_1 d_2 \dot{A}_2}{w_1},$$

$$D_0 v_{1,2} + w_1^2 u_{1,2} == -3 E^{2 I T_0 w_1} A_1^2 d_1 - 2 E^{I T_0 w_1 + I T_0 w_2} A_1 A_2 d_2 - E^{2 I T_0 w_2} A_2^2 d_3 - 3 E^{-I T_0 W - I t_1 + I T_0 w_1} A_1 d_1 L_1 - 6 E^{I T_0 W + I t_1 + I T_0 w_1} A_1 d_1 L_1 - 2 E^{-I T_0 W - I t_1 + I T_0 w_2} A_2 d_2 L_1 - 2 E^{I T_0 W + I t_1 + I T_0 w_2} A_2 d_2 L_1 - 6 d_1 L_1^2 - 3 E^{-2 I T_0 W - 2 I t_1} d_1 L_1^2 - 3 E^{2 I T_0 W + 2 I t_1} d_1 L_1^2 + 2 I E^{-I T_0 W - I t_1} W L_1 m_1 - 2 I E^{I T_0 W + I t_1} W L_1 m_1 - I E^{I T_0 w_1} A_1 m_1 w_1 - 6 A_1 d_1 \dot{A}_1 - E^{-I T_0 w_1 + I T_0 w_2} A_2 d_2 \dot{A}_1 - 6 E^{-I T_0 W - I t_1 - I T_0 w_1} d_1 L_1 \dot{A}_1 - 3 E^{I T_0 W + I t_1 - I T_0 w_1} d_1 L_1 \dot{A}_1 + I E^{-I T_0 w_1} m_1 w_1 \dot{A}_1 - 3 E^{-2 I T_0 w_1} d_1 \dot{A}_1^2 - E^{I T_0 w_1 - I T_0 w_2} A_1 d_2 \dot{A}_2 - 2 A_2 d_3 \dot{A}_2 - 2 E^{-I T_0 W - I t_1 - I T_0 w_2} d_2 L_1 \dot{A}_2 - 2 E^{I T_0 W + I t_1 - I T_0 w_2} d_2 L_1 \dot{A}_2 - 2 E^{-I T_0 w_1 - I T_0 w_2} d_2 \dot{A}_1 \dot{A}_2 - E^{-2 I T_0 w_2} d_3 \dot{A}_2^2,$$

$$D_0 u_{2,2} - v_{2,2} == E^{I T_0 w_2} A_2 m_2 - \frac{I E^{-I T_0 W - I t_2} f_2}{4 w_2} + \frac{I E^{I T_0 W + I t_2} f_2}{4 w_2} - \frac{I E^{2 I T_0 w_1} A_1^2 d_2}{2 w_2} + \frac{I E^{-2 I T_0 w_1} d_2 \dot{A}_1^2}{2 w_2} + E^{-I T_0 w_2} m_2 \dot{A}_2,$$

$$D_0 v_{2,2} + w_2^2 u_{2,2} == \frac{1}{4} E^{-I T_0 W - I t_2} f_2 + \frac{1}{4} E^{I T_0 W + I t_2} f_2 - \frac{1}{2} E^{2 I T_0 w_1} A_1^2 d_2 - 2 E^{I T_0 w_1 + I T_0 w_2} A_1 A_2 d_3 - 3 E^{2 I T_0 w_2} A_2^2 d_4 - 2 E^{-I T_0 W - I t_1 + I T_0 w_1} A_1 d_2 L_1 - 2 E^{I T_0 W + I t_1 + I T_0 w_1} A_1 d_2 L_1 - 2 E^{-I T_0 W - I t_1 + I T_0 w_2} A_2 d_3 L_1 - 2 E^{I T_0 W + I t_1 + I T_0 w_2} A_2 d_3 L_1 - 2 d_2 L_1^2 - E^{-2 I T_0 W - 2 I t_1} d_2 L_1^2 - E^{2 I T_0 W + 2 I t_1} d_2 L_1^2 - I E^{I T_0 w_2} A_2 m_2 w_2 - 2 A_1 d_2 \dot{A}_1 - 2 E^{-I T_0 w_1 + I T_0 w_2} A_2 d_3 \dot{A}_1 - 2 E^{-I T_0 W - I t_1 - I T_0 w_1} d_2 L_1 \dot{A}_1 - 2 E^{I T_0 W + I t_1 - I T_0 w_1} d_2 L_1 \dot{A}_1 - \frac{1}{2} E^{-2 I T_0 w_1} d_2 \dot{A}_1^2 - 2 E^{I T_0 w_1 - I T_0 w_2} A_1 d_3 \dot{A}_2 - 6 A_2 d_4 \dot{A}_2 - 2 E^{-I T_0 W - I t_1 - I T_0 w_2} d_3 L_1 \dot{A}_2 - 2 E^{I T_0 W + I t_1 - I T_0 w_2} d_3 L_1 \dot{A}_2 + I E^{-I T_0 w_2} m_2 w_2 \dot{A}_2 - 2 E^{-I T_0 w_1 - I T_0 w_2} d_3 \dot{A}_1 \dot{A}_2 - 3 E^{-2 I T_0 w_2} d_4 \dot{A}_2^2=$$

Next, we use the method of undetermined coefficients to determine the particular solutions of **order2Eqm**. To accomplish this, we first identify the form of the nonhomogeneous terms. To this end, we let

```
basicH = TableA9Ai@T1, T2D E^(I wi T0), Ȧi@T1, T2D E^(-I wi T0)=, 8i, 2<E •• Flatten
```

$8E^{I T_0 w_1} A_1@T_1, T_2D, E^{-I T_0 w_1} \dot{A}_1@T_1, T_2D, E^{I T_0 w_2} A_2@T_1, T_2D, E^{-I T_0 w_2} \dot{A}_2@T_1, T_2D<$

```
collectForm = JoinAbasicH, 9L1 E^(I W T0+I t1), L1 E^(-I W T0-I t1)=E
```

$8E^{I T_0 w_1} A_1@T_1, T_2D, E^{-I T_0 w_1} \dot{A}_1@T_1, T_2D, E^{I T_0 w_2} A_2@T_1, T_2D, E^{-I T_0 w_2} \dot{A}_2@T_1, T_2D, E^{I T_0 W + I t_1} L_1, E^{-I T_0 W - I t_1} L_1<$

Then, the possible form of the nonhomogeneous terms in **order2Eqm** is

```
possibleTerms = JoinAcollectForm, 9E^(I W T0+I t2), E^(-I W T0-I t2)=,
  Outer@Times, collectForm, collectFormD •• Flatten •• UnionE
```

```
:E^(I T0 w1) A1@T1, T2D, E^(-I T0 w1) Ȧ1@T1, T2D, E^(I T0 w2) A2@T1, T2D, E^(-I T0 w2) Ȧ2@T1, T2D,
 E^(I T0 W+I t1) L1, E^(-I T0 W-I t1) L1, E^(I T0 W+I t2), E^(-I T0 W-I t2), L1^2, E^(-2 I T0 W-2 I t1) L1^2, E^(2 I T0 W+2 I t1) L1^2,
 E^(-I T0 W-I t1+I T0 w1) L1 A1@T1, T2D, E^(I T0 W+I t1+I T0 w1) L1 A1@T1, T2D, E^(2 I T0 w1) A1@T1, T2D^2,
 E^(-I T0 W-I t1+I T0 w2) L1 A2@T1, T2D, E^(I T0 W+I t1+I T0 w2) L1 A2@T1, T2D, E^(I T0 w1+I T0 w2) A1@T1, T2D A2@T1, T2D,
 E^(2 I T0 w2) A2@T1, T2D^2, E^(-I T0 W-I t1-I T0 w1) L1 Ȧ1@T1, T2D, E^(I T0 W+I t1-I T0 w1) L1 Ȧ1@T1, T2D,
 A1@T1, T2D Ȧ1@T1, T2D, E^(-I T0 w1+I T0 w2) A2@T1, T2D Ȧ1@T1, T2D, E^(-2 I T0 w1) Ȧ1@T1, T2D^2,
 E^(-I T0 W-I t1-I T0 w2) L1 Ȧ2@T1, T2D, E^(I T0 W+I t1-I T0 w2) L1 Ȧ2@T1, T2D, E^(I T0 w1-I T0 w2) A1@T1, T2D Ȧ2@T1, T2D,
 A2@T1, T2D Ȧ2@T1, T2D, E^(-I T0 w1-I T0 w2) Ȧ1@T1, T2D Ȧ2@T1, T2D, E^(-2 I T0 w2) Ȧ2@T1, T2D^2>
```

Using the method of undetermined coefficients, we assume that the $u_{i,2}$ and $v_{i,2}$ are linear combinations of these possible terms, substitute the result into the governing equations, **order2Eqm**, equate the coefficients of each of the possible terms on both sides of each equation, and obtain a system of nonhomogeneous algebraic equations governing the unknown coefficients. Associated with each possible term is a pair of two algebraic equations. The solutions of all pairs, except those associated with resonance terms, are unique. The solutions of the pairs corresponding to the possible resonance terms are rendered unique by requiring them to be orthogonal to the components of **ccleftVec**. The possible resonance terms are proportional to $E^{I\,\omega_1 T_0}$, $E^{I\,\omega_2 T_0}$, and their complex conjugates; that is,

```
ResonantTerms@i_D :=
 I# •. 8a_ •; a =!= 0 -> 1< & •ž IE^(-I wi T0) possibleTerms •. expRule1@iD •. Exp@_T0 + _.D -> 0MM
   possibleTerms •• Union •• Rest
```

```
RT = Array@ResonantTerms, 2D
```

```
88E^(I T0 w1) A1@T1, T2D, E^(I T0 W+I t1-I T0 w1) L1 Ȧ1@T1, T2D, E^(-I T0 w1+I T0 w2) A2@T1, T2D Ȧ1@T1, T2D<,
 8E^(I T0 W+I t2), E^(I T0 W+I t1) L1, E^(2 I T0 w1) A1@T1, T2D^2, E^(I T0 w2) A2@T1, T2D<<
```

Hence, the coefficients of **RT** in the nonhomogeneous terms in **order2Eqm[[1]]** are

```
r1Rule =
 MapIndexed@r1,#2@@1DD -> Coefficient@order2Eqm@@1, 2DD, #1D &, RT@@1DDD •. Exp@_T0 + _.D -> 0
```

```
:r1,1 ® m1, r1,2 ® - (3 I d1)/w1, r1,3 ® - (I d2)/w1 >
```

and the coefficients of **RT** in the nonhomogeneous terms in **order2Eqm[[2]]** are

```
r2Rule =
 MapIndexed@r2,#2@@1DD -> Coefficient@order2Eqm@@2, 2DD, #1D &, RT@@1DDD •. Exp@_T0 + _.D -> 0
```

```
8r2,1 ® -I m1 w1, r2,2 ® -3 d1, r2,3 ® -d2<
```

It follows from

Table@ccleftVec@@1DD.8r$_{1,i}$, r$_{2,i}$<, 8i, 3<D •. r1Rule •. r2Rule

80, 0, 0<

that the parts **r1Rule** and **r2Rule** are orthogonal to the solution of the adjoint, the first component of **ccleftVec**, as they should.

Similarly, the coefficients of **RT** in the nonhomogeneous terms in **order2Eqm[[3]]** are

r3Rule = MapIndexed@r$_{3,\#2@@1DD}$ -> Coefficient@order2Eqm@@3, 2DD, #1D &, RT@@2DDD

$:r_{3,1} ® \frac{I f_2}{4 w_2}, r_{3,2} ® 0, r_{3,3} ® -\frac{I d_2}{2 w_2}, r_{3,4} ® m_2>$

and the coefficients of **RT** in the nonhomogeneous terms in **order2Eqm[[4]]** are

r4Rule =
MapIndexed@r$_{4,\#2@@1DD}$ -> Coefficient@order2Eqm@@4, 2DD, #1D &, RT@@2DDD •. Exp@_T$_0$ + _.D -> 0

$9r_{4,1} ® \frac{f_2}{4}, r_{4,2} ® 0, r_{4,3} ® -\frac{d_2}{2}, r_{4,4} ® -I m_2 w_2=$

Again, it follows from

Table@ccleftVec@@2DD.8r$_{3,i}$, r$_{4,i}$<, 8i, 4<D •. r3Rule •. r4Rule

80, 0, 0, 0<

that the vector 8$r_{3,i}$, $r_{4,i}$< is orthogonal to the solution of the adjoint, the second component of **ccleftVec**, as it should.

The complement of the resonance terms **RT** in **possibleTerms** yields the nonresonance terms; that is,

NRT = Complement@possibleTerms, Join@#, # •. conjugateRuleDD & •ž RT

$::E^{-I T_0 W-I t_2}, E^{I T_0 W+I t_2}, E^{-I T_0 W-I t_1} L_1, E^{I T_0 W+I t_1} L_1, L_1^2, E^{-2 I T_0 W-2 I t_1} L_1^2,$
$E^{2 I T_0 W+2 I t_1} L_1^2, E^{I T_0 W+I t_1+I T_0 w_1} L_1 A_1@T_1, T_2D, E^{2 I T_0 w_1} A_1@T_1, T_2D^2, E^{I T_0 w_2} A_2@T_1, T_2D,$
$E^{-I T_0 W-I t_1+I T_0 w_2} L_1 A_2@T_1, T_2D, E^{I T_0 W+I t_1+I T_0 w_2} L_1 A_2@T_1, T_2D, E^{I T_0 w_1+I T_0 w_2} A_1@T_1, T_2D A_2@T_1, T_2D,$
$E^{2 I T_0 w_2} A_2@T_1, T_2D^2, E^{-I T_0 W-I t_1-I T_0 w_1} L_1 \dot{A}_1@T_1, T_2D, A_1@T_1, T_2D \dot{A}_1@T_1, T_2D, E^{-2 I T_0 w_1} \dot{A}_1@T_1, T_2D^2,$
$E^{-I T_0 w_2} \dot{A}_2@T_1, T_2D, E^{-I T_0 W-I t_1-I T_0 w_2} L_1 \dot{A}_2@T_1, T_2D, E^{I T_0 W+I t_1-I T_0 w_2} L_1 \dot{A}_2@T_1, T_2D,$
$A_2@T_1, T_2D \dot{A}_2@T_1, T_2D, E^{-I T_0 w_1-I T_0 w_2} \dot{A}_1@T_1, T_2D \dot{A}_2@T_1, T_2D, E^{-2 I T_0 w_2} \dot{A}_2@T_1, T_2D^2>,$

$:L_1^2, E^{-2 I T_0 W-2 I t_1} L_1^2, E^{2 I T_0 W+2 I t_1} L_1^2, E^{I T_0 w_1} A_1@T_1, T_2D, E^{-I T_0 W-I t_1+I T_0 w_1} L_1 A_1@T_1, T_2D,$
$E^{I T_0 W+I t_1+I T_0 w_1} L_1 A_1@T_1, T_2D, E^{-I T_0 W-I t_1+I T_0 w_2} L_1 A_2@T_1, T_2D, E^{I T_0 W+I t_1+I T_0 w_2} L_1 A_2@T_1, T_2D,$
$E^{I T_0 w_1+I T_0 w_2} A_1@T_1, T_2D A_2@T_1, T_2D, E^{2 I T_0 w_2} A_2@T_1, T_2D^2, E^{-I T_0 w_1} \dot{A}_1@T_1, T_2D,$
$E^{-I T_0 W-I t_1-I T_0 w_1} L_1 \dot{A}_1@T_1, T_2D, E^{I T_0 W+I t_1-I T_0 w_1} L_1 \dot{A}_1@T_1, T_2D, A_1@T_1, T_2D \dot{A}_1@T_1, T_2D,$
$E^{-I T_0 w_1+I T_0 w_2} A_2@T_1, T_2D \dot{A}_1@T_1, T_2D, E^{-I T_0 W-I t_1-I T_0 w_2} L_1 \dot{A}_2@T_1, T_2D,$
$E^{I T_0 W+I t_1-I T_0 w_2} L_1 \dot{A}_2@T_1, T_2D, E^{I T_0 w_1-I T_0 w_2} A_1@T_1, T_2D \dot{A}_2@T_1, T_2D,$
$A_2@T_1, T_2D \dot{A}_2@T_1, T_2D, E^{-I T_0 w_1-I T_0 w_2} \dot{A}_1@T_1, T_2D \dot{A}_2@T_1, T_2D, E^{-2 I T_0 w_2} \dot{A}_2@T_1, T_2D^2>>$

To implement the method of undetermined coefficients, we associate with each possible resonance term an unknown coefficient using the rules

```
RTsymbolList1 = Table@G_{1,j}, 8j, Length@RT@@1DDD<D
```

8$G_{1,1}$, $G_{1,2}$, $G_{1,3}$<

```
RTsymbolList2 = Table@G_{3,j}, 8j, Length@RT@@2DDD<D
```

8$G_{3,1}$, $G_{3,2}$, $G_{3,3}$, $G_{3,4}$<

And, we associate with each possible nonresonance term two unknown coefficients using the rules

```
NRTsymbolList1@i_D = Table@L_{i,j}, 8j, Length@NRT@@1DDD<D
```

8$L_{i,1}$, $L_{i,2}$, $L_{i,3}$, $L_{i,4}$, $L_{i,5}$, $L_{i,6}$, $L_{i,7}$, $L_{i,8}$, $L_{i,9}$, $L_{i,10}$, $L_{i,11}$,
$L_{i,12}$, $L_{i,13}$, $L_{i,14}$, $L_{i,15}$, $L_{i,16}$, $L_{i,17}$, $L_{i,18}$, $L_{i,19}$, $L_{i,20}$, $L_{i,21}$, $L_{i,22}$, $L_{i,23}$<

```
NRTsymbolList2@i_D = Table@L_{i+2,j}, 8j, Length@NRT@@2DDD<D
```

8$L_{2+i,1}$, $L_{2+i,2}$, $L_{2+i,3}$, $L_{2+i,4}$, $L_{2+i,5}$, $L_{2+i,6}$, $L_{2+i,7}$, $L_{2+i,8}$, $L_{2+i,9}$, $L_{2+i,10}$, $L_{2+i,11}$,
$L_{2+i,12}$, $L_{2+i,13}$, $L_{2+i,14}$, $L_{2+i,15}$, $L_{2+i,16}$, $L_{2+i,17}$, $L_{2+i,18}$, $L_{2+i,19}$, $L_{2+i,20}$, $L_{2+i,21}$<

Accounting for the orthogonality conditions, 8$u_{1,2}$, $v_{1,2}$< must be orthogonal to **ccleftVec[[1]]** and 8$u_{2,2}$, $v_{2,2}$< must be orthogonal to **ccleftVec[[2]]**, for the resonance terms, **RT**, we express $u_{i,2}$ and $v_{i,2}$ in the following function form:

```
8ratio1, ratio2< = Table@-ccleftVec@@i, 1DD, 8i, 2<D
```

8-I w_1, -I w_2<

```
sol2Form = 8u_{1,2} -> Function@8T_0, T_1, T_2<,
      RTsymbolList1.RT@@1DD + HRTsymbolList1.RT@@1DD •. conjugateRuleL +
        NRTsymbolList1@1D.NRT@@1DD •• EvaluateD, v_{1,2} -> Function@8T_0, T_1, T_2<,
      ratio1 * RTsymbolList1.RT@@1DD + Hratio1 * RTsymbolList1.RT@@1DD •. conjugateRuleL +
        NRTsymbolList1@2D.NRT@@1DD •• EvaluateD, u_{2,2} -> Function@8T_0, T_1, T_2<,
      RTsymbolList2.RT@@2DD + HRTsymbolList2.RT@@2DD •. conjugateRuleL +
        NRTsymbolList2@1D.NRT@@2DD •• EvaluateD, v_{2,2} -> Function@8T_0, T_1, T_2<,
      ratio2 * RTsymbolList2.RT@@2DD + Hratio2 * RTsymbolList2.RT@@2DD •. conjugateRuleL +
        NRTsymbolList2@2D.NRT@@2DD •• EvaluateD<;
```

Substituting **sol2Form** into **order2Eqm** and collecting the coefficients of like **NRT**, we obtain two sets of algebraic equations as

```
eq611b = HCoefficient@Subtract žž # •. sol2Form, NRT@@1DDD •. Exp@_ T0 + _.D -> 0 & •ž
      order2Eqm@@81, 2<DD •• FlattenL == 0 •• Thread
```

$$8- I W L_{1,1} - L_{2,1} == 0,\ I W L_{1,2} - L_{2,2} == 0,\ -I W L_{1,3} - L_{2,3} == 0,\ I W L_{1,4} - L_{2,4} == 0,\ -L_{2,5} == 0,$$
$$-2 I W L_{1,6} - L_{2,6} == 0,\ 2 I W L_{1,7} - L_{2,7} == 0,\ I W L_{1,8} + I w_1 L_{1,8} - L_{2,8} == 0,\ 2 I w_1 L_{1,9} - L_{2,9} == 0,$$
$$I w_2 L_{1,10} - L_{2,10} == 0,\ -I W L_{1,11} + I w_2 L_{1,11} - L_{2,11} == 0,\ I W L_{1,12} + I w_2 L_{1,12} - L_{2,12} == 0,$$
$$I w_1 L_{1,13} + I w_2 L_{1,13} - L_{2,13} == 0,\ 2 I w_2 L_{1,14} - L_{2,14} == 0,\ -I W L_{1,15} - I w_1 L_{1,15} - L_{2,15} == 0,$$
$$-L_{2,16} == 0,\ -2 I w_1 L_{1,17} - L_{2,17} == 0,\ -I w_2 L_{1,18} - L_{2,18} == 0,\ -I W L_{1,19} - I w_2 L_{1,19} - L_{2,19} == 0,$$
$$I W L_{1,20} - I w_2 L_{1,20} - L_{2,20} == 0,\ -L_{2,21} == 0,\ -I w_1 L_{1,22} - I w_2 L_{1,22} - L_{2,22} == 0,$$
$$-2 I w_2 L_{1,23} - L_{2,23} == 0,\ w_1^2 L_{1,1} - I W L_{2,1} == 0,\ w_1^2 L_{1,2} + I W L_{2,2} == 0,$$
$$-2 I W m_1 + w_1^2 L_{1,3} - I W L_{2,3} == 0,\ 2 I W m_1 + w_1^2 L_{1,4} + I W L_{2,4} == 0,\ 6 d_1 + w_1^2 L_{1,5} == 0,$$
$$3 d_1 + w_1^2 L_{1,6} - 2 I W L_{2,6} == 0,\ 3 d_1 + w_1^2 L_{1,7} + 2 I W L_{2,7} == 0,\ 6 d_1 + w_1^2 L_{1,8} + I W L_{2,8} + I w_1 L_{2,8} == 0,$$
$$3 d_1 + w_1^2 L_{1,9} + 2 I w_1 L_{2,9} == 0,\ w_1^2 L_{1,10} + I w_2 L_{2,10} == 0,\ 2 d_2 + w_1^2 L_{1,11} - I W L_{2,11} + I w_2 L_{2,11} == 0,$$
$$2 d_2 + w_1^2 L_{1,12} + I W L_{2,12} + I w_2 L_{2,12} == 0,\ 2 d_2 + w_1^2 L_{1,13} + I w_1 L_{2,13} + I w_2 L_{2,13} == 0,$$
$$d_3 + w_1^2 L_{1,14} + 2 I w_2 L_{2,14} == 0,\ 6 d_1 + w_1^2 L_{1,15} - I W L_{2,15} - I w_1 L_{2,15} == 0,$$
$$6 d_1 + w_1^2 L_{1,16} == 0,\ 3 d_1 + w_1^2 L_{1,17} - 2 I w_1 L_{2,17} == 0,\ w_1^2 L_{1,18} - I w_2 L_{2,18} == 0,$$
$$2 d_2 + w_1^2 L_{1,19} - I W L_{2,19} - I w_2 L_{2,19} == 0,\ 2 d_2 + w_1^2 L_{1,20} + I W L_{2,20} - I w_2 L_{2,20} == 0,$$
$$2 d_3 + w_1^2 L_{1,21} == 0,\ 2 d_2 + w_1^2 L_{1,22} - I w_1 L_{2,22} - I w_2 L_{2,22} == 0,\ d_3 + w_1^2 L_{1,23} - 2 I w_2 L_{2,23} == 0<$$

```
eq611c = HCoefficient@Subtract žž # •. sol2Form, NRT@@2DDD •. Exp@_ T0 + _.D -> 0 & •ž
      order2Eqm@@83, 4<DD •• FlattenL == 0 •• Thread
```

$$8- L_{4,1} == 0,\ -2 I W L_{3,2} - L_{4,2} == 0,\ 2 I W L_{3,3} - L_{4,3} == 0,\ I w_1 L_{3,4} - L_{4,4} == 0,$$
$$-I W L_{3,5} + I w_1 L_{3,5} - L_{4,5} == 0,\ I W L_{3,6} + I w_1 L_{3,6} - L_{4,6} == 0,\ -I W L_{3,7} + I w_2 L_{3,7} - L_{4,7} == 0,$$
$$I W L_{3,8} + I w_2 L_{3,8} - L_{4,8} == 0,\ I w_1 L_{3,9} + I w_2 L_{3,9} - L_{4,9} == 0,\ 2 I w_2 L_{3,10} - L_{4,10} == 0,$$
$$-I w_1 L_{3,11} - L_{4,11} == 0,\ -I W L_{3,12} - I w_1 L_{3,12} - L_{4,12} == 0,\ I W L_{3,13} - I w_1 L_{3,13} - L_{4,13} == 0,\ -L_{4,14} == 0,$$
$$-I w_1 L_{3,15} + I w_2 L_{3,15} - L_{4,15} == 0,\ -I W L_{3,16} - I w_2 L_{3,16} - L_{4,16} == 0,\ I W L_{3,17} - I w_2 L_{3,17} - L_{4,17} == 0,$$
$$I w_1 L_{3,18} - I w_2 L_{3,18} - L_{4,18} == 0,\ -L_{4,19} == 0,\ -I w_1 L_{3,20} - I w_2 L_{3,20} - L_{4,20} == 0,$$
$$-2 I w_2 L_{3,21} - L_{4,21} == 0,\ 2 d_2 + w_2^2 L_{3,1} == 0,\ d_2 + w_2^2 L_{3,2} - 2 I W L_{4,2} == 0,\ d_2 + w_2^2 L_{3,3} + 2 I W L_{4,3} == 0,$$
$$w_2^2 L_{3,4} + I w_1 L_{4,4} == 0,\ 2 d_2 + w_2^2 L_{3,5} - I W L_{4,5} + I w_1 L_{4,5} == 0,\ 2 d_2 + w_2^2 L_{3,6} + I W L_{4,6} + I w_1 L_{4,6} == 0,$$
$$2 d_3 + w_2^2 L_{3,7} - I W L_{4,7} + I w_2 L_{4,7} == 0,\ 2 d_3 + w_2^2 L_{3,8} + I W L_{4,8} + I w_2 L_{4,8} == 0,$$
$$2 d_3 + w_2^2 L_{3,9} + I w_1 L_{4,9} + I w_2 L_{4,9} == 0,\ 3 d_4 + w_2^2 L_{3,10} + 2 I w_2 L_{4,10} == 0,\ w_2^2 L_{3,11} - I w_1 L_{4,11} == 0,$$
$$2 d_2 + w_2^2 L_{3,12} - I W L_{4,12} - I w_1 L_{4,12} == 0,\ 2 d_2 + w_2^2 L_{3,13} + I W L_{4,13} - I w_1 L_{4,13} == 0,\ 2 d_2 + w_2^2 L_{3,14} == 0,$$
$$2 d_3 + w_2^2 L_{3,15} - I w_1 L_{4,15} + I w_2 L_{4,15} == 0,\ 2 d_3 + w_2^2 L_{3,16} - I W L_{4,16} - I w_2 L_{4,16} == 0,$$
$$2 d_3 + w_2^2 L_{3,17} + I W L_{4,17} - I w_2 L_{4,17} == 0,\ 2 d_3 + w_2^2 L_{3,18} + I w_1 L_{4,18} - I w_2 L_{4,18} == 0,$$
$$6 d_4 + w_2^2 L_{3,19} == 0,\ 2 d_3 + w_2^2 L_{3,20} - I w_1 L_{4,20} - I w_2 L_{4,20} == 0,\ 3 d_4 + w_2^2 L_{3,21} - 2 I w_2 L_{4,21} == 0<$$

Solving **eq611b** for the **NRTsymbolList1**, we obtain

```
coef11 = Solve@eq611b, Array@NRTsymbolList1, 2D •• FlattenD@@1DD
```

$$\{L_{2,5}\to 0,\ L_{2,16}\to 0,\ L_{2,21}\to 0,\ L_{1,5}\to -\frac{6\,d_1}{w_1^2},\ L_{1,16}\to -\frac{6\,d_1}{w_1^2},\ L_{1,21}\to -\frac{2\,d_3}{w_1^2},\ L_{2,1}\to 0,\ L_{2,2}\to 0,$$

$$L_{2,3}\to -\frac{2\,W^2\,m_1}{W^2-w_1^2},\ L_{2,4}\to -\frac{2\,W^2\,m_1}{W^2-w_1^2},\ L_{2,6}\to -\frac{6\,I\,W\,d_1}{4\,W^2-w_1^2},\ L_{2,7}\to \frac{6\,I\,W\,d_1}{4\,W^2-w_1^2},\ L_{2,9}\to \frac{2\,I\,d_1}{w_1},$$

$$L_{2,10}\to 0,\ L_{2,14}\to -\frac{2\,I\,d_3\,w_2}{w_1^2-4\,w_2^2},\ L_{2,17}\to -\frac{2\,I\,d_1}{w_1},\ L_{2,18}\to 0,\ L_{2,23}\to \frac{2\,I\,d_3\,w_2}{w_1^2-4\,w_2^2},\ L_{1,1}\to 0,\ L_{1,2}\to 0,$$

$$L_{1,3}\to -\frac{2\,I\,W\,m_1}{W^2-w_1^2},\ L_{1,4}\to \frac{2\,I\,W\,m_1}{W^2-w_1^2},\ L_{1,6}\to \frac{3\,d_1}{4\,W^2-w_1^2},\ L_{1,7}\to \frac{3\,d_1}{4\,W^2-w_1^2},\ L_{1,9}\to \frac{d_1}{w_1^2},\ L_{1,10}\to 0,$$

$$L_{1,14}\to -\frac{d_3}{w_1^2-4\,w_2^2},\ L_{1,17}\to \frac{d_1}{w_1^2},\ L_{1,18}\to 0,\ L_{1,23}\to -\frac{d_3}{w_1^2-4\,w_2^2},\ L_{2,8}\to \frac{6\,I\,d_1\,(W+w_1)}{W\,(W+2\,w_1)},$$

$$L_{2,11}\to -\frac{2\,I\,d_2\,(W-w_2)}{W^2-w_1^2-2\,W\,w_2+w_2^2},\ L_{2,12}\to \frac{2\,I\,d_2\,(W+w_2)}{W^2-w_1^2+2\,W\,w_2+w_2^2},\ L_{2,13}\to \frac{2\,I\,d_2\,(w_1+w_2)}{w_2\,(2\,w_1+w_2)},$$

$$L_{2,15}\to -\frac{6\,I\,d_1\,(W+w_1)}{W\,(W+2\,w_1)},\ L_{2,19}\to -\frac{2\,I\,d_2\,(W+w_2)}{W^2-w_1^2+2\,W\,w_2+w_2^2},\ L_{2,20}\to \frac{2\,I\,d_2\,(W-w_2)}{W^2-w_1^2-2\,W\,w_2+w_2^2},$$

$$L_{2,22}\to -\frac{2\,I\,d_2\,(w_1+w_2)}{w_2\,(2\,w_1+w_2)},\ L_{1,8}\to \frac{6\,d_1}{W\,(W+2\,w_1)},\ L_{1,11}\to -\frac{2\,d_2}{-W^2+w_1^2+2\,W\,w_2-w_2^2},$$

$$L_{1,12}\to -\frac{2\,d_2}{-W^2+w_1^2-2\,W\,w_2-w_2^2},\ L_{1,13}\to \frac{2\,d_2}{w_2\,(2\,w_1+w_2)},\ L_{1,15}\to \frac{6\,d_1}{W\,(W+2\,w_1)},$$

$$L_{1,19}\to -\frac{2\,d_2}{-W^2+w_1^2-2\,W\,w_2-w_2^2},\ L_{1,20}\to -\frac{2\,d_2}{-W^2+w_1^2+2\,W\,w_2-w_2^2},\ L_{1,22}\to \frac{2\,d_2}{w_2\,(2\,w_1+w_2)}\}$$

Solving **eq611c** for the **NRTsymbolList2**, we obtain

```
coef12 = Solve@eq611c, Array@NRTsymbolList2, 2D •• FlattenD@@1DD
```

$$\{L_{4,1}\to 0,\ L_{4,14}\to 0,\ L_{4,19}\to 0,\ L_{3,1}\to -\frac{2\,d_2}{w_2^2},\ L_{3,14}\to -\frac{2\,d_2}{w_2^2},\ L_{3,19}\to -\frac{6\,d_4}{w_2^2},\ L_{4,2}\to -\frac{2\,I\,W\,d_2}{4\,W^2-w_2^2},$$

$$L_{4,3}\to \frac{2\,I\,W\,d_2}{4\,W^2-w_2^2},\ L_{4,4}\to 0,\ L_{4,10}\to \frac{2\,I\,d_4}{w_2},\ L_{4,11}\to 0,\ L_{4,21}\to -\frac{2\,I\,d_4}{w_2},\ L_{3,2}\to \frac{d_2}{4\,W^2-w_2^2},$$

$$L_{3,3}\to \frac{d_2}{4\,W^2-w_2^2},\ L_{3,4}\to 0,\ L_{3,10}\to \frac{d_4}{w_2^2},\ L_{3,11}\to 0,\ L_{3,21}\to \frac{d_4}{w_2^2},\ L_{4,5}\to -\frac{2\,I\,d_2\,(W-w_1)}{W^2-2\,W\,w_1+w_1^2-w_2^2},$$

$$L_{4,6}\to \frac{2\,I\,d_2\,(W+w_1)}{W^2+2\,W\,w_1+w_1^2-w_2^2},\ L_{4,7}\to -\frac{2\,I\,d_3\,(W-w_2)}{W\,(W-2\,w_2)},\ L_{4,8}\to \frac{2\,I\,d_3\,(W+w_2)}{W\,(W+2\,w_2)},\ L_{4,9}\to \frac{2\,I\,d_3\,(w_1+w_2)}{w_1\,(w_1+2\,w_2)},$$

$$L_{4,12}\to -\frac{2\,I\,d_2\,(W+w_1)}{W^2+2\,W\,w_1+w_1^2-w_2^2},\ L_{4,13}\to \frac{2\,I\,d_2\,(W-w_1)}{W^2-2\,W\,w_1+w_1^2-w_2^2},\ L_{4,15}\to -\frac{2\,I\,d_3\,(w_1-w_2)}{w_1\,(w_1-2\,w_2)},$$

$$L_{4,16}\to -\frac{2\,I\,d_3\,(W+w_2)}{W\,(W+2\,w_2)},\ L_{4,17}\to \frac{2\,I\,d_3\,(W-w_2)}{W\,(W-2\,w_2)},\ L_{4,18}\to \frac{2\,I\,d_3\,(w_1-w_2)}{w_1\,(w_1-2\,w_2)},\ L_{4,20}\to -\frac{2\,I\,d_3\,(w_1+w_2)}{w_1\,(w_1+2\,w_2)},$$

$$L_{3,5}\to \frac{2\,d_2}{W^2-2\,W\,w_1+w_1^2-w_2^2},\ L_{3,6}\to \frac{2\,d_2}{W^2+2\,W\,w_1+w_1^2-w_2^2},\ L_{3,7}\to \frac{2\,d_3}{W\,(W-2\,w_2)},\ L_{3,8}\to \frac{2\,d_3}{W\,(W+2\,w_2)},$$

$$L_{3,9}\to \frac{2\,d_3}{w_1\,(w_1+2\,w_2)},\ L_{3,12}\to \frac{2\,d_2}{W^2+2\,W\,w_1+w_1^2-w_2^2},\ L_{3,13}\to \frac{2\,d_2}{W^2-2\,W\,w_1+w_1^2-w_2^2},\ L_{3,15}\to \frac{2\,d_3}{w_1\,(w_1-2\,w_2)},$$

$$L_{3,16}\to \frac{2\,d_3}{W\,(W+2\,w_2)},\ L_{3,17}\to \frac{2\,d_3}{W\,(W-2\,w_2)},\ L_{3,18}\to \frac{2\,d_3}{w_1\,(w_1-2\,w_2)},\ L_{3,20}\to \frac{2\,d_3}{w_1\,(w_1+2\,w_2)}\}$$

Substituting **sol2Form** into **order2Eqm** and collecting the coefficients of like **RT**, we obtain two sets of algebraic equations as

```
eq1 = MapIndexed@Coefficient@order2Eqm@@1, 1DD •. sol2Form, #1D == r1, #2@@1DD &, RT@@1DDD •.
   Exp@_ T0 + _.D -> 0
```

$8 2\, I\, w_1\, G_{1,1} == r_{1,1},\ I\, W\, G_{1,2} == r_{1,2},\ I\, w_2\, G_{1,3} == r_{1,3} <$

```
eq2 = MapIndexed@Coefficient@order2Eqm@@3, 1DD •. sol2Form, #1D == r3, #2@@1DD &, RT@@2DDD •.
   Exp@_ T0 + _.D -> 0
```

$8 I\, W\, G_{3,1} + I\, w_2\, G_{3,1} == r_{3,1},\ I\, W\, G_{3,2} + I\, w_2\, G_{3,2} == r_{3,2},\ 2\, I\, w_1\, G_{3,3} + I\, w_2\, G_{3,3} == r_{3,3},\ 2\, I\, w_2\, G_{3,4} == r_{3,4} <$

Solving **eq1** and **eq2** for the **RTsymbolList1** and **RTsymbolList2**, we obtain

```
coef21 = Solve@eq1, RTsymbolList1D@@1DD •. r1Rule
```

$:G_{1,1} ® -\frac{I\, m_1}{2\, w_1},\ G_{1,2} ® -\frac{3\, d_1}{W\, w_1},\ G_{1,3} ® -\frac{d_2}{w_1\, w_2} >$

```
coef22 = Solve@eq2, RTsymbolList2D@@1DD •. r3Rule
```

$:G_{3,4} ® -\frac{I\, m_2}{2\, w_2},\ G_{3,1} ® \frac{f_2}{4\, w_2\, H W + w_2 L},\ G_{3,2} ® 0,\ G_{3,3} ® -\frac{d_2}{2\, w_2\, H 2\, w_1 + w_2 L} >$

Combining these coefficients using the function **Join** yields

```
coeffs = Join@coef11, coef12, coef21, coef22,
    coef21 •. conjugateRule, coef22 •. conjugateRuleD •. HomgRule@@1DD •. e -> 0L;
```

Then, we express the solution of the second-order equations in pure function form as

```
sol2 = sol2Form •. Function@8T0, T1, T2<, a_D :>
    Function@8T0, T1, T2< •• Evaluate, a •. coeffs •• Expand •• EvaluateD;
sol2 •. displayRule
```

$9u_{1,2} ® \text{FunctionA}8T_0,\ T_1,\ T_2<,$

$$-\frac{6\, d_1\, L_1^2}{w_1^2} + \frac{E^{-2 I T_0 W - 2 I t_1}\, d_1\, L_1^2}{5\, w_1^2} + \frac{E^{2 I T_0 W + 2 I t_1}\, d_1\, L_1^2}{5\, w_1^2} - \frac{4\, I\, E^{-I T_0 W - I t_1}\, L_1\, m_1}{3\, w_1} + \frac{4\, I\, E^{I T_0 W + I t_1}\, L_1\, m_1}{3\, w_1} -$$

$$\frac{3\, E^{-I T_0 W - I t_1 + I T_0 w_1}\, d_1\, L_1\, A_1}{2\, w_1^2} + \frac{3\, E^{I T_0 W + I t_1 + I T_0 w_1}\, d_1\, L_1\, A_1}{4\, w_1^2} - \frac{I\, E^{I T_0 w_1}\, m_1\, A_1}{2\, w_1} + \frac{E^{2 I T_0 w_1}\, d_1\, A_1^2}{w_1^2} -$$

$$\frac{2\, E^{-I T_0 W - I t_1 + I T_0 w_2}\, d_2\, L_1\, A_2}{w_1^2} + \frac{2\, E^{I T_0 W + I t_1 + I T_0 w_2}\, d_2\, L_1\, A_2}{15\, w_1^2} + \frac{E^{I T_0 w_1 + I T_0 w_2}\, d_2\, A_1\, A_2}{4\, w_1^2} + \frac{E^{2 I T_0 w_2}\, d_3\, A_2^2}{15\, w_1^2} +$$

$$\frac{3\, E^{-I T_0 W - I t_1 - I T_0 w_1}\, d_1\, L_1\, \dot{A}_1}{4\, w_1^2} - \frac{3\, E^{I T_0 W + I t_1 - I T_0 w_1}\, d_1\, L_1\, \dot{A}_1}{2\, w_1^2} + \frac{I\, E^{-I T_0 w_1}\, m_1\, \dot{A}_1}{2\, w_1} - \frac{6\, d_1\, A_1\, \dot{A}_1}{w_1^2} -$$

$$\frac{E^{-I T_0 w_1 + I T_0 w_2}\, d_2\, A_2\, \dot{A}_1}{2\, w_1^2} + \frac{E^{-2 I T_0 w_1}\, d_1\, \dot{A}_1^2}{w_1^2} + \frac{2\, E^{-I T_0 W - I t_1 - I T_0 w_2}\, d_2\, L_1\, \dot{A}_2}{15\, w_1^2} - \frac{2\, E^{I T_0 W + I t_1 - I T_0 w_2}\, d_2\, L_1\, \dot{A}_2}{w_1^2} -$$

$$\frac{E^{I T_0 w_1 - I T_0 w_2} d_2 A_1 \bar{A}_2}{2 w_1^2} - \frac{2 d_3 A_2 \bar{A}_2}{w_1^2} + \frac{E^{-I T_0 w_1 - I T_0 w_2} d_2 \bar{A}_1 \bar{A}_2}{4 w_1^2} + \frac{E^{-2 I T_0 w_2} d_3 \bar{A}_2^2}{15 w_1^2}\Big],$$

$$v_{1,2} \to \text{Function}\Big[\{T_0, T_1, T_2\}, -\frac{8}{3} E^{-I T_0 \Omega - I \tau_1} L_1 m_1 - \frac{8}{3} E^{I T_0 \Omega + I \tau_1} L_1 m_1 - \frac{4 I E^{-2 I T_0 \Omega - 2 I \tau_1} d_1 L_1^2}{5 w_1} +$$

$$\frac{4 I E^{2 I T_0 \Omega + 2 I \tau_1} d_1 L_1^2}{5 w_1} - \frac{1}{2} E^{I T_0 w_1} m_1 A_1 - \frac{3 I E^{-I T_0 \Omega - I \tau_1 + I T_0 w_1} d_1 L_1 A_1}{2 w_1} + \frac{9 I E^{I T_0 \Omega + I \tau_1 + I T_0 w_1} d_1 L_1 A_1}{4 w_1} +$$

$$\frac{2 I E^{2 I T_0 w_1} d_1 A_1^2}{w_1} + \frac{8 I E^{I T_0 \Omega + I \tau_1 + I T_0 w_2} d_2 L_1 A_2}{15 w_1} + \frac{3 I E^{I T_0 w_1 + I T_0 w_2} d_2 A_1 A_2}{4 w_1} + \frac{4 I E^{2 I T_0 w_2} d_3 A_2^2}{15 w_1} -$$

$$\frac{1}{2} E^{-I T_0 w_1} m_1 \bar{A}_1 - \frac{9 I E^{-I T_0 \Omega - I \tau_1 - I T_0 w_1} d_1 L_1 \bar{A}_1}{4 w_1} + \frac{3 I E^{I T_0 \Omega + I \tau_1 - I T_0 w_1} d_1 L_1 \bar{A}_1}{2 w_1} +$$

$$\frac{I E^{-I T_0 w_1 + I T_0 w_2} d_2 A_2 \bar{A}_1}{2 w_1} - \frac{2 I E^{-2 I T_0 w_1} d_1 \bar{A}_1^2}{w_1} - \frac{8 I E^{-I T_0 \Omega - I \tau_1 - I T_0 w_2} d_2 L_1 \bar{A}_2}{15 w_1} -$$

$$\frac{I E^{I T_0 w_1 - I T_0 w_2} d_2 A_1 \bar{A}_2}{2 w_1} - \frac{3 I E^{-I T_0 w_1 - I T_0 w_2} d_2 \bar{A}_1 \bar{A}_2}{4 w_1} - \frac{4 I E^{-2 I T_0 w_2} d_3 \bar{A}_2^2}{15 w_1}\Big],$$

$$u_{2,2} \to \text{Function}\Big[\{T_0, T_1, T_2\}, \frac{E^{-I T_0 \Omega - I \tau_2} f_2}{32 w_1^2} + \frac{E^{I T_0 \Omega + I \tau_2} f_2}{32 w_1^2} - \frac{d_2 L_1^2}{2 w_1^2} + \frac{E^{-2 I T_0 \Omega - 2 I \tau_1} d_2 L_1^2}{12 w_1^2} +$$

$$\frac{E^{2 I T_0 \Omega + 2 I \tau_1} d_2 L_1^2}{12 w_1^2} - \frac{2 E^{-I T_0 \Omega - I \tau_1 + I T_0 w_1} d_2 L_1 A_1}{3 w_1^2} + \frac{2 E^{I T_0 \Omega + I \tau_1 + I T_0 w_1} d_2 L_1 A_1}{5 w_1^2} - \frac{E^{2 I T_0 w_1} d_2 A_1^2}{16 w_1^2} -$$

$$\frac{E^{-I T_0 \Omega - I \tau_1 + I T_0 w_2} d_3 L_1 A_2}{2 w_1^2} + \frac{E^{I T_0 \Omega + I \tau_1 + I T_0 w_2} d_3 L_1 A_2}{6 w_1^2} - \frac{I E^{I T_0 w_2} m_2 A_2}{4 w_1} + \frac{2 E^{I T_0 w_1 + I T_0 w_2} d_3 A_1 A_2}{5 w_1^2} +$$

$$\frac{E^{2 I T_0 w_2} d_4 A_2^2}{4 w_1^2} + \frac{2 E^{-I T_0 \Omega - I \tau_1 - I T_0 w_1} d_2 L_1 \bar{A}_1}{5 w_1^2} - \frac{2 E^{I T_0 \Omega + I \tau_1 - I T_0 w_1} d_2 L_1 \bar{A}_1}{3 w_1^2} - \frac{d_2 A_1 \bar{A}_1}{2 w_1^2} -$$

$$\frac{2 E^{-I T_0 w_1 + I T_0 w_2} d_3 A_2 \bar{A}_1}{3 w_1^2} - \frac{E^{-2 I T_0 w_1} d_2 \bar{A}_1^2}{16 w_1^2} + \frac{E^{-I T_0 \Omega - I \tau_1 - I T_0 w_2} d_3 L_1 \bar{A}_2}{6 w_1^2} - \frac{E^{I T_0 \Omega + I \tau_1 - I T_0 w_2} d_3 L_1 \bar{A}_2}{2 w_1^2} +$$

$$\frac{I E^{-I T_0 w_2} m_2 \bar{A}_2}{4 w_1} - \frac{2 E^{I T_0 w_1 - I T_0 w_2} d_3 A_1 \bar{A}_2}{3 w_1^2} - \frac{3 d_4 A_2 \bar{A}_2}{2 w_1^2} + \frac{2 E^{-I T_0 w_1 - I T_0 w_2} d_3 \bar{A}_1 \bar{A}_2}{5 w_1^2} + \frac{E^{-2 I T_0 w_2} d_4 \bar{A}_2^2}{4 w_1^2}\Big],$$

$$v_{2,2} \to \text{Function}\Big[\{T_0, T_1, T_2\}, -\frac{I E^{-2 I T_0 \Omega - 2 I \tau_1} d_2 L_1^2}{3 w_1} + \frac{I E^{2 I T_0 \Omega + 2 I \tau_1} d_2 L_1^2}{3 w_1} + \frac{I E^{-I T_0 \Omega - I \tau_2} f_2 w_2}{32 w_1^2} -$$

$$\frac{I E^{I T_0 \Omega + I \tau_2} f_2 w_2}{32 w_1^2} + \frac{2 I E^{-I T_0 \Omega - I \tau_1 + I T_0 w_1} d_2 L_1 A_1}{3 w_1} + \frac{6 I E^{I T_0 \Omega + I \tau_1 + I T_0 w_1} d_2 L_1 A_1}{5 w_1} +$$

$$\frac{I E^{2 I T_0 w_1} d_2 w_2 A_1^2}{16 w_1^2} + \frac{2 I E^{I T_0 \Omega + I \tau_1 + I T_0 w_2} d_3 L_1 A_2}{3 w_1} - \frac{E^{I T_0 w_2} m_2 w_2 A_2}{4 w_1} + \frac{6 I E^{I T_0 w_1 + I T_0 w_2} d_3 A_1 A_2}{5 w_1} +$$

$$\frac{I E^{2 I T_0 w_2} d_4 A_2^2}{w_1} - \frac{6 I E^{-I T_0 \Omega - I \tau_1 - I T_0 w_1} d_2 L_1 \bar{A}_1}{5 w_1} - \frac{2 I E^{I T_0 \Omega + I \tau_1 - I T_0 w_1} d_2 L_1 \bar{A}_1}{3 w_1} -$$

$$\frac{2 I E^{-I T_0 w_1 + I T_0 w_2} d_3 A_2 \bar{A}_1}{3 w_1} - \frac{I E^{-2 I T_0 w_1} d_2 w_2 \bar{A}_1^2}{16 w_1^2} - \frac{2 I E^{-I T_0 \Omega - I \tau_1 - I T_0 w_2} d_3 L_1 \bar{A}_2}{3 w_1} -$$

$$\frac{E^{-I T_0 w_2} m_2 w_2 \bar{A}_2}{4 w_1} + \frac{2 I E^{I T_0 w_1 - I T_0 w_2} d_3 A_1 \bar{A}_2}{3 w_1} - \frac{6 I E^{-I T_0 w_1 - I T_0 w_2} d_3 \bar{A}_1 \bar{A}_2}{5 w_1} - \frac{I E^{-2 I T_0 w_2} d_4 \bar{A}_2^2}{w_1}\Big]=$$

▪ Third-Order Problem

Substituting the first- and second-order solutions into the third-order problem, **eqOrder[3]**, yields

```
order3Eq = eqOrder@3D •. sol1 •. sol2 •• ExpandAll;
```

Substituting the **expRule1** into the right-hand sides of **order3Eq** and collecting the terms that could produce secular terms, the terms proportional to $E^{I\,w_1 T_0}$ and $E^{I\,w_2\,T_0}$, we have

```
ST21 = Coefficient@order3Eq@@#, 2DD •. expRule1@1D, Exp@I w1 T0DD & •ž 81, 2<;
ST21 •. displayRule
```

$$: -HD_2A_1L + \frac{E^{I T_1 s_1} HD_1\dot{A}_1L\, A_2\, d_2}{2\, w_1^2} + \frac{3\, E^{I T_1 s_1 + I T_1 s_2 + I t_1} HD_1\dot{A}_1L\, d_1\, L_1}{2\, w_1^2} + \frac{I\, HD_1A_1L\, m_1}{2\, w_1} + \frac{E^{I T_1 s_1} HD_1A_2L\, d_2\, \dot{A}_1}{2\, w_1^2},$$

$$-6\, E^{-I T_1 s_2 - I t_1} A_1 A_2\, a_2\, L_1 - 24\, A_1\, a_1\, L_1^2 + \frac{1}{2} HD_1A_1L\, m_1 + A_1\, m_1^2 + \frac{27\, E^{-I T_1 s_2 - I t_1} A_1 A_2\, d_1\, d_2\, L_1}{2\, w_1^2} +$$

$$\frac{23\, E^{-I T_1 s_2 - I t_1} A_1 A_2\, d_2\, d_3\, L_1}{15\, w_1^2} + \frac{81\, A_1\, d_1^2\, L_1^2}{2\, w_1^2} + \frac{23\, A_1\, d_2^2\, L_1^2}{15\, w_1^2} - \frac{I\, E^{I T_1 s_1} HD_1\dot{A}_1L\, A_2\, d_2}{2\, w_1} -$$

$$\frac{3\, I\, E^{I T_1 s_1 + I T_1 s_2 + I t_1} HD_1\dot{A}_1L\, d_1\, L_1}{2\, w_1} - I\, HD_2A_1L\, w_1 - 12\, A_1^2\, a_1\, \dot{A}_1 + \frac{30\, A_1^2\, d_1^2\, \dot{A}_1}{w_1^2} - \frac{E^{I T_1 s_1 + I T_1 s_2 + I t_2} f_2\, d_2\, \dot{A}_1}{16\, w_1^2} +$$

$$\frac{9\, A_1^2\, d_2^2\, \dot{A}_1}{8\, w_1^2} - \frac{I\, E^{I T_1 s_1} HD_1A_2L\, d_2\, \dot{A}_1}{2\, w_1} - \frac{2\, I\, E^{I T_1 s_1} A_2\, d_2\, m_1\, \dot{A}_1}{w_1} - \frac{14\, I\, E^{I T_1 s_1 + I T_1 s_2 + I t_1} d_1\, L_1\, m_1\, \dot{A}_1}{w_1} +$$

$$\frac{I\, E^{I T_1 s_1} A_2\, d_2\, m_2\, \dot{A}_1}{2\, w_1} - 4\, A_1 A_2\, a_3\, \dot{A}_2 - 6\, E^{I T_1 s_2 + I t_1} A_1\, a_2\, L_1\, \dot{A}_2 + \frac{A_1 A_2\, d_2^2\, \dot{A}_2}{2\, w_1^2} + \frac{12\, A_1 A_2\, d_1\, d_3\, \dot{A}_2}{w_1^2} +$$

$$\frac{8\, A_1 A_2\, d_3^2\, \dot{A}_2}{15\, w_1^2} + \frac{3\, A_1 A_2\, d_2\, d_4\, \dot{A}_2}{w_1^2} + \frac{27\, E^{I T_1 s_2 + I t_1} A_1\, d_1\, d_2\, L_1\, \dot{A}_2}{2\, w_1^2} + \frac{23\, E^{I T_1 s_2 + I t_1} A_1\, d_2\, d_3\, L_1\, \dot{A}_2}{15\, w_1^2} >$$

```
ST22 = Coefficient@order3Eq@@#, 2DD •. expRule1@2D, Exp@I w2 T0DD & •ž 83, 4<;
ST22 •. displayRule
```

$$\Big\{ -(D_2 A_2) + \frac{E^{-I T_1 s_1} (D_1 A_1) A_1 d_2}{8 w_1^2} + \frac{I (D_1 A_2) m_2}{4 w_1},$$

$$-3 E^{-I T_1 s_2 - I t_1} A_2^2 a_4 L_1 - 4 A_2 a_3 L_1^2 - 3 E^{I T_1 s_2 + I t_1} a_2 L_1^3 + \frac{58 E^{-I T_1 s_2 - I t_1} A_2^2 d_2 d_3 L_1}{15 w_1^2} +$$

$$\frac{5 E^{-I T_1 s_2 - I t_1} A_2^2 d_3 d_4 L_1}{2 w_1^2} + \frac{56 A_2 d_2^2 L_1^2}{15 w_1^2} + \frac{12 A_2 d_1 d_3 L_1^2}{w_1^2} + \frac{2 A_2 d_3^2 L_1^2}{3 w_1^2} + \frac{3 A_2 d_2 d_4 L_1^2}{w_1^2} +$$

$$\frac{58 E^{I T_1 s_2 + I t_1} d_1 d_2 L_1^3}{5 w_1^2} + \frac{5 E^{I T_1 s_2 + I t_1} d_2 d_3 L_1^3}{6 w_1^2} + \frac{I E^{-I T_1 s_1} A_1^2 d_2 m_1}{w_1} - I (D_2 A_2) w_2 -$$

$$\frac{I E^{-I T_1 s_1} (D_1 A_1) A_1 d_2 w_2}{8 w_1^2} + \frac{I E^{I T_1 s_2 + I t_2} f_2 m_2 w_2}{16 w_1^2} - \frac{I E^{-I T_1 s_1} A_1^2 d_2 m_2 w_2}{8 w_1^2} + \frac{(D_1 A_2) m_2 w_2}{4 w_1} +$$

$$\frac{A_2 m_2^2 w_2}{2 w_1} - 4 A_1 A_2 a_3 \dot{A}_1 - 6 E^{I T_1 s_2 + I t_1} A_1 a_2 L_1 \dot{A}_1 + \frac{A_1 A_2 d_2^2 \dot{A}_1}{2 w_1^2} + \frac{12 A_1 A_2 d_1 d_3 \dot{A}_1}{w_1^2} +$$

$$\frac{8 A_1 A_2 d_3^2 \dot{A}_1}{15 w_1^2} + \frac{3 A_1 A_2 d_2 d_4 \dot{A}_1}{w_1^2} + \frac{27 E^{I T_1 s_2 + I t_1} A_1 d_1 d_2 L_1 \dot{A}_1}{2 w_1^2} + \frac{23 E^{I T_1 s_2 + I t_1} A_1 d_2 d_3 L_1 \dot{A}_1}{15 w_1^2} -$$

$$12 A_2^2 a_5 \dot{A}_2 - 6 E^{I T_1 s_2 + I t_1} A_2 a_4 L_1 \dot{A}_2 - 2 E^{2 I T_1 s_2 + 2 I t_1} a_3 L_1^2 \dot{A}_2 + \frac{58 A_2^2 d_3^2 \dot{A}_2}{15 w_1^2} + \frac{15 A_2^2 d_4^2 \dot{A}_2}{2 w_1^2} +$$

$$\frac{116 E^{I T_1 s_2 + I t_1} A_2 d_2 d_3 L_1 \dot{A}_2}{15 w_1^2} + \frac{5 E^{I T_1 s_2 + I t_1} A_2 d_3 d_4 L_1 \dot{A}_2}{w_1^2} + \frac{4 E^{2 I T_1 s_2 + 2 I t_1} d_2^2 L_1^2 \dot{A}_2}{w_1^2} -$$

$$\frac{2 E^{2 I T_1 s_2 + 2 I t_1} d_1 d_3 L_1^2 \dot{A}_2}{5 w_1^2} + \frac{E^{2 I T_1 s_2 + 2 I t_1} d_3^2 L_1^2 \dot{A}_2}{w_1^2} - \frac{E^{2 I T_1 s_2 + 2 I t_1} d_2 d_4 L_1^2 \dot{A}_2}{2 w_1^2} \Big\}$$

Then, the solvability conditions demand that **ST21** and **ST22** be orthogonal to solutions of their corresponding adjoints, the components of **ccleftVec**. The result is

```
SCond2 = 8ccleftVec@@1DD.ST21 == 0, ccleftVec@@2DD.ST22 == 0< •• ExpandAll;
SCond2 •. displayRule
```

$$\Big\{-6 E^{-I T_1 s_2 - I t_1} A_1 A_2 a_2 L_1 - 24 A_1 a_1 L_1^2 + A_1 m_1^2 + \frac{27 E^{-I T_1 s_2 - I t_1} A_1 A_2 d_1 d_2 L_1}{2 w_1^2} + \frac{23 E^{-I T_1 s_2 - I t_1} A_1 A_2 d_2 d_3 L_1}{15 w_1^2} + \frac{81 A_1 d_1^2 L_1^2}{2 w_1^2} + \frac{23 A_1 d_3^2 L_1^2}{15 w_1^2} - 2 I HD_2A_1L w_1 - 12 A_1^2 a_1 \dot{A}_1 + \frac{30 A_1^2 d_1^2 \dot{A}_1}{w_1^2} - \frac{E^{I T_1 s_1 + I T_1 s_2 + I t_2} f_2 d_2 \dot{A}_1}{16 w_1^2} + \frac{9 A_1^2 d_2^2 \dot{A}_1}{8 w_1^2} - \frac{2 I E^{I T_1 s_1} A_2 d_2 m_1 \dot{A}_1}{w_1} - \frac{14 I E^{I T_1 s_1 + I T_1 s_2 + I t_1} d_1 L_1 m_1 \dot{A}_1}{w_1} + \frac{I E^{I T_1 s_1} A_2 d_2 m_2 \dot{A}_1}{2 w_1} - 4 A_1 A_2 a_3 \dot{A}_2 - 6 E^{I T_1 s_2 + I t_1} A_1 a_2 L_1 \dot{A}_2 + \frac{A_1 A_2 d_2^2 \dot{A}_2}{2 w_1^2} + \frac{12 A_1 A_2 d_1 d_3 \dot{A}_2}{w_1^2} + \frac{8 A_1 A_2 d_3^2 \dot{A}_2}{15 w_1^2} + \frac{3 A_1 A_2 d_2 d_4 \dot{A}_2}{w_1^2} + \frac{27 E^{I T_1 s_2 + I t_1} A_1 d_1 d_2 L_1 \dot{A}_2}{2 w_1^2} + \frac{23 E^{I T_1 s_2 + I t_1} A_1 d_2 d_3 L_1 \dot{A}_2}{15 w_1^2} == 0,$$

$$-3 E^{-I T_1 s_2 - I t_1} A_2^2 a_4 L_1 - 4 A_2 a_3 L_1^2 - 3 E^{I T_1 s_2 + I t_1} a_2 L_1^3 + \frac{58 E^{-I T_1 s_2 - I t_1} A_2^2 d_2 d_3 L_1}{15 w_1^2} + \frac{5 E^{-I T_1 s_2 - I t_1} A_2^2 d_3 d_4 L_1}{2 w_1^2} + \frac{56 A_2 d_2^2 L_1^2}{15 w_1^2} + \frac{12 A_2 d_1 d_3 L_1^2}{w_1^2} + \frac{2 A_2 d_3^2 L_1^2}{3 w_1^2} + \frac{3 A_2 d_2 d_4 L_1^2}{w_1^2} + \frac{58 E^{I T_1 s_2 + I t_1} d_1 d_2 L_1^3}{5 w_1^2} + \frac{5 E^{I T_1 s_2 + I t_1} d_2 d_3 L_1^3}{6 w_1^2} + \frac{I E^{-I T_1 s_1} A_1^2 d_2 m_1}{w_1} - 2 I HD_2A_2L w_2 + \frac{I E^{I T_1 s_2 + I t_2} f_2 m_2 w_2}{16 w_1^2} - \frac{I E^{-I T_1 s_1} A_1^2 d_2 m_2 w_2}{8 w_1^2} + \frac{A_2 m_2^2 w_2}{2 w_1} - 4 A_1 A_2 a_3 \dot{A}_1 - 6 E^{I T_1 s_2 + I t_1} A_1 a_2 L_1 \dot{A}_1 + \frac{A_1 A_2 d_2^2 \dot{A}_1}{2 w_1^2} + \frac{12 A_1 A_2 d_1 d_3 \dot{A}_1}{w_1^2} + \frac{8 A_1 A_2 d_3^2 \dot{A}_1}{15 w_1^2} + \frac{3 A_1 A_2 d_2 d_4 \dot{A}_1}{w_1^2} + \frac{27 E^{I T_1 s_2 + I t_1} A_1 d_1 d_2 L_1 \dot{A}_1}{2 w_1^2} + \frac{23 E^{I T_1 s_2 + I t_1} A_1 d_2 d_3 L_1 \dot{A}_1}{15 w_1^2} - 12 A_2^2 a_5 \dot{A}_2 - 6 E^{I T_1 s_2 + I t_1} A_2 a_4 L_1 \dot{A}_2 - 2 E^{2 I T_1 s_2 + 2 I t_1} a_3 L_1^2 \dot{A}_2 + \frac{58 A_2^2 d_3^2 \dot{A}_2}{15 w_1^2} + \frac{15 A_2^2 d_4^2 \dot{A}_2}{2 w_1^2} + \frac{116 E^{I T_1 s_2 + I t_1} A_2 d_2 d_3 L_1 \dot{A}_2}{15 w_1^2} + \frac{5 E^{I T_1 s_2 + I t_1} A_2 d_3 d_4 L_1 \dot{A}_2}{w_1^2} + \frac{4 E^{2 I T_1 s_2 + 2 I t_1} d_2^2 L_1^2 \dot{A}_2}{w_1^2} - \frac{2 E^{2 I T_1 s_2 + 2 I t_1} d_1 d_3 L_1^2 \dot{A}_2}{5 w_1^2} + \frac{E^{2 I T_1 s_2 + 2 I t_1} d_3^2 L_1^2 \dot{A}_2}{w_1^2} - \frac{E^{2 I T_1 s_2 + 2 I t_1} d_2 d_4 L_1^2 \dot{A}_2}{2 w_1^2} == 0\Big\}$$

Solving **SCond2** for $D_2 A_1$ and $D_2 A_2$, we have

```
SCond2Rule1 = Solve[SCond2, {A1^(0,1)[T1, T2], A2^(0,1)[T1, T2]}][[1]] // ExpandAll;
SCond2Rule1 /. displayRule
```

$$\Big\{D_2A_1 \to -\frac{27 I E^{-I T_1 \sigma_2 - I \tau_1} A_1 A_2 \delta_1 \delta_2 \Lambda_1}{4 \omega_1^3} - \frac{23 I E^{-I T_1 \sigma_2 - I \tau_1} A_1 A_2 \delta_2 \delta_3 \Lambda_1}{30 \omega_1^3} - \frac{81 I A_1 \delta_1^2 \Lambda_1^2}{4 \omega_1^3} - \frac{23 I A_1 \delta_2^2 \Lambda_1^2}{30 \omega_1^3} +$$
$$\frac{3 I E^{-I T_1 \sigma_2 - I \tau_1} A_1 A_2 \alpha_2 \Lambda_1}{\omega_1} + \frac{12 I A_1 \alpha_1 \Lambda_1^2}{\omega_1} - \frac{I A_1 \mu_1^2}{2 \omega_1} - \frac{15 I A_1^2 \delta_1^2 \bar{A}_1}{\omega_1^3} + \frac{I E^{I T_1 \sigma_1 + I T_1 \sigma_2 + I \tau_2} \varphi_2 \delta_2 \bar{A}_1}{32 \omega_1^3} -$$
$$\frac{9 I A_1^2 \delta_2^2 \bar{A}_1}{16 \omega_1^3} - \frac{E^{I T_1 \sigma_1} A_2 \delta_2 \mu_1 \bar{A}_1}{\omega_1^2} - \frac{7 E^{I T_1 \sigma_1 + I T_1 \sigma_2 + I \tau_1} \delta_1 \Lambda_1 \mu_1 \bar{A}_1}{\omega_1^2} + \frac{E^{I T_1 \sigma_1} A_2 \delta_2 \mu_2 \bar{A}_1}{4 \omega_1^2} + \frac{6 I A_1^2 \alpha_1 \bar{A}_1}{\omega_1} -$$
$$\frac{I A_1 A_2 \delta_2^2 \bar{A}_2}{4 \omega_1^3} - \frac{6 I A_1 A_2 \delta_1 \delta_3 \bar{A}_2}{\omega_1^3} - \frac{4 I A_1 A_2 \delta_3^2 \bar{A}_2}{15 \omega_1^3} - \frac{3 I A_1 A_2 \delta_2 \delta_4 \bar{A}_2}{2 \omega_1^3} - \frac{27 I E^{I T_1 \sigma_2 + I \tau_1} A_1 \delta_1 \delta_2 \Lambda_1 \bar{A}_2}{4 \omega_1^3} -$$
$$\frac{23 I E^{I T_1 \sigma_2 + I \tau_1} A_1 \delta_2 \delta_3 \Lambda_1 \bar{A}_2}{30 \omega_1^3} + \frac{2 I A_1 A_2 \alpha_3 \bar{A}_2}{\omega_1} + \frac{3 I E^{I T_1 \sigma_2 + I \tau_1} A_1 \alpha_2 \Lambda_1 \bar{A}_2}{\omega_1},$$

$$D_2A_2 \to \frac{E^{I T_1 \sigma_2 + I \tau_2} \varphi_2 \mu_2}{32 \omega_1^2} - \frac{E^{-I T_1 \sigma_1} A_1^2 \delta_2 \mu_2}{16 \omega_1^2} - \frac{I A_2 \mu_2^2}{4 \omega_1} + \frac{3 I E^{-I T_1 \sigma_2 - I \tau_1} A_2^2 \alpha_4 \Lambda_1}{2 \omega_2} + \frac{2 I A_2 \alpha_3 \Lambda_1^2}{\omega_2} +$$
$$\frac{3 I E^{I T_1 \sigma_2 + I \tau_1} \alpha_2 \Lambda_1^3}{2 \omega_2} - \frac{29 I E^{-I T_1 \sigma_2 - I \tau_1} A_2^2 \delta_2 \delta_3 \Lambda_1}{15 \omega_1^2 \omega_2} - \frac{5 I E^{-I T_1 \sigma_2 - I \tau_1} A_2^2 \delta_3 \delta_4 \Lambda_1}{4 \omega_1^2 \omega_2} - \frac{28 I A_2 \delta_2^2 \Lambda_1^2}{15 \omega_1^2 \omega_2} -$$
$$\frac{6 I A_2 \delta_1 \delta_3 \Lambda_1^2}{\omega_1^2 \omega_2} - \frac{I A_2 \delta_3^2 \Lambda_1^2}{3 \omega_1^2 \omega_2} - \frac{3 I A_2 \delta_2 \delta_4 \Lambda_1^2}{2 \omega_1^2 \omega_2} - \frac{29 I E^{I T_1 \sigma_2 + I \tau_1} \delta_1 \delta_2 \Lambda_1^3}{5 \omega_1^2 \omega_2} - \frac{5 I E^{I T_1 \sigma_2 + I \tau_1} \delta_2 \delta_3 \Lambda_1^3}{12 \omega_1^2 \omega_2} +$$
$$\frac{E^{-I T_1 \sigma_1} A_1^2 \delta_2 \mu_1}{2 \omega_1 \omega_2} + \frac{2 I A_1 A_2 \alpha_3 \bar{A}_1}{\omega_2} + \frac{3 I E^{I T_1 \sigma_2 + I \tau_1} A_1 \alpha_2 \Lambda_1 \bar{A}_1}{\omega_2} - \frac{I A_1 A_2 \delta_2^2 \bar{A}_1}{4 \omega_1^2 \omega_2} - \frac{6 I A_1 A_2 \delta_1 \delta_3 \bar{A}_1}{\omega_1^2 \omega_2} -$$
$$\frac{4 I A_1 A_2 \delta_3^2 \bar{A}_1}{15 \omega_1^2 \omega_2} - \frac{3 I A_1 A_2 \delta_2 \delta_4 \bar{A}_1}{2 \omega_1^2 \omega_2} - \frac{27 I E^{I T_1 \sigma_2 + I \tau_1} A_1 \delta_1 \delta_2 \Lambda_1 \bar{A}_1}{4 \omega_1^2 \omega_2} - \frac{23 I E^{I T_1 \sigma_2 + I \tau_1} A_1 \delta_2 \delta_3 \Lambda_1 \bar{A}_1}{30 \omega_1^2 \omega_2} +$$
$$\frac{6 I A_2^2 \alpha_5 \bar{A}_2}{\omega_2} + \frac{3 I E^{I T_1 \sigma_2 + I \tau_1} A_2 \alpha_4 \Lambda_1 \bar{A}_2}{\omega_2} + \frac{I E^{2 I T_1 \sigma_2 + 2 I \tau_1} \alpha_3 \Lambda_1^2 \bar{A}_2}{\omega_2} - \frac{29 I A_2^2 \delta_3^2 \bar{A}_2}{15 \omega_1^2 \omega_2} - \frac{15 I A_2^2 \delta_4^2 \bar{A}_2}{4 \omega_1^2 \omega_2} -$$
$$\frac{58 I E^{I T_1 \sigma_2 + I \tau_1} A_2 \delta_2 \delta_3 \Lambda_1 \bar{A}_2}{15 \omega_1^2 \omega_2} - \frac{5 I E^{I T_1 \sigma_2 + I \tau_1} A_2 \delta_3 \delta_4 \Lambda_1 \bar{A}_2}{2 \omega_1^2 \omega_2} - \frac{2 I E^{2 I T_1 \sigma_2 + 2 I \tau_1} \delta_2^2 \Lambda_1^2 \bar{A}_2}{\omega_1^2 \omega_2} +$$
$$\frac{I E^{2 I T_1 \sigma_2 + 2 I \tau_1} \delta_1 \delta_3 \Lambda_1^2 \bar{A}_2}{5 \omega_1^2 \omega_2} - \frac{I E^{2 I T_1 \sigma_2 + 2 I \tau_1} \delta_3^2 \Lambda_1^2 \bar{A}_2}{2 \omega_1^2 \omega_2} + \frac{I E^{2 I T_1 \sigma_2 + 2 I \tau_1} \delta_2 \delta_4 \Lambda_1^2 \bar{A}_2}{4 \omega_1^2 \omega_2}\Big\}$$

Reconstitution

Using the method of reconstitution, $A_k' = \epsilon D_1 A_k + \epsilon^2 D_2 A_k + \cdots$, we combine the partial-differential equations **SCond1Rule1** and **SCond2Rule1** into the following two ordinary-differential equations governing the modulation of the complex-valued functions A_k :

```
moduEq = Table@2 I wk Ak¢, 8k, 2<D ==
    HTable@2 I wk dt@1D@ Ak@T1, T2DD, 8k, 2<D •. SCond1Rule1 •. SCond2Rule1 ••
      Collect@#, eD &L •• Thread;
moduEq •. displayRule
```

$$\Big\{2\,I\,w_1\,A_1' == 2\,I\,e\,w_1\left(-A_1 m_1+\frac{I\,E^{I T_1 s_1} A_2 d_2 \dot{A}_1}{w_1}+\frac{3\,I\,E^{I T_1 s_1+I T_1 s_2+I t_1} d_1 L_1 \dot{A}_1}{w_1}\right)+$$

$$2\,I\,e^2 w_1\left(-\frac{27\,I\,E^{-I T_1 s_2-I t_1} A_1 A_2 d_1 d_2 L_1}{4 w_1^3}-\frac{23\,I\,E^{-I T_1 s_2-I t_1} A_1 A_2 d_2 d_3 L_1}{30 w_1^3}-\right.$$

$$\frac{81\,I\,A_1 d_1^2 L_1^2}{4 w_1^3}-\frac{23\,I\,A_1 d_2^2 L_1^2}{30 w_1^3}+\frac{3\,I\,E^{-I T_1 s_2-I t_1} A_1 A_2 a_2 L_1}{w_1}+\frac{12\,I\,A_1 a_1 L_1^2}{w_1}-\frac{I\,A_1 m_1^2}{2 w_1}-$$

$$\frac{15\,I\,A_1^2 d_1^2 \dot{A}_1}{w_1^3}+\frac{I\,E^{I T_1 s_1+I T_1 s_2+I t_2} f_2 d_2 \dot{A}_1}{32 w_1^3}-\frac{9\,I\,A_1^2 d_2^2 \dot{A}_1}{16 w_1^3}-\frac{E^{I T_1 s_1} A_2 d_2 m_1 \dot{A}_1}{w_1^2}-$$

$$\frac{7\,E^{I T_1 s_1+I T_1 s_2+I t_1} d_1 L_1 m_1 \dot{A}_1}{w_1^2}+\frac{E^{I T_1 s_1} A_2 d_2 m_2 \dot{A}_1}{4 w_1^2}+\frac{6\,I\,A_1^2 a_1 \dot{A}_1}{w_1}-\frac{I\,A_1 A_2 d_2^2 \dot{A}_2}{4 w_1^3}-$$

$$\frac{6\,I\,A_1 A_2 d_1 d_3 \dot{A}_2}{w_1^3}-\frac{4\,I\,A_1 A_2 d_3^2 \dot{A}_2}{15 w_1^3}-\frac{3\,I\,A_1 A_2 d_2 d_4 \dot{A}_2}{2 w_1^3}-\frac{27\,I\,E^{I T_1 s_2+I t_1} A_1 d_1 d_2 L_1 \dot{A}_2}{4 w_1^3}-$$

$$\left.\frac{23\,I\,E^{I T_1 s_2+I t_1} A_1 d_2 d_3 L_1 \dot{A}_2}{30 w_1^3}+\frac{2\,I\,A_1 A_2 a_3 \dot{A}_2}{w_1}+\frac{3\,I\,E^{I T_1 s_2+I t_1} A_1 a_2 L_1 \dot{A}_2}{w_1}\right),$$

$$2\,I\,w_2\,A_2' == 2\,I\,e\left(-A_2 m_2-\frac{I\,E^{I T_1 s_2+I t_2} f_2}{4 w_2}+\frac{I\,E^{-I T_1 s_1} A_1^2 d_2}{2 w_2}\right) w_2+$$

$$2\,I\,e^2 w_2\left(\frac{E^{I T_1 s_2+I t_2} f_2 m_2}{32 w_1^2}-\frac{E^{-I T_1 s_1} A_1^2 d_2 m_2}{16 w_1^2}-\frac{I\,A_2 m_2^2}{4 w_1}+\frac{3\,I\,E^{-I T_1 s_2-I t_1} A_2^2 a_4 L_1}{2 w_2}+\frac{2\,I\,A_2 a_3 L_1^2}{w_2}+\right.$$

$$\frac{3\,I\,E^{I T_1 s_2+I t_1} a_2 L_1^3}{2 w_2}-\frac{29\,I\,E^{-I T_1 s_2-I t_1} A_2^2 d_2 d_3 L_1}{15 w_1^2 w_2}-\frac{5\,I\,E^{-I T_1 s_2-I t_1} A_2^2 d_3 d_4 L_1}{4 w_1^2 w_2}-\frac{28\,I\,A_2 d_2^2 L_1^2}{15 w_1^2 w_2}-$$

$$\frac{6\,I\,A_2 d_1 d_3 L_1^2}{w_1^2 w_2}-\frac{I\,A_2 d_3^2 L_1^2}{3 w_1^2 w_2}-\frac{3\,I\,A_2 d_2 d_4 L_1^2}{2 w_1^2 w_2}-\frac{29\,I\,E^{I T_1 s_2+I t_1} d_1 d_2 L_1^3}{5 w_1^2 w_2}-\frac{5\,I\,E^{I T_1 s_2+I t_1} d_2 d_3 L_1^3}{12 w_1^2 w_2}+$$

$$\frac{E^{-I T_1 s_1} A_1^2 d_2 m_1}{2 w_1 w_2}+\frac{2\,I\,A_1 A_2 a_3 \dot{A}_1}{w_2}+\frac{3\,I\,E^{I T_1 s_2+I t_1} A_1 a_2 L_1 \dot{A}_1}{w_2}-\frac{I\,A_1 A_2 d_2^2 \dot{A}_1}{4 w_1^2 w_2}-\frac{6\,I\,A_1 A_2 d_1 d_3 \dot{A}_1}{w_1^2 w_2}-$$

$$\frac{4\,I\,A_1 A_2 d_3^2 \dot{A}_1}{15 w_1^2 w_2}-\frac{3\,I\,A_1 A_2 d_2 d_4 \dot{A}_1}{2 w_1^2 w_2}-\frac{27\,I\,E^{I T_1 s_2+I t_1} A_1 d_1 d_2 L_1 \dot{A}_1}{4 w_1^2 w_2}-\frac{23\,I\,E^{I T_1 s_2+I t_1} A_1 d_2 d_3 L_1 \dot{A}_1}{30 w_1^2 w_2}+$$

$$\frac{6\,I\,A_2^2 a_5 \dot{A}_2}{w_2}+\frac{3\,I\,E^{I T_1 s_2+I t_1} A_2 a_4 L_1 \dot{A}_2}{w_2}+\frac{I\,E^{2 I T_1 s_2+2 I t_1} a_3 L_1^2 \dot{A}_2}{w_2}-\frac{29\,I\,A_2^2 d_3^2 \dot{A}_2}{15 w_1^2 w_2}-\frac{15\,I\,A_2^2 d_4^2 \dot{A}_2}{4 w_1^2 w_2}-$$

$$\frac{58\,I\,E^{I T_1 s_2+I t_1} A_2 d_2 d_3 L_1 \dot{A}_2}{15 w_1^2 w_2}-\frac{5\,I\,E^{I T_1 s_2+I t_1} A_2 d_3 d_4 L_1 \dot{A}_2}{2 w_1^2 w_2}-\frac{2\,I\,E^{2 I T_1 s_2+2 I t_1} d_2^2 L_1^2 \dot{A}_2}{w_1^2 w_2}+$$

$$\left.\frac{I\,E^{2 I T_1 s_2+2 I t_1} d_1 d_3 L_1^2 \dot{A}_2}{5 w_1^2 w_2}-\frac{I\,E^{2 I T_1 s_2+2 I t_1} d_3^2 L_1^2 \dot{A}_2}{2 w_1^2 w_2}+\frac{I\,E^{2 I T_1 s_2+2 I t_1} d_2 d_4 L_1^2 \dot{A}_2}{4 w_1^2 w_2}\right)\Big\}$$

Ÿ 6.1.2 The function MMS

According to the procedures described in the previous section, we can build a function named **MMS** (**M**ethod of **M**ultiple **S**cales) specifically for **eq61b**. A more general function (a Package) can be created by considering as arguments the governing equations, symbols for dependent variables, independent variable, excitation amplitudes and frequencies, and all other related quantities which allow the program to identify their respective meanings. We will use **MMS** to solve for different resonance cases in the following sections.

```
MMS@scaling_List, ResonanceCond : 8__Equal<D :=
 ModuleA8<,
      omgList = 8w1, w2<;
  omgRule = Solve@ResonanceCond, 8W, #< •• FlattenD@@1DD & •ž omgList •• Reverse;
  multiScales = 8ui_@tD -> ui žž timeScales, vi_@tD -> vi žž timeScales,
    Derivative@1D@u_D@tD -> dt@1D@u žž timeScalesD, t -> T0<;
      eqa = Heq61b •. scaling •. multiScales •. solRule •• TrigToExp •• ExpandAllL •.
    en_•;n>3 -> 0;
      eqEps = Rest@Thread@CoefficientList@Subtract žž #, eD == 0DD & •ž eqa •• Transpose;
      eqOrder@i_D :=
  I#@@1DD & •ž eqEps@@1DD •. fs_ -> 0 •. u_k_,1 -> uk,iM == I#@@1DD & •ž eqEps@@1DD •. fs_ -> 0 •.
      u_k_,1 -> uk,iM - H#@@1DD & •ž eqEps@@iDDL •• Thread;

      H* First-Order Problem *L
      linearSys = #@@1DD & •ž eqOrder@1D;
      expr1 = linearSys •. 9ui_,1 -> IP E^(I wi #1) &M, vi_,1 -> IQ E^(I wi #1) &M= •. Exp@a_D -> 1 ••
    Partition@#, 2D &;
      coefMat = Outer@D, #, 8P, Q<D & •ž expr1;
      hermitian@mat_?MatrixQD := mat •. conjugateRule •• Transpose;
      rightVec = # • #@@1DD & •ž HNullSpace@#D@@1DD & •ž coefMatL;
      leftVec = NullSpace@hermitian@#DD@@1DD & •ž coefMat;
      ccleftVec = leftVec •. conjugateRule;
      order1Eq = eqOrder@1D •. u_i_,1 -> Hui,1@#1D &L;
      sol1p =
  DSolve@order1Eq, 8u1,1@T0D, v1,1@T0D, u2,1@T0D, v2,1@T0D<, T0D@@1DD •. C@_D -> 0 •• Simplify;
      fRule = 9fi_ -> 2 Li Iwi^2 - W^2M=;
      sol1u =
  TableAui,1 -> FunctionA8T0, T1, T2<, Ai@T1, T2D Exp@I wi T0D + Ȧi@T1, T2D Exp@-I wi T0D +
      Hui,1@T0D •. sol1p •. fRule •• Simplify •• ExpandL •• EvaluateE, 8i, 2<E;
      sol1v = Table@vi,1 -> Function@8T0, T1, T2<,
     D@ui,1 žž timeScales, T0D •. sol1u •• EvaluateD, 8i, 2<D;
      sol1 = Join@sol1u, sol1vD;
```

```
H* Second-Order Problem *L
order2Eq = eqOrder@2D •. sol1 •• ExpandAll;
expRule1@i_D :=
Exp@a_D :> Exp@Expand@a •. omgRule@@iDDD •. e^n_· T0 :> timeScales@@n + 1DDD;
ST11 = Coefficient@order2Eq@@#, 2DD •. expRule1@1D, Exp@I w1 T0DD & •ž 81, 2<;
ST12 = Coefficient@order2Eq@@#, 2DD •. expRule1@2D, Exp@I w2 T0DD & •ž 83, 4<;
SCond1 = 8ccleftVec@@1DD.ST11 == 0, ccleftVec@@2DD.ST12 == 0<;
SCond1Rule1 = SolveASCond1, 9A1^H1,0L@T1, T2D, A2^H1,0L@T1, T2D=E@@1DD •• ExpandAll;
sigRule = Solve@ResonanceCond, 8s1, s2<D@@1DD;
expRule2 = Exp@a_D :> ExpAa •. 9T1 -> e T0, T2 -> e^2 T0= •. sigRule •• ExpandE;
SCond1Rule2 = SCond1Rule1 •. expRule2;
ccSCond1Rule2 = SCond1Rule2 •. conjugateRule;
order2Eqm = order2Eq •. SCond1Rule2 •. ccSCond1Rule2 •• ExpandAll;
IfAUnion@#@@2DD & •ž order2EqmD === 80<,
  sol2 = 8u1,2 -> H0 &L, v1,2 -> H0 &L, u2,2 -> H0 &L, v2,2 -> H0 &L<,
  basicH = TableA9Ai@T1, T2D E^(I wi T0), Ȧi@T1, T2D E^(-I wi T0)=, 8i, 2<E •• Flatten;
  collectForm =
 Join@basicH, If@list1 = List žž Plus žž H#@@2DD & •ž eqOrder@1DL •. c_ fi_ E^a_ -> Li E^a;
   Head@list1D === List, list1, 8<DD;
  possibleTerms = JoinAcollectForm,
  IfAlist1 = List žž Plus žž I#@@2DD & •ž eqOrder@2D •. u_i_,j_ -> H0 &LM •. c_ fi_ E^a_ -> E^a;
   Head@list1D === List, list1, 8<E,
  Outer@Times, collectForm, collectFormD •• Flatten •• UnionE;
  ResonantTerms@i_D :=
 I# •. 8a_ •; a =!= 0 -> 1< & •ž IE^(-I wi T0) possibleTerms •. expRule1@iD •. Exp@_ T0 + _.D -> 0MM
    possibleTerms •• Union •• Rest;
  RT = Array@ResonantTerms, 2D;
  r1Rule = MapIndexed@
   r1,#2@@1DD -> Coefficient@order2Eqm@@1, 2DD, #1D &, RT@@1DDD •. Exp@_ T0 + _.D -> 0;
 H* r2Rule=MapIndexed@r2,#2@@1DD->Coefficient@order2Eqm@@2,2DD,#1D &,
 RT@@1DDD•.Exp@_ T0+_.D->0; *L
  r3Rule = MapIndexed@r3,#2@@1DD -> Coefficient@order2Eqm@@3, 2DD, #1D &, RT@@2DDD •.
  Exp@_ T0 + _.D -> 0;
 H* r4Rule=MapIndexed@r4,#2@@1DD->Coefficient@order2Eqm@@4,2DD,#1D &,
 RT@@2DDD•.Exp@_ T0+_.D->0; *L
  NRT = Complement@possibleTerms, Join@#, # •. conjugateRuleDD & •ž RT;
  RTsymbolList1 = Table@G1,j, 8j, Length@RT@@1DDD<D;
  RTsymbolList2 = Table@G3,j, 8j, Length@RT@@2DDD<D;
  NRTsymbolList1@i_D = Table@Li,j, 8j, Length@NRT@@1DDD<D;
  NRTsymbolList2@i_D = Table@Li+2,j, 8j, Length@NRT@@2DDD<D;
  8ratio1, ratio2< = Table@-ccleftVec@@i, 1DD, 8i, 2<D;
  sol2Form = 8u1,2 -> Function@8T0, T1, T2<,
```

```
      RTsymbolList1.RT@@1DD + HRTsymbolList1.RT@@1DD •. conjugateRuleL +
        NRTsymbolList1@1D.NRT@@1DD •• EvaluateD, v1,2 -> Function@8T0, T1, T2<,
      ratio1 * RTsymbolList1.RT@@1DD + Hratio1 * RTsymbolList1.RT@@1DD •. conjugateRuleL +
        NRTsymbolList1@2D.NRT@@1DD •• EvaluateD, u2,2 -> Function@8T0, T1, T2<,
      RTsymbolList2.RT@@2DD + HRTsymbolList2.RT@@2DD •. conjugateRuleL +
        NRTsymbolList2@1D.NRT@@2DD •• EvaluateD, v2,2 -> Function@8T0, T1, T2<,
      ratio2 * RTsymbolList2.RT@@2DD + Hratio2 * RTsymbolList2.RT@@2DD •. conjugateRuleL +
        NRTsymbolList2@2D.NRT@@2DD •• EvaluateD<;
        eqb = HCoefficient@Subtract žž # •. sol2Form, NRT@@1DDD •. Exp@_ T0 + _.D -> 0 & •ž
        order2Eqm@@81, 2<DD •• FlattenL == 0 •• Thread;
        eqc = HCoefficient@Subtract žž # •. sol2Form, NRT@@2DDD •. Exp@_ T0 + _.D -> 0 & •ž
        order2Eqm@@83, 4<DD •• FlattenL == 0 •• Thread;
        coef11 = Solve@eqb, Array@NRTsymbolList1, 2D •• FlattenD@@1DD;
        coef12 = Solve@eqc, Array@NRTsymbolList2, 2D •• FlattenD@@1DD;
        eqd = MapIndexed@Coefficient@order2Eqm@@1, 1DD •. sol2Form, #1D == r1,#2@@1DD &,
     RT@@1DDD •. Exp@_ T0 + _.D -> 0;
        eqe = MapIndexed@Coefficient@order2Eqm@@3, 1DD •. sol2Form, #1D == r3,#2@@1DD &,
     RT@@2DDD •. Exp@_ T0 + _.D -> 0;
        coef21 = Solve@eqd, RTsymbolList1D@@1DD •. r1Rule;
        coef22 = Solve@eqe, RTsymbolList2D@@1DD •. r3Rule;
        coeffs = Join@coef11, coef12, coef21, coef22,
     coef21 •. conjugateRule, coef22 •. conjugateRuleD •. HomgRule@@1DD •. e -> 0L;
        sol2 = sol2Form •. Function@8T0, T1, T2<, a_D :>
     Function@8T0, T1, T2< •• Evaluate, a •. coeffs •• Expand •• EvaluateD
    E;

       H* Third-Order Problem *L
       order3Eq = eqOrder@3D •. sol1 •. sol2 •• ExpandAll;
       ST21 = Coefficient@order3Eq@@#, 2DD •. expRule1@1D, Exp@I w1 T0DD & •ž 81, 2<;
       ST22 = Coefficient@order3Eq@@#, 2DD •. expRule1@2D, Exp@I w2 T0DD & •ž 83, 4<;
       SCond2 = 8ccleftVec@@1DD.ST21 == 0, ccleftVec@@2DD.ST22 == 0< •• ExpandAll;
       SCond2Rule1 = SolveASCond2, 9A1^H0,1L@T1, T2D, A2^H0,1L@T1, T2D=E@@1DD •• ExpandAll;

       H* Reconstitution *L
       moduEq =
  Table@2 I wk Ak¢, 8k, 2<D == HTable@2 I wk dt@1D@ Ak@T1, T2DD, 8k, 2<D •. SCond1Rule1 •.
       SCond2Rule1 •• Collect@#, eD &L •• Thread;
       Print@"The second-order approximate solution:"D;
       PrintATableA
   ui@tD == Iui žž timeScales •. solRule •. e^3 -> 0 •. sol1 •. sol2 •. displayRuleM, 8i, 2<EE;
       IfAOr žž Table@HFi • fi •. scalingL === e, 8i, 2<D,
          Print@"where"D;
```

```
        PrintA"L_i== f_i / (2 Iw_i^2 - W^2M) "E
    E;
    Print@"\nThe modulation equations:"D;
    Print@moduEq •. displayRuleD
E
```

As an example, we check the case in Section 6.1.1:

```
scaling1 = 9m`_n_ -> e m_n, F_1 -> e f_1, F_2 -> e^2 f_2=;

ResonanceCond1 = 8w_2 == 2 w_1 + e s_1, W == w_2 + e s_2<;

MMS@scaling1, ResonanceCond1D •• Timing
```

The second-order approximate solution:

$$
:u_1[t] == e\,\left(E^{I T_0 w_1} A_1 + E^{-I T_0 W - I t_1} L_1 + E^{I T_0 W + I t_1} L_1 + E^{-I T_0 w_1} \dot{A}_1\right) + e^2 \left(\frac{E^{2 I T_0 w_1} A_1^2 d_1}{w_1^2} + \frac{E^{I T_0 w_1 + I T_0 w_2} A_1 A_2 d_2}{4 w_1^2} + \frac{E^{2 I T_0 w_2} A_2^2 d_3}{15 w_1^2} - \frac{3 E^{-I T_0 W - I t_1 + I T_0 w_1} A_1 d_1 L_1}{2 w_1^2} + \frac{3 E^{I T_0 W + I t_1 + I T_0 w_1} A_1 d_1 L_1}{4 w_1^2} - \frac{2 E^{-I T_0 W - I t_1 + I T_0 w_2} A_2 d_2 L_1}{w_1^2} + \frac{2 E^{I T_0 W + I t_1 + I T_0 w_2} A_2 d_2 L_1}{15 w_1^2} - \frac{6 d_1 L_1^2}{w_1^2} + \frac{E^{-2 I T_0 W - 2 I t_1} d_1 L_1^2}{5 w_1^2} + \frac{E^{2 I T_0 W + 2 I t_1} d_1 L_1^2}{5 w_1^2} - \frac{I E^{I T_0 w_1} A_1 m_1}{2 w_1} - \frac{4 I E^{-I T_0 W - I t_1} L_1 m_1}{3 w_1} + \frac{4 I E^{I T_0 W + I t_1} L_1 m_1}{3 w_1} - \frac{6 A_1 d_1 \dot{A}_1}{w_1^2} - \frac{E^{-I T_0 w_1 + I T_0 w_2} A_2 d_2 \dot{A}_1}{2 w_1^2} + \frac{3 E^{-I T_0 W - I t_1 - I T_0 w_1} d_1 L_1 \dot{A}_1}{4 w_1^2} - \frac{3 E^{I T_0 W + I t_1 - I T_0 w_1} d_1 L_1 \dot{A}_1}{2 w_1^2} + \frac{I E^{-I T_0 w_1} m_1 \dot{A}_1}{2 w_1} + \frac{E^{-2 I T_0 w_1} d_1 \dot{A}_1^2}{w_1^2} - \frac{E^{I T_0 w_1 - I T_0 w_2} A_1 d_2 \dot{A}_2}{2 w_1^2} - \frac{2 A_2 d_3 \dot{A}_2}{w_1^2} + \frac{2 E^{-I T_0 W - I t_1 - I T_0 w_2} d_2 L_1 \dot{A}_2}{15 w_1^2} - \frac{2 E^{I T_0 W + I t_1 - I T_0 w_2} d_2 L_1 \dot{A}_2}{w_1^2} + \frac{E^{-I T_0 w_1 - I T_0 w_2} d_2 \dot{A}_1 \dot{A}_2}{4 w_1^2} + \frac{E^{-2 I T_0 w_2} d_3 \dot{A}_2^2}{15 w_1^2} \right),
$$

$$
u_2[t] == e\,\left(E^{I T_0 w_2} A_2 + E^{-I T_0 w_2} \dot{A}_2\right) + e^2 \left(\frac{E^{-I T_0 W - I t_2} f_2}{32 w_1^2} + \frac{E^{I T_0 W + I t_2} f_2}{32 w_1^2} - \frac{E^{2 I T_0 w_1} A_1^2 d_2}{16 w_1^2} + \frac{2 E^{I T_0 w_1 + I T_0 w_2} A_1 A_2 d_3}{5 w_1^2} + \frac{E^{2 I T_0 w_2} A_2^2 d_4}{4 w_1^2} - \frac{2 E^{-I T_0 W - I t_1 + I T_0 w_1} A_1 d_2 L_1}{3 w_1^2} + \frac{2 E^{I T_0 W + I t_1 + I T_0 w_1} A_1 d_2 L_1}{5 w_1^2} - \frac{E^{-I T_0 W - I t_1 + I T_0 w_2} A_2 d_3 L_1}{2 w_1^2} + \frac{E^{I T_0 W + I t_1 + I T_0 w_2} A_2 d_3 L_1}{6 w_1^2} - \frac{d_2 L_1^2}{2 w_1^2} + \frac{E^{-2 I T_0 W - 2 I t_1} d_2 L_1^2}{12 w_1^2} + \frac{E^{2 I T_0 W + 2 I t_1} d_2 L_1^2}{12 w_1^2} - \frac{I E^{I T_0 w_2} A_2 m_2}{4 w_1} - \frac{A_1 d_2 \dot{A}_1}{2 w_1^2} - \frac{2 E^{-I T_0 w_1 + I T_0 w_2} A_2 d_3 \dot{A}_1}{3 w_1^2} + \frac{2 E^{-I T_0 W - I t_1 - I T_0 w_1} d_2 L_1 \dot{A}_1}{5 w_1^2} - \frac{2 E^{I T_0 W + I t_1 - I T_0 w_1} d_2 L_1 \dot{A}_1}{3 w_1^2} - \frac{E^{-2 I T_0 w_1} d_2 \dot{A}_1^2}{16 w_1^2} - \frac{2 E^{I T_0 w_1 - I T_0 w_2} A_1 d_3 \dot{A}_2}{3 w_1^2} - \frac{3 A_2 d_4 \dot{A}_2}{2 w_1^2} + \frac{E^{-I T_0 W - I t_1 - I T_0 w_2} d_3 L_1 \dot{A}_2}{6 w_1^2} - \frac{E^{I T_0 W + I t_1 - I T_0 w_2} d_3 L_1 \dot{A}_2}{2 w_1^2} + \frac{I E^{-I T_0 w_2} m_2 \dot{A}_2}{4 w_1} + \frac{2 E^{-I T_0 w_1 - I T_0 w_2} d_3 \dot{A}_1 \dot{A}_2}{5 w_1^2} + \frac{E^{-2 I T_0 w_2} d_4 \dot{A}_2^2}{4 w_1^2} \right)>
$$

where

$$L_i == \frac{f_i}{2 (w_1^2 - W^2 L)}$$

The modulation equations:

$$\{2 I w_1 A_1' == 2 I e w_1 \left(-A_1 m_1 + \frac{I E^{I T_1 s_1} A_2 d_2 \bar{A}_1}{w_1} + \frac{3 I E^{I T_1 s_1 + I T_1 s_2 + I t_1} d_1 L_1 \bar{A}_1}{w_1}\right) +$$

$$2 I e^2 w_1 \left(-\frac{27 I E^{-I T_1 s_2 - I t_1} A_1 A_2 d_1 d_2 L_1}{4 w_1^3} - \frac{23 I E^{-I T_1 s_2 - I t_1} A_1 A_2 d_2 d_3 L_1}{30 w_1^3} -\right.$$

$$\frac{81 I A_1 d_1^2 L_1^2}{4 w_1^3} - \frac{23 I A_1 d_3^2 L_1^2}{30 w_1^3} + \frac{3 I E^{-I T_1 s_2 - I t_1} A_1 A_2 a_2 L_1}{w_1} + \frac{12 I A_1 a_1 L_1^2}{w_1} - \frac{I A_1 m_1^2}{2 w_1} -$$

$$\frac{15 I A_1^2 d_1^2 \bar{A}_1}{w_1^3} + \frac{I E^{I T_1 s_1 + I T_1 s_2 + I t_2} f_2 d_2 \bar{A}_1}{32 w_1^3} - \frac{9 I A_1^2 d_2^2 \bar{A}_1}{16 w_1^3} - \frac{E^{I T_1 s_1} A_2 d_2 m_1 \bar{A}_1}{w_1^2} -$$

$$\frac{7 E^{I T_1 s_1 + I T_1 s_2 + I t_1} d_1 L_1 m_1 \bar{A}_1}{w_1^2} + \frac{E^{I T_1 s_1} A_2 d_2 m_2 \bar{A}_1}{4 w_1^2} + \frac{6 I A_1^2 a_1 \bar{A}_1}{w_1} - \frac{I A_1 A_2 d_2^2 \bar{A}_2}{4 w_1^3} -$$

$$\frac{6 I A_1 A_2 d_1 d_3 \bar{A}_2}{w_1^3} - \frac{4 I A_1 A_2 d_3^2 \bar{A}_2}{15 w_1^3} - \frac{3 I A_1 A_2 d_2 d_4 \bar{A}_2}{2 w_1^3} - \frac{27 I E^{I T_1 s_2 + I t_1} A_1 d_1 d_2 L_1 \bar{A}_2}{4 w_1^3} -$$

$$\left.\frac{23 I E^{I T_1 s_2 + I t_1} A_1 d_2 d_3 L_1 \bar{A}_2}{30 w_1^3} + \frac{2 I A_1 A_2 a_3 \bar{A}_2}{w_1} + \frac{3 I E^{I T_1 s_2 + I t_1} A_1 a_2 L_1 \bar{A}_2}{w_1}\right),$$

$$2 I w_2 A_2' == 2 I e \left(-A_2 m_2 - \frac{I E^{I T_1 s_2 + I t_2} f_2}{4 w_2} + \frac{I E^{-I T_1 s_1} A_1^2 d_2}{2 w_2}\right) w_2 +$$

$$2 I e^2 w_2 \left(\frac{E^{I T_1 s_2 + I t_2} f_2 m_2}{32 w_1^2} - \frac{E^{-I T_1 s_1} A_1^2 d_2 m_2}{16 w_1^2} - \frac{I A_2 m_2^2}{4 w_1} + \frac{3 I E^{-I T_1 s_2 - I t_1} A_2^2 a_4 L_1}{2 w_2} + \frac{2 I A_2 a_3 L_1^2}{w_2} +\right.$$

$$\frac{3 I E^{I T_1 s_2 + I t_1} a_2 L_1^3}{2 w_2} - \frac{29 I E^{-I T_1 s_2 - I t_1} A_2^2 d_2 d_3 L_1}{15 w_1^2 w_2} - \frac{5 I E^{-I T_1 s_2 - I t_1} A_2^2 d_3 d_4 L_1}{4 w_1^2 w_2} -$$

$$\frac{28 I A_2 d_2^2 L_1^2}{15 w_1^2 w_2} - \frac{6 I A_2 d_1 d_3 L_1^2}{w_1^2 w_2} - \frac{I A_2 d_3^2 L_1^2}{3 w_1^2 w_2} - \frac{3 I A_2 d_2 d_4 L_1^2}{2 w_1^2 w_2} - \frac{29 I E^{I T_1 s_2 + I t_1} d_1 d_2 L_1^3}{5 w_1^2 w_2} -$$

$$\frac{5 I E^{I T_1 s_2 + I t_1} d_2 d_3 L_1^3}{12 w_1^2 w_2} + \frac{E^{-I T_1 s_1} A_1^2 d_2 m_1}{2 w_1 w_2} + \frac{2 I A_1 A_2 a_3 \bar{A}_1}{w_2} + \frac{3 I E^{I T_1 s_2 + I t_1} A_1 a_2 L_1 \bar{A}_1}{w_2} -$$

$$\frac{I A_1 A_2 d_2^2 \bar{A}_1}{4 w_1^2 w_2} - \frac{6 I A_1 A_2 d_1 d_3 \bar{A}_1}{w_1^2 w_2} - \frac{4 I A_1 A_2 d_3^2 \bar{A}_1}{15 w_1^2 w_2} - \frac{3 I A_1 A_2 d_2 d_4 \bar{A}_1}{2 w_1^2 w_2} -$$

$$\frac{27 I E^{I T_1 s_2 + I t_1} A_1 d_1 d_2 L_1 \bar{A}_1}{4 w_1^2 w_2} - \frac{23 I E^{I T_1 s_2 + I t_1} A_1 d_2 d_3 L_1 \bar{A}_1}{30 w_1^2 w_2} + \frac{6 I A_2^2 a_5 \bar{A}_2}{w_2} +$$

$$\frac{3 I E^{I T_1 s_2 + I t_1} A_2 a_4 L_1 \bar{A}_2}{w_2} + \frac{I E^{2 I T_1 s_2 + 2 I t_1} a_3 L_1^2 \bar{A}_2}{w_2} - \frac{29 I A_2^2 d_3^2 \bar{A}_2}{15 w_1^2 w_2} - \frac{15 I A_2^2 d_4^2 \bar{A}_2}{4 w_1^2 w_2} -$$

$$\frac{58 I E^{I T_1 s_2 + I t_1} A_2 d_2 d_3 L_1 \bar{A}_2}{15 w_1^2 w_2} - \frac{5 I E^{I T_1 s_2 + I t_1} A_2 d_3 d_4 L_1 \bar{A}_2}{2 w_1^2 w_2} - \frac{2 I E^{2 I T_1 s_2 + 2 I t_1} d_2^2 L_1^2 \bar{A}_2}{w_1^2 w_2} +$$

$$\left.\frac{I E^{2 I T_1 s_2 + 2 I t_1} d_1 d_3 L_1^2 \bar{A}_2}{5 w_1^2 w_2} - \frac{I E^{2 I T_1 s_2 + 2 I t_1} d_3^2 L_1^2 \bar{A}_2}{2 w_1^2 w_2} + \frac{I E^{2 I T_1 s_2 + 2 I t_1} d_2 d_4 L_1^2 \bar{A}_2}{4 w_1^2 w_2}\right)\}$$

{882.98 Second, Null}

Ÿ 6.1.3 W » w_1 and w_2 » 2 w_1

In this case, we have a combination of a primary resonance of the first mode, a superharmonic resonance of the second mode, and a two-to-one internal resonance. In order to bring the effects of damping, forcing, and nonlinearity at the same order, we let

```
scaling3 = 9m̀n_ -> e mn, F1 -> e^2 f1, F2 -> e f2=;
```

To describe quantitatively the nearness of the resonances, we introduce the detuning parameters s_1 and s_2 defined according to

```
ResonanceCond3 = 8w2 == 2 w1 + e s1, W == w1 + e s2<;
```

Using **MMS**, we obtain the second-order approximate solution and the two equations governing the modulation of the complex-valued functions A_k:

```
MMS@scaling3, ResonanceCond3D •• Timing
```

```
The second-order approximate solution:
```

$$
\begin{aligned}
\{u_1[t] == {} & \epsilon\,(E^{I T_0 \omega_1} A_1 + E^{-I T_0 \omega_1} \bar{A}_1) + \\
& \epsilon^2 \Big(\frac{E^{-I T_0 \Omega - I \tau_1} f_1}{8 \omega_1^2} + \frac{E^{I T_0 \Omega + I \tau_1} f_1}{8 \omega_1^2} + \frac{E^{2 I T_0 \omega_1} A_1^2 \delta_1}{\omega_1^2} + \frac{E^{I T_0 \omega_1 + I T_0 \omega_2} A_1 A_2 \delta_2}{4 \omega_1^2} + \frac{E^{2 I T_0 \omega_2} A_2^2 \delta_3}{15 \omega_1^2} - \\
& \frac{2 E^{-I T_0 \Omega - I \tau_2 + I T_0 \omega_1} A_1 \delta_2 \Lambda_2}{\omega_1^2} + \frac{2 E^{I T_0 \Omega + I \tau_2 + I T_0 \omega_1} A_1 \delta_2 \Lambda_2}{3 \omega_1^2} - \frac{E^{-I T_0 \Omega - I \tau_2 + I T_0 \omega_2} A_2 \delta_3 \Lambda_2}{2 \omega_1^2} + \\
& \frac{E^{I T_0 \Omega + I \tau_2 + I T_0 \omega_2} A_2 \delta_3 \Lambda_2}{4 \omega_1^2} - \frac{2 \delta_3 \Lambda_2^2}{\omega_1^2} + \frac{E^{-2 I T_0 \Omega - 2 I \tau_2} \delta_3 \Lambda_2^2}{3 \omega_1^2} + \frac{E^{2 I T_0 \Omega + 2 I \tau_2} \delta_3 \Lambda_2^2}{3 \omega_1^2} - \\
& \frac{I E^{I T_0 \omega_1} A_1 \mu_1}{2 \omega_1} - \frac{6 A_1 \delta_1 \bar{A}_1}{\omega_1^2} - \frac{E^{-I T_0 \omega_1 + I T_0 \omega_2} A_2 \delta_2 \bar{A}_1}{2 \omega_1^2} + \frac{2 E^{-I T_0 \Omega - I \tau_2 - I T_0 \omega_1} \delta_2 \Lambda_2 \bar{A}_1}{3 \omega_1^2} - \\
& \frac{2 E^{I T_0 \Omega + I \tau_2 - I T_0 \omega_1} \delta_2 \Lambda_2 \bar{A}_1}{\omega_1^2} + \frac{I E^{-I T_0 \omega_1} \mu_1 \bar{A}_1}{2 \omega_1} + \frac{E^{-2 I T_0 \omega_1} \delta_1 \bar{A}_1^2}{\omega_1^2} - \frac{E^{I T_0 \omega_1 - I T_0 \omega_2} A_1 \delta_2 \bar{A}_2}{2 \omega_1^2} - \frac{2 A_2 \delta_3 \bar{A}_2}{\omega_1^2} + \\
& \frac{E^{-I T_0 \Omega - I \tau_2 - I T_0 \omega_2} \delta_3 \Lambda_2 \bar{A}_2}{4 \omega_1^2} - \frac{E^{I T_0 \Omega + I \tau_2 - I T_0 \omega_2} \delta_3 \Lambda_2 \bar{A}_2}{2 \omega_1^2} + \frac{E^{-I T_0 \omega_1 - I T_0 \omega_2} \delta_2 \bar{A}_1 \bar{A}_2}{4 \omega_1^2} + \frac{E^{-2 I T_0 \omega_2} \delta_3 \bar{A}_2^2}{15 \omega_1^2} \Big),
\end{aligned}
$$

$$
\begin{aligned}
u_2[t] == {} & \epsilon\,(E^{I T_0 \omega_2} A_2 + E^{-I T_0 \Omega - I \tau_2} \Lambda_2 + E^{I T_0 \Omega + I \tau_2} \Lambda_2 + E^{-I T_0 \omega_2} \bar{A}_2) + \\
& \epsilon^2 \Big(-\frac{E^{2 I T_0 \omega_1} A_1^2 \delta_2}{16 \omega_1^2} + \frac{2 E^{I T_0 \omega_1 + I T_0 \omega_2} A_1 A_2 \delta_3}{5 \omega_1^2} + \frac{E^{2 I T_0 \omega_2} A_2^2 \delta_4}{4 \omega_1^2} - \\
& \frac{E^{-I T_0 \Omega - I \tau_2 + I T_0 \omega_1} A_1 \delta_3 \Lambda_2}{2 \omega_1^2} - \frac{E^{I T_0 \Omega + I \tau_2 + I T_0 \omega_1} A_1 \delta_3 \Lambda_2}{8 \omega_1^2} - \frac{2 E^{-I T_0 \Omega - I \tau_2 + I T_0 \omega_2} A_2 \delta_4 \Lambda_2}{\omega_1^2} + \\
& \frac{6 E^{I T_0 \Omega + I \tau_2 + I T_0 \omega_2} A_2 \delta_4 \Lambda_2}{5 \omega_1^2} - \frac{3 \delta_4 \Lambda_2^2}{2 \omega_1^2} - \frac{3 E^{-2 I T_0 \Omega - 2 I \tau_2} \delta_4 \Lambda_2^2}{16 \omega_1^2} - \frac{3 E^{2 I T_0 \Omega + 2 I \tau_2} \delta_4 \Lambda_2^2}{16 \omega_1^2} - \\
& \frac{I E^{I T_0 \omega_2} A_2 \mu_2}{4 \omega_1} + \frac{2 I E^{-I T_0 \Omega - I \tau_2} \Lambda_2 \mu_2}{3 \omega_1} - \frac{2 I E^{I T_0 \Omega + I \tau_2} \Lambda_2 \mu_2}{3 \omega_1} - \frac{A_1 \delta_2 \bar{A}_1}{2 \omega_1^2} - \\
& \frac{2 E^{-I T_0 \omega_1 + I T_0 \omega_2} A_2 \delta_3 \bar{A}_1}{3 \omega_1^2} - \frac{E^{-I T_0 \Omega - I \tau_2 - I T_0 \omega_1} \delta_3 \Lambda_2 \bar{A}_1}{8 \omega_1^2} - \frac{E^{I T_0 \Omega + I \tau_2 - I T_0 \omega_1} \delta_3 \Lambda_2 \bar{A}_1}{2 \omega_1^2} - \\
& \frac{E^{-2 I T_0 \omega_1} \delta_2 \bar{A}_1^2}{16 \omega_1^2} - \frac{2 E^{I T_0 \omega_1 - I T_0 \omega_2} A_1 \delta_3 \bar{A}_2}{3 \omega_1^2} - \frac{3 A_2 \delta_4 \bar{A}_2}{2 \omega_1^2} + \frac{6 E^{-I T_0 \Omega - I \tau_2 - I T_0 \omega_2} \delta_4 \Lambda_2 \bar{A}_2}{5 \omega_1^2} - \\
& \frac{2 E^{I T_0 \Omega + I \tau_2 - I T_0 \omega_2} \delta_4 \Lambda_2 \bar{A}_2}{\omega_1^2} + \frac{I E^{-I T_0 \omega_2} \mu_2 \bar{A}_2}{4 \omega_1} + \frac{2 E^{-I T_0 \omega_1 - I T_0 \omega_2} \delta_3 \bar{A}_1 \bar{A}_2}{5 \omega_1^2} + \frac{E^{-2 I T_0 \omega_2} \delta_4 \bar{A}_2^2}{4 \omega_1^2} \Big)\}
\end{aligned}
$$

where

$$\Lambda_i == \frac{f_i}{2(\omega_1^2 - \Omega^2)}$$

The modulation equations:

$\{2\,I\,w_1\,A_1' ==$

$$2\,I\,e\,w_1\left(-A_1\,m_1-\frac{I\,E^{I\,T_1\,s_2+I\,t_1}\,f_1}{4\,w_1}+\frac{I\,E^{I\,T_1\,s_1-I\,T_1\,s_2-I\,t_2}\,A_2\,d_3\,L_2}{w_1}+\frac{I\,E^{I\,T_1\,s_1}\,A_2\,d_2\,\dot{A}_1}{w_1}\right)+2\,I\,e^2\,w_1$$

$$\left(\frac{I\,E^{I\,T_1\,s_1-I\,T_1\,s_2-I\,t_1}\,A_2\,f_1\,d_2}{8\,w_1^3}-\frac{5\,I\,E^{-I\,T_1\,s_2-I\,t_2}\,A_1^2\,d_1\,d_2\,L_2}{w_1^3}-\frac{9\,I\,E^{-I\,T_1\,s_2-I\,t_2}\,A_1^2\,d_2\,d_3\,L_2}{16\,w_1^3}-\frac{4\,I\,A_1\,d_2^2\,L_2^2}{3\,w_1^3}-\right.$$

$$\frac{6\,I\,A_1\,d_1\,d_3\,L_2^2}{w_1^3}-\frac{5\,I\,A_1\,d_3^2\,L_2^2}{8\,w_1^3}-\frac{3\,I\,A_1\,d_2\,d_4\,L_2^2}{2\,w_1^3}-\frac{5\,I\,E^{I\,T_1\,s_2+I\,t_2}\,d_2\,d_3\,L_2^3}{3\,w_1^3}-\frac{27\,I\,E^{I\,T_1\,s_2+I\,t_2}\,d_3\,d_4\,L_2^3}{16\,w_1^3}+$$

$$\frac{E^{I\,T_1\,s_2+I\,t_1}\,f_1\,m_1}{8\,w_1^2}-\frac{E^{I\,T_1\,s_1-I\,T_1\,s_2-I\,t_2}\,A_2\,d_3\,L_2\,m_1}{2\,w_1^2}-\frac{5\,E^{I\,T_1\,s_1-I\,T_1\,s_2-I\,t_2}\,A_2\,d_3\,L_2\,m_2}{12\,w_1^2}+$$

$$\frac{3\,I\,E^{-I\,T_1\,s_2-I\,t_2}\,A_1^2\,a_2\,L_2}{2\,w_1}+\frac{2\,I\,A_1\,a_3\,L_2^2}{w_1}+\frac{3\,I\,E^{I\,T_1\,s_2+I\,t_2}\,a_4\,L_2^3}{2\,w_1}-\frac{I\,A_1\,m_1^2}{2\,w_1}-\frac{15\,I\,A_1^2\,d_1^2\,\dot{A}_1}{w_1^3}-$$

$$\frac{9\,I\,A_1^2\,d_2^2\,\dot{A}_1}{16\,w_1^3}-\frac{10\,I\,E^{I\,T_1\,s_2+I\,t_2}\,A_1\,d_1\,d_2\,L_2\,\dot{A}_1}{w_1^3}-\frac{9\,I\,E^{I\,T_1\,s_2+I\,t_2}\,A_1\,d_2\,d_3\,L_2\,\dot{A}_1}{8\,w_1^3}-$$

$$\frac{2\,I\,E^{2\,I\,T_1\,s_2+2\,I\,t_2}\,d_2^2\,L_2^2\,\dot{A}_1}{w_1^3}+\frac{I\,E^{2\,I\,T_1\,s_2+2\,I\,t_2}\,d_1\,d_3\,L_2^2\,\dot{A}_1}{w_1^3}-\frac{I\,E^{2\,I\,T_1\,s_2+2\,I\,t_2}\,d_3^2\,L_2^2\,\dot{A}_1}{2\,w_1^3}-$$

$$\frac{3\,I\,E^{2\,I\,T_1\,s_2+2\,I\,t_2}\,d_2\,d_4\,L_2^2\,\dot{A}_1}{16\,w_1^3}-\frac{E^{I\,T_1\,s_1}\,A_2\,d_2\,m_1\,\dot{A}_1}{w_1^2}+\frac{E^{I\,T_1\,s_1}\,A_2\,d_2\,m_2\,\dot{A}_1}{4\,w_1^2}+$$

$$\frac{6\,I\,A_1^2\,a_1\,\dot{A}_1}{w_1}+\frac{3\,I\,E^{I\,T_1\,s_2+I\,t_2}\,A_1\,a_2\,L_2\,\dot{A}_1}{w_1}+\frac{I\,E^{2\,I\,T_1\,s_2+2\,I\,t_2}\,a_3\,L_2^2\,\dot{A}_1}{w_1}-\frac{I\,A_1\,A_2\,d_2^2\,\dot{A}_2}{4\,w_1^3}-$$

$$\frac{6\,I\,A_1\,A_2\,d_1\,d_3\,\dot{A}_2}{w_1^3}-\frac{4\,I\,A_1\,A_2\,d_3^2\,\dot{A}_2}{15\,w_1^3}-\frac{3\,I\,A_1\,A_2\,d_2\,d_4\,\dot{A}_2}{2\,w_1^3}-\frac{9\,I\,E^{I\,T_1\,s_2+I\,t_2}\,A_2\,d_2\,d_3\,L_2\,\dot{A}_2}{4\,w_1^3}-$$

$$\left.\frac{23\,I\,E^{I\,T_1\,s_2+I\,t_2}\,A_2\,d_3\,d_4\,L_2\,\dot{A}_2}{10\,w_1^3}+\frac{2\,I\,A_1\,A_2\,a_3\,\dot{A}_2}{w_1}+\frac{3\,I\,E^{I\,T_1\,s_2+I\,t_2}\,A_2\,a_4\,L_2\,\dot{A}_2}{w_1}\right),\ 2\,I\,w_2\,A_2' ==$$

$$2\,I\,e\left(-A_2\,m_2+\frac{I\,E^{-I\,T_1\,s_1}\,A_1^2\,d_2}{2\,w_2}+\frac{I\,E^{-I\,T_1\,s_1+I\,T_1\,s_2+I\,t_2}\,A_1\,d_3\,L_2}{w_2}+\frac{3\,I\,E^{-I\,T_1\,s_1+2\,I\,T_1\,s_2+2\,I\,t_2}\,d_4\,L_2^2}{2\,w_2}\right)w_2+$$

$$2\,I\,e^2\,w_2\left(-\frac{E^{-I\,T_1\,s_1}\,A_1^2\,d_2\,m_2}{16\,w_1^2}-\frac{E^{-I\,T_1\,s_1+I\,T_1\,s_2+I\,t_2}\,A_1\,d_3\,L_2\,m_2}{8\,w_1^2}-\frac{3\,E^{-I\,T_1\,s_1+2\,I\,T_1\,s_2+2\,I\,t_2}\,d_4\,L_2^2\,m_2}{16\,w_1^2}-\right.$$

$$\frac{I\,A_2\,m_2^2}{4\,w_1}+\frac{3\,I\,E^{-I\,T_1\,s_2-I\,t_2}\,A_1\,A_2\,a_4\,L_2}{w_2}+\frac{12\,I\,A_2\,a_5\,L_2^2}{w_2}+\frac{I\,E^{-I\,T_1\,s_1+I\,T_1\,s_2+I\,t_1}\,A_1\,f_1\,d_2}{8\,w_1^2\,w_2}+$$

$$\frac{I\,E^{-I\,T_1\,s_1+2\,I\,T_1\,s_2+I\,t_1+I\,t_2}\,f_1\,d_3\,L_2}{8\,w_1^2\,w_2}-\frac{9\,I\,E^{-I\,T_1\,s_2-I\,t_2}\,A_1\,A_2\,d_2\,d_3\,L_2}{4\,w_1^2\,w_2}-$$

$$\frac{23\,I\,E^{-I\,T_1\,s_2-I\,t_2}\,A_1\,A_2\,d_3\,d_4\,L_2}{10\,w_1^2\,w_2}-\frac{9\,I\,A_2\,d_3^2\,L_2^2}{4\,w_1^2\,w_2}-\frac{69\,I\,A_2\,d_4^2\,L_2^2}{10\,w_1^2\,w_2}+\frac{E^{-I\,T_1\,s_1}\,A_1^2\,d_2\,m_1}{2\,w_1\,w_2}+$$

$$\frac{E^{-I\,T_1\,s_1+I\,T_1\,s_2+I\,t_2}\,A_1\,d_3\,L_2\,m_1}{2\,w_1\,w_2}+\frac{2\,E^{-I\,T_1\,s_1+I\,T_1\,s_2+I\,t_2}\,A_1\,d_3\,L_2\,m_2}{3\,w_1\,w_2}+\frac{2\,E^{-I\,T_1\,s_1+2\,I\,T_1\,s_2+2\,I\,t_2}\,d_4\,L_2^2\,m_2}{w_1\,w_2}+$$

$$\frac{2\,I\,A_1\,A_2\,a_3\,\dot{A}_1}{w_2}+\frac{3\,I\,E^{I\,T_1\,s_2+I\,t_2}\,A_2\,a_4\,L_2\,\dot{A}_1}{w_2}-\frac{I\,A_1\,A_2\,d_2^2\,\dot{A}_1}{4\,w_1^2\,w_2}-\frac{6\,I\,A_1\,A_2\,d_1\,d_3\,\dot{A}_1}{w_1^2\,w_2}-$$

$$\frac{4\,I\,A_1\,A_2\,d_3^2\,\dot{A}_1}{15\,w_1^2\,w_2}-\frac{3\,I\,A_1\,A_2\,d_2\,d_4\,\dot{A}_1}{2\,w_1^2\,w_2}-\frac{9\,I\,E^{I\,T_1\,s_2+I\,t_2}\,A_2\,d_2\,d_3\,L_2\,\dot{A}_1}{4\,w_1^2\,w_2}-$$

$$\left.\frac{23\,I\,E^{I\,T_1\,s_2+I\,t_2}\,A_2\,d_3\,d_4\,L_2\,\dot{A}_1}{10\,w_1^2\,w_2}+\frac{6\,I\,A_2^2\,a_5\,\dot{A}_2}{w_2}-\frac{29\,I\,A_2^2\,d_3^2\,\dot{A}_2}{15\,w_1^2\,w_2}-\frac{15\,I\,A_2^2\,d_4^2\,\dot{A}_2}{4\,w_1^2\,w_2}\right)\}$$

{860.36 Second, Null}

6.1.4 $\Omega \approx \omega_1 + \omega_2$ and $\omega_2 \approx 2\,\omega_1$

In this case, we have a combination resonance and a two-to-one internal resonance. We let

```
scaling4 = {μn_ -> ε μn, F1 -> ε f1, F2 -> ε f2};

ResonanceCond4 = {ω2 == 2 ω1 + ε σ1, Ω == ω1 + ω2 + ε σ2};
```

Using **MMS**, we obtain the second-order approximate solution and the two equations governing the modulation of the complex-valued functions A_k:

```
MMS[scaling4, ResonanceCond4] // Timing
```

The second-order approximate solution:

$$\{u_1[t] == \epsilon\,(e^{I T_0 \omega_1} A_1 + e^{-I T_0 \Omega - I \tau_1} \Lambda_1 + e^{I T_0 \Omega + I \tau_1} \Lambda_1 + e^{-I T_0 \omega_1} \bar{A}_1) +$$

$$\epsilon^2 \Big(\frac{e^{2 I T_0 \omega_1} A_1^2 \delta_1}{\omega_1^2} + \frac{e^{I T_0 \omega_1 + I T_0 \omega_2} A_1^2 \delta_2}{4 \omega_1^2} + \frac{e^{2 I T_0 \omega_2} A_1^2 \delta_3}{15 \omega_1^2} + \frac{2 e^{-I T_0 \Omega - I \tau_1 + I T_0 \omega_1} A_1 \delta_1 \Lambda_1}{\omega_1^2} +$$

$$\frac{2 e^{I T_0 \Omega + I \tau_1 + I T_0 \omega_1} A_1 \delta_1 \Lambda_1}{5 \omega_1^2} - \frac{e^{-I T_0 \Omega - I \tau_1 + I T_0 \omega_2} A_1 \delta_2 \Lambda_1}{2 \omega_1^2} + \frac{e^{I T_0 \Omega + I \tau_1 + I T_0 \omega_2} A_1 \delta_2 \Lambda_1}{12 \omega_1^2} -$$

$$\frac{6 \delta_1 \Lambda_1^2}{\omega_1^2} + \frac{3 e^{-2 I T_0 \Omega - 2 I \tau_1} \delta_1 \Lambda_1^2}{35 \omega_1^2} + \frac{3 e^{2 I T_0 \Omega + 2 I \tau_1} \delta_1 \Lambda_1^2}{35 \omega_1^2} + \frac{2 e^{-I T_0 \Omega - I \tau_2 + I T_0 \omega_1} A_1 \delta_2 \Lambda_2}{3 \omega_1^2} +$$

$$\frac{2 e^{I T_0 \Omega + I \tau_2 + I T_0 \omega_1} A_1 \delta_2 \Lambda_2}{15 \omega_1^2} - \frac{e^{-I T_0 \Omega - I \tau_2 + I T_0 \omega_2} A_1 \delta_3 \Lambda_2}{2 \omega_1^2} + \frac{e^{I T_0 \Omega + I \tau_2 + I T_0 \omega_2} A_1 \delta_3 \Lambda_2}{12 \omega_1^2} +$$

$$\frac{2 e^{-2 I T_0 \Omega - I \tau_1 - I \tau_2} \delta_2 \Lambda_1 \Lambda_2}{35 \omega_1^2} - \frac{2 e^{I \tau_1 - I \tau_2} \delta_2 \Lambda_1 \Lambda_2}{\omega_1^2} - \frac{2 e^{-I \tau_1 + I \tau_2} \delta_2 \Lambda_1 \Lambda_2}{\omega_1^2} +$$

$$\frac{2 e^{2 I T_0 \Omega + I \tau_1 + I \tau_2} \delta_2 \Lambda_1 \Lambda_2}{35 \omega_1^2} - \frac{2 \delta_3 \Lambda_2^2}{\omega_1^2} + \frac{e^{-2 I T_0 \Omega - 2 I \tau_2} \delta_3 \Lambda_2^2}{35 \omega_1^2} + \frac{e^{2 I T_0 \Omega + 2 I \tau_2} \delta_3 \Lambda_2^2}{35 \omega_1^2} - \frac{I e^{I T_0 \omega_1} A_1 \mu_1}{2 \omega_1} -$$

$$\frac{3 I e^{-I T_0 \Omega - I \tau_1} \Lambda_1 \mu_1}{4 \omega_1} + \frac{3 I e^{I T_0 \Omega + I \tau_1} \Lambda_1 \mu_1}{4 \omega_1} - \frac{6 A_1 \delta_1 \bar{A}_1}{\omega_1^2} - \frac{e^{I T_0 \omega_1 - I T_0 \omega_2} A_1 \delta_2 \bar{A}_1}{2 \omega_1^2} -$$

$$\frac{e^{-I T_0 \omega_1 + I T_0 \omega_2} A_1 \delta_2 \bar{A}_1}{2 \omega_1^2} - \frac{2 A_1 \delta_3 \bar{A}_1}{\omega_1^2} + \frac{2 e^{-I T_0 \Omega - I \tau_1 - I T_0 \omega_1} \delta_1 \Lambda_1 \bar{A}_1}{5 \omega_1^2} + \frac{2 e^{I T_0 \Omega + I \tau_1 - I T_0 \omega_1} \delta_1 \Lambda_1 \bar{A}_1}{\omega_1^2} +$$

$$\frac{e^{-I T_0 \Omega - I \tau_1 - I T_0 \omega_2} \delta_2 \Lambda_1 \bar{A}_1}{12 \omega_1^2} - \frac{e^{I T_0 \Omega + I \tau_1 - I T_0 \omega_2} \delta_2 \Lambda_1 \bar{A}_1}{2 \omega_1^2} + \frac{2 e^{-I T_0 \Omega - I \tau_2 - I T_0 \omega_1} \delta_2 \Lambda_2 \bar{A}_1}{15 \omega_1^2} +$$

$$\frac{2 e^{I T_0 \Omega + I \tau_2 - I T_0 \omega_1} \delta_2 \Lambda_2 \bar{A}_1}{3 \omega_1^2} + \frac{e^{-I T_0 \Omega - I \tau_2 - I T_0 \omega_2} \delta_3 \Lambda_2 \bar{A}_1}{12 \omega_1^2} - \frac{e^{I T_0 \Omega + I \tau_2 - I T_0 \omega_2} \delta_3 \Lambda_2 \bar{A}_1}{2 \omega_1^2} +$$

$$\frac{I e^{-I T_0 \omega_1} \mu_1 \bar{A}_1}{2 \omega_1} + \frac{e^{-2 I T_0 \omega_1} \delta_1 \bar{A}_1^2}{\omega_1^2} + \frac{e^{-I T_0 \omega_1 - I T_0 \omega_2} \delta_2 \bar{A}_1^2}{4 \omega_1^2} + \frac{e^{-2 I T_0 \omega_2} \delta_3 \bar{A}_1^2}{15 \omega_1^2} \Big),$$

$$u_2[t] == \epsilon\,(e^{I T_0 \omega_2} A_2 + e^{-I T_0 \Omega - I \tau_2} \Lambda_2 + e^{I T_0 \Omega + I \tau_2} \Lambda_2 + e^{-I T_0 \omega_2} \bar{A}_2) +$$

$$\epsilon^2 \Big(-\frac{e^{2 I T_0 \omega_1} A_2^2 \delta_2}{16 \omega_1^2} + \frac{2 e^{I T_0 \omega_1 + I T_0 \omega_2} A_2^2 \delta_3}{5 \omega_1^2} + \frac{e^{2 I T_0 \omega_2} A_2^2 \delta_4}{4 \omega_1^2} - \frac{e^{-I T_0 \Omega - I \tau_1 + I T_0 \omega_1} A_2 \delta_2 \Lambda_1}{8 \omega_1^2} +$$

$$\frac{e^{I T_0 \Omega + I \tau_1 + I T_0 \omega_1} A_2 \delta_2 \Lambda_1}{6 \omega_1^2} - \frac{2 e^{-I T_0 \Omega - I \tau_1 + I T_0 \omega_2} A_2 \delta_3 \Lambda_1}{3 \omega_1^2} + \frac{2 e^{I T_0 \Omega + I \tau_1 + I T_0 \omega_2} A_2 \delta_3 \Lambda_1}{21 \omega_1^2} -$$

$$\frac{\delta_2 \Lambda_1^2}{2 \omega_1^2} + \frac{e^{-2 I T_0 \Omega - 2 I \tau_1} \delta_2 \Lambda_1^2}{32 \omega_1^2} + \frac{e^{2 I T_0 \Omega + 2 I \tau_1} \delta_2 \Lambda_1^2}{32 \omega_1^2} - \frac{e^{-I T_0 \Omega - I \tau_2 + I T_0 \omega_1} A_2 \delta_3 \Lambda_2}{8 \omega_1^2} +$$

$$\frac{E^{I T_0 W+I t_2+I T_0 w_1} A_2 d_3 L_2}{6 w_1^2} - \frac{2 E^{-I T_0 W-I t_2+I T_0 w_2} A_2 d_4 L_2}{w_1^2} + \frac{2 E^{I T_0 W+I t_2+I T_0 w_2} A_2 d_4 L_2}{7 w_1^2} +$$

$$\frac{E^{-2 I T_0 W-I t_1-I t_2} d_3 L_1 L_2}{16 w_1^2} - \frac{E^{I t_1-I t_2} d_3 L_1 L_2}{2 w_1^2} - \frac{E^{-I t_1+I t_2} d_3 L_1 L_2}{2 w_1^2} + \frac{E^{2 I T_0 W+I t_1+I t_2} d_3 L_1 L_2}{16 w_1^2} -$$

$$\frac{3 d_4 L_2^2}{2 w_1^2} + \frac{3 E^{-2 I T_0 W-2 I t_2} d_4 L_2^2}{32 w_1^2} + \frac{3 E^{2 I T_0 W+2 I t_2} d_4 L_2^2}{32 w_1^2} - \frac{I E^{I T_0 w_2} A_2 m_2}{4 w_1} -$$

$$\frac{6 I E^{-I T_0 W-I t_2} L_2 m_2}{5 w_1} + \frac{6 I E^{I T_0 W+I t_2} L_2 m_2}{5 w_1} - \frac{A_2 d_2 \dot{A}_2}{2 w_1^2} - \frac{2 E^{I T_0 w_1-I T_0 w_2} A_2 d_3 \dot{A}_2}{3 w_1^2} -$$

$$\frac{2 E^{-I T_0 w_1+I T_0 w_2} A_2 d_3 \dot{A}_2}{3 w_1^2} - \frac{3 A_2 d_4 \dot{A}_2}{2 w_1^2} + \frac{E^{-I T_0 W-I t_1-I T_0 w_1} d_2 L_1 \dot{A}_2}{6 w_1^2} - \frac{E^{I T_0 W+I t_1-I T_0 w_1} d_2 L_1 \dot{A}_2}{8 w_1^2} +$$

$$\frac{2 E^{-I T_0 W-I t_1-I T_0 w_2} d_3 L_1 \dot{A}_2}{21 w_1^2} - \frac{2 E^{I T_0 W+I t_1-I T_0 w_2} d_3 L_1 \dot{A}_2}{3 w_1^2} + \frac{E^{-I T_0 W-I t_2-I T_0 w_1} d_3 L_2 \dot{A}_2}{6 w_1^2} -$$

$$\frac{E^{I T_0 W+I t_2-I T_0 w_1} d_3 L_2 \dot{A}_2}{8 w_1^2} + \frac{2 E^{-I T_0 W-I t_2-I T_0 w_2} d_4 L_2 \dot{A}_2}{7 w_1^2} - \frac{2 E^{I T_0 W+I t_2-I T_0 w_2} d_4 L_2 \dot{A}_2}{w_1^2} +$$

$$\frac{I E^{-I T_0 w_2} m_2 \dot{A}_2}{4 w_1} - \frac{E^{-2 I T_0 w_1} d_2 \dot{A}_2^2}{16 w_1^2} + \frac{2 E^{-I T_0 w_1-I T_0 w_2} d_3 \dot{A}_2^2}{5 w_1^2} + \frac{E^{-2 I T_0 w_2} d_4 \dot{A}_2^2}{4 w_1^2} \Big\}\Big\}>$$

where

$$L_i == \frac{f_i}{2 (w_i^2 - W^2)}$$

The modulation equations:

$$:2 I w_1 A_1' ==$$

$$2 I e^2 w_1 \Big(-\frac{I E^{I T_1 s_1-I T_1 s_2-I t_1} A_2^2 d_2^2 L_1}{2 w_1^3} + \frac{I E^{I T_1 s_1-I T_1 s_2-I t_1} A_2^2 d_1 d_3 L_1}{5 w_1^3} - \frac{2 I E^{I T_1 s_1-I T_1 s_2-I t_1} A_2^2 d_3^2 L_1}{3 w_1^3} +$$

$$\frac{I E^{I T_1 s_1-I T_1 s_2-I t_1} A_2^2 d_2 d_4 L_1}{4 w_1^3} - \frac{54 I A_1 d_1^2 L_1^2}{5 w_1^3} - \frac{11 I A_1 d_2^2 L_1^2}{24 w_1^3} - \frac{13 I E^{I T_1 s_1-I T_1 s_2-I t_2} A_2^2 d_2 d_3 L_2}{30 w_1^3} -$$

$$\frac{7 I E^{I T_1 s_1-I T_1 s_2-I t_2} A_2^2 d_3 d_4 L_2}{4 w_1^3} - \frac{18 I E^{I t_1-I t_2} A_1 d_1 d_2 L_1 L_2}{5 w_1^3} - \frac{18 I E^{-I t_1+I t_2} A_1 d_1 d_2 L_1 L_2}{5 w_1^3} -$$

$$\frac{11 I E^{I t_1-I t_2} A_1 d_2 d_3 L_1 L_2}{24 w_1^3} - \frac{11 I E^{-I t_1+I t_2} A_1 d_2 d_3 L_1 L_2}{24 w_1^3} + \frac{4 I A_1 d_2^2 L_2^2}{5 w_1^3} - \frac{6 I A_1 d_1 d_3 L_2^2}{w_1^3} +$$

$$\frac{I A_1 d_3^2 L_2^2}{24 w_1^3} - \frac{3 I A_1 d_2 d_4 L_2^2}{2 w_1^3} + \frac{I E^{I T_1 s_1-I T_1 s_2-I t_1} A_2^2 a_3 L_1}{w_1} + \frac{12 I A_1 a_1 L_1^2}{w_1} +$$

$$\frac{3 I E^{I T_1 s_1-I T_1 s_2-I t_2} A_2^2 a_4 L_2}{2 w_1} + \frac{3 I E^{I t_1-I t_2} A_1 a_2 L_1 L_2}{w_1} + \frac{3 I E^{-I t_1+I t_2} A_1 a_2 L_1 L_2}{w_1} + \frac{2 I A_1 a_3 L_2^2}{w_1} -$$

$$\frac{I A_1 m_1^2}{2 w_1} - \frac{15 I A_1^2 d_1^2 \dot{A}_1}{w_1^3} - \frac{9 I A_1^2 d_2^2 \dot{A}_1}{16 w_1^3} - \frac{E^{I T_1 s_1} A_2 d_2 m_1 \dot{A}_1}{w_1^2} + \frac{E^{I T_1 s_1} A_2 d_2 m_2 \dot{A}_1}{4 w_1^2} + \frac{6 I A_1^2 a_1 \dot{A}_1}{w_1} +$$

$$\frac{9 I E^{I T_1 s_1+I T_1 s_2+I t_1} d_1^2 L_1 \dot{A}_1^2}{w_1^3} - \frac{3 I E^{I T_1 s_1+I T_1 s_2+I t_1} d_2^2 L_1 \dot{A}_1^2}{16 w_1^3} + \frac{3 I E^{I T_1 s_1+I T_1 s_2+I t_2} d_1 d_2 L_2 \dot{A}_1^2}{w_1^3} -$$

$$\frac{3 I E^{I T_1 s_1+I T_1 s_2+I t_2} d_2 d_3 L_2 \dot{A}_1^2}{16 w_1^3} + \frac{6 I E^{I T_1 s_1+I T_1 s_2+I t_1} a_1 L_1 \dot{A}_1^2}{w_1} + \frac{3 I E^{I T_1 s_1+I T_1 s_2+I t_2} a_2 L_2 \dot{A}_1^2}{2 w_1} -$$

$$\frac{I A_1 A_2 d_2^2 \dot{A}_2}{4 w_1^3} - \frac{6 I A_1 A_2 d_1 d_3 \dot{A}_2}{w_1^3} - \frac{4 I A_1 A_2 d_3^2 \dot{A}_2}{15 w_1^3} - \frac{3 I A_1 A_2 d_2 d_4 \dot{A}_2}{2 w_1^3} - \frac{5 E^{I T_1 s_2+I t_1} d_2 L_1 m_1 \dot{A}_2}{4 w_1^2} -$$

$$\frac{E^{I T_1 s_2+I t_2} d_3 L_2 m_1 \bar{A}_2}{2 w_1^2} - \frac{E^{I T_1 s_2+I t_1} d_2 L_1 m_2 \bar{A}_2}{4 w_1^2} - \frac{29 E^{I T_1 s_2+I t_2} d_3 L_2 m_2 \bar{A}_2}{20 w_1^2} + \frac{2 I A_1 A_2 a_3 \bar{A}_2}{w_1}\Bigg) +$$

$$2 I e w_1 \Bigg(-A_1 m_1 + \frac{I E^{I T_1 s_1} A_2 d_2 \bar{A}_1}{w_1} + \frac{I E^{I T_1 s_2+I t_1} d_2 L_1 \bar{A}_2}{w_1} + \frac{I E^{I T_1 s_2+I t_2} d_3 L_2 \bar{A}_2}{w_1}\Bigg),$$

$$2 I w_2 A_2' == 2 I e w_2 \Bigg(-A_2 m_2 + \frac{I E^{-I T_1 s_1} A_1^2 d_2}{2 w_2} + \frac{I E^{I T_1 s_2+I t_1} d_2 L_1 \bar{A}_1}{w_2} + \frac{I E^{I T_1 s_2+I t_2} d_3 L_2 \bar{A}_1}{w_2}\Bigg) +$$

$$2 I e^2 w_2 \Bigg(-\frac{E^{-I T_1 s_1} A_1^2 d_2 m_2}{16 w_1^2} - \frac{I A_2 m_2^2}{4 w_1} + \frac{2 I A_2 a_3 L_1^2}{w_2} + \frac{3 I E^{I t_1-I t_2} A_2 a_4 L_1 L_2}{w_2} +$$

$$\frac{3 I E^{-I t_1+I t_2} A_2 a_4 L_1 L_2}{w_2} + \frac{12 I A_2 a_5 L_2^2}{w_2} - \frac{5 I A_2 d_2^2 L_1^2}{12 w_1^2 w_2} - \frac{6 I A_2 d_1 d_3 L_1^2}{w_1^2 w_2} - \frac{4 I A_2 d_3^2 L_1^2}{7 w_1^2 w_2} -$$

$$\frac{3 I A_2 d_2 d_4 L_1^2}{2 w_1^2 w_2} - \frac{29 I E^{I t_1-I t_2} A_2 d_2 d_3 L_1 L_2}{12 w_1^2 w_2} - \frac{29 I E^{-I t_1+I t_2} A_2 d_2 d_3 L_1 L_2}{12 w_1^2 w_2} -$$

$$\frac{45 I E^{I t_1-I t_2} A_2 d_3 d_4 L_1 L_2}{14 w_1^2 w_2} - \frac{45 I E^{-I t_1+I t_2} A_2 d_3 d_4 L_1 L_2}{14 w_1^2 w_2} - \frac{29 I A_2 d_3^2 L_2^2}{12 w_1^2 w_2} -$$

$$\frac{135 I A_2 d_4^2 L_2^2}{14 w_1^2 w_2} + \frac{E^{-I T_1 s_1} A_1^2 d_2 m_1}{2 w_1 w_2} - \frac{E^{I T_1 s_2+I t_1} d_2 L_1 m_2 \bar{A}_1}{8 w_1^2} - \frac{E^{I T_1 s_2+I t_2} d_3 L_2 m_2 \bar{A}_1}{8 w_1^2} +$$

$$\frac{2 I A_1 A_2 a_3 \bar{A}_1}{w_2} - \frac{I A_1 A_2 d_2^2 \bar{A}_1}{4 w_1^2 w_2} - \frac{6 I A_1 A_2 d_1 d_3 \bar{A}_1}{w_1^2 w_2} - \frac{4 I A_1 A_2 d_3^2 \bar{A}_1}{15 w_1^2 w_2} - \frac{3 I A_1 A_2 d_2 d_4 \bar{A}_1}{2 w_1^2 w_2} -$$

$$\frac{5 E^{I T_1 s_2+I t_1} d_2 L_1 m_1 \bar{A}_1}{4 w_1 w_2} - \frac{E^{I T_1 s_2+I t_2} d_3 L_2 m_1 \bar{A}_1}{2 w_1 w_2} - \frac{6 E^{I T_1 s_2+I t_2} d_3 L_2 m_2 \bar{A}_1}{5 w_1 w_2} + \frac{6 I A_2^2 a_5 \bar{A}_2}{w_2} +$$

$$\frac{2 I E^{-I T_1 s_1+I T_1 s_2+I t_1} A_1 a_3 L_1 \bar{A}_2}{w_2} + \frac{3 I E^{-I T_1 s_1+I T_1 s_2+I t_2} A_1 a_4 L_2 \bar{A}_2}{w_2} - \frac{29 I A_2^2 d_3^2 \bar{A}_2}{15 w_1^2 w_2} -$$

$$\frac{15 I A_2^2 d_4^2 \bar{A}_2}{4 w_1^2 w_2} - \frac{I E^{-I T_1 s_1+I T_1 s_2+I t_1} A_1 d_2^2 L_1 \bar{A}_2}{w_1^2 w_2} + \frac{2 I E^{-I T_1 s_1+I T_1 s_2+I t_1} A_1 d_1 d_3 L_1 \bar{A}_2}{5 w_1^2 w_2} -$$

$$\frac{4 I E^{-I T_1 s_1+I T_1 s_2+I t_1} A_1 d_3^2 L_1 \bar{A}_2}{3 w_1^2 w_2} + \frac{I E^{-I T_1 s_1+I T_1 s_2+I t_1} A_1 d_2 d_4 L_1 \bar{A}_2}{2 w_1^2 w_2} -$$

$$\frac{13 I E^{-I T_1 s_1+I T_1 s_2+I t_2} A_1 d_2 d_3 L_2 \bar{A}_2}{15 w_1^2 w_2} - \frac{7 I E^{-I T_1 s_1+I T_1 s_2+I t_2} A_1 d_3 d_4 L_2 \bar{A}_2}{2 w_1^2 w_2}\Bigg)\Big\}$$

8192.266 Second, Null<

6.1.5 $\Omega \approx \omega_2$ and $\omega_2 \approx 3\omega_1$

In this case, we have a primary resonance of the second mode, a subharmonic resonance of the first mode, and a three-to-one internal resonance. We let

```
scaling5 = {μn_ -> e^2 μn, F1 -> e f1, F2 -> e^3 f2};
```

```
ResonanceCond5 = {w2 == 3 w1 + e^2 s1, W == w2 + e^2 s2};
```

Using **MMS**, we obtain the second-order approximate solution and the two equations governing the modulation of the complex-valued functions A_k:

```
MMS[scaling5, ResonanceCond5] // Timing
```

The second-order approximate solution:

$$\{u_1[t] == \epsilon\left(E^{I T_0 \omega_1} A_1 + E^{-I T_0 \Omega - I \tau_1}\Lambda_1 + E^{I T_0 \Omega + I \tau_1}\Lambda_1 + E^{-I T_0 \omega_1}\bar{A}_1\right) +$$

$$\epsilon^2\left(\frac{E^{2 I T_0 \omega_1} A_1^2 \delta_1}{\omega_1^2} + \frac{2 E^{I T_0 \omega_1 + I T_0 \omega_2} A_1 A_2 \delta_2}{15\,\omega_1^2} + \frac{E^{2 I T_0 \omega_2} A_2^2 \delta_3}{35\,\omega_1^2} + \frac{2 E^{-I T_0 \Omega - I \tau_1 + I T_0 \omega_1} A_1 \delta_1 \Lambda_1}{\omega_1^2} +\right.$$

$$\frac{2 E^{I T_0 \Omega + I \tau_1 + I T_0 \omega_1} A_1 \delta_1 \Lambda_1}{5\,\omega_1^2} - \frac{2 E^{-I T_0 \Omega - I \tau_1 + I T_0 \omega_2} A_2 \delta_2 \Lambda_1}{\omega_1^2} + \frac{2 E^{I T_0 \Omega + I \tau_1 + I T_0 \omega_2} A_2 \delta_2 \Lambda_1}{35\,\omega_1^2} -$$

$$\frac{6 \delta_1 \Lambda_1^2}{\omega_1^2} + \frac{3 E^{-2 I T_0 \Omega - 2 I \tau_1} \delta_1 \Lambda_1^2}{35\,\omega_1^2} + \frac{3 E^{2 I T_0 \Omega + 2 I \tau_1} \delta_1 \Lambda_1^2}{35\,\omega_1^2} - \frac{6 A_1 \delta_1 \bar{A}_1}{\omega_1^2} +$$

$$\frac{2 E^{-I T_0 \omega_1 + I T_0 \omega_2} A_2 \delta_2 \bar{A}_1}{3\,\omega_1^2} + \frac{2 E^{-I T_0 \Omega - I \tau_1 - I T_0 \omega_1} \delta_1 \Lambda_1 \bar{A}_1}{5\,\omega_1^2} + \frac{2 E^{I T_0 \Omega + I \tau_1 - I T_0 \omega_1} \delta_1 \Lambda_1 \bar{A}_1}{\omega_1^2} +$$

$$\frac{E^{-2 I T_0 \omega_1} \delta_1 \bar{A}_1^2}{\omega_1^2} + \frac{2 E^{I T_0 \omega_1 - I T_0 \omega_2} A_1 \delta_2 \bar{A}_2}{3\,\omega_1^2} - \frac{2 A_2 \delta_3 \bar{A}_2}{\omega_1^2} + \frac{2 E^{-I T_0 \Omega - I \tau_1 - I T_0 \omega_2} \delta_2 \Lambda_1 \bar{A}_2}{35\,\omega_1^2} -$$

$$\left.\frac{2 E^{I T_0 \Omega + I \tau_1 - I T_0 \omega_2} \delta_2 \Lambda_1 \bar{A}_2}{\omega_1^2} + \frac{2 E^{-I T_0 \omega_1 - I T_0 \omega_2} \delta_2 \bar{A}_1 \bar{A}_2}{15\,\omega_1^2} + \frac{E^{-2 I T_0 \omega_2} \delta_3 \bar{A}_2^2}{35\,\omega_1^2}\right),$$

$$u_2[t] == \epsilon\left(E^{I T_0 \omega_2} A_2 + E^{-I T_0 \omega_2}\bar{A}_2\right) + \epsilon^2\left(-\frac{E^{2 I T_0 \omega_1} A_1^2 \delta_2}{5\,\omega_1^2} + \frac{2 E^{I T_0 \omega_1 + I T_0 \omega_2} A_1 A_2 \delta_3}{7\,\omega_1^2} + \frac{E^{2 I T_0 \omega_2} A_2^2 \delta_4}{9\,\omega_1^2} -\right.$$

$$\frac{2 E^{-I T_0 \Omega - I \tau_1 + I T_0 \omega_1} A_1 \delta_2 \Lambda_1}{5\,\omega_1^2} + \frac{2 E^{I T_0 \Omega + I \tau_1 + I T_0 \omega_1} A_1 \delta_2 \Lambda_1}{7\,\omega_1^2} - \frac{2 E^{-I T_0 \Omega - I \tau_1 + I T_0 \omega_2} A_2 \delta_3 \Lambda_1}{9\,\omega_1^2} +$$

$$\frac{2 E^{I T_0 \Omega + I \tau_1 + I T_0 \omega_2} A_2 \delta_3 \Lambda_1}{27\,\omega_1^2} - \frac{2 \delta_2 \Lambda_1^2}{9\,\omega_1^2} + \frac{E^{-2 I T_0 \Omega - 2 I \tau_1} \delta_2 \Lambda_1^2}{27\,\omega_1^2} + \frac{E^{2 I T_0 \Omega + 2 I \tau_1} \delta_2 \Lambda_1^2}{27\,\omega_1^2} - \frac{2 A_1 \delta_2 \bar{A}_1}{9\,\omega_1^2} -$$

$$\frac{2 E^{-I T_0 \omega_1 + I T_0 \omega_2} A_2 \delta_3 \bar{A}_1}{5\,\omega_1^2} + \frac{2 E^{-I T_0 \Omega - I \tau_1 - I T_0 \omega_1} \delta_2 \Lambda_1 \bar{A}_1}{7\,\omega_1^2} - \frac{2 E^{I T_0 \Omega + I \tau_1 - I T_0 \omega_1} \delta_2 \Lambda_1 \bar{A}_1}{5\,\omega_1^2} -$$

$$\frac{E^{-2 I T_0 \omega_1} \delta_2 \bar{A}_1^2}{5\,\omega_1^2} - \frac{2 E^{I T_0 \omega_1 - I T_0 \omega_2} A_1 \delta_3 \bar{A}_2}{5\,\omega_1^2} - \frac{2 A_2 \delta_4 \bar{A}_2}{3\,\omega_1^2} + \frac{2 E^{-I T_0 \Omega - I \tau_1 - I T_0 \omega_2} \delta_3 \Lambda_1 \bar{A}_2}{27\,\omega_1^2} -$$

$$\left.\frac{2 E^{I T_0 \Omega + I \tau_1 - I T_0 \omega_2} \delta_3 \Lambda_1 \bar{A}_2}{9\,\omega_1^2} + \frac{2 E^{-I T_0 \omega_1 - I T_0 \omega_2} \delta_3 \bar{A}_1 \bar{A}_2}{7\,\omega_1^2} + \frac{E^{-2 I T_0 \omega_2} \delta_4 \bar{A}_2^2}{9\,\omega_1^2}\right)\}$$

where

$$\Lambda_i == \frac{f_i}{2\left(\omega_1^2 - \Omega^2\right)}$$

The modulation equations:

$$
\Big\{2\,i\,\omega_1 A_1' ==
2\,i\,\epsilon^2\omega_1\Bigg(-A_1\mu_1 - \frac{18\,i\,e^{-iT_2\sigma_2-i\tau_1}A_1A_2\delta_1\delta_2\Lambda_1}{5\omega_1^3} - \frac{106\,i\,e^{-iT_2\sigma_2-i\tau_1}A_1A_2\delta_2\delta_3\Lambda_1}{315\omega_1^3} - \frac{54\,i\,A_1\delta_1^2\Lambda_1^2}{5\omega_1^3} -
$$

$$
\frac{106\,i\,A_1\delta_2^2\Lambda_1^2}{315\omega_1^3} + \frac{3\,i\,e^{-iT_2\sigma_2-i\tau_1}A_1A_2\alpha_2\Lambda_1}{\omega_1} + \frac{12\,i\,A_1\alpha_1\Lambda_1^2}{\omega_1} - \frac{15\,i\,A_1^2\delta_1^2\bar{A}_1}{\omega_1^3} - \frac{19\,i\,A_1^2\delta_2^2\bar{A}_1}{45\omega_1^3} +
$$

$$
\frac{6\,i\,A_1^2\alpha_1\bar{A}_1}{\omega_1} + \frac{3\,i\,e^{iT_2\sigma_1}A_2\delta_1\delta_2\bar{A}_1^2}{\omega_1^3} - \frac{3\,i\,e^{iT_2\sigma_1}A_2\delta_2\delta_3\bar{A}_1^2}{5\omega_1^3} + \frac{9\,i\,e^{iT_2\sigma_1+iT_2\sigma_2+i\tau_1}\delta_1^2\Lambda_1\bar{A}_1^2}{\omega_1^3} -
$$

$$
\frac{3\,i\,e^{iT_2\sigma_1+iT_2\sigma_2+i\tau_1}\delta_2^2\Lambda_1\bar{A}_1^2}{5\omega_1^3} + \frac{3\,i\,e^{iT_2\sigma_1}A_2\alpha_2\bar{A}_1^2}{2\omega_1} + \frac{6\,i\,e^{iT_2\sigma_1+iT_2\sigma_2+i\tau_1}\alpha_1\Lambda_1\bar{A}_1^2}{\omega_1} + \frac{4\,i\,A_1A_2\delta_2^2\bar{A}_2}{5\omega_1^3} -
$$

$$
\frac{6\,i\,A_1A_2\delta_1\delta_3\bar{A}_2}{\omega_1^3} - \frac{4\,i\,A_1A_2\delta_3^2\bar{A}_2}{35\omega_1^3} - \frac{2\,i\,A_1A_2\delta_2\delta_4\bar{A}_2}{3\omega_1^3} - \frac{18\,i\,e^{iT_2\sigma_2+i\tau_1}A_1\delta_1\delta_2\Lambda_1\bar{A}_2}{5\omega_1^3} -
$$

$$
\frac{106\,i\,e^{iT_2\sigma_2+i\tau_1}A_1\delta_2\delta_3\Lambda_1\bar{A}_2}{315\omega_1^3} + \frac{2\,i\,A_1A_2\alpha_3\bar{A}_2}{\omega_1} + \frac{3\,i\,e^{iT_2\sigma_2+i\tau_1}A_1\alpha_2\Lambda_1\bar{A}_2}{\omega_1}\Bigg),
$$

$$
2\,i\,\omega_2 A_2' == 2\,i\,\epsilon^2\omega_2\Bigg(-A_2\mu_2 - \frac{i\,e^{iT_2\sigma_2+i\tau_2}f_2}{4\omega_2} + \frac{i\,e^{-iT_2\sigma_1}A_1^3\alpha_2}{2\omega_2} + \frac{3\,i\,e^{-iT_2\sigma_2-i\tau_1}A_2^2\alpha_4\Lambda_1}{2\omega_2} +
$$

$$
\frac{2\,i\,A_2\alpha_3\Lambda_1^2}{\omega_2} + \frac{3\,i\,e^{iT_2\sigma_2+i\tau_1}\alpha_2\Lambda_1^3}{2\omega_2} + \frac{i\,e^{-iT_2\sigma_1}A_1^3\delta_1\delta_2}{\omega_1^2\omega_2} - \frac{i\,e^{-iT_2\sigma_1}A_1^3\delta_2\delta_3}{5\omega_1^2\omega_2} -
$$

$$
\frac{69\,i\,e^{-iT_2\sigma_2-i\tau_1}A_2^2\delta_2\delta_3\Lambda_1}{35\omega_1^2\omega_2} - \frac{5\,i\,e^{-iT_2\sigma_2-i\tau_1}A_2^2\delta_3\delta_4\Lambda_1}{9\omega_1^2\omega_2} - \frac{68\,i\,A_2\delta_2^2\Lambda_1^2}{35\omega_1^2\omega_2} - \frac{6\,i\,A_2\delta_1\delta_3\Lambda_1^2}{\omega_1^2\omega_2} -
$$

$$
\frac{4\,i\,A_2\delta_3^2\Lambda_1^2}{27\omega_1^2\omega_2} - \frac{2\,i\,A_2\delta_2\delta_4\Lambda_1^2}{3\omega_1^2\omega_2} - \frac{207\,i\,e^{iT_2\sigma_2+i\tau_1}\delta_1\delta_2\Lambda_1^3}{35\omega_1^2\omega_2} - \frac{5\,i\,e^{iT_2\sigma_2+i\tau_1}\delta_2\delta_3\Lambda_1^3}{27\omega_1^2\omega_2} +
$$

$$
\frac{2\,i\,A_1A_2\alpha_3\bar{A}_1}{\omega_2} + \frac{3\,i\,e^{iT_2\sigma_2+i\tau_1}A_1\alpha_2\Lambda_1\bar{A}_1}{\omega_2} + \frac{4\,i\,A_1A_2\delta_2^2\bar{A}_1}{5\omega_1^2\omega_2} - \frac{6\,i\,A_1A_2\delta_1\delta_3\bar{A}_1}{\omega_1^2\omega_2} - \frac{4\,i\,A_1A_2\delta_3^2\bar{A}_1}{35\omega_1^2\omega_2} -
$$

$$
\frac{2\,i\,A_1A_2\delta_2\delta_4\bar{A}_1}{3\omega_1^2\omega_2} - \frac{18\,i\,e^{iT_2\sigma_2+i\tau_1}A_1\delta_1\delta_2\Lambda_1\bar{A}_1}{5\omega_1^2\omega_2} - \frac{106\,i\,e^{iT_2\sigma_2+i\tau_1}A_1\delta_2\delta_3\Lambda_1\bar{A}_1}{315\omega_1^2\omega_2} +
$$

$$
\frac{6\,i\,A_2^2\alpha_5\bar{A}_2}{\omega_2} + \frac{3\,i\,e^{iT_2\sigma_2+i\tau_1}A_2\alpha_4\Lambda_1\bar{A}_2}{\omega_2} + \frac{i\,e^{2iT_2\sigma_2+2i\tau_1}\alpha_3\Lambda_1^2\bar{A}_2}{\omega_2} - \frac{69\,i\,A_2^2\delta_3^2\bar{A}_2}{35\omega_1^2\omega_2} - \frac{5\,i\,A_2^2\delta_4^2\bar{A}_2}{3\omega_1^2\omega_2} -
$$

$$
\frac{138\,i\,e^{iT_2\sigma_2+i\tau_1}A_2\delta_2\delta_3\Lambda_1\bar{A}_2}{35\omega_1^2\omega_2} - \frac{10\,i\,e^{iT_2\sigma_2+i\tau_1}A_2\delta_3\delta_4\Lambda_1\bar{A}_2}{9\omega_1^2\omega_2} - \frac{2\,i\,e^{2iT_2\sigma_2+2i\tau_1}\delta_2^2\Lambda_1^2\bar{A}_2}{\omega_1^2\omega_2} +
$$

$$
\frac{3\,i\,e^{2iT_2\sigma_2+2i\tau_1}\delta_1\delta_3\Lambda_1^2\bar{A}_2}{35\omega_1^2\omega_2} - \frac{2\,i\,e^{2iT_2\sigma_2+2i\tau_1}\delta_3^2\Lambda_1^2\bar{A}_2}{9\omega_1^2\omega_2} + \frac{i\,e^{2iT_2\sigma_2+2i\tau_1}\delta_2\delta_4\Lambda_1^2\bar{A}_2}{9\omega_1^2\omega_2}\Bigg)\Big\}
$$

866.906 Second, Null}

6.1.6 $\Omega \approx \omega_2 - 2\omega_1$ and $\omega_2 \approx 3\omega_1$

In this case, we have a primary resonance of the first mode, a superharmonic resonance of the second mode, and a three-to-one internal resonance. We let

```
scaling6 = {μn_ -> ε^2 μn, F1 -> ε^3 f1, F2 -> ε f2};
```

```
ResonanceCond6 = {ω2 == 3 ω1 + ε^2 σ1, Ω == ω2 - 2 ω1 + ε^2 σ2};
```

Using **MMS**, we obtain the second-order approximate solution and the two equations governing the modulation of the complex-valued functions A_k:

```
MMS@scaling6, ResonanceCond6D •• Timing
```

The second-order approximate solution:

$$u_1[t] == \epsilon\left(E^{I T_0 \omega_1} A_1 + E^{-I T_0 \omega_1} \bar{A}_1\right) + \epsilon^2\left(\frac{E^{2 I T_0 \omega_1} A_1^2 d_1}{\omega_1^2} + \frac{2 E^{I T_0 \omega_1 + I T_0 \omega_2} A_1^2 d_2}{15 \omega_1^2} + \frac{E^{2 I T_0 \omega_2} A_1^2 d_3}{35 \omega_1^2} - \frac{2 E^{-I T_0 \Omega - I t_2 + I T_0 \omega_1} A_1 d_2 L_2}{\omega_1^2} + \frac{2 E^{I T_0 \Omega + I t_2 + I T_0 \omega_1} A_1 d_2 L_2}{3 \omega_1^2} + \frac{2 E^{-I T_0 \Omega - I t_2 + I T_0 \omega_2} A_1 d_3 L_2}{3 \omega_1^2} + \frac{2 E^{I T_0 \Omega + I t_2 + I T_0 \omega_2} A_1 d_3 L_2}{15 \omega_1^2} - \frac{2 d_3 L_2^2}{\omega_1^2} + \frac{E^{-2 I T_0 \Omega - 2 I t_2} d_3 L_2^2}{3 \omega_1^2} + \frac{E^{2 I T_0 \Omega + 2 I t_2} d_3 L_2^2}{3 \omega_1^2} - \frac{6 A_1 d_1 \bar{A}_1}{\omega_1^2} + \frac{2 E^{I T_0 \omega_1 - I T_0 \omega_2} A_1 d_2 \bar{A}_1}{3 \omega_1^2} + \frac{2 E^{-I T_0 \omega_1 + I T_0 \omega_2} A_1 d_2 \bar{A}_1}{3 \omega_1^2} - \frac{2 A_1 d_3 \bar{A}_1}{\omega_1^2} + \frac{2 E^{-I T_0 \Omega - I t_2 - I T_0 \omega_1} d_2 L_2 \bar{A}_1}{3 \omega_1^2} - \frac{2 E^{I T_0 \Omega + I t_2 - I T_0 \omega_1} d_2 L_2 \bar{A}_1}{\omega_1^2} + \frac{2 E^{-I T_0 \Omega - I t_2 - I T_0 \omega_2} d_3 L_2 \bar{A}_1}{15 \omega_1^2} + \frac{2 E^{I T_0 \Omega + I t_2 - I T_0 \omega_2} d_3 L_2 \bar{A}_1}{3 \omega_1^2} + \frac{E^{-2 I T_0 \omega_1} d_1 \bar{A}_1^2}{\omega_1^2} + \frac{2 E^{-I T_0 \omega_1 - I T_0 \omega_2} d_2 \bar{A}_1^2}{15 \omega_1^2} + \frac{E^{-2 I T_0 \omega_2} d_3 \bar{A}_1^2}{35 \omega_1^2}\right),$$

$$u_2[t] == \epsilon\left(E^{I T_0 \omega_2} A_2 + E^{-I T_0 \Omega - I t_2} L_2 + E^{I T_0 \Omega + I t_2} L_2 + E^{-I T_0 \omega_2} \bar{A}_2\right) + \epsilon^2\left(-\frac{E^{2 I T_0 \omega_1} A_2^2 d_2}{5 \omega_1^2} + \frac{2 E^{I T_0 \omega_1 + I T_0 \omega_2} A_2^2 d_3}{7 \omega_1^2} + \frac{E^{2 I T_0 \omega_2} A_2^2 d_4}{9 \omega_1^2} - \frac{2 E^{-I T_0 \Omega - I t_2 + I T_0 \omega_1} A_2 d_3 L_2}{9 \omega_1^2} - \frac{2 E^{I T_0 \Omega + I t_2 + I T_0 \omega_1} A_2 d_3 L_2}{5 \omega_1^2} - \frac{6 E^{-I T_0 \Omega - I t_2 + I T_0 \omega_2} A_2 d_4 L_2}{5 \omega_1^2} + \frac{6 E^{I T_0 \Omega + I t_2 + I T_0 \omega_2} A_2 d_4 L_2}{7 \omega_1^2} - \frac{2 d_4 L_2^2}{3 \omega_1^2} - \frac{3 E^{-2 I T_0 \Omega - 2 I t_2} d_4 L_2^2}{5 \omega_1^2} - \frac{3 E^{2 I T_0 \Omega + 2 I t_2} d_4 L_2^2}{5 \omega_1^2} - \frac{2 A_2 d_2 \bar{A}_2}{9 \omega_1^2} - \frac{2 E^{I T_0 \omega_1 - I T_0 \omega_2} A_2 d_3 \bar{A}_2}{5 \omega_1^2} - \frac{2 E^{-I T_0 \omega_1 + I T_0 \omega_2} A_2 d_3 \bar{A}_2}{5 \omega_1^2} - \frac{2 A_2 d_4 \bar{A}_2}{3 \omega_1^2} - \frac{2 E^{-I T_0 \Omega - I t_2 - I T_0 \omega_1} d_3 L_2 \bar{A}_2}{5 \omega_1^2} - \frac{2 E^{I T_0 \Omega + I t_2 - I T_0 \omega_1} d_3 L_2 \bar{A}_2}{9 \omega_1^2} + \frac{6 E^{-I T_0 \Omega - I t_2 - I T_0 \omega_2} d_4 L_2 \bar{A}_2}{7 \omega_1^2} - \frac{6 E^{I T_0 \Omega + I t_2 - I T_0 \omega_2} d_4 L_2 \bar{A}_2}{5 \omega_1^2} - \frac{E^{-2 I T_0 \omega_1} d_2 \bar{A}_2^2}{5 \omega_1^2} + \frac{2 E^{-I T_0 \omega_1 - I T_0 \omega_2} d_3 \bar{A}_2^2}{7 \omega_1^2} + \frac{E^{-2 I T_0 \omega_2} d_4 \bar{A}_2^2}{9 \omega_1^2}\right)$$

where

$$L_i == \frac{f_i}{2\left(\omega_i^2 - \Omega^2\right)}$$

The modulation equations:

$$2 I \omega_1 A_1' ==$$

$$2 I \epsilon^2 \omega_1\left(-A_1 \mu_1 - \frac{5 I E^{-I T_2 \sigma_1 - I T_2 \sigma_2 - I t_2} A_1^2 d_1 d_2 L_2}{\omega_1^3} - \frac{19 I E^{-I T_2 \sigma_1 - I T_2 \sigma_2 - I t_2} A_1^2 d_2 d_3 L_2}{45 \omega_1^3} - \frac{4 I A_1 d_2^2 L_2^2}{3 \omega_1^3} - \frac{6 I A_1 d_1 d_3 L_2^2}{\omega_1^3} + \frac{I E^{-I T_2 \sigma_1 - 2 I T_2 \sigma_2 - 2 I t_2} A_2 d_2 d_3 L_2^2}{\omega_1^3} - \frac{28 I A_1 d_3^2 L_2^2}{45 \omega_1^3} - \right.$$

$$\frac{2\,I\,A_1\,d_2\,d_4\,L_2^2}{3\,w_1^3}-\frac{9\,I\,E^{-I\,T_2\,s_1-2\,I\,T_2\,s_2-2\,I\,t_2}\,A_2\,d_3\,d_4\,L_2^2}{5\,w_1^3}-\frac{5\,I\,E^{I\,T_2\,s_1+I\,T_2\,s_2+I\,t_2}\,d_2\,d_3\,L_2^3}{3\,w_1^3}-$$

$$\frac{19\,I\,E^{I\,T_2\,s_1+I\,T_2\,s_2+I\,t_2}\,d_3\,d_4\,L_2^3}{15\,w_1^3}-\frac{I\,E^{I\,T_2\,s_1+I\,T_2\,s_2+I\,t_1}\,f_1}{4\,w_1}+\frac{3\,I\,E^{-I\,T_2\,s_1-I\,T_2\,s_2-I\,t_2}\,A_1^2\,a_2\,L_2}{2\,w_1}+$$

$$\frac{2\,I\,A_1\,a_3\,L_2^2}{w_1}+\frac{3\,I\,E^{-I\,T_2\,s_1-2\,I\,T_2\,s_2-2\,I\,t_2}\,A_2\,a_4\,L_2^2}{2\,w_1}+\frac{3\,I\,E^{I\,T_2\,s_1+I\,T_2\,s_2+I\,t_2}\,a_4\,L_2^3}{2\,w_1}-\frac{15\,I\,A_1^2\,d_1^2\,\dot{A}_1}{w_1^3}-$$

$$\frac{19\,I\,A_1^2\,d_2^2\,\dot{A}_1}{45\,w_1^3}-\frac{10\,I\,E^{I\,T_2\,s_1+I\,T_2\,s_2+I\,t_2}\,A_1\,d_1\,d_2\,L_2\,\dot{A}_1}{w_1^3}+\frac{4\,I\,E^{-I\,T_2\,s_2-I\,t_2}\,A_2\,d_2^2\,L_2\,\dot{A}_1}{3\,w_1^3}+$$

$$\frac{2\,I\,E^{-I\,T_2\,s_2-I\,t_2}\,A_2\,d_1\,d_3\,L_2\,\dot{A}_1}{w_1^3}-\frac{38\,I\,E^{I\,T_2\,s_1+I\,T_2\,s_2+I\,t_2}\,A_1\,d_2\,d_3\,L_2\,\dot{A}_1}{45\,w_1^3}-\frac{4\,I\,E^{-I\,T_2\,s_2-I\,t_2}\,A_2\,d_3^2\,L_2\,\dot{A}_1}{5\,w_1^3}-$$

$$\frac{6\,I\,E^{-I\,T_2\,s_2-I\,t_2}\,A_2\,d_2\,d_4\,L_2\,\dot{A}_1}{5\,w_1^3}-\frac{2\,I\,E^{2\,I\,T_2\,s_1+2\,I\,T_2\,s_2+2\,I\,t_2}\,d_2^2\,L_2^2\,\dot{A}_1}{w_1^3}+\frac{I\,E^{2\,I\,T_2\,s_1+2\,I\,T_2\,s_2+2\,I\,t_2}\,d_1\,d_3\,L_2^2\,\dot{A}_1}{w_1^3}-$$

$$\frac{2\,I\,E^{2\,I\,T_2\,s_1+2\,I\,T_2\,s_2+2\,I\,t_2}\,d_3^2\,L_2^2\,\dot{A}_1}{9\,w_1^3}-\frac{3\,I\,E^{2\,I\,T_2\,s_1+2\,I\,T_2\,s_2+2\,I\,t_2}\,d_2\,d_4\,L_2^2\,\dot{A}_1}{5\,w_1^3}+\frac{6\,I\,A_1^2\,a_1\,\dot{A}_1}{w_1}+$$

$$\frac{3\,I\,E^{I\,T_2\,s_1+I\,T_2\,s_2+I\,t_2}\,A_1\,a_2\,L_2\,\dot{A}_1}{w_1}+\frac{2\,I\,E^{-I\,T_2\,s_2-I\,t_2}\,A_2\,a_3\,L_2\,\dot{A}_1}{w_1}+\frac{I\,E^{2\,I\,T_2\,s_1+2\,I\,T_2\,s_2+2\,I\,t_2}\,a_3\,L_2^2\,\dot{A}_1}{w_1}+$$

$$\frac{3\,I\,E^{I\,T_2\,s_1}\,A_2\,d_1\,d_2\,\dot{A}_1^2}{w_1^3}-\frac{3\,I\,E^{I\,T_2\,s_1}\,A_2\,d_2\,d_3\,\dot{A}_1^2}{5\,w_1^3}+\frac{3\,I\,E^{I\,T_2\,s_1}\,A_2\,a_2\,\dot{A}_1^2}{2\,w_1}+\frac{4\,I\,A_1\,A_2\,d_2^2\,\dot{A}_2}{5\,w_1^3}-$$

$$\frac{6\,I\,A_1\,A_2\,d_1\,d_3\,\dot{A}_2}{w_1^3}-\frac{4\,I\,A_1\,A_2\,d_3^2\,\dot{A}_2}{35\,w_1^3}-\frac{2\,I\,A_1\,A_2\,d_2\,d_4\,\dot{A}_2}{3\,w_1^3}-\frac{6\,I\,E^{I\,T_2\,s_1+I\,T_2\,s_2+I\,t_2}\,A_2\,d_2\,d_3\,L_2\,\dot{A}_2}{5\,w_1^3}-$$

$$\left.\frac{106\,I\,E^{I\,T_2\,s_1+I\,T_2\,s_2+I\,t_2}\,A_2\,d_3\,d_4\,L_2\,\dot{A}_2}{105\,w_1^3}+\frac{2\,I\,A_1\,A_2\,a_3\,\dot{A}_2}{w_1}+\frac{3\,I\,E^{I\,T_2\,s_1+I\,T_2\,s_2+I\,t_2}\,A_2\,a_4\,L_2\,\dot{A}_2}{w_1}\right),$$

$$2\,I\,w_2\,A_2' == 2\,I\,e^2\,w_2\left(-A_2\,m_2+\frac{I\,E^{-I\,T_2\,s_1}\,A_1^3\,a_2}{2\,w_2}+\frac{I\,E^{I\,T_2\,s_2+I\,t_2}\,A_1^2\,a_3\,L_2}{w_2}+\frac{3\,I\,E^{-I\,T_2\,s_1-I\,T_2\,s_2-I\,t_2}\,A_1\,A_2\,a_4\,L_2}{w_2}+\right.$$

$$\frac{3\,I\,E^{I\,T_2\,s_1+2\,I\,T_2\,s_2+2\,I\,t_2}\,A_1\,a_4\,L_2^2}{2\,w_2}+\frac{12\,I\,A_2\,a_5\,L_2^2}{w_2}+\frac{2\,I\,E^{2\,I\,T_2\,s_1+3\,I\,T_2\,s_2+3\,I\,t_2}\,a_5\,L_2^3}{w_2}+$$

$$\frac{I\,E^{-I\,T_2\,s_1}\,A_1^3\,d_1\,d_2}{w_1^2\,w_2}-\frac{I\,E^{-I\,T_2\,s_1}\,A_1^3\,d_2\,d_3}{5\,w_1^2\,w_2}+\frac{2\,I\,E^{I\,T_2\,s_2+I\,t_2}\,A_1^2\,d_2^2\,L_2}{3\,w_1^2\,w_2}+\frac{I\,E^{I\,T_2\,s_2+I\,t_2}\,A_1^2\,d_1\,d_3\,L_2}{w_1^2\,w_2}-$$

$$\frac{6\,I\,E^{-I\,T_2\,s_1-I\,T_2\,s_2-I\,t_2}\,A_1\,A_2\,d_2\,d_3\,L_2}{5\,w_1^2\,w_2}-\frac{2\,I\,E^{I\,T_2\,s_2+I\,t_2}\,A_1^2\,d_3^2\,L_2}{5\,w_1^2\,w_2}-\frac{3\,I\,E^{I\,T_2\,s_2+I\,t_2}\,A_1^2\,d_2\,d_4\,L_2}{5\,w_1^2\,w_2}-$$

$$\frac{106\,I\,E^{-I\,T_2\,s_1-I\,T_2\,s_2-I\,t_2}\,A_1\,A_2\,d_3\,d_4\,L_2}{105\,w_1^2\,w_2}+\frac{I\,E^{I\,T_2\,s_1+2\,I\,T_2\,s_2+2\,I\,t_2}\,A_1\,d_2\,d_3\,L_2^2}{w_1^2\,w_2}-\frac{6\,I\,A_2\,d_3^2\,L_2^2}{5\,w_1^2\,w_2}-$$

$$\frac{9\,I\,E^{I\,T_2\,s_1+2\,I\,T_2\,s_2+2\,I\,t_2}\,A_1\,d_3\,d_4\,L_2^2}{5\,w_1^2\,w_2}-\frac{106\,I\,A_2\,d_4^2\,L_2^2}{35\,w_1^2\,w_2}+\frac{I\,E^{2\,I\,T_2\,s_1+3\,I\,T_2\,s_2+3\,I\,t_2}\,d_3^2\,L_2^3}{3\,w_1^2\,w_2}-$$

$$\frac{9\,I\,E^{2\,I\,T_2\,s_1+3\,I\,T_2\,s_2+3\,I\,t_2}\,d_4^2\,L_2^3}{5\,w_1^2\,w_2}+\frac{2\,I\,A_1\,A_2\,a_3\,\dot{A}_1}{w_2}+\frac{3\,I\,E^{I\,T_2\,s_1+I\,T_2\,s_2+I\,t_2}\,A_2\,a_4\,L_2\,\dot{A}_1}{w_2}+\frac{4\,I\,A_1\,A_2\,d_2^2\,\dot{A}_1}{5\,w_1^2\,w_2}-$$

$$\frac{6\,I\,A_1\,A_2\,d_1\,d_3\,\dot{A}_1}{w_1^2\,w_2}-\frac{4\,I\,A_1\,A_2\,d_3^2\,\dot{A}_1}{35\,w_1^2\,w_2}-\frac{2\,I\,A_1\,A_2\,d_2\,d_4\,\dot{A}_1}{3\,w_1^2\,w_2}-\frac{6\,I\,E^{I\,T_2\,s_1+I\,T_2\,s_2+I\,t_2}\,A_2\,d_2\,d_3\,L_2\,\dot{A}_1}{5\,w_1^2\,w_2}-$$

$$\left.\frac{106\,I\,E^{I\,T_2\,s_1+I\,T_2\,s_2+I\,t_2}\,A_2\,d_3\,d_4\,L_2\,\dot{A}_1}{105\,w_1^2\,w_2}+\frac{6\,I\,A_2^2\,a_5\,\dot{A}_2}{w_2}-\frac{69\,I\,A_2^2\,d_3^2\,\dot{A}_2}{35\,w_1^2\,w_2}-\frac{5\,I\,A_2^2\,d_4^2\,\dot{A}_2}{3\,w_1^2\,w_2}\right)>$$

872.154 Second, Null<

6.1.7 $\Omega \approx \omega_1 + \omega_2$ and $\omega_2 \approx \omega_1$

In this case, we have a subharmonic resonance of either mode and a one-to-one internal resonance. We let

```
scaling7 = {μ_n_ -> ε μ_n, F_1 -> ε f_1, F_2 -> ε f_2};

ResonanceCond7 = {ω_2 == ω_1 + ε σ_1, Ω == ω_1 + ω_2 + ε σ_2};
```

Using **MMS**, we obtain the second-order approximate solution and the two equations governing the modulation of the complex-valued functions A_k:

```
MMS[scaling7, ResonanceCond7] // Timing
```

The second-order approximate solution:

$$\Big\{u_1[t] == \epsilon\left(e^{iT_0\omega_1}A_1 + e^{-iT_0\Omega - i\tau_1}\Lambda_1 + e^{iT_0\Omega + i\tau_1}\Lambda_1 + e^{-iT_0\omega_1}\bar{A}_1\right) +$$

$$\epsilon^2\Big(\frac{e^{2iT_0\omega_1}A_1^2\delta_1}{\omega_1^2} + \frac{2e^{iT_0\omega_1+iT_0\omega_2}A_1^2\delta_2}{3\omega_1^2} + \frac{e^{2iT_0\omega_2}A_1^2\delta_3}{3\omega_1^2} - \frac{3e^{-iT_0\Omega-i\tau_1+iT_0\omega_1}A_1\delta_1\Lambda_1}{2\omega_1^2} +$$

$$\frac{3e^{iT_0\Omega+i\tau_1+iT_0\omega_1}A_1\delta_1\Lambda_1}{4\omega_1^2} - \frac{e^{-iT_0\Omega-i\tau_1+iT_0\omega_2}A_1\delta_2\Lambda_1}{2\omega_1^2} + \frac{e^{iT_0\Omega+i\tau_1+iT_0\omega_2}A_1\delta_2\Lambda_1}{4\omega_1^2} - \frac{6\delta_1\Lambda_1^2}{\omega_1^2} +$$

$$\frac{e^{-2iT_0\Omega-2i\tau_1}\delta_1\Lambda_1^2}{5\omega_1^2} + \frac{e^{2iT_0\Omega+2i\tau_1}\delta_1\Lambda_1^2}{5\omega_1^2} - \frac{e^{-iT_0\Omega-i\tau_2+iT_0\omega_1}A_1\delta_2\Lambda_2}{2\omega_1^2} + \frac{e^{iT_0\Omega+i\tau_2+iT_0\omega_1}A_1\delta_2\Lambda_2}{4\omega_1^2} -$$

$$\frac{e^{-iT_0\Omega-i\tau_2+iT_0\omega_2}A_1\delta_3\Lambda_2}{2\omega_1^2} + \frac{e^{iT_0\Omega+i\tau_2+iT_0\omega_2}A_1\delta_3\Lambda_2}{4\omega_1^2} + \frac{2e^{-2iT_0\Omega-i\tau_1-i\tau_2}\delta_2\Lambda_1\Lambda_2}{15\omega_1^2} -$$

$$\frac{2e^{i\tau_1-i\tau_2}\delta_2\Lambda_1\Lambda_2}{\omega_1^2} - \frac{2e^{-i\tau_1+i\tau_2}\delta_2\Lambda_1\Lambda_2}{\omega_1^2} + \frac{2e^{2iT_0\Omega+i\tau_1+i\tau_2}\delta_2\Lambda_1\Lambda_2}{15\omega_1^2} - \frac{2\delta_3\Lambda_2^2}{\omega_1^2} +$$

$$\frac{e^{-2iT_0\Omega-2i\tau_2}\delta_3\Lambda_2^2}{15\omega_1^2} + \frac{e^{2iT_0\Omega+2i\tau_2}\delta_3\Lambda_2^2}{15\omega_1^2} - \frac{i e^{iT_0\omega_1}A_1\mu_1}{2\omega_1} - \frac{4i e^{-iT_0\Omega-i\tau_1}\Lambda_1\mu_1}{3\omega_1} +$$

$$\frac{4i e^{iT_0\Omega+i\tau_1}\Lambda_1\mu_1}{3\omega_1} - \frac{6A_1\delta_1\bar{A}_1}{\omega_1^2} - \frac{2e^{iT_0\omega_1-iT_0\omega_2}A_1\delta_2\bar{A}_1}{\omega_1^2} - \frac{2e^{-iT_0\omega_1+iT_0\omega_2}A_1\delta_2\bar{A}_1}{\omega_1^2} -$$

$$\frac{2A_1\delta_3\bar{A}_1}{\omega_1^2} + \frac{3e^{-iT_0\Omega-i\tau_1-iT_0\omega_1}\delta_1\Lambda_1\bar{A}_1}{4\omega_1^2} - \frac{3e^{iT_0\Omega+i\tau_1-iT_0\omega_1}\delta_1\Lambda_1\bar{A}_1}{2\omega_1^2} +$$

$$\frac{e^{-iT_0\Omega-i\tau_1-iT_0\omega_2}\delta_2\Lambda_1\bar{A}_1}{4\omega_1^2} - \frac{e^{iT_0\Omega+i\tau_1-iT_0\omega_2}\delta_2\Lambda_1\bar{A}_1}{2\omega_1^2} + \frac{e^{-iT_0\Omega-i\tau_2-iT_0\omega_1}\delta_2\Lambda_2\bar{A}_1}{4\omega_1^2} -$$

$$\frac{e^{iT_0\Omega+i\tau_2-iT_0\omega_1}\delta_2\Lambda_2\bar{A}_1}{2\omega_1^2} + \frac{e^{-iT_0\Omega-i\tau_2-iT_0\omega_2}\delta_3\Lambda_2\bar{A}_1}{4\omega_1^2} - \frac{e^{iT_0\Omega+i\tau_2-iT_0\omega_2}\delta_3\Lambda_2\bar{A}_1}{2\omega_1^2} +$$

$$\frac{i e^{-iT_0\omega_1}\mu_1\bar{A}_1}{2\omega_1} + \frac{e^{-2iT_0\omega_1}\delta_1\bar{A}_1^2}{\omega_1^2} + \frac{2e^{-iT_0\omega_1-iT_0\omega_2}\delta_2\bar{A}_1^2}{3\omega_1^2} + \frac{e^{-2iT_0\omega_2}\delta_3\bar{A}_1^2}{3\omega_1^2}\Big),$$

$$u_2[t] == \epsilon\left(e^{iT_0\omega_2}A_2 + e^{-iT_0\Omega - i\tau_2}\Lambda_2 + e^{iT_0\Omega + i\tau_2}\Lambda_2 + e^{-iT_0\omega_2}\bar{A}_2\right) +$$

$$\epsilon^2\Big(\frac{e^{2iT_0\omega_1}A_2^2\delta_2}{3\omega_1^2} + \frac{2e^{iT_0\omega_1+iT_0\omega_2}A_2^2\delta_3}{3\omega_1^2} + \frac{e^{2iT_0\omega_2}A_2^2\delta_4}{\omega_1^2} - \frac{e^{-iT_0\Omega-i\tau_1+iT_0\omega_1}A_2\delta_2\Lambda_1}{2\omega_1^2} +$$

$$\frac{e^{iT_0\Omega+i\tau_1+iT_0\omega_1}A_2\delta_2\Lambda_1}{4\omega_1^2} - \frac{e^{-iT_0\Omega-i\tau_1+iT_0\omega_2}A_2\delta_3\Lambda_1}{2\omega_1^2} + \frac{e^{iT_0\Omega+i\tau_1+iT_0\omega_2}A_2\delta_3\Lambda_1}{4\omega_1^2} -$$

$$\frac{2\delta_2\Lambda_1^2}{\omega_1^2} + \frac{e^{-2iT_0\Omega-2i\tau_1}\delta_2\Lambda_1^2}{15\omega_1^2} + \frac{e^{2iT_0\Omega+2i\tau_1}\delta_2\Lambda_1^2}{15\omega_1^2} - \frac{e^{-iT_0\Omega-i\tau_2+iT_0\omega_1}A_2\delta_3\Lambda_2}{2\omega_1^2} +$$

$$\frac{E^{I T_0 W+I t_2+I T_0 w_1} A_2 d_3 L_2}{4 w_1^2} - \frac{3 E^{-I T_0 W-I t_2+I T_0 w_2} A_2 d_4 L_2}{2 w_1^2} + \frac{3 E^{I T_0 W+I t_2+I T_0 w_2} A_2 d_4 L_2}{4 w_1^2} +$$

$$\frac{2 E^{-2 I T_0 W-I t_1-I t_2} d_3 L_1 L_2}{15 w_1^2} - \frac{2 E^{I t_1-I t_2} d_3 L_1 L_2}{w_1^2} - \frac{2 E^{-I t_1+I t_2} d_3 L_1 L_2}{w_1^2} +$$

$$\frac{2 E^{2 I T_0 W+I t_1+I t_2} d_3 L_1 L_2}{15 w_1^2} - \frac{6 d_4 L_2^2}{w_1^2} + \frac{E^{-2 I T_0 W-2 I t_2} d_4 L_2^2}{5 w_1^2} + \frac{E^{2 I T_0 W+2 I t_2} d_4 L_2^2}{5 w_1^2} - \frac{I E^{I T_0 w_2} A_2 m_2}{2 w_1} -$$

$$\frac{4 I E^{-I T_0 W-I t_2} L_2 m_2}{3 w_1} + \frac{4 I E^{I T_0 W+I t_2} L_2 m_2}{3 w_1} - \frac{2 A_2 d_2 \dot{A}_2}{w_1^2} - \frac{2 E^{I T_0 w_1-I T_0 w_2} A_2 d_3 \dot{A}_2}{w_1^2} -$$

$$\frac{2 E^{-I T_0 w_1+I T_0 w_2} A_2 d_3 \dot{A}_2}{w_1^2} - \frac{6 A_2 d_4 \dot{A}_2}{w_1^2} + \frac{E^{-I T_0 W-I t_1-I T_0 w_1} d_2 L_1 \dot{A}_2}{4 w_1^2} - \frac{E^{I T_0 W+I t_1-I T_0 w_1} d_2 L_1 \dot{A}_2}{2 w_1^2} +$$

$$\frac{E^{-I T_0 W-I t_1-I T_0 w_2} d_3 L_1 \dot{A}_2}{4 w_1^2} - \frac{E^{I T_0 W+I t_1-I T_0 w_2} d_3 L_1 \dot{A}_2}{2 w_1^2} + \frac{E^{-I T_0 W-I t_2-I T_0 w_1} d_3 L_2 \dot{A}_2}{4 w_1^2} -$$

$$\frac{E^{I T_0 W+I t_2-I T_0 w_1} d_3 L_2 \dot{A}_2}{2 w_1^2} + \frac{3 E^{-I T_0 W-I t_2-I T_0 w_2} d_4 L_2 \dot{A}_2}{4 w_1^2} - \frac{3 E^{I T_0 W+I t_2-I T_0 w_2} d_4 L_2 \dot{A}_2}{2 w_1^2} +$$

$$\frac{I E^{-I T_0 w_2} m_2 \dot{A}_2}{2 w_1} + \frac{E^{-2 I T_0 w_1} d_2 \dot{A}_2^2}{3 w_1^2} + \frac{2 E^{-I T_0 w_1-I T_0 w_2} d_3 \dot{A}_2^2}{3 w_1^2} + \frac{E^{-2 I T_0 w_2} d_4 \dot{A}_2^2}{w_1^2} \Big\} >$$

where

$$L_i == \frac{f_i}{2 (w_i^2 - W^2)}$$

The modulation equations:

$$2 I w_1 A_1' == 2 I \epsilon^2 w_1 \Big\{ - \frac{81 I A_1 d_1^2 L_1^2}{4 w_1^3} - \frac{27 I E^{I T_1 s_1} A_2 d_1 d_2 L_1^2}{4 w_1^3} - \frac{9 I A_1 d_2^2 L_1^2}{4 w_1^3} - \frac{9 I E^{I T_1 s_1} A_2 d_2 d_3 L_1^2}{4 w_1^3} -$$

$$\frac{27 I E^{I t_1-I t_2} A_1 d_1 d_2 L_1 L_2}{4 w_1^3} - \frac{27 I E^{-I t_1+I t_2} A_1 d_1 d_2 L_1 L_2}{4 w_1^3} - \frac{7 I E^{I T_1 s_1+I t_1-I t_2} A_2 d_2^2 L_1 L_2}{4 w_1^3} -$$

$$\frac{5 I E^{I T_1 s_1-I t_1+I t_2} A_2 d_2^2 L_1 L_2}{2 w_1^3} - \frac{3 I E^{I T_1 s_1+I t_1-I t_2} A_2 d_1 d_3 L_1 L_2}{2 w_1^3} + \frac{3 I E^{I T_1 s_1-I t_1+I t_2} A_2 d_1 d_3 L_1 L_2}{4 w_1^3} -$$

$$\frac{9 I E^{I t_1-I t_2} A_1 d_2 d_3 L_1 L_2}{4 w_1^3} - \frac{9 I E^{-I t_1+I t_2} A_1 d_2 d_3 L_1 L_2}{4 w_1^3} - \frac{7 I E^{I T_1 s_1+I t_1-I t_2} A_2 d_3^2 L_1 L_2}{4 w_1^3} -$$

$$\frac{5 I E^{I T_1 s_1-I t_1+I t_2} A_2 d_3^2 L_1 L_2}{2 w_1^3} - \frac{3 I E^{I T_1 s_1+I t_1-I t_2} A_2 d_2 d_4 L_1 L_2}{2 w_1^3} + \frac{3 I E^{I T_1 s_1-I t_1+I t_2} A_2 d_2 d_4 L_1 L_2}{4 w_1^3} -$$

$$\frac{I A_1 d_2^2 L_2^2}{4 w_1^3} - \frac{6 I A_1 d_1 d_3 L_2^2}{w_1^3} - \frac{9 I E^{I T_1 s_1} A_2 d_2 d_3 L_2^2}{4 w_1^3} - \frac{I A_1 d_3^2 L_2^2}{4 w_1^3} - \frac{6 I A_1 d_2 d_4 L_2^2}{w_1^3} -$$

$$\frac{27 I E^{I T_1 s_1} A_2 d_3 d_4 L_2^2}{4 w_1^3} + \frac{12 I A_1 a_1 L_1^2}{w_1} + \frac{3 I E^{I T_1 s_1} A_2 a_2 L_1^2}{w_1} + \frac{3 I E^{I t_1-I t_2} A_1 a_2 L_1 L_2}{w_1} +$$

$$\frac{3 I E^{-I t_1+I t_2} A_1 a_2 L_1 L_2}{w_1} + \frac{2 I E^{I T_1 s_1+I t_1-I t_2} A_2 a_3 L_1 L_2}{w_1} + \frac{2 I E^{I T_1 s_1-I t_1+I t_2} A_2 a_3 L_1 L_2}{w_1} +$$

$$\frac{2 I A_1 a_3 L_2^2}{w_1} + \frac{3 I E^{I T_1 s_1} A_2 a_4 L_2^2}{w_1} - \frac{I A_1 m_1^2}{2 w_1} - \frac{15 I A_1^2 d_1^2 \dot{A}_1}{w_1^3} - \frac{10 I E^{I T_1 s_1} A_1 A_2 d_1 d_2 \dot{A}_1}{w_1^3} -$$

$$\frac{5 I A_1^2 d_2^2 \dot{A}_1}{3 w_1^3} - \frac{2 I E^{2 I T_1 s_1} A_2^2 d_2^2 \dot{A}_1}{w_1^3} + \frac{I E^{2 I T_1 s_1} A_2^2 d_1 d_3 \dot{A}_1}{w_1^3} - \frac{10 I E^{I T_1 s_1} A_1 A_2 d_2 d_3 \dot{A}_1}{3 w_1^3} -$$

$$\frac{2 I E^{2 I T_1 s_1} A_2^2 d_3^2 \dot{A}_1}{w_1^3} + \frac{I E^{2 I T_1 s_1} A_2^2 d_2 d_4 \dot{A}_1}{w_1^3} - \frac{7 E^{I T_1 s_1+I T_1 s_2+I t_1} d_1 L_1 m_1 \dot{A}_1}{w_1^2} -$$

$$\frac{E^{I T_1 s_1+I T_1 s_2+I t_2} d_2 L_2 m_1 \dot{A}_1}{w_1^2} - \frac{4 E^{I T_1 s_1+I T_1 s_2+I t_2} d_2 L_2 m_2 \dot{A}_1}{3 w_1^2} + \frac{6 I A_1^2 a_1 \dot{A}_1}{w_1} + \frac{3 I E^{I T_1 s_1} A_1 A_2 a_2 \dot{A}_1}{w_1} +$$

$$\frac{I E^{2 I T_1 s_1} A_2^2 a_3 \dot{A}_1}{w_1} - \frac{5 I E^{-I T_1 s_1} A_1^2 d_1 d_2 \dot{A}_2}{w_1^3} - \frac{4 I A_1 A_2 d_2^2 \dot{A}_2}{3 w_1^3} - \frac{6 I A_1 A_2 d_1 d_3 \dot{A}_2}{w_1^3} -$$

$$\frac{5 I E^{-I T_1 s_1} A_1^2 d_2 d_3 \dot{A}_2}{3 w_1^3} - \frac{5 I E^{I T_1 s_1} A_2^2 d_2 d_3 \dot{A}_2}{3 w_1^3} - \frac{4 I A_1 A_2 d_3^2 \dot{A}_2}{3 w_1^3} - \frac{6 I A_1 A_2 d_2 d_4 \dot{A}_2}{w_1^3} -$$

$$\frac{5 I E^{I T_1 s_1} A_2^2 d_3 d_4 \dot{A}_2}{w_1^3} - \frac{11 E^{I T_1 s_2+I t_1} d_2 L_1 m_1 \dot{A}_2}{6 w_1^2} - \frac{E^{I T_1 s_2+I t_2} d_3 L_2 m_1 \dot{A}_2}{2 w_1^2} - \frac{E^{I T_1 s_2+I t_1} d_2 L_1 m_2 \dot{A}_2}{2 w_1^2} -$$

$$\left.\frac{11 E^{I T_1 s_2+I t_2} d_3 L_2 m_2 \dot{A}_2}{6 w_1^2} + \frac{3 I E^{-I T_1 s_1} A_1^2 a_2 \dot{A}_2}{2 w_1} + \frac{2 I A_1 A_2 a_3 \dot{A}_2}{w_1} + \frac{3 I E^{I T_1 s_1} A_2^2 a_4 \dot{A}_2}{2 w_1}\right) +$$

$$2 I e w_1 \left(-A_1 m_1 + \frac{3 I E^{I T_1 s_1+I T_1 s_2+I t_1} d_1 L_1 \dot{A}_1}{w_1} + \frac{I E^{I T_1 s_1+I T_1 s_2+I t_2} d_2 L_2 \dot{A}_1}{w_1} + \right.$$

$$\left.\frac{I E^{I T_1 s_2+I t_1} d_2 L_1 \dot{A}_2}{w_1} + \frac{I E^{I T_1 s_2+I t_2} d_3 L_2 \dot{A}_2}{w_1}\right),$$

$$2 I w_2 A_2' == 2 I e w_2 \left(-A_2 m_2 + \frac{I E^{I T_1 s_2+I t_1} d_2 L_1 \dot{A}_1}{w_2} + \frac{I E^{I T_1 s_2+I t_2} d_3 L_2 \dot{A}_1}{w_2} + \right.$$

$$\left.\frac{I E^{-I T_1 s_1+I T_1 s_2+I t_1} d_3 L_1 \dot{A}_2}{w_2} + \frac{3 I E^{-I T_1 s_1+I T_1 s_2+I t_2} d_4 L_2 \dot{A}_2}{w_2}\right) +$$

$$2 I e^2 w_2 \left(-\frac{I A_2 m_2^2}{2 w_1} + \frac{3 I E^{-I T_1 s_1} A_1 a_2 L_1^2}{w_2} + \frac{2 I A_2 a_3 L_1^2}{w_2} + \frac{2 I E^{-I T_1 s_1+I t_1-I t_2} A_1 a_3 L_1 L_2}{w_2} + \right.$$

$$\frac{2 I E^{-I T_1 s_1-I t_1+I t_2} A_1 a_3 L_1 L_2}{w_2} + \frac{3 I E^{I t_1-I t_2} A_2 a_4 L_1 L_2}{w_2} + \frac{3 I E^{-I t_1+I t_2} A_2 a_4 L_1 L_2}{w_2} +$$

$$\frac{3 I E^{-I T_1 s_1} A_1 a_4 L_2^2}{w_2} + \frac{12 I A_2 a_5 L_2^2}{w_2} - \frac{27 I E^{-I T_1 s_1} A_1 d_1 d_2 L_1^2}{4 w_1^2 w_2} - \frac{I A_2 d_2^2 L_1^2}{4 w_1^2 w_2} - \frac{6 I A_2 d_1 d_3 L_1^2}{w_1^2 w_2} -$$

$$\frac{9 I E^{-I T_1 s_1} A_1 d_2 d_3 L_1^2}{4 w_1^2 w_2} - \frac{I A_2 d_3^2 L_1^2}{4 w_1^2 w_2} - \frac{6 I A_2 d_2 d_4 L_1^2}{w_1^2 w_2} - \frac{5 I E^{-I T_1 s_1+I t_1-I t_2} A_1 d_2^2 L_1 L_2}{2 w_1^2 w_2} -$$

$$\frac{7 I E^{-I T_1 s_1-I t_1+I t_2} A_1 d_2^2 L_1 L_2}{4 w_1^2 w_2} + \frac{3 I E^{-I T_1 s_1+I t_1-I t_2} A_1 d_1 d_3 L_1 L_2}{4 w_1^2 w_2} -$$

$$\frac{3 I E^{-I T_1 s_1-I t_1+I t_2} A_1 d_1 d_3 L_1 L_2}{2 w_1^2 w_2} - \frac{9 I E^{I t_1-I t_2} A_2 d_2 d_3 L_1 L_2}{4 w_1^2 w_2} - \frac{9 I E^{-I t_1+I t_2} A_2 d_2 d_3 L_1 L_2}{4 w_1^2 w_2} -$$

$$\frac{5 I E^{-I T_1 s_1+I t_1-I t_2} A_1 d_3^2 L_1 L_2}{2 w_1^2 w_2} - \frac{7 I E^{-I T_1 s_1-I t_1+I t_2} A_1 d_3^2 L_1 L_2}{4 w_1^2 w_2} + \frac{3 I E^{-I T_1 s_1+I t_1-I t_2} A_1 d_2 d_4 L_1 L_2}{4 w_1^2 w_2} -$$

$$\frac{3 I E^{-I T_1 s_1-I t_1+I t_2} A_1 d_2 d_4 L_1 L_2}{2 w_1^2 w_2} - \frac{27 I E^{I t_1-I t_2} A_2 d_3 d_4 L_1 L_2}{4 w_1^2 w_2} - \frac{27 I E^{-I t_1+I t_2} A_2 d_3 d_4 L_1 L_2}{4 w_1^2 w_2} -$$

$$\frac{9 I E^{-I T_1 s_1} A_1 d_2 d_3 L_2^2}{4 w_1^2 w_2} - \frac{9 I A_2 d_3^2 L_2^2}{4 w_1^2 w_2} - \frac{27 I E^{-I T_1 s_1} A_1 d_3 d_4 L_2^2}{4 w_1^2 w_2} - \frac{81 I A_2 d_4^2 L_2^2}{4 w_1^2 w_2} -$$

$$\frac{E^{I T_1 s_2+I t_1} d_2 L_1 m_2 \dot{A}_1}{2 w_1^2} - \frac{E^{I T_1 s_2+I t_2} d_3 L_2 m_2 \dot{A}_1}{2 w_1^2} + \frac{3 I E^{-I T_1 s_1} A_1^2 a_2 \dot{A}_1}{2 w_2} + \frac{2 I A_1 A_2 a_3 \dot{A}_1}{w_2} +$$

$$\frac{3 I E^{I T_1 s_1} A_2^2 a_4 \dot{A}_1}{2 w_2} - \frac{5 I E^{-I T_1 s_1} A_1^2 d_1 d_2 \dot{A}_1}{w_1^2 w_2} - \frac{4 I A_1 A_2 d_2^2 \dot{A}_1}{3 w_1^2 w_2} - \frac{6 I A_1 A_2 d_1 d_3 \dot{A}_1}{w_1^2 w_2} -$$

$$\frac{5 I E^{-I T_1 s_1} A_1^2 d_2 d_3 \dot{A}_1}{3 w_1^2 w_2} - \frac{5 I E^{I T_1 s_1} A_2^2 d_2 d_3 \dot{A}_1}{3 w_1^2 w_2} - \frac{4 I A_1 A_2 d_3^2 \dot{A}_1}{3 w_1^2 w_2} - \frac{6 I A_1 A_2 d_2 d_4 \dot{A}_1}{w_1^2 w_2} -$$

$$\frac{5 I E^{I T_1 s_1} A_2^2 d_3 d_4 \dot{A}_1}{w_1^2 w_2} - \frac{11 E^{I T_1 s_2+I t_1} d_2 L_1 m_1 \dot{A}_1}{6 w_1 w_2} - \frac{E^{I T_1 s_2+I t_2} d_3 L_2 m_1 \dot{A}_1}{2 w_1 w_2} -$$

$$\frac{4 E^{I T_1 s_2+I t_2} d_3 L_2 m_2 \dot{A}_1}{3 w_1 w_2} - \frac{E^{-I T_1 s_1+I T_1 s_2+I t_1} d_3 L_1 m_2 \dot{A}_2}{2 w_1^2} - \frac{3 E^{-I T_1 s_1+I T_1 s_2+I t_2} d_4 L_2 m_2 \dot{A}_2}{2 w_1^2} +$$

$$\frac{I E^{-2 I T_1 s_1} A_1^2 a_3 \dot{A}_2}{w_2} + \frac{3 I E^{-I T_1 s_1} A_1 A_2 a_4 \dot{A}_2}{w_2} + \frac{6 I A_2^2 a_5 \dot{A}_2}{w_2} - \frac{2 I E^{-2 I T_1 s_1} A_1^2 d_2^2 \dot{A}_2}{w_1^2 w_2} +$$

$$\frac{I E^{-2 I T_1 s_1} A_1^2 d_1 d_3 \dot{A}_2}{w_1^2 w_2} - \frac{10 I E^{-I T_1 s_1} A_1 A_2 d_2 d_3 \dot{A}_2}{3 w_1^2 w_2} - \frac{2 I E^{-2 I T_1 s_1} A_1^2 d_3^2 \dot{A}_2}{w_1^2 w_2} -$$

$$\frac{5 I A_2^2 d_3^2 \dot{A}_2}{3 w_1^2 w_2} + \frac{I E^{-2 I T_1 s_1} A_1^2 d_2 d_4 \dot{A}_2}{w_1^2 w_2} - \frac{10 I E^{-I T_1 s_1} A_1 A_2 d_3 d_4 \dot{A}_2}{w_1^2 w_2} - \frac{15 I A_2^2 d_4^2 \dot{A}_2}{w_1^2 w_2} -$$

$$\frac{4 E^{-I T_1 s_1+I T_1 s_2+I t_1} d_3 L_1 m_1 \dot{A}_2}{3 w_1 w_2} - \frac{E^{-I T_1 s_1+I T_1 s_2+I t_1} d_3 L_1 m_2 \dot{A}_2}{2 w_1 w_2} - \frac{11 E^{-I T_1 s_1+I T_1 s_2+I t_2} d_4 L_2 m_2 \dot{A}_2}{2 w_1 w_2} \Big\}\Big\}$$

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{194.159 Second, Null}
```

6.1.8 $\Omega \approx 2\omega_2 - \omega_1$ and $\omega_2 \approx \omega_1$

In this case, we have a primary resonance of either mode and a one-to-one internal resonance. We let

```
scaling8 = {μ_n_ -> ε^2 μ_n, F1 -> ε^3 f1, F2 -> ε^3 f2};
```

```
ResonanceCond8 = {ω2 == ω1 + ε^2 σ1, Ω == 2 ω2 - ω1 + ε^2 σ2};
```

Using **MMS**, we obtain the second-order approximate solution and the two equations governing the modulation of the complex-valued functions A_k:

```
MMS[scaling8, ResonanceCond8] // Timing
```

The second-order approximate solution:

$$\Big\{u_1[t] == e \left(E^{I T_0 w_1} A_1 + E^{-I T_0 w_1} \dot{A}_1\right) + e^2 \Big(\frac{E^{2 I T_0 w_1} A_1^2 d_1}{w_1^2} + \frac{2 E^{I T_0 w_1 + I T_0 w_2} A_1 A_2 d_2}{3 w_1^2} + \frac{E^{2 I T_0 w_2} A_2^2 d_3}{3 w_1^2} - \frac{6 A_1 d_1 \dot{A}_1}{w_1^2} - \frac{2 E^{-I T_0 w_1 + I T_0 w_2} A_2 d_2 \dot{A}_1}{w_1^2} +$$

$$\frac{E^{-2 I T_0 w_1} d_1 \dot{A}_1^2}{w_1^2} - \frac{2 E^{I T_0 w_1 - I T_0 w_2} A_1 d_2 \dot{A}_2}{w_1^2} - \frac{2 A_2 d_3 \dot{A}_2}{w_1^2} + \frac{2 E^{-I T_0 w_1 - I T_0 w_2} d_2 \dot{A}_1 \dot{A}_2}{3 w_1^2} + \frac{E^{-2 I T_0 w_2} d_3 \dot{A}_2^2}{3 w_1^2} \Big),$$

$$u_2[t] == e \left(E^{I T_0 w_2} A_2 + E^{-I T_0 w_2} \dot{A}_2\right) + e^2 \Big(\frac{E^{2 I T_0 w_1} A_1^2 d_2}{3 w_1^2} + \frac{2 E^{I T_0 w_1 + I T_0 w_2} A_1 A_2 d_3}{3 w_1^2} +$$

$$\frac{E^{2 I T_0 w_2} A_2^2 d_4}{w_1^2} - \frac{2 A_1 d_2 \dot{A}_1}{w_1^2} - \frac{2 E^{-I T_0 w_1 + I T_0 w_2} A_2 d_3 \dot{A}_1}{w_1^2} + \frac{E^{-2 I T_0 w_1} d_2 \dot{A}_1^2}{3 w_1^2} -$$

$$\frac{2 E^{I T_0 w_1 - I T_0 w_2} A_1 d_3 \dot{A}_2}{w_1^2} - \frac{6 A_2 d_4 \dot{A}_2}{w_1^2} + \frac{2 E^{-I T_0 w_1 - I T_0 w_2} d_3 \dot{A}_1 \dot{A}_2}{3 w_1^2} + \frac{E^{-2 I T_0 w_2} d_4 \dot{A}_2^2}{w_1^2} \Big)\Big\}$$

The modulation equations:

$$\Big\{2\,I\,\omega_1\,A_1' == 2\,I\,\epsilon^2\,\omega_1\Big(-A_1\mu_1 - \frac{I\,E^{2IT_2\sigma_1+IT_2\sigma_2+I\tau_1}f_1}{4\omega_1} - \frac{15\,I\,A_1^2\delta_1^2\bar{A}_1}{\omega_1^3} - \frac{10\,I\,E^{IT_2\sigma_1}A_1A_2\delta_1\delta_2\bar{A}_1}{\omega_1^3} - \frac{5\,I\,A_1^2\delta_2^2\bar{A}_1}{3\omega_1^3} -$$
$$\frac{2\,I\,E^{2IT_2\sigma_1}A_2^2\delta_2^2\bar{A}_1}{\omega_1^3} + \frac{I\,E^{2IT_2\sigma_1}A_2^2\delta_1\delta_3\bar{A}_1}{\omega_1^3} - \frac{10\,I\,E^{IT_2\sigma_1}A_1A_2\delta_2\delta_3\bar{A}_1}{3\omega_1^3} -$$
$$\frac{2\,I\,E^{2IT_2\sigma_1}A_2^2\delta_3^2\bar{A}_1}{\omega_1^3} + \frac{I\,E^{2IT_2\sigma_1}A_2^2\delta_2\delta_4\bar{A}_1}{\omega_1^3} + \frac{6\,I\,A_1^2\alpha_1\bar{A}_1}{\omega_1} + \frac{3\,I\,E^{IT_2\sigma_1}A_1A_2\alpha_2\bar{A}_1}{\omega_1} +$$
$$\frac{I\,E^{2IT_2\sigma_1}A_2^2\alpha_3\bar{A}_1}{\omega_1} - \frac{5\,I\,E^{-IT_2\sigma_1}A_1^2\delta_1\delta_2\bar{A}_2}{\omega_1^3} - \frac{4\,I\,A_1A_2\delta_2^2\bar{A}_2}{3\omega_1^3} - \frac{6\,I\,A_1A_2\delta_1\delta_3\bar{A}_2}{\omega_1^3} -$$
$$\frac{5\,I\,E^{-IT_2\sigma_1}A_1^2\delta_2\delta_3\bar{A}_2}{3\omega_1^3} - \frac{5\,I\,E^{IT_2\sigma_1}A_2^2\delta_2\delta_3\bar{A}_2}{3\omega_1^3} - \frac{4\,I\,A_1A_2\delta_3^2\bar{A}_2}{3\omega_1^3} - \frac{6\,I\,A_1A_2\delta_2\delta_4\bar{A}_2}{\omega_1^3} -$$
$$\frac{5\,I\,E^{IT_2\sigma_1}A_2^2\delta_3\delta_4\bar{A}_2}{\omega_1^3} + \frac{3\,I\,E^{-IT_2\sigma_1}A_1^2\alpha_2\bar{A}_2}{2\omega_1} + \frac{2\,I\,A_1A_2\alpha_3\bar{A}_2}{\omega_1} + \frac{3\,I\,E^{IT_2\sigma_1}A_2^2\alpha_4\bar{A}_2}{2\omega_1}\Big),$$

$$2\,I\,\omega_2\,A_2' == 2\,I\,\epsilon^2\,\omega_2\Big(-A_2\mu_2 - \frac{I\,E^{IT_2\sigma_1+IT_2\sigma_2+I\tau_2}f_2}{4\omega_2} + \frac{3\,I\,E^{-IT_2\sigma_1}A_1^2\alpha_2\bar{A}_1}{2\omega_2} + \frac{2\,I\,A_1A_2\alpha_3\bar{A}_1}{\omega_2} +$$
$$\frac{3\,I\,E^{IT_2\sigma_1}A_2^2\alpha_4\bar{A}_1}{2\omega_2} - \frac{5\,I\,E^{-IT_2\sigma_1}A_1^2\delta_1\delta_2\bar{A}_1}{\omega_1^2\omega_2} - \frac{4\,I\,A_1A_2\delta_2^2\bar{A}_1}{3\omega_1^2\omega_2} - \frac{6\,I\,A_1A_2\delta_1\delta_3\bar{A}_1}{\omega_1^2\omega_2} -$$
$$\frac{5\,I\,E^{-IT_2\sigma_1}A_1^2\delta_2\delta_3\bar{A}_1}{3\omega_1^2\omega_2} - \frac{5\,I\,E^{IT_2\sigma_1}A_2^2\delta_2\delta_3\bar{A}_1}{3\omega_1^2\omega_2} - \frac{4\,I\,A_1A_2\delta_3^2\bar{A}_1}{3\omega_1^2\omega_2} - \frac{6\,I\,A_1A_2\delta_2\delta_4\bar{A}_1}{\omega_1^2\omega_2} -$$
$$\frac{5\,I\,E^{IT_2\sigma_1}A_2^2\delta_3\delta_4\bar{A}_1}{\omega_1^2\omega_2} + \frac{I\,E^{-2IT_2\sigma_1}A_1^2\alpha_3\bar{A}_2}{\omega_2} + \frac{3\,I\,E^{-IT_2\sigma_1}A_1A_2\alpha_4\bar{A}_2}{\omega_2} + \frac{6\,I\,A_2^2\alpha_5\bar{A}_2}{\omega_2} -$$
$$\frac{2\,I\,E^{-2IT_2\sigma_1}A_1^2\delta_2^2\bar{A}_2}{\omega_1^2\omega_2} + \frac{I\,E^{-2IT_2\sigma_1}A_1^2\delta_1\delta_3\bar{A}_2}{\omega_1^2\omega_2} - \frac{10\,I\,E^{-IT_2\sigma_1}A_1A_2\delta_2\delta_3\bar{A}_2}{3\omega_1^2\omega_2} - \frac{2\,I\,E^{-2IT_2\sigma_1}A_1^2\delta_3^2\bar{A}_2}{\omega_1^2\omega_2} -$$
$$\frac{5\,I\,A_2^2\delta_3^2\bar{A}_2}{3\omega_1^2\omega_2} + \frac{I\,E^{-2IT_2\sigma_1}A_1^2\delta_2\delta_4\bar{A}_2}{\omega_1^2\omega_2} - \frac{10\,I\,E^{-IT_2\sigma_1}A_1A_2\delta_3\delta_4\bar{A}_2}{\omega_1^2\omega_2} - \frac{15\,I\,A_2^2\delta_4^2\bar{A}_2}{\omega_1^2\omega_2}\Big)\Big\}$$

```
{816.664 Second, Null}
```

6.2 Parametrically Excited Linearly Coupled Systems

6.2.1 Two-Pendulum Oscillator

As an example, we consider a parametrically excited two-pendulum oscillator with a three-to-one internal resonance:

```
eq621a = {q1''[t] + α q2''[t] + q1[t] == 1/2 α (q2[t] - q1[t])^2 q2''[t] + 1/6 q1[t]^3 +
    α (q2[t] - q1[t]) q2'[t]^2 - 2 F1 Ω^2 q1[t] Cos[Ω t], q2''[t] + q1''[t] + q2[t] ==
   1/2 (q2[t] - q1[t])^2 q1''[t] + 1/6 q2[t]^3 - (q2[t] - q1[t]) q1'[t]^2 - 2 F1 Ω^2 q2[t] Cos[Ω t]};
```

where

```
params = {α -> 16/25 + ε^2 σ1};
```

for small e. Here, the excitation frequency, W, is assumed to be close to twice the natural frequency of the second mode, w_2, which is three times the natural frequency of the first mode, w_1. Hence, we define the following lists:

```
omgList = 8w1, w2<;
```

```
ResonanceCond = 9W == 2 w2 + e^2 s2, w2 == 3 w1=;
```

```
omgRule = Solve@ResonanceCond, 8W, #< •• FlattenD@@1DD & •ž omgList •• Reverse
```

```
:8W ® e^2 s2 + 6 w1, w2 ® 3 w1<, :W ® e^2 s2 + 2 w2, w1 ® w2/3>>
```

In order to bring the effects of forcing and nonlinearity at the same order, we let

```
scaling = 9F1 -> e^2 f1=;
```

For a consistent expansion, we first transform **eq621a** into a system of four first-order equations. To this end, we introduce the two states v_1@*t*D and v_2@*t*D defined by

```
vel = 8q1¢@tD -> v1@tD, q2¢@tD -> v2@tD<;
```

Substituting the velocity and acceleration terms, using **vel**, into **eq621a** and combining the result with **vel**, we transform **eq621a** into the following set of four first-order equations:

```
eq621b = 8vel •. Rule -> Equal, eq621a •. D@vel, tD •. vel< •• Transpose •• Flatten
```

```
9q1¢@tD == v1@tD, q1@tD + v1¢@tD + a v2¢@tD ==
  - 2 W^2 Cos@t WD F1 q1@tD + 1/6 q1@tD^3 + a v2@tD^2 H- q1@tD + q2@tDL + 1/2 a H- q1@tD + q2@tDL^2 v2¢@tD,
 q2¢@tD == v2@tD, q2@tD + v1¢@tD + v2¢@tD ==
  - 2 W^2 Cos@t WD F1 q2@tD + 1/6 q2@tD^3 - v1@tD^2 H- q1@tD + q2@tDL + 1/2 H- q1@tD + q2@tDL^2 v1¢@tD=
```

We seek a uniform second-order expansion of the solution of **eq621b** in the form

```
solRule =
  9qi_ -> ISumAe^j qi,j@#1, #2, #3D, 8j, 3<E &M, vi_ -> ISumAe^j vi,j@#1, #2, #3D, 8j, 3<E &M=;
```

```
maxOrder = 2;
```

Using the time scales T_0, T_1, and T_2, we express the dependent variables and their time derivatives as

```
multiScales = 8qi_@tD -> qi žž timeScales, vi_@tD -> vi žž timeScales,
   Derivative@1D@u_D@tD -> dt@1D@u žž timeScalesD, t -> T0<;
```

Substituting the **params**, **scaling**, **multiScales**, and **solRule** into **eq621b**, expanding the result for small e, and discarding terms of order higher than e^3, we obtain

```
eq621c = Heq621b •. params •. scaling •. multiScales •. solRule •• TrigToExp •• ExpandAllL •.
   e^(n_•;n>3) -> 0;
```

Equating coefficients of like powers of e, we obtain

```
eqEps = Rest@Thread@CoefficientList@Subtract žž #, eD == 0DD & •ž eq621c •• Transpose;
```

To place the linear operator on one side and the nonhomogeneous terms on the other side, we define

```
eqOrder@i_D := I#@@1DD & •ž eqEps@@1DD •. fs_ -> 0 •. u_k_,1 -> uk,iM ==
   I#@@1DD & •ž eqEps@@1DD •. fs_ -> 0 •. u_k_,1 -> uk,iM - H#@@1DD & •ž eqEps@@iDDL •• Thread
```

Using the **eqOrder[i]** and **displayRule**, we rewrite **eqEps** in a concise way as

```
eqOrder@1D •. displayRule
eqOrder@2D •. displayRule
eqOrder@3D •. displayRule
```

$9D_0q_{1,1} - v_{1,1} == 0,\ D_0v_{1,1} + \frac{16}{25} HD_0v_{2,1}L + q_{1,1} == 0,\ D_0q_{2,1} - v_{2,1} == 0,\ D_0v_{1,1} + D_0v_{2,1} + q_{2,1} == 0=$

$9D_0q_{1,2} - v_{1,2} == -HD_1q_{1,1}L,\ D_0v_{1,2} + \frac{16}{25} HD_0v_{2,2}L + q_{1,2} == -HD_1v_{1,1}L - \frac{16}{25} HD_1v_{2,1}L,$
$D_0q_{2,2} - v_{2,2} == -HD_1q_{2,1}L,\ D_0v_{1,2} + D_0v_{2,2} + q_{2,2} == -HD_1v_{1,1}L - D_1v_{2,1}=$

$:D_0q_{1,3} - v_{1,3} == -HD_1q_{1,2}L - D_2q_{1,1},$
$D_0v_{1,3} + \frac{16}{25} HD_0v_{2,3}L + q_{1,3} == -HD_1v_{1,2}L - \frac{16}{25} HD_1v_{2,2}L - D_2v_{1,1} - \frac{16}{25} HD_2v_{2,1}L -$
$HD_0v_{2,1}L\ s_1 - E^{-I T_0 W} W^2 f_1 q_{1,1} - E^{I T_0 W} W^2 f_1 q_{1,1} - \frac{16}{25} v_{2,1}^2 q_{1,1} + \frac{8}{25} HD_0v_{2,1}L q_{1,1}^2 + \frac{q_{1,1}^3}{6} +$
$\frac{16}{25} v_{2,1}^2 q_{2,1} - \frac{16}{25} HD_0v_{2,1}L q_{1,1} q_{2,1} + \frac{8}{25} HD_0v_{2,1}L q_{2,1}^2,\ D_0q_{2,3} - v_{2,3} == -HD_1q_{2,2}L - D_2q_{2,1},$
$D_0v_{1,3} + D_0v_{2,3} + q_{2,3} == -HD_1v_{1,2}L - D_1v_{2,2} - D_2v_{1,1} - D_2v_{2,1} + v_{1,1}^2 q_{1,1} + \frac{1}{2} HD_0v_{1,1}L q_{1,1}^2 -$
$E^{-I T_0 W} W^2 f_1 q_{2,1} - E^{I T_0 W} W^2 f_1 q_{2,1} - v_{1,1}^2 q_{2,1} - HD_0v_{1,1}L q_{1,1} q_{2,1} + \frac{1}{2} HD_0v_{1,1}L q_{2,1}^2 + \frac{q_{2,1}^3}{6}>$

Ÿ First-Order Problem: Linear System

The first-order problem, **eqOrder[1]**, consists of a set of coupled linear homogeneous differential equations. Hence, the general solution is the solution of the homogeneous equations. To determine the general solution of the homogeneous set, we rewrite **eqOrder[1]** as

```
linearSys = #@@1DD & •ž eqOrder@1D;
linearSys •. displayRule
```

$9D_0q_{1,1} - v_{1,1},\ D_0v_{1,1} + \frac{16}{25} HD_0v_{2,1}L + q_{1,1},\ D_0q_{2,1} - v_{2,1},\ D_0v_{1,1} + D_0v_{2,1} + q_{2,1}=$

Next, we seek a solution of the **linearSys** in the form

```
assumedForm = 9q_{i_,1} -> Ic_i E^{I w #} &M, v_{i_,1} -> Id_i E^{I w #} &M=;
```

Substituting **assumedForm** into **linearSys** and collecting coefficients of c_i and d_i, we obtain the coefficient matrix as

```
coefMat = OuterAD, E^{-I w T0} linearSys •. assumedForm •• Expand, 8c1, d1, c2, d2<E
```

```
98I w, -1, 0, 0<, 91, I w, 0, 16 I w/25 =, 80, 0, I w, -1<, 80, I w, 1, I w<=
```

whose adjoint is defined by

```
hermitian@mat_? MatrixQD := mat •. conjugateRule •• Transpose
```

The natural frequencies of the two modes involved are

```
evals = Cases@w •. Solve@Det@coefMatD == 0, wD, n_ •; n > 0D
```

```
: •√5/3 , •√5 >
```

Hence, the right and left eigenvectors of **coefMat** corresponding to **evals** are

```
rightVec = # • #@@1DD & •ž HNullSpace@coefMat •. w -> #D@@1DD & •ž evalsL
```

```
::1, I •√5/3 , 5/4 , 5 I •√5/12 >, :1, I •√5 , - 5/4 , - 5 I •√5/4 >>
```

```
leftVec = NullSpace@hermitian@coefMatD •. w -> #D@@1DD & •ž evals
```

```
::- 3 I •√5/4 , 5/4 , - 3 I/•√5 , 1>, : I •√5/4 , - 5/4 , - I/•√5 , 1>>
```

whose complex conjugates are

```
ccleftVec = leftVec •. conjugateRule
```

```
:: 3 I •√5/4 , 5/4 , 3 I/•√5 , 1>, :- I •√5/4 , - 5/4 , I/•√5 , 1>>
```

Based on the **rightVec**, we can assume the general solution form as

```
sol1Form = Transpose@rightVecD.9A1@T1, T2D E^{I w1 T0}, A2@T1, T2D E^{I w2 T0}=
```

```
9E^{I T0 w1} A1@T1, T2D + E^{I T0 w2} A2@T1, T2D, 1/3 I •√5 E^{I T0 w1} A1@T1, T2D + I •√5 E^{I T0 w2} A2@T1, T2D,
5/4 E^{I T0 w1} A1@T1, T2D - 5/4 E^{I T0 w2} A2@T1, T2D, 5/12 I •√5 E^{I T0 w1} A1@T1, T2D - 5/4 I •√5 E^{I T0 w2} A2@T1, T2D=
```

where w_1 = •√5 ‘ 3 and w_2 = •√5. Therefore, the general solution of **eqOrder[1]** can be expressed in pure function form as

```
sol1 = {q1,1, v1,1, q2,1, v2,1} ->
    (Function[{T0, T1, T2}, # + (# /. conjugateRule)] // Evaluate) & /@ sol1Form // Thread;
sol1 /. displayRule
```

$$\{q_{1,1}\to \text{Function}[\{T_0,T_1,T_2\}, E^{I T_0 \omega_1}A_1 + E^{I T_0\omega_2}A_2 + E^{-I T_0\omega_1}\bar{A}_1 + E^{-I T_0\omega_2}\bar{A}_2],\ v_{1,1}\to \text{Function}\left[\{T_0,T_1,T_2\}, \frac{1}{3} I\sqrt{5}\, E^{I T_0\omega_1}A_1 + I\sqrt{5}\, E^{I T_0\omega_2}A_2 - \frac{1}{3} I\sqrt{5}\, E^{-I T_0\omega_1}\bar{A}_1 - I\sqrt{5}\, E^{-I T_0\omega_2}\bar{A}_2\right],$$
$$q_{2,1}\to \text{Function}\left[\{T_0,T_1,T_2\}, \frac{5}{4} E^{I T_0\omega_1}A_1 - \frac{5}{4} E^{I T_0\omega_2}A_2 + \frac{5}{4} E^{-I T_0\omega_1}\bar{A}_1 - \frac{5}{4} E^{-I T_0\omega_2}\bar{A}_2\right],$$
$$v_{2,1}\to \text{Function}\left[\{T_0,T_1,T_2\}, \frac{5}{12} I\sqrt{5}\, E^{I T_0\omega_1}A_1 - \frac{5}{4} I\sqrt{5}\, E^{I T_0\omega_2}A_2 - \frac{5}{12} I\sqrt{5}\, E^{-I T_0\omega_1}\bar{A}_1 + \frac{5}{4} I\sqrt{5}\, E^{-I T_0\omega_2}\bar{A}_2\right]\}$$

where A_1 and A_2 are to be determined from the solvability conditions at the next levels of approximation.

Second-Order Problem

Substituting the first-order solution **sol1** into the second-order problem, **eqOrder[2]**, yields

```
order2Eq = eqOrder[2] /. sol1 // ExpandAll;
order2Eq /. displayRule
```

$$\{D_0 q_{1,2} - v_{1,2} == -E^{I T_0\omega_1}(D_1A_1) - E^{I T_0\omega_2}(D_1A_2) - E^{-I T_0\omega_1}(D_1\bar{A}_1) - E^{-I T_0\omega_2}(D_1\bar{A}_2),$$
$$D_0 v_{1,2} + \frac{16}{25}(D_0 v_{2,2}) + q_{1,2} ==$$
$$-\frac{4 I E^{I T_0\omega_1}(D_1A_1)}{3\sqrt{5}} - \frac{1}{3} I\sqrt{5}\, E^{I T_0\omega_1}(D_1A_1) + \frac{4 I E^{I T_0\omega_2}(D_1A_2)}{\sqrt{5}} - I\sqrt{5}\, E^{I T_0\omega_2}(D_1A_2) +$$
$$\frac{4 I E^{-I T_0\omega_1}(D_1\bar{A}_1)}{3\sqrt{5}} + \frac{1}{3} I\sqrt{5}\, E^{-I T_0\omega_1}(D_1\bar{A}_1) - \frac{4 I E^{-I T_0\omega_2}(D_1\bar{A}_2)}{\sqrt{5}} + I\sqrt{5}\, E^{-I T_0\omega_2}(D_1\bar{A}_2),$$
$$D_0 q_{2,2} - v_{2,2} == -\frac{5}{4} E^{I T_0\omega_1}(D_1A_1) + \frac{5}{4} E^{I T_0\omega_2}(D_1A_2) - \frac{5}{4} E^{-I T_0\omega_1}(D_1\bar{A}_1) + \frac{5}{4} E^{-I T_0\omega_2}(D_1\bar{A}_2),$$
$$D_0 v_{1,2} + D_0 v_{2,2} + q_{2,2} ==$$
$$-\frac{3}{4} I\sqrt{5}\, E^{I T_0\omega_1}(D_1A_1) + \frac{1}{4} I\sqrt{5}\, E^{I T_0\omega_2}(D_1A_2) + \frac{3}{4} I\sqrt{5}\, E^{-I T_0\omega_1}(D_1\bar{A}_1) - \frac{1}{4} I\sqrt{5}\, E^{-I T_0\omega_2}(D_1\bar{A}_2)\}$$

For a uniform expansion, we choose $D_1 A_1$ and $D_1 A_2$ to eliminate the secular or small-divisor terms. To accomplish this, we first convert the small-divisor terms into secular terms using the rule

```
expRule1[i_] := Exp[a_] :> Exp[Expand[a /. omgRule[[i]]] /. ε^2 T0 -> T2]
```

To eliminate the terms that produce secular terms (i.e., determine the solvability conditions) from **order2Eq**, we collect the terms proportional to $E^{I\,\omega_1 T_0}$ and $E^{I\,\omega_2 T_0}$ and obtain

```
ST11 = Coefficient[order2Eq[[#, 2]] /. expRule1[1], E^(I ω1 T0)] & /@ Range[4]
```

$$\left\{-A_1^{(1,0)}[T_1, T_2],\ -\frac{3 I A_1^{(1,0)}[T_1, T_2]}{\sqrt{5}},\ -\frac{5}{4} A_1^{(1,0)}[T_1, T_2],\ -\frac{3}{4} I\sqrt{5}\, A_1^{(1,0)}[T_1, T_2]\right\}$$

```
ST12 = CoefficientAorder2Eq@@#, 2DD •. expRule1@2D, E^(I w2 T0)E & •ž Range@4D
```

$$9-A_2^{H1,0L}@T_1,\ T_2D,\ -\frac{I\,A_2^{H1,0L}@T_1,\ T_2D}{\sqrt{5}},\ \frac{5}{4}\,A_2^{H1,0L}@T_1,\ T_2D,\ \frac{1}{4}\,I\,\sqrt{5}\,A_2^{H1,0L}@T_1,\ T_2D=$$

Then, the solvability conditions demand that **ST11** and **ST12** be orthogonal to every solution of the corresponding adjoint problem, namely, the components of **ccleftVec**. The result is

```
SCond11 = ccleftVec@@1DD.ST11 == 0
```

$$-3\,I\,\sqrt{5}\,A_1^{H1,0L}@T_1,\ T_2D == 0$$

```
SCond12 = ccleftVec@@2DD.ST12 == 0
```

$$I\,\sqrt{5}\,A_2^{H1,0L}@T_1,\ T_2D == 0$$

Solving **SCond11** and **SCond12** for $D_1\,A_1$ and $D_1\,A_2$, respectively, yields

```
SCond11Rule = SolveASCond11, A1^(H1,0L)@T1, T2DE@@1DD
```

$$9A_1^{H1,0L}@T_1,\ T_2D \to 0=$$

```
SCond12Rule = SolveASCond12, A2^(H1,0L)@T1, T2DE@@1DD
```

$$9A_2^{H1,0L}@T_1,\ T_2D \to 0=$$

```
SCond1Rule = Join@SCond11Rule, SCond12RuleD;
```

whose complex conjugate is

```
ccSCond1Rule = SCond1Rule •. conjugateRule;
```

Substituting **SCond1Rule** and **ccSCond1Rule** into **order2Eq**, we obtain

```
order2Eqm = order2Eq •. SCond1Rule •. ccSCond1Rule •• ExpandAll;
order2Eqm •. displayRule
```

$$9D_0q_{1,2}-v_{1,2} == 0,\ D_0v_{1,2}+\frac{16}{25}\,HD_0v_{2,2}L+q_{1,2} == 0,\ D_0q_{2,2}-v_{2,2} == 0,\ D_0v_{1,2}+D_0v_{2,2}+q_{2,2} == 0=$$

which is a set of homogeneous equations. Then, we express the solution of the second-order equations in pure function form as

```
sol2 = 8q1,2 -> H0 &L, v1,2 -> H0 &L, q2,2 -> H0 &L, v2,2 -> H0 &L<;
```

Ÿ Third-Order Problem

Substituting the first- and second-order solutions into the third-order problem, **eqOrder[3]**, yields

```
order3Eq = eqOrder@3D •. sol1 •. sol2 •• ExpandAll;
```

Substituting the **expRule1** into the right-hand sides of **order3Eq** and collecting the terms that could produce secular terms, the terms proportional to $E^{I\,w_1 T_0}$ and $E^{I\,w_2 T_0}$, we have

```
ST21 = CoefficientA#@@2DD •. expRule1@1D, E^(I w1 T0)E & •ž order3Eq;
ST21 •. displayRule
```

```
9-HD2A1L, - 3 I HD2A1L/(• !!! 5) + 5/12 • !!! 5 A1 s1 w1 + 23/36 A1^2 A1• - A1^2 w1 A1•/(8 • !!! 5) + 11/12 A2 A1•^2 +
  3 A2 w1 A1•^2/(4 • !!! 5) + A2 w2 A1•^2/(8 • !!! 5) + 7/2 A1 A2 A2• - 27 A1 A2 w1 A2•/(4 • !!! 5) - 9 A1 A2 w2 A2•/(2 • !!! 5), - 5/4 HD2A1L,
  - 3/4 I • !!! 5 HD2A1L + 965 A1^2 A1•/1152 - 1/32 • !!! 5 A1^2 w1 A1• - 1175/384 A2 A1•^2 + 3/16 • !!! 5 A2 w1 A1•^2 -
  1/32 • !!! 5 A2 w2 A1•^2 - 35/64 A1 A2 A2• - 27/16 • !!! 5 A1 A2 w1 A2• + 9/8 • !!! 5 A1 A2 w2 A2•=
```

```
ST22 = CoefficientA#@@2DD •. expRule1@2D, E^(I w2 T0)E & •ž order3Eq;
ST22 •. displayRule
```

```
:-HD2A2L, - I HD2A2L/(• !!! 5) + A1^3/36 - A1^3 w1/(24 • !!! 5) - 5/4 • !!! 5 A2 s1 w2 - 3/2 A1 A2 A1• +
  3 A1 A2 w1 A1•/(2 • !!! 5) + A1 A2 w2 A1•/(4 • !!! 5) - 43/4 A2^2 A2• - E^(I T2 s2) W^2 f1 A2• + 243 A2^2 w2 A2•/(8 • !!! 5), 5/4 HD2A2L,
  1/4 I • !!! 5 HD2A2L + 535 A1^3/1152 - 1/96 • !!! 5 A1^3 w1 + 35/64 A1 A2 A1• + 3/8 • !!! 5 A1 A2 w1 A1• -
  1/16 • !!! 5 A1 A2 w2 A1• + 1315/128 A2^2 A2• + 5/4 E^(I T2 s2) W^2 f1 A2• - 243/32 • !!! 5 A2^2 w2 A2•>
```

Then, solvability conditions demand that **ST21** and **ST22** be orthogonal to their corresponding adjoints. The result is

```
SCond21 = SolveAccleftVec@@1DD.ST21 == 0, A1^H0,1L@T1, T2DE@@1DD •.
   9w1 -> • !!! 5 ' 3, w2 -> • !!! 5= •• ExpandAll;
SCond21 •. displayRule
```

```
:D2A1 ® - 25/432 I • !!! 5 A1 s1 - 353 I • !!! 5 A1^2 A1•/3456 + 11/128 I • !!! 5 A2 A1•^2 + 23/192 I • !!! 5 A1 A2 A2•>
```

```
SCond22 = SolveAccleftVec@@2DD.ST22 == 0, A2^H0,1L@T1, T2DE@@1DD •.
   9W -> 2 • !!! 5, w1 -> • !!! 5 ' 3, w2 -> • !!! 5= •• ExpandAll;
SCond22 •. displayRule
```

```
9D2A2 ® 11/128 I • !!! 5 A1^3 + 25/16 I • !!! 5 A2 s1 + 23/64 I • !!! 5 A1 A2 A1• - 1337/128 I • !!! 5 A2^2 A2• + 10 I • !!! 5 E^(I T2 s2) f1 A2•=
```

```
SCond2Rule = Join@SCond21, SCond22D;
```

Ÿ Reconstitution

Using the method of reconstitution, $A_k' = \epsilon D_1 A_k + \epsilon^2 D_2 A_k + \cdots$, we combine the partial-differential equations **SCond1Rule** and **SCond2Rule** into the following two ordinary-differential equations governing the modulation of the complex-valued functions A_k:

```
moduEq = Table@2 I wk Ak¢, 8k, 2<D ==
    HTable@2 I wk dt@1D@ Ak@T1, T2DD, 8k, 2<D •. SCond1Rule •. SCond2RuleL •• Thread;
moduEq •. displayRule
```

$$\left\{2 I \omega_1 A_1' == 2 I \epsilon^2 \omega_1 \left(-\frac{25}{432} I \sqrt{5}\, A_1 \sigma_1 - \frac{353 I \sqrt{5}\, A_1^2 \bar{A}_1}{3456} + \frac{11}{128} I \sqrt{5}\, A_2 \bar{A}_1^2 + \frac{23}{192} I \sqrt{5}\, A_1 A_2 \bar{A}_2\right),\right.$$

$$\left. 2 I \omega_2 A_2' == 2 I \epsilon^2 \omega_2 \left(\frac{11}{128} I \sqrt{5}\, A_1^3 + \frac{25}{16} I \sqrt{5}\, A_2 \sigma_1 + \frac{23}{64} I \sqrt{5}\, A_1 A_2 \bar{A}_1 - \frac{1337}{128} I \sqrt{5}\, A_2^2 \bar{A}_2 + 10 I \sqrt{5}\, E^{I T_2 \sigma_2} f_1 \bar{A}_2\right)\right\}$$

Ÿ 6.2.2 The function MMSC

According to the procedures described in the previous section, we build a function named **MMSC** (**M**ethod of **M**ultiple **S**cales for Linearly **C**oupled Systems) to automate the process.

```
MMSC@eqs_List, depVar_List, scaling_List, ResonanceCond : 8__Equal<D :=
 ModuleA8<,
       omgList = 8w1, w2<;
  omgRule = Solve@ResonanceCond, 8W, #< •• FlattenD@@1DD & •ž omgList •• Reverse;
       mydepVar = 8u1, v1, u2, v2<;
       solRule = 8ui_ -> HSum@e^j ui,j@#1, #2, #3D, 8j, 3<D &L,
    vi_ -> HSum@e^j vi,j@#1, #2, #3D, 8j, 3<D &L<;
       maxOrder = 2;
  multiScales = 8ui_@tD -> ui žž timeScales, vi_@tD -> vi žž timeScales,
    Derivative@1D@u_D@tD -> dt@1D@u žž timeScalesD, t -> T0<;
       eqa = Heqs •. Thread@depVar -> mydepVarD •. scaling •. multiScales •. solRule ••
      TrigToExp •• ExpandAllL •. e^Hn_ •; n > 3L -> 0;
       eqEps = Rest@Thread@CoefficientList@Subtract žž #, eD == 0DD & •ž eqa •• Transpose;
   eqOrder@i_D :=
  I#@@1DD & •ž eqEps@@1DD •. fs_ -> 0 •. u_k_,1 -> uk,iM == I#@@1DD & •ž eqEps@@1DD •. fs_ -> 0 •.
       u_k_,1 -> uk,iM - H#@@1DD & •ž eqEps@@iDDL •• Thread;

       H* First-Order Problem *L
       linearSys = #@@1DD & •ž eqOrder@1D;
       assumedForm = 8ui_,1 -> Hci E^HI w #L &L, vi_,1 -> Hdi E^HI w #L &L<;
       coefMat =
```

```
Outer@D, E^H-I w T0L linearSys •. assumedForm •• Expand, 8c1, d1, c2, d2<D;
      hermitian@mat_?MatrixQD := mat •. conjugateRule •• Transpose;
      evals = Cases@w •. Solve@Det@coefMatD == 0, wD, n_ •; n > 0D;
      values = Append@omgRule@@1DD •. e -> 0 •. w1 -> evals@@1DD, w1 -> evals@@1DDD;
      rightVec = # • #@@1DD & •ž HNullSpace@coefMat •. w -> #D@@1DD & •ž evalsL;
leftVec = NullSpace@hermitian@coefMatD •. w -> #D@@1DD & •ž evals;
      ccleftVec = leftVec •. conjugateRule;
      order1Eq = eqOrder@1D •. u_i_,1 -> Hui,1@#1D &L;
      sol1p =
 DSolve@order1Eq, 8u1,1@T0D, v1,1@T0D, u2,1@T0D, v2,1@T0D<, T0D@@1DD •. C@_D -> 0 •• Simplify;
      fRule = 9fi_ -> 2 Li Iwi^2 - W^2M=;
      sol1Form = H#@@2DD & •ž sol1pL + HH# + H# •. conjugateRuleLL & •ž
    HTranspose@rightVecD.8A1@T1, T2D E^HI w1 T0L, A2@T1, T2D E^HI w2 T0L<LL;
      sol1 = 8u1,1, v1,1, u2,1, v2,1< -> HFunction@8T0, T1, T2<, # •• EvaluateD & •ž
     sol1FormL •• Thread;

      H* Second-Order Problem *L
      order2Eq = eqOrder@2D •. sol1 •• ExpandAll;
      expRule1@i_D :=
 Exp@a_D :> Exp@Expand@a •. omgRule@@iDDD •. e^n_. T0 :> timeScales@@n + 1DDD;
      ST11 = Coefficient@#@@2DD •. expRule1@1D, E^HI w1 T0LD & •ž order2Eq;
      ST12 = Coefficient@#@@2DD •. expRule1@2D, E^HI w2 T0LD & •ž order2Eq;
      SCond1 = 8ccleftVec@@1DD.ST11 == 0, ccleftVec@@2DD.ST12 == 0<;
      SCond1Rule1 = SolveASCond1, 9A1^H1,0L@T1, T2D, A2^H1,0L@T1, T2D=E@@1DD •• ExpandAll;
      H* sigRule=Solve@ResonanceCond,8s1,s2<D@@1DD; *L
      sigRule = Solve@ResonanceCond, s2D@@1DD;
      expRule2 = Exp@a_D :> Exp@a •. 8T1 -> e T0, T2 -> e^2 T0< •. sigRule •• ExpandD;
      SCond1Rule2 = SCond1Rule1 •. expRule2;
      ccSCond1Rule2 = SCond1Rule2 •. conjugateRule;
      order2Eqm = order2Eq •. SCond1Rule2 •. ccSCond1Rule2 •• ExpandAll;
      IfAUnion@#@@2DD & •ž order2EqmD === 80<,
        sol2 = 8u1,2 -> H0 &L, v1,2 -> H0 &L, u2,2 -> H0 &L, v2,2 -> H0 &L<,
        Return@"The particular solution of the second order
    equations needs to be solved, which is not included in this function."D
      H*
         basicH=
   TableA9Ai@T1,T2D E^HI wi T0L,Ȧi@T1,T2D E^H-I wi T0L=,8i,2<E••Flatten;
         collectForm=Join@basicH,
    If@list1=List žž Plus žž H#@@2DD & •ž eqOrder@1DL•.c_ fi_ E^a_->Li E^a;
     Head@list1D===List,list1,8<DD;
         possibleTerms=JoinAcollectForm,
    IfAlist1=List žž Plus žž I#@@2DD & •ž eqOrder@2D•.u_i_,j_->H0 &LM•.
```

```
      c_ fi_ E^a_->E^a;Head@list1D===List,list1,8<E,
   Outer@Times,collectForm,collectFormD••Flatten••UnionE;
        ResonantTerms@i_D:=H#•.8a_•;a=!=0->1< & •ž
       HE^H-I wi T0L possibleTerms•.expRule1@iD•.Exp@_ T0+_.D->0LL
     possibleTerms••Union••Rest;
       RT=8Array@ResonantTerms,2D,Array@ResonantTerms,2D•.conjugateRule<••
   Flatten;
        NRT=Complement@possibleTerms,RTD;
        rRule=Table@MapIndexed@
     ri,#2@@1DD->Coefficient@order2Eqm@@i,2DD,#1D &,RTD,8i,4<D••Flatten;
        list1=Join@ccleftVec,Conjugate@ccleftVecDD;
      r4Rule=
  Table@Solve@list1@@jDD.Table@ri,j,8i,4<D==0,r4,jD@@1DD,8j,4<D••Flatten;
        RTsymbolList@i_D=Table@Gi,j,8j,Length@RTD<D;
        NRTsymbolList@i_D=Table@Li,j,8j,Length@NRTD<D;
        sol2Form=8u1,2,v1,2,u2,2,v2,2<->
    HFunction@8T0,T1,T2<,RTsymbolList@#D.RT+NRTsymbolList@#D.NRT••EvaluateD & •ž
      Range@4DL••Thread;
        coef1=Solve@Coefficient@Subtract žž #•.sol2Form,NRTD==0•.
         Exp@_ T0+_.D->0 & •ž order2Eqm,
      Array@NRTsymbolList,4D••FlattenD@@1DD•.values••ExpandAll;
        eq1=Table@MapIndexed@Coefficient@order2Eqm@@k,1DD•.sol2Form,#1D==
         rk,#2@@1DD &,RTD•.Exp@_ T0+_.D->0,8k,4<D•.r4Rule••Flatten;
     coef2=HSolve@eq1,Join žž Array@RTsymbolList,4DD@@1DD••SimplifyL•.values•.
    rRule••ExpandAll;
        sol2=sol2Form•.Function@8T0,T1,T2<,a_D:>
    Function@8T0,T1,T2<••Evaluate,a•.coef1•.coef2••Expand••EvaluateD    *L
  E;

     H* Third-Order Problem *L
     order3Eq = eqOrder@3D •. sol1 •. sol2 •• ExpandAll;
     ST21 = Coefficient@#@@2DD •. expRule1@1D, Exp@I w1 T0DD & •ž order3Eq;
     ST22 = Coefficient@#@@2DD •. expRule1@2D, Exp@I w2 T0DD & •ž order3Eq;
     SCond2 = 8ccleftVec@@1DD.ST21 == 0, ccleftVec@@2DD.ST22 == 0< •• ExpandAll;
     SCond2Rule1 =
  SolveASCond2, 9A1^H0,1L@T1, T2D, A2^H0,1L@T1, T2D=E@@1DD •. values •• ExpandAll;

     H* Reconstitution *L
     moduEq = Table@2 I wk Ak¢, 8k, 2<D ==
   HTable@2 I wk dt@1D@ Ak@T1, T2DD, 8k, 2<D •. HSCond1Rule1 •. valuesL •. SCond2Rule1 ••
    Collect@#, eD &L •• Thread;
     Print@"The second-order approximate solution:"D;
     Print@
```

```
    Table@ui@tD == Hui žž timeScales •. solRule •. e^3 -> 0 •. sol1 •. sol2 •. displayRuleL,
      8i, 2<D •. Thread@mydepVar -> depVarDD;
         IfAOr žž Table@HFi • fi •. scalingL === e, 8i, 2<D,
             Print@"where"D;
             PrintA"Li==fi•H2Hw2i-W^2LL"E
         E;
         Print@"\nThe modulation equations:"D;
         Print@moduEq •. displayRuleD
E
```

As an example, we check the case in Section 6.2.1:

```
eqtest = 9
         q1¢@tD == v1@tD,
         v1¢@tD + a v2¢@tD + q1@tD ==
    - 2 W2 Cos@t WD F1 q1@tD + 1/6 q1@tD3 + a v2@tD2 H- q1@tD + q2@tDL + 1/2 a H- q1@tD + q2@tDL2 v2¢@tD,
         q2¢@tD == v2@tD,
         v2¢@tD + v1¢@tD + q2@tD ==
    - 2 W2 Cos@t WD F1 q2@tD + 1/6 q2@tD3 - v1@tD2 H- q1@tD + q2@tDL + 1/2 H- q1@tD + q2@tDL2 v1¢@tD
    = •. 9a ® 16/25 + e2 s1=;

scaling1 = 9F1 -> e2 f1=;

ResonanceCond1 = 9w2 == 3 w1, W == 2 w2 + e2 s2=;

MMSC@eqtest, 8q1, v1, q2, v2<, scaling1, ResonanceCond1D •• Timing
```

```
The second-order approximate solution:

:q1@tD == e IEI T0 w1 A1 + EI T0 w2 A2 + E-I T0 w1 Ā1 + E-I T0 w2 Ā2M,
 q2@tD == e J5/4 EI T0 w1 A1 - 5/4 EI T0 w2 A2 + 5/4 E-I T0 w1 Ā1 - 5/4 E-I T0 w2 Ā2N>


The modulation equations:

:2 I w1 A1¢ == 2 I e2 w1 J- 25/432 I è!!!5 A1 s1 - (353 I è!!!5 A1^2 Ā1)/3456 + 11/128 I è!!!5 A2 Ā1^2 + 23/192 I è!!!5 A1 A2 Ā2N,
 2 I w2 A2¢ == 2 I e2 w2
   J11/128 I è!!!5 A1^3 + 25/16 I è!!!5 A2 s1 + 23/64 I è!!!5 A1 A2 Ā1 - 1337/128 I è!!!5 A2^2 Ā2 + 10 I è!!!5 EI T2 s2 f1 Ā2N>

87.691 Second, Null<
```

Chapter 7
Continuous Systems with Cubic Nonlinearities

Elastic systems, such as beams, plates, and shells, are usually modeled by partial-differential equations with specified boundary conditions. For small oscillations, the responses of such deformable bodies, continuous or distributed-parameter systems, can be adequately described by linear equations and boundary conditions. However, as the amplitude of oscillations increases, nonlinear effects in the governing equations, boundary conditions, or both come into play. The sources of nonlinearities may be geometric, inertial, material, or damping in nature.

Geometric nonlinearities may be due to one or more of the following: nonlinear relationships among the strains and the displacements, large rotations, free surfaces in fluids, time-dependent constraints, mid-plane stretching, and large curvatures. Nonlinear stretching of the midplane of a deformable body accompanies its transverse vibrations if it is supported in such a way as to restrict the movement of its ends and / or edges. If large-amplitude vibrations are accompanied by large changes in the curvature, it is necessary to employ a nonlinear relationship between the curvature and the displacement. Inertial nonlinearities are caused by concentrated and / or distributed masses, convective accelerations, and Corriolis and centripetal accelerations. Material nonlinearities occur whenever the constitutive relations are nonlinear, such as the case when the stresses are nonlinear functions of the strains; the resistive, inductive, and capacitive circuit elements are nonlinear; and the feedback control forces and moments in servomechanisms are nonlinear. Damping nonlinearities occur due to form drag and hysteresis.

Since exact solutions are, in general, not available for determining the dynamic responses of nonlinear continuous systems to external or parametric excitations, recourse has been made to approximate analyses by using either purely numerical techniques, or purely analytical techniques, or a combination of numerical and analytical techniques. Application of purely numerical techniques to such problems may be costly in terms of computation time and may not reveal some of their intricate and complicated responses. With purely analytical methods, it may be difficult to treat systems with inhomogeneities or complicated geometries. With a combination of numerical and analytical techniques, one can determine some of the complicated responses of systems with inhomogeneities and complicated shapes.

The numerical-analytic approaches can be divided into two groups: discretization and direct methods. In the discretization methods, one postulates the solution in the form

$$w[x,t]=\sum_{m=1}^{M} \phi_m(x)\, q_m(t)$$

where M is a finite integer. Then, one assumes the spatial functions $\phi_m(x)$, space discretization, or the temporal functions $q_m(t)$, time discretization. With this discretization, the $q_m(t)$ are usually taken to be harmonic and the method of harmonic balance is used to obtain a set of nonlinear boundary-value problems for the $\phi_m(x)$.

With space discretization, the $\phi_m(x)$ (and thus the spatial dependence) are assumed a priori. If the boundary conditions are homogeneous, the $\phi_m(x)$ are usually taken to be the eigenfunctions of the linearized problem. The method of weighted residuals or variational principles can then be used to determine a set of ordinary-differential equations governing the $q_m(t)$.

The obtained set of ordinary-differential equations can be studied using any of a great number of methods developed for discrete systems.

The most common implementation of weighted residuals is the Galerkin method. The great majority of recent studies of forced vibrations assume that the response can be expressed in terms of only the linear modes that are directly or indirectly excited (Nayfeh and Mook, 1979). For example, if a system is driven near the natural frequency of a linear mode and that mode is not involved in an internal resonance with other modes, the response is assumed to consist of only that mode. Such an approach is usually referred to as a single-mode approximation.

In the direct approach, a reduction method, such as the method of multiple scales, is applied directly to the governing partial-differential equations and associated boundary conditions and no assumptions are made a priori regarding the spatial or temporal dependence of the response. The principal advantage of this approach is in the treatment of the boundary conditions at higher orders. This approach has been used recently by Nayfeh and Nayfeh (1979), Nayfeh (1975, 1996), Nayfeh and Asfar (1986), Nayfeh and Bouguerra (1990), Pai and Nayfeh (1990), Raouf and Nayfeh (1990), Nayfeh, Nayfeh, and Mook (1992), Nayfeh and Nayfeh (1993, 1995), Nayfeh, Nayfeh, and Pakdemirli (1995), Pakdemirli, Nayfeh, and Nayfeh (1995), Chin and Nayfeh (1996), Nayfeh and Lacarbonara (1997, 1998), Lacarbonara, Nayfeh, and Kreider (1998), Nayfeh, Lacarbonara, and Chin (1999), and Rega, et al. (1999).

Some of the aforementioned studies show that the discretization and direct approaches yield the same results for systems with cubic nonlinearities provided that first-order approximations are sought. For systems with quadratic and cubic nonlinearities, the discretization approach might produce quantitative, and in some case qualitative, errors unless many modes are included in the discretized model. In other words, one has to include as many terms in the discretized model as needed for convergence. Since we are dealing with systems with cubic nonlinearities in this chapter, we will obtain first-order results using both the discretization and direct approaches.

à Preliminaries

```
Off@General::spell1, Integrate::generD
```

```
Needs@"Utilities`Notation`"D
```

To use the method of multiple scales, we introduce different time scales $T_0 = t$, $T_1 = \epsilon t$, and $T_2 = \epsilon^2 t$, symbolize them as

```
Symbolize@T0D; Symbolize@T1D; Symbolize@T2D;
```

and form a list of them as follows:

```
timeScales = 8T0, T1, T2<;
```

In terms of the time scales T_n, the time derivatives become

```
dt@0D@expr_D := expr; dt@1D@expr_D := SumAe^i D@expr, timeScales@@i + 1DDD, 8i, 0, 1<E;
dt@2D@expr_D := Hdt@1D@dt@1D@exprDD •• ExpandL •. e^(i_•;i>1) -> 0;
```

In the course of the analysis, we need the complex conjugates of A and G. We define them using the following rule:

```
conjugateRule = 9A -> Ȧ, Ȧ -> A, G -> Ġ, Ġ -> G, Complex@0, n_D -> Complex@0, -nD=;
```

To manipulate some complicated integrals without *Mathematica* being choked, we define the following rules:

```
intRule1 = 8int@fun_, arg2__D :> int@Expand@funD, arg2D<;
intRule2 = 8int@a_ + b_, arg2__D :> int@a, arg2D + int@b, arg2D,
   int@e^n_. fun_, arg2__D :> e^n int@fun, arg2D,
   int@a_ fun_, a1_, b1___D :> a int@fun, a1, b1D •; FreeQ@a, First@a1DD,
   int@int@a1_, a2_D fun_., a3__D :> int@a1, a2D int@fun, a3D •; FreeQ@Rest@a2D, First@a2DD<;
```

To represent some of the expressions in a more concise way, we introduce the following display rule:

```
displayRule = 9Derivative@a_, b__DAw_i_ E@x, __D :>
    SequenceFormAIfAarg1 = Times žž MapIndexedAD#1#2@@1DD-1 &, 8b<E; arg1 =!= 1, arg1, ""E,
     wi SequenceForm žž Table@"'",8a<DE,
   Derivative@a__D@Ai_D@__D :> SequenceFormATimes žž MapIndexedAD#1#2@@1DD &, 8a<E, AiE,
   Derivative@a__D@Ȧi_D@__D :> SequenceFormATimes žž MapIndexedAD#1#2@@1DD &, 8a<E, ȦiE,
   w_i_@x, __D -> wi, Ai_@__D -> Ai, Ȧi_@__D -> Ȧi,
   Exp@a_. + b_. Complex@0, m_D T0 + c_. Complex@0, n_D T0D :> Exp@a + Hm * b + n * cL I T0D,
   int -> Integrate=;
```

à 7.1 Solvability Conditions and the Concept of Adjoint

In directly attacking continuous systems, one often encounters nonhomogeneous boundary-value problems whose homogeneous parts have nontrivial solutions. Consequently, the nonhomogeneous problems have solutions only if solvability or consistency conditions are satisfied (Nayfeh, 1981). In this section, we describe how to determine such solvability conditions.

We start the discussion with the simple problem

```
eq1 = y''@xD + p^2 y@xD == p Sin@p xD;
bc1 = 8y@0D == b1, y@1D == b2<;
```

The corresponding homogeneous problem has the nontrivial solution

```
DSolve@8eq1@@1DD == 0<~Join~bc1 •. bi_ -> 0, y@xD, xD@@1DD
```

```
8y@xD ® - C@1D Sin@p xD<
```

Hence, the nonhomogeneous problem will not have a solution unless a solvability condition is satisfied. To determine this solvability condition, we use two approaches.

In the first approach, we find the general solution of **eq1** using the function **DSolve** as

```
yRule = DSolve@eq1, y@xD, xD@@1DD •• Simplify
```

```
:y@xD ® J- x/2 + C@2DN Cos@p xD - C@1D Sin@p xD>
```

where $C[1]$ and $C[2]$ are arbitrary constants. Imposing the boundary conditions **bc1**, we have

```
bc2 = bc1 •. Flatten@yRule •. 88x -> 0<, 8x -> 1<<D
```

```
9C@2D == b1, 1/2 - C@2D == b2=
```

These equations are inconsistent unless

```
Equal žž Plus žž HList žž # & •ž bc2L
```

```
1/2 == b1 + b2
```

which is the desired solvability condition.

In the second approach, instead of determining the general solution of the homogeneous differential equations and then enforcing the boundary conditions to determine the solvability conditions, we use the concept of adjoint as described next. This approach is attractive if one is not interested in determining the solution of the nonhomogeneous problem but interested only in determining the solvability conditions, as is the case in many applications of perturbation methods.

To determine the adjoint and then the solvability conditions, we need to perform integration by parts of some products. To accomplish this with *Mathematica*, we define a function named **intByParts** according to

```
intByParts@expr_ •; Head@exprD =!= Equal, depVar_D :=
 Module@8intRule1, intRule2, intRule3<,
       intRule1 = f_@terms_, x_D :> int@Expand@termsD, xD;
  intRule2 = int@term1_ + term2_, x_D :> int@term1, xD + int@term2, xD;
  intRule3 = c1_. int@term1_, x_D + c2_. int@term2_, x_D :> int@c1 term1 + c2 term2, xD;
  IBP@u_, v_, n_ •; n >= 1, range : 8x_, a_, b_<D := Hu Derivative@n - 1D@vD@xD •. x -> bL -
   Hu Derivative@n - 1D@vD@xD •. x -> aL - IBP@D@u, xD, v, n - 1, rangeD;
  IBP@u_, v_, 0, range : 8x_, a_, b_<D := int@u v@xD, rangeD;

  expr •. intRule1 ••. intRule2 •. int@u_ Derivative@n_D@depVarD@x_D, arg2_D :>
     IBP@u, depVar, n, arg2D ••. intRule3 •. int -> Integrate
 D
```

To determine the solvability condition of **eq1** and **bc1** with this approach, we multiply **eq1** by $u[x]$, integrate the result by parts from $x = 0$ to $x = 1$, and obtain

```
eq1a = intByParts@int@u@xD #, 8x, 0, 1<D, yD & •ž eq1
```

$$\int_0^1 (\pi^2 u[x]\, y[x] + y[x]\, u''[x])\, dx + y[0]\, u'[0] - y[1]\, u'[1] - u[0]\, y'[0] + u[1]\, y'[1] == \pi \int_0^1 \sin[\pi x]\, u[x]\, dx$$

To determine the adjoint, we set the coefficient of $y[x]$ in the integrand on the left-hand side of **eq1a** equal to zero; that is,

```
adjointEq1 = Cases@eq1a@@1DD, Integrate@expr_, _D :> Coefficient@expr, y@xDDD@@1DD == 0
```

```
p^2 u@xD + u″@xD == 0
```

To determine the adjoint boundary conditions, we consider the homogeneous problem (i.e., $f@xD = 0$, $b_1 = 0$, and $b_2 = 0$), use the **adjointEq1**, and obtain from **eq1a** that

```
bc1a = eq1a ê. Integrate -> H0 &L ê. Hbc1 ê. Equal -> Rule ê. b_i_ -> 0L
```

```
- u@0D y¢@0D + u@1D y¢@1D == 0
```

To determine the adjoint boundary conditions, we set each of the coefficients of $y'@0D$ and $y'@1D$ in **bc1a** equal to zero and obtain

```
adjointBC1 =
 Solve@Coefficient@bc1a@@1DD, 8y¢@0D, y¢@1D<D == 0, 8u@0D, u@1D<D@@1DD ê. Rule -> Equal
```

```
8u@0D == 0, u@1D == 0<
```

Therefore, the adjoint u is defined by the adjoint system consisting of **adjointEq1** and **adjointBC1**. Since they are the same as the homogeneous parts of **eq1** and **bc1**, the problem is said to be self-adjoint. Hence, $u@xD = \text{Sin}@xD$.

```
adjoint = 8u -> Function@x, Sin@p xDD<
```

```
8u ® Function@x, Sin@p xDD<
```

Once the adjoint problem has been defined, we return to the nonhomogeneous problem to determine the solvability condition. Substituting for the adjoint and the boundary conditions **bc1** into **eq1a** yields the solvability condition

```
SolvCond1 = eq1a@@2DD ==
  Heq1a@@1DD ê. Integrate -> H0 &L ê. Hbc1 ê. Equal -> RuleL ê. HadjointBC1 ê. Equal -> RuleLL
```

```
    1
p à   Sin@p xD u@xD â x == b1 u¢@0D - b2 u¢@1D
    0
```

```
SolvCondf = SolvCond1 ê. adjoint
```

```
p
¢¢¢ == p b1 + p b2
2
```

which is the same as the solvability condition obtained with the other method.

Ÿ 7.1.1 Hinged-Clamped Beam

In treating a hinged-clamped beam, we need to determine the adjoint of the boundary-value problem

```
eq1a = -w^2 f@xD - 2 P f″@xD + f^H4L@xD == f@xD;
```

```
bc1a = 8f@0D == 0, f″@0D == 0, f@1D == 0, f¢@1D == 0<;
```

Multiplying **eq1a** by *u*@*x*D and integrating the result by parts from $x = 0$ to $x = 1$, we obtain

```
eq1b = intByParts@int@u@xD #, 8x, 0, 1<D, fD & •ž eq1a
```

```
  1
à   H-w2 u@xD f@xD - 2 P f@xD u²@xD + f@xD uH4L@xDL â x - 2 P f@0D u¢@0D + 2 P f@1D u¢@1D +
  0
   2 P u@0D f¢@0D - 2 P u@1D f¢@1D - f¢@0D u²@0D + f¢@1D u²@1D + u¢@0D f²@0D - u¢@1D f²@1D +
                                                                    1
   f@0D uH3L@0D - f@1D uH3L@1D - u@0D fH3L@0D + u@1D fH3L@1D == à   f@xD u@xD â x
                                                                    0
```

To determine the adjoint equation, we set the coefficient of f@*x*D in the integrand on the left-hand side of **eq1b** equal to zero; that is,

```
adjointEq1 = Cases@eq1b@@1DD, Integrate@expr_, _D :> Coefficient@expr, f@xDDD@@1DD == 0
```

```
-w2 u@xD - 2 P u²@xD + uH4L@xD == 0
```

To determine the adjoint boundary conditions, we consider the homogeneous problem, use the **adjointEq1**, and obtain from **eq1b** that

```
bc1b = eq1b •. Integrate -> H0 &L •. Hbc1a •. Equal -> RuleL
```

```
2 P u@0D f¢@0D - f¢@0D u²@0D - u¢@1D f²@1D - u@0D fH3L@0D + u@1D fH3L@1D == 0
```

To determine the adjoint boundary conditions, we set each of the coefficients of f¢@0D, fH3L@0D, f²@1D, and fH3L@1D in **bc1b** equal to zero and obtain

```
adjointBC1 = SolveACoefficientAbc1b@@1DD, 9f¢@0D, fH3L@0D, f²@1D, fH3L@1D=E == 0,
    8u@0D, u²@0D, u@1D, u¢@1D<E@@1DD •. Rule -> Equal
```

```
8u²@0D == 0, u@1D == 0, u¢@1D == 0, u@0D == 0<
```

Therefore, the adjoint *u* is defined by the adjoint system consisting of **adjointEq1** and **adjointBC1**. Since they are the same as the homogeneous parts of **eq1a** and **bc1a**, the problem is said to be self-adjoint.

Once the adjoint problem has been defined, we return to the nonhomogeneous problem to determine the solvability condition. Substituting for the adjoint and the boundary conditions **bc1a** into **eq1b** yields the solvability condition

```
SolvCond1 = eq1b@@2DD ==
  Heq1b@@1DD •. Integrate -> H0 &L •. Hbc1a •. Equal -> RuleL •. HadjointBC1 •. Equal -> RuleLL
```

```
  1
à   f@xD u@xD â x == 0
  0
```

Ÿ 7.1.2 Cantilever Beam

The boundary-value problem for a cantilever beam can be written as

```
eq2a = -w2 f@xD + fH4L@xD == f@xD;
```

```
bc2a = 9f@0D == 0, f¢@0D == 0, f²@1D == 0, fH3L@1D == 0=;
```

To determine the adjoint of this problem, we multiply **eq2a** by *u*@*x*D, integrate the result by parts from *x* = 0 to *x* = 1, and obtain

```
eq2b = intByParts@int@u@xD #, 8x, 0, 1<D, fD & •ž eq2a
```

```
à 1 0 H-w2 u@xD f@xD + f@xD uH4L@xDL â x - f¢@0D u²@0D + f¢@1D u²@1D + u¢@0D f²@0D -
  u¢@1D f²@1D + f@0D uH3L@0D - f@1D uH3L@1D - u@0D fH3L@0D + u@1D fH3L@1D == à 1 0 f@xD u@xD â x
```

We set the coefficient of f@*x*D in the integrand on the left-hand side of **eq2b** equal to zero and obtain the adjoint equation as

```
adjointEq2 = Cases@eq2b@@1DD, Integrate@expr_, _D :> Coefficient@expr, f@xDDD@@1DD == 0
```

```
-w2 u@xD + uH4L@xD == 0
```

To determine the adjoint boundary conditions, we consider the homogeneous problem, use the **adjointEq2**, and obtain from **eq2b** that

```
bc2b = eq2b •. Integrate -> H0 &L •. Hbc2a •. Equal -> RuleL
```

```
f¢@1D u²@1D + u¢@0D f²@0D - f@1D uH3L@1D - u@0D fH3L@0D == 0
```

To determine the adjoint boundary conditions, we set each of the coefficients of f²@0D, fH3L@0D, f@1D, and f¢@1D in **bc2b** equal to zero and obtain

```
adjointBC2 = SolveACoefficientAbc2b@@1DD, 9f²@0D, fH3L@0D, f@1D, f¢@1D=E == 0,
    9u@0D, u¢@0D, u²@1D, uH3L@1D=E@@1DD •. Rule -> Equal
```

```
8u@0D == 0, u¢@0D == 0, u²@1D == 0, uH3L@1D == 0<
```

Therefore, the adjoint *u* is defined by the adjoint system consisting of **adjointEq2** and **adjointBC2**. Since they are the same as the homogeneous parts of **eq2a** and **bc2a**, the problem is said to be self-adjoint.

Once the adjoint problem has been defined, we return to the nonhomogeneous problem to determine the solvability condition. Substituting for the adjoint and the boundary conditions **bc2a** into **eq2b** yields the solvability condition

```
SolvCond2 = eq2b@@2DD ==
  Heq2b@@1DD •. Integrate -> H0 &L •. Hbc2a •. Equal -> RuleL •. HadjointBC2 •. Equal -> RuleLL
```

```
à 1 0 f@xD u@xD â x == 0
```

à 7.2 Hinged-Clamped Beam

Ÿ 7.2.1 EOM and BC's

We consider the nonlinear planar response of a hinged-clamped uniform prismatic beam to a harmonic axial load. The beam is subjected to a static axial load and one of its ends is restrained by a linear spring. We assume that the curvature and inertia nonlinearities are much smaller than the nonlinearity caused by the mid-plane stretching. The transverse deflection *w*@*x*, *t*D of the beam at the position *x* and time *t* is governed by the nondimensional integral-partial-differential equation (Nayfeh and Mook, 1979)

```
EOM = ¶t,t w@x, tD - 2 P ¶x,x w@x, tD + ¶x,x,x,x w@x, tD ==
  -2 e m@xD ¶t w@x, tD + 4 e F ¶x,x w@x, tD Cos@W tD + e a ¶x,x w@x, tD à_0^1 H¶x w@x, tDL^2 â x
```

```
w^H0,2L@x, tD - 2 P w^H2,0L@x, tD + w^H4,0L@x, tD ==
 -2 e m@xD w^H0,1L@x, tD + 4 F e Cos@t WD w^H2,0L@x, tD + a e jà_0^1 w^H1,0L@x, tD^2 â xz w^H2,0L@x, tD
                                                             k                          {
```

and homogeneous boundary conditions

```
BC = 8w@0, tD == 0, ¶x,x w@x, tD == 0 •. x -> 0, w@1, tD == 0, ¶x w@x, tD == 0 •. x -> 1<
```

```
8w@0, tD == 0, w^H2,0L@0, tD == 0, w@1, tD == 0, w^H1,0L@1, tD == 0<
```

We assume that the static axial load is such that the lowest two natural frequencies of the beam are in the ratio of three-to-one; that is, there is a three-to-one internal resonance between the second and first modes. Moreover, we assume that neither of these two modes is involved in an internal resonance with any other mode. We consider three parametric resonances, namely, principal parametric resonance of the first mode, principal parametric resonance of the second mode, and combination parametric resonance of the first and second modes.

Ÿ 7.2.2 Direct Attack of the Continuous Problem

In this section, we directly attack the integral-partial-differential equation **EOM** and associated boundary conditions **BC** and seek a first-order uniform expansion of their solution in the form

```
solRule = w -> IEvaluateASumAe^j wj@#1, #2, #3D, 8j, 0, 1<EE &M
```

```
w ® Hw0@#1, #2, #3D + e w1@#1, #2, #3D &L
```

where the first independent variable stands for *x* and the last two independent variables stand for the two time scales T_0 and T_1. Substituting the **solRule** into **EOM**, transforming the total time derivatives into partial derivatives in terms of T_0 and T_1, expanding the result for small e, discarding terms of order higher than e, and using the **intRule2** to simplify the expansions of the integrands, we obtain

```
eq722a =
  HHJoin@8EOM<, BCD •. Integrate -> int •. 8w@x_, tD -> w@x, T0, T1D, Derivative@m_, n_D@wD@
        x_, tD -> dt@nD@D@w@x, T0, T1D, 8x, m<DD, t -> T0< •.
      solRule •• ExpandAllL ••. intRule2 •• ExpandAllL •. e^(n_•;n>1) -> 0;
```

Equating coefficients of like powers of e in **eq722a**, we obtain

```
eqEps = Thread@CoefficientList@Subtract žž #, eD == 0D & •ž eq722a •• Transpose;
eqEps •. displayRule
```

:9- 2 P Hw0''L + w0'''' + D0^2 w0 == 0, w0@0, T0, T1D == 0,
w0^H2,0,0L@0, T0, T1D == 0, w0@1, T0, T1D == 0, w0^H1,0,0L@1, T0, T1D == 0=,
: - 4 F Cos@T0 WD Hw0''L - a Hw0'L^2 Hw0''L - 2 P Hw1''L + w1'''' + D0^2 w1 + 2 HD0 D1w0L + 2 HD0w0L m@xD == 0,
w1@0, T0, T1D == 0, w1^H2,0,0L@0, T0, T1D == 0, w1@1, T0, T1D == 0, w1^H1,0,0L@1, T0, T1D == 0>>

Because in the presence of damping, all modes that are not directly excited by the forcing or indirectly excited by the internal resonance will decay with time (Nayfeh and Mook, 1979), the solution of **eqEps[[1]]** can be expressed in terms of the lowest two linear free-vibration modes; that is,

```
sol0 =
  w0 -> FunctionA8x, T0, T1<, SumAAi@T1D fi@xD Exp@I wi T0D + Ȧi@T1D fi@xD Exp@- I wi T0D, 8i, 2<EE;
```

where w_1 and w_2 are the natural frequencies of these modes. For later use, we define the list

```
omgList = 8w1, w2<;
```

One can easily show that the mode shapes $f_i@xD$ are orthogonal. We assume that these modes are normalized so that

Ù0^1 fi@xD fj@xD â x -> dij

where d_{ij} is the Kronecker delta function.

Substituting **sol0** into the first-order equation, **eqEps[[2,1]]**, and using **intRule1** and **intRule2**, we obtain

```
order1Eq = HeqEps@@1, 1, 1DD •. w0 -> w1L ==
   HHeqEps@@1, 1, 1DD •. w0 -> w1L - HSubtract žž eqEps@@2, 1DD •. sol0 •• TrigToExp ••
        ExpandL •. intRule1 ••. intRule2 •• ExpandL;
order1Eq •. displayRule
```

```
-2 P Hw1''L + w1'''' + D0^2 w1 == -2 I E^(I T0 w1) HD1A1L w1 f1@xD + 2 I E^(-I T0 w1) HD1Ȧ1L w1 f1@xD -
  2 I E^(I T0 w1) A1 w1 m@xD f1@xD + 2 I E^(-I T0 w1) w1 Ȧ1 m@xD f1@xD - 2 I E^(I T0 w2) HD1A2L w2 f2@xD +
  2 I E^(-I T0 w2) HD1Ȧ2L w2 f2@xD - 2 I E^(I T0 w2) A2 w2 m@xD f2@xD + 2 I E^(-I T0 w2) w2 Ȧ2 m@xD f2@xD +
  2 E^(I T0 H-W+w1L) F A1 f1^2@xD + 2 E^(I T0 HW+w1L) F A1 f1^2@xD + E^(3 I T0 w1) a ∫_0^1 f1¢@xD^2 â x A1^3 f1^2@xD +
  2 E^(I T0 H2 w1+w2L) a ∫_0^1 f1¢@xD f2¢@xD â x A1^2 A2 f1^2@xD + E^(I T0 Hw1+2 w2L) a ∫_0^1 f2¢@xD^2 â x A1 A2^2 f1^2@xD +
  2 E^(I T0 H-W-w1L) F Ȧ1 f1^2@xD + 2 E^(I T0 HW-w1L) F Ȧ1 f1^2@xD + 3 E^(I T0 w1) a ∫_0^1 f1¢@xD^2 â x A1^2 Ȧ1 f1^2@xD +
  4 E^(I T0 w2) a ∫_0^1 f1¢@xD f2¢@xD â x A1 A2 Ȧ1 f1^2@xD + E^(I T0 H-w1+2 w2L) a ∫_0^1 f2¢@xD^2 â x A2^2 Ȧ1 f1^2@xD +
  3 E^(-I T0 w1) a ∫_0^1 f1¢@xD^2 â x A1 Ȧ1^2 f1^2@xD + 2 E^(I T0 H-2 w1+w2L) a ∫_0^1 f1¢@xD f2¢@xD â x A2 Ȧ1^2 f1^2@xD +
  E^(-3 I T0 w1) a ∫_0^1 f1¢@xD^2 â x Ȧ1^3 f1^2@xD + 2 E^(I T0 H2 w1-w2L) a ∫_0^1 f1¢@xD f2¢@xD â x A1^2 Ȧ2 f1^2@xD +
  2 E^(I T0 w1) a ∫_0^1 f2¢@xD^2 â x A1 A2 Ȧ2 f1^2@xD + 4 E^(-I T0 w2) a ∫_0^1 f1¢@xD f2¢@xD â x A1 Ȧ1 Ȧ2 f1^2@xD +
  2 E^(-I T0 w1) a ∫_0^1 f2¢@xD^2 â x A2 Ȧ1 Ȧ2 f1^2@xD + 2 E^(I T0 H-2 w1-w2L) a ∫_0^1 f1¢@xD f2¢@xD â x Ȧ1^2 Ȧ2 f1^2@xD +
  E^(I T0 Hw1-2 w2L) a ∫_0^1 f2¢@xD^2 â x A1 Ȧ2^2 f1^2@xD + E^(I T0 H-w1-2 w2L) a ∫_0^1 f2¢@xD^2 â x Ȧ1 Ȧ2^2 f1^2@xD +
  2 E^(I T0 H-W+w2L) F A2 f2^2@xD + 2 E^(I T0 HW+w2L) F A2 f2^2@xD + E^(I T0 H2 w1+w2L) a ∫_0^1 f1¢@xD^2 â x A1^2 A2 f2^2@xD +
  2 E^(I T0 Hw1+2 w2L) a ∫_0^1 f1¢@xD f2¢@xD â x A1 A2^2 f2^2@xD + E^(3 I T0 w2) a ∫_0^1 f2¢@xD^2 â x A2^3 f2^2@xD +
  2 E^(I T0 w2) a ∫_0^1 f1¢@xD^2 â x A1 A2 Ȧ1 f2^2@xD + 2 E^(I T0 H-w1+2 w2L) a ∫_0^1 f1¢@xD f2¢@xD â x A2^2 Ȧ1 f2^2@xD +
  E^(I T0 H-2 w1+w2L) a ∫_0^1 f1¢@xD^2 â x A2 Ȧ1^2 f2^2@xD + 2 E^(I T0 H-W-w2L) F Ȧ2 f2^2@xD + 2 E^(I T0 HW-w2L) F Ȧ2 f2^2@xD +
  E^(I T0 H2 w1-w2L) a ∫_0^1 f1¢@xD^2 â x A1^2 Ȧ2 f2^2@xD + 4 E^(I T0 w1) a ∫_0^1 f1¢@xD f2¢@xD â x A1 A2 Ȧ2 f2^2@xD +
  3 E^(I T0 w2) a ∫_0^1 f2¢@xD^2 â x A2^2 Ȧ2 f2^2@xD + 2 E^(-I T0 w2) a ∫_0^1 f1¢@xD^2 â x A1 Ȧ1 Ȧ2 f2^2@xD +
  4 E^(-I T0 w1) a ∫_0^1 f1¢@xD f2¢@xD â x A2 Ȧ1 Ȧ2 f2^2@xD + E^(I T0 H-2 w1-w2L) a ∫_0^1 f1¢@xD^2 â x Ȧ1^2 Ȧ2 f2^2@xD +
  2 E^(I T0 Hw1-2 w2L) a ∫_0^1 f1¢@xD f2¢@xD â x A1 Ȧ2^2 f2^2@xD + 3 E^(-I T0 w2) a ∫_0^1 f2¢@xD^2 â x A2 Ȧ2^2 f2^2@xD +
  2 E^(I T0 H-w1-2 w2L) a ∫_0^1 f1¢@xD f2¢@xD â x Ȧ1 Ȧ2^2 f2^2@xD + E^(-3 I T0 w2) a ∫_0^1 f2¢@xD^2 â x Ȧ2^3 f2^2@xD
```

It follows from **eqEps[[2]]** that the first-order boundary conditions are

```
order1BC = eqEps[[2]] // Rest
```

$$\{w_1[0,T_0,T_1]==0,\ w_1^{(2,0,0)}[0,T_0,T_1]==0,\ w_1[1,T_0,T_1]==0,\ w_1^{(1,0,0)}[1,T_0,T_1]==0\}$$

■ Principal Parametric Resonance of the First Mode

In this case, $\Omega \approx 2\omega_1$. To describe the nearness of the internal and principal parametric resonances, we introduce the two detuning parameters σ_1 and σ_2 defined by

```
ResonanceConds = {ω2 == 3 ω1 + ε σ1, Ω == 2 ω1 + ε σ2};
```

and define the following rules:

```
OmgRule = Solve[ResonanceConds, Drop[omgList, {#}]~Join~{Ω}][[1]] & /@ {1, 2}
```

$$\left\{\{\omega_2 \to \epsilon\sigma_1 + 3\omega_1,\ \Omega \to \epsilon\sigma_2 + 2\omega_1\},\ \left\{\omega_1 \to \tfrac{1}{3}(-\epsilon\sigma_1 + \omega_2),\ \Omega \to \tfrac{1}{3}(-2\epsilon\sigma_1 + 3\epsilon\sigma_2 + 2\omega_2)\right\}\right\}$$

```
expRule[i_] := Exp[arg_] :> Exp[Expand[arg /. OmgRule[[i]]] /. ε T0 -> T1]
```

We substitute **ResonanceConds** into the right-hand side of **order1Eq** and obtain the source of secular terms as

```
ST = Table[Coefficient[order1Eq[[2]] /. expRule[i], Exp[I ωi T0]], {i, 2}];
ST /. displayRule
```

$$\begin{aligned}\Big\{&-2 i (D_1A_1)\omega_1 f_1[x] - 2 i A_1\omega_1 \mu[x] f_1[x] + 2 e^{i T_1\sigma_2} F \bar A_1 f_1''[x] + \\ &3\alpha\left(\int_0^1 f_1'[x]^2\,dx\right) A_1^2 \bar A_1 f_1''[x] + 2 e^{i T_1\sigma_1}\alpha\left(\int_0^1 f_1'[x] f_2'[x]\,dx\right) A_2 \bar A_1^2 f_1''[x] + \\ &2\alpha\left(\int_0^1 f_2'[x]^2\,dx\right) A_1 A_2 \bar A_2 f_1''[x] + 2 e^{i T_1\sigma_1 - i T_1\sigma_2} F A_2 f_2''[x] + \\ &e^{i T_1\sigma_1}\alpha\left(\int_0^1 f_1'[x]^2\,dx\right) A_2 \bar A_1^2 f_2''[x] + 4\alpha\left(\int_0^1 f_1'[x] f_2'[x]\,dx\right) A_1 A_2 \bar A_2 f_2''[x],\\ &-2 i (D_1A_2)\omega_2 f_2[x] - 2 i A_2\omega_2 \mu[x] f_2[x] + 2 e^{-i T_1\sigma_1 + i T_1\sigma_2} F A_1 f_1''[x] + \\ &e^{-i T_1\sigma_1}\alpha\left(\int_0^1 f_1'[x]^2\,dx\right) A_1^3 f_1''[x] + 4\alpha\left(\int_0^1 f_1'[x] f_2'[x]\,dx\right) A_1 A_2 \bar A_1 f_1''[x] + \\ &2\alpha\left(\int_0^1 f_1'[x]^2\,dx\right) A_1 A_2 \bar A_1 f_2''[x] + 3\alpha\left(\int_0^1 f_2'[x]^2\,dx\right) A_2^2 \bar A_2 f_2''[x]\Big\}\end{aligned}$$

Because the homogeneous part of **order1Eq** and **order1BC** has a nontrivial solution, the corresponding nonhomogeneous problem has a solution only if solvability conditions are satisfied. It follows from Section 7.1.1 that the homogeneous problem is self-adjoint and that the solvability conditions demand that **ST** be orthogonal to $f_1[x]$ and $f_2[x]$, respectively. Imposing these conditions, we have

```
SCond =
  Table[int[φj[x] ST[[j]]], {x, 0, 1}] /. intRule1 //. intRule2 /. int -> Integrate, {j, 2}] ==
    0 // Thread;
SCond /. displayRule
```

$$\Big\{2\,e^{i T_1 \sigma_1 - i T_1 \sigma_2} f \Big(\int_0^1 \phi_1[x]\,\phi_2''[x]\,dx\Big) A_2 - 2 i \Big(\int_0^1 \phi_1[x]^2\,dx\Big) (D_1 A_1)\,\omega_1 - 2 i \Big(\int_0^1 \mu[x]\,\phi_1[x]^2\,dx\Big) A_1\,\omega_1 + 2\,e^{i T_1 \sigma_2} f \Big(\int_0^1 \phi_1[x]\,\phi_1''[x]\,dx\Big) \bar{A}_1 + 3\alpha \Big(\int_0^1 \phi_1'[x]^2\,dx\Big)\Big(\int_0^1 \phi_1[x]\,\phi_1''[x]\,dx\Big) A_1^2 \bar{A}_1 + 2\,e^{i T_1 \sigma_1} \alpha \Big(\int_0^1 \phi_1'[x]\,\phi_2'[x]\,dx\Big)\Big(\int_0^1 \phi_1[x]\,\phi_1''[x]\,dx\Big) A_2 \bar{A}_1^2 + e^{i T_1 \sigma_1} \alpha \Big(\int_0^1 \phi_1'[x]^2\,dx\Big)\Big(\int_0^1 \phi_1[x]\,\phi_2''[x]\,dx\Big) A_2 \bar{A}_1^2 + 2\alpha \Big(\int_0^1 \phi_2'[x]^2\,dx\Big)\Big(\int_0^1 \phi_1[x]\,\phi_1''[x]\,dx\Big) A_1 A_2 \bar{A}_2 + 4\alpha \Big(\int_0^1 \phi_1'[x]\,\phi_2'[x]\,dx\Big)\Big(\int_0^1 \phi_1[x]\,\phi_2''[x]\,dx\Big) A_1 A_2 \bar{A}_2 == 0,$$

$$2\,e^{-i T_1 \sigma_1 + i T_1 \sigma_2} f \Big(\int_0^1 \phi_2[x]\,\phi_1''[x]\,dx\Big) A_1 + e^{-i T_1 \sigma_1} \alpha \Big(\int_0^1 \phi_1'[x]^2\,dx\Big)\Big(\int_0^1 \phi_2[x]\,\phi_1''[x]\,dx\Big) A_1^3 - 2 i \Big(\int_0^1 \phi_2[x]^2\,dx\Big) (D_1 A_2)\,\omega_2 - 2 i \Big(\int_0^1 \mu[x]\,\phi_2[x]^2\,dx\Big) A_2\,\omega_2 + 4\alpha \Big(\int_0^1 \phi_1'[x]\,\phi_2'[x]\,dx\Big)\Big(\int_0^1 \phi_2[x]\,\phi_1''[x]\,dx\Big) A_1 A_2 \bar{A}_1 + 2\alpha \Big(\int_0^1 \phi_1'[x]^2\,dx\Big)\Big(\int_0^1 \phi_2[x]\,\phi_2''[x]\,dx\Big) A_1 A_2 \bar{A}_1 + 3\alpha \Big(\int_0^1 \phi_2'[x]^2\,dx\Big)\Big(\int_0^1 \phi_2[x]\,\phi_2''[x]\,dx\Big) A_2^2 \bar{A}_2 == 0\Big\}$$

To simplify the notation in the solvability conditions, we use the orthonormality of the mode shapes and define the following parameters:

```
notationRule = {∫_0^1 φi_[x]^2 dx -> 1, ∫_0^1 μ[x] φi_[x]^2 dx -> μi, f ∫_0^1 φi_[x] φj_''[x] dx -> -ωi σi,j};
```

To identify the forms of the nonlinear terms in the solvability conditions, we first identify all of the possible forms of the nonhomogeneous terms in the first-order problem. To this end, we let

```
basicTerms = Table[{Ai[T1] E^(I ωi T0), Āi[T1] E^(-I ωi T0)}, {i, 2}] // Flatten
```

$$\{e^{i T_0 \omega_1} A_1[T_1],\ e^{-i T_0 \omega_1} \bar{A}_1[T_1],\ e^{i T_0 \omega_2} A_2[T_1],\ e^{-i T_0 \omega_2} \bar{A}_2[T_1]\}$$

Then, all of the possible forms of the nonlinear terms in the first-order problem are given by

```
cubicTerms = Nest[Outer[Times, basicTerms, #] &, basicTerms, 2] // Flatten // Union;
cubicTerms /. displayRule
```

$$\{e^{3 i T_0 \omega_1} A_1^3,\ e^{i T_0 (2\omega_1+\omega_2)} A_1^2 A_2,\ e^{i T_0 (\omega_1+2\omega_2)} A_1 A_2^2,\ e^{3 i T_0 \omega_2} A_2^3,\ e^{i T_0 \omega_1} A_1^2 \bar{A}_1,$$
$$e^{i T_0 \omega_2} A_1 A_2 \bar{A}_1,\ e^{i T_0 (-\omega_1+2\omega_2)} A_2^2 \bar{A}_1,\ e^{-i T_0 \omega_1} A_1 \bar{A}_1^2,\ e^{i T_0 (-2\omega_1+\omega_2)} A_2 \bar{A}_1^2,\ e^{-3 i T_0 \omega_1} \bar{A}_1^3,$$
$$e^{i T_0 (2\omega_1-\omega_2)} A_1^2 \bar{A}_2,\ e^{i T_0 \omega_1} A_1 A_2 \bar{A}_2,\ e^{i T_0 \omega_2} A_2^2 \bar{A}_2,\ e^{-i T_0 \omega_2} A_1 \bar{A}_1 \bar{A}_2,\ e^{-i T_0 \omega_1} A_2 \bar{A}_1 \bar{A}_2,$$
$$e^{i T_0 (-2\omega_1-\omega_2)} \bar{A}_1^2 \bar{A}_2,\ e^{i T_0 (\omega_1-2\omega_2)} A_1 \bar{A}_2^2,\ e^{-i T_0 \omega_2} A_2 \bar{A}_2^2,\ e^{i T_0 (-\omega_1-2\omega_2)} \bar{A}_1 \bar{A}_2^2,\ e^{-3 i T_0 \omega_2} \bar{A}_2^3\}$$

Out of these terms, only the terms that may lead to secular terms appear in the solvability conditions, which can be identified according to

```
secularTerms =
 I E^(-I w# T0) cubicTerms •. expRule@#D •. Exp@_ T0 + _.D -> 0 •• Union •• RestM & •ž 81, 2<
```

: : A1@T1D^2 Ȧ1@T1D, E^(I T1 s1) A2@T1D Ȧ1@T1D^2, A1@T1D A2@T1D Ȧ2@T1D>,
 8E^(-I T1 s1) A1@T1D^3, A1@T1D A2@T1D Ȧ1@T1D, A2@T1D^2 Ȧ2@T1D<>

Next, we define the following parameters:

```
symbolList = 8-8 w1 8g11, d1, g12<, -8 w2 8d2, g21, g22<<
```

88-8 g11 w1, -8 d1 w1, -8 g12 w1<, 8-8 d2 w2, -8 g21 w2, -8 g22 w2<<

and express **SCond** in a more concise form as

```
eqMod = ExpandA- 1/w# HHSCond@@#, 1DD •. notationRule •. Thread@secularTerms@@#DD -> 0DL +
        symbolList@@#DD.secularTerms@@#DDLE == 0 & •ž 81, 2<
```

:2 I m1 A1@T1D + 2 E^(I T1 s1 - I T1 s2) f s1,2 A2@T1D + 2 E^(I T1 s2) f s1,1 Ȧ1@T1D + 8 g11 A1@T1D^2 Ȧ1@T1D +
 8 E^(I T1 s1) d1 A2@T1D Ȧ1@T1D^2 + 8 g12 A1@T1D A2@T1D Ȧ2@T1D + 2 I A1¢@T1D == 0,
 2 E^(-I T1 s1 + I T1 s2) f s2,1 A1@T1D + 8 E^(-I T1 s1) d2 A1@T1D^3 + 2 I m2 A2@T1D +
 8 g21 A1@T1D A2@T1D Ȧ1@T1D + 8 g22 A2@T1D^2 Ȧ2@T1D + 2 I A2¢@T1D == 0>

where

```
-symbolList@@1DD -> H-Coefficient@SCond@@1, 1DD, #D & •ž secularTerms@@1DDL •• Thread ••
 TableForm
```

8 g11 w1 ® -3 a IÙ_0^1 f1¢@xD^2 â xM Ù_0^1 f1@xD f1²@xD â x
8 d1 w1 ® -2 a IÙ_0^1 f1¢@xD f2¢@xD â xM Ù_0^1 f1@xD f1²@xD â x - a IÙ_0^1 f1¢@xD^2 â xM Ù_0^1 f1@xD f2²@xD â x
8 g12 w1 ® -2 a IÙ_0^1 f2¢@xD^2 â xM Ù_0^1 f1@xD f1²@xD â x - 4 a IÙ_0^1 f1¢@xD f2¢@xD â xM Ù_0^1 f1@xD f2²@xD â x

```
-symbolList@@2DD -> H-Coefficient@SCond@@2, 1DD, #D & •ž secularTerms@@2DDL •• Thread ••
 TableForm
```

8 d2 w2 ® -a IÙ_0^1 f1¢@xD^2 â xM Ù_0^1 f2@xD f1²@xD â x
8 g21 w2 ® -4 a IÙ_0^1 f1¢@xD f2¢@xD â xM Ù_0^1 f2@xD f1²@xD â x - 2 a IÙ_0^1 f1¢@xD^2 â xM Ù_0^1 f2@xD f2²@xD â x
8 g22 w2 ® -3 a IÙ_0^1 f2¢@xD^2 â xM Ù_0^1 f2@xD f2²@xD â x

Modulation Equations in Polar Form

The complex-valued solvability conditions can be expressed in real-valued form by introducing the polar transformation

```
ruleA = 9Ai_ -> j 1/2 ai@#D Exp@I qi@#DD &y, Ȧi_ -> j 1/2 ai@#D Exp@-I qi@#DD &y=;
```

into **eqMod** and obtain

```
expr1 = Expand@eqMod@@#, 1DD Exp@-I q#@T1DD •. ruleAD & •ž 81, 2<
```

```
8I m1 a1@T1D + E^(I T1 s2-2 I q1@T1D) f s1,1 a1@T1D + g11 a1@T1D^3 + E^(I T1 s1-I T1 s2-I q1@T1D+I q2@T1D) f s1,2 a2@T1D +
  E^(I T1 s1-3 I q1@T1D+I q2@T1D) d1 a1@T1D^2 a2@T1D + g12 a1@T1D a2@T1D^2 + I a1¢@T1D - a1@T1D q1¢@T1D,
 E^(-I T1 s1+I T1 s2+I q1@T1D-I q2@T1D) f s2,1 a1@T1D + E^(-I T1 s1+3 I q1@T1D-I q2@T1D) d2 a1@T1D^3 +
  I m2 a2@T1D + g21 a1@T1D^2 a2@T1D + g22 a2@T1D^3 + I a2¢@T1D - a2@T1D q2¢@T1D<
```

Next, we separate the real and imaginary parts of **expr1** to obtain the equations governing the modulation of the amplitudes a_i and phases q_i. To accomplish this, we define the rule

```
realRule = 8Re@s_D -> s, Im@s_D -> 0<;
```

Then, the equations governing the amplitudes of motion correspond to the imaginary parts of **expr1**; that is,

```
ampEq = Solve@ComplexExpand@Im@#DD == 0 •. realRule & •ž expr1, 8a1¢@T1D, a2¢@T1D<D@@1DD •.
   Rule -> Equal •• ExpandAll
```

```
8a1¢@T1D == -m1 a1@T1D - f Sin@T1 s2 - 2 q1@T1DD s1,1 a1@T1D -
   f Sin@T1 s1 - T1 s2 - q1@T1D + q2@T1DD s1,2 a2@T1D - Sin@T1 s1 - 3 q1@T1D + q2@T1DD d1 a1@T1D^2 a2@T1D,
 a2¢@T1D == f Sin@T1 s1 - T1 s2 - q1@T1D + q2@T1DD s2,1 a1@T1D +
   Sin@T1 s1 - 3 q1@T1D + q2@T1DD d2 a1@T1D^3 - m2 a2@T1D<
```

Moreover, the equations governing the phases of motion correspond to the real parts of **expr1**; that is,

```
phaseEq = Solve@ComplexExpand@Re@#DD == 0 •. realRule & •ž expr1, 8q1¢@T1D, q2¢@T1D<D@@1DD •.
   Rule -> Equal •• ExpandAll
```

```
:q1¢@T1D ==
  f Cos@T1 s2 - 2 q1@T1DD s1,1 + g11 a1@T1D^2 + (f Cos@T1 s1 - T1 s2 - q1@T1D + q2@T1DD s1,2 a2@T1D)/(a1@T1D) +
   Cos@T1 s1 - 3 q1@T1D + q2@T1DD d1 a1@T1D a2@T1D + g12 a2@T1D^2,
 q2¢@T1D == g21 a1@T1D^2 + (f Cos@T1 s1 - T1 s2 - q1@T1D + q2@T1DD s2,1 a1@T1D)/(a2@T1D) +
   (Cos@T1 s1 - 3 q1@T1D + q2@T1DD d2 a1@T1D^3)/(a2@T1D) + g22 a2@T1D^2>
```

These modulation equations are nonautonomous because they depend explicitly on T_1.

To determine an autonomous set of modulation equations, we start with **expr1** and identify the independent phase arguments in it. To accomplish this, we first identify all possible phase arguments; that is,

```
expTerms = I Cases@expr1, Exp@arg_D -> arg, InfinityD •• Expand
```

```
8-T1 s2 + 2 q1@T1D, -T1 s1 + T1 s2 + q1@T1D - q2@T1D,
 -T1 s1 + 3 q1@T1D - q2@T1D, T1 s1 - T1 s2 - q1@T1D + q2@T1D, T1 s1 - 3 q1@T1D + q2@T1D<
```

Out of these arguments, only two are independent because

```
Outer@D, expTerms, 8q1@T1D, q2@T1D<D •• RowReduce
```

```
881, 0<, 80, 1<, 80, 0<, 80, 0<, 80, 0<<
```

Denoting these two independent arguments by g_1 and g_2, we have

```
gammaList = 82 g1@T1D, g2@T1D< == expTerms@@81, 2<DD •• Thread
```

```
82 g1@T1D == -T1 s2 + 2 q1@T1D, g2@T1D == -T1 s1 + T1 s2 + q1@T1D - q2@T1D<
```

which can be solved for q_1 and q_2 to obtain

```
thetaRule = Solve@gammaList, 8q1@T1D, q2@T1D<D@@1DD •• ExpandAll
```

```
          T1 s2                                  3 T1 s2
9q1@T1D ® ¢¢¢¢¢¢ + g1@T1D, q2@T1D ® -T1 s1 + ¢¢¢¢¢¢¢¢ + g1@T1D - g2@T1D=
            2                                      2
```

Substituting for the q_i in **expr1** yields

```
expr2 = expr1 •. Table@qi -> HEvaluate@thetaRule@@i, 2DD •. T1 -> #D &L, 8i, 2<D •.
   Exp@arg_D :> Exp@Expand@argDD •• Expand
```

```
             1
9I m1 a1@T1D - ¢¢ s2 a1@T1D + E^(-2 I g1@T1D) f s1,1 a1@T1D + g11 a1@T1D^3 + E^(-I g2@T1D) f s1,2 a2@T1D +
             2
  E^(-2 I g1@T1D-I g2@T1D) d1 a1@T1D^2 a2@T1D + g12 a1@T1D a2@T1D^2 + I a1¢@T1D - a1@T1D g1¢@T1D,
                                                                                       3
 E^(I g2@T1D) f s2,1 a1@T1D + E^(2 I g1@T1D+I g2@T1D) d2 a1@T1D^3 + I m2 a2@T1D + s1 a2@T1D - ¢¢ s2 a2@T1D +
                                                                                       2
  g21 a1@T1D^2 a2@T1D + g22 a2@T1D^3 + I a2¢@T1D - a2@T1D g1¢@T1D + a2@T1D g2¢@T1D=
```

Then, the equations governing the amplitudes of motion correspond to the imaginary parts of **expr2**; that is,

```
HampEq = Solve@ComplexExpand@Im@#DD == 0 •. realRule & •ž expr2, 8a1¢@T1D, a2¢@T1D<D@@1DD •.
     Rule -> Equal •• ExpandAllL •. f_@T1D -> f
```

```
8a1¢ == Sin@2 g1 + g2D a1^2 a2 d1 - a1 m1 + f Sin@2 g1D a1 s1,1 + f Sin@g2D a2 s1,2,
 a2¢ == -Sin@2 g1 + g2D a1^3 d2 - a2 m2 - f Sin@g2D a1 s2,1<
```

Moreover, the equations governing the phases of motion correspond to the real parts of **expr2**; that is,

```
HphaseEq = Solve@ComplexExpand@Re@#DD == 0 •. realRule & •ž expr2, 8g1¢@T1D, g2¢@T1D<D@@1DD •.
    Rule -> Equal •• ExpandAllL •. f_@T1D -> f
```

```
:g2¢ == a1^2 g11 + a2^2 g12 - a1^2 g21 - a2^2 g22 + Cos@2 g1 + g2D a1 a2 d1 -

   Cos@2 g1 + g2D a1^3 d2                                    f Cos@g2D a2 s1,2     f Cos@g2D a1 s2,1
   ----------------------- - s1 + s2 + f Cos@2 g1D s1,1 + ------------------ - ------------------ ,
             a2                                                   a1                    a2

                                                                  s2                        f Cos@g2D a2 s1,2
 g1¢ == a1^2 g11 + a2^2 g12 + Cos@2 g1 + g2D a1 a2 d1 - ---- + f Cos@2 g1D s1,1 + ------------------ >
                                                                  2                                a1
```

The modulation equations are autonomous because they are independent of the independent variable T_1.

Modulation Equations in Cartesian Form and Symmetry Property

In performing bifurcation analyses of the dynamics of the system, one might find it more convenient, in some cases, to represent the modulation equations in Cartesian rather than polar form. As a byproduct, one can also obtain the symmetries of the system.To this end, we introduce the Cartesian transformation

```
cartRule =
  9Ak_ -> j 1/2 Hpk@#D - I qk@#DL Exp@I lk@#DD &z, Ȧk_ -> j 1/2 Hpk@#D + I qk@#DL Exp@-I lk@#DD &z=;
realRule = 8Im@x_D -> 0, Re@x_D -> x, lk_¢@T1D -> nk<;
```

where the l_i are chosen to render the resulting modulation equations autonomous. Substituting the **cartRule** into **eqMod** yields

```
eqModCart@k_D := Exp@-I lk@T1DD Subtract žž eqMod@@kDD •. cartRule •• Expand;
```

Next, we choose the l_i to render **eqModCart** autonomous. To accomplish this, we identify its independent arguments by identifying first all of its arguments. They are given by

```
list1 =
  -I HCases@#, Exp@x_D -> x, InfinityD & •ž Array@eqModCart, 2D •• Flatten •• UnionL •• Expand
```

```
8T1 s2 - 2 l1@T1D, -T1 s1 + T1 s2 + l1@T1D - l2@T1D,
 -T1 s1 + 3 l1@T1D - l2@T1D, T1 s1 - T1 s2 - l1@T1D + l2@T1D, T1 s1 - 3 l1@T1D + l2@T1D<
```

To determine the number of independent relations in **list1**, we use the function **RowReduce** and obtain

```
Outer@D, list1, 8 l1@T1D, l2@T1D<D •• RowReduce
```

```
881, 0<, 80, 1<, 80, 0<, 80, 0<, 80, 0<<
```

Hence, there are only two linearly independent arguments in **list1**. We choose the first two to determine the l_i. For an invariant set of modulation equations, we set each of these relations to be an even multiple of p. Solving the resulting relations for the l_i, we obtain

```
list1 = list1@@81, 2<DD;
```

```
lambdaRule = Solve@list1 + 8m1, m2< 2 p == 0 •• Thread, Table@ li@T1D, 8i, 2<DD@@1DD
```

```
9l1@T1D ® 1/2 H2 p m1 + T1 s2L, l2@T1D ® 1/2 H2 p m1 + 4 p m2 - 2 T1 s1 + 3 T1 s2L=
```

where the m_i are positive or negative integers. Letting $n_i = l_i'$ yields

```
D@lambdaRule, T1D •. realRule
```

```
9n1 ® s2/2, n2 ® 1/2 H-2 s1 + 3 s2L=
```

Then, separating the real and imaginary parts of **eqModCart**, we obtain the following Cartesian form of the modulation equations:

```
Heqs = Solve@
    Flatten@Table@ComplexExpand@8Im@#D, Re@#D<D & •ž HeqModCart@kD •. Exp@_D -> 1L ••.
        realRule, 8k, 2<DD == 0 •• Thread, Table@8pk'@T1D, qk'@T1D<, 8k, 2<D ••
     FlattenD@@1DD •. Rule -> EqualL •. f_@T1D -> f •• TableForm
```

```
p1' == p12 q1 g11 + q13 g11 + p22 q1 g12 + q1 q22 g12 - 2 p1 p2 q1 d1 + p12 q2 d1 - q12 q2 d1 - p1 m1 - q1 n1 - f q1 s1,1 + f q2
q1' == -p13 g11 - p1 q12 g11 - p1 p22 g12 - p1 q22 g12 - p12 p2 d1 + p2 q12 d1 - 2 p1 q1 q2 d1 - q1 m1 + p1 n1 - f p1 s1,1 - f p
p2' == p12 q2 g21 + q12 q2 g21 + p22 q2 g22 + q23 g22 + 3 p12 q1 d2 - q13 d2 - p2 m2 - q2 n2 + f q1 s2,1
q2' == -p12 p2 g21 - p2 q12 g21 - p23 g22 - p2 q22 g22 - p13 d2 + 3 p1 q12 d2 - q2 m2 + p2 n2 - f p1 s2,1
```

The symmetry property of the system can also be obtained as follows:

```
phase = Flatten@
   Table@Mod@#@@2DD & •ž lambdaRule •. T1 -> 0, 2 pD, 8m1, 0, 10<, 8m2, 0, 10<D, 1D •• Union
```

```
880, 0<, 8p, p<<
```

```
rotMat@a_D = 88Cos@aD, Sin@aD<, 8-Sin@aD, Cos@aD<<;
```

```
symmetryList = Join žž Table@rotMat@#@@iDDD.8pi, qi<, 8i, 2<D & •ž phase
```

```
88p1, q1, p2, q2<, 8-p1, -q1, -p2, -q2<<
```

Hence, if 8p_1, q_1, p_2, q_2< is a solution of the modulation equations, then 8- p_1, - q_1, - p_2, -q_2< is also another solution of these equations. If these two sets of solutions are the same, then the system response is symmetric; otherwise, it will be asymmetric.

à The Functions **PolarForm** and **CartesianForm**

Following the procedures described in the preceding section, we build two functions, **PolarForm** and **CartesianForm**, to automate the process provided that the complex dependent variables in the modulation equations are written in the form S_i, $i = 1$, ..., neq and the corresponding complex conjugates are $\ddot{S}_i$. The functions can be slightly modified to allow a more general form for the modulation equations. To obtain the polar form or Cartesian form of the complex modulation equations, we simply provide the list of modulation equations and S as two required inputs in **PolarForm** or **CartesianForm**, respectively. If $S = A$, the list of modulation equations is the only required input.

```
PolarForm@eqs_List, S_Symbol: AD :=
  ModuleA8neq = Length@eqsD, t<,
    t = Cases@eqs, Si_@a_D -> a, ¥D@@1DD;
    polarRule = 9Sk_ -> 1/2 ak@#D Exp@I bk@#DD &, Ṡk_ -> 1/2 ak@#D Exp@-I bk@#DD &=;
    realRule = 8Im@x_D -> 0, Re@x_D -> x<;
    eqModPolar@k_D := Exp@-I bk@tDD Subtract žž eqs@@kDD •. polarRule •• Expand;
    list1 = I Cases@Array@eqModPolar, neqD, Exp@a_D -> a, ¥D •• Expand •• Union;
    newLists@list_, n_D :=
      With@8v = Table@Unique@D, 8n<D<,
        ReplaceList@list,
  Append@Flatten@8___, Pattern@#, _D< & •ž vD, ___D -> vDD;
    betaList = Table@bi@tD, 8i, neq<D;
    list2 = 8<;
    Scan@
 If@NullSpace@Outer@Coefficient, #, betaListDD === 8<, list2 = -#; Return@DD &,
 newLists@list1, neqDD;
    If@list2 === 8<, Print@
  "Autonomous system is not possible!!!\nThe phase list is:"D; Return@list1D
    D;
 betaRule1 = Solve@list3 = Table@gi@tD, 8i, neq<D == list2 •• Thread, betaListD@@1DD;
    betaRule2 =
 Table@bi -> Function@t •• Evaluate, bi@tD •. betaRule1 •• EvaluateD, 8i, neq<D;
    Do@eq0@iD = eqModPolar@iD •. betaRule2 •. Exp@a_D :> Exp@Expand@aDD;
  eq@iD =
  Solve@ComplexExpand@Im@eq0@iDDD == 0 •. realRule, ai¢@tDD@@1, 1DD •. Rule -> Equal;
  eqRe@iD = ComplexExpand@Re@eq0@iDDD == 0 •. realRule, 8i, neq<
    D;
 gammapList = Table@gi¢@tD, 8i, neq<D;
    eqList = Solve@Array@eqRe, neqD, gammapListD@@1DD •. Rule -> Equal •• ExpandAll;
 Do@eq@i + neqD = eqList@@iDD, 8i, neq<D;
 8Array@eq, 2 * neqD, list3<
E
```

```
CartesianForm@eqs_List, S_Symbol: AD :=
    ModuleA8neq = Length@eqsD, t<,
        t = Cases@eqs, Si_@a_D -> a, ¥D@@1DD;
        cartRule = 9Sk_ -> H1 • 2 Hpk@#D - I qk@#DL Exp@I lk@#DD &L,
    Ṡk_ -> H1 • 2 Hpk@#D + I qk@#DL Exp@- I lk@#DD &L=;
        realRule = 8Im@x_D -> 0, Re@x_D -> x, lk_¢@tD -> nk<;
        eqModCart@k_D := Exp@- I lk@tDD Subtract žž eqs@@kDD •. cartRule •• Expand;
        eqs1 = 8ComplexExpand@Im@#DD == 0, ComplexExpand@Re@#DD == 0< & •ž
      HArray@eqModCart, neqD •. E^_ -> 1L ••. realRule •• Flatten;
      eqs2 = Solve@eqs1, Table@8pk¢@tD, qk¢@tD<, 8k, neq<D •• FlattenD@@1DD •.
    Rule -> Equal;
        list1 = I Cases@Array@eqModCart, neqD, Exp@a_D -> a, ¥D •• Expand •• Union;
        newLists@list_, n_D :=
            With@8v = Table@Unique@D, 8n<D<,
                ReplaceList@list,
     Append@Flatten@8___, Pattern@#, _D< & •ž vD, ___D -> vDD;
      lambdaList = Table@li@tD, 8i, neq<D;
      list2 = 8<;
        Scan@
   If@NullSpace@Outer@Coefficient, #, lambdaListDD === 8<, list2 = -#; Return@DD &,
   newLists@list1, neqDD;
        If@list2 === 8<, Print@
    "Autonomous system is not possible!!!\nThe phase list is:"D; Return@list1D
        D;
        lambdaRule = Solve@list2 + 2 Table@mi, 8i, neq<D p == 0 •• Thread, lambdaListD@@1DD;
        iterList = Table@8mi, 0, 10<, 8i, neq<D;
        phase =
   Flatten@Table@Mod@Select@#, ! FreeQ@#, pD &D & •ž HlambdaList •. lambdaRule •• ExpandL,
       2 pD •• Evaluate, Sequence žž iterList •• EvaluateD, neq - 1D •• Union;
        rotMat@q_D := 88Cos@qD, -Sin@qD<, 8Sin@qD, Cos@qD<<;
        symmetryList = Flatten@Table@rotMat@#@@kDDD.8pk, qk<, 8k, 2<DD & •ž phase;
      8eqs2, D@lambdaRule, tD •. realRule, symmetryList<
 E
```

Ÿ Principal Parametric Resonance of the Second Mode

In this case, $\Omega \approx \omega_2$. To describe the nearness of the internal and principal parametric resonances, we introduce the two detuning parameters σ_1 and σ_2 defined by

```
ResonanceConds = 8w2 == 3 w1 + e s1, W == 2 w2 + e s2<;
```

and define the following rules:

```
OmgRule = Solve@ResonanceConds, Drop@omgList, 8#<D ~Join~ 8W<D@@1DD & •ž 81, 2<
```

```
98w2 ® e s1 + 3 w1, W ® 2 e s1 + e s2 + 6 w1<, 9w1 ® 1/3 H-e s1 + w2L, W ® e s2 + 2 w2==
```

```
expRule@i_D := Exp@arg_D :> Exp@Expand@arg •. OmgRule@@iDDD •. e T0 -> T1D
```

We substitute **ResonanceConds** into the right-hand side of **order1Eq** and obtain the source of secular terms as

```
ST = Table@Coefficient@order1Eq@@2DD •. expRule@iD, Exp@I wi T0DD, 8i, 2<D;
ST •. displayRule
```

```
:-2 I HD1A1L w1 f1@xD - 2 I A1 w1 m@xD f1@xD + 3 a ∫0^1 f1¢@xD^2 â x A1^2 Ā1 f1²@xD +
  2 E^(I T1 s1) a ∫0^1 f1¢@xD f2¢@xD â x A2 Ā1^2 f1²@xD + 2 a ∫0^1 f2¢@xD^2 â x A1 A2 Ā2 f1²@xD +
  E^(I T1 s1) a ∫0^1 f1¢@xD^2 â x A2 Ā1^2 f2²@xD + 4 a ∫0^1 f1¢@xD f2¢@xD â x A1 A2 Ā2 f2²@xD,
 -2 I HD1A2L w2 f2@xD - 2 I A2 w2 m@xD f2@xD + E^(-I T1 s1) a ∫0^1 f1¢@xD^2 â x A1^3 f1²@xD +
  4 a ∫0^1 f1¢@xD f2¢@xD â x A1 A2 Ā1 f1²@xD + 2 a ∫0^1 f1¢@xD^2 â x A1 A2 Ā1 f2²@xD +
  2 E^(I T1 s2) F Ā2 f2²@xD + 3 a ∫0^1 f2¢@xD^2 â x A2^2 Ā2 f2²@xD>
```

As discussed in the case of principal parametric resonance of the first mode, the solvability conditions of **order1Eq** and **order1BC** demand that **ST** be orthogonal to $f_1[x]$ and $f_2[x]$, respectively. Imposing these conditions, we have

```
SCond =
  Table@int@fj@xD ST@@jDD, 8x, 0, 1<D •. intRule1 ••. intRule2 •. int -> Integrate, 8j, 2<D ==
    0 •• Thread;
SCond •. displayRule
```

```
:-2 I Ùà 1 f1@xD^2 â x HD1A1L w1 -
   2 I Ùà 1 m@xD f1@xD^2 â x A1 w1 + 3 a Ùà 1 f1¢@xD^2 â x Ùà 1 f1@xD f1²@xD â x A1^2 A1• +
   2 E^(I T1 s1) a Ùà 1 f1¢@xD f2¢@xD â x Ùà 1 f1@xD f1²@xD â x A2 A1•^2 + E^(I T1 s1) a Ùà 1 f1¢@xD^2 â x
   Ùà 1 f1@xD f2²@xD â x A2 A1•^2 + 2 a Ùà 1 f2¢@xD^2 â x Ùà 1 f1@xD f1²@xD â x A1 A2 A2• +
   4 a Ùà 1 f1¢@xD f2¢@xD â x Ùà 1 f1@xD f2²@xD â x A1 A2 A2• == 0,
 E^(-I T1 s1) a Ùà 1 f1¢@xD^2 â x Ùà 1 f2@xD f1²@xD â x A1^3 - 2 I Ùà 1 f2@xD^2 â x HD1A2L w2 -
   2 I Ùà 1 m@xD f2@xD^2 â x A2 w2 + 4 a Ùà 1 f1¢@xD f2¢@xD â x Ùà 1 f2@xD f1²@xD â x A1 A2 A1• +
   2 a Ùà 1 f1¢@xD^2 â x Ùà 1 f2@xD f2²@xD â x A1 A2 A1• + 2 E^(I T1 s2) F Ùà 1 f2@xD f2²@xD â x A2• +
   3 a Ùà 1 f2¢@xD^2 â x Ùà 1 f2@xD f2²@xD â x A2^2 A2• == 0>
```

Using the notations introduced in the preceding section, we can express **SCond** in a more concise form as

```
eqMod = ExpandA- 1/w# HHSCond@@#, 1DD •. notationRule •. Thread@secularTerms@@#DD -> 0DL +
        symbolList@@#DD.secularTerms@@#DDLE == 0 & •ž 81, 2<
```

```
:2 I m1 A1@T1D + 8 g11 A1@T1D^2 A1•@T1D +
   8 E^(I T1 s1) d1 A2@T1D A1•@T1D^2 + 8 g12 A1@T1D A2@T1D A2•@T1D + 2 I A1¢@T1D == 0,
 8 E^(-I T1 s1) d2 A1@T1D^3 + 2 I m2 A2@T1D + 8 g21 A1@T1D A2@T1D A1•@T1D +
   2 E^(I T1 s2) f s2,2 A2•@T1D + 8 g22 A2@T1D^2 A2•@T1D + 2 I A2¢@T1D == 0>
```

where

```
-symbolList@@1DD -> H-Coefficient@SCond@@1, 1DD, #D & •ž secularTerms@@1DDL •• Thread ••
 TableForm
```

```
8 g11 w1 ® -3 a IÙ0^1 f1¢@xD^2 â xM Ù0^1 f1@xD f1²@xD â x
8 d1 w1 ® -2 a IÙ0^1 f1¢@xD f2¢@xD â xM Ù0^1 f1@xD f1²@xD â x - a IÙ0^1 f1¢@xD^2 â xM Ù0^1 f1@xD f2²@xD â x
8 g12 w1 ® -2 a IÙ0^1 f2¢@xD^2 â xM Ù0^1 f1@xD f1²@xD â x - 4 a IÙ0^1 f1¢@xD f2¢@xD â xM Ù0^1 f1@xD f2²@xD â x
```

```
-symbolList@@2DD -> H-Coefficient@SCond@@2, 1DD, #D & •ž secularTerms@@2DDL •• Thread ••
 TableForm
```

```
8 d2 w2 ® - a IÙ0^1 f1¢@xD^2 â xM Ù0^1 f2@xD f1²@xD â x
8 g21 w2 ® - 4 a IÙ0^1 f1¢@xD f2¢@xD â xM Ù0^1 f2@xD f1²@xD â x - 2 a IÙ0^1 f1¢@xD^2 â xM Ù0^1 f2@xD f2²@xD â x
8 g22 w2 ® - 3 a IÙ0^1 f2¢@xD^2 â xM Ù0^1 f2@xD f2²@xD â x
```

Modulation Equations in Polar Form

Using **eqMod** and the function **PolarForm** defined in the preceding section, we obtain the modulation equations in polar form and the definitions for g_i as

```
PolarForm@eqModD
```

```
::a1¢@T1D == - m1 a1@T1D - Sin@g1@T1DD d1 a1@T1D^2 a2@T1D,
  a2¢@T1D == Sin@g1@T1DD d2 a1@T1D^3 - m2 a2@T1D - f Sin@g2@T1DD s2,2 a2@T1D,
  g1¢@T1D == s1 + f Cos@g2@T1DD s2,2 - 3 g11 a1@T1D^2 + g21 a1@T1D^2 +
    Cos@g1@T1DD d2 a1@T1D^3 / a2@T1D - 3 Cos@g1@T1DD d1 a1@T1D a2@T1D - 3 g12 a2@T1D^2 + g22 a2@T1D^2,
  g2¢@T1D == s2 - 2 f Cos@g2@T1DD s2,2 - 2 g21 a1@T1D^2 - 2 Cos@g1@T1DD d2 a1@T1D^3 / a2@T1D - 2 g22 a2@T1D^2>,
 8g1@T1D == T1 s1 - 3 b1@T1D + b2@T1D, g2@T1D == T1 s2 - 2 b2@T1D<>
```

Modulation Equations in Cartesian Form and Symmetry Property

Using **eqMod** and the function **CartesianForm** defined in the preceding section, we obtain the modulation equations in Cartesian form, the definitions for n_i, and the symmetry property as

```
CartesianForm[eqModD]
```

$$\Big\{\Big\{p_1'[T_1] == -\mu_1 p_1[T_1] - \nu_1 q_1[T_1] + \gamma_{11} p_1[T_1]^2 q_1[T_1] - 2\,\delta_1 p_1[T_1] p_2[T_1] q_1[T_1] + \gamma_{12} p_2[T_1]^2 q_1[T_1] + \gamma_{11} q_1[T_1]^3 + \delta_1 p_1[T_1]^2 q_2[T_1] - \delta_1 q_1[T_1]^2 q_2[T_1] + \gamma_{12} q_1[T_1] q_2[T_1]^2,$$

$$q_1'[T_1] == \nu_1 p_1[T_1] - \gamma_{11} p_1[T_1]^3 - \delta_1 p_1[T_1]^2 p_2[T_1] - \gamma_{12} p_1[T_1] p_2[T_1]^2 - \mu_1 q_1[T_1] - \gamma_{11} p_1[T_1] q_1[T_1]^2 + \delta_1 p_2[T_1] q_1[T_1]^2 - 2\,\delta_1 p_1[T_1] q_1[T_1] q_2[T_1] - \gamma_{12} p_1[T_1] q_2[T_1]^2,$$

$$p_2'[T_1] == -\mu_2 p_2[T_1] + 3\,\delta_2 p_1[T_1]^2 q_1[T_1] - \delta_2 q_1[T_1]^3 - \nu_2 q_2[T_1] - f\, s_{2,2}\, q_2[T_1] + \gamma_{21} p_1[T_1]^2 q_2[T_1] + \gamma_{22} p_2[T_1]^2 q_2[T_1] + \gamma_{21} q_1[T_1]^2 q_2[T_1] + \gamma_{22} q_2[T_1]^3,$$

$$q_2'[T_1] == -\delta_2 p_1[T_1]^3 + \nu_2 p_2[T_1] - f\, s_{2,2}\, p_2[T_1] - \gamma_{21} p_1[T_1]^2 p_2[T_1] - \gamma_{22} p_2[T_1]^3 + 3\,\delta_2 p_1[T_1] q_1[T_1]^2 - \gamma_{21} p_2[T_1] q_1[T_1]^2 - \mu_2 q_2[T_1] - \gamma_{22} p_2[T_1] q_2[T_1]^2\Big\},$$

$$\Big\{\nu_1 \to \frac{1}{6}(2\,\sigma_1 + \sigma_2),\ \nu_2 \to \frac{\sigma_2}{2}\Big\},\ \Big\{\{p_1, q_1, p_2, q_2\},\ \Big\{\frac{p_1}{2} - \frac{\sqrt{3}\, q_1}{2},\ \frac{\sqrt{3}\, p_1}{2} + \frac{q_1}{2},\ -p_2,\ -q_2\Big\},$$

$$\Big\{-\frac{p_1}{2} - \frac{\sqrt{3}\, q_1}{2},\ \frac{\sqrt{3}\, p_1}{2} - \frac{q_1}{2},\ p_2,\ q_2\Big\},\ \{-p_1, -q_1, -p_2, -q_2\},$$

$$\Big\{-\frac{p_1}{2} + \frac{\sqrt{3}\, q_1}{2},\ -\frac{1}{2}\sqrt{3}\, p_1 - \frac{q_1}{2},\ p_2,\ q_2\Big\},\ \Big\{\frac{p_1}{2} + \frac{\sqrt{3}\, q_1}{2},\ -\frac{1}{2}\sqrt{3}\, p_1 + \frac{q_1}{2},\ -p_2,\ -q_2\Big\}\Big\}\Big\}$$

Combination Parametric Resonance of the Two Modes

In this case, $\Omega \approx \omega_1 + \omega_2$. To describe the nearness of the internal and combination parametric resonances, we introduce the two detuning parameters σ_1 and σ_2 defined by

```
ResonanceConds = {ω2 == 3 ω1 + ε σ1, Ω == ω1 + ω2 + ε σ2};
```

and define the following rules:

```
OmgRule = Solve[ResonanceConds, Drop[omgList, {#}~Join~{Ω}][[1]]] & /@ {1, 2}
```

$$\Big\{\{\omega_2 \to \epsilon\,\sigma_1 + 3\,\omega_1,\ \Omega \to \epsilon\,\sigma_1 + \epsilon\,\sigma_2 + 4\,\omega_1\},\ \Big\{\omega_1 \to \frac{1}{3}(-\epsilon\,\sigma_1 + \omega_2),\ \Omega \to \frac{1}{3}(-\epsilon\,\sigma_1 + 3\,\epsilon\,\sigma_2 + 4\,\omega_2)\Big\}\Big\}$$

```
expRule[i_] := Exp[arg_] :> Exp[Expand[arg /. OmgRule[[i]]] /. ε T0 -> T1]
```

We substitute **ResonanceConds** into the right-hand side of **order1Eq** and obtain the source of secular terms as

```
ST = Table@Coefficient@order1Eq@@2DD •. expRule@iD, Exp@I wi T0DD, 8i, 2<D;
ST •. displayRule
```

```
:-2 I HD1A1L w1 f1@xD - 2 I A1 w1 m@xD f1@xD + 3 a ‡ f1¢@xD2 â x A12 A1 f1@xD +
   2 EI T1 s1 a ‡ f1¢@xD f2¢@xD â x A2 A12 f1@xD + 2 a ‡ f2¢@xD2 â x A1 A2 A2 f1@xD +
   EI T1 s1 a ‡ f1¢@xD2 â x A2 A12 f2@xD + 2 EI T1 s2 F A2 f2@xD + 4 a ‡ f1¢@xD f2¢@xD â x A1 A2 A2 f2@xD,
 -2 I HD1A2L w2 f2@xD - 2 I A2 w2 m@xD f2@xD + E-I T1 s1 a ‡ f1¢@xD2 â x A13 f1@xD +
   2 EI T1 s2 F A1 f1@xD + 4 a ‡ f1¢@xD f2¢@xD â x A1 A2 A1 f1@xD +
   2 a ‡ f1¢@xD2 â x A1 A2 A1 f2@xD + 3 a ‡ f2¢@xD2 â x A22 A2 f2@xD>
```

The solvability conditions demand that **ST** be orthogonal to solutions of the adjoint homogeneous problem. Since the problem is self-adjoint, we have

```
SCond =
  Table@int@fj@xD ST@@jDD, 8x, 0, 1<D •. intRule1 ••. intRule2 •. int -> Integrate, 8j, 2<D ==
    0 •• Thread;
SCond •. displayRule
```

```
:-2 I ‡ f1@xD2 â x HD1A1L w1 - 2 I ‡ m@xD f1@xD2 â x A1 w1 +
   3 a ‡ f1¢@xD2 â x ‡ f1@xD f1@xD â x A12 A1 + 2 EI T1 s1 a ‡ f1¢@xD f2¢@xD â x
    ‡ f1@xD f1@xD â x A2 A12 + EI T1 s1 a ‡ f1¢@xD2 â x ‡ f1@xD f2@xD â x A2 A12 +
   2 EI T1 s2 F ‡ f1@xD f2@xD â x A2 + 2 a ‡ f2¢@xD2 â x ‡ f1@xD f1@xD â x A1 A2 A2 +
   4 a ‡ f1¢@xD f2¢@xD â x ‡ f1@xD f2@xD â x A1 A2 A2 == 0,
 E-I T1 s1 a ‡ f1¢@xD2 â x ‡ f2@xD f1@xD â x A13 - 2 I ‡ f2@xD2 â x HD1A2L w2 -
   2 I ‡ m@xD f2@xD2 â x A2 w2 + 2 EI T1 s2 F ‡ f2@xD f1@xD â x A1 +
   4 a ‡ f1¢@xD f2¢@xD â x ‡ f2@xD f1@xD â x A1 A2 A1 + 2 a ‡ f1¢@xD2 â x
    ‡ f2@xD f2@xD â x A1 A2 A1 + 3 a ‡ f2¢@xD2 â x ‡ f2@xD f2@xD â x A22 A2 == 0>
```

Using the notations introduced in the preceding section, we can express **SCond** in a more concise form as

```
eqMod = ExpandA- 1/w# HHSCond@@#, 1DD •. notationRule •. Thread@secularTerms@@#DD -> 0DL +
        symbolList@@#DD.secularTerms@@#DDLE == 0 & •ž 81, 2<
```

:2 I m1 A1@T1D + 8 g11 A1@T1D^2 Ȧ1@T1D + 8 E^{I T1 s1} d1 A2@T1D Ȧ1@T1D^2 +
2 E^{I T1 s2} f s1,2 Ȧ2@T1D + 8 g12 A1@T1D A2@T1D Ȧ2@T1D + 2 I A1¢@T1D == 0,
8 E^{-I T1 s1} d2 A1@T1D^3 + 2 I m2 A2@T1D + 2 E^{I T1 s2} f s2,1 Ȧ1@T1D +
8 g21 A1@T1D A2@T1D Ȧ1@T1D + 8 g22 A2@T1D^2 Ȧ2@T1D + 2 I A2¢@T1D == 0>

where

```
-symbolList@@1DD -> H-Coefficient@SCond@@1, 1DD, #D & •ž secularTerms@@1DDL •• Thread ••
 TableForm
```

8 g11 w1 ® -3 a IÙ_0^1 f1¢@xD^2 â xM Ù_0^1 f1@xD f1²@xD â x
8 d1 w1 ® -2 a IÙ_0^1 f1¢@xD f2¢@xD â xM Ù_0^1 f1@xD f1²@xD â x - a IÙ_0^1 f1¢@xD^2 â xM Ù_0^1 f1@xD f2²@xD â x
8 g12 w1 ® -2 a IÙ_0^1 f2¢@xD^2 â xM Ù_0^1 f1@xD f1²@xD â x - 4 a IÙ_0^1 f1¢@xD f2¢@xD â xM Ù_0^1 f1@xD f2²@xD â x

```
-symbolList@@2DD -> H-Coefficient@SCond@@2, 1DD, #D & •ž secularTerms@@2DDL •• Thread ••
 TableForm
```

8 d2 w2 ® -a IÙ_0^1 f1¢@xD^2 â xM Ù_0^1 f2@xD f1²@xD â x
8 g21 w2 ® -4 a IÙ_0^1 f1¢@xD f2¢@xD â xM Ù_0^1 f2@xD f1²@xD â x - 2 a IÙ_0^1 f1¢@xD^2 â xM Ù_0^1 f2@xD f2²@xD â x
8 g22 w2 ® -3 a IÙ_0^1 f2¢@xD^2 â xM Ù_0^1 f2@xD f2²@xD â x

Modulation Equations in Polar Form

Using **eqMod** and the function **PolarForm**, we obtain the modulation equations in polar form and the definitions for g_i as

```
PolarForm@eqModD
```

::a1¢@T1D == -m1 a1@T1D - f Sin@g2@T1DD s1,2 a2@T1D - Sin@g1@T1DD d1 a1@T1D^2 a2@T1D,
a2¢@T1D == -f Sin@g2@T1DD s2,1 a1@T1D + Sin@g1@T1DD d2 a1@T1D^3 - m2 a2@T1D,
g1¢@T1D == s1 - 3 g11 a1@T1D^2 + g21 a1@T1D^2 + (f Cos@g2@T1DD s2,1 a1@T1D)/(a2@T1D) + (Cos@g1@T1DD d2 a1@T1D^3)/(a2@T1D) -
(3 f Cos@g2@T1DD s1,2 a2@T1D)/(a1@T1D) - 3 Cos@g1@T1DD d1 a1@T1D a2@T1D - 3 g12 a2@T1D^2 + g22 a2@T1D^2,
g2¢@T1D == s2 - g11 a1@T1D^2 - g21 a1@T1D^2 - (f Cos@g2@T1DD s2,1 a1@T1D)/(a2@T1D) - (Cos@g1@T1DD d2 a1@T1D^3)/(a2@T1D) -
(f Cos@g2@T1DD s1,2 a2@T1D)/(a1@T1D) - Cos@g1@T1DD d1 a1@T1D a2@T1D - g12 a2@T1D^2 - g22 a2@T1D^2>,
8g1@T1D == T1 s1 - 3 b1@T1D + b2@T1D, g2@T1D == T1 s2 - b1@T1D - b2@T1D<>

Modulation Equations in Cartesian Form and Symmetry Property

Using **eqMod** and the function **CartesianForm**, we obtain the modulation equations in Cartesian form, the definitions for n$_i$, and the symmetry property as

```
CartesianForm@eqModD
```

98p¢1@T1D == - m1 p1@T1D - n1 q1@T1D + g11 p1@T1D2 q1@T1D - 2 d1 p1@T1D p2@T1D q1@T1D + g12 p2@T1D2 q1@T1D +
g11 q1@T1D3 - f s1,2 q2@T1D + d1 p1@T1D2 q2@T1D - d1 q1@T1D2 q2@T1D + g12 q1@T1D q2@T1D2,
q¢1@T1D == n1 p1@T1D - g11 p1@T1D3 - f s1,2 p2@T1D - d1 p1@T1D2 p2@T1D - g12 p1@T1D p2@T1D2 -
m1 q1@T1D - g11 p1@T1D q1@T1D2 + d1 p2@T1D q1@T1D2 - 2 d1 p1@T1D q1@T1D q2@T1D - g12 p1@T1D q2@T1D2,
p¢2@T1D == - m2 p2@T1D - f s2,1 q1@T1D + 3 d2 p1@T1D2 q1@T1D - d2 q1@T1D3 - n2 q2@T1D +
g21 p1@T1D2 q2@T1D + g22 p2@T1D2 q2@T1D + g21 q1@T1D2 q2@T1D + g22 q2@T1D3,
q¢2@T1D == - f s2,1 p1@T1D - d2 p1@T1D3 + n2 p2@T1D - g21 p1@T1D2 p2@T1D - g22 p2@T1D3 +
3 d2 p1@T1D q1@T1D2 - g21 p2@T1D q1@T1D2 - m2 q2@T1D - g22 p2@T1D q2@T1D2<,
9n1 ® 1/4 H s1 + s2L, n2 ® 1/4 H- s1 + 3 s2L=, 88p1, q1, p2, q2<, 8- q1, p1, q2, - p2<,
8- p1, - q1, - p2, - q2<, 8q1, - p1, - q2, p2<<=

Ÿ 7.2.3 Discretization of the Continuous Problem

As an alternative, we apply the method of multiple scales to the discretized system of **EOM** and **BC**. To determine the discretized form, we expand *w*@*x*, *t*D in terms of the linear mode shapes f$_m$@*x*D as

```
wRule1 = w -> FunctionA8x, t<,  ∑(m=1 to ¥) um@tD fm@xDE;
```

To simplify the computation, we temporarily drop the Ú sign in **wRule1**, as long as we know the repeated *m* represents a summation index, and rewrite it as

```
wRule2 = w -> Function@8x, t<, um@tD fm@xDD;
```

where the *u$_m$*@*t*D are the generalized coordinates. Substituting **wRule2** into **EOM**, multiplying the result with f$_n$@*x*D, and integrating the outcome from $x = 0$ to $x = 1$ using **intRule1** and **intRule2**, we obtain the following discretized form of the equations describing the response of the beam:

```
eq723a = int@fn@xD H# •. Integrate -> int •. wRule2L, 8x, 0, 1<D •. intRule1 ••. intRule2 •.
    int -> Integrate & •ž EOM
```

- 2 P ∫0^1 fn@xD f''m@xD â x um@tD + ∫0^1 fn@xD fm^H4L@xD â x um@tD + ∫0^1 fm@xD fn@xD â x u''m@tD ==
4 F e Cos@t WD ∫0^1 fn@xD f''m@xD â x um@tD +
a e ∫0^1 f¢m@xD2 â x ∫0^1 fn@xD f''m@xD â x um@tD3 - 2 e ∫0^1 m@xD fm@xD fn@xD â x u¢m@tD

where the damping is assumed to be modal. The nonlinear term, in general, should be rewritten as

```
NT = ICoefficientAeq723a@@2DD, um@tD^3E •. f¢m@xD^2 -> fk¢@xD f{¢@xDM um@tD uk@tD u{@tD
```

```
a e jà^1_0 f¢k@xD f¢{@xD â xy jà^1_0 fn@xD f²m@xD â xy uk@tD um@tD u{@tD
    k                        z{ k                     z{
```

Using the orthonormality of the mode shapes f_i@xD and the results from the corresponding eigenvalue problem, we define the following rules:

```
notationRule1 =
 9à^1_0 fm@xD fn@xD â x -> dm,n, à^1_0 fn@xD fm^H4L@xD â x -> 2 P à^1_0 fn@xD f²m@xD â x + wm^2 dm,n=;

notationRule2 = 9à^1_0 m@xD fm@xD fn@xD â x -> mm dm,n, à^1_0 fn@xD f²m@xD â x -> -gn,m,
   jà^1_0 f¢k@xD f¢{@xD â xy jà^1_0 fn@xD f²m@xD â xy -> -gk,{ gn,m, a_ dm,n :> Ha •. m -> nL=;
   k                        z{ k                     z{
```

and then rewrite **eq723a** as

```
EOM1 =
  Expand@eq723a@@1DD •. notationRule1D == Ieq723a@@2DD •. um@tD^3 -> 0M + NT ••. notationRule2
```

```
wn^2 un@tD + u²n@tD == -4 F e Cos@t WD gn,m um@tD - a e gk,{ gn,m uk@tD um@tD u{@tD - 2 e mn u¢n@tD
```

where *m*, *k*, and { represent summation indicies.

Using the method of multiple scales, we seek a first-order uniform expansion in the form

```
solRule = un_ -> jâ^1_{j=0} e^j un,j@#1, #2D &y;
                 k                             z{
```

Transforming the total time derivatives in **EOM1** into partial derivatives in terms of T_0 and T_1, substituting the **solRule** into **EOM1**, expanding the result for small e, and discarding terms of order higher than e, we obtain

```
eq723b = HEOM1 •. 8us_@tD -> us@T0, T1D, Derivative@n_D@us_D@tD -> dt@nD@us@T0, T1DD, t -> T0< •.
     solRule •• ExpandAllL •. e^n_•;n>1 -> 0
```

```
wn^2 un,0@T0, T1D + e wn^2 un,1@T0, T1D + 2 e un,0^H1,1L@T0, T1D + un,0^H2,0L@T0, T1D + e un,1^H2,0L@T0, T1D ==
 -4 F e Cos@T0 WD gn,m um,0@T0, T1D -
  a e gk,{ gn,m uk,0@T0, T1D um,0@T0, T1D u{,0@T0, T1D - 2 e mn un,0^H1,0L@T0, T1D
```

Equating coefficients of like powers of e in **eq723b**, we obtain

```
eqEps = CoefficientList@#, eD & •ž eq723b •• Thread
```

$$9\mathrm{w}_n^2\, u_{n,0}@T_0,\ T_1D + u_{n,0}^{H2,0L}@T_0,\ T_1D == 0,$$
$$\mathrm{w}_n^2\, u_{n,1}@T_0,\ T_1D + 2\,u_{n,0}^{H1,1L}@T_0,\ T_1D + u_{n,1}^{H2,0L}@T_0,\ T_1D == -4\,F\,\mathrm{Cos}@T_0\,\mathrm{W}D\,g_{n,m}\,u_{m,0}@T_0,\ T_1D\ -$$
$$a\,g_{k,\{}\,g_{n,m}\,u_{k,0}@T_0,\ T_1D\,u_{m,0}@T_0,\ T_1D\,u_{\{,0}@T_0,\ T_1D - 2\,m_n\,u_{n,0}^{H1,0L}@T_0,\ T_1D=$$

Because in the presence of damping, all modes that are not directly or indirectly excited decay with time, all of the $u_{n,0}$ decay with time except $u_{1,0}$ and $u_{2,0}$. Then it follows from **eqEps[[1]]** that

```
sol0 = u_{i_,0} -> FunctionA8T_0, T_1<, A_i@T_1D Exp@I w_i T_0D + Ȧ_i@T_1D Exp@-I w_i T_0DE;
```

Taking account of the summation indicies, we define

```
sumRule = 9g_{k,{} g_{n,m} u_{k,0}@a__D u_{m,0}@a__D u_{{,0}@a__D -> â_{k=1}^{2} â_{m=1}^{2} â_{{=1}^{2} g_{k,{} g_{n,m} u_{k,0}@aD u_{m,0}@aD u_{{,0}@aD,
    g_{n,m} u_{m,0}@a__D -> â_{m=1}^{2} g_{n,m} u_{m,0}@aD=;
```

Hence, we rewrite **eqEps[[2]]** as

```
Clear@order1EqD

order1Eq@n_D = HeqEps@@1, 1DD •. u_{n,0} -> u_{n,1}L ==
    HeqEps@@1, 1DD •. u_{n,0} -> u_{n,1}L - HSubtract žž eqEps@@2DDL •. sumRule
```

$$\mathrm{w}_n^2\, u_{n,1}@T_0,\ T_1D + u_{n,1}^{H2,0L}@T_0,\ T_1D == -4\,F\,\mathrm{Cos}@T_0\,\mathrm{W}D\,Hg_{n,1}\,u_{1,0}@T_0,\ T_1D + g_{n,2}\,u_{2,0}@T_0,\ T_1DL\ -$$
$$a\,Hg_{1,1}\,g_{n,1}\,u_{1,0}@T_0,\ T_1D^3 + g_{1,2}\,g_{n,1}\,u_{1,0}@T_0,\ T_1D^2\,u_{2,0}@T_0,\ T_1D\ +$$
$$g_{2,1}\,g_{n,1}\,u_{1,0}@T_0,\ T_1D^2\,u_{2,0}@T_0,\ T_1D + g_{1,1}\,g_{n,2}\,u_{1,0}@T_0,\ T_1D^2\,u_{2,0}@T_0,\ T_1D\ +$$
$$g_{2,2}\,g_{n,1}\,u_{1,0}@T_0,\ T_1D\,u_{2,0}@T_0,\ T_1D^2 + g_{1,2}\,g_{n,2}\,u_{1,0}@T_0,\ T_1D\,u_{2,0}@T_0,\ T_1D^2\ +$$
$$g_{2,1}\,g_{n,2}\,u_{1,0}@T_0,\ T_1D\,u_{2,0}@T_0,\ T_1D^2 + g_{2,2}\,g_{n,2}\,u_{2,0}@T_0,\ T_1D^3L - 2\,m_n\,u_{n,0}^{H1,0L}@T_0,\ T_1D - 2\,u_{n,0}^{H1,1L}@T_0,\ T_1D$$

Substituting **sol0** into **order1Eq[1]** and **order1Eq[2]** yields

```
eqOrder1 = Array@order1Eq, 2D •. sol0 •• TrigToExp •• ExpandAll;
```

à Principal Parametric Resonance of the First Mode

```
ResonanceConds = 8w_2 == 3 w_1 + e s_1, W == 2 w_1 + e s_2<;

OmgRule = Solve@ResonanceConds, Drop@omgList, 8#<D~Join~8W<D@@1DD & •ž 81, 2<
```

$$98\mathrm{w}_2 \text{®}\ e\ s_1 + 3\,\mathrm{w}_1,\ \mathrm{W} \text{®}\ e\ s_2 + 2\,\mathrm{w}_1<,\ 9\mathrm{w}_1 \text{®}\ \frac{1}{3}\ H-e\ s_1 + \mathrm{w}_2L,\ \mathrm{W} \text{®}\ \frac{1}{3}\ H-2\,e\ s_1 + 3\,e\ s_2 + 2\,\mathrm{w}_2L==$$

```
expRule@i_D := Exp@arg_D :> Exp@Expand@arg •. OmgRule@@iDDD •. e T_0 -> T_1D
```

We substitute **ResonanceConds** into the right-hand sides of **eqOrder1** and obtain the solvability condition as

```
SCond = Table@Coefficient@-eqOrder1@@i, 2DD •. expRule@iD, Exp@I wi T0DD == 0, 8i, 2<D;
SCond •. displayRule
```

$$:2\,I\,HD_1A_1L\,w_1 + 2\,I\,A_1\,m_1\,w_1 + 2\,E^{I\,T_1\,s_2}\,F\,\dot{A}_1\,g_{1,1} + 3\,a\,A_1^2\,\dot{A}_1\,g_{1,1}^2 + 2\,E^{I\,T_1\,s_1 - I\,T_1\,s_2}\,F\,A_2\,g_{1,2} + 2\,E^{I\,T_1\,s_1}\,a\,A_2\,\dot{A}_1^2\,g_{1,1}\,g_{1,2} + 2\,a\,A_1\,A_2\,\dot{A}_2\,g_{1,2}^2 + E^{I\,T_1\,s_1}\,a\,A_2\,\dot{A}_1^2\,g_{1,1}\,g_{2,1} + 2\,a\,A_1\,A_2\,\dot{A}_2\,g_{1,2}\,g_{2,1} + 2\,a\,A_1\,A_2\,\dot{A}_2\,g_{1,1}\,g_{2,2} == 0,$$

$$2\,I\,HD_1A_2L\,w_2 + 2\,I\,A_2\,m_2\,w_2 + 2\,E^{-I\,T_1\,s_1 + I\,T_1\,s_2}\,F\,A_1\,g_{2,1} + E^{-I\,T_1\,s_1}\,a\,A_1^3\,g_{1,1}\,g_{2,1} + 2\,a\,A_1\,A_2\,\dot{A}_1\,g_{1,2}\,g_{2,1} + 2\,a\,A_1\,A_2\,\dot{A}_1\,g_{2,1}^2 + 2\,a\,A_1\,A_2\,\dot{A}_1\,g_{1,1}\,g_{2,2} + 3\,a\,A_2^2\,\dot{A}_2\,g_{2,2}^2 == 0>$$

which is in agreement with that obtained by direct approach.

à Principal Parametric Resonance of the Second Mode

```
ResonanceConds = 8w2 == 3 w1 + e s1, W == 2 w2 + e s2<;
```

```
OmgRule = Solve@ResonanceConds, Drop@omgList, 8#<D ~Join~ 8W<D@@1DD & •ž 81, 2<
```

$$98w_2 ® e\,s_1 + 3\,w_1,\ W ® 2\,e\,s_1 + e\,s_2 + 6\,w_1<,\ 9w_1 ® \frac{1}{3}\,H-e\,s_1 + w_2L,\ W ® e\,s_2 + 2\,w_2==$$

```
expRule@i_D := Exp@arg_D :> Exp@Expand@arg •. OmgRule@@iDDD •. e T0 -> T1D
```

We substitute **ResonanceConds** into the right-hand sides of **eqOrder1** and obtain the solvability condition as

```
SCond = Table@Coefficient@-eqOrder1@@i, 2DD •. expRule@iD, Exp@I wi T0DD == 0, 8i, 2<D;
SCond •. displayRule
```

$$:2\,I\,HD_1A_1L\,w_1 + 2\,I\,A_1\,m_1\,w_1 + 3\,a\,A_1^2\,\dot{A}_1\,g_{1,1}^2 + 2\,E^{I\,T_1\,s_1}\,a\,A_2\,\dot{A}_1^2\,g_{1,1}\,g_{1,2} + 2\,a\,A_1\,A_2\,\dot{A}_2\,g_{1,2}^2 + E^{I\,T_1\,s_1}\,a\,A_2\,\dot{A}_1^2\,g_{1,1}\,g_{2,1} + 2\,a\,A_1\,A_2\,\dot{A}_2\,g_{1,2}\,g_{2,1} + 2\,a\,A_1\,A_2\,\dot{A}_2\,g_{1,1}\,g_{2,2} == 0,$$

$$2\,I\,HD_1A_2L\,w_2 + 2\,I\,A_2\,m_2\,w_2 + E^{-I\,T_1\,s_1}\,a\,A_1^3\,g_{1,1}\,g_{2,1} + 2\,a\,A_1\,A_2\,\dot{A}_1\,g_{1,2}\,g_{2,1} + 2\,a\,A_1\,A_2\,\dot{A}_1\,g_{2,1}^2 + 2\,E^{I\,T_1\,s_2}\,F\,\dot{A}_2\,g_{2,2} + 2\,a\,A_1\,A_2\,\dot{A}_1\,g_{1,1}\,g_{2,2} + 3\,a\,A_2^2\,\dot{A}_2\,g_{2,2}^2 == 0>$$

which is in agreement with that obtained by direct approach.

à Combination Parametric Resonance of the Two Modes

```
ResonanceConds = 8w2 == 3 w1 + e s1, W == w1 + w2 + e s2<;
```

```
OmgRule = Solve@ResonanceConds, Drop@omgList, 8#<D ~Join~ 8W<D@@1DD & •ž 81, 2<
```

$$98w_2 ® e\,s_1 + 3\,w_1,\ W ® e\,s_1 + e\,s_2 + 4\,w_1<,\ 9w_1 ® \frac{1}{3}\,H-e\,s_1 + w_2L,\ W ® \frac{1}{3}\,H-e\,s_1 + 3\,e\,s_2 + 4\,w_2L==$$

```
expRule@i_D := Exp@arg_D :> Exp@Expand@arg •. OmgRule@@iDDD •. e T0 -> T1D
```

We substitute **ResonanceConds** into the right-hand sides of **eqOrder1** and obtain the solvability condition as

```
SCond = Table@Coefficient@-eqOrder1@@i, 2DD •. expRule@iD, Exp@I wi T0DD == 0, 8i, 2<D;
SCond •. displayRule
```

```
:2 I HD1A1L w1 + 2 I A1 m1 w1 + 3 a A1^2 Ā1 g1,1^2 + 2 E^(I T1 s2) F Ā2 g1,2 + 2 E^(I T1 s1) a A2 Ā1^2 g1,1 g1,2 +
   2 a A1 A2 Ā2 g1,2^2 + E^(I T1 s1) a A2 Ā1^2 g1,1 g2,1 + 2 a A1 A2 Ā2 g1,2 g2,1 + 2 a A1 A2 Ā2 g1,1 g2,2 == 0,
 2 I HD1A2L w2 + 2 I A2 m2 w2 + 2 E^(I T1 s2) F Ā1 g2,1 + E^(-I T1 s1) a A1^3 g1,1 g2,1 +
   2 a A1 A2 Ā1 g1,2 g2,1 + 2 a A1 A2 Ā1 g2,1^2 + 2 a A1 A2 Ā1 g1,1 g2,2 + 3 a A2^2 Ā2 g2,2^2 == 0>
```

which is in agreement with that obtained by direct approach.

Ÿ 7.2.4 Method of Time-Averaged Lagrangian

As a second alternative, we derive the modulation equations by using the method of time-averaged Lagrangian. The nondimensional Lagrangian of the beam can be expressed as (Nayfeh, 1998)

```
Lagr1 = 1/2 ∫_0^1 H¶t w@x, tDL^2 âx - 1/2 ∫_0^1 H¶x,x w@x, tDL^2 âx -
   HP + 2 e F Cos@W tDL ∫_0^1 H¶x w@x, tDL^2 âx - 1/4 e a (∫_0^1 H¶x w@x, tDL^2 âx)^2 + C@tD;
```

where $C@tD$ is independent of w. Transforming the total time derivatives into partial derivatives in terms of T_0 and T_1, we modify **Lagr1** as

```
Lagr2 = Lagr1 •. Integrate -> int •. 8w@x_, tD -> w@x, T0, T1D,
   Derivative@m_, n_D@wD@x_, tD -> dt@nD@D@w@x, T0, T1D, 8x, m<DD, t -> T0<
```

```
C@T0D + 1/2 intBHe w^H0,0,1L@x, T0, T1D + w^H0,1,0L@x, T0, T1DL^2, 8x, 0, 1<F -
 HP + 2 F e Cos@T0 WDL intBw^H1,0,0L@x, T0, T1D^2, 8x, 0, 1<F -
 1/4 a e intBw^H1,0,0L@x, T0, T1D^2, 8x, 0, 1<F^2 - 1/2 intBw^H2,0,0L@x, T0, T1D^2, 8x, 0, 1<F
```

To this end, we let

```
solRule = w -> I EvaluateASumAAi@#3D fi@#1D E^(I wi #2) + Āi@#3D fi@#1D E^(-I wi #2), 8i, 2<EE &M
```

```
w ® HE^(I #2 w1) A1@#3D f1@#1D + E^(I #2 w2) A2@#3D f2@#1D + E^(-I #2 w1) f1@#1D Ā1@#3D + E^(-I #2 w2) f2@#1D Ā2@#3D &L
```

where the first independent variable (#1) stands for x and the last two independent variables (#2 and #3) stand for the two time scales T_0 and T_1. Substituting **solRule** into **Lagr2**, using **intRule1** and **intRule2** to simplify the expansions of the integrands, and collecting the coefficients of e, we obtain

```
order1Lagr =
  Lagr2 •. solRule •. intRule1 ••. intRule2 •• TrigToExp •• Expand •• Coefficient@#, eD &;
```

Principal Parametric Resonance of the First Mode

```
ResonanceConds = {ω2 == 3 ω1 + ε σ1, Ω == 2 ω1 + ε σ2} /. Equal -> Rule
```

$$\{\omega_2 \to \epsilon\,\sigma_1 + 3\,\omega_1,\ \Omega \to \epsilon\,\sigma_2 + 2\,\omega_1\}$$

Using **ResonanceConds**, we obtain the slowly varying terms from **order1Lagr** as

```
TAL = order1Lagr /. Exp[a_] :> Exp[Expand[a /. ResonanceConds] /. ε T0 -> T1] /.
    Exp[_ T0 + _.] -> 0 /. int -> Integrate
```

$$-E^{-I T_1 \sigma_2} F \left(\int_0^1 \phi_1'[x]^2\, dx\right) A_1[T_1]^2 - 2 E^{I T_1 \sigma_1 - I T_1 \sigma_2} F \left(\int_0^1 \phi_1'[x]\,\phi_2'[x]\, dx\right) A_2[T_1]\,\bar{A}_1[T_1] -$$
$$E^{I T_1 \sigma_2} F \left(\int_0^1 \phi_1'[x]^2\, dx\right) \bar{A}_1[T_1]^2 - \frac{3}{2}\,\alpha \left(\int_0^1 \phi_1'[x]^2\, dx\right)^2 A_1[T_1]^2\,\bar{A}_1[T_1]^2 -$$
$$E^{I T_1 \sigma_1}\,\alpha \left(\int_0^1 \phi_1'[x]^2\, dx\right)\left(\int_0^1 \phi_1'[x]\,\phi_2'[x]\, dx\right) A_2[T_1]\,\bar{A}_1[T_1]^3 -$$
$$2 E^{-I T_1 \sigma_1 + I T_1 \sigma_2} F \left(\int_0^1 \phi_1'[x]\,\phi_2'[x]\, dx\right) A_1[T_1]\,\bar{A}_2[T_1] -$$
$$E^{-I T_1 \sigma_1}\,\alpha \left(\int_0^1 \phi_1'[x]^2\, dx\right)\left(\int_0^1 \phi_1'[x]\,\phi_2'[x]\, dx\right) A_1[T_1]^3\,\bar{A}_2[T_1] -$$
$$4\,\alpha \left(\int_0^1 \phi_1'[x]\,\phi_2'[x]\, dx\right)^2 A_1[T_1]\,A_2[T_1]\,\bar{A}_1[T_1]\,\bar{A}_2[T_1] -$$
$$2\,\alpha \left(\int_0^1 \phi_1'[x]^2\, dx\right)\left(\int_0^1 \phi_2'[x]^2\, dx\right) A_1[T_1]\,A_2[T_1]\,\bar{A}_1[T_1]\,\bar{A}_2[T_1] -$$
$$\frac{3}{2}\,\alpha \left(\int_0^1 \phi_2'[x]^2\, dx\right)^2 A_2[T_1]^2\,\bar{A}_2[T_1]^2 -$$
$$I \left(\int_0^1 \phi_1[x]^2\, dx\right) \omega_1\,\bar{A}_1[T_1]\,A_1'[T_1] - I \left(\int_0^1 \phi_2[x]^2\, dx\right) \omega_2\,\bar{A}_2[T_1]\,A_2'[T_1] +$$
$$I \left(\int_0^1 \phi_1[x]^2\, dx\right) \omega_1\,A_1[T_1]\,\bar{A}_1'[T_1] + I \left(\int_0^1 \phi_2[x]^2\, dx\right) \omega_2\,A_2[T_1]\,\bar{A}_2'[T_1]$$

The Euler-Lagrange equations corresponding to the **TAL** can be written as

```
eqMod1 = D[D[TAL, Ā1'[T1]], T1] - D[TAL, Ā1[T1]] == 0
```

$$2 E^{I T_1 \sigma_1 - I T_1 \sigma_2} F \left(\int_0^1 \phi_1'[x]\,\phi_2'[x]\, dx\right) A_2[T_1] + 2 E^{I T_1 \sigma_2} F \left(\int_0^1 \phi_1'[x]^2\, dx\right) \bar{A}_1[T_1] +$$
$$3\,\alpha \left(\int_0^1 \phi_1'[x]^2\, dx\right)^2 A_1[T_1]^2\,\bar{A}_1[T_1] + 3 E^{I T_1 \sigma_1}\,\alpha \left(\int_0^1 \phi_1'[x]^2\, dx\right)\left(\int_0^1 \phi_1'[x]\,\phi_2'[x]\, dx\right) A_2[T_1]\,\bar{A}_1[T_1]^2 +$$
$$4\,\alpha \left(\int_0^1 \phi_1'[x]\,\phi_2'[x]\, dx\right)^2 A_1[T_1]\,A_2[T_1]\,\bar{A}_2[T_1] +$$
$$2\,\alpha \left(\int_0^1 \phi_1'[x]^2\, dx\right)\left(\int_0^1 \phi_2'[x]^2\, dx\right) A_1[T_1]\,A_2[T_1]\,\bar{A}_2[T_1] + 2 I \left(\int_0^1 \phi_1[x]^2\, dx\right) \omega_1\,A_1'[T_1] == 0$$

and

```
eqMod2 = D[D[TAL, A2bar'[T1]], T1] - D[TAL, A2bar[T1]] == 0
```

$$2 E^{-I T_1 \sigma_1 + I T_1 \sigma_2} F \left(\int_0^1 \phi_1'[x]\,\phi_2'[x]\,dx\right) A_1[T_1] + E^{-I T_1 \sigma_1} \alpha \left(\int_0^1 \phi_1'[x]^2\,dx\right)\left(\int_0^1 \phi_1'[x]\,\phi_2'[x]\,dx\right) A_1[T_1]^3 +$$
$$4\alpha \left(\int_0^1 \phi_1'[x]\,\phi_2'[x]\,dx\right)^2 A_1[T_1]\,A_2[T_1]\,\bar{A}_1[T_1] +$$
$$2\alpha \left(\int_0^1 \phi_1'[x]^2\,dx\right)\left(\int_0^1 \phi_2'[x]^2\,dx\right) A_1[T_1]\,A_2[T_1]\,\bar{A}_1[T_1] +$$
$$3\alpha \left(\int_0^1 \phi_2'[x]^2\,dx\right)^2 A_2[T_1]^2\,\bar{A}_2[T_1] + 2 I \left(\int_0^1 \phi_2[x]^2\,dx\right) \omega_2\, A_2'[T_1] == 0$$

Adding linear viscous damping to **eqMod1** and **eqMod2** and performing integration by parts yields the same modulation equations as obtained by directly attacking the partial-differential system.

Principal Parametric Resonance of the Second Mode

```
ResonanceConds = Solve[{ω2 == 3 ω1 + ε σ1, Ω == 2 ω2 + ε σ2}, {ω2, Ω}][[1]]
```

$\{\omega_2 \to \epsilon\,\sigma_1 + 3\,\omega_1,\ \Omega \to 2\,\epsilon\,\sigma_1 + \epsilon\,\sigma_2 + 6\,\omega_1\}$

Using **ResonanceConds**, we obtain the slowly varying terms from **order1Lagr** as

```
TAL = order1Lagr /. Exp[a_] :> Exp[Expand[a /. ResonanceConds] /. ε T0 -> T1] /.
   Exp[_ T0 + _.] -> 0 /. int -> Integrate
```

$$-E^{-I T_1 \sigma_2} F \left(\int_0^1 \phi_2'[x]^2\,dx\right) A_2[T_1]^2 - \frac{3}{2}\alpha \left(\int_0^1 \phi_1'[x]^2\,dx\right)^2 A_1[T_1]^2\,\bar{A}_1[T_1]^2 -$$
$$E^{I T_1 \sigma_1} \alpha \left(\int_0^1 \phi_1'[x]^2\,dx\right)\left(\int_0^1 \phi_1'[x]\,\phi_2'[x]\,dx\right) A_2[T_1]\,\bar{A}_1[T_1]^3 -$$
$$E^{-I T_1 \sigma_1} \alpha \left(\int_0^1 \phi_1'[x]^2\,dx\right)\left(\int_0^1 \phi_1'[x]\,\phi_2'[x]\,dx\right) A_1[T_1]^3\,\bar{A}_2[T_1] -$$
$$4\alpha \left(\int_0^1 \phi_1'[x]\,\phi_2'[x]\,dx\right)^2 A_1[T_1]\,A_2[T_1]\,\bar{A}_1[T_1]\,\bar{A}_2[T_1] -$$
$$2\alpha \left(\int_0^1 \phi_1'[x]^2\,dx\right)\left(\int_0^1 \phi_2'[x]^2\,dx\right) A_1[T_1]\,A_2[T_1]\,\bar{A}_1[T_1]\,\bar{A}_2[T_1] -$$
$$E^{I T_1 \sigma_2} F \left(\int_0^1 \phi_2'[x]^2\,dx\right) \bar{A}_2[T_1]^2 - \frac{3}{2}\alpha \left(\int_0^1 \phi_2'[x]^2\,dx\right)^2 A_2[T_1]^2\,\bar{A}_2[T_1]^2 -$$
$$I \left(\int_0^1 \phi_1[x]^2\,dx\right) \omega_1\,\bar{A}_1[T_1]\,A_1'[T_1] - I \left(\int_0^1 \phi_2[x]^2\,dx\right) \omega_2\,\bar{A}_2[T_1]\,A_2'[T_1] +$$
$$I \left(\int_0^1 \phi_1[x]^2\,dx\right) \omega_1\,A_1[T_1]\,\bar{A}_1'[T_1] + I \left(\int_0^1 \phi_2[x]^2\,dx\right) \omega_2\,A_2[T_1]\,\bar{A}_2'[T_1]$$

The Euler-Lagrange equations corresponding to the **TAL** can be written as

eqMod1 = D[D[AL, $\bar{A}_1'[T_1]$], T_1] - D[AL, $\bar{A}_1[T_1]$] == 0

$$3\,\alpha \left(\int_0^1 f_1'[x]^2\, dx\right)^2 A_1[T_1]^2\, \bar{A}_1[T_1] + 3\, e^{i T_1 \sigma_1}\, \alpha \left(\int_0^1 f_1'[x]^2\, dx\right)\left(\int_0^1 f_1'[x]\, f_2'[x]\, dx\right) A_2[T_1]\, \bar{A}_1[T_1]^2 +$$
$$4\,\alpha \left(\int_0^1 f_1'[x]\, f_2'[x]\, dx\right)^2 A_1[T_1]\, A_2[T_1]\, \bar{A}_2[T_1] +$$
$$2\,\alpha \left(\int_0^1 f_1'[x]^2\, dx\right)\left(\int_0^1 f_2'[x]^2\, dx\right) A_1[T_1]\, A_2[T_1]\, \bar{A}_2[T_1] + 2\, i \left(\int_0^1 f_1[x]^2\, dx\right) \omega_1\, A_1'[T_1] == 0$$

and

eqMod2 = D[D[AL, $\bar{A}_2'[T_1]$], T_1] - D[AL, $\bar{A}_2[T_1]$] == 0

$$e^{-i T_1 \sigma_1}\, \alpha \left(\int_0^1 f_1'[x]^2\, dx\right)\left(\int_0^1 f_1'[x]\, f_2'[x]\, dx\right) A_1[T_1]^3 + 4\,\alpha \left(\int_0^1 f_1'[x]\, f_2'[x]\, dx\right)^2 A_1[T_1]\, A_2[T_1]\, \bar{A}_1[T_1] +$$
$$2\,\alpha \left(\int_0^1 f_1'[x]^2\, dx\right)\left(\int_0^1 f_2'[x]^2\, dx\right) A_1[T_1]\, A_2[T_1]\, \bar{A}_1[T_1] + 2\, e^{i T_1 \sigma_2}\, \Phi \left(\int_0^1 f_2'[x]^2\, dx\right) \bar{A}_2[T_1] +$$
$$3\,\alpha \left(\int_0^1 f_2'[x]^2\, dx\right)^2 A_2[T_1]^2\, \bar{A}_2[T_1] + 2\, i \left(\int_0^1 f_2[x]^2\, dx\right) \omega_2\, A_2'[T_1] == 0$$

Adding linear viscous damping to **eqMod1** and **eqMod2** and performing integration by parts yields the same modulation equations as obtained by directly attacking the partial-differential system.

■ Combination Parametric Resonance of the Two Modes

ResonanceConds = Solve[{ω_2 == 3 ω_1 + $\epsilon\, \sigma_1$, Ω == ω_1 + ω_2 + $\epsilon\, \sigma_2$}, {ω_2, Ω}][[1]]

$\{\omega_2 \to \epsilon\, \sigma_1 + 3\, \omega_1,\ \Omega \to \epsilon\, \sigma_1 + \epsilon\, \sigma_2 + 4\, \omega_1\}$

Using **ResonanceConds**, we obtain the slowly varying terms from **order1Lagr** as

```
TAL = order1Lagr •. Exp@a_D :> Exp@Expand@a •. ResonanceCondsD •. e T0 -> T1D •.
   Exp@_ T0 + _.D -> 0 •. int -> Integrate
```

```
-2 E^(-I T1 s2) F ∫_0^1 f1¢@xD f2¢@xD â x A1@T1D A2@T1D - 3/2 a (∫_0^1 f1¢@xD^2 â x)^2 A1@T1D^2 A1•@T1D^2 -
E^(I T1 s1) a ∫_0^1 f1¢@xD^2 â x ∫_0^1 f1¢@xD f2¢@xD â x A2@T1D A1•@T1D^3 -
E^(-I T1 s1) a ∫_0^1 f1¢@xD^2 â x ∫_0^1 f1¢@xD f2¢@xD â x A1@T1D^3 A2•@T1D -
2 E^(I T1 s2) F ∫_0^1 f1¢@xD f2¢@xD â x A1•@T1D A2•@T1D - 4 a (∫_0^1 f1¢@xD f2¢@xD â x)^2 A1@T1D A2@T1D A1•@T1D A2•@T1D -
2 a ∫_0^1 f1¢@xD^2 â x ∫_0^1 f2¢@xD^2 â x A1@T1D A2@T1D A1•@T1D A2•@T1D -
3/2 a (∫_0^1 f2¢@xD^2 â x)^2 A2@T1D^2 A2•@T1D^2 -
I ∫_0^1 f1@xD^2 â x w1 A1•@T1D A1¢@T1D - I ∫_0^1 f2@xD^2 â x w2 A2•@T1D A2¢@T1D +
I ∫_0^1 f1@xD^2 â x w1 A1@T1D A1•¢@T1D + I ∫_0^1 f2@xD^2 â x w2 A2@T1D A2•¢@T1D
```

The Euler-Lagrange equations corresponding to the **TAL** can be written as

```
eqMod1 = DADATAL, A1•¢@T1DE, T1E - DATAL, A1•@T1DE == 0
```

```
3 a (∫_0^1 f1¢@xD^2 â x)^2 A1@T1D^2 A1•@T1D + 3 E^(I T1 s1) a ∫_0^1 f1¢@xD^2 â x ∫_0^1 f1¢@xD f2¢@xD â x A2@T1D A1•@T1D^2 +
  2 E^(I T1 s2) F ∫_0^1 f1¢@xD f2¢@xD â x A2•@T1D + 4 a (∫_0^1 f1¢@xD f2¢@xD â x)^2 A1@T1D A2@T1D A2•@T1D +
  2 a ∫_0^1 f1¢@xD^2 â x ∫_0^1 f2¢@xD^2 â x A1@T1D A2@T1D A2•@T1D + 2 I ∫_0^1 f1@xD^2 â x w1 A1¢@T1D == 0
```

and

```
eqMod2 = DADATAL, A2•¢@T1DE, T1E - DATAL, A2•@T1DE == 0
```

```
E^(-I T1 s1) a ∫_0^1 f1¢@xD^2 â x ∫_0^1 f1¢@xD f2¢@xD â x A1@T1D^3 +
  2 E^(I T1 s2) F ∫_0^1 f1¢@xD f2¢@xD â x A1•@T1D + 4 a (∫_0^1 f1¢@xD f2¢@xD â x)^2 A1@T1D A2@T1D A1•@T1D +
  2 a ∫_0^1 f1¢@xD^2 â x ∫_0^1 f2¢@xD^2 â x A1@T1D A2@T1D A1•@T1D +
  3 a (∫_0^1 f2¢@xD^2 â x)^2 A2@T1D^2 A2•@T1D + 2 I ∫_0^1 f2@xD^2 â x w2 A2¢@T1D == 0
```

Adding linear viscous damping to **eqMod1** and **eqMod2** and performing integration by parts yields the same modulation equations as obtained by directly attacking the partial-differential system.

7.3 Cantilever Beam

7.3.1 EOM and BC's

We consider the nonlinear nonplanar response of an inextensional cantilever beam to an external excitation of one of its flexural modes. The lowest torsional frequencies of the beam considered are much higher than the frequencies of the excited flexural modes so that the torsional inertia can be neglected. We assume that the beam is uniform and homogeneous. The transverse deflections $v[x, t]$ and $w[x, t]$ of the beam at the position x and time t are governed by the nondimensional integral-partial-differential equation (Crespo da Silva and Glynn, 1978)

$$
\begin{aligned}
\mathbf{EOM1} = \Big\{ & \partial_{t,t} v[x,t] + \beta_y\, \partial_{x,x,x,x} v[x,t] == -2\,\epsilon\, \mu[x]\, \partial_t v[x,t] + \epsilon\,(1-\beta_y) \\
& \partial_x\Big(\partial_{x,x} w[x,t] \int_1^x \partial_{x,x} v[x,t]\, \partial_{x,x} w[x,t]\, dx - \partial_{x,x,x} w[x,t] \int_0^x \partial_{x,x} v[x,t]\, \partial_x w[x,t]\, dx\Big) - \\
& \epsilon\, \frac{(1-\beta_y)^2}{\beta_\gamma}\, \partial_{x,x}\Big(\partial_{x,x} w[x,t] \int_0^x \int_1^x \partial_{x,x} v[x,t]\, \partial_{x,x} w[x,t]\, dx\, dx\Big) - \\
& \epsilon\, \beta_y\, \partial_x\big(\partial_x v[x,t]\, \partial_x(\partial_x v[x,t]\, \partial_{x,x} v[x,t] + \partial_x w[x,t]\, \partial_{x,x} w[x,t])\big) - \\
& \epsilon\, \frac{1}{2}\, \partial_x\Big(\partial_x v[x,t] \int_1^x \partial_{t,t}\Big(\int_0^x \big((\partial_x v[x,t])^2 + (\partial_x w[x,t])^2\big)\, dx\Big)\, dx\Big) - \epsilon\, F[x]\, \mathrm{Cos}[\Omega\, t], \\
& \partial_{t,t} w[x,t] + \partial_{x,x,x,x} w[x,t] == -2\,\epsilon\, \mu[x]\, \partial_t w[x,t] - \epsilon\,(1-\beta_y) \\
& \partial_x\Big(\partial_{x,x} v[x,t] \int_1^x \partial_{x,x} v[x,t]\, \partial_{x,x} w[x,t]\, dx - \partial_{x,x,x} v[x,t] \int_0^x \partial_{x,x} w[x,t]\, \partial_x v[x,t]\, dx\Big) - \\
& \epsilon\, \frac{(1-\beta_y)^2}{\beta_\gamma}\, \partial_{x,x}\Big(\partial_{x,x} v[x,t] \int_0^x \int_1^x \partial_{x,x} v[x,t]\, \partial_{x,x} w[x,t]\, dx\, dx\Big) - \\
& \epsilon\, \partial_x\big(\partial_x w[x,t]\, \partial_x(\partial_x v[x,t]\, \partial_{x,x} v[x,t] + \partial_x w[x,t]\, \partial_{x,x} w[x,t])\big) - \\
& \epsilon\, \frac{1}{2}\, \partial_x\Big(\partial_x w[x,t] \int_1^x \partial_{t,t}\Big(\int_0^x \big((\partial_x v[x,t])^2 + (\partial_x w[x,t])^2\big)\, dx\Big)\, dx\Big)\Big\};
\end{aligned}
$$

along with the homogeneous boundary conditions

$$
\begin{aligned}
\mathbf{BC1} = \{ & v[0,t] == 0,\ \partial_x v[x,t] == 0\ /.\ x \to 0, \\
& \partial_{x,x} v[x,t] == 0\ /.\ x \to 1,\ \partial_{x,x,x} v[x,t] == 0\ /.\ x \to 1\}
\end{aligned}
$$

$$
\{v[0,t] == 0,\ v^{(1,0)}[0,t] == 0,\ v^{(2,0)}[1,t] == 0,\ v^{(3,0)}[1,t] == 0\}
$$

$$
\mathbf{BC = Join[BC1,\ BC1\ /.\ v \to w]}
$$

$$
\begin{aligned}
\{ & v[0,t] == 0,\ v^{(1,0)}[0,t] == 0,\ v^{(2,0)}[1,t] == 0,\ v^{(3,0)}[1,t] == 0, \\
& w[0,t] == 0,\ w^{(1,0)}[0,t] == 0,\ w^{(2,0)}[1,t] == 0,\ w^{(3,0)}[1,t] == 0\}
\end{aligned}
$$

The spatial derivatives outside the integrals on the right-hand sides of **EOM1** result in lengthy expressions. To simplify the symbolic computation, we first define some operators as

$$\text{op1}[v\_, w\_] = \partial_{x,x} w[x,t] \int_1^x \partial_{x,x} v[x,t]\, \partial_{x,x} w[x,t]\, dx - \partial_{x,x,x} w[x,t] \int_0^x \partial_{x,x} v[x,t]\, \partial_x w[x,t]\, dx \text{ /. Integrate -> int[1]}$$

$$\text{int}[1][v^{(2,0)}[x,t]\, w^{(2,0)}[x,t], \{x, 1, x\}]\, w^{(2,0)}[x,t] - \text{int}[1][w^{(1,0)}[x,t]\, v^{(2,0)}[x,t], \{x, 0, x\}]\, w^{(3,0)}[x,t]$$

$$\text{op2}[v\_, w\_] = \text{HoldForm}\left[\partial_{x,x} w[x,t] \int_0^x \int_1^x \partial_{x,x} v[x,t]\, \partial_{x,x} w[x,t]\, dx\, dx\right] \text{ /. Integrate -> int[2] // ReleaseHold}$$

$$\text{int}[2][v^{(2,0)}[x,t]\, w^{(2,0)}[x,t], \{x, 0, x\}, \{x, 1, x\}]\, w^{(2,0)}[x,t]$$

$$\text{op3}[v\_, w\_] = \partial_x v[x,t] \int_1^x \partial_{t,t}\left(\int_0^x \left((\partial_x v[x,t])^2 + (\partial_x w[x,t])^2\right) dx\right) dx \text{ /. Integrate -> int[1]}$$

$$\text{int}[1]\Big[\text{int}[1]\big[2\, v^{(1,1)}[x,t]^2 + 2\, w^{(1,1)}[x,t]^2 + 2\, v^{(1,0)}[x,t]\, v^{(1,2)}[x,t] + 2\, w^{(1,0)}[x,t]\, w^{(1,2)}[x,t], \{x, 0, x\}\big], \{x, 1, x\}\Big]\, v^{(1,0)}[x,t]$$

where int[*i*] indicates the whole expression, including the integrand and the term multiplied by int[*i*][expr], will later be wrapped by the *i*th spatial derivative in the solvability condition.

Then we rewrite **EOM1** as

$$\begin{aligned}\text{EOM} = \{&\partial_{t,t} v[x,t] + \beta_y\, \partial_{x,x,x,x} v[x,t] == \\ &-2\epsilon\, \mu[x]\, \partial_t v[x,t] + \epsilon (1-\beta_y)\, \text{op1}[v, w] - \epsilon \frac{(1-\beta_y)^2}{\beta_\gamma}\, \text{op2}[v, w] - \\ &\epsilon \beta_y\, \partial_x(\partial_x v[x,t]\, \partial_x(\partial_x v[x,t]\, \partial_{x,x} v[x,t] + \partial_x w[x,t]\, \partial_{x,x} w[x,t])) - \\ &\epsilon \frac{1}{2}\, \text{op3}[v, w] - \epsilon F[x] \cos[\Omega t],\; \partial_{t,t} w[x,t] + \partial_{x,x,x,x} w[x,t] == \\ &-2\epsilon\, \mu[x]\, \partial_t w[x,t] - \epsilon (1-\beta_y)\, \text{op1}[w, v] - \epsilon \frac{(1-\beta_y)^2}{\beta_\gamma}\, \text{op2}[w, v] - \\ &\epsilon\, \partial_x(\partial_x w[x,t]\, \partial_x(\partial_x v[x,t]\, \partial_{x,x} v[x,t] + \partial_x w[x,t]\, \partial_{x,x} w[x,t])) - \epsilon \frac{1}{2}\, \text{op3}[w, v]\};\end{aligned}$$

It follows from **EOM1** and **BC** that the linear undamped natural frequencies and their corresponding mode shapes are given by

$$\{\hat{\omega}_{1,m} == \lambda_m^2 \sqrt{\beta_y},\; \omega_{2,n} == \lambda_n^2,$$
$$\phi_i[x] == \cosh[\lambda_i x] - \cos[\lambda_i x] + \frac{\cos[\lambda_i] + \cosh[\lambda_i]}{\sin[\lambda_i] + \sinh[\lambda_i]} (\sin[\lambda_i x] - \sinh[\lambda_i x]) \; (*\; i=m,n\; *)\};$$

where $\hat{\omega}_{1,m}$ and $\omega_{2,n}$ are the natural frequencies in the *y* and *z* directions and the λ_i are the roots of

$$1 + \cos[\lambda_i] \cosh[\lambda_i] == 0$$

We consider the case of one-to-one internal resonance between the *m*th mode in the *y* or *v* direction and the *n*th mode in the *z* or *w* direction; that is, $\omega_{1,m} \approx \omega_{2,n}$. To express the nearness of these frequencies quantitatively, we let

```
betayRule = 8by -> 1 + d0 + e d1<;
```

so that $\omega_{1,m} = \lambda_m^2 \sqrt{1+\delta_0} = \lambda_n^2 = \omega_{2,n}$. The beam has a near-square cross-section when $\delta_0 = 0$. We assume neither of these two modes is involved in an internal resonance with any other mode. Moreover, we consider a primary resonance of the flexural mode in the *y* direction.

To manipulate some complicated integrals, we define the following rules:

```
intRule3 = 8int@i_D@fun_, arg2__D :> int@iD@Expand@funD, arg2D<;
intRule4 = 8int@i_D@a_ + b_, arg2__D :> int@iD@a, arg2D + int@iD@b, arg2D,
   int@i_D@en_. fun_, arg2__D :> en int@iD@fun, arg2D,
   int@i_D@a_ fun_, a1_, b1___D :> a int@iD@fun, a1, b1D •; FreeQ@a, First@a1DD,
   int@i_D@int@i_D@a1_, a2_D fun_., a3__D :> int@iD@a1, a2D int@iD@fun, a3D •;
     FreeQ@Rest@a2D, First@a2DD, int@i_D@int@i_D@a1_, a2_D, a3__D :> int@iD@a1, a3, a2D<;
```

7.3.2 Direct Attack of the Continuous Problem

In this section, we directly attack the integral-partial-differential equation **EOM** and associated boundary conditions **BC** and seek a first-order uniform expansion of their solution in the form

```
solRule = 9v -> IEvaluateASumAej vj@#1, #2, #3D, 8j, 0, 1<EE &M,
  w -> IEvaluateASumAej wj@#1, #2, #3D, 8j, 0, 1<EE &M=

8v ® Hv0@#1, #2, #3D + e v1@#1, #2, #3D &L, w ® Hw0@#1, #2, #3D + e w1@#1, #2, #3D &L<
```

where the first independent variable stands for *x* and the last two independent variables stand for the two time scales T_0 and T_1. Transforming the total time derivatives in **EOM** into partial derivatives in terms of T_0 and T_1, substituting the **solRule** and **betayRule** into **EOM** and **BC**, expanding the result for small ϵ, discarding terms of order higher than ϵ, and using **intRule3** and **intRule4** to simplify the expansions of the integrands, we obtain

```
Ieq732a =
    HHJoin@EOM, BCD •. 8v@x_, tD -> v@x, T0, T1D, w@x_, tD -> w@x, T0, T1D, Derivative@m_, n_D@
                 w_D@x_, tD -> dt@nD@D@w@x, T0, T1D, 8x, m<DD, t -> T0< •.
            solRule •. betayRule •• ExpandAllL •. intRule3 ••.
      intRule4 •• ExpandAllL •. en_•;n>1 -> 0;M •• Timing

812.629 Second, Null<
```

Equating coefficients of like powers of ϵ in **eq732a**, we obtain

```
eqEps = Thread@CoefficientList@Subtract žž #, eD == 0D & •ž eq732a •• Transpose;
```

```
eqEps@@1DD •. displayRule
```

```
9v0'''' + D0^2v0 + Hv0''''L d0 == 0, w0'''' + D0^2w0 == 0, v0@0, T0, T1D == 0,
 v0^H1,0,0L@0, T0, T1D == 0, v0^H2,0,0L@1, T0, T1D == 0, v0^H3,0,0L@1, T0, T1D == 0,
 w0@0, T0, T1D == 0, w0^H1,0,0L@0, T0, T1D == 0, w0^H2,0,0L@1, T0, T1D == 0, w0^H3,0,0L@1, T0, T1D == 0=
```

In the presence of damping, all modes that are not directly excited by the forcing or indirectly excited by the internal resonance will decay with time. Hence, the solution of **eqEps[[1]]** can be expressed in terms of the two excited linear free-vibration modes; that is,

```
sol0 = 9v0 -> FunctionA8x, T0, T1<, A1@T1D fm@xD Exp@I w1,m T0D + Ȧ1@T1D fm@xD Exp@-I w1,m T0DE,
   w0 -> FunctionA8x, T0, T1<, A2@T1D fn@xD Exp@I w2,n T0D + Ȧ2@T1D fn@xD Exp@-I w2,n T0DE=;
```

One can easily show that the mode shapes $f_i@xD$ are orthogonal. We assume that these modes are normalized so that

```
Ù0^1 fi@xD fj@xD â x -> dij
```

where d_{ij} is the Kronecker delta function.

Substituting **sol0** into the first-order equations, **eqEps[[2,1]]** and **eqEps[[2,2]]**, and using **intRule3** and **intRule4**, we obtain

```
Clear@order1EqD
```

```
order1Eq@1D = HeqEps@@1, 1, 1DD •. v0 -> v1L ==
   HHeqEps@@1, 1, 1DD •. v0 -> v1L - HSubtract žž eqEps@@2, 1DD •. sol0 •• TrigToExp ••
         ExpandL •. intRule3 ••. intRule4 •• ExpandL;
```

```
order1Eq@2D = HeqEps@@1, 2, 1DD •. w0 -> w1L ==
   HHeqEps@@1, 2, 1DD •. w0 -> w1L - HSubtract žž eqEps@@2, 2DD •. sol0 •• TrigToExp ••
         ExpandL •. intRule3 ••. intRule4 •• ExpandL;
```

It follows from **eqEps[[2]]** that the first-order boundary conditions are

```
order1BC = Drop@eqEps@@2DD, 2D
```

```
9v1@0, T0, T1D == 0, v1^H1,0,0L@0, T0, T1D == 0, v1^H2,0,0L@1, T0, T1D == 0, v1^H3,0,0L@1, T0, T1D == 0,
 w1@0, T0, T1D == 0, w1^H1,0,0L@0, T0, T1D == 0, w1^H2,0,0L@1, T0, T1D == 0, w1^H3,0,0L@1, T0, T1D == 0=
```

We consider the case of primary resonance of the flexural mode in the *y* direction, $W » w_{1,m}$. To describe the nearness of the primary resonance, we introduce the detuning parameter s by

```
omgList = 8w1,m, w2,n<;
```

```
ResonanceConds = 8w2,n == w1,m, W == w1,m + e s<;
```

and define the following rules:

```
OmgRule = Solve@ResonanceConds, Drop@omgList, 8#<D ~ Join ~ 8W<D@@1DD & •ž 81, 2<
```

```
88w2,n ® w1,m, W ® e s + w1,m<, 8w1,m ® w2,n, W ® e s + w2,n<<
```

```
expRule@i_D := Exp@arg_D :> Exp@Expand@arg •. OmgRule@@iDDD •. e T0 -> T1D
```

We substitute **ResonanceConds** into the right-hand side of **order1Eq[1]** and **order1Eq[2]** and obtain the source of secular terms as

```
ST = Table@Coefficient@order1Eq@iD@@2DD •. expRule@iD, Exp@I omgList@@iDD T0DD, 8i, 2<D;
```

The solvability conditions of **order1Eq[i]** and **order1BC** demand that **ST[[i]]** be orthogonal to $f_i@xD$. Imposing these conditions, we have

```
SCond1 = int@fm@xD ST@@1DD, 8x, 0, 1<D == 0 •. intRule1 ••. intRule2;
```

```
SCond2 = int@fn@xD ST@@2DD, 8x, 0, 1<D == 0 •. intRule1 ••. intRule2;
```

Recall that the *i*th spatial derivative should be recovered back from int@iD as stated in the preceding section. Hence, we define

```
intRule5@mode_D =
  int@mode a_ int@i_D@b_, c__D, d_D :> int@mode HoldForm@D@a int@b, cD, 8x, i<DD, dD;
```

and rewrite **SCond1** and **SCond2** as

```
SCond = 8SCond1 •. intRule5@fm@xDD, SCond2 •. intRule5@fn@xDD<;
```

To simplify the notation in the solvability conditions, we use the orthonormality of the mode shapes and define the following parameters:

```
notationRule = 9intAfi_@xD^2, 8x, 0, 1<E -> 1, intAfi_@xD fi_^H4L@xD, 8x, 0, 1<E -> li^4,
  intAm@xD fi_@xD^2, 8x, 0, 1<E -> mi, int@F@xD fi_@xD, 8x, 0, 1<D -> w1,i f=;
```

To identify the forms of the nonlinear terms in the solvability conditions, we first identify all of the possible forms of the nonhomogeneous terms in the first-order problem. To this end, we let

```
basicTerms = TableA9Ai@T1D E^(I omgList@@iDD T0), Ȧi@T1D E^(-I omgList@@iDD T0)=, 8i, 2<E •• Flatten
```

```
8E^(I T0 w1,m) A1@T1D, E^(-I T0 w1,m) Ȧ1@T1D, E^(I T0 w2,n) A2@T1D, E^(-I T0 w2,n) Ȧ2@T1D<
```

Then, all of the possible forms of the nonlinear terms in the first-order problem are given by

```
cubicTerms = Nest@Outer@Times, basicTerms, #D &, basicTerms, 2D •• Flatten •• Union;
cubicTerms •. displayRule
```

```
:E^(3 I T0 w1,m) A1^3, E^(I T0 H2 w1,m+w2,nL) A1^2 A2, E^(I T0 Hw1,m+2 w2,nL) A1 A2^2, E^(3 I T0 w2,n) A2^3, E^(I T0 w1,m) A1^2 Ȧ1,
 E^(I T0 w2,n) A1 A2 Ȧ1, E^(I T0 H-w1,m+2 w2,nL) A2^2 Ȧ1, E^(-I T0 w1,m) A1 Ȧ1^2, E^(I T0 H-2 w1,m+w2,nL) A2 Ȧ1^2, E^(-3 I T0 w1,m) Ȧ1^3,
 E^(I T0 H2 w1,m-w2,nL) A1^2 Ȧ2, E^(I T0 w1,m) A1 A2 Ȧ2, E^(I T0 w2,n) A2^2 Ȧ2, E^(-I T0 w2,n) A1 Ȧ1 Ȧ2, E^(-I T0 w1,m) A2 Ȧ1 Ȧ2,
 E^(I T0 H-2 w1,m-w2,nL) Ȧ1^2 Ȧ2, E^(I T0 Hw1,m-2 w2,nL) A1 Ȧ2^2, E^(-I T0 w2,n) A2 Ȧ2^2, E^(I T0 H-w1,m-2 w2,nL) Ȧ1 Ȧ2^2, E^(-3 I T0 w2,n) Ȧ2^3>
```

Out of these terms, only the terms that may lead to secular terms appear in the solvability conditions, which can be identified according to

```
secularTerms =
  IE^(-I omgList@@#DD T0) cubicTerms •. expRule@#D •. Exp@_ T0 + _.D -> 0 •• Union •• RestM & •ž 81, 2<
```

```
88A1@T1D^2 A1•@T1D, A1@T1D A2@T1D A1•@T1D, A2@T1D^2 A1•@T1D, A1@T1D^2 A2•@T1D,
  A1@T1D A2@T1D A2•@T1D, A2@T1D^2 A2•@T1D<, 8A1@T1D^2 A1•@T1D, A1@T1D A2@T1D A1•@T1D,
  A2@T1D^2 A1•@T1D, A1@T1D^2 A2•@T1D, A1@T1D A2@T1D A2•@T1D, A2@T1D^2 A2•@T1D<<
```

Next, we define the following parameters:

```
coef = Table@Coefficient@SCond@@i, 1DD, secularTerms@@iDDD, 8i, 2<D •• Collect@#, d0D &;
```

```
Clear@symbolListD
```

```
symbolList@1D = MapIndexed@If@#1 =!= 0, 8 w1,m a1,#2@@1DD, 0D &, coef@@1DDD
```

```
88 a1,1 w1,m, 0, 8 a1,3 w1,m, 0, 8 a1,5 w1,m, 0<
```

```
symbolList@2D = MapIndexed@If@#1 =!= 0, 8 w2,n a2,#2@@1DD, 0D &, coef@@2DDD
```

```
80, 8 a2,2 w2,n, 0, 8 a2,4 w2,n, 0, 8 a2,6 w2,n<
```

and express **SCond** in a more concise form as

```
eqMod =
  ExpandA- 1/omgList@@#DD HHSCond@@#, 1DD •. notationRule •. Thread@secularTerms@@#DD -> 0DL +
        symbolList@#D.secularTerms@@#DDLE == 0 & •ž 81, 2<
```

```
: 1/2 E^(I T1 s) f + 2 I mm A1@T1D + (d1,1^4 A1@T1D)/w1,m - 8 a1,1 A1@T1D^2 A1•@T1D -
   8 a1,3 A2@T1D^2 A1•@T1D - 8 a1,5 A1@T1D A2@T1D A2•@T1D + 2 I A1¢@T1D == 0, 2 I mn A2@T1D -
   8 a2,2 A1@T1D A2@T1D A1•@T1D - 8 a2,4 A1@T1D^2 A2•@T1D - 8 a2,6 A2@T1D^2 A2•@T1D + 2 I A2¢@T1D == 0>
```

where

```
HsymbolList@1D -> coef@@1DD •• Thread •• Union •• RestL •. int -> Integrate •• Timing
```

:29.683 Second, :8 a1,1 w1, m ®

-3 à_0^1 f m@xD f ² m@xD³ â x - 12 à_0^1 f m@xD f ¢ m@xD f ² m@xD f H3L m@xD â x - 3 à_0^1 f m@xD f ¢ m@xD² f H4L m@xD â x +

j -3 à_0^1 f m@xD f ² m@xD³ â x - 12 à_0^1 f m@xD f ¢ m@xD f ² m@xD f H3L m@xD â x - 3 à_0^1 f m@xD f ¢ m@xD² f H4L m@xD â x ¥ d0 +
k {

2 j à_0^1 ¶8x, 1< J f ¢ m@xD à_1^x à_0^x f ¢ m@xD² â x â xN f m@xD â x ¥ w²1, m,
k {

8 a1,3 w1, m ® - à_0^1 f m@xD f ² m@xD f ² n@xD² â x - à_0^1 f m@xD f ¢ n@xD f ² m@xD f H3L n@xD â x -

3 à_0^1 f m@xD f ¢ m@xD f ² n@xD f H3L n@xD â x - à_0^1 f m@xD f ¢ m@xD f ¢ n@xD f H4L n@xD â x +

j à_0^1 ¶8x, 1< J f H3L n@xD à_0^x f ¢ n@xD f ² m@xD â xN f m@xD â x - à_0^1 ¶8x, 1< J f ² n@xD à_1^x f ² m@xD f ² n@xD â xN f m@xD â x -
k

à_0^1 f m@xD f ² m@xD f ² n@xD² â x - à_0^1 f m@xD f ¢ n@xD f ² m@xD f H3L n@xD â x -

3 à_0^1 f m@xD f ¢ m@xD f ² n@xD f H3L n@xD â x - à_0^1 f m@xD f ¢ m@xD f ¢ n@xD f H4L n@xD â x ¥ d0 -
{

I Ù_0^1 ¶8x, 2< H f ² n@xD Ù_0^x Ù_1^x f ² m@xD f ² n@xD â x â xL f m@xD â xM d²0
ÅÅ +
bg

2 j à_0^1 ¶8x, 1< J f ¢ m@xD à_1^x à_0^x f ¢ n@xD² â x â xN f m@xD â x ¥ w²2, n,
k {

8 a1,5 w1, m ® -2 à_0^1 f m@xD f ² m@xD f ² n@xD² â x - 2 à_0^1 f m@xD f ¢ n@xD f ² m@xD f H3L n@xD â x -

6 à_0^1 f m@xD f ¢ m@xD f ² n@xD f H3L n@xD â x - 2 à_0^1 f m@xD f ¢ m@xD f ¢ n@xD f H4L n@xD â x +

j 2 à_0^1 ¶8x, 1< J f H3L n@xD à_0^x f ¢ n@xD f ² m@xD â xN f m@xD â x - 2 à_0^1 ¶8x, 1< J f ² n@xD à_1^x f ² m@xD f ² n@xD â xN f m@xD
k

â x - 2 à_0^1 f m@xD f ² m@xD f ² n@xD² â x - 2 à_0^1 f m@xD f ¢ n@xD f ² m@xD f H3L n@xD â x -

6 à_0^1 f m@xD f ¢ m@xD f ² n@xD f H3L n@xD â x - 2 à_0^1 f m@xD f ¢ m@xD f ¢ n@xD f H4L n@xD â x ¥ d0 -
{

2 I Ù_0^1 ¶8x, 2< H f ² n@xD Ù_0^x Ù_1^x f ² m@xD f ² n@xD â x â xL f m@xD â xM d²0
ÅÅ >>
bg

```
HsymbolList@2D -> coef@@2DD •• Thread •• Union •• RestL •. int -> Integrate •• Timing
```

```
:22.282 Second, :8 a2,2 w2,n ® -2 à_0^1 fn@xD f″m@xD^2 f″n@xD â x -
    6 à_0^1 fn@xD f¢n@xD f″m@xD f^H3L_m@xD â x - 2 à_0^1 fn@xD f¢m@xD f″n@xD f^H3L_m@xD â x -
    2 à_0^1 fn@xD f¢m@xD f¢n@xD f^H4L_m@xD â x + j -2 à_0^1 ¶8x,1< J f^H3L_m@xD à_0^x f¢m@xD f″n@xD â xN fn@xD â x +
      2 à_0^1 ¶8x,1< J f″m@xD à_1^x f″m@xD f″n@xD â xN fn@xD â x z d0 -
    2 IÙ_0^1 ¶8x,2< Hf″m@xD Ù_0^x Ù_1^x f″m@xD f″n@xD â x â xL fn@xD â xM d0^2
    ─────────────────────────────────────────────────────────────────── , 8 a2,4 w2,n ®
                                  bg
   -à_0^1 fn@xD f″m@xD^2 f″n@xD â x - 3 à_0^1 fn@xD f¢n@xD f″m@xD f^H3L_m@xD â x - à_0^1 fn@xD f¢m@xD f″n@xD f^H3L_m@xD â x -
    à_0^1 fn@xD f¢m@xD f¢n@xD f^H4L_m@xD â x + j -à_0^1 ¶8x,1< J f^H3L_m@xD à_0^x f¢m@xD f″n@xD â xN fn@xD â x +
      à_0^1 ¶8x,1< J f″m@xD à_1^x f″m@xD f″n@xD â xN fn@xD â x z d0 -
    IÙ_0^1 ¶8x,2< Hf″m@xD Ù_0^x Ù_1^x f″m@xD f″n@xD â x â xL fn@xD â xM d0^2
    ─────────────────────────────────────────────────────────────────── +
                                  bg
    2 j à_0^1 ¶8x,1< J f¢n@xD à_1^x à_0^x f¢m@xD^2 â x â xN fn@xD â x z w1,m^2,
  8 a2,6 w2,n ® -3 à_0^1 fn@xD f″n@xD^3 â x - 12 à_0^1 fn@xD f¢n@xD f″n@xD f^H3L_n@xD â x -
    3 à_0^1 fn@xD f¢n@xD^2 f^H4L_n@xD â x + 2 j à_0^1 ¶8x,1< J f¢n@xD à_1^x à_0^x f¢n@xD^2 â x â xN fn@xD â x z w2,n^2>>
```

Modulation Equations in Polar Form

Using **eqMod** and the function **PolarForm** defined in the preceding section, we obtain the modulation equations in polar form and the definitions for g_i as

```
PolarForm@eqModD
```

```
::a¢1@T1D == 1/2 H-f Sin@g1@T1DD - 2 mm a1@T1D + 2 Sin@g2@T1DD a1,3 a1@T1D a2@T1D^2L,
  a¢2@T1D == -mn a2@T1D - Sin@g2@T1DD a2,4 a1@T1D^2 a2@T1D,
  g¢2@T1D == - d1 l^4_m / w1,m - f Cos@g1@T1DD / a1@T1D + 2 a1,1 a1@T1D^2 - 2 a2,2 a1@T1D^2 -
    2 Cos@g2@T1DD a2,4 a1@T1D^2 + 2 Cos@g2@T1DD a1,3 a2@T1D^2 + 2 a1,5 a2@T1D^2 - 2 a2,6 a2@T1D^2,
  g¢1@T1D == s - d1 l^4_m / (2 w1,m) - f Cos@g1@T1DD / (2 a1@T1D) + a1,1 a1@T1D^2 + Cos@g2@T1DD a1,3 a2@T1D^2 + a1,5 a2@T1D^2>,
 8g1@T1D == T1 s - b1@T1D, g2@T1D == -2 b1@T1D + 2 b2@T1D<>
```

Modulation Equations in Cartesian Form and Symmetry Property

Using **eqMod** and the function **CartesianForm** defined in the preceding section, we obtain the modulation equations in Cartesian form, the definitions for ν_i, and the symmetry property as

```
CartesianForm[eqMod]
```

$$\Big\{\Big\{p_1'[T_1] == -\mu_m\, p_1[T_1] - \nu_1\, q_1[T_1] + \frac{\delta_1 \lambda_m^4\, q_1[T_1]}{2\,\omega_{1,m}} - \alpha_{1,1}\, p_1[T_1]^2 q_1[T_1] + \alpha_{1,3}\, p_2[T_1]^2 q_1[T_1] - \alpha_{1,5}\, p_2[T_1]^2 q_1[T_1] - \alpha_{1,1}\, q_1[T_1]^3 - 2\,\alpha_{1,3}\, p_1[T_1]\, p_2[T_1]\, q_2[T_1] - \alpha_{1,3}\, q_1[T_1]\, q_2[T_1]^2 - \alpha_{1,5}\, q_1[T_1]\, q_2[T_1]^2,\ q_1'[T_1] ==$$

$$-\frac{f}{2} + \nu_1\, p_1[T_1] - \frac{\delta_1 \lambda_m^4\, p_1[T_1]}{2\,\omega_{1,m}} + \alpha_{1,1}\, p_1[T_1]^3 + \alpha_{1,3}\, p_1[T_1]\, p_2[T_1]^2 + \alpha_{1,5}\, p_1[T_1]\, p_2[T_1]^2 - \mu_m\, q_1[T_1] + \alpha_{1,1}\, p_1[T_1]\, q_1[T_1]^2 + 2\,\alpha_{1,3}\, p_2[T_1]\, q_1[T_1]\, q_2[T_1] - \alpha_{1,3}\, p_1[T_1]\, q_2[T_1]^2 + \alpha_{1,5}\, p_1[T_1]\, q_2[T_1]^2,$$

$$p_2'[T_1] == -\mu_n\, p_2[T_1] - 2\,\alpha_{2,4}\, p_1[T_1]\, p_2[T_1]\, q_1[T_1] - \nu_2\, q_2[T_1] - \alpha_{2,2}\, p_1[T_1]^2 q_2[T_1] + \alpha_{2,4}\, p_1[T_1]^2 q_2[T_1] - \alpha_{2,6}\, p_2[T_1]^2 q_2[T_1] - \alpha_{2,2}\, q_1[T_1]^2 q_2[T_1] - \alpha_{2,4}\, q_1[T_1]^2 q_2[T_1] - \alpha_{2,6}\, q_2[T_1]^3,$$

$$q_2'[T_1] == \nu_2\, p_2[T_1] + \alpha_{2,2}\, p_1[T_1]^2 p_2[T_1] + \alpha_{2,4}\, p_1[T_1]^2 p_2[T_1] + \alpha_{2,6}\, p_2[T_1]^3 + \alpha_{2,2}\, p_2[T_1]\, q_1[T_1]^2 - \alpha_{2,4}\, p_2[T_1]\, q_1[T_1]^2 - \mu_n\, q_2[T_1] + 2\,\alpha_{2,4}\, p_1[T_1]\, q_1[T_1]\, q_2[T_1] + \alpha_{2,6}\, p_2[T_1]\, q_2[T_1]^2\Big\},$$

$$\{\nu_1 \to \sigma,\ \nu_2 \to \sigma\},\ \{\{p_1, q_1, p_2, q_2\}, \{p_1, q_1, -p_2, -q_2\}\}\Big\}$$

7.3.3 Discretization of the Continuous Problem

As an alternative, we apply the method of multiple scales to the discretized system of **EOM** and **BC**. To determine the discretized form, we express $v[x, t]$ and $w[x, t]$ in terms of the linear mode shapes $\phi_m[x]$ and $\phi_n[x]$, respectively, as

```
discretRule = {v -> Function[{x, t}, q_m[t] φ_m[x]], w -> Function[{x, t}, q_n[t] φ_n[x]]};

modes = {φ_m[x], φ_n[x]};
```

where the $q_i[t]$ are the generalized coordinates. We consider only $q_m[t]$ and $q_n[t]$ because in the presence of damping, all other modes that are not directly or indirectly excited decay with time. Substituting **discretRule** into each of **EOM**, multiplying the results with $\phi_m[x]$ and $\phi_n[x]$, respectively, and integrating the outcome from $x = 0$ to $x = 1$ using **intRule3**, **intRule4**, **intRule1**, **intRule2**, and **intRule5**, we obtain the following discretized form of the equations describing the response of the beam:

```
eq733a =
 Table[int[modes[[k]] # /. discretRule, {x, 0, 1}] /. intRule3 //. intRule4 /. intRule1 //.
      intRule2 /. intRule5[modes[[k]]] //.
    int[a_, b__] -> HoldForm[Integrate[a, b]] & /@ EOM[[k]], {k, 2}]
```

$$\Big\{\Big(\int_0^1 \phi_m[x]\, \phi_m^{(4)}[x]\, dx\Big)\beta_y\, q_m[t] + \Big(\int_0^1 \phi_m[x]^2\, dx\Big)\ddot{q}_m[t] ==$$

$$-\epsilon \operatorname{Cos}[t\,\Omega] \int_0^1 F[x]\, \phi_m[x]\, dx - \epsilon \Big(\int_0^1 \phi_m[x]\, \phi_m''[x]^3\, dx\Big)\beta_y\, q_m[t]^3 -$$

$$4\,e\left(\int_0^1 f_m[x]\,f'_m[x]\,f''_m[x]\,f_m^{(3)}[x]\,dx\right) b_y\,q_m[t]^3 - e\left(\int_0^1 f_m[x]\,f'_m[x]^2\,f_m^{(4)}[x]\,dx\right) b_y\,q_m[t]^3 -$$

$$e\left(\int_0^1 \P\{x,1\}\left(f_n^{(3)}[x]\int_0^x f'_n[x]\,f''_m[x]\,dx\right) f_m[x]\,dx\right) q_m[t]\,q_n[t]^2 +$$

$$e\left(\int_0^1 \P\{x,1\}\left(f''_n[x]\int_1^x f''_m[x]\,f''_n[x]\,dx\right) f_m[x]\,dx\right) q_m[t]\,q_n[t]^2 +$$

$$e\left(\int_0^1 \P\{x,1\}\left(f_n^{(3)}[x]\int_0^x f'_n[x]\,f''_m[x]\,dx\right) f_m[x]\,dx\right) b_y\,q_m[t]\,q_n[t]^2 -$$

$$e\left(\int_0^1 \P\{x,1\}\left(f''_n[x]\int_1^x f''_m[x]\,f''_n[x]\,dx\right) f_m[x]\,dx\right) b_y\,q_m[t]\,q_n[t]^2 -$$

$$e\left(\int_0^1 f_m[x]\,f''_m[x]\,f''_n[x]^2\,dx\right) b_y\,q_m[t]\,q_n[t]^2 -$$

$$e\left(\int_0^1 f_m[x]\,f'_n[x]\,f''_m[x]\,f_n^{(3)}[x]\,dx\right) b_y\,q_m[t]\,q_n[t]^2 - 3\,e\left(\int_0^1 f_m[x]\,f'_m[x]\,f''_n[x]\,f_n^{(3)}[x]\,dx\right)$$

$$b_y\,q_m[t]\,q_n[t]^2 - e\left(\int_0^1 f_m[x]\,f'_m[x]\,f'_n[x]\,f_n^{(4)}[x]\,dx\right) b_y\,q_m[t]\,q_n[t]^2 -$$

$$\frac{e\left(\int_0^1 \P\{x,2\}\left(f''_n[x]\int_0^x\int_1^x f''_m[x]\,f''_n[x]\,dx\,dx\right) f_m[x]\,dx\right) q_m[t]\,q_n[t]^2}{b_g} +$$

$$\frac{2\,e\left(\int_0^1 \P\{x,2\}\left(f''_n[x]\int_0^x\int_1^x f''_m[x]\,f''_n[x]\,dx\,dx\right) f_m[x]\,dx\right) b_y\,q_m[t]\,q_n[t]^2}{b_g} -$$

$$\frac{e\left(\int_0^1 \P\{x,2\}\left(f''_n[x]\int_0^x\int_1^x f''_m[x]\,f''_n[x]\,dx\,dx\right) f_m[x]\,dx\right) b_y^2\,q_m[t]\,q_n[t]^2}{b_g} -$$

$$2\,e\left(\int_0^1 m[x]\,f_m[x]^2\,dx\right) q'_m[t] - e\left(\int_0^1 \P\{x,1\}\left(f'_m[x]\int_1^x\int_0^x f'_m[x]^2\,dx\,dx\right) f_m[x]\,dx\right) q_m[t]\,q'_m[t]^2 -$$

$$e\left(\int_0^1 \P\{x,1\}\left(f'_m[x]\int_1^x\int_0^x f'_n[x]^2\,dx\,dx\right) f_m[x]\,dx\right) q_m[t]\,q'_n[t]^2 -$$

$$e\left(\int_0^1 \P\{x,1\}\left(f'_m[x]\int_1^x\int_0^x f'_m[x]^2\,dx\,dx\right) f_m[x]\,dx\right) q_m[t]^2\,q''_m[t] -$$

$$e\left(\int_0^1 \P\{x,1\}\left(f'_m[x]\int_1^x\int_0^x f'_n[x]^2\,dx\,dx\right) f_m[x]\,dx\right) q_m[t]\,q_n[t]\,q''_n[t],$$

$$\left(\int_0^1 f_n[x]\,f_n^{(4)}[x]\,dx\right) q_n[t] + \left(\int_0^1 f_n[x]^2\,dx\right) q''_n[t] ==$$

$$e\left(\int_0^1 \P\{x,1\}\left(f_m^{(3)}[x]\int_0^x f'_m[x]\,f''_n[x]\,dx\right) f_n[x]\,dx\right) q_m[t]^2\,q_n[t] -$$

$$e\left(\int_0^1 \P\{x,1\}\left(f''_m[x]\int_1^x f''_m[x]\,f''_n[x]\,dx\right) f_n[x]\,dx\right) q_m[t]^2\,q_n[t] -$$

$$e\left(\int_0^1 f_n[x]\,f''_m[x]^2\,f''_n[x]\,dx\right) q_m[t]^2\,q_n[t] - 3\,e\left(\int_0^1 f_n[x]\,f'_n[x]\,f''_m[x]\,f_m^{(3)}[x]\,dx\right) q_m[t]^2\,q_n[t] -$$

$$e\left(\int_0^1 f_n[x]\,f'_m[x]\,f''_n[x]\,f_m^{(3)}[x]\,dx\right) q_m[t]^2\,q_n[t] -$$

$$e\left(\int_0^1 f_n[x]\,f'_m[x]\,f'_n[x]\,f_m^{(4)}[x]\,dx\right) q_m[t]^2\,q_n[t] -$$

$$e\left(\int_0^1 \P\{x,1\}\left(f_m^{(3)}[x]\int_0^x f'_m[x]\,f''_n[x]\,dx\right) f_n[x]\,dx\right) b_y\,q_m[t]^2\,q_n[t] +$$

```
e ∫₀¹ ¶8x,1< J f'm²@xD ∫₁ˣ f'm²@xD f'n²@xD â xN fn@xD â x bY qm@tD² qn@tD -

e I∫₀¹ ¶8x,2< H f'm²@xD ∫₀ˣ ∫₁ˣ f'm²@xD f'n²@xD â x â xL fn@xD â xM qm@tD² qn@tD
─────────────────────────────────────────────────────────────────── +
                              bg

2 e I∫₀¹ ¶8x,2< H f'm²@xD ∫₀ˣ ∫₁ˣ f'm²@xD f'n²@xD â x â xL fn@xD â xM bY qm@tD² qn@tD
──────────────────────────────────────────────────────────────────────── -
                              bg

e I∫₀¹ ¶8x,2< H f'm²@xD ∫₀ˣ ∫₁ˣ f'm²@xD f'n²@xD â x â xL fn@xD â xM bY² qm@tD² qn@tD
─────────────────────────────────────────────────────────────────────── -
                              bg

e ∫₀¹ fn@xD f'n²@xD³ â x qn@tD³ - 4 e ∫₀¹ fn@xD f¢n@xD f'n²@xD fn^H3L@xD â x qn@tD³ -

e ∫₀¹ fn@xD f¢n@xD² fn^H4L@xD â x qn@tD³ -

e ∫₀¹ ¶8x,1< J f¢n@xD ∫₁ˣ ∫₀ˣ f¢m@xD² â x â xN fn@xD â x qn@tD q¢m@tD² -

2 e ∫₀¹ m@xD fn@xD² â x q¢n@tD - e ∫₀¹ ¶8x,1< J f¢n@xD ∫₁ˣ ∫₀ˣ f¢n@xD² â x â xN fn@xD â x qn@tD q¢n@tD² -

e ∫₀¹ ¶8x,1< J f¢n@xD ∫₁ˣ ∫₀ˣ f¢m@xD² â x â xN fn@xD â x qm@tD qn@tD q'm²@tD -

e ∫₀¹ ¶8x,1< J f¢n@xD ∫₁ˣ ∫₀ˣ f¢n@xD² â x â xN fn@xD â x qn@tD² q'n²@tD>
```

where the damping is assumed to be modal. Using the orthonormality of the mode shapes f_i@xD, we define

```
notationRule = 9intAfi_@xD², 8x, 0, 1<E -> 1, intAfi_@xD fi_^H4L@xD, 8x, 0, 1<E -> li⁴,
   intAm@xD fi_@xD², 8x, 0, 1<E -> mi, int@F@xD fi_@xD, 8x, 0, 1<D -> w1,i f= •.
  int@a_, b__D -> HoldForm@Integrate@a, bDD
```

```
:∫₀¹ fi_@xD² â x ® 1, ∫₀¹ fi_@xD fi_^H4L@xD â x ® li⁴, ∫₀¹ m@xD fi_@xD² â x ® mi, ∫₀¹ F@xD fi_@xD â x ® f w1,i>
```

To simplify the expression for **eq733a**, we first find possible nonlinear terms as follows:

```
basicTerms = Table@D@qm@tD, 8t, i<D, 8i, 0, 2<D
```

```
8qm@tD, q¢m@tD, q'm²@tD<
```

```
linearTerms = Join@basicTerms, basicTerms •. m -> nD
```

```
8qm@tD, q¢m@tD, q'm²@tD, qn@tD, q¢n@tD, q'n²@tD<
```

```
cubicTerms = e Nest@Outer@Times, linearTerms, #D &, linearTerms, 2D •• Flatten •• Union;
```

The coefficients corresponding to **cubicTerms** in **eq733a** are obtained as

```
coefList = Coefficient@eq733a@@#, 2DD, cubicTermsD & •ž 81, 2<;
```

We replace these coefficients by $a_{i,\,j}$ as

```
symbolList = MapIndexed@If@#1 =!= 0, aSequence žž #2, 0D &, coefList, 82<D;
```

Using **notationRule**, **symbolList**, and **cubicTerms**, we can rewrite **eq733a** as

```
EOM1 = Table@eq733a@@k, 1DD == Heq733a@@k, 2DD •. Thread@cubicTerms -> 0DL +
    symbolList@@kDD.cubicTerms •. notationRule, 8k, 2<D
```

```
8by 1m^4 qm@tD + qm''@tD == -f e Cos@t WD w1,m + e a1,1 qm@tD^3 + e a1,3 qm@tD qn@tD^2 - 2 e mm qm'@tD +
   e a1,8 qm@tD qm'@tD^2 + e a1,17 qm@tD qn'@tD^2 + e a1,21 qm@tD^2 qm''@tD + e a1,37 qm@tD qn@tD qn''@tD,
 1n^4 qn@tD + qn''@tD == e a2,2 qm@tD^2 qn@tD + e a2,4 qn@tD^3 + e a2,9 qn@tD qm'@tD^2 -
   2 e mn qn'@tD + e a2,18 qn@tD qn'@tD^2 + e a2,22 qm@tD qn@tD qm''@tD + e a2,38 qn@tD^2 qn''@tD<
```

Using the method of multiple scales, we seek a first-order uniform expansion in the form

```
solRule = qn_ -> ikj Sum[e^j qn,j@#1, #2D, {j=0,1}] &yz{;
```

Transforming the total time derivatives in **EOM1** into partial derivatives in terms of T_0 and T_1, substituting the **solRule** and **betayRule** into **EOM1**, expanding the result for small e, and discarding terms of order higher than e, we obtain

```
eq733b = HEOM1 •. 8qs_@tD -> qs@T0, T1D, Derivative@n_D@qs_D@tD -> dt@nD@qs@T0, T1DD, t -> T0< •.
     solRule •. betayRule •• ExpandAllL •. e^(n_•;n>1) -> 0;
```

Equating coefficients of like powers of e in **eq733b**, we obtain

```
eqEps = Thread@CoefficientList@Subtract žž #, eD == 0D & •ž eq733b •• Transpose
```

```
:91m^4 qm,0@T0, T1D + d0 1m^4 qm,0@T0, T1D + qm,0^H2,0L@T0, T1D == 0, 1n^4 qn,0@T0, T1D + qn,0^H2,0L@T0, T1D == 0=,
 :f Cos@T0 WD w1,m + d1 1m^4 qm,0@T0, T1D - a1,1 qm,0@T0, T1D^3 + 1m^4 qm,1@T0, T1D + d0 1m^4 qm,1@T0, T1D -
   a1,3 qm,0@T0, T1D qn,0@T0, T1D^2 + 2 mm qm,0^H1,0L@T0, T1D - a1,8 qm,0@T0, T1D qm,0^H1,0L@T0, T1D^2 -
   a1,17 qm,0@T0, T1D qn,0^H1,0L@T0, T1D^2 + 2 qm,0^H1,1L@T0, T1D - a1,21 qm,0@T0, T1D^2 qm,0^H2,0L@T0, T1D +
   qm,1^H2,0L@T0, T1D - a1,37 qm,0@T0, T1D qn,0@T0, T1D qn,0^H2,0L@T0, T1D == 0,
  - a2,2 qm,0@T0, T1D^2 qn,0@T0, T1D - a2,4 qn,0@T0, T1D^3 + 1n^4 qn,1@T0, T1D -
   a2,9 qn,0@T0, T1D qm,0^H1,0L@T0, T1D^2 + 2 mn qn,0^H1,0L@T0, T1D - a2,18 qn,0@T0, T1D qn,0^H1,0L@T0, T1D^2 +
   2 qn,0^H1,1L@T0, T1D - a2,22 qm,0@T0, T1D qn,0@T0, T1D qm,0^H2,0L@T0, T1D -
   a2,38 qn,0@T0, T1D^2 qn,0^H2,0L@T0, T1D + qn,1^H2,0L@T0, T1D == 0>>
```

Then it follows from **eqEps[[1]]** that

```
sol0 = 9qm,0 -> FunctionA8T0, T1<, A1@T1D Exp@I w1,m T0D + Ā1@T1D Exp@-I w1,m T0DE,
   qn,0 -> FunctionA8T0, T1<, A2@T1D Exp@I w2,n T0D + Ā2@T1D Exp@-I w2,n T0DE=;
```

Substituting **sol0** into **eqEps[[2]]** yields

```
order1Eq = H#@@1DD & •ž eqEps@@1DD •. qk_,0 -> qk,1L == H#@@1DD & •ž eqEps@@1DD •. qk_,0 -> qk,1L -
     H#@@1DD & •ž eqEps@@2DDL •. sol0 •• TrigToExp •• ExpandAll •• Thread;
```

We consider the case of primary resonance of the flexural mode in the *y* direction, $W » w_{1,m}$. To describe the nearness of the primary resonance, we introduce the detuning parameter s and define

```
omgList = 8w1,m, w2,n<;
```

```
ResonanceConds = 8w2,n == w1,m, W == w1,m + e s<;
```

```
OmgRule = Solve@ResonanceConds, Drop@omgList, 8#<D ~Join~ 8W<D@@1DD & •ž 81, 2<
```

```
88w2,n ® w1,m, W ® e s + w1,m<, 8w1,m ® w2,n, W ® e s + w2,n<<
```

```
expRule@i_D := Exp@arg_D :> Exp@Expand@arg •. OmgRule@@iDDD •. e T0 -> T1D
```

We substitute **ResonanceConds** into the right-hand sides of **order1Eq** and obtain the solvability condition as

```
SCond =
  Table@Coefficient@-order1Eq@@i, 2DD •. expRule@iD, Exp@I omgList@@iDD T0DD == 0, 8i, 2<D;
SCond •. displayRule
```

$$9A_1\, d_1\, l_m^4 - 3 A_1^2 \dot{A}_1\, a_{1,1} - A_2^2 \dot{A}_1\, a_{1,3} - 2 A_1 A_2 \dot{A}_2\, a_{1,3} + \frac{1}{2} E^{I T_1 s} f\, w_{1,m} + 2 I\, HD_1A_1L\, w_{1,m} + 2 I A_1 m_m\, w_{1,m} - A_1^2 \dot{A}_1\, a_{1,8}\, w_{1,m}^2 + 3 A_1^2 \dot{A}_1\, a_{1,21}\, w_{1,m}^2 + A_2^2 \dot{A}_1\, a_{1,17}\, w_{2,n}^2 - 2 A_1 A_2 \dot{A}_2\, a_{1,17}\, w_{2,n}^2 + A_2^2 \dot{A}_1\, a_{1,37}\, w_{2,n}^2 + 2 A_1 A_2 \dot{A}_2\, a_{1,37}\, w_{2,n}^2 == 0,$$

$$-2 A_1 A_2 \dot{A}_1\, a_{2,2} - A_1^2 \dot{A}_2\, a_{2,2} - 3 A_2^2 \dot{A}_2\, a_{2,4} - 2 A_1 A_2 \dot{A}_1\, a_{2,9}\, w_{1,m}^2 + A_1^2 \dot{A}_2\, a_{2,9}\, w_{1,m}^2 + 2 A_1 A_2 \dot{A}_1\, a_{2,22}\, w_{1,m}^2 + A_1^2 \dot{A}_2\, a_{2,22}\, w_{1,m}^2 + 2 I\, HD_1A_2L\, w_{2,n} + 2 I A_2 m_n\, w_{2,n} - A_2^2 \dot{A}_2\, a_{2,18}\, w_{2,n}^2 + 3 A_2^2 \dot{A}_2\, a_{2,38}\, w_{2,n}^2 == 0=$$

where

 # -> H# •. a$_{i_,j_}$:> coefList@@i, jDD •. betayRule •. e -> 0 •• Expand •• Collect@#, d$_0$D &L & •ž HsymbolList •• Flatten •• Union •• RestL

```
:a1,1 ® -à_0^1 fm@xD f²m@xD³ â x - 4 à_0^1 fm@xD f¢m@xD f²m@xD fmH3L@xD â x - à_0^1 fm@xD f¢m@xD² fmH4L@xD â x +
   j -à_0^1 fm@xD f²m@xD³ â x - 4 à_0^1 fm@xD f¢m@xD f²m@xD fmH3L@xD â x - à_0^1 fm@xD f¢m@xD² fmH4L@xD â xž d0,
   k                                                                                                          {
a1,3 ® -à_0^1 fm@xD f²m@xD f²n@xD² â x - à_0^1 fm@xD f¢n@xD f²m@xD fnH3L@xD â x -
   3 à_0^1 fm@xD f¢m@xD f²n@xD fnH3L@xD â x - à_0^1 fm@xD f¢m@xD f¢n@xD fnH4L@xD â x +
   jà_0^1 ¶8x,1< JfnH3L@xD à_0^x f¢n@xD f²m@xD â xN fm@xD â x - à_0^1 ¶8x,1< Jf²n@xD à_1^x f²m@xD f²n@xD â xN fm@xD â x -
   k
     à_0^1 fm@xD f²m@xD f²n@xD² â x - à_0^1 fm@xD f¢n@xD f²m@xD fnH3L@xD â x -
     3 à_0^1 fm@xD f¢m@xD f²n@xD fnH3L@xD â x - à_0^1 fm@xD f¢m@xD f¢n@xD fnH4L@xD â xž d0 -
                                                                                       {
   IÙ_0^1 ¶8x,2< Hf²n@xD Ù_0^x Ù_1^x f²m@xD f²n@xD â x â xL fm@xD â xM d0²
   ¢¢¢¢¢¢¢¢¢¢¢¢¢¢¢¢¢¢¢¢¢¢¢¢¢¢¢¢¢¢¢¢¢¢¢¢¢¢¢¢¢¢¢¢¢¢¢¢¢¢¢¢¢¢¢¢¢¢¢¢¢¢¢¢¢¢¢¢¢¢¢¢¢¢ ,
                              bg
a1,8 ® -à_0^1 ¶8x,1< Jf¢m@xD à_1^x à_0^x f¢m@xD² â x â xN fm@xD â x,
a1,17 ® -à_0^1 ¶8x,1< Jf¢m@xD à_1^x à_0^x f¢n@xD² â x â xN fm@xD â x,
a1,21 ® -à_0^1 ¶8x,1< Jf¢m@xD à_1^x à_0^x f¢m@xD² â x â xN fm@xD â x,
a1,37 ® -à_0^1 ¶8x,1< Jf¢m@xD à_1^x à_0^x f¢n@xD² â x â xN fm@xD â x,
a2,2 ® -à_0^1 fn@xD f²m@xD² f²n@xD â x - 3 à_0^1 fn@xD f¢n@xD f²m@xD fmH3L@xD â x -
   à_0^1 fn@xD f¢m@xD f²n@xD fmH3L@xD â x - à_0^1 fn@xD f¢m@xD f¢n@xD fmH4L@xD â x +
   j -à_0^1 ¶8x,1< JfmH3L@xD à_0^x f¢m@xD f²n@xD â xN fn@xD â x + à_0^1 ¶8x,1< Jf²m@xD à_1^x f²m@xD f²n@xD â xN fn@xD â xž
   k                                                                                                                       {
        IÙ_0^1 ¶8x,2< Hf²m@xD Ù_0^x Ù_1^x f²m@xD f²n@xD â x â xL fn@xD â xM d0²
   d0 - ¢¢¢¢¢¢¢¢¢¢¢¢¢¢¢¢¢¢¢¢¢¢¢¢¢¢¢¢¢¢¢¢¢¢¢¢¢¢¢¢¢¢¢¢¢¢¢¢¢¢¢¢¢¢¢¢¢¢¢¢¢¢¢¢¢¢¢¢¢¢¢¢ ,
                                   bg
a2,4 ® -à_0^1 fn@xD f²n@xD³ â x - 4 à_0^1 fn@xD f¢n@xD f²n@xD fnH3L@xD â x - à_0^1 fn@xD f¢n@xD² fnH4L@xD â x,
a2,9 ® -à_0^1 ¶8x,1< Jf¢n@xD à_1^x à_0^x f¢m@xD² â x â xN fn@xD â x,
a2,18 ® -à_0^1 ¶8x,1< Jf¢n@xD à_1^x à_0^x f¢n@xD² â x â xN fn@xD â x,
a2,22 ® -à_0^1 ¶8x,1< Jf¢n@xD à_1^x à_0^x f¢m@xD² â x â xN fn@xD â x,
a2,38 ® -à_0^1 ¶8x,1< Jf¢n@xD à_1^x à_0^x f¢n@xD² â x â xN fn@xD â x>
```

which is in agreement with that obtained by direct approach.

Ÿ 7.3.4 Method of Time-Averaged Lagrangian

As a second alternative, we derive the modulation equations by using the method of time-averaged Lagrangian. The nondimensional Lagrangian of the beam can be expressed as (Crespo da Silva and Glynn, 1978; Arafat, Nayfeh, and Chin, 1998)

```
Lagr1 =
  HoldA‡_0^1 ikjj1/2 ikjikj∂_t ‡_0^x 1/2 IH∂_x v@x, tDL^2 + H∂_x w@x, tDL^2M âxy{^2 + H∂_t v@x, tDL^2 + H∂_t w@x, tDL^2y{ -
       e H1 - b_yL J∂_{x,x} v@x, tD ∂_{x,x} w@x, tD ‡_0^x ∂_{x,x} v@x, tD ∂_x w@x, tD âxN -
       1/2 IH∂_{x,x} w@x, tDL^2 + e H∂_x w@x, tDL^2 H∂_{x,x} w@x, tDL^2M - 1/2 b_y IH∂_{x,x} v@x, tDL^2 +
          e H∂_x v@x, tDL^2 H∂_{x,x} v@x, tDL^2 + e 2 ∂_x v@x, tD ∂_{x,x} v@x, tD ∂_x w@x, tD ∂_{x,x} w@x, tDM -
       e (H1 - b_yL^2)/(2 b_g) ikjJ‡_1^x ∂_{x,x} v@x, tD ∂_{x,x} w@x, tD âxN^2 + 2 ∂_{x,x} v@x, tD ∂_{x,x} w@x, tD
          ‡_0^x ‡_1^x ∂_{x,x} v@x, tD ∂_{x,x} w@x, tD âx âxy{ - e F@xD Cos@W tD v@x, tDy{ âxE •.
    Integrate -> int •. HoldPattern@∂_t int@a_, b_DD :> int@∂_t a, bD •• ReleaseHold;
```

where **Hold** and **int** are used to keep **Integrate** from being evaluated. Transforming the total time derivatives into partial derivatives in terms of T_0 and T_1, we modify **Lagr1** as

```
Lagr2 = Lagr1 •. 8v@x_, tD -> v@x, T_0, T_1D, w@x_, tD -> w@x, T_0, T_1D,
    Derivative@m_, n_D@w_D@x_, tD -> dt@nD@D@w@x, T_0, T_1D, 8x, m<DD, t -> T_0<;
```

To this end, we let

```
solRule = 9v -> FunctionA8x, T_0, T_1<, A_1@T_1D f_m@xD Exp@I w_{1,m} T_0D + Ā_1@T_1D f_m@xD Exp@-I w_{1,m} T_0DE,
   w -> FunctionA8x, T_0, T_1<, A_2@T_1D f_n@xD Exp@I w_{2,n} T_0D + Ā_2@T_1D f_n@xD Exp@-I w_{2,n} T_0DE=;
```

and define

```
intRule6 = 8int@c1_. int@a1__D^n_., b1_D :>
    c1 int@int@a1D^n •. intRule1 ••. intRule2 •. e -> 0, b1D •; FreeQ@c1, First@b1DD,
   int@c1_ int@a1__D^n_., b1_D :> int@c1 int@a1D^n •. intRule1 ••. intRule2 •. e -> 0, b1D<;
```

Substituting **solRule** and **betayRule** into **Lagr2**, using **intRule1**, **intRule2**, and **intRule6** to simplify the expansions of the integrands, and collecting the coefficients of e, we obtain

```
Horder1Lagr =
    Lagr2 •. solRule •. betayRule •. intRule1 ••. intRule2 ••. intRule6 •. intRule1 ••.
          intRule2 ••. intRule6 •. intRule1 ••. intRule2 ••
       TrigToExp •• Expand •• Coefficient@#, eD &;L •• Timing
```

```
814.951 Second, Null<
```

```
ResonanceConds = 8w2,n == w1,m , W == w1,m + e s< •. Equal -> Rule
```

```
8w2,n ® w1,m, W ® e s + w1,m<
```

```
notationRule1 = 9intAfi_@xD2, 8x, 0, 1<E -> 1,
  int@F@xD fi_@xD, 8x, 0, 1<D -> f w1,i, intAfi_2@xD2, 8x, 0, 1<E -> li4=;
```

Using **ResonanceConds** and **notationRule1**, we obtain the slowly varying terms from **order1Lagr** as

```
TAL = order1Lagr •. Exp@a_D :> Exp@Expand@a •. ResonanceCondsD •. e T0 -> T1D •.
    Exp@_ T0 + _.D -> 0 •. notationRule1;
```

The Euler-Lagrange equations corresponding to the **TAL** can be written as

```
eqMod1 =
 DADATAL, A1•¢@T1DE, T1E - DATAL, A1•@T1DE == 0 ••. int@a_, b__D -> HoldForm@Integrate@a, bDD
```

```
1
€€€ EI T1 s f w1,m + d1 l4m A1@T1D + 6 ià^1 f¢m@xD2 f²m@xD2 â xy A1@T1D2 A•1@T1D +
2                                     k 0               z {

  6 ià^1 f¢m@xD2 f²m@xD2 â xy d0 A1@T1D2 A•1@T1D - 2 ià^1 Jà^x f¢m@xD2 â xN^2 â xy w²1,m A1@T1D2 A•1@T1D +
    k 0                  z {                          k 0   0                    z {

  2 ià^1 f¢m@xD f¢n@xD f²m@xD f²n@xD â xy A2@T1D2 A•1@T1D -
    k 0                                z {

  2 ià^1 Jà^x f¢n@xD f²m@xD â xN f²m@xD f²n@xD â xy d0 A2@T1D2 A•1@T1D +
    k 0   0                                     z {

  2 ià^1 f¢m@xD f¢n@xD f²m@xD f²n@xD â xy d0 A2@T1D2 A•1@T1D +
    k 0                                z {

   IÙ0^1 HÙ1^x f²m@xD f²n@xD â xL^2 â xM d0² A2@T1D2 A•1@T1D
  €€€€€€€€€€€€€€€€€€€€€€€€€€€€€€€€€€€€€€€€€€€€€€€€€€€€€€€€€€€€€€€€€€€ +
                                  bg

   2 IÙ0^1 HÙ0^x Ù1^x f²m@xD f²n@xD â x â xL f²m@xD f²n@xD â xM d0² A2@T1D2 A•1@T1D
  €€€€€€€€€€€€€€€€€€€€€€€€€€€€€€€€€€€€€€€€€€€€€€€€€€€€€€€€€€€€€€€€€€€€€€€€€€€€€€€€€€€€€€€€€€€€€€€€€ -
                                          bg

  2 ià^1 Jà^x f¢m@xD2 â xN à^x f¢n@xD2 â x â xy w1,m w2,n A2@T1D2 A•1@T1D +
    k 0   0               0                  z {

  4 ià^1 f¢m@xD f¢n@xD f²m@xD f²n@xD â xy A1@T1D A2@T1D A•2@T1D -
    k 0                                z {

  4 ià^1 Jà^x f¢n@xD f²m@xD â xN f²m@xD f²n@xD â xy d0 A1@T1D A2@T1D A•2@T1D +
    k 0   0                                     z {

  4 ià^1 f¢m@xD f¢n@xD f²m@xD f²n@xD â xy d0 A1@T1D A2@T1D A•2@T1D +
    k 0                                z {

   2 IÙ0^1 HÙ1^x f²m@xD f²n@xD â xL^2 â xM d0² A1@T1D A2@T1D A•2@T1D
  €€€€€€€€€€€€€€€€€€€€€€€€€€€€€€€€€€€€€€€€€€€€€€€€€€€€€€€€€€€€€€€€€€€€€€€€€€€€€€€€ +
                                   bg

   4 IÙ0^1 HÙ0^x Ù1^x f²m@xD f²n@xD â x â xL f²m@xD f²n@xD â xM d0² A1@T1D A2@T1D A•2@T1D
  €€€€€€€€€€€€€€€€€€€€€€€€€€€€€€€€€€€€€€€€€€€€€€€€€€€€€€€€€€€€€€€€€€€€€€€€€€€€€€€€€€€€€€€€€€€€€€€€€€€€€€€€ + 2 I w1,m A¢1@T1D == 0
                                              bg
```

and

```
eqMod2 =
 D[D[L, A2'[T1]], T1] - D[L, A2[T1]] == 0 //. int[a_, b__] -> HoldForm[Integrate[a, b]]
```

$$4\left(\int_0^1 f'_m(x)\, f'_n(x)\, f''_m(x)\, f''_n(x)\,dx\right) A_1(T_1)\, A_2(T_1)\, \dot A_1(T_1) -$$

$$4\left(\int_0^1 \left(\int_0^x f'_n(x)\, f''_m(x)\,dx\right) f''_m(x)\, f''_n(x)\,dx\right) d_0\, A_1(T_1)\, A_2(T_1)\, \dot A_1(T_1) +$$

$$4\left(\int_0^1 f'_m(x)\, f'_n(x)\, f''_m(x)\, f''_n(x)\,dx\right) d_0\, A_1(T_1)\, A_2(T_1)\, \dot A_1(T_1) +$$

$$\frac{2\left(\int_0^1 \left(\int_1^x f''_m(x)\, f''_n(x)\,dx\right)^2 dx\right) d_0^2\, A_1(T_1)\, A_2(T_1)\, \dot A_1(T_1)}{\beta_\gamma} +$$

$$\frac{4\left(\int_0^1 \left(\int_0^x \int_1^x f''_m(x)\, f''_n(x)\,dx\,dx\right) f''_m(x)\, f''_n(x)\,dx\right) d_0^2\, A_1(T_1)\, A_2(T_1)\, \dot A_1(T_1)}{\beta_\gamma} +$$

$$2\left(\int_0^1 f'_m(x)\, f'_n(x)\, f''_m(x)\, f''_n(x)\,dx\right) A_1(T_1)^2\, \dot A_2(T_1) -$$

$$2\left(\int_0^1 \left(\int_0^x f'_n(x)\, f''_m(x)\,dx\right) f''_m(x)\, f''_n(x)\,dx\right) d_0\, A_1(T_1)^2\, \dot A_2(T_1) +$$

$$2\left(\int_0^1 f'_m(x)\, f'_n(x)\, f''_m(x)\, f''_n(x)\,dx\right) d_0\, A_1(T_1)^2\, \dot A_2(T_1) +$$

$$\frac{\left(\int_0^1 \left(\int_1^x f''_m(x)\, f''_n(x)\,dx\right)^2 dx\right) d_0^2\, A_1(T_1)^2\, \dot A_2(T_1)}{\beta_\gamma} +$$

$$\frac{2\left(\int_0^1 \left(\int_0^x \int_1^x f''_m(x)\, f''_n(x)\,dx\,dx\right) f''_m(x)\, f''_n(x)\,dx\right) d_0^2\, A_1(T_1)^2\, \dot A_2(T_1)}{\beta_\gamma} -$$

$$2\left(\int_0^1 \left(\int_0^x f'_m(x)^2\,dx\right) \int_0^x f'_n(x)^2\,dx\,dx\right) \omega_{1,m}\, \omega_{2,n}\, A_1(T_1)^2\, \dot A_2(T_1) +$$

$$6\left(\int_0^1 f'_n(x)^2\, f''_n(x)^2\,dx\right) A_2(T_1)^2\, \dot A_2(T_1) -$$

$$2\left(\int_0^1 \left(\int_0^x f'_n(x)^2\,dx\right)^2 dx\right) \omega_{2,n}^2\, A_2(T_1)^2\, \dot A_2(T_1) + 2\, i\, \omega_{2,n}\, A'_2(T_1) == 0$$

Adding linear viscous damping to **eqMod1** and **eqMod2** and performing integration by parts yields the same modulation equations as obtained by directly attacking the partial-differential system.

Chapter 8
Continuous Systems with Quadratic and Cubic Nonlinearities

In this chapter, we use the method of multiple scales to determine second-order uniform asymptotic solutions of continuous or distributed-parameter systems with quadratic and cubic nonlinearities. We consider some of the internal resonances arising from the cubic nonlinearities, namely, one-to-one and three-to-one internal resonances. Two-to-one internal resonances are treated in Chapter 9. We directly attack these continuous systems because treatment of reduced-order models obtained with the Galerkin procedure might lead to erroneous results. We consider buckled beams in Section 8.1, circular cylindrical shells in Section 8.2, and near-square plates in Section 8.3.

à Preliminaries

```
Off@General::spell1, Integrate::generD
```

```
Needs@"Utilities`Notation`"D
```

To determine second-order uniform asymptotic expansions of the solutions of buckled beams, circular cylindrical shells, and near-square plates by using the method of multiple scales, we introduce the three time scales $T_0 = t$, $T_1 = \epsilon t$, and $T_2 = \epsilon^2 t$, where ϵ is a small nondimensional parameter and the order of the amplitude of oscillations. Moreover, we symbolize these scales according to

```
Symbolize@T0D; Symbolize@T1D; Symbolize@T2D;
```

```
timeScales = 8T0, T1, T2<;
```

In terms of the time scales T_0, T_1, and T_2, the time derivatives can be expressed as

```
dt@0D@expr_D := expr; dt@1D@expr_D := SumAe^i D@expr, timeScales@@i + 1DDD, 8i, 0, 2<E;
dt@2D@expr_D := Hdt@1D@dt@1D@exprDD •• ExpandL •. e^(i_•;i>3) -> 0;
```

In the course of the analysis, we need the complex conjugates of A and other variables. We define them by using the following rule:

```
conjugateRule = 9A -> Ā, Ā -> A, Complex@m_, n_D -> Complex@m, -nD=;
```

To speed up manipulations of the integrals involved in the governing equations with *Mathematica*, we introduce the rules

```
intRule1 = 8int@fun_, arg2__D :> int@Expand@funD, arg2D<;
intRule2 = 8int@a_ + b_, arg2__D :> int@a, arg2D + int@b, arg2D,
   int@e^n_. fun_, arg2__D :> e^n int@fun, arg2D,
   int@a_ fun_, a1_, b1___D :> a int@fun, a1, b1D •; FreeQ@a, First@a1DD,
   int@int@a1_, a2_D fun_., a3__D :> int@a1, a2D int@fun, a3D •; FreeQ@Rest@a2D, First@a2DD<;
```

To display the outputs in easily read expressions, we introduce the rules

```
displayRule = 9Derivative@a_, b__DAw_i_E@_, T0, __D :>
    SequenceFormAIfAarg1 = Times žž MapIndexedAD#1#2@@1DD-1 &, 8b<E; arg1 =!= 1, arg1, ""E,
     wi^SequenceForm žž Table@"'",8a<DE,
   Derivative@a__D@Ai_D@__D :> SequenceFormATimes žž MapIndexedAD#1#2@@1DD &, 8a<E, AiE,
   Derivative@a__D@Ȧi_D@__D :> SequenceFormATimes žž MapIndexedAD#1#2@@1DD &, 8a<E, ȦiE,
   w_i_@_, T0, __D -> wi, Ai_@__D -> Ai, Ȧi_@__D -> Ȧi,
   Exp@a_. + b_. Complex@0, m_D T0 + c_. Complex@0, n_D T0D -> Exp@a + Hm * b + n * cL I T0D=;
```

à 8.1 Buckled Beams

In this section, we consider the nonlinear transverse response of a buckled beam possessing either a one-to-one or a three-to-one internal resonance to a principal parametric resonance of the higher mode. The analysis assumes a unimodal buckled deflection.

The equation governing nonlinear transverse vibrations of a parametrically excited homogeneous beam is (Nayfeh and Mook, 1979)

```
eq81a =
   m wt,t + c wt + Es Is wx,x,x,x + jP - fp Cos@W tD - (Es A)/(2 {) à_0^{ Hwx L^2 â xz wx,x == 0 •. wm__ :> ¶m w@x, tD;
```

where P is a static axial load, f_p Cos@W tD is an axial harmonic excitation, w@x, tD is the transverse deflection of the beam, x is the distance along the undeflected beam, and t is time. Other dimensional parameters are the damping coefficient c and the beam length {, mass per unit length m, cross-sectional area A, modulus of elasticity E_s, and cross-sectional area moment of inertia I_s. We note that the subscript s is used with I and E so that *Mathematica* will not confuse them with $\sqrt{-1}$ and the exponential function . Equation **eq81a** is supplemented with the four linear homogeneous boundary conditions

```
bc81a = 8B1@w@0, tDD == 0, B2@w@0, tDD == 0, B3@w@{, tDD == 0, B4@w@{, tDD == 0<;
```

where the B_i are linear homogeneous operators.

It is good practice to cast the governing equations and boundary conditions in nondimensional form. Thus, we introduce the following nondimensional variables:

```
T = $ (m {^4)/(Es Is) ;
```

```
nondimRule = 9w -> H{ w@#1 • {, #2 • TD &L, x -> {x, t -> T t, W -> W • T,
    P -> P Es Is • {^2, fp -> f Es Is • {^2, c -> c • !!!!!!!!! Es Is m ' {^2, A -> 2 a Is • {^2=;
```

where the tilde indicates a nondimensional variable. Substituting **nondimRule** into **eq81a** and dropping the tildes, we obtain

```
eq81b = J {^3 / (k Es Is) eq81a@@1DD •. Integrate -> int •. nondimRule •. s_ -> s •.
        int@a_, 8x {, __<D -> { int@a, 8x, 0, 1<D •• Expand •• PowerExpandN == 0;
```

```
eq81b •.
 int ->
  Integrate
```

```
c w^H0,1L@x, tD + w^H0,2L@x, tD + P w^H2,0L@x, tD -
  f Cos@t WD w^H2,0L@x, tD - a J∫_0^1 w^H1,0L@x, tD^2 â xN w^H2,0L@x, tD + w^H4,0L@x, tD == 0
```

Ÿ 8.1.1 Postbuckling Deflection

The first step in analyzing the nonlinear vibrations of a buckled beam is the determination of the critical Euler buckling loads P_k and their corresponding mode shapes $\phi_k[x]$. We symbolize the P_k as

```
Symbolize@PkD;
```

Dropping the time derivative and neglecting the forcing and nonlinear terms in **eq81b**, we define the linear buckling problem as

```
buckEq = eq81b •. int -> H0 &L •. 8P -> Pk, f -> 0, w -> Hfk@#1D &L<
```

```
Pk f''k@xD + f^H4L_k@xD == 0
```

Solving **buckEq** for $\phi_k^{(4)}[x]$ yields

```
buckRule = SolveAbuckEq, f^H4L_k@xDE@@1DD
```

```
9f^H4L_k@xD ® -Pk f''k@xD=
```

Next, we increase the axial load P beyond P_k and assume that the postbuckling displacement is $b\,\phi_k$, where b is a nondimensional measure of the buckling level. Substituting this displacement into **eq81b**, dropping the forcing term and the time derivative, and using **intRule2** and **buckRule**, we obtain

```
bEq = Factor@eq81b@@1DD •. 8f -> 0, w -> Hb fk@#1D &L< •. intRule2 •. buckRuleD == 0
```

```
-b H-P + Pk + b^2 a int@f'k@xD^2, 8x, 0, 1<DL f''k@xD == 0
```

It follows from **bEq** that either $b = 0$ (unbuckled case) or b is given by

```
(bRule = b^2 -> (b^2 /. Solve[bEq, b][[2]])) /. int -> Integrate
```

$$b^2 \to \frac{P - P_k}{\alpha \int_0^1 \phi_k'[x]^2\, dx}$$

And solving **bEq** for P yields

```
(PRule = Solve[bEq, P][[1]]) /. int -> Integrate
```

$$\left\{P \to P_k + b^2 \alpha \int_0^1 \phi_k'[x]^2\, dx\right\}$$

Because P is greater than P_k, **PRule** yields the amplitude of the kth buckling mode.

Having solved the postbuckling problem, we assume that the beam deflection is the sum of the static buckled displacement $b\, \phi_k[x]$, corresponding to the kth buckled mode, and a time-dependent relative deflection $u[x, t]$; that is,

```
wRule = {w -> (b φk[#1] + u[#1, #2] &)};
```

Substituting **wRule** into **eq81b** and using **intRule1**, **intRule2**, **buckRule**, and **PRule**, we find that the nonlinear transverse vibrations of the beam around its buckled configuration are given by

```
EOM = eq81b /. wRule /. intRule1 //. intRule2 /. buckRule /. PRule // ExpandAll;
EOM /. int -> Integrate
```

$$-b\, f \cos[t\, \Omega]\, \phi_k''[x] - 2 b^2 \alpha \left(\int_0^1 \phi_k'[x]\, u^{(1,0)}[x, t]\, dx\right) \phi_k''[x] - b\, \alpha \left(\int_0^1 u^{(1,0)}[x, t]^2\, dx\right) \phi_k''[x] + c\, u^{(0,1)}[x, t] + u^{(0,2)}[x, t] + P_k\, u^{(2,0)}[x, t] - f \cos[t\, \Omega]\, u^{(2,0)}[x, t] - 2 b\, \alpha \left(\int_0^1 \phi_k'[x]\, u^{(1,0)}[x, t]\, dx\right) u^{(2,0)}[x, t] - \alpha \left(\int_0^1 u^{(1,0)}[x, t]^2\, dx\right) u^{(2,0)}[x, t] + u^{(4,0)}[x, t] == 0$$

```
BC = bc81a /. {w -> u, ℓ -> 1}
```

$$\{B_1[u[0, t]] == 0,\ B_2[u[0, t]] == 0,\ B_3[u[1, t]] == 0,\ B_4[u[1, t]] == 0\}$$

In the following analysis, we focus on hinged-hinged, first mode-buckled beams. Then, the boundary conditions for the static deflection $\phi_k[x]$ can be written as

```
bBC = {φk[0] == 0, φk''[0] == 0, φk[1] == 0, φk''[1] == 0};
```

and the boundary conditions for the dynamic deflection $u[x, t]$ can be written as

```
BC1 = {u[0, t] == 0, ∂x,x u[x, t] == 0 /. x -> 0, u[1, t] == 0, ∂x,x u[x, t] == 0 /. x -> 1}
```

$$\{u[0, t] == 0,\ u^{(2,0)}[0, t] == 0,\ u[1, t] == 0,\ u^{(2,0)}[1, t] == 0\}$$

For the case of first buckled mode, it follows from **buckEq** and **bBC** that the first critical buckling load and associated mode-shape are

```
bucklingLoad = Pk -> p2;
modeshape = 8fk -> HSin@p #D &L<;
```

where the buckling modeshape was normalized so that f_k@1 • 2D = 1.

Ÿ Linear Vibrations

The next step in analyzing the nonlinear vibrations of a buckled beam is the calculation of its linear natural frequencies and corresponding modeshapes. To this end, we linearize **EOM** in *u*@*x*, *t*D, drop the forcing, damping, and nonlinear terms, and obtain

```
linearEq = EOM •. 8c -> 0, f -> 0< •. u -> He u@#1, #2D &L •. intRule2 •. en_•;n>1 -> 0 •. e -> 1;
linearEq •. int -> Integrate
```

```
-2 b2 a ‡à 1 0 f¢k@xD uH1,0L@x, tD â x¥ f²k@xD + uH0,2L@x, tD + Pk uH2,0L@x, tD + uH4,0L@x, tD == 0
```

The boundary conditions are the same as **BC1**, which we list below as

```
linearBC = BC1
```

```
8u@0, tD == 0, uH2,0L@0, tD == 0, u@1, tD == 0, uH2,0L@1, tD == 0<
```

Then, using separation of variables, we seek the solution of **linearEq** and **linearBC** in the form

```
uSol = u -> Function@8x, t<, F@xD Exp@I w tDD;
```

where w is the nondimensional linear natural frequency and F@*x*D is the corresponding modeshape. Substituting **uSol** into **linearEq** and **linearBC** and using **intRule2** and **Thread**, we obtain the equation governing F@*x*D as

```
FEq = linearEq •. uSol •. intRule2 •• ThreadAE-I w t #, EqualE & •• ExpandAll;
FEq •. int -> Integrate
```

```
-w2 F@xD + Pk F²@xD - 2 b2 a ‡à 1 0 F¢@xD f¢k@xD â x¥ f²k@xD + FH4L@xD == 0
```

and the boundary conditions as

```
FBC = linearBC •. u -> HF@#1D &L
```

```
8F@0D == 0, F²@0D == 0, F@1D == 0, F²@1D == 0<
```

To find the general solution of **FEq** and **FBC**, we follow Nayfeh, Kreider, and Anderson (1995) and treat the definite integral as a constant *B*; thus, we have a nonhomogeneous linear ordinary-differential equation. Hence, its general solution can be expressed as a linear combination of homogeneous and particular solutions. Because f_k@*x*D is a solution of **buckEq**, the particular solution is proportional to f_k²@*x*D. Consequently, we express the general solution of **FEq** and **FBC** as

```
cList = Table@ci, 8i, 5<D;
```

```
shapeFunc = 8Cos@l1 xD, Sin@l1 xD, Cosh@l2 xD, Sinh@l2 xD, fk²@xD<;
```

```
FSol = F -> Function@x, cList.shapeFunc •. modeshape •• EvaluateD
```

```
F ® Function@x, Cos@x l1D c1 + Sin@x l1D c2 + Cosh@x l2D c3 + Sinh@x l2D c4 - p^2 Sin@p xD c5D
```

where the l_i satisfy the characteristic equation

```
charEq =
 FEq •. b -> 0 •. bucklingLoad •. F -> HExp@l #D &L •• ThreadAE^(-l x) #, EqualE & •• ExpandAll
```

```
p^2 l^2 + l^4 - w^2 == 0
```

```
lRule = 9- l1^2, l2^2= -> l l^2 •. Solve@charEq, lD •• UnionM •• Thread
```

```
:- l1^2 ® 1/2 (-p^2 - Sqrt[p^4 + 4 w^2]), l2^2 ® 1/2 (-p^2 + Sqrt[p^4 + 4 w^2])>
```

Substituting **bucklingLoad**, **modeshape**, and **FSol** into the left-hand sides of **FBC** and **FEq** and using **intRule1** and **intRule2**, we obtain

```
expr1 = Append@#@@1DD & •ž FBC •. FSol,
  FEq@@1DD •. bucklingLoad •. modeshape •. FSol •. intRule1 ••. intRule2 •. int -> IntegrateD
```

```
:c1 + c3, -c1 l1^2 + c3 l2^2, Cos@l1D c1 + Sin@l1D c2 + Cosh@l2D c3 + Sinh@l2D c4,
 -Cos@l1D c1 l1^2 - Sin@l1D c2 l1^2 + Cosh@l2D c3 l2^2 + Sinh@l2D c4 l2^2,
 -p^6 Sin@p xD c5 - w^2 HCos@x l1D c1 + Sin@x l1D c2 + Cosh@x l2D c3 + Sinh@x l2D c4 - p^2 Sin@p xD c5L +
  Cos@x l1D c1 l1^4 + Sin@x l1D c2 l1^4 + Cosh@x l2D c3 l2^4 + Sinh@x l2D c4 l2^4 +
  p^2 Hp^4 Sin@p xD c5 - Cos@x l1D c1 l1^2 - Sin@x l1D c2 l1^2 + Cosh@x l2D c3 l2^2 + Sinh@x l2D c4 l2^2L +
  2 b^2 p^2 a Sin@p xD (- 1/2 p^4 c5 + (p Sin@l1D c2 l1^2)/(Hp - l1L Hp + l1L) - p c1 l1 (- l1/(Hp - l1L Hp + l1L) - (Cos@l1D l1)/(Hp - l1L Hp + l1L)) -
    (p Sinh@l2D c4 l2^2)/(Hp - I l2L Hp + I l2L) + p c3 l2 (- l2/(Hp - I l2L Hp + I l2L) - (Cosh@l2D l2)/(Hp - I l2L Hp + I l2L)))>
```

The coefficient matrix of **cList** in **expr1** can be obtained as

```
coefMat = Outer@Coefficient, expr1, cListD
```

```
:81, 0, 1, 0, 0<, 8-l1^2, 0, l2^2, 0, 0<, 8Cos@l1D, Sin@l1D, Cosh@l2D, Sinh@l2D, 0<,
 8-Cos@l1D l1^2, -Sin@l1D l1^2, Cosh@l2D l2^2, Sinh@l2D l2^2, 0<, :-w^2 Cos@x l1D -
   p^2 Cos@x l1D l1^2 + Cos@x l1D l1^4 + (2 b^2 p^3 a Sin@p xD l1^2)/(Hp - l1L Hp + l1L) + (2 b^2 p^3 a Cos@l1D Sin@p xD l1^2)/(Hp - l1L Hp + l1L),
  -w^2 Sin@x l1D - p^2 Sin@x l1D l1^2 + Sin@x l1D l1^4 + (2 b^2 p^3 a Sin@p xD Sin@l1D l1^2)/(Hp - l1L Hp + l1L), -w^2 Cosh@x l2D +
   p^2 Cosh@x l2D l2^2 - (2 b^2 p^3 a Sin@p xD l2^2)/(Hp - I l2L Hp + I l2L) - (2 b^2 p^3 a Cosh@l2D Sin@p xD l2^2)/(Hp - I l2L Hp + I l2L) + Cosh@x l2D l2^4,
  -w^2 Sinh@x l2D + p^2 Sinh@x l2D l2^2 - (2 b^2 p^3 a Sin@p xD Sinh@l2D l2^2)/(Hp - I l2L Hp + I l2L) + Sinh@x l2D l2^4,
  -b^2 p^6 a Sin@p xD + p^2 w^2 Sin@p xD>>
```

Substituting **charEq** into **coefMat[[5]]** and eliminating the common factor Sin@p *x*D, we have

```
coefMat@@5DD = Coefficient@coefMat@@5DD, Sin@p xDD
```

```
:(2 b^2 p^3 a l1^2)/(Hp - l1L Hp + l1L) + (2 b^2 p^3 a Cos@l1D l1^2)/(Hp - l1L Hp + l1L), (2 b^2 p^3 a Sin@l1D l1^2)/(Hp - l1L Hp + l1L),
 -(2 b^2 p^3 a l2^2)/(Hp - I l2L Hp + I l2L) - (2 b^2 p^3 a Cosh@l2D l2^2)/(Hp - I l2L Hp + I l2L), -(2 b^2 p^3 a Sinh@l2D l2^2)/(Hp - I l2L Hp + I l2L), -b^2 p^6 a + p^2 w^2>
```

Setting the determinant of **coefMat** equal to zero yields the following characteristic equation for the natural frequencies:

```
Factor@Det@coefMatDD == 0
```

```
p^2 Hb^2 p^4 a - w^2L Sin@l1D Sinh@l2D Hl1^2 + l2^2L^2 == 0
```

There are two possibilities: b^2 p^4 a - w^2 = 0 and **Sin**@l$_1$D = 0. Hence, either

```
wSol1 = SolveAb^2 p^4 a - w^2 == 0, wE@@2DD
```

```
9w ® b p^2 √a=
```

or

```
wSol2 = SolveA- l1^2 + Hn pL^2 == 0 •. lRule, wE@@2DD
```

```
:w ® n √(-1 + n^2) p^2>
```

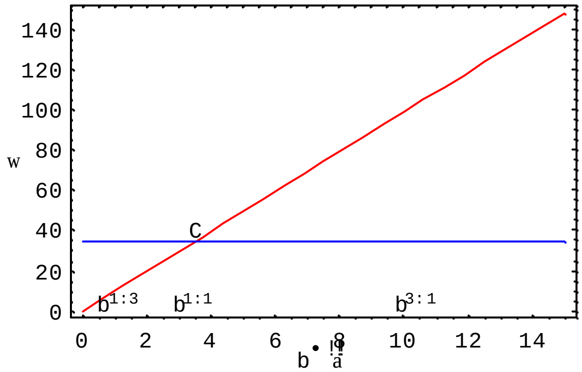
for $n > 1$. The lowest five natural frequencies are

```
frequencies = 8w1 -> Hw •. wSol1L, w2 -> Hw •. wSol2 •. n -> 2L,
  w3 -> Hw •. wSol2 •. n -> 3L, w4 -> Hw •. wSol2 •. n -> 4L, w5 -> Hw •. wSol2 •. n -> 5L<
```

```
9w1 ® b p^2 √a, w2 ® 2 √3 p^2, w3 ® 6 √2 p^2, w4 ® 4 √15 p^2, w5 ® 10 √6 p^2=
```

The first two frequencies vs. the buckling level are plotted as follows:

```
PlotAw •. 9wSol1 •. b • !!! a -> s, wSol2 •. n -> 2= •• Evaluate,
  8s, 0, 15<, PlotStyle -> 8RGBColor@1, 0, 0D, RGBColor@0, 0, 1D<,
  Frame -> True, FrameLabel -> 9"b • !!! a", "w"=, RotateLabel -> False,
  Epilog -> 9TextA"C", 92 • !!! 3, 2 • !! 3 p^2 + 5=E, TextA"b^1:3", 92 ' • !! 3, 3=E,
   TextA"b^1:1", 92 • !! 3, 3=E, TextA"b^3:1", 96 • !!! 3, 3=E=E;
```



where the low-frequency mode (red line) is symmetric and the second mode (blue line) is antisymmetric. As the buckling level increases from zero, the frequency ω_1 of the first mode increases from zero and crosses the frequency ω_2 of the second mode at point C. Hence, there are two possible three-to-one internal resonances: $\omega_2 = 3\omega_1$ when $b/\sqrt{\alpha} = b^{1:3}$ and $\omega_1 = 3\omega_2$ when $b^{3:1}$. There is also a possible one-to-one internal resonance $\omega_2 = \omega_1$ when $b/\sqrt{\alpha} = b^{1:1}$. Whether or not these candidates for internal resonances will be activated depends on the corresponding modeshapes. In this section, we consider the case of one-to-one internal resonance between the first and second modes. In Section 8.1.4, we consider three-to-one internal resonances between the first two modes. In Sections 8.1.5 and 8.1.6, we consider one-to-one and three-to-one internal resonances between the first and third modes and the first and fouth modes, respectively.

In the next section, we consider a combination of a one-to-one internal resonance between the lowest two modes and a principal parametric resonance of the second mode; that is,

```
omgList = 8w1, w2<;
ResonanceConds = 9w2 == w1 + e^2 s1, W == 2 w2 + e^2 s2=;
```

where the σ_i are detuning parameters that describe the nearness of the resonances. The buckling level for which the natural frequencies of the first and second modes are equal (i.e., $\omega_2 = \omega_1$) is given by

```
values1 = Solve@w1 == w2 •. frequencies, bD@@1DD
```

$$\left\{b \to \frac{2\sqrt{3}}{\sqrt{\alpha}}\right\}$$

The corresponding λ_m are

```
lRuleN = Solve@lRule •. w -> w1 •. frequencies •. values1 •. Rule -> Equal, 8l1, l2<D@@4DD
```

$$\{\lambda_1 \to 2\pi,\ \lambda_2 \to \sqrt{3}\,\pi\}$$

In order that the influence of the damping and nonlinearity balance the influence of the forcing, we scale the damping and forcing terms as

```
scaling = 9c -> 2 e^2 m , f -> e^2 f=;
```

Substituting for the natural frequency w_1 back into **coefMat**, we determine the modeshape as

```
NullSpace@coefMat •. w -> w1 •. frequenciesD@@1DD.shapeFunc •. modeshape
```

```
- p^2 Sin@p xD
```

which we normalize as

```
shape1 = c Sin@p xD •. SolveA∫_0^1 c^2 Sin@p xD^2 âx == 1, cE@@2DD
```

```
• √2 Sin@p xD
```

Substituting w_2 into **coefMat**, we determine the modeshape as

```
NullSpace@coefMat •. lRuleN •. w -> w2 •. frequenciesD@@1DD.shapeFunc •. lRuleN
```

```
Sin@2 p xD
```

which we also normalize as

```
shape2 = c Sin@2 p xD •. SolveA∫_0^1 c^2 Sin@2 p xD^2 âx == 1, cE@@2DD
```

```
• √2 Sin@2 p xD
```

In a similar fashion, we find that the next three normalized modeshapes are

```
shape3 = • √2 Sin@3 p xD;

shape4 = • √2 Sin@4 p xD;

shape5 = • √2 Sin@5 p xD;
```

To treat a one-to-one internal resonance between the first and second modes, we define f_k and the first two eigenmodes of the buckled beam as

```
modeshapes = Join@modeshape,
   8F1 -> Function@x, shape1 •• EvaluateD, F2 -> Function@x, shape2 •• EvaluateD<D;
```

Ÿ 8.1.2 Perturbation Analysis

We use the method of multiple scales to directly attack **EOM** and **BC1**. To transform the time derivatives in **EOM** in terms of the scales T_0, T_1, and T_2, we let

```
multiScales = 8u@x_, tD -> u@x, T0, T1, T2D,
   Derivative@m_, n_D@u_D@x_, tD -> dt@nD@D@u@x, T0, T1, T2D, 8x, m<DD, t -> T0<;
```

Then, we seek a second-order approximate solution in the form

```
solRule = u -> I EvaluateASumAe^j uj@#1, #2, #3, #4D, 8j, 3<EE &M
```

```
u ® He u1@#1, #2, #3, #4D + e^2 u2@#1, #2, #3, #4D + e^3 u3@#1, #2, #3, #4D &L
```

where #1 stands for x and #2, #3, and #4 stand for T_0, T_1, and T_2, respectively.

Substituting **multiScales**, **solRule**, and **scaling** into **EOM** and **BC1**, expanding the result for small e, and discarding terms of order higher than e^3, we obtain

```
eq81c = HJoin@8EOM<, BC1D •. multiScales •. solRule •. scaling •. intRule1 ••. intRule2 ••
     TrigToExp •• ExpandAllL •. e^(n_•;n>3) -> 0;
```

Equating coefficients of like powers of e, we obtain

```
eqEps = Thread@CoefficientList@Subtract žž #, eD == 0D & •ž eq81c •• Transpose •• Rest;
eqEps •. displayRule
```

```
98Pk Hu1''L + u1'''' + D0^2 u1 - 2 b^2 a int@Hu1'L fk¢@xD, 8x, 0, 1<D fk²@xD == 0,
  u1 == 0, u1'' == 0, u1 == 0, u1'' == 0<,
 9-2 b a int@Hu1'L fk¢@xD, 8x, 0, 1<D Hu1''L + Pk Hu2''L + u2'''' + D0^2 u2 + 2 HD0 D1 u1L - (1/2) b E^(-I T0 W) f fk²@xD -
   (1/2) b E^(I T0 W) f fk²@xD - b a intBHu1'L^2, 8x, 0, 1<F fk²@xD -
   2 b^2 a int@Hu2'L fk¢@xD, 8x, 0, 1<D fk²@xD == 0, u2 == 0, u2'' == 0, u2 == 0, u2'' == 0=,
 9-(1/2) E^(-I T0 W) f Hu1''L - (1/2) E^(I T0 W) f Hu1''L - a intBHu1'L^2, 8x, 0, 1<F Hu1''L -
   2 b a int@Hu2'L fk¢@xD, 8x, 0, 1<D Hu1''L - 2 b a int@Hu1'L fk¢@xD, 8x, 0, 1<D Hu2''L + Pk Hu3''L +
   u3'''' + 2 m HD0 u1L + D0^2 u3 + 2 HD0 D1 u2L + D1^2 u1 + 2 HD0 D2 u1L - 2 b a int@Hu1'L Hu2'L, 8x, 0, 1<D fk²@xD -
   2 b^2 a int@Hu3'L fk¢@xD, 8x, 0, 1<D fk²@xD == 0, u3 == 0, u3'' == 0, u3 == 0, u3'' == 0==
```

Ÿ First-Order Solution

Because in the presence of damping all of the modes that are not directly or indirectly excited decay with time, the solution of **eqEps[[1]]** is taken to consist of the two modes that might be involved in the internal resonance; that is,

```
sol1 = 9u1 -> FunctionA8x, T0, T1, T2<,
     SumAAi@T1, T2D Fi@xD Exp@I wi T0D + Ȧi@T1, T2D Fi@xD Exp@-I wi T0D, 8i, 2<E •• EvaluateE=;
```

where the F_i are the eigenmodes, which satisfy the orthonormality condition $\int_0^1 F_n(x) F_m(x)\, dx = d_{n,m}$.

Ÿ Second-Order Solution

Substituting **sol1** into **eqEps[[2,1]]** and using **intRule1** and **intRule2** yields

```
order2Eq =
 HeqEps@@1, 1, 1DD •. u1 -> u2L == HeqEps@@1, 1, 1DD •. u1 -> u2L - eqEps@@2, 1, 1DD •. sol1 •.
    intRule1 ••. intRule2 •• ExpandAll; order2Eq •. displayRule
```

```
Pk Hu2''L + u2'''' + D0^2 u2 - 2 b^2 a int@Hu2'L fk¢@xD, 8x, 0, 1<D fk²@xD ==
 - 2 I E^(I T0 w1) HD1A1L w1 F1@xD + 2 I E^(-I T0 w1) HD1Ȧ1L w1 F1@xD -
  2 I E^(I T0 w2) HD1A2L w2 F2@xD + 2 I E^(-I T0 w2) HD1Ȧ2L w2 F2@xD + 1/2 f b E^(-I T0 W) f fk²@xD +
  1/2 f b E^(I T0 W) f fk²@xD + b E^(2 I T0 w1) a int@F1¢@xD^2, 8x, 0, 1<D A1^2 fk²@xD +
  2 b E^(I T0 Hw1+w2L) a int@F1¢@xD F2¢@xD, 8x, 0, 1<D A1 A2 fk²@xD +
  b E^(2 I T0 w2) a int@F2¢@xD^2, 8x, 0, 1<D A2^2 fk²@xD + 2 b a int@F1¢@xD^2, 8x, 0, 1<D A1 Ȧ1 fk²@xD +
  2 b E^(I T0 H-w1+w2L) a int@F1¢@xD F2¢@xD, 8x, 0, 1<D A2 Ȧ1 fk²@xD +
  b E^(-2 I T0 w1) a int@F1¢@xD^2, 8x, 0, 1<D Ȧ1^2 fk²@xD +
  2 b E^(I T0 Hw1-w2L) a int@F1¢@xD F2¢@xD, 8x, 0, 1<D A1 Ȧ2 fk²@xD + 2 b a int@F2¢@xD^2, 8x, 0, 1<D A2 Ȧ2 fk²@xD +
  2 b E^(I T0 H-w1-w2L) a int@F1¢@xD F2¢@xD, 8x, 0, 1<D Ȧ1 Ȧ2 fk²@xD +
  b E^(-2 I T0 w2) a int@F2¢@xD^2, 8x, 0, 1<D Ȧ2^2 fk²@xD + 2 b E^(2 I T0 w1) a int@fk¢@xD F1¢@xD, 8x, 0, 1<D A1^2 F1²@xD +
  2 b E^(I T0 Hw1+w2L) a int@fk¢@xD F2¢@xD, 8x, 0, 1<D A1 A2 F1²@xD +
  4 b a int@fk¢@xD F1¢@xD, 8x, 0, 1<D A1 Ȧ1 F1²@xD +
  2 b E^(I T0 H-w1+w2L) a int@fk¢@xD F2¢@xD, 8x, 0, 1<D A2 Ȧ1 F1²@xD +
  2 b E^(-2 I T0 w1) a int@fk¢@xD F1¢@xD, 8x, 0, 1<D Ȧ1^2 F1²@xD +
  2 b E^(I T0 Hw1-w2L) a int@fk¢@xD F2¢@xD, 8x, 0, 1<D A1 Ȧ2 F1²@xD +
  2 b E^(I T0 H-w1-w2L) a int@fk¢@xD F2¢@xD, 8x, 0, 1<D Ȧ1 Ȧ2 F1²@xD +
  2 b E^(I T0 Hw1+w2L) a int@fk¢@xD F1¢@xD, 8x, 0, 1<D A1 A2 F2²@xD +
  2 b E^(2 I T0 w2) a int@fk¢@xD F2¢@xD, 8x, 0, 1<D A2^2 F2²@xD +
  2 b E^(I T0 H-w1+w2L) a int@fk¢@xD F1¢@xD, 8x, 0, 1<D A2 Ȧ1 F2²@xD +
  2 b E^(I T0 Hw1-w2L) a int@fk¢@xD F1¢@xD, 8x, 0, 1<D A1 Ȧ2 F2²@xD +
  4 b a int@fk¢@xD F2¢@xD, 8x, 0, 1<D A2 Ȧ2 F2²@xD +
  2 b E^(I T0 H-w1-w2L) a int@fk¢@xD F1¢@xD, 8x, 0, 1<D Ȧ1 Ȧ2 F2²@xD +
  2 b E^(-2 I T0 w2) a int@fk¢@xD F2¢@xD, 8x, 0, 1<D Ȧ2^2 F2²@xD
```

To collect the terms that might lead to secular terms from the right-hand side of **order2Eq**, we define the rules:

```
OmgRule = Solve@ResonanceConds, Complement@omgList, 8#<D ~Join~ 8W<D@@1DD & •ž omgList
```

```
88w2 ® e^2 s1 + w1, W ® 2 e^2 s1 + e^2 s2 + 2 w1<, 8w1 ® - e^2 s1 + w2, W ® e^2 s2 + 2 w2<<
```

```
expRule1@i_D := Exp@arg_D :> ExpAExpand@arg •. OmgRule@@iDDD •. e^2 T0 -> T2E
```

Collecting the terms that may lead to secular terms, the terms proportional to $E^{I\,w_i\,T_0}$, we have

```
ST1 = CoefficientAorder2Eq@@2DD •. expRule1@#D, E^(I w# T0)E & •ž 81, 2<;
ST1 •. displayRule
```

```
8-2 I HD1A1L w1 F1@xD - 2 I E^(I T2 s1) HD1A2L w2 F2@xD, -2 I E^(-I T2 s1) HD1A1L w1 F1@xD - 2 I HD1A2L w2 F2@xD<
```

Because the homogeneous parts of **order2Eq** and corresponding boundary conditions have a nontrivial solution, the corresponding nonhomogeneous problem has a solution only if solvability conditions are satisfied. These conditions demand that **ST1** be orthogonal to every solution of the adjoint homogeneous problem. In this case, the problem is self-adjoint and hence solutions of the adjoint problem are given by the F_i@xD. Demanding that **ST1[[i]]** be orthogonal to the F_i@xD, we obtain the solvability conditions

```
SCond1 = TableAà_0^1 ST1@@kDD Fk@xD âx == 0, 8k, 2<E •. modeshapes
```

```
9-2 I w1 A1^H1,0L@T1, T2D == 0, -2 I w2 A2^H1,0L@T1, T2D == 0=
```

```
SCond1Rule = SolveASCond1, 9A1^H1,0L@T1, T2D, A2^H1,0L@T1, T2D=E@@1DD
```

```
9A1^H1,0L@T1, T2D ® 0, A2^H1,0L@T1, T2D ® 0=
```

whose complex conjugate is

```
ccSCond1Rule = SCond1Rule •. conjugateRule
```

```
:Ȧ1^H1,0L@T1, T2D ® 0, Ȧ2^H1,0L@T1, T2D ® 0>
```

Substituting **SCond1Rule** and **ccSCond1Rule** into **order2Eq** and using **modeshapes**, we have

```
order2Eqm = order2Eq •. SCond1Rule •. ccSCond1Rule •. modeshapes •. int -> Integrate •.
   Integrate -> int;
order2Eqm •. displayRule
```

```
Pk Hu2''L + u2'''' + D0^2u2 + 2 b^2 p^3 a int@Cos@p xD Hu2'L, 8x, 0, 1<D Sin@p xD ==
 - 1/2 b E^(-I T0 W) f p^2 Sin@p xD - 1/2 b E^(I T0 W) f p^2 Sin@p xD - 3 b E^(2 I T0 w1) p^4 a Sin@p xD A1^2 -
  8 b E^(I T0 Hw1+w2L) p^4 a Sin@2 p xD A1 A2 - 4 b E^(2 I T0 w2) p^4 a Sin@p xD A2^2 -
  6 b p^4 a Sin@p xD A1 Ȧ1 - 8 b E^(I T0 H-w1+w2L) p^4 a Sin@2 p xD A2 Ȧ1 -
  3 b E^(-2 I T0 w1) p^4 a Sin@p xD Ȧ1^2 - 8 b E^(I T0 Hw1-w2L) p^4 a Sin@2 p xD A1 Ȧ2 -
  8 b p^4 a Sin@p xD A2 Ȧ2 - 8 b E^(I T0 H-w1-w2L) p^4 a Sin@2 p xD Ȧ1 Ȧ2 - 4 b E^(-2 I T0 w2) p^4 a Sin@p xD Ȧ2^2
```

The associated boundary conditions are

```
order2BC = eqEps@@2DD •• Rest
```

```
9u2@0, T0, T1, T2D == 0, u2^H2,0,0,0L@0, T0, T1, T2D == 0,
 u2@1, T0, T1, T2D == 0, u2^H2,0,0,0L@1, T0, T1, T2D == 0=
```

We use the method of undetermined coefficients to determine a particular solution of **order2Eqm** and **order2BC**. To accomplish this, we first determine the forms of the nonhomogeneous terms (i.e., the terms on the right-hand side of **order2Eqm**) as

```
rhsTerms = Cases@order2Eqm@@2DD, #D & •ž 9a_ f_@b_ xD :> f@b xD •; FreeQ@a, T0D,
      _ E^(a_ T0+b_)· f_@c_ xD -> E^(a T0+b) f@c xD= •• Flatten •• Union
```

```
8Sin@p xD, E^(-I T0 W) Sin@p xD, E^(I T0 W) Sin@p xD, E^(-2 I T0 w1) Sin@p xD,
 E^(2 I T0 w1) Sin@p xD, E^(-2 I T0 w2) Sin@p xD, E^(2 I T0 w2) Sin@p xD, E^(-I T0 w1 - I T0 w2) Sin@2 p xD,
 E^(I T0 w1 - I T0 w2) Sin@2 p xD, E^(-I T0 w1 + I T0 w2) Sin@2 p xD, E^(I T0 w1 + I T0 w2) Sin@2 p xD<
```

which all satisfy the **order2BC**. Since we have only even spatial derivatives on the left-hand side of **order2Eqm**, we seek a particular solution as a linear combination of the **rhsTerms** as follows:

```
symbolList = Table@a_i, 8i, Length@rhsTermsD<D
```

```
8a1, a2, a3, a4, a5, a6, a7, a8, a9, a10, a11<
```

```
sol2Form = u2 -> Function@8x, T0, T1, T2<, symbolList.rhsTerms •• EvaluateD;
```

Substituting **sol2Form** into **order2Eqm**, equating coefficients of like terms, and solving for the **symbolList**, we have

```
symbolRule =
 Solve@Coefficient@Subtract žž order2Eqm •. sol2Form •. intRule1 ••. intRule2 •. int ->
        Integrate, rhsTermsD == 0 •. Exp@_ T0 + _.D -> 0 •• Thread, symbolListD@@1DD
```

$$:a_1 \to -\frac{2\,(3\,b\,p^2\,a\,A_1[T_1,T_2]\,\dot{A}_1[T_1,T_2]+4\,b\,p^2\,a\,A_2[T_1,T_2]\,\dot{A}_2[T_1,T_2])}{p^2-P_k+b^2\,p^2\,a},$$

$$a_2 \to -\frac{b\,f\,p^2}{2\,(p^4-p^2\,P_k+b^2\,p^4\,a-W^2)},\quad a_3 \to -\frac{b\,f\,p^2}{2\,(p^4-p^2\,P_k+b^2\,p^4\,a-W^2)},$$

$$a_4 \to -\frac{3\,b\,p^4\,a\,\dot{A}_1[T_1,T_2]^2}{p^4-p^2\,P_k+b^2\,p^4\,a-4\,w_1^2},\quad a_5 \to -\frac{3\,b\,p^4\,a\,A_1[T_1,T_2]^2}{p^4-p^2\,P_k+b^2\,p^4\,a-4\,w_1^2},$$

$$a_6 \to -\frac{4\,b\,p^4\,a\,\dot{A}_2[T_1,T_2]^2}{p^4-p^2\,P_k+b^2\,p^4\,a-4\,w_2^2},\quad a_7 \to -\frac{4\,b\,p^4\,a\,A_2[T_1,T_2]^2}{p^4-p^2\,P_k+b^2\,p^4\,a-4\,w_2^2},$$

$$a_8 \to -\frac{8\,b\,p^4\,a\,\dot{A}_1[T_1,T_2]\,\dot{A}_2[T_1,T_2]}{16\,p^4-4\,p^2\,P_k-w_1^2-2\,w_1\,w_2-w_2^2},\quad a_9 \to -\frac{8\,b\,p^4\,a\,A_1[T_1,T_2]\,\dot{A}_2[T_1,T_2]}{16\,p^4-4\,p^2\,P_k-w_1^2+2\,w_1\,w_2-w_2^2},$$

$$a_{10} \to -\frac{8\,b\,p^4\,a\,A_2[T_1,T_2]\,\dot{A}_1[T_1,T_2]}{16\,p^4-4\,p^2\,P_k-w_1^2+2\,w_1\,w_2-w_2^2},\quad a_{11} \to -\frac{8\,b\,p^4\,a\,A_1[T_1,T_2]\,A_2[T_1,T_2]}{16\,p^4-4\,p^2\,P_k-w_1^2-2\,w_1\,w_2-w_2^2}>$$

Substituting for the parameter values in **symbolRule** yields

```
symbolRuleN = symbolRule /. Ω -> 2 ω2 /. frequencies /. bucklingLoad /. values1
```

$$\left\{a_1 \to -\frac{6\sqrt{3}\,\pi^2\sqrt{\alpha}\,A_1[T_1,T_2]\,\bar{A}_1[T_1,T_2] + 8\sqrt{3}\,\pi^2\sqrt{\alpha}\,A_2[T_1,T_2]\,\bar{A}_2[T_1,T_2]}{6\pi^2},\ a_2 \to \frac{f}{12\sqrt{3}\,\pi^2\sqrt{\alpha}},\right.$$

$$a_3 \to \frac{f}{12\sqrt{3}\,\pi^2\sqrt{\alpha}},\ a_4 \to \frac{\sqrt{\alpha}\,A_1[T_1,T_2]^2}{2\sqrt{3}},\ a_5 \to \frac{\sqrt{\alpha}\,\bar{A}_1[T_1,T_2]^2}{2\sqrt{3}},\ a_6 \to \frac{2\sqrt{\alpha}\,A_2[T_1,T_2]^2}{3\sqrt{3}},$$

$$a_7 \to \frac{2\sqrt{\alpha}\,\bar{A}_2[T_1,T_2]^2}{3\sqrt{3}},\ a_8 \to \frac{4\sqrt{\alpha}\,\bar{A}_1[T_1,T_2]\,\bar{A}_2[T_1,T_2]}{3\sqrt{3}},\ a_9 \to -\frac{4\sqrt{\alpha}\,A_1[T_1,T_2]\,\bar{A}_2[T_1,T_2]}{\sqrt{3}},$$

$$\left.a_{10} \to -\frac{4\sqrt{\alpha}\,A_2[T_1,T_2]\,\bar{A}_1[T_1,T_2]}{\sqrt{3}},\ a_{11} \to \frac{4\sqrt{\alpha}\,A_1[T_1,T_2]\,A_2[T_1,T_2]}{3\sqrt{3}}\right\}$$

Substituting **symbolRuleN** into **sol2Form** yields

```
sol2 = sol2Form /. symbolRuleN
```

$$u_2 \to \mathrm{Function}\Big[\{x, T_0, T_1, T_2\},$$

$$\frac{\mathrm{Sin}[\pi x]\left(-\left(6\sqrt{3}\,\pi^2\sqrt{\alpha}\,A_1[T_1,T_2]\,\bar{A}_1[T_1,T_2] + 8\sqrt{3}\,\pi^2\sqrt{\alpha}\,A_2[T_1,T_2]\,\bar{A}_2[T_1,T_2]\right)\right)}{6\pi^2} +$$

$$\frac{E^{-I T_0 \Omega}\,\mathrm{Sin}[\pi x]\,f}{12\sqrt{3}\,\pi^2\sqrt{\alpha}} + \frac{E^{I T_0 \Omega}\,\mathrm{Sin}[\pi x]\,f}{12\sqrt{3}\,\pi^2\sqrt{\alpha}} + \frac{E^{-2 I T_0 \omega_1}\,\mathrm{Sin}[\pi x]\left(\sqrt{\alpha}\,\bar{A}_1[T_1,T_2]^2\right)}{2\sqrt{3}} +$$

$$\frac{E^{2 I T_0 \omega_1}\,\mathrm{Sin}[\pi x]\left(\sqrt{\alpha}\,A_1[T_1,T_2]^2\right)}{2\sqrt{3}} + \frac{E^{-2 I T_0 \omega_2}\,\mathrm{Sin}[\pi x]\left(2\sqrt{\alpha}\,\bar{A}_2[T_1,T_2]^2\right)}{3\sqrt{3}} +$$

$$\frac{E^{2 I T_0 \omega_2}\,\mathrm{Sin}[\pi x]\left(2\sqrt{\alpha}\,A_2[T_1,T_2]^2\right)}{3\sqrt{3}} + \frac{E^{-I T_0 \omega_1 - I T_0 \omega_2}\,\mathrm{Sin}[2\pi x]\left(4\sqrt{\alpha}\,\bar{A}_1[T_1,T_2]\,\bar{A}_2[T_1,T_2]\right)}{3\sqrt{3}} +$$

$$\frac{E^{I T_0 \omega_1 - I T_0 \omega_2}\,\mathrm{Sin}[2\pi x]\left(-4\sqrt{\alpha}\,A_1[T_1,T_2]\,\bar{A}_2[T_1,T_2]\right)}{\sqrt{3}} +$$

$$\frac{E^{-I T_0 \omega_1 + I T_0 \omega_2}\,\mathrm{Sin}[2\pi x]\left(-4\sqrt{\alpha}\,A_2[T_1,T_2]\,\bar{A}_1[T_1,T_2]\right)}{\sqrt{3}} +$$

$$\frac{E^{I T_0 \omega_1 + I T_0 \omega_2}\,\mathrm{Sin}[2\pi x]\left(4\sqrt{\alpha}\,A_1[T_1,T_2]\,A_2[T_1,T_2]\right)}{3\sqrt{3}}\Big]$$

Solvability Conditions

Substituting **sol1** and **sol2** into **eqEps[[3,1]]** and using **intRule1** and **intRule2** yields

```
order3Eq = (eqEps[[1, 1]] /. u1 -> u3L) ==
    ((eqEps[[1, 1]] /. u1 -> u3L) - eqEps[[3, 1]] /. sol1 /. sol2 /. intRule1 //. intRule2 /.
        modeshapes /. int -> Integrate // Expand); // Timing
```

```
{810.475 Second, Null}
```

Collecting the terms that may lead to secular terms, the terms proportional to $E^{I\, w_i\, T_0}$, we have

```
ST2 = CoefficientAorder3Eq@@2DD •. expRule1@#D, E^(I w# T0) E & •ž 81, 2<;
ST2 •. displayRule
```

: - • !!! 2 HD$_1^2$A$_1$L Sin@p xD - • !!! 2 E$^{I T_2 s_1}$ HD$_1^2$A$_2$L Sin@2 p xD -
2 I • !!! 2 HD$_2$A$_1$L Sin@p xD w$_1$ - 2 I • !!! 2 m Sin@p xD A$_1$ w$_1$ - 2 I • !!! 2 E$^{I T_2 s_1}$ HD$_2$A$_2$L Sin@2 p xD w$_2$ -
2 I • !!! 2 E$^{I T_2 s_1}$ m Sin@2 p xD A$_2$ w$_2$ - $\frac{E^{2 I T_2 s_1 + I T_2 s_2} f p^2 Sin@p xD \dot{A}_1}{• !!! 2}$ -
$\frac{b E^{2 I T_2 s_1 + I T_2 s_2} f p^2 • !! a Sin@p xD \dot{A}_1}{2 • !!! 6}$ - 3 • !!! 2 p^4 a Sin@p xD A$_1^2$ $\dot{A}_1$ + 5 $ $\frac{3}{2}$ b p^4 a$^{3•2}$ Sin@p xD A$_1^2$ $\dot{A}_1$ -
8 • !!! 2 E$^{I T_2 s_1}$ p^4 a Sin@2 p xD A$_1$ A$_2$ $\dot{A}_1$ + $\frac{68}{3}$ $ $\frac{2}{3}$ b E$^{I T_2 s_1}$ p^4 a$^{3•2}$ Sin@2 p xD A$_1$ A$_2$ $\dot{A}_1$ -
4 • !!! 2 E$^{2 I T_2 s_1}$ p^4 a Sin@p xD A$_2^2$ $\dot{A}_1$ + 14 $ $\frac{2}{3}$ b E$^{2 I T_2 s_1}$ p^4 a$^{3•2}$ Sin@p xD A$_2^2$ $\dot{A}_1$ -
2 • !!! 2 E$^{I T_2 s_1 + I T_2 s_2}$ f p^2 Sin@2 p xD $\dot{A}_2$ - $\frac{1}{3}$ $ $\frac{2}{3}$ b E$^{I T_2 s_1 + I T_2 s_2}$ f p^2 • !! a Sin@2 p xD $\dot{A}_2$ -
4 • !!! 2 E$^{-I T_2 s_1}$ p^4 a Sin@2 p xD A$_1^2$ $\dot{A}_2$ + 14 $ $\frac{2}{3}$ b E$^{-I T_2 s_1}$ p^4 a$^{3•2}$ Sin@2 p xD A$_1^2$ $\dot{A}_2$ -
8 • !!! 2 p^4 a Sin@p xD A$_1$ A$_2$ $\dot{A}_2$ + $\frac{68}{3}$ $ $\frac{2}{3}$ b p^4 a$^{3•2}$ Sin@p xD A$_1$ A$_2$ $\dot{A}_2$ -
48 • !!! 2 E$^{I T_2 s_1}$ p^4 a Sin@2 p xD A$_2^2$ $\dot{A}_2$ + $\frac{40}{3}$ $ $\frac{2}{3}$ b E$^{I T_2 s_1}$ p^4 a$^{3•2}$ Sin@2 p xD A$_2^2$ $\dot{A}_2$,
- • !!! 2 E$^{-I T_2 s_1}$ HD$_1^2$A$_1$L Sin@p xD - • !!! 2 HD$_1^2$A$_2$L Sin@2 p xD - 2 I • !!! 2 E$^{-I T_2 s_1}$ HD$_2$A$_1$L Sin@p xD w$_1$ -
2 I • !!! 2 E$^{-I T_2 s_1}$ m Sin@p xD A$_1$ w$_1$ - 2 I • !!! 2 HD$_2$A$_2$L Sin@2 p xD w$_2$ - 2 I • !!! 2 m Sin@2 p xD A$_2$ w$_2$ -
$\frac{E^{I T_2 s_1 + I T_2 s_2} f p^2 Sin@p xD \dot{A}_1}{• !!! 2}$ - $\frac{b E^{I T_2 s_1 + I T_2 s_2} f p^2 • !!! a Sin@p xD \dot{A}_1}{2 • !!! 6}$ -
3 • !!! 2 E$^{-I T_2 s_1}$ p^4 a Sin@p xD A$_1^2$ $\dot{A}_1$ + 5 $ $\frac{3}{2}$ b E$^{-I T_2 s_1}$ p^4 a$^{3•2}$ Sin@p xD A$_1^2$ $\dot{A}_1$ -
8 • !!! 2 p^4 a Sin@2 p xD A$_1$ A$_2$ $\dot{A}_1$ + $\frac{68}{3}$ $ $\frac{2}{3}$ b p^4 a$^{3•2}$ Sin@2 p xD A$_1$ A$_2$ $\dot{A}_1$ -
4 • !!! 2 E$^{I T_2 s_1}$ p^4 a Sin@p xD A$_2^2$ $\dot{A}_1$ + 14 $ $\frac{2}{3}$ b E$^{I T_2 s_1}$ p^4 a$^{3•2}$ Sin@p xD A$_2^2$ $\dot{A}_1$ -
2 • !!! 2 E$^{I T_2 s_2}$ f p^2 Sin@2 p xD $\dot{A}_2$ - $\frac{1}{3}$ $ $\frac{2}{3}$ b E$^{I T_2 s_2}$ f p^2 • !!! a Sin@2 p xD $\dot{A}_2$ -
4 • !!! 2 E$^{-2 I T_2 s_1}$ p^4 a Sin@2 p xD A$_1^2$ $\dot{A}_2$ + 14 $ $\frac{2}{3}$ b E$^{-2 I T_2 s_1}$ p^4 a$^{3•2}$ Sin@2 p xD A$_1^2$ $\dot{A}_2$ -
8 • !!! 2 E$^{-I T_2 s_1}$ p^4 a Sin@p xD A$_1$ A$_2$ $\dot{A}_2$ + $\frac{68}{3}$ $ $\frac{2}{3}$ b E$^{-I T_2 s_1}$ p^4 a$^{3•2}$ Sin@p xD A$_1$ A$_2$ $\dot{A}_2$ -
48 • !!! 2 p^4 a Sin@2 p xD A$_2^2$ $\dot{A}_2$ + $\frac{40}{3}$ $ $\frac{2}{3}$ b p^4 a$^{3•2}$ Sin@2 p xD A$_2^2$ $\dot{A}_2$>

Demanding that **ST2[[i]]** be orthogonal to the $F_i@xD$, we obtain the solvability conditions

```
SCond2 =
  Table@int@ST2@@kDD Fk@xD, 8x, 0, 1<D == 0, 8k, 2<D •. intRule1 ••. intRule2 •. modeshapes •.
     values1 •. int -> Integrate •. D@SCond1Rule, T1D;
SCond2 •. displayRule
```

```
9-2 I HD2A1L w1 - 2 I m A1 w1 - E^(2 I T2 s1+I T2 s2) f p^2 A1 + 12 p^4 a A1^2 A1 +
   24 E^(2 I T2 s1) p^4 a A2^2 A1 + 112/3 p^4 a A1 A2 A2 == 0, -2 I HD2A2L w2 - 2 I m A2 w2 +
   112/3 p^4 a A1 A2 A1 - 8/3 E^(I T2 s2) f p^2 A2 + 24 E^(-2 I T2 s1) p^4 a A1^2 A2 - 64/3 p^4 a A2^2 A2 == 0=
```

Ÿ 8.1.3 The Function MMSDirect11

Collecting the commands in the preceding section, we build a function **MMSDirect11**, specifically for the nonlinear transverse vibrations of a hinged-hinged beam around its first buckled mode (**EOM** and **BC1**), to determine the solvability conditions for different combinations of **frequencies**, **modeshapes**, and **ResonanceConds** in which the first mode might be involved in either a one-to-one or a three-to-one internal resonance with one of the other modes.

```
MMSDirect11@frequencies_, modeshapes_, ResonanceConds_D :=
    ModuleA8<,
        multiScales = 8u@x_, tD -> u@x, T0, T1, T2D,
    Derivative@m_, n_D@u_D@x_, tD -> dt@nD@D@u@x, T0, T1, T2D, 8x, m<DD, t -> T0<;
  solRule = u -> IEvaluateASumAe^j uj@#1, #2, #3, #4D, 8j, 3<EE &M;
        scaling = 9c -> 2 e^2 m , f -> e^2 f=;
  eqs = HJoin@8EOM<, BC1D •. multiScales •. solRule •. scaling •. intRule1 ••. intRule2 ••
       TrigToExp •• ExpandAllL •. e^(n_•;n>3) -> 0;
  eqEps = Thread@CoefficientList@Subtract žž #, eD == 0D & •ž eqs •• Transpose •• Rest;
  sol1 = 9u1 -> FunctionA8x, T0, T1, T2<,
      SumAAi@T1, T2D Fi@xD Exp@I wi T0D + Ai@T1, T2D Fi@xD Exp@-I wi T0D, 8i, 2<E •• EvaluateE=;
  order2Eq = HeqEps@@1, 1, 1DD •. u1 -> u2L == HeqEps@@1, 1, 1DD •. u1 -> u2L - eqEps@@2, 1, 1DD •.
       sol1 •. intRule1 ••. intRule2 •• ExpandAll;
        omgList = 8w1, w2<;
  OmgRule =
   Solve@ResonanceConds, Complement@omgList, 8#<D~Join~8W<D@@1DD & •ž omgList;
  expRule1@i_D := Exp@arg_D :> ExpAExpand@arg •. OmgRule@@iDDD •. e^2 T0 -> T2E;
  ST1 = CoefficientAorder2Eq@@2DD •. expRule1@#D, E^(I w# T0)E & •ž 81, 2<;
  SCond1 = TableAà_0^1 ST1@@kDD Fk@xD â x == 0, 8k, 2<E •. modeshapes;
  SCond1Rule = SolveASCond1, 9A1^H1,0L@T1, T2D, A2^H1,0L@T1, T2D=E@@1DD;
  ccSCond1Rule = SCond1Rule •. conjugateRule;
  order2Eqm = order2Eq •. SCond1Rule •. ccSCond1Rule •. modeshapes •. int -> Integrate •.
    Integrate -> int;
```

```
    rhsTerms = Cases@order2Eqm@@2DD, #D & •ž 9a_ f_@b_ xD :> f@b xD •; FreeQ@a, T0D,
          _ Ea_ T0+b_. f_@c_ xD -> Ea T0+b f@c xD= •• Flatten •• Union;
    symbolList = Table@ai, 8i, Length@rhsTermsD<D;
    sol2Form = u2 -> Function@8x, T0, T1, T2<, symbolList.rhsTerms •• EvaluateD;
    symbolRule =
     Solve@Coefficient@Subtract žž order2Eqm •. sol2Form •. intRule1 ••. intRule2 •.
            int -> Integrate, rhsTermsD == 0 •. Exp@_ T0 + _.D -> 0 •• Thread, symbolListD@@1DD;
      values1 = Solve@Select@ResonanceConds, FreeQ@#, WD &D •. e -> 0 •. frequencies, bD ••
       Flatten;
    symbolRuleN = symbolRule •. W -> 2 w2 •. frequencies •. bucklingLoad •. values1;
    sol2 = sol2Form •. symbolRuleN;
    order3Eq = HeqEps@@1, 1, 1DD •. u1 -> u3L ==
      HHeqEps@@1, 1, 1DD •. u1 -> u3L - eqEps@@3, 1, 1DD •. sol1 •. sol2 •. intRule1 ••. intRule2 •.
          modeshapes •. int -> Integrate •• ExpandL;
    ST2 = CoefficientAorder3Eq@@2DD •. expRule1@#D, EI w# T0E & •ž 81, 2<;
    SCond2 =
     Table@int@ST2@@kDD Fk@xD, 8x, 0, 1<D == 0, 8k, 2<D •. intRule1 ••. intRule2 •. modeshapes •.
        values1 •. int -> Integrate •. D@SCond1Rule, T1D;
      Return@SCond2 •. Ai_ -> HAi@#2D &L •. f_@T1, T2D -> f@T2DD
  E
```

Ÿ 8.1.4 Three-to-One Internal Resonances Between the First Two Modes

In this section, we consider the case $\omega_2 \approx 3\omega_1$ and $\Omega \approx 2\,\omega_2$. To describe the nearness of the resonances, we introduce the detuning parameters σ_i defined by

```
ResonanceConds1 = 9w2 == 3 w1 + e^2 s1, W == 2 w2 + e^2 s2=;
```

Using **MMSDirect11**, we obtain the solvability conditions

```
MMSDirect11@frequencies, modeshapes, ResonanceConds1D •• Timing
```

```
921.501 Second,
```

$$\Big\{-2 I \mu \omega_1 A_1[T_2] + 12 \pi^4 \alpha A_1[T_2]^2 \bar{A}_1[T_2] + \frac{624}{35} \pi^4 \alpha A_1[T_2] A_2[T_2] \bar{A}_2[T_2] - 2 I \omega_1 A_1'[T_2] == 0,$$

$$-2 I \mu \omega_2 A_2[T_2] + \frac{624}{35} \pi^4 \alpha A_1[T_2] A_2[T_2] \bar{A}_1[T_2] -$$

$$\frac{72}{35} E^{I T_2 \sigma_2} f \pi^2 \bar{A}_2[T_2] - \frac{576}{35} \pi^4 \alpha A_2[T_2]^2 \bar{A}_2[T_2] - 2 I \omega_2 A_2'[T_2] == 0\Big\}\Big\}$$

which indicates that this three-to-one internal resonance is not activated.

Next, we consider the case in which the frequency of the first mode is three times that of the second mode; that is, the buckling load is above the first crossover value. To accomplish this, we redefine the frequencies and denote the frequency of the first mode by ω_2 and that of the second mode by ω_1 as follows:

```
frequencies2 = 8w2 -> Hw •. wSol1L, w1 -> Hw •. wSol2 •. n -> 2L<
```

```
9w2 ® b p^2 • !!! ä, w1 ® 2 • !! 3 p^2=
```

Moreover, we redefine the modeshapes according to

```
modeshapes2 = Join@modeshape,
   8F1 -> Function@x, shape2 •• EvaluateD, F2 -> Function@x, shape1 •• EvaluateD<D;
```

To describe the nearness of the resonances, we introduce the two detuning parameters s_i defined by

```
ResonanceConds2 = 9w2 == 3 w1 + e^2 s1, W == 2 w2 + e^2 s2=;
```

Using **MMSDirect11**, we obtain the solvability conditions

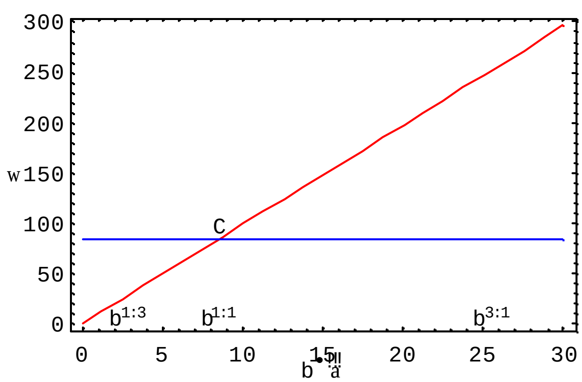
```
MMSDirect11@frequencies2, modeshapes2, ResonanceConds2D •• Timing
```

```
921.721 Second,
 9-2 I m w1 A1@T2D + 64/5 p^4 a A1@T2D^2 A1•@T2D - 496/5 p^4 a A1@T2D A2@T2D A2•@T2D - 2 I w1 A1¢@T2D == 0,
  -2 I m w2 A2@T2D - 496/5 p^4 a A1@T2D A2@T2D A1•@T2D -
   E^(I T2 s2) f p^2 A2•@T2D + 12 p^4 a A2@T2D^2 A2•@T2D - 2 I w2 A2¢@T2D == 0==
```

Again, this three-to-one internal resonance is not activated.

Ÿ 8.1.5 One-to-One and Three-to-One Internal Resonances Between the First and Third Modes

Variations of the first and third natural frequencies with the buckling level are plotted as follows:



It follows from this figure that the frequencies of the first and third modes are equal when $b \bullet \sqrt{\ddot{a}} = b^{1:1}$, the frequency of the third mode is three times that of the first mode when $b \bullet \sqrt{\ddot{a}} = b^{1:3}$, and the frequency of the first mode is three times that of the third mode when $b \bullet \sqrt{\ddot{a}} = b^{3:1}$. Next, we use the function **MMSDirect11** to treat these three cases.

Ÿ One-to-One Internal Resonance

We denote the frequency of the third mode by w_2 and that of the first mode by w_1 as follows:

```
frequencies3 = 8w1 -> Hw •. wSol1L, w2 -> Hw •. wSol2 •. n -> 3L<
```

$9w_1 ® b\, p^2 • \sqrt{a},\; w_2 ® 6 • \sqrt{2}\; p^2=$

Moreover, we define f_k and the first and third eigenmodes of the buckled beam as

```
modeshapes3 = Join@modeshape,
   8F1 -> Function@x, shape1 •• EvaluateD, F2 -> Function@x, shape3 •• EvaluateD<D;
```

We consider a combination of a one-to-one internal resonance between the first and third modes and a principal parametric resonance of the third mode; that is,

```
ResonanceConds3 = 9w2 == w1 + e^2 s1, W == 2 w2 + e^2 s2=;
```

Using **MMSDirect11**, we obtain the solvability conditions

```
MMSDirect11@frequencies3, modeshapes3, ResonanceConds3D •• Timing
```

826.809 Second,

$8-2\,I\,m\,w_1 A_1@T_2D - E^{2\,I\,T_2\,s_1+I\,T_2\,s_2}\, f\, p^2\, \dot{A}_1@T_2D + 12\, p^4\, a\, A_1@T_2D^2\, \dot{A}_1@T_2D + 144\, E^{2\,I\,T_2\,s_1}\, p^4\, a\, A_2@T_2D^2\, \dot{A}_1@T_2D +$
$\quad 144\, p^4\, a\, A_1@T_2D\, A_2@T_2D\, \dot{A}_2@T_2D - 2\,I\,w_1\, A_1^{¢}@T_2D == 0,$
$-2\,I\,m\,w_2 A_2@T_2D + 144\, p^4\, a\, A_1@T_2D\, A_2@T_2D\, \dot{A}_1@T_2D - 6\, E^{I\,T_2\,s_2}\, f\, p^2\, \dot{A}_2@T_2D +$
$\quad 144\, E^{-2\,I\,T_2\,s_1}\, p^4\, a\, A_1@T_2D^2\, \dot{A}_2@T_2D - 108\, p^4\, a\, A_2@T_2D^2\, \dot{A}_2@T_2D - 2\,I\,w_2\, A_2^{¢}@T_2D == 0<<$

which indicates that the one-to-one internal resonance between the first and third modes is activated.

Ÿ Three-to-One Internal Resonances

To treat the case in which the natural frequency of the third mode is approximately three times that of the first mode using the function **MMSDirect11**, we denote these frequencies by w_2 and w_1, respectively, according to

```
frequencies4 = 8w1 -> Hw •. wSol1L, w2 -> Hw •. wSol2 •. n -> 3L<
```

$9w_1 ® b\, p^2 • \sqrt{a},\; w_2 ® 6 • \sqrt{2}\; p^2=$

Moreover, we define f_k and the first and third eigenmodes of the buckled beam as

```
modeshapes4 = Join@modeshape,
   8F1 -> Function@x, shape1 •• EvaluateD, F2 -> Function@x, shape3 •• EvaluateD<D;
```

We consider a combination of a three-to-one internal resonance between the first and third modes and a principal parametric resonance of the third mode; that is,

```
ResonanceConds4 = 9w2 == 3 w1 + e^2 s1, W == 2 w2 + e^2 s2=;
```

Using **MMSDirect11**, we obtain the solvability conditions

```
MMSDirect11@frequencies4, modeshapes4, ResonanceConds4D •• Timing
```

```
921.651 Second,
```

$$9 - 2\,I\,m\,w_1\,A_1@T_2D + 12\,p^4\,a\,A_1@T_2D^2\,\dot{\bar{A}}_1@T_2D + \frac{1584}{35}\,p^4\,a\,A_1@T_2D\,A_2@T_2D\,\dot{\bar{A}}_2@T_2D - 2\,I\,w_1\,A_1^{¢}@T_2D == 0,$$

$$-2\,I\,m\,w_2\,A_2@T_2D + \frac{1584}{35}\,p^4\,a\,A_1@T_2D\,A_2@T_2D\,\dot{\bar{A}}_1@T_2D -$$

$$\frac{162}{35}\,E^{I\,T_2\,s_2}\,f\,p^2\,\dot{\bar{A}}_2@T_2D - \frac{2916}{35}\,p^4\,a\,A_2@T_2D^2\,\dot{\bar{A}}_2@T_2D - 2\,I\,w_2\,A_2^{¢}@T_2D == 0==$$

which indicates that this three-to-one internal resonance between the first and third modes is not activated.

When the frequency of first mode is approximately three times that of the third mode, we define the frequencies as

```
frequencies5 = 8w2 -> Hw •. wSol1L, w1 -> Hw •. wSol2 •. n -> 3L<
```

$$9w_2 ® b\,p^2\,\sqrt{a},\ w_1 ® 6\,\sqrt{2}\,p^2=$$

Moreover, we define f_k and the third and first eigenmodes of the buckled beam as

```
modeshapes5 = Join@modeshape,
   8F1 -> Function@x, shape3 •• EvaluateD, F2 -> Function@x, shape1 •• EvaluateD<D;
```

We consider a combination of a three-to-one internal resonance between the first and third modes and a principal parametric resonance of the third mode; that is,

```
ResonanceConds5 = 9w2 == 3 w1 + e^2 s1, W == 2 w2 + e^2 s2=;
```

Using **MMSDirect11**, we obtain the solvability conditions

```
MMSDirect11@frequencies5, modeshapes5, ResonanceConds5D •• Timing
```

```
921.712 Second,
```

$$9 - 2\,I\,m\,w_1\,A_1@T_2D + \frac{324}{5}\,p^4\,a\,A_1@T_2D^2\,\dot{\bar{A}}_1@T_2D - \frac{2736}{5}\,p^4\,a\,A_1@T_2D\,A_2@T_2D\,\dot{\bar{A}}_2@T_2D - 2\,I\,w_1\,A_1^{¢}@T_2D == 0,$$

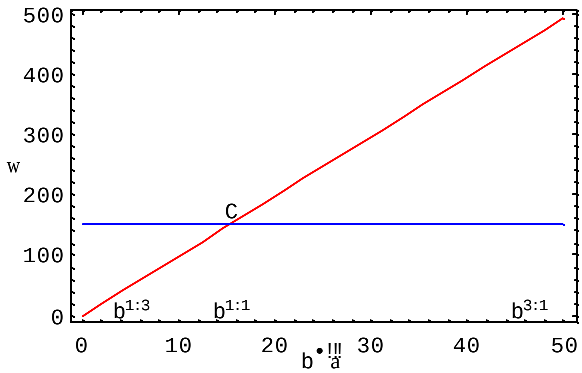
$$-2\,I\,m\,w_2\,A_2@T_2D - \frac{2736}{5}\,p^4\,a\,A_1@T_2D\,A_2@T_2D\,\dot{\bar{A}}_1@T_2D -$$

$$E^{I\,T_2\,s_2}\,f\,p^2\,\dot{\bar{A}}_2@T_2D + 12\,p^4\,a\,A_2@T_2D^2\,\dot{\bar{A}}_2@T_2D - 2\,I\,w_2\,A_2^{¢}@T_2D == 0==$$

which indicates that this three-to-one internal resonance between the first and third modes is not activated.

Ÿ 8.1.6 One-to-One and Three-to-One Internal Resonances Between the First and Fourth Modes

Variations of the first and fourth natural frequencies with the buckling level are plotted as follows:



Again, there exist buckling levels for which the frequencies of the first and fourth modes are in the ratio of 1:1, 3:1, and 1:3. We use the function **MMSDirect11** to treat these cases.

Ÿ One-to-One Internal Resonance

When the frequencies of the first and fourth modes are approximately equal, we denote them by w_1 and w_2, respectively, according to

```
frequencies6 = 8w1 -> Hw •. wSol1L, w2 -> Hw •. wSol2 •. n -> 4L<
```

```
9w1 ® b p^2 • !!! a , w2 ® 4 • !!!! 15 p^2=
```

Moreover, we define f_k and the first and fourth eigenmodes of the buckled beam as

```
modeshapes6 = Join@modeshape,
   8F1 -> Function@x, shape1 •• EvaluateD, F2 -> Function@x, shape4 •• EvaluateD<D;
```

We consider a combination of a one-to-one internal resonance between the first and fourth modes and a principal parametric resonance of the fourth mode; that is,

```
ResonanceConds6 = 9w2 == w1 + e^2 s1, W == 2 w2 + e^2 s2=;
```

Using **MMSDirect11**, we obtain the solvability conditions

```
MMSDirect11@frequencies6, modeshapes6, ResonanceConds6D •• Timing
```

```
929.522 Second,
 9-2 I m w1 A1@T2D - E^(2 I T2 s1+I T2 s2) f p^2 A1•@T2D + 12 p^4 a A1@T2D^2 A1•@T2D + 480 E^(2 I T2 s1) p^4 a A2@T2D^2 A1•@T2D +
   1216/3 p^4 a A1@T2D A2@T2D A2•@T2D - 2 I w1 A1¢@T2D == 0,
  -2 I m w2 A2@T2D + 1216/3 p^4 a A1@T2D A2@T2D A1•@T2D - 32/3 E^(I T2 s2) f p^2 A2•@T2D +
   480 E^(-2 I T2 s1) p^4 a A1@T2D^2 A2•@T2D - 1024/3 p^4 a A2@T2D^2 A2•@T2D - 2 I w2 A2¢@T2D == 0==
```

which indicates that this one-to-one internal resonance between the first and fourth modes is activated.

Ÿ Three-to-One Internal Resonances

When the frequency of the fourth mode is approximately three times that of the first mode, we denote these frequencies by w_2 and w_1, respectively, as

```
frequencies7 = 8w1 -> Hw •. wSol1L, w2 -> Hw •. wSol2 •. n -> 4L<
```

$9w_1 ® b p^2 • \sqrt{a}, w_2 ® 4 • \sqrt{15} p^2=$

Moreover, we define f_k and the first and fourth eigenmodes of the buckled beam as

```
modeshapes7 = Join@modeshape,
   8F1 -> Function@x, shape1 •• EvaluateD, F2 -> Function@x, shape4 •• EvaluateD<D;
```

We consider a combination of a three-to-one internal resonance between the first and fourth modes and a principal parametric resonance of the fourth mode; that is,

```
ResonanceConds7 = 9w2 == 3 w1 + e^2 s1, W == 2 w2 + e^2 s2=;
```

Using **MMSDirect11**, we obtain the solvability conditions

```
MMSDirect11@frequencies7, modeshapes7, ResonanceConds7D •• Timing
```

922.653 Second,

$9-2 I m w_1 A_1@T_2D + 12 p^4 a A_1@T_2D^2 \dot{A}_1@T_2D + \frac{3264}{35} p^4 a A_1@T_2D A_2@T_2D \dot{A}_2@T_2D - 2 I w_1 A_1'@T_2D == 0,$

$-2 I m w_2 A_2@T_2D + \frac{3264}{35} p^4 a A_1@T_2D A_2@T_2D \dot{A}_1@T_2D -$

$\frac{288}{35} E^{I T_2 s_2} f p^2 \dot{A}_2@T_2D - \frac{9216}{35} p^4 a A_2@T_2D^2 \dot{A}_2@T_2D - 2 I w_2 A_2'@T_2D == 0==$

which indicates that this three-to-one internal resonance between the first and fourth modes is not activated.

When the frequency of the first mode is approximately three times that of the fourth mode, we denote these frequencies by w_2 and w_1, respectively, as

```
frequencies8 = 8w2 -> Hw •. wSol1L, w1 -> Hw •. wSol2 •. n -> 4L<
```

$9w_2 ® b p^2 • \sqrt{a}, w_1 ® 4 • \sqrt{15} p^2=$

Moreover, we define f_k and the fourth and first eigenmodes of the buckled beam as

```
modeshapes8 = Join@modeshape,
   8F1 -> Function@x, shape4 •• EvaluateD, F2 -> Function@x, shape1 •• EvaluateD<D;
```

We consider a combination of a three-to-one internal resonance between the first and fourth modes and a principal parametric resonance of the fourth mode; that is,

```
ResonanceConds8 = 9w2 == 3 w1 + e^2 s1, W == 2 w2 + e^2 s2=;
```

Using **MMSDirect11**, we obtain the solvability conditions

```
MMSDirect11@frequencies8, modeshapes8, ResonanceConds8D •• Timing
```

```
922.072 Second,
 9-2 I m w1 A1@T2D + 1024/5 p^4 a A1@T2D^2 Ȧ1@T2D - 8896/5 p^4 a A1@T2D A2@T2D Ȧ2@T2D - 2 I w1 A1¢@T2D == 0,
  -2 I m w2 A2@T2D - 8896/5 p^4 a A1@T2D A2@T2D Ȧ1@T2D -
   E^(I T2 s2) f p^2 Ȧ2@T2D + 12 p^4 a A2@T2D^2 Ȧ2@T2D - 2 I w2 A2¢@T2D == 0==
```

which indicates again that this three-to-one internal resonance between the first and fourth modes is not activated.

à 8.2 Circular Cylindrical Shells

Following McIvor (1966) and Raouf and Nayfeh (1990), we write the equations of motion governing the dynamic response of an infinitely long cylindrical shell to a primary-resonance excitation of one of its two orthogonal flexural modes as

```
EOM = 9w - yq + wt,t + a^2 Hw + 2 wq,q + wq,q,q,qL ==
    - wq^2/2 - w wq^2 - yt^2 + w yt^2 - 2 w yq + 1/2 wq^2 yq + yq^2 - w yq^2 + f e^3 Cos@n qD Cos@W tD H1 - w + yqL -
     w wq,q - w^2 wq,q + 3/2 wq^2 wq,q + yq wq,q + w yq wq,q - yq^2 wq,q + wq yq,q + w wq yq,q -
     2 wq yq yq,q + a^2 j-3 w^2 - 11 wq^2/2 - 11 w wq,q + 4 yq wq,q - 6 wq,q^2 + 4 wq yq,q - 8 wq wq,q,q +
        8 yq,q wq,q,q + w yq,q,q + 5 wq,q yq,q,q - 4 w wq,q,q,q + 4 yq wq,q,q,q + wq yq,q,q,qz,
   wq + yt,t - yq,q == 2 w wq + wq^3/2 + 2 wt yt - 2 w wt yt - 2 wq yq + 2 w wq yq + wq wq,q +
     w wq wq,q - 2 wq yq wq,q + 2 w yt,t - w^2 yt,t - 2 w yq,q + w^2 yq,q - wq^2 yq,q +
     a^2 Hwq wq,q - w wq,q,q - wq,q wq,q,q + wq wq,q,q,qL= •. 8w -> w@q, tD, Hs : w È yLm__ :> ¶m s@q, tD<;
```

We seek a second-order uniform asymptotic expansion of the solution of **EOM** in the form

```
solRule = 9w -> IEvaluateASumAe^j wj@#1, #2, #3, #4D, 8j, 3<EE &M,
   y -> IEvaluateASumAe^j yj@#1, #2, #3, #4D, 8j, 3<EE &M=
```

```
8w ® He w1@#1, #2, #3, #4D + e^2 w2@#1, #2, #3, #4D + e^3 w3@#1, #2, #3, #4D &L,
 y ® He y1@#1, #2, #3, #4D + e^2 y2@#1, #2, #3, #4D + e^3 y3@#1, #2, #3, #4D &L<
```

where #1 stands for q and #2, #3, and #4 stand for T_0, T_1, and T_2, respectively. Transforming the time derivatives in **EOM** in terms of the scales T_0, T_1, and T_2, substituting **solRule** into **EOM**, expanding the result for small e, and discarding terms of order higher than e^3, we obtain

```
eq82a = HEOM •. 8w@x_, tD -> w@x, T0, T1, T2D, y@x_, tD -> y@x, T0, T1, T2D,
        Derivative@m_, n_D@w_D@x_, tD -> dt@nD@D@w@x, T0, T1, T2D, 8x, m<DD,
        t -> T0< •. solRule •• ExpandAllL •. e^(n_•;n>3) -> 0;
```

Equating coefficients of like powers of e, we have

```
eqEps = Rest@Thread@CoefficientList@Subtract žž #, eD == 0DD & •ž eq82a •• Transpose;
eqEps •. displayRule
```

```
982 a^2 Hw1''L + a^2 Hw1''''L - y1' + D0^2w1 + w1 + a^2 w1 == 0, w1' - y1'' + D0^2y1 == 0<,
 9 1/2 Hw1'L^2 + 11/2 a^2 Hw1'L^2 + 6 a^2 Hw1''L^2 + 8 a^2 Hw1'L Hw1'''L + 2 a^2 Hw2''L + a^2 Hw2''''L -
   Hw1''L Hy1'L - 4 a^2 Hw1''L Hy1'L - 4 a^2 Hw1''''L Hy1'L - Hy1'L^2 - Hw1'L Hy1''L - 4 a^2 Hw1'L Hy1''L -
   8 a^2 Hw1'''L Hy1''L - 5 a^2 Hw1''L Hy1'''L - a^2 Hw1'L Hy1''''L - y2' + HD0y1L^2 + D0^2w2 + 2 HD0 D1w1L +
   Hw1''L w1 + 11 a^2 Hw1''L w1 + 4 a^2 Hw1''''L w1 + 2 Hy1'L w1 - a^2 Hy1'''L w1 + 3 a^2 w1^2 + w2 + a^2 w2 == 0,
  - Hw1'L Hw1''L - a^2 Hw1'L Hw1''L + a^2 Hw1''L Hw1'''L - a^2 Hw1'L Hw1''''L + w2' + 2 Hw1'L Hy1'L - y2'' -
   2 HD0w1L HD0y1L + D0^2y2 + 2 HD0 D1y1L - 2 Hw1'L w1 + a^2 Hw1'''L w1 + 2 Hy1''L w1 - 2 HD0^2y1L w1 == 0=,
 9 - f Cos@n qD Cos@T0 WD - 3/2 Hw1'L^2 Hw1''L + Hw1'L Hw2'L + 11 a^2 Hw1'L Hw2'L + 8 a^2 Hw1'''L Hw2'L +
   12 a^2 Hw1''L Hw2''L + 8 a^2 Hw1'L Hw2'''L + 2 a^2 Hw3''L + a^2 Hw3''''L - 1/2 Hw1'L^2 Hy1'L - Hw2''L Hy1'L -
   4 a^2 Hw2''L Hy1'L - 4 a^2 Hw2''''L Hy1'L + Hw1''L Hy1'L^2 - Hw2'L Hy1''L - 4 a^2 Hw2'L Hy1''L - 8 a^2 Hw2'''L Hy1''L +
   2 Hw1'L Hy1'L Hy1''L - 5 a^2 Hw2''L Hy1'''L - a^2 Hw2'L Hy1''''L - Hw1''L Hy2'L - 4 a^2 Hw1''L Hy2'L -
   4 a^2 Hw1''''L Hy2'L - 2 Hy1'L Hy2'L - Hw1'L Hy2''L - 4 a^2 Hw1'L Hy2''L - 8 a^2 Hw1'''L Hy2''L -
   5 a^2 Hw1''L Hy2'''L - a^2 Hw1'L Hy2''''L - y3' + 2 HD0y1L HD0y2L + D0^2w3 + 2 HD0y1L HD1y1L +
   2 HD0 D1w2L + D1^2w1 + 2 HD0 D2w1L + Hw1'L^2 w1 + Hw2''L w1 + 11 a^2 Hw2''L w1 + 4 a^2 Hw2''''L w1 -
   Hw1''L Hy1'L w1 + Hy1'L^2 w1 - Hw1'L Hy1''L w1 + 2 Hy2'L w1 - a^2 Hy2'''L w1 - HD0y1L^2 w1 + Hw1''L w1^2 +
   Hw1''L w2 + 11 a^2 Hw1''L w2 + 4 a^2 Hw1''''L w2 + 2 Hy1'L w2 - a^2 Hy1'''L w2 + 6 a^2 w1 w2 + w3 + a^2 w3 == 0,
  - 1/2 Hw1'L^3 - Hw1''L Hw2'L - a^2 Hw1''L Hw2'L - a^2 Hw1''''L Hw2'L - Hw1'L Hw2''L - a^2 Hw1'L Hw2''L +
   a^2 Hw1'''L Hw2''L + a^2 Hw1''L Hw2'''L - a^2 Hw1'L Hw2''''L + w3' + 2 Hw1'L Hw1''L Hy1'L +
   2 Hw2'L Hy1'L + Hw1'L^2 Hy1''L + 2 Hw1'L Hy2'L - y3'' - 2 HD0w2L HD0y1L - 2 HD0w1L HD0y2L + D0^2y3 -
   2 HD0y1L HD1w1L - 2 HD0w1L HD1y1L + 2 HD0 D1y2L + D1^2y1 + 2 HD0 D2y1L - Hw1'L Hw1''L w1 -
   2 Hw2'L w1 + a^2 Hw2'''L w1 - 2 Hw1'L Hy1'L w1 + 2 Hy2''L w1 + 2 HD0w1L HD0y1L w1 - 2 HD0^2y2L w1 -
   4 HD0 D1y1L w1 - Hy1''L w1^2 + HD0^2y1L w1^2 - 2 Hw1'L w2 + a^2 Hw1'''L w2 + 2 Hy1''L w2 - 2 HD0^2y1L w2 == 0==
```

Ÿ 8.2.1 First-Order Solution

To determine the solution of the first-order problem, **eqEps[[1]]**, we use the method of separation of variables and let

```
svRule = 9w1 -> IA E^(I m #1) E^(I wm #2) &M, y1 -> IB E^(I m #1) E^(I wm #2) &M=;
```

Substituting the **svRule** into **eqEps[[1]]**, we obtain the characteristic equations

```
eqEps1lhs = ExpandAE^(-I m q-I wm T0) #@@1DD •. svRuleE & •ž eqEps@@1DD
```

```
8A - I B m + A a^2 - 2 A m^2 a^2 + A m^4 a^2 - A w_m^2, I A m + B m^2 - B w_m^2<
```

Setting the determinant of the coefficient matrix in **eqEps1lhs** equal to zero yields the following equation governing the natural frequencies w_m of the shell:

```
HOuter@D, eqEps1lhs, 8A, B<D •• Det •• Collect@#, wmD &L == 0
```

```
m^2 a^2 - 2 m^4 a^2 + m^6 a^2 + H-1 - m^2 - a^2 + 2 m^2 a^2 - m^4 a^2L w_m^2 + w_m^4 == 0
```

It follows from **eqEps1lhs** that the ratio $G_m = I \frac{B}{A}$ of the amplitudes of y_1 and w_1 is given by

```
GRule = Gm -> FactorAI B/A •. Solve@eqEps1lhs@@1DD == 0, BD@@1DDE
```

```
Gm ® (1 + a^2 - 2 m^2 a^2 + m^4 a^2 - w_m^2)/m
```

Hence, the solution of **eqEps[[1]]** can be expressed in terms of the linear free-vibration modes

```
modeShapes = ComplexExpand@8Re@#D, Im@#D<D & •ž 8Exp@I n qD, -I Gn Exp@I n qD<
```

```
88Cos@n qD, Sin@n qD<, 8Sin@n qD Gn, -Cos@n qD Gn<<
```

as

```
sol1Form =
  modeShapes.TableAAi,n@T1, T2D Exp@I wn T0D + Ȧi,n@T1, T2D Exp@-I wn T0D, 8i, 2<E •• Expand
```

```
8E^(I T0 wn) Cos@n qD A1,n@T1, T2D + E^(I T0 wn) Sin@n qD A2,n@T1, T2D +
  E^(-I T0 wn) Cos@n qD Ȧ1,n@T1, T2D + E^(-I T0 wn) Sin@n qD Ȧ2,n@T1, T2D, E^(I T0 wn) Sin@n qD Gn A1,n@T1, T2D -
  E^(I T0 wn) Cos@n qD Gn A2,n@T1, T2D + E^(-I T0 wn) Sin@n qD Gn Ȧ1,n@T1, T2D - E^(-I T0 wn) Cos@n qD Gn Ȧ2,n@T1, T2D<
```

It follows from the **sol1Form** that all of the flexural modes are degenerate because two linearly independent eigenfunctions, namely Cos@n qD and Sin@n qD, correspond to the same frequency w_n for each n.

We consider the case of a primary-resonance excitation of the Cos@n qD mode corresponding to the frequency w_n; it is called the driven mode. Because of the degeneracy, this mode is involved in a one-to-one internal resonance with the Sin@n qD mode, which is called the companion mode. We assume that neither of these two modes is involved in an internal resonance with any other mode. Therefore, in the presence of viscous damping, out of the infinite number of modes present in w_1 and y_1, only the driven mode Cos@n qD and its companion mode Sin@n qD will contribute to the steady-state response (Nayfeh and Mook, 1979). Consequently, we take the solution of the first-order problem to consist only of these two modes and express it as

```
sol1 = 8w1 -> Function@8q, T0, T1, T2<, sol1Form@@1DD •• EvaluateD,
   y1 -> Function@8q, T0, T1, T2<, sol1Form@@2DD •• EvaluateD<;
```

To describe the nearness of the primary resonance, we introduce the detuning parameter s defined by

```
ResonanceCond = 9W == wn + e2 s=;
```

where the detuning is ordered at e^2 because secular terms appear first at order e^3.

Ÿ 8.2.2 Second-Order Solution

Substituting the **sol1** into the second-order problem, **eqEps[[2]]**, we obtain

```
order2Eq =
  Table@HeqEps@@1, i, 1DD •. 8w1 -> w2, y1 -> y2<L == HeqEps@@1, i, 1DD •. 8w1 -> w2, y1 -> y2<L -
     HSubtract žž eqEps@@2, iDD •. sol1 •• ExpandL, 8i, 2<D;
```

Collecting the terms that may lead to secular terms, the terms proportional to $E^{I\, w_n\, T_0}$, we have

```
ST1 = CoefficientA#@@2DD, EI wn T0E & •ž order2Eq
```

```
9-2 I Cos@n qD wn A1,n^H1,0L@T1, T2D - 2 I Sin@n qD wn A2,n^H1,0L@T1, T2D,
 -2 I Sin@n qD Gn wn A1,n^H1,0L@T1, T2D + 2 I Cos@n qD Gn wn A2,n^H1,0L@T1, T2D=
```

Because the homogeneous parts of **order2Eq** have a nontrivial solution, the corresponding nonhomogeneous equations have a solution only if solvability conditions are satisfied. These conditions demand that **ST1** be orthogonal to every solution of the adjoint homogeneous equations. In this case, these equations are self-adjoint and hence solutions of the adjoint equations are given by the transpose of the **modeShapes**. Demanding that **ST1** be orthogonal to the transpose of the **modeShapes**, we obtain the solvability conditions

```
SCond1 =
 SolveAHint@#, 8q, 0, 2 p<D & •ž HTranspose@modeShapesD.ST1L •. intRule1 ••. intRule2 •.
      8int@Cos@s_D Sin@s_D, _D -> 0, int@f_@n qD^2, arg_D -> p<L == 0 ••
   Thread, 9A1,n^H1,0L@T1, T2D, A2,n^H1,0L@T1, T2D=E@@1DD
```

```
9A1,n^H1,0L@T1, T2D ® 0, A2,n^H1,0L@T1, T2D ® 0=
```

Having determined the solvability conditions of the second-order equations, **order2Eq**, we use a combination of the superposition principle and the method of undetermined coefficients to determine a particular solution of them. To this end, we identify the forms of all of the possible terms on the right-hand sides of **order2Eq** as follows:

```
collectForm = OuterATimes, 8Cos@n qD, Sin@n qD<,
   TableA9Ai,n@T1, T2D EI wn T0, Ȧi,n@T1, T2D E-I wn T0=, 8i, 2<E •• FlattenE •• Flatten
```

```
8EI T0 wn Cos@n qD A1,n@T1, T2D, E-I T0 wn Cos@n qD Ȧ1,n@T1, T2D,
 EI T0 wn Cos@n qD A2,n@T1, T2D, E-I T0 wn Cos@n qD Ȧ2,n@T1, T2D, EI T0 wn Sin@n qD A1,n@T1, T2D,
 E-I T0 wn Sin@n qD Ȧ1,n@T1, T2D, EI T0 wn Sin@n qD A2,n@T1, T2D, E-I T0 wn Sin@n qD Ȧ2,n@T1, T2D<
```

```
possibleTerms = Flatten@If@Head@#D === Plus, List žž #, #D & •ž
      HOuter@Times, collectForm, collectFormD •• Flatten •• Union •• TrigReduce ••
        ExpandLD •. c_? NumberQ form_ -> form •• Union;
possibleTerms •. displayRule
```

:$E^{2 I T_0 w_n} A_{1,n}^2$, $E^{2 I T_0 w_n}$ Cos@2 n qD $A_{1,n}^2$, $E^{2 I T_0 w_n}$ Sin@2 n qD $A_{1,n}^2$, $E^{2 I T_0 w_n} A_{1,n} A_{2,n}$, $E^{2 I T_0 w_n}$ Cos@2 n qD $A_{1,n} A_{2,n}$, $E^{2 I T_0 w_n}$ Sin@2 n qD $A_{1,n} A_{2,n}$, $E^{2 I T_0 w_n} A_{2,n}^2$, $E^{2 I T_0 w_n}$ Cos@2 n qD $A_{2,n}^2$, $E^{2 I T_0 w_n}$ Sin@2 n qD $A_{2,n}^2$, $A_{1,n} \dot{A}_{1,n}$, Cos@2 n qD $A_{1,n} \dot{A}_{1,n}$, Sin@2 n qD $A_{1,n} \dot{A}_{1,n}$, $A_{2,n} \dot{A}_{1,n}$, Cos@2 n qD $A_{2,n} \dot{A}_{1,n}$, Sin@2 n qD $A_{2,n} \dot{A}_{1,n}$, $E^{-2 I T_0 w_n} \dot{A}_{1,n}^2$, $E^{-2 I T_0 w_n}$ Cos@2 n qD $\dot{A}_{1,n}^2$, $E^{-2 I T_0 w_n}$ Sin@2 n qD $\dot{A}_{1,n}^2$, $A_{1,n} \dot{A}_{2,n}$, Cos@2 n qD $A_{1,n} \dot{A}_{2,n}$, Sin@2 n qD $A_{1,n} \dot{A}_{2,n}$, $A_{2,n} \dot{A}_{2,n}$, Cos@2 n qD $A_{2,n} \dot{A}_{2,n}$, Sin@2 n qD $A_{2,n} \dot{A}_{2,n}$, $E^{-2 I T_0 w_n} \dot{A}_{1,n} \dot{A}_{2,n}$, $E^{-2 I T_0 w_n}$ Cos@2 n qD $\dot{A}_{1,n} \dot{A}_{2,n}$, $E^{-2 I T_0 w_n}$ Sin@2 n qD $\dot{A}_{1,n} \dot{A}_{2,n}$, $E^{-2 I T_0 w_n} \dot{A}_{2,n}^2$, $E^{-2 I T_0 w_n}$ Cos@2 n qD $\dot{A}_{2,n}^2$, $E^{-2 I T_0 w_n}$ Sin@2 n qD $\dot{A}_{2,n}^2$>

```
possibleTerms •• Length
```

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We note that the complex conjugate of each term on the right-hand sides of **order2Eq** is also a term on their right-hand sides. Therefore, we use this fact to reduce the algebra and calculate only the particular solutions corresponding to almost half of the terms because the particular solutions corresponding to the remaining terms are the complex conjugates of those already calculated. Then, we express the right-hand side of **order2Eq[[i]]** as the sum of two terms, **tempList[i]** and its complex conjugate, for $i = 1$ and 2, according to

```
DoAeq2rhsList = List žž order2Eq@@i, 2DD; tempList@iD = eq2rhsList;
    IfAterm = # •. conjugateRule;
   MemberQ@tempList@iD, termD, tempList@iD = Complement@tempList@iD, 8#<D;
   IfA# === term, AppendToAtempList@iD, 1/2 #EEE & •ž Reverse@eq2rhsListD, 8i, 2<E
```

Next, we determine the coefficients of the **possibleTerms** in each of the **tempList[i]** according to

```
Do@rhs = Plus žž tempList@iD •. SCond1 •• TrigReduce •• Expand;
 coeffs@iD = Coefficient@rhs, possibleTermsD •. HCos È SinL@2 n qD -> 0, 8i, 2<D
```

and denote them by $c_{i,\,j}$ as

```
symbolList1 = MapIndexed@If@#1 =!= 0, cSequence žž #2, 0D &, Array@coeffs, 2D, 82<D
```

$\{\{c_{1,1}, c_{1,2}, 0, 0, 0, c_{1,6}, c_{1,7}, c_{1,8}, 0, c_{1,10}, c_{1,11}, 0, 0, 0, c_{1,15}, 0, 0, 0, 0, 0, 0, c_{1,22}, c_{1,23}, 0, 0, 0, 0, 0, 0, 0\}, \{0, 0, c_{2,3}, 0, c_{2,5}, 0, 0, 0, c_{2,9}, 0, 0, c_{2,12}, 0, c_{2,14}, 0, 0, 0, 0, 0, 0, 0, 0, 0, c_{2,24}, 0, 0, 0, 0, 0, 0\}\}$

```
symbolList2@s_D = Plus žž symbolList1 •. c -> s;
```

where $c_{i,\,j}$ = **coeffs[i][[j]]**. Moreover, we associate with the jth term in the **symbolList1[[i]]** the undetermined coefficient $d_{i,\,j}$ in w_2 and the undetermined coefficient $e_{i,\,j}$ in y_2 by replacing the $c_{i,\,j}$ in **symbolList1[[i]]** with $d_{i,\,j}$ and $e_{i,\,j}$ according to

```
sol2temp = 8w2 -> Function@8q, T0, T1, T2<, symbolList2@dD.possibleTerms •• EvaluateD,
   y2 -> Function@8q, T0, T1, T2<, symbolList2@eD.possibleTerms •• EvaluateD<;
```

Substituting the **sol2temp** into the left-hand sides of the second-order equations, **order2Eq**, setting the coefficient of the *j*th term in **possibleTerms** in the *i*th equation equal to $c_{i,\,j}$, and solving the resulting algebraic equations, we obtain

```
deRule = Solve@
   Flatten@Table@Coefficient@order2Eq@@i, 1DD •. sol2temp, possibleTermsD •. f_@2 n qD -> 0,
      8i, 2<DD == Flatten@symbolList1D •• Thread •• Union,
   8symbolList2@dD, symbolList2@eD< •• Flatten •• Union •• RestD@@1DD
```

```
Solve::svars : Equations may not give solutions for all "solve" variables.
```

$$\Big\{e_{1,1}\to 0,\ e_{1,7}\to 0,\ d_{1,10}\to \frac{c_{1,10}}{1+a^2},\ d_{1,22}\to \frac{c_{1,22}}{1+a^2},\ e_{1,11}\to 0,\ e_{1,15}\to 0,$$

$$e_{1,23}\to 0,\ e_{2,12}\to -\frac{-2\,n\,c_{1,11}-c_{2,12}}{4\,n^2(-1+4\,n^2)^2\,a^2}+\frac{c_{2,12}}{4\,n^2},\ e_{2,14}\to \frac{c_{2,14}}{4\,n^2}+\frac{-2\,n\,c_{1,15}+c_{2,14}}{4\,n^2(-1+4\,n^2)^2\,a^2},$$

$$e_{2,24}\to -\frac{-2\,n\,c_{1,23}-c_{2,24}}{4\,n^2(-1+4\,n^2)^2\,a^2}+\frac{c_{2,24}}{4\,n^2},\ d_{1,1}\to \frac{c_{1,1}}{1+a^2-4\,w_n^2},\ d_{1,7}\to \frac{c_{1,7}}{1+a^2-4\,w_n^2},\ e_{1,2}\to 0,\ e_{1,6}\to 0,$$

$$e_{1,8}\to 0,\ e_{2,3}\to \frac{c_{2,3}}{4\,n^2-4\,w_n^2}-\frac{2\,n\,(-(4\,n^2-4\,w_n^2)\,c_{1,2}-2\,n\,c_{2,3})}{(4\,n^2-4\,w_n^2)(-4\,n^2+(4\,n^2-4\,w_n^2)(1+a^2-8\,n^2\,a^2+16\,n^4\,a^2-4\,w_n^2))},$$

$$e_{2,5}\to \frac{c_{1,6}}{2\,n}+\frac{(1+a^2-8\,n^2\,a^2+16\,n^4\,a^2-4\,w_n^2)((4\,n^2-4\,w_n^2)\,c_{1,6}-2\,n\,c_{2,5})}{2\,n(4\,n^2-(4\,n^2-4\,w_n^2)(1+a^2-8\,n^2\,a^2+16\,n^4\,a^2-4\,w_n^2))},$$

$$e_{2,9}\to -\frac{c_{1,8}}{2\,n}-\frac{(1+a^2-8\,n^2\,a^2+16\,n^4\,a^2-4\,w_n^2)((4\,n^2-4\,w_n^2)\,c_{1,8}+2\,n\,c_{2,9})}{2\,n(4\,n^2-(4\,n^2-4\,w_n^2)(1+a^2-8\,n^2\,a^2+16\,n^4\,a^2-4\,w_n^2))},$$

$$d_{1,11}\to -\frac{-2\,n\,c_{1,11}-c_{2,12}}{2\,n(-1+4\,n^2)^2\,a^2},\ d_{1,15}\to -\frac{-2\,n\,c_{1,15}+c_{2,14}}{2\,n(-1+4\,n^2)^2\,a^2},\ d_{1,23}\to -\frac{-2\,n\,c_{1,23}-c_{2,24}}{2\,n(-1+4\,n^2)^2\,a^2},$$

$$d_{2,12}\to 0,\ d_{2,14}\to 0,\ d_{2,24}\to 0,\ d_{1,2}\to -\frac{-(4\,n^2-4\,w_n^2)\,c_{1,2}-2\,n\,c_{2,3}}{-4\,n^2+(4\,n^2-4\,w_n^2)(1+a^2-8\,n^2\,a^2+16\,n^4\,a^2-4\,w_n^2)},$$

$$d_{1,6}\to -\frac{(4\,n^2-4\,w_n^2)\,c_{1,6}-2\,n\,c_{2,5}}{4\,n^2-(4\,n^2-4\,w_n^2)(1+a^2-8\,n^2\,a^2+16\,n^4\,a^2-4\,w_n^2)},$$

$$d_{1,8}\to -\frac{(4\,n^2-4\,w_n^2)\,c_{1,8}+2\,n\,c_{2,9}}{4\,n^2-(4\,n^2-4\,w_n^2)(1+a^2-8\,n^2\,a^2+16\,n^4\,a^2-4\,w_n^2)},\ d_{2,3}\to 0,\ d_{2,5}\to 0,\ d_{2,9}\to 0\Big\}$$

To simplify the expressions in the second-order solution, we introduce the notation

```
paperNotationRule = Thread[(symbolList2[c] // Union // Rest) ->
    {c2, c1, 2 c1, c2, -c1, -c4, -1/2 c3, -c3, -c4, 1/2 c3, d1, -2 d1, -d1, -1/2 d2, d2, 1/2 d2}] ~Join~
  {(1 + a^2 - 8 n^2 a^2 + 16 n^4 a^2 - 4 ω_n^2) -> (D_n + 4 n^2)/(4 n^2 - 4 ω_n^2), Power[-1 + 4 n^2, -2] Power[a, -2] -> 4 n^2/Q_n}
```

$$\Big\{c_{1,1}\to c_2,\ c_{1,2}\to c_1,\ c_{1,6}\to 2c_1,\ c_{1,7}\to c_2,\ c_{1,8}\to -c_1,\ c_{1,10}\to -c_4,\ c_{1,11}\to -\frac{c_3}{2},\ c_{1,15}\to -c_3,$$
$$c_{1,22}\to -c_4,\ c_{1,23}\to \frac{c_3}{2},\ c_{2,3}\to d_1,\ c_{2,5}\to -2d_1,\ c_{2,9}\to -d_1,\ c_{2,12}\to -\frac{d_2}{2},\ c_{2,14}\to d_2,$$
$$c_{2,24}\to \frac{d_2}{2},\ 1+a^2-8n^2a^2+16n^4a^2-4\omega_n^2\to \frac{4n^2+D_n}{4n^2-4\omega_n^2},\ \frac{1}{(-1+4n^2)^2a^2}\to \frac{4n^2}{Q_n}\Big\}$$

Then, we express the second-order solution as

```
w2Sol = symbolList2[d].possibleTerms /. deRule //. paperNotationRule;
w2Sol /. displayRule
```

$$\frac{e^{2iT_0\omega_n}c_2A_{1,n}^2}{1+a^2-4\omega_n^2}-\frac{e^{2iT_0\omega_n}\cos[2n\theta](-2nd_1-c_1(4n^2-4\omega_n^2))A_{1,n}^2}{D_n}+$$
$$\frac{e^{2iT_0\omega_n}\sin[2n\theta](4nd_1+2c_1(4n^2-4\omega_n^2))A_{1,n}A_{2,n}}{D_n}+\frac{e^{2iT_0\omega_n}c_2A_{2,n}^2}{1+a^2-4\omega_n^2}+$$
$$\frac{e^{2iT_0\omega_n}\cos[2n\theta](-2nd_1-c_1(4n^2-4\omega_n^2))A_{2,n}^2}{D_n}-\frac{c_4A_{1,n}\dot{A}_{1,n}}{1+a^2}-\frac{2n\cos[2n\theta]\left(nc_3+\frac{d_2}{2}\right)A_{1,n}\dot{A}_{1,n}}{Q_n}-$$
$$\frac{2n\sin[2n\theta](2nc_3+d_2)A_{2,n}\dot{A}_{1,n}}{Q_n}-\frac{c_4A_{2,n}\dot{A}_{2,n}}{1+a^2}-\frac{2n\cos[2n\theta]\left(-nc_3-\frac{d_2}{2}\right)A_{2,n}\dot{A}_{2,n}}{Q_n}$$

```
y2Sol = symbolList2[e].possibleTerms /. deRule //. paperNotationRule /. e_i__ -> 0;
y2Sol /. displayRule
```

$$e^{2iT_0\omega_n}\sin[2n\theta]\left(\frac{d_1}{4n^2-4\omega_n^2}-\frac{2n(-2nd_1-c_1(4n^2-4\omega_n^2))}{D_n(4n^2-4\omega_n^2)}\right)A_{1,n}^2+$$
$$e^{2iT_0\omega_n}\cos[2n\theta]\left(\frac{c_1}{n}-\frac{(4n^2+D_n)(4nd_1+2c_1(4n^2-4\omega_n^2))}{2nD_n(4n^2-4\omega_n^2)}\right)A_{1,n}A_{2,n}+$$
$$e^{2iT_0\omega_n}\sin[2n\theta]\left(\frac{c_1}{2n}+\frac{(4n^2+D_n)(-2nd_1-c_1(4n^2-4\omega_n^2))}{2nD_n(4n^2-4\omega_n^2)}\right)A_{2,n}^2+$$
$$\sin[2n\theta]\left(-\frac{d_2}{8n^2}-\frac{nc_3+\frac{d_2}{2}}{Q_n}\right)A_{1,n}\dot{A}_{1,n}+$$
$$\cos[2n\theta]\left(\frac{d_2}{4n^2}+\frac{2nc_3+d_2}{Q_n}\right)A_{2,n}\dot{A}_{1,n}+\sin[2n\theta]\left(\frac{d_2}{8n^2}-\frac{-nc_3-\frac{d_2}{2}}{Q_n}\right)A_{2,n}\dot{A}_{2,n}$$

We express this solution in pure function form as

```
sol2 = {w2 -> Function[{θ, T0, T1, T2}, w2Sol + (w2Sol /. conjugateRule) // Evaluate],
   y2 -> Function[{θ, T0, T1, T2}, y2Sol + (y2Sol /. conjugateRule) // Evaluate]};
```

Ÿ 8.2.3 Solvability Conditions

Substituting the **sol1** and **sol2** into **eqEps[[3]]** yields

```
order3Eq =
  Table@HeqEps@@1, i, 1DD •. 8w1 -> w3, y1 -> y3<L == HeqEps@@1, i, 1DD •. 8w1 -> w3, y1 -> y3<L -
     HSubtract žž eqEps@@3, iDD •. sol1 •. sol2 •• TrigToExp •• ExpandL, 8i, 2<D; •• Timing
```

```
890.5 Second, Null<
```

To convert the terms that lead to small-divisor terms in the third-order equations, **order3Eq**, into terms that lead to secular terms, we introduce the rule

```
expRule1 = Exp@arg_D :> ExpAExpand@arg •. HResonanceCond •. Equal -> RuleLD •. e^2 T0 -> T2E;
```

Then, the sources of secular terms in **order3Eq**, the terms proportional to Exp@± *I n* q + *I* w$_n$ T_0D, are

```
ST2 =
  Expand@Coefficient@#@@2DD •. Exp@c1_. + Complex@0, s_ •; Abs@sD != 1D n qD -> 0 •. expRule1,
      Exp@I wn T0DD •. Exp@c1_. + Complex@0, m_D n qD ->
      HCos@n qD + I m Sin@n qDL Exp@c1DD & •ž order3Eq; •• Timing
```

```
864.343 Second, Null<
```

These terms consist of linear and cubic terms in the complex-valued amplitudes A_i. The cubic terms are proportional to

```
cubicTerms =
  Flatten@If@Head@#D === Plus, List žž #, #D & •ž HNest@Outer@Times, collectForm, #D &,
             collectForm, 2D •• Flatten •• Union •• TrigReduce •• ExpandLD •.
      8f_@s_ n qD -> 0, Exp@c1_. + Complex@0, s_ •; s != 1D wn T0D -> 0< •.
     Exp@_D -> 1 •. c_? NumberQ form_ -> form •• Union •• Rest;
cubicTerms •. displayRule
```

```
8Cos@n qD A²1,n Ȧ1,n, Sin@n qD A²1,n Ȧ1,n, Cos@n qD A1,n A2,n Ȧ1,n, Sin@n qD A1,n A2,n Ȧ1,n,
 Cos@n qD A²2,n Ȧ1,n, Sin@n qD A²2,n Ȧ1,n, Cos@n qD A²1,n Ȧ2,n, Sin@n qD A²1,n Ȧ2,n,
 Cos@n qD A1,n A2,n Ȧ2,n, Sin@n qD A1,n A2,n Ȧ2,n, Cos@n qD A²2,n Ȧ2,n, Sin@n qD A²2,n Ȧ2,n<
```

And, their coefficients in **ST2** are

```
coef2 = Outer@Coefficient, ST2, cubicTermsD;
```

which we denote by

```
symbolList3 = MapIndexed@If@#1 =!= 0, gSequence žž #2, 0D &, coef2, 82<D;
```

```
notationRule = HsymbolList3 •• Flatten •• Union •• RestL ->
   8E2, E1, E3, E3, E1, E2, -G2, G1, -G3, G3, -G1, G2< •• Thread
```

```
8g1,1 ® E2, g1,4 ® E1, g1,5 ® E3, g1,8 ® E3, g1,9 ® E1, g1,12 ® E2,
 g2,2 ® -G2, g2,3 ® G1, g2,6 ® -G3, g2,7 ® G3, g2,10 ® -G1, g2,11 ® G2<
```

In terms of this notation, **ST2** can be rewritten as

```
ST2New = H# •. Thread@cubicTerms -> 0D & •ž ST2L + symbolList3.cubicTerms •. notationRule •.
   D@SCond1, T1D;
ST2New •. displayRule
```

```
9 1/2 E^(I T2 s) f Cos@n qD - 2 I Cos@n qD HD2A1,nL wn - 2 I HD2A2,nL Sin@n qD wn +
   Cos@n qD E2 A1,n^2 A1,n' + Sin@n qD E1 A1,n A2,n A1,n' + Cos@n qD E3 A2,n^2 A1,n' + Sin@n qD E3 A1,n^2 A2,n' +
   Cos@n qD E1 A1,n A2,n A2,n' + Sin@n qD E2 A2,n^2 A2,n', 2 I Cos@n qD HD2A2,nL Gn wn -
   2 I HD2A1,nL Sin@n qD Gn wn - Sin@n qD G2 A1,n^2 A1,n' + Cos@n qD G1 A1,n A2,n A1,n' -
   Sin@n qD G3 A2,n^2 A1,n' + Cos@n qD G3 A1,n^2 A2,n' - Sin@n qD G1 A1,n A2,n A2,n' + Cos@n qD G2 A2,n^2 A2,n=
```

The solvability conditions of **order3Eq** demand that the vector **ST2New** be orthogonal to the transpose of the **modeShapes**; that is,

```
SCond2 = ExpandA 1/p int@#, 8q, 0, 2 p<D & •ž HTranspose@modeShapesD.ST2NewL •. intRule1 ••.
      intRule2 •. 8int@Cos@s_D Sin@s_D, _D -> 0, int@f_@n qD^2, _D -> p<E == 0 •• Thread;
SCond2 •. displayRule
```

```
9 1/2 E^(I T2 s) f - 2 I HD2A1,nL wn - 2 I HD2A1,nL Gn^2 wn + E2 A1,n^2 A1,n' -
   G2 Gn A1,n^2 A1,n' + E3 A2,n^2 A1,n' - G3 Gn A2,n^2 A1,n' + E1 A1,n A2,n A2,n' - G1 Gn A1,n A2,n A2,n' == 0,
 -2 I HD2A2,nL wn - 2 I HD2A2,nL Gn^2 wn + E1 A1,n A2,n A1,n' - G1 Gn A1,n A2,n A1,n' +
   E3 A1,n^2 A2,n' - G3 Gn A1,n^2 A2,n' + E2 A2,n^2 A2,n' - G2 Gn A2,n^2 A2,n' == 0=
```

à 8.3 Near-Square Plates

We consider the nonlinear transverse vibrations of a near-square simply supported plate that is initially flat. We introduce a Cartesian coordinate system $x\,y\,z$, with the x and y axes lying in the undisturbed plane of the middle surface of the plate, and let u, v, and w denote the displacements of the middle surface in the x, y, and z directions, respectively. Then, the equations of motion are (Chu and Herrmann, 1956)

```
Needs@"Calculus`VectorAnalysis`"D
```

```
BH@wD = Biharmonic@w@x, y, tD, Cartesian@x, y, zDD
```

```
w^H0,4,0L@x, y, tD + 2 w^H2,2,0L@x, y, tD + w^H4,0,0L@x, y, tD
```

```
EOM = 9
        ux,x + wx wx,x + n Hvx,y + wy wx,yL + 1/2 H1 - nL Huy,y + vx,y + wx wy,y + wy wx,yL == cp^-2 ut,t,
        vy,y + wy wy,y + n Hux,y + wx wx,yL + 1/2 H1 - nL Hvx,x + ux,y + wy wx,x + wx wx,yL == cp^-2 vt,t,
        1/12 h^2 BH@wD - N1^i wx,x - 2 N12^i wx,y - N2^i wy,y - ux wx,x -
      1/2 wx^2 wx,x - vy wy,y - 1/2 wy^2 wy,y - n jk vy wx,x + 1/2 wy^2 wx,x + ux wy,y + 1/2 wx^2 wy,y y{ -
      H1 - nL Huy + vx + wx wyL wx,y == - cp^-2 Hwx ut,t + wy vt,t + wt,tL + F I1 - n^2M / (Es h)
    = •. Hs : u È v È wLm__ :> ¶m s@x, y, tD;
```

and the boundary conditions are

```
BC = 88u@x, y, tD == 0, w@x, y, tD == 0, ¶x,x w@x, y, tD == 0< •. 88x -> 0<, 8x -> a<<,
    8v@x, y, tD == 0, w@x, y, tD == 0, ¶y,y w@x, y, tD == 0< •. 88y -> 0<, 8y -> b<<< •• Flatten
```

```
8u@0, y, tD == 0, w@0, y, tD == 0, w^H2,0,0L@0, y, tD == 0, u@a, y, tD == 0,
 w@a, y, tD == 0, w^H2,0,0L@a, y, tD == 0, v@x, 0, tD == 0, w@x, 0, tD == 0,
 w^H0,2,0L@x, 0, tD == 0, v@x, b, tD == 0, w@x, b, tD == 0, w^H0,2,0L@x, b, tD == 0<
```

where $c_p^2 = E_s$•HrH1 - n^{2}LL, E_s is Young's modulus, n is Poisson's ratio, r and h are the density and thickness of the plate, respectively, F is the transverse load, the N_k^i are the uniformly distributed in-plane applied edge loads, and a and b are the edge lengths, with b » a. EOM are usually called the dynamic analog of the von Karman equations because they reduce to von Karman's equations (1910) in the absence of the time derivatives.

For thin plates, h • a is very small, which we denote by • !! e . Next, we introduce the aspect ratio $c = a$ • b, define the rule

```
scaleRule1 = 9h^2 -> e a^2, a -> c b=;
```

and nondimensionalize the variables as

```
charT = • !!!!!!!! 12 • e  a • cp;     H* characteristic time *L

nondimRule = 9Hdv : u È v È wL -> Ha dv@#1 • a, #2 • b, #3 • charTD &L,
    F -> F Es h • I12 a I1 - n^2MM, x -> a x, y -> b y, t -> charT t=;
```

Substituting nondimRule and scaleRule1 into EOM and BC and considering the case of no in-plane edge loads, we obtain

```
EOM1 = Ia HSubtract žž #L •. Nk_^i -> 0 •. nondimRule ••. scaleRule1 •• ExpandM == 0 & •ž EOM;

BC1 = BC •. nondimRule •. c_ h_@a__, tD :> h@a, tD
```

```
8u@0, y, tD == 0, w@0, y, tD == 0, w^H2,0,0L@0, y, tD == 0, u@1, y, tD == 0,
 w@1, y, tD == 0, w^H2,0,0L@1, y, tD == 0, v@x, 0, tD == 0, w@x, 0, tD == 0,
 w^H0,2,0L@x, 0, tD == 0, v@x, 1, tD == 0, w@x, 1, tD == 0, w^H0,2,0L@x, 1, tD == 0<
```

To write **EOM1** in a more concise way, we introduce the display rule

```
display1@expr_D := expr •. 8Derivative@a_, b_, c_D@w_D@x, y, tD :>
   Subscript@w, Sequence žž Join@Table@x, 8a<D, Table@y, 8b<D, Table@t, 8c<DDD<
```

and obtain

```
EOM1@@1DD •• display1
EOM1@@2DD •• display1
EOM1@@3DD •• display1
```

$$-\frac{1}{12} e\, u_{t,t} + u_{x,x} + \frac{1}{2} c^2 u_{y,y} - \frac{1}{2} c^2 n\, u_{y,y} + \frac{1}{2} c\, v_{x,y} + \frac{1}{2} c\, n\, v_{x,y} + w_x w_{x,x} + \frac{1}{2} c^2 w_y w_{x,y} + \frac{1}{2} c^2 n\, w_y w_{x,y} + \frac{1}{2} c^2 w_x w_{y,y} - \frac{1}{2} c^2 n\, w_x w_{y,y} == 0$$

$$\frac{1}{2} c\, u_{x,y} + \frac{1}{2} c\, n\, u_{x,y} - \frac{1}{12} e\, v_{t,t} + \frac{v_{x,x}}{2} - \frac{1}{2} n\, v_{x,x} + c^2 v_{y,y} + \frac{1}{2} c\, w_y w_{x,x} - \frac{1}{2} c\, n\, w_y w_{x,x} + \frac{1}{2} c\, w_x w_{x,y} + \frac{1}{2} c\, n\, w_x w_{x,y} + c^3 w_y w_{y,y} == 0$$

$$-\frac{F}{12} + \frac{1}{12} e\, w_x u_{t,t} + \frac{1}{12} c\, e\, w_y v_{t,t} + \frac{1}{12} e\, w_{t,t} - u_x w_{x,x} - c\, n\, v_y w_{x,x} - \frac{1}{2} w_x^2 w_{x,x} - \frac{1}{2} c^2 n\, w_y^2 w_{x,x} - c^2 u_y w_{x,y} + c^2 n\, u_y w_{x,y} - c\, v_x w_{x,y} + c\, n\, v_x w_{x,y} - c^2 w_x w_y w_{x,y} + c^2 n\, w_x w_y w_{x,y} - c^2 n\, u_x w_{y,y} - c^3 v_y w_{y,y} - \frac{1}{2} c^2 n\, w_x^2 w_{y,y} - \frac{1}{2} c^4 w_y^2 w_{y,y} + \frac{1}{12} e\, w_{x,x,x,x} + \frac{1}{6} c^2 e\, w_{x,x,y,y} + \frac{1}{12} c^4 e\, w_{y,y,y,y} == 0$$

We use the method of multiple scales to directly attack **EOM1** and **BC1**. To transform the time derivatives in **EOM1** in terms of the scales T_0 and T_1, we let

```
multiScales = 8Hdv : u È v È wL@x_, y_, tD -> dv@x, y, T0, T1D,
   Derivative@m_, n_, o_D@u_D@x_, y_, tD -> dt@oD@D@u@x, y, T0, T1D, 8x, m<, 8y, n<DD<;
```

Then, we seek a second-order approximate solution in the form

```
solRule = 9Hdv : u È vL -> I EvaluateASumAe^(j+1) dvj@#1, #2, #3, #4D, 8j, 2<EE &M,
   w -> I EvaluateASumAe^j wj@#1, #2, #3, #4D, 8j, 3<EE &M=
```

```
8dv : u È v ® He^2 dv1@#1, #2, #3, #4D + e^3 dv2@#1, #2, #3, #4D &L,
 w ® He w1@#1, #2, #3, #4D + e^2 w2@#1, #2, #3, #4D + e^3 w3@#1, #2, #3, #4D &L<
```

where #1, #2, #3, and #4 stand for x, y, T_0, and T_1, respectively.

We introduce the detuning s_1 to describe the nearness of the edge lengths a and b, consider the case of primary resonance, and hence let

```
scaleRule2 = 9c -> 1 + e s1, F -> e^3 F@x, yD Cos@W T0D=;
```

Substituting **multiScales**, **solRule**, and **scaleRule2** into **EOM1** and **BC1**, expanding the result for small e, and discarding terms of order higher than e^3, we obtain

```
eq83a =
  HEOM1 •. multiScales •. solRule •. scaleRule2 •• TrigToExp •• ExpandAllL •. e^(n_•;n>3) -> 0;

bc83a = BC1 •. multiScales •. solRule;
```

Equating coefficients of like powers of e in **eq83a** and **bc83a** yields

```
eqEps =
  Thread@CoefficientList@Subtract žž #, eD == 0D & •ž eq83a •• Transpose •• Rest •• Rest;

bcEpsuv =
 Thread@CoefficientList@Subtract žž #, eD == 0D & •ž Select@bc83a, FreeQ@#, wD &D ••
   Transpose •• Rest •• Rest

88u1@0, y, T0, T1D == 0, u1@1, y, T0, T1D == 0, v1@x, 0, T0, T1D == 0, v1@x, 1, T0, T1D == 0<,
 8u2@0, y, T0, T1D == 0, u2@1, y, T0, T1D == 0, v2@x, 0, T0, T1D == 0, v2@x, 1, T0, T1D == 0<<

bcEpsw = Thread@CoefficientList@Subtract žž #, eD == 0D & •ž
   Select@bc83a, ! FreeQ@#, wD &D •• Transpose •• Rest

99w1@0, y, T0, T1D == 0, w1^H2,0,0,0L@0, y, T0, T1D == 0,
  w1@1, y, T0, T1D == 0, w1^H2,0,0,0L@1, y, T0, T1D == 0, w1@x, 0, T0, T1D == 0,
  w1^H0,2,0,0L@x, 0, T0, T1D == 0, w1@x, 1, T0, T1D == 0, w1^H0,2,0,0L@x, 1, T0, T1D == 0=,
 9w2@0, y, T0, T1D == 0, w2^H2,0,0,0L@0, y, T0, T1D == 0, w2@1, y, T0, T1D == 0,
  w2^H2,0,0,0L@1, y, T0, T1D == 0, w2@x, 0, T0, T1D == 0, w2^H0,2,0,0L@x, 0, T0, T1D == 0,
  w2@x, 1, T0, T1D == 0, w2^H0,2,0,0L@x, 1, T0, T1D == 0=,
 9w3@0, y, T0, T1D == 0, w3^H2,0,0,0L@0, y, T0, T1D == 0, w3@1, y, T0, T1D == 0,
  w3^H2,0,0,0L@1, y, T0, T1D == 0, w3@x, 0, T0, T1D == 0, w3^H0,2,0,0L@x, 0, T0, T1D == 0,
  w3@x, 1, T0, T1D == 0, w3^H0,2,0,0L@x, 1, T0, T1D == 0==
```

Next, we introduce the display rule

```
displayRule1 = 9Derivative@a_, b_, c__DAw_i_E@x_, y_, T0, __D :>
   SequenceFormAIfAarg1 = Times žž MapIndexedAD^#1_#2@@1DD-1 &, 8c<E; arg1 =!= 1, arg1, ""E,
    Subscript@w, Sequence žž Join@8i<, Table@x, 8a<D, Table@y, 8b<DDDE,
  Derivative@a__D@A_i_D@__D :> SequenceFormATimes žž MapIndexedAD^#1_#2@@1DD &, 8a<E, A_iE,
  Derivative@a__D@Ȧ_i_D@__D :> SequenceFormATimes žž MapIndexedAD^#1_#2@@1DD &, 8a<E, Ȧ_iE,
  w_i_@__, T0, __D -> w_i, A_i_@__D -> A_i, Ȧ_i_@__D -> Ȧ_i=;
```

and write **eqEps** in a more concise way as

```
HeqEpsm = {eqEps[[1, 3]], eqEps[[1, {1, 2}]], eqEps[[2, 3]]} /. displayRule1
```

$$\Big\{\tfrac{1}{12}(w_{1,x,x,x,x})+\tfrac{1}{6}(w_{1,x,x,y,y})+\tfrac{1}{12}(w_{1,y,y,y,y})+\tfrac{1}{12}(D_0^2 w_1)==0,$$
$$\Big\{u_{1,x,x}+\tfrac{1}{2}(u_{1,y,y})-\tfrac{1}{2}\nu(u_{1,y,y})+\tfrac{1}{2}(v_{1,x,y})+\tfrac{1}{2}\nu(v_{1,x,y})+(w_{1,x})(w_{1,x,x})+$$
$$\tfrac{1}{2}(w_{1,y})(w_{1,x,y})+\tfrac{1}{2}\nu(w_{1,y})(w_{1,x,y})+\tfrac{1}{2}(w_{1,x})(w_{1,y,y})-\tfrac{1}{2}\nu(w_{1,x})(w_{1,y,y})==0,$$
$$\tfrac{1}{2}(u_{1,x,y})+\tfrac{1}{2}\nu(u_{1,x,y})+\tfrac{1}{2}(v_{1,x,x})-\tfrac{1}{2}\nu(v_{1,x,x})+v_{1,y,y}+\tfrac{1}{2}(w_{1,y})(w_{1,x,x})-$$
$$\tfrac{1}{2}\nu(w_{1,y})(w_{1,x,x})+\tfrac{1}{2}(w_{1,x})(w_{1,x,y})+\tfrac{1}{2}\nu(w_{1,x})(w_{1,x,y})+(w_{1,y})(w_{1,y,y})==0\Big\},$$
$$-\tfrac{1}{24}E^{-I T_0 \Omega}F[x,y]-\tfrac{1}{24}E^{I T_0 \Omega}F[x,y]-(u_{1,x})(w_{1,x,x})-\nu(v_{1,y})(w_{1,x,x})-\tfrac{1}{2}(w_{1,x})^2(w_{1,x,x})-$$
$$\tfrac{1}{2}\nu(w_{1,y})^2(w_{1,x,x})-(u_{1,y})(w_{1,x,y})+\nu(u_{1,y})(w_{1,x,y})-(v_{1,x})(w_{1,x,y})+\nu(v_{1,x})(w_{1,x,y})-$$
$$(w_{1,x})(w_{1,y})(w_{1,x,y})+\nu(w_{1,x})(w_{1,y})(w_{1,x,y})-\nu(u_{1,x})(w_{1,y,y})-(v_{1,y})(w_{1,y,y})-$$
$$\tfrac{1}{2}\nu(w_{1,x})^2(w_{1,y,y})-\tfrac{1}{2}(w_{1,y})^2(w_{1,y,y})+\tfrac{1}{12}(w_{2,x,x,x,x})+\tfrac{1}{6}(w_{2,x,x,y,y})+$$
$$\tfrac{1}{12}(w_{2,y,y,y,y})+\tfrac{1}{12}(D_0^2 w_2)+\tfrac{1}{6}(D_0 D_1 w_1)+\tfrac{1}{3}(w_{1,x,x,y,y})s_1+\tfrac{1}{3}(w_{1,y,y,y,y})s_1==0\Big\}$$

▼ 8.3.1 First-Order Solution

Linear vibrations are governed by **eqEpsm[[1]]** and **bcEpsw[[1]]**; that is,

```
order1Eq = eqEpsm[[1]] /. displayRule1
```

$$\tfrac{1}{12}(w_{1,x,x,x,x})+\tfrac{1}{6}(w_{1,x,x,y,y})+\tfrac{1}{12}(w_{1,y,y,y,y})+\tfrac{1}{12}(D_0^2 w_1)==0$$

```
order1BC = bcEpsw[[1]]
```

$$\{w_1[0,y,T_0,T_1]==0,\ w_1^{(2,0,0,0)}[0,y,T_0,T_1]==0,$$
$$w_1[1,y,T_0,T_1]==0,\ w_1^{(2,0,0,0)}[1,y,T_0,T_1]==0,\ w_1[x,0,T_0,T_1]==0,$$
$$w_1^{(0,2,0,0)}[x,0,T_0,T_1]==0,\ w_1[x,1,T_0,T_1]==0,\ w_1^{(0,2,0,0)}[x,1,T_0,T_1]==0\}$$

It follows from **order1Eq** and **order1BC** that the linear modeshapes and associated frequencies are given by

```
modeshapes = {φ_{m_,n_} :> (Sin[m π #1] Sin[n π #2] &)};
frequencies = {ω_{m_,n_} :> (m^2 + n^2) π^2};
```

To investigate the case of one-to-one internal resonance between the *mn*th and *nm*th modes ($m \neq n$), we let

```
w1sol = {w_1 -> Function[{x, y, T_0, T_1}, A_1[T_1] E^(I ω_{m,n} T_0) φ_{m,n}[x, y] +
     Ā_1[T_1] E^(-I ω_{m,n} T_0) φ_{m,n}[x, y] + A_2[T_1] E^(I ω_{m,n} T_0) φ_{n,m}[x, y] + Ā_2[T_1] E^(-I ω_{m,n} T_0) φ_{n,m}[x, y]]};
```

Ÿ 8.3.2 Second-Order Solution

Substituting the **w1sol** into **eqEpsm[[2]]**, we obtain

```
order2Equ = HeqEpsm@@2, 1, 1DD •. w1 -> H0 &LL ==
    HHeqEpsm@@2, 1, 1DD •. w1 -> H0 &LL - eqEpsm@@2, 1, 1DD •. w1sol •. modeshapes ••
       TrigReduce •• ExpandL; •• Timing
```

```
84.39 Second, Null<
```

```
order2Eqv = HeqEpsm@@2, 2, 1DD •. w1 -> H0 &LL ==
    HHeqEpsm@@2, 2, 1DD •. w1 -> H0 &LL - eqEpsm@@2, 2, 1DD •. w1sol •. modeshapes ••
       TrigReduce •• ExpandL; •• Timing
```

```
84.28 Second, Null<
```

The associated boundary conditions are

```
order2BC = bcEpsuv@@1DD
```

```
8u1@0, y, T0, T1D == 0, u1@1, y, T0, T1D == 0, v1@x, 0, T0, T1D == 0, v1@x, 1, T0, T1D == 0<
```

To obtain particular solutions for u_1 and v_1, we use the method of undetermined coefficients. To accomplish this, we first look for all possible terms that appear on the right-hand sides of **order2Equ** and **order2Eqv** as follows:

```
possibleTerms1 =
  CasesA8order2Equ@@2DD, order2Eqv@@2DD<, _ Sin@a_D E^b_ :> Sin@aD E^b, ¥E •• Union;
```

```
possibleTerms2 = possibleTerms1 •. Exp@_D -> 1 •• Union;
```

```
possibleTerms = Join@possibleTerms1, possibleTerms2D
```

```
8E^(-2 I T0 ωm,n) Sin@2 m p xD, E^(2 I T0 ωm,n) Sin@2 m p xD, E^(-2 I T0 ωm,n) Sin@2 n p xD, E^(2 I T0 ωm,n) Sin@2 n p xD,
 E^(-2 I T0 ωm,n) Sin@2 m p yD, E^(2 I T0 ωm,n) Sin@2 m p yD, E^(-2 I T0 ωm,n) Sin@2 n p yD, E^(2 I T0 ωm,n) Sin@2 n p yD,
 E^(-2 I T0 ωm,n) Sin@2 n p x - 2 m p yD, E^(2 I T0 ωm,n) Sin@2 n p x - 2 m p yD, E^(-2 I T0 ωm,n) Sin@2 n p x + 2 m p yD,
 E^(2 I T0 ωm,n) Sin@2 n p x + 2 m p yD, E^(-2 I T0 ωm,n) Sin@2 m p x - 2 n p yD, E^(2 I T0 ωm,n) Sin@2 m p x - 2 n p yD,
 E^(-2 I T0 ωm,n) Sin@m p x - n p x - m p y - n p yD, E^(2 I T0 ωm,n) Sin@m p x - n p x - m p y - n p yD,
 E^(-2 I T0 ωm,n) Sin@m p x + n p x - m p y - n p yD, E^(2 I T0 ωm,n) Sin@m p x + n p x - m p y - n p yD,
 E^(-2 I T0 ωm,n) Sin@m p x - n p x + m p y - n p yD, E^(2 I T0 ωm,n) Sin@m p x - n p x + m p y - n p yD,
 E^(-2 I T0 ωm,n) Sin@m p x + n p x + m p y - n p yD, E^(2 I T0 ωm,n) Sin@m p x + n p x + m p y - n p yD,
 E^(-2 I T0 ωm,n) Sin@m p x - n p x - m p y + n p yD, E^(2 I T0 ωm,n) Sin@m p x - n p x - m p y + n p yD,
 E^(-2 I T0 ωm,n) Sin@m p x + n p x - m p y + n p yD, E^(2 I T0 ωm,n) Sin@m p x + n p x - m p y + n p yD,
 E^(-2 I T0 ωm,n) Sin@m p x - n p x + m p y + n p yD, E^(2 I T0 ωm,n) Sin@m p x - n p x + m p y + n p yD,
 E^(-2 I T0 ωm,n) Sin@m p x + n p x + m p y + n p yD, E^(2 I T0 ωm,n) Sin@m p x + n p x + m p y + n p yD,
 E^(-2 I T0 ωm,n) Sin@2 m p x + 2 n p yD, E^(2 I T0 ωm,n) Sin@2 m p x + 2 n p yD, Sin@2 m p xD, Sin@2 n p xD,
 Sin@2 m p yD, Sin@2 n p yD, Sin@2 n p x - 2 m p yD, Sin@2 n p x + 2 m p yD, Sin@2 m p x - 2 n p yD,
 Sin@m p x - n p x - m p y - n p yD, Sin@m p x + n p x - m p y - n p yD, Sin@m p x - n p x + m p y - n p yD,
 Sin@m p x + n p x + m p y - n p yD, Sin@m p x - n p x - m p y + n p yD, Sin@m p x + n p x - m p y + n p yD,
 Sin@m p x - n p x + m p y + n p yD, Sin@m p x + n p x + m p y + n p yD, Sin@2 m p x + 2 n p yD<
```

which satisfy **order2BC**. We then define the following symbol lists to represent the undetermined coefficients.

```
usymbolList = Table@c_i, 8i, Length@possibleTermsD<D
```

```
8c1, c2, c3, c4, c5, c6, c7, c8, c9, c10, c11, c12, c13, c14, c15, c16,
 c17, c18, c19, c20, c21, c22, c23, c24, c25, c26, c27, c28, c29, c30, c31, c32,
 c33, c34, c35, c36, c37, c38, c39, c40, c41, c42, c43, c44, c45, c46, c47, c48<
```

```
vsymbolList = usymbolList •. c -> d
```

```
8d1, d2, d3, d4, d5, d6, d7, d8, d9, d10, d11, d12, d13, d14, d15, d16,
 d17, d18, d19, d20, d21, d22, d23, d24, d25, d26, d27, d28, d29, d30, d31, d32,
 d33, d34, d35, d36, d37, d38, d39, d40, d41, d42, d43, d44, d45, d46, d47, d48<
```

Then, we express the solution as

```
u1sol = u1 -> Function@8x, y, T0, T1<, usymbolList.possibleTerms •• EvaluateD;
v1sol = v1 -> Function@8x, y, T0, T1<, vsymbolList.possibleTerms •• EvaluateD;
```

Substituting **u1sol** and **v1sol** into **order2Equ** and **order2Eqv**, collecting coefficients of **possibleTerms**, and solving for the **usymbolList** and **vsymbolList**, we obtain

```
algEq1 = Coefficient@Subtract žž order2Equ •. u1sol •. v1sol, possibleTermsD == 0 •.
     Exp@_ T0D -> 0 •• Thread; •• Timing
```

```
867.45 Second, Null<
```

```
algEq2 = Coefficient@Subtract žž order2Eqv •. u1sol •. v1sol, possibleTermsD == 0 •.
    Exp@_ T0D -> 0 •• Thread; •• Timing
```

865.91 Second, Null<

```
symbolRule1 = Solve@Join@algEq1, algEq2D, Join@usymbolList, vsymbolListDD@@1DD •• Simplify
```

:c_1 ® $-\frac{p\,Hm^2 - n^2\,nL\,\dot{A}_1@T_1D^2}{16\,m}$, c_2 ® $-\frac{p\,Hm^2 - n^2\,nL\,A_1@T_1D^2}{16\,m}$, c_3 ® $-\frac{p\,Hn^2 - m^2\,nL\,\dot{A}_2@T_1D^2}{16\,n}$,

c_4 ® $-\frac{p\,Hn^2 - m^2\,nL\,A_2@T_1D^2}{16\,n}$, c_{33} ® $-\frac{p\,Hm^2 - n^2\,nL\,A_1@T_1D\,\dot{A}_1@T_1D}{8\,m}$, c_{34} ® $-\frac{p\,Hn^2 - m^2\,nL\,A_2@T_1D\,\dot{A}_2@T_1D}{8\,n}$,

d_5 ® $-\frac{p\,Hm^2 - n^2\,nL\,\dot{A}_2@T_1D^2}{16\,m}$, d_6 ® $-\frac{p\,Hm^2 - n^2\,nL\,A_2@T_1D^2}{16\,m}$, d_7 ® $-\frac{p\,Hn^2 - m^2\,nL\,\dot{A}_1@T_1D^2}{16\,n}$,

d_8 ® $-\frac{p\,Hn^2 - m^2\,nL\,A_1@T_1D^2}{16\,n}$, d_{35} ® $-\frac{p\,Hm^2 - n^2\,nL\,A_2@T_1D\,\dot{A}_2@T_1D}{8\,m}$, d_{36} ® $-\frac{p\,Hn^2 - m^2\,nL\,A_1@T_1D\,\dot{A}_1@T_1D}{8\,n}$,

c_5 ® 0, c_6 ® 0, c_7 ® 0, c_8 ® 0, c_{35} ® 0, c_{36} ® 0, d_1 ® 0, d_2 ® 0, d_3 ® 0, d_4 ® 0,

d_{33} ® 0, d_{34} ® 0, c_9 ® $\frac{1}{32}\,n\,p\,\dot{A}_2@T_1D^2$, c_{10} ® $\frac{1}{32}\,n\,p\,A_2@T_1D^2$, c_{11} ® $\frac{1}{32}\,n\,p\,\dot{A}_2@T_1D^2$,

c_{12} ® $\frac{1}{32}\,n\,p\,A_2@T_1D^2$, c_{13} ® $\frac{1}{32}\,m\,p\,\dot{A}_1@T_1D^2$, c_{14} ® $\frac{1}{32}\,m\,p\,A_1@T_1D^2$, c_{31} ® $\frac{1}{32}\,m\,p\,\dot{A}_1@T_1D^2$,

c_{32} ® $\frac{1}{32}\,m\,p\,A_1@T_1D^2$, c_{37} ® $\frac{1}{16}\,n\,p\,A_2@T_1D\,\dot{A}_2@T_1D$, c_{38} ® $\frac{1}{16}\,n\,p\,A_2@T_1D\,\dot{A}_2@T_1D$,

c_{39} ® $\frac{1}{16}\,m\,p\,A_1@T_1D\,\dot{A}_1@T_1D$, c_{48} ® $\frac{1}{16}\,m\,p\,A_1@T_1D\,\dot{A}_1@T_1D$, d_9 ® $-\frac{1}{32}\,m\,p\,\dot{A}_2@T_1D^2$,

d_{10} ® $-\frac{1}{32}\,m\,p\,A_2@T_1D^2$, d_{11} ® $\frac{1}{32}\,m\,p\,\dot{A}_2@T_1D^2$, d_{12} ® $\frac{1}{32}\,m\,p\,A_2@T_1D^2$, d_{13} ® $-\frac{1}{32}\,n\,p\,\dot{A}_1@T_1D^2$,

d_{14} ® $-\frac{1}{32}\,n\,p\,A_1@T_1D^2$, d_{31} ® $\frac{1}{32}\,n\,p\,\dot{A}_1@T_1D^2$, d_{32} ® $\frac{1}{32}\,n\,p\,A_1@T_1D^2$, d_{37} ® $-\frac{1}{16}\,m\,p\,A_2@T_1D\,\dot{A}_2@T_1D$,

d_{38} ® $\frac{1}{16}\,m\,p\,A_2@T_1D\,\dot{A}_2@T_1D$, d_{39} ® $-\frac{1}{16}\,n\,p\,A_1@T_1D\,\dot{A}_1@T_1D$, d_{48} ® $\frac{1}{16}\,n\,p\,A_1@T_1D\,\dot{A}_1@T_1D$,

c_{15} ® $\frac{1}{32}\,Hm - nL\,p\,H-1 + nL\,\dot{A}_1@T_1D\,\dot{A}_2@T_1D$, c_{16} ® $\frac{1}{32}\,Hm - nL\,p\,H-1 + nL\,A_1@T_1D\,A_2@T_1D$,

c_{21} ® $\frac{1}{32}\,Hm + nL\,p\,H-1 + nL\,\dot{A}_1@T_1D\,\dot{A}_2@T_1D$, c_{22} ® $\frac{1}{32}\,Hm + nL\,p\,H-1 + nL\,A_1@T_1D\,A_2@T_1D$,

c_{25} ® $\frac{1}{32}\,Hm + nL\,p\,H-1 + nL\,\dot{A}_1@T_1D\,\dot{A}_2@T_1D$, c_{26} ® $\frac{1}{32}\,Hm + nL\,p\,H-1 + nL\,A_1@T_1D\,A_2@T_1D$,

c_{27} ® $\frac{1}{32}\,Hm - nL\,p\,H-1 + nL\,\dot{A}_1@T_1D\,\dot{A}_2@T_1D$, c_{28} ® $\frac{1}{32}\,Hm - nL\,p\,H-1 + nL\,A_1@T_1D\,A_2@T_1D$,

c_{40} ® $\frac{1}{32}\,Hm - nL\,p\,H-1 + nL\,HA_2@T_1D\,\dot{A}_1@T_1D + A_1@T_1D\,\dot{A}_2@T_1DL$,

c_{43} ® $\frac{1}{32}\,Hm + nL\,p\,H-1 + nL\,HA_2@T_1D\,\dot{A}_1@T_1D + A_1@T_1D\,\dot{A}_2@T_1DL$,

c_{45} ® $\frac{1}{32}\,Hm + nL\,p\,H-1 + nL\,HA_2@T_1D\,\dot{A}_1@T_1D + A_1@T_1D\,\dot{A}_2@T_1DL$,

c_{46} ® $\frac{1}{32}\,Hm - nL\,p\,H-1 + nL\,HA_2@T_1D\,\dot{A}_1@T_1D + A_1@T_1D\,\dot{A}_2@T_1DL$,

d_{15} ® $-\frac{1}{32}\,Hm + nL\,p\,H-1 + nL\,\dot{A}_1@T_1D\,\dot{A}_2@T_1D$, d_{16} ® $-\frac{1}{32}\,Hm + nL\,p\,H-1 + nL\,A_1@T_1D\,A_2@T_1D$,

$d_{21} ® \frac{1}{32} Hm - nL\, p\, H-1 + nL\, \dot{A}_1@T_1D\, \dot{A}_2@T_1D$, $d_{22} ® \frac{1}{32} Hm - nL\, p\, H-1 + nL\, A_1@T_1D\, A_2@T_1D$,

$d_{25} ® -\frac{1}{32} Hm - nL\, p\, H-1 + nL\, \dot{A}_1@T_1D\, \dot{A}_2@T_1D$, $d_{26} ® -\frac{1}{32} Hm - nL\, p\, H-1 + nL\, A_1@T_1D\, A_2@T_1D$,

$d_{27} ® \frac{1}{32} Hm + nL\, p\, H-1 + nL\, \dot{A}_1@T_1D\, \dot{A}_2@T_1D$, $d_{28} ® \frac{1}{32} Hm + nL\, p\, H-1 + nL\, A_1@T_1D\, A_2@T_1D$,

$d_{40} ® -\frac{1}{32} Hm + nL\, p\, H-1 + nL\, HA_2@T_1D\, \dot{A}_1@T_1D + A_1@T_1D\, \dot{A}_2@T_1DL$,

$d_{43} ® \frac{1}{32} Hm - nL\, p\, H-1 + nL\, HA_2@T_1D\, \dot{A}_1@T_1D + A_1@T_1D\, \dot{A}_2@T_1DL$,

$d_{45} ® -\frac{1}{32} Hm - nL\, p\, H-1 + nL\, HA_2@T_1D\, \dot{A}_1@T_1D + A_1@T_1D\, \dot{A}_2@T_1DL$,

$d_{46} ® \frac{1}{32} Hm + nL\, p\, H-1 + nL\, HA_2@T_1D\, \dot{A}_1@T_1D + A_1@T_1D\, \dot{A}_2@T_1DL$,

$c_{17} ® -\frac{p\, Hm^2\, H-1+nL + n^2\, H-1+nL - 2\, m\, n\, H1+nLL\, \dot{A}_1@T_1D\, \dot{A}_2@T_1D}{32\, Hm+nL}$,

$c_{18} ® -\frac{p\, Hm^2\, H-1+nL + n^2\, H-1+nL - 2\, m\, n\, H1+nLL\, A_1@T_1D\, A_2@T_1D}{32\, Hm+nL}$,

$c_{19} ® -\frac{p\, Hm^2\, H-1+nL + n^2\, H-1+nL + 2\, m\, n\, H1+nLL\, \dot{A}_1@T_1D\, \dot{A}_2@T_1D}{32\, Hm-nL}$,

$c_{20} ® -\frac{p\, Hm^2\, H-1+nL + n^2\, H-1+nL + 2\, m\, n\, H1+nLL\, A_1@T_1D\, A_2@T_1D}{32\, Hm-nL}$,

$c_{23} ® -\frac{p\, Hm^2\, H-1+nL + n^2\, H-1+nL + 2\, m\, n\, H1+nLL\, \dot{A}_1@T_1D\, \dot{A}_2@T_1D}{32\, Hm-nL}$,

$c_{24} ® -\frac{p\, Hm^2\, H-1+nL + n^2\, H-1+nL + 2\, m\, n\, H1+nLL\, A_1@T_1D\, A_2@T_1D}{32\, Hm-nL}$,

$c_{29} ® -\frac{p\, Hm^2\, H-1+nL + n^2\, H-1+nL - 2\, m\, n\, H1+nLL\, \dot{A}_1@T_1D\, \dot{A}_2@T_1D}{32\, Hm+nL}$,

$c_{30} ® -\frac{p\, Hm^2\, H-1+nL + n^2\, H-1+nL - 2\, m\, n\, H1+nLL\, A_1@T_1D\, A_2@T_1D}{32\, Hm+nL}$,

$c_{41} ® -\frac{p\, Hm^2\, H-1+nL + n^2\, H-1+nL - 2\, m\, n\, H1+nLL\, HA_2@T_1D\, \dot{A}_1@T_1D + A_1@T_1D\, \dot{A}_2@T_1DL}{32\, Hm+nL}$,

$c_{42} ® -\frac{p\, Hm^2\, H-1+nL + n^2\, H-1+nL + 2\, m\, n\, H1+nLL\, HA_2@T_1D\, \dot{A}_1@T_1D + A_1@T_1D\, \dot{A}_2@T_1DL}{32\, Hm-nL}$,

$c_{44} ® -\frac{p\, Hm^2\, H-1+nL + n^2\, H-1+nL + 2\, m\, n\, H1+nLL\, HA_2@T_1D\, \dot{A}_1@T_1D + A_1@T_1D\, \dot{A}_2@T_1DL}{32\, Hm-nL}$,

$c_{47} ® -\frac{p\, Hm^2\, H-1+nL + n^2\, H-1+nL - 2\, m\, n\, H1+nLL\, HA_2@T_1D\, \dot{A}_1@T_1D + A_1@T_1D\, \dot{A}_2@T_1DL}{32\, Hm+nL}$,

$d_{17} ® \frac{p\, Hm^2\, H-1+nL + n^2\, H-1+nL - 2\, m\, n\, H1+nLL\, \dot{A}_1@T_1D\, \dot{A}_2@T_1D}{32\, Hm+nL}$,

$d_{18} ® \frac{p\, Hm^2\, H-1+nL + n^2\, H-1+nL - 2\, m\, n\, H1+nLL\, A_1@T_1D\, A_2@T_1D}{32\, Hm+nL}$,

$d_{19} ® -\frac{p\, Hm^2\, H-1+nL + n^2\, H-1+nL + 2\, m\, n\, H1+nLL\, \dot{A}_1@T_1D\, \dot{A}_2@T_1D}{32\, Hm-nL}$,

$$d_{20} \to -\frac{\pi\,(m^2(-1+n)+n^2(-1+n)+2\,m\,n\,(1+n))\,A_1[T_1]\,A_2[T_1]}{32\,(m-n)},$$

$$d_{23} \to \frac{\pi\,(m^2(-1+n)+n^2(-1+n)+2\,m\,n\,(1+n))\,\dot{A}_1[T_1]\,\dot{A}_2[T_1]}{32\,(m-n)},$$

$$d_{24} \to \frac{\pi\,(m^2(-1+n)+n^2(-1+n)+2\,m\,n\,(1+n))\,A_1[T_1]\,A_2[T_1]}{32\,(m-n)},$$

$$d_{29} \to -\frac{\pi\,(m^2(-1+n)+n^2(-1+n)-2\,m\,n\,(1+n))\,\dot{A}_1[T_1]\,\dot{A}_2[T_1]}{32\,(m+n)},$$

$$d_{30} \to -\frac{\pi\,(m^2(-1+n)+n^2(-1+n)-2\,m\,n\,(1+n))\,A_1[T_1]\,A_2[T_1]}{32\,(m+n)},$$

$$d_{41} \to \frac{\pi\,(m^2(-1+n)+n^2(-1+n)-2\,m\,n\,(1+n))\,(A_2[T_1]\,\dot{A}_1[T_1]+A_1[T_1]\,\dot{A}_2[T_1])}{32\,(m+n)},$$

$$d_{42} \to -\frac{\pi\,(m^2(-1+n)+n^2(-1+n)+2\,m\,n\,(1+n))\,(A_2[T_1]\,\dot{A}_1[T_1]+A_1[T_1]\,\dot{A}_2[T_1])}{32\,(m-n)},$$

$$d_{44} \to \frac{\pi\,(m^2(-1+n)+n^2(-1+n)+2\,m\,n\,(1+n))\,(A_2[T_1]\,\dot{A}_1[T_1]+A_1[T_1]\,\dot{A}_2[T_1])}{32\,(m-n)},$$

$$d_{47} \to -\frac{\pi\,(m^2(-1+n)+n^2(-1+n)-2\,m\,n\,(1+n))\,(A_2[T_1]\,\dot{A}_1[T_1]+A_1[T_1]\,\dot{A}_2[T_1])}{32\,(m+n)}\}$$

• 8.3.3 Solvability Conditions

Substituting the **w1sol**, **u1sol**, and **v1sol** into **eqEpsm[[3]]**, we obtain

```
order3Eq = (order1Eq[[1]] /. w1 -> w2) ==
    ((order1Eq[[1]] /. w1 -> w2) - eqEpsm[[3, 1]] /. w1sol /. u1sol /. v1sol /. modeshapes //
      Expand); // Timing
```

{84.18 Second, Null}

The associated boundary conditions are

```
order3BC = bcEpsw[[2]]
```

$\{w_2[0, y, T_0, T_1] == 0,\ w_2^{(2,0,0,0)}[0, y, T_0, T_1] == 0,$
$w_2[1, y, T_0, T_1] == 0,\ w_2^{(2,0,0,0)}[1, y, T_0, T_1] == 0,\ w_2[x, 0, T_0, T_1] == 0,$
$w_2^{(0,2,0,0)}[x, 0, T_0, T_1] == 0,\ w_2[x, 1, T_0, T_1] == 0,\ w_2^{(0,2,0,0)}[x, 1, T_0, T_1] == 0\}$

To describe the nearness of the primary resonance, we introduce the detuning parameter σ_2 defined by

```
ResonanceCond = {Ω -> ω_{m,n} + ε σ2};
```

Because the homogeneous parts of **order3Eq** and **order3BC** have a nontrivial solution, the corresponding nonhomogeneous problem has a solution only if solvability conditions are satisfied. These conditions demand that the right-hand side of **order3Eq** be orthogonal to every solution of the adjoint homogeneous problem. In this case, the problem is self-adjoint.

To determine the solvability conditions, we collect the terms that may lead to secular terms, the terms proportional to $E^{I \, \omega_{m,n} T_0}$. To this end, we use the rule

```
expRule = Exp@a_D :> Exp@Expand@a •. ResonanceCondD •. e T0 -> T1D;
```

and obtain

```
ST = CoefficientAorder3Eq@@2DD •. expRule, E^(I wm,n T0)E •• TrigReduce; •• Timing
```

```
8132.53 Second, Null<
```

Among all the spatial-dependent non-forcing terms in **ST**, only the following terms are not orthogonal to the **modeshapes**:

```
terms1 = Cases@ST, Cos@m p x + a_. n p y •; Abs@aD == 1D, ¥D •• Union
```

```
8Cos@m p x - n p yD, Cos@m p x + n p yD<
```

```
terms2 = Cases@ST, Cos@n p x + a_. m p y •; Abs@aD == 1D, ¥D •• Union
```

```
8Cos@n p x - m p yD, Cos@n p x + m p yD<
```

The coefficients of these terms in **ST** are

```
terms1Coef = Coefficient@ST, terms1D;
```

```
terms2Coef = Coefficient@ST, terms2D;
```

The projection of **terms1** onto Sin@*m* p *x*D Sin@*n* p *y*D can be obtained as

```
trigRule = 8Cos@2 _. pD -> 1, Sin@_. pD -> 0<;
```

```
proj1Coef = à_0^1 à_0^1 Sin@m p xD Sin@n p yD # â x â y & •ž terms1 •. trigRule
```

```
9 1/4, - 1/4 =
```

and the projection of **terms2** onto Sin@*n* p *x*D Sin@*m* p *y*D can be obtained as

```
proj2Coef = à_0^1 à_0^1 Sin@n p xD Sin@m p yD # â x â y & •ž terms2 •. trigRule
```

```
9 1/4, - 1/4 =
```

The forcing term in **ST** is

```
f0 = ST •. Cos@_D -> 0
```

```
1/24 E^(I T1 s2) F@x, yD
```

Then, the solvability conditions demand that **ST** be orthogonal to every linear eigenfunction, **modeshapes**; that is,

```
intRule3 = 8int@a_ fun_, a1_, b1_D :> a int@fun, a1, b1D •; FreeQ@a, First@a1DÈ First@b1DD<;

SCond1 =
  Expand@24 Hint@Sin@m p xD Sin@n p yD f0, 8x, 0, 1<, 8y, 0, 1<D + proj1Coef.terms1Coef •.
        symbolRule1 ••. intRule3 •. int -> IntegrateLD == 0;

SCond2 =
  Expand@24 Hint@Sin@n p xD Sin@m p yD f0, 8x, 0, 1<, 8y, 0, 1<D + proj2Coef.terms2Coef •.
        symbolRule1 ••. intRule3 •. int -> IntegrateLD == 0;
```

To express these solvability conditions in a more readable form, we define

```
basicTerms = 9A1@T1D E^(I wm,n T0), Ȧ1@T1D E^(-I wm,n T0), A2@T1D E^(I wm,n T0), Ȧ2@T1D E^(-I wm,n T0)=;

cubicTerms =
 I E^(-I wm,n T0) Nest@Outer@Times, basicTerms, #D &, basicTerms, 2D •• FlattenM •. Exp@_ T0D -> 0 ••
   Union •• Rest

8A1@T1D^2 Ȧ1@T1D, A1@T1D A2@T1D Ȧ1@T1D, A2@T1D^2 Ȧ1@T1D,
 A1@T1D^2 Ȧ2@T1D, A1@T1D A2@T1D Ȧ2@T1D, A2@T1D^2 Ȧ2@T1D<
```

Then, we collect the coefficients of **cubicTerms** from **SCond1** and **SCond2** and obtain

```
coef1 = Coefficient@SCond1@@1DD, cubicTermsD •• Simplify

9 9/8 p^4 H-4 m^2 n^2 n + m^4 H-3 + n^2L + n^4 H-3 + n^2LL, 0,
 3/8 p^4 H4 m^2 n^2 H-2 + n^2L + m^4 H-1 - 2 n + n^2L + n^4 H-1 - 2 n + n^2LL, 0,
 3/4 p^4 H4 m^2 n^2 H-2 + n^2L + m^4 H-1 - 2 n + n^2L + n^4 H-1 - 2 n + n^2LL, 0=

coef2 = Coefficient@SCond2@@1DD, cubicTermsD •• Simplify

90, 3/4 p^4 H4 m^2 n^2 H-2 + n^2L + m^4 H-1 - 2 n + n^2L + n^4 H-1 - 2 n + n^2LL,
 0, 3/8 p^4 H4 m^2 n^2 H-2 + n^2L + m^4 H-1 - 2 n + n^2L + n^4 H-1 - 2 n + n^2LL,
 0, 9/8 p^4 H-4 m^2 n^2 n + m^4 H-3 + n^2L + n^4 H-3 + n^2LL=
```

Hence, **SCond1** and **SCond2** can be rewritten and combined as

```
SCond = 8Collect@SCond1@@1DD •. Thread@cubicTerms -> 0D, s1 A1@T1DD + coef1.cubicTerms == 0,
   Collect@SCond2@@1DD •. Thread@cubicTerms -> 0D, s1 A2@T1DD + coef2.cubicTerms == 0<;
SCond •. displayRule1
```

```
:E^(I T1 s2) à_0^1 à_0^1 F@x, yD Sin@m p xD Sin@n p yD â y â x +
   H- 2 m^2 n^2 p^4 - 2 n^4 p^4L A1 s1 + 9/8 p^4 H- 4 m^2 n^2 n + m^4 H- 3 + n^2L + n^4 H- 3 + n^2LL A1^2 Ȧ1 +
   3/8 p^4 H4 m^2 n^2 H- 2 + n^2L + m^4 H- 1 - 2 n + n^2L + n^4 H- 1 - 2 n + n^2LL A2^2 Ȧ1 +
   3/4 p^4 H4 m^2 n^2 H- 2 + n^2L + m^4 H- 1 - 2 n + n^2L + n^4 H- 1 - 2 n + n^2LL A1 A2 Ȧ2 - I HD1A1L w_{m,n} == 0,
 E^(I T1 s2) à_0^1 à_0^1 F@x, yD Sin@n p xD Sin@m p yD â y â x + H- 2 m^4 p^4 - 2 m^2 n^2 p^4L A2 s1 +
   3/4 p^4 H4 m^2 n^2 H- 2 + n^2L + m^4 H- 1 - 2 n + n^2L + n^4 H- 1 - 2 n + n^2LL A1 A2 Ȧ1 +
   3/8 p^4 H4 m^2 n^2 H- 2 + n^2L + m^4 H- 1 - 2 n + n^2L + n^4 H- 1 - 2 n + n^2LL A1^2 Ȧ2 +
   9/8 p^4 H- 4 m^2 n^2 n + m^4 H- 3 + n^2L + n^4 H- 3 + n^2LL A2^2 Ȧ2 - I HD1A2L w_{m,n} == 0>
```

Ÿ 8.3.4 Mixed Approach

Alternatively, we can use a mixed approach, a combination of discretization and direct approaches, to attack the problem, **EOM1** and **BC1**. We first define

```
scaleRule3 = 9Hdv : u È vL -> I e^2 dv@#1, #2, #3D &M, w -> He w@#1, #2, #3D &L=;
```

Substituting **scaleRule2** and **scaleRule3** into **EOM1**, expanding the result for small e, and discarding terms of order higher than e^3, we obtain

```
eq83b = HEOM1 •. scaleRule2 •. scaleRule3 •• TrigToExp •• ExpandAllL •. e^(n_•;n>3) -> 0;
```

To write **eq83b** in a more concise way, we introduce the dispaly rule

```
display2@expr_D := expr •. 8Derivative@a_, b_, c_D@w_D@x, y, tD :>
    Subscript@w, Sequence žž Join@Table@x, 8a<D, Table@y, 8b<D, Table@t, 8c<DDD<
```

Hence,

```
IuEOM = CoefficientAeq83b@@1, 1DD, e^2E == 0M •• display2
```

$$u_{x,x} + \frac{u_{y,y}}{2} - \frac{1}{2} n\, u_{y,y} + \frac{v_{x,y}}{2} + \frac{1}{2} n\, v_{x,y} + w_x w_{x,x} + \frac{1}{2} w_y w_{x,y} + \frac{1}{2} n\, w_y w_{x,y} + \frac{1}{2} w_x w_{y,y} - \frac{1}{2} n\, w_x w_{y,y} == 0$$

```
IvEOM = CoefficientAeq83b@@2, 1DD, e^2E == 0M •• display2
```

$$\frac{u_{x,y}}{2} + \frac{1}{2} n\, u_{x,y} + \frac{v_{x,x}}{2} - \frac{1}{2} n\, v_{x,x} + v_{y,y} + \frac{1}{2} w_y w_{x,x} - \frac{1}{2} n\, w_y w_{x,x} + \frac{1}{2} w_x w_{x,y} + \frac{1}{2} n\, w_x w_{x,y} + w_y w_{y,y} == 0$$

```
HwEOM = eq83b@@3DD •• ThreadAe^-2 #, EqualE & •• ExpandAllM •• display2
```

$-\frac{1}{24} E^{-I T_0 W} e F@x, yD - \frac{1}{24} E^{I T_0 W} e F@x, yD + \frac{w_{t,t}}{12} - e u_x w_{x,x} -$
$e n v_y w_{x,x} - \frac{1}{2} e w_x^2 w_{x,x} - \frac{1}{2} e n w_y^2 w_{x,x} - e u_y w_{x,y} + e n u_y w_{x,y} - e v_x w_{x,y} + e n v_x w_{x,y} -$
$e w_x w_y w_{x,y} + e n w_x w_y w_{x,y} - e n u_x w_{y,y} - e v_y w_{y,y} - \frac{1}{2} e n w_x^2 w_{y,y} - \frac{1}{2} e w_y^2 w_{y,y} +$
$\frac{1}{12} w_{x,x,x,x} + \frac{1}{6} w_{x,x,y,y} + \frac{1}{3} e s_1 w_{x,x,y,y} + \frac{1}{12} w_{y,y,y,y} + \frac{1}{3} e s_1 w_{y,y,y,y} == 0$

Linear vibrations are governed by **wEOM** with e = 0; that is,

```
Horder1Eqw = wEOM •. e -> 0L •• display2
```

$\frac{w_{t,t}}{12} + \frac{1}{12} w_{x,x,x,x} + \frac{1}{6} w_{x,x,y,y} + \frac{1}{12} w_{y,y,y,y} == 0$

subject to the boundary conditions **BC1**. It follows from **order1Eqw** and **BC1** that the linear modeshapes and associated frequencies are given by

```
f_{m,n}@x, yD •. modeshapes
```

Sin@m p xD Sin@n p yD

```
w_{m,n} •. frequencies
```

$Hm^2 + n^2L\ p^2$

To investigate the case of one-to-one internal resonance between the *mn*th and *nm*th modes (*m* ¹ *n*), we let

```
w1sol = 8w -> Function@8x, y, t<, h@tD Sin@m p xD Sin@n p yD + z@tD Sin@n p xD Sin@m p yDD<;
```

Substituting the **w1sol** into **uEOM** and **vEOM**, we obtain

```
order2Equ =
  HuEOM@@1DD •. w -> H0 &LL == HHuEOM@@1DD •. w -> H0 &LL - uEOM@@1DD •. w1sol •• TrigReduceL;
order2Equ •• display2
```

$u_{x,x} + \frac{u_{y,y}}{2} - \frac{1}{2} n u_{y,y} + \frac{v_{x,y}}{2} + \frac{1}{2} n v_{x,y} ==$

$\frac{1}{16} H4 n^3 p^3 Sin@2 n p xD z@tD^2 - 4 m^2 n p^3 n Sin@2 n p xD z@tD^2 -$
$2 m^2 n p^3 Sin@2 n p x - 2 m p yD z@tD^2 - 2 n^3 p^3 Sin@2 n p x - 2 m p yD z@tD^2 -$
$2 m^2 n p^3 Sin@2 n p x + 2 m p yD z@tD^2 - 2 n^3 p^3 Sin@2 n p x + 2 m p yD z@tD^2 +$
$m^3 p^3 Sin@m p x - n p x - m p y - n p yD z@tD h@tD - m^2 n p^3 Sin@m p x - n p x - m p y - n p yD z@tD h@tD +$
$m n^2 p^3 Sin@m p x - n p x - m p y - n p yD z@tD h@tD - n^3 p^3 Sin@m p x - n p x - m p y - n p yD z@tD h@tD -$
$m^3 p^3 n Sin@m p x - n p x - m p y - n p yD z@tD h@tD + m^2 n p^3 n Sin@m p x - n p x - m p y - n p yD$
$z@tD h@tD - m n^2 p^3 n Sin@m p x - n p x - m p y - n p yD z@tD h@tD +$
$n^3 p^3 n Sin@m p x - n p x - m p y - n p yD z@tD h@tD - m^3 p^3 Sin@m p x + n p x - m p y - n p yD z@tD h@tD -$
$3 m^2 n p^3 Sin@m p x + n p x - m p y - n p yD z@tD h@tD -$

```
3 m n^2 p^3 Sin@m p x + n p x - m p y - n p yD z@tD h@tD -
n^3 p^3 Sin@m p x + n p x - m p y - n p yD z@tD h@tD + m^3 p^3 n Sin@m p x + n p x - m p y - n p yD z@tD h@tD -
m^2 n p^3 n Sin@m p x + n p x - m p y - n p yD z@tD h@tD -
m n^2 p^3 n Sin@m p x + n p x - m p y - n p yD z@tD h@tD +
n^3 p^3 n Sin@m p x + n p x - m p y - n p yD z@tD h@tD - m^3 p^3 Sin@m p x - n p x + m p y - n p yD z@tD h@tD +
3 m^2 n p^3 Sin@m p x - n p x + m p y - n p yD z@tD h@tD -
3 m n^2 p^3 Sin@m p x - n p x + m p y - n p yD z@tD h@tD +
n^3 p^3 Sin@m p x - n p x + m p y - n p yD z@tD h@tD + m^3 p^3 n Sin@m p x - n p x + m p y - n p yD z@tD h@tD +
m^2 n p^3 n Sin@m p x - n p x + m p y - n p yD z@tD h@tD - m n^2 p^3 n Sin@m p x - n p x + m p y - n p yD
  z@tD h@tD - n^3 p^3 n Sin@m p x - n p x + m p y - n p yD z@tD h@tD +
m^3 p^3 Sin@m p x + n p x + m p y - n p yD z@tD h@tD + m^2 n p^3 Sin@m p x + n p x + m p y - n p yD z@tD h@tD +
m n^2 p^3 Sin@m p x + n p x + m p y - n p yD z@tD h@tD + n^3 p^3 Sin@m p x + n p x + m p y - n p yD z@tD h@tD -
m^3 p^3 n Sin@m p x + n p x + m p y - n p yD z@tD h@tD - m^2 n p^3 n Sin@m p x + n p x + m p y - n p yD
  z@tD h@tD - m n^2 p^3 n Sin@m p x + n p x + m p y - n p yD z@tD h@tD -
n^3 p^3 n Sin@m p x + n p x + m p y - n p yD z@tD h@tD - m^3 p^3 Sin@m p x - n p x - m p y + n p yD z@tD h@tD +
3 m^2 n p^3 Sin@m p x - n p x - m p y + n p yD z@tD h@tD -
3 m n^2 p^3 Sin@m p x - n p x - m p y + n p yD z@tD h@tD +
n^3 p^3 Sin@m p x - n p x - m p y + n p yD z@tD h@tD + m^3 p^3 n Sin@m p x - n p x - m p y + n p yD z@tD h@tD +
m^2 n p^3 n Sin@m p x - n p x - m p y + n p yD z@tD h@tD -
m n^2 p^3 n Sin@m p x - n p x - m p y + n p yD z@tD h@tD -
n^3 p^3 n Sin@m p x - n p x - m p y + n p yD z@tD h@tD + m^3 p^3 Sin@m p x + n p x - m p y + n p yD z@tD h@tD +
m^2 n p^3 Sin@m p x + n p x - m p y + n p yD z@tD h@tD +
m n^2 p^3 Sin@m p x + n p x - m p y + n p yD z@tD h@tD + n^3 p^3 Sin@m p x + n p x - m p y + n p yD z@tD h@tD -
m^3 p^3 n Sin@m p x + n p x - m p y + n p yD z@tD h@tD - m^2 n p^3 n Sin@m p x + n p x - m p y + n p yD
  z@tD h@tD - m n^2 p^3 n Sin@m p x + n p x - m p y + n p yD z@tD h@tD -
n^3 p^3 n Sin@m p x + n p x - m p y + n p yD z@tD h@tD + m^3 p^3 Sin@m p x - n p x + m p y + n p yD z@tD h@tD -
m^2 n p^3 Sin@m p x - n p x + m p y + n p yD z@tD h@tD +
m n^2 p^3 Sin@m p x - n p x + m p y + n p yD z@tD h@tD - n^3 p^3 Sin@m p x - n p x + m p y + n p yD z@tD h@tD -
m^3 p^3 n Sin@m p x - n p x + m p y + n p yD z@tD h@tD + m^2 n p^3 n Sin@m p x - n p x + m p y + n p yD
  z@tD h@tD - m n^2 p^3 n Sin@m p x - n p x + m p y + n p yD z@tD h@tD +
n^3 p^3 n Sin@m p x - n p x + m p y + n p yD z@tD h@tD - m^3 p^3 Sin@m p x + n p x + m p y + n p yD z@tD h@tD -
3 m^2 n p^3 Sin@m p x + n p x + m p y + n p yD z@tD h@tD -
3 m n^2 p^3 Sin@m p x + n p x + m p y + n p yD z@tD h@tD -
n^3 p^3 Sin@m p x + n p x + m p y + n p yD z@tD h@tD + m^3 p^3 n Sin@m p x + n p x + m p y + n p yD z@tD h@tD -
m^2 n p^3 n Sin@m p x + n p x + m p y + n p yD z@tD h@tD -
m n^2 p^3 n Sin@m p x + n p x + m p y + n p yD z@tD h@tD + n^3 p^3 n Sin@m p x + n p x + m p y + n p yD
  z@tD h@tD + 4 m^3 p^3 Sin@2 m p xD h@tD^2 - 4 m n^2 p^3 n Sin@2 m p xD h@tD^2 -
2 m^3 p^3 Sin@2 m p x - 2 n p yD h@tD^2 - 2 m n^2 p^3 Sin@2 m p x - 2 n p yD h@tD^2 -
2 m^3 p^3 Sin@2 m p x + 2 n p yD h@tD^2 - 2 m n^2 p^3 Sin@2 m p x + 2 n p yD h@tD^2L
```

```
order2Eqv =
  HvEOM@@1DD •. w -> H0 &LL == HHvEOM@@1DD •. w -> H0 &LL - vEOM@@1DD •. w1sol •• TrigReduceL;
order2Eqv •• display2
```

$\frac{u_{x,y}}{2} + \frac{1}{2}\, n\, u_{x,y} + \frac{v_{x,x}}{2} - \frac{1}{2}\, n\, v_{x,x} + v_{y,y}$ ==

$\frac{1}{16}$ H4 m^3 p^3 Sin@2 m p yD z@tD^2 - 4 m n^2 p^3 n Sin@2 m p yD z@tD^2 +
2 m^3 p^3 Sin@2 n p x - 2 m p yD z@tD^2 + 2 m n^2 p^3 Sin@2 n p x - 2 m p yD z@tD^2 -
2 m^3 p^3 Sin@2 n p x + 2 m p yD z@tD^2 - 2 m n^2 p^3 Sin@2 n p x + 2 m p yD z@tD^2 -
m^3 p^3 Sin@m p x - n p x - m p y - n p yD z@tD h@tD - m^2 n p^3 Sin@m p x - n p x - m p y - n p yD z@tD h@tD -
m n^2 p^3 Sin@m p x - n p x - m p y - n p yD z@tD h@tD - n^3 p^3 Sin@m p x - n p x - m p y - n p yD z@tD h@tD +
m^3 p^3 n Sin@m p x - n p x - m p y - n p yD z@tD h@tD + m^2 n p^3 n Sin@m p x - n p x - m p y - n p yD
z@tD h@tD + m n^2 p^3 n Sin@m p x - n p x - m p y - n p yD z@tD h@tD +
n^3 p^3 n Sin@m p x - n p x - m p y - n p yD z@tD h@tD + m^3 p^3 Sin@m p x + n p x - m p y - n p yD z@tD h@tD +
3 m^2 n p^3 Sin@m p x + n p x - m p y - n p yD z@tD h@tD +
3 m n^2 p^3 Sin@m p x + n p x - m p y - n p yD z@tD h@tD +
n^3 p^3 Sin@m p x + n p x - m p y - n p yD z@tD h@tD - m^3 p^3 n Sin@m p x + n p x - m p y - n p yD z@tD h@tD +
m^2 n p^3 n Sin@m p x + n p x - m p y - n p yD z@tD h@tD +
m n^2 p^3 n Sin@m p x + n p x - m p y - n p yD z@tD h@tD -
n^3 p^3 n Sin@m p x + n p x - m p y - n p yD z@tD h@tD - m^3 p^3 Sin@m p x - n p x + m p y - n p yD z@tD h@tD +
3 m^2 n p^3 Sin@m p x - n p x + m p y - n p yD z@tD h@tD -
3 m n^2 p^3 Sin@m p x - n p x + m p y - n p yD z@tD h@tD +
n^3 p^3 Sin@m p x - n p x + m p y - n p yD z@tD h@tD + m^3 p^3 n Sin@m p x - n p x + m p y - n p yD z@tD h@tD +
m^2 n p^3 n Sin@m p x - n p x + m p y - n p yD z@tD h@tD - m n^2 p^3 n Sin@m p x - n p x + m p y - n p yD
z@tD h@tD - n^3 p^3 n Sin@m p x - n p x + m p y - n p yD z@tD h@tD +
m^3 p^3 Sin@m p x + n p x + m p y - n p yD z@tD h@tD - m^2 n p^3 Sin@m p x + n p x + m p y - n p yD z@tD h@tD +
m n^2 p^3 Sin@m p x + n p x + m p y - n p yD z@tD h@tD - n^3 p^3 Sin@m p x + n p x + m p y - n p yD z@tD h@tD -
m^3 p^3 n Sin@m p x + n p x + m p y - n p yD z@tD h@tD + m^2 n p^3 n Sin@m p x + n p x + m p y - n p yD
z@tD h@tD - m n^2 p^3 n Sin@m p x + n p x + m p y - n p yD z@tD h@tD +
n^3 p^3 n Sin@m p x + n p x + m p y - n p yD z@tD h@tD + m^3 p^3 Sin@m p x - n p x - m p y + n p yD z@tD h@tD -
3 m^2 n p^3 Sin@m p x - n p x - m p y + n p yD z@tD h@tD +
3 m n^2 p^3 Sin@m p x - n p x - m p y + n p yD z@tD h@tD -
n^3 p^3 Sin@m p x - n p x - m p y + n p yD z@tD h@tD - m^3 p^3 n Sin@m p x - n p x - m p y + n p yD z@tD h@tD -
m^2 n p^3 n Sin@m p x - n p x - m p y + n p yD z@tD h@tD +
m n^2 p^3 n Sin@m p x - n p x - m p y + n p yD z@tD h@tD +
n^3 p^3 n Sin@m p x - n p x - m p y + n p yD z@tD h@tD - m^3 p^3 Sin@m p x + n p x - m p y + n p yD z@tD h@tD +
m^2 n p^3 Sin@m p x + n p x - m p y + n p yD z@tD h@tD -
m n^2 p^3 Sin@m p x + n p x - m p y + n p yD z@tD h@tD + n^3 p^3 Sin@m p x + n p x - m p y + n p yD z@tD h@tD +
m^3 p^3 n Sin@m p x + n p x - m p y + n p yD z@tD h@tD - m^2 n p^3 n Sin@m p x + n p x - m p y + n p yD
z@tD h@tD + m n^2 p^3 n Sin@m p x + n p x - m p y + n p yD z@tD h@tD -
n^3 p^3 n Sin@m p x + n p x - m p y + n p yD z@tD h@tD + m^3 p^3 Sin@m p x - n p x + m p y + n p yD z@tD h@tD +
m^2 n p^3 Sin@m p x - n p x + m p y + n p yD z@tD h@tD +
m n^2 p^3 Sin@m p x - n p x + m p y + n p yD z@tD h@tD + n^3 p^3 Sin@m p x - n p x + m p y + n p yD z@tD h@tD -
m^3 p^3 n Sin@m p x - n p x + m p y + n p yD z@tD h@tD - m^2 n p^3 n Sin@m p x - n p x + m p y + n p yD
z@tD h@tD - m n^2 p^3 n Sin@m p x - n p x + m p y + n p yD z@tD h@tD -
n^3 p^3 n Sin@m p x - n p x + m p y + n p yD z@tD h@tD - m^3 p^3 Sin@m p x + n p x + m p y + n p yD z@tD h@tD -
3 m^2 n p^3 Sin@m p x + n p x + m p y + n p yD z@tD h@tD -
3 m n^2 p^3 Sin@m p x + n p x + m p y + n p yD z@tD h@tD -
n^3 p^3 Sin@m p x + n p x + m p y + n p yD z@tD h@tD + m^3 p^3 n Sin@m p x + n p x + m p y + n p yD z@tD h@tD -

```
m^2 n p^3 n Sin@m p x + n p x + m p y + n p yD z@tD h@tD -
m n^2 p^3 n Sin@m p x + n p x + m p y + n p yD z@tD h@tD + n^3 p^3 n Sin@m p x + n p x + m p y + n p yD
 z@tD h@tD + 4 n^3 p^3 Sin@2 n p yD h@tD^2 - 4 m^2 n p^3 n Sin@2 n p yD h@tD^2 +
2 m^2 n p^3 Sin@2 m p x - 2 n p yD h@tD^2 + 2 n^3 p^3 Sin@2 m p x - 2 n p yD h@tD^2 -
2 m^2 n p^3 Sin@2 m p x + 2 n p yD h@tD^2 - 2 n^3 p^3 Sin@2 m p x + 2 n p yD h@tD^2L
```

To obtain the particular solutions for *u* and *v*, we use the method of undetermined coefficients. To accomplish this, we first look for all possible terms that appear on the right-hand sides of **order2Equ** and **order2Eqv** as follows:

```
possibleTerms =
 Cases@8order2Equ@@2DD, order2Eqv@@2DD<, #, ¥D & •ž 9_ Sin@a_D h1_@tD^2 -> Sin@aD h1@tD^2,
    _ Sin@a_D h1_@tD h2_@tD -> Sin@aD h1@tD h2@tD= •• Flatten •• Union
```

```
8Sin@2 n p xD z@tD^2, Sin@2 m p yD z@tD^2, Sin@2 n p x - 2 m p yD z@tD^2, Sin@2 n p x + 2 m p yD z@tD^2,
 Sin@m p x - n p x - m p y - n p yD z@tD h@tD, Sin@m p x + n p x - m p y - n p yD z@tD h@tD,
 Sin@m p x - n p x + m p y - n p yD z@tD h@tD, Sin@m p x + n p x + m p y - n p yD z@tD h@tD,
 Sin@m p x - n p x - m p y + n p yD z@tD h@tD, Sin@m p x + n p x - m p y + n p yD z@tD h@tD,
 Sin@m p x - n p x + m p y + n p yD z@tD h@tD, Sin@m p x + n p x + m p y + n p yD z@tD h@tD,
 Sin@2 m p xD h@tD^2, Sin@2 n p yD h@tD^2, Sin@2 m p x - 2 n p yD h@tD^2, Sin@2 m p x + 2 n p yD h@tD^2<
```

which satisfy **BC1**. We then define the following symbol lists to represent the undetermined coefficients.

```
usymbolList = Table@c_i, 8i, Length@possibleTermsD<D
```

```
8c_1, c_2, c_3, c_4, c_5, c_6, c_7, c_8, c_9, c_10, c_11, c_12, c_13, c_14, c_15, c_16<
```

```
vsymbolList = usymbolList •. c -> d
```

```
8d_1, d_2, d_3, d_4, d_5, d_6, d_7, d_8, d_9, d_10, d_11, d_12, d_13, d_14, d_15, d_16<
```

Then, we express the solution as

```
usol = u -> Function@8x, y, t<, usymbolList.possibleTerms •• EvaluateD;
vsol = v -> Function@8x, y, t<, vsymbolList.possibleTerms •• EvaluateD;
```

Substituting **usol** and **vsol** into **order2Equ** and **order2Eqv**, collecting coefficients of **possibleTerms**, and solving for **usymbol-List** and **vsymbolList**, we obtain

```
algEq1 = Coefficient@Subtract žž order2Equ •. usol •. vsol, possibleTermsD == 0 •• Thread;
```

```
algEq2 = Coefficient@Subtract žž order2Eqv •. usol •. vsol, possibleTermsD == 0 •• Thread;
```

```
HsymbolRule2 =
  Solve@Join@algEq1, algEq2D, Join@usymbolList, vsymbolListDD@@1DD •• SimplifyL •• Timing
```

$:2.48\ \text{Second},\ :c_1 ® -\frac{p\ Hn^2 - m^2\ nL}{16\ n},\ c_{13} ® -\frac{p\ Hm^2 - n^2\ mL}{16\ m},\ d_2 ® -\frac{p\ Hm^2 - n^2\ mL}{16\ m},\ d_{14} ® -\frac{p\ Hn^2 - m^2\ nL}{16\ n},$

$c_2 ® 0,\ c_{14} ® 0,\ d_1 ® 0,\ d_{13} ® 0,\ c_3 ® \frac{n\ p}{32},\ c_4 ® \frac{n\ p}{32},\ c_{15} ® \frac{m\ p}{32},\ c_{16} ® \frac{m\ p}{32},\ d_3 ® -\frac{m\ p}{32},$

$d_4 ® \frac{m\ p}{32},\ d_{15} ® -\frac{n\ p}{32},\ d_{16} ® \frac{n\ p}{32},\ c_7 ® -\frac{p\ Hm^2\ H-1 + nL + n^2\ H-1 + nL + 2\ m\ n\ H1 + nLL}{32\ Hm - nL},$

$c_{12} ® -\frac{p\ Hm^2\ H-1 + nL + n^2\ H-1 + nL - 2\ m\ n\ H1 + nLL}{32\ Hm + nL},$

$d_7 ® -\frac{p\ Hm^2\ H-1 + nL + n^2\ H-1 + nL + 2\ m\ n\ H1 + nLL}{32\ Hm - nL},$

$d_{12} ® -\frac{p\ Hm^2\ H-1 + nL + n^2\ H-1 + nL - 2\ m\ n\ H1 + nLL}{32\ Hm + nL},$

$c_5 ® \frac{1}{32}\ Hm - nL\ p\ H-1 + nL,\ c_6 ® -\frac{p\ Hm^2\ H-1 + nL + n^2\ H-1 + nL - 2\ m\ n\ H1 + nLL}{32\ Hm + nL},$

$c_8 ® \frac{1}{32}\ Hm + nL\ p\ H-1 + nL,\ c_9 ® -\frac{p\ Hm^2\ H-1 + nL + n^2\ H-1 + nL + 2\ m\ n\ H1 + nLL}{32\ Hm - nL},$

$c_{10} ® \frac{1}{32}\ Hm + nL\ p\ H-1 + nL,\ c_{11} ® \frac{1}{32}\ Hm - nL\ p\ H-1 + nL,$

$d_5 ® -\frac{1}{32}\ Hm + nL\ p\ H-1 + nL,\ d_6 ® \frac{p\ Hm^2\ H-1 + nL + n^2\ H-1 + nL - 2\ m\ n\ H1 + nLL}{32\ Hm + nL},$

$d_8 ® \frac{1}{32}\ Hm - nL\ p\ H-1 + nL,\ d_9 ® \frac{p\ Hm^2\ H-1 + nL + n^2\ H-1 + nL + 2\ m\ n\ H1 + nLL}{32\ Hm - nL},$

$d_{10} ® -\frac{1}{32}\ Hm - nL\ p\ H-1 + nL,\ d_{11} ® \frac{1}{32}\ Hm + nL\ p\ H-1 + nL>>$

Ÿ Solvability Conditions

We introduce two time scales T_0 and T_1 and seek a first-order uniform asymptotic expansion of the solution of **wEOM** in the form

```
solRule = 9w -> FunctionA8x, y, T0, T1<, SumAe^(j-1) wj@x, y, T0, T1D, 8j, 2<EE=;
```

Transforming the time derivatives in **wEOM** in terms of the scales T_0 and T_1, substituting **solRule** into **wEOM**, expanding the result for small e, and discarding terms of order higher than e, we obtain

```
eq83c =
  HwEOM •. 8Hs : u È v È wL@x, y, tD -> s@x, y, T0, T1D, Derivative@a_, b_, c_D@h_D@x, y, tD ->
      dt@cD@D@h@x, y, T0, T1D, 8x, a<, 8y, b<DD< •. solRule •• ExpandAllL •. e^(n_•;n>1) -> 0;
```

Equating coefficients of like powers of e, we have

```
eqEps = CoefficientList@eq83c@@1DD, eD == 0 •• Thread;
```

Accounting for the two time scales T_0 and T_1, we create a rule for h@*t*D and z@*t*D

```
ampRule = 9h@tD -> A1@T1D E^(I wm,n T0) + Ā1@T1D E^(-I wm,n T0), z@tD -> A2@T1D E^(I wm,n T0) + Ā2@T1D E^(-I wm,n T0)=;
```

and rewrite **w1sol**, **usol**, and **vsol** as

```
sol1 = 8w1 -> Function@8x, y, T0, T1<, w@x, y, tD •. w1sol •. ampRule •• EvaluateD,
   u -> Function@8x, y, T0, T1<, u@x, y, tD •. usol •. ampRule •• EvaluateD,
   v -> Function@8x, y, T0, T1<, v@x, y, tD •. vsol •. ampRule •• EvaluateD<;
```

Substituting the **sol1** into the second-order problem, **eqEps[[2]]**, we obtain

```
order2Eqw = HeqEps@@1, 1DD •. w1 -> w2L == HeqEps@@1, 1DD •. w1 -> w2L - eqEps@@2, 1DD •. sol1 ••
     ExpandAll; •• Timing
```

```
811.09 Second, Null<
```

Collecting the terms that may lead to secular terms, the terms proportional to E^(I wm,n T0), we have

```
ST = CoefficientAorder2Eqw@@2DD •. expRule, E^(I wm,n T0)E •• TrigReduce; •• Timing
```

```
8156.76 Second, Null<
```

Among all the spatial-dependent non-forcing terms in **ST**, only the following terms are not orthogonal to the **modeshapes**:

```
terms1 = Cases@ST, Cos@m p x + a_. n p y •; Abs@aD == 1D, ¥D •• Union
```

```
8Cos@m p x - n p yD, Cos@m p x + n p yD<
```

```
terms2 = Cases@ST, Cos@n p x + a_. m p y •; Abs@aD == 1D, ¥D •• Union
```

```
8Cos@n p x - m p yD, Cos@n p x + m p yD<
```

The coefficients of these terms in **ST** are

```
terms1Coef = Coefficient@ST, terms1D;
```

```
terms2Coef = Coefficient@ST, terms2D;
```

The projection of **terms1** onto Sin@*m* p *x*D Sin@*n* p *y*D can be obtained as

```
p1Coef = à_0^1 à_0^1 Sin@m p xD Sin@n p yD # â x â y & •ž terms1 •. trigRule
```

```
9 1/4, - 1/4 =
```

and the projection of **terms2** onto Sin@*n* p *x*D Sin@*m* p *y*D can be obtained as

```
p2Coef = à  à  Sin@n p xD Sin@m p yD # â x â y & •ž terms2 •. trigRule
```

```
9 1/4 , - 1/4 =
```

The forcing term in **ST** is

```
f0 = ST •. Cos@_D -> 0
```

```
1/24 E^(I T1 s2) F@x, yD
```

Then, the solvability conditions demand that **ST** be orthogonal to every linear eigenfunction, **modeshapes**; that is,

```
SC1 = Expand@24 Hint@Sin@m p xD Sin@n p yD f0, 8x, 0, 1<, 8y, 0, 1<D + p1Coef.terms1Coef •.
        symbolRule2 ••. intRule3 •. int -> IntegrateLD == 0;
```

```
SC2 = Expand@24 Hint@Sin@n p xD Sin@m p yD f0, 8x, 0, 1<, 8y, 0, 1<D + p2Coef.terms2Coef •.
        symbolRule2 ••. intRule3 •. int -> IntegrateLD == 0;
```

They agree with the results obtained in the previous section:

```
8SC1, SC2< === 8SCond1, SCond2<
```

```
True
```

Chapter 9
Higher Approximations of Continuous Systems Having Two-to-One Internal Resonances

In this chapter, we use the method of multiple scales to determine second-order uniform asymptotic expansions of the solutions of continuous systems with quadratic and cubic nonlinearities possessing two-to-one internal resonances. Higher-order treatments of such systems lead to inconsistent results if the time derivatives in their governing equations are expressed in second-order rather than first-order form. Therefore, we express the time derivatives in the governing equations in first-order form before treating them with the method of multiple scales. We describe the methodology using two examples: two-mode interactions in hinged-hinged buckled beams and four-mode interactions in suspended cables.

à Preliminaries

```
Off@General::spell1, Integrate::generD
```

To determine second-order uniform asymptotic expansions of the solutions of buckled beams and suspended cables by using the method of multiple scales, taking into account two-to-one internal resonances, we introduce the three time scales $T_0 = t$, $T_1 = \epsilon t$, and $T_2 = \epsilon^2 t$, where ϵ is a small nondimensional parameter and the order of the amplitude of oscillations. Moreover, we symbolize these scales according to

```
Needs@"Utilities`Notation`"D
```

```
Symbolize@T0D; Symbolize@T1D; Symbolize@T2D;
```

```
timeScales = 8T0, T1, T2<;
```

In terms of the time scales T_0, T_1, and T_2, the time derivatives can be expressed as

```
dt@0D@expr_D := expr; dt@1D@expr_D := SumAe^i D@expr, timeScales@@i + 1DDD, 8i, 0, 2<E;
```

To speed up manipulations of the integrals involved in the governing equations with *Mathematica*, we introduce the rules

```
intRule1 = 8int@fun_, arg2__D :> int@Expand@funD, arg2D<;
intRule2 = 8int@a_ + b_, arg2__D :> int@a, arg2D + int@b, arg2D,
   int@e^n_. fun_, arg2__D :> e^n int@fun, arg2D,
   int@a_ fun_, a1_, b1___D :> a int@fun, a1, b1D /; FreeQ@a, First@a1DD,
   int@int@a1_, a2_D fun_., a3__D :> int@a1, a2D int@fun, a3D /; FreeQ@Rest@a2D, First@a2DD<;
```

In the ensuing analysis, we express the solutions in terms of complex-variable functions and hence we introduce the complex-conjugate rule

```
conjugateRule = 9A_s_ :> Ȧ_s, Ȧ_s_ :> A_s, Complex@m_, n_D :> Complex@m, -nD=;
```

To display the outputs in easily read expressions, we introduce the rules

```
displayRule = 9Derivative@a_, b__DAw_i__E@_, T0, __D :>
   SequenceFormAIfAarg1 = Times žž MapIndexedAD#1#2@@1DD-1 &, 8b<E; arg1 =!= 1, arg1, ""E,
    wi SequenceForm žž Table@"'",8a<D E,
  Derivative@a__D@Ai_D@__D :> SequenceFormATimes žž MapIndexedAD#1#2@@1DD &, 8a<E, AiE,
  Derivative@a__D@Ȧi_D@__D :> SequenceFormATimes žž MapIndexedAD#1#2@@1DD &, 8a<E, ȦiE,
  w_i__@_, T0, __D -> wi, Ai_@__D -> Ai, Ȧi_@__D -> Ȧi,
  Exp@a_. + b_. Complex@0, m_D T0 + c_. Complex@0, n_D T0D -> Exp@a + Hm * b + n * cL I T0D=;
```

à 9.1 Two-Mode Interactions in Buckled Beams

In this section, we consider the nonlinear response of a hinged-hinged buckled beam possessing a two-to-one internal resonance to a principal parametric resonance of the higher mode. The analysis assumes a unimodal static buckled deflection. The nondimensional equation of motion is (Section 8.1.1)

```
EOM = u_{t,t} + u_{x,x,x,x} + P_k u_{x,x} - 2 b^2 a ∫_0^1 u_x φ'_k@xD âx φ''_k@xD ==
   -c u_t + b a ∫_0^1 u_x^2 âx φ''_k@xD + 2 b a ∫_0^1 u_x φ'_k@xD âx u_{x,x} +
    a ∫_0^1 u_x^2 âx u_{x,x} + Cos@t ΩD f u_{x,x} + b Cos@t ΩD f φ''_k@xD •. u_m__ :> ∂_m u@x, tD;
```

and the associated boundary conditions are

```
BC = 9u@0, tD == 0, u^H2,0L@0, tD == 0, u@1, tD == 0, u^H2,0L@1, tD == 0=;
```

where P_k is the kth critical Euler buckling load, ϕ_k is the kth buckling mode shape, and b is the nondimensional buckling level.

Because the two-to-one internal resonance is activated by the quadratic nonlinearities, as indicated in Chapter 5, second- and higher-order treatments of such systems lead to inconsistent results if the time derivatives in their governing equations are expressed in second-order rather than first-order form. Therefore, we express the time derivatives in first-order form by letting $v[x, t] == \partial_t u[x, t]$, adding it to **EOM**, and obtaining

```
EOM1 = 8∂_t u@x, tD - v@x, tD == 0, EOM •. ∂_{t,t} u@x, tD -> ∂_t v@x, tD<;
```

We use the method of multiple scales to directly attack **EOM1** and **BC**. To transform the time derivatives in **EOM1** in terms of the scales T_0, T_1, and T_2, we define

```
multiScales = 8u@x_, tD -> u@x, T0, T1, T2D, v@x, tD -> v@x, T0, T1, T2D,
   Derivative@m_, n_D@u_D@x_, tD -> dt@nD@D@u@x, T0, T1, T2D, 8x, m<DD, t -> T0<;
```

Then, we seek a second-order approximate solution of **EOM1** and **BC** in the form

```
solRule = h : u È v -> I EvaluateASumAe^j hj@#1, #2, #3, #4D, 8j, 3<EE &M;
```

where #1 stands for x and #2, #3, and #4 stand for T_0, T_1, and T_2, respectively.

We consider the case of a two-to-one internal resonance between the lowest two modes and a principal parametric resonance of the second mode; that is,

```
omgList = 8w1, w2<;
ResonanceConds = 8w2 == 2 w1 + e s1, W == 2 w2 + e s2<;
```

where the s_i are detuning parameters that describe the nearness of the resonances. Because the influence of the two-to-one internal resonance appears at $O(e)$, we scale the damping and forcing terms as

```
scaling = 8c -> 2 e m, f -> e f<;
```

Substituting **multiScales**, **solRule**, and **scaling** into **EOM1**, expanding the result for small e, and discarding terms of order higher than e^3, we obtain

```
eq91a =
  HEOM1 •. Integrate -> int •. multiScales •. solRule •. scaling •. intRule1 ••. intRule2 ••
     TrigToExp •• ExpandAllL •. e^n_•;n>3 -> 0;
```

Equating coefficients of like powers of e, we obtain

```
eqEps = Thread@CoefficientList@Subtract žž #, eD == 0D & •ž eq91a •• Transpose •• Rest;
eqEps •. displayRule
```

```
99D0u1 - v1 == 0, u1'''' + D0v1 + Hu1''L Pk - 1/2 b E^(-I T0 W) f fk^2@xD -
    1/2 b E^(I T0 W) f fk^2@xD - 2 b^2 a int@Hu1'L fk¢@xD, 8x, 0, 1<D fk^2@xD == 0=,
 9D0u2 + D1u1 - v2 == 0, - 1/2 E^(-I T0 W) f Hu1''L - 1/2 E^(I T0 W) f Hu1''L -
    2 b a int@Hu1'L fk¢@xD, 8x, 0, 1<D Hu1''L + u2'''' + 2 m HD0u1L + D0v2 + D1v1 + Hu2''L Pk -
    b a intBHu1'L^2, 8x, 0, 1<F fk^2@xD - 2 b^2 a int@Hu2'L fk¢@xD, 8x, 0, 1<D fk^2@xD == 0=,
 9D0u3 + D1u2 + D2u1 - v3 == 0, - a intBHu1'L^2, 8x, 0, 1<F Hu1''L -
    2 b a int@Hu2'L fk¢@xD, 8x, 0, 1<D Hu1''L - 1/2 E^(-I T0 W) f Hu2''L - 1/2 E^(I T0 W) f Hu2''L -
    2 b a int@Hu1'L fk¢@xD, 8x, 0, 1<D Hu2''L + u3'''' + 2 m HD0u2L + D0v3 + 2 m HD1u1L + D1v2 + D2v1 +
    Hu3''L Pk - 2 b a int@Hu1'L Hu2'L, 8x, 0, 1<D fk^2@xD - 2 b^2 a int@Hu3'L fk¢@xD, 8x, 0, 1<D fk^2@xD == 0==
```

Ÿ 9.1.1 First-Order Solution

The homogeneous part of the first-order equations, eqEps[[1]], can be written as

```
linearSys = #@@1DD & •ž eqEps@@1DD •. f -> 0;
linearSys •. displayRule
```

8D0u1 - v1, u1'''' + D0v1 + Hu1''L Pk - 2 b2 a int@Hu1'L fk¢@xD, 8x, 0, 1<D fk²@xD<

Because in the presence of damping all of the modes that are not directly or indirectly excited decay with time (Nayfeh and Mook, 1979), the solution of eqEps[[1]] is taken to consist of the lowest two modes; that is,

```
solForm = Y@xD Exp@I W T0D + Y@xD Exp@-I W T0D +
   SumAAi@T1, T2D Fi@xD Exp@I wi T0D + Ȧi@T1, T2D Fi@xD Exp@-I wi T0D, 8i, 2<E;
```

```
sol1 = 8u1 -> Function@8x, T0, T1, T2<, solForm •• EvaluateD,
  v1 -> Function@8x, T0, T1, T2<, D@solForm, T0D •• EvaluateD<
```

8u1 ® Function@8x, T0, T1, T2<, E-I T0 W Y@xD + EI T0 W Y@xD + EI T0 w1 A1@T1, T2D F1@xD +
 EI T0 w2 A2@T1, T2D F2@xD + E-I T0 w1 F1@xD Ȧ1@T1, T2D + E-I T0 w2 F2@xD Ȧ2@T1, T2DD,
 v1 ® Function@8x, T0, T1, T2<, -I E-I T0 W W Y@xD + I EI T0 W W Y@xD + I EI T0 w1 w1 A1@T1, T2D F1@xD +
 I EI T0 w2 w2 A2@T1, T2D F2@xD - I E-I T0 w1 w1 F1@xD Ȧ1@T1, T2D - I E-I T0 w2 w2 F2@xD Ȧ2@T1, T2DD<

where the F_i are the eigenmodes, which satisfy the orthonormality condition $\int_0^1 F_n(x) F_m(x)\,dx = \delta_{n,m}$, and Y is uniquely determined by the following boundary-value problem:

```
eq91b = CoefficientA
   eqEps@@1, 2, 1DD •. Isol1 •. 9Ai_ -> H0 &L, Ȧi_ -> H0 &L=M •. intRule1 ••. intRule2,
   EI W T0E == 0
```

-W2 Y@xD + Pk Y²@xD - 1/2 b f fk²@xD - 2 b2 a int@Y¢@xD fk¢@xD, 8x, 0, 1<D fk²@xD + YH4L@xD == 0

```
bc91b = BC •. u -> Y
```

8Y@0, tD == 0, YH2,0L@0, tD == 0, Y@1, tD == 0, YH2,0L@1, tD == 0<

In what follows, we consider nonlinear vibrations of a first-mode buckled beam; hence, the critical buckling load is

```
bucklingLoad = Pk -> p^2;
```

We note that, as the buckling level increases from zero, the symmetric mode is the first mode. However, as the buckling level exceeds the first cross-over value, the symmetric mode becomes the second mode. Hence, the second natural frequency ω_2 depends on the buckling level. For buckling levels greater than the cross-over value, the first static buckled deflection and the first two normalized mode shapes and associated natural frequencies are

```
modeshapes = 9f_k -> HSin@p #D &L, F_1 -> I • !! 2 Sin@2 p #D &M, F_2 -> I • !! 2 Sin@p #D &M=;
frequencies = 9w_2 -> b p^2 • !! a, w_1 -> 2 • !! 3 p^2=;
```

The buckling level at which $w_2 = 2\,w_1$ is

```
values1 = Solve@w_2 == 2 w_1 •. frequencies, bD@@1DD
```

```
:b ® (4 • !! 3) / (• !! a) >
```

Substituting the **bucklingLoad** and **modeshapes** into **eq91b** yields

```
eq91c = eq91b •. bucklingLoad •. modeshapes
```

```
1/2 b f p^2 Sin@p xD +
  2 b^2 p^2 a int@p Cos@p xD Y'@xD, 8x, 0, 1<D Sin@p xD - W^2 Y@xD + p^2 Y''@xD + Y^H4L@xD == 0
```

Inspection of **eq91c** and **bc91b** suggests that Y@xD has the form

```
Ysol = Y -> HG Sin@p #D &L;
```

Substituting **Ysol** into **eq91c**, collecting the coefficient of Sin@p xD, solving the resulting equation for G, and simplifying it using **frequencies** and **values1**, we obtain

```
GRule = Solve@Coefficient@eq91c@@1DD •. Ysol •. int -> Integrate, Sin@p xDD == 0, GD@@1DD •.
    W -> 2 w_2 •. frequencies •. values1
```

```
:G ® f / (24 • !! 3 p^2 • !! a) >
```

Because the buckled-beam problem is self-adjoint, we have

```
adjoint = 88- I w_1 F_1@xD, F_1@xD<, 8- I w_2 F_2@xD, F_2@xD<<;
```

whose complex conjugate is

```
adjointC = adjoint •. conjugateRule
```

```
88I w_1 F_1@xD, F_1@xD<, 8I w_2 F_2@xD, F_2@xD<<
```

Ÿ 9.1.2 Second-Order Solution

Substituting **sol1**, **Ysol**, and some of the parameter values into **eqEps[[2]]** yields

```
order2Eq =
  HlinearSys •. u_1 -> u2L == HHlinearSys •. u_1 -> u2L - H#@@1DD & •ž eqEps@@2DDL •. sol1 •.
            Ysol •. GRule •. intRule1 ••. intRule2 •. modeshapes •.
        int -> Integrate •• ExpandL •. bucklingLoad •. values1 •• Thread;
order2Eq •. displayRule
```

```
:D0u2 - v2 == - • 2 E^(I T0 w2) HD1A2L Sin@p xD - • 2 E^(-I T0 w2) HD1Ȧ2L Sin@p xD -
   • 2 E^(I T0 w1) HD1A1L Sin@2 p xD - • 2 E^(-I T0 w1) HD1Ȧ1L Sin@2 p xD,
 p^2 Hu2''L + u2'''' + D0v2 - 96 int@Hu2'L fk@xD, 8x, 0, 1<D fk@xD ==
  - f^2 Sin@p xD/(16 • 3 • a) - E^(-2 I T0 W) f^2 Sin@p xD/(32 • 3 • a) - E^(2 I T0 W) f^2 Sin@p xD/(32 • 3 • a) + I E^(-I T0 W) f m W Sin@p xD/(12 • 3 p^2 • a) -
   I E^(I T0 W) f m W Sin@p xD/(12 • 3 p^2 • a) - 8/3 • 2 E^(I T0 H-W+w1L) f p^2 Sin@2 p xD A1 - 8/3 • 2 E^(I T0 HW+w1L) f p^2 Sin@2 p xD A1 -
   16 • 3 E^(2 I T0 w1) p^4 • a Sin@p xD A1^2 - • 2 E^(I T0 H-W+w2L) f p^2 Sin@p xD A2 - • 2 E^(I T0 HW+w2L) f p^2 Sin@p xD A2 -
   32 • 3 E^(I T0 Hw1+w2L) p^4 • a Sin@2 p xD A1 A2 - 12 • 3 E^(2 I T0 w2) p^4 • a Sin@p xD A2^2 -
   I • 2 E^(I T0 w1) HD1A1L Sin@2 p xD w1 + I • 2 E^(-I T0 w1) HD1Ȧ1L Sin@2 p xD w1 -
   2 I • 2 E^(I T0 w1) m Sin@2 p xD A1 w1 - I • 2 E^(I T0 w2) HD1A2L Sin@p xD w2 + I • 2 E^(-I T0 w2) HD1Ȧ2L Sin@p xD w2 -
   2 I • 2 E^(I T0 w2) m Sin@p xD A2 w2 - 8/3 • 2 E^(I T0 H-W-w1L) f p^2 Sin@2 p xD Ȧ1 -
   8/3 • 2 E^(I T0 HW-w1L) f p^2 Sin@2 p xD Ȧ1 - 32 • 3 p^4 • a Sin@p xD A1 Ȧ1 -
   32 • 3 E^(I T0 H-w1+w2L) p^4 • a Sin@2 p xD A2 Ȧ1 + 2 I • 2 E^(-I T0 w1) m Sin@2 p xD w1 Ȧ1 -
   16 • 3 E^(-2 I T0 w1) p^4 • a Sin@p xD Ȧ1^2 - • 2 E^(I T0 H-W-w2L) f p^2 Sin@p xD Ȧ2 -
   • 2 E^(I T0 HW-w2L) f p^2 Sin@p xD Ȧ2 - 32 • 3 E^(I T0 Hw1-w2L) p^4 • a Sin@2 p xD A1 Ȧ2 -
   24 • 3 p^4 • a Sin@p xD A2 Ȧ2 + 2 I • 2 E^(-I T0 w2) m Sin@p xD w2 Ȧ2 -
   32 • 3 E^(I T0 H-w1-w2L) p^4 • a Sin@2 p xD Ȧ1 Ȧ2 - 12 • 3 E^(-2 I T0 w2) p^4 • a Sin@p xD Ȧ2^2>
```

In order to collect the terms that may lead to secular terms from the right-hand sides of **order2Eq**, we define the rules

```
OmgRule = Solve@ResonanceConds, Complement@omgList, 8#<D ~Join~ 8W<D@@1DD & •ž omgList
```

```
98w2 ® e s1 + 2 w1, W ® 2 e s1 + e s2 + 4 w1<, 9w1 ® 1/2 H-e s1 + w2L, W ® e s2 + 2 w2==
```

```
expRule1@i_D := Exp@arg_D :> Exp@Expand@arg •. OmgRule@@iDDD •. e T0 -> T1D
```

Collecting the terms that may lead to secular terms, the terms proportional to $E^{I\, w_1\, T_0}$, we have

```
ST11 = CoefficientA#@@2DD & •ž order2Eq •. expRule1@1D, E^(I w1 T0)E;
ST11 •. displayRule
```

```
9- • 2 HD1A1L Sin@2 p xD,
 -I • 2 HD1A1L Sin@2 p xD w1 - 2 I • 2 m Sin@2 p xD A1 w1 - 32 • 3 E^(I T1 s1) p^4 • a Sin@2 p xD A2 Ȧ1=
```

```
ST12 = CoefficientA#@@2DD & •ž order2Eq •. expRule1@2D, E^(I w2 T0) E;
ST12 •. displayRule
```

```
9- 2 HD1A2L Sin@p xD, -16 3 E^(-I T1 s1) p^4 a Sin@p xD A1^2 -
   I 2 HD1A2L Sin@p xD w2 - 2 I 2 m Sin@p xD A2 w2 - 2 E^(I T1 s2) f p^2 Sin@p xD Ā2=
```

Demanding that **ST11** be orthogonal to the **adjointC[[1]]**, we obtain the solvability condition

```
SCond11 =
 SolveAint@adjointC@@1DD.ST11, 8x, 0, 1<D == 0 •. intRule1 ••. intRule2 •. modeshapes •.
     int -> Integrate, A1^H1,0L@T1, T2DE@@1DD •• ExpandAll
```

```
:A1^H1,0L@T1, T2D ® -m A1@T1, T2D + (8 I 6 E^(I T1 s1) p^4 a A2@T1, T2D Ā1@T1, T2D)/w1 >
```

Demanding that **ST12** be orthogonal to the **adjointC[[2]]**, we obtain the solvability condition

```
SCond12 =
 SolveAint@adjointC@@2DD.ST12, 8x, 0, 1<D == 0 •. intRule1 ••. intRule2 •. modeshapes •.
     int -> Integrate, A2^H1,0L@T1, T2DE@@1DD •• ExpandAll
```

```
:A2^H1,0L@T1, T2D ® (4 I 6 E^(-I T1 s1) p^4 a A1@T1, T2D^2)/w2 - m A2@T1, T2D + (I E^(I T1 s2) f p^2 Ā2@T1, T2D)/(2 w2) >
```

Hence, we have the solvability conditions

```
SCond1 = Join@SCond11, SCond12D;
```

whose complex conjugates are

```
ccSCond1 = SCond1 •. conjugateRule;
```

To remove $D_1 A_1$ and $D_1 A_2$ from the right-hand sides of **order2Eq**, we first define the rules

```
sigRule = Solve@ResonanceConds, 8s1, s2<D@@1DD
```

```
9s1 ® -(2 w1 - w2)/e , s2 ® -(-W + 2 w2)/e =
```

```
expRule2 = Exp@a_D :> Exp@a •. sigRule •. T1 -> e T0 •• ExpandD;
```

Substituting **SCond1**, **ccSCond1**, and **expRule2** into **order2Eq**, we obtain

```
order2Eqm = order2Eq •. SCond1 •. ccSCond1 •. expRule2 •• ExpandAll;
order2Eqm •. displayRule
```

```
9D0u2 - v2 == • 2 E^(I T0 w1) m Sin@2 p xD A1 + • 2 E^(I T0 w2) m Sin@p xD A2 -
   (8 I • 3 E^(2 I T0 w1) p^4 • a Sin@p xD A1^2)/w2 + (I E^(I T0 H-W+w2L) f p^2 Sin@p xD A2)/(• 2 w2) +
   • 2 E^(-I T0 w1) m Sin@2 p xD Ā1 - (16 I • 3 E^(I T0 H-w1+w2L) p^4 • a Sin@2 p xD A2 Ā1)/w1 +
   (8 I • 3 E^(-2 I T0 w1) p^4 • a Sin@p xD Ā1^2)/w2 + • 2 E^(-I T0 w2) m Sin@p xD Ā2 +
   (16 I • 3 E^(I T0 Hw1-w2L) p^4 • a Sin@2 p xD A1 Ā2)/w1 - (I E^(I T0 HW-w2L) f p^2 Sin@p xD Ā2)/(• 2 w2),
 p^2 Hu2''L + u2'''' + D0v2 - 96 int@Hu2'L fk@xD, 8x, 0, 1<D fk@xD ==
  - (f^2 Sin@p xD)/(16 • 3 • a) - (E^(-2 I T0 W) f^2 Sin@p xD)/(32 • 3 • a) - (E^(2 I T0 W) f^2 Sin@p xD)/(32 • 3 • a) + (I E^(-I T0 W) f m W Sin@p xD)/(12 • 3 p^2 • a) -
   (I E^(I T0 W) f m W Sin@p xD)/(12 • 3 p^2 • a) - 8/3 • 2 E^(I T0 H-W+w1L) f p^2 Sin@2 p xD A1 - 8/3 • 2 E^(I T0 HW+w1L) f p^2 Sin@2 p xD A1 -
   8 • 3 E^(2 I T0 w1) p^4 • a Sin@p xD A1^2 + (E^(I T0 H-W+w2L) f p^2 Sin@p xD A2)/(• 2) - • 2 E^(I T0 H-W+w2L) f p^2 Sin@p xD A2 -
   • 2 E^(I T0 HW+w2L) f p^2 Sin@p xD A2 - 32 • 3 E^(I T0 Hw1+w2L) p^4 • a Sin@2 p xD A1 A2 -
   12 • 3 E^(2 I T0 w2) p^4 • a Sin@p xD A2^2 - I • 2 E^(I T0 w1) m Sin@2 p xD A1 w1 -
   I • 2 E^(I T0 w2) m Sin@p xD A2 w2 - 8/3 • 2 E^(I T0 H-W-w1L) f p^2 Sin@2 p xD Ā1 -
   8/3 • 2 E^(I T0 HW-w1L) f p^2 Sin@2 p xD Ā1 - 32 • 3 p^4 • a Sin@p xD A1 Ā1 -
   16 • 3 E^(I T0 H-w1+w2L) p^4 • a Sin@2 p xD A2 Ā1 + I • 2 E^(-I T0 w1) m Sin@2 p xD w1 Ā1 -
   8 • 3 E^(-2 I T0 w1) p^4 • a Sin@p xD Ā1^2 - • 2 E^(I T0 H-W-w2L) f p^2 Sin@p xD Ā2 + (E^(I T0 HW-w2L) f p^2 Sin@p xD Ā2)/(• 2) -
   • 2 E^(I T0 HW-w2L) f p^2 Sin@p xD Ā2 - 16 • 3 E^(I T0 Hw1-w2L) p^4 • a Sin@2 p xD A1 Ā2 -
   24 • 3 p^4 • a Sin@p xD A2 Ā2 + I • 2 E^(-I T0 w2) m Sin@p xD w2 Ā2 -
   32 • 3 E^(I T0 H-w1-w2L) p^4 • a Sin@2 p xD Ā1 Ā2 - 12 • 3 E^(-2 I T0 w2) p^4 • a Sin@p xD Ā2^2=
```

We use the method of undetermined coefficients to determine the solution of **order2Eqm** and their associated boundary conditions. To implement this, we first determine all possible forms of the terms on the right-hand sides of **order2Eqm** as follows:

```
sol2Form = TableACases@order2Eqm@@k, 2DD, #D & •ž 9_ E^(a_ T0+b_.) f_@c_ xD -> E^(a T0+b) f@c xD,
      a_ f_@b_ xD :> f@b xD •; FreeQ@a, T0D=, 8k, 2<E •• Flatten •• Union
```

```
8Sin@p xD, E^(-I T0 W) Sin@p xD, E^(I T0 W) Sin@p xD, E^(-2 I T0 W) Sin@p xD, E^(2 I T0 W) Sin@p xD,
 E^(-2 I T0 w1) Sin@p xD, E^(2 I T0 w1) Sin@p xD, E^(-I T0 w2) Sin@p xD, E^(I T0 w2) Sin@p xD, E^(-2 I T0 w2) Sin@p xD,
 E^(2 I T0 w2) Sin@p xD, E^(-I T0 W-I T0 w2) Sin@p xD, E^(I T0 W-I T0 w2) Sin@p xD, E^(-I T0 W+I T0 w2) Sin@p xD,
 E^(I T0 W+I T0 w2) Sin@p xD, E^(-I T0 w1) Sin@2 p xD, E^(I T0 w1) Sin@2 p xD, E^(-I T0 W-I T0 w1) Sin@2 p xD,
 E^(I T0 W-I T0 w1) Sin@2 p xD, E^(-I T0 W+I T0 w1) Sin@2 p xD, E^(I T0 W+I T0 w1) Sin@2 p xD, E^(-I T0 w1-I T0 w2) Sin@2 p xD,
 E^(I T0 w1-I T0 w2) Sin@2 p xD, E^(-I T0 w1+I T0 w2) Sin@2 p xD, E^(I T0 w1+I T0 w2) Sin@2 p xD<
```

where all the spatial dependent functions satisfy the hinged-hinged boundary conditions. We note that **sol2Form** consists of (a) the terms that are proportional to $E^{-I w_1 T_0}$, $E^{I w_1 T_0}$, $E^{-I w_2 T_0}$, $E^{I w_2 T_0}$ and (b) the remainder of the terms. The positions of the former terms, which are orthogonal to the adjoint, in **sol2Form** are

```
pos@1D =
 HPosition@# sol2Form •. expRule1@1D •. Exp@_. + _ T0D -> 0, a_ •; a =!= 0, 1D •• Flatten ••
     RestL & •ž 9E^(-I w1 T0), E^(I w1 T0)=
```

```
8817, 24<, 816, 23<<
```

```
pos@2D =
 HPosition@# sol2Form •. expRule1@2D •. Exp@_. + _ T0D -> 0, a_ •; a =!= 0, 1D •• Flatten ••
     RestL & •ž 9E^(-I w2 T0), E^(I w2 T0)=
```

```
887, 9, 13<, 86, 8, 14<<
```

Next, we seek the solution of u_2 and v_2 in two parts. First, we consider the part of **sol2Form** that is not related to the secular terms

```
sol2Forma = Delete@sol2Form, 8#< & •ž Flatten@Array@pos, 2DDD
```

```
8Sin@p xD, E^(-I T0 W) Sin@p xD, E^(I T0 W) Sin@p xD, E^(-2 I T0 W) Sin@p xD, E^(2 I T0 W) Sin@p xD,
 E^(-2 I T0 w2) Sin@p xD, E^(2 I T0 w2) Sin@p xD, E^(-I T0 W-I T0 w2) Sin@p xD, E^(I T0 W+I T0 w2) Sin@p xD,
 E^(-I T0 W-I T0 w1) Sin@2 p xD, E^(I T0 W-I T0 w1) Sin@2 p xD, E^(-I T0 W+I T0 w1) Sin@2 p xD,
 E^(I T0 W+I T0 w1) Sin@2 p xD, E^(-I T0 w1-I T0 w2) Sin@2 p xD, E^(I T0 w1+I T0 w2) Sin@2 p xD<
```

and define the undetermined coefficients as

```
uSymbola = Table@y_i, 8i, Length@sol2FormaD<D;
vSymbola = uSymbola •. y -> h;
```

The general solution in terms of the **sol2Forma** can be written as

```
sol2a = 8u2 -> Function@8x, T0, T1, T2<, uSymbola.sol2Forma •• EvaluateD,
   v2 -> Function@8x, T0, T1, T2<, vSymbola.sol2Forma •• EvaluateD<;
```

Second, we consider the part of **sol2Form** that is related to the secular terms

```
sol2Formb1 = sol2Form@@pos@1D@@1DDDD
```

```
8E^(I T0 w1) Sin@2 p xD, E^(-I T0 w1+I T0 w2) Sin@2 p xD<
```

```
sol2Formb2 = sol2Form@@pos@2D@@1DDDD
```

```
8E^(2 I T0 w1) Sin@p xD, E^(I T0 w2) Sin@p xD, E^(I T0 W-I T0 w2) Sin@p xD<
```

and define the undetermined coefficients as

```
uSymbolb1 = Table@ys1,i, 8i, Length@sol2Formb1D<D;
uSymbolb2 = Table@ys2,i, 8i, Length@sol2Formb2D<D;
```

where the undetermined coefficients for v_2 are related to **uSymbolb1** and **uSymbolb2** due to the orthogonality condition. Therefore, the general solution in terms of the **sol2Formb1** and **sol2Formb2** can be written as

```
sol2b =
  8u2 -> Function@8x, T0, T1, T2<, uSymbolb1.sol2Formb1 + uSymbolb2.sol2Formb2 •• EvaluateD,
   v2 -> Function@8x, T0, T1, T2<,
     - I w1 uSymbolb1.sol2Formb1 - I w2 uSymbolb2.sol2Formb2 •• EvaluateD<;
```

The total solution is the sum of **sol2a**, **sol2b**, and the complex conjugate of **sol2b**.

Next, we solve for these undetermined coefficients. Substituting **sol2a** into **order2Eqm** and equating the coefficients of **sol2Forma**, we have

```
algEqa =
  Flatten@Coefficient@Subtract žž # •. sol2a •. intRule1 ••. intRule2 •. modeshapes •. int ->
        Integrate, sol2FormaD & •ž order2EqmD == 0 •. Exp@_ T0 + _.D -> 0 •• Thread;
```

Substituting some of the parameter values into **algEqa** and solving for **uSymbola** and **vSymbola**, we obtain

```
symbolaRule =
 Solve@algEqa, uSymbola~Join~vSymbolaD@@1DD •. W -> 2 w2 •. frequencies •. values1
```

$:h_1 ® 0,\ y_1 ® -\frac{\sqrt{3}\, f^2 + 1536\sqrt{3}\, p^4\, a\, A_1@T_1,\ T_2D\, \dot{A}_1@T_1,\ T_2D + 1152\sqrt{3}\, p^4\, a\, A_2@T_1,\ T_2D\, \dot{A}_2@T_1,\ T_2D}{2304\, p^4 \sqrt{ä}},$

$h_2 ® -\frac{f\, m}{9\sqrt{3}\, p^2\sqrt{ä}},\ h_3 ® -\frac{f\, m}{9\sqrt{3}\, p^2\sqrt{ä}},\ h_4 ® -\frac{I\, f^2}{1440\, p^2\sqrt{ä}},\ h_5 ® \frac{I\, f^2}{1440\, p^2\sqrt{ä}},$

$h_6 ® -2\, I\, p^2\sqrt{ä}\, \dot{A}_2@T_1,\ T_2D^2,\ h_7 ® 2\, I\, p^2\sqrt{ä}\, A_2@T_1,\ T_2D^2,\ y_2 ® -\frac{I\, f\, m}{216\, p^4\sqrt{ä}},$

$y_3 ® \frac{I\, f\, m}{216\, p^4\sqrt{ä}},\ y_4 ® \frac{f^2}{23040\sqrt{3}\, p^4\sqrt{ä}},\ y_5 ® \frac{f^2}{23040\sqrt{3}\, p^4\sqrt{ä}},\ y_6 ® \frac{\sqrt{ä}\, \dot{A}_2@T_1,\ T_2D^2}{4\sqrt{3}},$

$y_7 ® \frac{\sqrt{ä}\, A_2@T_1,\ T_2D^2}{4\sqrt{3}},\ h_8 ® -\frac{1}{16}\, I\, \sqrt{\frac{3}{2}}\, f\, \dot{A}_2@T_1,\ T_2D,\ h_9 ® \frac{1}{16}\, I\, \sqrt{\frac{3}{2}}\, f\, A_2@T_1,\ T_2D,$

$h_{10} ® -\frac{5\, I\, f\, \dot{A}_1@T_1,\ T_2D}{9\sqrt{6}},\ h_{11} ® \frac{I\, f\, \dot{A}_1@T_1,\ T_2D}{\sqrt{6}},\ h_{12} ® -\frac{I\, f\, A_1@T_1,\ T_2D}{\sqrt{6}},\ h_{13} ® \frac{5\, I\, f\, A_1@T_1,\ T_2D}{9\sqrt{6}},$

$h_{14} ® -6\, I\, p^2\sqrt{ä}\, \dot{A}_1@T_1,\ T_2D\, \dot{A}_2@T_1,\ T_2D,\ h_{15} ® 6\, I\, p^2\sqrt{ä}\, A_1@T_1,\ T_2D\, A_2@T_1,\ T_2D,\ y_8 ® \frac{f\, \dot{A}_2@T_1,\ T_2D}{192\sqrt{2}\, p^2},$

$y_9 ® \frac{f\, A_2@T_1,\ T_2D}{192\sqrt{2}\, p^2},\ y_{10} ® \frac{f\, \dot{A}_1@T_1,\ T_2D}{54\sqrt{2}\, p^2},\ y_{11} ® \frac{f\, \dot{A}_1@T_1,\ T_2D}{18\sqrt{2}\, p^2},\ y_{12} ® \frac{f\, A_1@T_1,\ T_2D}{18\sqrt{2}\, p^2},$

$y_{13} ® \frac{f\, A_1@T_1,\ T_2D}{54\sqrt{2}\, p^2},\ y_{14} ® \frac{\sqrt{ä}\, \dot{A}_1@T_1,\ T_2D\, \dot{A}_2@T_1,\ T_2D}{\sqrt{3}},\ y_{15} ® \frac{\sqrt{ä}\, A_1@T_1,\ T_2D\, A_2@T_1,\ T_2D}{\sqrt{3}}>$

Substituting **sol2b** into either of the two equations in **order2Eqm**, equating the coefficients of **sol2Formb1** and **sol2Formb2**, and solving for **uSymbolb1** and **uSymbolb2**, we have

```
symbolbRule =
 Solve@Coefficient@Subtract žž order2Eqm@@1DD •. sol2b, sol2Formb1~Join~sol2Formb2D ==
      0 •. Exp@_ T0 + _.D -> 0 •• Thread,
    uSymbolb1~Join~uSymbolb2D@@1DD •. W -> 2 w2 •. frequencies •. values1
```

$:ys_{1,1} ® -\frac{I\, m\, A_1@T_1,\ T_2D}{2\sqrt{6}\, p^2},\ ys_{1,2} ® -\frac{2\sqrt{ä}\, A_2@T_1,\ T_2D\, \dot{A}_1@T_1,\ T_2D}{\sqrt{3}},$

$ys_{2,2} ® -\frac{I\, m\, A_2@T_1,\ T_2D}{4\sqrt{6}\, p^2},\ ys_{2,3} ® -\frac{f\, \dot{A}_2@T_1,\ T_2D}{96\sqrt{2}\, p^2},\ ys_{2,1} ® -\frac{\sqrt{ä}\, A_1@T_1,\ T_2D^2}{4\sqrt{3}}>$

Substituting these symbol values into **sol2a** and **sol2b** yields

```
sol2aForm = 8u2@x, T0, T1, T2D, v2@x, T0, T1, T2D< •. sol2a •. symbolaRule;
sol2bForm = 8u2@x, T0, T1, T2D, v2@x, T0, T1, T2D< •. sol2b •. symbolbRule;
```

```
sol2 = 8u2 ® Function@8x, T0, T1, T2<,
    sol2aForm@@1DD + sol2bForm@@1DD + Hsol2bForm@@1DD •. conjugateRuleL •• Expand ••
     EvaluateD, v2 ® Function@8x, T0, T1, T2<, sol2aForm@@2DD + sol2bForm@@2DD +
      Hsol2bForm@@2DD •. conjugateRuleL •• Expand •• EvaluateD<
```

:u_2 ® FunctionB8x, T_0, T_1, T_2<,

$$-\frac{f^2 Sin@p xD}{768 \sqrt{3} p^4 \sqrt{a}} + \frac{E^{-2 I T_0 W} f^2 Sin@p xD}{23040 \sqrt{3} p^4 \sqrt{a}} + \frac{E^{2 I T_0 W} f^2 Sin@p xD}{23040 \sqrt{3} p^4 \sqrt{a}} - \frac{I E^{-I T_0 W} f m Sin@p xD}{216 p^4 \sqrt{a}} +$$

$$\frac{I E^{I T_0 W} f m Sin@p xD}{216 p^4 \sqrt{a}} + \frac{E^{-I T_0 W+I T_0 w_1} f Sin@2 p xD A_1@T_1, T_2D}{18 \sqrt{2} p^2} + \frac{E^{I T_0 W+I T_0 w_1} f Sin@2 p xD A_1@T_1, T_2D}{54 \sqrt{2} p^2} -$$

$$\frac{I E^{I T_0 w_1} m Sin@2 p xD A_1@T_1, T_2D}{2 \sqrt{6} p^2} - \frac{E^{2 I T_0 w_1} \sqrt{a} Sin@p xD A_1@T_1, T_2D^2}{4 \sqrt{3}} -$$

$$\frac{E^{-I T_0 W+I T_0 w_2} f Sin@p xD A_2@T_1, T_2D}{96 \sqrt{2} p^2} + \frac{E^{I T_0 W+I T_0 w_2} f Sin@p xD A_2@T_1, T_2D}{192 \sqrt{2} p^2} -$$

$$\frac{I E^{I T_0 w_2} m Sin@p xD A_2@T_1, T_2D}{4 \sqrt{6} p^2} + \frac{E^{I T_0 w_1+I T_0 w_2} \sqrt{a} Sin@2 p xD A_1@T_1, T_2D A_2@T_1, T_2D}{\sqrt{3}} +$$

$$\frac{E^{2 I T_0 w_2} \sqrt{a} Sin@p xD A_2@T_1, T_2D^2}{4 \sqrt{3}} + \frac{E^{-I T_0 W-I T_0 w_1} f Sin@2 p xD \dot{A}_1@T_1, T_2D}{54 \sqrt{2} p^2} +$$

$$\frac{E^{I T_0 W-I T_0 w_1} f Sin@2 p xD \dot{A}_1@T_1, T_2D}{18 \sqrt{2} p^2} + \frac{I E^{-I T_0 w_1} m Sin@2 p xD \dot{A}_1@T_1, T_2D}{2 \sqrt{6} p^2} -$$

$$\frac{2 \sqrt{a} Sin@p xD A_1@T_1, T_2D \dot{A}_1@T_1, T_2D}{\sqrt{3}} - \frac{2 E^{-I T_0 w_1+I T_0 w_2} \sqrt{a} Sin@2 p xD A_2@T_1, T_2D \dot{A}_1@T_1, T_2D}{\sqrt{3}} -$$

$$\frac{E^{-2 I T_0 w_1} \sqrt{a} Sin@p xD \dot{A}_1@T_1, T_2D^2}{4 \sqrt{3}} + \frac{E^{-I T_0 W-I T_0 w_2} f Sin@p xD \dot{A}_2@T_1, T_2D}{192 \sqrt{2} p^2} -$$

$$\frac{E^{I T_0 W-I T_0 w_2} f Sin@p xD \dot{A}_2@T_1, T_2D}{96 \sqrt{2} p^2} + \frac{I E^{-I T_0 w_2} m Sin@p xD \dot{A}_2@T_1, T_2D}{4 \sqrt{6} p^2} -$$

$$\frac{2 E^{I T_0 w_1-I T_0 w_2} \sqrt{a} Sin@2 p xD A_1@T_1, T_2D \dot{A}_2@T_1, T_2D}{\sqrt{3}} - \frac{1}{2} \sqrt{3} \sqrt{a} Sin@p xD A_2@T_1, T_2D \dot{A}_2@T_1, T_2D +$$

$$\frac{E^{-I T_0 w_1-I T_0 w_2} \sqrt{a} Sin@2 p xD \dot{A}_1@T_1, T_2D \dot{A}_2@T_1, T_2D}{\sqrt{3}} + \frac{E^{-2 I T_0 w_2} \sqrt{a} Sin@p xD \dot{A}_2@T_1, T_2D^2}{4 \sqrt{3}} F,$$

v_2 ® FunctionB8x, T_0, T_1, T_2<, $-\frac{I E^{-2 I T_0 W} f^2 Sin@p xD}{1440 p^2 \sqrt{a}} + \frac{I E^{2 I T_0 W} f^2 Sin@p xD}{1440 p^2 \sqrt{a}} -$

$$\frac{E^{-I T_0 W} f m Sin@p xD}{9 \sqrt{3} p^2 \sqrt{a}} - \frac{E^{I T_0 W} f m Sin@p xD}{9 \sqrt{3} p^2 \sqrt{a}} - \frac{I E^{-I T_0 W+I T_0 w_1} f Sin@2 p xD A_1@T_1, T_2D}{\sqrt{6}} +$$

$$\frac{5 I E^{I T_0 W+I T_0 w_1} f Sin@2 p xD A_1@T_1, T_2D}{9 \sqrt{6}} - \frac{E^{I T_0 w_1} m Sin@2 p xD w_1 A_1@T_1, T_2D}{2 \sqrt{6} p^2} +$$

$$\frac{I E^{2 I T_0 w_1} \sqrt{a} Sin@p xD w_2 A_1@T_1, T_2D^2}{4 \sqrt{3}} + \frac{1}{16} I \sqrt{\frac{3}{2}} E^{I T_0 W+I T_0 w_2} f Sin@p xD A_2@T_1, T_2D -$$

```
I E^(-I T0 W+I T0 w2) f Sin@p xD w2 A2@T1, T2D      E^(I T0 w2) m Sin@p xD w2 A2@T1, T2D
---------------------------------------------  -  -------------------------------------  +
           96 • !! 2 p^2                                    4 • !! 6 p^2

6 I E^(I T0 w1+I T0 w2) p^2 • !! a Sin@2 p xD A1@T1, T2D A2@T1, T2D + 2 I E^(2 I T0 w2) p^2 • !! a Sin@p xD A2@T1, T2D^2 -

5 I E^(-I T0 W-I T0 w1) f Sin@2 p xD A1'@T1, T2D      I E^(I T0 W-I T0 w1) f Sin@2 p xD A1'@T1, T2D
-----------------------------------------------  +  ---------------------------------------------  -
              9 • !! 6                                            • !! 6

E^(-I T0 w1) m Sin@2 p xD w1 A1'@T1, T2D      2 I E^(-I T0 w1+I T0 w2) • !! a Sin@2 p xD w1 A2@T1, T2D A1'@T1, T2D
----------------------------------------  +  -----------------------------------------------------------------  -
           2 • !! 6 p^2                                                  • !! 3

I E^(-2 I T0 w1) • !! a Sin@p xD w2 A1'@T1, T2D^2      1         3
-------------------------------------------------  -  ---- I $  --- E^(-I T0 W-I T0 w2) f Sin@p xD A2'@T1, T2D +
                 4 • !! 3                              16         2

I E^(I T0 W-I T0 w2) f Sin@p xD w2 A2'@T1, T2D      E^(-I T0 w2) m Sin@p xD w2 A2'@T1, T2D
---------------------------------------------  -  --------------------------------------  -
              96 • !! 2 p^2                                 4 • !! 6 p^2

2 I E^(I T0 w1-I T0 w2) • !! a Sin@2 p xD w1 A1@T1, T2D A2'@T1, T2D
------------------------------------------------------------------  - 6 I E^(-I T0 w1-I T0 w2) p^2 • !! a
                          • !! 3

 Sin@2 p xD A1'@T1, T2D A2'@T1, T2D - 2 I E^(-2 I T0 w2) p^2 • !! a Sin@p xD A2'@T1, T2D^2 F>
```

Ÿ 9.1.3 Solvability Conditions

Substituting **sol1**, **Ysol**, **sol2**, and some of the parameter values into **eqEps[[3]]** yields

```
order3Eq = HlinearSys •. u_1 -> u3L ==
      HHlinearSys •. u_1 -> u3L - H#@@1DD & •ž eqEps@@3DDL •. sol1 •. Ysol •. GRule •. sol2 •.
          intRule1 ••. intRule2 •• ExpandL •. values1 •• Thread; •• Timing
```

```
821.141 Second, Null<
```

Collecting the terms that may lead to secular terms, the terms proportional to $E^{I\,w_i\,T_0}$, we have

```
ST21 =
  CoefficientA#@@2DD & •ž order3Eq •. expRule1@1D, E^(I w1 T0)E •. modeshapes •. int -> Integrate;
```

```
ST22 =
  CoefficientA#@@2DD & •ž order3Eq •. expRule1@2D, E^(I w2 T0)E •. modeshapes •. int -> Integrate;
```

Demanding that **ST21** be orthogonal to the **adjointC[[1]]**, we obtain the solvability condition

```
SCond21 = SolveAHint@adjointC@@1DD.ST21, 8x, 0, 1<D •. intRule1 ••. intRule2L == 0,
        A1H0,1L@T1, T2DE@@1DD •. modeshapes •.
      int -> Integrate •. SCond1 •. frequencies •. values1 •• ExpandAll;
SCond21 •. displayRule
```

$$\left\{D_2A_1 \to \frac{13\, I\, \phi^2 A_1}{648 \sqrt{3}\, \pi^2} - \frac{I\, \mu^2 A_1}{4 \sqrt{3}\, \pi^2} + I \sqrt{3}\, \pi^2 \alpha A_1^2 \bar{A}_1 - \sqrt{\frac{3}{2}}\, E^{I T_1 \sigma_1} \sqrt{\alpha}\, \mu A_2 \bar{A}_1 - \frac{8\, I\, \pi^2 \alpha A_1 A_2 \bar{A}_2}{\sqrt{3}} + \frac{31\, I\, E^{I T_1 \sigma_1 + I T_1 \sigma_2} \phi \sqrt{\alpha}\, \bar{A}_1 \bar{A}_2}{72 \sqrt{2}}\right\}$$

Demanding that **ST22** be orthogonal to the **adjointC[[2]]**, we obtain the solvability condition

```
SCond22 = SolveAHint@adjointC@@2DD.ST22, 8x, 0, 1<D •. intRule1 ••. intRule2L == 0,
        A2H0,1L@T1, T2DE@@1DD •. modeshapes •. int -> Integrate •.
      SCond1 •. W -> 2 w2 •. frequencies •. values1 •• ExpandAll;
SCond22 •. displayRule
```

$$\left\{D_2A_2 \to \frac{1}{4}\sqrt{\frac{3}{2}}\, E^{-I T_1 \sigma_1} \sqrt{\alpha}\, \mu A_1^2 - \frac{19\, I\, \phi^2 A_2}{9216 \sqrt{3}\, \pi^2} - \frac{I\, \mu^2 A_2}{8 \sqrt{3}\, \pi^2} - \frac{4\, I\, \pi^2 \alpha A_1 A_2 \bar{A}_1}{\sqrt{3}} + \frac{31\, I\, E^{I T_1 \sigma_1 + I T_1 \sigma_2} \phi \sqrt{\alpha}\, \bar{A}_1^2}{288 \sqrt{2}} - \frac{5\, E^{I T_1 \sigma_2} \phi\, \mu \bar{A}_2}{288\, \pi^2} - \frac{1}{2} I \sqrt{3}\, \pi^2 \alpha A_2^2 \bar{A}_2\right\}$$

Because in the absence of damping the system is conservative, the modulation equations, solvability conditions, must satisfy symmetry conditions. To check for these symmetries, we let

```
formList1a = 9Ā1@T1, T2D Ā2@T1, T2D, A1@T1, T2D A2@T1, T2D Ā2@T1, T2D=;
```

```
a1coefs = 8s11, s12< -> Coefficient@H2 w1 IL SCond21@@1, 2DD, formList1aD •• Thread
```

$$\left\{s_{11} \to -\frac{31\, E^{I T_1 \sigma_1 + I T_1 \sigma_2} \phi \sqrt{\alpha}\, \omega_1}{36 \sqrt{2}},\ s_{12} \to \frac{16\, \pi^2 \alpha\, \omega_1}{\sqrt{3}}\right\}$$

```
formList2a = 9A1@T1, T2D A2@T1, T2D Ā1@T1, T2D, Ā1@T1, T2D^2=;
```

```
a2coefs = 8s21, s22< -> Coefficient@H2 w2 IL SCond22@@1, 2DD, formList2aD •• Thread
```

$$\left\{s_{21} \to \frac{8\, \pi^2 \alpha\, \omega_2}{\sqrt{3}},\ s_{22} \to -\frac{31\, E^{I T_1 \sigma_1 + I T_1 \sigma_2} \phi \sqrt{\alpha}\, \omega_2}{144 \sqrt{2}}\right\}$$

The symmetry condition demands that $s_{12} = s_{21}$, which is true because

```
s12 - s21 •. a1coefs •. a2coefs •. frequencies •. values1
```

0

à **9.2 Four-Mode Interactions in Suspended Cables**

We determine a second-order uniform asymptotic expansion of the three-dimensional response of a suspended cable to a transverse harmonic distributed excitation of one of the first two vertical or horizontal modes when their frequencies are in the ratio of either two-to-one or one-to-one and no other internal resonance is activated. Benedettini, Rega, and Alaggio (1995) derived nondimensional equations governing three-dimensional finite-amplitude vibrations of a suspended homogeneous elastic cable excited by harmonic distributed vertical and horizontal loads. The time derivatives are expressed in second-order form. As discussed in Chapter 5, second- and higher-order treatments of such equations lead to inconsistent results unless the time derivatives are expressed in first-order rather than second-order form. Therefore, we rewrite these equations as

```
d_{i_Integer,j_Integer} := If@i == j, 1, 0D

eq1 = Table@u_{j,t} - v_j == 0, 8j, 2<D;

eq2 = TableAv_{j,t} + 2 m_j v_j - u_{j,x,x} - a Hb y''@xD d_{1,j} + u_{j,x,x}L à_0^1 jb y'@xD u_{1,x} + 1/2 Iu_{1,x}^2 + u_{2,x}^2 M z âx ==
    P_j@xD Cos@W t + t_j D, 8j, 2<E;

EOM = Transpose@8eq1, eq2<D •. 9u_{s_,m__} :> ¶_m u_s@x, tD, v_{s_} :> v_s@x, tD= •• Flatten
```

```
:-v_1@x, tD + u_1^{H0,1L}@x, tD == 0, 2 m_1 v_1@x, tD + v_1^{H0,1L}@x, tD - u_1^{H2,0L}@x, tD -
   a j à_0^1 jb y'@xD u_1^{H1,0L}@x, tD + 1/2 Ju_1^{H1,0L}@x, tD^2 + u_2^{H1,0L}@x, tD^2 N z âx z Ib y''@xD + u_1^{H2,0L}@x, tDM ==
  Cos@t W + t_1 D P_1@xD, -v_2@x, tD + u_2^{H0,1L}@x, tD == 0, 2 m_2 v_2@x, tD + v_2^{H0,1L}@x, tD - u_2^{H2,0L}@x, tD -
   a j à_0^1 jb y'@xD u_1^{H1,0L}@x, tD + 1/2 Ju_1^{H1,0L}@x, tD^2 + u_2^{H1,0L}@x, tD^2 N z âx z u_2^{H2,0L}@x, tD ==
  Cos@t W + t_2 D P_2@xD>
```

```
yRule = 8y -> H4 # H1 - #L &L<;
```

Moreover, we write the associated boundary conditions as

```
BC = 8u_j@0, tD == 0, u_j@1, tD == 0<;
```

Here, the index $j = 1$ refers to the vertical component of the displacement and $j = 2$ refers to the horizontal component of the displacement; the m_j are the viscous damping coefficients; the P_j@xD are the distributed amplitudes of the excitation with the frequency W; $d_{i,j}$ denotes the Kronecker delta; and $b\, y$@xD, where y@xD = $4x$H1 - xL, defined in the **yRule**, is the initial static configuration. The spatial and temporal variables were nondimensionalized using the span of the cable { and the characteristic time { • !!!!!!! ·•E, where · and E are the cable density and modulus of elasticity, respectively.

We seek an asymptotic expansion of the response of the cable when its span is such that the first vertical and horizontal natural frequencies are in the ratio of two-to-one; that is, $b^2 a$ » p^2 • 16. When $b^2 a$ = p^2 • 16, the natural frequencies are

```
params = 9w_{j_•;j!=3} :> 2 p, w_3 -> p, b^2 a -> p^2 • 16=;
```

and the corresponding normalized modeshapes are

```
modeshapes = 9F1 -> ˘ $ ¶¶¶¶ H1 - Cos@2 p #DL &ˇ,
                    ˘  2                    ˇ
                    k  3                    {
   F2 -> I • !!! 2 Sin@2 p #D &M, F3 -> I • !!! 2 Sin@p #D &M, F4 -> I • !!! 2 Sin@2 p #D &M=;
```

We use the method of multiple scales and seek a second-order uniform asymptotic expansion of the response of the cable to a primary-resonance excitation of the first vertical mode in the form

```
multiScales = 9u_j_@x, tD -> uj@x, T0, T1, T2D,
   Derivative@m_, n_D@u_D@x, tD -> dt@nD@D@u@x, T0, T1, T2D, 8x, m<DD, t -> T0=;
```

```
solRule = 9ui_ -> IEvaluateASumAe^j ui,j@#1, #2, #3, #4D, 8j, 3<EE &M,
   vi_ -> IEvaluateASumAe^j vi,j@#1, #2, #3, #4D, 8j, 3<EE &M=;
```

Taking into account the resonance conditions, we scale the damping and forcing terms as

```
scaling = 9mj_ -> e^2 mj, Pj_@xD -> e^3 Pj@xD=;
```

where the forcing has been scaled at order e^3 so that its influence first appears at the same order as the nonlinear shift in the frequencies in the absence of the internal resonance.

We consider the case in which the orders of magnitude of the forcing levels for both the vertical and horizontal motions are the same. Otherwise, a primary resonance of the vertical modes (current study) could also initiate a subharmonic resonance of order one-half of the first horizontal mode and a primary resonance of the second horizontal mode at different levels of approximation.

Substituting the **multiScales**, **solRule**, and **scaling** into **EOM**, expanding the result for small e, and discarding terms of order higher than e^3, we obtain

```
eq92a =
  HHEOM •. Integrate -> int •. multiScales •. solRule •. scaling •• ExpandAllL ••. intRule2 ••
     ExpandAllL •. e^n_•;n>3 -> 0;
```

Equating coefficients of like powers of e in **eq92a**, we obtain

```
eqEps =
  Rest@Thread@CoefficientList@Subtract žž #, eD == 0DD & •ž eq92a •• Transpose •• TrigToExp;
```

Ÿ 9.2.1 First-Order Solution

The homogeneous parts of the first-order equations are

```
HlinearSys = #@@1DD & •ž eqEps@@1DDL •. displayRule
```

```
8D0u1,1 - v1,1, -Hu''1,1L + D0v1,1 - b^2 a int@Hu'1,1L y¢@xD, 8x, 0, 1<D y''@xD,
 D0u2,1 - v2,1, -Hu''2,1L + D0v2,1<
```

The first-order problems are identical to the linear eigenvalue problems. To account for the interactions arising from the multiple internal resonances and the primary resonance, we include the lowest two modes in each plane; the other modes will decay due to damping. Hence, the solution of **eqEps[[1]]** can be expressed as

```
sol1u = 9u1,1 -> FunctionA8x, T0, T1, T2<,
     SumAAi@T1, T2D Fi@xD Exp@I wi T0D + Ȧi@T1, T2D Fi@xD Exp@- I wi T0D, 8i, 1, 2<E •• EvaluateE,
   u2,1 -> FunctionA8x, T0, T1, T2<,
     SumAAi@T1, T2D Fi@xD Exp@I wi T0D + Ȧi@T1, T2D Fi@xD Exp@- I wi T0D, 8i, 3, 4<E •• EvaluateE=;
```

```
sol1v = Table@
   vk,1 ® Function@8x, T0, T1, T2<, D@uk,1@x, T0, T1, T2D, T0D •. sol1u •• EvaluateD, 8k, 2<D;
```

```
sol1 = Join@sol1u, sol1vD
```

```
8u1,1 ® Function@8x, T0, T1, T2<, E^(I T0 w1) A1@T1, T2D F1@xD +
     E^(I T0 w2) A2@T1, T2D F2@xD + E^(-I T0 w1) F1@xD Ȧ1@T1, T2D + E^(-I T0 w2) F2@xD Ȧ2@T1, T2DD,
 u2,1 ® Function@8x, T0, T1, T2<, E^(I T0 w3) A3@T1, T2D F3@xD + E^(I T0 w4) A4@T1, T2D F4@xD +
     E^(-I T0 w3) F3@xD Ȧ3@T1, T2D + E^(-I T0 w4) F4@xD Ȧ4@T1, T2DD,
 v1,1 ® Function@8x, T0, T1, T2<, I E^(I T0 w1) w1 A1@T1, T2D F1@xD + I E^(I T0 w2) w2 A2@T1, T2D F2@xD -
     I E^(-I T0 w1) w1 F1@xD Ȧ1@T1, T2D - I E^(-I T0 w2) w2 F2@xD Ȧ2@T1, T2DD,
 v2,1 ® Function@8x, T0, T1, T2<, I E^(I T0 w3) w3 A3@T1, T2D F3@xD + I E^(I T0 w4) w4 A4@T1, T2D F4@xD -
     I E^(-I T0 w3) w3 F3@xD Ȧ3@T1, T2D - I E^(-I T0 w4) w4 F4@xD Ȧ4@T1, T2DD<
```

where F_1@xD and F_2@xD are the lowest symmetric and antisymmetric vertical eigenmodes, and F_3@xD and F_4@xD are the lowest symmetric and antisymmetric horizontal eigenmodes. The F_n@xD are orthogonal; they are normalized so that they satisfy the orthonormality condition $Ù_0^1$ F_n@xD F_m@xD â x = $d_{n,m}$.

Because the problem is self-adjoint, we have

```
adjoint = Table@8- I wk Fk@xD, Fk@xD<, 8k, 4<D;
```

whose complex conjugate is

```
adjointC = adjoint •. conjugateRule
```

```
88I w1 F1@xD, F1@xD<, 8I w2 F2@xD, F2@xD<, 8I w3 F3@xD, F3@xD<, 8I w4 F4@xD, F4@xD<<
```

Ÿ 9.2.2 Second-Order Solution

Substituting **sol1** into **eqEps[[2]]** yields

```
order2Eq = IlinearSys •. u_{k_,1} -> u_{k,2}M ==
    IIlinearSys •. u_{k_,1} -> u_{k,2}M - H#@@1DD & •ž eqEps@@2DDL •. sol1 •. intRule1 ••. intRule2 •.
        modeshapes •. yRule •. int -> Integrate •• ExpandM •• Thread;
order2Eq •. displayRule
```

```
:D0u1,2 - v1,2 == - $ 2/3 E^(I T0 w1) HD1A1L + $ 2/3 E^(I T0 w1) Cos@2 p xD HD1A1L - $ 2/3 E^(-I T0 w1) HD1Ȧ1L +
   $ 2/3 E^(-I T0 w1) Cos@2 p xD HD1Ȧ1L - • 2 E^(I T0 w2) HD1A2L Sin@2 p xD - • 2 E^(-I T0 w2) HD1Ȧ2L Sin@2 p xD,
 -Hu''1,2L + D0v1,2 - b^2 a int@Hu'1,2L y¢@xD, 8x, 0, 1<D y²@xD ==
  - 16/3 b E^(2 I T0 w1) p^2 a A1^2 + 64/3 b E^(2 I T0 w1) p^2 a Cos@2 p xD A1^2 - 64 b E^(I T0 Hw1+w2L) p^2 a Sin@2 p xD A1 A2 / (• 3) -
   16 b E^(2 I T0 w2) p^2 a A2^2 - 4 b E^(2 I T0 w3) p^2 a A3^2 - 16 b E^(2 I T0 w4) p^2 a A4^2 - I $ 2/3 E^(I T0 w1) HD1A1L w1 +
   I $ 2/3 E^(I T0 w1) Cos@2 p xD HD1A1L w1 + I $ 2/3 E^(-I T0 w1) HD1Ȧ1L w1 - I $ 2/3 E^(-I T0 w1) Cos@2 p xD HD1Ȧ1L w1 -
   I • 2 E^(I T0 w2) HD1A2L Sin@2 p xD w2 + I • 2 E^(-I T0 w2) HD1Ȧ2L Sin@2 p xD w2 - 32/3 b p^2 a A1 Ȧ1 +
   128/3 b p^2 a Cos@2 p xD A1 Ȧ1 - 64 b E^(I T0 H-w1+w2L) p^2 a Sin@2 p xD A2 Ȧ1 / (• 3) - 16/3 b E^(-2 I T0 w1) p^2 a Ȧ1^2 +
   64/3 b E^(-2 I T0 w1) p^2 a Cos@2 p xD Ȧ1^2 - 64 b E^(I T0 Hw1-w2L) p^2 a Sin@2 p xD A1 Ȧ2 / (• 3) -
   32 b p^2 a A2 Ȧ2 - 64 b E^(I T0 H-w1-w2L) p^2 a Sin@2 p xD Ȧ1 Ȧ2 / (• 3) - 16 b E^(-2 I T0 w2) p^2 a Ȧ2^2 -
   8 b p^2 a A3 Ȧ3 - 4 b E^(-2 I T0 w3) p^2 a Ȧ3^2 - 32 b p^2 a A4 Ȧ4 - 16 b E^(-2 I T0 w4) p^2 a Ȧ4^2,
 D0u2,2 - v2,2 == - • 2 E^(I T0 w3) HD1A3L Sin@p xD - • 2 E^(-I T0 w3) HD1Ȧ3L Sin@p xD -
   • 2 E^(I T0 w4) HD1A4L Sin@2 p xD - • 2 E^(-I T0 w4) HD1Ȧ4L Sin@2 p xD,
 -Hu''2,2L + D0v2,2 == - 16 b E^(I T0 Hw1+w3L) p^2 a Sin@p xD A1 A3 / (• 3) - 64 b E^(I T0 Hw1+w4L) p^2 a Sin@2 p xD A1 A4 / (• 3) -
   I • 2 E^(I T0 w3) HD1A3L Sin@p xD w3 + I • 2 E^(-I T0 w3) HD1Ȧ3L Sin@p xD w3 -
   I • 2 E^(I T0 w4) HD1A4L Sin@2 p xD w4 + I • 2 E^(-I T0 w4) HD1Ȧ4L Sin@2 p xD w4 -
   16 b E^(I T0 H-w1+w3L) p^2 a Sin@p xD A3 Ȧ1 / (• 3) - 64 b E^(I T0 H-w1+w4L) p^2 a Sin@2 p xD A4 Ȧ1 / (• 3) -
   16 b E^(I T0 Hw1-w3L) p^2 a Sin@p xD A1 Ȧ3 / (• 3) - 16 b E^(I T0 H-w1-w3L) p^2 a Sin@p xD Ȧ1 Ȧ3 / (• 3) -
   64 b E^(I T0 Hw1-w4L) p^2 a Sin@2 p xD A1 Ȧ4 / (• 3) - 64 b E^(I T0 H-w1-w4L) p^2 a Sin@2 p xD Ȧ1 Ȧ4 / (• 3) >
```

We consider the case of primary resonance of the first vertical mode, a two-to-one internal resonance between the first vertical and horizontal modes, a one-to-one internal resonance between the first and second vertical modes, and a one-to-one internal resonance between the first vertical and second horizontal modes. In order to collect the terms that may lead to secular terms from the right-hand sides of **order2Eq**, we define the rules

```
omgList = Table@wi, 8i, 4<D;
ResonanceConds = 8W == w1 + e s1, w2 == w1 + e s2, 2 w3 == w1 + e s3, w4 == w1 + e s4<;
```

```
OmgRule = Solve@ResonanceConds, Complement@omgList, 8#<D~Join~8W<D@@1DD & •ž omgList
```

$$\left\{\left\{\omega_2 \to \epsilon\sigma_2+\omega_1,\ \omega_3 \to \frac{1}{2}(\epsilon\sigma_3+\omega_1),\ \omega_4 \to \epsilon\sigma_4+\omega_1,\ \Omega \to \epsilon\sigma_1+\omega_1\right\},\right.$$
$$\left\{\omega_1 \to -\epsilon\sigma_2+\omega_2,\ \omega_3 \to \frac{1}{2}(-\epsilon\sigma_2+\epsilon\sigma_3+\omega_2),\ \omega_4 \to -\epsilon\sigma_2+\epsilon\sigma_4+\omega_2,\ \Omega \to \epsilon\sigma_1-\epsilon\sigma_2+\omega_2\right\},$$
$$\{\omega_1 \to -\epsilon\sigma_3+2\omega_3,\ \omega_2 \to \epsilon\sigma_2-\epsilon\sigma_3+2\omega_3,\ \omega_4 \to -\epsilon\sigma_3+\epsilon\sigma_4+2\omega_3,\ \Omega \to \epsilon\sigma_1-\epsilon\sigma_3+2\omega_3\},$$
$$\left.\left\{\omega_1 \to -\epsilon\sigma_4+\omega_4,\ \omega_2 \to \epsilon\sigma_2-\epsilon\sigma_4+\omega_4,\ \omega_3 \to \frac{1}{2}(\epsilon\sigma_3-\epsilon\sigma_4+\omega_4),\ \Omega \to \epsilon\sigma_1-\epsilon\sigma_4+\omega_4\right\}\right\}$$

```
expRule1@i_D := Exp@arg_D :> Exp@Expand@arg •. OmgRule@@iDDD •. e T0 -> T1D
```

Collecting the terms that may lead to secular terms, the terms proportional to $E^{I\omega_i T_0}$, from the equations governing the vertical modes, we have

```
ST1v = CoefficientA#@@2DD & •ž order2Eq@@81, 2<DD •. expRule1@#D, E^(I w# T0)E & •ž 81, 2<;
ST1v •. displayRule
```

$$\left\{\left\{-\sqrt{\frac{2}{3}}(D_1A_1)+\sqrt{\frac{2}{3}}\cos[2\pi x](D_1A_1)-\sqrt{2}\,E^{IT_1\sigma_2}(D_1A_2)\sin[2\pi x],\ -4bE^{IT_1\sigma_3}\pi^2\alpha A_3^2-\right.\right.$$
$$\left.I\sqrt{\frac{2}{3}}(D_1A_1)\omega_1+I\sqrt{\frac{2}{3}}\cos[2\pi x](D_1A_1)\omega_1-I\sqrt{2}\,E^{IT_1\sigma_2}(D_1A_2)\sin[2\pi x]\omega_2\right\},$$
$$\left\{-\sqrt{\frac{2}{3}}E^{-IT_1\sigma_2}(D_1A_1)+\sqrt{\frac{2}{3}}E^{-IT_1\sigma_2}\cos[2\pi x](D_1A_1)-\sqrt{2}(D_1A_2)\sin[2\pi x],\right.$$
$$-4bE^{-IT_1\sigma_2+IT_1\sigma_3}\pi^2\alpha A_3^2-I\sqrt{\frac{2}{3}}E^{-IT_1\sigma_2}(D_1A_1)\omega_1+$$
$$\left.\left.I\sqrt{\frac{2}{3}}E^{-IT_1\sigma_2}\cos[2\pi x](D_1A_1)\omega_1-I\sqrt{2}(D_1A_2)\sin[2\pi x]\omega_2\right\}\right\}$$

Collecting the terms that may lead to secular terms from the equations governing the horizontal modes, we have

```
ST1h = CoefficientA#@@2DD & •ž order2Eq@@83, 4<DD •. expRule1@#D, E^(I w# T0)E & •ž 83, 4<;
ST1h •. displayRule
```

$$\left\{\left\{-\sqrt{2}(D_1A_3)\sin[\pi x],\ -I\sqrt{2}(D_1A_3)\sin[\pi x]\omega_3-\frac{16bE^{-IT_1\sigma_3}\pi^2\alpha\sin[\pi x]A_1\bar{A}_3}{\sqrt{3}}\right\},\right.$$
$$\left.\left\{-\sqrt{2}(D_1A_4)\sin[2\pi x],\ -I\sqrt{2}(D_1A_4)\sin[2\pi x]\omega_4\right\}\right\}$$

Demanding that **ST1v[[i]]** be orthogonal to the **adjointC[[i]]**, we obtain the solvability conditions

```
SCond1v =
 SolveAHint@adjointC@@#DD.ST1v@@#DD, 8x, 0, 1<D •. intRule1 ••. intRule2 •. modeshapes •.
        params •. int -> IntegrateL == 0, A#H1,0L@T1, T2DE@@1DD & •ž 81, 2< •• Flatten;
SCond1v •. displayRule
```

:D1A1 ® I $ 2/3 b E^{I T1 s3} p a A3^2, D1A2 ® 0>

Demanding that **ST1h[[i]]** be orthogonal to the **adjointC[[i+2]]**, we obtain the solvability conditions

```
SCond1h =
 SolveAHint@adjointC@@# + 2DD.ST1h@@#DD, 8x, 0, 1<D •. intRule1 ••. intRule2 •. modeshapes •.
        params •. int -> IntegrateL == 0, A#+2H1,0L@T1, T2DE@@1DD & •ž 81, 2< •• Flatten;
SCond1h •. displayRule
```

:D1A3 ® 4 I $ 2/3 b E^{-I T1 s3} p a A1 Ā3, D1A4 ® 0>

```
SCond1 = Join@SCond1v, SCond1hD;
```

whose complex conjugates are

```
ccSCond1 = SCond1 •. conjugateRule;
```

Next, we use the solvability conditions to eliminate the $D_1 A_i$ from the right-hand sides of **order2Eq**. To this end, we define the rules

```
sigRule = Solve@ResonanceConds, Table@si, 8i, 4<DD@@1DD
```

9s1 ® - (-W + w1)/e, s2 ® - (w1 - w2)/e, s3 ® - (w1 - 2 w3)/e, s4 ® - (w1 - w4)/e =

```
expRule2 = Exp@a_D :> Exp@a •. sigRule •. T1 -> e T0 •• ExpandD;
```

Substituting **SCond1**, **ccSCond1**, and **expRule2** into **order2Eq**, we obtain

```
order2Eqm = order2Eq •. SCond1 •. ccSCond1 •. expRule2;
```

To determine the solution of **order2Eqm** and the associated boundary conditions, we use the method of undetermined coefficients. To accomplish this, we first determine the forms of the terms on the right-hand sides of **order2Eqm** by picking up the x dependent or x and T_0 dependent terms by defining the rule

```
baits = 9_ E^(a_ T0+b_.) f_@c_ xD :> E^(a T0+b) f@c xD, a_ f_@b_ xD :> f@b xD •; FreeQ@a, T0D=;
```

Then, we collect the solution forms for the horizontal modes according to

```
sol2Formh = Table@Cases@order2Eqm@@k, 2DD, #D & •ž baits, 8k, 3, 4<D •• Flatten •• Union
```

```
8E^(-I T0 w1 - I T0 w3) Sin@p xD, E^(I T0 w1 - I T0 w3) Sin@p xD, E^(-I T0 w1 + I T0 w3) Sin@p xD, E^(I T0 w1 + I T0 w3) Sin@p xD,
 E^(-I T0 w1 - I T0 w4) Sin@2 p xD, E^(I T0 w1 - I T0 w4) Sin@2 p xD, E^(-I T0 w1 + I T0 w4) Sin@2 p xD, E^(I T0 w1 + I T0 w4) Sin@2 p xD<
```

Each of these terms satisfies the boundary conditions. However, not all of the solution forms

```
sol2Formv = Table@Cases@order2Eqm@@k, 2DD, #D & •ž baits, 8k, 2<D •• Flatten •• Union
```

```
8Cos@2 p xD, E^(-2 I T0 w1) Cos@2 p xD, E^(2 I T0 w1) Cos@2 p xD, E^(-2 I T0 w3) Cos@2 p xD, E^(2 I T0 w3) Cos@2 p xD,
 E^(-I T0 w1 - I T0 w2) Sin@2 p xD, E^(I T0 w1 - I T0 w2) Sin@2 p xD, E^(-I T0 w1 + I T0 w2) Sin@2 p xD, E^(I T0 w1 + I T0 w2) Sin@2 p xD<
```

for the vertical modes satisfy the boundary conditions. Besides, there exist terms that are not functions of x on the right-hand sides of **order2Eqm[[{1,2}]]**, namely

```
#@@2DD & •ž order2Eqm@@81, 2<DD •. Thread@sol2Formv -> 0D •. displayRule
```

$$9-\frac{2}{3} I b E^{2 I T_0 w_3} p\, a A_3^2+\frac{2}{3} I b E^{-2 I T_0 w_3} p\, a \dot{A}_3^2,$$

$$-\frac{16}{3} b E^{2 I T_0 w_1} p^2 a A_1^2-16 b E^{2 I T_0 w_2} p^2 a A_2^2-4 b E^{2 I T_0 w_3} p^2 a A_3^2-16 b E^{2 I T_0 w_4} p^2 a A_4^2+$$

$$\frac{2}{3} b E^{2 I T_0 w_3} p\, a A_3^2 w_1-\frac{32}{3} b p^2 a A_1 \dot{A}_1-\frac{16}{3} b E^{-2 I T_0 w_1} p^2 a \dot{A}_1^2-32 b p^2 a A_2 \dot{A}_2-16 b E^{-2 I T_0 w_2} p^2 a \dot{A}_2^2-$$

$$8 b p^2 a A_3 \dot{A}_3-4 b E^{-2 I T_0 w_3} p^2 a \dot{A}_3^2+\frac{2}{3} b E^{-2 I T_0 w_3} p\, a w_1 \dot{A}_3^2-32 b p^2 a A_4 \dot{A}_4-16 b E^{-2 I T_0 w_4} p^2 a \dot{A}_4^2 =$$

Hence, we need to either consider some algebraic forms or combined algebraic and sinusoidal forms for x to manually adjust the forms in **sol2Formv** or use a more general approach to let *Mathematica* pick up the right forms. The latter approach is used here.

Ÿ Vertical Modes

Instead of looking for the x dependent forms, we seek all of the T_0 dependent forms as

```
sol2Formv1 =
 Join@81<, Table@Cases@order2Eqm@@k, 2DD, Exp@_D, ¥D, 8k, 2<D •• Flatten •• UnionD
```

```
81, E^(-2 I T0 w1), E^(2 I T0 w1), E^(-2 I T0 w2), E^(2 I T0 w2), E^(-I T0 w1 - I T0 w2),
 E^(I T0 w1 - I T0 w2), E^(-I T0 w1 + I T0 w2), E^(I T0 w1 + I T0 w2), E^(-2 I T0 w3), E^(2 I T0 w3), E^(-2 I T0 w4), E^(2 I T0 w4)<
```

where the first argument in **sol2Formv1** is included to account for the T_0 independent terms.

We define the undetermined coefficients for $u_{1,2}$ and $v_{1,2}$ as

```
uSymbolv = Table@y1,i@xD, 8i, Length@sol2Formv1D<D;
vSymbolv = uSymbolv •. y -> h;
```

and write the general solution in the form

```
sol2v = 8u1,2 -> Function@8x, T0, T1, T2<, uSymbolv.sol2Formv1 •• EvaluateD,
   v1,2 -> Function@8x, T0, T1, T2<, vSymbolv.sol2Formv1 •• EvaluateD<;
```

Substituting **sol2v** into **order2Eqm[[1]]** and equating the coefficients of the **sol2Formv1** on both sides yields

```
expr1 = Subtract žž order2Eqm@@1DD •. sol2v;
```

```
eqv1 = Join@8expr1 == 0 •. Exp@_D -> 0<, Coefficient@expr1, Rest@sol2Formv1DD == 0 •• ThreadD
```

```
9- h1,1@xD == 0, - h1,2@xD - 2 I w1 y1,2@xD == 0, -h1,3@xD + 2 I w1 y1,3@xD == 0,
 - h1,4@xD - 2 I w2 y1,4@xD == 0, -h1,5@xD + 2 I w2 y1,5@xD == 0,
 - h1,6@xD - I w1 y1,6@xD - I w2 y1,6@xD == 0, - h1,7@xD + I w1 y1,7@xD - I w2 y1,7@xD == 0,
 - h1,8@xD - I w1 y1,8@xD + I w2 y1,8@xD == 0, - h1,9@xD + I w1 y1,9@xD + I w2 y1,9@xD == 0,
 - 2/3 I b p a A3'@T1, T2D^2 + 2/3 I b p a Cos@2 p xD A3'@T1, T2D^2 - h1,10@xD - 2 I w3 y1,10@xD == 0,
 2/3 I b p a A3@T1, T2D^2 - 2/3 I b p a Cos@2 p xD A3@T1, T2D^2 - h1,11@xD + 2 I w3 y1,11@xD == 0,
 - h1,12@xD - 2 I w4 y1,12@xD == 0, - h1,13@xD + 2 I w4 y1,13@xD == 0=
```

Solving for **vSymbolv** from **eqv1**, we obtain

```
eta1Sol = Solve@eqv1, vSymbolvD@@1DD
```

```
9h1,1@xD ® 0, h1,2@xD ® - 2 I w1 y1,2@xD, h1,3@xD ® 2 I w1 y1,3@xD, h1,4@xD ® - 2 I w2 y1,4@xD,
 h1,5@xD ® 2 I w2 y1,5@xD, h1,6@xD ® - I Hw1 y1,6@xD + w2 y1,6@xDL, h1,7@xD ® I Hw1 y1,7@xD - w2 y1,7@xDL,
 h1,8@xD ® - I Hw1 y1,8@xD - w2 y1,8@xDL, h1,9@xD ® I Hw1 y1,9@xD + w2 y1,9@xDL,
 h1,10@xD ® 2/3 I J- b p a A3'@T1, T2D^2 + b p a Cos@2 p xD A3'@T1, T2D^2 - 3 w3 y1,10@xDN,
 h1,11@xD ® - 2/3 I H- b p a A3@T1, T2D^2 + b p a Cos@2 p xD A3@T1, T2D^2 - 3 w3 y1,11@xDL,
 h1,12@xD ® - 2 I w4 y1,12@xD, h1,13@xD ® 2 I w4 y1,13@xD=
```

Substituting **sol2v** and **eta1Sol** into **order2Eqm[[2]]** and equating the coefficients of the **sol2Formv1** on both sides yields

```
expr2 = Subtract žž order2Eqm@@2DD •. sol2v •. eta1Sol •. intRule1 ••. intRule2 •.
    int@y¢@xD y1,k_¢@xD, _D -> bk •. yRule;
```

```
eqv2 = Join@8expr2 == 0 •. Exp@_D -> 0<,
   Coefficient@expr2, Rest@sol2Formv1DD == 0 •• ThreadD •. params
```

$$: \frac{p^2 b_1}{2} + \frac{32}{3} b\, p^2\, a\, A_1@T_1,\ T_2D\, \dot{A}_1@T_1,\ T_2D - \frac{128}{3} b\, p^2\, a\, Cos@2\, p\, xD\, A_1@T_1,\ T_2D\, \dot{A}_1@T_1,\ T_2D + 32\, b\, p^2\, a\, A_2@T_1,\ T_2D\, \dot{A}_2@T_1,\ T_2D + 8\, b\, p^2\, a\, A_3@T_1,\ T_2D\, \dot{A}_3@T_1,\ T_2D + 32\, b\, p^2\, a\, A_4@T_1,\ T_2D\, \dot{A}_4@T_1,\ T_2D - y^{2}_{1,1}@xD == 0,$$

$$\frac{p^2 b_2}{2} + \frac{16}{3} b\, p^2\, a\, \dot{A}_1@T_1,\ T_2D^2 - \frac{64}{3} b\, p^2\, a\, Cos@2\, p\, xD\, \dot{A}_1@T_1,\ T_2D^2 - 16\, p^2\, y_{1,2}@xD - y^{2}_{1,2}@xD == 0,$$

$$\frac{p^2 b_3}{2} + \frac{16}{3} b\, p^2\, a\, A_1@T_1,\ T_2D^2 - \frac{64}{3} b\, p^2\, a\, Cos@2\, p\, xD\, A_1@T_1,\ T_2D^2 - 16\, p^2\, y_{1,3}@xD - y^{2}_{1,3}@xD == 0,$$

$$\frac{p^2 b_4}{2} + 16\, b\, p^2\, a\, \dot{A}_2@T_1,\ T_2D^2 - 16\, p^2\, y_{1,4}@xD - y^{2}_{1,4}@xD == 0,$$

$$\frac{p^2 b_5}{2} + 16\, b\, p^2\, a\, A_2@T_1,\ T_2D^2 - 16\, p^2\, y_{1,5}@xD - y^{2}_{1,5}@xD == 0,$$

$$\frac{p^2 b_6}{2} + \frac{64\, b\, p^2\, a\, Sin@2\, p\, xD\, \dot{A}_1@T_1,\ T_2D\, \dot{A}_2@T_1,\ T_2D}{\sqrt{3}} - 16\, p^2\, y_{1,6}@xD - y^{2}_{1,6}@xD == 0,$$

$$\frac{p^2 b_7}{2} + \frac{64\, b\, p^2\, a\, Sin@2\, p\, xD\, A_1@T_1,\ T_2D\, \dot{A}_2@T_1,\ T_2D}{\sqrt{3}} - y^{2}_{1,7}@xD == 0,$$

$$\frac{p^2 b_8}{2} + \frac{64\, b\, p^2\, a\, Sin@2\, p\, xD\, A_2@T_1,\ T_2D\, \dot{A}_1@T_1,\ T_2D}{\sqrt{3}} - y^{2}_{1,8}@xD == 0,$$

$$\frac{p^2 b_9}{2} + \frac{64\, b\, p^2\, a\, Sin@2\, p\, xD\, A_1@T_1,\ T_2D\, A_2@T_1,\ T_2D}{\sqrt{3}} - 16\, p^2\, y_{1,9}@xD - y^{2}_{1,9}@xD == 0,$$

$$\frac{p^2 b_{10}}{2} + \frac{4}{3} b\, p^2\, a\, \dot{A}_3@T_1,\ T_2D^2 + \frac{8}{3} b\, p^2\, a\, Cos@2\, p\, xD\, \dot{A}_3@T_1,\ T_2D^2 - 4\, p^2\, y_{1,10}@xD - y^{2}_{1,10}@xD == 0,$$

$$\frac{p^2 b_{11}}{2} + \frac{4}{3} b\, p^2\, a\, A_3@T_1,\ T_2D^2 + \frac{8}{3} b\, p^2\, a\, Cos@2\, p\, xD\, A_3@T_1,\ T_2D^2 - 4\, p^2\, y_{1,11}@xD - y^{2}_{1,11}@xD == 0,$$

$$\frac{p^2 b_{12}}{2} + 16\, b\, p^2\, a\, \dot{A}_4@T_1,\ T_2D^2 - 16\, p^2\, y_{1,12}@xD - y^{2}_{1,12}@xD == 0,$$

$$\frac{p^2 b_{13}}{2} + 16\, b\, p^2\, a\, A_4@T_1,\ T_2D^2 - 16\, p^2\, y_{1,13}@xD - y^{2}_{1,13}@xD == 0>$$

Next, we determine the solutions of the boundary-value problems: **eqv2** and the associated boundary conditions. First, we obtain the information about the solution forms **uSymbolv** according to

```
yForm = DSolve@8#@@1DD, #@@2DD == 0 •. x -> 0, #@@2DD == 0 •. x -> 1<, #@@2DD, xD@@1DD •.
       C@i_D -> 0 & •ž Transpose@8eqv2, uSymbolv<D •• Flatten •• ExpandAll;
yForm •. displayRule
```

$$
\begin{aligned}
\{ & y_{1,1}[x] \to -\tfrac{1}{4} p^2 x b_1 + \tfrac{1}{4} p^2 x^2 b_1 - \tfrac{32}{3} b a A_1 \dot{A}_1 - \tfrac{16}{3} b p^2 x a A_1 \dot{A}_1 + \\
& \quad \tfrac{16}{3} b p^2 x^2 a A_1 \dot{A}_1 + \tfrac{32}{3} b a \mathrm{Cos}[2 p x] A_1 \dot{A}_1 - 16 b p^2 x a A_2 \dot{A}_2 + 16 b p^2 x^2 a A_2 \dot{A}_2 - \\
& \quad 4 b p^2 x a A_3 \dot{A}_3 + 4 b p^2 x^2 a A_3 \dot{A}_3 - 16 b p^2 x a A_4 \dot{A}_4 + 16 b p^2 x^2 a A_4 \dot{A}_4, \\
& y_{1,2}[x] \to \tfrac{b_2}{32} - \tfrac{1}{32} \mathrm{Cos}[4 p x] b_2 + \tfrac{1}{3} b a \dot{A}_1^2 - \tfrac{16}{9} b a \mathrm{Cos}[2 p x] \dot{A}_1^2 + \tfrac{13}{9} b a \mathrm{Cos}[4 p x] \dot{A}_1^2, \\
& y_{1,3}[x] \to \tfrac{1}{3} b a A_1^2 - \tfrac{16}{9} b a \mathrm{Cos}[2 p x] A_1^2 + \tfrac{13}{9} b a \mathrm{Cos}[4 p x] A_1^2 + \tfrac{b_3}{32} - \tfrac{1}{32} \mathrm{Cos}[4 p x] b_3, \\
& y_{1,4}[x] \to \tfrac{b_4}{32} - \tfrac{1}{32} \mathrm{Cos}[4 p x] b_4 + b a \dot{A}_2^2 - b a \mathrm{Cos}[4 p x] \dot{A}_2^2, \\
& y_{1,5}[x] \to b a A_2^2 - b a \mathrm{Cos}[4 p x] A_2^2 + \tfrac{b_5}{32} - \tfrac{1}{32} \mathrm{Cos}[4 p x] b_5, \\
& y_{1,6}[x] \to \tfrac{b_6}{32} - \tfrac{1}{32} \mathrm{Cos}[4 p x] b_6 + \tfrac{16 b a \mathrm{Sin}[2 p x] \dot{A}_1 \dot{A}_2}{3 \sqrt{3}}, \\
& y_{1,7}[x] \to -\tfrac{1}{4} p^2 x b_7 + \tfrac{1}{4} p^2 x^2 b_7 - \tfrac{16 b a \mathrm{Sin}[2 p x] A_1 \dot{A}_2}{\sqrt{3}}, \\
& y_{1,8}[x] \to -\tfrac{1}{4} p^2 x b_8 + \tfrac{1}{4} p^2 x^2 b_8 - \tfrac{16 b a \mathrm{Sin}[2 p x] A_2 \dot{A}_1}{\sqrt{3}}, \\
& y_{1,9}[x] \to \tfrac{16 b a \mathrm{Sin}[2 p x] A_1 A_2}{3 \sqrt{3}} + \tfrac{b_9}{32} - \tfrac{1}{32} \mathrm{Cos}[4 p x] b_9, \\
& y_{1,10}[x] \to \tfrac{b_{10}}{8} - \tfrac{1}{8} \mathrm{Cos}[2 p x] b_{10} + \tfrac{1}{3} b a \dot{A}_3^2 - \tfrac{1}{3} b a \mathrm{Cos}[2 p x] \dot{A}_3^2 + \tfrac{2}{3} b p x a \mathrm{Sin}[2 p x] \dot{A}_3^2, \\
& y_{1,11}[x] \to \tfrac{1}{3} b a A_3^2 - \tfrac{1}{3} b a \mathrm{Cos}[2 p x] A_3^2 + \tfrac{2}{3} b p x a \mathrm{Sin}[2 p x] A_3^2 + \tfrac{b_{11}}{8} - \tfrac{1}{8} \mathrm{Cos}[2 p x] b_{11}, \\
& y_{1,12}[x] \to \tfrac{b_{12}}{32} - \tfrac{1}{32} \mathrm{Cos}[4 p x] b_{12} + b a \dot{A}_4^2 - b a \mathrm{Cos}[4 p x] \dot{A}_4^2, \\
& y_{1,13}[x] \to b a A_4^2 - b a \mathrm{Cos}[4 p x] A_4^2 + \tfrac{b_{13}}{32} - \tfrac{1}{32} \mathrm{Cos}[4 p x] b_{13} \}
\end{aligned}
$$

Then, according to **yForm**, we pick up the spatial function forms for **uSymbolv** as

```
bait1 =
  8a_. x^n_. :> x^n •; FreeQ@a, xD, a_ f_@b_ xD :> f@b xD •; FreeQ@a, xD, a_ x f_@b_ xD :> x f@b xD<;

funcList = Table@
  Prepend@Cases@yForm@@i, 2DD, #D & •ž bait1 •• Flatten •• Union, 1D, 8i, Length@yFormD<D
```

```
{{1, x, x^2, Cos[2 p x]}, {1, Cos[2 p x], Cos[4 p x]}, {1, Cos[2 p x], Cos[4 p x]},
 {1, Cos[4 p x]}, {1, Cos[4 p x]}, {1, Cos[4 p x], Sin[2 p x]}, {1, x, x^2, Sin[2 p x]},
 {1, x, x^2, Sin[2 p x]}, {1, Cos[4 p x], Sin[2 p x]}, {1, Cos[2 p x], x Sin[2 p x]},
 {1, Cos[2 p x], x Sin[2 p x]}, {1, Cos[4 p x]}, {1, Cos[4 p x]}}
```

We assume the solution for $y_{1,i}$ as a linear combination of **funcList[[i]]**, substitute it into **yForm[[i]]**, equate both sides, solve for the undetermined coefficients, and obtain the solution for $y_{1,i}$ as

```
psi1Sol = Table@
        form = funcList@@kDD;
        cList = Table@ci, 8i, Length@formD<D;
        sol = cList.form; algExpr =
   sol - yForm@@k, 2DD •. bk_ -> Integrate@Expand@y¢@xD D@sol, xD •. yRuleD, 8x, 0, 1<D;
        algEq = Append@Coefficient@algExpr, Rest@formDD,
      algExpr •. Thread@Rest@formD -> 0DD == 0 •• Thread;
     y1,k -> Function@x, sol •. Solve@algEq, cListD@@1DD •• Expand •• EvaluateD,
    8k, Length@yFormD<D
```

```
Solve::svars : Equations may not give solutions for all "solve" variables.

Solve::svars : Equations may not give solutions for all "solve" variables.
```

```
:y1,1 ® FunctionBx, -32/3 b a A1@T1, T2D A1'@T1, T2D + (48 b p^2 x a A1@T1, T2D A1'@T1, T2D)/(3 + p^2) -
   (48 b p^2 x^2 a A1@T1, T2D A1'@T1, T2D)/(3 + p^2) + 32/3 b a Cos@2 p xD A1@T1, T2D A1'@T1, T2D -
   (48 b p^2 x a A2@T1, T2D A2'@T1, T2D)/(3 + p^2) + (48 b p^2 x^2 a A2@T1, T2D A2'@T1, T2D)/(3 + p^2) -
   (12 b p^2 x a A3@T1, T2D A3'@T1, T2D)/(3 + p^2) + (12 b p^2 x^2 a A3@T1, T2D A3'@T1, T2D)/(3 + p^2) -
   (48 b p^2 x a A4@T1, T2D A4'@T1, T2D)/(3 + p^2) + (48 b p^2 x^2 a A4@T1, T2D A4'@T1, T2D)/(3 + p^2)F, y1,2 ® FunctionAx,
  4/9 b a A1'@T1, T2D^2 - 16/9 b a Cos@2 p xD A1'@T1, T2D^2 + 4/3 b a Cos@4 p xD A1'@T1, T2D^2E, y1,3 ®
   FunctionAx, 4/9 b a A1@T1, T2D^2 - 16/9 b a Cos@2 p xD A1@T1, T2D^2 + 4/3 b a Cos@4 p xD A1@T1, T2D^2E,
 y1,4 ® FunctionAx, 4/3 b a A2'@T1, T2D^2 - 4/3 b a Cos@4 p xD A2'@T1, T2D^2E,
 y1,5 ® FunctionAx, 4/3 b a A2@T1, T2D^2 - 4/3 b a Cos@4 p xD A2@T1, T2D^2E,
 y1,6 ® FunctionBx, (16 b a Sin@2 p xD A1'@T1, T2D A2'@T1, T2D)/(3 è!!!3)F,
 y1,7 ® FunctionBx, -(16 b a Sin@2 p xD A1@T1, T2D A2'@T1, T2D)/(è!!!3)F,
 y1,8 ® FunctionBx, -(16 b a Sin@2 p xD A2@T1, T2D A1'@T1, T2D)/(è!!!3)F,
 y1,9 ® FunctionBx, (16 b a Sin@2 p xD A1@T1, T2D A2@T1, T2D)/(3 è!!!3)F,
 y1,10 ® FunctionAx, -c2 + Cos@2 p xD c2 + 2/3 b p x a Sin@2 p xD A3'@T1, T2D^2E,
 y1,11 ® FunctionAx, -c2 + Cos@2 p xD c2 + 2/3 b p x a Sin@2 p xD A3@T1, T2D^2E,
 y1,12 ® FunctionAx, 4/3 b a A4'@T1, T2D^2 - 4/3 b a Cos@4 p xD A4'@T1, T2D^2E,
 y1,13 ® FunctionAx, 4/3 b a A4@T1, T2D^2 - 4/3 b a Cos@4 p xD A4@T1, T2D^2E>
```

We note that all of the spatial dependent functions satisfy the boundary conditions.

The **sol2Formv1** consists of two parts: a part is related to the secular terms, which consists of terms proportional to $E^{\pm I\, w_1\, T_0}$ and $E^{\pm I\, w_2\, T_0}$ and the other part consists of terms that are not related to the secular terms. The positions of the former are

```
pos@1D =
 HPosition@# sol2Formv1 •. expRule1@1D •. Exp@_. + _ T0D -> 0, a_ •; a =!= 0, 1D •• Flatten ••
    RestL & •ž 9E^-I w1 T0, E^I w1 T0=

8811<, 810<<
```

```
pos@2D =
 HPosition@# sol2Formv1 •. expRule1@2D •. Exp@_. + _ T0D -> 0, a_ •; a =!= 0, 1D •• Flatten ••
     RestL & •ž 9E^(-I w2 T0), E^(I w2 T0)=
```

```
8811<, 810<<
```

Since **pos[1]** and **pos[2]** are the same, we simply define

```
8pos1, pos1cc< = pos@1D •• Flatten
```

```
811, 10<
```

Next, we augment $y_{1,pos1}$ and $h_{1,pos1}$ by a linear combination of all possible homogeneous solutions. The unknown coefficients can be determined by the orthogonality conditions. ($y_{1,pos1cc}$ and $h_{1,pos1cc}$ are simply the complex conjugates of $y_{1,pos1}$ and $h_{1,pos1}$, respectively.) First, we express their solution forms using **psi1Sol** and **eta1Sol** as

```
STFormv =
 8y1, pos1@xD + Sum@c1,j Fj@xD, 8j, 2<D, h1, pos1@xD + Sum@c1,j I wj Fj@xD, 8j, 2<D< •. eta1Sol •.
    psi1Sol •. ci_ -> 0 •. modeshapes •. params •• Expand
```

```
:$ (2/3) c1,1 - $ (2/3) Cos@2 p xD c1,1 + • 2 Sin@2 p xD c1,2 + (2/3) b p x a Sin@2 p xD A3@T1, T2D^2,

 2 I $ (2/3) p c1,1 - 2 I $ (2/3) p Cos@2 p xD c1,1 + 2 I • 2 p Sin@2 p xD c1,2 +

 (2/3) I b p a A3@T1, T2D^2 - (2/3) I b p a Cos@2 p xD A3@T1, T2D^2 + (4/3) I b p^2 x a Sin@2 p xD A3@T1, T2D^2>
```

Demanding that **STFormv** be orthogonal to the **adjointC[[i]]**, we obtain the unknown coefficients

```
c1Rule =
 Solve@int@adjointC@@#DD.STFormv, 8x, 0, 1<D == 0 •. modeshapes •. params •. intRule1 ••.
     intRule2 •. int -> Integrate, c1,#D & •ž 81, 2< •• Flatten
```

```
:c1,1 ® 0, c1,2 ® - (b p a A3@T1, T2D^2)/(3 • 2)>
```

Then, we adjust **psi1Sol** with

```
psi1Sol@@pos1DD = y1, pos1 -> Function@x, STFormv@@1DD •. c1Rule •• EvaluateD
```

```
y1,11 ® FunctionAx, - (1/3) b p a Sin@2 p xD A3@T1, T2D^2 + (2/3) b p x a Sin@2 p xD A3@T1, T2D^2E
```

```
psi1Sol@@pos1ccDD =
 y1, pos1cc -> Function@x, STFormv@@1DD •. c1Rule •. conjugateRule •• EvaluateD
```

```
y1,10 ® FunctionAx, - (1/3) b p a Sin@2 p xD Ā3@T1, T2D^2 + (2/3) b p x a Sin@2 p xD Ā3@T1, T2D^2E
```

Ÿ Horizontal Modes

The **sol2Formh** consists of two parts: a part is related to the secular terms, which consists of terms proportional to $E^{\pm I\,\omega_3\,T_0}$ and $E^{\pm I\,\omega_4\,T_0}$ and the other part consists of terms that are not related to the secular terms. The positions of the former are

```
pos@3D =
 HPosition@# sol2Formh •. expRule1@3D •. Exp@_. + _ T0D -> 0, a_ •; a =!= 0, 1D •• Flatten ••
     RestL & •ž 9E^(-I w3 T0), E^(I w3 T0)=
```

```
882<, 83<<
```

```
pos@4D =
 HPosition@# sol2Formh •. expRule1@4D •. Exp@_. + _ T0D -> 0, a_ •; a =!= 0, 1D •• Flatten ••
     RestL & •ž 9E^(-I w4 T0), E^(I w4 T0)=
```

```
88<, 8<<
```

Hence, we assume that the solution for $u_{2,2}$ and $v_{2,2}$ consists of two parts corresponding to the parts in **sol2Formh**. The part, which is not related to the secular terms, is

```
sol2Formha = Delete@sol2Formh, pos@3DD
```

```
8E^(-I T0 w1 - I T0 w3) Sin@p xD, E^(I T0 w1 + I T0 w3) Sin@p xD, E^(-I T0 w1 - I T0 w4) Sin@2 p xD,
 E^(I T0 w1 - I T0 w4) Sin@2 p xD, E^(-I T0 w1 + I T0 w4) Sin@2 p xD, E^(I T0 w1 + I T0 w4) Sin@2 p xD<
```

We define the corresponding undetermined coefficients as

```
uSymbolha = Table@y2,i, 8i, Length@sol2FormhaD<D;
vSymbolha = uSymbolha •. y -> h;
```

Hence, the general solution in terms of the **sol2Formha** can be written as

```
sol2ha = 8u2,2 -> Function@8x, T0, T1, T2<, uSymbolha.sol2Formha •• EvaluateD,
   v2,2 -> Function@8x, T0, T1, T2<, vSymbolha.sol2Formha •• EvaluateD<;
```

The part of **sol2Formh**, which is related to the secular terms, is

```
sol2Formhb = sol2Formh@@pos@3D@@1DDDD
```

```
8E^(I T0 w1 - I T0 w3) Sin@p xD<
```

and define the undetermined coefficients as

```
uSymbolhb = Table@ys2,i, 8i, Length@sol2FormhbD<D;
```

where the undetermined coefficients for $v_{2,2}$ are related to the **uSymbolhb** due to the orthogonality condition. Therefore, the general solution in terms of the **sol2Formhb** can be written as

```
sol2hb = 8u2,2 -> Function@8x, T0, T1, T2<, uSymbolhb.sol2Formhb •• EvaluateD,
   v2,2 -> Function@8x, T0, T1, T2<, -I w3 uSymbolhb.sol2Formhb •• EvaluateD<;
```

The total solution is a combination of **sol2ha**, **sol2hb**, and the complex conjugate of **sol2hb**.

Substituting **sol2ha** into **order2Eqm[[{3,4}]]** and equating the coefficients of **sol2Formha** on both sides, we have

```
algEqa =
  Flatten@Coefficient@Subtract žž # •. sol2ha, sol2FormhaD & •ž order2Eqm@@83, 4<DDD == 0 ••
   Thread;
```

Solving for the **uSymbolha** and the **vSymbolha** yields

```
symbolaRule = Solve@algEqa, uSymbolha~Join~vSymbolhaD@@1DD •. params
```

```
:y2,1 ® 2 b a A1'@T1, T2D A3'@T1, T2D / √3 , y2,2 ® 2 b a A1@T1, T2D A3@T1, T2D / √3 ,
 y2,3 ® 16 b a A1'@T1, T2D A4'@T1, T2D / (3 √3) , y2,4 ® - 16 b a A1@T1, T2D A4'@T1, T2D / √3 ,
 y2,5 ® - 16 b a A4@T1, T2D A1'@T1, T2D / √3 , y2,6 ® 16 b a A1@T1, T2D A4@T1, T2D / (3 √3) ,
 h2,1 ® -2 I √3 b p a A1'@T1, T2D A3'@T1, T2D, h2,2 ® 2 I √3 b p a A1@T1, T2D A3@T1, T2D,
 h2,3 ® - 64 I b p a A1'@T1, T2D A4'@T1, T2D / (3 √3) , h2,4 ® 0, h2,5 ® 0, h2,6 ® 64 I b p a A1@T1, T2D A4@T1, T2D / (3 √3) >
```

Substituting **sol2hb** into either of **order2Eqm[[{3,4}]]**, equating the coefficients of **sol2Formhb** on both sides, and solving for the **uSymbolhb**, we have

```
symbolbRule =
 Solve@Coefficient@Subtract žž order2Eqm@@3DD •. sol2hb, sol2FormhbD == 0 •• Thread,
    uSymbolhbD@@1DD •. params
```

```
:ys2,1 ® - 4 b a A1@T1, T2D A3'@T1, T2D / √3 >
```

Substituting **symbolaRule** and **symbolbRule** into **sol2ha** and **sol2hb** yields

```
sol2haForm = 8u2,2@x, T0, T1, T2D, v2,2@x, T0, T1, T2D< •. sol2ha •. symbolaRule;
sol2hbForm = 8u2,2@x, T0, T1, T2D, v2,2@x, T0, T1, T2D< •. sol2hb •. symbolbRule;
```

```
sol2h = {u2,2 -> Function[{x, T0, T1, T2},
    sol2haForm[[1]] + sol2hbForm[[1]] + (sol2hbForm[[1]] /. conjugateRule) // Expand //
     Evaluate], v2,2 -> Function[{x, T0, T1, T2}, sol2haForm[[2]] +
      sol2hbForm[[2]] + (sol2hbForm[[2]] /. conjugateRule) // Expand // Evaluate]}
```

$$\Big\{u_{2,2} \to \text{Function}\Big[\{x, T_0, T_1, T_2\}, \frac{2 \beta e^{I T_0 \omega_1 + I T_0 \omega_3} \alpha \sin[\pi x] A_1[T_1, T_2] A_3[T_1, T_2]}{\sqrt{3}} + \frac{16 \beta e^{I T_0 \omega_1 + I T_0 \omega_4} \alpha \sin[2 \pi x] A_1[T_1, T_2] A_4[T_1, T_2]}{3\sqrt{3}} - \frac{4 \beta e^{-I T_0 \omega_1 + I T_0 \omega_3} \alpha \sin[\pi x] A_3[T_1, T_2] \bar{A}_1[T_1, T_2]}{\sqrt{3}} - \frac{16 \beta e^{-I T_0 \omega_1 + I T_0 \omega_4} \alpha \sin[2 \pi x] A_4[T_1, T_2] \bar{A}_1[T_1, T_2]}{\sqrt{3}} - \frac{4 \beta e^{I T_0 \omega_1 - I T_0 \omega_3} \alpha \sin[\pi x] A_1[T_1, T_2] \bar{A}_3[T_1, T_2]}{\sqrt{3}} + \frac{2 \beta e^{-I T_0 \omega_1 - I T_0 \omega_3} \alpha \sin[\pi x] \bar{A}_1[T_1, T_2] \bar{A}_3[T_1, T_2]}{\sqrt{3}} - \frac{16 \beta e^{I T_0 \omega_1 - I T_0 \omega_4} \alpha \sin[2 \pi x] A_1[T_1, T_2] \bar{A}_4[T_1, T_2]}{\sqrt{3}} + \frac{16 \beta e^{-I T_0 \omega_1 - I T_0 \omega_4} \alpha \sin[2 \pi x] \bar{A}_1[T_1, T_2] \bar{A}_4[T_1, T_2]}{3\sqrt{3}}\Big],$$

$$v_{2,2} \to \text{Function}\Big[\{x, T_0, T_1, T_2\}, 2 I \sqrt{3}\, \beta e^{I T_0 \omega_1 + I T_0 \omega_3} \pi \alpha \sin[\pi x] A_1[T_1, T_2] A_3[T_1, T_2] + \frac{64 I \beta e^{I T_0 \omega_1 + I T_0 \omega_4} \pi \alpha \sin[2 \pi x] A_1[T_1, T_2] A_4[T_1, T_2]}{3\sqrt{3}} - \frac{4 I \beta e^{-I T_0 \omega_1 + I T_0 \omega_3} \alpha \sin[\pi x] \omega_3 A_3[T_1, T_2] \bar{A}_1[T_1, T_2]}{\sqrt{3}} + \frac{4 I \beta e^{I T_0 \omega_1 - I T_0 \omega_3} \alpha \sin[\pi x] \omega_3 A_1[T_1, T_2] \bar{A}_3[T_1, T_2]}{\sqrt{3}} - 2 I \sqrt{3}\, \beta e^{-I T_0 \omega_1 - I T_0 \omega_3} \pi \alpha \sin[\pi x] \bar{A}_1[T_1, T_2] \bar{A}_3[T_1, T_2] - \frac{64 I \beta e^{-I T_0 \omega_1 - I T_0 \omega_4} \pi \alpha \sin[2 \pi x] \bar{A}_1[T_1, T_2] \bar{A}_4[T_1, T_2]}{3\sqrt{3}}\Big]\Big\}$$

9.2.3 Solvability Conditions

Substituting **sol1**, **sol2v**, and **sol2h** into **eqEps[[3]]** yields

```
order3Eq =
   IlinearSys •. u_k_,1 -> uk,3M == IIlinearSys •. u_k_,1 -> uk,3M - H#@@1DD & •ž eqEps@@3DDL •.
                  sol1 •. sol2v •. eta1Sol •. psi1Sol •. sol2h •. intRule1 ••. intRule2 •.
            modeshapes •. yRule •. int -> Integrate •• ExpandM •• Thread; •• Timing
```

```
843.242 Second, Null<
```

Collecting the terms that may lead to secular terms, the terms proportional to $E^{I\, w_i\, T_0}$, from the equations governing the vertical modes, we have

```
ST2v = CoefficientA#@@2DD & •ž order3Eq@@81, 2<DD •. expRule1@#D, E^(I w# T0) E & •ž 81, 2<;
```

Collecting the terms that may lead to secular terms from the equations governing the horizontal modes, we have

```
ST2h = CoefficientA#@@2DD & •ž order3Eq@@83, 4<DD •. expRule1@#D, E^(I w# T0) E & •ž 83, 4<;
```

Demanding that **ST2v[[i]]** be orthogonal to the **adjointC[[i]]**, we obtain the solvability conditions

```
SCond2v =
 SolveAHint@adjointC@@#DD.ST2v@@#DD, 8x, 0, 1<D •. intRule1 ••. intRule2 •. modeshapes •.
      params •. int -> IntegrateL == 0,
    A#^H0,1L@T1, T2DE@@1DD & •ž 81, 2< •• Flatten •• ExpandAll;
SCond2v •. displayRule
```

$$
\Big\{D_2A_1 \to -\frac{I\, E^{I T_1 \sigma_1 + I \tau_1} \int_0^1 (P_1[x] - \mathrm{Cos}[2 \pi x] P_1[x])\, dx}{4 \sqrt{6}\, \pi} - A_1 \mu_1 + \frac{2}{3} I \pi^3 \alpha A_1^2 \bar{A}_1 -
$$

$$
\frac{224}{9} I \beta^2 \pi \alpha^2 A_1^2 \bar{A}_1 + \frac{256 I \beta^2 \pi \alpha^2 A_1^2 \bar{A}_1}{3+\pi^2} + \frac{64 I \beta^2 \pi^3 \alpha^2 A_1^2 \bar{A}_1}{9+3\pi^2} + \frac{2}{3} I E^{2 I T_1 \sigma_2} \pi^3 \alpha A_2^2 \bar{A}_1 -
$$

$$
\frac{352}{9} I \beta^2 E^{2 I T_1 \sigma_2} \pi \alpha^2 A_2^2 \bar{A}_1 + \frac{2}{3} I E^{2 I T_1 \sigma_4} \pi^3 \alpha A_4^2 \bar{A}_1 - \frac{352}{9} I \beta^2 E^{2 I T_1 \sigma_4} \pi \alpha^2 A_4^2 \bar{A}_1 +
$$

$$
\frac{4}{3} I \pi^3 \alpha A_1 A_2 \bar{A}_2 - \frac{256}{9} I \beta^2 \pi \alpha^2 A_1 A_2 \bar{A}_2 - \frac{256 I \beta^2 \pi \alpha^2 A_1 A_2 \bar{A}_2}{3+\pi^2} - \frac{64 I \beta^2 \pi^3 \alpha^2 A_1 A_2 \bar{A}_2}{9+3\pi^2} +
$$

$$
\frac{1}{3} I \pi^3 \alpha A_1 A_3 \bar{A}_3 - \frac{4}{3} I \beta^2 \pi \alpha^2 A_1 A_3 \bar{A}_3 - \frac{64 I \beta^2 \pi \alpha^2 A_1 A_3 \bar{A}_3}{3+\pi^2} - \frac{16 I \beta^2 \pi^3 \alpha^2 A_1 A_3 \bar{A}_3}{9+3\pi^2} +
$$

$$
\frac{4}{3} I \pi^3 \alpha A_1 A_4 \bar{A}_4 - \frac{256}{9} I \beta^2 \pi \alpha^2 A_1 A_4 \bar{A}_4 - \frac{256 I \beta^2 \pi \alpha^2 A_1 A_4 \bar{A}_4}{3+\pi^2} - \frac{64 I \beta^2 \pi^3 \alpha^2 A_1 A_4 \bar{A}_4}{9+3\pi^2},
$$

$$
D_2A_2 \to -\frac{I\, E^{I T_1 \sigma_1 - I T_1 \sigma_2 + I \tau_1} \int_0^1 \mathrm{Sin}[2 \pi x] P_1[x]\, dx}{4 \sqrt{2}\, \pi} - A_2 \mu_1 + \frac{4}{3} I \pi^3 \alpha A_1 A_2 \bar{A}_1 -
$$

$$
\frac{1024}{9} I \beta^2 \pi \alpha^2 A_1 A_2 \bar{A}_1 + \frac{64 I \beta^2 \pi^3 \alpha^2 A_1 A_2 \bar{A}_1}{3+\pi^2} + \frac{2}{3} I E^{-2 I T_1 \sigma_2} \pi^3 \alpha A_1^2 \bar{A}_2 -
$$

$$
\frac{352}{9} I \beta^2 E^{-2 I T_1 \sigma_2} \pi \alpha^2 A_1^2 \bar{A}_2 + 6 I \pi^3 \alpha A_2^2 \bar{A}_2 + \frac{32}{3} I \beta^2 \pi \alpha^2 A_2^2 \bar{A}_2 - \frac{64 I \beta^2 \pi^3 \alpha^2 A_2^2 \bar{A}_2}{3+\pi^2} +
$$

$$
2 I E^{-2 I T_1 \sigma_2 + 2 I T_1 \sigma_4} \pi^3 \alpha A_4^2 \bar{A}_2 + \frac{32}{3} I \beta^2 E^{-2 I T_1 \sigma_2 + 2 I T_1 \sigma_4} \pi \alpha^2 A_4^2 \bar{A}_2 +
$$

$$
I \pi^3 \alpha A_2 A_3 \bar{A}_3 - \frac{16 I \beta^2 \pi^3 \alpha^2 A_2 A_3 \bar{A}_3}{3+\pi^2} + 4 I \pi^3 \alpha A_2 A_4 \bar{A}_4 - \frac{64 I \beta^2 \pi^3 \alpha^2 A_2 A_4 \bar{A}_4}{3+\pi^2}\Big\}
$$

Demanding that **ST2h[[i]]** be orthogonal to the **adjointC[[i+2]]**, we obtain the solvability conditions

```
SCond2h =
 SolveAHint@adjointC@@# + 2DD.ST2h@@#DD, 8x, 0, 1<D •. intRule1 ••. intRule2 •. modeshapes •.
        params •. int -> IntegrateL == 0,
     A_{#+2}^{H0,1L}@T1, T2DE@@1DD & •ž 81, 2< •• Flatten •• ExpandAll;
SCond2h •. displayRule
```

$$\{D_2A_3 \to -A_3\, m_2 + \frac{2}{3}\, I\, p^3\, a\, A_1\, A_3\, \dot{A}_1 - \frac{136}{3}\, I\, b^2\, p\, a^2\, A_1\, A_3\, \dot{A}_1 + \frac{32\, I\, b^2\, p^3\, a^2\, A_1\, A_3\, \dot{A}_1}{3+p^2} + 2\, I\, p^3\, a\, A_2\, A_3\, \dot{A}_2 - \frac{32\, I\, b^2\, p^3\, a^2\, A_2\, A_3\, \dot{A}_2}{3+p^2} + \frac{3}{4}\, I\, p^3\, a\, A_3^2\, \dot{A}_3 - \frac{4}{3}\, I\, b^2\, p\, a^2\, A_3^2\, \dot{A}_3 - \frac{8\, I\, b^2\, p^3\, a^2\, A_3^2\, \dot{A}_3}{3+p^2} + 2\, I\, p^3\, a\, A_3\, A_4\, \dot{A}_4 - \frac{32\, I\, b^2\, p^3\, a^2\, A_3\, A_4\, \dot{A}_4}{3+p^2},$$

$$D_2A_4 \to -\frac{I\, E^{I T_1 s_1 - I T_1 s_4 + I t_2} \int_0^1 \mathrm{Sin}[2\, p\, x]\, P_2[x]\, dx}{4\sqrt{2}\, p} - A_4\, m_2 + \frac{4}{3}\, I\, p^3\, a\, A_1\, A_4\, \dot{A}_1 - \frac{1024}{9}\, I\, b^2\, p\, a^2\, A_1\, A_4\, \dot{A}_1 + \frac{64\, I\, b^2\, p^3\, a^2\, A_1\, A_4\, \dot{A}_1}{3+p^2} + 4\, I\, p^3\, a\, A_2\, A_4\, \dot{A}_2 - \frac{64\, I\, b^2\, p^3\, a^2\, A_2\, A_4\, \dot{A}_2}{3+p^2} + I\, p^3\, a\, A_3\, A_4\, \dot{A}_3 - \frac{16\, I\, b^2\, p^3\, a^2\, A_3\, A_4\, \dot{A}_3}{3+p^2} + \frac{2}{3}\, I\, E^{-2 I T_1 s_4}\, p^3\, a\, A_1^2\, \dot{A}_4 - \frac{352}{9}\, I\, b^2\, E^{-2 I T_1 s_4}\, p\, a^2\, A_1^2\, \dot{A}_4 + 2\, I\, E^{2 I T_1 s_2 - 2 I T_1 s_4}\, p^3\, a\, A_2^2\, \dot{A}_4 + \frac{32}{3}\, I\, b^2\, E^{2 I T_1 s_2 - 2 I T_1 s_4}\, p\, a^2\, A_2^2\, \dot{A}_4 + 6\, I\, p^3\, a\, A_4^2\, \dot{A}_4 + \frac{32}{3}\, I\, b^2\, p\, a^2\, A_4^2\, \dot{A}_4 - \frac{64\, I\, b^2\, p^3\, a^2\, A_4^2\, \dot{A}_4}{3+p^2}\}$$

We can rewrite **SCond2v** and **SCond2h** in compact form by collecting the coefficients of the same form as follows:

```
collectForm = TableA9Ai@T1, T2D E^(I wi T0), Ȧi@T1, T2D E^(-I wi T0)=, 8i, 4<E •• Flatten;

cubicTerms = Nest@Outer@Times, collectForm, #D &, collectForm, 2D •• Flatten •• Union;

cubicST@i_D := E^(-I wi T0) cubicTerms •. expRule1@iD •. Exp@_ T0 + _.D -> 0 •• Union •• Rest
```

```
moduEq = MapIndexed@Hterm = cubicST@#2@@1DDD; coef = Coefficient@#1@@2DD, termD;
    #1@@1DD == H#1@@2DD •. Thread@term -> 0DL + coef.termL &, Join@SCond2v, SCond2hDD;
moduEq •. displayRule
```

$$\Big\{D_2A_1 == -\frac{i\, e^{i T_1 \sigma_1 + i \tau_1} \int_0^1 (P_1[x] - \mathrm{Cos}[2\pi x] P_1[x])\, dx}{4\sqrt{6}\,\pi} - A_1 \mu_1 + \left(\frac{2}{3} i \pi^3 \alpha - \frac{224}{3} i \beta^2 \pi \alpha^2 + \frac{256 i \beta^2 \pi \alpha^2}{3+\pi^2} + \frac{64 i \beta^2 \pi^3 \alpha^2}{9+3\pi^2}\right) A_1^2 \dot{A}_1 +$$
$$e^{2 i T_1 \sigma_2}\left(\frac{2}{3} i \pi^3 \alpha - \frac{352}{9} i \beta^2 \pi \alpha^2\right) A_2^2 \dot{A}_1 + e^{2 i T_1 \sigma_4}\left(\frac{2}{3} i \pi^3 \alpha - \frac{352}{9} i \beta^2 \pi \alpha^2\right) A_4^2 \dot{A}_1 +$$
$$\left(\frac{4}{3} i \pi^3 \alpha - \frac{256}{9} i \beta^2 \pi \alpha^2 - \frac{256 i \beta^2 \pi \alpha^2}{3+\pi^2} - \frac{64 i \beta^2 \pi^3 \alpha^2}{9+3\pi^2}\right) A_1 A_2 \dot{A}_2 +$$
$$\left(\frac{1}{3} i \pi^3 \alpha - \frac{4}{3} i \beta^2 \pi \alpha^2 - \frac{64 i \beta^2 \pi \alpha^2}{3+\pi^2} - \frac{16 i \beta^2 \pi^3 \alpha^2}{9+3\pi^2}\right) A_1 A_3 \dot{A}_3 +$$
$$\left(\frac{4}{3} i \pi^3 \alpha - \frac{256}{9} i \beta^2 \pi \alpha^2 - \frac{256 i \beta^2 \pi \alpha^2}{3+\pi^2} - \frac{64 i \beta^2 \pi^3 \alpha^2}{9+3\pi^2}\right) A_1 A_4 \dot{A}_4,$$

$$D_2A_2 == -\frac{i\, e^{i T_1 \sigma_1 - i T_1 \sigma_2 + i \tau_1} \int_0^1 \mathrm{Sin}[2\pi x] P_1[x]\, dx}{4\sqrt{2}\,\pi} - A_2 \mu_1 +$$
$$\left(\frac{4}{3} i \pi^3 \alpha - \frac{1024}{9} i \beta^2 \pi \alpha^2 + \frac{64 i \beta^2 \pi^3 \alpha^2}{3+\pi^2}\right) A_1 A_2 \dot{A}_1 + e^{-2 i T_1 \sigma_2}\left(\frac{2}{3} i \pi^3 \alpha - \frac{352}{9} i \beta^2 \pi \alpha^2\right) A_1^2 \dot{A}_2 +$$
$$\left(6 i \pi^3 \alpha + \frac{32}{3} i \beta^2 \pi \alpha^2 - \frac{64 i \beta^2 \pi^3 \alpha^2}{3+\pi^2}\right) A_2^2 \dot{A}_2 + e^{-2 i T_1 \sigma_2 + 2 i T_1 \sigma_4}\left(2 i \pi^3 \alpha + \frac{32}{3} i \beta^2 \pi \alpha^2\right) A_4^2 \dot{A}_2 +$$
$$\left(i \pi^3 \alpha - \frac{16 i \beta^2 \pi^3 \alpha^2}{3+\pi^2}\right) A_2 A_3 \dot{A}_3 + \left(4 i \pi^3 \alpha - \frac{64 i \beta^2 \pi^3 \alpha^2}{3+\pi^2}\right) A_2 A_4 \dot{A}_4,$$

$$D_2A_3 == -A_3 \mu_2 + \left(\frac{2}{3} i \pi^3 \alpha - \frac{136}{3} i \beta^2 \pi \alpha^2 + \frac{32 i \beta^2 \pi^3 \alpha^2}{3+\pi^2}\right) A_1 A_3 \dot{A}_1 + \left(2 i \pi^3 \alpha - \frac{32 i \beta^2 \pi^3 \alpha^2}{3+\pi^2}\right) A_2 A_3 \dot{A}_2 +$$
$$\left(\frac{3}{4} i \pi^3 \alpha - \frac{4}{3} i \beta^2 \pi \alpha^2 - \frac{8 i \beta^2 \pi^3 \alpha^2}{3+\pi^2}\right) A_3^2 \dot{A}_3 + \left(2 i \pi^3 \alpha - \frac{32 i \beta^2 \pi^3 \alpha^2}{3+\pi^2}\right) A_3 A_4 \dot{A}_4,$$

$$D_2A_4 == -\frac{i\, e^{i T_1 \sigma_1 - i T_1 \sigma_4 + i \tau_2} \int_0^1 \mathrm{Sin}[2\pi x] P_2[x]\, dx}{4\sqrt{2}\,\pi} - A_4 \mu_2 +$$
$$\left(\frac{4}{3} i \pi^3 \alpha - \frac{1024}{9} i \beta^2 \pi \alpha^2 + \frac{64 i \beta^2 \pi^3 \alpha^2}{3+\pi^2}\right) A_1 A_4 \dot{A}_1 + \left(4 i \pi^3 \alpha - \frac{64 i \beta^2 \pi^3 \alpha^2}{3+\pi^2}\right) A_2 A_4 \dot{A}_2 +$$
$$\left(i \pi^3 \alpha - \frac{16 i \beta^2 \pi^3 \alpha^2}{3+\pi^2}\right) A_3 A_4 \dot{A}_3 + e^{-2 i T_1 \sigma_4}\left(\frac{2}{3} i \pi^3 \alpha - \frac{352}{9} i \beta^2 \pi \alpha^2\right) A_1^2 \dot{A}_4 +$$
$$e^{2 i T_1 \sigma_2 - 2 i T_1 \sigma_4}\left(2 i \pi^3 \alpha + \frac{32}{3} i \beta^2 \pi \alpha^2\right) A_2^2 \dot{A}_4 + \left(6 i \pi^3 \alpha + \frac{32}{3} i \beta^2 \pi \alpha^2 - \frac{64 i \beta^2 \pi^3 \alpha^2}{3+\pi^2}\right) A_4^2 \dot{A}_4\Big\}$$

These solvability conditions satisfy the symmetry properties because

```
formList = TableAAj@T1, T2D TableAAi@T1, T2D Ȧi@T1, T2D, 8i, 4<E, 8j, 4<E;
formList •. displayRule
```

$$\{\{A_1^2 \dot{A}_1, A_1 A_2 \dot{A}_2, A_1 A_3 \dot{A}_3, A_1 A_4 \dot{A}_4\}, \{A_1 A_2 \dot{A}_1, A_2^2 \dot{A}_2, A_2 A_3 \dot{A}_3, A_2 A_4 \dot{A}_4\},$$
$$\{A_1 A_3 \dot{A}_1, A_2 A_3 \dot{A}_2, A_3^2 \dot{A}_3, A_3 A_4 \dot{A}_4\}, \{A_1 A_4 \dot{A}_1, A_2 A_4 \dot{A}_2, A_3 A_4 \dot{A}_3, A_4^2 \dot{A}_4\}\}$$

```
notationList = Table@s_{i,j}, 8i, 4<, 8j, 4<D
```

88$s_{1,1}$, $s_{1,2}$, $s_{1,3}$, $s_{1,4}$<, 8$s_{2,1}$, $s_{2,2}$, $s_{2,3}$, $s_{2,4}$<, 8$s_{3,1}$, $s_{3,2}$, $s_{3,3}$, $s_{3,4}$<, 8$s_{4,1}$, $s_{4,2}$, $s_{4,3}$, $s_{4,4}$<<

```
coefRule = MapThread@Rule, 8notationList,
    Coefficient@H2 w_# IL moduEq@@#, 2DD, formList@@#DDD & •ž Range@4D<, 2D •• Flatten;
```

```
testList = Table@s_{i,j} - s_{j,i}, 8i, 4<, 8j, i, 4<D •• Flatten •• Union •• Rest
```

8$s_{1,2} - s_{2,1}$, $s_{1,3} - s_{3,1}$, $s_{2,3} - s_{3,2}$, $s_{1,4} - s_{4,1}$, $s_{2,4} - s_{4,2}$, $s_{3,4} - s_{4,3}$<

```
testList •. coefRule •. params •• Factor
```

80, 0, 0, 0, 0, 0<

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