Transseries in Difference and Differential Equations

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Chapter 1

Introduction

The problems studied in this thesis concern nonlinear analytic difference and differential equations in a neighbourhood of ∞ in the complex plane $\mathbb{C}$. In fact, we will study difference and differential systems of the form $y(x+1) = f(x, y(x))$ and $y'(x) + f(x, y(x)) = 0$ respectively, where $y = y(x) \in \mathbb{C}^n$. The nonlinear function $f(x, y) = f(x, y_1, y_2, \ldots, y_n)$ is assumed to have the form $f(x, y) = \Lambda(x)y + g(x, y)$, where Λ is an $n \times n$-matrix that is holomorphic for x in a neighbourhood of ∞ and where g is an n-vector valued function that is holomorphic for x in a neighbourhood of ∞ and y in a neighbourhood of 0 such that $g(x, y) = O(x^{-2}) + O(|y|^2)$ as $x \to \infty$ and $|y| \to 0$.

To fix the ideas, let us assume that $n = 1$ and that we are dealing with the equation

$$y(x+1) = e^{-\mu}(1 + x^{-1})^a y(x) + g(x, y(x)), \tag{1.0.1}$$

where $\mu \not\equiv 0 \bmod 2\pi i$ (i.e. $e^{\mu} \neq 1$) and $a \in \mathbb{C}$. Let $\mathcal{U}$ denote the algebra of formal expressions $\sum_{j=0}^{\infty} f_j(x)\, e^{-\sigma_j x}$ satisfying the following two properties.

(i) Every f_j can be written as $f_j = p_j g_j$ with $g_j \in \mathbb{C}((x^{-1}))[\{x^c\}_{c \in \mathbb{C}}]$ and p_j a 1-periodic $\mathbb{C}$-valued function of x;

(ii) Every σ_j belongs to $\mathbb{N} \cdot \mu$.

We note that the second condition implies that $\mathcal{U}$ indeed is an algebra. Moreover, if τ denotes the shift operator defined by $(\tau f)(x) := f(x+1)$, then τ acts naturally on $\mathcal{U}$ by

$$\tau\big(\textstyle\sum_{j=0}^{\infty} f_j(x) e^{-\sigma_j x}\big) = \sum_{j=0}^{\infty} \big(\tau(f_j)(x)\, e^{-\sigma_j}\big) e^{-\sigma_j x} \quad \text{and} \quad \tau(f_j) = p_j\, \tau(g_j).$$

Here one should observe that if g_j equals $g_j(x) = a(x)x^c$, with $a \in \mathbb{C}((x^{-1}))$ and $c \in \mathbb{C}$, then $(\tau g_j)(x) = a(x+1)(1+x^{-1})^c x^c$ and $a(x+1)(1+x^{-1})^c$ belongs to $\mathbb{C}((x^{-1}))$. With the algebra $\mathcal{U}$ we associate the space

$$\mathrm{FSol}(\Delta) := \{y \in \mathcal{U} \mid y \text{ is a solution of (1.0.1)}\}.$$

Here Δ denotes the equation (1.0.1). The special form of (1.0.1) implies that there exists a unique formal solution $\hat{y}_0 \in x^{-1}\mathbb{C}[[x^{-1}]]$ of this equation and obviously this solution belongs to $\mathrm{FSol}(\Delta)$.

With the nonlinear equation (1.0.1) we associate the linear difference equation (in this thesis also referred to as the *normal form* corresponding to (1.0.1)) defined by

$$z(x+1) = e^{-\mu}(1+x^{-1})^a z(x). \tag{1.0.2}$$

We denote the space of its formal solutions in $\mathcal{U}$ by $\mathrm{FSol}(\Delta_{norm})$ and we will show that the general solution of (1.0.2) in $\mathcal{U}$ equals $z(x) = c(x)e^{-\mu x}x^a$, with c an arbitrary 1-periodic scalar function. Indeed, substituting $z(x) = c(x)e^{-\mu x}x^a$ in the normal form we obtain $c(x+1) = c(x)$. Hence,

$$\mathrm{FSol}(\Delta_{norm}) = \{c(x)e^{-\mu x}x^a \mid \text{for all } c \text{ with } c(x+1) = c(x)\}.$$

The general solution in the class of holomorphic functions on the Riemann surface of the logarithm is $z(x) = c(x)e^{-\mu x}x^a$, with c an arbitrary 1-periodic scalar holomorphic function.

The aim is to construct a (at first formal) transformation $y = \hat{T}(x,z)$ that formally transforms the nonlinear difference equation into its normal form and we require the transformation to be of the form

$$y = \hat{T}(x,z) := \sum_{k=0}^{\infty} \hat{y}_k(x) z^k,$$

where $\hat{y}_0$ is the unique formal power series solution of the nonlinear equation (1.0.1) found above and where $\hat{y}_k$, $k \geqslant 1$, are certain unknown expressions that we require to be elements of $\mathbb{C}[[x^{-1}]]$. It turns out that the $\hat{y}_k$, $k \geqslant 1$, have to satisfy linear difference equations of the form $y(x+1) = a_k(x)y(x)+t_k(x)$, where a_k is a convergent power series in the 'variables' x^{-1} and $\hat{y}_0$ and where t_k is some polynomial expression in $\hat{y}_{k'}$ with $0 \leqslant k' < k$. These equations can (at least formally) be solved, and thus there does exist a formal transformation $\hat{T}$. In particular we obtain a map $\hat{T} : \mathrm{FSol}(\Delta_{norm}) \to \mathrm{FSol}(\Delta)$, transforming formal solutions of the normal form into formal solutions of the original difference equation. Now, when we substitute a general element of $\mathrm{FSol}(\Delta_{norm})$ into the formal transformation one obtains the following *formal integral* of the difference equation

$$\hat{y}(x) = \sum_{k=0}^{\infty} c^k(x) e^{-k\mu x} x^{ka} \hat{y}_k(x) \in \mathrm{FSol}(\Delta) \tag{1.0.3}$$

(cf. [Eca85, CNP93]). The right-hand side of (1.0.3) belongs to the class of so-called *transseries* (cf. [Eca92]).

The next problem is to associate a holomorphic transformation with the given formal transformation $\hat{T}$ on a suitable sector S_1, in such a way that this holomorphic transformation has $\hat{T}$ as asymptotic expansion in a certain sense. For this we first need to associate holomorphic functions y_k with the constructed formal expressions $\hat{y}_k$ on the same sector S_1, having $\hat{y}_k$ as asymptotic expansion on S_1. If we succeed in this, then the series $\sum_{k=0}^{\infty} y_k(x)z^k$ turns out to be a convergent power series in z for every value of x in a neighbourhood U of ∞ in S_1. In other words: $T(x,z) := \sum_{k=0}^{\infty} y_k(x)z^k$ is a holomorphic transformation on $U \times \Delta(0,\rho)$ for some $\rho > 0$.

The formal power series $\hat{y}_k$, $k \in \mathbb{N}$, that occur here are Borel summable. This Borel summation method can be explained as follows. First one applies the formal Borel transform $\hat{\mathcal{B}}$ to the formal series $\hat{y}_k$. The resulting formal power series $\hat{\mathcal{B}}\hat{y}_k$ converges in a neighbourhood of 0 and has an analytic continuation along every half line $[0, \infty e^{i\theta})$, except for countably many. The directions of those exceptional half lines are called singular directions and are due to one or more singularities of $\hat{\mathcal{B}}\hat{y}_k$ on this half line. If θ is not a singular direction and $\theta \neq \pm\pi/2$, then this analytic continuation is such that the Laplace transform of $\hat{\mathcal{B}}\hat{y}_k$ with integration along the half line with direction θ exists. This Laplace transform produces a unique 'asymptotic lift' y_k of $\hat{y}_k$ on a sector with opening larger than π and bisecting direction $-\theta$. This asymptotic lift is called the *Borel sum* of $\hat{y}_k$ in the direction $-\theta$ (in for example [CNP93] this is called the Borel sum in direction θ).

In our case the singular directions of the formal solution $\hat{y}_0$ are given by $\arg(\mu + 2l\pi i)$, $l \in \mathbb{Z}$, and using the equation for $\hat{y}_k$ one easily infers that the singular directions of $\hat{y}_k$ are given by $\arg((1-j)\mu + 2l\pi i)$ for every $j \in \{0, 2, 3, \ldots, k\}$ and $l \in \mathbb{Z}$. In the case θ_- and θ_+ are two consecutive singular directions in the right half plane of the set of all $\hat{y}_k$, the Borel summation method gives Borel sums y_k of $\hat{y}_k$ that all are holomorphic in a neighbourhood of ∞ in the sector $S_1 := \{x \in \mathbb{C}^* \mid -\pi/2 - \theta_+ < \arg x < \pi/2 - \theta_-\}$. Note that S_1 contains the positive real axis. From this it can be shown that $\hat{T}$ can be lifted to a unique holomorphic transformation of the type described above.

However, this does not automatically ensure convergence of the expression obtained by replacing $\hat{y}_k$ by their Borel sums y_k in the right-hand side of (1.0.3). In the following we will look at the convergence of this expression on certain sectors S_1 containing the positive real axis. As already mentioned, the *transformation* T converges provided that $|z|$ is small enough. Hence, if $c \not\equiv 0$, the corresponding transseries converges provided that $z(x) = c(x)e^{-\mu x}x^a$ tends to 0 as $x \to \infty$ in S_1, which implies $\Re\mu > 0$. It turns out that given a solution y of (1.0.1) such that y is 'small' on some sub-sector S_2 of S_1 containing the positive real axis, then there exists a unique convergent transseries such that y equals the sum of this transseries on S_2.

In the more general case where we study a system of difference equations

$$y(x+1) = \Lambda(x)y(x) + g(x, y(x)), \tag{1.0.4}$$

with Λ an $n \times n$-matrix ($n > 1$) of the form $\Lambda(x) = A_0(1 + x^{-1})^{A_1}$ and A_0 in diagonal form, the algebra $\mathcal{U}$ has to be replaced by formal expressions $\sum_{j=0}^{\infty} f_j(x)\, e^{-\sigma_j x}$ satisfying the following two properties.

(i) Every f_j can be written as $f_j = p_j g_j$ with $g_j \in \mathbb{C}^n((x^{-1}))[\{x^c\}_{c\in\mathbb{C}}, \log x]$ and p_j a 1-periodic $\mathbb{C}$-valued function of x;

(ii) There exist nonzero complex numbers $\mu_1, \mu_2, \ldots, \mu_n$, such that every σ_j belongs to $\mathbb{N} \cdot \mu_1 + \mathbb{N} \cdot \mu_2 + \cdots + \mathbb{N} \cdot \mu_n$.

With the nonlinear system of difference equations (1.0.4) one associates the *normal form*

$$z(x+1) = \Lambda(x)z(x). \tag{1.0.5}$$

It may be shown that (1.0.5) is uniquely solvable in $\mathcal{U}^n$, and this formal solution can actually be replaced by a holomorphic solution by replacing the 1-periodic functions by holomorphic 1-periodic functions.

Like in the 1-dimensional case it is possible to construct a formal transformation of the form $y = \hat{T}(x, z) = \sum_{\mathbf{k}\in\mathbb{N}^n} \hat{y}_{\mathbf{k}}(x) z^{\mathbf{k}}$ that formally transforms (1.0.4) into the normal form (1.0.5). However, the convergence of the corresponding transseries depends on the position of the eigenvalues of $\Lambda(\infty) = A_0$. Therefore it is assumed that the eigenvalues of A_0, which for convenience will be denoted by $e^{-\mu_j}$, $j = 1, 2, \ldots, n$, are ordered in such a way that $\Re\mu_j > 0$ for $j = 1, 2, \ldots, p$ and $\Re\mu_j \leqslant 0$ for $j \in \{p+1, p+2, \ldots, n\}$. In the case that both A_0 and A_1 are in diagonal form, it has been shown by Braaksma in [Bra01] that only the 'partial' formal transformation

$$\hat{T}_1(x, u_1, \ldots, u_p) := \hat{T}(x, u_1, \ldots, u_p, 0, \ldots, 0)$$

can be lifted to a holomorphic expression T_1: the formal series $\hat{T}_1(x, u)$ turns out to be Borel summable with respect to x in some sector S_1 containing the positive real axis, uniformly in u (provided that $|u|$ is small enough). Consequently, the Borel sum $T_1(x, u)$ of $\hat{T}_1(x, u)$ exists for x in a neighbourhood of ∞ in S_1 and u in a neighbourhood of 0 and the original difference equation, restricted to the manifold defined by $y = T_1(x, u)$, transforms into the so-called *semi-canonical form* $u(x+1) = \Lambda(x)u(x)$. However, the main result in [Bra01] is that given a solution y of (1.0.4) and given a sector $S_2 \subset S_1$ containing the positive real axis such that y is 'small' on S_2, there exists a unique convergent transseries such that y equals the sum of this transseries.

Similar results have been obtained by Costin in [Cos95, Cos98] for the analogous nonlinear rank 1 differential equation

$$y'(x) + \Lambda(x)y(x) + g(x, y(x)) = 0, \tag{1.0.6}$$

under the assumption that $\Lambda(x) = A_0 - x^{-1}A_1$, with both A_0 and A_1 diagonal matrices. Such systems have been studied also by Malmquist ([Malm40, Malm41]) and Iwano ([Iwa57, Iwa59]), compare also [Was87], chapter IX. Costin derived convergent transseries representations for 'small' solutions on sectors, resurgence relations and balanced averages. An important notion in his work is that of staircase distributions. In [CC01] Costin and Costin showed that at the boundary of a maximal sector where a solution of the system considered in [Cos98] is 'small', singularities of this solution occur that are situated in nearly periodic arrays.

One part of this thesis (chapters 4 and 5) is concerned with generalisations of the assumptions on Λ for both differential and difference equations: we derive similar results concerning the analytic reduction to a semi-canonical form for these generalised equations. In chapter 2 we extend Braaksma's study of the difference equation (1.0.4), while in chapter 3 we will extend the results in [CC01] to systems of difference equations.

Very general results concerning nonlinear meromorphic differential and difference equations of the type considered in this thesis have been formulated by Écalle (cf. [Eca85, Eca92]). His treatment includes the study of formal integrals, resurgence relations, alien derivations, the associated bridge equation, holomorphic invariants, accelero-summation and medianisation. An instructional treatment of Écalle's work for scalar differential equations has been given by Candelpergher, Nosmas and Pham ([CNP93]). They consider the construction of the formal integral and its resurgence properties, alien derivations, the bridge equation, Stokes transition and analytic classification.

A more detailed outline of the thesis is given in the next section. In section 1.2 we introduce some notations and terminology, while in section 1.3 and section 1.4 the notion of Borel summability and multisummability will be discussed. In section 1.5 we study in detail two examples that might clarify some of the above.

1.1 General Outline of the Thesis

In chapter 2 we consider difference equations of only one level, namely *level one*. This means that the matrix $\Lambda(x) = A_0(1+x^{-1})^{A_1}$ has the form $\Lambda(x) = \oplus_{j=1}^{r} e^{-\mu_j}(1+x^{-1})^{\mathtt{M}_j}$, with $\mu_j \not\equiv 0 \bmod 2\pi i$, and we assume that each $\mathtt{M}_j$ is a diagonal matrix with complex numbers on the diagonal. In fact, this chapter contains results for difference equations analogous to the results Costin gave in [Cos95, Cos98] for rank 1 differential equations. As the existence and unicity of the transformation $\hat{T}$ and the asymptotic lift to T_1 both are discussed by Braaksma in [Bra01], these results will only be summarised. The main focus of this chapter will be on the behaviour of the Borel transform of $\hat{y}_{\mathbf{k}}$ near and on singular rays. In this study we use the theory of Costin's staircase distributions ([Cos98]). The convolution equations on singular rays, solutions on singular rays, resurgence relations, Stokes transition and balanced averages will be discussed.

Chapter 3 contains results about difference equations analogous to the results Costin and Costin obtained in [CC01] for differential equations. We consider a special type of the class of equations that we studied in chapter 2, namely those with Λ of the form $\Lambda(x) = \operatorname{diag}\{e^{-\mu_1}(1+x^{-1})^{a_1}, \ldots, e^{-\mu_n}(1+x^{-1})^{a_n}\}$. The actual transseries solution will be studied in more detail. A first result is that the solution y can be extended to a larger region than the one obtained in chapter 2. Loosely speaking this region cannot be enlarged in the sense that singularities occur near the border of this extended region. It turns out that the singular points are grouped together in nearly periodic arrays.

In chapter 4 we will generalise the results in [Bra01] concerning analytic reduction to a semi-canonical form to that case where we admit nilpotent matrices in the linear part of the equation we take into consideration. In this chapter we will study both differential and difference equations. In the differential case we take $\Lambda(x) = \oplus_{j=1}^{r}(\mu_j \mathtt{I}_{n_j} - x^{-1}\mathtt{M}_j)$, while in the case of difference equations we take $\Lambda(x) = \oplus_{j=1}^{r} e^{-\mu_j}(1+x^{-1})^{\mathtt{M}_j}$. For each $j \in \{1, 2, \ldots, r\}$ we only assume that two eigenvalues of $\mathtt{M}_j$ do not differ by a nonzero integer, which makes it possible to compute formal solutions of the equations for $\hat{y}_{\mathbf{k}}$. Apart from this assumption this is the most general form for the matrices $\mathtt{M}_j$.

The last chapter deals with differential equations $y'(x) + \Lambda(x)y(x) + g(x, y(x)) = 0$ in the case that Λ has a pole at $x = \infty$. Formal solutions of such equations are not Borel summable, but in [Bra92, RS94, Bal94] it is shown that such formal solutions are multisummable. We first consider differential equations as above with three levels. In fact, we will first restrict ourselves to equations with the coefficient Λ of the form

$$\begin{aligned}\Lambda(x) \;=\; & \operatorname{diag}\{\omega_m x^{q_1-1} + \lambda_m x^{q_2-1} + \mu_m x^{q_3-1} - a_m x^{-1}\}_{m=1}^{n_1} \\ & \oplus \operatorname{diag}\{\lambda_m x^{q_2-1} + \mu_m x^{q_3-1} - a_m x^{-1}\}_{m=n_1+1}^{n_1+n_2} \\ & \oplus \operatorname{diag}\{\mu_m x^{q_3-1} - a_m x^{-1}\}_{m=n_1+n_2+1}^{n},\end{aligned}$$

where $0 < q_3 < q_2 < q_1 < \infty$, $n = n_1 + n_2 + n_3$, $q_j, n_j \in \mathbb{N}$, $j = 1, 2, 3$. In section 5.6 we will give a generalisation to r levels, with $r \in \mathbb{N}_+$.

In appendix A we discuss the theory on staircase distributions, first introduced by Costin in [Cos98]. Most of the results we prove in this appendix can be found in Costin's article. In appendix B we give an existence theorem for a class of linear difference equations that will be used in chapter 3 and chapter 4.

1.2 Some Notations and Terminology

1.2.1 Regions in the Complex Plane

In this thesis an open disc with centre $x \in \mathbb{C}$ and radius $\rho > 0$ will be denoted by $\Delta(x, \rho)$, while its closure will be denoted by $\overline{\Delta}(x, \rho)$. A sector (on the Riemann surface of the logarithm) is defined to be a set of the form

$$S(\theta, \alpha) := \{x \in \mathbb{C}^* \mid \theta - \alpha/2 < \arg x < \theta + \alpha/2\},$$

where $\mathbb{C}^* = \mathbb{C}\backslash\{0\}$ and where $\theta \in \mathbb{R}$ and $\alpha > 0$. We shall refer to θ and α as the *bisecting direction* and the *opening* of $S(\theta, \alpha)$ respectively. Given a direction $\theta \in \mathbb{R}$, the half line $\{x \in \mathbb{C}^* \mid \arg x = \theta\}$ will also be denoted by its direction θ. A closed[1] sub-sector $\overline{S}_1 \subset S(\theta, \alpha)$ is a set of the form

$$\overline{S}_1 := \{x \in S(\theta, \alpha) \mid \beta_1 \leqslant \arg x \leqslant \beta_2\},$$

where $\theta - \alpha/2 < \beta_1 \leqslant \beta_2 < \theta + \alpha/2$. A neighbourhood of ∞ in a sector $S(\theta, \alpha)$ will be defined by

$$\big\{x \in S(\theta, \alpha) \bigm| |x| > r(\arg x)\big\},$$

[1] Here *closed* means closed with respect to the induced topology of $\mathbb{C}$ on $\mathbb{C}^*$.

where r is some positive valued continuous function on $(\theta - \alpha/2, \theta + \alpha/2)$. In some cases r can be taken a constant function, in other cases r is a function which tends to ∞ as $\arg x \to \theta \pm \alpha/2$. A neighbourhood of 0 in $S(\theta, \alpha)$ is defined similarly, but with $>$ replaced by $<$.

1.2.2 Asymptotics

Given a sequence $\{f_m\}_{m=0}^{\infty}$ of complex numbers, the series $\hat{f}(x) := \sum_{m=0}^{\infty} f_m x^{-m}$ is called a formal power series (in x^{-1}), the term 'formal' emphasising that we do not restrict the numbers f_m in any way. Thus the radius of convergence of the series $\sum_{m=0}^{\infty} f_m x^{-m}$ may well be equal to zero. The set of such formal power series is denoted by $\mathbb{C}[[x^{-1}]]$. Now, if f is holomorphic in a neighbourhood of ∞ in a sector S and if $\hat{f}(x) = \sum_{m=0}^{\infty} f_m x^{-m} \in \mathbb{C}[[x^{-1}]]$ is a formal series, we say that $f(x)$ asymptotically equals $\hat{f}(x)$ as $x \to \infty$ in S ($f(x) \sim \hat{f}(x)$ as $x \to \infty$ in S) if, for every nonnegative integer N and every closed sub-sector $\overline{S}_1$ of S, we have

$$f(x) = \sum_{m=0}^{N-1} f_m x^{-m} + O(x^{-N}), \quad \text{as } x \to \infty \text{ in } \overline{S}_1.$$

More precisely, for every nonnegative integer N and every closed sub-sector $\overline{S}_1$ there exists a positive constant C depending on both N and $\overline{S}_1$, such that

$$|f(x) - \textstyle\sum_{m=0}^{N-1} f_m x^{-m}| \leqslant C|x|^{-N}, \qquad \text{for all } x \in \overline{S}_1.$$

If the constant C can be chosen in the form $C = cN!K^N$, for some positive constants c and K independent of N, we say that $f(x)$ asymptotically equals $\hat{f}(x)$ of *Gevrey order 1*. In that case we write $f(x) \sim_1 \hat{f}(x)$ as $x \to \infty$ in S.

Similarly, given a function g holomorphic in a neighbourhood of 0 in a sector S and a formal series $\hat{g}(t) := \sum_{m=0}^{\infty} g_m t^m \in \mathbb{C}[[t]]$, we say that $g(t)$ asymptotically equals $\hat{g}(t)$ as $t \to 0$ in S ($g(t) \sim \hat{g}(t)$ as $t \to 0$ in S) if, for every nonnegative integer N and every closed bounded sub-sector $\overline{S}_1$ of S, we have $g(t) = \sum_{m=0}^{N-1} g_m t^m + O(t^N)$ as $t \to 0$ in $\overline{S}_1$. If the difference $g(t) - \sum_{m=0}^{N-1} g_m t^m$ can be estimated by $cN!K^N|t|^N$ for $t \in \overline{S}_1$, then $g(t)$ asymptotically equals $\hat{g}(t)$ of Gevrey order 1 and this is expressed as $g(t) \sim_1 \hat{g}(t)$ as $t \to 0$ in S.

1.2.3 Multi-indices

The notation $\mathbb{N}$ will be used for the set of natural numbers *including* 0. The notation $\mathbb{N}_+$ will be reserved for the set $\{1, 2, 3, \ldots\}$. Elements of the set $\mathbb{N}^n$, with $n \in \mathbb{N}_+$, will be called multi-indices. These multi-indices are ordered in the following way: if both $\mathbf{k} = (k_1, k_2, \ldots, k_n)$ and $\mathbf{l} = (l_1, l_2, \ldots, l_n)$ are elements of $\mathbb{N}^n$, then we will write $\mathbf{k} \preceq \mathbf{l}$ if $k_j \leqslant l_j$ for every $j \in \{1, 2, \ldots, n\}$. If moreover $\mathbf{k} \neq \mathbf{l}$ (meaning $k_j \neq l_j$ for some j), then we will use the notation $\mathbf{k} \prec \mathbf{l}$. In these cases we will also write $\mathbf{l} \succeq \mathbf{k}$ and $\mathbf{l} \succ \mathbf{k}$ respectively.

The j^{th} unit vector in $\mathbb{C}^n$ will be denoted by $\mathbf{e}_j$. For a multi-index $\mathbf{k} \in \mathbb{N}^n$ the notation $|\mathbf{k}|$ (also referred to as the *length* of $\mathbf{k}$) will be used for the sum of the components of $\mathbf{k}$, i.e.

$|\mathbf{k}| = \sum_{j=1}^{n} k_j$. For $r \in \mathbb{N}_+$ we define $\mathbb{N}_r^n := \{\mathbf{k} \in \mathbb{N}^n \mid |\mathbf{k}| \geqslant r\}$. For two multi-indices $\mathbf{k}$ and $\mathbf{l}$, the notation $\binom{\mathbf{k}}{\mathbf{l}}$ will be used for the expression $\prod_{j=1}^{n} \binom{k_j}{l_j}$. If $z = (z_1, z_2, \ldots, z_n) \in \mathbb{C}^n$, then $z^{\mathbf{k}}$ is defined by $z_1^{k_1} z_2^{k_2} \cdots z_n^{k_n}$. Moreover, the map $\langle \cdot, \cdot \rangle : \mathbb{N}^n \times \mathbb{C}^n \to \mathbb{C}$ is defined by the bilinear form $\langle \mathbf{k}, z \rangle := \sum_{j=1}^{n} k_j z_j$.

1.3 Borel Summability

In this section we will give a concise review of the notion of Borel summability. For a detailed exposition of this theory we refer to the papers [MR91, MalR92, Mal95]. In [Bal94, Bal00] Balser also gives a complete and detailed overview of Borel summability. However, he uses slightly different definitions from those we will give in this thesis.

1.3.1 The Borel Transform

Let f be a holomorphic function in a neighbourhood U of ∞ in the sector $S(-\theta, \alpha + \pi)$, where $\alpha > 0$, and assume that $f(x) = O(x^\delta)$ as $x \to \infty$ in U, where $\delta \in \mathbb{R}$. Then the Borel transform of f is defined by

$$(\mathcal{B}_\gamma f)(t) := \frac{1}{2\pi i} \int_\gamma f(x) e^{tx} \mathrm{d}x, \tag{1.3.1}$$

for $\pi/2 - \theta_2 < \arg t < -\pi/2 - \theta_1$ if γ is an unbounded contour in U with two limiting directions θ_1 and θ_2 with $\theta_2 - \theta_1 > \pi$. See figure 1 (in which the shaded part denotes the sector $S(-\theta, \alpha + \pi)$). By deformation of γ, $\mathcal{B}_\gamma f$ can be continued analytically to $S(\theta, \alpha)$. If

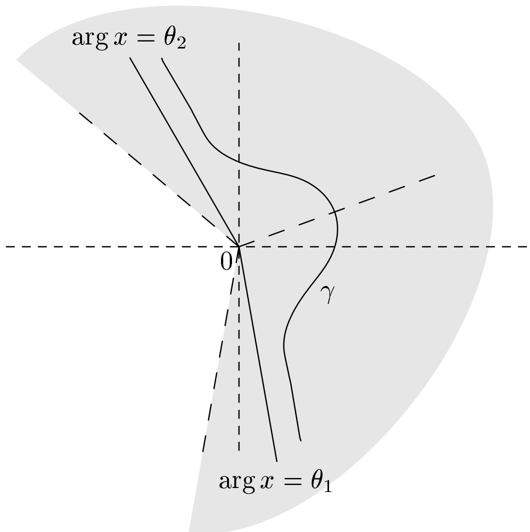

Figure 1.1: *The contour γ.*

we denote $\mathcal{B}f$ to be the Borel transform of f in $S(\theta, \alpha)$, then $\mathcal{B}f$ is of at most exponential growth in $S(\theta,\alpha)$, meaning that for any closed sub-sector $\overline{S}_1$ of $S(\theta,\alpha)$ there exists a positive constant M such that $\sup_{t\in\overline{S}_1} e^{-M|t|}|(\mathcal{B}f)(t)| < \infty$.

The formal Borel transform of a convergent or divergent series $\hat{f}(x) = \sum_{m=0}^{\infty} c_m x^{-m-r}$, with $r \in \mathbb{C}$, is defined by

$$(\hat{\mathcal{B}}\hat{f})(t) := \sum_{m=0}^{\infty} c_m \frac{t^{m+r-1}}{\Gamma(m+r)}. \tag{1.3.2}$$

Let $\hat{g}$ be a series of the same form as $\hat{f}$, but with r replaced by s. If $\hat{f}$ and $\hat{g}$ have (formal) Borel transforms F and G, $\Re r > 0$ or $r = 0$ and $\Re s > 0$ or $s = 0$, and $t \mapsto t^{1-r}F(t)$ and $t \mapsto t^{1-s}G(t)$ are convergent in some disc $\Delta(0,\rho)$, $\rho > 0$, then

$$\hat{\mathcal{B}}(\hat{f}\hat{g}) = \hat{f}(\infty)G + \hat{g}(\infty)F + F * G,$$

where $(F * G)(t) := \int_0^t F(t-\sigma)G(\sigma)\,\mathrm{d}\sigma$ for all $t \in \Delta(0,\rho)$. Moreover, if $r = 1$ and $f(x) \sim \hat{f}(x)$ as $x \to \infty$ in $S(-\theta, \alpha+\pi)$, then $(\mathcal{B}f)(t) \sim (\hat{\mathcal{B}}\hat{f})(t)$ as $t \to 0$ in $S(\theta,\alpha)$.

1.3.2 The Laplace Transform

Let F be a holomorphic function of at most exponential growth in $S(\theta,\alpha)$. Moreover, assume that $F(t) = O(t^{\varepsilon-1})$ as $t \to 0$ in $S(\theta,\alpha)$ for some $\varepsilon > 0$. Then the Laplace transform of f in the direction $\psi \in (\theta - \alpha/2, \theta + \alpha/2)$ is defined by

$$(\mathcal{L}_\psi F)(x) := \int_0^{\infty e^{i\psi}} F(t)\, e^{-xt}\,\mathrm{d}t, \tag{1.3.3}$$

with integration along $\arg t = \psi$. It is easily seen that this integral converges for $\Re(xe^{i\psi})$ sufficiently large, thus $\mathcal{L}_\psi F$ is holomorphic in a neighbourhood of ∞ in $S(-\psi,\pi)$. Moreover, it is easy to see that varying $\psi \in (\theta-\alpha/2, \theta+\alpha/2)$ gives rise to analytic continuations of this Laplace transform. So we end up with a Laplace transform $\mathcal{L}F$ that is holomorphic in a neighbourhood of ∞ in $S(-\theta,\alpha+\pi)$.

Given a convergent or divergent series $\hat{F}(t) = \sum_{m=0}^{\infty} c_m t^m \in \mathbb{C}[[t]]$, the formal Laplace transform of $\hat{F}$ is defined by

$$(\hat{\mathcal{L}}\hat{F})(x) := \sum_{m=0}^{\infty} c_m \Gamma(m+1)\, x^{-m-1} \in x^{-1}\mathbb{C}[[x^{-1}]]. \tag{1.3.4}$$

If $F(t) \sim \hat{F}(t)$ as $t \to 0$ in $S(\theta,\alpha)$, and the above assumptions on F are satisfied, then $(\mathcal{L}F)(x) \sim (\hat{\mathcal{L}}\hat{F})(x)$ as $x \to \infty$ in $S(-\theta,\alpha+\pi)$. In that case we have $(\mathcal{B}\mathcal{L})F = F$. On the other hand, if f is holomorphic in a neighbourhood of ∞ in a sector $S(-\theta,\alpha+\pi)$ and $f(x) \sim \sum_{m=0}^{\infty} c_m x^{-m-1}$ as $x \to \infty$ in $S(-\theta,\alpha+\pi)$, then $(\mathcal{L}\mathcal{B})f = f$.

If F and G are holomorphic and of at most exponential growth in $S(\theta,\alpha)$ and if both are of order $O(t^{\varepsilon-1})$ as $t \to 0$ in $S(\theta,\alpha)$ for some $\varepsilon > 0$, then the convolution product of F and G is defined, holomorphic and of at most exponential growth in $S(\theta,\alpha)$. Moreover, $F * G$ is of order $O(t^{2\varepsilon-1})$ as $t \to 0$ in $S(d,\alpha)$. Hence, $\mathcal{L}(F * G)$ is defined and holomorphic in a neighbourhood of ∞ in $S(-\theta,\alpha+\pi)$. It is known that $\mathcal{L}(F * G) = \mathcal{L}F \cdot \mathcal{L}G$ in this neighbourhood.

1.3.3 Definition of Borel Summability

Given a formal series $\hat{f}(x) = \sum_{m=0}^{\infty} c_m x^{-m-1} \in x^{-1}\mathbb{C}[[x^{-1}]]$, we say that $\hat{f}$ is *Borel summable in the direction* $-\theta$ if $\hat{\mathcal{B}}\hat{f}$ converges in a neighbourhood of the origin, with sum F, and F can be analytically continued in a sector $S(\theta, \delta)$ for some small positive δ and has at most exponential growth in this sector. If $\hat{f}$ is Borel summable in the direction $-\theta$, then we call $f := \mathcal{L}F$ the *Borel sum* of $\hat{f}$. Note that this Borel sum is holomorphic in a neighbourhood of ∞ in $S := S(-\theta, \pi + \delta)$ and it asymptotically equals $\hat{f}$ as $x \to \infty$ in S. If $\hat{f} \in x^{-1}\mathbb{C}[[x^{-1}]]$ is Borel summable in all but countably many directions, we just call $\hat{f}$ Borel summable.

The set of Borel summable series is closed under addition, multiplication and differentiation, i.e. if both $\hat{f}$ and $\hat{g}$ are Borel summable (in the direction $-\theta$) with Borel sums f and g respectively, then also $\hat{f} + \hat{g}$, $\hat{f}\hat{g}$ and $\hat{f}'$ are Borel summable (in the direction $-\theta$) with sums $f + g$, fg and f' respectively. Moreover, if $\hat{f}$ is Borel summable with Borel sum f and $\hat{f}(\infty) \neq 0$, then $\hat{f}^{-1}$ is Borel summable with Borel sum f^{-1}. If $\hat{f} \in \mathbb{C}\{x^{-1}\}$ is some convergent series in x^{-1}, then $\hat{f}$ is Borel summable in every direction $-\theta$ and its Borel sum equals the sum of the convergent series.

A formal series $\hat{f}(x) := \sum_{m=0}^{\infty} c_m x^{-m}$ is Borel summable if $\hat{g}(x) := \sum_{m=1}^{\infty} c_m x^{-m}$ is Borel summable and in that case the Borel sum of $\hat{f}$ is defined by $c_0 + g$, where g is the Borel sum of $\hat{g}$. If $\hat{f}(x) = x^{-r}\hat{g}(x)$, with $\Re r > 0$, and $\hat{g}(x) = \sum_{m=0}^{\infty} c_m x^{-m}$, then $\hat{f}$ is called Borel summable if $\hat{g}$ is Borel summable and its Borel sum equals $f(x) = x^{-r}g(x)$, where g is the Borel sum of $\hat{g}$. It is not hard to prove that this definition is equivalent to the following: the series $\hat{f}(x) := \sum_{m=0}^{\infty} c_m x^{-m-r}$ is Borel summable if $t \mapsto t^{1-r}(\hat{\mathcal{B}}\hat{f})(t)$ converges in a neighbourhood of the origin and the sum F of the series $\hat{\mathcal{B}}\hat{f}$ can be analytically continued in a certain sector and is of at most exponential growth in that sector. If so, then its Borel sum equals $f := \mathcal{L}F$.

Example 1.3.1 (Euler's equation) The differential equation $y'(x) - y(x) + x^{-1} = 0$ is known as *Euler's equation*[2]. A formal substitution of a series $\sum_{m=0}^{\infty} \alpha_m x^{-m}$ leads to[3]

$$-\sum_{m=2}^{\infty}(m-1)\alpha_{m-1}x^{-m} - \sum_{m=0}^{\infty} \alpha_m x^{-m} + x^{-1} = 0.$$

Comparing coefficients of x^{-m}, $m \geqslant 0$, gives $\alpha_0 = 0$, $\alpha_1 = 1$ and $\alpha_m = -(m-1)\alpha_{m-1}$ for $m \geqslant 2$, and thus $\alpha_0 = 0$ and $\alpha_m = (-1)^{m-1}(m-1)!$ for $m \in \{1, 2, 3, \ldots\}$. Hence, Euler's equations possesses a unique *formal series solution* $\hat{y}(x) := \sum_{m=0}^{\infty}(-1)^m m!\, x^{-m-1}$.

Taking a formal Borel transform of this formal solution gives the series $\sum_{m=0}^{\infty}(-1)^m t^m$, which converges in the disc $\Delta(0, 1)$. Moreover, its sum can be analytically extended as the function $t \mapsto (1+t)^{-1}$ in every sector S not containing the negative real axis. Obviously, the

[2]This is Euler's equation written down for x in a neighbourhood of ∞. The equivalent form near 0 looks like $t^2\tilde{y}'(t) + \tilde{y}(t) - t = 0$.

[3]We just substitute the series $\sum_{m=0}^{\infty} \alpha_m x^{-m}$ into the equation and interchange summation and differentiation without verifying whether it is allowed are not.

latter function is of at most exponential growth in such a sector, so $\hat{y}$ is Borel summable in every direction $-\theta$, with $\theta \neq \pi$ and its Borel sum equals the Laplace transform of $(1+t)^{-1}$ (where we can integrate along any half line starting at 0, except for the negative real axis).

By means of variation of constants one easily obtains the following general holomorphic solution of Euler's equation:

$$y(x) = c\, e^{x} + \int_{x}^{\infty} \frac{e^{x-\sigma}}{\sigma}\, \mathrm{d}\sigma, \quad c \in \mathbb{C}. \tag{1.3.5}$$

Substituting $\sigma = x(1+t)$ and taking $c = 0$ we see that $y(x) = \int_0^\infty e^{-xt} \frac{\mathrm{d}t}{1+t}$, which is exactly the Borel sum of $\hat{y}$. So, the Borel sum of $\hat{y}$ indeed is a holomorphic solution of Euler's equation.

Another way to reach the same conclusion is by transforming Euler's equation into a corresponding equation by applying the (formal) Borel transform. This gives

$$-tY - Y + 1 = 0,$$

where $Y := \hat{\mathcal{B}}\hat{y}$. The latter equation has the unique solution $Y(t) = \frac{1}{1+t}$. As the Borel and Laplace operator are each others inverse, the Laplace transform of Y is a (holomorphic) solution of Euler's equation. Moreover, from the theory above one immediately deduces that this holomorphic solution asymptotically equals $\sum_{m=0}^{\infty}(-1)^m m!\, x^{-m-1}$ as $x \to \infty$ in any sector not containing the directions $\pm\frac{\pi}{2}$ mod 2π. So the formal solution $\hat{y}$ is lifted to an actual solution of Euler's equation.

1.4 Multisummability

Again we only give a brief overview of the definitions of the generalised Laplace and Borel transforms, acceleration operators and the notion of multisummability, which are due to Écalle (cf. [Eca85, Eca90, Eca91, Eca92]). In this section we will follow the paper [MR91] of Martinet and Ramis. However, equivalent forms of multisummability have been given by Malgrange and Ramis in [MalR92, Mal95] and by Balser in [Bal92, Bal94, Bal00] (the latter with slightly different definitions).

In the remaining part of this chapter q will always denote a positive number.

1.4.1 The Borel Transform of Order q

Let f be a holomorphic function in a neighbourhood U of ∞ in the sector $S(-\theta, \alpha + \frac{\pi}{q})$, where $\alpha > 0$, and assume that $f(x) = O(x^\delta)$ as $x \to \infty$ in U, where $\delta \in \mathbb{R}$. Then the Borel transform of order q of f is defined by

$$(\mathcal{B}_q f)(t) := \frac{1}{2\pi i} \int_\gamma f(x)\, e^{(tx)^q}\, \mathrm{d}(x^q), \tag{1.4.1}$$

for a certain unbounded contour γ in U, which may depend on q. Here $\mathrm{d}(x^q)$ is a short-hand notation for $q x^{q-1}\mathrm{d}x$. The integration contour γ has the same form as the path of integration in the definition of the ordinary Borel transform (cf. figure 1.1), but now with limiting directions θ_1 and θ_2 with $\theta_2 - \theta_1 > \pi/q$ and then $\frac{\pi}{2q} - \theta_2 < \arg t < -\frac{\pi}{2q} - \theta_1$. Again changes of γ give rise to analytic continuations and by varying γ one obtains a Borel transform of order q of f, which is holomorphic and of exponential growth of order $\leqslant q$ in $S(\theta, \alpha)$, the latter meaning that for any closed sub-sector $\overline{S}_1$ of $S(\theta, \alpha)$ there exists a positive constant M such that $\sup_{t \in \overline{S}_1} e^{-M|t|^q} |(\mathcal{B}_q f)(t)| < \infty$.

Given a convergent or divergent series $\hat{f}(x) = \sum_{m=0}^{\infty} c_m x^{-m-1} \in x^{-1}\mathbb{C}[[x^{-1}]]$, its formal Borel transform of order q is defined by

$$(\hat{\mathcal{B}}_q \hat{f})(t) := \sum_{m=0}^{\infty} c_m \frac{t^{m+1-q}}{\Gamma(\frac{m+1}{q})}. \tag{1.4.2}$$

If $f(x) \sim \hat{f}(x)$ as $x \to \infty$ in $S(-\theta, \alpha + \frac{\pi}{q})$, then $(\mathcal{B}_q f)(t) \sim (\hat{\mathcal{B}}_q \hat{f})(t)$ as $t \to 0$ in $S(\theta, \alpha)$. Note that this has to be interpreted as $t^{q-1}(\mathcal{B}_q f)(t) \sim \sum_{m=0}^{\infty} c_m \frac{t^m}{\Gamma(\frac{m+1}{q})}$ as $t \to 0$ in $S(\theta, \alpha)$.

Now let f and g be holomorphic functions in a neighbourhood of ∞ in $S(-\theta, \alpha + \frac{\pi}{q})$ and assume that both are $O(x^{-\varepsilon})$ as $x \to \infty$ in $S(-\theta, \alpha + \frac{\pi}{q})$ for some $\varepsilon > 0$. If we denote $F = \mathcal{B}_q f$ and $G = \mathcal{B}_q g$, then one can prove that

$$\mathcal{B}_q(f \cdot g) = F *_q G, \tag{1.4.3}$$

where the so-called *q-convolution* $F *_q G$ is defined by

$$(F *_q G)(t) := \int_0^t F\big((t^q - \sigma^q)^{1/q}\big) G(\sigma)\, \mathrm{d}(\sigma^q).$$

A trivial, but useful example of q-convolution is the following:

$$t^{r-q} *_q t^{s-q} = B\Big(\frac{r}{q}, \frac{s}{q}\Big) t^{r+s-q}, \quad \Re r, \Re s > 0, \tag{1.4.4}$$

where B is the beta function.

1.4.2 The Laplace Transform of Order q

Let F be a holomorphic function which is of exponential growth of order $\leqslant q$ in $S(\theta, \alpha)$. Moreover, assume that $F(t) = O(t^{\varepsilon - q})$ as $t \to 0$ in $S(\theta, \alpha)$ for some $\varepsilon > 0$. Then the Laplace transform of order q of F is defined by

$$(\mathcal{L}_q F)(x) := \int_0^{\infty e^{i\psi}} F(t)\, e^{-(xt)^q} \mathrm{d}(t^q), \tag{1.4.5}$$

for arbitrary $\psi \in (\theta - \alpha/2, \theta + \alpha/2)$. Obviously this integral converges for $\Re(x^q e^{iq\psi})$ sufficiently large, so a priori the integral represents a holomorphic function in a neighbourhood of ∞ in $S(-\psi, \pi/q)$. By varying ψ in the interval above, we end up with a Laplace transform $\mathcal{L}_q F$ that is holomorphic in a neighbourhood of ∞ in $S(-\theta, \alpha + \frac{\pi}{q})$.

Given a convergent or divergent series $\hat{F}(t) = \sum_{m=0}^{\infty} c_m t^{m+1-q}$, the formal Laplace transform of order q of $\hat{F}$ is defined by

$$(\hat{\mathcal{L}}_q \hat{F})(x) := \sum_{m=0}^{\infty} c_m \, \Gamma\Big(\frac{m+1}{q}\Big)\, x^{-m-1}. \tag{1.4.6}$$

Again, if $F(t) \sim \hat{F}(t)$ as $t \to 0$ on $S(\theta, \alpha)$ and the above assumptions on F are satisfied, then $(\mathcal{L}_q F)(x) \sim (\hat{\mathcal{L}}_q \hat{F})(x)$ as $x \to \infty$ in $S(-\theta, \alpha + \frac{\pi}{q})$ and in that case $\mathcal{B}_q \mathcal{L}_q F = F$. If, on the other hand, f is a holomorphic function in a neighbourhood of ∞ in the sector $S(-\theta, \alpha + \frac{\pi}{q})$ and $f(x) \sim \sum_{m=0}^{\infty} c_m x^{-m-1}$ as $x \to \infty$ in this sector, then $\mathcal{L}_q \mathcal{B}_q f = f$.

If F and G are holomorphic and of exponential growth of order $\leqslant q$ in $S(\theta, \alpha)$ and $F(t)$, $G(t) = O(t^{\varepsilon - q})$ as $t \to 0$ on $S(\theta, \alpha)$ for some $\varepsilon > 0$, then the q-convolution of F and G is defined and

$$\mathcal{L}_q(F *_q G) = \mathcal{L}_q F \cdot \mathcal{L}_q G \tag{1.4.7}$$

in a neighbourhood of ∞ in $S(-\theta, \alpha + \pi/q)$.

1.4.3 Definition of Multisummability

In order to give a proper definition of multisummability we first have to introduce another integral operator: Écalle's *acceleration operator*. For $q' > q > 0$ this acceleration operator $\mathcal{A}_{q',q}$ is defined by

$$\mathcal{A}_{q',q} := \mathcal{B}_{q'} \mathcal{L}_q.$$

This operator makes sense on the space of functions $F : S(\theta, \alpha) \to \mathbb{C}$ which are of exponential growth of order $\leqslant q$ in $S(\theta, \alpha)$ and which satisfy $F(t) = O(t^{\varepsilon - q})$ as $t \to 0$ in $S(\theta, \alpha)$ for some $\varepsilon > 0$. Écalle has shown that this operator can be extended to functions with the same conditions except that 'exponential growth of order $\leqslant q$' is replaced by 'exponential growth of order $\leqslant \kappa$', where $\frac{1}{\kappa} := \frac{1}{q} - \frac{1}{q'}$ i.e. $\kappa = \frac{q'q}{q'-q}$. For such a function F, the accelerate $\mathcal{A}_{q',q}F$ is holomorphic in a neighbourhood of 0 in $S(\theta, \alpha + \pi/\kappa)$ and can be represented by

$$(\mathcal{A}_{q',q} F)(t) = t^{-q'} \int_0^{\infty e^{i\psi}} C_{q'/q}\big((\xi/t)^q\big)\, F(\xi)\, \mathrm{d}(\xi^q), \tag{1.4.8}$$

where $\psi \in (\theta - \alpha/2, \theta + \alpha/2)$. Here $C_{q'/q}$ is a special case of Écalle's function C_α, defined for $\alpha > 1$ by

$$C_\alpha(t) := \sum_{m=1}^{\infty} \frac{(-t)^m}{m!\, \Gamma(-m/\alpha)}, \quad t \in \mathbb{C}. \tag{1.4.9}$$

If $F(t) \sim \hat{F}(t) := \sum_{m=0}^{\infty} c_m t^{m+1-q}$ as $t \to 0$ on $S(\theta, \alpha)$, then

$$(\mathcal{A}_{q',q} F)(t) \sim \sum_{m=0}^{\infty} c_m \frac{\Gamma(\frac{m+1}{q})}{\Gamma(\frac{m+1}{q'})} t^{m+1-q'} =: (\hat{\mathcal{A}}_{q',q} \hat{F})(t),$$

as $t \to 0$ in $S(\theta, \alpha + \pi/\kappa)$. Moreover, if F and G are functions for which the accelerate is defined, then one can prove that

$$\mathcal{A}_{q',q}(F *_q G) = (\mathcal{A}_{q',q}F) *_{q'} (\mathcal{A}_{q',q}G).$$

Definition 1.4.1 (Multisummability) *Let $\hat{f} \in x^{-1}\mathbb{C}[[x^{-1}]]$ and let r be a natural number. Let $\mathbf{q} = (q_1, q_2, \ldots, q_r)$, $0 < q_r < q_{r-1} < \cdots < q_1$, $\boldsymbol{\theta} = (\theta_1, \theta_2, \ldots, \theta_r) \in \mathbb{R}^r$ and let $S_j = S(-\theta_j, \alpha_j + \pi/q_j)$, $\alpha_j > 0$, $j = 1, 2, \ldots, r$. Moreover, define*

$$q_0 := \infty, \quad \kappa_j = \left(q_j^{-1} - q_{j-1}^{-1}\right)^{-1}, \quad j = 1, 2, \ldots, r.$$

Then $\hat{f}$ is said to be $\mathbf{q}$-summable in the multi-direction $-\boldsymbol{\theta}$ (or $\mathbf{q}$-summable on the multi-sector $\mathbf{S} := (S_1, S_2, \ldots, S_r)$) if

(i) *For all $j \in \{2, 3, \ldots, r\}$: $S_{j-1} \subset S_j$.*

(ii) *The series $\hat{\mathcal{B}}_{q_r}\hat{f}$ is convergent in $\Delta(0,\rho)\backslash\{0\}$ for some $\rho > 0$. Denote g_r the sum of this convergent series.*

(iii) *For $j = r, r-1, \ldots, 1$ respectively, the function g_j can be analytically continued and is of exponential growth of order $\leqslant \kappa_j$ in $S(\theta_j, \alpha_j)$. Here g_{j-1}, $j = r, r-1, \ldots, 2$, is defined in a neighbourhood of 0 in $S(\theta_j, \alpha_j + \pi/\kappa_j)$ by $g_{j-1} := \mathcal{A}_{q_{j-1},q_j} g_j$.*

If these conditions are fulfilled, then the $\mathbf{q}$-sum (or multi-sum) of the formal series $\hat{f}$ is defined by $S_{\mathbf{q},\boldsymbol{\theta}}\hat{f} := \mathcal{L}_{q_1} g_1$.

This sum is holomorphic in a neighbourhood U of ∞ in S_1 and it satisfies $(S_{\mathbf{q},\boldsymbol{\theta}}\hat{f})(x) \sim \hat{f}(x)$ as $x \to \infty$ in U. The summation operator $S_{\mathbf{q},\boldsymbol{\theta}}$ is injective. Moreover, Borel summability coincides with 1-summability, since $\mathcal{B} = \mathcal{B}_1$ and $\mathcal{L} = \mathcal{L}_1$. If $\hat{f} \in x^{-1}\mathbb{C}[[x^{-1}]]$ is $\mathbf{q}$-summable in all but countably many multi-directions, we just call $\hat{f}$ $\mathbf{q}$-summable.

As in the case of Borel summability, the set of multisummable power series is closed under addition, multiplication and differentiation and every series that converges in a neighbourhood of ∞ is $\mathbf{q}$-summable in every multi-direction with multi-sum equal to the sum of the convergent series. A series $\hat{f}(x) := \sum_{m=0}^{\infty} c_m x^{-m}$ is $\mathbf{q}$-summable in the multi-direction $-\boldsymbol{\theta}$ if $\hat{g}(x) := \sum_{m=1}^{\infty} c_m x^{-m}$ is $\mathbf{q}$-summable in the multi-direction $-\boldsymbol{\theta}$ and it that case $S_{\mathbf{q},\boldsymbol{\theta}}\hat{f} = c_0 + S_{\mathbf{q},\boldsymbol{\theta}}\hat{g}$.

A proof of the following lemma can be found in [Bal94]. Another proof, that uses cohomology arguments, is given by Malgrange and Ramis in [MalR92].

Lemma 1.4.2 *For natural n let $h \in \mathbb{C}\{x^{-1}, y_1, \ldots, y_n\}$. Choosing arbitrary formal power series $\hat{f}_1, \hat{f}_2, \ldots, \hat{f}_n$ in x^{-1} with vanishing constant terms, one can (formally) define a power series*

$$\hat{g}(x) := h\big(x, \hat{f}_1(x), \ldots, \hat{f}_n(x)\big).$$

Let $\mathbf{q} := (q_1, q_2, \ldots, q_r)$ with $1/2 < q_r < q_{r-1} < \cdots < q_1$ and assume that the formal series $\hat{f}_1, \hat{f}_2, \ldots, \hat{f}_r$ are $\mathbf{q}$-summable in the multi-direction $-\boldsymbol{\theta} \in \mathbb{R}^r$. Then $\hat{g}$ is $\mathbf{q}$-summable in the multi-direction $-\boldsymbol{\theta}$.

1.5 Two Illustrative Examples

In this section we illustrate the occurrence of transseries and singularities at Stokes lines of small solutions in examples of a Riccati differential equation and a Riccati difference equation. However, these equations may also be discussed without the use of transseries.

Example 1.5.1 In [CK02] Costin and Kruskal considered the Bernoulli differential equation $y' - y + y^2 = 0$. Following a suggestion of M. van der Put we consider here the differential equation

$$y'(x) + y(x) + y^2(x) + x^{-2} = 0 \tag{1.5.1}$$

for x in a neighbourhood of ∞ and we will restrict ourselves to complex valued solutions y which tend to 0 as $x \to \infty$.

If we write $y = \frac{f'}{f}$ or $f' = yf$ for some function f, then $f'' = y'f + y^2 f$. Using (1.5.1) we see that f satisfies the equation

$$f''(x) + f'(x) + x^{-2} f(x) = 0. \tag{1.5.2}$$

In general (1.5.1) is called the *Riccati equation* of (1.5.2). From [Olv74], sections 1 and 2 (in particular theorems 2.1 and 2.2) in chapter 7, we conclude that there exist formal series $\hat{f}_0, \hat{f}_1 \in \mathbb{C}[[x^{-1}]]$ and holomorphic solutions f_0 and g_0 with $f_0(x) \sim \hat{f}_0(x)$ as $x \to \infty$ in $S(\pi, 3\pi)$ and $e^x g_0(x) \sim \hat{f}_1(x)$ as $x \to \infty$ in $S(0, 3\pi)$. Moreover, the constant terms in $\hat{f}_0$ and $\hat{f}_1$ may be chosen arbitrary, and we will give them the value 1. With this prescription both the formal solutions $\hat{f}_0$ and $\hat{f}_1$ and the holomorphic solutions f_0 and g_0 turn out to be unique. Now observe that if $e^{-x}\hat{f}_1(x)$ satisfies (1.5.2), then $\hat{f}_1$ satisfies $f_1''(x) - f_1'(x) + x^{-2} f_1(x) = 0$, so $\hat{f}_1(x) = \hat{f}_0(-x)$.

Substitution of $\hat{f}_0(x) = \sum_{m=0}^{\infty} c_m x^{-m}$ in (1.5.2) gives

$$\sum_{m=3}^{\infty}(m-1)(m-2)c_{m-2}\, x^{-m} - \sum_{m=2}^{\infty}(m-1)c_{m-1}\, x^{-m} + \sum_{m=2}^{\infty} c_{m-2}\, x^{-m} = 0$$

and comparing coefficients of x^{-m}, $m \geqslant 0$, leads to the recurrence relation

$$c_m = \frac{m^2 - m + 1}{m}\, c_{m-1}, \quad m \geqslant 1.$$

Given the prescription $c_0 = 1$, the other coefficients are determined uniquely. In fact, if $\zeta = e^{\pm \pi i/3}$, then $c_m = \frac{(m-\zeta)(m-1+\zeta)}{m}\, c_{m-1}$ and thus $c_m = \frac{(\zeta)_m (1-\zeta)_m}{m!}$ [4]. So (1.5.2) possesses the following two formal solutions:

$$\hat{f}_0(x) = \sum_{m=0}^{\infty} \frac{(\zeta)_m (1-\zeta)_m}{m!}\, x^{-m} \quad \text{and} \quad \hat{g}_0(x) = e^{-x} \sum_{m=0}^{\infty} (-1)^m \frac{(\zeta)_m (1-\zeta)_m}{m!}\, x^{-m}.$$

Hence, one may conclude that $\alpha \hat{f}_0(x) - \beta e^{-x} \hat{f}_1(x)$, with α and β arbitrary complex constants, again is a formal solution of (1.5.2). If we define z to be the general holomorphic

[4] Here we used *Pochhammer's symbol*: $(a)_0 = 1$, $(a)_m = a(a+1)\cdots(a+m-1)$, $m \in \mathbb{N}_+$.

solution of $z'(x) + z(x) = 0$, i.e. $z(x) = \beta e^{-x}$, $\beta \in \mathbb{C}$, then the formal solution of (1.5.2) described above can be written as $\alpha \hat{f}_0(x) - z(x)\hat{f}_1(x)$. Now if we choose $\alpha = 0$, then the corresponding formal solution of (1.5.1) equals $\hat{y}(x) = \frac{(z(x)\hat{f}_1(x))'}{z(x)\hat{f}_1(x)} = \frac{\hat{f}_1'(x)}{\hat{f}_1(x)} - 1$. In that case $\hat{y}(\infty) \neq 0$, which means that if $y(x)$ is a solution of (1.5.1) that asymptotically equals $\hat{y}(x)$ as $x \to \infty$, then $y(x) \not\to 0$ as $x \to \infty$. Since we only consider solutions y that tend to 0 as $x \to \infty$, we thus have $\alpha \neq 0$ and by taking $\alpha = 1$ we get

$$\hat{y}(x) = \frac{\hat{f}_0'(x) - z(x)\big(\hat{f}_1'(x) - \hat{f}_1(x)\big)}{\hat{f}_0(x) - z(x)\hat{f}_1(x)} \tag{1.5.3}$$

as formal solution of (1.5.1). By construction, the *formal* transformation

$$y = \hat{P}(x,z) := \frac{\hat{f}_0'(x) - z\big(\hat{f}_1'(x) - \hat{f}_1(x)\big)}{\hat{f}_0(x) - z\hat{f}_1(x)}$$

transforms (1.5.1) into the *normal form* $z'(x) + z(x) = 0$. Now, the expression $\hat{P}(x,z)$ can formally be rewritten as a power series in z,

$$\hat{P}(x,z) = \sum_{k=0}^{\infty} \hat{y}_k(x)\, z^k,$$

with $\hat{y}_0(x) = \frac{\hat{f}_0'(x)}{\hat{f}_0(x)}$ and $\hat{y}_k(x) = \left(\frac{\hat{f}_1(x)}{\hat{f}_0(x)}\right)^k \left(\frac{\hat{f}_0'(x)}{\hat{f}_0(x)} - \frac{\hat{f}_1'(x)}{\hat{f}_1(x)} + 1\right)$ for $k \geqslant 1$. When we substitute the solution $z(x) = \beta e^{-x}$ of the normal form $z'(x) + z(x) = 0$ into the expansion of $\hat{P}(x,z)$, we obtain an example of a *transseries*, i.e. a formal exponential series solution of (1.5.1):

$$\hat{y}(x) = \sum_{k=0}^{\infty} \beta^k e^{-kx} \hat{y}_k(x). \tag{1.5.4}$$

As pointed out in the beginning of this example we can conclude from [Olv74] that the formal transformation $y = \hat{P}(x,z)$ can be lifted to an actual transformation, because there exist lifts $f_0(x)$ of $\hat{f}_0(x)$ and $f_1(x) := e^x g_0(x)$ of $\hat{f}_1(x)$. So, the *transformation*

$$y = P(x,z), \quad \text{with } P(x,z) := \frac{f_0'(x) - z\big(f_1'(x) - f_1(x)\big)}{f_0(x) - zf_1(x)},$$

is well defined for x in a neighbourhood of ∞ in $S := S(\pi/2, 2\pi)$ and z in a neighbourhood of 0, and transforms the differential equation (1.5.1) in a neighbourhood of ∞ in S into the normal form $z'(x) + z(x) = 0$. Moreover, if we define

$$y(x) := \frac{f_0'(x) - \beta e^{-x}(f_1'(x) - f_1(x))}{f_0(x) - \beta e^{-x} f_1(x)}, \tag{1.5.5}$$

then, for each $\beta \in \mathbb{C}$, y is an actual *solution* of (1.5.1), which is holomorphic in a neighbourhood of ∞ in the region $R := S \backslash \{x \in \mathbb{C}^* \mid f_0(x) = \beta e^{-x} f_1(x)\}$.

We can prove that both $\hat{f}_0$ and $\hat{f}_1$ are Borel summable: the formal Borel transform of $x^{-\zeta}\hat{f}_0(x)$, with $\zeta = e^{\pm\pi i/3}$, equals

$$\hat{\mathcal{B}}[x^{-\zeta}\hat{f}_0](t) = \sum_{m=0}^{\infty} \frac{c_m}{\Gamma(m+\zeta)} t^{m+\zeta-1} = \sum_{m=0}^{\infty} \frac{(1-\zeta)_m}{m!\Gamma(\zeta)} t^{m+\zeta-1} = \frac{\{t(1-t)\}^{\zeta-1}}{\Gamma(\zeta)},$$

where the last equality follows from the fact that $\frac{(1-\zeta)_m}{m!} = (-1)^m\binom{\zeta-1}{m}$. The function $t \mapsto \{t(1-t)\}^{\zeta-1}\Gamma^{-1}(\zeta)$ is holomorphic for $t \neq 0, 1$, integrable at $t = 0$ and of at most exponential growth in every sector not containing the positive real axis. So $x^{-\zeta}\hat{f}_0(x)$, and thus $\hat{f}_0$, is Borel summable and the Borel sum of $\hat{f}_0$ is holomorphic in a neighbourhood of ∞ in $S(\pi, 3\pi)$. Since f_0 is the unique solution of (1.5.2) with $f_0(x) \sim \hat{f}_0(x)$ as $x \to \infty$ in $S(\pi, 3\pi)$, the Borel sum of $\hat{f}_0$ coincides with f_0, so

$$f_0(x) = \frac{x^\zeta}{\Gamma(\zeta)} \int_0^{\infty e^{i\theta}} e^{-xt}\{t(1-t)\}^{\zeta-1}\mathrm{d}t, \quad |\arg x + \theta| < \frac{\pi}{2}, \quad -2\pi < \theta < 0. \qquad (1.5.6)$$

Obviously, $f_0(xe^{\pi i})$ is the Borel sum of $\hat{f}_1(x)$ on $S(0, 3\pi)$ and coincides with $f_1(x) = e^x g_0(x)$. Hence, $e^{-x}f_0(xe^{\pi i})$ is a solution of (1.5.2).

Another way to reach the last conclusion is by observing that in (1.5.6) it is allowed to take $\theta = 0$ and $\theta = -2\pi$, provided that $\arg(1-t) = \pi$, respectively $-\pi$, for $t > 1$. In particular for $x > 0$, $\arg x = 0$, we have

$$f_0(x) = \frac{x^\zeta}{\Gamma(\zeta)}\left\{\int_0^1 e^{-xt}\{t(1-t)\}^{\zeta-1}\mathrm{d}t + \int_1^\infty e^{-xt}\{te^{\pi i}(t-1)\}^{\zeta-1}\mathrm{d}t\right\}$$

and

$$f_0(xe^{2\pi i}) = \frac{(xe^{2\pi i})^\zeta}{\Gamma(\zeta)}\left\{\int_0^1 e^{-xt}\{e^{-2\pi i}t(1-t)\}^{\zeta-1}\mathrm{d}t + \int_1^\infty e^{-xt}\{e^{-2\pi i}te^{-\pi i}(t-1)\}^{\zeta-1}\mathrm{d}t\right\}.$$

Since the two integrals from $t = 0$ to $t = 1$ coincide we get

$$\begin{aligned} f_0(x) - f_0(xe^{2\pi i}) &= \frac{x^\zeta}{\Gamma(\zeta)}\{e^{\pi i(\zeta-1)} - e^{-\pi i(\zeta-1)}\}\int_1^\infty e^{-xt}\{t(t-1)\}^{\zeta-1}\mathrm{d}t \\ &= 2i\sin(\pi\zeta)\frac{x^\zeta}{\Gamma(\zeta)}e^{-x}\int_0^{\infty e^{-\pi i}} e^{xs}\{(1-s)e^{\pi i}s\}^{\zeta-1}\mathrm{d}s \\ &= -2i\sin(\pi\zeta)e^{-x}f_0(xe^{\pi i}), \end{aligned}$$

where the substitution $t = 1 + e^{\pi i}s$ is used to obtain the second equality. By analytic continuation we then deduce that

$$f_0(x) - f_0(xe^{2\pi i}) = -2i\sin(\pi\zeta)e^{-x}f_0(xe^{\pi i}), \quad |\arg x| < \frac{\pi}{2}. \qquad (1.5.7)$$

Since the left-hand side of the preceding formula is a solution of (1.5.2), the expression $e^{-x}f_0(xe^{\pi i})$ also is a solution of this equation. The formula (1.5.7) is a connection formula for f_0.

There also is a remarkable correspondence with modified Bessel functions, as follows: the integral representation of the modified Bessel function of the third kind[5], K_ν, is given by (cf. [Wat44], p. 206)

$$K_\nu(x) = \sqrt{\frac{\pi}{2x}}\frac{e^{-x}}{\Gamma(\nu+1/2)}\int_0^\infty e^{-u}u^{\nu-1/2}\left(1+\frac{u}{2x}\right)^{\nu-1/2}\mathrm{d}u$$

[5]In some textbooks this function is also called *Macdonald's function*.

and if we choose $\nu = \zeta - 1/2$ and put $u = e^{\pi i}tx$, then one easily infers the following correspondence with f_0

$$\begin{cases} e^{-x} f_0(xe^{\pi i}) = \sqrt{\frac{x}{\pi}}\, e^{-x/2}\, K_{\zeta-1/2}(x/2) \\ f_0(x) = \sqrt{\frac{xe^{-\pi i}}{\pi}}\, e^{-x/2}\, K_{\zeta-1/2}(xe^{-\pi i}/2). \end{cases} \tag{1.5.8}$$

So the asymptotic expansion of f_0 also gives an asymptotic expansion for K_ν (which coincides with the one Watson gives in [Wat44], p. 207) and the connection formula (1.5.7) gives a connection formula for K_ν.

Finally, we mention that the differential equation (1.5.2) can be reduced to a modified Bessel equation of order $\nu = \zeta - 1/2$, by means of the substitution $f(x) = \sqrt{x}e^{-x/2}u(\pm ix/2)$. Then u satisfies a modified Bessel equation with as general solution the so-called cylinder functions $C_{\zeta-1/2}$ (compare [Wat44], p. 98, formulae (15) and (16) with $\varphi(x) = e^{-x}$ and $\psi(x) = \pm\frac{ix}{2}$). For these cylinder functions one can choose the functions $K_{\zeta-1/2}(e^{\pm\pi i/2}t)$ and with this particular choice we get, via (1.5.8), $f_0(x)$ and $e^{-x}f_0(xe^{\pi i})$, except for a constant factor, as solutions for (1.5.2).

As both $\hat{f}_0$ and $\hat{f}_1$ are Borel summable, the same holds for each $\hat{y}_k$ and their Borel sums

$$y_0(x) = \frac{f_0'(x)}{f_0(x)} \quad \text{and} \quad y_k(x) = \left(\frac{f_1(x)}{f_0(x)}\right)^k \left(\frac{f_0'(x)}{f_0(x)} - \frac{f_1'(x)}{f_1(x)} + 1\right), \quad k \geqslant 1,$$

are holomorphic in a neighbourhood of ∞ in S, but of course this does not guarantee convergence of the series $\sum_{k=0}^{\infty} \beta^k e^{-kx} y_k(x)$. In fact, one can prove that the formal series $\hat{P}(x,z)$ is Borel summable with respect to x, *provided that* $|z|$ *is small enough*, and then its sum equals $\sum_{k=0}^{\infty} y_k(x) z^k$. This statement is shown in a more general context by Braaksma in [Bra01]. Hence, we only are allowed to write y, as given in (1.5.5), in the form of a convergent transseries $\sum_{k=0}^{\infty} \beta^k e^{-kx} y_k(x)$ if βe^{-x} decreases as $x \to \infty$, which is the case on $R \cap \{x \in \mathbb{C}^* \mid \Re x > 0\}$.

From (1.5.5) we see that y might be singular at those points $x \in \mathbb{C}$ satisfying the equation $f_0(x) = \beta e^{-x} f_1(x)$. Then $e^x = \beta f_0(-x) f_0^{-1}(x)$ and thus

$$x = \ln\beta + \ln\bigl(f_0(-x)f_0^{-1}(x)\bigr) + 2n\pi i, \quad n \in \mathbb{Z}. \tag{1.5.9}$$

This equation can be rewritten as

$$\bigl[x - \ln\bigl(f_0(-x)f_0^{-1}(x)\bigr)\bigr]^{-1} = (\ln\beta + 2n\pi i)^{-1} =: \gamma_n, \quad n \in \mathbb{Z}.$$

Putting $x = t^{-1}$ we infer that the latter equation is equivalent to

$$t = \gamma_n\bigl(1 + \psi(t)\bigr) := h(t), \quad \text{with} \quad \psi(t) = -t\ln\Bigl(\frac{f_0(-t^{-1})}{f_0(t^{-1})}\Bigr). \tag{1.5.10}$$

One should notice that ψ is holomorphic for t in a neighbourhood of 0 in $\tilde{S} := S(-\pi/2, 2\pi)$. Hence, ψ certainly is holomorphic $G := \{t \in \mathbb{C}^* \mid 0 < |t| < r,\ -3\pi/2 + \varepsilon < \arg t < \pi/2 - \varepsilon\}$ for some $r > 0$ and $\varepsilon > 0$. From the asymptotic behaviour of f_0 and f_1 we easily deduce that $\psi(t) = o(t)$ as $t \to 0$ in $\tilde{S}$, so $\psi'(t) = o(1)$ as $t \to 0$ in $\tilde{S}$, and $K := \sup_{t\in G} |\psi'(t)| < \infty$. From this we easily infer that $t \mapsto h(t)$ is a contraction on G provided that γ_n belongs to some subset G_1 of G, $|\gamma_n| < 1/K$ and r sufficiently small (i.e. so small that $h(t) \in G$ for $t \in G$). The requirement $\gamma_n \in G_1$, $|\gamma_n| < 1/K$, can only be fulfilled for $n \in \mathbb{N}$, n large enough. Hence, (1.5.10) has a unique solution $t = t_n$ for every $n \in \mathbb{N}$ large enough. So, if we define $x_n := t_n^{-1}$, then $x = x_n$ is a solution of (1.5.9), provided that $n \in \mathbb{N}$ is large enough.

As $\ln\big(f_0(-x)f_0^{-1}(x)\big) = O(x^{-1})$, $x \to \infty$ in S, we deduce that $x_n = \ln\beta + 2n\pi i + O(x_n^{-1})$ and using this we see that the solution y of (1.5.1) might be singular at points $x = x_n$ with

$$x_n = \ln\beta + 2n\pi i + O(n^{-1}), \quad n \in \mathbb{N},\ n \to \infty. \tag{1.5.11}$$

Thus we conclude that if $\beta \neq 0$, the solution y might be singular at a distance at most $O(n^{-1})$ of $\ln\beta + 2n\pi i$, as $n \in \mathbb{N}$, $n \to \infty$. Here we recognise a more general result of O. Costin and R.D. Costin on the formation of singularities of solutions of nonlinear differential equations in so-called *Stokes directions*[6]. For more details on this result the reader is referred to [CC01], theorem 2.

Example 1.5.2 As an analogue of the preceding example we consider the difference equation

$$y(x+1) + y(x)y(x+1) = e^{-1}y(x) + x^{-2} \tag{1.5.12}$$

for x in a neighbourhood of ∞. Again we will only consider complex valued solutions y such that $y(x) \to 0$ as $x \to \infty$. Notice that the linearised equation $z(x+1) = e^{-1}z(x)$ now has as general holomorphic solution $z(x) = \beta(x)e^{-x}$ with β an arbitrary 1-periodic holomorphic function, which is the analogue of the general solution of the linearised differential equation in the preceding example. This is the reason to put e^{-1} in front of $y(x)$ in the right-hand side of (1.5.12).

If we write $y(x) = \frac{f(x+1)}{f(x)} - 1$ for some function f (compare [MT51], p. 346), then a straightforward calculation shows that f satisfies the following second order linear difference equation

$$f(x+2) = (e^{-1}+1)f(x+1) + (x^{-2} - e^{-1})f(x). \tag{1.5.13}$$

As in the preceding example we first try to find formal solutions $\hat{f}_0(x) \in \mathbb{C}[[x^{-1}]]$ and $\hat{g}_0(x) = e^{-x}\hat{f}_1(x)$, with $\hat{f}_1 \in \mathbb{C}[[x^{-1}]]$.

For $a \in \mathbb{C}$ and $m \in \mathbb{N}_+$ we have $(a+x)^{-m} = x^{-m}(1+ax^{-1})^{-m} = \sum_{k=0}^{\infty}\binom{-m}{k}a^k x^{-m-k}$. So substituting the series $\sum_{m=0}^{\infty} c_m x^{-m}$ in (1.5.13) and comparing coefficients of x^{-m}, $m \geqslant 0$, we find the relations

$$\begin{cases} c_0 &= (e^{-1}+1)c_0 - e^{-1}c_0 \\ c_1 &= (e^{-1}+1)c_1 - e^{-1}c_1 \\ \sum_{k=1}^{m}\binom{-k}{m-k}2^{m-k}c_k &= \sum_{k=1}^{m}\binom{-k}{m-k}(1+e^{-1})c_k - e^{-1}c_m + c_{m-2}, \quad m \geqslant 2. \end{cases}$$

[6]If σ is a singular direction (in our case $\sigma = 0$), then $-\sigma \pm \pi/2$ is called a *Stokes direction* (in our case the imaginary axis). In [CC01] however, Costin and Costin use the term *anti-Stokes direction* for such a half line.

The first two equations are trivially satisfied and the third equation is equivalent to

$$\sum_{k=1}^{m-1} \binom{-k}{m-k} 2^{m-k} c_k = \sum_{k=1}^{m-1} \binom{-k}{m-k} (1+e^{-1}) c_k + c_{m-2}, \tag{1.5.14}$$

which for $m = 2$ reduces to $-2c_1 = -(1+e^{-1})c_1 + c_0$. As in the preceding example we prescribe $c_0 := 1$. Then c_1 is determined uniquely. Now assume that c_m has been found for $m \in \{0, 1, \ldots, \ell - 1\}$ for some $\ell \geqslant 2$, then c_ℓ can be determined uniquely from (1.5.14) with $m = \ell + 1$.

To find the other formal solution we first observe that $\hat{f}_1$ satisfies

$$f_1(x+2) = (1+e) f_1(x+1) + \big(e^2 x^{-2} - e\big) f_1(x) \tag{1.5.15}$$

and in a similar way as before we can construct a unique formal solution $\hat{f}_1 \in \mathbb{C}^n[[x^{-1}]]$ of (1.5.15), if we prescribe its constant term to be equal to 1. Hence, $\alpha(x)\hat{f}_0(x) - \beta(x) e^{-x} \hat{f}_1(x)$, with α and β holomorphic 1-periodic functions, also is a formal solution of (1.5.13). In the following we assume that α is invertible. Then without loss of generality we can assume that $\alpha \equiv 1$ and

$$\hat{y}(x) = \frac{\hat{f}_0(x+1) - e^{-1} z(x) \hat{f}_1(x+1)}{\hat{f}_0(x) - z(x)\hat{f}_1(x)} - 1$$

is a formal solution of (1.5.12). By construction the formal transformation

$$y = \hat{P}(x,z) := \frac{\hat{f}_0(x+1) - e^{-1} z \hat{f}_1(x+1)}{\hat{f}_0(x) - z\hat{f}_1(x)} - 1$$

transforms the difference equation (1.5.12) into the normal form $z(x+1) = e^{-1} z(x)$ and, as in the preceding example, the expression $\hat{P}(x,z)$ can formally be expanded as

$$\hat{P}(x,z) = \sum_{k=0}^{\infty} \hat{y}_k(x)\, z^k,$$

with $\hat{y}_0(x) = \frac{\hat{f}_0(x+1)}{\hat{f}_0(x)} - 1$ and $\hat{y}_k(x) = \Big(\frac{\hat{f}_1(x)}{\hat{f}_0(x)}\Big)^k \Big(\frac{\hat{f}_0(x+1)}{\hat{f}_0(x)} - e^{-1}\frac{\hat{f}_1(x+1)}{\hat{f}_1(x)}\Big)$ for $k \geqslant 1$. When we substitute the solution $z(x) = \beta(x) e^{-x}$ of the normal form $z(x+1) = e^{-1} z(x)$ into the expansion of $\hat{P}(x,z)$, we obtain a transseries solution of (1.5.12):

$$\hat{y}(x) = \sum_{k=0}^{\infty} \beta^k(x) e^{-kx} \hat{y}_k(x). \tag{1.5.16}$$

To show the Borel summability of for example $\hat{f}_0$, we put $F(x) := \big(f_0(x), f_0(x+1)\big)^t$ and we see that F satisfies

$$F(x+1) = A(x)F(x), \qquad \text{with } A(x) = \begin{pmatrix} 0 & 1 \\ x^{-2} - e^{-1} & e^{-1} + 1 \end{pmatrix}.$$

The constant matrix $A(\infty)$ has two different eigenvalues, namely $\lambda = e^{-1}$ and $\lambda = 1$. If Q is a matrix with the property $QA(\infty)Q^{-1} = \operatorname{diag}\{1, e^{-1}\}$, then $\tilde{F} := QF$ satisfies $\tilde{F}(x+1) = QA(x)Q^{-1}\tilde{F}(x)$, which is in the form that Braaksma considered in [Bra80]. Using theorem 2 and 3 from this article one may show that $\hat{f}_0$ is Borel summable in every direction $-\theta$, with $\theta \neq \pm\frac{\pi}{2}, \arg(1 + 2l\pi i)$, $l \in \mathbb{Z}$ (cf. also [Hor16, Hor18]). In a similar way one might observe that $\hat{f}_1$ is Borel summable in every direction $-\theta$, with $\theta \neq \pm\frac{\pi}{2}, \arg(-1 + 2l\pi i)$, $l \in \mathbb{Z}$. Note that if f_0 and f_1 are the Borel sums of $\hat{f}_0$ and $\hat{f}_1$ respectively, then in particular f_0 exists and is holomorphic in a neighbourhood of ∞ in $S(\pi, 2\pi)$ and f_1 exists and is holomorphic in a neighbourhood of ∞ in $S(0, 2\pi)$.

In the following S denotes the sector $S(\pi/2, \pi)$. Then both f_0 and f_1 are holomorphic in a neighbourhood of ∞ in S and $f_0(x) \sim \hat{f}_0(x)$ and $f_1(x) \sim \hat{f}_1(x)$ as $x \to \infty$ in S. Hence, the *transformation*

$$y = P(x, z) := \frac{f_0(x+1) - e^{-1}zf_1(x+1)}{f_0(x) - zf_1(x)} - 1$$

is well defined for x in a neighbourhood of ∞ in S and z in a neighbourhood of 0 and transforms the difference equation (1.5.12) into the normal form $z(x+1) = e^{-1}z(x)$. Moreover, if we define

$$y(x) := \frac{f_0(x+1) - e^{-1}\beta(x)e^{-x}f_1(x+1)}{f_0(x) - \beta(x)e^{-x}f_1(x)} - 1, \tag{1.5.17}$$

then y is an actual *solution* of (1.5.12), which is holomorphic in a neighbourhood of ∞ in the region $R := S \backslash \{x \in \mathbb{C}^* \mid f_0(x) = \beta(x)e^{-x}f_1(x)\}$.

As both $\hat{f}_0$ and $\hat{f}_1$ are Borel summable, the same holds for each $\hat{y}_k$ and their Borel sums

$$y_0(x) = \frac{f_0(x+1)}{f_0(x)} - 1 \quad \text{and} \quad y_k(x) = \left(\frac{f_1(x)}{f_0(x)}\right)^k \left(\frac{f_0(x+1)}{f_0(x)} - e^{-1}\frac{f_1(x+1)}{f_1(x)}\right), \quad k \geqslant 1,$$

are holomorphic in a neighbourhood of ∞ in S. However, as in the preceding example, this does not guarantee convergence of the corresponding transseries. Again one can prove that the formal series $\hat{P}(x, z)$ is Borel summable with respect to x, *provided that* $|z|$ *is small enough*, and then its sum equals $\sum_{k=0}^{\infty} y_k(x)z^k$, which is shown by Braaksma in [Bra01]. Hence, we only are allowed to write the solution y, as given in (1.5.17), in the form of a convergent transseries, if $\beta(x)e^{-x}$ decreases as $x \to \infty$.

Now let us assume that $\beta(x)e^{-x}$ indeed decreases to 0 as $x \to \infty$ in S, and thus in every sub-sector S_1 of S, and assume that the Fourier expansion of β is given by $\sum_{h=-\infty}^{\infty} \beta_h e^{2\pi i h x}$. Then

$$\beta_h = \int_0^1 \beta(x+t)e^{-(x+t)}e^{(1-2\pi i h)(x+t)}\mathrm{d}t,$$

if $[x, x+1] \in S_1$. Let us write $S_1 = \{x \in \mathbb{C}^* \mid \varphi_- < \arg x < \varphi_+\}$, with $0 < \varphi_- < \varphi_+ < \pi$. The integral above tends to 0 as $x \to \infty$, $\arg x = \varphi$, if $\cos\varphi + 2\pi h \sin\varphi < 0$. Now, let h_- be equal to $\frac{-1}{2\pi}\cot\varphi_-$ if $\frac{-1}{2\pi}\cot\varphi_- \in \mathbb{Z}$ or the smallest integer larger than $\frac{-1}{2\pi}\cot\varphi_-$ if

$\frac{-1}{2\pi}\cot\varphi_- \notin \mathbb{Z}$. Then $\cos\varphi + 2\pi h\sin\varphi < 0$ holds for all $\varphi \in (\varphi_-, \varphi_+)$ if $h < h_-$. Hence, $\beta_h = 0$ if $h < h_-$ and we see that β necessarily is of the form

$$\beta(x) = \sum_{h=h_-}^{\infty} \beta_h e^{2\pi i h x} = \beta_{h_-} e^{2\pi i h_- x}\left\{1 + \sum_{h=h_-+1}^{\infty} \beta_{h_-}^{-1}\beta_h e^{2\pi i(h-h_-)x}\right\}$$

and this latter sum is exponentially small on S. So $\beta(x) = \beta_{h_-}e^{2\pi i h_- x}\big(1 + O(x^{-1})\big)$ as $x \to \infty$ in S.

From (1.5.17) we conclude that y might be singular at those points $x \in \mathbb{C}$ satisfying the equation $f_0(x) = \beta(x)e^{-x}f_1(x)$ and this equation can be rewritten as

$$x = \ln\big(\beta(x)\big) + \ln\big(f_1(x)f_0^{-1}(x)\big) + 2n\pi i, \quad n \in \mathbb{Z}. \tag{1.5.18}$$

In the following we restrict x to S, because there we know the asymptotic behaviour of f_0, f_1 and β. Observing that $\ln\big(\beta(x)\big) = \ln\beta_{h_-} + 2\pi i h_- x + O(x^{-1})$ and $\ln\big(f_1(x)f_0^{-1}(x)\big) = O(x^{-1})$, one may show (in a similar way as in the preceding example) that (1.5.18) has a unique solution $x = x_n$ for every $n \in \mathbb{N}$ large enough. As in the preceding example we obtain

$$x_n = \frac{\ln\beta_{h_-}}{1-2\pi i h_-} + \frac{2n\pi i}{1-2\pi i h_-} + O(n^{-1}), \quad n \in \mathbb{N}, \quad n \to \infty. \tag{1.5.19}$$

We conclude that the solution y of equation (1.5.12) might be singular at a distance at most $O(n^{-1})$ of $(1-2\pi i h_-)^{-1}\big(\ln\beta_{h_-} + 2n\pi i\big)$, as $n \in \mathbb{N}$, $n \to \infty$.

We will end this example with the remark that we are not really restricted to the upper half plane. The same study can be done in for example the lower half plane. In this thesis we will consider sectors containing the positive real axis. Of course this can also be achieved here by taking the Borel sum f_0 of $\hat{f}_0$ on $\{x \in \mathbb{C}^* \mid -\pi/2 - \theta_+ < \arg x < \pi/2 - \theta_-\}$, with θ_- and θ_+ two consecutive singular directions in the right half plane. In that case we can prove that if the 1-periodic function β has the property $\beta(x)e^{-x} \to 0$ as $x \to \infty$, then β necessarily must be a trigonometric polynomial (cf. [Bra01] for more precise results).

Chapter 2

Resurgence Properties for Difference Equations

2.1 Introduction

In this chapter we will consider difference equations in the following prepared form

$$y(x+1) = \Lambda(x)y(x) + g(x, y(x)), \tag{2.1.1}$$

where

$$\Lambda(x) = \bigoplus_{j=1}^{r} e^{-\mu_j}(1 + x^{-1})^{\mathtt{M}_j}. \tag{2.1.2}$$

Here $r \in \mathbb{N}_+$, $n = n_1 + n_2 + \cdots + n_r$, $n_j \in \mathbb{N}$, and $\mathtt{M}_j$ is an $n_j \times n_j$-diagonal matrix with complex numbers a_m, $m = n_1 + \cdots + n_{j-1} + 1, \ \ldots \ , n_1 + \cdots + n_j$, on the diagonal.

We assume g to be a holomorphic $\mathbb{C}^n$-valued function of (x, y) in a neighbourhood of $(\infty, 0)$, such that $g(x, y) = O(x^{-2}) + O(|y|^2)$ as $x \to \infty$ and $y \to 0$. We assume μ_j, $j = 1, 2, \ldots, r$, to be complex numbers such that

$$\mu_j \not\equiv (k_1\mu_1 + k_2\mu_2 + \cdots + k_r\mu_r) \bmod 2\pi i, \quad \text{for every} \ (k_1, k_2, \ldots, k_r) \in \mathbb{N}^r \backslash \{\mathbf{e}_j\}, \tag{2.1.3}$$

and we assume the existence of a positive integer $r_1 \leqslant r$ such that

$$\Re\mu_j > 0, \ j \in \{1, 2, \ldots, r_1\} \quad \text{and} \quad \Re\mu_j \leqslant 0, \ j \in \{r_1 + 1, r_1 + 2, \ldots, r\}. \tag{2.1.4}$$

If we define $\mathcal{J}_j := \{n_1 + \cdots + n_{j-1} + 1, \ldots, n_1 + \cdots + n_j\}$, $j \in \{1, 2, \ldots, r\}$, then we finally assume that $a_k - a_l \notin \mathbb{Z}\backslash\{0\}$ if $k, l \in \mathcal{J}_j$, $j = 1, 2, \ldots, r$. The requirement on $\mathtt{M}_j$, $j \in \{1, 2, \ldots, r\}$, being in diagonal form will be generalised in chapter 4.

Like in the introduction of chapter 1 we introduce an algebra $\mathcal{U}$ consisting of formal expressions $\sum_{j=0}^{\infty} f_j(x)\, e^{-\sigma_j x}$ satisfying the following two properties.

(i) Every f_j can be written as $f_j = p_j g_j$ with $g_j \in \mathbb{C}^n((x^{-1}))[\{x^c\}_{c\in\mathbb{C}}]$ and p_j a 1-periodic $\mathbb{C}$-valued function of x;

(ii) Every σ_j belongs to $\mathbb{N} \cdot \mu_1 + \mathbb{N} \cdot \mu_2 + \cdots + \mathbb{N} \cdot \mu_r$.

With the algebra $\mathcal{U}$ we associate the space

$$\mathrm{FSol}(\Delta) := \{y = (y_1, y_2, \ldots, y_n) \in \mathcal{U}^n \mid y \text{ is a solution of } (2.1.1)\}.$$

Given the difference equation (2.1.1), it is easy to construct a (unique) formal power series solution $\hat{y}_0(x) := \sum_{m=1}^{\infty} \alpha_m x^{-m} \in x^{-1}\mathbb{C}^n[[x^{-1}]]$. This is simply done by substituting such a series in the equation and deriving a recurrence relation for the coefficients α_m, $m \in \{1, 2, 3, \ldots\}$ (compare examples 1.5.1 and 1.5.2). As $\Lambda(\infty)$ does not possess the eigenvalue 1, this recurrence relation can be solved uniquely and it turns out that $\alpha_1 = 0$. Hence, there exists a unique formal solution $\hat{y}_0 \in x^{-2}\mathbb{C}[[x^{-1}]]$ of (2.1.1) and obviously this solution belongs to $\mathrm{FSol}(\Delta)$.

With the nonlinear equation (2.1.1) one associates the normal form

$$z(x+1) = \Lambda(x) z(x). \tag{2.1.5}$$

If we denote the linear vector space of its formal solutions in $\mathcal{U}$ by $\mathrm{FSol}(\Delta_{norm})$, then it is easily seen that $\mathrm{FSol}(\Delta_{norm})$ can be written as

$$\left\{ \sum_{j=1}^{r} e^{-\mu_j x} \sum_{h \in \mathcal{J}_j} C_h(x)\, x^{a_h}\, \mathbf{e}_h \mid C_h \text{ is } \mathbb{C}\text{-valued and 1-periodic for all } h = 1, 2, \ldots, n \right\}.$$

The general solution of (2.1.5) in the class of holomorphic functions on the Riemann surface of the logarithm is obtained by replacing each C_h by an arbitrary 1-periodic holomorphic function.

First we will consider a formal transformation $y = \hat{T}(x, z)$ that formally transforms the nonlinear difference equation into its normal form and we require the transformation to be of the form

$$y = \hat{T}(x, z) := \sum_{\mathbf{k} \in \mathbb{N}^n} \hat{y}_{\mathbf{k}}(x)\, z^{\mathbf{k}} =: \hat{y}_0(x) + \hat{P}(x, z), \tag{2.1.6}$$

with $\hat{y}_{\mathbf{k}}$, $\mathbf{k} \succ 0$, a formal series in $\mathbb{C}^n[[x^{-1}]]$. In section 2.2 we will prove that the formal transformation $\hat{T}$ does exist and $\hat{T}$ is unique if one normalises the formal series $\hat{y}_{\mathbf{e}_j}$ by $\hat{y}_{\mathbf{e}_j}(\infty) = \mathbf{e}_j$, $j = 1, 2, \ldots, n$. Now, when we substitute a general element of $\mathrm{FSol}(\Delta_{norm})$ into the formal transformation one obtains the following *formal integral* of the difference equation (cf. [Eca85, CNP93])

$$\hat{y}(x) = \sum_{\mathbf{k} \in \mathbb{N}^n} C^{\mathbf{k}}(x)\, e^{-\langle \mathbf{k}, \boldsymbol{\mu} \rangle x}\, x^{\langle \mathbf{k}, \mathbf{a} \rangle}\, \hat{y}_{\mathbf{k}}(x). \tag{2.1.7}$$

Here C denotes the vector with components C_h, $1 \leqslant h \leqslant n$, $\boldsymbol{\mu} = \sum_{j=1}^{r} \mu_j \sum_{h \in \mathcal{J}_j} \mathbf{e}_h$ and $\mathbf{a} = (a_1, a_2, \ldots, a_n)$. The right-hand side of (2.1.7) is an example of a so-called *transseries* (compare [Eca92]).

The next problem is to associate a holomorphic transformation T with the formal transformation $\hat{T}$. For the class of equations we consider in the present chapter, this problem is tackled by Braaksma in [Bra01] and is summarised in section 2.3 of the thesis. This lifting of $\hat{T}$ to a holomorphic expression amounts to the association of holomorphic functions $y_{\mathbf{k}}$ with the constructed formal expressions $\hat{y}_{\mathbf{k}}$. In [Bra01] it is shown that each $\hat{y}_{\mathbf{k}}$ is Borel summable and their Borel sums $y_{\mathbf{k}}$ are holomorphic in a neighbourhood of ∞ in a certain sector S_1 of opening larger than π and containing the positive real axis. However, this does not automatically ensure convergence of the series $\sum_{\mathbf{k}\in\mathbb{N}^n} C^{\mathbf{k}}(x)\, e^{-\langle\mathbf{k},\boldsymbol{\mu}\rangle x}\, x^{\langle\mathbf{k},\mathbf{a}\rangle}\, y_{\mathbf{k}}(x)$. It is shown in [Bra01] that the formal series

$$\hat{T}_1(x,u) := \hat{T}(x,u) \quad \text{with } u = \sum_{j=1}^{r_1} \sum_{h\in\mathcal{J}_j} z_h \mathbf{e}_h,$$

is Borel summable with respect to x on S_1, uniformly in u (provided that $|u|$ is small enough). Consequently, the Borel sum $T_1(x,u)$ of $\hat{T}_1(x,u)$ exists for x in a neighbourhood of ∞ in S_1 and u in a neighbourhood of 0 and the original difference equation, restricted to the manifold defined by $y = T_1(x,u)$, transforms into the so-called *semi-canonical form* $u(x+1) = \Lambda(x)u(x)$. However, the main result in [Bra01] is that given a solution y of (2.1.1) and given a sector $S_2 \subset S_1$ containing the positive real axis such that y is 'small' on S_2, there exists a unique convergent transseries such that y equals the sum of this transseries (compare theorem 2.3.6).

The main focus of this chapter will be on the behaviour of the Borel transform $\tilde{Y}_{\mathbf{k}}$ of $x^{\langle\mathbf{k},\tilde{\mathbf{a}}\rangle}\hat{y}_{\mathbf{k}}$ near (or on) the so-called singular rays. Here the factor $x^{\langle\mathbf{k},\tilde{\mathbf{a}}\rangle}$ is put in front of $\hat{y}_{\mathbf{k}}$ just for convenience. Under some additional assumptions we will show in section 2.4 that the behaviour of $Y_0 := \tilde{Y}_0$ near each μ_j might be quite nasty in the sense that Y_0 might not be integrable near μ_j. Hence, the classical Laplace transform of Y_0, with integration along the half line with direction $\arg \mu_j$, may not be defined and thus we might not be able to transform Y_0 back into a solution of the original difference equation. This is one of the reasons to introduce *staircase distributions* in section 2.5.1. The theory of these distributions is due to Costin (cf. [Cos98]) and is described in detail in appendix A. Using these staircase distributions we will prove in section 2.5.3 that each $\tilde{Y}_{\mathbf{k}}$ can be extended to singular rays $\arg t = \arg \mu_j$ in the sense of distributions and by means of a generalised Laplace transform, with integration along a singular half line, one can transform these distributions into solutions of the corresponding difference equations (compare proposition 2.5.11).

In section 2.6 we will derive so-called *resurgence relations* between the expressions $\tilde{Y}_{\mathbf{k}}$. These resurgence relations have been first discovered by Écalle in a much more general context. They describe in a certain sense the behaviour of $\tilde{Y}_{\mathbf{k}}$ near singular points *different* from the origin by the behaviour of other $\tilde{Y}_{\mathbf{k}'}$, $\mathbf{k}' \succ \mathbf{k}$, *near* the origin. In [Cos95, Cos98] Costin derived such relations for rank 1 differential equations and Braaksma showed some particular relations for the analogous difference equation in [Bra01], using sectorial Laplace integrals. The latter resurgence relations will be generalised in section 2.6 of the thesis, by means of the method Costin used in [Cos98]. In fact, we first derive the existence of a decomposition of $\tilde{Y}_{\mathbf{k}}$ on a singular half line (lemma 2.6.2). Using this decomposition and the extended Laplace transform on the space of staircase distributions we will derive the resurgence relations (proposition 2.6.8).

Let S be some sector containing the positive real axis and let y be a 'small' solution of (2.1.1) on S. Section 2.7 then is concerned with the two possible convergent transseries representations of y that both hold on S and that depend on the sector $S_1 \supset S$ in which the Borel sums of each of the $\hat{y}_{\mathbf{k}}$'s are taken. We will deduce a relation between two 1-periodic functions in the convergent transseries corresponding to y for two choices of $S_1 \supset S$ and this relation is called the *Stokes transition*.

Finally, in section 2.8 we give some additional results among which the so-called *balanced averages* (cf. [Eca85, Cos98]). Using these balanced averages we will show that if the difference equation (2.1.1) in a certain sense is real, then it possesses certain transseries solutions that are real when restricted to the positive real axis and these transseries can be summed to actual solutions y that are real-valued on the positive real axis. In section 2.8.2 we will determine the asymptotic behaviour of the coefficients α_m of $\hat{y}_0$ as $m \to \infty$ using some particular resurgence relations.

2.2 Formal Reduction to a Normal Form

In this section we will show that there indeed exists a formal transformation $\hat{T}$ that transforms the difference equation (2.1.1) into the normal form (2.1.5). As already mentioned we require the formal transformation to be of the form

$$y = \hat{T}(x,z) = \sum_{\mathbf{k}\in\mathbb{N}^n} \hat{y}_{\mathbf{k}}(x)\, z^{\mathbf{k}} = \hat{y}_0(x) + \hat{P}(x,z),$$

with $\hat{y}_{\mathbf{k}}$, $\mathbf{k} \succ 0$, a formal series in $\mathbb{C}^n[[x^{-1}]]$ and our aim is to actually compute each $\hat{y}_{\mathbf{k}}$. As g is holomorphic in a neighbourhood of $(\infty, 0) \in \mathbb{C} \times \mathbb{C}^n$, it can be expanded as

$$g(x,y) = \sum_{\mathbf{l}\in\mathbb{N}^n} g_{\mathbf{l}}(x) y^{\mathbf{l}}, \tag{2.2.1}$$

for $|x|$ large enough and $|y|$ small enough. The assumptions on g then imply $g_{\mathbf{l}}(x) = O(x^{-2})$ as $x \to \infty$, if $|\mathbf{l}| \leqslant 1$. Let us define $d(x,w) := g(x, \hat{y}_0 + w) - g(x, \hat{y}_0) - D_y g(x, \hat{y}_0)\, w$, so

$$d(x,w) = \sum_{\mathbf{j}\in\mathbb{N}_2^n} d_{\mathbf{j}}(x) w^{\mathbf{j}}, \quad d_{\mathbf{j}}(x) := \sum_{\mathbf{l}\succeq\mathbf{j}} \binom{\mathbf{l}}{\mathbf{j}} g_{\mathbf{l}}(x) \hat{y}_0^{\mathbf{l}-\mathbf{j}}(x) \text{ if } \mathbf{j} \in \mathbb{N}^n, \tag{2.2.2}$$

where $\binom{\mathbf{l}}{\mathbf{j}} = \prod_{m=1}^n \binom{l_m}{j_m}$.

Now if (2.1.5) holds, then for $\mathbf{k} \in \mathbb{N}^n$ we have $z^{\mathbf{k}}(x+1) = e^{-\langle \mathbf{k}, \boldsymbol{\mu}\rangle}(1+x^{-1})^{\langle \mathbf{k}, \mathbf{a}\rangle} z^{\mathbf{k}}(x)$. Therefore the requirement that (2.1.6) reduces the difference equation (2.1.1) to the normal form (2.1.5) translates into

$$\sum_{\mathbf{k}\neq 0} \left\{ e^{-\langle \mathbf{k}, \boldsymbol{\mu}\rangle}(1+x^{-1})^{\langle \mathbf{k}, \mathbf{a}\rangle} \hat{y}_{\mathbf{k}}(x+1) - \Lambda_1(x)\, \hat{y}_{\mathbf{k}}(x) \right\} z^{\mathbf{k}} = d\big(x, \hat{P}(x,z)\big),$$

where

$$\Lambda_1(x) := \Lambda(x) + \operatorname{col}\{d_{\mathbf{e}_1}(x), d_{\mathbf{e}_2}(x), \ldots, d_{\mathbf{e}_n}(x)\}. \tag{2.2.3}$$

Thus $\Lambda_1(x) - \Lambda(x) = D_y g(x, \hat{y}_0(x)) = O(x^{-2})$ as $x \to \infty$.

By comparing coefficients of $z^{\mathbf{k}}$, $\mathbf{k} \succ 0$, it follows that $\hat{y}_{\mathbf{k}}$ has to be a formal solution of the following linear equation

$$e^{-\langle \mathbf{k}, \boldsymbol{\mu} \rangle} (1 + x^{-1})^{\langle \mathbf{k}, \mathbf{a} \rangle} y(x+1) = \Lambda_1(x)\, y(x) + t_{\mathbf{k}}(x), \quad \mathbf{k} \neq 0, \tag{2.2.4}$$

where $t_{\mathbf{k}}$ is a polynomial in $\hat{y}_{\mathbf{k}'}$ with $\mathbf{k}' \prec \mathbf{k}$. More precisely,

$$t_{\mathbf{k}}(x) = \sum_{2 \leqslant |\mathbf{j}| \leqslant |\mathbf{k}|} d_{\mathbf{j}}(x) \sum_{(\mathbf{i}_{mp};\mathbf{k})} \prod_{m=1}^{n} \prod_{p=1}^{j_m} (\hat{y}_{\mathbf{i}_{mp}})_m(x), \tag{2.2.5}$$

where $\sum_{(\mathbf{i}_{mp};\mathbf{k})}$ denotes the sum over all $\mathbf{i}_{mp} \succ 0$ with $1 \leqslant p \leqslant j_m$, $1 \leqslant m \leqslant n$ and $\sum_{m=1}^{n} \sum_{p=1}^{j_m} \mathbf{i}_{mp} = \mathbf{k}$. The definition of $t_{\mathbf{k}}$ implies that it satisfies the following homogeneity condition

$$t_{\mathbf{k}}\big(\{C^{\mathbf{k}'} \hat{y}_{\mathbf{k}'}\}_{0 \prec \mathbf{k}' \prec \mathbf{k}}\big) = C^{\mathbf{k}} t_{\mathbf{k}}\big(\{\hat{y}_{\mathbf{k}'}\}_{0 \prec \mathbf{k}' \prec \mathbf{k}}\big)$$

for any $C \in \mathbb{C}^n$. Moreover, from (2.2.5) it is clear that $t_{\mathbf{k}} = 0$ if $|\mathbf{k}| = 1$.

Proposition 2.2.1 *For each $\mathbf{k} \succ 0$ the difference equation (2.2.4) possesses a formal solution $\hat{y}_{\mathbf{k}} \in \mathbb{C}^n[[x^{-1}]]$. For $j \in \{1, 2, \ldots, n\}$ the series $\hat{y}_{\mathbf{e}_j}$ is unique if we prescribe the constant term in $\hat{y}_{\mathbf{e}_j}$ to be equal to $\mathbf{e}_j$. Then also the series $\hat{y}_{\mathbf{k}}$ for $|\mathbf{k}| > 1$ are unique.*

Proof. The proof is given with induction on $|\mathbf{k}|$. For $|\mathbf{k}| = 1$, say $\mathbf{k} = \mathbf{e}_1$, we substitute a representing series $\sum_{m=0}^{\infty} y_{\mathbf{e}_1,m} x^{-m}$ for $\hat{y}_{\mathbf{e}_1}$ into the corresponding equation

$$e^{-\mu_1} y(x+1) = (1 + x^{-1})^{-a_1} \Lambda(x) y(x) + \sum_{j=1}^{n} (1 + x^{-1})^{-a_1} d_{\mathbf{e}_j}(x)\, y_j(x).$$

Now, $(1 + x^{-1})^{-a_1} \Lambda(x)$ can be expanded as $\hat{\Lambda}_0 + \hat{\Lambda}_1 x^{-1} + O(x^{-2})$, with $\hat{\Lambda}_0 = \oplus_{j=1}^{r} e^{-\mu_j} \mathtt{I}_{n_j}$ and $\hat{\Lambda}_1 = \oplus_{j=1}^{r} e^{-\mu_j} (\Lambda_j - a_1 \mathtt{I}_{n_j})$. Moreover, one should observe that $d_{\mathbf{e}_j}(x) = O(x^{-2})$ for each $j \in \{1, 2, \ldots, n\}$. Comparing coefficients of x^{-m}, $m \geqslant 0$, then gives the following recurrence relation:

$$\begin{cases} e^{-\mu_1}\, y_{\mathbf{e}_1,0} &= \hat{\Lambda}_0\, y_{\mathbf{e}_1,0} \\ e^{-\mu_1} \{ y_{\mathbf{e}_1,m} - (m-1) y_{\mathbf{e}_1,m-1} \} &= \hat{\Lambda}_0 y_{\mathbf{e}_1,m} + \hat{\Lambda}_1 y_{\mathbf{e}_1,m-1} + \varphi_m(y_{\mathbf{e}_1,0}, \ldots, y_{\mathbf{e}_1,m-2}),\ m \geqslant 1, \end{cases}$$

where $\varphi_m(y_{\mathbf{e}_1,0}, \ldots, y_{\mathbf{e}_1,m-2})$ is an expression only depending on $y_{\mathbf{e}_1,k}$, $0 \leqslant k \leqslant m-2$. The equations above can be solved step by step, by prescribing $y_{\mathbf{e}_1,0} := \mathbf{e}_1$, since from the second equation with $m = 1$ we deduce that the last $n - n_1$ components of $y_{\mathbf{e}_1,1}$ equal 0, while the first n_1 components of $y_{\mathbf{e}_1,1}$ are undetermined yet. From this equation it also is clear that at least $(y_{\mathbf{e}_1,0})_1$ is a priori free to choose, but it is given the value 1 by prescription. The second equation with $m = 2$ then gives uniquely both the first n_1 components of $y_{\mathbf{e}_1,1}$ and the last $n - n_1$ components of $y_{\mathbf{e}_1,2}$. We can proceed in this way and find all the coefficients $y_{\mathbf{e}_1,m}$.

Next suppose that $\hat{y}_{\mathbf{k}}$ is constructed for all multi-indices $\mathbf{k} \in \mathbb{N}^n$ with $1 \leqslant |\mathbf{k}| \leqslant \ell$ for some natural number $\ell \geqslant 1$, then take $\mathbf{k} \in \mathbb{N}_1^n$ with $|\mathbf{k}| = \ell + 1$. Obviously $t_{\mathbf{k}} \in \mathbb{C}^n[[x^{-1}]]$ and as $e^{\langle \mathbf{k}, \boldsymbol{\mu} \rangle} \hat{\Lambda}_0$ does not have eigenvalue 1 it follows that (2.2.4) possesses a unique formal solution $\hat{y}_{\mathbf{k}}$ in $\mathbb{C}^n[[x^{-1}]]$. ■

2.3 Analytic Reduction on a Manifold

In this section we will summarise the results Braaksma obtained in [Bra01]. To that end we put $p := \sum_{j=1}^{r_1} n_j$, where r_1 is as in (2.1.4), and we will make the following identification

$$\mathbb{N}^p \cong \{\mathbf{k} \in \mathbb{N}^n \mid k_j = 0,\ j = p+1, p+2, \ldots, n\}. \tag{2.3.1}$$

With this identification it is clear what is meant by $\hat{y}_\mathbf{k}$, $\mathbf{k} \in \mathbb{N}^p$.

Braaksma's main result is the analytic reduction of (2.1.1) to a semi-canonical form. This reduction to only a semi-canonical form is due to the fact that the formal series $\hat{P}(x,z) = \sum_{\mathbf{k}\neq 0} \hat{y}_\mathbf{k}(x) z^\mathbf{k}$ in general diverges. Therefore he introduces a transformation P which is related to $\hat{P}$ and which transforms (2.1.1) into (2.1.5) on a sub-manifold:

$$y = y_0(x) + P(x,z) := y_0(x) + P_1(x,u) + v, \quad P_1(x,u) := \sum_{\mathbf{k}\in\mathbb{N}_1^p} y_\mathbf{k}(x) u^\mathbf{k}, \tag{2.3.2}$$

where $y_\mathbf{k}$, $\mathbf{k} \in \mathbb{N}^p$, are Borel sums of $\hat{y}_\mathbf{k}$ (cf. section 2.3.1) and where

$$z = \sum_{m=1}^{n} z_m \mathbf{e}_m, \quad u := \sum_{m=1}^{p} z_m \mathbf{e}_m \quad \text{and} \quad v := \sum_{m=p+1}^{n} z_m \mathbf{e}_m.$$

The proofs of the statements made in this section can be found in [Bra01]. For complete proofs in more general settings the reader is referred to the chapters 4 and 5 of this thesis.

2.3.1 Borel Summability of $\hat{y}_\mathbf{k}$

First we will consider the (formal) Borel transform applied to (2.1.1). To that end we define

$$\begin{aligned} \Lambda_0 &:= \Lambda(\infty) = \bigoplus_{j=1}^{r} e^{-\mu_j} \mathrm{I}_{n_j}, \\ A(t) &:= (\hat{\mathcal{B}}\Lambda)(t) =: \operatorname{diag}\{a_1(t), a_2(t), \ldots, a_n(t)\}, \end{aligned}$$

where $a_h(t) = e^{-\mu_j} \sum_{m=1}^{\infty} \binom{a_h}{m} \frac{t^{m-1}}{(m-1)!}$, $h \in \mathcal{J}_j$, $j \in \{1,2,\ldots,r\}$. Moreover, we define

$$g_{\mathbf{l},0} := g_\mathbf{l}(\infty), \quad G_\mathbf{l}(t) := (\hat{\mathcal{B}} g_\mathbf{l})(t), \quad \text{for } \mathbf{l} \in \mathbb{N}^n$$

and for $Y \in \mathbb{C}^n[[t]]$ and $\mathbf{j} \in \mathbb{N}^n$ we define

$$D_\mathbf{j}(Y) := G_\mathbf{j} + \sum_{\mathbf{l}\succ\mathbf{j}} \binom{\mathbf{l}}{\mathbf{j}} \left[g_{\mathbf{l},0} Y^{*(\mathbf{l}-\mathbf{j})} + G_\mathbf{l} * Y^{*(\mathbf{l}-\mathbf{j})} \right], \tag{2.3.3}$$

so that $D_\mathbf{j}(Y_0) = \hat{\mathcal{B}} d_\mathbf{j}$ if $Y_0 = \hat{\mathcal{B}} \hat{y}_0$. Moreover, $D(Y)$ is defined to be the $n \times n$-matrix with columns $D_{\mathbf{e}_j}(Y)$, $j = 1,2,\ldots,n$. Here $Y^{*\mathbf{l}} = Y_1^{*l_1} * Y_2^{*l_2} * \cdots * Y_n^{*l_n}$ and $Y_j^{*l_j}$ denotes the convolution of l_j factors Y_j.

From the properties of g, together with Cauchy's inequality, we deduce the existence of some positive constants K and c such that for all $\mathbf{l} \in \mathbb{N}^n$

$$|g_{\mathbf{l},0}| \leqslant Kc^{|\mathbf{l}|}, \quad \sup_{t\in\mathbb{C}} e^{-c|t|}|G_{\mathbf{l}}(t)| \leqslant Kc^{|\mathbf{l}|} \quad \text{and} \quad \sup_{t\in\mathbb{C}} e^{-c|t|}|A(t)| \leqslant K. \tag{2.3.4}$$

Applying the formal Borel transform to (2.1.1) with $y = \hat{y}_0$ we obtain

$$(e^{-t} - \Lambda_0)Y_0 = A * Y_0 + D_0(Y_0). \tag{2.3.5}$$

Proposition 2.3.1 *The equation (2.3.5) has a unique solution Y_0 that is holomorphic in the maximal star domain $\mathcal{O}$ with centre 0, which does not contain any singular point $\mu_j + 2l\pi i$, $j \in \{1, 2, \ldots, r\}$ and $l \in \mathbb{Z}$. This solution is the sum of the convergent series $\hat{\mathcal{B}}\hat{y}_0$ in a disc around 0. Let θ_0^- and θ_0^+ be two consecutive singular directions, i.e. directions which contain a singular point. If $0 < \varepsilon < \frac{1}{2}(\theta_0^+ - \theta_0^-)$, then there exists a positive constant M such that $Y_0(t) = O(1)e^{M|t|}$ for all $t \in \mathbb{C}^*$ with $\theta_0^- + \varepsilon \leqslant \arg t \leqslant \theta_0^+ - \varepsilon$.*

The formal solution $\hat{y}_0$ is Borel summable in every direction $-\theta$, $\theta \neq \arg(\mu_j + 2l\pi i)$, $j \in \{1, 2, \ldots, r\}$ and $l \in \mathbb{Z}$.

Remark 2.3.2 From this proposition it follows that the formal expression $D_{\mathbf{j}}(Y_0)$, $\mathbf{j} \in \mathbb{N}^n$, corresponds to a holomorphic function in $\mathcal{O}$. This holomorphic function will also be denoted by $D_{\mathbf{j}}(Y_0)$.

In the following we will see that each $\hat{w}_{\mathbf{k}}(x) := x^{-|\mathbf{k}|}\hat{y}_{\mathbf{k}}(x)$ is Borel summable in all but countably many directions. From section 1.3.3 it is clear that this is equivalent to each $\hat{y}_{\mathbf{k}}$ being Borel summable in all but countably many directions. As in [Bra01] we define $\mathbf{b} := \mathbf{a} + \sum_{m=1}^n \mathbf{e}_m$, i.e. each b_m equals $a_m + 1$. From (2.2.4) we then deduce that $\hat{w}_{\mathbf{k}}$ satisfies the equation

$$e^{-\langle \mathbf{k},\boldsymbol{\mu}\rangle}(1 + x^{-1})^{\langle \mathbf{k},\mathbf{b}\rangle}\, w(x+1) = \Lambda_1(x)\, w(x) + u_{\mathbf{k}}(x), \quad \mathbf{k} \neq 0, \tag{2.3.6}$$

where $u_{\mathbf{k}}(x) := x^{-|\mathbf{k}|}t_{\mathbf{k}}(x)$. Defining

$$W_{\mathbf{k}} := \hat{\mathcal{B}}\hat{w}_{\mathbf{k}}, \quad B := A + D(Y_0), \quad U_{\mathbf{k}} := \hat{\mathcal{B}}u_{\mathbf{k}}, \quad \beta_{\mathbf{k}} := \hat{\mathcal{B}}(1 + x^{-1})^{\langle \mathbf{k},\mathbf{b}\rangle} \tag{2.3.7}$$

we obtain, by applying the formal Borel transform to (2.3.6) with $w = \hat{w}_{\mathbf{k}}$,

$$(e^{-t-\langle \mathbf{k},\boldsymbol{\mu}\rangle} - \Lambda_0)W_{\mathbf{k}} \;=\; B * W_{\mathbf{k}} \;-\; e^{-\langle \mathbf{k},\boldsymbol{\mu}\rangle}\beta_{\mathbf{k}} * e^{-t}W_{\mathbf{k}} \;+\; U_{\mathbf{k}}. \tag{2.3.8}$$

Note that $U_{\mathbf{k}} = 0$ if $\mathbf{k}$ has length 1, while for $\mathbf{k} \in \mathbb{N}_2^n$ we have

$$U_{\mathbf{k}} = \sum_{2\leqslant|\mathbf{j}|\leqslant|\mathbf{k}|} \big(g_{\mathbf{j},0} + D_{\mathbf{j}}(Y_0)*\big) \sum_{(\mathbf{i}_{mp};\mathbf{k})} \prod_{m=1}^{n}{}^{*} \prod_{p=1}^{j_m}{}^{*} (W_{\mathbf{i}_{mp}})_m, \tag{2.3.9}$$

where $\prod^*$ means the corresponding convolution product. The definition of $W_{\mathbf{k}}$ and $U_{\mathbf{k}}$ have to be interpreted at first formally, but they are given an analytic meaning recursively in the following proposition (cf. [Cos98], theorem 1, for the case of differential equations).

Proposition 2.3.3 *Let* $\mathbf{k} \in \mathbb{N}^n$, $\mathbf{k} \neq 0$. *Then* $t \mapsto t^{-|\mathbf{k}|+1} W_{\mathbf{k}}(t)$ *exists and is holomorphic in the maximal star domain with centre* 0, *which does not contain any of the singular points* $\mu_j - \langle \mathbf{k}', \boldsymbol{\mu} \rangle + 2l\pi i \neq 0$ *where* $j \in \{1, 2, \ldots, r\}$, $\mathbf{k}' \preceq \mathbf{k}$ *and* $l \in \mathbb{Z}$. *If* $\overline{S}$ *is a closed sector with vertex* 0, *not containing any of those singular points, then there exists a positive constant* M, *which may depend on* $\mathbf{k}$ *and* $\overline{S}$, *such that* $\sup_{t \in \overline{S}} e^{-M|t|} |W_{\mathbf{k}}(t)| < \infty$.

The formal solutions $\hat{w}_{\mathbf{k}}$ *and* $\hat{y}_{\mathbf{k}}$, $|\mathbf{k}| \geqslant 1$, *are Borel summable in every direction* $-\theta$, $\theta \neq \arg\big(\mu_j - \langle \mathbf{k}', \boldsymbol{\mu} \rangle + 2l\pi i\big)$, $j \in \{1, 2, \ldots, r\}$, $\mathbf{k}' \preceq \mathbf{k}$ *and* $l \in \mathbb{Z}$.

Remark 2.3.4 The rays through the (nonzero) singular points $\mu_j - \langle \mathbf{k}', \boldsymbol{\mu} \rangle + 2l\pi i$, where $j \in \{1, 2, \ldots, r\}$, $\mathbf{k}' \preceq \mathbf{k}$ and $l \in \mathbb{Z}$, are called the *singular directions* of $W_{\mathbf{k}}$ and of $Y_{\mathbf{k}} := \hat{\mathcal{B}} \hat{y}_{\mathbf{k}} = W_{\mathbf{k}}^{(|\mathbf{k}|)}$. We will also refer to these singular directions as the singular directions of $\hat{w}_{\mathbf{k}}$ and $\hat{y}_{\mathbf{k}}$.

2.3.2 Reduction on a Manifold

Since $\Re \mu_j > 0$ if $j \in \{1, 2, \ldots, r_1\}$, the number of singular points $\mu_j - \langle \mathbf{k}, \boldsymbol{\mu} \rangle$ in the right half plane, with $j \in \{1, 2, \ldots, r\}$ and $\mathbf{k} \in \mathbb{N}^p$, is finite. Hence, the singular directions of $\hat{y}_{\mathbf{k}}$, $\mathbf{k} \in \mathbb{N}^p$, in the right half plane form a discrete set, so it makes sense to consider consecutive singular directions of $\hat{y}_{\mathbf{k}}$, $\mathbf{k} \in \mathbb{N}^p$.

In the following we take θ_- and θ_+ to be two such consecutive singular directions in the right half plane.

Proposition 2.3.5 *Let* $S := \{t \in \mathbb{C}^* \mid \arg t \in (\theta_-, \theta_+)\}$ *and let* $\overline{S'}$ *be a closed sub-sector of* S. *Moreover, let* $0 < \rho < \min\{|\mu| \mid \mu \in \mathcal{Q}\}$, *where* $\mathcal{Q}$ *is the set of singular points of* $\hat{y}_{\mathbf{k}}$, $\mathbf{k} \in \mathbb{N}^p$, *and define* $V := \overline{S'} \cup \overline{\Delta}(0, \rho)$. *Then there exist positive constants* R *and* K *such that*

$$|W_{\mathbf{k}}(t)| \leqslant K^{|\mathbf{k}|} \frac{|t|^{|\mathbf{k}|-1}}{(|\mathbf{k}|-1)!} e^{R|t|} \tag{2.3.10}$$

for all $\mathbf{k} \in \mathbb{N}^p \backslash \{0\}$ *and all* $t \in V$.

Using (2.3.10) one may prove Borel summability of the formal series $\hat{P}_1(x, u)$ defined in the introduction of this section: if $0 < \varepsilon < \frac{1}{2}(\theta_+ - \theta_-)$ and S_1 is the sector defined by $S_1 := \big\{x \in \mathbb{C}^* \;\big|\; \arg x \in (-\pi/2 - \theta_+ + \varepsilon, \pi/2 - \theta_- - \varepsilon)\big\}$, then there exist positive constants δ and $\tilde{\rho}$, which in general depend on ε, such that $\hat{P}_1(x, u)$ is Borel summable with respect to x on S_1 for $|u| \leqslant \delta$ with sum $P_1(x, u) := \sum_{\mathbf{k} \in \mathbb{N}_1^p} y_{\mathbf{k}}(x) u^{\mathbf{k}}$. This sum converges uniformly for $|u| \leqslant \delta$ if $x \in S_1$, $|x| \geqslant \tilde{\rho}$.

Now, Braaksma's main result reads:

Theorem 2.3.6 *Let* $\mathcal{M}$ *be the manifold defined by* $y = y_0(x) + P_1(x, u)$, *with* x *in a neighbourhood of* ∞ *in* S_1 *and* u *in a neighbourhood of* 0 *in* $\mathbb{C}^p$. *Then on* $\mathcal{M}$ *the difference equation (2.1.1) is transformed into* $u(x+1) = \Lambda(x) u(x)$.

Let S_2 *be a sub-sector of* S_1 *containing the positive real axis and let* y *be a solution of (2.1.1) such that* $y(x) = O(x^{-2})$ *as* $x \to \infty$ *on* S_2. *Then* y *belongs to the restriction of* $\mathcal{M}$

to S_2. Moreover, there exists a unique 1-periodic trigonometric polynomial C, with values in $\mathbb{C}^p$, such that

$$y(x) = \sum_{\mathbf{k}\in\mathbb{N}^p} C^{\mathbf{k}}(x)\, e^{-\langle\mathbf{k},\boldsymbol{\mu}\rangle x}\, x^{\langle\mathbf{k},\mathbf{a}\rangle}\, y_{\mathbf{k}}(x), \quad x \in S_2. \tag{2.3.11}$$

Here $y(x) - y_0(x)$ and $C_h(x)e^{-\mu_j x}$, $h \in \mathcal{J}_j$, $j \in \{1, 2, \dots, r_1\}$, are exponentially small uniformly in a neighbourhood of ∞ in any closed sub-sector of S_2.

Conversely, if C is a p-vector valued 1-periodic trigonometric polynomial such that $C_h(x)e^{-\mu_j x} \to 0$ as $x \to \infty$ in S_2 for every $h \in \mathcal{J}_j$, $j \in \{1, 2, \dots, r_1\}$, then the sum in (2.3.11) converges in a neighbourhood of ∞ in S_2, defines a solution of (2.1.1) which is holomorphic in this neighbourhood and this solution has the property $y(x) \sim \hat{y}_0(x)$ as $x \to \infty$ in S_2.

Remark 2.3.7

1. From the statement just above theorem 2.3.6 it is easily seen that the right-hand side of (2.3.11) converges even in $\{x \in S_1 \mid |C_j(x)e^{-\mu_j x}x^{a_j}| \leqslant \delta,\ j = 1, 2, \dots, p\}\backslash\Delta(0, \tilde{\rho})$, provided that δ is small enough and $\tilde{\rho}$ is large enough. Hence, y can be analytically continued into this region and remains a solution of the original difference equation.

2. The results in this section and the results that follow are also valid in the case that for every $j \in \{1, 2, \dots, r\}$ we assume

   $$\mu_j \not\equiv (k_1\mu_1 + k_2\mu_2 + \cdots + k_{r_1}\mu_{r_1}) \bmod 2\pi i, \quad \text{for every } (k_1, k_2, \dots, k_{r_1}) \in \mathbb{N}^{r_1}\backslash\{\mathbf{e}_j\},$$

   instead of the corresponding assumption made in the introduction of this chapter. In fact, (2.1.3) was only necessary in proving the existence of the formal transformation $y = \hat{T}(x, z)$.

2.4 Properties of Y_0 near Singular Points

From now on we assume that the line-segment $(0, \mu_j)$, $j = 1, 2, \dots, r$, does not contain any singular point $\mu_h + 2l\pi i$ of Y_0, where $h \in \{1, 2, \dots, r\}$ and $l \in \mathbb{Z}$. Later on, in section 2.5.3, we will include this assumption in hypothesis **H**. Moreover, for each $h \in \{1, 2, \dots, r\}$ we assume that $a_k = a_l$ whenever $k, l \in \mathcal{J}_h$ and we assume $\Re a_k \notin \mathbb{Z}$ for all $k \in \{1, 2, \dots, n\}$. Hence, by introducing new quantities a_h, $h = 1, 2, \dots, r$, we can assume that Λ has the form

$$\Lambda(x) = \bigoplus_{h=1}^{r} e^{-\mu_h}(1 + x^{-1})^{a_h \mathrm{I}_{n_h}},$$

with $a_h \in \mathbb{C}$, $\Re a_h \notin \mathbb{Z}$. We have already discussed the existence and uniqueness of a formal solution $\hat{y}_0(x) = \sum_{m=2}^{\infty} \alpha_m x^{-m} \in x^{-2}\mathbb{C}^n[[x^{-1}]]$ of (2.1.1).

Through the transformation $\tilde{y} = y - \sum_{m=2}^{N-1} \alpha_m x^{-m}$ we obtain an equation for $\tilde{y}$ of the form (2.1.1), with the inhomogeneous term $g(x, 0)$ of order $O(x^{-N})$ as $x \to \infty$ for arbitrary $N \in \mathbb{Z}_{\geqslant 1}$. Next take $N_1 > \max\{-\Re a_h \mid 1 \leqslant h \leqslant r\}$ and let N above be equal to $N_1 + 2$, then through the transformation $y^* = x^{N_1}\tilde{y}$ we obtain an equation for y^* of the form (2.1.1) with the properties mentioned before and

$$\Re a_h > 0, \quad h = 1, 2, \ldots, r.$$

Therefore it is sufficient to consider (2.1.1) with this last condition.

In this section we first want to determine the behaviour of Y_0 near the singular points μ_j, $j \in \{1, 2, \ldots, r\}$. After that we will derive a so-called *resurgence relation* (cf. [Eca85]). This relation will be generalised in section 2.6.2. Moreover, the resurgence relation that we will find in this section will help us in determining the asymptotic behaviour of the coefficients α_m of the formal solution $\hat{y}_0$ as $m \to \infty$ (cf. section 2.8.2).

2.4.1 Behaviour of Y_0 near Singular Points

To determine the behaviour of Y_0 near the singular points μ_j, $j \in \{1, 2, \ldots, r\}$, we use a method analogous to the one Costin used in [Cos98], section 2.2.2. To that end we first derive some properties of a function H, which is a truncation of Y_0.

Choose $\varepsilon_j \in (0, |\mu_j|/3)$ so small that the distance of $(0, \mu_j)$ to the singular points of Y_0 different from μ_j is at least $2\varepsilon_j$. Then define the function H by $H(t) := Y_0(t)$ if t belongs to the maximal open subset T of the disc $\Delta(0, |\mu_j| - \varepsilon_j)$, which has distance ε_j to the boundary of the star domain $\mathcal{O}$ as defined in proposition 2.3.1, and $H(t) := 0$ otherwise.

Obviously, the function H is holomorphic in T. Hence, for all multi-indices $\mathbf{l} \succ 0$, the convolution product $H^{*\mathbf{l}}$ is holomorphic in T and satisfies $|H^{*\mathbf{l}}(t)| \leqslant M^{|\mathbf{l}|}|t|^{|\mathbf{l}|-1}/(|\mathbf{l}|-1)!$ for all $t \in T$, where $M = \sup_{t\in T} |H(t)|$. This implies that $D_0(H) := G_0 + \sum_{\mathbf{l}\succ 0} \big[g_{\mathbf{l},0} + G_{\mathbf{l}}*\big] H^{*\mathbf{l}}$ is holomorphic in T. As in [Bra01] we now may prove:

Lemma 2.4.1 *For $j \in \{1, 2, \ldots, r\}$ let ε_j be as above and define $\eta_j := \varepsilon_j e^{i\theta_j}$, where $\theta_j := \arg \mu_j$. If $\mathbf{l} \in \mathbb{N}_2^n$, then $H^{*\mathbf{l}}(\mu_j - t)$ can be extended as a function of t from $(0, \eta_j]$ to a holomorphic function in $\Delta(0, \varepsilon_j)$ with continuous boundary values on the circle $|t| = \varepsilon_j$. The same holds for $D_0(H)(\mu_j - t)$.*

The following result may be derived by means of the method Braaksma used in [Bra01], section 5. This method uses so-called sectorial Laplace integrals. However, we will give another proof, which is related to the work of Costin (cf. [Cos98], proposition 26 (iii)).

Proposition 2.4.2 *Let $j \in \{1, 2, \ldots, r\}$ and let ε_j and η_j be as in lemma 2.4.1. Moreover, assume that $(0, \mu_j)$ does not contain any singular point of Y_0. Then there exist unique complex constants C_h, $h \in \mathcal{J}_j$, and holomorphic functions $u_1, u_2 : \Delta(\mu_j, \varepsilon_j) \to \mathbb{C}^n$, such that Y_0 can be analytically continued on the Riemann surface of $\log(\mu_j - t)$ by*

$$(\mu_j - t)^{-a_j-1} \sum_{h\in\mathcal{J}_j} C_h \mathbf{e}_h + (\mu_j - t)^{-a_j} u_1(t) + u_2(t)$$

for $0 < |\mu_j - t| < \varepsilon_j$. Here the branch of $(\mu_j - t)^{-a_j}$ is chosen which corresponds to $\arg(\mu_j - t) = \theta_j$ if $t \in (\mu_j - \eta_j, \mu_j)$.

PROOF. By renumbering we may assume that $j = 1$. In the following we will omit the index 1 in ε_1 and η_1.

Let us first introduce a natural number m_1 such that $\Re a_1 - m_1 \in (-1, 0)$ (remember that we assumed $\Re a_1 > 0$, $\Re a_1 \notin \mathbb{N}$) and put $\tilde{a}_1 := a_1 - m_1$. Let $\Psi_0(t) := Y_0(t) - H(t)$ and define $\Psi(t) := \Psi_0(\mu_1 - t)$. Then with $t = \mu_1 - s$, $s \in (0, \eta]$, the equation (2.3.5) can be rewritten as

$$(e^{s-\mu_1} - \Lambda_0)\Psi(s) = -\int_\eta^s A(\sigma - s)\Psi(\sigma)\mathrm{d}\sigma - (e^{s-\mu_1} - \Lambda_0)H(\mu_1 - s) + \int_0^{\mu_1-\eta} A(\mu_1 - s - \sigma)H(\sigma)\mathrm{d}\sigma + D_0(H + \Psi_0)(\mu_1 - s).$$

From the definition of D_0 it follows that

$$D_0(H + \Psi_0) - D_0(H) = \sum_{\mathbf{j}\in\mathbb{N}_2^n} g_{\mathbf{j},0}\Psi_0^{*\mathbf{j}} + \sum_{\mathbf{j}\in\mathbb{N}_1^n} D_{\mathbf{j}}(H) * \Psi_0^{*\mathbf{j}} \tag{2.4.1}$$

and it is easily seen that $\Psi_0^{*\mathbf{j}}(t) = 0$ for $t \in (0, \mu_1)$ and $\mathbf{j} \in \mathbb{N}_2^n$, so that

$$D_0(H + \Psi_0)(\mu_1 - s) = D_0(H)(\mu_1 - s) - \int_\eta^s D(H)(\sigma - s)\Psi(\sigma)\mathrm{d}\sigma.$$

The function $D(H)$ is holomorphic in T and, as $H(0) = Y_0(0) = 0$, we have $D(H)(0) = 0$. Hence, the equation for Ψ can be rewritten as

$$(e^{s-\mu_1} - \Lambda_0)\Psi(s) = -\int_\eta^s B(\sigma - s)\Psi(\sigma)\mathrm{d}\sigma + P(s), \tag{2.4.2}$$

where B is as in (2.3.7) and where P is some holomorphic function in $\Delta(0, \varepsilon)$ with continuous boundary values on the circle $|t| = \varepsilon$. The integral equation above is singular at $s = 0$, since the first n_1 components of the left-hand side behave like $s\, e^{-\mu_1}\Psi_h(s)$, $h = 1, 2, \ldots, n_1$, near $s = 0$.

We will solve (2.4.2) in $\{s \in \overline{\Delta}(0, \varepsilon) \mid |\arg s - \theta_1| < \pi\}$, where $\theta_1 = \arg \mu_1$. Put $\psi(s) := (\mathcal{P}_\eta^{m_1+1}\Psi)(s)$, where $\mathcal{P}_\eta$ is defined by $(\mathcal{P}_\eta\Psi)(s) := \int_\eta^s \Psi(\sigma)\mathrm{d}\sigma$. Then the equation (2.4.2) can be written as

$$(e^{s-\mu_1} - \Lambda_0)\psi^{(m_1+1)}(s) = -\int_\eta^s B(\sigma - s)\psi^{(m_1+1)}(\sigma)\mathrm{d}\sigma + P(s). \tag{2.4.3}$$

Our aim is to integrate this equation m_1 times in order to get a first order differential equation, which is more easy to handle. With induction we easily prove that

$$\mathcal{P}_\eta^{m_1}\left[e^\sigma\psi^{(m_1+1)}(\sigma)\right](s) = \sum_{k=0}^{m_1}(-1)^k\binom{m_1}{k}\mathcal{P}_\eta^k\left[e^\sigma\psi'(\sigma)\right](s),$$

where $\mathcal{P}_\eta^0$ denotes the identity operator, and for $k \geqslant 1$ we have

$$\mathcal{P}_\eta^k \left[e^\sigma \psi'(\sigma)\right](s) = \mathcal{P}_\eta^{k-1} \left[e^\sigma \psi(\sigma)\right](s) - \mathcal{P}_\eta^k \left[e^\sigma \psi(\sigma)\right](s),$$

so integrating the left-hand side of (2.4.3) m_1 times from η to s gives

$$e^{s-\mu_1}\psi'(s) - \Lambda_0 \psi'(s) - m_1 e^{s-\mu_1}\psi(s) - e^{-\mu_1} \sum_{k=1}^{m_1} (-1)^k \binom{m_1+1}{k+1} \mathcal{P}_\eta^k \left[e^\sigma \psi(\sigma)\right](s).$$

By integrating by parts m_1 times the integral in the right-hand side of (2.4.3) we obtain

$$\int_\eta^s B(\sigma - s)\psi^{(m_1+1)}(\sigma)\mathrm{d}\sigma \;=\; \sum_{k=0}^{m_1-1} B_k\, \psi^{(m_1-k)}(s) \;+ (-1)^{m_1} \int_\eta^s B^{(m_1)}(\sigma - s)\psi'(\sigma)\mathrm{d}\sigma,$$

where $B_k = (-1)^k B^{(k)}(0)$ and integrating the right-hand side of the latter formula m_1 times we get, by using Fubini's theorem, $\int_\eta^s B(\sigma - s)\psi'(\sigma)\mathrm{d}\sigma$.

As $\mathcal{P}_\eta^k \left[e^\sigma \psi(\sigma)\right](s) = \int_\eta^s \frac{(s-\sigma)^{k-1}}{(k-1)!}\, e^\sigma \psi(\sigma)\mathrm{d}\sigma$ we finally obtain after integrating (2.4.3) m_1 times

$$\begin{aligned}(e^{s-\mu_1} - \Lambda_0)\psi'(s) \;&=\; -\int_\eta^s B(\sigma - s)\psi'(\sigma)\mathrm{d}\sigma \;+\; \mathcal{P}_\eta^{m_1}[P](s) \;+ \\ & m_1 e^{s-\mu_1}\psi(s) + e^{-\mu_1} \sum_{k=1}^{m_1} (-1)^k \binom{m_1+1}{k+1} \int_\eta^s \frac{(s-\sigma)^{k-1}}{(k-1)!}\, e^\sigma \psi(\sigma)\mathrm{d}\sigma.\end{aligned} \tag{2.4.4}$$

However, we still want to write (2.4.4) into a somewhat more convenient form in order to solve the equation using the contraction mapping principle. As $D(H)(0) = 0$ we get, by integration by parts, $\int_\eta^s B(\sigma - s)\psi'(\sigma)\mathrm{d}\sigma = A(0)\psi(s) - \int_\eta^s B'(\sigma - s)\psi(\sigma)\mathrm{d}\sigma$. So (2.4.4) can be written as

$$(e^{s-\mu_1} - \Lambda_0)\psi'(s) \;=\; m_1 e^{s-\mu_1}\psi(s) - A(0)\psi(s) + \int_\eta^s C(\sigma, s)\psi(\sigma)\mathrm{d}\sigma + \mathcal{P}_\eta^{m_1}[P](s), \tag{2.4.5}$$

where $C(\sigma, s)$ is the $n \times n$-matrix defined by

$$C(\sigma, s) := e^{-\mu_1} \sum_{k=1}^{m_1} (-1)^k \binom{m_1+1}{k+1} \frac{(s-\sigma)^{k-1}}{(k-1)!}\, e^\sigma + B'(\sigma - s). \tag{2.4.6}$$

As $D(H)$, and thus B, is holomorphic in T, the function C is holomorphic in both variables σ and s, provided that $\sigma - s \in T$.

The following step is to split (2.4.5) after the first n_1 components. We will use the notation $\psi = (\psi^{[1]}, \psi^\perp)$, where $\psi^{[1]}$ denotes the projection of ψ on the space spanned by the first n_1 unit vectors of $\mathbb{C}^n$ and $\psi^\perp$ denotes the projection of ψ on the space spanned by the last $n - n_1$ unit vectors of $\mathbb{C}^n$. A similar splitting is used for $n \times n$-matrices after the first n_1 rows, i.e. the matrix consisting of the first n_1 rows of an $n \times n$-matrix M is denoted by $M^{[1]}$, while the matrix consisting of the last $n - n_1$ rows is denoted by $M^{\perp}$. This gives

$$e^{-\mu_1}\big\{(e^s - 1)(\psi^{[1]})'(s) + a_1\psi^{[1]}(s) - m_1 e^s \psi^{[1]}(s)\big\} = \int_\eta^s C^{[1]}(\sigma, s)\psi(\sigma)\mathrm{d}\sigma + \mathcal{P}_\eta^{m_1}\big[P^{[1]}\big](s)$$

$$\big(e^{s-\mu_1} - (\Lambda_0)^{\perp\perp}\big)(\psi^{\perp})'(s) = E(s)\psi^{\perp}(s) + \int_\eta^s C^{\perp}(\sigma, s)\psi(\sigma)\mathrm{d}\sigma + \mathcal{P}_\eta^{m_1}[P^{\perp}](s),$$

where $(\Lambda_0)^{\perp\perp} = \oplus_{j=2}^r e^{-\mu_j}\mathtt{I}_{n_j}$ and $E(s) = m_1 e^{s-\mu_1}\mathtt{I}_{n-n_1} - \oplus_{j=2}^r a_j e^{-\mu_j}\mathtt{I}_{n_j}$. If we define the functions α and $\tilde{\alpha}$ by

$$\alpha(s) = \begin{cases} e^{\mu_1}\frac{s}{e^s-1} & \text{if } s \neq 0 \\ e^{\mu_1} & \text{if } s = 0 \end{cases} \quad \text{and} \quad \tilde{\alpha}(s) = \begin{cases} \frac{\tilde{a}_1}{s} - \frac{\tilde{a}_1}{e^s-1} + m_1 & \text{if } s \neq 0 \\ \frac{\tilde{a}_1}{2} + m_1 & \text{if } s = 0 \end{cases},$$

then an easy computation shows that $e^{-\mu_1}\big\{(e^s - 1)(\psi^{[1]})'(s) + a_1\psi^{[1]}(s) - m_1 e^s\psi^{[1]}(s)\big\}$ equals $\frac{1}{\alpha(s)}\big\{s^{1-\tilde{a}_1}\frac{\mathrm{d}}{\mathrm{d}s}(s^{\tilde{a}_1}\psi^{[1]}(s)) - s\tilde{\alpha}(s)\psi^{[1]}(s)\big\}$. So we may rewrite the equations for $\psi^{[1]}$ and $\psi^{\perp}$ as

$$\begin{aligned} \psi^{[1]}(s) = s^{-\tilde{a}_1}\int_\eta^s \tau^{\tilde{a}_1-1}\big\{\tau\tilde{\alpha}(\tau)\psi^{[1]}(\tau) + \alpha(\tau)\mathcal{P}_\eta^{m_1}\big[P^{[1]}\big](\tau)\big\}\,\mathrm{d}\tau\, + \\ s^{-\tilde{a}_1}\int_\eta^s \tau^{\tilde{a}_1-1}\alpha(\tau)\int_\eta^\tau C^{[1]}(\sigma,\tau)\psi(\sigma)\,\mathrm{d}\sigma\,\mathrm{d}\tau \end{aligned} \tag{2.4.7a}$$

$$\begin{aligned} \psi^{\perp}(s) = \int_\eta^s \big(e^{\tau-\mu_1} - (\Lambda_0)^{\perp\perp}\big)^{-1}\big\{E(\tau)\psi^{\perp}(\tau) + \mathcal{P}_\eta^{m_1}[P^{\perp}](\tau)\big\}\,\mathrm{d}\tau\, + \\ \int_\eta^s \big(e^{\tau-\mu_1} - (\Lambda_0)^{\perp\perp}\big)^{-1}\int_\eta^\tau C^{\perp}(\sigma,\tau)\psi(\sigma)\,\mathrm{d}\sigma\,\mathrm{d}\tau. \end{aligned} \tag{2.4.7b}$$

We consider (2.4.7) in the space $\mathcal{V}_{\tilde{a}_1,0}$ of n-vector valued functions ψ that can be written as $\psi(s) = s^{-\tilde{a}_1}u_1(s) + u_2(s)$, with u_i, $i = 1, 2$, holomorphic in $\Delta(0, \varepsilon)$ and continuous up to $\overline{\Delta}(0, \varepsilon)$. We take the branch of $s^{-\tilde{a}_1}$ with $\theta_1 - \pi < \arg s \leqslant \theta_1 + \pi$ and we endow $\mathcal{V}_{\tilde{a}_1,0}$ with the norm $\|\psi\| := \sup_{|s|\leqslant\varepsilon}\{|u_1(s)|, |u_2(s)|\}$, which makes $\mathcal{V}_{\tilde{a}_1,0}$ into a Banach space. We consider on this space the operator $\mathcal{T}$ that maps any $\psi \in \mathcal{V}_{\tilde{a}_1,0}$ to the n-vector with as components the right-hand sides of (2.4.7a) and (2.4.7b) and we will verify that $\mathcal{T}$ defines a contraction on $\mathcal{V}_{\tilde{a}_1,0}$.

First we remark that if f is continuous in $\overline{\Delta}(0, \varepsilon)$ and holomorphic in the interior, and $a \in \mathbb{C}$ is a complex number with $\Re a > -1$, then

$$\int_\eta^s \tau^a f(\tau)\mathrm{d}\tau = s^{a+1}\int_0^1 \tau^a f(\tau s)\mathrm{d}\tau - \eta^{a+1}\int_0^1 \tau^a f(\tau\eta)\mathrm{d}\tau. \tag{2.4.8}$$

This formula implies that if $\psi \in \mathcal{V}_{\tilde{a}_1,0}$, then $s \mapsto s^{-\tilde{a}_1}\int_\eta^s \tau^{\tilde{a}_1}\tilde{\alpha}(\tau)\psi^{[1]}(\tau)\mathrm{d}\tau$ also is of the form $s^{-\tilde{a}_1}u_1(s) + u_2(s)$ for n_1-vector valued functions u_1 and u_2 and its norm is bounded by $O\big(\varepsilon^{\Re\tilde{a}_1+1}\big)\|\psi\|$, as $\varepsilon \to 0$. Moreover, $s \mapsto s^{-\tilde{a}_1}\int_\eta^s \tau^{\tilde{a}_1-1}\alpha(\tau)\mathcal{P}_\eta^{m_1}\big[P^{[1]}\big](\tau)\mathrm{d}\tau$ belongs to

$\mathcal{V}_{\tilde{a}_1,0}$ as can be seen by using partial integration. For the double integral in (2.4.7a) we use Fubini to get

$$\int_\eta^s \tau^{\tilde{a}_1-1}\alpha(\tau)\int_\eta^\tau C^{[1]}(\sigma,\tau)\psi(\sigma)\mathrm{d}\sigma\,\mathrm{d}\tau = \int_\eta^s \psi(\sigma)\int_\sigma^s \tau^{\tilde{a}_1-1}\alpha(\tau)\,C^{[1]}(\sigma,\tau)\,\mathrm{d}\tau\,\mathrm{d}\sigma \qquad (2.4.9)$$

and for this last integral with respect to τ we use an analogue of (2.4.8): if g is holomorphic in $(\sigma,\tau)\in\Delta(0,\varepsilon_1)\times\Delta(0,\varepsilon_1)$ with $\varepsilon_1>\varepsilon$, then $\tilde{a}_1\int_\sigma^s \tau^{\tilde{a}_1-1}g(\sigma,\tau)\mathrm{d}\tau$ can be written as

$$s^{\tilde{a}_1}g(\sigma,s)-\sigma^{\tilde{a}_1}g(\sigma,\sigma)-\int_0^1 \tau^{\tilde{a}_1}\Big\{s^{1+\tilde{a}_1}\frac{\partial g}{\partial\tau}(\sigma,\tau s)-\sigma^{1+\tilde{a}_1}\frac{\partial g}{\partial\tau}(\sigma,\tau\sigma)\Big\}\mathrm{d}\tau,$$

which is of the form $s^{\tilde{a}_1}g_1(\sigma,s)+\sigma^{\tilde{a}_1}g_2(\sigma)$. It is easily seen that g_1 is holomorphic in both variables for $(\sigma,s)\in\Delta(0,\varepsilon_1)\times\Delta(0,\varepsilon_1)$ and g_2 is holomorphic in $\Delta(0,\varepsilon_1)$. So the right-hand side of (2.4.9) can be written as $s^{\tilde{a}_1}$ times

$$\int_\eta^s g_1(\sigma,s)\psi(\sigma)\mathrm{d}\sigma + s^{-\tilde{a}_1}\int_\eta^s \sigma^{\tilde{a}_1}g_2(\sigma)\psi(\sigma)\mathrm{d}\sigma,$$

and this expression is an element of $\mathcal{V}_{\tilde{a}_1,0}$ if $\psi\in\mathcal{V}_{\tilde{a}_1,0}$. Using (2.4.8) we may show that its norm is bounded by $O\big(\varepsilon^{\Re\tilde{a}_1+1}\big)\|\psi\|$, as $\varepsilon\to 0$.

We recall that ε was chosen so small that the distance of $(0,\mu_1)$ to the singular points of Y_0 different from μ_1 is at least 2ε. So the inverse of $e^{s-\mu_1}-(\Lambda_0)^{\perp\perp}$ is an $(n-n_1)\times(n-n_1)$-matrix, which is holomorphic in $\Delta(0,\varepsilon)$ and continuous up to the boundary. As the last integral in (2.4.7b) can be written as

$$\int_\eta^s\int_\sigma^s \big(e^{\tau-\mu_1}-(\Lambda_0)^{\perp\perp}\big)^{-1}C^\perp(\sigma,\tau)\,\mathrm{d}\tau\,\psi(\sigma)\mathrm{d}\sigma,$$

we conclude that $\psi^\perp$ satisfies

$$\psi^\perp(s)=\int_\eta^s\big\{\tilde{C}^\perp(\sigma,s)\psi(\sigma)+\big(e^{\sigma-\mu_1}-(\Lambda_0)^{\perp\perp}\big)^{-1}\mathcal{P}_\eta^{m_1}[P^\perp](\sigma)\big\}\mathrm{d}\sigma,$$

where $\tilde{C}^\perp$ is some function holomorphic in both variables σ and s. The right-hand side of the latter equality is easily seen to be an element of $\mathcal{V}_{\tilde{a}_1,0}$ if $\psi\in\mathcal{V}_{\tilde{a}_1,0}$. Moreover, $\mathcal{T}$ defines a contraction on $\mathcal{V}_{\tilde{a}_1,0}$ if ε is chosen sufficiently small. Thus we get a unique solution $\psi\in\mathcal{V}_{\tilde{a}_1,0}$ of (2.4.5). If we write $\psi(s)=s^{-\tilde{a}_1}u_1(s)+u_2(s)$, then it is easily seen that $(u_1)_j(0)=0$ for all $j\in\{n_1+1,n_1+2,\dots,n\}$ by substituting this form into (2.4.5) and taking the limit for $s\to 0$.

Since $\Psi=\psi^{(m_1+1)}$ we also have a solution $\Psi\in\mathcal{V}_{a_1+1,0}$ of (2.4.2). Since $\Psi_0(t)=Y_0(t)$ for $t\in[\mu_1-\eta,\mu_1)$, we have $\Psi(t)=Y_0(\mu_1-t)$ if $t\in(0,\eta]$ and so $Y_0(\mu_1-t)\in\mathcal{V}_{a_1+1,0}$. Hence,

$$Y_0(t)=(\mu_1-t)^{-a_1-1}\sum_{h=1}^{n_1}C_h\mathbf{e}_h+(\mu_1-t)^{-a_1}\tilde{u}_1(t)+\tilde{u}_2(t),$$

with $\tilde{u}_j$, $j=1,2$, holomorphic in $\Delta(\mu_1,\varepsilon)$ and C_h, $h=1,2,\dots,n_1$, certain uniquely determined complex constants. ■

2.4.2 A Resurgence Property of Y_0

The solution ψ of (2.4.4), found in the preceding section, has holomorphic boundary values ψ^+ and ψ^- on the lower and upper side of the interval $(-\eta, 0)$ respectively and by taking limits one may deduce that both ψ^+ and ψ^- are solutions of (2.4.4) on $(-\eta,\eta)$ if we define $\psi^+(s) = \psi^-(s) = \psi(s)$ on $[0,\eta)$. Let us denote $Y_0^+(\mu_1 + t) := (\psi^+)^{(m_1+1)}(-t)$ and $Y_0^-(\mu_1+t) := (\psi^-)^{(m_1+1)}(-t)$ if $t \in (0,\eta)$ and similarly for the other singular points μ_j, $j = 2, 3, \ldots, r$. Then we have (cf. [Cos98], theorem 4, for the case of differential equations):

Proposition 2.4.3 *Under the assumptions of the preceding proposition there exist unique complex constants s_h, $h \in \mathcal{J}_j$, such that*

$$Y_0^+(\mu_j + t) - Y_0^-(\mu_j + t) = \sum_{h\in\mathcal{J}_j} s_h \tilde{Y}_{\mathbf{e}_h}^{(m_j)}(t), \tag{2.4.10}$$

for $\arg t = \theta_j = \arg \mu_j$ *and* $|t|$ *sufficiently small. Here* $\tilde{Y}_{\mathbf{e}_h} = \mathcal{B}\big[x^{\tilde{a}_j} y_{\mathbf{e}_h}(x)\big]$*,* m_j *is such that* $\Re a_j - m_j \in (-1,0)$ *and* $\tilde{a}_j = a_j - m_j$*. Moreover,*

$$Y_0(\mu_j + t) \equiv \frac{1}{1 - e^{-2\pi i \tilde{a}_j}} \sum_{h\in\mathcal{J}_j} s_h\, \tilde{Y}_{\mathbf{e}_h}^{(m_j)}(t) \bmod \mathbb{C}^n\{t\} \tag{2.4.11}$$

for $\theta_j \leqslant \arg t \leqslant \theta_j + 2\pi$*,* $|t|$ *sufficiently small, provided that we choose* $Y_0(\mu_j+t) = Y_0^+(\mu_j+t)$ *if* $\arg t = \theta_j$ *and* $Y_0(\mu_j + t) = Y_0^-(\mu_j + t)$ *if* $\arg t = \theta_j + 2\pi$*.*

Proof. Again we only give the proof for $j = 1$. Let $v(s) := \psi^+(-s) - \psi^-(-s)$. Then $v(s) = 0$ on $(-\eta, 0]$ and $v(s) = s^{-\tilde{a}_1} u_1(-s)(e^{\pi i \tilde{a}_1} - e^{-\pi i \tilde{a}_1})$ on $(0,\eta)$.

Now define $\tilde{\beta}_{\mathbf{e}_1} = \mathcal{B}[(1 + x^{-1})^{m_1}]$, then $\tilde{\beta}_{\mathbf{e}_1}(s) = \sum_{k=1}^{m_1} \binom{m_1}{k} \frac{s^{k-1}}{(k-1)!}$ and obviously

$$\big(\tilde{\beta}_{\mathbf{e}_1} * e^{-s} v'\big)(s) = \sum_{k=1}^{m_1} \binom{m_1}{k} \mathcal{P}^k\big[e^{-\sigma} v'(\sigma)\big](s),$$

where $\mathcal{P}$ is defined as $(\mathcal{P}v)(s) = (1 * v)(s) = \int_0^s v(\sigma)\mathrm{d}\sigma$. Using this we obtain in a similar way as in the proof of proposition 2.4.2

$$\big(\tilde{\beta}_{\mathbf{e}_1} * e^{-s} v'\big)(s) = m_1 e^{-s} v(s) + \sum_{k=1}^{m_1} \binom{m_1+1}{k+1} \int_0^s \frac{(s-\sigma)^{k-1}}{(k-1)!} e^{-\sigma} v(\sigma)\mathrm{d}\sigma$$

and from this together with (2.4.4), we deduce that v satisfies

$$(e^{-s-\mu_1} - \Lambda_0) v' = B * v' - e^{-\mu_1} \tilde{\beta}_{\mathbf{e}_1} * e^{-s} v'. \tag{2.4.12}$$

Since $v'(s) = s^{-\tilde{a}_1 - 1}\tilde{u}(s)$ with $\tilde{u} \in \mathbb{C}^n\{s\}$, we have $\hat{\mathcal{L}} v' \in x^{\tilde{a}_1}\mathbb{C}^n[[x^{-1}]]$. Taking a formal Laplace transform of (2.4.12), and using (2.2.3) and the definition of $B = A + D(Y_0)$ in section 2.3.1, we see that $\hat{\mathcal{L}} v'$ is a formal solution of

$$e^{-\mu_1}(1 + x^{-1})^{m_1} y(x+1) = \Lambda_1(x) y(x), \tag{2.4.13}$$

which is exactly the equation for $x^{\tilde{a}_1}\hat{y}_{\mathbf{e}_h}(x)$, $h=1,2,\dots,n_1$. It is easily seen that the system $\{\vartheta_1,\vartheta_2,\dots,\vartheta_n\}$ defined by

$$\begin{cases} \vartheta_h(x) & := & x^{\tilde{a}_1}\hat{y}_{\mathbf{e}_h}(x), & h=1,\dots,n_1 \\ \vartheta_h(x) & := & e^{(\mu_1-\mu_2)x}x^{m_2-m_1}x^{\tilde{a}_2}\hat{y}_{\mathbf{e}_h}(x), & h=n_1+1,\dots,n_1+n_2 \\ \vdots & \vdots & \vdots & \\ \vartheta_h(x) & := & e^{(\mu_1-\mu_r)x}x^{m_r-m_1}x^{\tilde{a}_r}\hat{y}_{\mathbf{e}_h}(x), & h=n-n_r+1,\dots,n \end{cases}$$

forms a formal fundamental system of (2.4.13).

Since $\hat{\mathcal{L}}v' \in x^{\tilde{a}_1}\mathbb{C}^n[[x^{-1}]]$, we conclude to the existence of constants $c_1,c_2,\dots,c_{n_1}$ such that $(\hat{\mathcal{L}}v')(x)=\sum_{h=1}^{n_1}c_h\vartheta_h(x)$ and therefore $v'(s)=\sum_{h=1}^{n_1}c_h\tilde{Y}_{\mathbf{e}_h}(s)$. In particular there exist n_1 complex constants $s_1,s_2,\dots,s_{n_1}$ such that

$$Y_0^+(\mu_1+t)-Y_0^-(\mu_1+t)=\sum_{h=1}^{n_1}s_h\tilde{Y}_{\mathbf{e}_h}^{(m_1)}(t),$$

for $\arg t=\theta_1$ and $|t|$ sufficiently small. In fact, $s_h=(-1)^{m_1+1}c_h$. The proof of the behaviour of Y_0 near μ_1, as given in (2.4.11), is part of the proof of proposition 2.6.8. ■

Remark 2.4.4 We will briefly sketch another proof of proposition 2.4.3. Once we have (2.4.12) it can be rewritten as

$$(e^{-s-\mu_1}-\Lambda_0)v'(s)=-m_1e^{-s-\mu_1}v(s)+A(0)v(s)+\int_0^s C(-\sigma,-s)v(\sigma)\mathrm{d}\sigma,$$

with C defined in (2.4.6). We split this equation similarly as in the proof of proposition 2.4.2 and using that $e^{-\mu_1}\big\{(e^{-s}-1)(v^{[1]})'(s)-a_1v^{[1]}(s)+m_1e^{-s}v^{[1]}(s)\big\}$ can be rewritten as $\frac{-1}{\alpha(-s)}\big\{s^{1-\tilde{a}_1}\frac{\mathrm{d}}{\mathrm{d}s}(s^{\tilde{a}_1}v^{[1]}(s))+s\tilde{\alpha}(-s)v^{[1]}(s)\big\}$, with α and $\tilde{\alpha}$ as in the proof of proposition 2.4.2, we obtain

$$\begin{aligned} v^{[1]}(s) = -s^{-\tilde{a}_1}\int_0^s \tau^{\tilde{a}_1}\tilde{\alpha}(-\tau)v^{[1]}(\tau)\,\mathrm{d}\tau + s^{-\tilde{a}_1}\mathbf{c} + \\ - s^{-\tilde{a}_1}\int_0^s \tau^{\tilde{a}_1-1}\alpha(-\tau)\int_0^\tau C^{[1]}(-\sigma,-\tau)v(\sigma)\,\mathrm{d}\sigma\,\mathrm{d}\tau \end{aligned}$$

$$\begin{aligned} v^{\perp}(s) = -\int_0^s \big(e^{-\tau-\mu_1}-(\Lambda_0)^{\perp\perp}\big)^{-1}E(-\tau)v^{\perp}(\tau)\,\mathrm{d}\tau + \\ \int_0^s \big(e^{-\tau-\mu_1}-(\Lambda_0)^{\perp\perp}\big)^{-1}\int_0^\tau C^{\perp}(-\sigma,-\tau)v(\sigma)\,\mathrm{d}\sigma\,\mathrm{d}\tau, \end{aligned}$$

where $\mathbf{c}\in\mathbb{C}^{n_1}$ is a constant vector. Similarly as in the proof of proposition 2.4.2, this equation can be solved in the subspace $\mathcal{V}_{\tilde{a}_1}$ of $\mathcal{V}_{\tilde{a}_1,0}$ consisting of functions of the form $s^{-\tilde{a}_1}u(s)$ with u holomorphic in $\Delta(0,\varepsilon)$ and continuous up to $\overline{\Delta}(0,\varepsilon)$, together with the norm $\sup_{|s|\leqslant\varepsilon}|u(s)|$. Thus we obtain a unique solution v of (2.4.12) in $\mathcal{V}_{\tilde{a}_1}$.

Now, for $h = 1, 2, \ldots, n_1$, let $v_{(h)}$ correspond to the solution with $\mathbf{c} = 1/\Gamma(1-\tilde{a}_1)\mathbf{e}_h$, then for a general constant vector $\mathbf{c}$ we have the solution $v(s) = \sum_{h=1}^{n_1} c_h \Gamma(1-\tilde{a}_1) v_{(h)}(s)$. On the other hand we recognise in (2.4.12) the equations for $\tilde{Y}_{\mathbf{e}_h}$, $h = 1, 2, \ldots, n_1$, and by construction we have

$$t^{\tilde{a}_1+1}\tilde{Y}_{\mathbf{e}_h}(t)\,\Big|_{t=0} \;=\; \frac{1}{\Gamma(-\tilde{a}_1)}\,\mathbf{e}_h \;=\; t^{\tilde{a}_1+1}v_h'(t)\,\Big|_{t=0},$$

so $\tilde{Y}_{\mathbf{e}_h}(t) = v_{(h)}'(t)$. This gives another proof of the resurgence relation (2.4.10).

2.5 Convolution Equations on Singular Directions

As we saw in the preceding section, the solution Y_0 of (2.3.5) behaves near the singular point μ_j like

$$Y_0(t) = (\mu_j - t)^{-a_j-1}\sum_{h\in\mathcal{J}_j} C_h\mathbf{e}_h + (\mu_j - t)^{-a_j}u_1(t) + u_2(t),$$

with u_i, $i = 1, 2$, holomorphic in a neighbourhood of μ_j. As $\Re a_j > 0$, the solution Y_0 is in general not integrable on the ray $[0, \infty e^{i\theta_j})$. Hence, the Laplace transform of Y_0 in the direction θ_j may not be defined and we can't transform Y_0 to a solution y_0 of (2.1.1) on the ray $[0, \infty e^{i\theta_j})$. To tackle this problem we introduce Costin's *staircase distributions* (cf. [Cos98]) and solve (2.3.5) on such a singular ray in the sense of distributions. The usual Laplace transform extends to this space of staircase distributions and this extended Laplace transform then gives an actual solution of (2.1.1) on such a singular ray.

2.5.1 Staircase Distributions

In this section we only give an overview of the theory of staircase distributions. A complete exposition of this theory can be found in [Cos98] or in appendix A.

Definition 2.5.1 *A distribution $f \in \mathcal{D}'(0,\infty)$ is called a* staircase distribution *of order $m \in \mathbb{N}_{\geqslant 1}$ if for all $k \in \mathbb{N}$ there exists a function $F_k \in L^1(0, k+1)$ such that the restriction of $\mathcal{P}^{mk}f$ to $\mathcal{D}(0, k+1)$ equals F_k. We will denote the set of staircase distributions of order m by $\mathcal{D}'_m$.*

Obviously, if f is a staircase distribution of order m, then the restriction of f to $\mathcal{D}(0, k+1)$ equals $F_k^{(mk)}$. It turns out that each $f \in \mathcal{D}'_m$ can be decomposed as $f = \sum_{k=0}^{\infty}(\Delta_k(f))^{(mk)}$, with unique $\Delta_k(f) \in L^1(0,\infty)$, supported in $[k, k+1]$ and, conversely, any series of the form $\sum_{k=0}^{\infty}\Delta_k^{(mk)}$, with Δ_k an L^1-function supported in $[k, k+1]$, represents an element in $\mathcal{D}'_m$. Moreover, the space $\mathcal{D}'_m$ is a commutative algebra with respect to addition and convolution.

In the sequel ν will always be a positive number larger than m. Let L^1_ν be the space of functions $f \in L^1_{loc}(0,\infty)$ such that $\|f\|_\nu := \int_0^\infty e^{-\nu t}|f(t)|\mathrm{d}t$ is finite. Using this norm we define a family of norms $\|\cdot\|_{m,\nu}$, depending on the parameter ν, on the set of staircase distributions by

$$\|f\|_{m,\nu} := C_m \sum_{k=0}^{\infty} \nu^{mk} \|\Delta_k\|_{\nu},$$

where C_m is a certain positive constant independent of ν. This constant C_m is specified in appendix A. Defining $\mathcal{D}'_{m,\nu} := \left\{ f \in \mathcal{D}'_m \;\middle|\; \|f\|_{m,\nu} < \infty \right\}$ one may show that $\mathcal{D}'_{m,\nu}$ defines a Banach algebra with respect to addition and convolution (cf. appendix A). In fact, $\|f * \tilde{f}\|_{m,\nu} \leqslant \|f\|_{m,\nu}\|\tilde{f}\|_{m,\nu}$ for $f,\ \tilde{f} \in \mathcal{D}'_{m,\nu}$. In the appendix we also show that $L^1_{\nu_0} \subset \mathcal{D}'_{m,\nu}$ for every $\nu > \nu_0$. An important property of staircase distributions is formulated in the following definition and theorem.

Definition 2.5.2 *The Laplace transform of a staircase distribution $f \in \mathcal{D}'_{m,\nu}$ is defined by*

$$(\mathcal{L}f)(x) := \sum_{k=0}^{\infty} x^{mk} \int_0^{\infty} \Delta_k(t) e^{-xt} \mathrm{d}t.$$

Theorem 2.5.3 *Let $f \in \mathcal{D}'_{m,\nu}$. Then the function $x \mapsto (\mathcal{L}f)(x)$ is holomorphic in some neighbourhood U of ∞ in $S(0,\pi)$ depending on m and ν. Moreover, U may be chosen such that for $x \in U$ we have $|(\mathcal{L}f)(x)| \leqslant \|f\|_{m,\nu}$. The Laplace transform is a continuous and injective operator on $\mathcal{D}'_{m,\nu}$.*

Using density arguments one may prove (cf. appendix A) that this generalised Laplace transform takes convolution into ordinary multiplication. If $f \in \mathcal{D}'_{m,\nu}$, then $e^{-t}f \in \mathcal{D}'_{m,\nu}$ and its Laplace transform equals $(\mathcal{L}f)(x+1)$. If $\alpha > 0$, then $f(t-\alpha)\mathbb{1}_{[\alpha,\infty)} \in \mathcal{D}'_{m,\nu}$ and its Laplace transform equals $e^{-\alpha x}$ times the Laplace transform of f (here $\mathbb{1}_{[\alpha,\infty)}$ denotes the indicator-function on $[\alpha,\infty)$). If $f \in \mathcal{D}'_{m,\nu}$, then $\mathcal{P}f \in \mathcal{D}'_{m,\nu}$ and $\mathcal{L}(\mathcal{P}f) = x^{-1}\mathcal{L}f$.

We denote the set of staircase distributions on an arbitrary half line $\arg t = \theta$, $\theta \in \mathbb{R}$, by $\mathcal{D}'_m(\theta)$. These staircase distributions have similar properties as those defined on the positive real line. To define a family of norms, we first introduce the space $L^1_\nu(\theta)$, consisting of functions $f \in L^1_{loc}(0, \infty e^{i\theta})$ such that $\|f\|_{\nu,\theta} := \int_0^\infty e^{-\nu t}|f(te^{i\theta})|\mathrm{d}t$ is finite. On $\mathcal{D}'_m(\theta)$ we now define norms $\|\cdot\|_{m,\nu,\theta}$ by

$$\|f\|_{m,\nu,\theta} := C_m \sum_{k=0}^{\infty} \nu^{mk} \|\Delta_k\|_{\nu,\theta}$$

and we use the notation $\mathcal{D}'_{m,\nu}(\theta)$ for the set of distributions f in $\mathcal{D}'_m(\theta)$ with $\|f\|_{m,\nu,\theta} < \infty$. If we choose a different interval length $\ell > 0$ instead of $\ell = 1$ in the partition used in the decomposition of staircase distributions, we will use the notation $\mathcal{D}'_{m,\nu}(\ell,\theta)$ instead of $\mathcal{D}'_{m,\nu}(\theta) = \mathcal{D}'_{m,\nu}(1,\theta)$ and the corresponding norm will be denoted by $\|\cdot\|_{m,\nu,\theta,\ell}$.

2.5.2 Some Auxiliary Results

Lemma 2.5.4 *Fix $m \in \mathbb{N}_{\geqslant 1}$ and $\theta \in \mathbb{R}$. Let c be the constant appearing in (2.3.4). Then there exist positive constants ν_0 and K such that for all $\mathbf{l} \succeq 0$ and $\nu \geqslant \nu_0$ we have*

$$\|G_{\mathbf{l}}\|_{m,\nu,\theta} \leqslant \frac{K}{\nu} c^{|\mathbf{l}|}, \quad \|A\|_{m,\nu,\theta} \leqslant \frac{K}{\nu}. \tag{2.5.1}$$

There exist positive constants δ_0 and K_1 such that for all $\nu \geqslant \nu_0$ and $Y \in \mathcal{D}'_{m,\nu}(\theta)$, with $\|Y\|_{m,\nu,\theta} \leqslant \delta \leqslant \delta_0$, we have

$$\|D_0(Y)\|_{m,\nu,\theta} \leqslant \big(\delta^2 + \nu^{-1}\big)K_1, \quad \|D_{\mathbf{j}}(Y)\|_{m,\nu,\theta} \leqslant \big(\delta + \nu^{-1}\big)K_1^{|\mathbf{j}|+1}, \ \mathbf{j} \in \mathbb{N}^n \backslash \{0\}.$$

If also $Z \in \mathcal{D}'_{m,\nu}(\theta)$ with $\|Z\|_{m,\nu,\theta} \leqslant \delta$, then there exists a constant K_2 such that

$$\|D_0(Y) - D_0(Z)\|_{m,\nu,\theta} \leqslant K_2\big(\delta + \nu^{-1}\big)\|Y - Z\|_{m,\nu,\theta}$$

and for $\mathbf{j} \neq 0$

$$\|D_{\mathbf{j}}(Y) - D_{\mathbf{j}}(Z)\|_{m,\nu,\theta} \leqslant K_2^{|\mathbf{j}|+1}\|Y - Z\|_{m,\nu,\theta}.$$

Moreover, the constants K, K_1 and K_2 are independent of θ, ν and δ.

Proof. Take $\nu_1 > c$. Using (2.3.4) we easily infer that there exists a positive constant $\tilde{K}$ such that $\int_0^\infty e^{-\nu t}|G_{\mathbf{l}}(te^{i\theta})|\mathrm{d}t \leqslant \frac{\tilde{K}}{\nu} c^{|\mathbf{l}|}$ for all $\nu \geqslant \nu_1$. Using (A.4.1) we then deduce

$$\|G_{\mathbf{l}}\|_{m,\nu,\theta} \ \leqslant \ 2\,C_m \int_0^\infty e^{-\frac{\nu}{2}t}|G_{\mathbf{l}}(te^{i\theta})|\mathrm{d}t \ \leqslant \ \frac{4\,C_m\tilde{K}}{\nu} c^{|\mathbf{l}|},$$

provided that $\nu \geqslant 2\nu_1$ is large enough. This proves the first statement in the lemma. The estimate for $\|A\|_{m,\nu,\theta}$ may be proved similarly. Further, one should observe that for $j \in \mathbb{N}$ we have $\sum_{l=j}^\infty \binom{l}{j} t^{l-j} = (1-t)^{-j-1}$ if $0 < t < 1$ and the same sum without the term with $l = j$ equals $(1-t)^{-j-1} - 1$, which does not exceed $(j+1)t(1-t)^{-j-1}$. Hence, for $0 < t < 1$ we have $\sum_{\mathbf{l} \succ \mathbf{j}} \binom{\mathbf{l}}{\mathbf{j}} t^{|\mathbf{l}|-|\mathbf{j}|} \leqslant (|\mathbf{j}|+1)t(1-t)^{-|\mathbf{j}|-n}$. Moreover, the number of multi-indices $\mathbf{l} \in \mathbb{N}^n$ such that $|\mathbf{l}| = h$ equals the coefficient of x^h in the expansion of $(1-x)^{-n}$ around 0. This coefficient is equal to $\binom{n+h-1}{h}$, which can be majorized by 2^{n+h-1}. Hence, $\sum_{|\mathbf{l}|=h} 1 \leqslant 2^{n+h-1}$. From these observations, together with (2.3.4) and the definition of $D_{\mathbf{j}}$ in (2.3.3), we obtain for $\mathbf{j} \in \mathbb{N}^n \backslash \{0\}$

$$\|D_{\mathbf{j}}(Y)\|_{m,\nu,\theta} \ \leqslant \ Kc^{|\mathbf{j}|}\left\{\nu^{-1} + (1+\nu^{-1})\sum_{\mathbf{l} \succ \mathbf{j}} \binom{\mathbf{l}}{\mathbf{j}} (c\delta)^{|\mathbf{l}|-|\mathbf{j}|}\right\} \ \leqslant \ (\delta + \nu^{-1})K_1^{|\mathbf{j}|+1},$$

for a suitable K_1 if δ_0 is small enough. The estimate for $\|D_0(Y)\|_{m,\nu,\theta}$ follows from (2.3.3), (2.3.4) and by using that $g_{\mathbf{l},0} = 0$ for $|\mathbf{l}| \leqslant 1$. Using (2.4.1) with H and Ψ_0 replaced by Z and $Y - Z$, together with the previous estimates, we obtain in a similar way as above

$$\begin{aligned}\|D_0(Y) - D_0(Z)\|_{m,\nu,\theta} &\leqslant K \sum_{\mathbf{j} \in \mathbb{N}_2^n} \big(c\|Y - Z\|_{m,\nu,\theta}\big)^{|\mathbf{j}|} + K_1(\delta + \nu^{-1}) \sum_{\mathbf{j} \in \mathbb{N}_1^n} \big(K_1\|Y - Z\|_{m,\nu,\theta}\big)^{|\mathbf{j}|} \\ &\leqslant K_2(\delta + \nu^{-1})\|Y - Z\|_{m,\nu,\theta},\end{aligned}$$

for some positive constant K_2 if we choose δ_0 small enough. To prove the last estimate we observe, by writing $Y = Y - Z + Z$, that

$$D_{\mathbf{j}}(Y) - D_{\mathbf{j}}(Z) = \sum_{\mathbf{r}\in\mathbb{N}_1^n} \binom{\mathbf{r}+\mathbf{j}}{\mathbf{j}} \big(g_{\mathbf{r}+\mathbf{j},0} + D_{\mathbf{r}+\mathbf{j}}(Z)*\big)(Y-Z)^{*\mathbf{r}},$$

which can be estimated in a similar way as above. ■

For $\sigma \in \mathbb{R}$ we define the map $\rho_\sigma : L^1_\nu(\theta) \to L^1_\nu(\theta - \sigma)$ by $(\rho_\sigma f)(t) = e^{i\sigma} f(te^{i\sigma})$. From this definition one easily deduces that $\|\rho_\sigma f\|_{\nu,\theta-\sigma} = \|f\|_{\nu,\theta}$. Moreover, if $f, \tilde{f} \in L^1_\nu(\theta)$, then $\rho_\sigma(f * \tilde{f}) = (\rho_\sigma f) * (\rho_\sigma \tilde{f})$. This can be generalised to staircase distributions, as follows. If $f \in L^1_\nu(\theta)$, then in the sense of distributions we have $\big(f(te^{i\sigma})\big)' = e^{i\sigma} f'(te^{i\sigma})$. Hence, if $f \in \mathcal{D}'_m(\theta)$ is a staircase distribution with decomposition $\{\Delta_k\}_{k\in\mathbb{N}}$, then

$$\rho_\sigma f := e^{i\sigma} f(\cdot e^{i\sigma}) = e^{i\sigma} \sum_{k=0}^{\infty} \Delta_k^{(mk)}(\cdot e^{i\sigma}) = \sum_{k=0}^{\infty} e^{-imk\sigma} \big(\rho_\sigma \Delta_k\big)^{(mk)}.$$

This immediately implies that if $f \in \mathcal{D}'_{m,\nu}(\theta)$, then $\rho_\sigma f \in \mathcal{D}'_{m,\nu}(\theta - \sigma)$ and the corresponding norms equal each other. Finally, if $f, \tilde{f} \in \mathcal{D}'_{m,\nu}(\theta)$ we have

$$\rho_\sigma(f * \tilde{f}) = \rho_\sigma f * \rho_\sigma \tilde{f}.$$

Lemma 2.5.5 *Let φ be a function holomorphic in a neighbourhood of ∞. Then there exist positive constants K and ν_0 such that for all $\theta \in \mathbb{R}$, $\sigma \in [-\pi, \pi]$ and $\nu \geqslant \nu_0$ we have*

$$\|\rho_\sigma(\mathcal{B}\varphi) - \mathcal{B}\varphi\|_{\nu,\theta} \leqslant \frac{K}{\nu}\,|\sigma|.$$

Proof. If φ_k denotes the coefficient of x^{-k} in the Taylor expansion of φ, then $|\varphi_k| \leqslant \tilde{K} c^k$ for some positive constants $\tilde{K}$ and c. Hence,

$$|\rho_\sigma(\mathcal{B}\varphi)(t) - (\mathcal{B}\varphi)(t)| \leqslant \sum_{k=0}^{\infty} \frac{|\varphi_{k+1}|}{k!} |t|^k |e^{i\sigma(k+1)} - 1|.$$

Using that $|e^{i\sigma(k+1)} - 1| \leqslant (k+1)|\sigma| \leqslant 2^k|\sigma|$, the proof is easily completed. ■

Lemma 2.5.6 *Let $m \in \mathbb{N}_{\geqslant 1}$ and let c be the constant as it appears in (2.3.4). Then there exist positive constants ν_0 and K such that for all $\mathbf{l} \succ 0$, $\theta \in \mathbb{R}$, $\sigma \in [-\pi, \pi]$ and $\nu \geqslant \nu_0$ we have*

$$\|\rho_\sigma G_{\mathbf{l}} - G_{\mathbf{l}}\|_{m,\nu,\theta} \leqslant \frac{K}{\nu}\, c^{|\mathbf{l}|}|\sigma|, \quad \|\rho_\sigma A - A\|_{m,\nu,\theta} \leqslant \frac{K}{\nu}\,|\sigma|. \tag{2.5.2}$$

Moreover, if $\theta_0 \in \mathbb{R}$ is fixed, then there exist positive δ_0 and K_1 such that for all $\mathbf{j} \in \mathbb{N}^n$, $\nu \geqslant \nu_0$ and $Y \in \mathcal{D}'_{m,\nu}(\theta)$, with $\|Y\|_{m,\nu,\theta} \leqslant \delta \leqslant \delta_0$ for $\theta = \theta_0$ and $\theta = \theta_0 + \sigma$, we have

$$\|\rho_\sigma D_{\mathbf{j}}(Y) - D_{\mathbf{j}}(Y)\|_{m,\nu,\theta_0} \leqslant K_1^{|\mathbf{j}|+1}\,\nu^{-1}|\sigma| + \|D_{\mathbf{j}}(\rho_\sigma Y) - D_{\mathbf{j}}(Y)\|_{m,\nu,\theta_0}.$$

Moreover, the constant K_1 is independent of σ.

PROOF. The statements in (2.5.2) can be shown in a similar way as we proved (2.5.1), using the preceding lemma. To prove the estimate for $\|\rho_\sigma D_{\mathbf{j}}(Y) - D_{\mathbf{j}}(Y)\|_{m,\nu,\theta_0}$ we use (2.3.3) to obtain

$$\rho_\sigma D_{\mathbf{j}}(Y) = D_{\mathbf{j}}(\rho_\sigma Y) + \rho_\sigma G_{\mathbf{j}} - G_{\mathbf{j}} + \sum_{\mathbf{l} \succ \mathbf{j}} \binom{\mathbf{l}}{\mathbf{j}} (\rho_\sigma G_{\mathbf{l}} - G_{\mathbf{l}}) * (\rho_\sigma Y)^{*(\mathbf{l}-\mathbf{j})}$$

and we observe that $\|\rho_\sigma Y\|_{m,\nu,\theta_0} = \|Y\|_{m,\nu,\theta_0+\sigma}$. Since $\|\rho_\sigma G_{\mathbf{l}} - G_{\mathbf{l}}\|_{m,\nu,\theta} \leqslant \frac{K}{\nu} c^{|\mathbf{l}|} |\sigma|$, the proof is easily completed. ■

Let us abbreviate

$$U_1(t) := (1 - e^{-t+\mu})^{-1}, \quad U_2(t) := e^{-t} \tag{2.5.3}$$

and observe that $(e^{-t} - e^{-\mu})^{-1} = -e^{\mu}\, U_1(t)$. Moreover, for $a \in \mathbb{R}_+$ we will use the notation

$$\mathcal{D}'_{m,\nu}\big(ae^{i\theta}, \infty e^{i\theta}\big) := \big\{ f \in \mathcal{D}'_{m,\nu}(\theta) \;\big|\; \Delta_k(f)(x) = 0 \;\text{ if }\; xe^{-i\theta} < a,\; k \in \mathbb{N} \big\}.$$

Obviously, $\mathcal{D}'_{m,\nu}\big(ae^{i\theta}, \infty e^{i\theta}\big)$ is a Banach algebra with respect to addition and convolution. Using these notations we may prove the following lemma. However, since the proof is far from easy and quite laborious, we have decided to postpone it till the end of this chapter.

Lemma 2.5.7 *Let $-\pi/2 < \theta_- < \theta_+ < \pi/2$ and let $I \subset (\theta_-, \theta_+)$ be a compact interval. Take $\delta > 0$ so small that $I \pm \delta \subset (\theta_-, \theta_+)$ and let U_1 and U_2 be as in (2.5.3).*

(i) *For $\mu \in \mathbb{C}^*$, $\arg \mu \in (\theta_-, \theta_+)$, or $\mu = 0$, we take $a \in \mathbb{R}_+$ so large that $a\cos(\theta+\sigma) > \Re\mu$ for all $\theta \in I$ and all $\sigma \in [-\delta, \delta]$. Then there exists a positive ν_0 such that for all $\theta \in I$ and $\sigma \in [-\delta, \delta]$ the operator defined by multiplication with $U_1(\cdot e^{i\sigma})$ maps $\mathcal{D}'_{m,\nu}\big(ae^{i\theta}, \infty e^{i\theta}\big)$ into itself, provided that $\nu \geqslant \nu_0$. This operator is bounded by a positive constant independent of θ and σ.*

(ii) *For $\mu \in \mathbb{C}^*$ such that $\arg(\mu + 2l\pi i) \notin (\theta_-, \theta_+)$ for all $l \in \mathbb{Z}$, there exists a positive ν_0 such that for all $\theta \in I$ and $\sigma \in [-\delta, \delta]$ the operator defined by multiplication with $U_1(\cdot e^{i\sigma})$ maps $\mathcal{D}'_{m,\nu}(\theta)$ into itself, provided that $\nu \geqslant \nu_0$. This operator is bounded by a positive constant independent of θ and σ.*

(iii) *There exists a positive ν_0 such that for all $\theta \in I$ and $\sigma \in [-\delta, \delta]$ the operator defined by multiplication with $U_2(\cdot e^{i\sigma})$ maps $\mathcal{D}'_{m,\nu}(\theta)$ into itself, provided that $\nu \geqslant \nu_0$. This operator is bounded by a positive constant independent of θ and σ.*

(iv) *There exists a positive ν_0 such that for all $\theta \in I$ and $\sigma \in [-\delta, \delta]$ the operator defined by multiplication with $U_2(\cdot e^{i\sigma}) - U_2$ maps $\mathcal{D}'_{m,\nu}(\theta)$ into itself, provided that $\nu \geqslant \nu_0$. This operator is bounded by $C|\sigma|$, where C is some positive constant independent of θ and σ.*

Remark 2.5.8 If we take a different interval length ℓ instead of $\ell = 1$ in the partition used in the decomposition of staircase distributions, the lemma above still holds.

Lemma 2.5.9 *Let $\theta, \sigma \in \mathbb{R}$ and assume f to be a staircase distribution which belongs to both $\mathcal{D}'_{m,\nu}(\theta)$ and $\mathcal{D}'_{m,\nu}(\theta+\sigma)$. Moreover, let $U(t)$ be either e^{-t} or $(1-e^{-t+\mu})^{-1}$. Then*

$$\rho_\sigma(Uf) = U(\cdot e^{i\sigma})(\rho_\sigma f).$$

Here U and $U(\cdot e^{i\sigma})$ denote the operators defined by multiplication with $U(t)$ and $U(te^{i\sigma})$ respectively.

Proof. The proof simply follows by a tedious, but straightforward calculation that only involves definitions. ■

2.5.3 Solutions on Singular Rays

Let $j \in \{1,2,\dots,r_1\}$. For $\varepsilon_j > 0$ small enough we have seen that on the interval $[0, \mu_j + \varepsilon_j e^{i\theta_j})$, where $\theta_j = \arg \mu_j$, we had to deal with expressions $Y_0^\pm$ (defined in section 2.4.2), which were holomorphic solutions of (2.3.5) on $[0,\mu_j)$. However, from the proof of proposition 2.4.2 we conclude that on $(\mu_j - \varepsilon_j e^{i\theta_j}, \mu_j + \varepsilon_j e^{i\theta_j})$ both Y_0^+ and Y_0^- can be written as m_j^{th} derivatives of L^1-functions. Hence, both Y_0^+ and Y_0^- are solutions of (2.3.5) on $[0, \mu_j + \varepsilon_j e^{i\theta_j})$ in the sense of distributions and obviously they belong to $\mathcal{D}'_{m_j,\nu}\big(\ell_j, (0, \mu_j + \varepsilon_j e^{i\theta_j})\big)$ for every $\nu > 0$, where ℓ_j is some interval length in between 0 and $|\mu_j|$. Here $\mathcal{D}'_{m_j,\nu}\big(\ell_j, (0, ae^{i\theta_j})\big)$, $a > 0$, is defined as the set of those $f \in \mathcal{D}'_{m_j,\nu}(\ell_j, \theta_j)$ with $\Delta_k(f)(x) = 0$ if $xe^{-i\theta_j} \geqslant a$. In the following we take $\ell_j \in (|\mu_j|/2, |\mu_j|)$.

With such a singular direction $\theta_j \in (-\pi/2, \pi/2)$ we associate two other directions θ_{j-} and θ_{j+} in such a way that θ_{j-}, θ_j and θ_{j+} are three consecutive singular directions in the right half plane of the set of all $\hat{y}_{\mathbf{k}}$, $\mathbf{k} \in \mathbb{N}^p$. Using this notation we assume that hypothesis

$$(\mathbf{H}) \qquad \begin{cases} \arg\big(\mu_m - \langle \mathbf{k}, \boldsymbol{\mu}\rangle + 2l\pi i\big) \neq \theta_j \text{ if } \mathbf{k} \in \mathbb{N}^p,\ m \in \{1,2,\dots,r\},\ l \in \mathbb{Z} \\ \qquad\qquad \text{except if } \mathbf{k} = 0,\ m = j,\ l = 0 \end{cases}$$

is satisfied. Moreover, in the following we will only concentrate on the direction θ_1, although we may generalise the results also to directions θ_j, $j \in \{2,3,\dots,r_1\}$. We will omit the index 1 in ε_1 and $\ell_1 \in (|\mu_1|/2, |\mu_1|)$.

For convenience (and as a generalisation of the definitions in proposition 2.4.3), we now define for $\mathbf{k} \in \mathbb{N}^p$

$$\tilde{Y}_{\mathbf{k}} := \mathcal{B}\big[x^{\langle \mathbf{k}, \tilde{\mathbf{a}}\rangle} y_{\mathbf{k}}(x)\big], \quad \text{where } \tilde{\mathbf{a}} = \textstyle\sum_{j=1}^r \tilde{a}_j \sum_{h \in \mathcal{J}_j} \mathbf{e}_h, \tag{2.5.4}$$

with $\tilde{a}_j$, $j = 1,2,\dots,r$, defined by $\tilde{a}_j = a_j - m_j$. As each $\hat{y}_{\mathbf{k}}$ is Borel summable, the same holds for the formal series $x^{\langle \mathbf{k}, \tilde{\mathbf{a}}\rangle} \hat{y}_{\mathbf{k}}(x)$, $\mathbf{k} \in \mathbb{N}^p$, and thus the functions $t \mapsto t^{\langle \mathbf{k}, \tilde{\mathbf{a}}\rangle + 1} \tilde{Y}_{\mathbf{k}}(t)$, $\mathbf{k} \in \mathbb{N}^p$, are holomorphic in a neighbourhood of the origin.

Now we first observe that equation (2.3.5) can be rewritten as $Y_0 = \mathcal{M}_0(Y_0)$, where

$$\mathcal{M}_0(Y_0) := (e^{-t} - \Lambda_0)^{-1}\big(A * Y_0 + D_0(Y_0)\big),$$

while the equation for $\tilde{Y}_{\mathbf{k}}$ reads

$$(e^{-t-\langle \mathbf{k},\boldsymbol{\mu}\rangle} - \Lambda_0)\tilde{Y}_{\mathbf{k}} = B * \tilde{Y}_{\mathbf{k}} - e^{-\langle \mathbf{k},\boldsymbol{\mu}\rangle}\tilde{\beta}_{\mathbf{k}} * e^{-t}\tilde{Y}_{\mathbf{k}} + \tilde{T}_{\mathbf{k}}, \tag{2.5.5}$$

with $\tilde{\beta}_{\mathbf{k}} := \mathcal{B}[(1+x^{-1})^{\langle \mathbf{k},\mathbf{m}\rangle}] = \sum_{j=1}^{\langle \mathbf{k},\mathbf{m}\rangle}\binom{\langle \mathbf{k},\mathbf{m}\rangle}{j}\frac{t^{j-1}}{(j-1)!}$, $\mathbf{m} = \sum_{j=1}^{r} m_j \sum_{h\in\mathcal{J}_j}\mathbf{e}_h$ (m_j as in proposition 2.4.3), B as in (2.3.7) and $\tilde{T}_{\mathbf{k}}$ is defined as $U_{\mathbf{k}}$ in (2.3.9), but with $W_{\mathbf{i}_{mp}}$ replaced by $\tilde{Y}_{\mathbf{i}_{mp}}$. The equation for $\tilde{Y}_{\mathbf{k}}$ can be rewritten as $\tilde{Y}_{\mathbf{k}} = \mathcal{M}_{\mathbf{k}}(\tilde{Y}_{\mathbf{k}})$ with

$$\mathcal{M}_{\mathbf{k}}(\tilde{Y}_{\mathbf{k}}) = \left(e^{-t-\langle \mathbf{k},\boldsymbol{\mu}\rangle} - \Lambda_0\right)^{-1}\left[B * \tilde{Y}_{\mathbf{k}} - e^{-\langle \mathbf{k},\boldsymbol{\mu}\rangle}\tilde{\beta}_{\mathbf{k}} * e^{-t}\tilde{Y}_{\mathbf{k}} + \tilde{T}_{\mathbf{k}}\right].$$

Lemma 2.5.10 *Let θ_{1-}, $\theta_1 = \arg\mu_1$ and θ_{1+} be three consecutive singular directions of the set of all $\hat{y}_{\mathbf{k}}$, $\mathbf{k}\in\mathbb{N}^p$. Then the equation $Y = \mathcal{M}_0(Y)$ possesses a unique solution in $L^1(0,\eta e^{i\theta_1})$ for any $\eta\in(0,|\mu_1|)$. This solution coincides with the solution Y_0 found in proposition 2.3.1. Moreover, the equation $Y = \mathcal{M}_{\mathbf{e}_h}(Y)$, $h\in\mathcal{J}_j$, $j = 1,2,\ldots,r_1$, has a unique solution in $L^1(0,\eta e^{i\theta})$ for any $\theta\in(\theta_{1-},\theta_{1+})$ provided that η is small enough and provided that we prescribe $Y(t) = t^{-\tilde{a}_j-1}\left[\left(\Gamma(-\tilde{a}_j)\right)^{-1}\mathbf{e}_h + o(1)\right]$ as $t\to 0$. This solution coincides with the holomorphic solution $\tilde{Y}_{\mathbf{e}_h}$ mentioned above.*

Proof. Let us first consider the Banach space $L^1_\nu(0,\eta e^{i\theta_1})$, with the corresponding norm $\|Y\|_{L^1_\nu(0,\eta e^{i\theta_1})} := \int_0^\eta e^{-\nu t}|Y(te^{i\theta_1})|\mathrm{d}t$. In the proof of lemma 2.5.4 we already saw that there exists a positive ν_0 such that $\|A\|_{m_1,\nu}\leqslant\frac{K}{\nu}$ for some constant K, provided that $\nu\geqslant\nu_0$. So $\|A\|_{L^1_\nu(0,\eta e^{i\theta_1})}\leqslant\frac{K}{\nu}$, if $\nu\geqslant\nu_0$ and thus

$$\|A * Y\|_{L^1_\nu(0,\eta e^{i\theta_1})} \leqslant \frac{K}{\nu}\|Y\|_{L^1_\nu(0,\eta e^{i\theta_1})},\ \nu\geqslant\nu_0,$$

for every $Y\in L^1_\nu(0,\eta e^{i\theta_1})$. Similarly as in lemma 2.5.4 we may prove the existence of positive constants δ_0 and K_1 such that for all $\nu\geqslant\nu_0$ and $Y\in L^1_\nu(0,\eta e^{i\theta_1})$ with $\|Y\|_{L^1_\nu(0,\eta e^{i\theta_1})}\leqslant\delta\leqslant\delta_0$ we have $\|D_0(Y)\|_{L^1_\nu(0,\eta e^{i\theta_1})}\leqslant(\delta^2+\nu^{-1})K_1$. If also $Z\in L^1_\nu(0,\eta e^{i\theta_1})$ with $\|Z\|_{L^1_\nu(0,\eta e^{i\theta_1})}\leqslant\delta$ then $\|D_0(Y)-D_0(Z)\|_{L^1_\nu(0,\eta e^{i\theta_1})}\leqslant K_2(\delta+\nu^{-1})\|Y-Z\|_{L^1_\nu(0,\eta e^{i\theta_1})}$ for some positive constant K_2.

Using these observations one easily deduces that $\mathcal{M}_0$ defines a contraction on the ball $\{Y\in L^1_\nu(0,\eta e^{i\theta_1})\mid\|Y\|_{L^1_\nu(0,\eta e^{i\theta_1})}\leqslant r\}$, provided that r is small enough and ν is large enough. Hence, there exists a unique solution of $Y = \mathcal{M}_0(Y)$ in $L^1_\nu(0,\eta e^{i\theta_1})$ and as the latter space is equivalent to $L^1(0,\eta e^{i\theta_1})$ we thus have a unique L^1-solution of $Y = \mathcal{M}_0(Y)$. Evidently this solution coincides with the solution Y_0 found in proposition 2.3.1.

Next take a multi-index of length 1. For simplicity we assume $\mathbf{k} = \mathbf{e}_1$ and we consider the equation

$$Y = \left(e^{-t-\mu_1} - \Lambda_0\right)^{-1}\left[B * Y - e^{-\mu_1}\tilde{\beta}_{\mathbf{e}_1} * e^{-t}Y\right]. \tag{2.5.6}$$

Let $\eta\in(0,|\mu_1|-\varepsilon)$, where $\varepsilon = \varepsilon_1$ is defined in the introduction of this section and let $\theta\in(\theta_{1-},\theta_{1+})$. We want to show that (2.5.6) possesses a unique solution $Y\in L^1(0,\eta e^{i\theta})$ with the behaviour $Y(t) = t^{-\tilde{a}_1-1}\left[\left(\Gamma(-\tilde{a}_1)\right)^{-1}\mathbf{e}_1 + o(1)\right]$ as $t\to 0$, provided that η is small enough. Now, if $Y \in L^1(0,\eta e^{i\theta})$ behaves like $Y(t) = t^{-\tilde{a}_1-1}\big[\big(\Gamma(-\tilde{a}_1)\big)^{-1}\mathbf{e}_1 + o(1)\big]$ as $t \to 0$, then $v := \mathcal{P}Y$ is absolutely continuous and as in remark 2.4.4 we may show that v satisfies

$$(e^{-s-\mu_1} - \Lambda_0)v'(s) = -m_1 e^{-s-\mu_1}v(s) + A(0)v(s) + \int_0^s C(-\sigma,-s)v(\sigma)\mathrm{d}\sigma,$$

with C defined in (2.4.6). Splitting this equation after the first n_1 components, we obtain in a similar way as in remark 2.4.4

$$v^{[1]}(s) = -s^{-\tilde{a}_1}\int_0^s \tau^{\tilde{a}_1}\tilde{\alpha}(-\tau)v^{[1]}(\tau)\,\mathrm{d}\tau + \frac{s^{-\tilde{a}_1}}{\Gamma(1-\tilde{a}_1)}\mathbf{e}_1 + \\ - s^{-\tilde{a}_1}\int_0^s \tau^{\tilde{a}_1-1}\alpha(-\tau)\int_0^\tau C^{[1]}(-\sigma,-\tau)v(\sigma)\,\mathrm{d}\sigma\,\mathrm{d}\tau$$

$$v^{\perp}(s) = -\int_0^s \big(e^{-\tau-\mu_1} - (\Lambda_0)^{\perp\perp}\big)^{-1}E(-\tau)v^{\perp}(\tau)\,\mathrm{d}\tau + \\ \int_0^s \big(e^{-\tau-\mu_1} - (\Lambda_0)^{\perp\perp}\big)^{-1}\int_0^\tau C^{\perp}(-\sigma,-\tau)v(\sigma)\,\mathrm{d}\sigma\,\mathrm{d}\tau,$$

where we used the same notation as in (2.4.7).

Conversely, using the contraction mapping principle it is easily seen that this equation has a unique continuous solution v on $(0,\eta e^{i\theta})$, provided that η is small enough and this solution behaves like $v(s) = s^{-\tilde{a}_1}\big[\big(\Gamma(1-\tilde{a}_1)\big)^{-1}\mathbf{e}_1 + o(1)\big]$ as $s \to 0$. Moreover, from the integral equation for v it is clear that v is differentiable at every point $s \in (0,\eta e^{i\theta})$ and its derivative v' behaves like $v'(s) = s^{-\tilde{a}_1-1}\big[\big(\Gamma(-\tilde{a}_1)\big)^{-1}\mathbf{e}_1 + o(1)\big]$ as $s \to 0$. Hence, v' belongs to $L^1(0,\eta e^{i\theta})$ and solves (2.5.6). To show uniqueness, assume that we have two L^1-solutions Y_1 and Y_2 of (2.5.6), with the prescribed behaviour near $t = 0$, then both $v_1 := \mathcal{P}Y_1$ and $v_2 := \mathcal{P}Y_2$ are absolutely continuous solutions of the equation for v. Since absolute continuity implies continuity, both are continuous solutions of the equation for v, and thus they equal each other. Hence, $Y_1 = Y_2$. Since $\tilde{Y}_{\mathbf{e}_1}$ is a solution of (2.5.6), behaves like $\tilde{Y}_{\mathbf{e}_1}(t) = t^{-\tilde{a}_1-1}\big[\big(\Gamma(-\tilde{a}_1)\big)^{-1}\mathbf{e}_1 + o(1)\big]$ as $t \to 0$ and belongs to $L^1(0,\eta e^{i\theta})$, it has to coincide with the solution of (2.5.6) just found. ■

Proposition 2.5.11 *Let $I \subset (\theta_{1-},\theta_{1+})$ be a compact interval containing θ_1 in its interior and let $\varepsilon = \varepsilon_1$ be as in the introduction of this section.*

Given a solution Y of (2.3.5) in $\mathcal{D}'_{m_1,\nu_0}\big(\ell,(0,\mu_1+\varepsilon e^{i\theta_1})\big)$ for some $\nu_0 > 0$ and defining $W_0 = Y$ on $[0,\mu_1+\varepsilon e^{i\theta_1})$, $W_0(t) = Y_0(t)$ if $\arg t \in I\backslash\{\theta_1\}$, $|t| < |\mu_1| + \varepsilon$, and $W_0(t) = 0$ if $\arg t \in I$, $|t| \geqslant |\mu_1| + \varepsilon$, there exists a positive $\nu_1 > \nu_0$ such that for every $\theta \in I$, W_0 extends uniquely to a solution of (2.3.5) in $\mathcal{D}'_{m_1,\nu}(\ell,\theta)$, provided that $\nu \geqslant \nu_1$. This extension coincides with Y_0 except on $[\mu_1,\infty e^{i\theta_1})$. If $Y = Y_0^{\pm}$, as defined in the introduction of section 2.4.2, then also the extension in $\mathcal{D}'_{m_1,\nu}(\ell,\theta_1)$ will be denoted by $Y_0^{\pm}$.

For all $\mathbf{k} \in \mathbb{N}^p\backslash\{0\}$ and all $\theta \in I\backslash\{\theta_1\}$ the equation $Y = \mathcal{M}_{\mathbf{k}}(Y)$ possesses a unique solution in $\mathcal{D}'_{m_1,\nu}(\ell,\theta)$, provided that $\nu \geqslant \nu_1$ and provided that for the multi-indices $\mathbf{k} = \mathbf{e}_h$, $h \in \mathcal{J}_j$, $j = 1,2,\ldots,r_1$, we prescribe $Y(t) = t^{-\tilde{a}_j-1}\big[\big(\Gamma(-\tilde{a}_j)\big)^{-1}\mathbf{e}_h + o(1)\big]$ as $t \to 0$. These *solutions coincide with the holomorphic solutions* $\tilde{Y}_{\mathbf{k}}$, $\mathbf{k} \in \mathbb{N}_1^p$, *defined in (2.5.4). If, for* $\mathbf{k} \in \mathbb{N}^p \backslash \{0\}$, *we define recursively equations for* $\tilde{Y}_{\mathbf{k}}^{\pm}$, *corresponding to* $\tilde{Y}_{\mathbf{k}'}^{\pm}$ *with* $\mathbf{k}' \prec \mathbf{k}$, *on* $\arg t = \theta_1$, *then these equations have unique solutions* $\tilde{Y}_{\mathbf{k}}^{\pm}$ *in* $\mathcal{D}'_{m_1,\nu}(\ell, \theta_1)$ *for* $\nu \geqslant \nu_1$.

Moreover, for $\nu \geqslant \nu_1$ *we have* $\lim_{\sigma \to 0} \|\tilde{Y}_{\mathbf{k}}(\cdot e^{i\sigma}) - \tilde{Y}_{\mathbf{k}}\|_{m_1,\nu,\theta,\ell} = 0$ *for all* θ *in the part of* I *where* $\theta \geqslant \theta_1$ *and the part where* $\theta \leqslant \theta_1$ *respectively, provided that we choose* $\tilde{Y}_{\mathbf{k}} = \tilde{Y}_{\mathbf{k}}^{+}$ *and* $\tilde{Y}_{\mathbf{k}} = \tilde{Y}_{\mathbf{k}}^{-}$ *if* $\theta = \theta_1$ *in the first part and the second part respectively. Here we have to choose* σ *in the limit process such that both* θ *and* $\theta + \sigma$ *belong to the same part of* I.

There exist positive constants K *and* $\nu_2 > \nu_1$ *such that for all* $\nu \geqslant \nu_2$, $0 \prec \mathbf{k} \in \mathbb{N}^p$ *and half lines* $\theta \in I$ *we have*

$$\|\tilde{Y}_{\mathbf{k}}\|_{m_1,\nu,\theta,\ell} \leqslant K^{|\mathbf{k}|}. \tag{2.5.7}$$

If $\theta = \theta_1$ *we may choose either* $\tilde{Y}_{\mathbf{k}} = \tilde{Y}_{\mathbf{k}}^{+}$ *or* $\tilde{Y}_{\mathbf{k}} = \tilde{Y}_{\mathbf{k}}^{-}$ *in (2.5.7).*

The functions $\tilde{Y}_{\mathbf{k}}$, $\mathbf{k} \in \mathbb{N}^p$, *are Laplace transformable in* $\mathcal{D}'_{m_1,\nu}(\ell, \theta)$ *for any* $\theta \in I$ *and* $y_{\mathbf{k}}(x) = x^{-\langle \mathbf{k}, \tilde{\mathbf{a}} \rangle}(\mathcal{L}_\theta \tilde{Y}_{\mathbf{k}})(x)$ *is a solution of (2.1.1) if* $\mathbf{k} = 0$ *and of (2.2.4) if* $\mathbf{k} \succ 0$. *Here again we may choose either* $\tilde{Y}_{\mathbf{k}} = \tilde{Y}_{\mathbf{k}}^{+}$ *or* $\tilde{Y}_{\mathbf{k}} = \tilde{Y}_{\mathbf{k}}^{-}$ *if* $\theta = \theta_1$ *(cf. [Cos98], lemma 20).*

Remark 2.5.12

1. Since the interval length $\ell \in (|\mu_1|/2, |\mu_1|)$ remains unchanged we will omit in the following the index ℓ in the norm $\|\cdot\|_{m_1,\nu,\theta,\ell}$.

2. The statement $\lim_{\sigma \to 0} \|\tilde{Y}_{\mathbf{k}}(\cdot e^{i\sigma}) - \tilde{Y}_{\mathbf{k}}\|_{m_1,\nu,\theta} = 0$ shows that $\tilde{Y}_{\mathbf{k}}$ varies continuously on half lines $\arg t = \theta$ in the $\mathcal{D}'_{m_1,\nu}(\ell, \cdot)$-topology, provided that we 'stay' on one side of the singular ray $\arg t = \theta_1$. This immediately implies that the norm $\|\tilde{Y}_{\mathbf{k}}\|_{m_1,\nu,\theta}$ is continuous in θ on both $I \cap (\theta_{1-}, \theta_1]$ and $I \cap [\theta_1, \theta_{1+})$, provided that we make the correct choice of $\tilde{Y}_{\mathbf{k}}$ on the singular ray $\arg t = \theta_1$.

Proof of proposition 2.5.11. We first assume that I is chosen in such a way that $(|\mu_1| + \varepsilon)\cos\theta > \Re\mu_1$ for all $\theta \in I$. Let W_0 be as in the proposition, then obviously $W_0 \in L^1[0, \infty e^{i\theta}) \subset L^1_\nu(\theta)$, $\nu > 0$, for all $\theta \in I \backslash \{\theta_1\}$. Hence, for all $\nu > \nu_0$ and all $\theta \in I$ the expression W_0 belongs to $\mathcal{D}'_{m_1,\nu}\big(\ell, (0, (|\mu_1| + \varepsilon)e^{i\theta})\big)$.

If both θ and $\theta + \sigma$ belong to either $I \cap (\theta_{1-}, \theta_1]$ or $I \cap [\theta_1, \theta_{1+})$, then the integral $\int_0^{\ell e^{i\theta}} e^{-\nu|t|}|Y_0(te^{i\sigma}) - Y_0(t)||\mathrm{d}t|$ tends to 0 as $\sigma \to 0$. In the case $\theta \neq \theta_1$, the expression $\int_{\ell e^{i\theta}}^{(|\mu_1|+\varepsilon)e^{i\theta}} e^{-\nu|t|}|(\mathcal{P}^{m_1}Y_0)(te^{i\sigma}) - (\mathcal{P}^{m_1}Y_0)(t)||\mathrm{d}t|$ also tends to 0 as $\sigma \to 0$, provided that $|\mu_1| + \varepsilon \leqslant 2\ell$. If $\theta = \theta_1$, then for small but positive σ, $(\mathcal{P}^{m_1}Y_0)(\mu_1 + te^{i\sigma}) - (\mathcal{P}^{m_1}Y_0^{+})(\mu_1 + t)$ can be written as $(te^{i\sigma})^{-\tilde{a}_1 - 1}u_1(te^{i\sigma}) + u_2(te^{i\sigma}) - t^{-\tilde{a}_1 - 1}u_1(t) - u_2(t)$ for some holomorphic functions u_1 and u_2 and again $\int_{\ell e^{i\theta_1}}^{\mu_1 + \varepsilon e^{i\theta_1}} e^{-\nu|t|}|(\mathcal{P}^{m_1}Y_0)(te^{i\sigma}) - (\mathcal{P}^{m_1}Y_0^{+})(t)||\mathrm{d}t|$ tends to 0 as $\sigma \downarrow 0$. A similar statement may be deduced when choosing Y_0^{-} instead of Y_0^{+} on the singular ray. Altogether this implies that

$$\|W_0(\cdot e^{i\sigma}) - W_0\|_{m_1,\nu,\theta} \to 0 \quad \text{as} \quad \sigma \to 0, \tag{2.5.8}$$

provided that we choose $Y = Y_0^{-}$, $\sigma < 0$, respectively $Y = Y_0^{+}$, $\sigma > 0$, if $\arg t = \theta_1$. Hence, $\|W_0\|_{m_1,\nu,\theta}$ is continuous in θ separately in $I \cap (\theta_{1-}, \theta_1]$ and $I \cap [\theta_1, \theta_{1+})$ if we make the correct choice on the singular ray $\arg t = \theta_1$. From this, together with proposition A.3.7, we conclude that $\|W_0\|_{m_1,\nu,\theta}$ tends to 0 as $\nu \to \infty$, uniformly in $\theta \in I$. If, on the singular ray, we choose Y different from $Y_0^{\pm}$, then also $\|W_0\|_{m_1,\nu,\theta_1} = \|Y\|_{m_1,\nu,\theta_1}$ tends to 0 as $\nu \to \infty$.

We will solve $W = \tilde{\mathcal{M}}_0(W)$ in $\mathcal{D}'_{m_1,\nu}(\ell,\theta)$ for each $\theta \in I$, where

$$\begin{cases} \tilde{\mathcal{M}}_0(W) := 0 & \text{on } \mathcal{D}'_{m_1,\nu}\big(\ell,(0,(|\mu_1|+\varepsilon)e^{i\theta})\big) \\ \tilde{\mathcal{M}}_0(W) := \mathcal{M}_0(W_0+W) & \text{on } \mathcal{D}'_{m_1,\nu}\big(\ell,((|\mu_1|+\varepsilon)e^{i\theta},\infty e^{i\theta})\big), \end{cases}$$

because then $Y = W_0 + W$ is a solution of (2.3.5). Here $\mathcal{M}_0$ is the integral operator defined in the introduction of this section. Using lemma 2.5.7, together with remark 2.5.8, we deduce that multiplication with $U(t) := (e^{-t} - \Lambda_0)^{-1}$ is a bounded operator on the set of staircase distributions $\mathcal{D}'_{m_1,\nu}\big(\ell,\big((|\mu_1|+\varepsilon)e^{i\theta},\infty e^{i\theta}\big)\big)$. Assume this multiplication operator to be bounded by C. Let K, K_1, K_2 and δ_0 be the constants as they appear in lemma 2.5.4 and lemma 2.5.6. Without loss of generality we may assume that $\delta_0 < 1$. Define $K_3 = \max\{K_1, K_2\}$ and assume r to be so small that $r < \delta_0/2$ and $4CK_3 r < 1/2$. Moreover, assume ν to be so large that $C(K+K_3)/\nu < r/2$ and $\|W_0\|_{m_1,\nu,\theta} \leqslant r$. Then, if $W \in \mathcal{D}'_{m_1,\nu}(\ell,\theta)$ and $\|W\|_{m_1,\nu,\theta} \leqslant r$ we have, with lemma 2.5.4,

$$\|\tilde{\mathcal{M}}_0(W)\|_{m_1,\nu,\theta} \leqslant C\big\{2Kr\nu^{-1} + \big((2r)^2 + \nu^{-1}\big)K_1\big\} \leqslant C\big\{(K+K_1)\nu^{-1} + r\cdot 4K_1 r\big\} \leqslant r.$$

Hence, $\tilde{\mathcal{M}}_0$ maps the ball $\{W \in \mathcal{D}'_{m_1,\nu}(\ell,\theta) \mid \|W\|_{m_1,\nu,\theta} \leqslant r\}$ into itself. Moreover, if both W and $\tilde{W}$ belong to this ball, then

$$\|\tilde{\mathcal{M}}_0(W) - \tilde{\mathcal{M}}_0(\tilde{W})\|_{m_1,\nu,\theta} \leqslant C\big\{K\nu^{-1} + \big(2r+\nu^{-1}\big)K_2\big\}\|W-\tilde{W}\|_{m_1,\nu,\theta} \leqslant \frac{1}{2}\|W-\tilde{W}\|_{m_1,\nu,\theta}.$$

Hence, $\tilde{\mathcal{M}}_0$ defines a contraction on the Banach space $\mathcal{D}'_{m_1,\nu}(\ell,\theta)$ and thus there is a unique solution W of (2.3.5) in $\mathcal{D}'_{m_1,\nu}(\ell,\theta)$. This solution extends the given solution W_0.

On nonsingular rays θ the function Y_0 is of at most exponential growth, and thus it belongs to $L^1_\nu(\theta)$ for some ν large enough. So proposition A.4.1 implies that $Y_0 \in \mathcal{D}'_{m_1,\nu}(\ell,\theta)$ for every $\theta \neq \theta_1$, provided that ν is large enough. Hence, $Y = Y_0$ on nonsingular rays. On the singular ray $\arg t = \theta_1$ we know that Y_0 is the unique solution in $L^1(0,\eta e^{i\theta_1})$ for every $\eta \in (0,|\mu_1|)$, and thus $Y = Y_0$ also on $[0,\mu_1)$. We conclude that $Y = Y_0$ except on $[\mu_1,\infty e^{i\theta_1})$. Note that both r and ν can be chosen independently of $\theta \in I$.

To prove the continuity of Y_0 in $\arg t$ in the $\mathcal{D}'_{m_1,\nu}(\ell,\cdot)$-topology on $I \cap (\theta_{1-},\theta_1]$, we first remark that this already holds for W_0, provided that we choose $Y = Y_0^-$ on $[0,\mu_1+\varepsilon e^{i\theta_1})$, compare (2.5.8). Let θ,σ be such that both θ and $\theta+\sigma$ belong to this part of I, then we will show that the expression $\|\rho_\sigma W - W\|_{m_1,\nu,\theta}$ tends to 0 as $\sigma \to 0$. This obviously is equivalent to $\|W(\cdot e^{i\sigma}) - W\|_{m_1,\nu,\theta} \to 0$ as $\sigma \to 0$. Now, the equation $(\rho_\sigma - 1)W = (\rho_\sigma - 1)\tilde{\mathcal{M}}_0(W)$ restricted to $\mathcal{D}'_{m_1,\nu}\big(\ell,((|\mu_1|+\varepsilon)e^{i\theta},\infty e^{i\theta})\big)$ may be written as

$$\begin{aligned} (\rho_\sigma - 1)W \;=\; & \big(U(te^{i\sigma}) - U(t)\big)\big\{A*(W_0+W) + D_0(W_0+W)\big\} \;+ \\ & U(te^{i\sigma})\big\{\rho_\sigma(A)*\rho_\sigma(W_0+W) - A*(W_0+W)\big\} \;+ \\ & U(te^{i\sigma})\big\{\rho_\sigma(D_0(W_0+W)) - D_0(W_0+W)\big\}. \end{aligned}$$

Since $U(te^{i\sigma}) - U(t)$ equals $(e^{-t} - e^{-te^{i\sigma}})U(t)U(te^{i\sigma})$, multiplication with this function is a bounded operator on staircase distributions and its bound tends to 0 as $\sigma \to 0$. Using lemmas 2.5.4, 2.5.6 and 2.5.7 one may show that

$$\|(\rho_\sigma - 1)W\|_{m_1,\nu,\theta} \leqslant \{CK\nu^{-1} + CK_2(2r + \nu^{-1})\}\|(\rho_\sigma - 1)(W_0 + W)\|_{m_1,\nu,\theta} + o(1),$$

where $o(1)$ here means an expression that tends to 0 as $\sigma \to 0$. Using (2.5.8) we then obtain

$$\|(\rho_\sigma - 1)W\|_{m_1,\nu,\theta} \leqslant \{CK\nu^{-1} + CK_2(2r + \nu^{-1})\}\|(\rho_\sigma - 1)W\|_{m_1,\nu,\theta} + o(1)$$

and as $C(K+K_2)\nu^{-1} + 2CK_2r$ is smaller than 1, we conclude that $\|(\rho_\sigma - 1)W\|_{m_1,\nu,\theta}$ tends to 0 as $\sigma \to 0$. This implies the continuity of $Y = W_0 + W$ on $I \cap (\theta_{1-}, \theta_1]$, provided that we choose $Y = Y_0^-$ on the singular ray $\arg t = \theta_1$. The continuity on the part of I where $\theta \geqslant \theta_1$ can be shown in a similar way.

If $\mathbf{k} \in \mathbb{N}^p$ has length 1, then we define $W_\mathbf{k}(t) := \tilde{Y}_\mathbf{k}(t)$ on the Riemann surface above $0 < |t| \leqslant \eta$, where η is so small that the equation $Y = \mathcal{M}_\mathbf{k}(Y)$ has $\tilde{Y}_\mathbf{k}$ as unique solution in $L^1(0, \eta e^{i\theta})$ (compare lemma 2.5.10). As $\mathcal{D}'_{m_1,\nu}(\ell, (0, \eta e^{i\theta}))$ coincides with $L^1(0, \eta e^{i\theta})$, the function $W_\mathbf{k}$ is the unique solution of $Y = \mathcal{M}_\mathbf{k}(Y)$ in $\mathcal{D}'_{m_1,\nu}(\ell, (0, \eta e^{i\theta}))$. In a similar way as above we may use the contraction mapping principle to find a unique solution $W \in \mathcal{D}'_{m_1,\nu}(\theta)$ of (2.5.5), which extends the given $W_\mathbf{k}$, provided that ν is large enough. Here we have used that $\|e^{-\langle \mathbf{k},\boldsymbol{\mu}\rangle}\tilde{\beta}_\mathbf{k}\|_{m_1,\nu,\theta}$, $\mathbf{k} \in \mathbb{N}^p$, $|\mathbf{k}| = 1$, can be made arbitrarily small. This follows from (A.4.1), together with the fact that for a general multi-index $\mathbf{k} \in \mathbb{N}_1^p$ we have

$$\|e^{-\langle \mathbf{k},\boldsymbol{\mu}\rangle}\tilde{\beta}_\mathbf{k}\|_{\nu,\theta} \leqslant e^{-\Re(\langle \mathbf{k},\boldsymbol{\mu}\rangle)} \sum_{j=1}^{\langle \mathbf{k},\mathbf{m}\rangle} \binom{\langle \mathbf{k},\mathbf{m}\rangle}{j} \nu^{-j} \leqslant \nu^{-1} \langle \mathbf{k},\mathbf{m}\rangle\, e^{-\Re(\langle \mathbf{k},\boldsymbol{\mu}\rangle)} \bigl(1 + \nu^{-1}\bigr)^{\langle \mathbf{k},\mathbf{m}\rangle}.$$

Therefore $\nu\|e^{-\langle \mathbf{k},\boldsymbol{\mu}\rangle}\tilde{\beta}_\mathbf{k}\|_{\nu,\theta}$ can be estimated by a constant independent of ν and $\mathbf{k}$, provided that ν is large enough. Again ν can be chosen independently of $\theta \in I$. The continuity in $\arg t$ may be obtained in a similar way as above.

Next we consider $Y = \mathcal{M}_\mathbf{k}(Y)$ with $\mathbf{k} \in \mathbb{N}_2^p$. As $\arg(\mu_j - \langle \mathbf{k},\boldsymbol{\mu}\rangle + 2l\pi i) \notin (\theta_{1-}, \theta_{1+})$ for all $\mathbf{k} \in \mathbb{N}_2^p$, $j \in \{1, 2, \ldots, r\}$ and $l \in \mathbb{Z}$ we conclude from lemma 2.5.7 that multiplication with $(e^{-t-\langle \mathbf{k},\boldsymbol{\mu}\rangle} - \Lambda_0)^{-1}$, working on the set of staircase distributions, is a bounded operator. Moreover, for $j \in \{1, 2, \ldots, r\}$ we have $\Re(\mu_j - \langle \mathbf{k},\boldsymbol{\mu}\rangle) \geqslant 0$ for only finitely many $\mathbf{k} \in \mathbb{N}_2^p$, and the proof of lemma 2.5.7(ii) in fact implies that multiplication with $(e^{-t-\langle \mathbf{k},\boldsymbol{\mu}\rangle} - \Lambda_0)^{-1}$ is bounded by a constant independent of $\mathbf{k} \in \mathbb{N}^p$. Now the equation $Y_\mathbf{k} = \mathcal{M}_\mathbf{k}(Y)$ can be solved directly in the Banach space $\mathcal{D}'_{m_1,\nu}(\ell, \theta)$, by choosing ν large enough and its solution coincides with $\tilde{Y}_\mathbf{k}$ on nonsingular rays. The fact that ν can be chosen to be independent of both $\mathbf{k} \in \mathbb{N}^p$ and $\theta \in I$ follows from the fact that the norm of $e^{-\langle \mathbf{k},\boldsymbol{\mu}\rangle}\tilde{\beta}_\mathbf{k}$ can be majorized by ν^{-1} times a constant independent of ν and $\mathbf{k}$. The continuity simply follows by estimating $(\rho_\sigma - 1)\tilde{Y}_\mathbf{k} = (\rho_\sigma - 1)\mathcal{M}_\mathbf{k}\tilde{Y}_\mathbf{k}$.

Finally to prove (2.5.7) we first define $f_\mathbf{k} := \max_{\theta \in I} \|\tilde{Y}_\mathbf{k}\|_{m_1,\nu_1,\theta}$ if $|\mathbf{k}| = 1$, where ν_1 is as in the proposition. For arbitrary $\mathbf{k} \in \mathbb{N}_2^p$ we now make the following remarks: From the small estimates for $\|A\|_{m_1,\nu,\theta}$ and $\|D_{\mathbf{j}}(Y_0)\|_{m_1,\nu,\theta}$ for $|\mathbf{j}| = 1$ (cf. lemma 2.5.4) we deduce that the norm of $\big(A + D(Y_0)\big) * \tilde{Y}_{\mathbf{k}}$ can be estimated by a constant times $\|\tilde{Y}_{\mathbf{k}}\|_{m_1,\nu,\theta}$. This constant can be made arbitrarily small, uniformly in $\theta \in I$, by choosing ν large enough. Moreover, the norm of $e^{-\langle \mathbf{k},\boldsymbol{\mu}\rangle}\tilde{\beta}_{\mathbf{k}}$ in the $\mathcal{D}'_{m_1,\nu}(\ell,\cdot)$-topology is bounded by ν^{-1} times a constant which is independent of ν and $\mathbf{k}$. Hence, the norm of $e^{-\langle \mathbf{k},\boldsymbol{\mu}\rangle}\tilde{\beta}_{\mathbf{k}} * e^{-t}\tilde{Y}_{\mathbf{k}}$ can be majorized by a constant times $\nu^{-1}\|\tilde{Y}_{\mathbf{k}}\|_{m_1,\nu,\theta}$. So we conclude the existence of positive constants C and $\nu_2 > \nu_1$, both independent of $\mathbf{k}$ and θ, such that for $\mathbf{k} \in \mathbb{N}_2^p$ we have $\|\tilde{Y}_{\mathbf{k}}\|_{m_1,\nu,\theta} \leqslant C\|\tilde{T}_{\mathbf{k}}\|_{m_1,\nu,\theta}$, for all $\nu \geqslant \nu_2$.

We define $f_{\mathbf{k}}$ for $|\mathbf{k}| > 1$ recursively as follows. Suppose that for some integer $r > 1$ we have $\|\tilde{Y}_{\mathbf{k}}\|_{m_1,\nu_2,\theta} \leqslant f_{\mathbf{k}}$ for all $\mathbf{k} \in \mathbb{N}^p$, $1 \leqslant |\mathbf{k}| < r$, where $f_{\mathbf{k}}$ are certain θ-independent constants. For a multi-index $\mathbf{k}$ of length r we then have

$$\|\tilde{T}_{\mathbf{k}}\|_{m_1,\nu_2,\theta} \leqslant \sum_{2\leqslant|\mathbf{j}|\leqslant|\mathbf{k}|} \big(|g_{\mathbf{j},0}| + \|D_{\mathbf{j}}(Y_0)\|_{m_1,\nu_2,\theta}\big) \sum_{(\mathbf{i}_{mp};\mathbf{k})} \prod_{m=1}^{n} \prod_{p=1}^{j_m} \|\tilde{Y}_{\mathbf{i}_{mp}}\|_{m_1,\nu_2,\theta},$$

which in turn can be estimated by $\tilde{K}\sum_{2\leqslant|\mathbf{j}|\leqslant|\mathbf{k}|} K_1^{|\mathbf{j}|} \sum_{(\mathbf{i}_{mp};\mathbf{k})} \prod_{m=1}^{n}\prod_{p=1}^{j_m} f_{\mathbf{i}_{mp}}$, with $\tilde{K}$ some constant independent of $\mathbf{k}$, ν and θ, and K_1 the constant found in lemma 2.5.4.

The number of multi-indices $\mathbf{j} \in \mathbb{N}^n$ such that $|\mathbf{j}| = h$ is equal to $\binom{n+h-1}{h}$, which is bounded by 2^{n+h-1} (cf. the proof of lemma 2.5.4). Consequently we have

$$\|\tilde{T}_{\mathbf{k}}\|_{m_1,\nu_2,\theta} \leqslant 2^{n-1}\tilde{K}\sum_{h=2}^{|\mathbf{k}|}(2K_1)^h \sum_{(\mathbf{i}_m;\mathbf{k})} \prod_{m=1}^{h} f_{\mathbf{i}_m},$$

where $\sum_{(\mathbf{i}_m;\mathbf{k})}$ denotes the sum over all $\mathbf{i}_m \in \mathbb{N}^p$ with $\mathbf{i}_m \succ 0$ and $\sum_{m=1}^{h}\mathbf{i}_m = \mathbf{k}$. Now let $M = 2^{n-1}C\tilde{K}$ and define $f_{\mathbf{k}} := M\sum_{h=2}^{|\mathbf{k}|}(2K_1)^h\sum_{(\mathbf{i}_m;\mathbf{k})}\prod_{m=1}^{h} f_{\mathbf{i}_m}$. Then we have $\|\tilde{Y}_{\mathbf{k}}\|_{m_1,\nu_2,\theta} \leqslant f_{\mathbf{k}}$ for all $\theta \in I$, $|\mathbf{k}| = r$, and therefore the recursive definition of $f_{\mathbf{k}}$.

Next define $f_1 : \mathbb{C}^p \to \mathbb{C}$ by $f_1(v) = \sum_{m=1}^{p} f_{\mathbf{e}_m} v_m$, then it is easy to verify that the formal series $\hat{f}(v) := \sum_{\mathbf{i}\in\mathbb{N}_2^p} f_{\mathbf{i}} v^{\mathbf{i}}$ satisfies

$$f(v) = M\sum_{h=2}^{\infty}\{2K_1(f_1(v)+f(v))\}^h = M\frac{[2K_1(f_1(v)+f(v))]^2}{1-2K_1(f_1(v)+f(v))}.$$

This equation has a unique holomorphic solution f with $f(v) = M[2K_1 f_1(v)]^2(1+O(f_1))$ as $f_1 \to 0$. So the formal series $\hat{f}$ converges and there exists a positive constant $K > 0$ such that $f_{\mathbf{k}} < K^{|\mathbf{k}|}$. This proves the proposition in case the compact interval I is such that $(|\mu_1| + \varepsilon)\cos\theta > \Re\mu_1$ for all $\theta \in I$.

For a general compact subset I of $(\theta_{1-}, \theta_{1+})$ one can write I as a union of at most three compact subsets I_1, I_2 and I_3, where I_1 is such that $\theta_1 \in I_1$ and $(|\mu_1| + \varepsilon)\cos\theta > \Re\mu_1$ for all $\theta \in I_1$, $I_2 \subset (\theta_{1-}, \theta_1)\backslash I_1$ and $I_3 \subset (\theta_1, \theta_{1+})\backslash I_1$. Then on I_2 and I_3 one may apply the same method but now in $L^1_\nu(\theta)$ and we use proposition A.4.1 to complete the proof. ∎

2.6 Higher Order Resurgence Relations

In this section we want to prove some higher order resurgence relations in the sense of Écalle (cf. proposition 2.6.4). The proof of these relations is similar to the one Costin gave in [Cos98].

2.6.1 Two Decomposition Lemmas

The first lemma gives an expression for the general solution of (2.3.5) on the singular ray $\{t \in \mathbb{C}^* \mid \arg t = \theta_1\}$ (cf. [Cos98], proposition 23). From this general solution we already obtain a generalisation of the resurgence relation (2.4.10). Moreover, it turns out that this general solution will help us to find a decomposition of $\tilde{Y}_{\mathbf{k}}^{\pm}$ on this singular ray (lemma 2.6.2). The latter decomposition helps in proving the higher order resurgence relations.

Lemma 2.6.1 *The general solution Y of (2.3.5) in $\mathcal{D}'_{m_1,\nu}(\ell,\theta_1)$, $\ell \in (|\mu_1|/2, |\mu_1|)$, can be written as*

$$Y = Y_0^+ + \sum_{\mathbf{j}\in\mathbb{N}_1^{n_1}} \mathbf{C}^{\mathbf{j}} \big\{\tilde{Y}_{\mathbf{j}}^+(t-|\mathbf{j}|\mu_1)\mathbf{1}_{[|\mathbf{j}|\mu_1,\infty e^{i\theta_1})}\big\}^{(m_1|\mathbf{j}|)}, \tag{2.6.1}$$

where $\mathbf{C} \in \mathbb{C}^{n_1}$ is an arbitrary vector of constants, provided that ν is large enough. A similar statement holds with all $+$-signs replaced by $-$-signs.
In particular $Y_0^{\pm} = Y_0^{\mp} + \sum_{\mathbf{j}\in\mathbb{N}_1^{n_1}} (\pm\mathbf{s})^{\mathbf{j}} \big\{\tilde{Y}_{\mathbf{j}}^{\mp}(t-|\mathbf{j}|\mu_1)\mathbf{1}_{[|\mathbf{j}|\mu_1,\infty e^{i\theta_1})}\big\}^{(m_1|\mathbf{j}|)}$, where $\mathbf{s}$ is the vector with components s_h, $h = 1, 2, \ldots, n_1$, found in proposition 2.4.3.

Proof. First we observe that for $\mathbf{j} \in \mathbb{N}^{n_1}$ the expression $\big(\tilde{Y}_{\mathbf{j}}^+(t-|\mathbf{j}|\mu_1)\mathbf{1}_{[|\mathbf{j}|\mu_1,\infty e^{i\theta_1})}\big)^{(m_1|\mathbf{j}|)}$ belongs to $\mathcal{D}'_{m_1,\nu}(\ell,\theta_1)$. This is an immediate consequence of proposition A.6.1. This proposition also shows that $\|(\tilde{Y}_{\mathbf{j}}^+(t-|\mathbf{j}|\mu_1)\mathbf{1}_{[|\mathbf{j}|\mu_1,\infty e^{i\theta_1})})^{(m_1|\mathbf{j}|)}\|_{m_1,\nu,\theta_1} \leqslant CK(\nu)^{|\mathbf{j}|}\|\tilde{Y}_{\mathbf{j}}^+\|_{m_1,\nu,\theta_1}$, where $\lim_{\nu\to\infty} K(\nu) = 0$. Together with (2.5.7) this proves convergence of the right-hand side of (2.6.1) in the set of staircase distributions $\mathcal{D}'_{m_1,\nu}(\ell,\theta_1)$ for ν large enough.

Let us first assume that the right-hand side of (2.6.1) indeed is a solution of (2.3.5) for any choice of C. To see that there are no other solutions, it suffices by proposition 2.5.11 to check that the restriction of the right-hand side of (2.6.1) to $\mathcal{D}(0, \mu_1 + \varepsilon e^{i\theta_1})$ is the general solution of (2.3.5) in $\mathcal{D}'_{m_1,\nu}\big(\ell, (0, \mu_1 + \varepsilon e^{i\theta_1})\big)$ for $\varepsilon > 0$ sufficiently small. Without loss of generality we may assume that $[0, \mu_1 + \varepsilon e^{i\theta_1}) \subset [0, 2\ell e^{i\theta_1})$.

Now Y_0^+ is a solution of (2.3.5) in $\mathcal{D}'_{m_1,\nu}\big(\ell, (0, \mu_1 + \varepsilon e^{i\theta_1})\big)$ by construction. If Y is another solution in $\mathcal{D}'_{m_1,\nu}\big(\ell, (0, \mu_1 + \varepsilon e^{i\theta_1})\big)$ and $\Psi := Y - Y_0^+$, then $\Psi = 0$ on $[0, \mu_1)$ and $\mathcal{P}^{m_1}\Psi \in L^1(0, \mu_1 + \varepsilon e^{i\theta_1})$. Using (2.4.1) it is easily seen that Ψ satisfies the equation

$$(e^{-t} - \Lambda_0)\Psi = B * \Psi, \tag{2.6.2}$$

where $B = A + D(Y_0)$. Here we used that for every $\mathbf{l} \in \mathbb{N}_2^n$ we have $\Psi^{*\mathbf{l}} = 0$ in $\mathcal{D}'_{m_1,\nu}\big(\ell, (0, \mu_1 + \varepsilon e^{i\theta_1})\big)$. Put $\psi(s) := \mathcal{P}^{m_1+1}\Psi$, then $\psi = 0$ on $[0, \mu_1)$ and ψ is absolutely continuous on $[0, \mu_1 + \varepsilon e^{i\theta_1})$. Applying $\mathcal{P}^{m_1}$ to both left-hand and right-hand side of (2.6.2) and defining $v(s) := \psi(\mu_1 + s)$ we obtain, in a similar way as in the proof of proposition 2.4.2,

$$(e^{-s-\mu_1} - \Lambda_0)v'(s) = -m_1 e^{-s-\mu_1} v(s) + A(0)v(s) + \int_0^s C(-\sigma,-s)v(\sigma)\mathrm{d}\sigma,$$

with C defined in (2.4.6). Splitting this equation after the first n_1 components we obtain

$$\frac{\mathrm{d}}{\mathrm{d}s}\big(s^{\tilde{a}_1}(v^{[1]})(s) = -s^{\tilde{a}_1}\tilde{\alpha}(-s)v^{[1]}(s) - s^{\tilde{a}_1-1}\alpha(-s)\int_0^s C^{[1]}(-\sigma,-s)v(\sigma)\,\mathrm{d}\sigma \tag{2.6.3a}$$

$$\frac{\mathrm{d}}{\mathrm{d}s}v^{\perp}(s) = \big(e^{-s-\mu_1} - (\Lambda_0)^{\perp\perp}\big)^{-1}\Big\{-E(-s)v^{\perp}(s) + \int_0^s C^{\perp}(-\sigma,-s)v(\sigma)\,\mathrm{d}\sigma\Big\}, \tag{2.6.3b}$$

where we used the same notation as in (2.4.7). Integrating the latter equation from η to s, with $\eta \in (0, \varepsilon e^{i\theta_1})$, we get

$$\begin{aligned} s^{\tilde{a}_1}v^{[1]}(s) - \eta^{\tilde{a}_1}v^{[1]}(\eta) = &-\int_\eta^s \tau^{\tilde{a}_1}\tilde{\alpha}(-\tau)v^{[1]}(\tau)\,\mathrm{d}\tau\ + \\ &- \int_\eta^s \tau^{\tilde{a}_1-1}\alpha(-\tau)\int_0^\tau C^{[1]}(-\sigma,-\tau)v(\sigma)\,\mathrm{d}\sigma\,\mathrm{d}\tau \end{aligned} \tag{2.6.4a}$$

$$\begin{aligned} v^{\perp}(s) - v^{\perp}(\eta) = &-\int_\eta^s \big(e^{-\tau-\mu_1} - (\Lambda_0)^{\perp\perp}\big)^{-1}E(-\tau)v^{\perp}(\tau)\,\mathrm{d}\tau\ + \\ &\int_\eta^s \big(e^{-\tau-\mu_1} - (\Lambda_0)^{\perp\perp}\big)^{-1}\int_0^\tau C^{\perp}(-\sigma,-\tau)v(\sigma)\,\mathrm{d}\sigma\,\mathrm{d}\tau. \end{aligned} \tag{2.6.4b}$$

From (2.6.4a) it follows that $v^{[1]}(s) = s^{-\tilde{a}_1}\big(\mathbf{c} + o(1)\big)$ as $s \to 0$ for some constant vector $\mathbf{c} \in \mathbb{C}^{n_1}$. Moreover, (2.6.4) also implies that v' exists on $(0, \varepsilon e^{i\theta_1})$ and $(v^{[1]})'$ behaves like $(v^{[1]})'(s) = s^{-\tilde{a}_1-1}\big(-\tilde{a}_1\mathbf{c} + o(1)\big)$ as $s \to 0$. Hence, we can let η in (2.6.4) tend to 0 to obtain the integral equation as given in remark 2.4.4, but now with v absolutely continuous. Combining remark 2.4.4 with lemma 2.5.10, one easily infers that Ψ may be written as a linear combination of $\big(\tilde{Y}_{\mathbf{e}_h}(t-\mu_1)\mathbb{1}_{[\mu_1,\mu_1+\varepsilon e^{i\theta_1})}\big)^{(m_1)}$, $h = 1, 2, \ldots, n_1$. So indeed the restriction of the right-hand side of (2.6.1) to $\mathcal{D}(0, \mu_1 + \varepsilon e^{i\theta_1})$ is the general solution of (2.3.5) in $\mathcal{D}'_{m_1,\nu}\big(\ell, (0, \mu_1 + \varepsilon e^{i\theta_1})\big)$.

So in fact we only have to check that the right-hand side of (2.6.1) indeed is a solution of (2.3.5) in $\mathcal{D}'_{m_1,\nu}(\ell, \theta_1)$. Let us write for short

$$Y = Y_0^+ + \sum_{\mathbf{j}\in\mathbb{N}_1^{n_1}} \big(W_{\mathbf{j}}(t - |\mathbf{j}|\mu_1)\mathbb{1}_{[|\mathbf{j}|\mu_1,\infty e^{i\theta_1})}\big)^{(m_1|\mathbf{j}|)}, \tag{2.6.5}$$

where $W_{\mathbf{j}} = \mathbf{C}^{\mathbf{j}}\tilde{Y}_{\mathbf{j}}^+$. Now, if $\mathbf{j}_1$ and $\mathbf{j}_2$ are two multi-indices in $\mathbb{N}^{n_1}$ with length N_1 and N_2 respectively, then we have

$$\begin{aligned} \big(W_{\mathbf{j}_1}(t - N_1\mu_1)\mathbb{1}_{[N_1\mu_1,\infty e^{i\theta_1})}\big)^{(m_1N_1)} * \big(W_{\mathbf{j}_2}(t - N_2\mu_1)\mathbb{1}_{[N_2\mu_1,\infty e^{i\theta_1})}\big)^{(m_1N_2)} = \\ \big((W_{\mathbf{j}_1} * W_{\mathbf{j}_2})(t - (N_1+N_2)\mu_1)\mathbb{1}_{[(N_1+N_2)\mu_1,\infty e^{i\theta_1})}\big)^{(m_1N_1+m_1N_2)}. \end{aligned} \tag{2.6.6}$$

By substituting the right-hand side of (2.6.5) in $D_0(Y) = G_0 + \sum_{\mathbf{l}\succ 0}\big(g_{\mathbf{l},0} + G_{\mathbf{l}} *\big)Y^{*\mathbf{l}}$, and using (2.6.6), we obtain after a tedious but straightforward calculation

$$D_0(Y) \;=\; D_0(Y_0^+) + \sum_{\mathbf{j}\in\mathbb{N}_1^{n_1}} \left\{(D(Y_0^+) * W_{\mathbf{j}})(t - |\mathbf{j}|\mu_1)\mathbb{1}_{[|\mathbf{j}|\mu_1,\infty e^{i\theta_1})}\right\}^{(m_1|\mathbf{j}|)} +$$

$$\sum_{\mathbf{j}\in\mathbb{N}_2^{n_1}} \sum_{2\leqslant|\mathbf{l}|\leqslant|\mathbf{j}|} \Big\{\big(g_{\mathbf{l},0} + D_{\mathbf{l}}(Y_0^+)*\big) \sum_{(\mathbf{i}_{mp};\mathbf{j})} \prod_{m=1}^{n}{}^{*} \prod_{p=1}^{l_m}{}^{*} (W_{\mathbf{i}_{mp}})_m (t - |\mathbf{j}|\mu_1)\mathbb{1}_{[|\mathbf{j}|\mu_1,\infty e^{i\theta_1})}\Big\}^{(m_1|\mathbf{j}|)},$$

which can be written more conveniently as

$$D_0(Y) = \sum_{N=0}^{\infty} \big(D_{0,N}(t - N\mu_1)\mathbb{1}_{[N\mu_1,\infty e^{i\theta_1})}\big)^{(m_1 N)}, \tag{2.6.7}$$

with $D_{0,0} = D_0(Y_0^+)$ and for $N \geqslant 1$

$$D_{0,N} = \sum_{\substack{\mathbf{j}\in\mathbb{N}^{n_1}\\|\mathbf{j}|=N}} \Big\{D(Y_0^+) * W_{\mathbf{j}} + \sum_{2\leqslant|\mathbf{l}|\leqslant|\mathbf{j}|} \big(g_{\mathbf{l},0} + D_{\mathbf{l}}(Y_0^+)*\big) \sum_{(\mathbf{i}_{mp};\mathbf{j})} \prod_{m=1}^{n}{}^{*} \prod_{p=1}^{l_m}{}^{*} (W_{\mathbf{i}_{mp}})_m\Big\}.$$

Hence, the equation $(e^{-t} - \Lambda_0)Y = A * Y + D_0(Y)$ is satisfied if for $N = 1, 2, 3, \ldots$ the $W_{\mathbf{j}}$'s satisfy

$$\sum_{\substack{\mathbf{j}\in\mathbb{N}^{n_1}\\|\mathbf{j}|=N}} (e^{-t} - \Lambda_0)\big(W_{\mathbf{j}}(t - N\mu_1)\mathbb{1}_{[N\mu_1,\infty e^{i\theta_1})}\big)^{(m_1 N)} =$$

$$\sum_{\substack{\mathbf{j}\in\mathbb{N}^{n_1}\\|\mathbf{j}|=N}} \big((A * W_{\mathbf{j}})(t - N\mu_1)\mathbb{1}_{[N\mu_1,\infty e^{i\theta_1})}\big)^{(m_1 N)} + \big(D_{0,N}(t - N\mu_1)\mathbb{1}_{[N\mu_1,\infty e^{i\theta_1})}\big)^{(m_1 N)},$$

which obviously is equivalent to

$$\sum_{\substack{\mathbf{j}\in\mathbb{N}^{n_1}\\|\mathbf{j}|=N}} (e^{-t-N\mu_1} - \Lambda_0)W_{\mathbf{j}}^{(m_1 N)} = \sum_{\substack{\mathbf{j}\in\mathbb{N}^{n_1}\\|\mathbf{j}|=N}} (A * W_{\mathbf{j}})^{(m_1 N)} + D_{0,N}^{(m_1 N)}$$

in $\mathcal{D}'_{m_1,\nu}(\ell, \theta_1)$. Using Leibniz's formula we deduce $e^{-t}W_{\mathbf{j}}^{(m_1 N)} = \sum_{k=0}^{m_1 N} \binom{m_1 N}{k} \big[e^{-t}W_{\mathbf{j}}\big]^{(k)}$ and thus $\mathcal{P}^{m_1 N}\big[e^{-t}W_{\mathbf{j}}^{(m_1 N)}\big] = e^{-t}W_{\mathbf{j}} + \sum_{k=1}^{m_1 N} \binom{m_1 N}{k}\mathcal{P}^k\big[e^{-t}W_{\mathbf{j}}\big]$. Since $\mathcal{P}^k\big[e^{-t}W_{\mathbf{j}}\big]$ can be written as $\int_0^t \frac{(t-s)^{k-1}}{(k-1)!}e^{-s}W_{\mathbf{j}}(s)\mathrm{d}s$ we have $\mathcal{P}^{m_1 N}\big[e^{-t}W_{\mathbf{j}}^{(m_1 N)}\big] = e^{-t}W_{\mathbf{j}} + \tilde{\beta}_{\mathbf{j}} * e^{-t}W_{\mathbf{j}}$, provided that $\mathbf{j} \in \mathbb{N}^{n_1}$ and $|\mathbf{j}| = N$. So applying $\mathcal{P}^{m_1 N}$ to the equation for the $W_{\mathbf{j}}$'s with $\mathbf{j} \in \mathbb{N}^{n_1}$, $|\mathbf{j}| = N$, we finally obtain

$$\sum_{\substack{\mathbf{j}\in\mathbb{N}^{n_1}\\|\mathbf{j}|=N}} (e^{-t-N\mu_1} - \Lambda_0)W_{\mathbf{j}} = \sum_{\substack{\mathbf{j}\in\mathbb{N}^{n_1}\\|\mathbf{j}|=N}} A * W_{\mathbf{j}} \;+\; D_{0,N} \;-\; e^{-N\mu_1} \sum_{\substack{\mathbf{j}\in\mathbb{N}^{n_1}\\|\mathbf{j}|=N}} \tilde{\beta}_{\mathbf{j}} * e^{-t}W_{\mathbf{j}}.$$

From (2.5.5) it follows that this equation is satisfied, since $W_{\mathbf{j}} = C^{\mathbf{j}}\tilde{Y}_{\mathbf{j}}^{+}$. To show the last part of the lemma we simply compare left-hand side with right-hand side of (2.6.1) on $[0, \mu_1 + \varepsilon e^{i\theta_1})$ and we apply (2.4.10). ■

Lemma 2.6.2 *Let θ_{1-} and θ_{1+} be as in the introduction of section 2.5.3. Let $I \subset (\theta_{1-}, \theta_{1+})$ be an arbitrary compact interval containing θ_1 in its interior. For $\mathbf{k} \in \mathbb{N}^p$ there exists a decomposition of $\tilde{Y}_{\mathbf{k}}^{+}$ on the singular ray $\arg t = \theta_1$ of the form*

$$\tilde{Y}_{\mathbf{k}}^{+} = \sum_{\mathbf{j}\in\mathbb{N}^{n_1}} \left(\tilde{Y}_{\mathbf{k},\mathbf{j}}^{-}(t - |\mathbf{j}|\mu_1)\mathbf{1}_{[|\mathbf{j}|\mu_1,\infty e^{i\theta_1})}\right)^{(m_1|\mathbf{j}|)}, \tag{2.6.8}$$

with each $\tilde{Y}_{\mathbf{k},\mathbf{j}}^{-} \in \mathcal{D}'_{m_1,\nu}(\ell, \theta_1)$ satisfying the following properties:

(i) *The function $t \mapsto t^{\langle \mathbf{k}+\mathbf{j},\tilde{\mathbf{a}}\rangle+1}\tilde{Y}_{\mathbf{k},\mathbf{j}}^{-}(t)$ is holomorphic in a neighbourhood of the origin. Moreover, $\tilde{Y}_{\mathbf{k},\mathbf{j}}^{-}$ can be analytically extended to a holomorphic function of at most exponential growth in $\{t \in \mathbb{C}^* \mid \arg t \in (\theta_{1-}, \theta_1)\}$.*

(ii) *There exist positive constants K and ν_2, both independent of $\mathbf{k}$ and $\mathbf{j}$, such that for all $\nu \geqslant \nu_2$ and all $\theta \in I \cap (\theta_{1-}, \theta_1]$, we also have an extension of $\tilde{Y}_{\mathbf{k},\mathbf{j}}^{-}$ in $\mathcal{D}'_{m_1,\nu}(\ell, \theta)$ and*

$$\|\tilde{Y}_{\mathbf{k},\mathbf{j}}^{-}\|_{m_1,\nu,\theta} \leqslant K^{|\mathbf{k}|+|\mathbf{j}|}, \quad \mathbf{k}+\mathbf{j} \succ 0. \tag{2.6.9}$$

On the rays $\arg t = \theta \in (\theta_{1-}, \theta_1)$ both extensions coincide. Moreover, on these rays $\tilde{Y}_{\mathbf{k},0}^{-}$ coincides with $\tilde{Y}_{\mathbf{k}}$, while on the singular ray θ_1 we have $\tilde{Y}_{\mathbf{k},0}^{-} = \tilde{Y}_{\mathbf{k}}^{-}$.

(iii) *For $\nu \geqslant \nu_2$ we have $\lim_{\sigma\to 0} \|\tilde{Y}_{\mathbf{k},\mathbf{j}}^{-}(\cdot e^{i\sigma}) - \tilde{Y}_{\mathbf{k},\mathbf{j}}^{-}\|_{m_1,\nu,\theta} = 0$ for all $\theta \in I \cap (\theta_{1-}, \theta_1]$, provided that σ is chosen such that $\theta + \sigma$ belongs to $I \cap (\theta_{1-}, \theta_1]$.*

A similar statement holds for $\tilde{Y}_{\mathbf{k}}^{-}$, with $\tilde{Y}_{\mathbf{k},\mathbf{j}}^{-}$ replaced by certain expressions $\tilde{Y}_{\mathbf{k},\mathbf{j}}^{+}$.

Remark 2.6.3

1. For $\mathbf{k} \in \mathbb{N}^p$ and $\mathbf{j} \in \mathbb{N}^{n_1}$ an expression like $\langle \mathbf{k}+\mathbf{j}, \tilde{\mathbf{a}}\rangle$ has to be interpreted as $\sum_{h\in\mathcal{J}_1}(k_h + j_h)\tilde{a}_1 + \sum_{j=2}^{r_1}\sum_{h\in\mathcal{J}_j} k_h\tilde{a}_j$. Here $\mathcal{J}_j$, $j = 1, 2, \ldots, r_1$, is defined as the set $\{n_1 + \cdots + n_{j-1} + 1, \ldots, n_1 + \cdots + n_j\}$.

2. From (2.6.9), together with proposition A.6.1 one may conclude that the right-hand side of (2.6.8) indeed converges in the space of staircase distributions $\mathcal{D}'_{m_1,\nu}(\ell, \theta)$, provided that ν is large enough.

Proof of lemma 2.6.2. We will give the proof, with exception of (2.6.9), by using induction on $|\mathbf{k}|$. We will show (2.6.9) at the end. First we take $\mathbf{k} = 0$. In the preceding lemma we found a decomposition of $\tilde{Y}_0^{+} = Y_0^{+}$ of the form (2.6.8) with $\tilde{Y}_{0,\mathbf{j}}^{-} = \mathbf{s}^{\mathbf{j}}\tilde{Y}_{\mathbf{j}}^{-}$. However, in a neighbourhood of the origin we have $\tilde{Y}_{\mathbf{j}}^{-} = \tilde{Y}_{\mathbf{j}}$ and using proposition 2.5.11 we immediately obtain the properties of $\tilde{Y}_{0,\mathbf{j}}^{-}$ listed in the lemma.

Now suppose the statement to be true for all multi-indices $\mathbf{k} \in \mathbb{N}^p$ with $|\mathbf{k}| < \ell$ for some $\ell \in \mathbb{N}_+$. Then take a multi-index $\mathbf{k}$ of length ℓ. By definition $\tilde{Y}_{\mathbf{k}}^+$ satisfies

$$(e^{-t-\langle \mathbf{k},\boldsymbol{\mu}\rangle} - \Lambda_0)\tilde{Y}_{\mathbf{k}}^+ \;=\; \big(A + D(Y_0^+)\big) * \tilde{Y}_{\mathbf{k}}^+ \;-\; e^{-\langle \mathbf{k},\boldsymbol{\mu}\rangle}\tilde{\beta}_{\mathbf{k}} * e^{-t}\tilde{Y}_{\mathbf{k}}^+ \;+\; \tilde{T}_{\mathbf{k}}^+, \tag{2.6.10}$$

where $\tilde{T}_{\mathbf{k}}^+ := \sum_{2\leqslant|\mathbf{j}|\leqslant|\mathbf{k}|}\big(g_{\mathbf{j},0} + D_{\mathbf{j}}(Y_0^+)*\big)\sum_{(\mathbf{i}_{mp};\mathbf{k})}\prod_{m=1}^{*n}\prod_{p=1}^{*j_m}(\tilde{Y}_{\mathbf{i}_{mp}}^+)_m$. Using lemma 2.6.1 we have $D_{\mathbf{j}}(Y_0^+) = D_{\mathbf{j}}(Y_0^- + R)$, where $R = \sum_{\mathbf{r}\in\mathbb{N}_1^{n_1}}\mathbf{s}^{\mathbf{r}}\big\{\tilde{Y}_{\mathbf{r}}^-(t - |\mathbf{r}|\mu_1)1_{[|\mathbf{r}|\mu_1,\infty e^{i\theta_1})}\big\}^{(m_1|\mathbf{r}|)}$. Hence,

$$\begin{aligned}
D_{\mathbf{j}}(Y_0^+) &= G_{\mathbf{j}} + \sum_{\mathbf{k}\succ\mathbf{j}}\binom{\mathbf{k}}{\mathbf{j}}(g_{\mathbf{k},0} + G_{\mathbf{k}}*)\sum_{0\preceq\mathbf{l}\preceq\mathbf{k}-\mathbf{j}}\binom{\mathbf{k}-\mathbf{j}}{\mathbf{l}}(Y_0^-)^{*(\mathbf{k}-\mathbf{j}-\mathbf{l})} * R^{*\mathbf{l}}\\
&= D_{\mathbf{j}}(Y_0^-) + \sum_{\mathbf{k}\succ\mathbf{j}}\sum_{0\prec\mathbf{l}\preceq\mathbf{k}-\mathbf{j}}\binom{\mathbf{k}}{\mathbf{j}}\binom{\mathbf{k}-\mathbf{j}}{\mathbf{l}}(g_{\mathbf{k},0} + G_{\mathbf{k}}*)(Y_0^-)^{*(\mathbf{k}-\mathbf{j}-\mathbf{l})} * R^{*\mathbf{l}}\\
&= D_{\mathbf{j}}(Y_0^-) + \sum_{\mathbf{l}\succ 0}\binom{\mathbf{l}+\mathbf{j}}{\mathbf{l}}\big(g_{\mathbf{l}+\mathbf{j},0} + D_{\mathbf{l}+\mathbf{j}}(Y_0^-)*\big)R^{*\mathbf{l}}.
\end{aligned}$$

Using (2.6.6) the expression $R^{*\mathbf{l}}$ can be expanded as

$$R^{*\mathbf{l}} = \sum_{\mathbf{r}\succ 0}\mathbf{s}^{\mathbf{r}}\sum_{(\mathbf{i}_{mp};\mathbf{r})}\left\{\left(\prod_{m=1}^{n}{}^{*}\prod_{p=1}^{l_m}{}^{*}\{(\tilde{Y}_{\mathbf{i}_{mp}}^-)_m\right)(t - |\mathbf{r}|\mu_1)1_{[|\mathbf{r}|\mu_1,\infty e^{i\theta_1})}\right\}^{(m_1|\mathbf{r}|)},$$

so $D_{\mathbf{j}}(Y_0^+)$, $\mathbf{j} \in \mathbb{N}_1^n$, can be written as

$$D_{\mathbf{j}}(Y_0^+) = \sum_{\mathbf{r}\in\mathbb{N}^{n_1}}\big(D_{\mathbf{j},\mathbf{r}}(t - |\mathbf{r}|\mu_1)1_{[|\mathbf{r}|\mu_1,\infty e^{i\theta_1})}\big)^{(m_1|\mathbf{r}|)}, \tag{2.6.11}$$

with $D_{\mathbf{j},0} = D_{\mathbf{j}}(Y_0^-)$ and for $\mathbf{r} \in \mathbb{N}_1^{n_1}$

$$D_{\mathbf{j},\mathbf{r}} = \mathbf{s}^{\mathbf{r}}\sum_{1\leqslant|\mathbf{l}|\leqslant|\mathbf{r}|}\binom{\mathbf{l}+\mathbf{j}}{\mathbf{l}}\big(g_{\mathbf{l}+\mathbf{j},0} + D_{\mathbf{l}+\mathbf{j}}(Y_0^-)*\big)\sum_{(\mathbf{i}_{mp};\mathbf{r})}\prod_{m=1}^{n}{}^{*}\prod_{p=1}^{l_m}{}^{*}(\tilde{Y}_{\mathbf{i}_{mp}}^-)_m.$$

Using (2.6.6) and (2.6.8) (with $|\mathbf{k}| < \ell$) we deduce that $\sum_{(\mathbf{i}_{mp};\mathbf{k})}\prod_{m=1}^{*n}\prod_{p=1}^{*j_m}(\tilde{Y}_{\mathbf{i}_{mp}}^+)_m$ can be written as

$$\sum_{\mathbf{r}\in\mathbb{N}^{n_1}}\left\{\left(\sum_{(\mathbf{i}_{mp};\mathbf{k})}\sum_{(\mathbf{q}_{mp}:\mathbf{r})}{}'\prod_{m=1}^{n}{}^{*}\prod_{p=1}^{j_m}{}^{*}(\tilde{Y}_{\mathbf{i}_{mp},\mathbf{q}_{mp}}^-)_m\right)(t - |\mathbf{r}|\mu_1)1_{[|\mathbf{r}|\mu_1,\infty e^{i\theta_1})}\right\}^{(m_1|\mathbf{r}|)},$$

where $\sum'_{(\mathbf{q}_{mp}:\mathbf{r})}$ denotes the sum over all multi-indices $\mathbf{q}_{mp} \in \mathbb{N}^{n_1}$ with $1 \leqslant p \leqslant j_m$, $1 \leqslant m \leqslant n$ and $\sum_{m=1}^{n}\sum_{p=1}^{j_m}\mathbf{q}_{mp} = \mathbf{r}$. Using this expansion, (2.6.11) and again (2.6.6), we obtain the following decomposition of $\tilde{T}_{\mathbf{k}}^+$:

$$\tilde{T}_{\mathbf{k}}^+ = \sum_{\mathbf{r}\in\mathbb{N}^{n_1}}\big(T_{\mathbf{k},\mathbf{r}}(t - |\mathbf{r}|\mu_1)1_{[|\mathbf{r}|\mu_1,\infty e^{i\theta})}\big)^{(m_1|\mathbf{r}|)}, \tag{2.6.12}$$

with $T_{\mathbf{k},0} = \tilde{T}_{\mathbf{k}}^-$ (which is defined as $\tilde{T}_{\mathbf{k}}^+$ with all $+$-signs replaced by $-$-signs) and for $\mathbf{r} \in \mathbb{N}_1^{n_1}$

$$T_{\mathbf{k},\mathbf{r}} = \sum_{2\leqslant|\mathbf{j}|\leqslant|\mathbf{k}|} \big(g_{\mathbf{j},0} + D_{\mathbf{j}}(Y_0^-)*\big) \sum_{(\mathbf{i}_{mp};\mathbf{k})} \sum_{(\mathbf{q}_{mp}:\mathbf{r})}' \prod_{m=1}^{n}{}^{*} \prod_{p=1}^{j_m}{}^{*} (\tilde{Y}^-_{\mathbf{i}_{mp},\mathbf{q}_{mp}})_m$$
$$+ \sum_{2\leqslant|\mathbf{j}|\leqslant|\mathbf{k}|} \sum_{0\preceq\mathbf{l}\prec\mathbf{r}} D_{\mathbf{j},\mathbf{r}-\mathbf{l}} * \sum_{(\mathbf{i}_{mp};\mathbf{k})} \sum_{(\mathbf{q}_{mp}:\mathbf{l})}' \prod_{m=1}^{n}{}^{*} \prod_{p=1}^{j_m}{}^{*} (\tilde{Y}^-_{\mathbf{i}_{mp},\mathbf{q}_{mp}})_m.$$

We will now proceed as in the proof of the preceding proposition. The requirement that $\tilde{Y}_{\mathbf{k}}^+$ satisfies (2.6.10) first gives the following equation for $\tilde{Y}^-_{\mathbf{k},0}$:

$$(e^{-t-\langle\mathbf{k},\boldsymbol{\mu}\rangle} - \Lambda_0)\tilde{Y}^-_{\mathbf{k},0} = \big(A + D(Y_0^-)\big) * \tilde{Y}^-_{\mathbf{k},0} - e^{-\langle\mathbf{k},\boldsymbol{\mu}\rangle}\tilde{\beta}_{\mathbf{k}} * e^{-t}\tilde{Y}^-_{\mathbf{k},0} + \tilde{T}^-_{\mathbf{k}},$$

which is exactly the equation for $\tilde{Y}^-_{\mathbf{k}}$. Hence, we shall take $\tilde{Y}^-_{\mathbf{k},0} = \tilde{Y}^-_{\mathbf{k}}$ on the singular ray $\arg t = \theta_1$. Taking $\tilde{Y}^-_{\mathbf{k},0} = \tilde{Y}_{\mathbf{k}}$ on the rays $\arg t = \theta \in (\theta_{1-}, \theta_1)$ one easily infers that this choice satisfies the required properties. Next assume that we found $\tilde{Y}^-_{\mathbf{k},\mathbf{r}}$ for all multi-indices $|\mathbf{r}|$ of length $\leqslant N-1$ for some $N \geqslant 1$. With a similar argument as in the proof of the preceding lemma we see that $W_{\mathbf{k},N} := \sum_{\substack{\mathbf{r}\in\mathbb{N}^{n_1}\\|\mathbf{r}|=N}} \tilde{Y}^-_{\mathbf{k},\mathbf{r}}$ should satisfy

$$(e^{-t-\langle\mathbf{k},\boldsymbol{\mu}\rangle-N\mu_1} - \Lambda_0)W^{(m_1N)}_{\mathbf{k},N} = \big(A + D(Y_0^-)\big) * W^{(m_1N)}_{\mathbf{k},N} - e^{-\langle\mathbf{k},\boldsymbol{\mu}\rangle-N\mu_1}\tilde{\beta}_{\mathbf{k}} * e^{-t}W^{(m_1N)}_{\mathbf{k},N} +$$
$$\sum_{\substack{\mathbf{r}\in\mathbb{N}^{n_1}\\|\mathbf{r}|=N}} T^{(m_1N)}_{\mathbf{k},\mathbf{r}} + \sum_{\substack{\mathbf{r}\in\mathbb{N}^{n_1}\\|\mathbf{r}|=N}} \sum_{|\mathbf{j}|=1} \sum_{0\preceq\mathbf{l}\prec\mathbf{r}} \big(D_{\mathbf{j},\mathbf{r}-\mathbf{l}} * (\tilde{Y}^-_{\mathbf{k},\mathbf{l}})^{*\mathbf{j}}\big)^{(m_1N)}$$

in $\mathcal{D}'_{m_1,\nu}\big(\ell, (N\mu_1, \infty e^{i\theta_1})\big)$. Applying $\mathcal{P}^{m_1N}$ to both sides of this equation we obtain

$$(e^{-t-\langle\mathbf{k}+N\mathbf{e}_1,\boldsymbol{\mu}\rangle} - \Lambda_0)W_{\mathbf{k},N} = \big(A + D(Y_0^-)\big) * W_{\mathbf{k},N} - e^{-\langle\mathbf{k}+N\mathbf{e}_1,\boldsymbol{\mu}\rangle}\tilde{\beta}_{\mathbf{k}+N\mathbf{e}_1} * e^{-t}W_{\mathbf{k},N} +$$
$$\sum_{\substack{\mathbf{r}\in\mathbb{N}^{n_1}\\|\mathbf{r}|=N}} T_{\mathbf{k},\mathbf{r}} + \sum_{\substack{\mathbf{r}\in\mathbb{N}^{n_1}\\|\mathbf{r}|=N}} \sum_{|\mathbf{j}|=1} \sum_{0\preceq\mathbf{l}\prec\mathbf{r}} D_{\mathbf{j},\mathbf{r}-\mathbf{l}} * (\tilde{Y}^-_{\mathbf{k},\mathbf{l}})^{*\mathbf{j}}.$$

Here we have used that $\sum_{h=1}^{m_1N}\binom{m_1N}{h}\frac{t^{h-1}}{(h-1)!} + \sum_{h=0}^{m_1N}\binom{m_1N}{h}\mathcal{P}^h\tilde{\beta}_{\mathbf{k}} = \tilde{\beta}_{\mathbf{k}+N\mathbf{e}_1} = \tilde{\beta}_{\mathbf{k}+\mathbf{r}}$.

Now, this equation is satisfied if each $\tilde{Y}^-_{\mathbf{k},\mathbf{r}}$ satisfies

$$(e^{-t-\langle\mathbf{k}+\mathbf{r},\boldsymbol{\mu}\rangle} - \Lambda_0)\tilde{Y}^-_{\mathbf{k},\mathbf{r}} = \big(A + D(Y_0^-)\big) * \tilde{Y}^-_{\mathbf{k},\mathbf{r}} - e^{-\langle\mathbf{k}+\mathbf{r},\boldsymbol{\mu}\rangle}\tilde{\beta}_{\mathbf{k}+\mathbf{r}} * e^{-t}\tilde{Y}^-_{\mathbf{k},\mathbf{r}} +$$
$$T_{\mathbf{k},\mathbf{r}} + \sum_{|\mathbf{j}|=1} \sum_{0\preceq\mathbf{l}\prec\mathbf{r}} D_{\mathbf{j},\mathbf{r}-\mathbf{l}} * (\tilde{Y}^-_{\mathbf{k},\mathbf{l}})^{*\mathbf{j}}$$

and the latter equation can be written in the more general form:

$$(e^{-t-\langle\mathbf{k}+\mathbf{r},\boldsymbol{\mu}\rangle} - \Lambda_0)\tilde{Y}^-_{\mathbf{k},\mathbf{r}} = \big(A + D(\tilde{Y}^-_{0,0})\big) * \tilde{Y}^-_{\mathbf{k},\mathbf{r}} - e^{-\langle\mathbf{k}+\mathbf{r},\boldsymbol{\mu}\rangle}\tilde{\beta}_{\mathbf{k}+\mathbf{r}} * e^{-t}\tilde{Y}^-_{\mathbf{k},\mathbf{r}} +$$
$$T^{gen}_{\mathbf{k},\mathbf{r}} + \sum_{|\mathbf{j}|=1} \sum_{0\preceq\mathbf{l}\prec\mathbf{r}} D^{gen}_{\mathbf{j},\mathbf{r}-\mathbf{l}} * (\tilde{Y}^-_{\mathbf{k},\mathbf{l}})^{*\mathbf{j}}, \tag{2.6.13}$$

where $D^{gen}_{\mathbf{j},\mathbf{r}}$ is defined as $D_{\mathbf{j},\mathbf{r}}$, but with Y_0^- and $\tilde{Y}^-_{\mathbf{i}_{mp}}$ replaced by $\tilde{Y}^-_{0,0}$ and $\tilde{Y}^-_{\mathbf{i}_{mp},0}$ respectively and where $T^{gen}_{\mathbf{k},\mathbf{r}}$ is defined as $T_{\mathbf{k},\mathbf{r}}$, but with Y_0^- and $D_{\mathbf{j},\mathbf{r}}$ replaced by $\tilde{Y}^-_{0,0}$ and $D^{gen}_{\mathbf{j},\mathbf{r}}$ respectively.

One easily deduces that there exists a solution $\tilde{Y}^-_{\mathbf{k},\mathbf{r}}$ of (2.6.13) such that $t^{\langle \mathbf{k}+\mathbf{r},\tilde{\mathbf{a}}\rangle+1}\tilde{Y}^-_{\mathbf{k},\mathbf{r}}(t)$ is holomorphic in a neighbourhood of the origin (for example by using the contraction mapping principle together with the Banach space $\mathcal{V}$ consisting of n-vector valued functions Y such that $t \mapsto t^{\langle \mathbf{k}+\mathbf{r},\tilde{\mathbf{a}}\rangle+1}Y(t)$ is continuous on $\overline{\Delta}(0,\varepsilon)$ and holomorphic in the interior for some $\varepsilon > 0$, endowed with the norm $\|Y\| := \sup\big\{|t^{\langle \mathbf{k}+\mathbf{r},\tilde{\mathbf{a}}\rangle+1}Y(t)| \bigm| |t| \leqslant \varepsilon\big\}$). As in [Bra80] we may extend this solution to a holomorphic solution that is of at most exponential growth in the sector bounded by $\arg t = \theta_{1-}$ and $\arg t = \theta_1$. Moreover, as in the proof of proposition 2.5.11, there also exists a solution $\tilde{Y}^-_{\mathbf{k},\mathbf{r}}$ in $\mathcal{D}'_{m_1,\nu}(\ell,\theta)$, $\theta \in I$, for ν large enough, say $\nu \geqslant \nu_0$, and this ν_0 can be chosen independent of $\mathbf{k}$, $\mathbf{r}$ and θ. Obviously, both solutions coincide on rays $\theta \in (\theta_{1-},\theta_1)$. The continuity of $\tilde{Y}^-_{\mathbf{k},\mathbf{r}}$ in $\arg t$ in the $\mathcal{D}'_{m_1,\nu}(\ell,\cdot)$-topology also follows in a similar way as in the proof of proposition 2.5.11.

To prove (2.6.9) one should distinguish the cases where I is a small compact interval containing θ_1 in its interior and I an arbitrary compact interval containing no singular directions (compare proposition 2.5.11). As both cases are proven in exactly the same way, we forget about this distinction. In the following $\|\cdot\|_\nu$ either denotes $\|\cdot\|_{\nu,\theta}$ or $\|\cdot\|_{m_1,\nu,\theta}$. First observe that for each $\mathbf{j} \in \mathbb{N}_1^n$ and $\mathbf{r} \in \mathbb{N}^{n_1}$, the expression $\|D^{gen}_{\mathbf{j},\mathbf{r}}\|_\nu$ can be estimated by $\tilde{K}^{|\mathbf{j}|+|\mathbf{r}|}$ for some $\tilde{K}$ large enough, provided that ν is large enough. Here we used (2.3.4), (2.5.7) and the estimate for the norm of $D_{\mathbf{j}}(Y)$ found in lemma 2.5.4.

Now we start proving (2.6.9) for a fixed $\mathbf{k} \in \mathbb{N}^p$ of length 1 and, as in the proof of proposition 2.5.11, we first define $f_{\mathbf{k},0} := \|\tilde{Y}^-_{\mathbf{k},0}\|_{\nu_0}$. From (2.6.13) it is easy to deduce the existence of positive constants C and $\nu_1 > \nu_0$, both independent of $\mathbf{r}$ and θ, such that

$$\|\tilde{Y}^-_{\mathbf{k},\mathbf{r}}\|_\nu \leqslant C\Big\|\sum_{|\mathbf{j}|=1}\sum_{0\preceq\mathbf{l}\prec\mathbf{r}} D^{gen}_{\mathbf{j},\mathbf{r}-\mathbf{l}} * (\tilde{Y}^-_{\mathbf{k},\mathbf{l}})^{*\mathbf{j}}\Big\|_\nu$$

for all $\nu \geqslant \nu_1$ and all $\theta \in I \cap (\theta_-,\theta_1]$. Now let $N \geqslant 1$ and suppose that $\|\tilde{Y}^-_{\mathbf{k},\mathbf{r}}\|_{\nu_1}$ can be estimated by $f_{\mathbf{k},\mathbf{r}}$ for all $\mathbf{r} \in \mathbb{N}^{n_1}$, $|\mathbf{r}| \leqslant N-1$, then for a multi-index $\mathbf{r}$ of length N we will derive a similar estimate for $\|\tilde{Y}^-_{\mathbf{k},\mathbf{r}}\|_{\nu_1}$ with some $f_{\mathbf{k},\mathbf{r}}$ which can be defined recursively:

$$\|\tilde{Y}^-_{\mathbf{k},\mathbf{r}}\|_{\nu_1} \leqslant M_1 \sum_{0\preceq\mathbf{l}\prec\mathbf{r}} \tilde{K}^{|\mathbf{r}|-|\mathbf{l}|} f_{\mathbf{k},\mathbf{l}} =: f_{\mathbf{k},\mathbf{r}},$$

where M_1 is some positive constant, independent of $\mathbf{r}$ and θ. Hence, $f_{\mathbf{k}}(u) := \sum_{\mathbf{r}\in\mathbb{N}_1^{n_1}} f_{\mathbf{k},\mathbf{r}}u^{\mathbf{r}}$ is a solution of

$$f_{\mathbf{k}}(u) = M_1\big(f_{\mathbf{k},0} + f_{\mathbf{k}}(u)\big)\sum_{\mathbf{r}\in\mathbb{N}_1^{n_1}} \tilde{K}^{|\mathbf{r}|}u^{\mathbf{r}}.$$

The latter equation obviously has a solution $f_{\mathbf{k}}(u)$ which is holomorphic in a neighbourhood of $0 \in \mathbb{C}^{n_1}$, and thus there exists a constant K such that $f_{\mathbf{k},\mathbf{r}} \leqslant K^{|\mathbf{r}|}$. This proves (2.6.9) in case the length of $\mathbf{k}$ equals 1. Moreover, the function $f_1 : \mathbb{C}^{n_1} \times \mathbb{C}^p \to \mathbb{C}$ defined by

$f_1(u,v) := \sum_{|\mathbf{i}|=1} \sum_{|\mathbf{r}|\geqslant 0} f_{\mathbf{i},\mathbf{r}}\, u^{\mathbf{r}} v^{\mathbf{i}}$ represents a holomorphic function in a neighbourhood of $(0,0) \in \mathbb{C}^{n_1} \times \mathbb{C}^p$.

For arbitrary multi-indices $\mathbf{k} \in \mathbb{N}_2^p$ and $\mathbf{r} \in \mathbb{N}^{n_1}$ we deduce from (2.6.13)

$$\|\tilde{Y}^-_{\mathbf{k},\mathbf{r}}\|_\nu \leqslant C \Big\| T^{gen}_{\mathbf{k},\mathbf{r}} + \sum_{|\mathbf{j}|=1} \sum_{0 \preceq \mathbf{l} \prec \mathbf{r}} D^{gen}_{\mathbf{j},\mathbf{r}-\mathbf{l}} * (\tilde{Y}^-_{\mathbf{k},\mathbf{l}})^{*\mathbf{j}} \Big\|_\nu$$

for $\nu \geqslant \nu_1$, with possibly enlarged C and ν_1. Now we suppose that $\|\tilde{Y}^-_{\mathbf{k}',\mathbf{r}'}\|_{\nu_1}$, $\mathbf{k}' \prec \mathbf{k}$, $\mathbf{r}' \succeq 0$, can be estimated by $f_{\mathbf{k}',\mathbf{r}'}$, while $\|\tilde{Y}^-_{\mathbf{k},\mathbf{r}'}\|_{\nu_1}$, $0 \preceq \mathbf{r}' \prec \mathbf{r}$, can be estimated by $f_{\mathbf{k},\mathbf{r}'}$. We will show the same for $\|\tilde{Y}^-_{\mathbf{k},\mathbf{r}}\|_{\nu_1}$ by estimating the norm of $T^{gen}_{\mathbf{k},\mathbf{r}} + \sum_{|\mathbf{j}|=1} \sum_{0 \preceq \mathbf{l} \prec \mathbf{r}} D^{gen}_{\mathbf{j},\mathbf{r}-\mathbf{l}} * (\tilde{Y}^-_{\mathbf{k},\mathbf{l}})^{*\mathbf{j}}$. In a similar way and with the same notation as in the proof of proposition 2.5.11 we obtain

$$\|T^{gen}_{\mathbf{k},\mathbf{r}}\| \leqslant const. \cdot \sum_{h=2}^{|\mathbf{k}|} (2\tilde{K})^h \sum_{0 \preceq \mathbf{l} \preceq \mathbf{r}} \tilde{K}^{|\mathbf{r}|-|\mathbf{l}|} \sum_{(\mathbf{i}_m;\mathbf{k})} \sum_{(\mathbf{q}_m;\mathbf{l})}{}' \prod_{m=1}^{h} f_{\mathbf{i}_m,\mathbf{q}_m}.$$

Here $\sum'_{(\mathbf{q}_m;\mathbf{l})}$ denotes the sum over all $\mathbf{q}_m \succeq 0$ with $\sum_{m=1}^h \mathbf{q}_m = \mathbf{l}$. Hence,

$$\|\tilde{Y}^-_{\mathbf{k},\mathbf{r}}\|_{\nu_1} \leqslant M_2 \sum_{h=2}^{|\mathbf{k}|} (2\tilde{K})^h \sum_{0 \preceq \mathbf{l} \preceq \mathbf{r}} \tilde{K}^{|\mathbf{r}|-|\mathbf{l}|} \sum_{(\mathbf{i}_m;\mathbf{k})} \sum_{(\mathbf{q}_m;\mathbf{l})}{}' \prod_{m=1}^{h} f_{\mathbf{i}_m,\mathbf{q}_m} + M_2 \sum_{0 \preceq \mathbf{l} \prec \mathbf{r}} \tilde{K}^{|\mathbf{r}|-|\mathbf{l}|} f_{\mathbf{k},\mathbf{l}} =: f_{\mathbf{k},\mathbf{r}}$$

and we see that $\hat{f}(u,v) := \sum_{\mathbf{k}\in\mathbb{N}_2^p} \sum_{\mathbf{r}\in\mathbb{N}^{n_1}} f_{\mathbf{k},\mathbf{r}}\, u^{\mathbf{r}} v^{\mathbf{k}}$ is a formal solution of

$$f(u,v) \;=\; M_2\, \varphi(u) \sum_{h=2}^{\infty} \{2\tilde{K}(f_1(u,v) + f(u,v))\}^h \;+\; M_2\, \tilde{\varphi}(u) f(u,v),$$

where $\varphi(u) = \sum_{\mathbf{r}\in\mathbb{N}^{n_1}} \tilde{K}^{|\mathbf{r}|} u^{\mathbf{r}}$ and $\tilde{\varphi}(u) = \sum_{\mathbf{r}\in\mathbb{N}_1^{n_1}} \tilde{K}^{|\mathbf{r}|} u^{\mathbf{r}}$. This equation can be rewritten as $f(u,v) = M_2\, \varphi(u) \big(1 - M_2\, \tilde{\varphi}(u)\big)^{-1} \cdot \frac{[2\tilde{K}(f_1(u,v)+f(u,v))]^2}{1-2\tilde{K}(f_1(u,v)+f(u,v))}$ and it has a unique holomorphic solution f with $f(u,v) = M_2\, \varphi(u)\big(1 - M_2\, \tilde{\varphi}(u)\big)^{-1} [2\tilde{K} f_1(u,v)]^2 (1 + O(f_1))$ as $f_1 \to 0$. So the formal series $\hat{f}$ converges in a neighbourhood of $(0,0) \in \mathbb{C}^{n_1} \times \mathbb{C}^p$ and there exists a positive constant K such that $f_{\mathbf{k},\mathbf{r}} \leqslant K^{|\mathbf{k}|+|\mathbf{r}|}$. This completes the proof of the lemma. ■

2.6.2 Higher Order Resurgence Relations

Using the decomposition lemma above we can prove higher order resurgence relations. These relations have also been found by Costin in his study of rank one differential equations in [Cos98], theorem 4(i). The proof we give is analogous to the proof Costin gave for the differential case.

Proposition 2.6.4 *For $\mathbf{k} \in \mathbb{N}^p$ we have $\tilde{Y}^{\mp}_{\mathbf{k},\mathbf{j}} = (\pm\mathbf{s})^{\mathbf{j}} \binom{\mathbf{k}_1+\mathbf{j}}{\mathbf{j}} \tilde{Y}^{\mp}_{\mathbf{k}+\mathbf{j}}$ (cf. lemma 2.6.2), so on the singular ray $\arg t = \theta_1$ we have*

$$\tilde{Y}^{\pm}_{\mathbf{k}} - \tilde{Y}^{\mp}_{\mathbf{k}} = \sum_{\mathbf{j}\in\mathbb{N}_1^{n_1}} (\pm\mathbf{s})^{\mathbf{j}} \binom{\mathbf{k}_1+\mathbf{j}}{\mathbf{j}} \Big(\tilde{Y}^{\mp}_{\mathbf{k}+\mathbf{j}}(t - |\mathbf{j}|\mu_1)\, 1_{[|\mathbf{j}|\mu_1, \infty e^{i\theta_1})}\Big)^{(m_1|\mathbf{j}|)}, \qquad (2.6.14)$$

where $\mathbf{k}_1 := \sum_{h=1}^{n_1} \langle \mathbf{k}, \mathbf{e}_h \rangle \mathbf{e}_h$.

PROOF. We already proved the statement in the case $\mathbf{k} = 0$ (cf. lemma 2.6.1). For $\mathbf{k} \in \mathbb{N}_1^p$ we only give the proof for $\tilde{Y}_{\mathbf{k}}^{+} - \tilde{Y}_{\mathbf{k}}^{-}$. A similar proof, with obvious modifications, may be given for $\tilde{Y}_{\mathbf{k}}^{-} - \tilde{Y}_{\mathbf{k}}^{+}$.

Let θ_{1-}, θ_1 and θ_{1+} be as in the introduction of section 2.5.3. With these directions we associate the sectors $S^+ := \{x \in \mathbb{C}^* \mid -\pi/2 - \theta_{1+} + \varepsilon < \arg x < \pi/2 - \theta_1 - \varepsilon\}$ and $S^- := \{x \in \mathbb{C}^* \mid -\pi/2 - \theta_1 + \varepsilon < \arg x < \pi/2 - \theta_{1-} - \varepsilon\}$, with $\varepsilon > 0$ so small that both S^+ and S^- contain the positive real axis. Let $y_{\mathbf{k}}^{\pm}$ denote the Borel sum of $\hat{y}_{\mathbf{k}}$ on $S^{\pm}$. Moreover, we define the sector S by $S := \{x \in \mathbb{C}^* \mid \phi_- < \arg x < \phi_+\}$, with ϕ_- and ϕ_+ chosen such that $-\pi/2 - \theta_1 + \varepsilon < \phi_- < 0 < \phi_+ < \pi/2 - \theta_1 - \varepsilon$. Then obviously $S \subset S^+ \cap S^-$. In addition we assume $\phi_+ - \phi_-$ to be so small that for each $j \in \{1, 2, \ldots, r_1\}$ the function $x \mapsto e^{-\mu_j x}$ is exponentially small on every closed sub-sector of S. Note that this last requirement is equivalent to choosing $\phi_- < 0 < \phi_+$ in such a way that $-\pi/2 - \phi_- < \theta_j < \pi/2 - \phi_+$ for every $j \in \{1, 2, \ldots, r_1\}$.

Next take an arbitrary vector $C \in \mathbb{C}^p$, then $x \mapsto C_h e^{-\mu_j x}$, $h \in \mathcal{J}_j$, $j \in \{1, 2, \ldots, r_1\}$, is exponentially small on every closed sub-sector of S and if we define $\tilde{C} := C + \sum_{h=1}^{n_1} s_h \mathbf{e}_h$, then the same holds for $\tilde{C}$. Due to theorem 2.3.6 $y^+(x) := \sum_{\mathbf{k} \in \mathbb{N}^p} C^{\mathbf{k}} e^{-\langle \mathbf{k}, \boldsymbol{\mu} \rangle x} x^{\langle \mathbf{k}, \mathbf{a} \rangle} y_{\mathbf{k}}^{+}(x)$ and $y^-(x) := \sum_{\mathbf{k} \in \mathbb{N}^p} \tilde{C}^{\mathbf{k}} e^{-\langle \mathbf{k}, \boldsymbol{\mu} \rangle x} x^{\langle \mathbf{k}, \mathbf{a} \rangle} y_{\mathbf{k}}^{-}(x)$ both are solutions of (2.1.1) that are holomorphic in a neighbourhood of ∞ in S.

Remember that $\tilde{\mathbf{a}} = \sum_{j=1}^{r} \sum_{h \in \mathcal{J}_j} \tilde{a}_j \mathbf{e}_h$, with $\tilde{a}_j = a_j - m_j$, $j = 1, 2, \ldots, r$ (cf. proposition 2.4.3). From this, together with proposition 2.5.11, we will deduce that y^+ and y^- can also be written as

$$y^+(x) := \sum_{\mathbf{k} \in \mathbb{N}^p} C^{\mathbf{k}} e^{-\langle \mathbf{k}, \boldsymbol{\mu} \rangle x} x^{\langle \mathbf{k}, \mathbf{m} \rangle} (\mathcal{L}_{\theta_1} \tilde{Y}_{\mathbf{k}}^{+})(x) \text{ and } y^-(x) := \sum_{\mathbf{k} \in \mathbb{N}^p} \tilde{C}^{\mathbf{k}} e^{-\langle \mathbf{k}, \boldsymbol{\mu} \rangle x} x^{\langle \mathbf{k}, \mathbf{m} \rangle} (\mathcal{L}_{\theta_1} \tilde{Y}_{\mathbf{k}}^{-})(x),$$

$x \in S$, where $\mathcal{L}_{\theta_1}$ means the Laplace integral in the sense of distributions with as path of integration the singular ray $\arg t = \theta_1$. In fact, for every $\theta \in (\theta_1, \theta_{1+})$ we have in the classical sense $y_{\mathbf{k}}^{+}(x) = x^{-\langle \mathbf{k}, \tilde{\mathbf{a}} \rangle} (\mathcal{L}_\theta \tilde{Y}_{\mathbf{k}})(x)$ and the Laplace transform of $\tilde{Y}_{\mathbf{k}}^{+}$ in the sense of distributions along the half line $(0, \infty e^{i\theta})$ coincides with the classical Laplace transform. Moreover, in the sense of distributions we have $(\mathcal{L}_\theta \tilde{Y}_{\mathbf{k}})(x) = (\mathcal{L}_{\theta_1}[\rho_{\theta - \theta_1} \tilde{Y}_{\mathbf{k}}^{+}])(x e^{i(\theta - \theta_1)})$ and thus $\left|(\mathcal{L}_\theta \tilde{Y}_{\mathbf{k}})(x) - (\mathcal{L}_{\theta_1} \tilde{Y}_{\mathbf{k}}^{+})(x)\right|$ can be estimated by

$$\left|(\mathcal{L}_{\theta_1} \tilde{Y}_{\mathbf{k}}^{+})(x) - (\mathcal{L}_{\theta_1}[\rho_{\theta - \theta_1} \tilde{Y}_{\mathbf{k}}^{+}])(x)\right| + \left|(\mathcal{L}_{\theta_1}[\rho_{\theta - \theta_1} \tilde{Y}_{\mathbf{k}}^{+}])(x) - (\mathcal{L}_{\theta_1}[\rho_{\theta - \theta_1} \tilde{Y}_{\mathbf{k}}^{+}])(x e^{i(\theta - \theta_1)})\right|.$$

Using this, together with theorem A.5.2, we conclude that $\left|(\mathcal{L}_\theta \tilde{Y}_{\mathbf{k}})(x) - (\mathcal{L}_{\theta_1} \tilde{Y}_{\mathbf{k}}^{+})\right|$ can be majorized by $\|\tilde{Y}_{\mathbf{k}}^{+} - \rho_{\theta - \theta_1} \tilde{Y}_{\mathbf{k}}^{+}\|_{m_1, \nu, \theta_1} + \left|(\mathcal{L}_{\theta_1}[\rho_{\theta - \theta_1} \tilde{Y}_{\mathbf{k}}^{+}])(x) - (\mathcal{L}_{\theta_1}[\rho_{\theta - \theta_1} \tilde{Y}_{\mathbf{k}}^{+}])(x e^{i(\theta - \theta_1)})\right|$ and by proposition 2.5.11 the latter expression tends to 0 as $\theta \downarrow \theta_1$. This proves that $x^{\langle \mathbf{k}, \tilde{\mathbf{a}} \rangle} y_{\mathbf{k}}^{+}(x)$ can be written as $(\mathcal{L}_{\theta_1} \tilde{Y}_{\mathbf{k}}^{+})(x)$. A similar reasoning shows that $x^{\langle \mathbf{k}, \tilde{\mathbf{a}} \rangle} y_{\mathbf{k}}^{-}(x) = (\mathcal{L}_{\theta_1} \tilde{Y}_{\mathbf{k}}^{-})(x)$. Using (2.5.7) and theorem A.5.2 we deduce that $|(\mathcal{L}_{\theta_1} \tilde{Y}_{\mathbf{k}}^{\pm})(x)| \leqslant K^{|\mathbf{k}|}$ and thus the expressions for $y^{\pm}$, given above, indeed converge in a neighbourhood of ∞ in S.

By using the preceding lemma, y^+ can be rewritten as

$$y^+(x) = \mathcal{L}_{\theta_1} \tilde{Y}_0^{-} + \sum_{\mathbf{k} \succ 0} x^{\langle \mathbf{k}, \mathbf{m} \rangle} e^{-\langle \mathbf{k}, \boldsymbol{\mu} \rangle x} \sum_{\substack{\mathbf{k}' \in \mathbb{N}^p, \mathbf{j} \in \mathbb{N}^{n_1} \\ \mathbf{k}' + \mathbf{j} = \mathbf{k}}} C^{\mathbf{k}'} (\mathcal{L}_{\theta_1} \tilde{Y}_{\mathbf{k}', \mathbf{j}}^{-})(x)$$

and thus

$$y^+(x) - y^-(x) = \sum_{\mathbf{k}\succ 0} x^{\langle \mathbf{k},\mathbf{m}\rangle} e^{-\langle \mathbf{k},\boldsymbol{\mu}\rangle x} \Big\{ \sum_{\substack{\mathbf{k}'\in\mathbb{N}^p,\, \mathbf{j}\in\mathbb{N}^{n_1}\\ \mathbf{k}'+\mathbf{j}=\mathbf{k}}} C^{\mathbf{k}'}\, (\mathcal{L}_{\theta_1}\tilde{Y}^-_{\mathbf{k}',\mathbf{j}})(x) - \tilde{C}^{\mathbf{k}}\, (\mathcal{L}_{\theta_1}\tilde{Y}^-_{\mathbf{k}})(x) \Big\}.$$

Since $\tilde{Y}^-_{\mathbf{k},0} = \tilde{Y}^-_{\mathbf{k}}$ we deduce from the particular choice of $\tilde{C}$ that the terms with $|\mathbf{k}| = 1$ cancel and thus

$$y^+(x) - y^-(x) = \sum_{\mathbf{k}\in\mathbb{N}^p_2} x^{\langle \mathbf{k},\mathbf{m}\rangle} e^{-\langle \mathbf{k},\boldsymbol{\mu}\rangle x} (\mathcal{L}_{\theta_1} U_{\mathbf{k}})(x),$$

where $U_{\mathbf{k}} = -\tilde{C}^{\mathbf{k}}\, \tilde{Y}^-_{\mathbf{k}} + \sum_{\substack{\mathbf{k}'\in\mathbb{N}^p,\, \mathbf{j}\in\mathbb{N}^{n_1}\\ \mathbf{k}'+\mathbf{j}=\mathbf{k}}} C^{\mathbf{k}'}\, \tilde{Y}^-_{\mathbf{k}',\mathbf{j}}$.

Now, with $\mathbf{k} \in \mathbb{N}^p$ we associate $\mathbf{p}_{\mathbf{k}} := (|\mathbf{k}_1|, |\mathbf{k}_2|, \ldots, |\mathbf{k}_{r_1}|) \in \mathbb{N}^{r_1}$, where $\mathbf{k}_j$ is defined by $\sum_{h\in\mathcal{J}_j} \langle \mathbf{k}, \mathbf{e}_h\rangle \mathbf{e}_h$, $j = 1, 2, \ldots, r_1$. Moreover, if we define $\mathbf{m}' := (m_1, m_2, \ldots, m_{r_1})$ and $\boldsymbol{\mu}' := (\mu_1, \mu_2, \ldots, \mu_{r_1})$, then $\langle \mathbf{k}, \mathbf{m}\rangle = \langle \mathbf{p}, \mathbf{m}'\rangle$ for every $\mathbf{k} \in \mathbb{N}^p$ with $\mathbf{p}_{\mathbf{k}} = \mathbf{p}$ and similarly with $\mathbf{m}$ and $\mathbf{m}'$ replaced by $\boldsymbol{\mu}$ and $\boldsymbol{\mu}'$ respectively. With these notations $y^+ - y^-$ can also be written as

$$y^+(x) - y^-(x) = \sum_{\mathbf{p}\in\mathbb{N}^{r_1}_2} x^{\langle \mathbf{p},\mathbf{m}'\rangle} e^{-\langle \mathbf{p},\boldsymbol{\mu}'\rangle x} (\mathcal{L}_{\theta_1} V_{\mathbf{p}})(x),$$

with $V_{\mathbf{p}} = \sum_{\substack{\mathbf{k}\in\mathbb{N}^p\\ \mathbf{p}_{\mathbf{k}}=\mathbf{p}}} U_{\mathbf{k}}$. Now $V_{\mathbf{p}}$ can more conveniently be written as

$$V_{\mathbf{p}} = \sum_{\substack{\mathbf{k}\in\mathbb{N}^p\\ \mathbf{p}_{\mathbf{k}}=\mathbf{p}}} \sum_{0\preceq \mathbf{j}\preceq \mathbf{k}_1} C^{\mathbf{k}-\mathbf{j}} U_{\mathbf{k},\mathbf{j}}, \quad U_{\mathbf{k},\mathbf{j}} = \tilde{Y}^-_{\mathbf{k}-\mathbf{j},\mathbf{j}} - \mathbf{s}^{\mathbf{j}} \binom{\mathbf{k}_1}{\mathbf{j}} \tilde{Y}^-_{\mathbf{k}}.$$

We want to prove that each $U_{\mathbf{k},\mathbf{j}} = 0$. To that end we assume the existence of a nonempty subset $\mathcal{I}$ of $\mathbb{N}^{r_1}_2$ consisting of multi-indices $\mathbf{p}$ with the property $V_{\mathbf{p}} \neq 0$. As both $t \mapsto t^{\langle \mathbf{k},\tilde{\mathbf{a}}\rangle+1}\tilde{Y}_{\mathbf{k}}(t)$ and $t \mapsto t^{\langle \mathbf{k},\tilde{\mathbf{a}}\rangle+1}\tilde{Y}^-_{\mathbf{k}-\mathbf{j},\mathbf{j}}(t)$ are holomorphic in a neighbourhood of the origin for each $\mathbf{k} \in \mathbb{N}^p_2$ and $\mathbf{j} \in \mathbb{N}^{n_1}$, $0 \preceq \mathbf{j} \preceq \mathbf{k}_1$, we deduce the same for $t \mapsto t^{\langle \mathbf{k},\tilde{\mathbf{a}}\rangle+1}U_{\mathbf{k},\mathbf{j}}(t)$. Hence, $t \mapsto t^{\langle \mathbf{p},\tilde{\mathbf{a}}'\rangle+1}V_{\mathbf{p}}(t)$ is holomorphic in a neighbourhood of the origin, for each $\mathbf{p} \in \mathcal{I}$ (here $\tilde{\mathbf{a}}' = (\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_{r_1})$). Proposition A.5.4 then implies that $(\mathcal{L}_{\theta_1} V_{\mathbf{p}})(x)$ asymptotically equals the term-by-term Laplace transform of the Puiseux series of $V_{\mathbf{p}}$ at the origin, which is nonzero by assumption. Hence, the Laplace transform of $V_{\mathbf{p}}$ asymptotically equals $x^{\langle \mathbf{p},\tilde{\mathbf{a}}'\rangle}$ times a (nonzero) series in $\mathbb{C}^n[[x^{-1}]]$. If this latter series is of the form $x^{-M_{\mathbf{p}}}\big(\alpha_{\mathbf{p}} + O(x^{-1})\big)$ for certain $M_{\mathbf{p}} \in \mathbb{N}$ and $\alpha_{\mathbf{p}} \in \mathbb{C}^n\backslash\{0\}$, then for $x \to \infty$ in S, $(\mathcal{L}_{\theta_1} V_{\mathbf{p}})(x)$ behaves like $x^{\langle \mathbf{p},\tilde{\mathbf{a}}'\rangle - M_{\mathbf{p}}}\,\big(\alpha_{\mathbf{p}} + O(x^{-1})\big)$. Using the abbreviation $y(x) := y^+(x) - y^-(x)$ we thus obtain the following asymptotic behaviour for y:

$$y(x) \;\sim\; \sum_{\mathbf{p}\in\mathcal{I}} x^{\langle \mathbf{p},\mathbf{a}'\rangle - M_{\mathbf{p}}}\, e^{-\langle \mathbf{p},\boldsymbol{\mu}'\rangle x}\, \big(\alpha_{\mathbf{p}} + O(x^{-1})\big), \quad \text{as } x \to \infty \text{ in } S, \tag{2.6.15}$$

where $\mathbf{a}' = (a_1, a_2, \ldots, a_{r_1})$.

On the other hand both $y^+(x)$ and $y^-(x)$ are solutions of (2.1.1) that asymptotically equal $\hat{y}_0(x)$. So the difference $y(x) = y^+(x) - y^-(x)$ is of order $O(x^{-2})$ as $x \to \infty$ in S and satisfies

$$y(x+1) = \Lambda(x)y(x) + g\big(x, y^+(x)\big) - g\big(x, y^-(x)\big).$$

Writing $y^+ = y + y^-$, we easily infer that $g\big(x, y^+(x)\big) - g\big(x, y^-(x)\big)$ can be written as $A(x)y(x)$ with some particular A depending on y^+ and y^- and of order $O(x^{-2})$ as $x \to \infty$ in S. So theorem B.0.2 implies that $y(x) = \sum_{j=1}^{r} \sum_{h \in \mathcal{J}_j} \tilde{C}_h(x) e^{-\mu_j x} x^{a_j} \big(\mathbf{e}_h + O(x^{-1})\big)$ as $x \to \infty$ in S for certain 1-periodic functions $\tilde{C}_h$. However, in [Bra01] it is shown that the 1-periodic functions $\tilde{C}_h$ have to be equal to 1-periodic trigonometric polynomials and then one may deduce, as in example 1.5.2, that each $\tilde{C}_h$ behaves like a constant times $e^{2\pi i r_h x}$, as $x \to \infty$ in $S \cap \{x \in \mathbb{C} \mid \Im x > 0\}$ for certain $r_h \in \mathbb{Z}$. So the two possible asymptotic expansions match only if each $e^{2\pi i r_h - \mu_j}$ equals $e^{-\langle \mathbf{p}, \boldsymbol{\mu}' \rangle}$ for some $\mathbf{p} \in \mathcal{I} \subset \mathbb{N}_2^{r_1}$. The latter is impossible due to the assumptions on the μ_j's as formulated in the introduction of this chapter (compare also remark 2.3.7). Hence, $V_{\mathbf{p}} = 0$ for all $\mathbf{p} \in \mathbb{N}_2^{r_1}$. Now varying the constant vector C gives $U_{\mathbf{k},\mathbf{j}} = 0$ for all $\mathbf{k} \in \mathbb{N}_2^p$ and $\mathbf{j} \in \mathbb{N}^{n_1}$. ■

In the following we will show that it is possible to make analytic continuations of the solutions $\tilde{Y}_{\mathbf{k}}$, $\mathbf{k} \in \mathbb{N}^p$, across the singular ray $\arg t = \theta_1$. To that end we first introduce some notation. Let us define

$$\mathcal{W} = \big\{ t \in \mathbb{C}^* \;\big|\; \arg t \in (\theta_{1-}, \theta_{1+}) \text{ and } t \neq k\mu_1, \forall\, k \in \mathbb{N} \big\} \tag{2.6.16}$$

and let $\mathcal{W}^+ := \{t \in \mathcal{W} \mid \arg t \in (\theta_1, \theta_{1+})\}$ and $\mathcal{W}^- := \{t \in \mathcal{W} \mid \arg t \in (\theta_{1-}, \theta_1)\}$.

Lemma 2.6.5 *Let $I \subset (\theta_{1-}, \theta_{1+})$ be a compact interval and let $f^+ \in \mathcal{D}'_{m_1,\nu}(\ell, \theta_1)$, with $\ell \in (|\mu_1|/2, |\mu_1|)$, be a staircase distribution. Let f be a function holomorphic in $\mathcal{W}^+$ such that on any ray $\arg t = \theta \in I \cap (\theta_1, \theta_{1+})$ the function f_θ, defined by $f_\theta(t) := f(|t|e^{i\theta})$, $t \in \mathcal{W}$, belongs to $\mathcal{D}'_{m_1,\nu}(\ell, \theta)$ and that $\lim_{\theta \downarrow \theta_1} f_\theta = f^+$ holds in the $\mathcal{D}'_{m_1,\nu}(\ell, \cdot)$-topology. Assume in addition that there exists a decomposition*

$$f^+ = \sum_{j=0}^{\infty} \big(f_j(t - j\mu_1) \mathbf{1}_{[j\mu_1, \infty e^{i\theta_1})}\big)^{(m_1 j)},$$

where for each j, $f_j \in \mathcal{D}'_{m_1,\nu}(\ell, \theta_1)$. Finally assume that each f_j extends analytically into $\mathcal{W}^-$ in the following way: there exists a function g_j, holomorphic in $\mathcal{W}^-$, such that on any ray $\arg t = \theta \in I \cap (\theta_{1-}, \theta_1)$ the function $g_{j,\theta}$ belongs to $\mathcal{D}'_{m_1,\nu}(\ell, \theta)$ and that $\lim_{\theta \uparrow \theta_1} g_{j,\theta} = f_j$ holds in the $\mathcal{D}'_{m_1,\nu}(\ell, \cdot)$-topology.
Then for each $l \in \mathbb{N}$ the function f can be analytically continued across $(l\mu_1, (l+1)\mu_1)$ into $\mathcal{W}^- \backslash \overline{\Delta}(0, l|\mu_1|)$.

Proof. Restricting f^+ to $\mathcal{D}(0, (l+1)\mu_1)$ we have

$$f^+ = \sum_{j=0}^{l} \big(f_j(t - j\mu_1) \mathbf{1}_{[j\mu_1, \infty e^{i\theta_1})}\big)^{(m_1 j)}. \tag{2.6.17}$$

The assumptions in the lemma imply the existence of a positive integer N such that both $\mathcal{P}^{N+1}f^+$ and $\sum_{j=0}^{l}\mathcal{P}^{N+1-m_1j}\big[\big(f_j(t-j\mu_1)\mathbf{1}_{[j\mu_1,\infty e^{i\theta_1})}\big)\big]$ are continuous on $(l\mu_1,(l+1)\mu_1)$. Note that N depends on m_1, l and the step size ℓ in the definition of $\mathcal{D}'_{m_1,\nu}(\ell,\theta_1)$, so it is possible to calculate the minimal N and in the discussion that follows we will take this minimal N. From (2.6.17) we now deduce

$$\mathcal{P}^{N+1}f^+=\sum_{j=0}^{l}\mathcal{P}^{N+1-m_1j}\big[\big(f_j(t-j\mu_1)\mathbf{1}_{[j\mu_1,\infty e^{i\theta_1})}\big)\big]. \tag{2.6.18}$$

By assumption f_θ converges to f^+ as $\theta\downarrow\theta_1$ in the $\mathcal{D}'_{m_1,\nu}(\ell,\cdot)$-topology and we want to show that this implies that $(\mathcal{P}^{N+1}f)(s)$ converges to $(\mathcal{P}^{N+1}f^+)(t)$ as $s\to t$, $\arg s\geqslant\theta_1$, uniformly on $(l\mu_1,(l+1)\mu_1)$. To that end we suppose that the distributions f^+ and f_θ restricted to $\mathcal{D}(0,(l+1)\mu_1)$ have decompositions $\sum_{j=0}^{h}\Delta_j^{(m_1j)}$ and $\sum_{j=0}^{h}\Delta_{j,\theta}^{(m_1j)}$ respectively, for some $h\geqslant l$. For $t\in(l\mu_1,(l+1)\mu_1)$ we then have $(\mathcal{P}^{N+1}f)(te^{i(\theta-\theta_1)})=e^{(N+1)i(\theta-\theta_1)}(\mathcal{P}^{N+1}f_\theta)(t)$, so

$$\begin{aligned}\big|(\mathcal{P}^{N+1}f)(te^{i(\theta-\theta_1)})-(\mathcal{P}^{N+1}f^+)(t)\big| &\leqslant \big|e^{(N+1)i(\theta-\theta_1)}-1\big|\big|(\mathcal{P}^{N+1}f_\theta)(t)\big|+\\ &\quad\big|(\mathcal{P}^{N+1}f_\theta)(t)-(\mathcal{P}^{N+1}f^+)(t)\big|.\end{aligned}$$

The last term in the right-hand side can be estimated by

$$\begin{aligned}\big|(\mathcal{P}^{N+1}f_\theta)(t)-(\mathcal{P}^{N+1}f^+)(t)\big| &\leqslant \int_0^{|t|}\big|(\mathcal{P}^{N}f_\theta)(se^{i\theta_1})-(\mathcal{P}^{N}f^+)(se^{i\theta_1})\big|\mathrm{d}s\\ &\leqslant \sum_{j=0}^{h}\int_0^{|t|}\big|\mathcal{P}^{N-m_1j}[\Delta_{j,\theta}-\Delta_j](se^{i\theta_1})\big|\mathrm{d}s.\end{aligned}$$

Moreover, if $t\in(l\mu_1,(l+1)\mu_1)$ and $s\in[0,|t|]$ we have $1\leqslant e^{\nu(l+1)|\mu_1|}\cdot e^{-\nu s}$, so the latter expression is less than

$$C\sum_{j=0}^{h}\int_0^{|t|}e^{-\nu s}\big|\mathcal{P}^{N-m_1j}[\Delta_{j,\theta}-\Delta_j](se^{i\theta_1})\big|\mathrm{d}s \;\leqslant\; C\sum_{j=0}^{h}\big\|\mathcal{P}^{N-m_1j}[\Delta_{j,\theta}-\Delta_j]\big\|_{\nu,\theta_1},$$

with $C=e^{\nu(l+1)|\mu_1|}$. As $\|\mathcal{P}^{N-m_1j}[\Delta_{j,\theta}-\Delta_j]\|_{\nu,\theta_1}\leqslant\nu^{m_1j-N}\|\Delta_{j,\theta}-\Delta_j\|_{\nu,\theta_1}$ we then obtain

$$\big|(\mathcal{P}^{N+1}f_\theta)(t)-(\mathcal{P}^{N+1}f^+)(t)\big|\;\leqslant\;\frac{C}{\nu^N}\|f_\theta-f^+\|_{m_1,\nu,\theta_1},$$

which by assumption tends to 0 as $\theta\downarrow\theta_1$. Similarly one may prove that

$$|e^{(N+1)i(\theta-\theta_1)}-1||(\mathcal{P}^{N+1}f_\theta)(t)|\leqslant\frac{C(N+1)}{\nu^N}(\theta-\theta_1)\|f_\theta\|_{m_1,\nu,\theta_1},$$

which also tends to 0 as $\theta\downarrow\theta_1$. From this we infer that the function $(\mathcal{P}^{N+1}f)(te^{i(\theta-\theta_1)})$ converges to $(\mathcal{P}^{N+1}f^+)(t)$ as $\theta\downarrow\theta_1$ uniformly on $(l\mu_1,(l+1)\mu_1)$, so the left-hand side of (2.6.18) is the limit of $\mathcal{P}^{N+1}f$ on $(l\mu_1,(l+1)\mu_1)$.

Define $\tilde{g}_j : \mathcal{W}^- \to \mathbb{C}$ by $g_j(t - j|\mu_1|e^{i \arg t})$ if $t \in \mathcal{W}^-$ and $|t| > j|\mu_1|$, while for $t \in \mathcal{W}^-$, $|t| \leqslant j|\mu_1|$, we define $\tilde{g}_j(t) := 0$. Then the same proof as above, together with proposition A.6.1, shows that $(\mathcal{P}^{N-m_1 j+1}\tilde{g}_j)(te^{i(\theta-\theta_1)}) \to \mathcal{P}^{N-m_1 j+1}\big[f_j(t - j|\mu_1|)\mathbf{1}_{[j|\mu_1|,\infty e^{i\theta_1})}\big]$ as $\theta \uparrow \theta_1$. So the right-hand side of (2.6.18) is the limit on $(l\mu_1, (l+1)\mu_1)$ of a function which is holomorphic in $\mathcal{W}^- \backslash \overline{\Delta}(0, l|\mu_1|)$.

Let us define the function h on $\mathcal{W}^+ \cup (l\mu_1, (l+1)\mu_1) \cup \big(\mathcal{W}^- \backslash \overline{\Delta}(0, l|\mu_1|)\big)$ by

$$h(t) := \begin{cases} (\mathcal{P}^{N+1} f)(t) & \text{if } t \in \mathcal{W}^+ \\ (\mathcal{P}^{N+1} f^+)(t) & \text{if } t \in (l\mu_1, (l+1)\mu_1) \\ \sum_{j=0}^{l} \mathcal{P}^{N+1-m_1 j}\big[\big(g_j(t - j|\mu_1|e^{i \arg t})\big)\big] & \text{if } t \in \mathcal{W}^- \backslash \overline{\Delta}(0, l|\mu_1|), \end{cases}$$

where in the latter expression we used the convention $g_j(t - j|\mu_1|e^{i \arg t}) = 0$ if $|t| \leqslant j|\mu_1|$. Using Cauchy's theorem we deduce that $\int_\gamma h(t)\mathrm{d}t = 0$ for every triangle γ which, together with its interior, is in $\mathcal{W}^+ \cup (l\mu_1, (l+1)\mu_1) \cup \big(\mathcal{W}^- \backslash \overline{\Delta}(0, l|\mu_1|)\big)$. From Morera's theorem we now conclude that h is holomorphic in $\mathcal{W}^+ \cup (l\mu_1, (l+1)\mu_1) \cup \big(\mathcal{W}^- \backslash \overline{\Delta}(0, l|\mu_1|)\big)$ and thus $\mathcal{P}^{N+1} f$ extends analytically through $(l\mu_1, (l+1)\mu_1)$ into $\mathcal{W}^- \backslash \overline{\Delta}(0, l|\mu_1|)$. Obviously the same holds for f. ■

Remark 2.6.6 In an analogous way one may prove the following statement.
Let f^- be a staircase distribution in $\mathcal{D}'_{m_1,\nu}(\ell, \theta_1)$ and let f be a function which is holomorphic in $\mathcal{W}^-$, such that on any ray $\arg t = \theta \in I \cap (\theta_{1^-}, \theta_1)$ the function f_θ belongs to $\mathcal{D}'_{m_1,\nu}(\ell, \theta)$ and that $\lim_{\theta\uparrow\theta_1} f_\theta = f^-$ holds in the $\mathcal{D}'_{m_1,\nu}(\ell, \cdot)$-topology. Assume in addition that there exists a decomposition

$$f^- = \sum_{j=0}^{\infty} \big(f_j(t - j\mu_1)\mathbf{1}_{[j\mu_1,\infty e^{i\theta_1})}\big)^{(m_1 j)},$$

where for each j, f_j belongs to $\mathcal{D}'_{m_1,\nu}(\ell, \theta_1)$ and can be analytically extended into $\mathcal{W}^+$ in a similar way as described in the preceding lemma. Then for each $l \in \mathbb{N}$ the function f can be analytically continued across $(l\mu_1, (l+1)\mu_1)$ into $\mathcal{W}^+ \backslash \overline{\Delta}(0, l|\mu_1|)$.

For $l \in \mathbb{N}$ we define $\tilde{Y}_{\mathbf{k}}^{+l-}$ to be the analytic continuation of $\tilde{Y}_{\mathbf{k}}$ from $\mathcal{W}^+$ across $(l\mu_1, (l+1)\mu_1)$ into $\mathcal{W}^- \backslash \overline{\Delta}(0, l|\mu_1|)$ and similarly $\tilde{Y}_{\mathbf{k}}^{-l+}$ is the analytic continuation of $\tilde{Y}_{\mathbf{k}}$ from $\mathcal{W}^-$ across $(l\mu_1, (l+1)\mu_1)$ into $\mathcal{W}^+ \backslash \overline{\Delta}(0, l|\mu_1|)$. The fact that these continuations exist, follows from proposition 2.6.4, proposition 2.5.11 and the results just obtained. Moreover, we can give exact formulas for these continuations. For example:

$$\tilde{Y}_{\mathbf{k}}^{+l-}(t) = \begin{cases} \tilde{Y}_{\mathbf{k}}(t) & \text{if } t \in \mathcal{W}^+, \\ \sum\limits_{0 \leqslant |\mathbf{j}| \leqslant l} \mathbf{s}^{\mathbf{j}} \binom{\mathbf{k}_1+\mathbf{j}}{\mathbf{j}} \big(\tilde{Y}_{\mathbf{k}+\mathbf{j}}^-(t - |\mathbf{j}|\mu_1)\mathbf{1}_{[|\mathbf{j}|\mu_1,\infty e^{i\theta_1})}\big)^{(m_1|\mathbf{j}|)} & \text{if } t \in (l\mu_1, (l+1)\mu_1), \\ \sum\limits_{0 \leqslant |\mathbf{j}| \leqslant l} \mathbf{s}^{\mathbf{j}} \binom{\mathbf{k}_1+\mathbf{j}}{\mathbf{j}} \big(\tilde{Y}_{\mathbf{k}+\mathbf{j}}(t - |\mathbf{j}||\mu_1|e^{i \arg t})\big)^{(m_1|\mathbf{j}|)} & \text{if } t \in \mathcal{W}^- \backslash \overline{\Delta}(0, l|\mu_1|). \end{cases}$$

Here we use the convention that if $\mathbf{j} \in \mathbb{N}^{n_1}$, $|\mathbf{j}| = j$, then $\tilde{Y}_{\mathbf{k}+\mathbf{j}}(t - j|\mu_1|e^{i\arg t}) = 0$ if $|t| \leqslant j|\mu_1|$.

However, if $|\mathbf{j}| = j$, then $t \mapsto \left(\tilde{Y}_{\mathbf{k}+\mathbf{j}}(t - j|\mu_1|e^{i\arg t})\right)^{(m_1 j)}$ can be analytically continued from $\mathcal{W}^-\backslash\overline{\Delta}(0, j|\mu_1|)$ across $(k\mu_1, (k+1)\mu_1)$ into $\mathcal{W}^+\backslash\overline{\Delta}(0, k|\mu_1|)$ for every $k \geqslant j$. This follows from propositions 2.6.4 and 2.5.11 and remark 2.6.6. Hence, $\tilde{Y}_{\mathbf{k}}^{+l-}$ can also be defined on $(k\mu_1, (k+1)\mu_1)$ for all $k \geqslant l+1$ by

$$\tilde{Y}_{\mathbf{k}}^{+l-}(t) = \begin{cases} \tilde{Y}_{\mathbf{k}}(t) & \text{if } t \in \mathcal{W}^+, \\ \sum\limits_{0\leqslant|\mathbf{j}|\leqslant l} \mathbf{s}^{\mathbf{j}} \binom{\mathbf{k}_1+\mathbf{j}}{\mathbf{j}} \left(\tilde{Y}_{\mathbf{k}+\mathbf{j}}^-(t - |\mathbf{j}|\mu_1) 1_{[|\mathbf{j}|\mu_1, \infty e^{i\theta_1})}\right)^{(m_1|\mathbf{j}|)} & \text{if } t \in (l\mu_1, \infty e^{i\theta_1})\backslash\mu_1\mathbb{N}, \\ \sum\limits_{0\leqslant|\mathbf{j}|\leqslant l} \mathbf{s}^{\mathbf{j}} \binom{\mathbf{k}_1+\mathbf{j}}{\mathbf{j}} \left(\tilde{Y}_{\mathbf{k}+\mathbf{j}}(t - |\mathbf{j}||\mu_1|e^{i\arg t})\right)^{(m_1|\mathbf{j}|)} & \text{if } t \in \mathcal{W}^-\backslash\overline{\Delta}(0, l|\mu_1|). \end{cases}$$

In exactly the same way we can prove that

$$\tilde{Y}_{\mathbf{k}}^{-l+}(t) = \begin{cases} \tilde{Y}_{\mathbf{k}}(t) & \text{if } t \in \mathcal{W}^-, \\ \sum\limits_{0\leqslant|\mathbf{j}|\leqslant l} (-\mathbf{s})^{\mathbf{j}} \binom{\mathbf{k}_1+\mathbf{j}}{\mathbf{j}} \left(\tilde{Y}_{\mathbf{k}+\mathbf{j}}^+(t - |\mathbf{j}|\mu_1) 1_{[|\mathbf{j}|\mu_1, \infty e^{i\theta_1})}\right)^{(m_1|\mathbf{j}|)} & \text{if } t \in (l\mu_1, \infty e^{i\theta_1})\backslash\mu_1\mathbb{N}, \\ \sum\limits_{0\leqslant|\mathbf{j}|\leqslant l} (-\mathbf{s})^{\mathbf{j}} \binom{\mathbf{k}_1+\mathbf{j}}{\mathbf{j}} \left(\tilde{Y}_{\mathbf{k}+\mathbf{j}}(t - |\mathbf{j}||\mu_1|e^{i\arg t})\right)^{(m_1|\mathbf{j}|)} & \text{if } t \in \mathcal{W}^+\backslash\overline{\Delta}(0, l|\mu_1|) \end{cases}$$

and we conclude that on the singular ray $\arg t = \theta_1$ we have

$$\tilde{Y}_{\mathbf{k}}^{\pm l\mp}(t) - \tilde{Y}_{\mathbf{k}}^{\pm(l-1)\mp}(t) = \sum_{\mathbf{j}\in\mathbb{N}^{n_1}, |\mathbf{j}|=l} (\pm\mathbf{s})^{\mathbf{j}} \binom{\mathbf{k}_1+\mathbf{j}}{\mathbf{j}} \left(\tilde{Y}_{\mathbf{k}+\mathbf{j}}^{\mp}(t - l\mu_1) 1_{[l\mu_1, \infty e^{i\theta_1})}\right)^{(m_1 l)}, \quad (2.6.19)$$

for all $\mathbf{k} \in \mathbb{N}^p$ and $l \in \mathbb{N}_+$, provided that the left-hand side is defined: $t \in (l\mu_1, \infty e^{i\theta_1})\backslash\mu_1\mathbb{N}$.

Remark 2.6.7 If $\gamma : [0,1] \to \mathcal{W}$ is some path in $\mathcal{W}$ starting at 0 such that $\frac{\mathrm{d}}{\mathrm{d}t}|\gamma(t)| > 0$, then each $\tilde{Y}_{\mathbf{k}}$ can be analytically continued along this path. Let us denote this continuation by $AC_\gamma\tilde{Y}_{\mathbf{k}}$. If $\gamma(t) \in \mathcal{W}^+$ for all $t \in [0,1]$ or $\gamma(t) \in \mathcal{W}^-$ for all $t \in [0,1]$, then $AC_\gamma\tilde{Y}_{\mathbf{k}} = \tilde{Y}_{\mathbf{k}}$. However, if γ crosses the singular ray $\arg t = \theta_1$ only once from above and below, between $l\mu_1$ and $(l+1)\mu_1$, then $AC_\gamma\tilde{Y}_{\mathbf{k}} = \tilde{Y}_{\mathbf{k}}^{+l-}$ and $AC_\gamma\tilde{Y}_{\mathbf{k}} = \tilde{Y}_{\mathbf{k}}^{-l+}$ respectively.

From (2.6.19) one can determine the behaviour of $\tilde{Y}_{\mathbf{k}}$ near the singular points $l\mu_1$, $l \in \mathbb{Z}_{\geqslant 1}$, since (2.6.19) can also be written as

$$\tilde{Y}_{\mathbf{k}}^{\pm l\mp}(t+l\mu_1) - \tilde{Y}_{\mathbf{k}}^{\pm(l-1)\mp}(t+l\mu_1) = \sum_{\mathbf{j}\in\mathbb{N}^{n_1}, |\mathbf{j}|=l} (\pm\mathbf{s})^{\mathbf{j}} \binom{\mathbf{k}_1+\mathbf{j}}{\mathbf{j}} \left(\tilde{Y}_{\mathbf{k}+\mathbf{j}}^{\mp}(t) 1_{[0, \infty e^{i\theta_1})}\right)^{(m_1 l)}, \quad (2.6.20)$$

if $\arg t = \theta_1$, $t \notin \mu_1\mathbb{N}$. For $\arg t = \theta_1$ and $|t|$ small enough we know that both $\tilde{Y}_{\mathbf{k}+\mathbf{j}}^+$ and $\tilde{Y}_{\mathbf{k}+\mathbf{j}}^-$ are equal to $\tilde{Y}_{\mathbf{k}+\mathbf{j}}$ and the function $t \mapsto t^{\langle\mathbf{k}+\mathbf{j}, \tilde{\mathbf{a}}\rangle+1}\tilde{Y}_{\mathbf{k}+\mathbf{j}}(t)$ is holomorphic in a neighbourhood of the origin, say in $\Delta(0, \varepsilon)$ for some $\varepsilon > 0$.

In the following let β be a shorthand notation for $-\langle \mathbf{k}+\mathbf{j}, \tilde{\mathbf{a}}\rangle$, which is the same as $-\langle \mathbf{k}+l\mathbf{e}_1, \tilde{\mathbf{a}}\rangle$ in case $\mathbf{j} \in \mathbb{N}^{n_1}$, $|\mathbf{j}| = l$. So $t \mapsto t^{-\beta+1}\tilde{Y}_{\mathbf{k}+\mathbf{j}}(t)$ is holomorphic near $t=0$. Let us first assume that $\beta \notin \mathbb{Z}$. In that case we define the function $V^+_{\mathbf{k},l}$ on $\{t \in \mathbb{C}^* \mid \arg t \in [\theta_1, \theta_1+2\pi],\ |t| < \varepsilon\}$ by

$$V^+_{\mathbf{k},l}(t) := \tilde{Y}^{+l-}_{\mathbf{k}}(t+l\mu_1) - \frac{1}{1-e^{2\pi i\beta}} \sum_{|\mathbf{j}|=l} \mathbf{s}^{\mathbf{j}} \binom{\mathbf{k}_1+\mathbf{j}}{\mathbf{j}} \tilde{Y}^{(m_1 l)}_{\mathbf{k}+\mathbf{j}}(t).$$

For $t \in \mathbb{C}^*$ with $|t| < \varepsilon$ and $\arg t = \theta_1$ we have $\tilde{Y}^{+l-}_{\mathbf{k}}(te^{2\pi i}+l\mu_1) = \tilde{Y}^{+(l-1)-}_{\mathbf{k}}(t+l\mu_1)$, so $V^+_{\mathbf{k},l}(t) - V^+_{\mathbf{k},l}(te^{2\pi i}) = 0$. Hence, $V^+_{\mathbf{k},l}$ is holomorphic in $\Delta(0,\varepsilon)$. On the other hand, if $\beta \in \mathbb{Z}$ (and thus $\beta \in \mathbb{N}$), we replace the definition of $V^+_{\mathbf{k},l}$ by

$$V^+_{\mathbf{k},l}(t) := \tilde{Y}^{+l-}_{\mathbf{k}}(t+l\mu_1) + \frac{\log t}{2\pi i} \sum_{|\mathbf{j}|=l} \mathbf{s}^{\mathbf{j}} \binom{\mathbf{k}_1+\mathbf{j}}{\mathbf{j}} \tilde{Y}^{(m_1 l)}_{\mathbf{k}+\mathbf{j}}(t)$$

and again we deduce that $V^+_{\mathbf{k},l} \in \mathbb{C}^n\{t\}$. So for t in a neighbourhood of 0 we have

$$\tilde{Y}^{+l-}_{\mathbf{k}}(t+l\mu_1) \equiv \begin{cases} \frac{1}{1-e^{2\pi i\beta}} \sum\limits_{|\mathbf{j}|=l} \mathbf{s}^{\mathbf{j}} \binom{\mathbf{k}_1+\mathbf{j}}{\mathbf{j}} \tilde{Y}^{(m_1 l)}_{\mathbf{k}+\mathbf{j}}(t) \bmod \mathbb{C}^n\{t\}, & \text{if } \beta \notin \mathbb{Z} \\ -\frac{\log t}{2\pi i} \sum\limits_{|\mathbf{j}|=l} \mathbf{s}^{\mathbf{j}} \binom{\mathbf{k}_1+\mathbf{j}}{\mathbf{j}} \tilde{Y}^{(m_1 l)}_{\mathbf{k}+\mathbf{j}}(t) \bmod \mathbb{C}^n\{t\}, & \text{if } \beta \in \mathbb{Z}. \end{cases}$$

With the same reasoning we obtain a similar statement for $\tilde{Y}^{-l+}_{\mathbf{k}}(t+l\mu_1)$ and t in a neighbourhood of the origin. This proves the following proposition.

Proposition 2.6.8 *For every* $\mathbf{k} \in \mathbb{N}^p$ *and* $l \in \mathbb{Z}_{\geqslant 1}$ *we have*

$$\tilde{Y}^{\pm l\mp}_{\mathbf{k}}(t) - \tilde{Y}^{\pm(l-1)\mp}_{\mathbf{k}}(t) = \sum_{\mathbf{j}\in\mathbb{N}^{n_1}, |\mathbf{j}|=l} (\pm\mathbf{s})^{\mathbf{j}} \binom{\mathbf{k}_1+\mathbf{j}}{\mathbf{j}} \left(\tilde{Y}^{\mp}_{\mathbf{k}+\mathbf{j}}(t-l\mu_1)\mathbb{1}_{[l\mu_1,\infty e^{i\theta_1})}\right)^{(m_1 l)}, \quad (2.6.21)$$

on $(l\mu_1, \infty e^{i\theta_1})\backslash\mu_1\mathbb{N}$*. Moreover, for* t *in a neighbourhood of* 0 *we have*

$$\tilde{Y}^{\pm l\mp}_{\mathbf{k}}(t+l\mu_1) \equiv \begin{cases} \frac{1}{1-e^{\mp 2\pi i(\langle \mathbf{k}+l\mathbf{e}_1, \tilde{\mathbf{a}}\rangle)}} \sum\limits_{|\mathbf{j}|=l} (\pm\mathbf{s})^{\mathbf{j}} \binom{\mathbf{k}_1+\mathbf{j}}{\mathbf{j}} \tilde{Y}^{(m_1 l)}_{\mathbf{k}+\mathbf{j}}(t) \bmod \mathbb{C}^n\{t\}, & \text{if } \langle \mathbf{k}+l\mathbf{e}_1, \tilde{\mathbf{a}}\rangle \notin \mathbb{Z} \\ \mp\frac{\log t}{2\pi i} \sum\limits_{|\mathbf{j}|=l} (\pm\mathbf{s})^{\mathbf{j}} \binom{\mathbf{k}_1+\mathbf{j}}{\mathbf{j}} \tilde{Y}^{(m_1 l)}_{\mathbf{k}+\mathbf{j}}(t) \bmod \mathbb{C}^n\{t\}, & \text{if } \langle \mathbf{k}+l\mathbf{e}_1, \tilde{\mathbf{a}}\rangle \in \mathbb{Z}. \end{cases}$$

Remark 2.6.9 Without any difficulty one may generalise the results of this section to other singular rays $\arg t = \theta_j = \arg \mu_j$, $j \in \{2, 3, \ldots, r_1\}$. However, in order not to get lost in the quite heavy formulas, we won't write down these generalisations.

2.7 Stokes Transition

In this section the transition of convergent transseries at *Stokes directions* will be discussed. Note that in this thesis a direction θ is called a Stokes direction if $\theta = -\sigma \pm \pi/2$, with σ a singular direction. (Some authors, among them Costin and Costin in [CC01], use the term *anti-Stokes direction* instead. Those authors then refer to the singular direction as the Stokes direction.) We assume that S is some sector containing the positive real axis and we assume y to be a solution of (2.1.1) such that $y(x) = O(x^{-2})$ as $x \to \infty$ in S. We then look for the relation between two possible convergent transseries associated with this solution.

Let θ_{1-}, θ_1 and θ_{1+} be as in the introduction of section 2.5.3. With these directions we associate the sectors $S^+ := \{x \in \mathbb{C}^* \mid -\pi/2 - \theta_{1+} + \varepsilon < \arg x < \pi/2 - \theta_1 - \varepsilon\}$ and $S^- := \{x \in \mathbb{C}^* \mid -\pi/2 - \theta_1 + \varepsilon < \arg x < \pi/2 - \theta_{1-} - \varepsilon\}$, with $\varepsilon > 0$ so small that both S^+ and S^- contain the positive real axis, and we assume that S is a subset of $S^+ \cap S^-$. Let $y^{\pm}_{\mathbf{k}}$ denote the Borel sum of $\hat{y}_{\mathbf{k}}$ on $S^{\pm}$. Due to theorem 2.3.6 we then have two convergent transseries for y, i.e. there exist two 1-periodic trigonometric polynomials $C^{\pm}$, with values in $\mathbb{C}^p$, such that

$$y(x) = F_{\theta_1^+}(x, C^+(x)) = F_{\theta_1^-}(x, C^-(x)), \quad x \in S, \tag{2.7.1}$$

where

$$F_{\theta_1^{\pm}}(x, C) := \sum_{\mathbf{k} \in \mathbb{N}^p} C^{\mathbf{k}}\, e^{-\langle \mathbf{k}, \boldsymbol{\mu} \rangle x}\, x^{\langle \mathbf{k}, \mathbf{a} \rangle}\, y^{\pm}_{\mathbf{k}}(x).$$

Like in the proof of proposition 2.6.4, these convergent transseries may be written as

$$y(x) = \sum_{\mathbf{k} \in \mathbb{N}^p} (C^+)^{\mathbf{k}}(x)\, e^{-\langle \mathbf{k}, \boldsymbol{\mu} \rangle x}\, x^{\langle \mathbf{k}, \mathbf{m} \rangle}\, (\mathcal{L}_{\theta_1} \tilde{Y}^+_{\mathbf{k}})(x) = \sum_{\mathbf{k} \in \mathbb{N}^p} (C^-)^{\mathbf{k}}(x)\, e^{-\langle \mathbf{k}, \boldsymbol{\mu} \rangle x}\, x^{\langle \mathbf{k}, \mathbf{m} \rangle}\, (\mathcal{L}_{\theta_1} \tilde{Y}^-_{\mathbf{k}})(x),$$

$x \in S$, where $\mathcal{L}_{\theta_1}$ means the Laplace integral in the sense of distributions with as path of integration the singular ray $\arg t = \theta_1$.

Using (2.6.14) we can rewrite $\sum_{\mathbf{k} \in \mathbb{N}^p} (C^+)^{\mathbf{k}}(x)\, e^{-\langle \mathbf{k}, \boldsymbol{\mu} \rangle x}\, x^{\langle \mathbf{k}, \mathbf{m} \rangle}\, (\mathcal{L}_{\theta_1} \tilde{Y}^+_{\mathbf{k}})(x)$ as

$$\sum_{\mathbf{k} \in \mathbb{N}^p} \sum_{\substack{\mathbf{k}' \in \mathbb{N}^p, \mathbf{j} \in \mathbb{N}^{n_1} \\ \mathbf{k}' + \mathbf{j} = \mathbf{k}}} \binom{\mathbf{k}_1}{\mathbf{j}} \mathbf{s}^{\mathbf{j}}\, (C^+)^{\mathbf{k}'}(x)\, e^{-\langle \mathbf{k}, \boldsymbol{\mu} \rangle x}\, x^{\langle \mathbf{k}, \mathbf{m} \rangle}\, (\mathcal{L}_{\theta_1} \tilde{Y}^-_{\mathbf{k}})(x).$$

Moreover, $\sum_{\substack{\mathbf{k}' \in \mathbb{N}^p, \mathbf{j} \in \mathbb{N}^{n_1} \\ \mathbf{k}' + \mathbf{j} = \mathbf{k}}} \binom{\mathbf{k}_1}{\mathbf{j}} \mathbf{s}^{\mathbf{j}}\, (C^+)^{\mathbf{k}'}(x) = \big(C^+(x) + \sum_{h \in \mathcal{J}_1} s_h \mathbf{e}_h\big)^{\mathbf{k}}$ and the uniqueness of the convergent transseries then implies $C^-(x) = C^+(x) + \sum_{h=1}^{n_1} s_h \mathbf{e}_h$. This proves the following proposition.

Proposition 2.7.1 *Let y be a solution of (2.1.1) such that $y(x) = O(x^{-2})$ as $x \to \infty$ in a sector S that contains the positive real axis. Let θ_{j-}, θ_j and θ_{j+}, $j = 1, 2, \ldots, r_1$, be three consecutive singular directions in the right half plane of the set of all $\hat{y}_{\mathbf{k}}$, $\mathbf{k} \in \mathbb{N}^p$, and define the sectors $S^+ := \{x \in \mathbb{C}^* \mid -\pi/2 - \theta_{j+} + \varepsilon_j < \arg x < \pi/2 - \theta_j - \varepsilon_j\}$ and $S^- := \{x \in \mathbb{C}^* \mid -\pi/2 - \theta_j + \varepsilon_j < \arg x < \pi/2 - \theta_{j-} - \varepsilon_j\}$, with $\varepsilon_j > 0$ so small that both*

S^+ and S^- contain the positive real axis. Let $y^{\pm}_{\mathbf{k}}$ denote the Borel sum of $\hat{y}_{\mathbf{k}}$ on $S^{\pm}$ and assume that $S \subset S^+ \cap S^-$. Then, given two convergent transseries for y,

$$y(x) = F_{\theta_j^+}(x, C^+(x)) = F_{\theta_j^-}(x, C^-(x)), \quad x \in S$$

we have

$$F_{\theta_j^+}(x, C^+(x)) = F_{\theta_j^-}(x, C^+(x) + \textstyle\sum_{h\in\mathcal{J}_j} s_h \mathbf{e}_h). \tag{2.7.2}$$

2.8 Some Additional Results

In this section we want to prove some additional results. The first result concerns so-called *balanced averages* (cf. [Cos98]). With these balanced averages we will show that if (2.1.1) is real in the sense that $g_{\mathbf{l}}(x) \in \mathbb{R}^n$ for all $\mathbf{l} \in \mathbb{N}^n$ and $x \in \mathbb{R}$ and that for each $j \in \{1, 2, \ldots, r\}$ we have $e^{-\mu_j}, a_j \in \mathbb{R}$, then (2.1.1) possesses certain transseries solutions that are real when restricted to the positive real axis and these transseries can be summed to actual solutions y of (2.1.1) that are real-valued when restricted to the positive real axis. In section 2.8.2 we will determine the behaviour of the coefficients α_m in the formal solution $\hat{y}_0$ as $m \to \infty$, while in section 2.8.3 we will give an analogue for differential equations of the results obtained in this chapter.

2.8.1 Balanced Averages

We already found solutions Y_0 and $\tilde{Y}_{\mathbf{k}}$, $\mathbf{k} \in \mathbb{N}_1^p$, of (2.3.5) and (2.5.5) respectively, in a sector $\{t \in \mathbb{C}^* \mid \arg t \in (\theta_{1-}, \theta_{1+})\}$, where θ_{1-}, θ_1 and θ_{1+} are three consecutive singular directions. Note that for $t \in \mathbb{C}^*$ with $\arg t = \theta_1$, each $\tilde{Y}_{\mathbf{k}}(te^{i(\theta-\theta_1)})$ has limits $\tilde{Y}^{\pm}_{\mathbf{k}}(t)$ in the $\mathcal{D}'_{m_1,\nu}(\ell, \cdot)$-topology as θ tends to θ_1 (cf. proposition 2.5.11). Moreover, $\tilde{Y}^{\pm}_{\mathbf{k}}$ has a decomposition in terms of $\tilde{Y}^{\mp}_{\mathbf{k}+\mathbf{j}}$, $\mathbf{j} \in \mathbb{N}^{n_1}$ (cf. proposition 2.6.4).

Now, let $\mathcal{W}$ and $\mathcal{W}^{\pm}$ be defined as in (2.6.16) and let us consider the space $\mathcal{F}(\mathcal{W})$ of functions f which are holomorphic in both $\mathcal{W}^+$ and $\mathcal{W}^-$ and such that on $\arg t = \theta_1$ the function $f(te^{i(\theta-\theta_1)})$ has a limit $f^+(t)$ in the $\mathcal{D}'_{m_1,\nu}(\ell, \cdot)$-topology as $\theta \downarrow \theta_1$, while $f(te^{i(\theta-\theta_1)})$ has a limit $f^-(t)$ as $\theta \uparrow \theta_1$. Moreover, we assume that on the singular ray $\arg t = \theta_1$ we have decompositions

$$f^{\pm} = \sum_{j=0}^{\infty} \left(f_j^{\mp}(t - j\mu_1)\mathbb{1}_{[j\mu_1, \infty e^{i\theta_1})}\right)^{(m_1 j)}, \tag{2.8.1}$$

where each $f_j^{\mp}$ belongs to $\mathcal{D}'_{m_1,\nu}(\ell, \theta_1)$ and extends analytically into $\mathcal{W}^{\mp}$ in the way as described in lemma 2.6.5. Finally we assume the existence of a constant C, independent of ν, such that $\|(f_j^{\mp}(t-j\mu_1)\mathbb{1}_{[j\mu_1,\infty e^{i\theta_1})})^{(m_1 j)}\|_{m_1,\nu,\theta_1} \leqslant C\, K(\nu)^j$, where $\lim_{\nu\to\infty} K(\nu) = 0$. The decomposition in (2.8.1) is unique, which can be proved as follows: suppose for example that f^+ has two decompositions, namely $f^+ = \sum_{j=0}^{\infty}\left(f_j^-(t - j\mu_1)\mathbb{1}_{[j\mu_1,\infty e^{i\theta_1})}\right)^{(m_1 j)}$ and $f^+ = \sum_{j=0}^{\infty}\left(\tilde{f}_j^-(t - j\mu_1)\mathbb{1}_{[j\mu_1,\infty e^{i\theta_1})}\right)^{(m_1 j)}$. We already saw in lemma 2.6.5 that for $l \in \mathbb{N}$ the function f can be analytically continued from $\mathcal{W}^+$ across $(l\mu_1,(l+1)\mu_1)$ into $\mathcal{W}^-\backslash\overline{\Delta}(0,l|\mu_1|)$ and for $t\in(l\mu_1,\infty e^{i\theta_1})\backslash\mu_1\mathbb{N}$ we have, using the notation introduced after lemma 2.6.5,

$$f^{+l-}(t)=\sum_{j=0}^{l}\left(f_j^-(t-j\mu_1)\mathbb{1}_{[j\mu_1,\infty e^{i\theta_1})}\right)^{(m_1j)}=\sum_{j=0}^{l}\left(\tilde{f}_j^-(t-j\mu_1)\mathbb{1}_{[j\mu_1,\infty e^{i\theta_1})}\right)^{(m_1j)}.$$

For $l=0$ this implies $f_0^-=\tilde{f}_0^-$. Now suppose that $f_j^-=\tilde{f}_j^-$ for all $j\in\{0,1,\dots,l-1\}$, where $l\geqslant 1$. Then

$$\left(f_l^-(t-l\mu_1)\mathbb{1}_{[l\mu_1,\infty e^{i\theta_1})}\right)^{(m_1l)}=\left(\tilde{f}_l^-(t-l\mu_1)\mathbb{1}_{[l\mu_1,\infty e^{i\theta_1})}\right)^{(m_1l)}$$

and with an analogue of lemma A.1.1 we deduce that

$$f_l^-(t-l\mu_1)\mathbb{1}_{[l\mu_1,\infty e^{i\theta_1})}=\tilde{f}_l^-(t-l\mu_1)\mathbb{1}_{[l\mu_1,\infty e^{i\theta_1})}+p_l(t),$$

where p_l is a polynomial of degree at most m_1l-1. As $p_l(t)=0$ for all $t\in(0,l\mu_1)$ it has to be identically equal to zero and thus $f_l^-=\tilde{f}_l^-$.

For $\alpha\in\mathbb{C}$ we now define the operators $A_\alpha^\pm:\mathcal{F}(\mathcal{W})\to\mathcal{D}'_{m_1,\nu}(\ell,\theta_1)$ by

$$A_\alpha^\pm f:=\sum_{j=0}^{\infty}\alpha^j\left(f_j^\pm(t-j\mu_1)\mathbb{1}_{[j\mu_1,\infty e^{i\theta_1})}\right)^{(m_1j)}. \tag{2.8.2}$$

Lemma 2.8.1 *The operators $A_\alpha^\pm$ satisfy the following properties:*

(i) *Each $\tilde{Y}_{\mathbf{k}}$ belongs to $\mathcal{F}(\mathcal{W})$ and $A_\alpha^+\tilde{Y}_{\mathbf{k}}=A_{1-\alpha}^-\tilde{Y}_{\mathbf{k}}$.*

(ii) *If φ is holomorphic in a neighbourhood of ∞, then $\mathcal{B}\varphi\in\mathcal{F}(\mathcal{W})$ and $A_\alpha^\pm(\mathcal{B}\varphi)=\mathcal{B}\varphi$.*

(iii) *If $f\in\mathcal{F}(\mathcal{W})$, then $e^{-t}f\in\mathcal{F}(\mathcal{W})$ and $A_\alpha^\pm(e^{-t}f)=e^{-t}A_\alpha^\pm f$.*

(iv) *If both f and g belong to $\mathcal{F}(\mathcal{W})$, then $f*g\in\mathcal{F}(\mathcal{W})$ and $A_\alpha^\pm(f*g)=A_\alpha^\pm f*A_\alpha^\pm g$.*

Proof. Due to the assumptions, convergence of the right-hand side of (2.8.2) is ensured in $\mathcal{D}'_{m_1,\nu}(\ell,\theta_1)$ for ν large enough.

Using proposition 2.5.11, lemma 2.6.2 and proposition 2.6.4, the first statement in (i) becomes trivial (the proof of the required estimate is given in proposition A.6.1). To prove the second statement in (i) we first observe that

$$A_\alpha^+\tilde{Y}_{\mathbf{k}}=\sum_{\mathbf{j}\in\mathbb{N}^{n_1}}\alpha^{|\mathbf{j}|}(-\mathbf{s})^{\mathbf{j}}\binom{\mathbf{k}_1+\mathbf{j}}{\mathbf{j}}\left(\tilde{Y}^+_{\mathbf{k}+\mathbf{j}}(t-|\mathbf{j}|\mu_1)\mathbb{1}_{[|\mathbf{j}|\mu_1,\infty e^{i\theta_1})}\right)^{(m_1|\mathbf{j}|)} \tag{2.8.3}$$

and using proposition 2.6.4 this can be rewritten as

$$\sum_{\mathbf{r}\in\mathbb{N}^{n_1}}\mathbf{s}^{\mathbf{r}}\sum_{0\preceq\mathbf{j}\preceq\mathbf{r}}(-\alpha)^{|\mathbf{j}|}\binom{\mathbf{k}_1+\mathbf{j}}{\mathbf{j}}\binom{\mathbf{k}_1+\mathbf{r}}{\mathbf{k}_1+\mathbf{j}}\left(\tilde{Y}^-_{\mathbf{k}+\mathbf{r}}(t-|\mathbf{r}|\mu_1)\mathbb{1}_{[|\mathbf{r}|\mu_1,\infty e^{i\theta_1})}\right)^{(m_1|\mathbf{r}|)},$$

which coincides with $A^-_{1-\alpha}\tilde{Y}_{\mathbf{k}}$. If φ is holomorphic in a neighbourhood of ∞, then $\mathcal{B}\varphi$ belongs to $\mathcal{D}'_{m_1,\nu}(\ell,\theta)$ for all $\theta\in\mathbb{R}$ and if $\arg t=\theta_1$ we deduce from lemma 2.5.5, together with proposition A.4.1 that $(\mathcal{B}\varphi)(te^{i(\theta-\theta_1)})$ tends to $(\mathcal{B}\varphi)(t)$ in the $\mathcal{D}'_{m_1,\nu}(\ell,\cdot)$-topology if $\theta\to\theta_1$. Hence, on the ray $\arg t=\theta_1$ we have decompositions $(\mathcal{B}\varphi)^{\pm}=\mathcal{B}\varphi$, which proves (ii). The third statement follows by a straightforward use of the definitions. Finally, to prove (iv) we can assume that $f=g$, since $(f+g)*(f+g)=f*f+g*g+2(f*g)$. Now if $f\in\mathcal{F}(\mathcal{W})$, then also $f*f\in\mathcal{F}(\mathcal{W})$ and by (2.6.6) we obtain the following decomposition on the singular ray $\arg t=\theta_1$:

$$f^{\pm}*f^{\pm}=\sum_{N=0}^{\infty}\Big(\Big[\sum_{j=0}^{N}f_j^{\mp}*f_{N-j}^{\mp}\Big](t-N\mu_1)1_{[N\mu_1,\infty e^{i\theta_1})}\Big)^{(m_1N)}.$$

From this decomposition it follows that $A^{\pm}_{\alpha}(f*f)=(A^{\pm}_{\alpha}f)^{*2}$. ■

Remark 2.8.2 The choice $\alpha=1/2$ is a rather special one, since in this case $A^+_{1/2}$ coincides with $A^-_{1/2}$ when applying it to the solutions $\tilde{Y}_{\mathbf{k}}$.

Now let us assume that (2.1.1) is real in the sense that $g_{\mathbf{l}}(x)\in\mathbb{R}^n$ if $x\in\mathbb{R}$ for all $\mathbf{l}\in\mathbb{N}^n$ (compare (2.2.1)) and that for each $j\in\{1,2,\ldots,r\}$ we have $e^{-\mu_j}\in\mathbb{R}$ and $a_j\in\mathbb{R}$. Then (2.1.1) possesses a formal solution $\hat{y}_0(x)=\sum_{r=2}^{\infty}\alpha_r x^{-r}$ with $\alpha_r\in\mathbb{R}^n$. Using proposition 2.2.1 it then is obvious that $\hat{y}_{\mathbf{k}}$, with $|\mathbf{k}|=1$, has real coefficients and proceeding with induction on $|\mathbf{k}|$ also $\hat{y}_{\mathbf{k}}$, with $|\mathbf{k}|>1$, turns out to have real coefficients. Assume in addition that $\mu_1>0$, so that $\theta_1=\arg\mu_1=0$. As $\hat{y}_{\mathbf{k}}$ has real coefficients, the same holds for $x^{\langle\mathbf{k},\tilde{\mathbf{a}}\rangle}\hat{y}_{\mathbf{k}}(x)$ and it then is obvious that $\tilde{Y}_{\mathbf{k}}(t)=\overline{\tilde{Y}_{\mathbf{k}}(\overline{t})}$ for all t in a neighbourhood of the origin. Hence, this equality also holds for $t\in\mathcal{W}^*$, where $\mathcal{W}^*$ is the region defined by $\{t\in\mathbb{C}^*\mid 0<|\arg t|<\min\{-\theta_{1-},\theta_{1+}\}\}$. This implies that $\lim_{\arg t\uparrow 0}\tilde{Y}_{\mathbf{k}}(t)=\lim_{\arg t\downarrow 0}\overline{\tilde{Y}_{\mathbf{k}}(t)}$ in the $\mathcal{D}'_{m_1,\nu}(\ell,\cdot)$-topology and thus $\tilde{Y}^-_{\mathbf{k}}=\overline{\tilde{Y}^+_{\mathbf{k}}}$ on the singular ray $\arg t=0$. Using proposition 2.6.4 and the uniqueness of the decomposition we now deduce that

$$\sum_{|\mathbf{j}|=l}\overline{\mathbf{s}}^{\mathbf{j}}\binom{\mathbf{k}_1+\mathbf{j}}{\mathbf{j}}\Big(\overline{\tilde{Y}^-_{\mathbf{k+j}}}(t-l\mu_1)1_{[l\mu_1,\infty)}\Big)^{(m_1l)}=\sum_{|\mathbf{j}|=l}(-\mathbf{s})^{\mathbf{j}}\binom{\mathbf{k}_1+\mathbf{j}}{\mathbf{j}}\Big(\tilde{Y}^+_{\mathbf{k+j}}(t-l\mu_1)1_{[l\mu_1,\infty)}\Big)^{(m_1l)}$$

for each $l\in\mathbb{N}$. Hence, using (2.8.3), we obtain

$$\overline{A^-_{\alpha}\tilde{Y}_{\mathbf{k}}}=A^{\pm}_{\overline{\alpha}}\tilde{Y}_{\mathbf{k}}=A^-_{1-\overline{\alpha}}\tilde{Y}_{\mathbf{k}}$$

and similarly $\overline{A^+_{\alpha}\tilde{Y}_{\mathbf{k}}}=A^+_{1-\overline{\alpha}}\tilde{Y}_{\mathbf{k}}$. So $A^{\pm}_{\alpha}$ is reality-preserving if and only if $\alpha=1-\overline{\alpha}$, that is if and only if $\Re\alpha=1/2$.

In the following we will take $\alpha=1/2$, since then $A^+_{1/2}=A^-_{1/2}$ is a reality-preserving operator, when applying it to the solutions $\tilde{Y}_{\mathbf{k}}$.

Definition 2.8.3 *We call* $\tilde{Y}^{ba}_{\mathbf{k}}:=A^{\pm}_{1/2}\tilde{Y}_{\mathbf{k}}$ *the* balanced average *of* $\tilde{Y}_{\mathbf{k}}$.

Note that, due to (2.6.14) and (2.6.19), $\tilde{Y}_{\mathbf{k}}^{ba}$ can also be written as

$$\tilde{Y}_{\mathbf{k}}^{ba} = \tilde{Y}_{\mathbf{k}}^{\pm} + \sum_{l=1}^{\infty} 2^{-l} \big(\tilde{Y}_{\mathbf{k}}^{\mp l \pm} - \tilde{Y}_{\mathbf{k}}^{\mp(l-1)\pm}\big) \mathbb{1}_{[l\mu_1, \infty e^{i\theta_1})},$$

compare also [Cos98].

Proposition 2.8.4 *The expressions Y_0^{ba} and $\tilde{Y}_{\mathbf{k}}^{ba}$, $\mathbf{k} \in \mathbb{N}_1^p$, solve the equations (2.3.5) and (2.5.5) in $\mathcal{D}'_{m_1,\nu}(\ell) = \mathcal{D}'_{m_1,\nu}(\ell, \theta_1)$.*

Proof. The fact that Y_0^{ba} solves (2.3.5) on the singular ray $\arg t = 0$ immediately follows from proposition 2.6.1. Now for $\mathbf{j} \in \mathbb{N}_1^n$ the function $D_{\mathbf{j}}(Y_0)$ belongs to $\mathcal{F}(\mathcal{W})$ (the continuity follows from the lemmas 2.5.4 and 2.5.6). In fact, we have $D_{\mathbf{j}}(Y_0)^{\pm} = D_{\mathbf{j}}(Y_0^{\pm})$ and we can find a decomposition of $D_{\mathbf{j}}(Y_0^{\pm})$ in a similar way as in the proof of lemma 2.6.2 (in particular see (2.6.11)). From this it is easy to conclude that $D_{\mathbf{j}}(Y_0^{ba}) = A_{1/2}^{\pm} D_{\mathbf{j}}(Y_0)$. Applying $A_{1/2}^{\pm}$ to equation (2.5.5) and using the lemma above, it is easy to complete the proof of this proposition. ■

We now deduce that $\tilde{Y}_{\mathbf{k}}^{ba}$, $\mathbf{k} \in \mathbb{N}_1^p$, satisfies an estimate of the form (2.5.7) for ν is large enough. In fact, proposition A.6.1 implies that $\|(\tilde{Y}_{\mathbf{k}+\mathbf{j}}^{+}(t - |\mathbf{j}|\mu_1)\mathbb{1}_{[|\mathbf{j}|\mu_1,\infty)})^{(m_1|\mathbf{j}|)}\|_{m_1,\nu}$ is bounded by a constant times $K(\nu)^{|\mathbf{j}|}$ times $\|\tilde{Y}_{\mathbf{k}+\mathbf{j}}^{+}\|_{m_1,\nu}$ and from (2.5.7) we conclude that $\|\tilde{Y}_{\mathbf{k}+\mathbf{j}}^{+}\|_{m_1,\nu} \leqslant \tilde{K}^{|\mathbf{k}|+|\mathbf{j}|}$ for some $\tilde{K} > 0$. These observations together finally imply that there exists a positive constant K_1, independent of $\mathbf{k}$, such that $\|\tilde{Y}_{\mathbf{k}}^{ba}\|_{m_1,\nu} \leqslant K_1^{|\mathbf{k}|}$, provided that ν is large enough. Theorem A.5.2 then implies $\big|\big(\mathcal{L}\tilde{Y}_{\mathbf{k}}^{ba}\big)(x)\big| \leqslant K_1^{|\mathbf{k}|}$ for x in a neighbourhood of ∞ in a sector S of opening π and containing the positive real axis (compare the results in section A.5).

Using these observations it is easy to verify the following proposition (compare [Cos98], theorem 3(iii)).

Proposition 2.8.5 *Let us assume that (2.1.1) is real in the sense that $g(x, y)$ is real analytic and that for each $j \in \{1, 2, \ldots, r\}$ we have $e^{-\mu_j} \in \mathbb{R}$ and $a_j \in \mathbb{R}$. If C is a p-vector valued 1-periodic trigonometric polynomial such that $C(x) \in \mathbb{R}^p$ if $x \in \mathbb{R}_+$ and such that $C_h(x)e^{-\mu_j x} \to 0$ as $x \to \infty$ in S for every $h \in \mathcal{J}_j$, $j \in \{1, 2, \ldots, r_1\}$, then the sum*

$$\sum_{\mathbf{k}\in\mathbb{N}^p} C^{\mathbf{k}}(x)\, e^{-\langle\mathbf{k},\boldsymbol{\mu}\rangle x}\, x^{\langle\mathbf{k},\mathbf{m}\rangle}\, (\mathcal{L}\tilde{Y}_{\mathbf{k}}^{ba})(x)$$

converges in a neighbourhood of ∞ in S, defines a solution of (2.1.1) which is holomorphic in this neighbourhood and is real-valued on the positive real axis. Moreover, this solution asymptotically equals $\hat{y}_0(x)$ as $x \to \infty$ in S. Hence, the transseries solution $\sum_{\mathbf{k}\in\mathbb{N}^p} C^{\mathbf{k}}(x)\, e^{-\langle\mathbf{k},\boldsymbol{\mu}\rangle x}\, x^{\langle\mathbf{k},\mathbf{a}\rangle}\, \hat{y}_{\mathbf{k}}(x)$ can be summed to an actual solution y of (2.1.1), which is real when restricted to the positive real axis.

Remark 2.8.6 The results presented in this section correspond to the so-called *generalised Borel summation operator* $\sum^{1/2}$, introduced by Costin in [Cos98] for differential equations, in the sense that if $y(x) = \sum_{\mathbf{k}\in\mathbb{N}^p} C^{\mathbf{k}}(x)\, e^{-\langle\mathbf{k},\boldsymbol{\mu}\rangle x}\, x^{\langle\mathbf{k},\mathbf{m}\rangle}\, (\mathcal{L}\tilde{Y}_{\mathbf{k}}^{ba})(x)$, then y can also be written as $y(x) = \sum^{1/2} \sum_{\mathbf{k}\in\mathbb{N}^p} C^{\mathbf{k}}(x)\, e^{-\langle\mathbf{k},\boldsymbol{\mu}\rangle x}\, x^{\langle\mathbf{k},\mathbf{a}\rangle}\, \hat{y}_{\mathbf{k}}(x)$ (compare also [CK02]).

2.8.2 Asymptotic Behaviour of the Coefficients of $\hat{y}_0$

In this section we put $\rho := \min_{j\in\{1,2,\ldots,r\}} |\mu_j|$ and we may assume without loss of generality that $|\mu_j + 2l\pi i| > \rho$ for all $j \in \{1,2,\ldots,r\}$ and all $l \in \mathbb{Z}\backslash\{0\}$, since the μ_j, as given in the definition of Λ (cf. (2.1.2)), are determined modulo $2\pi i$.

In this case one may deduce from proposition 2.6.8 an asymptotic expansion for the coefficients in the formal series solution $\hat{y}_0(x) = \sum_{m=2}^{\infty} \alpha_m x^{-m}$ of (2.1.1), as in [CK99] (see proposition 2.8.10). Since $\alpha_m = Y_0^{(m-1)}(0)$, its behaviour for $m \to \infty$ can be derived from the corresponding Cauchy formula, given by $Y_0^{(m-1)}(0) = \frac{(m-1)!}{2\pi i} \oint_\gamma \frac{Y_0(t)}{t^m}\,\mathrm{d}t$, with γ a positively orientated Jordan contour around 0 lying in the maximal star domain with centre 0 which does not contain any of the singular points $\mu_j + 2l\pi i$, $j \in \{1,2,\ldots,r\}$ and $l \in \mathbb{Z}$. We may deform γ by pushing it towards the circle with centre 0 and radius $\rho + \varepsilon$ for some small $\varepsilon > 0$. Then γ looks like the contour as drawn in figure 2.1.

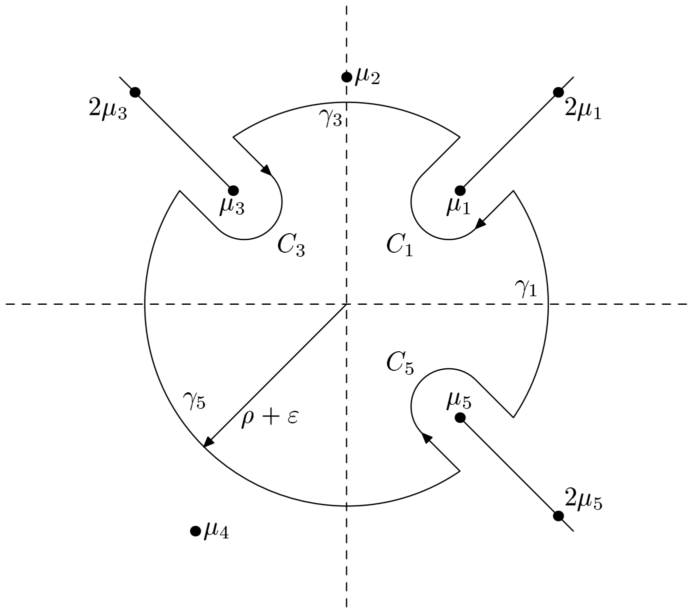

Figure 2.1: *The Cauchy-contour γ split in parts C_j and γ_j.*

Lemma 2.8.7 *For each μ_j with $|\mu_j| = \rho$, let C_j be the part of γ around μ_j (cf. figure 2.1). Then*

$$\alpha_m = \frac{(m-1)!}{2\pi i} \sum_j \int_{C_j} \frac{Y_0(t)}{t^m}\,\mathrm{d}t + O\big((m-1)!(\rho+\varepsilon)^{-m+1}\big) \quad \text{as } m \to \infty,$$

where the sum is taken over those j with $|\mu_j| = \rho$.

Proof. As $\alpha_m - \frac{(m-1)!}{2\pi i}\sum_j \int_{C_j} \frac{Y_0(t)}{t^m}\,\mathrm{d}t = \frac{(m-1)!}{2\pi i}\sum_j \int_{\gamma_j} \frac{Y_0(t)}{t^m}\,\mathrm{d}t$, with γ_j parts of the circle with centre 0 and radius $\rho+\varepsilon$, the proof easily follows from the boundedness of Y_0 on each γ_j. ■

Lemma 2.8.8 *For $\beta \notin \mathbb{Z}$, $m \in \mathbb{N}_+$ and $\Re(m-\beta) > 0$ we have*

$$\int_\infty^{(0^-)} t^{\beta-1}(1+t)^{-m}\mathrm{d}t = (e^{-2\pi i\beta} - 1)B(\beta, m-\beta),$$

where in the right-hand side B denotes the beta function and in the left-hand side the path of integration starts at ∞, goes around 0 in the negative sense and ends at $\infty e^{-2\pi i}$. Moreover, $B(\beta, m-\beta) = \frac{\Gamma(\beta)\Gamma(m-\beta)}{\Gamma(m)}$.

The proof of this lemma can be found in for example [HTF53], p. 15, formula (9) with the substitution $t = \frac{s}{1+s}$.

We now concentrate on the terms $\frac{(m-1)!}{2\pi i}\int_{C_j}\frac{Y_0(t)}{t^m}\,\mathrm{d}t$ for those j with $|\mu_j| = \rho$. Such a contour C_j can be deformed to a loop around μ_j, so the integral can be replaced by $\frac{(m-1)!}{2\pi i}\int_{\varepsilon e^{i\theta_j}}^{(0^-)}\frac{Y_0(\mu_j+t)}{(\mu_j+t)^m}\,\mathrm{d}t$, $\varepsilon > 0$ small enough, where the path of integration starts at $\varepsilon e^{i\theta_j}$, goes around 0 in the negative sense and ends at $\varepsilon e^{i\theta_j - 2\pi i}$. Similarly as in proposition 2.4.3 one may show that for $j \in \{1, 2, \ldots, r\}$ we have

$$Y_0(\mu_j + t) \equiv \frac{-1}{1 - e^{2\pi i\tilde{a}_j}}\sum_{h\in\mathcal{J}_j} s_h\,\tilde{Y}_{\mathbf{e}_h}^{(m_j)}(t) \bmod \mathbb{C}^n\{t\}$$

for $|t| < \varepsilon$, $\theta_j - 2\pi \leqslant \arg t \leqslant \theta_j$, provided that we choose $Y_0(\mu_j + t) = Y_0^+(\mu_j + t)$ if $\arg t = \theta_j - 2\pi$ and $Y_0(\mu_j + t) = Y_0^-(\mu_j + t)$ if $\arg t = \theta_j$, compare also proposition 2.6.8. Hence,

$$\begin{aligned}\frac{(m-1)!}{2\pi i}\int_{\varepsilon e^{i\theta_j}}^{(0^-)}\frac{Y_0(\mu_j+t)}{(\mu_j+t)^m}\,\mathrm{d}t &= \frac{(m-1)!}{2\pi i(e^{2\pi i\tilde{a}_j}-1)}\sum_{h\in\mathcal{J}_j} s_h\int_{\varepsilon e^{i\theta_j}}^{(0^-)}\frac{t^{-a_j-1}(\mathbf{e}_h + O(t))}{(\mu_j+t)^m}\,\mathrm{d}t\\ &= \frac{(m-1)!}{2\pi i(e^{2\pi i\tilde{a}_j}-1)}\mu_j^{-a_j-m}\sum_{h\in\mathcal{J}_j} s_h\int_{\varepsilon/\rho}^{(0^-)}\frac{t^{-a_j-1}(\mathbf{e}_h + O(t))}{(1+t)^m}\,\mathrm{d}t.\end{aligned}$$

From lemma 2.8.8 we now deduce

$$\int_{\varepsilon/\rho}^{(0^-)}\frac{t^{-a_j-1}}{(1+t)^m}\,\mathrm{d}t = (e^{2\pi i a_j} - 1)B(-a_j, m+a_j) - (e^{2\pi i a_j} - 1)\int_{\varepsilon/\rho}^{\infty}\frac{t^{-a_j-1}}{(1+t)^m}\,\mathrm{d}t$$

and the latter integral is bounded by a constant times $(1+\varepsilon/\rho)^{-m}$ and hence is of order $O\big(e^{-\frac{\varepsilon}{\rho}m}\big)$ as $m \to \infty$.

Lemma 2.8.9 *Let H be a holomorphic function in a neighbourhood of 0 containing the disc $\overline{\Delta}(0, \varepsilon/\rho)$, and suppose that $H(t) = O(t)$ as $t \to 0$. Then*

$$\int_{\varepsilon/\rho}^{(0^-)}\frac{t^{-a_j-1}H(t)}{(1+t)^m}\,\mathrm{d}t = O(m^{\Re a_j - 1}) \text{ as } m \to \infty.$$

Proof. To start the proof we make a particular choice of the integration contour: as it is sufficient to assume $m > \rho/\varepsilon$ we start at $t = \varepsilon/\rho$ and go to $1/m$ along the lower cut of the positive real axis, then we continue towards the upper cut anticlockwise along the circle with radius $1/m$, centred at the origin, and finally we go from $t = \frac{e^{-2\pi i}}{m}$ towards $t = \frac{\varepsilon}{\rho}e^{-2\pi i}$ along the upper cut of the positive real axis. Then the integral can be rewritten as

$$im^{a_j}\int_0^{-2\pi}\frac{e^{-i\sigma a_j}H(m^{-1}e^{i\sigma})}{(1+m^{-1}e^{i\sigma})^m}\,\mathrm{d}\sigma+\int_{m^{-1}}^{\varepsilon/\rho}t^{-a_j-1}\tilde{H}(t)\,e^{-m\log(1+t)}\,\mathrm{d}t, \qquad (2.8.4)$$

where $\tilde{H}(t) = e^{2\pi i a_j}H(te^{-2\pi i}) - H(t)$, and so $\tilde{H}$ also is of order $O(t)$ as $t \to 0$. Since $|1+m^{-1}e^{i\varphi}|^m \geqslant (1-m^{-1})^m$ is bounded from below by a positive constant independent of m, the first integral in (2.8.4) is of order $O(m^{\Re a_j - 1})$ as $m \to \infty$. Moreover, for $t > 0$ small enough we have $e^{-m\log(1+t)} \leqslant e^{-\frac{1}{2}mt}$, and by the substitution $s = mt$ we see that the second integral in (2.8.4) also is of order $O(m^{\Re a_j - 1})$ as $m \to \infty$. ■

Proposition 2.8.10 *Given the formal series solution $\hat{y}_0(x) = \sum_{m=2}^{\infty}\alpha_m x^{-m}$ of (2.1.1), we have*

$$\alpha_m = \frac{1}{2\pi i}\sum_j \Gamma(m+a_j)\Gamma(-a_j)\,\mu_j^{-m-a_j}\sum_{h\in\mathcal{J}_j}s_h\big(\mathbf{e}_h + O(m^{-1})\big), \quad m \to \infty, \qquad (2.8.5)$$

where $\sum_j$ is the sum taken over those j with $|\mu_j| = \rho$.

Proof. Using that $\frac{\Gamma(m)}{\Gamma(m+a_j)} \sim m^{-a_j}$ for $m \to \infty$, we obtain by combining the results above

$$\alpha_m = \frac{1}{2\pi i}\sum_j \Gamma(m+a_j)\Gamma(-a_j)\,\mu_j^{-m-a_j}\sum_{h\in\mathcal{J}_j}s_h\big[\mathbf{e}_h + O(m^{-1})\big] + O\big((m-1)!(\rho+\varepsilon)^{-m+1}\big),$$

as $m \to \infty$. However, $O\big((m-1)!(\rho+\varepsilon)^{-m+1}\big)\mu_j^{m+a_j}\big(\Gamma(m+a_j)\big)^{-1} = O\big(m^{-\Re a_j}e^{-\frac{\varepsilon}{\rho}m}\big)$, as $m \to \infty$, for arbitrary j such that $|\mu_j| = \rho$. So (2.8.5) holds. ■

2.8.3 A Remark on the Analogue for Differential Equations

The analogue of (2.1.1) for the differential case reads

$$y'(x) + \Lambda(x)y(x) + g(x, y(x)) = 0,$$

where $\Lambda(x) = \oplus_{j=1}^{r}(\mu_j \mathtt{I}_{n_j} - x^{-1}\mathtt{M}_j)$. As before $r \in \mathbb{N}_+$, $n = n_1 + n_2 + \cdots + n_r$, $n_j \in \mathbb{N}$, and $\mathtt{M}_j$ is an $n_j \times n_j$ diagonal matrix with a_m, $m \in \mathcal{J}_j$, on the diagonal. However, in the differential case we assume the μ_j's to be complex numbers such that

$$\mu_j \neq (k_1\mu_1 + k_2\mu_2 + \cdots + k_r\mu_r), \quad \text{for every} \quad (k_1, k_2, \ldots, k_r) \in \mathbb{N}^r \backslash \{\mathbf{e}_j\}.$$

In addition, we make the same assumptions on a_m, $m = 1, 2, \ldots, n$, and on g as in the introduction of this chapter. The corresponding results concerning the formal and analytic reduction to the (semi-)canonical form

$$z'(x) + \Lambda(x)z(x) = 0$$

can be shown in a similar way as in [Bra01], section 7. For complete proofs in more general settings the reader is referred to the chapters 4 and 5.

The case $r = n$ (and thus $\Lambda(x) = \operatorname{diag}\{\mu_j - a_j x^{-1}\}_{j=1}^{n}$) is exhaustively studied by Costin in [Cos95, Cos98]. In this case he proved similar resurgence relations to the ones we proved in the case of difference equations. If, in the case $r < n$, we assume that Λ has the form $\Lambda(x) = \oplus_{j=1}^{r}(\mu_j - x^{-1}a_j)\mathrm{I}_{n_j}$ (compare the beginning of section 2.4, where we made a similar assumption in the case of difference equations), then by modifying the proofs we gave in sections 2.5 and 2.6, one may derive similar resurgence relations to those found in section 2.5.

2.9 Proof of Lemma 2.5.7

Let us first recall the lemma.

Lemma 2.9.1 *Let $-\pi/2 < \theta_- < \theta_+ < \pi/2$ and let $I \subset (\theta_-, \theta_+)$ be a compact interval. Take $\delta > 0$ so small that $I \pm \delta \subset (\theta_-, \theta_+)$ and let U_1 and U_2 be as in (2.5.3).*

(i) *For $\mu \in \mathbb{C}^*$, $\arg \mu \in (\theta_-, \theta_+)$, or $\mu = 0$, we take $a \in \mathbb{R}_+$ so large that $a\cos(\theta+\sigma) > \Re\mu$ for all $\theta \in I$ and all $\sigma \in [-\delta, \delta]$. Then there exists a positive ν_0 such that for all $\theta \in I$ and $\sigma \in [-\delta, \delta]$ the operator defined by multiplication with $U_1(\cdot e^{i\sigma})$ maps $\mathcal{D}'_{m,\nu}\big(ae^{i\theta}, \infty e^{i\theta}\big)$ into itself, provided that $\nu \geqslant \nu_0$. This operator is bounded by a positive constant independent of θ and σ.*

(ii) *For $\mu \in \mathbb{C}^*$ such that $\arg(\mu + 2l\pi i) \notin (\theta_-, \theta_+)$ for all $l \in \mathbb{Z}$, there exists a positive ν_0 such that for all $\theta \in I$ and $\sigma \in [-\delta, \delta]$ the operator defined by multiplication with $U_1(\cdot e^{i\sigma})$ maps $\mathcal{D}'_{m,\nu}(\theta)$ into itself, provided that $\nu \geqslant \nu_0$. This operator is bounded by a positive constant independent of θ and σ.*

(iii) *There exists a positive ν_0 such that for all $\theta \in I$ and $\sigma \in [-\delta, \delta]$ the operator defined by multiplication with $U_2(\cdot e^{i\sigma})$ maps $\mathcal{D}'_{m,\nu}(\theta)$ into itself, provided that $\nu \geqslant \nu_0$. This operator is bounded by a positive constant independent of θ and σ.*

(iv) *There exists a positive ν_0 such that for all $\theta \in I$ and $\sigma \in [-\delta, \delta]$ the operator defined by multiplication with $U_2(\cdot e^{i\sigma}) - U_2$ maps $\mathcal{D}'_{m,\nu}(\theta)$ into itself, provided that $\nu \geqslant \nu_0$. This operator is bounded by $C|\sigma|$, where C is some positive constant independent of θ and σ.*

Proof. For $f = \sum_{k=0}^{\infty} \Delta_k^{(mk)} \in \mathcal{D}'_{m,\nu}(\theta)$ a staircase distribution, we obviously have $U(te^{i\sigma})f = \sum_{k=0}^{\infty} U(te^{i\sigma})\Delta_k^{(mk)}$. Hence, it is enough to find a k-independent upperbound for the norm of multiplication with $U(te^{i\sigma})$ as operator working on the space $\mathcal{D}'_{m,\nu,k}(\theta)$ defined by

$$\mathcal{D}'_{m,\nu,k}(\theta) := \big\{ f \in \mathcal{D}'_{m,\nu}(\theta) \;\big|\; \Delta_i \equiv 0 \text{ for all } i \neq k \big\}.$$

To prove (i) we assume k to be so large that $\mathcal{D}'_{m,\nu}\big(ae^{i\theta}, \infty e^{i\theta}\big) \cap \mathcal{D}'_{m,\nu,k}(\theta)$ is nonempty and we take $f \in \mathcal{D}'_{m,\nu}\big(ae^{i\theta}, \infty e^{i\theta}\big) \cap \mathcal{D}'_{m,\nu,k}(\theta)$. Then f can be written as $f = \Delta_k^{(mk)}$ and by construction Δ_k is supported in $[ke^{i\theta}, (k+1)e^{i\theta}] \cap [ae^{i\theta}, \infty e^{i\theta})$. Let $b_k = \max\{a, k\}$. If $g_{k,j}(t) := U_1^{(mk-j)}(te^{i\sigma})\Delta_k(t)$, then $g_{k,j}$ is supported in $[b_k e^{i\theta}, (k+1)e^{i\theta}]$ and we will show that

$$U_1(te^{i\sigma})\Delta_k^{(mk)} = \sum_{j=0}^{mk} (-e^{i\sigma})^{mk-j} \binom{mk}{j} g_{k,j}^{(j)}. \tag{2.9.1}$$

To prove (2.9.1) we expand $g_{k,j}^{(j)}$ in the the right-hand side of (2.9.1) to obtain

$$\sum_{j=0}^{mk} (-e^{i\sigma})^{mk-j} \binom{mk}{j} g_{k,j}^{(j)} = \sum_{j=0}^{mk} (-1)^{mk-j} \binom{mk}{j} \sum_{l=0}^{j} e^{i\sigma(mk-l)} \binom{j}{l} U_1^{(mk-l)}(te^{i\sigma})\Delta_k^{(l)}$$

and interchanging the order of summation we easily infer that the latter expression can be written as

$$\sum_{l=0}^{mk} (-e^{i\sigma})^{mk-l} \binom{mk}{l} U_1^{(mk-l)}(te^{i\sigma})\Delta_k^{(l)} \sum_{j=0}^{mk-l} (-1)^j \binom{mk-l}{j}.$$

Since $\sum_{j=0}^{mk-l} (-1)^j \binom{mk-l}{j} = 0$ except if $l = mk$ we obtain (2.9.1).

Let $\mathcal{P}_\theta : \mathcal{D}'_{m,\nu}(\theta) \to \mathcal{D}'_{m,\nu}(\theta)$ be the operator defined by $\mathcal{P}_\theta f = \mathcal{H}_\theta * f$, where $\mathcal{H}_\theta$ is the (generalised) Heaviside one step function defined in section A.6. Then $\mathcal{P}_\theta$ satisfies the same properties as $\mathcal{P} = \mathcal{P}_0$ and if no confusion can arise we will not display the dependence of $\mathcal{H}_\theta$ and $\mathcal{P}_\theta$ on θ. With this definition we then have $g_{k,j}^{(j)} = \big(\mathcal{P}^{mk-j} g_{k,j}\big)^{(mk)}$ in distributional sense, with $\mathcal{P}^{mk-j} g_{k,j}$, $j \in \{0, 1, \ldots, mk-1\}$, supported in $[b_k e^{i\theta}, \infty e^{i\theta}) \subset [ke^{i\theta}, \infty e^{i\theta})$. Moreover, lemma A.3.1 justifies that $g_{k,j}^{(j)}$, with $j \in \{0, 1, \ldots, mk-1\}$, can be written as $g_{k,j}^{(j)} = \sum_{l=k}^{\infty} \tilde{\Delta}_{l,j}^{(ml)}$, where for $l \geqslant k$ we have

$$\tilde{\Delta}_{l,j} = G_{l,j} 1_{[le^{i\theta}, (l+1)e^{i\theta}]}, \quad \text{with} \quad G_{l,j} = \mathcal{P}^m\big(G_{l-1,j} 1_{[le^{i\theta}, \infty e^{i\theta})}\big), \;\; G_{k,j} = \mathcal{P}^{mk-j} g_{k,j}. \tag{2.9.2}$$

So (2.9.1) can be rewritten as

$$U_1(te^{i\sigma})\Delta_k^{(mk)} = \big(U_1(te^{i\sigma})\Delta_k\big)^{(mk)} + \sum_{j=0}^{mk-1} (-e^{i\sigma})^{mk-j} \binom{mk}{j} \sum_{l=k}^{\infty} \tilde{\Delta}_{l,j}^{(ml)}$$

and this latter sum can be split into

$$\sum_{j=0}^{mk-1} (-e^{i\sigma})^{mk-j} \binom{mk}{j} \tilde{\Delta}_{k,j}^{(mk)} \;+\; \sum_{j=0}^{mk-1} (-e^{i\sigma})^{mk-j} \binom{mk}{j} \sum_{l=k+1}^{\infty} \tilde{\Delta}_{l,j}^{(ml)}. \tag{2.9.3}$$

By Cauchy's formula we have for $j \in \{0, 1, \ldots, mk\}$

$$U_1^{(mk-j)}(te^{i\sigma}) = \frac{(mk-j)!}{2\pi i} \oint_\gamma \frac{(1-e^{\mu-s})^{-1}}{(s-te^{i\sigma})^{mk-j+1}} \,\mathrm{d}s, \tag{2.9.4}$$

where γ is an arbitrary positively orientated Jordan contour around $te^{i\sigma}$, contained in a simply connected domain in which U_1 is holomorphic. The latter integral can be rewritten as $\frac{(mk-j)!}{2\pi i} \oint_{\tilde\gamma} \frac{(1-e^{\mu-te^{i\sigma}-s})^{-1}}{s^{mk-j+1}} \,\mathrm{d}s$, where $\tilde\gamma = \{s - te^{i\sigma} \mid s \in \gamma\}$. We take $\tilde\gamma$ to be the circle around 0 with radius $\rho(\Re(te^{i\sigma}) - \Re\mu)$ for some fixed $\rho \in (0,1)$. As we only work with t in $(ae^{i\theta}, \infty e^{i\theta})$, there exists a positive constant c such that

$$|U_1^{(mk-j)}(te^{i\sigma})| \leqslant c\,(mk-j)!\big(\rho(\Re(te^{i\sigma}) - \Re\mu)\big)^{-(mk-j)}$$

and due to our assumption this c can be chosen independently of $\theta \in I$ and $\sigma \in [-\delta, \delta]$.

Let $\tilde\theta$ and $\tilde\sigma$ be so that $\cos(\tilde\theta + \tilde\sigma) = \min\{\cos(\theta+\sigma) \mid \theta \in I, |\sigma| \leqslant \delta\} > 0$ and define $\tilde\rho = \rho\cos(\tilde\theta + \tilde\sigma)$ and $\tilde\mu = \Re\mu / \cos(\tilde\theta + \tilde\sigma)$, then

$$|U_1^{(mk-j)}(te^{i\sigma})| \leqslant c\,(mk-j)!\big(\tilde\rho(|t| - \tilde\mu)\big)^{-(mk-j)}, \quad j \in \{0, 1, \ldots, mk\}. \tag{2.9.5}$$

Observe that this upperbound is independent of $\theta \in I$ and $\sigma \in [-\delta, \delta]$. Using this we directly obtain $\big\|(U_1(te^{i\sigma})\Delta_k)^{(mk)}\big\|_{m,\nu,\theta} \leqslant c\,\|\Delta_k\|_{m,\nu,\theta}$. To prove that both sums in (2.9.3) satisfy the appropriate estimate, we may assume that $k \geqslant 1$ (note that a priori k might be equal to 0 if $0 < a < 1$).

As $|(\mathcal{P}_\theta^m F)(t)| \leqslant \big(\mathcal{P}_0^m(|F_\theta|)\big)(|t|)$, where $F_\theta : \mathbb{R} \to \mathbb{C}$ is defined by $F_\theta(s) = F(se^{i\theta})$, we obtain for every $l \geqslant k$, $t \in [l, l+1]$ and $j \in \{0, 1, \ldots, mk-1\}$

$$\begin{aligned} |G_{l,j}(te^{i\theta})| &\leqslant \mathcal{P}_0^{ml-j}\big(|g_{k,j}(se^{i\theta})|\big)(t) \\ &\leqslant c\,\frac{(mk-j)!}{(ml-j-1)!} \int_{b_k}^t (t-s)^{ml-j-1}\big(\tilde\rho(s-\tilde\mu)\big)^{-(mk-j)} |\Delta_k(se^{i\theta})| \mathrm{d}s \\ &\leqslant c\,\frac{(mk-j)!}{(ml-j-1)!} \frac{(l+1-k)^{ml-j-1}}{\big(\tilde\rho(b_k-\tilde\mu)\big)^{mk-j}} \int_{b_k}^t |\Delta_k(se^{i\theta})| \mathrm{d}s. \end{aligned}$$

Hence, using Fubini's theorem on interchanging the order of integration, we deduce

$$\big\|\tilde\Delta_{k,j}\big\|_{\nu,\theta} \leqslant \frac{c}{\nu} \|\Delta_k\|_{\nu,\theta} (mk-j)\big(\tilde\rho(b_k - \tilde\mu)\big)^{-(mk-j)}$$

and we conclude that in $\mathcal{D}'_{m,\nu}(\theta)$ the norm of $\sum_{j=0}^{mk-1}(-e^{i\sigma})^{mk-j}\binom{mk}{j}\tilde\Delta_{k,j}^{(mk)}$ can be bounded by

$$\frac{c}{\nu}\|f\|_{m,\nu,\theta} \sum_{j=0}^{mk-1} (mk-j)\binom{mk}{j}\big(\tilde\rho(b_k - \tilde\mu)\big)^{-(mk-j)}.$$

The latter sum equals $\frac{mk}{\tilde\rho(b_k-\tilde\mu)}\big(1 + \frac{1}{\tilde\rho(b_k-\tilde\mu)}\big)^{mk-1}$ and since $b_k = k$ for k large enough, this expression is bounded by a constant independent of k.

In case $l > k$ we have

$$\big\|\tilde{\Delta}_{l,j}\big\|_{\nu,\theta} \leqslant c\,\frac{(mk-j)!}{(ml-j-1)!}\,\frac{(l+1-k)^{ml-j-1}}{\big(\tilde{\rho}(b_k-\tilde{\mu})\big)^{mk-j}}\,\frac{e^{-\nu(l-k-1)}}{\nu}\|\Delta_k\|_{\nu,\theta}$$

and using $\frac{(mk-j-1)!}{(ml-j-1)!} \leqslant \frac{1}{(ml-mk-1)!}$, the norm of $\sum_{j=0}^{mk-1}(-e^{i\sigma})^{mk-j}\binom{mk}{j}\tilde{\Delta}_{l,j}^{(ml)}$ in $\mathcal{D}'_{m,\nu}(\theta)$ is less than

$$\frac{cmk}{\nu\tilde{\rho}(b_k-\tilde{\mu})}\|f\|_{m,\nu,\theta}\,\frac{(l+1-k)^{m(l-k)}}{\big(m(l-k)-1\big)!}\,e^{-\nu(l-k-1)}\nu^{ml-mk}\Big(1+\frac{l+1-k}{\tilde{\rho}(b_k-\tilde{\mu})}\Big)^{mk-1}.$$

As before the quotient $\frac{mk}{\tilde{\rho}(b_k-\tilde{\mu})}$ is bounded by a constant independent of k. Defining $n = l-k$, the proof is complete if we can show that

$$\sum_{n=1}^{\infty}\frac{(n+1)^{mn}}{(mn-1)!}\,e^{-\nu(n-1)}\nu^{mn}\Big(1+\frac{n+1}{\tilde{\rho}(b_k-\tilde{\mu})}\Big)^{mk-1} \tag{2.9.6}$$

is bounded by a constant independent of k.

Using Stirling's formula[1] we get $\frac{(n+1)^{mn}}{(mn-1)!} \leqslant const.\cdot\frac{\sqrt{n}\,e^{mn}}{m^{mn}}$. Moreover, $\big(1+\frac{n+1}{\tilde{\rho}(b_k-\tilde{\mu})}\big)^{mk-1}$ can be majorized by $\exp\big[mk\frac{n+1}{\tilde{\rho}(b_k-\tilde{\mu})}\big]$ and as $\frac{k}{\tilde{\rho}(b_k-\tilde{\mu})}$ is bounded by a constant independent of k, say β, we can estimate (2.9.6) by a constant times $\sum_{n=0}^{\infty}\sqrt{n+1}\,e^{-\nu n}\Big(\frac{e^{m+m\beta}\nu^m}{m^m}\Big)^{n+1}$. Now $\nu^{m(n+1)} \leqslant \nu^{2mn}$ and $\sqrt{n+1} \leqslant n+1 \leqslant 2^n$, so (2.9.6) can be estimated by a constant (which is independent of k and ν) times

$$\sum_{n=0}^{\infty}\Big(\frac{2e^{(1+\beta)m-\nu}\nu^{2m}}{m^m}\Big)^n,$$

which converges for ν large enough. This proves (i).

To prove (ii) we distinguish two cases, namely $\Re\mu \geqslant 0$ and $\Re\mu < 0$. First let us assume $\Re\mu < 0$ and fix $\beta \in (0, -\Re\mu)$. In order to derive a Cauchy estimate on $U_1^{(mk-j)}(te^{i\sigma})$ similar to the one given in (2.9.5), we take the parametrisation $s = te^{i\sigma} + \rho\,\Re(te^{i\sigma})\,e^{i\varphi}$, $0 \leqslant \varphi < 2\pi$, $\rho \in (0,1)$, in (2.9.4) and we obtain

$$|U_1^{(mk-j)}(te^{i\sigma})| \leqslant (1-e^{-\beta})^{-1}\,(mk-j)!(\tilde{\rho}|t|)^{-(mk-j)}, \quad |t| > 0,$$

where $\tilde{\rho}$ is independent of $\theta \in I$ and $\sigma \in [-\delta,\delta]$. Now we may continue as in the proof of (i), by distinguishing the cases $k = 0$ and $k \geqslant 1$. Next we consider the case $\Re\mu \geqslant 0$. From the assumption $\arg(\mu + 2l\pi i) \notin (\theta_-, \theta_+)$ for all $l \in \mathbb{Z}$, we conclude the existence of a positive constant c such that $|U_1(te^{i\sigma})| \leqslant c$ for all $t \in \mathbb{C}^*$, $\arg t \in I$ and $\sigma \in [-\delta,\delta]$. Hence, if $f = \Delta_0$ we have $\|U_1(te^{i\sigma})\Delta_0\|_{m,\nu,\theta} \leqslant c\|\Delta_0\|_{m,\nu,\theta}$. Thus without loss of generality we may take $f = \Delta_k^{(mk)}$ with $k \geqslant 1$, so we only work with $t \in \mathbb{C}^*$, $\arg t \in I$ and $|t| \geqslant 1$.

[1]Stirling's formula: $\Gamma(n) = \sqrt{2\pi}\,n^{n-1/2}e^{-n}[1+O(n^{-1})]$ as $n \to \infty$.

Again we want to derive a Cauchy estimate on $U_1^{(mk-j)}(te^{i\sigma})$. To that end we observe that $\frac{\Re(te^{i\sigma})-\Re\mu}{\Re(te^{i\sigma})} = 1 - \frac{\Re\mu/\cos(\arg t+\sigma)}{|t|} \to 1$ uniformly as $t \to \infty$ in $\{t \in \mathbb{C}^* \mid \arg t \in I\}$, so there exists a positive R such that $\Re(te^{i\sigma} - \mu) \geqslant \delta_1|t|$ for all $t \in \mathbb{C}^*$, $\arg t \in I$ and $|t| > R$, where δ_1 is some positive constant independent of $\theta \in I$ and $\sigma \in [-\delta, \delta]$. Therefore if $t \in \mathbb{C}^*$, $\arg t \in I$ and $|t| > R$, we take the parametrisation $s = te^{i\sigma} + \rho(\Re(te^{i\sigma}) - \Re\mu)e^{i\varphi}$, $0 \leqslant \varphi < 2\pi$, $\rho \in (0,1)$, to obtain from (2.9.4)

$$|U_1^{(mk-j)}(te^{i\sigma})| \leqslant c_1\,(mk-j)!(\rho\,\delta_1|t|)^{-(mk-j)},$$

where c_1 is some constant independent of $\theta \in I$ and $\sigma \in [-\delta, \delta]$. For $t \in \mathbb{C}^*$, $\arg t \in I$ and $1 \leqslant |t| \leqslant R$, we take the parametrisation $s = te^{i\sigma}+\delta_2|t|e^{i\varphi}$, $\varphi \in [0, 2\pi)$, in (2.9.4). Taking δ_2 small enough, the set $\{te^{i\sigma}+\delta_2|t|e^{i\varphi} \mid t \in \mathbb{C}^*,\ \arg t \in I,\ 1 \leqslant |t| \leqslant R,\ |\sigma| \leqslant \delta,\ 0 \leqslant \varphi < 2\pi\}$ is contained in a closed bounded subset of $\{s \in \mathbb{C}^* \mid \theta_- < \arg s < \theta_+\}$ outside a disc around 0. Since $\arg(\mu + 2l\pi i) \notin (\theta_-, \theta_+)$ for all $l \in \mathbb{Z}$, the function $(1 - e^{\mu-s})^{-1}$ is bounded for all s described above and thus

$$|U_1^{(mk-j)}(te^{i\sigma})| \leqslant c_2\,(mk-j)!(\delta_2|t|)^{-(mk-j)},$$

where c_2 is some constant independent of $\theta \in I$ and $\sigma \in [-\delta, \delta]$. Altogether we have for $t \in \mathbb{C}^*$, $\arg t \in I$, $|t| \geqslant 1$ and $\sigma \in [-\delta, \delta]$

$$|U_1^{(mk-j)}(te^{i\sigma})| \leqslant c_3\,(mk-j)!(\delta_3|t|)^{-(mk-j)},$$

where c_3 and δ_3 are some positive constants independent of $\theta \in I$ and $\sigma \in [-\delta, \delta]$ and we can continue as in (i) to complete the proof of (ii).

For the proof of (iii) we use that $U_2^{(mk-j)}(te^{i\sigma}) = (-1)^{mk-j}U_2(te^{i\sigma})$ and instead of (2.9.5) we now have the estimate

$$|U_2^{(mk-j)}(te^{i\sigma})| = e^{-|t|\cos(\arg t+\sigma)} \leqslant (mk-j)!\,\big(|t|\cos(\arg t + \sigma)\big)^{-(mk-j)}, \quad |t| > 0,$$

for arbitrary $j \in \{0, 1, \ldots, mk\}$. Now we can continue as in (i), by distinguishing the cases $f = \Delta_0$ and $f = \Delta_k^{(mk)}$, $k \geqslant 1$.

Finally, for the proof of (iv) we define U by $U(t) := U_2(te^{i\sigma}) - U_2(t)$. Then analogous to (2.9.1) we have

$$U(t)\Delta_k^{(mk)} = \sum_{j=0}^{mk}(-1)^{mk-j}\binom{mk}{j}\big[U^{(mk-j)}\Delta_k\big]^{(j)}.$$

First observe that $U(t) = -t\int_1^{e^{i\sigma}} e^{-\alpha t}\mathrm{d}\alpha$ and with the substitution $\alpha = e^{i\varphi}$ we obtain the following estimate for U if $t \in \mathbb{C}^*$, $\arg t \in I$

$$|U(t)| \ \leqslant\ |t|\int_0^{\sigma} e^{-|t|\cos(\arg t+\varphi)}|\mathrm{d}\varphi| \ \leqslant\ |t|e^{-\delta_1|t|}|\sigma|,$$

where δ_1 is some positive constant independent of θ and σ. Hence, $|U(t)| \leqslant \frac{1}{\delta_1}|\sigma|$ and for $j \in \{0, 1, \dots, mk-1\}$ we have

$$\begin{aligned} U^{(mk-j)}(t) &= (-1)^{mk-j}\big(e^{i\sigma(mk-j)}U_2(te^{i\sigma}) - U_2(t)\big) \\ &= (-1)^{mk-j}\big(e^{i\sigma(mk-j)} - 1\big)U_2(te^{i\sigma}) + (-1)^{mk-j}U(t), \end{aligned}$$

which implies (using that $mk-j \leqslant 2^{mk-j}$ and $|t|e^{-\frac{\delta_1}{2}|t|} \leqslant 2/\delta_1$)

$$\begin{aligned} |U^{(mk-j)}(t)| &\leqslant (mk-j)\,|\sigma|\,e^{-\delta_1|t|} \;+\; |t|e^{-\delta_1|t|}|\sigma| \\ &\leqslant \big(1+2/\delta_1\big)\,|\sigma|\,(mk-j)!\,\Big(\frac{\delta_1}{2}|t|\Big)^{-(mk-j)}, \quad |t|>0. \end{aligned}$$

Again we may continue as in (i) by distinguishing the cases $f = \Delta_0$ and $f = \Delta_k^{(mk)}$, $k \geqslant 1$. This proves (iv) and therefore the lemma. ∎

Chapter 3

Formation of Singularities near Stokes Lines

3.1 Introduction

In the present chapter we will consider a special type of difference equations of the class that we studied in chapter 2

$$y(x+1) = \Lambda(x)y(x) + g(x, y(x)), \tag{3.1.1}$$

where

$$\Lambda(x) = \operatorname{diag}\{e^{-\mu_1}(1+x^{-1})^{a_1}, \dots, e^{-\mu_n}(1+x^{-1})^{a_n}\}. \tag{3.1.2}$$

As before we assume g to be a holomorphic $\mathbb{C}^n$-valued function of (x, y) in a neighbourhood of $(\infty, 0)$, such that $g(x, y) = O(x^{-2}) + O(|y|^2)$ as $x \to \infty$ and $y \to 0$. We assume μ_j, $j = 1, 2, \dots, n$, to be complex numbers such that

$$\mu_j \not\equiv (k_1\mu_1 + k_2\mu_2 + \cdots + k_p\mu_p) \bmod 2\pi i, \quad \text{for every} \quad (k_1, k_2, \dots, k_p) \in \mathbb{N}^p \backslash \{\mathbf{e}_j\}, \tag{3.1.3}$$

where $p \leqslant n$ is such that

$$\Re\mu_j > 0, \; j \in \{1, 2, \dots, p\} \quad \text{and} \quad \Re\mu_j \leqslant 0, \; j \in \{p+1, p+2, \dots, n\} \tag{3.1.4}$$

(cf. remark 2.3.7). Moreover, we assume $a_j \in \mathbb{C}$ for all $j \in \{1, 2, \dots, n\}$. However, in the introduction of section 2.4 we already saw that without loss of generality it may be assumed that $\Re a_j > 0$ for all $j \in \{1, 2, \dots, n\}$.

In chapter 2 we have seen that (3.1.1) possesses a formal solution $\hat{y}_0 \in x^{-2}\mathbb{C}^n[[x^{-1}]]$. To explain what this chapter is about, let us assume that y is a holomorphic solution of (3.1.1) such that $y(x) \sim \hat{y}_0(x)$ as $x \to \infty$ in a sector $\{x \in \mathbb{C}^* \mid \varphi_- < \arg x < \varphi_+\}$ of opening smaller than π and containing the positive real axis. Moreover, let this sector be chosen in such a way that φ_+ cannot be enlarged (i.e. the asymptotic behaviour does not hold in the sector $\{x \in \mathbb{C}^* \mid \varphi_- < \arg x < \varphi_+ + \varepsilon\}$, for any $\varepsilon > 0$). Then φ_+ is a Stokes direction, which we assume to correspond with $C_1(x)e^{-\mu_1 x}$. Here C_1 is the 1-periodic function that appears in the convergent transseries representation of y on $\{x \in \mathbb{C}^* \mid \varphi_- < \arg x < \varphi_+\}$.

In the present chapter we will show that the solution y admits a 'two-scale' asymptotic expansion of the form $\sum_{m=0}^{\infty} F_m(\xi(x))x^{-m}$ in a closed region with as upper boundary the half line given by $\{x \in \mathbb{C}^* \mid \arg(x-a) = \varphi_+\}$ for some $a > 0$, instead of the asymptotic expansion $y(x) \sim \hat{y}_0(x)$ that only holds in closed sub-sectors of $\{x \in \mathbb{C}^* \mid \varphi_- < \arg x < \varphi_+\}$. Here $\xi = \xi(x)$ is a small parameter defined by $\xi(x) = C_1(x)e^{-\mu_1 x}x^{a_1}$ and the functions F_m turn out to be uniquely determined by the convergent transseries representation of the solution y of the original difference equation. The method to prove this can be described as *transasymptotic matching* in the sense that the asymptotic expansion $\sum_{m=0}^{\infty} F_m(\xi(x))x^{-m}$ of y matches the convergent transseries corresponding to y in a sector where $y(x)$ asymptotically equals $\hat{y}_0(x)$ as $x \to \infty$. We will prove that this 'two-scale' asymptotic expansion is of Gevrey-type. Then by a suitable truncation of this series we get an estimate for y with exponential accuracy.

Under some additional assumptions we will prove that the solution y can be extended to a larger region, being a part of a half plane parallel to the direction φ_+ (cf. theorem 3.5.6). Moreover, in this extended region the solution y typically develops singularities. The corresponding singular points turn out to lie close to the half line with direction φ_+ and form a nearly periodic array (cf. theorem 3.5.7). These singular points are movable since they depend on the particular solution y of the difference equation. Of course similar results can be obtained for φ_-.

3.2 Preliminaries

In section 2.3 of the preceding chapter we have seen that (3.1.1) possesses a unique formal solution $\hat{y}_0 \in x^{-2}\mathbb{C}^n[[x^{-1}]]$, which is Borel summable. Its Borel transform Y_0 exists in the maximal star domain with centre 0, which does not contain any singular point $\mu_j + 2l\pi i$, $j \in \{1, 2, \ldots, n\}$ and $l \in \mathbb{Z}$. Moreover, if (3.1.3) is satisfied with p replaced by n, then the difference equation (3.1.1) can formally be reduced to the normal form $z(x+1) = \Lambda(x)z(x)$ by means of the transformation

$$y = \sum_{\mathbf{k}\in\mathbb{N}^n} \hat{y}_{\mathbf{k}}(x)\, z^{\mathbf{k}}, \tag{3.2.1}$$

with $\hat{y}_{\mathbf{k}}$, $\mathbf{k} \in \mathbb{N}_1^n$, a formal series in $\mathbb{C}^n[[x^{-1}]]$. Substituting the general solution of the normal form (in the class of holomorphic functions on the Riemann surface of the logarithm) into (3.2.1) we obtain the following transseries solution of (3.1.1):

$$\hat{y}(x) = \sum_{\mathbf{k}\in\mathbb{N}^n} C^{\mathbf{k}}(x)\, e^{-\langle \mathbf{k},\boldsymbol{\mu}\rangle x}\, x^{\langle \mathbf{k},\mathbf{a}\rangle}\, \hat{y}_{\mathbf{k}}(x),$$

with C a 1-periodic holomorphic function of x, $\boldsymbol{\mu} = (\mu_1, \mu_2, \ldots, \mu_n)$ and $\mathbf{a} = (a_1, a_2, \ldots, a_n)$. The formal expressions $\hat{y}_{\mathbf{k}}$, $\mathbf{k} \in \mathbb{N}_1^n$, again are Borel summable and their Borel transforms exist and are holomorphic in the maximal star domain with centre 0, which does not contain any of the singular points $\mu_j - \langle \mathbf{k}', \boldsymbol{\mu} \rangle + 2l\pi i \neq 0$ where $j \in \{1, 2, \ldots, n\}$, $\mathbf{k}' \preceq \mathbf{k}$ and $l \in \mathbb{Z}$.

In order to get an analytic reduction, we restricted ourselves in the preceding chapter to multi-indices in $\mathbb{N}^p$ instead of multi-indices in $\mathbb{N}^n$ (and thus the assumption (3.1.3) suffices, compare remark 2.3.7). More precisely, if we denote $\hat{\mathcal{B}}[x^{-|\mathbf{k}|}\hat{y}_{\mathbf{k}}(x)]$ by $W_{\mathbf{k}}$, $\mathbf{k} \in \mathbb{N}^p$, then we have (see chapter 2, in particular proposition 2.3.5 and theorem 2.3.6):

> Define $\mathcal{Q}$ to be the set of singular points of all $\hat{y}_{\mathbf{k}}$, $\mathbf{k} \in \mathbb{N}^p$. Let θ_- and θ_+ be two consecutive singular directions in the right half plane of the set of singular directions corresponding to $\mathcal{Q}$. Let $S := \{t \in \mathbb{C}^* \mid \arg t \in (\theta_-, \theta_+)\}$ and define $\overline{S'}$ to be an arbitrary closed sub-sector of S. Moreover, let ρ be a positive number smaller than $\min\{|\mu| \mid \mu \in \mathcal{Q}\}$, and define $V := \overline{S'} \cup \overline{\Delta}(0, \rho)$. Then there exist positive constants R and K such that
>
> $$|W_{\mathbf{k}}(t)| \leqslant K^{|\mathbf{k}|} \frac{|t|^{|\mathbf{k}|-1}}{(|\mathbf{k}|-1)!} e^{R|t|} \tag{3.2.2}$$
>
> for all $\mathbf{k} \in \mathbb{N}^p \backslash \{0\}$ and all $t \in V$.

and

> Let $S_1 := \{x \in \mathbb{C}^* \mid \arg x \in (-\pi/2 - \theta_+ + \varepsilon, \pi/2 - \theta_- - \varepsilon)\}$, where $0 < \varepsilon < \frac{1}{2}(\theta_+ - \theta_-)$. Let S_2 be a sub-sector of S_1 containing the positive real axis and let y be a solution of (3.1.1) such that $y(x) = O(x^{-2})$ as $x \to \infty$ on S_2. Then there exists a unique 1-periodic trigonometric polynomial C, with values in $\mathbb{C}^p$, such that
>
> $$y(x) = \sum_{\mathbf{k} \in \mathbb{N}^p} C^{\mathbf{k}}(x)\, e^{-\langle \mathbf{k}, \boldsymbol{\mu} \rangle x}\, x^{\langle \mathbf{k}, \mathbf{a} \rangle}\, y_{\mathbf{k}}(x), \quad x \in S_2. \tag{3.2.3}$$
>
> Here $y(x) - y_0(x)$ and $C_j(x)e^{-\mu_j x}$, $j \in \{1, 2, \ldots, p\}$, are exponentially small uniformly in a neighbourhood of ∞ in any closed sub-sector of S_2.

In fact, from the statement just above theorem 2.3.6 one may conclude that the right-hand side of (3.2.3) converges even in $\{x \in S_1 \mid |C_j(x)e^{-\mu_j x}x^{a_j}| \leqslant \delta,\ j = 1, 2, \ldots, p\} \backslash \Delta(0, \tilde{\rho})$, provided that δ is small enough and $\tilde{\rho}$ is large enough. Hence, y can be analytically continued into this region and remains a solution of the original difference equation (see also remark 2.3.7).

3.3 Transasymptotic Matching

Let us now consider a solution y of (3.1.1) such that $y(x) = O(x^{-2})$ as $x \to \infty$ in a sector $S_2 := \{x \in \mathbb{C}^* \mid \varphi_- < \arg x < \varphi_+\}$, with $\varphi_- < 0 < \varphi_+ < \varphi_- + \pi$, in such a way that φ_+ cannot be enlarged. Then φ_+ is a Stokes direction, which we will prove in the next section.

3.3.1 Behaviour Near a Stokes Line

Let θ_- and θ_+ be two consecutive singular directions in the right half plane of the set of all $\hat{y}_{\mathbf{k}}$, $\mathbf{k} \in \mathbb{N}^p$, in such a way that $S_2 \subset S_1 := \{x \in \mathbb{C}^* \mid \arg x \in (-\pi/2-\theta_+ + \varepsilon, \pi/2-\theta_- - \varepsilon)\}$, with $0 < \varepsilon < \frac{1}{2}(\theta_+ - \theta_-)$. Note that $S_2 \subset S_1$ can always be arranged by taking ε small enough: choose θ_+ to be the smallest singular direction in the right half plane that is larger than $-\pi/2 - \varphi_-$. Then $\theta_- \leqslant -\pi/2 - \varphi_-$ and thus $\varphi_+ < \varphi_- + \pi \leqslant \pi/2 - \theta_-$.

Then on S_2 we can represent y by a convergent transseries: there exists a unique 1-periodic trigonometric polynomial C, with values in $\mathbb{C}^p$, such that

$$y(x) = \sum_{\mathbf{k}\in\mathbb{N}^p} C^{\mathbf{k}}(x)\, e^{-\langle \mathbf{k},\boldsymbol{\mu}\rangle x}\, x^{\langle \mathbf{k},\mathbf{a}\rangle}\, y_{\mathbf{k}}(x). \tag{3.3.1}$$

This trigonometric polynomial has the property that $C_j(x)e^{-\mu_j x}$, $j \in \{1,2,\ldots,p\}$, is exponentially small, uniformly in a neighbourhood of ∞ in any closed sub-sector of S_2. Since each $y_{\mathbf{k}}$ exists on S_1, and thus on S_2, the fact that φ_+ in the definition of S_2 cannot be enlarged is due to one or more expressions $C_j(x)e^{-\mu_j x}$ that will explode for $x \to \infty$ when crossing the half line $\arg x = \varphi_+$.

Now let us write $C_j(x) = \sum_{h=h_{j-}}^{h_{j+}} \gamma_{j,h}\, e^{2\pi i h x}$, with $\gamma_{j,h_{j\pm}} \neq 0$ for each $j \in \{1,2,\ldots,p\}$. Then $C_j(x) \sim \gamma_{j,h_{j-}}\, e^{2\pi i h_{j-} x}$ as $\Im x \to \infty$ (compare example 1.5.2), so

$$C_j(x)e^{-\mu_j x} \sim \gamma_{j,h_{j-}}\, e^{(2\pi i h_{j-} - \mu_j)x}, \qquad \text{as } \Im x \to \infty. \tag{3.3.2}$$

Hence, there exists a $j \in \{1,2,\ldots,p\}$ such that $\varphi_+ = \pi/2 - \arg(\mu_j - 2\pi i h_{j-})$ and thus φ_+ is a Stokes line. By rearranging the μ_j's we can ensure that $\varphi_+ = \pi/2 - \arg(\mu_1 - 2\pi i h_{1-})$. Let us assume that

$$\arg(\mu_1 - 2\pi i h_{1-}) \not\equiv \arg(\mu_j - 2\pi i h_{j-}) \bmod 2\pi, \qquad \text{for } j \in \{2,3,\ldots,p\}, \tag{3.3.3}$$

then $C_j(x)e^{-\mu_j x}$ is exponentially small in a neighbourhood of the ray $\arg x = \varphi_+$ for every $j \in \{2,3,\ldots,p\}$.

For $\varepsilon_1 > 0$ let the set $\mathcal{E}$ be defined by

$$\mathcal{E} := \{x \in S_1 \mid |C_j(x)e^{-\mu_j x}| \leqslant e^{-\varepsilon_1 |x|} \text{ for } j = 2,3,\ldots,p\}. \tag{3.3.4}$$

If we choose ε_1 small enough, then $\mathcal{E}$ extends beyond the Stokes line $\arg x = \varphi_+$, because in the definition of $\mathcal{E}$ we only require $C_j(x)e^{-\mu_j x}$ to be exponentially small for $j \in \{2,3,\ldots,p\}$ and there is no such condition for $j = 1$. Note that the Borel sums $y_{\mathbf{k}}$ of $\hat{y}_{\mathbf{k}}$, $\mathbf{k} \in \mathbb{N}^p$, are holomorphic in a neighbourhood of ∞ in S_1 and thus in a neighbourhood of ∞ in $\mathcal{E}$.

If we define a new variable ξ by

$$\xi(x) := C_1(x)e^{-\mu_1 x}x^{a_1}, \tag{3.3.5}$$

(compare [CC01]), then the convergent transseries mentioned above can formally be rewritten as

$$\sum_{k=0}^{\infty} y_{k\mathbf{e}_1}(x)\, \xi^k(x) + \sum_{\substack{\mathbf{k}\in\mathbb{N}^p \\ \mathbf{k}\neq|\mathbf{k}|\mathbf{e}_1}} C^{\mathbf{k}}(x)\, e^{-\langle \mathbf{k},\boldsymbol{\mu}\rangle x}\, x^{\langle \mathbf{k},\mathbf{a}\rangle}\, y_{\mathbf{k}}(x).$$

Using this new variable ξ we define $\mathcal{E}_\delta := \{x \in \mathcal{E} \mid |\xi(x)| \leqslant \delta\}$ for $\delta > 0$, compare (3.3.4). From remark 2.3.7 we then conclude that the domain of definition of the convergent transseries $\sum_{\mathbf{k}\in\mathbb{N}^p} C^{\mathbf{k}}(x)\, e^{-\langle\mathbf{k},\boldsymbol{\mu}\rangle x}\, x^{\langle\mathbf{k},\mathbf{a}\rangle}\, y_{\mathbf{k}}(x)$ can be enlarged to $\mathcal{E}_\delta$, provided that $|x|$ is large enough. So y can be analytically continued into this region.

Proposition 3.3.1 *There exist positive δ_1 and r_1 such that the series*

(i) $\sum_{k=0}^{\infty} y_{k\mathbf{e}_1}(x)\, \xi^k(x)$ *converges for $x \in \mathcal{E}_{\delta_1}$ and $|x| \geqslant r_1$;*

(ii) $\sum_{\substack{\mathbf{k}\in\mathbb{N}^p \\ \mathbf{k}\neq|\mathbf{k}|\mathbf{e}_1}} C^{\mathbf{k}}(x)\, e^{-\langle\mathbf{k},\boldsymbol{\mu}\rangle x}\, x^{\langle\mathbf{k},\mathbf{a}\rangle}\, y_{\mathbf{k}}(x)$ *is exponentially small on $\mathcal{E}_{\delta_1}\backslash\Delta(0, r_1)$.*

PROOF. Fix $\theta \in (\theta_- + \varepsilon/2, \theta_+ - \varepsilon/2)$ and let $\eta = \sin(\varepsilon/2)$. Then for $x \in \mathbb{C}^*$ with $-\theta - \pi/2 + \varepsilon/2 < \arg x < -\theta + \pi/2 - \varepsilon/2$ and $|x| \geqslant 2R/\eta$ we have $\Re(xe^{i\theta}) - R \geqslant \frac{\eta}{2}|x|$ and for these values of x the Borel sum $w_{\mathbf{k}}(x) = (\mathcal{L}_\theta W_{\mathbf{k}})(x)$, $\mathbf{k} \in \mathbb{N}^p\backslash\{0\}$, can be estimated (using (3.2.2)) by

$$|w_{\mathbf{k}}(x)| \;\leqslant\; \frac{K^{|\mathbf{k}|}}{(|\mathbf{k}|-1)!} \int_0^\infty \sigma^{|\mathbf{k}|-1} e^{-\frac{\eta}{2}|x|\sigma} \mathrm{d}\sigma \;\leqslant\; \Big(\frac{2K}{\eta}\Big)^{|\mathbf{k}|} x^{-|\mathbf{k}|}. \tag{3.3.6}$$

By varying $\theta \in (\theta_- + \varepsilon/2, \theta_+ - \varepsilon/2)$, we obtain (3.3.6) for all $x \in S_1$, $|x| \geqslant r_1 := 2R/\eta$. Since $y_{\mathbf{k}}(x) = x^{-|\mathbf{k}|} w_{\mathbf{k}}(x)$, we thus have $|y_{\mathbf{k}}(x)| \leqslant (2K\eta^{-1})^{|\mathbf{k}|}$, for all $\mathbf{k} \in \mathbb{N}^p\backslash\{0\}$, $x \in S_1$, $|x| \geqslant r_1$. This proves the first statement.

The expression in the second statement can be written as

$$\sum_{j=2}^{p} C_j(x) e^{-\mu_j x} x^{a_j} \sum_{\mathbf{k}\in\mathbb{N}^p} C^{\mathbf{k}}(x)\, e^{-\langle\mathbf{k},\boldsymbol{\mu}\rangle x}\, x^{\langle\mathbf{k},\mathbf{a}\rangle}\, y_{\mathbf{k}+\mathbf{e}_j}(x)$$

and due to the definition of $\mathcal{E}$, there exist positive constants α_j, $j = 2, 3, \ldots, p$, such that $C_j(x) e^{-\mu_j x} x^{a_j} = O(e^{-\alpha_j |x|})$ as $x \to \infty$ in $\mathcal{E}$. Using this it is easy to prove the second statement (it may be necessary to take a smaller δ_1 and a larger r_1 than in the proof of the first statement). ■

3.3.2 Gevrey Asymptotics

From proposition 2.3.3 we deduce that $W_{\mathbf{k}}(t) = \sum_{m=0}^{\infty} \frac{W_{\mathbf{k}}^{(m)}(0)}{m!} t^m$ in a small disc around the origin and from section 1.3.2 we then conclude that $w_{\mathbf{k}}(x)$ asymptotically equals $\sum_{m=0}^{\infty} W_{\mathbf{k}}^{(m)}(0) x^{-m-1}$ as $x \to \infty$ in the sector bounded by the rays $\arg x = -\pi/2 - \theta_+$ and $\arg x = \pi/2 - \theta_-$. Moreover, if $|\mathbf{k}| \geqslant 2$ then $W_{\mathbf{k}}^{(m)}(0) = 0$ for all $m \in \{0, 1, \ldots, |\mathbf{k}| - 2\}$. On the other hand we know that $w_{\mathbf{k}}(x) = x^{-|\mathbf{k}|} y_{\mathbf{k}}(x) \sim x^{-|\mathbf{k}|} \hat{y}_{\mathbf{k}}(x) := \sum_{m=0}^{\infty} y_{\mathbf{k},m}\, x^{-m-|\mathbf{k}|}$, so $W_{\mathbf{k}}^{(|\mathbf{k}|+m-1)}(0) = y_{\mathbf{k},m}$ for all $\mathbf{k} \in \mathbb{N}_1^p$ and all $m \in \mathbb{N}$. If $\mathbf{k} = 0$, then $W_0 = Y_0$ and in that case $Y_0^{(m-1)}(0) = y_{0,m}$ for all $m \in \mathbb{N}_{\geqslant 2}$.

For $m \in \mathbb{N}$ let us define

$$F_m(\xi) := \sum_{k=0}^{\infty} y_{k\mathbf{e}_1,m} \xi^k. \tag{3.3.7}$$

Theorem 3.3.2 *There exists a* $\delta_2 > 0$ *such that the power series* $F_m(\xi) = \sum_{k=0}^{\infty} y_{k\mathbf{e}_1,m}\xi^k$, $m = 0, 1, 2, \ldots$, *converges for* $|\xi| \leqslant \delta_2$. *Moreover* $y(x) \sim_1 \sum_{m=0}^{\infty} F_m\big(\xi(x)\big)x^{-m}$, *as* $x \to \infty$ *in* $\mathcal{E}_{\delta_2}$. *More precisely, there exist positive constants* r_2, K_2 *and* c_2 *such that*

$$|F_m(\xi)| \leqslant K_2\, m!\, c_2^m, \quad \textit{if } m \in \mathbb{N} \textit{ and } |\xi| \leqslant \delta_2$$

and

$$\Big|y(x) - \textstyle\sum_{m=0}^{N-1} F_m\big(\xi(x)\big)x^{-m}\Big| \leqslant K_2\, N!\, c_2^N\, |x|^{-N}, \quad \textit{if } N \in \mathbb{N} \textit{ and } x \in \mathcal{E}_{\delta_2},\ |x| \geqslant r_2.$$

To prove this theorem we will utilise the following lemmas:

Lemma 3.3.3 *Let* $\overline{S_1'}$ *be a closed sub-sector of the sector* $S = \{t \in \mathbb{C}^* \mid \arg t \in (\theta_-, \theta_+)\}$. *Let* $0 < \rho_1 < \rho$, *where* ρ *is as in section 3.2 and define* $V_1 := \overline{S_1'} \cup \overline{\Delta}(0, \rho_1)$. *Then there exist positive constants* R_1, K_1 *and* c_1 *such that*

$$|W_{\mathbf{k}}^{(|\mathbf{k}|+N-1)}(t)| \leqslant K_1^{|\mathbf{k}|+1}\, \frac{N!}{c_1^N}\, e^{R_1|t|} \tag{3.3.8}$$

for all $\mathbf{k} \in \mathbb{N}^p \backslash \{0\}$, $N \in \mathbb{N}$ *and* $t \in V_1$. *Moreover, for every* $\mathbf{k} \in \mathbb{N}^p \backslash \{0\}$ *and* $N \in \mathbb{N}$ *we have*

$$\left|W_{\mathbf{k}}(t) - \sum_{m=0}^{|\mathbf{k}|+N-2} W_{\mathbf{k}}^{(m)}(0)\frac{t^m}{m!}\right| \leqslant K_1^{|\mathbf{k}|+1}\, \frac{N!}{c_1^N}\, \frac{|t|^{|\mathbf{k}|+N-1}}{(|\mathbf{k}|+N-1)!}\, e^{R_1|t|} \tag{3.3.9}$$

for all $t \in V_1$. *If* $\mathbf{k} = 0$, *then (3.3.8) and (3.3.9) hold for* $N \in \mathbb{N}_+$.

Proof. Fix $\rho_2 \in (\rho_1, \rho)$, let $\overline{S_2'}$ be a closed sub-sector of S containing $\overline{S_1'}$ in its interior and define $V_2 = \overline{S_2'} \cup \overline{\Delta}(0, \rho_2)$.

If $t \in V_2$, $t \neq 0$, and $\mathbf{k} \neq 0$, Cauchy's formula yields $W_{\mathbf{k}}^{(|\mathbf{k}|-1)}(t) = \frac{(|\mathbf{k}|-1)!}{2\pi i} \oint \frac{W_{\mathbf{k}}(s)}{(s-t)^{|\mathbf{k}|}}\,\mathrm{d}s$, where we take as path of integration the circle $s = t + |t|\eta e^{i\varphi}$, $0 \leqslant \varphi < 2\pi$, $\eta > 0$. By choosing η small enough we see $s = t + |t|\eta e^{i\varphi}$, $0 \leqslant \varphi < 2\pi$, belongs to the union V (as described in the introduction of section 3.2) for every $t \in V_2$ and using (3.2.2) one easily infers that $|W_{\mathbf{k}}(s)| \leqslant \frac{K}{(|\mathbf{k}|-1)!}\big(K(1+\eta)|t|\big)^{|\mathbf{k}|-1} e^{R(1+\eta)|t|}$ for all s on the path of integration. Hence,

$$|W_{\mathbf{k}}^{(|\mathbf{k}|-1)}(t)| \leqslant K\left(\frac{K(1+\eta)}{\eta}\right)^{|\mathbf{k}|-1} e^{R(1+\eta)|t|}$$

for all $t \in V_2$ and $\mathbf{k} \in \mathbb{N}_1^p$. If $\mathbf{k} = 0$, then we can estimate W_0 on V_2 by $Ke^{R|t|}$ with possibly enlarged K and R. Now, if $t \in V_1$, $t \neq 0$, $\mathbf{k} \in \mathbb{N}_1^p$ and $N \in \mathbb{N}$ we can write in a similar way as above $W_{\mathbf{k}}^{(|\mathbf{k}|+N-1)}(t) = \frac{N!}{2\pi i}\oint \frac{W_{\mathbf{k}}^{(|\mathbf{k}|-1)}(s)}{(s-t)^{N+1}}\,\mathrm{d}s$, but now we take as path of integration the circle $s = t + c_1 e^{i\varphi}$, $0 \leqslant \varphi < 2\pi$. By choosing c_1 small enough we can ensure that $t \in V_1$ implies $s \in V_2$, and thus there exist positive constants R_1 and K_1 such that (3.3.8) holds. If $\mathbf{k} = 0$ and $N \in \mathbb{N}_+$, then $W_0^{(N-1)}(t) = \frac{N!}{2\pi i}\oint \frac{W_0(s)}{(s-t)^{N+1}}\,\mathrm{d}s$, and again (3.3.8) follows. For $\mathbf{k} \in \mathbb{N}_1^p$ and $N \in \mathbb{N}$, Taylor's formula gives

$$W_{\mathbf{k}}(t) - \sum_{m=0}^{|\mathbf{k}|+N-2} W_{\mathbf{k}}^{(m)}(0)\frac{t^m}{m!} = \frac{1}{(|\mathbf{k}|+N-2)!}\int_0^t (t-\sigma)^{|\mathbf{k}|+N-2}\, W_{\mathbf{k}}^{(|\mathbf{k}|+N-1)}(\sigma)\mathrm{d}\sigma$$

and by applying (3.3.8) we obtain (3.3.9). It is easily seen that (3.3.9) also holds in the case $\mathbf{k} = 0$ and $N \in \mathbb{N}_+$. ■

Lemma 3.3.4 *For every $\mathbf{k} \in \mathbb{N}^p$ we have $y_{\mathbf{k}}(x) \sim_1 \sum_{m=0}^{\infty} y_{\mathbf{k},m} x^{-m}$ as $x \to \infty$ in S_1. More precisely, if R_1, K_1 and c_1 are as in the preceding lemma and $\eta := \sin(\varepsilon/2)$, then*

$$\left| y_{\mathbf{k}}(x) - \sum_{m=0}^{N-1} y_{\mathbf{k},m}\, x^{-m} \right| \leqslant K_1 \Big(\frac{2K_1}{\eta}\Big)^{|\mathbf{k}|} N! \,\Big(\frac{2}{\eta\, c_1}\Big)^N |x|^{-N} \tag{3.3.10}$$

for all $x \in S_1$, $|x| \geqslant 2R_1/\eta$, and all $N \in \mathbb{N}$.

Proof. We use the preceding lemma with $\overline{S_1'} = \{t \in \mathbb{C}^* \mid \arg t \in [\theta_- + \varepsilon/2, \theta_+ - \varepsilon/2]\}$. Fix $\theta \in [\theta_- + \varepsilon/2, \theta_+ - \varepsilon/2]$, then for $x \in \mathbb{C}^*$ with $-\theta - \pi/2 + \varepsilon/2 < \arg x < -\theta + \pi/2 - \varepsilon/2$ and $|x| \geqslant 2R_1/\eta$ we have $\Re(xe^{i\theta}) - R_1 \geqslant \frac{\eta}{2}|x|$ and for these values of x we have for $\mathbf{k} \in \mathbb{N}_1^p$ and $N \in \mathbb{N}$, or $\mathbf{k} = 0$ and $N \in \mathbb{N}_+$

$$\begin{aligned}\left| w_{\mathbf{k}}(x) - \sum_{m=0}^{N-1} y_{\mathbf{k},m}\, x^{-m-|\mathbf{k}|} \right| &= \left| \int_0^{\infty e^{i\theta}} \Big\{ W_{\mathbf{k}}(t) - \sum_{m=0}^{|\mathbf{k}|+N-2} W_{\mathbf{k}}^{(m)}(0) \frac{t^m}{m!} \Big\} e^{-xt} \mathrm{d}t \right| \\ &\leqslant K_1^{|\mathbf{k}|+1} \Big(\frac{2}{\eta}\Big)^{|\mathbf{k}|+N} \frac{N!}{c_1^N} |x|^{-|\mathbf{k}|-N}.\end{aligned}$$

So (3.3.10) holds for $x \in \mathbb{C}^*$ with $-\theta - \pi/2 + \varepsilon/2 < \arg x < -\theta + \pi/2 - \varepsilon/2$ and $|x| \geqslant 2R_1/\eta$. By varying $\theta \in (\theta_- + \varepsilon/2, \theta_+ - \varepsilon/2)$, we obtain (3.3.10) for all $x \in S_1$, $|x| \geqslant 2R_1/\eta$. This proves the lemma. ■

Proof of theorem 3.3.2. From (3.3.8) we conclude that $|y_{\mathbf{k},m}| \leqslant K_1^{|\mathbf{k}|+1}\, m!\, c_1^{-m}$. So $F_m(\xi)$ converges for $|\xi| < K_1^{-1}$ and the desired estimate for $F_m(\xi)$ follows immediately.

In proposition 3.3.1 we proved that $\sum_{\mathbf{k} \in \mathbb{N}^p, \mathbf{k} \neq |\mathbf{k}|\mathbf{e}_1} C^{\mathbf{k}}(x)\, e^{-\langle \mathbf{k}, \boldsymbol{\mu} \rangle x}\, x^{\langle \mathbf{k}, \mathbf{a} \rangle}\, y_{\mathbf{k}}(x)$ is exponentially small in a neighbourhood of ∞ in $\mathcal{E}_{\delta_1}$ and thus this series can be estimated by a constant times $e^{-\alpha|x|}$ for some positive α. As $e^{-\alpha|x|} \leqslant N!(\alpha|x|)^{-N}$ for every $N \in \mathbb{N}$ it suffices to show the last estimate in theorem 3.3.2 with $y(x)$ replaced by $\sum_{k=0}^{\infty} y_{k\mathbf{e}_1}(x)\, \xi^k(x)$. This series can be rewritten as

$$\sum_{k=0}^{\infty} y_{k\mathbf{e}_1}(x)\, \xi^k(x) = \sum_{k=0}^{\infty} \sum_{m=0}^{N-1} y_{k\mathbf{e}_1,m}\, x^{-m}\, \xi^k(x) + \sum_{k=0}^{\infty} \Big(y_{k\mathbf{e}_1}(x) - \sum_{m=0}^{N-1} y_{k\mathbf{e}_1,m}\, x^{-m} \Big) \xi^k(x).$$

The first series in the right-hand side of this equality coincides with $\sum_{m=0}^{N-1} F_m\big(\xi(x)\big) x^{-m}$, whereas according to the lemma above the last series in the right-hand side can be estimated by

$$K_1\, N! \,\Big(\frac{2}{\eta\, c_1}\Big)^N |x|^{-N} \sum_{k=0}^{\infty} \Big(\frac{2K_1}{\eta}\Big)^k |\xi(x)|^k,$$

provided that $|x| \geqslant r_2 := 2R_1/\eta$. The series $\sum_{k=0}^{\infty} \big(2K_1\eta^{-1}\big)^k \xi^k$ converges for $|\xi|$ small enough, say for $|\xi| \leqslant \delta_2$. This proves theorem 3.3.2. ■

3.4 Analytic Continuation of F_m

Theorem 3.3.2 implies that the functions F_m exist and are holomorphic in a small disc around the origin in the ξ-plane. In this section we will show that these functions can be analytically extended into a larger region under the condition that

$$a_1 = 0. \tag{3.4.1}$$

In remark 3.4.1 we motivate this assumption on a_1.

3.4.1 On the Equations for F_m

Theorem 3.3.2 implies that $y(x) - \sum_{j=0}^{m} F_j\big(\xi(x)\big)x^{-j} = O(x^{-m-1})$ as $x \to \infty$ in $\mathcal{E}_{\delta_2}$, for every $m \in \mathbb{N}$. Since y is a solution of $y(x+1) = \Lambda(x)y(x) + g(x, y(x))$ we thus have

$$\sum_{j=0}^{m} F_j\big(\xi(x+1)\big)(x+1)^{-j} - \Lambda(x)\sum_{j=0}^{m} F_j\big(\xi(x)\big)x^{-j} - g\Big(x, \sum_{j=0}^{m} F_j\big(\xi(x)\big)x^{-j}\Big) = O(x^{-m-1}), \tag{3.4.2}$$

as $x \to \infty$ in $\mathcal{E}_{\delta_2}$. Our aim is to find a recursive set of equations for the functions F_m, $m \in \mathbb{N}$. To that end we first observe that $\xi(x+1) = \lambda_1\xi$, with $\lambda_1 = e^{-\mu_1}$. If we fix $\xi(x) := \xi$ for some $\xi \in \Delta(0, \delta_2)$ we obtain, by using that $C_1(x)e^{-\mu_1 x} = \gamma_{1,h_{1-}}e^{(2\pi i h_{1-} - \mu_1)x}\big(1 + O(x^{-1})\big)$ as $\Im x \to \infty$,

$$x_l = (\mu_1 - 2\pi i h_{1-})^{-1}\big[\ln\gamma_{1,h_{1-}} - \ln\xi + 2l\pi i + O(x_l^{-1})\big], \quad l \in \mathbb{Z}. \tag{3.4.3}$$

Observe that for $l \in \mathbb{N}$ large enough $(\mu_1 - 2\pi i h_{1-})^{-1}\big(\ln\gamma_{1,h_{1-}} - \ln\xi + 2l\pi i\big)$ belongs to $\mathcal{E}_{\delta_2}$, because $\mathcal{E}$ contains the Stokes line $\arg x = \pi/2 - \arg(\mu_1 - 2\pi i h_{1-})$.

With the fixed choice of $\xi(x)$, (3.4.2) reduces to

$$\sum_{j=0}^{m} F_j(\lambda_1\xi)(x+1)^{-j} - \Lambda(x)\sum_{j=0}^{m} F_j(\xi)x^{-j} - g\Big(x, \sum_{j=0}^{m} F_j(\xi)x^{-j}\Big) = O(x^{-m-1}), \tag{3.4.4}$$

as $x \to \infty$ in $\mathcal{E}_{\delta_2}$ such that $\xi(x) = \xi$. It is easily seen than $\sum_{j=0}^{m} F_j(\lambda_1\xi)(x+1)^{-j}$ can be expanded as $F_0(\lambda_1\xi) + \sum_{j=1}^{m}\sum_{k=1}^{j}\binom{-k}{j-k}F_k(\lambda_1\xi)x^{-j} + O(x^{-m-1})$, Moreover, if we expand Λ as $\Lambda(x) = \sum_{j=0}^{\infty}\Lambda_j x^{-j}$ then $\Lambda(x)\sum_{j=0}^{m} F_j(\xi)x^{-j} = \sum_{j=0}^{m}\sum_{k=0}^{j}\Lambda_{j-k}F_k(\xi)x^{-j} + O(x^{-m-1})$. Hence, (3.4.4) is equivalent to

$$\begin{aligned} F_0(\lambda_1\xi) + \sum_{j=1}^{m}\sum_{k=1}^{j}\binom{-k}{j-k}F_k(\lambda_1\xi)x^{-j} \; - \; &\sum_{j=0}^{m}\sum_{k=0}^{j}\Lambda_{j-k}F_k(\xi)x^{-j} + \\ &- g\Big(x, \sum_{j=0}^{m} F_j(\xi)x^{-j}\Big) = O(x^{-m-1}). \end{aligned} \tag{3.4.5}$$

For $m=0$ we have $g\big(x, F_0(\xi)\big) = g(\infty, F_0(\xi)) + O(x^{-1})$, while for $m \geqslant 1$ a straightforward calculation shows that

$$g\Big(x, \sum_{j=0}^{m} F_j(\xi)x^{-j}\Big) \;=$$

$$\sum_{j=0}^{m} \frac{1}{j!}\frac{\mathrm{d}^j}{\mathrm{d}z^j}\, g\Big(z^{-1}, \sum_{k=0}^{m-1} F_k(\xi)z^k\Big)\Big|_{z=0} x^{-j} + \partial_y g\big(\infty, F_0(\xi)\big)F_m(\xi)x^{-m} + O(x^{-m-1}).$$

Here $\partial_y g(\infty, F_0)$ is a shorthand notation for the $n\times n$-matrix with as columns $\frac{\partial}{\partial y_j} g(\infty, F_0)$, $j = 1, 2, \ldots, n$.

In order to get the equations for F_m we successively take $m = 0, 1, 2, \ldots$ and at each step we let l in (3.4.3) tend to $+\infty$. For $m = 0$ we then obtain from (3.4.5)

$$F_0(\lambda_1\xi) = \Lambda_0 F_0(\xi) + g\big(\infty, F_0(\xi)\big). \tag{3.4.6}$$

With induction on m one easily proves that F_m satisfies the following linear q-difference equation

$$F_m(\lambda_1\xi) = \big(\Lambda_0 + \partial_y g(\infty, F_0(\xi)\big)F_m(\xi) + R_m(\xi), \tag{3.4.7}$$

with

$$R_m(\xi) = -\sum_{k=1}^{m-1}\binom{-k}{m-k}F_k(\lambda_1\xi) + \sum_{k=0}^{m-1}\Lambda_{m-k}F_k(\xi) + \frac{1}{m!}\frac{\mathrm{d}^m}{\mathrm{d}z^m}\, g\Big(z^{-1}, \sum_{k=0}^{m-1}F_k(\xi)z^k\Big)\Big|_{z=0}. \tag{3.4.8}$$

Here we used that for $j \in \{1, 2, \ldots, m-1\}$ we have

$$\frac{1}{j!}\frac{\mathrm{d}^j}{\mathrm{d}z^j}\, g\Big(z^{-1}, \sum_{k=0}^{m-1}F_k(\xi)z^k\Big)\Big|_{z=0} - \frac{1}{j!}\frac{\mathrm{d}^j}{\mathrm{d}z^j}\, g\Big(z^{-1}, \sum_{k=0}^{j-1}F_k(\xi)z^k\Big)\Big|_{z=0} = \partial_y g(\infty, F_0)F_j.$$

Remark 3.4.1 If $a_1 \neq 0$ then $\xi(x+1) = \lambda_1(1+x^{-1})^{a_1}\xi(x)$. If we follow the procedure described above to obtain the equations for F_m, $m \in \mathbb{N}$, and only consider those values of $x \in \mathcal{E}_\delta$ with $\xi(x) = \xi$ for some fixed value of $\xi \in \Delta(0, \delta_2)$, then we still have to expand $(1+x^{-1})^{a_1}$ and thus $F_m(\xi(x+1))$, $m \in \mathbb{N}$. This obviously is more complicated and therefore this case is excluded in the thesis.

3.4.2 Analytic Continuation of F_m

Fix ϱ_1 and ϱ_2 such that g is continuous on $\{(x,y) \in \mathbb{C}\times\mathbb{C}^n \mid |x| \geqslant \varrho_1,\ |y| \leqslant \varrho_2\}$ and holomorphic in the interior of this set. Let us assume that F_0 can be analytically continued on the Riemann surface $\mathcal{R}$ above $\mathbb{C}\backslash\Xi$, where $\Xi \subset \mathbb{C}$ equals the set of singular points of F_0. Since F_0 is holomorphic in the disc $\Delta(0, \delta_2)$, we have $\mathrm{dist}(\Xi, 0) \geqslant \delta_2$.

Let $\mathcal{D} \subset \mathcal{R}$ be an open, relatively compact and connected subset with the following properties:

1. The function F_0 is holomorphic in an $\varepsilon_{\mathcal{D}}$-neighbourhood of $\mathcal{D}$ with $\varepsilon_{\mathcal{D}} > 0$;
2. There exists $\varrho_3 < \varrho_2$ such that $\sup_{\xi\in\mathcal{D}} |F_0(\xi)| \leqslant \varrho_3$;
3. The matrix function $\left|\big(\Lambda_0 + \partial_y g(\infty, F_0(\xi))\big)^{-1}\right|$ is bounded on $\mathcal{D}$ by some constant C;
4. The set $\pi(\mathcal{D})$ contains the disc $\Delta(0,\delta_2)$, where π denotes the projection of $\mathcal{D}$ on $\mathbb{C}$.

Proposition 3.4.2 *For each $m \in \mathbb{N}_+$ the function F_m can be analytically continued in $\mathcal{D}$.*

Proof. As $\lambda_1 = e^{-\mu_1}$ we have $|\lambda_1| < 1$ and for convenience we first assume that $\pi(\mathcal{D})$ is contained in $\Delta(0, \delta_2/|\lambda_1|)$. One should note that F_0 exists and is holomorphic on $\mathcal{D}$, which is part of the assumption. Now assume that $m \geqslant 1$ and that $F_1, F_2, \ldots, F_{m-1}$ are holomorphic in $\mathcal{D}$. The equation for F_m can be written as

$$F_m(\xi) = \big(\Lambda_0 + \partial_y g(\infty, F_0(\xi))\big)^{-1}\big[F_m(\lambda_1\xi) - R_m(\xi)\big]. \tag{3.4.9}$$

As R_m only depends on F_0 up to F_{m-1}, and thus is holomorphic in $\mathcal{D}$, the equation above defines the extension of F_m, since $\xi \in \mathcal{D}$ implies $|\lambda_1\xi| < \delta_2$.

Finally, suppose that $\pi(\mathcal{D}) \not\subset \Delta(0, \delta_2/|\lambda_1|)$. Since $\pi(\mathcal{D})$ is contained in $\Delta(0, r)$ for some $r > 0$, we can obtain the statement in the proposition by repeating the procedure above finitely many times. In fact, if $r \leqslant |\lambda_1|^{-l}\delta_2$ for some $l \in \mathbb{N}_+$, we need at most l repetitions of the procedure. ∎

3.4.3 Estimates for F_m

In this section we will prove that each F_m is bounded on $\mathcal{D}$, as formulated in the following proposition.

Proposition 3.4.3 *There exists a positive constant B such that $|F_m(\xi)| \leqslant m!B^m$ for all $\xi \in \mathcal{D}$ and all $m \in \mathbb{N}_+$.*

To prove this proposition we will utilise the following lemma taken from [CC01].

Lemma 3.4.4 *Given $m \in \mathbb{N}_+$ and a formal series $\sum_{k=0}^{\infty} a_k z^k \in \mathbb{C}^n[[z]]$. Assume that $\varrho_2 - |a_0| > 0$ and assume that there exists a positive constant B such that $|a_k| \leqslant k!B^k$, $k = 1, 2, \ldots, m-1$, then for m large enough we have*

$$\left| \frac{1}{m!}\frac{\mathrm{d}^m}{\mathrm{d}z^m}\, g\Big(z^{-1}, \sum_{k=0}^{m-1} a_k z^k\Big)\,\Big|_{z=0} \right| \leqslant \frac{1}{4C}\, m!\, B^m. \tag{3.4.10}$$

Proof of proposition 3.4.3. First we remark that again it is sufficient to consider the case where $\pi(\mathcal{D})$ is contained in the disc $\Delta(0, \delta_2/|\lambda_1|)$ (compare the last observation in the proof of proposition 3.4.2). The function F_1 satisfies

$$F_1(\xi) = \big(\Lambda_0 + \partial_y g(\infty, F_0(\xi))\big)^{-1}\Big[F_1(\lambda_1\xi) - \frac{\mathrm{d}}{\mathrm{d}z}\, g\Big(z^{-1}, F_0(\xi)\Big)\,\Big|_{z=0}\Big],$$

compare (3.4.7) and (3.4.8). The fact that $\big|\big(\Lambda_0 + \partial_y g(\infty, F_0(\xi))\big)^{-1}\big|$ is bounded on $\mathcal{D}$ by C is part of the assumptions made above. Moreover, if $\xi \in \mathcal{D}$ then $|\lambda_1 \xi| < \delta_2$, so theorem 3.3.2 implies that $|F_1(\lambda_1\xi)|$ is bounded by $K_2\, c_2$ and also $\frac{\mathrm{d}}{\mathrm{d}z}\, g\big(z^{-1}, F_0(\xi)\big)\,\big|_{z=0}$ is trivially bounded by a constant. Hence, the statement in proposition 3.4.3 holds for $m = 1$. Now suppose $m \geqslant 2$ and assume that $|F_l(\xi)| \leqslant l!B^l$ for all $l \in \{1, 2, \ldots, m-1\}$, then by (3.4.9)

$$|F_m(\xi)| \leqslant \big|\big(\Lambda_0 + \partial_y g(\infty, F_0(\xi))\big)^{-1}\big|\big(|F_m(\lambda_1\xi)| + |R_m(\xi)|\big).$$

If $\xi \in \mathcal{D}$ then $|\lambda_1\xi| < \delta_2$, so theorem 3.3.2 asserts that $|F_m(\lambda_1\xi)| \leqslant K_2\, m!\, c_2^m$. As $m \geqslant 1$ we may assume that in this inequality $K_2 < \frac{1}{4C}$ by choosing c_2 large enough. Hence, we only have to estimate R_m (compare (3.4.8)). Since $\binom{-k}{m-k}k! = (-1)^{m-k} k \frac{(m-1)!}{(m-k)!}$, the first term in the right-hand side of (3.4.8) can be estimated by

$$\Big|\sum_{k=1}^{m-1}\binom{-k}{m-k}F_k(\lambda_1\xi)\Big| \leqslant K_2(m-1)!\sum_{k=1}^{m-1}\frac{k\, c_2^k}{(m-k)!} \leqslant K_2\,(m-1)!\,(2\,c_2)^m\sum_{k=0}^{\infty}\frac{1}{k!}.$$

As Λ is a diagonal matrix with on the diagonal $e^{-\mu_j}(1 + x^{-1})^{a_j}$, the terms Λ_k in the expansion of Λ can be majorized by $\max_{j=1,2,\ldots,n}\{|e^{-\mu_j}\binom{a_j}{k}|\}$. But $|\binom{a_j}{k}| \leqslant \max\{1, |a_j|^k\}$, so there exists a constant $a \geqslant 1$ with $|\Lambda_k| \leqslant a^k$ for all $k \geqslant 1$. Hence,

$$\Big|\sum_{k=0}^{m-1}\Lambda_{m-k}F_k(\xi)\Big| \leqslant |\Lambda_m F_0(\xi)| + \Big|\sum_{k=1}^{m-1}\Lambda_{m-k}F_k(\xi)\Big| \leqslant \varrho_3 a^m + (m-1)!\sum_{k=1}^{m-1}a^{m-k}B^k.$$

By choosing B large enough we can achieve that $\varrho_3 a^m < B^m$, while for B larger than $2a$ we have $\sum_{k=1}^{m-1} a^{m-k}B^k \leqslant B^m\sum_{k=1}^{\infty}2^{-k} = B^m$. So, for B large enough the expression $\big|\sum_{k=0}^{m-1}\Lambda_{m-k}F_k(\xi)\big|$ can be estimated by $B^m\big(1 + (m-1)!\big)$.

The estimates just obtained, together with lemma 3.4.4 then imply that

$$|F_m(\xi)| \leqslant \big[1/2 + m^{-1}(C + e/4) + C/m!\big]\, m!\, B^m$$

and $1/2 + m^{-1}(C + e/4) + C/m!$ is less than 1 for m large enough. So a priori the statement in proposition 3.4.3 holds for m large enough. However, by enlarging B (if necessary) we can ensure that the statement holds for any $m \in \mathbb{N}_+$. ■

3.5 Singularity Analysis

For $N \geqslant 1$ we define the function $d_N(x) := y(x) - \sum_{k=0}^{N-1} F_k\big(\xi(x)\big)x^{-k}$. Theorem 3.3.2 says that $|d_N(x)| \leqslant K_2 N! c_2^N |x|^{-N}$ for all $x \in \mathcal{E}_{\delta_2}$, with $|x| \geqslant r_2$. In this section we first want to show that y can be analytically continued into the region $\xi^{-1}(\mathcal{D})$ and satisfies in this region a similar Gevrey estimate as above. This will finally result in the connection of the singular points of y with the singular points of F_0 (compare theorem 3.5.7).

We first observe that by choosing φ_- in the definition of S_2 somewhat larger (i.e. closer to zero), we can ensure that $|C_j(x)e^{-\mu_j x}| \leqslant e^{-\varepsilon_1|x|}$, $j = 1, 2, \ldots, p$, on the ray $\arg x = \varphi_-$. Hence, the Gevrey estimate $|d_N(x)| \leqslant K_2 N! c_2^N |x|^{-N}$ holds on

$$\mathcal{E}_{\delta_2}\backslash\Delta(0, r_2) = \left\{x \in S_1 \;\middle|\; \arg x \geqslant \varphi_-,\ |C_1(x)e^{-\mu_1 x}| \leqslant \delta_2 \text{ and } |x| \geqslant r_2\right\}. \tag{3.5.1}$$

It is easily seen that d_N satisfies the equation

$$w(x+1) = \Lambda(x)w(x) + f(x, d_N(x)), \quad x \in \mathcal{E}_{\delta_2}, \tag{3.5.2}$$

with

$$f(x,d) = g\Big(x, d+\sum_{m=0}^{N-1} F_m\big(\xi(x)\big)x^{-m}\Big) - \sum_{m=0}^{N-1} F_m\big(\xi(x+1)\big)(x+1)^{-m} + \Lambda(x)\sum_{m=0}^{N-1} F_m\big(\xi(x)\big)x^{-m}. \tag{3.5.3}$$

In the following we will omit the subscript N in d_N. As in appendix B one may show that a particular solution $\tilde{d}$ of (3.5.2) can be written as $\tilde{d}(x) = -\Lambda^{-1}(x)f(x, d(x)) + I(x)$, where

$$\begin{aligned} I_h(x) \;&=\; e^{\mu_h}(1+x^{-1})^{-a_h}\int_{\Gamma_+(x)} e^{\mu_h(\sigma-x)}\left(\frac{x+1}{\sigma+1}\right)^{a_h}\frac{e^{2\pi i m_+(\sigma-x)}}{e^{2\pi i(\sigma-x)}-1}\,f_h(\sigma, d(\sigma))\,\mathrm{d}\sigma \\ &-\; e^{\mu_h}(1+x^{-1})^{-a_h}\int_{\Gamma_-(x)} e^{\mu_h(\sigma-x)}\left(\frac{x+1}{\sigma+1}\right)^{a_h}\frac{e^{2\pi i m_-(\sigma-x)}}{e^{2\pi i(\sigma-x)}-1}\,f_h(\sigma, d(\sigma))\,\mathrm{d}\sigma \\ &+\; e^{\mu_h}(1+x^{-1})^{-a_h}\sum_{j=m_-}^{m_+-1}\int_{c_{jh}}^{x+\alpha} e^{\mu_h(\sigma-x)}\left(\frac{x+1}{\sigma+1}\right)^{a_h} e^{2\pi i j(\sigma-x)}\,f_h(\sigma, d(\sigma))\,\mathrm{d}\sigma, \end{aligned} \tag{3.5.4}$$

$h = 1, 2, \ldots, n$. Here $x \in \mathcal{E}_{\delta_2}$, $|x|$ large enough, $0 < \alpha < 1$ and $\Gamma_\pm(x)$ is the half line from $\sigma = x + \alpha$ to $\sigma = \infty e^{i\varphi_\pm}$. Moreover, the integer m_- is chosen in such a way that $\frac{1}{2\pi}(\Re\mu_h \cot\varphi_- - \Im\mu_h) \leqslant m_- < 1 + \frac{1}{2\pi}(\Re\mu_h \cot\varphi_- - \Im\mu_h)$ and m_+ is chosen such that $m_+ = m_-$ if $m_- > \frac{1}{2\pi}(\Re\mu_h \cot\varphi_+ - \Im\mu_h)$ and m_+ is the smallest integer strictly larger than $\frac{1}{2\pi}(\Re\mu_h \cot\varphi_+ - \Im\mu_h)$ if $m_- \leqslant \frac{1}{2\pi}(\Re\mu_h \cot\varphi_+ - \Im\mu_h)$. The constants c_{jh} are suitably chosen numbers (possibly infinite) in the given neighbourhood of ∞ in $\mathcal{E}_{\delta_2}$. Note that the numbers $m_\pm$, and thus the finite sum, depend on h. The fact that $f(\sigma, d)$ is well defined on the paths of integration will be shown in the following section.

3.5.1 A Useful Lemma

Lemma 3.5.1 *As before let g be continuous on $\{(x, y) \in \mathbb{C}\times\mathbb{C}^n \mid |x| \geqslant \varrho_1,\ |y| \leqslant \varrho_2\}$ and holomorphic in the interior and let $\sup_{\xi\in\mathcal{D}} |F_0(\xi)| \leqslant \varrho_3 < \varrho_2$. Then there exists a constant C such that $f(x, d)$, defined in (3.5.3), is well defined for $x \in \xi^{-1}(\mathcal{D})$, $|x| \geqslant CN$, and $|d| \leqslant (\varrho_2 - \varrho_3)/2$. Moreover, there exist constants $B_1, B_2 > 0$ such that*

$$|f(x,0)| \leqslant B_1^N N! |x|^{-N} \quad \textit{and} \quad |f(x,d_1) - f(x,d_2)| \leqslant B_2|d_1 - d_2|$$

for all $x \in \xi^{-1}(\mathcal{D})$, $|x| \geqslant CN$, and $|d_j| \leqslant (\varrho_2 - \varrho_3)/2$, $j = 1, 2$.

PROOF. A tedious but straightforward calculation shows that if $\xi = \xi(x)$, then

$$\begin{aligned} f(x,0) &= \sum_{m=N}^{\infty} \frac{1}{m!}\frac{\mathrm{d}^m}{\mathrm{d}z^m}\, g\Big(z^{-1}, \sum_{k=0}^{N-1} F_k(\xi) z^k\Big)\Big|_{z=0} x^{-m} \\ &\quad + \sum_{m=N}^{\infty}\sum_{k=0}^{N-1} \Lambda_{m-k} F_k(\xi) x^{-m} - \sum_{m=N}^{\infty}\sum_{k=1}^{N-1} \binom{-k}{m-k} F_k(\lambda_1 \xi) x^{-m} \end{aligned} \tag{3.5.5}$$

and

$$f(x,d) - f(x,0) = g\Big(x, d + \sum_{k=0}^{N-1} F_k(\xi) x^{-k}\Big) - g\Big(x, \sum_{k=0}^{N-1} F_k(\xi) x^{-k}\Big).$$

Assume that $|x| \geqslant \tilde{C}N$, for some positive constant $\tilde{C}$. From proposition 3.4.3 we then deduce that $\sum_{k=1}^{N-1} |F_k(\xi)||x|^{-k} \leqslant \sum_{k=1}^{N-1} k! B^k (\tilde{C}N)^{-k}$. Choosing $\tilde{C} \geqslant B$, the latter sum can be estimated by $\sum_{k=1}^{N-1} k! N^{-k}$, which in turn is bounded by $N^{-1} + N \cdot 2N^{-2}$, since $k! N^{-k} \leqslant 2N^{-2}$ for all $k \in \{2,3,\ldots,N-1\}$. Hence, $\sum_{k=1}^{N-1} |F_k(\xi)||x|^{-k}$ tends to 0 as $N \to \infty$, uniformly for $\xi \in \mathcal{D}$ and $|x| \geqslant \tilde{C}N$.

Using that $|F_0(\xi)| \leqslant \varrho_3 < \varrho_2$, $\xi \in \mathcal{D}$, we deduce that $g\big(x, d + \sum_{k=0}^{N-1} F_k(\xi) x^{-k}\big)$ is defined for $x \in \xi^{-1}(\mathcal{D})$, $|x| \geqslant \tilde{C}N$, N large enough and d small enough. By enlarging $\tilde{C}$ we can get rid of the condition 'N large enough', so that $g\big(x, d + \sum_{k=0}^{N-1} F_k(\xi) x^{-k}\big)$ is defined for $x \in \xi^{-1}(\mathcal{D})$, $|x| \geqslant \tilde{C}N$, and $|d|$ small enough. Moreover, we assume $\tilde{C}$ to be so large that $g\big(x, d + \sum_{k=0}^{N-1} F_k(\xi) x^{-k}\big)$ is defined for $x \in \xi^{-1}(\mathcal{D})$, $|x| \geqslant \tilde{C}N$, and $|d| \leqslant (\varrho_2 - \varrho_3)/2$. This in turn proves that $f(x,d) - f(x,0)$ is defined for these values of x and d.

To prove the first estimate in the lemma we note that $\sum_{k=1}^{N-1} k! \big|\binom{-k}{m-k}\big| = \sum_{k=1}^{N-1} k \frac{(m-1)!}{(m-k)!}$. Hence,

$$\begin{aligned} \sum_{k=1}^{N-1} k! \Big|\binom{-k}{m-k}\Big| &= 1 + 2(m-1) + \cdots + (N-1)(m-1)(m-2)\cdots(m-N+2) \\ &\leqslant m^{N-2}(1 + 2 + \cdots + N-1) = \frac{(N-1)N}{2} m^{N-2} \leqslant \frac{N!}{2} e^m \end{aligned}$$

and using proposition 3.4.3 we then deduce that

$$\sum_{m=N}^{\infty}\sum_{k=1}^{N-1} \Big|\binom{-k}{m-k}\Big| |F_k(\lambda_1 \xi)||x|^{-m} \leqslant (Be)^N N! \, |x|^{-N}, \quad \text{if } x \in \xi^{-1}(\mathcal{D}),\ |x| \geqslant 2e. \tag{3.5.6}$$

In the proof of proposition 3.4.3 we saw that $|\Lambda_k| \leqslant a^k$, $k \geqslant 1$, for some $a \geqslant 1$ and this implies that $\sum_{k=0}^{N-1} |\Lambda_{m-k} F_k(\xi)| \leqslant \varrho_3 a^m + \sum_{k=1}^{N-1} k! a^{m-k} B^k$, which in turn is bounded by $\{\varrho_3 + N(N-1)! B^{N-1}\} a^m$. The latter expression can be estimated by $(\varrho_3 + 1) N! B^N a^m$ and thus

$$\sum_{m=N}^{\infty}\sum_{k=0}^{N-1} |\Lambda_{m-k} F_k(\xi)||x|^{-m} \leqslant 2(\varrho_3 + 1)(Ba)^N N! \, |x|^{-N}, \quad \text{if } x \in \xi^{-1}(\mathcal{D}),\ |x| \geqslant 2a. \tag{3.5.7}$$

If we denote $\|g\|_\infty := \sup_{\{|x|\geqslant \varrho_1, |y|\leqslant \varrho_2\}} |g(x,y)|$, then

$$\frac{1}{m!}\frac{\mathrm{d}^m}{\mathrm{d}z^m}\, g\Big(z^{-1}, \sum_{k=0}^{N-1} F_k(\xi) z^k\Big)\,\Big|_{z=0} = \frac{1}{2\pi i}\oint_{|\sigma|=(\tilde{C}N)^{-1}} \frac{g\big(\sigma^{-1}, \sum_{k=0}^{N-1} F_k(\xi)\sigma^k\big)}{\sigma^{m+1}}\,\mathrm{d}\sigma$$

is bounded by $\|g\|_\infty (\tilde{C}N)^m$. Hence,

$$\Big|\sum_{m=N}^{\infty} \frac{1}{m!}\frac{\mathrm{d}^m}{\mathrm{d}z^m} g\Big(z^{-1}, \sum_{k=0}^{N-1} F_k(\xi) z^k\Big)\,\Big|_{z=0} x^{-m}\Big| \leqslant 2\|g\|_\infty (\tilde{C}N)^N |x|^{-N}, \tag{3.5.8}$$

provided that $x \in \xi^{-1}(\mathcal{D})$ with $|x| \geqslant 2\tilde{C}N =: CN$. Combining (3.5.5), (3.5.6), (3.5.7), (3.5.8) and using that $N^N \leqslant N!\, e^N$ we obtain the Gevrey estimate for $|f(x,0)|$, provided that $x \in \xi^{-1}(\mathcal{D})$ and $|x| \geqslant CN$.

Finally, to prove the second estimate in the lemma we write

$$f(x,d_1) - f(x,d_2) = g\Big(x, d_1 - d_2 + d_2 + \sum_{k=0}^{N-1} F_k(\xi) x^{-k}\Big) - g\Big(x, d_2 + \sum_{k=0}^{N-1} F_k(\xi) x^{-k}\Big)$$

and the expression in the right-hand side can be estimated by a constant times $|d_1 - d_2|$, provided that $x \in \xi^{-1}(\mathcal{D})$, $|x| \geqslant CN$, and $|d_j| \leqslant (\varrho_2 - \varrho_3)/2$, $j = 1, 2$. ■

Remark 3.5.2 In the following we need the constant B_2 in the preceding lemma to be small. More precisely, if $M = \max_{\{|x|\geqslant CN\}} |\Lambda^{-1}(x)|$, the constant B_2 has to satisfy $MB_2 < 1$ (compare the end of section 3.5.3). This requirement can always be satisfied if we assume that $(x,y) \mapsto xg(x,y)$ is holomorphic in the poly-disc $\{(x,y) \in \mathbb{C}\times\mathbb{C}^n \mid |x| > \varrho_1,\ |y| < \varrho_2\}$, by taking $|x|$ large enough (and thus by choosing C sufficiently large).

3.5.2 Estimate for I

Remember that the set $\mathcal{E}_{\delta_2}$ is defined by

$$\mathcal{E}_{\delta_2} = \big\{x \in S_1 \;\big|\; \arg x \geqslant \varphi_-,\ |C_1(x)e^{-\mu_1 x}| \leqslant \delta_2\big\},$$

compare (3.5.1). Let δ be some small positive number, then with $\mathcal{E}_{\delta_2}$ we associate the set $\tilde{\mathcal{E}}_{\delta,\delta_2}$ defined by

$$\tilde{\mathcal{E}}_{\delta,\delta_2} := \big\{x \in \mathbb{C}^* \;\big|\; \varphi_+ - \delta \leqslant \arg x \leqslant \varphi_+,\ |e^{-(\mu_1 - 2\pi i h_{1-})x}| \leqslant \delta_2/(2|\gamma_{1,h_{1-}}|)\big\},$$

which can also be written as $\tilde{\mathcal{E}}_{\delta,\delta_2} = \{x \in \mathbb{C}^* \mid \varphi_+ - \delta \leqslant \arg x \leqslant \varphi_+,\ \Re(xe^{i(\pi/2-\varphi_+)}) \geqslant \nu_1\}$, with $\nu_1 := |\mu_1 - 2\pi i h_{1-}|^{-1}\ln\big(2|\gamma_{1,h_{1-}}|/\delta_2\big)$. By choosing δ_2 small enough we can ensure that $\nu_1 > 0$. Then it is easily seen that $\tilde{\mathcal{E}}_{\delta,\delta_2}$ is bounded by the half line $\arg x = \varphi_+ - \delta$ and that part of the ray $\Re(xe^{i(\pi/2-\varphi_+)}) = \nu_1$, where $\arg x \geqslant \varphi_+ - \delta$, compare figure 3.1. From the asymptotic behaviour of $C_1(x)e^{-\mu_1 x}$ in the upper half plane one easily deduces that $x \in \tilde{\mathcal{E}}_{\delta,\delta_2}$, $|x|$ large enough, implies $x \in \mathcal{E}_{\delta_2}$. In fact, there exists a constant R_0, which without loss of generality can be chosen larger than r_2 (compare theorem 3.3.2), such that

$$\tilde{\mathcal{E}}_{\delta,\delta_2} \backslash \Delta(0, R_0) \subset \mathcal{E}_{\delta_2}. \tag{3.5.9}$$

In the following we will specify the constants c_{jh} in the last integral given in (3.5.4) for those $h \in \{1, 2, \ldots, n\}$ with $m_+ > m_-$. By definition these values of m_+ satisfy $\frac{1}{2\pi}(\Re\mu_h \cot\varphi_+ - \Im\mu_h) < m_+ \leqslant \frac{1}{2\pi}(\Re\mu_h \cot\varphi_+ - \Im\mu_h) + 1$. Hence, $j \in \{m_-, \ldots, m_+ - 1\}$ implies $\frac{1}{2\pi}(\Re\mu_h \cot\varphi_- - \Im\mu_h) \leqslant j \leqslant \frac{1}{2\pi}(\Re\mu_h \cot\varphi_+ - \Im\mu_h)$. So if we define $\lambda_{jh} := \mu_h + 2\pi i j$, then $\Re[\lambda_{jh} e^{i\varphi_\pm}] \geqslant 0$ and therefore $\arg\lambda_{jh} \in [-\pi/2 - \varphi_-, \pi/2 - \varphi_+]$ for $j \in \{m_-, \ldots, m_+ - 1\}$. By a little change in φ_- we can ensure that $\arg\lambda_{jh} \neq -\pi/2 - \varphi_-$ and we conclude that if the finite sum in (3.5.4) is present for some $h \in \{1, 2, \ldots, n\}$, then

$$\arg\lambda_{jh} \in (-\pi/2 - \varphi_-, \pi/2 - \varphi_+], \quad \text{for all } j \in \{m_-, \ldots, m_+ - 1\}.$$

Moreover, we assume the constant C in lemma 3.5.1 to be larger than R_0.

- If $\arg\lambda_{jh} = \pi/2 - \varphi_+$, then we define $c_{jh} = \infty e^{i\varphi_+}$ and the path of integration is just equal to $\Gamma_+(x)$;
- If $\arg\lambda_{jh} \in (-\pi/2 - \varphi_-, \pi/2 - \varphi_+)$ we take $c_{jh} = \nu_1 e^{i(\varphi_+ - \pi/2)} + CNe^{i\varphi_+}$, so that $c_{jh} \in \mathcal{E}_{\delta_2}$, $|c_{jh}| \geqslant CN$.

Lemma 3.5.3 *Let $\tilde{d}(x) = -\Lambda^{-1}(x) f(x, d(x)) + I(x)$, where $I(x)$ is defined in (3.5.4). Then there exist positive constants $B_3 > \max\{B_1, B_2\}$ and $C_* > C$ such that for any $N \in \mathbb{N}_+$*

$$|I(x)| \leqslant B_3^N \,(N-1)!\, |x|^{1-N} \quad \text{and} \quad |\tilde{d}(x)| \leqslant B_3^N \,(N-1)!\, |x|^{1-N}, \quad x \in \tilde{\mathcal{E}}_{\delta,\delta_2},\ |x| \geqslant C_* N,$$

provided that δ in the definition of $\tilde{\mathcal{E}}_{\delta,\delta_2}$ is small enough.

Remark 3.5.4

1. Taking $C_* > C$ large enough we can ensure that $x \in \tilde{\mathcal{E}}_{\delta,\delta_2} \backslash \Delta(0, C_* N)$ implies that $x + \alpha$ belongs to $\mathcal{E}_{\delta_2} \backslash \Delta(0, CN)$ for any $N \in \mathbb{N}_+$, where α is as in (3.5.4). With the choice of c_{jh} given above we then conclude that the paths of integration in (3.5.4) belong to $\mathcal{E}_{\delta_2} \backslash \Delta(0, CN)$ if $x \in \tilde{\mathcal{E}}_{\delta,\delta_2} \backslash \Delta(0, C_* N)$.
2. As both d and $\tilde{d}$ are holomorphic solutions of (3.5.2), the difference $d - \tilde{d}$ satisfies $w(x+1) = \Lambda(x) w(x)$, and thus

$$d(x) - \tilde{d}(x) = \sum_{j=1}^{n} \tilde{C}_j(x) e^{-\mu_j x} x^{a_j} \mathbf{e}_j,$$

for certain 1-periodic holomorphic functions $\tilde{C}_j(x)$. In theorem 3.3.2 we have proved that $d(x) = O(x^{-N})$ as $x \to \infty$ in $\tilde{\mathcal{E}}_{\delta,\delta_2}$, which, together with $\tilde{d}(x) = O(x^{1-N})$, implies that each $\tilde{C}_j(x) e^{-\mu_j x}$ is exponentially small on $\tilde{\mathcal{E}}_{\delta,\delta_2}$ for $|x|$ large enough.

PROOF OF LEMMA 3.5.3. Since $C > R_0 > r_2$ we conclude from theorem 3.3.2 that $|d(\sigma)| \leqslant K_2 N! c_2^N |\sigma|^{-N}$ for all $\sigma \in \mathcal{E}_{\delta_2} \backslash \Delta(0, CN)$. By taking C large enough we can ensure that $|d(\sigma)| \leqslant (\varrho_2 - \varrho_3)/2$ and lemma 3.5.1 then implies the existence of some $B_4 > 0$ such that

$$|f(\sigma, d(\sigma))| \leqslant B_4^N N! |\sigma|^{-N}, \quad \sigma \in \mathcal{E}_{\delta_2} \backslash \Delta(0, CN). \tag{3.5.10}$$

Since $|\sigma|^{-1} \leqslant (CN)^{-1}$ and $\tilde{\mathcal{E}}_{\delta,\delta_2} \backslash \Delta(0, C_* N) \subset \mathcal{E}_{\delta_2} \backslash \Delta(0, CN)$ for all $C_* > C$, it is enough to prove the estimate for $I(x)$.

Let $\Gamma_+(x)$ be the path of integration as defined in (3.5.4), then for all $\sigma \in \Gamma_+(x)$ and $h \in \{1, 2, \ldots, n\}$ we have $\left| \left(\frac{x+1}{\sigma+1} \right)^{a_h} \right| \leqslant const. \cdot |x|^{\Re a_h} |\sigma|^{-\Re a_h}$ and $|e^{2\pi i(\sigma - x)} - 1| \geqslant const. > 0$. Here *const.* means some positive constant, which may be different in different places. Using (3.5.10), together with the fact that the function $x \mapsto e^{\mu_h}(1 + x^{-1})^{-a_h}$ is bounded by some positive constant, we conclude that there exists a constant $K_1 > 0$, independent of N, such that for all $x \in \tilde{\mathcal{E}}_{\delta,\delta_2}$ with $|x| \geqslant C_* N$ we have

$$\begin{aligned} &\left| e^{\mu_h}(1 + x^{-1})^{-a_h} \int_{\Gamma_+(x)} e^{\mu_h(\sigma - x)} \left(\frac{x+1}{\sigma+1} \right)^{a_h} \frac{e^{2\pi i m_+(\sigma - x)}}{e^{2\pi i(\sigma - x)} - 1} f_h(\sigma, d(\sigma)) \, \mathrm{d}\sigma \right| \\ \leqslant\ & K_1 B_4^N N! \, |x|^{\Re a_h} \int_{x+\alpha}^{\infty e^{i\varphi_+}} |e^{(\mu_h + 2\pi i m_+)(\sigma - x)}| |\sigma|^{-N - \Re a_h} |\mathrm{d}\sigma|. \end{aligned}$$

For every $\sigma \in \Gamma_+(x)$ we have $|\sigma| \geqslant |x| - \alpha$. Hence, $|\sigma|^{-N-\Re a_h} \leqslant (|x| - \alpha)^{-N-\Re a_h}$. As $|x| \geqslant 1$, the latter expression is bounded by $|x|^{-N-\Re a_h}(1 - \alpha)^{-N-\Re a_h}$. Hence,

$$\begin{aligned} &\left| e^{\mu_h}(1 + x^{-1})^{-a_h} \int_{\Gamma_+(x)} e^{\mu_h(\sigma - x)} \left(\frac{x+1}{\sigma+1} \right)^{a_h} \frac{e^{2\pi i m_+(\sigma - x)}}{e^{2\pi i(\sigma - x)} - 1} f_h(\sigma, d(\sigma)) \, \mathrm{d}\sigma \right| \\ \leqslant\ & K_1 (1 - \alpha)^{-\Re a_h} \big(B_4/(1 - \alpha) \big)^N N! \, |x|^{-N} \int_{x+\alpha}^{\infty e^{i\varphi_+}} |e^{(\mu_h + 2\pi i m_+)(\sigma - x)}| |\mathrm{d}\sigma| \end{aligned}$$

and the latter integral can be estimated by a constant independent of N and x, because of the choice of m_+ in the introduction of section 3.5. This gives a Gevrey type estimate of $e^{\mu_h}(1 + x^{-1})^{-a_h} \int_{\Gamma_+(x)} e^{\mu_h(\sigma - x)} \left(\frac{x+1}{\sigma+1} \right)^{a_h} \frac{e^{2\pi i m_+(\sigma - x)}}{e^{2\pi i(\sigma - x)} - 1} f_h(\sigma, d(\sigma)) \, \mathrm{d}\sigma$ on $\tilde{\mathcal{E}}_{\delta,\delta_2} \backslash \Delta(0, C_* N)$.

Let $\Gamma_-(x)$ be the path of integration as defined in (3.5.4), then for all $\sigma \in \Gamma_-(x)$ we have $|e^{2\pi i(\sigma - x)} - 1| \geqslant const. \cdot |e^{2\pi i(\sigma - x)}|$. Using (3.5.10) we deduce that there exists a constant $K_2 > 0$, independent of N, such that

$$\begin{aligned} &\left| e^{\mu_h}(1 + x^{-1})^{-a_h} \int_{\Gamma_-(x)} e^{\mu_h(\sigma - x)} \left(\frac{x+1}{\sigma+1} \right)^{a_h} \frac{e^{2\pi i m_-(\sigma - x)}}{e^{2\pi i(\sigma - x)} - 1} f_h(\sigma, d(\sigma)) \, \mathrm{d}\sigma \right| \\ \leqslant\ & K_2 B_4^N N! \, |x|^{\Re a_h} \int_{x+\alpha}^{\infty e^{i\varphi_-}} |e^{(\mu_h + 2\pi i(m_- - 1))(\sigma - x)}| |\sigma|^{-N - \Re a_h} |\mathrm{d}\sigma|. \end{aligned}$$

In a similar way as above we can continue to prove a Gevrey type estimate for the expression $e^{\mu_h}(1 + x^{-1})^{-a_h} \int_{\Gamma_-(x)} e^{\mu_h(\sigma - x)} \left(\frac{x+1}{\sigma+1} \right)^{a_h} \frac{e^{2\pi i m_-(\sigma - x)}}{e^{2\pi i(\sigma - x)} - 1} f_h(\sigma, d(\sigma)) \, \mathrm{d}\sigma$ on $\tilde{\mathcal{E}}_{\delta,\delta_2} \backslash \Delta(0, C_* N)$.

Finally we have to look at the integrals $\int_{c_{jh}}^{x+\alpha}$ in the two different cases mentioned in the introduction of this section. If $\arg\lambda_{jh} = \pi/2 - \varphi_+$, then $c_{jh} = \infty e^{i\varphi_+}$ and in that case $e^{\lambda_{jh}(\sigma-x)}$ is bounded by a constant for all σ on the path of integration and all $x \in \tilde{\mathcal{E}}_{\delta,\delta_2}\backslash\Delta(0, C_*N)$. Thus there exists a positive constant K_3, independent of N, such that for $x \in \tilde{\mathcal{E}}_{\delta,\delta_2}\backslash\Delta(0, C_*N)$ we have by using (3.5.10)

$$\begin{aligned} &\left| e^{\mu_h}(1+x^{-1})^{-a_h}\int_{x+\alpha}^{\infty e^{i\varphi_+}} e^{\mu_h(\sigma-x)}\left(\frac{x+1}{\sigma+1}\right)^{a_h} e^{2\pi i j(\sigma-x)}\, f_h(\sigma, d(\sigma))\,\mathrm{d}\sigma\right| \\ \leqslant\;& K_3 B_4^N N!\,|x|^{\Re a_h}\int_{x+\alpha}^{\infty e^{i\varphi_+}} |\sigma|^{-N-\Re a_h}|\mathrm{d}\sigma|. \end{aligned}$$

The path of integration in the latter integral can be taken equal to the sum of the circular arc between $x+\alpha$ and $|x+\alpha|e^{i\varphi_+}$ and the straight line from $|x+\alpha|e^{i\varphi_+}$ to $\infty e^{i\varphi_+}$. The integral with the circular arc as path of integration is bounded by a positive constant times $|x+\alpha|^{-N-\Re a_h}$ and $|x+\alpha|^{-N-\Re a_h} \geqslant |x|^{-N-\Re a_h}(1-\alpha)^{-N-\Re a_h}$. The integral with the straight line as integration contour is equal to $\frac{|x+\alpha|^{1-N-\Re a_h}}{N+\Re a_h - 1}$. Altogether this implies that there exist a positive constant K_4, independent of N, such that

$$\begin{aligned} &\left| e^{\mu_h}(1+x^{-1})^{-a_h}\int_{c_{jh}}^{x+\alpha} e^{\mu_h(\sigma-x)}\left(\frac{x+1}{\sigma+1}\right)^{a_h} e^{2\pi i j(\sigma-x)}\, f_h(\sigma, d(\sigma))\mathrm{d}\sigma\right| \\ \leqslant\;& K_4\big(B_4/(1-\alpha)\big)^N (N-1)!|x|^{1-N}. \end{aligned}$$

If $\arg\lambda_{jh} \in (-\pi/2-\varphi_-, \pi/2-\varphi_+)$ we take $c_{jh} = \nu_1 e^{i(\varphi_+-\pi/2)} + CNe^{i\varphi_+}$. In that case the integral $\int_{c_{jh}}^{x+\alpha}$ can be written as

$$\int_{c_{jh}}^{x+\alpha} = \int_{\tilde{x}}^{x+\alpha} + \int_{c_{jh}}^{\tilde{x}}, \tag{3.5.11}$$

where $\tilde{x}$ is the point on $\nu_1 e^{i(\varphi_+-\pi/2)} + \mathbb{R}_+ e^{i\varphi_+}$ with radius $|\tilde{x}| = |x+\alpha|$, compare figure 3.1. By construction of C_* we can ensure that $x \in \tilde{\mathcal{E}}_{\delta,\delta_2}\backslash\Delta(0, C_*N)$ implies $\tilde{x} \in \mathcal{E}_{\delta_2}\backslash\Delta(0, CN)$. For convenience we take C_* so large that $x \in \tilde{\mathcal{E}}_{\delta,\delta_2}\backslash\Delta(0, C_*N)$ implies $\Im(\tilde{x}e^{i(\pi/2-\varphi_+)}) > CN$. The reason for this will become clear later on. In the first integral in the right-hand side of (3.5.11) we take as path of integration the circular arc between $\tilde{x}$ and $x+\alpha$. With this choice we can make sure that the path of integration belongs to $\mathcal{E}_{\delta_2}\backslash\Delta(0, CN)$. Moreover, the straight line between c_{jh} and $\tilde{x}$ belongs to $\mathcal{E}_{\delta_2}\backslash\Delta(0, CN)$.

From (3.5.10) one may conclude that it is sufficient to consider the integrals in (3.5.11) with integrand $|e^{\lambda_{jh}(\sigma-x)}\sigma^{-m}|$, where $m := N + \Re a_h$. On the circular arc between $\tilde{x}$ and $x+\alpha$, we have $|\sigma| \geqslant |x| - \alpha$, so the first integral may be estimated by

$$\begin{aligned} \int_{\tilde{x}}^{x+\alpha} |e^{\lambda_{jh}(\sigma-x)}\sigma^{-m}||\mathrm{d}\sigma| &\leqslant |x|^{-m}(1-\alpha)^{-m}\int_{\tilde{x}}^{x+\alpha}|e^{\lambda_{jh}(\sigma-x)}||\mathrm{d}\sigma| \\ &= |e^{\lambda_{jh}\alpha}||x|^{-m}(1-\alpha)^{-m}\int_{\tilde{x}}^{x+\alpha}|e^{\lambda_{jh}(\sigma-x-\alpha)}||\mathrm{d}\sigma|. \end{aligned}$$

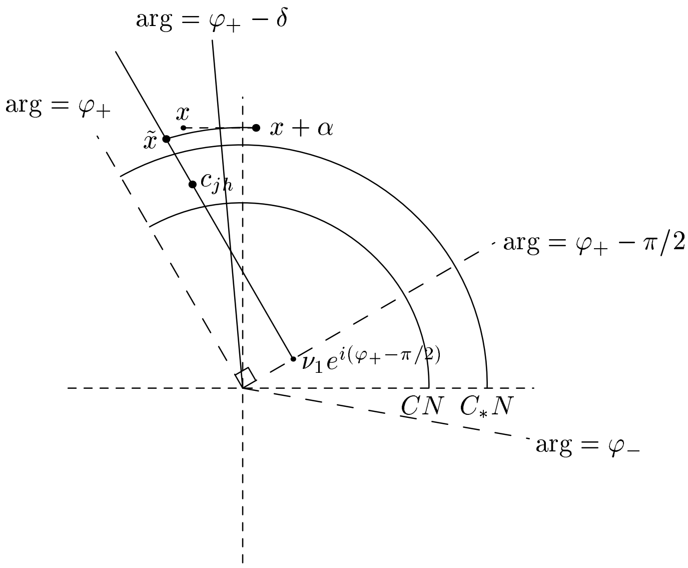

Figure 3.1: *The set* $\tilde{\mathcal{E}}_{\delta,\delta_2}$ *together with the choice of* c_{jh}.

Now

$$\begin{aligned} \Re[\lambda_{jh}(\sigma - x - \alpha)] &= |\lambda_{jh}||x+\alpha|\{\cos(\arg\lambda_{jh} + \arg\sigma) - \cos(\arg\lambda_{jh} + \arg(x+\alpha))\} \\ &= -2|\lambda_{jh}||x+\alpha|\sin(\arg\lambda_{jh} + [\arg\sigma + \arg(x+\alpha)]/2)\sin([\arg\sigma - \arg(x+\alpha)]/2). \end{aligned}$$

Due to the bounds on $\arg x$ for $x \in \tilde{\mathcal{E}}_{\delta,\delta_2}\backslash\Delta(0, C_*N)$, one easily infers that on the circular arc between $\tilde{x}$ and $x+\alpha$ we have $0 < \arg\sigma - \arg(x+\alpha) < \varepsilon$ for some small but positive ε. Hence, $\sin([\arg\sigma - \arg(x+\alpha)]/2) \geqslant \mathit{const.} \cdot (\arg\sigma - \arg(x+\alpha))$. On the other hand the numbers $\arg\lambda_{jh} + [\arg\sigma + \arg(x+\alpha)]/2$ belong to a closed subset of $(0,\pi)$ provided that $\varphi_+ - \varphi_- > \pi/2$ and $\arg x \in (\varphi_+ - \delta, \varphi_+)$ with δ small enough. Hence, there exists a constant $c > 0$ such that $\Re[\lambda_{jh}(\sigma - x - \alpha)] \leqslant -c|x+\alpha|(\arg\sigma - \arg(x+\alpha))$ and we conclude that

$$\int_{\tilde{x}}^{x+\alpha} |e^{\lambda_{jh}(\sigma - x)}\sigma^{-m}||\mathrm{d}\sigma| \ \leqslant\ \frac{|e^{\lambda_{jh}\alpha}|}{c}(1-\alpha)^{-m}|x|^{-m}. \tag{3.5.12}$$

To estimate the integral $\int_{c_{jh}}^{\tilde{x}} |e^{\lambda_{jh}(\sigma-x)}\sigma^{-m}||\mathrm{d}\sigma|$ we introduce the number $\eta := \Re[\lambda_{jh}e^{i\varphi_+}]$, so $\eta > 0$, and with $\lambda := \lambda_{jh}e^{i(\varphi_+ - \pi/2)}$, $y := xe^{i(\pi/2-\varphi_+)}$ and $\tilde{y} := \tilde{x}e^{i(\pi/2-\varphi_+)}$ we have

$$\int_{c_{jh}}^{\tilde{x}} |e^{\lambda_{jh}(\sigma-x)}\sigma^{-m}||\mathrm{d}\sigma| = e^{\Re[\lambda\nu_1]}\int_{CN}^{\Im\tilde{y}} e^{\Re[\lambda(it-y)]}|it+\nu_1|^{-m}\mathrm{d}t.$$

Now $\Re[\lambda i] = \eta$, $|it+\nu_1| \geqslant t$ and $\Re[\lambda y] = |\lambda||y|\cos(\arg\lambda_{jh} + \arg x) \geqslant \eta|y| = \eta|x|$. Hence, the latter integral is bounded by $\int_{CN}^{\Im\tilde{y}} e^{\eta(t-|x|)-m\ln t}\mathrm{d}t$.

Now let us assume C to be so large that $CN \geqslant 2m\eta^{-1} = 2(N + \Re a_h)\eta^{-1}$ and let us define the function $g(t) := \eta t - m \ln t$. Then

$$\int_{c_{jh}}^{\tilde{x}} |e^{\lambda_{jh}(\sigma - x)}\sigma^{-m}||\mathrm{d}\sigma| \leqslant e^{\Re[\lambda\nu_1]}e^{-\eta|x|}\int_{CN}^{\Im\tilde{y}} e^{g(t)}\mathrm{d}t.$$

The derivative g' is increasing and has a zero at $t = m\eta^{-1}$. Since $\Im\tilde{y} > CN > 2m\eta^{-1}$ we have $g(\Im\tilde{y}) - g(t) \geqslant g'(CN)(\Im\tilde{y} - t) \geqslant \frac{\eta}{2}(\Im\tilde{y} - t)$ for all $t \in [CN, \Im\tilde{y}]$, so that

$$\int_{CN}^{\Im\tilde{y}} e^{g(t)}\mathrm{d}t \leqslant \frac{2}{\eta}\, e^{g(\Im\tilde{y})}.$$

As $|x| - \nu_1 - \alpha \leqslant |\tilde{y} - \nu_1| = |\Im(\tilde{y})| \leqslant |\tilde{y}| \leqslant |x| + \alpha$ and $\exp[g(\Im\tilde{y})] = \exp[\eta\,\Im\tilde{y}]\,(\Im\tilde{y})^{-m}$, we see that $\exp[g(\Im\tilde{y})]$ is bounded by $e^{\eta\alpha}e^{\eta|x|}\,|x|^{-m}(1 - \frac{\nu_1+\alpha}{CN})^{-m}$. Since $m = N + \Re a_h$, the expression $(1 - \frac{\nu_1+\alpha}{CN})^{-m}$ may be estimated by a constant independent of N, and thus there exists a constant $K_5 > 0$ such that

$$e^{\Re[\lambda\nu_1]}e^{-\eta|x|}\int_{CN}^{\Im\tilde{y}} e^{g(t)}\mathrm{d}t \leqslant K_5|x|^{-m}. \tag{3.5.13}$$

From (3.5.10), (3.5.12) and (3.5.13) we conclude that there exists a constant K_6 such that

$$\begin{aligned}&\left| e^{\mu_h}(1 + x^{-1})^{-a_h}\int_{c_{jh}}^{x+\alpha} e^{\mu_h(\sigma - x)}\left(\frac{x+1}{\sigma+1}\right)^{a_h} e^{2\pi i j(\sigma - x)}\, f_h(\sigma, d(\sigma))\,\mathrm{d}\sigma\right| \\ &\leqslant\;\; K_6\big(B_4/(1-\alpha)\big)^N N!|x|^{-N},\end{aligned}$$

provided that $x \in \tilde{\mathcal{E}}_{\delta,\delta_2}\backslash\Delta(0, C_*N)$. This finally proves a Gevrey type estimate for $I(x)$ in the region $\tilde{\mathcal{E}}_{\delta,\delta_2}\backslash\Delta(0, C_*N)$ and therefore the lemma. ■

3.5.3 Analytic Continuation of y into $\xi^{-1}(\mathcal{D})$

Due to the assumptions on $\mathcal{D}$, as listed in the introduction of section 3.4.2, there exists a positive $r > \delta_2$ such that $\Delta(0, \delta_2) \subset \pi(\mathcal{D}) \subset \Delta(0, r)$. We assume r to be so small that $\ln\big(2|\gamma_{1,h_{1-}}|/r\big) > 0$, where $\gamma_{1,h_{1-}}$ is as in (3.3.2).

Proposition 3.5.5 *The solution y of the difference equation (3.1.1) can be analytically continued into $\xi^{-1}(\mathcal{D})\backslash\Delta(0, C_*)$, provided that C_* is large enough.*

Proof. For convenience we first assume that $\pi(\mathcal{D}) \subset \Delta(0, \delta_3)$, where $\delta_3 \in (\delta_2, r)$ is such that $\nu_2 := |\mu_1 - 2\pi i h_{1-}|^{-1}\ln\big(2|\gamma_{1,h_{1-}}|/\delta_3\big) > 0$ satisfies $0 < \nu_2 < \nu_1 < \nu_2 + \alpha\sin\varphi_+$, with α as in (3.5.4). We define the set $\mathcal{S}$ by

$$\mathcal{S} := \big\{x \in \xi^{-1}(\mathcal{D}) \;\big|\; \arg x \in (\varphi_+ - \delta, \varphi_+),\ \nu_2 < \Re(xe^{i(\pi/2-\varphi_+)}) < \nu_1 + \delta,\ |x| > C_*N\big\}. \tag{3.5.14}$$

From (3.5.4) and remark 3.5.4 it follows that

$$d(x) = -\Lambda^{-1}(x) f(x, d(x)) + I(x) + \sum_{j=1}^{n} \tilde{C}_j(x) e^{-\mu_j x} x^{a_j} \mathbf{e}_j. \tag{3.5.15}$$

Now if $x \in \mathcal{S}$, then $\Re[(x+\alpha)e^{i(\pi/2-\varphi_+)}] > \nu_1$ and all paths of integration belong to the set $\mathcal{E}_{\delta_2} \backslash \Delta(0, CN)$. There the function d is known and satisfies $|d(\sigma)| \leqslant K_2 N! c_2^N |\sigma|^{-N}$. Using this estimate for d, one may prove in a similar way as in the proof of lemma 3.5.3 that $I(x)$ satisfies a Gevrey-like estimate if $x \in \mathcal{S}$. As $|f(x,0)| \leqslant B_1^N N! |x|^{-N}$ for all $x \in \xi^{-1}(\mathcal{D})$ with $|x| \geqslant CN$ (lemma 3.5.1), one easily deduces that the integral equation (3.5.15) can be written as

$$d(x) = (\mathcal{T}d)(x) := -\Lambda^{-1}(x)[f(x, d(x)) - f(x, 0)] + b(x), \tag{3.5.16}$$

with $b(x)$ satisfying $|b(x)| \leqslant B_3^N (N-1)!\, |x|^{1-N}$ for all $x \in \mathcal{S}$, where B_3 is the constant as it appears in lemma 3.5.3. Moreover, lemma 3.5.1, together with remark 3.5.2, implies that

$$|\Lambda^{-1}(x)[f(x, d_1) - f(x, d_2)]| \leqslant MB_2 |d_1 - d_2|,$$

provided that $|d_j| \leqslant (\varrho_2 - \varrho_3)/2$, $j = 1, 2$, and $x \in \xi^{-1}(\mathcal{D}) \backslash \Delta(0, CN)$. As in remark 3.5.2 we will assume that $MB_2 < 1$.

Let us consider the integral operator $\mathcal{T}$ on the Banach space $\mathcal{V}_N$ consisting of functions $d : \mathcal{S} \mapsto \mathbb{C}^n$, such that $x \mapsto x^{N-1} d(x)$ is holomorphic in and bounded on $\mathcal{S}$, endowed with the norm $\|d\| := \sup_{x \in S} |d(x)| |x|^{N-1}$. Lemma 3.5.1 then implies that the function $f(x, d)$ is defined for all $x \in \mathcal{S}$ and $d \in \mathcal{V}$, with $\|d\| \leqslant \frac{\varrho_2 - \varrho_3}{2} (C_* N)^{N-1}$. We will show that $\mathcal{T}$ maps the ball $\mathcal{V}_N := \{d \in \mathcal{V} \mid \|d\| \leqslant \frac{\varrho_2 - \varrho_3}{2} (C_* N)^{N-1}\}$ into itself and defines a contraction. Using the remarks above, we easily deduce that $d \in \mathcal{V}_N$ implies

$$|(\mathcal{T}d)(x)| \leqslant \big\{ MB_2 \|d\| + B_3^N (N-1)! \big\} |x|^{1-N},$$

so that $\|\mathcal{T}d\| \leqslant MB_2 \|d\| + B_3^N (N-1)!$. By choosing C_* large enough we can ensure that $B_3^N \leqslant \frac{1}{2}(1 - MB_2)(\varrho_2 - \varrho_3) C_*^{N-1}$, provided that $N \geqslant 2$. Thus $\|\mathcal{T}d\| \leqslant \frac{1}{2}(\varrho_2 - \varrho_3)(C_* N)^{N-1}$ and $\mathcal{T}$ maps $\mathcal{V}_N$ into itself.

If both d_1 and d_2 belong to $\mathcal{V}_N$, then $(\mathcal{T}d_1 - \mathcal{T}d_2)(x) = \Lambda^{-1}(x)\big[f(x, d_2) - f(x, d_1)\big]$, so that $\|\mathcal{T}d_1 - \mathcal{T}d_2\| \leqslant MB_2 \|d_1 - d_2\|$. Hence, $\mathcal{T}$ defines a contraction on $\mathcal{V}_N$ and thus there exists a unique fixed point of the equation $d = \mathcal{T}d$.

Due to unicity, this fixed point must be equal to $y(x) - \sum_{m=0}^{N-1} F_m\big(\xi(x)\big) x^{-m}$ on the set $\big\{x \in \xi^{-1}(\mathcal{D}) \bigm| \arg x \in (\varphi_+ - \delta, \varphi_+),\ \nu_1 < \Re(x e^{i(\pi/2 - \varphi_+)}) < \nu_1 + \delta,\ |x| > C_* N\big\}$. By taking $N = 2$, we thus see that y can be analytically continued on $\xi^{-1}(\mathcal{D}) \backslash \Delta(0, 2C_*)$.

In case $\pi(\mathcal{D}) \not\subset \Delta(0, \delta_3)$ we can reach the statement in the proposition by repeating the procedure above finitely many times (compare the proof of proposition 3.4.2). ■

Combining theorem 3.3.2 with the results just obtained we have for every $N \geqslant 2$

$$\Big| y(x) - \sum_{m=0}^{N-1} F_m\big(\xi(x)\big) x^{-m} \Big| \leqslant \frac{\varrho_2 - \varrho_3}{2} (C_* N)^{N-1} |x|^{1-N},$$

for all $x \in \xi^{-1}(\mathcal{D})$, $|x| \geqslant C_* N$. Using proposition 3.4.3 this inequality can be rewritten as

$$\begin{aligned}\Big|y(x) - \sum_{m=0}^{N-2} F_m\big(\xi(x)\big)x^{-m}\Big| &\leqslant \frac{\varrho_2 - \varrho_3}{2}(C_*N)^{N-1}|x|^{1-N} + \big|F_{N-1}\big(\xi(x)\big)\big||x|^{1-N} \\ &\leqslant \frac{\varrho_2 - \varrho_3}{2}(C_*N)^{N-1}|x|^{1-N} + (N-1)!B^{N-1}|x|^{1-N}.\end{aligned}$$

From this, together with the fact that $N^{N-1} \leqslant (N-1)!\, e^N$, we conclude that there exists a constant $\tilde{C}_*$ such that for every $N \in \mathbb{N}_+$ we have

$$\Big|y(x) - \sum_{m=0}^{N-1} F_m\big(\xi(x)\big)x^{-m}\Big| \leqslant N!\, \tilde{C}_*^N\, |x|^{-N} \tag{3.5.17}$$

for all $x \in \xi^{-1}(\mathcal{D})$, $|x| \geqslant (N+1)C_*$. In the following we will refer to $y(x) - \sum_{m=0}^{N-1} F_m\big(\xi(x)\big)$ as $d(x)$, also on $\xi^{-1}(\mathcal{D})$.

3.5.4 Gevrey Estimate of the Analytic Continuation

In the preceding section we have seen that $|d(x)| \leqslant N!\, \tilde{C}_*^N\, |x|^{-N}$ for all $x \in \xi^{-1}(\mathcal{D})$ with $|x| \geqslant (N+1)C_*$, $N \geqslant 1$. However, d is holomorphic for all $x \in \xi^{-1}(\mathcal{D})$, $|x| \geqslant 2C_*$. In order to obtain the Gevrey estimate for d we write for $x \in \xi^{-1}(\mathcal{D})$ with $2C_* \leqslant |x| \leqslant (N+1)C_*$

$$|d(x)| \leqslant \big|y(x) - F_0\big(\xi(x)\big)\big| + \sum_{m=1}^{N-1}\big|F_m\big(\xi(x)\big)\big||x|^{-m}. \tag{3.5.18}$$

The first expression in the right-hand side is bounded by $\tilde{C}_*|x|^{-1}$, which in turn can be estimated by $\tilde{C}_*\big((N+1)C_*\big)^{N-1}|x|^{-N}$. As $(N+1)^{N-1} \leqslant (N-1)!\, e^{N+1}$, the first term in the right-hand side satisfies the proper Gevrey estimate. Using proposition 3.4.3 the second term in the right-hand side of (3.5.18) can be majorized by

$$\sum_{m=1}^{N-1}\big|F_m\big(\xi(x)\big)\big||x|^{-m} \leqslant \sum_{m=1}^{N-1} m!B^m|x|^{-m} \leqslant |x|^{-N}(N+1)^N \max\{B, C_*\}^N \sum_{m=1}^{N-1} 1,$$

where we used that $|x|^{-m} \leqslant |x|^{-N}\big((N+1)C_*\big)^{N-m}$. In a similar way as above it can be shown that the latter term in this inequality satisfies the proper Gevrey estimate. Altogether, this proves the following theorem.

Theorem 3.5.6 *Let y be a solution of (3.1.1) such that $y(x) = O(x^{-2})$ as $x \to \infty$ in $\{x \in \mathbb{C}^* \mid \varphi_- < \arg x < \varphi_+\}$ and suppose that φ_+ cannot be enlarged. Then φ_+ is a Stokes line and we assume it to be equal to $\pi/2 - \arg(\mu_1 - 2\pi i h_{1-})$. Moreover, we assume that $a_1 = 0$ and $\arg(\mu_1 - 2\pi i h_{1-}) \not\equiv \arg(\mu_j - 2\pi i h_{j-}) \bmod 2\pi$ for every $j \in \{2, 3, \ldots, p\}$, where h_{j-}, $j = 1, 2, \ldots, p$, is as in (3.3.2). Let $\mathcal{D}$ be as in the introduction of section 3.4.2 and* *assume that* $\pi(\mathcal{D}) \subset \Delta(0,r)$ *with* r *so small that* $\ln\bigl(2|\gamma_{1,h_{1-}}|/r\bigr) > 0$. *Finally, let us assume that* $MB_2 < 1$, *where* $M = \max_{\{|x|\geqslant CN\}} |\Lambda^{-1}(x)|$ *and where* B_2 *is as in lemma 3.5.1.*

Then there exists a positive R *such that the solution* y *of the difference equation (3.1.1) exists and is holomorphic in* $\xi^{-1}(\mathcal{D})\backslash\Delta(0,R)$ *and it has the asymptotic representation*

$$y(x) \sim_1 \sum_{m=0}^{\infty} F_m\bigl(\xi(x)\bigr)x^{-m}$$

as $x \to \infty$ *in* $\xi^{-1}(\mathcal{D})$. *More precisely, there exists a positive constant* c_3 *such that*

$$\Bigl|y(x) - \sum_{m=0}^{N-1} F_m\bigl(\xi(x)\bigr)x^{-m}\Bigr| \leqslant N!\, c_3^N |x|^{-N}, \quad N \in \mathbb{N}_+,$$

for all $x \in \xi^{-1}(\mathcal{D})$, $|x| \geqslant R$.

3.5.5 Formation of Singularities

In this section we will prove the following theorem:

Theorem 3.5.7 *Let the assumptions in theorem 3.5.6 be fulfilled and assume that* F_0 *has an isolated singular point at* $\xi = \xi_0 \in \Xi$. *Moreover, assume that* $\pi(\mathcal{D})$ *contains a punctured neighbourhood of* ξ_0. *Then the solution* y *of (3.1.1), given by (3.3.1), is singular at a distance at most* $o(1)$ *of* $x_n \in \xi^{-1}(\{\xi_0\}) \cap \xi^{-1}(\mathcal{D})$, *as* $x_n \to \infty$. *The collection* $\{x_n\}_{n=1}^{\infty}$ *forms a nearly periodic array*

$$x_n = -\frac{\ln\xi_0}{\mu_1 - 2\pi i h_{1-}} + \frac{\ln\gamma_{1,h_{1-}}}{\mu_1 - 2\pi i h_{1-}} + \frac{2n\pi i}{\mu_1 - 2\pi i h_{1-}} + O\bigl(n^{-1}\bigr)$$

as $n \to \infty$, *where* h_{1-} *and* $\gamma_{1,h_{1-}}$ *are as in (3.3.2).*

The proof of the theorem utilises the following lemma, taken from [CC01].

Lemma 3.5.8 *Let* $r > 0$ *and assume* f *to be some holomorphic function on the universal covering of* $\Delta(0,r)\backslash\{0\}$. *Moreover, assume that for any circle around zero* $\mathcal{C} \subset \Delta(0,r)\backslash\{0\}$ *and any* g *holomorphic in* $\Delta(0,r)$ *we have* $\oint_{\mathcal{C}} f(\xi)g(\xi)\mathrm{d}\xi = 0$. *Then* f *in fact is holomorphic in* $\Delta(0,r)$.

Proof of theorem 3.5.7. The definition of ξ implies that $\xi(x) = \xi_0$ is equivalent to

$$\mu_1 x = -\ln\xi_0 + \ln\bigl(C_1(x)\bigr) + 2n\pi i, \quad n \in \mathbb{Z}. \tag{3.5.19}$$

Recall that $C_1(x) = \gamma_{1,h_{1-}} e^{2\pi i h_{1-} x}\bigl(1 + O(x^{-1})\bigr)$ as $\Im x \to \infty$. Using this the equation above can be rewritten as

$$(\mu_1 - 2\pi i h_{1-})x - \ln\left(\frac{C_1(x)}{\gamma_{1,h_{1-}} e^{2\pi i h_{1-} x}}\right) = -\ln\xi_0 + \ln\gamma_{1,h_{1-}} + 2n\pi i, \quad n \in \mathbb{Z},$$

and putting $x = t^{-1}$ we see that the latter equation is equivalent to

$$t = \zeta_n\big(1 + \psi(t)\big) := h(t), \tag{3.5.20}$$

where

$$\zeta_n := \frac{\mu_1 - 2\pi i h_{1-}}{-\ln\xi_0 + \ln\gamma_{1,h_{1-}} + 2n\pi i} \quad \text{and} \quad \psi(t) := -\frac{t}{\mu_1 - 2\pi i h_{1-}} \ln\left(\frac{C_1(t^{-1})}{\gamma_{1,h_{1-}} e^{2\pi i h_{1-} t^{-1}}}\right).$$

As in example 1.5.1 we introduce the region G, but now defined by

$$G := \{t \in \mathbb{C}^* \mid 0 < |t| < r, \ -\pi + \delta < \arg t < -\delta\}$$

for some $r > 0$ and $\delta > 0$. Then, from the asymptotic behaviour of C_1 we easily deduce that $\psi(t) = o(t)$ and $\psi'(t) = o(1)$ as $t \to 0$ in G and thus $K := \sup_{t\in G} |\psi'(t)| < \infty$. From this we easily infer that $t \mapsto h(t)$ is a contraction on G provided that ζ_n belongs to some subset G_1 of G, $|\zeta_n| < 1/K$ and r sufficiently small.

Since $\arg(\mu_1 - 2\pi i h_{1-}) \in (-\pi/2, \pi/2)$, the requirement $\zeta_n \in G_1$, $|\zeta_n| < 1/K$, can only be arranged for $n \in \mathbb{N}$, n large enough. Hence, (3.5.20) has a unique solution $t = t_n$ for every $n \in \mathbb{N}$ large enough. So, if we define $x_n := t_n^{-1}$, then (3.5.19) has a unique solution x_n, provided that $n \in \mathbb{N}$ is large enough. As $\ln\big(C_1(x)e^{-2\pi i h_{1-} x}/\gamma_{1,h_{1-}}\big) = O(x^{-1})$ for $\Im x \to \infty$, we deduce that $x_n = (\mu_1 - 2\pi i h_{1-})^{-1}(-\ln\xi_0 + \ln\gamma_{1,h_{1-}} + 2n\pi i) + O(x_n^{-1})$ and using this we see that the array $\{x_n\}_{n\in\mathbb{N}}$ has the property stated in the theorem.

Finally, the proof that y indeed is singular near $x = x_n$ is given by Costin and Costin in [CC01], but we will give the proof for completeness. Since we assume F_0 to be singular at $\xi = \xi_0$, lemma 3.5.8 implies that there exists a circle $\mathcal{C}$ around ξ_0 and a function g, holomorphic in $\Delta(\xi_0, r)$ for some $r > 0$, such that $\oint_{\mathcal{C}} F_0(\xi)g(\xi)\mathrm{d}\xi = 1$. Using that $\xi'(x)$ can be written as $(C_1'(x)/C_1(x) - \mu_1)\xi(x)$, together with the asymptotic behaviour of C_1, we easily deduce that $\xi'(x)/\xi(x) = 2\pi i h_{1-} - \mu_1 + O(x^{-1})$ as $\Im x \to \infty$. Then using the preceding theorem we have for large x_n

$$\begin{aligned}
\oint_{\xi^{-1}(\mathcal{C})} y(x)\, g(\xi(x))\, \xi(x)\, \mathrm{d}x &= \oint_{\mathcal{C}} \big(F_0(\xi) + O(x^{-1})\big)\, g(\xi)\, \frac{\xi(x)}{\xi'(x)}\, \mathrm{d}\xi \\
&= \oint_{\mathcal{C}} \big(F_0(\xi) + O(x^{-1})\big)\, g(\xi)\, \big((2\pi i h_{1-} - \mu_1)^{-1} + O(x^{-1})\big)\, \mathrm{d}\xi \\
&= (2\pi i h_{1-} - \mu_1)^{-1} + O(x^{-1}) \neq 0.
\end{aligned}$$

It follows from lemma 3.5.8 that for large enough x_n the function y is not holomorphic inside $\xi^{-1}(\mathcal{C})$. Since the radius of $\mathcal{C}$ can be taken $o(1)$, theorem 3.5.7 follows. ■

Chapter 4

Extension to Nilpotent Cases

In chapter 2 we made the assumption that the matrix Λ, defined in (2.1.2), is in diagonal form. In this chapter we will see what happens when we take the $\mathtt{M}_j$'s, appearing in (2.1.2), to be more general $n_j \times n_j$-matrices. For each $j \in \{1, 2, \ldots, r\}$ we assume that two eigenvalues of $\mathtt{M}_j$ do not differ by a nonzero integer. Apart from this assumption this is the most general form for the matrices $\mathtt{M}_j$. In this chapter we will give the generalisation of the formal and analytic reduction to the normal and the semi-canonical form respectively, for both differential and difference equations.

4.1 Introduction

In this chapter we study both

$$y'(x) + \Lambda(x)y(x) + g(x, y(x)) = 0, \quad \text{with } \Lambda(x) = \bigoplus_{j=1}^{r} (\mu_j \mathtt{I}_{n_j} - x^{-1}\mathtt{M}_j) \tag{4.1.1}$$

and

$$y(x+1) = \Lambda(x)y(x) + g(x, y(x)), \quad \text{with } \Lambda(x) = \bigoplus_{j=1}^{r} e^{-\mu_j}(1 + x^{-1})^{\mathtt{M}_j}, \tag{4.1.2}$$

where $r \in \mathbb{N}_+$, $n = n_1 + n_2 + \cdots + n_r$, $n_j \in \mathbb{N}$, and $\mathtt{M}_j$ an $n_j \times n_j$-matrix with eigenvalues a_m, $m = n_1 + \cdots + n_{j-1} + 1, \ \ldots \ , \ n_1 + \cdots + n_j$.

We assume g to be a holomorphic $\mathbb{C}^n$-valued function of (x, y) in a neighbourhood of $(\infty, 0)$, such that $g(x, y) = O(x^{-2}) + O(|y|^2)$ as $x \to \infty$ and $y \to 0$. If for $j \in \{1, 2, \ldots, r\}$ we define $\mathcal{J}_j := \{n_1 + \cdots + n_{j-1} + 1, \ldots, n_1 + \cdots + n_j\}$, then we assume $a_k - a_l \notin \mathbb{Z}\backslash\{0\}$ if $k, l \in \mathcal{J}_j$, $j = 1, 2, \ldots, r$. Moreover we assume the μ_j's to be complex numbers such that for every $(k_1, k_2, \ldots, k_r) \in \mathbb{N}^r\backslash\{\mathbf{e}_j\}$

$$\begin{cases} \mu_j \neq k_1\mu_1 + k_2\mu_2 + \cdots + k_r\mu_r & \text{in the differential case,} \\ \mu_j \not\equiv (k_1\mu_1 + k_2\mu_2 + \cdots + k_r\mu_r) \bmod 2\pi i & \text{in the difference case.} \end{cases}$$

In the case of difference equations we assume in addition the existence of a positive integer r_1 such that

$$\Re\mu_j > 0,\ j \in \{1, 2, \ldots, r_1\} \quad \text{and} \quad \Re\mu_j \leqslant 0,\ j \in \{r_1 + 1, r_1 + 2, \ldots, r\}.$$

For $j \in \{1, 2, \ldots, r\}$ we define Q_j an $n_j \times n_j$ invertible matrix that transforms $\mathtt{M}_j$ into its Jordan canonical form: $Q_j \mathtt{M}_j Q_j^{-1} = \operatorname{diag}\{a_m,\ m \in \mathcal{J}_j\} + \mathtt{N}_{n_j}$, where $\mathtt{N}_{n_j}$ is an $n_j \times n_j$ nilpotent matrix of the form $(b_{k,l}^{(j)})$, $1 \leqslant k, l \leqslant n_j$, $b_{k,k+1}^{(j)} \in \{0, 1\}$ for $1 \leqslant k \leqslant n_j - 1$ and $b_{k,l}^{(j)} = 0$ otherwise[1]. If $Q := \oplus_{j=1}^r Q_j$, then through the transformation $\tilde{y} := Q\, y$ we see that the equations (4.1.1) and (4.1.2) are transformed into similar equations but now with

$$\mathtt{M}_j = \operatorname{diag}\{a_m,\ m \in \mathcal{J}_j\} + \mathtt{N}_{n_j}.$$

To prove this one should observe that $\tilde{y}' = Q\, y'$ and that if Λ is as in (4.1.2), then $Q\Lambda(x)Q^{-1} = \oplus_{j=1}^r e^{-\mu_j}(1 + x^{-1})^{Q_j \mathtt{M}_j Q_j^{-1}}$.

Like in chapter 2 we easily deduce the existence of a unique formal series solution $\hat{y}_0(x) := \sum_{m=1}^\infty y_{0,m} x^{-m} \in x^{-1}\mathbb{C}^n[[x^{-1}]]$ of (4.1.1) and again α_1 turns out to be equal to 0. So in fact $\hat{y}_0 \in x^{-2}\mathbb{C}^n[[x^{-1}]]$. Also the difference equation (4.1.2) possesses a (different) unique formal series solution $\hat{y}_0 \in x^{-2}\mathbb{C}^n[[x^{-1}]]$. Our aim in this chapter is to study the transformation

$$y = \hat{T}(x, z) := \sum_{\mathbf{k} \in \mathbb{N}^n} \hat{y}_{\mathbf{k}}(x)\, z^{\mathbf{k}}$$

associated with the formal solution $\hat{y}_0$, which (at first formally, in the sense as described in the introduction of chapter 1) transforms the differential equation (4.1.1) or the difference equation (4.1.2) into a corresponding linear equation. In the differential case this linearised equation reads

$$z'(x) + \Lambda(x) z(x) = 0. \tag{4.1.3}$$

However, in the case of difference equations we first reduce (4.1.2) into a linear equation of the form

$$\vartheta(x + 1) = \Lambda(x)(1 + x^{-1})^{-\mathtt{N}}\big(\mathtt{I} + x^{-1}\mathtt{N}\big)\vartheta(x), \tag{4.1.4}$$

where $\mathtt{I} = \mathtt{I}_n$ is the $n \times n$ identity matrix and $\mathtt{N} := \oplus_{j=1}^r \mathtt{N}_{n_j}$. With this latter form, which is quite close to the normal form we are used to, it is more easy to see what the equations for the formal series appearing in the transformation $\hat{T}$ look like (compare section 4.5.1). Besides this formal transformation, the analytic analogue will be discussed. Also the correspondence with solutions $y(x)$ that behave like $\hat{y}_0(x)$ as $x \to \infty$ in certain sectors will be studied.

After the linearisation to an equation corresponding to (4.1.4) on a sub-manifold in the case of difference equations, we further reduce to a semi-canonical form corresponding to the normal form

$$z(x + 1) = \Lambda(x) z(x). \tag{4.1.5}$$

[1]So each $\mathtt{M}_j$ is a direct sum with the number of terms equal to the number of different eigenvalues of $\mathtt{M}_j$. In particular, if all eigenvalues have multiplicity one, then each $\mathtt{N}_{n_j}$ equals the $n_j \times n_j$-matrix with all entries equal to zero. If $b_{k,k+1}^{(j)} = 1$, then $a_{n_1+\cdots+n_{j-1}+k} = a_{n_1+\cdots+n_{j-1}+k+1}$.

Since both equations are linear, there only is a linear transformation involved between these two equations (compare also section 4.5.4).

Moreover, in section 4.4 we will derive a simple resurgence relation in the case of the differential equations, while in section 4.5.5 its analogue for the difference equations will be deduced. These relations are generalisations of those found in section 2.4.2.

4.2 Formal Reduction in the Differential Case

We will use the following decomposition of multi-indices $\mathbf{k} \in \mathbb{N}^n$ in r 'sub multi-indices':

$$\mathbf{k} = (\mathbf{k}_1, \mathbf{k}_2, \ldots, \mathbf{k}_r), \tag{4.2.1}$$

where $\mathbf{k}_j$, $j = 1, 2, \ldots, r$, equals $\mathbf{k}_j = \sum_{h \in \mathcal{J}_j} \langle \mathbf{k}, \mathbf{e}_h \rangle \mathbf{e}_h$. Moreover, we will use the notation $z = (z^{[1]}, z^{[2]}, \ldots, z^{[r]})^t$ to denote a corresponding decomposition for vectors $z \in \mathbb{C}^n$.

One should observe that the normal form (4.1.3) has as general holomorphic solution in the class of holomorphic functions on the Riemann surface of the logarithm

$$z(x) = \oplus_{j=1}^{r} e^{-\mu_j x} x^{\mathtt{M}_j} C^{[j]}, \quad \mathtt{M}_j = \operatorname{diag}\{a_m,\ m \in \mathcal{J}_j\} + \mathtt{N}_{n_j}, \tag{4.2.2}$$

with $C \in \mathbb{C}^n$ an arbitrary constant vector. So we obtain the following transseries of the differential equation (4.1.1)

$$\hat{y}(x) = \sum_{\mathbf{k} \in \mathbb{N}^n} e^{-\langle \mathbf{k}, \boldsymbol{\mu} \rangle x}\, x^{\langle \mathbf{k}, \mathbf{a} \rangle}\, (x^{\mathtt{N}} C)^{\mathbf{k}} \hat{y}_{\mathbf{k}}(x), \quad \mathtt{N} = \oplus_{j=1}^{r} \mathtt{N}_j,$$

where $\boldsymbol{\mu} = \sum_{j=1}^{r} \mu_j \sum_{h \in \mathcal{J}_j} \mathbf{e}_h$ and $\mathbf{a} = (a_1, a_2, \ldots, a_n)$.

Moreover, if (4.1.3) holds, then for $\mathbf{k} \in \mathbb{N}^n$ we have

$$\frac{\mathrm{d}}{\mathrm{d}x} z^{\mathbf{k}}(x) = \sum_{j=1}^{r} \prod_{h \neq j} \big(z^{[h]}(x)\big)^{\mathbf{k}_h} \frac{\mathrm{d}}{\mathrm{d}x} \big\{ \big(z^{[j]}(x)\big)^{\mathbf{k}_j} \big\} = \sum_{j=1}^{r} \sum_{h \in \mathcal{J}_j} k_h\, z_h'(x)\, z^{\mathbf{k} - \mathbf{e}_h}(x).$$

For $j \in \{1, 2, \ldots, r\}$ the function $z^{[j]}$ satisfies $w'(x) + \big(\mu_j \mathtt{I}_{n_j} - x^{-1} \mathtt{M}_j\big) w(x) = 0$. So if we define $\mathcal{J}_j' := \{n_1 + \cdots + n_{j-1} + 1, \ldots, n_1 + \cdots + n_j - 1\} = \mathcal{J}_j \backslash \{n_1 + \cdots + n_j\}$, then the latter expression can be written as

$$(-\langle \mathbf{k}, \boldsymbol{\mu} \rangle + \langle \mathbf{k}, \mathbf{a} \rangle x^{-1}) z^{\mathbf{k}}(x) \;+\; \sum_{j=1}^{r} \sum_{h \in \mathcal{J}_j'} \alpha_h\, k_h\, x^{-1} z^{\mathbf{k} - \mathbf{e}_h + \mathbf{e}_{h+1}}(x),$$

where $\alpha_h = b^{(j)}_{h - n_1 - \cdots - n_{j-1}, h+1-n_1-\cdots-n_{j-1}}$, the element of $\mathtt{N}_{n_j}$ at the $(h - n_1 - \cdots - n_{j-1})^{\text{th}}$ row and $(h + 1 - n_1 - \cdots - n_{j-1})^{\text{th}}$ column (cf. the introduction of this chapter). Hence, the requirement that $y = \hat{y}_0(x) + \sum_{\mathbf{k} \in \mathbb{N}_1^n} \hat{y}_{\mathbf{k}}(x) z^{\mathbf{k}}$ reduces (4.1.1) to its normal form is equivalent to the condition that $\hat{y}_{\mathbf{k}}$, $\mathbf{k} \in \mathbb{N}_1^n$, is a formal solution of the following linear equation

$$y'(x) + \big(\Lambda_1(x) - \langle \mathbf{k}, \boldsymbol{\mu} \rangle + \langle \mathbf{k}, \mathbf{a} \rangle x^{-1}\big)\, y(x) + t_{\mathbf{k}}(x) + \\ + \sum_{j=1}^{r} \sum_{\substack{h \in \mathcal{J}_j' \\ k_{h+1} \neq 0}} \alpha_h\,(k_h + 1)\, x^{-1}\, \hat{y}_{\mathbf{k}+\mathbf{e}_h-\mathbf{e}_{h+1}}(x) = 0, \qquad (4.2.3)$$

where Λ_1 and $t_{\mathbf{k}}$ are defined in (2.2.3) and (2.2.5) respectively.

We will show that the equations (4.2.3) possess formal solutions, by means of induction on $\mathbf{k}$. To that end we have to make sure that for a fixed multi-index $\mathbf{k}$ the expressions $\hat{y}_{\mathbf{k}+\mathbf{e}_h-\mathbf{e}_{h+1}}$, $h \in \mathcal{J}_j'$, $k_{h+1} \neq 0$, $j \in \{1, 2, \ldots, r\}$, are known. In other words: we need to have an ordering on multi-indices in $\mathbb{N}^n$ such that for a fixed multi-index $\mathbf{k}$, the multi-indices $\mathbf{k} + \mathbf{e}_h - \mathbf{e}_{h+1}$, $h \in \mathcal{J}_j'$, $k_{h+1} \neq 0$, $j \in \{1, 2, \ldots, r\}$, are smaller than $\mathbf{k}$ with respect to that ordering.

Definition 4.2.1 *Let $p \in \mathbb{N}$ and suppose that $\mathbf{k}, \mathbf{l} \in \mathbb{N}^n$ have length p. Then $\mathbf{l} \prec_p \mathbf{k}$ if and only if there exists a $j \in \{1, 2, \ldots, n\}$ such that $l_j < k_j$, while $l_i = k_i$ for all $i \in \{j+1, j+2, \ldots, n\}$.*

Example 4.2.2 If $n = 3$, then the multi-indices in $\mathbb{N}^3$ of length p, $p \in \mathbb{N}$, are ordered with respect to $\prec_p$ as follows:

$$(p,0,0) \prec_p (p-1,1,0) \prec_p (p-2,2,0) \prec_p \cdots \prec_p (0,p,0) \prec_p (p-1,0,1) \prec_p \\ \prec_p (p-2,1,1) \prec_p \cdots\cdots \prec_p (2,0,p-2) \prec_p (1,1,p-2) \prec_p (0,2,p-2) \prec_p \\ \prec_p (1,0,p-1) \prec_p (0,1,p-1) \prec_p (0,0,p).$$

Definition 4.2.3 *Let $r \in \mathbb{N}_+$ and fix $\mathbf{p} := (p_1, p_2, \ldots, p_r) \in \mathbb{N}^r$. Let $\mathbf{k}$ and $\mathbf{l}$ be multi-indices in $\mathbb{N}^n$ such that $|\mathbf{k}_j| = |\mathbf{l}_j| = p_j$ for all $j \in \{1, 2, \ldots, r\}$. Then $\mathbf{l} \prec_{\mathbf{p}} \mathbf{k}$ if and only if there exists a $j \in \{1, 2, \ldots, r\}$ such that $\mathbf{l}_j \prec_{p_j} \mathbf{k}_j$ (see the definition above), while $\mathbf{l}_i = \mathbf{k}_i$ for all $i \in \{j+1, j+2, \ldots, r\}$.*

Example 4.2.4 Let $r = 3$, $\mathbf{p} = (2, 1, 1)$ and suppose that $n_1 = n_2 = n_3 = 2$, then with respect to $\prec_{\mathbf{p}}$ the following multi-indices in $\mathbb{N}^6$ are arranged in increasing order:

$$(2,0,1,0,1,0),\ \ (1,1,1,0,1,0),\ \ (0,2,1,0,1,0),\ \ (2,0,0,1,1,0),\ \ (1,1,0,1,1,0), \\ (0,2,0,1,1,0),\ \ (2,0,1,0,0,1),\ \ (1,1,1,0,0,1),\ \ (0,2,1,0,0,1),\ \ (2,0,0,1,0,1), \\ (1,1,0,1,0,1),\ \ (0,2,0,1,0,1).$$

Now, with a multi-index $\mathbf{k} \in \mathbb{N}^n$ we associate $\mathbf{p}_{\mathbf{k}} := (|\mathbf{k}_1|, |\mathbf{k}_2|, \ldots, |\mathbf{k}_r|) \in \mathbb{N}^r$ (see also the proof of proposition 2.6.4). Then for a fixed $\mathbf{p} \in \mathbb{N}^r$ one easily verifies that if $\mathbf{p}_{\mathbf{k}} = \mathbf{p}$ for some $\mathbf{k} \in \mathbb{N}^n$, then $\mathbf{p}_{\mathbf{k}+\mathbf{e}_h-\mathbf{e}_{h+1}} = \mathbf{p}$ and $\mathbf{k} + \mathbf{e}_h - \mathbf{e}_{h+1} \prec_{\mathbf{p}} \mathbf{k}$ for all $h \in \mathcal{J}_j'$ with $k_{h+1} \neq 0$ and $j \in \{1, 2, \ldots, r\}$.

Proposition 4.2.5 *For each* $\mathbf{k} \succ 0$ *the differential equation (4.2.3) possesses a formal solution* $\hat{y}_{\mathbf{k}} \in \mathbb{C}^n[[x^{-1}]]$. *For* $j \in \{1, 2, \ldots, n\}$ *the series* $\hat{y}_{\mathbf{e}_j}$ *is unique if we prescribe the constant term in* $\hat{y}_{\mathbf{e}_j}$ *to be equal to* $\mathbf{e}_j$. *Then also the series* $\hat{y}_{\mathbf{k}}$ *for* $|\mathbf{k}| > 1$ *are unique.*

PROOF. The proof is given with 'double' induction. First we use induction on the length of $\mathbf{p} \in \mathbb{N}_1^r$ and at each step of this induction procedure we use induction on the set $\{\mathbf{k} \in \mathbb{N}^n \mid \mathbf{p_k} = \mathbf{p}\}$ with respect to $\prec_{\mathbf{p}}$. If $|\mathbf{p}| = 1$ one may construct formal solutions as follows.

Suppose that $\mathtt{M}_1$ contains a Jordan block J_q of order $q \in \{1, 2, \ldots, n_1\}$ such that J_q is the sub-matrix of $\oplus_{j=1}^r \mathtt{M}_j$ formed by the elements in the rows and columns with index $1, 2, \ldots, q$. So $a_1 = a_2 = \cdots = a_q := a$ and thus J_q can be written as $J_q = a\mathtt{I}_q + \mathtt{N}_q$, with $\mathtt{N}_q$ the $q \times q$ nilpotent matrix with ones on the first super diagonal and zeros elsewhere. For $\mathbf{k} = \mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_q$ we write $\hat{y}_{\mathbf{k}}(x) := \sum_{m=0}^{\infty} y_{\mathbf{k},m} x^{-m}$ and we define $\gamma_{\mathbf{k},m}$ to be the vector consisting of the components of $y_{\mathbf{k},m}$ in the first till q^{th} row. From (4.2.3) we deduce that $\hat{y}_{\mathbf{e}_1}$ is a formal solution of

$$y'(x) + \big(\Lambda(x) - \mu_1 + a_1 x^{-1}\big) y(x) + \sum_{j=1}^{n} d_{\mathbf{e}_j}(x) y_j(x) = 0, \tag{4.2.4}$$

while $\hat{y}_{\mathbf{e}_h}$, $h = 2, 3, \ldots, q$, is a formal solution of

$$y'(x) + \big(\Lambda(x) - \mu_1 + a_1 x^{-1}\big) y(x) + \sum_{j=1}^{n} d_{\mathbf{e}_j}(x) y_j(x) + x^{-1} \hat{y}_{\mathbf{e}_{h-1}}(x) = 0. \tag{4.2.5}$$

Substitution of the series for $\hat{y}_{\mathbf{k}}$, $\mathbf{k} = \mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_q$, in the corresponding differential equations and considering coefficients of x^{-1} we obtain

$$\begin{cases} \mathtt{N}_q \gamma_{\mathbf{e}_1,0} &= 0 \\ \mathtt{N}_q \gamma_{\mathbf{e}_h,0} &= \gamma_{\mathbf{e}_{h-1},0}, \qquad h = 2, 3, \ldots, q. \end{cases}$$

Now the first equation corresponding to $\mathbf{k} = \mathbf{e}_1$ has a solution $\gamma_{\mathbf{e}_1,0} = \mathbf{e}_1 \in \mathbb{C}^q$ and the equation corresponding to $h = 2, 3, \ldots, q$ has a solution $\gamma_{\mathbf{e}_h,0} = \mathbf{e}_h$ as can be verified by recursion. The equations obtained from substitution of the formal series in the corresponding differential equations and comparing coefficients of x^{-m-1}, $m \geqslant 1$, may be solved recursively, using that $\mathtt{N}_q + m\mathtt{I}_q$ is invertible. In this manner and using the assumptions $\mu_j \neq \mu_1$ for any $j = 2, 3, \ldots, r$ and $a_k - a_1 \notin \mathbb{Z}$ for any $k \in \{q+1, q+2, \ldots, n_1\}$, we may verify that we can construct formal solutions $\hat{y}_{\mathbf{k}}$, $\mathbf{k} = \mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_q$. Repeating this procedure for each Jordan block present in $\oplus_{j=1}^r \mathtt{M}_j$ one may derive formal solutions $\hat{y}_{\mathbf{k}}$, $|\mathbf{k}| = 1$, in case we choose the constant term in this series equal to $\mathbf{k}$.

Next we take a multi-index $\mathbf{p} \in \mathbb{N}^r$ of length $\ell \geqslant 2$ and fix some $\mathbf{k}$ with $\mathbf{p_k} = \mathbf{p}$. Suppose that we have found $\hat{y}_{\mathbf{k}'}$ for those $\mathbf{k}'$ with $|\mathbf{k}'| < |\mathbf{k}|$ and those $\mathbf{k}'$ with $\mathbf{p}_{\mathbf{k}'} = \mathbf{p}$, but smaller than $\mathbf{k}$ with respect to $\prec_{\mathbf{p}}$. Then the equation for $\hat{y}_{\mathbf{k}}$ can be written as

$$y'(x) + \big(\Lambda(x) - \langle \mathbf{k}, \boldsymbol{\mu} \rangle + \langle \mathbf{k}, \mathbf{a} \rangle x^{-1}\big) y(x) + \sum_{j=1}^{n} d_{\mathbf{e}_j}(x) y_j(x) = \textit{known},$$

where *known* means some series in $\mathbb{C}^n[[x^{-1}]]$ with known coefficients. Now substituting a formal series, deriving a recursive relation for the coefficients in this series and using the assumption that $\mu_j \neq \langle \mathbf{k}, \boldsymbol{\mu} \rangle$ if $\mathbf{k} \in \mathbb{N}_2^n$ for all $j \in \{1, 2, \ldots, r\}$, one easily verifies that the coefficients can be determined uniquely. ■

4.3 Analytic Reduction in the Differential Case

In this section we will follow the outline given in section 2.3. We first prove that each $\hat{y}_{\mathbf{k}}$, $\mathbf{k} \in \mathbb{N}^n$, is Borel summable in all but countably many directions. After that we will show that we can reduce (4.1.1) to a semi-canonical form, in a similar way as Braaksma did in [Bra01].

4.3.1 Borel Summability of $\hat{y}_{\mathbf{k}}$

Taking a formal Borel transform of (4.1.1) we obtain

$$(t - \Lambda_0) Y_0 = A * Y_0 + D_0(Y_0), \tag{4.3.1}$$

where $Y_0 = \hat{\mathcal{B}}\hat{y}_0$, $\Lambda_0 = \oplus_{j=1}^r \mu_j \mathtt{I}_{n_j}$, $A = -\oplus_{j=1}^r \mathtt{M}_j$ and $D_0(Y_0)$ is as in (2.3.3). The proof that proposition 2.3.1 remains valid with $l = 0$ follows from [Bra80]: first one solves (4.3.1) in a full neighbourhood of the origin (cf. section 4.3 in [Bra80]), which implies that $Y_0 = \hat{\mathcal{B}}\hat{y}_0$ converges. Then if $\theta \neq \arg \mu_j$, $j = 1, 2, \ldots, r$, one constructs, as in section 4.4 of that paper, a unique analytic continuation of Y_0 in a small sector with bisecting direction θ not containing any singular direction $\arg \mu_j$, $j = 1, 2, \ldots, r$. Moreover, as in section 4.5 in [Bra80] it can be shown that $Y_0(t) = O(1)e^{M|t|}$ on every closed sub-sector, for some positive M, which may depend on the choice of the closed sub-sector. We conclude that $\hat{y}_0$ is Borel summable in every direction $-\theta$, $\theta \neq \arg \mu_j$, $j = 1, 2, \ldots, r$.

As in section 2.3 we put $\hat{w}_{\mathbf{k}}(x) := x^{-|\mathbf{k}|}\hat{y}_{\mathbf{k}}(x)$, $\mathbf{k} \succ 0$, and we see that $\hat{w}_{\mathbf{k}}$ is a formal solution of

$$w'(x) + \left(\Lambda_1(x) - \langle \mathbf{k}, \boldsymbol{\mu} \rangle + \langle \mathbf{k}, \mathbf{b} \rangle x^{-1}\right) w(x) + u_{\mathbf{k}}(x) + \\ + \sum_{j=1}^{r} \sum_{\substack{h \in \mathcal{J}_j' \\ k_{h+1} \neq 0}} \alpha_h \left(k_h + 1\right) x^{-1}\, \hat{w}_{\mathbf{k}+\mathbf{e}_h - \mathbf{e}_{h+1}}(x) = 0, \tag{4.3.2}$$

where $\mathbf{b} := \mathbf{a} + (1, 1, \ldots, 1)$ and $u_{\mathbf{k}}(x) := x^{-|\mathbf{k}|} t_{\mathbf{k}}(x)$. Taking a formal Borel transform we obtain the following convolution equation for $W_{\mathbf{k}} = \hat{\mathcal{B}}\hat{w}_{\mathbf{k}}$:

$$(t + \langle \mathbf{k}, \boldsymbol{\mu} \rangle - \Lambda_0) W_{\mathbf{k}} \;=\; (B + \langle \mathbf{k}, \mathbf{b} \rangle) * W_{\mathbf{k}} \;+\; U_{\mathbf{k}} \;+\; \sum_{j=1}^{r} \sum_{\substack{h \in \mathcal{J}_j' \\ k_{h+1} \neq 0}} \alpha_h \left(k_h + 1\right) * W_{\mathbf{k}+\mathbf{e}_h - \mathbf{e}_{h+1}}.$$

Here $U_{\mathbf{k}}$ is as in (2.3.9) and $B = -\oplus_{j=1}^r \mathtt{M}_j + D(Y_0)$ with $D(Y_0)$ the $n \times n$-matrix with columns $D_{\mathbf{e}_j}(Y_0)$, $j = 1, 2, \ldots, n$ (cf. (2.3.3)).

Proposition 4.3.1 *Let* $\mathbf{k} \in \mathbb{N}^n$, $\mathbf{k} \neq 0$. *Then* $t \mapsto t^{-|\mathbf{k}|+1} W_{\mathbf{k}}(t)$ *exists and is holomorphic in the maximal star domain with centre* 0, *which does not contain any of the singular points* $\mu_j - \langle \mathbf{k}', \boldsymbol{\mu} \rangle \neq 0$ *where* $j \in \{1, 2, \ldots, r\}$ *and* $\mathbf{k}' \in \mathbb{N}^n$ *with* $\mathbf{k}' \preceq \mathbf{k}$ *or* $\mathbf{k}' \prec_{\mathbf{p_k}} \mathbf{k}$ [2]. *If* $\overline{S}$ *is a closed sector with vertex* 0, *not containing any of those singular points, then there exists a positive constant* M, *which may depend on* $\mathbf{k}$ *and* $\overline{S}$, *such that* $\sup_{t \in \overline{S}} e^{-M|t|} |W_{\mathbf{k}}(t)| < \infty$.

PROOF. We give the proof of this proposition using the 'double' induction method described in proposition 4.2.5. If $\mathbf{p} \in \mathbb{N}^r$ has length 1, say $\mathbf{p} = \mathbf{e}_1$, then the differential equation for $\hat{w}_{\mathbf{e}_1}$ and the corresponding convolution equation are homogeneous and the assertions follow in this case from [Bra80]. If $\mathbf{k} = \mathbf{e}_2$, then we have the same convolution equation, with the inhomogeneous term $\alpha_1 * W_{\mathbf{e}_1}$ added. This inhomogeneous term is holomorphic in the maximal star domain with centre 0, not containing any of the singular points μ_j and $\mu_j - \mu_1 \neq 0$, $j \in \{1, 2, \ldots, r\}$. Again the assertions follow from [Bra80]. We can continue in this way, showing the proposition to be true for every $\mathbf{k} \in \mathbb{N}^n$ with length 1.

Next we take a multi-index $\mathbf{p} \in \mathbb{N}^r$ of length $\ell \geqslant 2$ and fix some $\mathbf{k}$ with $\mathbf{p_k} = \mathbf{p}$. Suppose that we have proved the assertions in the proposition for $W_{\mathbf{k}'}$ for those $\mathbf{k}'$ with $|\mathbf{k}'| < |\mathbf{k}|$ and those $\mathbf{k}'$ with $\mathbf{p}_{\mathbf{k}'} = \mathbf{p}$ and $\mathbf{k}' \prec_{\mathbf{p}} \mathbf{k}$. Since $U_{\mathbf{k}}$ only involves $W_{\mathbf{k}'}$'s with $\mathbf{k}' \prec \mathbf{k}$, we see that $t \mapsto t^{-|\mathbf{k}|+1} U_{\mathbf{k}}(t)$ is holomorphic in the star domain mentioned in the proposition and obviously the same holds for $t \mapsto t^{-|\mathbf{k}|+1} \sum_{j=1}^{r} \sum_{h \in \mathcal{J}_j', k_{h+1} \neq 0} \alpha_h \, (k_h+1) * W_{\mathbf{k}+\mathbf{e}_h-\mathbf{e}_{h+1}}(t)$. Now we are ready to apply the results from [Bra80] again and the assertions in the proposition follow. ∎

4.3.2 Estimates for $W_{\mathbf{k}}$

Let $\theta \in \mathbb{R}$ and define $i(\theta) := \big\{ j \in \{1, 2, \ldots, r\} \;\big|\; \Re(\mu_j e^{-i\theta}) > 0 \big\}$. Moreover, define $I(\theta) := \{ \mathbf{k} \in \mathbb{N}^n \mid \mathbf{k}_j = 0 \text{ if } j \notin i(\theta) \}$. With these definitions there are only finitely many singular points of $\hat{y}_{\mathbf{k}}$, $\mathbf{k} \in I(\theta)$, in the half plane $H(\theta) := \{ t \in \mathbb{C} \mid \Re(te^{-i\theta}) > 0 \}$ and it makes sense to consider two consecutive singular directions of $\hat{y}_{\mathbf{k}}$, $\mathbf{k} \in I(\theta)$, in $H(\theta)$. With a direction $\theta \in \mathbb{R}$ we associate two such consecutive singular directions θ_- and θ_+ in $H(\theta)$.

Lemma 4.3.2 *Let* $S := \{ t \in \mathbb{C}^* \mid \arg t \in (\theta_-, \theta_+) \}$ *and let* $\overline{S'}$ *be a closed sub-sector of* S. *Moreover, let* $0 < \rho < \min\big\{ |\mu_j - \langle \mathbf{k}, \boldsymbol{\mu} \rangle| \neq 0 \;\big|\; j = 1, 2, \ldots, r, \; \mathbf{k} \in I(\theta) \big\}$ *and define* $V = \overline{S'} \cup \overline{\Delta}(0, \rho)$. *Then there exist positive constants* K_1 *and* c_1 *such that*

$$|B(t)| \leqslant K_1 \, e^{c_1 |t|} \quad \textit{and} \quad |D_{\mathbf{j}}(Y_0)(t)| \leqslant K_1^{|\mathbf{j}|+1} \, e^{c_1 |t|}, \tag{4.3.3}$$

for all $t \in V$ *and all* $\mathbf{j} \in \mathbb{N}_1^n$.

PROOF. Since $B = -\oplus_{j=1}^{r} \mathtt{M}_j + D(Y_0)$ the first estimate follows from the second one by enlarging K_1, so it is sufficient to prove the estimate for $|D_{\mathbf{j}}(Y_0)|$. Now remember that

[2] In this case we assume that $\mathbf{p}_{\mathbf{k}'} = \mathbf{p_k}$.

$D_{\mathbf{j}}(Y_0) = G_{\mathbf{j}} + \sum_{\mathbf{l}\succ\mathbf{j}} \binom{\mathbf{l}}{\mathbf{j}}(g_{\mathbf{l},0} + G_{\mathbf{l}}*)Y_0^{*(\mathbf{l}-\mathbf{j})}$. From the Borel summability of $\hat{y}_0$ we deduce the existence of positive constants K_2 and c_2 such that $|Y_0(t)| \leqslant K_2\, e^{c_2|t|}$ for all $t \in V$. Then with induction one can prove that $\left|Y_0^{*\mathbf{l}}(t)\right| \leqslant K_2^{|\mathbf{l}|}\, \frac{|t|^{|\mathbf{l}|-1}}{(|\mathbf{l}|-1)!}\, e^{c_2|t|}$, for all $t \in V$ and all $\mathbf{l} \in \mathbb{N}_1^n$. By enlarging K_2 and c_2 if necessary we may also assume that $|g_{\mathbf{l},0}| \leqslant K_2^{|\mathbf{l}|+1}$ and $|G_{\mathbf{l}}(t)| \leqslant K_2^{|\mathbf{l}|+1}\, e^{c_2|t|}$ (compare (2.3.4)). From this we deduce that if $\mathbf{j} \in \mathbb{N}_1^n$ and $\mathbf{l} \succ \mathbf{j}$, then

$$|G_{\mathbf{l}} * Y_0^{*(\mathbf{l}-\mathbf{j})}(t)| \leqslant K_2^{2|\mathbf{l}|+1-|\mathbf{j}|}\, \frac{|t|^{|\mathbf{l}|-|\mathbf{j}|}}{(|\mathbf{l}|-|\mathbf{j}|)!}\, e^{c_2|t|},$$

for all $t \in V$. Using this estimate it is easy to prove (4.3.3). ■

Proposition 4.3.3 *Let V be as in the preceding lemma. Then there exist positive constants R and K such that*

$$|W_{\mathbf{k}}(t)| \leqslant K^{|\mathbf{k}|}\, \frac{|t|^{|\mathbf{k}|-1}}{(|\mathbf{k}|-1)!}\, e^{R|t|}$$

for all $\mathbf{k} \in I(\theta)\backslash\{0\}$ and all $t \in V$.

Proof. First observe that the equation for $W_{\mathbf{k}}$ can more conveniently be written as $W = Q_{\mathbf{k}}W$, with

$$\begin{aligned} Q_{\mathbf{k}}W = {} & (t + \langle \mathbf{k}, \boldsymbol{\mu}\rangle - \Lambda_0)^{-1}\big\{(B + \langle \mathbf{k}, \mathbf{b}\rangle) * W + U_{\mathbf{k}}\big\} + \\ & (t + \langle \mathbf{k}, \boldsymbol{\mu}\rangle - \Lambda_0)^{-1} \sum_{j=1}^{r} \sum_{\substack{h\in\mathcal{J}_j' \\ k_{h+1}\neq 0}} \alpha_h\,(k_h+1) * W_{\mathbf{k}+\mathbf{e}_h-\mathbf{e}_{h+1}}. \end{aligned} \tag{4.3.4}$$

Now as in [Bra01] we define for $\ell \in \mathbb{N}_+$ and $R > 0$ the Banach space $\mathcal{V}_{\ell,R}$ consisting of continuous functions $W : V \to \mathbb{C}^n$ which are holomorphic in the interior of V, such that $\|W\|_{\ell,R} := \sup_{t\in V} |W(t)|/\zeta_{\ell,R}(t) < \infty$, where $\zeta_{\ell,R}(t) := \frac{|t|^{\ell-1}}{(\ell-1)!}\, e^{R|t|}$. If $|\mathbf{k}| = 1$, then the Borel summability of $\hat{w}_{\mathbf{k}}$ implies the existence of a positive M such that $W_{\mathbf{k}} \in \mathcal{V}_{1,M}$ (cf. proposition 4.3.1).

For $\mathbf{k} \in I(\theta)$, $|\mathbf{k}| \geqslant 2$, we will use the 'double' induction described in proposition 4.2.5 to solve the convolution equation for $W_{\mathbf{k}}$ in $\mathcal{V}_{|\mathbf{k}|,R}$ for some suitable $R \geqslant M$. From the definition of V it follows that there exists a positive constant K_0 such that

$$\left|(t + \langle \mathbf{k}, \boldsymbol{\mu}\rangle - \Lambda_0)^{-1}\right| \leqslant K_0\, |\mathbf{k}|^{-1}$$

for all $t \in V$, $\mathbf{k} \in I(\theta)$, $|\mathbf{k}| \geqslant 2$. Moreover, from the preceding lemma it follows that for all $R \geqslant c_1$, all integers $\ell \geqslant 2$ and all $W \in \mathcal{V}_{\ell,R}$ we have

$$|(B * W)(t)| \leqslant K_1\|W\|_{\ell,R} \int_0^t \frac{|s|^{\ell-1}}{(\ell-1)!}\, e^{c_1|t-s|+R|s|}\mathrm{d}s \leqslant \frac{K_1}{R-c_1}\|W\|_{\ell,R}\, \frac{|t|^{\ell-1}}{(\ell-1)!}\, e^{R|t|}$$

and similarly

$$|(1 * W)(t)| \leqslant \frac{\|W\|_{\ell,R}}{R}\, \frac{|t|^{\ell-1}}{(\ell-1)!}\, e^{R|t|}.$$

So there exists a positive $R_1 > \max\{c_1, M\}$ such that for all $R \geqslant R_1$, all integers $\ell > 1$, all $\mathbf{k} \in \mathbb{N}^n$ with $|\mathbf{k}| = \ell$ and all $W \in \mathcal{V}_{\ell,R}$ we have

$$\left\|(t + \langle \mathbf{k}, \boldsymbol{\mu}\rangle - \Lambda_0)^{-1}\{(B + \langle \mathbf{k}, \mathbf{b}\rangle) * W\}\right\|_{\ell,R} \leqslant \frac{1}{2}\|W\|_{\ell,R}. \tag{4.3.5}$$

In the following we choose $R \geqslant R_1 + 1$ and we consider the equations $W_{\mathbf{k}} = Q_{\mathbf{k}} W_{\mathbf{k}}$ in $\mathcal{V}_{|\mathbf{k}|,R}$. Take a multi-index $\mathbf{p} \in \mathbb{N}^r$, $|\mathbf{p}| \geqslant 2$, and fix $\mathbf{k} \in I(\theta)$ with $\mathbf{p}_{\mathbf{k}} = \mathbf{p}$. Note that this makes sense if we pick multi-indices $\mathbf{p} \in \mathbb{N}^r$ with $p_j \neq 0$ if and only if $j \in i(\theta)$. Let us denote this set of multi-indices by $I'(\theta)$. Now assume that we have found solutions $W_{\mathbf{k}'} \in \mathcal{V}_{|\mathbf{k}'|,R}$ for those $\mathbf{k}' \in I(\theta)$ with $|\mathbf{k}'| < |\mathbf{k}|$ and those $\mathbf{k}' \in I(\theta)$ with $\mathbf{p}_{\mathbf{k}'} = \mathbf{p}$ and $\mathbf{k}' \prec_{\mathbf{p}} \mathbf{k}$.

Then $t \mapsto (t + \langle \mathbf{k}, \boldsymbol{\mu}\rangle - \Lambda_0)^{-1} \sum_{j=1}^{r} \sum_{h \in \mathcal{J}_j', k_{h+1} \neq 0} \alpha_h\,(k_h + 1) * W_{\mathbf{k}+\mathbf{e}_h-\mathbf{e}_{h+1}}(t)$ belongs to $\mathcal{V}_{|\mathbf{k}|,R}$ and its norm can be estimated by

$$\frac{K_0}{R\,|\mathbf{k}|} \sum_{j=1}^{r} \sum_{h \in \mathcal{J}_j', k_{h+1} \neq 0} (k_h + 1)\|W_{\mathbf{k}+\mathbf{e}_h-\mathbf{e}_{h+1}}\|_{|\mathbf{k}|,R}.$$

In [Bra01] it is shown that $t \mapsto U_{\mathbf{k}}(t)$ belongs to $\mathcal{V}_{|\mathbf{k}|,R}$ and using the lemma above we deduce that

$$\|U_{\mathbf{k}}\|_{|\mathbf{k}|,R} \leqslant 2^n K_1 \sum_{h=2}^{|\mathbf{k}|} (2K_1)^h \sum_{(\mathbf{i}_m;\mathbf{k})} \prod_{m=1}^{h} \|W_{\mathbf{i}_m}\|_{|\mathbf{i}_m|,R},$$

where $\sum_{(\mathbf{i}_m;\mathbf{k})}$ denotes the sum over all $\mathbf{i}_m \in I(\theta)$ with $\mathbf{i}_m \succ 0$ and $\sum_{m=1}^{h} \mathbf{i}_m = \mathbf{k}$. Hence, $Q_{\mathbf{k}}$ maps $\mathcal{V}_{|\mathbf{k}|,R}$ into itself and defines a contraction on $\mathcal{V}_{|\mathbf{k}|,R}$ because of (4.3.5). Therefore there exists a unique fixed point $W_{\mathbf{k}}$ of $W = Q_{\mathbf{k}} W$ in the space $\mathcal{V}_{|\mathbf{k}|,R}$.

To prove the estimate for $W_{\mathbf{k}}$ we define for $\mathbf{p} \in I'(\theta)$, $|\mathbf{p}| = 1$,

$$f_{\mathbf{p}} := \max\{\|W_{\mathbf{k}}\|_{1,R} \mid \mathbf{k} \in I(\theta),\ \mathbf{p}_{\mathbf{k}} = \mathbf{p}\}.$$

For technical reasons (cf. (4.3.6) and (4.3.7)), we take $R \geqslant \max\{R_1 + 1, 4K_0(1+n)\}$. Now suppose that for some integer $\ell \geqslant 2$ we are given $f_{\mathbf{p}} > 0$, for all $\mathbf{p} \in I'(\theta)$ with $|\mathbf{p}| < \ell$, such that

$$\max\{\|W_{\mathbf{k}}\|_{|\mathbf{k}|,R} \mid \mathbf{k} \in I(\theta),\ \mathbf{p}_{\mathbf{k}} = \mathbf{p}\} \leqslant f_{\mathbf{p}}.$$

If $\mathbf{p}$ is a multi-index of length ℓ and $\mathbf{k}$ is a multi-index in $I(\theta)$, with $\mathbf{p}_{\mathbf{k}} = \mathbf{p}$, we have

$$\sum_{j=1}^{r} \sum_{\substack{h \in \mathcal{J}_j' \\ k_{h+1} \neq 0}} (k_h + 1)\|W_{\mathbf{k}+\mathbf{e}_h-\mathbf{e}_{h+1}}\|_{|\mathbf{k}|,R} \leqslant (|\mathbf{k}| + n) \max_{\mathbf{k} \in I(\theta), \mathbf{p}_{\mathbf{k}} = \mathbf{p}} \|W_{\mathbf{k}}\|_{|\mathbf{k}|,R}, \tag{4.3.6}$$

so that from (4.3.4) and the estimates above we may deduce that

$$\|W_{\mathbf{k}}\|_{|\mathbf{k}|,R} \;\leqslant\; \frac{1}{2}\|W_{\mathbf{k}}\|_{|\mathbf{k}|,R} + \frac{1}{4} \max_{\mathbf{k} \in I(\theta), \mathbf{p}_{\mathbf{k}} = \mathbf{p}} \|W_{\mathbf{k}}\|_{|\mathbf{k}|,R} + K_0\|U_{\mathbf{k}}\|_{|\mathbf{k}|,R}. \tag{4.3.7}$$

Now $\sum_{(\mathbf{i}_m;\mathbf{k})} 1 \leqslant \prod_{m=1}^{n} \binom{k_m+h-1}{k_m} \leqslant 2^{|\mathbf{k}|+n(h-1)}$ and as $|\mathbf{k}| = |\mathbf{p}|$ we can estimate $\|U_{\mathbf{k}}\|_{|\mathbf{k}|,R}$ by

$$\|U_{\mathbf{k}}\|_{|\mathbf{k}|,R} \leqslant K_1 \sum_{h=2}^{|\mathbf{p}|} (2^{n+1}K_1)^h \sum_{(\mathbf{q}_m;\mathbf{p})} \prod_{m=1}^{h} 2^{|\mathbf{q}_m|} f_{\mathbf{q}_m},$$

where $\sum_{(\mathbf{q}_m;\mathbf{p})}$ stands for the sum over all $0 \prec \mathbf{q}_m \in I'(\theta)$ with $\sum_{m=1}^{h} \mathbf{q}_m = \mathbf{p}$. Next taking the maximum of (4.3.7) over all $\mathbf{k} \in I(\theta)$ with $\mathbf{p}_{\mathbf{k}} = \mathbf{p}$, we obtain

$$\max_{\mathbf{k}\in I(\theta),\mathbf{p}_{\mathbf{k}}=\mathbf{p}} \|W_{\mathbf{k}}\|_{|\mathbf{k}|,R} \leqslant 4K_0K_1 \sum_{h=2}^{|\mathbf{p}|} (2^{n+1}K_1)^h \sum_{(\mathbf{q}_m;\mathbf{p})} \prod_{m=1}^{h} 2^{|\mathbf{q}_m|} f_{\mathbf{q}_m} =: f_{\mathbf{p}}.$$

We can complete the proof in a similar way as in proposition 2.5.11, by defining the function f_1 by $f_1(v) := \sum_{j\in i(\theta)} 2f_{\mathbf{e}_j} v_j$ and the formal series $\hat{f}$ by $\hat{f}(v) := \sum_{\mathbf{i}\in I'(\theta),|\mathbf{i}|\geqslant 2} 2^{|\mathbf{i}|} f_{\mathbf{i}}\, v^{\mathbf{i}}$. ■

4.3.3 Analytic Reduction on a Manifold

As in the preceding section we take $\theta \in \mathbb{R}$ and we define θ_- and θ_+ to be two consecutive singular directions in $H(\theta)$ of the set of all $\hat{y}_{\mathbf{k}}$, $\mathbf{k} \in I(\theta)$. Moreover, let $0 < \varepsilon < \frac{1}{2}(\theta_+ - \theta_-)$ and define $S_1 := \{x \in \mathbb{C}^* \mid \arg x \in (-\pi/2 - \theta_+ + \varepsilon, \pi/2 - \theta_- - \varepsilon)\}$.

As in section 2.3 we write

$$y = y_0(x) + P(x,z) := y_0(x) + P_1(x,u) + v, \quad P_1(x,u) := \sum_{\mathbf{k}\in I(\theta)\setminus\{0\}} y_{\mathbf{k}}(x)u^{\mathbf{k}},$$

where $y_{\mathbf{k}}$, $\mathbf{k} \in I(\theta)$, are Borel sums of $\hat{y}_{\mathbf{k}}$ and where

$$z = \sum_{m=1}^{n} z_m \mathbf{e}_m, \quad u := \sum_{m\in i(\theta)} z_m \mathbf{e}_m \quad \text{and} \quad v := z - u.$$

Theorem 4.3.4 *Let S_1 be as above. Then there exist positive constants δ and $\tilde{\rho}$ such that $P_1(x,u)$ converges uniformly for $|u| \leqslant \delta$ if $x \in S_1$, $|x| \geqslant \tilde{\rho}$, and $P_1(x,u)$ is the Borel sum of $\sum_{\mathbf{k}\in I(\theta)\setminus\{0\}} \hat{y}_{\mathbf{k}}(x)u^{\mathbf{k}}$ with respect to x on S_1 for $|u| \leqslant \delta$.*

Moreover, by means of $y = y_0(x) + P(x,z)$ the differential equation (4.1.1) is, in a neighbourhood of ∞ in S_1, transformed into

$$z'(x) + \Lambda(x)z(x) = \Lambda_2\big(x, z(x)\big)v(x), \tag{4.3.8}$$

where $\Lambda_2(x,z) = O(x^{-2}) + O(|z|)$ as $x \to \infty$ in S_1 and $z \to 0$.

Remark 4.3.5 In general the δ and $\tilde{\rho}$ in this theorem depend on the value of ε in the definition of the sector S_1.

PROOF OF THEOREM 4.3.4. The statement about the convergence of the series $P_1(x,u)$ is shown in proposition 3.3.1. To prove Borel summability we observe that the formal Borel transform of $x^{-1}\sum_{\mathbf{k}\in I(\theta)\setminus\{0\}}\hat{y}_{\mathbf{k}}(x)u^{\mathbf{k}}$, with respect to x, is equal to $\sum_{\mathbf{k}\in I(\theta)\setminus\{0\}}u^{\mathbf{k}}\,\tilde{Y}_{\mathbf{k}}(t)$, where $\tilde{Y}_{\mathbf{k}}=\hat{\mathcal{B}}(x^{-1}\hat{y}_{\mathbf{k}})$. Now $\tilde{Y}_{\mathbf{k}}(t)=W_{\mathbf{k}}^{(|\mathbf{k}|-1)}(t)$, for every t belonging to the domain in which $W_{\mathbf{k}}$ is holomorphic. Let V_1 be the union of the sector $\{t\in\mathbb{C}^*\mid \arg t\in(\theta_-+\varepsilon,\theta_+-\varepsilon)\}$ and the disc $\Delta(0,\rho/2)$, where ρ is as in lemma 4.3.2 and define $\eta:=\sin(\varepsilon/2)$.

If $t\in V_1$, $t\neq 0$, Cauchy's formula gives $\tilde{Y}_{\mathbf{k}}(t)=\frac{(|\mathbf{k}|-1)!}{2\pi i}\oint\frac{W_{\mathbf{k}}(s)}{(s-t)^{|\mathbf{k}|}}\,\mathrm{d}s$, where we take as path of integration the circle $s=t+|t|\eta e^{i\varphi}$, $0\leqslant\varphi<2\pi$. On this circle we see that s belongs to the union V of the sector $\{t\in\mathbb{C}^*\mid \arg t\in[\theta_-+\varepsilon/2,\theta_+-\varepsilon/2]\}$ and the disc $\overline{\Delta}(0,\rho)$, so proposition 4.3.3 implies $|W_{\mathbf{k}}(s)|\leqslant\frac{K}{(|\mathbf{k}|-1)!}\big(K(1+\eta)|t|\big)^{|\mathbf{k}|-1}e^{R(1+\eta)|t|}$. Hence,

$$|\tilde{Y}_{\mathbf{k}}(t)|\leqslant K\left(\frac{K(1+\eta)}{\eta}\right)^{|\mathbf{k}|-1}e^{R(1+\eta)|t|}$$

for all $t\in V_1$ (including $t=0$). Therefore $\sum_{\mathbf{k}\in I(\theta)\setminus\{0\}}u^{\mathbf{k}}\tilde{Y}_{\mathbf{k}}(t)$ converges uniformly for $|u|\leqslant\delta$ and $t\in V_1$, provided that δ is small enough. Moreover, its sum can be estimated by $\tilde{K}e^{\tilde{R}|t|}$ for some positive constants $\tilde{K}$ and $\tilde{R}$, which shows that $x^{-1}\sum_{\mathbf{k}\in I(\theta)\setminus\{0\}}\hat{y}_{\mathbf{k}}(x)u^{\mathbf{k}}$ is Borel summable with respect to x on S_1, provided that $|u|\leqslant\delta$. From this the summability property of $\sum_{\mathbf{k}\in I(\theta)\setminus\{0\}}\hat{y}_{\mathbf{k}}(x)u^{\mathbf{k}}$ follows.

If $w(x):=y(x)-y_0(x)$ with y a solution of (4.1.1), it is not so hard to verify that w satisfies the differential equation $w'(x)+\Lambda_1(x)w(x)+d(x,w)=0$, where Λ_1 and $d=d(x,w)$ are defined in (2.2.3) and (2.2.2) respectively. If $w(x)=P\big(x,z(x)\big)$ we thus obtain

$$\frac{\mathrm{d}}{\mathrm{d}x}P\big(x,z(x)\big)+\Lambda_1(x)P\big(x,z(x)\big)+d\Big(x,P\big(x,z(x)\big)\Big)=0. \tag{4.3.9}$$

By definition of $P(x,z)$ we have

$$\frac{\mathrm{d}}{\mathrm{d}x}P\big(x,z(x)\big)=\frac{\partial}{\partial x}P_1(x,u)\big|_{u=u(x)}\;+\;D_uP_1(x,u)\big|_{u=u(x)}u'(x)\;+\;v'(x) \tag{4.3.10}$$

and from (4.2.3) we deduce

$$\begin{aligned}&\frac{\partial}{\partial x}P_1(x,u)+\Lambda_1(x)P_1(x,u)+d\big(x,P_1(x,u)\big)\\ =\;&\sum_{\mathbf{k}\in I(\theta)\setminus\{0\}}\big(\langle\mathbf{k},\boldsymbol{\mu}\rangle-\langle\mathbf{k},\mathbf{a}\rangle x^{-1}\big)y_{\mathbf{k}}(x)u^{\mathbf{k}}\\ &\qquad-\sum_{\mathbf{k}\in I(\theta)\setminus\{0\}}\sum_{j\in i(\theta)}\sum_{h\in\mathcal{J}_j',k_{h+1}\neq 0}\alpha_h\,(k_h+1)\,x^{-1}\,\hat{y}_{\mathbf{k}+\mathbf{e}_h-\mathbf{e}_{h+1}}(x)\,u^{\mathbf{k}}\\ =\;&D_uP_1(x,u)\}\cdot\{\Lambda(x)u\}.\end{aligned}$$

Combining the last formula with (4.3.9) and (4.3.10) we obtain

$$\begin{aligned}D_zP\big(x,z(x)\big)\{z'(x)+\Lambda(x)z(x)\}=\big(\Lambda(x)-\Lambda_1(x)\big)v(x)\;+\;d\Big(x,P_1\big(x,u(x)\big)\Big)\;+\\ -\,d\Big(x,P\big(x,z(x)\big)\Big).\end{aligned} \tag{4.3.11}$$

One should observe that

$$\begin{aligned} d\big(x, P(x,z)\big) - d\big(x, P_1(x,u)\big) - \big(\Lambda(x) - \Lambda_1(x)\big)v \\ = \sum_{|\mathbf{j}|\geqslant 2} d_{\mathbf{j}}(x) \sum_{0\prec \mathbf{h}\preceq \mathbf{j}} \binom{\mathbf{j}}{\mathbf{h}} \big\{P_1(x,u)\big\}^{\mathbf{j}-\mathbf{h}} v^{\mathbf{h}} - \big(\Lambda(x) - \Lambda_1(x)\big)v \\ = \tilde{\Lambda}_2(x,z)v, \end{aligned}$$

where $\tilde{\Lambda}_2(x,z) = O(x^{-2}) + O(|z|)$ as $x \to \infty$ in S_1 and $z \to 0$. Now $D_zP(\infty,z)|_{z=0} = \mathtt{I}$, so the inverse of $D_zP(x,z)$ exists in a neighbourhood of $x = \infty$ in S_1 and a neighbourhood of $z = 0$. By multiplying both sides of (4.3.11) by the inverse of $D_zP(x,z)$ we obtain (4.3.8), which proves the theorem. ■

Theorem 4.3.6 *Let $\mathcal{M}$ be the manifold defined by $y = y_0(x) + P_1(x,u)$, with x in a neighbourhood of ∞ in S_1 and u in a neighbourhood of 0. Then on $\mathcal{M}$ the differential equation (4.1.1) is transformed into $u'(x) + \Lambda(x)u(x) = 0$.*

Let S_2 be a sub-sector of S_1 containing the direction $-\theta$ and let y be a solution of (4.1.1) such that $y(x) = O(x^{-2})$ as $x \to \infty$ on S_2. Then y belongs to the restriction of $\mathcal{M}$ to S_2. Moreover, there exists a unique $C \in \mathbb{C}^n$ with $C_h = 0$ unless $h \in \mathcal{J}_j$ with j such that $e^{-\mu_j x} \to 0$ as $x \to \infty$ in S_2, and

$$y(x) = \sum_{\mathbf{k}\in I(\theta)} e^{-\langle \mathbf{k},\boldsymbol{\mu}\rangle x} x^{\langle \mathbf{k},\mathbf{a}\rangle} \prod_{j\in i(\theta)} \big(x^{\mathtt{N}_{n_j}} C^{[j]}\big)^{\mathbf{k}_j} y_{\mathbf{k}}(x), \quad x \in S_2. \tag{4.3.12}$$

Conversely, if $C \in \mathbb{C}^n$ is such that $C_h = 0$ unless $h \in \mathcal{J}_j$ with j such that $e^{-\mu_j x} \to 0$ as $x \to \infty$ in S_2, then the sum in (4.3.12) converges in a neighbourhood of ∞ in S_2, defines a solution of (4.1.1) which is holomorphic in this neighbourhood and has the property $y(x) \sim \hat{y}_0(x)$ as $x \to \infty$ in S_2.

Proof. The first assertion follows at once from the preceding theorem, if we take $v = 0$.

Next assume that y is a solution of (4.1.1) such that $y(x) = O(x^{-2})$ as $x \to \infty$ on S_2. Then put $w(x) = y(x) - y_0(x) = O(x^{-2})$ on S_2 and let z, u and v be associated with these solutions as in the previous theorem. As $D_zP(\infty,z)|_{z=0} = \mathtt{I}$, the implicit function theorem implies that the function $z \mapsto P(x,z)$ is invertible for x in a neighbourhood of ∞ in S_2 and z in a neighbourhood of the origin. So we can conclude that $z(x)$, and thus $u(x)$ and $v(x)$, also are $O(x^{-2})$ as $x \to \infty$ in S_2.

From (4.3.8) we deduce an equation for v which involves the known function u. In fact, v satisfies an equation of the form $\vartheta'(x) + \big(\Lambda(x) + A(x)\big)\vartheta(x)$, with $A(x) = O(x^{-2})$ as $x \to \infty$ in S_2. Now, let S_3 be a sub-sector of S_2 with opening less than π and containing the direction $-\theta$. It is well known that a system of the form just described possesses a fundamental system of solutions $V(x) = \big(\mathtt{I} + O(x^{-1})\big)\big(\oplus_{j\notin i(\theta)} e^{-\mu_j x} x^{\mathtt{M}_j}\big)$, as $x \to \infty$ in S_3 (cf. [Was87, HS99]). Hence, there exists a constant vector $C \in \mathbb{C}^p$, with $p = \sum_{j\notin i(\theta)} \sum_{h\in\mathcal{J}_j} 1$, such that $v(x) = V(x)C$, as $x \to \infty$ in S_3. However, $v(x) = O(x^{-2})$ as $x \to \infty$ in S_3 and $\Re(\mu_j e^{-i\theta}) \leqslant 0$ if $j \notin i(\theta)$, so we can conclude that $C = 0$. Hence,

$v \equiv 0$ on S_3 and thus $v \equiv 0$ on S_2 and it follows that y belongs to the restriction of $\mathcal{M}$ to S_2.

The equation for u then reduces to (4.1.3), which has as general solution (4.2.2), where we only take those j that belong to $i(\theta)$. This proves (4.3.12). The last statement is a consequence of the first part of theorem 4.3.4. ■

4.4 A Resurgence Relation

Let $j \in \{1, 2, \ldots, r\}$ and assume that the line-segment $(0, \mu_j)$ does not contain any singular points μ_h of Y_0, with $h \in \{1, 2, \ldots, r\}$. Moreover, for each $h \in \{1, 2, \ldots, r\}$ we assume that $a_k = a_l$ whenever $k, l \in \mathcal{J}_h$ and we assume $\Re a_h \notin \mathbb{Z}$ for all $h \in \{1, 2, \ldots, n\}$. Hence, by introducing new quantities a_h, $h = 1, 2, \ldots, r$, we can assume that Λ has the form

$$\Lambda(x) = \bigoplus_{h=1}^{r} (\mu_h \mathtt{I}_{n_h} - x^{-1}\mathtt{M}_h), \quad \mathtt{M}_h = a_h \mathtt{I}_{n_h} + \mathtt{N}_{n_h},$$

with $a_h \in \mathbb{C}$, $\Re a_h \notin \mathbb{Z}$. Moreover, as in the introduction of section 2.4, one can assume that $\Re a_h > 0$ for all $h = 1, 2, \ldots, r$. In this section we study the behaviour of Y_0 near the singular points μ_j, $j = 1, 2, \ldots, r$, from which we can derive an analogue of the resurgence relation found in proposition 2.4.3.

Proposition 4.4.1 *Let $j \in \{1, 2, \ldots, r\}$ be such that $(0, \mu_j)$ does not contain any singular point of Y_0 and let Y_0^+ and Y_0^- be related to Y_0 in the way described in the introduction of section 2.4.2. Then there exists a vector $\mathbf{s} = (s_1, s_2, \ldots, s_n) \in \mathbb{C}^n$ such that*

$$Y_0^+(\mu_j + t) - Y_0^-(\mu_j + t) = \sum_{h \in \mathcal{J}_j} \left\{ \mathcal{B}\left[(x^{\mathtt{N}_{n_j}} \mathbf{s}^{[j]})^{\mathbf{e}_{h-(n_1+\cdots+n_{j-1})}} x^{\tilde{a}_j} y_{\mathbf{e}_h}(x) \right] \right\}^{(m_j)}$$

for $\arg t = \theta_j = \arg \mu_j$ and $|t|$ sufficiently small. Here m_j is such that $\Re a_j - m_j \in (-1, 0)$ and $\tilde{a}_j = a_j - m_j$.

Proof. By renumbering we may assume that $j = 1$. Choose $\varepsilon \in (0, |\mu_1|/3)$ so small that the distance of $(0, \mu_1)$ to the singular points of Y_0 different from μ_1 is at least 2ε. Then define the function H by $H(t) := Y_0(t)$ if t belongs to the maximal open subset T of the disc $\Delta(0, |\mu_1| - \varepsilon)$, which has distance ε to the boundary of the star domain in which Y_0 is holomorphic and $H(t) := 0$ otherwise. Let Ψ_0 and Ψ be as in the beginning of the proof of proposition 2.4.2. If $\eta = \varepsilon e^{i\theta_1}$ and $t = \mu_1 - s$, $s \in (0, \eta]$, then (4.3.1) can be written as

$$(\mu_1 - s - \Lambda_0)\Psi(s) = -\int_{\eta}^{s} \left\{ A + D(\sigma - s) \right\} \Psi(\sigma) \mathrm{d}\sigma + P(s),$$

where $D = D(H)$ is holomorphic in T, $A = -\bigoplus_{j=1}^{r} \mathtt{M}_j$ and where P is some function holomorphic in $\Delta(0, \varepsilon)$ with continuous boundary values on the circle $|t| = \varepsilon$ (compare the proof of proposition 2.4.2). Note that in this case $\Lambda_0 = \oplus_{j=1}^{r} \mu_j \mathtt{I}_{n_j}$, so this integral equation is singular at $s = 0$ in the first n_1 components.

We now define $\psi(s) := (\mathcal{P}_\eta^{m_1+1}\Psi)(s)$, where $\mathcal{P}_\eta$ is defined by $(\mathcal{P}_\eta\Psi)(s) := \int_\eta^s \Psi(\sigma)\mathrm{d}\sigma$, then

$$(\mu_1 - s - \Lambda_0)\psi^{(m_1+1)}(s) = -\int_\eta^s \big\{A + D(\sigma - s)\big\}\psi^{(m_1+1)}(\sigma)\mathrm{d}\sigma + P(s).$$

It is easily seen that $\mathcal{P}_\eta^{m_1}\big[\sigma\psi^{(m_1+1)}(\sigma)\big](s) = s\psi'(s) - m_1\psi(s)$, so integrating the above equation m_1 times we obtain in a similar way as in the proof of proposition 2.4.2

$$(\mu_1 - s - \Lambda_0)\psi' + m_1\psi = -\int_\eta^s \big(A + D(\sigma - s)\big)\psi'(\sigma)\mathrm{d}\sigma + \mathcal{P}_\eta^{m_1}[P](s). \tag{4.4.1}$$

Splitting this equation after the first n_1 components yields (with the same notation as in the proof of proposition 2.4.2):

$$-s\frac{\mathrm{d}\psi^{[1]}}{\mathrm{d}s} - (\mathtt{M}_1 - m_1)\psi^{[1]} = \int_\eta^s (D^{[1]})'(\sigma - s)\psi(\sigma)\mathrm{d}\sigma + \mathcal{P}_\eta^{m_1}\big[P^{[1]}\big](s)$$

$$\big(\mu_1 - s - (\Lambda_0)^{\perp\perp}\big)\frac{\mathrm{d}\psi^{\perp}}{\mathrm{d}s} = E\psi^{\perp}(s) + \int_\eta^s (D')^{\perp}(\sigma - s)\psi(\sigma)\mathrm{d}\sigma + \mathcal{P}_\eta^{m_1}[P^{\perp}](s),$$

where $(\Lambda_0)^{\perp\perp} = \oplus_{j=2}^{r}\,\mu_j\mathtt{I}_{n_j}$ and $E = -m_1\mathtt{I}_{n-n_1} + \oplus_{j=2}^{r}\,\mathtt{M}_j$. If we define $\tilde{\mathtt{M}}_1 := \mathtt{M}_1 - m_1\mathtt{I}_{n_1}$, then $s\frac{\mathrm{d}\psi^{[1]}}{\mathrm{d}s} + (\mathtt{M}_1 - m_1)\psi^{[1]} = s^{1-\tilde{\mathtt{M}}_1}\big(s^{\tilde{\mathtt{M}}_1}\psi^{[1]}(s)\big)'$. Using this we obtain

$$\psi^{[1]}(s) = -s^{-\tilde{\mathtt{M}}_1}\int_\eta^s \tau^{\tilde{\mathtt{M}}_1 - 1}\Big\{\int_\eta^\tau (D^{[1]})'(\sigma - \tau)\psi(\sigma)\mathrm{d}\sigma + \mathcal{P}_\eta^{m_1}\big[P^{[1]}\big](\tau)\Big\}\,\mathrm{d}\tau \tag{4.4.2a}$$

$$\psi^{\perp}(s) = \int_\eta^s \big(\mu_1 - \tau - (\Lambda_0)^{\perp\perp}\big)^{-1}\Big\{E\psi^{\perp}(\tau) + \int_\eta^\tau (D')^{\perp}(\sigma - \tau)\psi(\sigma)\mathrm{d}\sigma + \mathcal{P}_\eta^{m_1}[P^{\perp}](\tau)\Big\}\,\mathrm{d}\tau. \tag{4.4.2b}$$

We consider these equations in the space $\mathcal{V}_{\tilde{a}_1,\log,0}$ of n-vector valued functions ψ that can be written as $\psi(s) = s^{-\tilde{a}_1}\sum_{k=0}^{n_1-1}(\log s)^k u_k(s) + u_{n_1}(s)$, with u_k, $k = 0, 1, \ldots, n_1$, continuous on $\overline{\Delta}(0,\varepsilon)$ and holomorphic in the interior $\Delta(0,\varepsilon)$. We choose the branches of $s^{-\tilde{a}_1}$ and $\log s$ with $\theta_1 - \pi < \arg s \leqslant \theta_1 + \pi$ and we endow $\mathcal{V}_{\tilde{a}_1,\log,0}$ with the norm $\|\psi\|$ defined by $\|\psi\| := \sup\big\{\big|u_k(s)\big|,\ |s| \leqslant \varepsilon,\ k = 0, 1, \ldots, n_1\big\}$. Moreover, we consider the operator $\mathcal{T}$ that maps any $\psi \in \mathcal{V}_{\tilde{a}_1,\log,0}$ to the n-vector with components the right-hand sides of (4.4.2a) and (4.4.2b) and we will verify that $\mathcal{T}$ defines a contraction on $\mathcal{V}_{\tilde{a}_1,\log,0}$.

First we observe that if f is an n_1-vector valued function that is holomorphic in $\Delta(0,\varepsilon)$ and continuous on $\overline{\Delta}(0,\varepsilon)$, then

$$\begin{aligned} s^{-\tilde{\mathtt{M}}_1}\int_\eta^s \tau^{\mathtt{N}_{n_1}}(\log\tau)^k f(\tau)\mathrm{d}\tau &= s^{-\tilde{\mathtt{M}}_1}\int_\eta^0 \tau^{\mathtt{N}_{n_1}}(\log\tau)^k f(\tau)\mathrm{d}\tau + s^{-\tilde{\mathtt{M}}_1}\int_0^s \tau^{\mathtt{N}_{n_1}}\,(\log\tau)^k f(\tau)\mathrm{d}\tau \\ &= s^{-\tilde{a}_1}\sum_{k=0}^{n_1-1}\frac{(-\mathtt{N}_{n_1})^k}{k!}(\log s)^k\int_\eta^0 \tau^{\mathtt{N}_{n_1}}(\log\tau)^k f(\tau)\mathrm{d}\tau + \\ &\quad\ s^{1-\tilde{a}_1}\int_0^1 \sigma^{\mathtt{N}_{n_1}}(\log s + \log\sigma)^k f(s\sigma)\mathrm{d}\sigma. \end{aligned}$$

Hence, $s^{-\tilde{\mathtt{M}}_1}\int_\eta^s \tau^{\mathtt{N}_{n_1}}(\log\tau)^k f(\tau)\mathrm{d}\tau$ equals $s^{-\tilde{a}_1}$ times a polynomial in $\log s$ of degree at most n_1-1. The coefficients of this polynomial are holomorphic functions in $\Delta(0,\varepsilon)$, having continuous boundary values on the circle $|t|=\varepsilon$. From this observation one may deduce, in a similar way as in the proof of proposition 2.4.2, that $\mathcal{T}$ defines a contraction on $\mathcal{V}_{\tilde{a}_1,\log,0}$ if ε is chosen sufficiently small. Now the behaviour of Y_0 near μ_1 follows in a similar way as in the proof of proposition 2.4.2. In fact,

$$Y_0(t) = (\mu_1-t)^{-a_1-1}\sum_{k=0}^{n_1-1}\bigl(\log(\mu_1-t)\bigr)^k\tilde{u}_k(t)+\tilde{u}_{n_1}(t) \tag{4.4.3}$$

for $0<|\mu_1-t|<\varepsilon$ and certain functions $\tilde{u}_j$ holomorphic in $\Delta(\mu_1,\varepsilon)$.

To prove the resurgence relation we introduce $v(s):=\psi^+(-s)-\psi^-(-s)$, where ψ^+ and ψ^- are the boundary values of ψ on the lower and upper side of the interval $(-\eta,0)$ respectively. By taking limits one may deduce that both ψ^+ and ψ^- are solutions of (4.4.1) on $(-\eta,\eta)$ if we define $\psi^+(s)=\psi^-(s)=\psi(s)$ on $[0,\eta)$. Then $v(s)=0$ on $(-\eta,0]$ and $v(s)=s^{-\tilde{a}_1}\sum_{k=0}^{n_1-1}(\log s)^k w_k(s)$ on $(0,\eta)$ for certain functions w_k holomorphic in $\Delta(0,\varepsilon)$. From (4.4.1) we easily infer that v' satisfies

$$(s+\mu_1-\Lambda_0)v' = (m_1+A+D)*v'. \tag{4.4.4}$$

Note that, for $\alpha\in\mathbb{C}$ and $\Re\alpha>-1$, we have

$$\mathcal{L}\bigl[s^\alpha(\log s)^k\bigr](x) = \Bigl(\frac{\mathrm{d}}{\mathrm{d}\alpha}\Bigr)^k\mathcal{L}[s^\alpha](x) = \sum_{j=0}^k\binom{k}{j}\Gamma^{(k-j)}(\alpha+1)\,x^{-\alpha-1}\,(-\log x)^j,$$

so $\hat{\mathcal{L}}v$ exists and can be written as a polynomial in $\log x$ of degree $\leqslant n_1-1$, with coefficients in $x^{\tilde{a}_1-1}\mathbb{C}^n[[x^{-1}]]$. As $v(0)=0$ we have $(\hat{\mathcal{L}}v')(x)=x(\hat{\mathcal{L}}v)(x)\in x^{\tilde{a}_1}\mathbb{C}^n[[x^{-1}]][\log x]$ and by taking a formal Laplace transform of (4.4.4) we see that $\hat{\mathcal{L}}v'$ is a formal solution of

$$y'(x)+\bigl(\Lambda_1(x)-\mu_1+m_1x^{-1}\bigr)y(x)=0. \tag{4.4.5}$$

Now for $h=1,2,\dots,n_1$ we define $u_h(x):=\sum_{l=1}^{n_1}(x^{\mathtt{N}_{n_1}})_{l,h}\,\hat{y}_{\mathbf{e}_l}(x)$, where $(x^{\mathtt{N}_{n_1}})_{l,h}$ denotes the element of $x^{\mathtt{N}_{n_1}}$ at the l^{th} row and h^{th} column. As for $l\in\{1,2,\dots,n_1-1\}$ we have

$$\frac{\mathrm{d}}{\mathrm{d}x}(x^{\mathtt{N}_{n_1}})_{l,h} = x^{-1}\bigl(\mathtt{N}_{n_1}x^{\mathtt{N}_{n_1}}\bigr)_{l,h} = x^{-1}\alpha_l(x^{\mathtt{N}_{n_1}})_{l+1,h},$$

we easily infer that each u_h is a solution of $y'(x)+\bigl(\Lambda_1(x)-\mu_1+a_1x^{-1}\bigr)y(x)=0$. Hence, $\sum_{l=1}^{n_1}(x^{\mathtt{N}_{n_1}})_{l,h}\,x^{\tilde{a}_1}\hat{y}_{\mathbf{e}_l}(x)$ satisfies (4.4.5) and similarly we deduce that for $j\in\{2,3,\dots,r\}$ and $h\in\{1,2,\dots,n_j\}$ the expressions $\sum_{l\in\mathcal{J}_j}(x^{\mathtt{N}_{n_j}})_{l-(n_1+\dots+n_{j-1}),h}\,x^{\tilde{a}_j}\hat{y}_{\mathbf{e}_l}(x)$ satisfy the differential equation $y'(x)+\bigl(\Lambda_1(x)-\mu_j+m_jx^{-1}\bigr)y(x)=0$. So when for $j\in\{1,2,\dots,r\}$ and $h\in\{1,2,\dots,n_j\}$ we define

$$\vartheta_{j,h}(x):=e^{(\mu_1-\mu_j)x}x^{m_j-m_1}\sum_{l\in\mathcal{J}_j}(x^{\mathtt{N}_{n_j}})_{l-(n_1+\dots+n_{j-1}),h}\,x^{\tilde{a}_j}\hat{y}_{\mathbf{e}_l}(x),$$

we see that $\{\vartheta_{j,h} \mid j=1,2,\ldots,r,\ h=1,2,\ldots,n_j\}$ forms a formal fundamental system of (4.4.5).

Since we have shown that $\hat{\mathcal{L}}v' \in x^{\tilde{a}_1}\mathbb{C}^n[[x^{-1}]][\log x]$, we can conclude to the existence of n_1 constants $c_1, c_2, \ldots, c_{n_1}$ such that

$$(\hat{\mathcal{L}}v')(x) = \sum_{h=1}^{n_1} c_h \vartheta_{1,h}(x),$$

and thus $v'(s) = \sum_{h=1}^{n_1} c_h(\mathcal{B}\vartheta_{1,h})(s)$. Hence, the analogue of (2.4.10) becomes

$$Y_0^+(\mu_1+t) - Y_0^-(\mu_1+t) = (-1)^{m_1+1}v^{(m_1+1)}(t) = \sum_{h=1}^{n_1} s_h(\mathcal{B}\vartheta_{1,h})^{(m_1)}(t),$$

for $\arg t = \theta_1$ and $|t|$ sufficiently small. Here $s_h := (-1)^{m_1+1}c_h$. Finally, we observe that if $\mathbf{s}^{[1]} := (s_1, s_2, \ldots, s_{n_1})^t$, then this resurgence relation can also be written as

$$Y_0^+(\mu_1+t) - Y_0^-(\mu_1+t) = \sum_{h=1}^{n_1} \left\{\mathcal{B}\left[(x^{\mathtt{N}_{n_1}}\mathbf{s}^{[1]})^{\mathbf{e}_h}\, x^{\tilde{a}_1} y_{\mathbf{e}_h}(x)\right]\right\}^{(m_1)}.$$

This proves the proposition. ■

4.5 The Analogue for Difference Equations

In this section we want to give the analogue of the results obtained in the preceding sections for difference equations of the form

$$y(x+1) = \Lambda(x)y(x) + g(x, y(x)), \tag{4.5.1}$$

with $\Lambda(x) = \oplus_{j=1}^r e^{-\mu_j}(1+x^{-1})^{\mathtt{M}_j}$, $\mathtt{M}_j = \operatorname{diag}\{a_m,\ m \in \mathcal{J}_j\} + \mathtt{N}_{n_j}$, $j = 1, 2, \ldots, r$.

Instead of a direct formal reduction to the normal form $z(x+1) = \Lambda(x)z(x)$, we now first transform (4.5.1) into an 'almost' normal form

$$\vartheta(x+1) = \Lambda(x)(1+x^{-1})^{-\mathtt{N}}\big(\mathtt{I} + x^{-1}\mathtt{N}\big)\vartheta(x), \tag{4.5.2}$$

by means of $y = \hat{y}_0(x) + \sum_{\mathbf{k}\in\mathbb{N}_1^n} \tilde{y}_{\mathbf{k}}(x)\vartheta^{\mathbf{k}}$ (compare remark 4.5.2). Here $\mathtt{N} = \oplus_{j=1}^r \mathtt{N}_{n_j}$, $\hat{y}_0$ is the formal solution of (4.5.1) and the expressions $\tilde{y}_{\mathbf{k}}$, $\mathbf{k} \in \mathbb{N}_1^n$, at first have to be interpreted as formal series. Later on we will give them an analytic meaning.

In section 4.5.3 we will lift a part of this transformation to a holomorphic transformation, which transforms (4.5.1) into an 'almost' semi-canonical form, i.e. the form (4.5.2) in which we take a part of the ϑ equal to 0. In section 4.5.4, we will reduce this almost semi-canonical form into the form we are used to

$$u(x+1) = \Lambda(x)u(x). \tag{4.5.3}$$

Finally, in section 4.5.5 we will prove that proposition 4.4.1 remains valid in the case of difference equations.

4.5.1 Formal Reduction to an Almost Normal Form

If $\vartheta = \vartheta(x)$ satisfies (4.5.2) and if $\boldsymbol{\mu} = \sum_{j=1}^{r} \mu_j \sum_{h\in\mathcal{J}_j} \mathbf{e}_h$ and $\mathbf{a} = (a_1, a_2, \ldots, a_n)$, then for $\mathbf{l} \in \mathbb{N}_1^n$ we have

$$\vartheta^{\mathbf{l}}(x+1) = e^{-\langle \mathbf{l},\boldsymbol{\mu}\rangle}(1+x^{-1})^{\langle \mathbf{l},\mathbf{a}\rangle} \prod_{j=1}^{r} \big[(\mathtt{I}_{n_j} + x^{-1}\mathtt{N}_{n_j})\vartheta^{[j]}\big]^{\mathbf{l}_j}.$$

If α_h, $h \in \mathcal{J}_j' = \{n_1 + \cdots + n_{j-1} + 1, \ldots, n_1 + \cdots + n_j - 1\}$, denotes the element of $\mathtt{N}_{n_j}$ at the $(h - n_1 - \cdots - n_{j-1})^{\text{th}}$ row and $(h + 1 - n_1 - \cdots - n_{j-1})^{\text{th}}$ column, then the product above can be expanded as

$$\sum_{i_1=0}^{l_1} \cdots \sum_{i_{n_1-1}=0}^{l_{n_1-1}} \quad \sum_{i_{n_1+1}=0}^{l_{n_1+1}} \cdots \sum_{i_{n_1+n_2-1}=0}^{l_{n_1+n_2-1}} \quad \cdots \quad \sum_{i_{n-n_r+1}=0}^{l_{n-n_r+1}} \cdots \sum_{i_{n-1}=0}^{l_{n-1}} \; \prod_{j=1}^{r} \prod_{h\in\mathcal{J}_j} \binom{l_h}{i_h} \Big(\frac{\alpha_h}{x}\Big)^{i_h} \vartheta_h^{l_h - i_h + i_{h-1}},$$

where we used the convention $i_0 = i_{n_1} = i_{n_1+n_2} = \cdots = i_n = 0$. Hence, the infinite sum $\sum_{\mathbf{l}\in\mathbb{N}_1^n} \tilde{y}_{\mathbf{l}}(x+1)\vartheta^{\mathbf{l}}(x+1)$ can be rewritten as

$$\sum_{\mathbf{k}\in\mathbb{N}_1^n} \big[e^{-\langle \mathbf{k},\boldsymbol{\mu}\rangle}(1+x^{-1})^{\langle \mathbf{k},\mathbf{a}\rangle}\tilde{y}_{\mathbf{k}}(x+1) + s_{\mathbf{k}}(x)\big]\vartheta^{\mathbf{k}}(x),$$

with

$$s_{\mathbf{k}}(x) = \sum_{\mathbf{l}} e^{-\langle \mathbf{l},\boldsymbol{\mu}\rangle}(1+x^{-1})^{\langle \mathbf{l},\mathbf{a}\rangle} \prod_{j=1}^{r} \prod_{h\in\mathcal{J}_j} \binom{l_h}{i_h} \Big(\frac{\alpha_h}{x}\Big)^{i_h} \tilde{y}_{\mathbf{l}}(x+1),$$

where we sum over those $\mathbf{l} \in \mathbb{N}_1^n$ with $\mathbf{l} \neq \mathbf{k}$ and for all $j \in \{1, 2, \ldots, r\}$

$$l_h - i_h + i_{h-1} = k_h, \quad h \in \mathcal{J}_j,$$

for certain $i_h \in \{0, 1, \ldots, l_h\}$ with the convention $i_0 = i_{n_1} = \cdots = i_n = 0$. Observe that $\mathbf{l} \neq \mathbf{k}$ simply means that we cannot take every i_h equal to 0. From this we deduce that for all $j \in \{1, 2, \ldots, r\}$ we have $|\mathbf{l}_j| = |\mathbf{k}_j|$ and $i_h \leqslant k_{h+1}$ for all $h \in \mathcal{J}_j'$. If $\sum_{\mathbf{i}}'$ is a shorthand notation for the sum over all $0 \neq \mathbf{i} = (i_1, \ldots, i_{n_1-1}, i_{n_1+1}, \ldots, i_{n_1+n_2-1}, \ldots, i_{n-n_r+1}, \ldots, i_{n-1})$ with i_h running from 0 to k_{h+1}, then $s_{\mathbf{k}}$ can be rewritten as

$$s_{\mathbf{k}}(x) = e^{-\langle \mathbf{k},\boldsymbol{\mu}\rangle} \sum_{\mathbf{i}}{}' (1+x^{-1})^{\langle \mathbf{l},\mathbf{a}\rangle} \prod_{j=1}^{r} \prod_{h\in\mathcal{J}_j} \binom{l_h}{i_h} \Big(\frac{\alpha_h}{x}\Big)^{i_h} \tilde{y}_{\mathbf{l}}(x+1), \tag{4.5.4}$$

where $\mathbf{l} = (l_1, l_2, \ldots, l_n) \neq \mathbf{k}$ is the fixed multi-index defined by

$$l_h = k_h + i_h - i_{h-1}, \quad h \in \mathcal{J}_j, \tag{4.5.5}$$

for every $j \in \{1, 2, \ldots, r\}$.

As $y = \hat{y}_0(x) + \sum_{\mathbf{k}\in\mathbb{N}_1^n} \tilde{y}_{\mathbf{k}}(x)\vartheta^{\mathbf{k}}$ should transform (4.5.1) into (4.5.2), we now easily deduce that each $\tilde{y}_{\mathbf{k}}$, $\mathbf{k} \succ 0$, has to be a formal solution of

$$e^{-\langle \mathbf{k},\boldsymbol{\mu}\rangle} (1+x^{-1})^{\langle \mathbf{k},\mathbf{a}\rangle}\, y(x+1) = \Lambda_1(x)\, y(x) + t_{\mathbf{k}}(x) - s_{\mathbf{k}}(x), \tag{4.5.6}$$

where Λ_1 is as in (2.2.3), $t_{\mathbf{k}}$ is as in (2.2.5) and $s_{\mathbf{k}}$ is as in (4.5.4). Using the same notation as in section 4.2 we see that if $\mathbf{k} \in \mathbb{N}_1^n$ and $\mathbf{p}_{\mathbf{k}} = \mathbf{p}$ for some $\mathbf{p} \in \mathbb{N}_1^r$, then $s_{\mathbf{k}}$ only depends on $\tilde{y}_{\mathbf{l}}$ with $\mathbf{l} \prec_{\mathbf{p}} \mathbf{k}$. So we can use the same 'double' induction method as used in the proof of proposition 4.2.5 in order to show that this proposition remains valid in the case of difference equations.

Proposition 4.5.1 *For each $\mathbf{k} \succ 0$ the difference equation (4.5.6) possesses a formal solution $\tilde{y}_{\mathbf{k}} \in \mathbb{C}^n[[x^{-1}]]$. For $j \in \{1, 2, \ldots, n\}$ the series $\hat{y}_{\mathbf{e}_j}$ is unique if we prescribe the constant term in $\hat{y}_{\mathbf{e}_j}$ to be equal to $\mathbf{e}_j$. Then also the series $\hat{y}_{\mathbf{k}}$ for $|\mathbf{k}| > 1$ are unique.*

Remark 4.5.2 Suppose that $r = 1$ and that $\mathtt{M}_1$ is an $n\times n$-matrix with only one eigenvalue, say a, and with corresponding 1-dimensional eigenspace. Then without loss of generality we may assume that $\mathtt{M}_1 = a\mathtt{I} + \mathtt{N}$, where $\mathtt{N}$ only has ones on the first super diagonal. In this particular case we have $(1+x^{-1})^{\mathtt{N}} z = \sum_{j=1}^n \sum_{h=j}^n \frac{\log(1+x^{-1})^{h-j}}{(h-j)!} z_h\, \mathbf{e}_j$. So if we had taken the normal form $z(x+1) = \Lambda(x)z(x)$ instead of (4.5.2), then we'd had to expand $\left[(1+x^{-1})^{\mathtt{N}} z\right]^{\mathbf{1}}$ instead of $\left[(\mathtt{I} + x^{-1}\mathtt{N})\vartheta\right]^{\mathbf{1}}$ and obviously this is more complicated. Therefore we first reduce to $\vartheta(x+1) = \Lambda(x)(1+x^{-1})^{-\mathtt{N}}(\mathtt{I} + x^{-1}\mathtt{N})\vartheta(x)$.

4.5.2 Properties of $W_{\mathbf{k}}$

From [Bra80] it follows that proposition 2.3.1 remains valid, so $\hat{y}_0$ is Borel summable in every direction $-\theta$, $\theta \neq \arg(\mu_j + 2l\pi i)$, $j \in \{1, 2, \ldots, r\}$ and $l \in \mathbb{Z}$. To prove the Borel summability of $\tilde{y}_{\mathbf{k}}$ for $\mathbf{k} \succ 0$ we put $\hat{w}_{\mathbf{k}}(x) := x^{-|\mathbf{k}|}\tilde{y}_{\mathbf{k}}(x)$ and we see that $\hat{w}_{\mathbf{k}}$ satisfies the following linear difference equation

$$e^{-\langle \mathbf{k},\boldsymbol{\mu}\rangle} (1+x^{-1})^{\langle \mathbf{k},\mathbf{b}\rangle}\, w(x+1) = \Lambda_1(x)\, w(x) + u_{\mathbf{k}}(x) - \tilde{u}_{\mathbf{k}}(x), \tag{4.5.7}$$

where $\mathbf{b} = \mathbf{a} + (1, 1, \ldots, 1)$, $u_{\mathbf{k}}(x) = x^{-|\mathbf{k}|}t_{\mathbf{k}}(x)$ and $\tilde{u}_{\mathbf{k}}(x) = x^{-|\mathbf{k}|}s_{\mathbf{k}}(x)$. An easy computation shows that

$$\tilde{u}_{\mathbf{k}}(x) = e^{-\langle \mathbf{k},\boldsymbol{\mu}\rangle} \sum_{\mathbf{i}}{}' (1+x^{-1})^{\langle \mathbf{l},\mathbf{b}\rangle} \prod_{j=1}^{r} \prod_{h\in\mathcal{J}_j} \binom{l_h}{i_h} \left(\frac{\alpha_h}{x}\right)^{i_h} \hat{w}_{\mathbf{l}}(x+1), \tag{4.5.8}$$

where $\mathbf{l} = (l_1, l_2, \ldots, l_n) \neq \mathbf{k}$ is the multi-index defined in (4.5.5) and where $\sum_{\mathbf{i}}'$ denotes the sum over all $0 \neq \mathbf{i} = (i_1, \ldots, i_{n_1-1}, i_{n_1+1}, \ldots, i_{n_1+n_2-1}, \ldots, i_{n-n_r+1}, \ldots, i_{n-1})$ with i_h running from 0 to k_{h+1}.

Taking a formal Borel transform on both sides of (4.5.7) gives the following convolution equation for $W_{\mathbf{k}} = \hat{\mathcal{B}}\hat{w}_{\mathbf{k}}$

$$(e^{-t-\langle \mathbf{k},\boldsymbol{\mu}\rangle} - \Lambda_0)W_{\mathbf{k}} \;=\; B * W_{\mathbf{k}} \,-\, e^{-\langle \mathbf{k},\boldsymbol{\mu}\rangle}\beta_{\mathbf{k}} * e^{-t}W_{\mathbf{k}} \,+\, U_{\mathbf{k}} \,-\, \tilde{U}_{\mathbf{k}},$$

with $\Lambda_0 = \bigoplus_{j=1}^{r} e^{-\mu_j} \mathrm{I}_{n_j}$, B and $\beta_{\mathbf{k}}$ as in (2.3.7), $U_{\mathbf{k}}$ as in (2.3.9) and

$$\tilde{U}_{\mathbf{k}} = e^{-\langle \mathbf{k}, \boldsymbol{\mu} \rangle} \sum_{\mathbf{i}}{}' \mathcal{B}\Big[(1+x^{-1})^{\langle \mathbf{l},\mathbf{b}\rangle} \prod_{j=1}^{r} \prod_{h\in\mathcal{J}_j} \binom{l_h}{i_h} \Big(\frac{\alpha_h}{x}\Big)^{i_h}\Big] * e^{-t} W_{\mathbf{l}}. \qquad (4.5.9)$$

With slight modifications in the proof of proposition 4.3.1 and using [Bra80] one can prove the following proposition.

Proposition 4.5.3 *Let $\mathbf{k} \in \mathbb{N}^n$, $\mathbf{k} \neq 0$. Then $t \mapsto t^{-|\mathbf{k}|+1} W_{\mathbf{k}}(t)$ exists and is holomorphic in the maximal star domain with centre 0, which does not contain any of the singular points $\mu_j - \langle \mathbf{k}', \boldsymbol{\mu}\rangle + 2l\pi i \neq 0$ where $j \in \{1,2,\dots,r\}$, $l \in \mathbb{Z}$ and $\mathbf{k}' \in \mathbb{N}^n$ with $\mathbf{k}' \preceq \mathbf{k}$ or $\mathbf{k}' \prec_{\mathbf{p_k}} \mathbf{k}$. If $\overline{S}$ is a closed sector with vertex 0, not containing any of those singular points, then there exists a positive constant M, which may depend on $\mathbf{k}$ and $\overline{S}$, such that* $\sup_{t\in\overline{S}} e^{-M|t|} |W_{\mathbf{k}}(t)| < \infty$.

Since we assume $\Re\mu_j > 0$ for all $j \in \{1,2,\dots,r_1\}$ (cf. the introduction of this chapter), it makes sense to consider consecutive singular directions of the set of all $\tilde{y}_{\mathbf{k}}$, $\mathbf{k} \in \mathbb{N}^p$, in the right half plane. Here p is, as in chapter 2, defined by $p = n_1 + n_2 + \cdots + n_{r_1}$. Let θ_- and θ_+ be two such singular directions in the right half plane.

Lemma 4.5.4 *Let V be as in proposition 2.3.5. Then there exist positive constants K_1 and c_1 such that for all $t \in V$ and all $\mathbf{k} \in \mathbb{N}_2^p$ we have*

$$|e^{-\langle \mathbf{k},\boldsymbol{\mu}\rangle} \beta_{\mathbf{k}}(t)| \leqslant K_1\, e^{c_1|t|}$$

and

$$e^{-\Re(\langle \mathbf{k},\boldsymbol{\mu}\rangle)} \sum_{\mathbf{i}}{}' \Big|\mathcal{B}\Big[(1+x^{-1})^{\langle \mathbf{l},\mathbf{b}\rangle} \prod_{j=1}^{r_1} \prod_{h\in\mathcal{J}_j} \binom{l_h}{i_h}\Big(\frac{\alpha_h}{x}\Big)^{i_h}\Big](t)\Big| \leqslant K_1\, e^{c_1|t|}.$$

Proof. The first estimate is shown by Braaksma in [Bra01]. To prove the second estimate we define

$$\varphi_{\mathbf{k}}(x) := \big|(1+x^{-1})^{\langle \mathbf{k},\mathbf{b}\rangle}\big| + \sum_{\mathbf{i}}{}' \big|(1+x^{-1})^{\langle \mathbf{l},\mathbf{b}\rangle}\big| \prod_{j=1}^{r_1} \prod_{h\in\mathcal{J}_j} \binom{l_h}{i_h} |x|^{-i_h}, \quad \mathbf{k} \in \mathbb{N}^p,$$

then $\sum_{\mathbf{k}\in\mathbb{N}^p} \varphi_{\mathbf{k}}(x) \vartheta^{\mathbf{k}} = \sum_{\mathbf{l}\in\mathbb{N}^p} \big|(1+x^{-1})^{\langle \mathbf{l},\mathbf{b}\rangle}\big| \prod_{j=1}^{r_1} \big[(\mathrm{I}_{n_j} + |x|^{-1}\tilde{\mathrm{N}}_{n_j})\vartheta^{[j]}\big]^{\mathbf{l}_j}$, where $\tilde{\mathrm{N}}_{n_j}$ is the $n_j \times n_j$ nilpotent matrix with only ones on the first super diagonal. For $j \in \{1,2,\dots,r_1\}$ we now put $\vartheta_h = \zeta_j$ for all $h \in \mathcal{J}_j$, then this identity can be rewritten in the form

$$\sum_{\mathbf{p}\in\mathbb{N}^{r_1}} \sum_{\mathbf{k}:\mathbf{p_k}=\mathbf{p}} \varphi_{\mathbf{k}}(x)\, \zeta^{\mathbf{p}} = \sum_{\mathbf{p}\in\mathbb{N}^{r_1}} \Big\{ \sum_{\mathbf{l}:\mathbf{p_l}=\mathbf{p}} \big|(1+x^{-1})^{\langle \mathbf{l},\mathbf{b}\rangle}\big| \prod_{j=1}^{r_1} (1+|x|^{-1})^{p_j - l_{n_1+\cdots+n_j}} \Big\} \zeta^{\mathbf{p}}.$$

As each $\varphi_{\mathbf{k}}$ is positive we have for a multi-index $\mathbf{k} \in \mathbb{N}^p$ with $\mathbf{p}_{\mathbf{k}} = \mathbf{p}$:

$$\varphi_{\mathbf{k}}(x) \leqslant \sum_{\mathbf{l}:\mathbf{p}_{\mathbf{l}}=\mathbf{p}} \left|(1+x^{-1})^{\langle \mathbf{l},\mathbf{b}\rangle}\right| \prod_{j=1}^{r_1} (1+|x|^{-1})^{p_j - l_{n_1+\cdots+n_j}}.$$

First we observe that, if $|x| \geqslant 2$, then $|\log(1+x^{-1})| \leqslant \frac{2}{|x|}$. So if $\mathbf{p}_{\mathbf{l}} = \mathbf{p}$ and b is defined by $b := \max_{j=1,2,\ldots,n} |b_j|$, we obtain $|(1+x^{-1})^{\langle \mathbf{l},\mathbf{b}\rangle}| \leqslant e^{2b|\mathbf{p}||x|^{-1}}$. Moreover,

$$\sum_{\mathbf{l}:\mathbf{p}_{\mathbf{l}}=\mathbf{p}} \prod_{j=1}^{r_1} (1+|x|^{-1})^{p_j - l_{n_1+\cdots+n_j}} \leqslant \prod_{j=1}^{r_1} \sum_{\substack{\mathbf{l}_j \in \mathbb{N}^{n_j} \\ |\mathbf{l}_j| = p_j}} (1+|x|^{-1})^{p_j - l_{n_1+\cdots+n_j}}.$$

Now the sum in the right-hand side of this equality, with the index j omitted, equals

$$\sum_{\substack{\mathbf{l}\in\mathbb{N}^n \\ |\mathbf{l}|=p}} (1+|x|^{-1})^{p-l_n} = \sum_{h=0}^{p} \sum_{\substack{l_1,l_2,\ldots,l_{n-1}\geqslant 0 \\ l_1+l_2+\cdots+l_{n-1}=h}} (1+|x|^{-1})^h = \sum_{h=0}^{p} \binom{h+n-2}{h} (1+|x|^{-1})^h,$$

which can be estimated by $\mathit{const.} \cdot p^n (1+|x|^{-1})^p$ provided that $p \geqslant 1$ and $n \geqslant 2$. This follows from the fact that $\binom{h+n-2}{h} \sim \frac{h^{n-2}}{\Gamma(n-1)}$ as $h \to \infty$. Combining these results we find for $\mathbf{k} \in \mathbb{N}_2^p$, $\mathbf{p}_{\mathbf{k}} = \mathbf{p}$, and $|x| \geqslant 2$

$$\varphi_{\mathbf{k}}(x) \leqslant \mathit{const.} \cdot |\mathbf{p}|^n e^{2b|\mathbf{p}||x|^{-1}} (1+|x|^{-1})^{|\mathbf{p}|} \leqslant \mathit{const.} \cdot |\mathbf{p}|^n e^{2(b+1)|\mathbf{p}||x|^{-1}}.$$

Now let $\delta := \frac{1}{2}\min\{\Re\mu_j \mid j = 1,2,\ldots,r_1\}$, then $\delta > 0$ and $e^{-\Re(\langle \mathbf{k},\boldsymbol{\mu}\rangle)} \leqslant e^{-2\delta|\mathbf{k}|}$, so if $|x| \geqslant 2$, then

$$e^{-\Re(\langle \mathbf{k},\boldsymbol{\mu}\rangle)} \varphi_{\mathbf{k}}(x) \leqslant \frac{\mathit{const.} \cdot n!}{\delta^n} e^{-|\mathbf{k}|(\delta - 2(b+1)|x|^{-1})}.$$

Now $e^{-\langle \mathbf{k},\boldsymbol{\mu}\rangle}(\mathcal{B}\varphi_{\mathbf{k}})(t) = \frac{1}{2\pi i}\int_\gamma e^{tx} e^{-\langle \mathbf{k},\boldsymbol{\mu}\rangle}\varphi_{\mathbf{k}}(x)\mathrm{d}x$, where γ is the contour given in figure 1.1. This path of integration may be chosen in such a way that $|x| \geqslant M$ on γ for some positive M. Now from the estimate above we see that by choosing M large enough we obtain $e^{-\Re(\langle \mathbf{k},\boldsymbol{\mu}\rangle)}\varphi_{\mathbf{k}}(x) \leqslant \mathit{const.} \cdot n!\,\delta^{-n}$ for all $x \in \gamma$. Hence, $|e^{-\langle \mathbf{k},\boldsymbol{\mu}\rangle}(\mathcal{B}\varphi_{\mathbf{k}})(t)| \leqslant K_1 e^{c_1|t|}$ for some positive K_1 and c_1. ∎

Proposition 4.5.5 *Let V be as in proposition 2.3.5. Then there exist positive constants R and K such that*

$$|W_{\mathbf{k}}(t)| \leqslant K^{|\mathbf{k}|} \frac{|t|^{|\mathbf{k}|-1}}{(|\mathbf{k}|-1)!} e^{R|t|}$$

for all $\mathbf{k} \in \mathbb{N}^p \backslash \{0\}$ and all $t \in V$.

PROOF. As $B = \mathcal{B}\Lambda + D(Y_0)$, and Λ is holomorphic in a neighbourhood of ∞, we see that lemma 4.3.2 remains valid with this new B. From the definition of V it follows that there exists a positive constant K_0 such that $|(e^{-t-\langle \mathbf{k},\boldsymbol{\mu}\rangle} - \Lambda_0)^{-1}| \leqslant K_0$ for all $t \in V$ and all $\mathbf{k} \in \mathbb{N}_2^p$. Using lemma 4.5.4, the proof of this proposition can be completed in a similar way as the proof of proposition 4.3.3. ∎

4.5.3 Analytic Reduction to an Almost Semi-Canonical Form

Let θ_- and θ_+ be two consecutive singular directions of the set of all $\tilde{y}_{\mathbf{k}}$, $\mathbf{k} \in \mathbb{N}^p$, in the right half plane. Take a positive ε smaller than $\frac{1}{2}(\theta_+ - \theta_-)$ and define the sector S_1 by $S_1 := \{x \in \mathbb{C}^* \mid \arg x \in (-\pi/2 - \theta_+ + \varepsilon, \pi/2 - \theta_- - \varepsilon)\}$. Similarly to (2.3.2) we write

$$y = y_0(x) + \tilde{P}(x, \vartheta) := y_0(x) + \tilde{P}_1(x, \tilde{u}) + \tilde{v}, \quad \tilde{P}_1(x, \tilde{u}) := \sum_{\mathbf{k} \in \mathbb{N}_1^p} \tilde{y}_{\mathbf{k}}(x) \tilde{u}^{\mathbf{k}},$$

where $\tilde{y}_{\mathbf{k}}$, $\mathbf{k} \in \mathbb{N}^p$, now denotes the Borel sum of the corresponding formal series and where

$$\vartheta = \sum_{m=1}^{n} \vartheta_m \mathbf{e}_m, \quad \tilde{u} := \sum_{m=1}^{p} \vartheta_m \mathbf{e}_m \quad \text{and} \quad \tilde{v} := \vartheta - \tilde{u} = \sum_{m=p+1}^{n} \vartheta_m \mathbf{e}_m.$$

Theorem 4.5.6 *There exist positive constants δ and $\tilde{\rho}$ such that $\tilde{P}_1(x, \tilde{u})$ converges uniformly for $|\tilde{u}| \leqslant \delta$ if $x \in S_1$, $|x| \geqslant \tilde{\rho}$, and $\tilde{P}_1(x, \tilde{u})$ is the Borel sum of the corresponding formal series with respect to x on S_1 for $|\tilde{u}| \leqslant \delta$.*

Moreover, by means of $y = y_0(x) + \tilde{P}(x, \vartheta)$ the difference equation (4.1.2) is, in a neighbourhood of ∞ in S_1, transformed into

$$\vartheta(x+1) = \Lambda(x)(1 + x^{-1})^{-\mathtt{N}}(\mathtt{I} + x^{-1}\mathtt{N})\vartheta(x) + \Lambda_2\big(x, \vartheta(x)\big)\tilde{v}(x), \tag{4.5.10}$$

where $\Lambda_2(x, \vartheta) = O(x^{-2}) + O(|\vartheta|)$ as $x \to \infty$ in S_1 and $\vartheta \to 0$.

Proof. The first part is shown in a similar way as the corresponding theorem in the differential case (compare also the proof of proposition 3.3.1).

To prove the second part we again define $w(x) := y(x) - y_0(x)$, with y being a solution of (4.1.2). Then w satisfies the difference equation $w(x+1) = \Lambda_1(x)w(x) + d(x, w)$, where Λ_1 and d are as in (2.2.3) and (2.2.2) respectively. If $w(x) = \tilde{P}\big(x, \vartheta(x)\big)$ we thus obtain

$$\tilde{P}\big(x+1, \vartheta(x+1)\big) = \Lambda_1(x)\tilde{P}\big(x, \vartheta(x)\big) + d\Big(x, \tilde{P}\big(x, \vartheta(x)\big)\Big). \tag{4.5.11}$$

By definition we have

$$\tilde{P}_1\big(x+1, \Lambda(x)(1 + x^{-1})^{-\mathtt{N}}(\mathtt{I} + x^{-1}\mathtt{N})\tilde{u}\big) = \sum_{\mathbf{k} \in \mathbb{N}_1^p} \tilde{y}_{\mathbf{k}}(x+1)\big[\Lambda(x)(1 + x^{-1})^{-\mathtt{N}}(\mathtt{I} + x^{-1}\mathtt{N})\tilde{u}\big]^{\mathbf{k}}$$

and as in section 4.5.1 we deduce that the right-hand side of the latter equation can be rewritten as

$$\sum_{\mathbf{k} \in \mathbb{N}_1^p} \big[e^{-\langle \mathbf{k}, \boldsymbol{\mu} \rangle}(1 + x^{-1})^{\langle \mathbf{k}, \mathbf{a} \rangle}\tilde{y}_{\mathbf{k}}(x+1) + s_{\mathbf{k}}(x)\big]\tilde{u}^{\mathbf{k}}(x).$$

Using (4.5.6) we then obtain

$$\tilde{P}_1\big(x+1, \Lambda(x)(1 + x^{-1})^{-\mathtt{N}}(\mathtt{I} + x^{-1}\mathtt{N})\tilde{u}\big) = \Lambda_1(x)\tilde{P}_1(x, \tilde{u}) + d\big(x, \tilde{P}_1(x, \tilde{u})\big), \tag{4.5.12}$$

hence

$$\begin{aligned}&\tilde{P}\big(x+1,\Lambda(x)(1+x^{-1})^{-\mathtt{N}}(\mathtt{I}+x^{-1}\mathtt{N})\vartheta\big)\\&\quad=\tilde{P}_1\big(x+1,\Lambda(x)(1+x^{-1})^{-\mathtt{N}}(\mathtt{I}+x^{-1}\mathtt{N})\tilde{u}\big)+\Lambda(x)(1+x^{-1})^{-\mathtt{N}}(\mathtt{I}+x^{-1}\mathtt{N})\tilde{v}\\&\quad=\Lambda_1(x)\tilde{P}(x,\vartheta)+d\big(x,\tilde{P}_1(x,\tilde{u})\big)+\big[\Lambda(x)(1+x^{-1})^{-\mathtt{N}}(\mathtt{I}+x^{-1}\mathtt{N})-\Lambda_1(x)\big]\tilde{v}.\end{aligned}$$

From the latter equation one may deduce an expression for $\Lambda_1(x)\tilde{P}(x,\vartheta)$ and substituting this expression in (4.5.11) we finally obtain

$$\tilde{P}\big(x+1,\vartheta(x+1)\big)=\tilde{P}\big(x+1,\Lambda(x)(1+x^{-1})^{-\mathtt{N}}(\mathtt{I}+x^{-1}\mathtt{N})\vartheta(x)\big)+r_1\big(x,\vartheta(x)\big),\quad(4.5.13)$$

where $r_1(x,\vartheta)=d\big(x,\tilde{P}(x,\vartheta)\big)-d\big(x,\tilde{P}_1(x,\tilde{u})\big)-\big[\Lambda(x)(1+x^{-1})^{-\mathtt{N}}(\mathtt{I}+x^{-1}\mathtt{N})-\Lambda_1(x)\big]\tilde{v}$.

Since $\tilde{P}_1(x,\tilde{u})$ has linear part $\sum_{m=1}^{p}\tilde{y}_{\mathbf{e}_m}(x)\tilde{u}_m$ which behaves like $\tilde{u}$ as $x\to\infty$ in S_1, the implicit function theorem implies that the function $\vartheta\mapsto\tilde{P}(x,\vartheta)$ is invertible in a neighbourhood of ∞ in S_1, say with inverse $\tilde{Q}$. Applying this inverse to both sides of (4.5.13) we obtain

$$\vartheta(x+1)=\Lambda(x)(1+x^{-1})^{-\mathtt{N}}(\mathtt{I}+x^{-1}\mathtt{N})\vartheta(x)+r\big(x,\vartheta(x)\big),$$

where

$$\begin{aligned}r(x,\vartheta)=\tilde{Q}\big(x+1,\big[\tilde{P}\big(x+1,\,\Lambda(x)(1+x^{-1})^{-\mathtt{N}}(\mathtt{I}+x^{-1}\mathtt{N})\vartheta\big)+r_1(x,\vartheta)\big]\big)\;-&\\ \tilde{Q}\big(x+1,\tilde{P}\big(x+1,\Lambda(x)(1+x^{-1})^{-\mathtt{N}}(\mathtt{I}+x^{-1}\mathtt{N})\vartheta\big)\big).&\end{aligned}$$

Since $(1+x^{-1})^{-\mathtt{N}}(\mathtt{I}+x^{-1}\mathtt{N})-\mathtt{I}=O(x^{-2})$ we deduce, as in the proof of the corresponding theorem for differential equations, that $r_1(x,\vartheta)$ can be written as $\tilde{\Lambda}_2(x,\vartheta)\tilde{v}$, for some $\tilde{\Lambda}_2$ of the form $\tilde{\Lambda}_2(x,\vartheta)=O(x^{-2})+O(|\vartheta|)$ as $x\to\infty$ in S_1 and $\vartheta\to0$. From this we conclude that $r(x,\vartheta)$ can be written as $\Lambda_2(x,\vartheta)\tilde{v}$ as in the statement of the theorem. ∎

Corollary 4.5.7 *On the manifold defined by $y=y_0(x)+\tilde{P}_1(x,\tilde{u})$ the difference equation (4.1.2) is transformed into $\tilde{u}(x+1)=\Lambda(x)(1+x^{-1})^{-\mathtt{N}}(\mathtt{I}+x^{-1}\mathtt{N})\tilde{u}(x)$.*

4.5.4 Analytic Reduction on a Manifold

If $\vartheta=\mathtt{A}z$ transforms $\vartheta(x+1)=\Lambda(x)(1+x^{-1})^{-\mathtt{N}}(\mathtt{I}+x^{-1}\mathtt{N})\vartheta(x)$ into $z(x+1)=\Lambda(x)z(x)$, then $\mathtt{A}$ satisfies

$$\mathtt{A}(x+1)\Lambda(x)=\Lambda(x)(1+x^{-1})^{-\mathtt{N}}(\mathtt{I}+x^{-1}\mathtt{N})\mathtt{A}(x).$$

We recall that $\Lambda(x)=\oplus_{j=1}^{r}e^{-\mu_j}(1+x^{-1})^{\mathtt{M}_j}$, with $\mathtt{M}_j=\mathrm{diag}\{a_m,\ m\in\mathcal{J}_j\}+\mathtt{N}_{n_j}$ in Jordan canonical form. So each $\mathtt{M}_j$ also is a direct sum with the number of terms equal to the number of different eigenvalues of $\mathtt{M}_j$. If this number equals l_j, then each $\mathtt{M}_j$ can be written as $\mathtt{M}_j=\oplus_{k=1}^{l_j}\mathtt{M}_{j,k}$, where $\mathtt{M}_{j,k}=a_{j,k}\mathtt{I}_{j,k}+\mathtt{N}_{j,k}$. Here $a_{j,k}$ is an eigenvalue of $\mathtt{M}_j$, $\mathtt{I}_{j,k}$ is an identity-matrix and $\mathtt{N}_{j,k}$ is a nilpotent matrix.

If we assume that $\mathtt{A}$ is block-diagonal with the same partition as Λ, so $\mathtt{A} = \oplus_{j=1}^{r}\mathtt{A}_j$, $\mathtt{A}_j = \oplus_{k=1}^{l_j}\mathtt{A}_{j,k}$, then we easily deduce that the equation for $\mathtt{A}$ is equivalent to each block of $\mathtt{A}$ satisfying the equation

$$\mathtt{A}_{j,k}(x+1)(1+x^{-1})^{\mathtt{N}_{j,k}} = (\mathtt{I}_{j,k} + x^{-1}\mathtt{N}_{j,k})\mathtt{A}_{j,k}(x). \tag{4.5.14}$$

In the following we will omit the indices j and k.

Substituting a formal series $\mathtt{A}(x) = \sum_{m=0}^{\infty} \mathtt{A}_m x^{-m}$ in (4.5.14) and comparing coefficients we obtain

$$\begin{cases} \mathtt{A}_0 = \mathtt{A}_0, \\ \mathtt{A}_1 + \mathtt{A}_0\mathtt{N} = \mathtt{A}_1 + \mathtt{N}\mathtt{A}_0, \\ \mathtt{A}_m - (m-1)\mathtt{A}_{m-1} + \mathtt{A}_{m-1}\mathtt{N} + \varphi_m(\mathtt{A}_0, \dots, \mathtt{A}_{m-2}) = \mathtt{A}_m + \mathtt{N}\mathtt{A}_{m-1}, \quad m \geqslant 2, \end{cases}$$

where $\varphi_m(\mathtt{A}_0, \dots, \mathtt{A}_{m-2})$ is an expression only depending on $\mathtt{A}_k$, $0 \leqslant k \leqslant m-2$. If we prescribe $\mathtt{A}_0$ to be the identity-matrix, then the first two equations are trivially satisfied. Now assume that we have found $\mathtt{A}_0$ up to $\mathtt{A}_{m-2}$ for some $m \geqslant 2$, then the third equation can be written as

$$\mathtt{A}_{m-1}\big[\mathtt{N} - (m-1)\mathtt{I}\big] - \mathtt{N}\mathtt{A}_{m-1} = -\varphi_m(\mathtt{A}_0, \dots, \mathtt{A}_{m-2}).$$

As for $m \geqslant 2$, $\mathtt{N} - (m-1)\mathtt{I}$ and $\mathtt{N}$ have no common eigenvalues, this system has a unique solution $\mathtt{A}_{m-1}$ (compare [Was87], theorem 4.1). Hence, the equation (4.5.14) possesses a unique formal solution with the constant coefficient equal to the identity-matrix for every $k \in \{1, 2, \dots, l_j\}$ and every $j \in \{1, 2, \dots, r\}$.

To show that each of these formal series is Borel summable, we first observe that (4.5.14) is equivalent to $\mathtt{A}(x+1) = (\mathtt{I} + x^{-1}\mathtt{N})\mathtt{A}(x)(1+x^{-1})^{-\mathtt{N}}$, where again we omit the indices j and k. In vector notation this equation can be written as $v(x+1) = B(x)v(x)$, with $B(\infty) = \mathtt{I}$. Braaksma showed in [Bra80], section 3, that the Borel transform of v converges in a neighbourhood of the origin, can be analytically continued in every sector not containing the imaginary axis and is of at most exponential growth there. Hence, the formal solution $\sum_{m=0}^{\infty} \mathtt{A}_m x^{-m}$ is Borel summable in every direction $-\theta$, $\theta \neq \pm\frac{\pi}{2}$.

Hence, we deduce the existence of a transformation $\vartheta = \mathtt{A}z$, with $\mathtt{A}$ holomorphic in a neighbourhood of ∞ in $S(0, 2\pi)$, that transforms the equation for ϑ into the normal form $z(x+1) = \Lambda(x)z(x)$.

Now we introduce a similar splitting of the vector z as we had for ϑ

$$z = \sum_{m=1}^{n} z_m \mathbf{e}_m, \quad u = \sum_{m=1}^{p} z_m \mathbf{e}_m \quad \text{and} \quad v = \sum_{m=p+1}^{n} z_m \mathbf{e}_m.$$

Let $\mathcal{M}$ be the manifold defined by $y = y_0(x) + P_1(x,u)$, $P_1(x,u) = \tilde{P}_1(x, \mathtt{A}u)$, with x in a neighbourhood of ∞ in S_1 and u in a neighbourhood of 0. Here $\tilde{P}_1$ and S_1 are defined in the introduction of section 4.5.3. Now, $P_1(x,u) = \tilde{P}_1(x, \mathtt{A}u)$ can be rewritten as

$$P_1(x,u) = \sum_{\mathbf{k} \in \mathbb{N}_1^p} y_{\mathbf{k}}(x) u^{\mathbf{k}}$$

for certain functions $y_{\mathbf{k}}$, which are related to the Borel sums $\tilde{y}_{\mathbf{k}'}$, $\mathbf{k}' \in \mathbb{N}_1^p$, involving the elements of $\mathtt{A}$. An immediate consequence of theorem 4.5.6 is that $P_1(x,u)$ converges uniformly for u in a neighbourhood of 0 and x in a neighbourhood of ∞ in S_1. Moreover corollary 4.5.7 implies that on $\mathcal{M}$ the original difference equation (4.1.2) is transformed into the semi-canonical form $u(x+1) = \Lambda(x)u(x)$.

Theorem 4.5.8 *On the manifold $\mathcal{M}$ the difference equation (4.1.2) is transformed into $u(x+1) = \Lambda(x)u(x)$.*

Let S_2 be a sub-sector of S_1 containing the positive real axis and let y be a solution of (4.1.2) such that $y(x) = O(x^{-2})$ as $x \to \infty$ on S_2. Then y belongs to the restriction of $\mathcal{M}$ to S_2. Moreover, there exists a unique 1-periodic trigonometric polynomial C, with values in $\mathbb{C}^p$, such that

$$y(x) = \sum_{\mathbf{k}\in\mathbb{N}^p} e^{-\langle\mathbf{k},\boldsymbol{\mu}\rangle x}\, x^{\langle\mathbf{k},\mathbf{a}\rangle} \prod_{j=1}^{r_1}\bigl(x^{\mathtt{N}_{n_j}} C^{[j]}(x)\bigr)^{\mathbf{k}_j} y_{\mathbf{k}}(x), \quad x \in S_2. \tag{4.5.15}$$

Here $C^{[j]}(x)e^{-\mu_j x}$, $j \in \{1,2,\ldots,r_1\}$, is exponentially small, uniformly in a neighbourhood of ∞ in any closed sub-sector of S_2.

Conversely, if C is a p-vector valued 1-periodic trigonometric polynomial such that $C^{[j]}(x)e^{-\mu_j x} \to 0$ as $x \to \infty$ in S_2 for every $j \in \{1,2,\ldots,r_1\}$, then the sum in (4.5.15) converges in a neighbourhood of ∞ in S_2, defines a solution of (4.1.2) which is holomorphic in this neighbourhood and has the property $y(x) \sim \hat{y}_0(x)$ as $x \to \infty$ in S_2.

Proof. Assume that y is a solution of (4.1.2), $y(x) = O(x^{-2})$ as $x \to \infty$ on S_2. Then put $w(x) = y(x) - y_0(x) = O(x^{-2})$ on S_2 and let ϑ, $\tilde{u}$ and $\tilde{v}$ be associated with these solutions as in the introduction of section 4.5.3. Moreover, define u by $\tilde{u}(x) = \mathtt{A}(x)u(x)$. As $\vartheta \mapsto \tilde{P}(x,\vartheta)$ is invertible for x in a neighbourhood of ∞ in S_2 and ϑ in a neighbourhood of the origin (cf. the proof of theorem 4.5.6), we can conclude that $\vartheta(x)$, and thus $\tilde{u}(x)$ and $\tilde{v}(x)$, also are $O(x^{-2})$ as $x \to \infty$ in S_2.

From (4.5.10) one may conclude that $\tilde{v}$ satisfies an equation of the form (B.0.3), with $A(x) = \Lambda_2\bigl(x,\vartheta(x)\bigr) + \Lambda(x)\{(1+x^{-1})^{-\mathtt{N}}(\mathtt{I}+x^{-1}\mathtt{N}) - \mathtt{I}\}$. Now, let S_3 be a sub-sector of S_2 with opening less than π and containing the positive real axis. Then theorem B.0.2 implies that there exists a 1-periodic $(n-p)$-vector valued holomorphic function C such that $\tilde{v}(x) = V(x)C(x)$, as $x \to \infty$ in S_3, where $V(x) = \bigl(\mathtt{I}+O(x^{-1})\bigr)\bigl(\oplus_{j=r_1+1}^{r} e^{-\mu_j x}x^{\mathtt{M}_j}\bigr)$. As $\tilde{v}(x) = O(x^{-2})$ we may conclude in a similar way as Braaksma did in [Bra01] that $\tilde{v} \equiv 0$ on S_3 and thus on S_2 (compare also the proof of theorem 4.3.6). The equation for u then reduces to $u(x+1) = \Lambda(x)u(x)$, which has the general solution $u(x) = \oplus_{j=1}^{r_1} e^{-\mu_j x}x^{\mathtt{M}_j}C^{[j]}(x)$ for some n_j-vector valued 1-periodic functions $C^{[j]}(x)$. The proof is completed by using lemma 8 in [Bra01]. ∎

4.5.5 A Resurgence Relation

Let $j \in \{1,2,\ldots,r\}$ and assume that the line-segment $(0,\mu_j)$ does not contain any singular point $\mu_h + 2l\pi i$ of $Y_0 = \mathcal{B}y_0$, with $h \in \{1,2,\ldots,r\}$ and $l \in \mathbb{Z}$. Moreover, for each

$h \in \{1, 2, \ldots, r\}$ we assume that $a_k = a_l$ whenever $k, l \in \mathcal{J}_h$ and we assume $\Re a_h \notin \mathbb{Z}$ for all $h \in \{1, 2, \ldots, n\}$. Hence, by introducing new quantities a_h, $h = 1, 2, \ldots, r$, we can assume that Λ has the form

$$\Lambda(x) = \bigoplus_{h=1}^{r} e^{-\mu_h}(1 + x^{-1})^{\mathtt{M}_h}, \quad \mathtt{M}_h = a_h \mathtt{I}_{n_h} + \mathtt{N}_{n_h},$$

with $a_h \in \mathbb{C}$, $\Re a_h \notin \mathbb{Z}$. Moreover, one can again assume that $\Re a_h > 0$ for all $h = 1, 2, \ldots, r$. Let m_h be the integer chosen in such a way that $\Re a_h - m_h \in (-1, 0)$ and put $\tilde{a}_h = a_h - m_h$.

By substitution of $y = y_0(x) + P_1(x, u)$ into the original difference equation (4.1.2) and using $u(x + 1) = \Lambda(x)u(x)$, one can show that, for example $y_{\mathbf{e}_h}$, with $h \in \{1, 2, \ldots, n_1\}$, has to satisfy

$$e^{-\mu_1}(1 + x^{-1})^{a_1} \sum_{l=1}^{n_1} \big((1 + x^{-1})^{\mathtt{N}_{n_1}}\big)_{l,h} y_{\mathbf{e}_l}(x + 1) = \Lambda_1(x) y_{\mathbf{e}_h}(x), \tag{4.5.16}$$

in which $\big((1+x^{-1})^{\mathtt{N}_{n_1}}\big)_{l,h}$ denotes the element of $(1+x^{-1})^{\mathtt{N}_{n_1}}$ at the l^{th} row and h^{th} column. Using this we will prove that the analogue of proposition 4.4.1 holds.

Proposition 4.5.9 *Let $j \in \{1, 2, \ldots, r\}$ be such that $(0, \mu_j)$ does not contain any singular point of Y_0 and let Y_0^+ and Y_0^- be related to Y_0 in the way described in the introduction of section 2.4.2. Then there exists a vector $\mathbf{s} = (s_1, s_2, \ldots, s_n) \in \mathbb{C}^n$ such that*

$$Y_0^+(\mu_j + t) - Y_0^-(\mu_j + t) = \sum_{h \in \mathcal{J}_j} \big\{\mathcal{B}\big[(x^{\mathtt{N}_{n_j}} \mathbf{s}^{[j]})^{\mathbf{e}_{h-(n_1+\cdots+n_{j-1})}} x^{\tilde{a}_j} y_{\mathbf{e}_h}(x)\big]\big\}^{(m_j)}$$

for $\arg t = \theta_j = \arg \mu_j$ and $|t|$ sufficiently small.

Proof. By renumbering we may assume that $j = 1$. The proof of this proposition in fact is a combination of the proofs of propositions 2.4.2 and 2.4.3 and the proof of proposition 4.4.1.

Let H be as in the introduction of section 2.4.1, and let Ψ_0 and Ψ be as in the beginning of the proof of proposition 2.4.2. If $\psi = \mathcal{P}_\eta^{m_1+1}\Psi$, with $(\mathcal{P}_\eta\Psi)(s) := \int_\eta^s \Psi(\sigma)\mathrm{d}\sigma$, then ψ satisfies

$$\begin{aligned}(e^{s-\mu_1} - \Lambda_0)\psi'(s) \;&=\; -\int_\eta^s B(\sigma - s)\psi'(\sigma)\mathrm{d}\sigma + \mathcal{P}_\eta^{m_1}[P](s) + \\ &m_1 e^{s-\mu_1}\psi(s) + e^{-\mu_1}\sum_{k=1}^{m_1}(-1)^k \binom{m_1+1}{k+1} \int_\eta^s \frac{(s-\sigma)^{k-1}}{(k-1)!} e^{\sigma}\psi(\sigma)\mathrm{d}\sigma,\end{aligned} \tag{4.5.17}$$

where we used similar notation as in the proof of proposition 2.4.2. In fact, the proof of this statement is just a copy of the first part of the proof of proposition 2.4.2. Now splitting (4.5.17) after the first n_1 components, we obtain (in a similar way as in the proof of proposition 2.4.2)

$$\psi^{[1]}(s) = s^{-\tilde{\mathtt{M}}_1}\int_\eta^s \tau^{\tilde{\mathtt{M}}_1-1}\big\{\tau\tilde{\alpha}(\tau)\psi^{[1]}(\tau)+\alpha(\tau)\mathcal{P}_\eta^{m_1}\big[P^{[1]}\big](\tau)\big\}\,\mathrm{d}\tau\; + \\ s^{-\tilde{\mathtt{M}}_1}\int_\eta^s \tau^{\tilde{\mathtt{M}}_1-1}\alpha(\tau)\int_\eta^\tau C^{[1]}(\sigma,\tau)\psi(\sigma)\,\mathrm{d}\sigma\,\mathrm{d}\tau$$

$$\psi^{\perp}(s) = \int_\eta^s \big(e^{\tau-\mu_1}-(\Lambda_0)^{\perp\perp}\big)^{-1}\big\{E(\tau)\psi^{\perp}(\tau)+\mathcal{P}_\eta^{m_1}[P^{\perp}](\tau)\big\}\,\mathrm{d}\tau\; + \\ \int_\eta^s \big(e^{\tau-\mu_1}-(\Lambda_0)^{\perp\perp}\big)^{-1}\int_\eta^\tau C^{\perp}(\sigma,\tau)\psi(\sigma)\,\mathrm{d}\sigma\,\mathrm{d}\tau,$$

where $\tilde{\mathtt{M}}_1 = \mathtt{M}_1 - m_1\mathtt{I}_{n_1}$,

$$\alpha(s) = \begin{cases} e^{\mu_1}\frac{s}{e^s-1} & \text{if } s\neq 0 \\ e^{\mu_1} & \text{if } s=0 \end{cases} \quad\text{and}\quad \tilde{\alpha}(s) = \begin{cases} \tilde{\mathtt{M}}_1\big(\frac{1}{s}-\frac{1}{e^s-1}\big)+m_1 & \text{if } s\neq 0 \\ \frac{1}{2}\tilde{\mathtt{M}}_1+m_1 & \text{if } s=0 \end{cases}.$$

Moreover, $(\Lambda_0)^{\perp\perp} = \oplus_{j=2}^r e^{-\mu_j}\mathtt{I}_{n_j}$, $E(\tau) = m_1 e^{\tau-\mu_1}\mathtt{I}_{n-n_1} - \oplus_{j=2}^r e^{-\mu_j}\mathtt{M}_j$ and $C = C(\sigma,\tau)$ is as in (2.4.6).

These equations can be uniquely solved in the space $\mathcal{V}_{\tilde{a}_1,\log,0}$ of n-vector valued functions ψ which can be written as $\psi(s) = s^{-\tilde{a}_1}\sum_{k=0}^{n_1-1}(\log s)^k u_k(s)+u_{n_1}(s)$, with u_k, $k=0,1,\ldots,n_1$, continuous on $\overline{\Delta}(0,\varepsilon)$ and holomorphic in the interior $\Delta(0,\varepsilon)$. We choose the branch of $s^{-\tilde{a}_1}$ and $\log s$ with $\theta_1-\pi < \arg s \leqslant \theta_1+\pi$ and we endow $\mathcal{V}_{\tilde{a}_1,\log,0}$ with the norm $\|\psi\| := \sup\big\{\big|u_k(s)\big|,\ |s|\leqslant\varepsilon,\ k=0,1,\ldots,n_1\big\}$.

To prove the resurgence relation we introduce $v(s) := \psi^+(-s)-\psi^-(-s)$, where ψ^+ and ψ^- are the boundary values of ψ on the lower and upper side of the interval $(-\eta,0)$ respectively. By taking limits one may deduce that both ψ^+ and ψ^- are solutions of (4.5.17) on $(-\eta,\eta)$ if we define $\psi^+(s)=\psi^-(s)=\psi(s)$ on $[0,\eta)$. Then $v(s)=0$ on $(-\eta,0]$ and $v(s) = s^{-\tilde{a}_1}\sum_{k=0}^{n_1-1}(\log s)^k w_k(s)$ on $(0,\eta)$ for certain functions w_k holomorphic in $\Delta(0,\varepsilon)$. In the proof of proposition 4.4.1 we saw that the formal Laplace transform of v' exists and in a similar way as in proposition 2.4.3 we deduce that $\hat{\mathcal{L}}v'$ is a formal solution of

$$e^{-\mu_1}(1+x^{-1})^{m_1}y(x+1) = \Lambda_1(x)y(x). \tag{4.5.19}$$

For $h=1,2,\ldots,n_1$ we define $u_h(x) := \sum_{l=1}^{n_1}(x^{\mathtt{N}_{n_1}})_{l,h}\,\hat{y}_{\mathsf{e}_l}(x)$. Using (4.5.16) together with the fact that for $l\in\{1,2,\ldots,n_1-1\}$ we have

$$\big((1+x)^{\mathtt{N}_{n_1}}\big)_{l,h} = \big((1+x^{-1})^{\mathtt{N}_{n_1}}x^{\mathtt{N}_{n_1}}\big)_{l,h} = \sum_{j=1}^{n_1}\big((1+x^{-1})^{\mathtt{N}_{n_1}}\big)_{l,j}\big(x^{\mathtt{N}_{n_1}}\big)_{j,h},$$

we easily infer that each u_h is a solution of $e^{-\mu_1}(1+x^{-1})^{a_1}y(x+1) = \Lambda_1(x)y(x)$. Hence, $\sum_{l=1}^{n_1}(x^{\mathtt{N}_{n_1}})_{l,h}\,x^{\tilde{a}_1}\hat{y}_{\mathsf{e}_l}(x)$ satisfies (4.5.19) and in a similar way one may deduce that for $j\in\{2,3,\ldots,r\}$ and $h\in\{1,2,\ldots,n_j\}$ the expressions $\sum_{l\in\mathcal{J}_j}(x^{\mathtt{N}_{n_j}})_{l-(n_1+\cdots+n_{j-1}),h}\,x^{\tilde{a}_j}\hat{y}_{\mathsf{e}_l}(x)$ satisfy the difference equation $e^{-\mu_j}(1+x^{-1})^{m_j}y(x+1) = \Lambda_1(x)y(x)$.

So when for $j \in \{1, 2, \ldots, r\}$ and $h \in \{1, 2, \ldots, n_j\}$ we define

$$\vartheta_{j,h}(x) := e^{(\mu_1 - \mu_j)x} x^{m_j - m_1} \sum_{l \in \mathcal{J}_j} (x^{\mathtt{N}_{n_j}})_{l-(n_1+\cdots+n_{j-1}),h}\, x^{\tilde{a}_j}\, \hat{y}_{\mathbf{e}_l}(x),$$

we see that $\{\vartheta_{j,h} \mid j = 1, 2, \ldots, r,\ h = 1, 2, \ldots, n_j\}$ forms a formal fundamental system of (4.5.19). Now one may continue as in the proof of proposition 4.4.1 to complete the proof of this proposition. ■

Chapter 5

Multisummability of Transseries for Differential Equations

Until now we considered differential (and difference) equations of only one level, which is called *level one*: the μ_j's appearing in the definition of Λ are always nonzero. In this chapter our aim is to study the formal reduction to a normal form corresponding to differential equations with more than one positive level. Also its analytic analogue and the correspondence with 'small' solutions will be discussed. In this study it turns out that the method of Borel summability is no longer applicable, but a more general summation method, called *multisummability* (cf. section 1.4), can be applied to tackle the problems in this chapter.

5.1 Introduction

In the present chapter we will restrict ourselves to differential equations of the following type with r levels

$$y'(x) + \Lambda(x)y(x) + g(x, y(x)) = 0, \quad \text{with } \Lambda(x) = \bigoplus_{j=1}^{r} \big(x^{q_j-1}\mathtt{M}_j + x^{q_{j+1}-1}\tilde{\mathtt{M}}_j(x)\big), \tag{5.1.1}$$

where $r \in \mathbb{N}_+$, $0 := q_{r+1} < q_r < q_{r-1} < \cdots < q_1$, $q_j \in \mathbb{N}$, $n = n_1 + n_2 + \cdots + n_r$, $n_j \in \mathbb{N}$, and both $\mathtt{M}_j$ and $\tilde{\mathtt{M}}_j(x)$ are $n_j \times n_j$-matrices. If for $j \in \{1, 2, \ldots, r\}$ we define $\mathcal{J}_j := \{n_1 + \cdots + n_{j-1} + 1, \ldots, n_1 + \cdots + n_j\}$, then

$$\begin{cases} \mathtt{M}_j &= \operatorname{diag}\big\{\mu_m^{(j)},\ m \in \mathcal{J}_j\big\} \\ \tilde{\mathtt{M}}_j(x) &= \operatorname{diag}\big\{\mu_m^{(j+1)},\ m \in \mathcal{J}_j\big\} + \\ & \quad + x^{q_{j+2}-q_{j+1}} \operatorname{diag}\big\{\mu_m^{(j+2)},\ m \in \mathcal{J}_j\big\} + \\ & \quad \ddots \\ & \quad + x^{q_r-q_{j+1}} \operatorname{diag}\big\{\mu_m^{(r)},\ m \in \mathcal{J}_j\big\} + \\ & \quad + x^{-q_{j+1}} \operatorname{diag}\big\{-a_m,\ m \in \mathcal{J}_j\big\}. \end{cases} \tag{5.1.2}$$

Now, if $r = 1$ and $q_1 = 1$, then $\Lambda(x) = \operatorname{diag}\{\mu_1^{(1)} - a_1 x^{-1}, \mu_2^{(1)} - a_2 x^{-1}, \ldots, \mu_n^{(1)} - a_n x^{-1}\}$, and we obtain a differential equation of level 1 that we studied in the preceding chapter (compare also [Cos95, Cos98]). Later on we will specify the conditions on $\mu_m^{(j)}$. We assume g to be a holomorphic $\mathbb{C}^n$-valued function of (x, y) in a neighbourhood of $(\infty, 0)$, such that $g(x, y) = O(x^{-2}) + O(|y|^2)$ as $x \to \infty$ and $y \to 0$.

We first consider (5.1.1) with three levels. However, all the statements we prove here can without any difficulty be generalised to the case with more than three levels (see section 5.6). The case of two levels in fact is included in the case $r = 3$, for example by taking $n_1, n_2 > 0$ and $n_3 = 0$. Thus we first consider (5.1.1) with

$$\begin{aligned}\Lambda(x) \;=\; & \operatorname{diag}\{\omega_m x^{q_1-1} + \lambda_m x^{q_2-1} + \mu_m x^{q_3-1} - a_m x^{-1},\ m \in \mathcal{J}_1\} \\ & \oplus \operatorname{diag}\{\lambda_m x^{q_2-1} + \mu_m x^{q_3-1} - a_m x^{-1},\ m \in \mathcal{J}_2\} \\ & \oplus \operatorname{diag}\{\mu_m x^{q_3-1} - a_m x^{-1},\ m \in \mathcal{J}_3\},\end{aligned}$$

where $0 := q_4 < q_3 < q_2 < q_1 < q_0 := \infty$, $q_j \in \mathbb{N}_+$ for $j \in \{1, 2, 3\}$. As before we take g a holomorphic $\mathbb{C}^n$-valued function of (x, y) in a neighbourhood of $(\infty, 0)$ such that $g(x, y) = O(x^{-2}) + O(|y|^2)$ as $x \to \infty$ and $y \to 0$. We assume ω_m, $m \in \mathcal{J}_1$, λ_m, $m \in \mathcal{J}_1 \cup \mathcal{J}_2$, and μ_m, $m \in \mathcal{J}_1 \cup \mathcal{J}_2 \cup \mathcal{J}_3$, to be complex numbers such that the sets $\{\omega_m \mid m \in \mathcal{J}_1\}$, $\{\lambda_m \mid m \in \mathcal{J}_1 \cup \mathcal{J}_2\}$ and $\{\mu_m \mid m \in \mathcal{J}_1 \cup \mathcal{J}_2 \cup \mathcal{J}_3\}$ are linearly independent over $\mathbb{Z}$. In the following we will use the convention

$$\omega_m := 0, \quad m = n_1 + 1, \ldots, n \quad \text{and} \quad \lambda_m := 0, \quad m = n_1 + n_2 + 1, \ldots, n \tag{5.1.3}$$

and in that case Λ can be written as

$$\Lambda(x) = \operatorname{diag}\{\omega_m x^{q_1-1} + \lambda_m x^{q_2-1} + \mu_m x^{q_3-1} - a_m x^{-1}\}_{m=1}^n.$$

It is easily seen that (5.1.1) is equivalent to

$$\Big(\bigoplus_{j=1}^{3} x^{-q_j} \mathrm{I}_{n_j}\Big) x \frac{\mathrm{d}y}{\mathrm{d}x} + \Big(\bigoplus_{j=1}^{3} A_{0,j}\Big) y + \tilde{g}(x, y) = 0, \tag{5.1.4}$$

where $A_{0,1} = \operatorname{diag}\{\omega_m, m \in \mathcal{J}_1\}$, $A_{0,2} = \operatorname{diag}\{\lambda_m, m \in \mathcal{J}_2\}$, $A_{0,3} = \operatorname{diag}\{\mu_m, m \in \mathcal{J}_3\}$ and $\tilde{g}$ some holomorphic function for x in a neighbourhood of ∞ and y in a neighbourhood of 0 and $\tilde{g}(x, y) = O(x^{-2}) + O(x^{-1}|y|) + O(|y|^2)$ as $x \to \infty$ and $y \to 0$. Substituting a series $\hat{y}_0(x) = \sum_{m=1}^{\infty} y_{0,m} x^{-m}$ into this equation one obtains

$$(\oplus_{j=1}^{3} A_{0,j})\, y_{0,m} = \zeta_m, \quad m \geqslant 1,$$

where ζ_m only depends on $y_{0,k}$ with $1 \leqslant k \leqslant m - 1$. As $A_{0,j}$, $j = 1, 2, 3$, is invertible (if $n_j > 0$) we uniquely determine $y_{0,m}$ for $m \in \mathbb{N}_+$. Moreover, since $\zeta_1 = 0$ we have $y_{0,1} = 0$ and thus (5.1.1) has a unique formal solution $\hat{y}_0 \in x^{-2}\mathbb{C}^n[[x^{-1}]]$.

As in chapter 2 and 4 we will study the formal transformation (in the sense as described in the introduction of chapter 1, but now with an algebra consisting of formal expressions

$\sum_{j=0}^{\infty} f_j(x)\, e^{-\frac{\sigma_{1,j}}{q_1}x^{q_1}-\frac{\sigma_{2,j}}{q_2}x^{q_2}-\frac{\sigma_{1,j}}{q_3}x^{q_3}}$ with $f_j \in \mathbb{C}^n((x^{-1}))[\{x^c\}_{c\in\mathbb{C}}]$, $\sigma_{1,j} \in \mathbb{N}\cdot\omega_1+\cdots+\mathbb{N}\cdot\omega_n$, $\sigma_{2,j} \in \mathbb{N}\cdot\lambda_1+\cdots+\mathbb{N}\cdot\lambda_n$ and $\sigma_{3,j} \in \mathbb{N}\cdot\mu_1+\cdots+\mathbb{N}\cdot\mu_n$, endowed with natural differentiation)

$$y = \hat{T}(x,z) := \sum_{\mathbf{k}\in\mathbb{N}^n} \hat{y}_{\mathbf{k}}(x)\, z^{\mathbf{k}}$$

associated with the formal solution $\hat{y}_0$, which formally transforms the differential equation to the normal form

$$z'(x) + \Lambda(x)z(x) = 0. \tag{5.1.5}$$

Besides this formal transformation, the analytic analogue will be discussed. This lifting of $\hat{T}$ to a holomorphic expression again amounts to the association of holomorphic functions $y_{\mathbf{k}}$ with the formal series $\hat{y}_{\mathbf{k}}$ that will be constructed in section 5.2. It is shown by Braaksma in [Bra92] that the formal expressions $\hat{y}_{\mathbf{k}}$ are multisummable and their sums are holomorphic in a neighbourhood of ∞ in a certain sector S of opening larger than π/q_1, where q_1 corresponds to the highest level in the original differential equation (cf. also section 5.3). However, as in chapter 2 and 4 this does not guarantee convergence of the corresponding transseries which in this case have the form

$$\sum_{\mathbf{k}\in\mathbb{N}^n} C^{\mathbf{k}} e^{-\frac{\langle\mathbf{k},\boldsymbol{\omega}\rangle}{q_1}x^{q_1}-\frac{\langle\mathbf{k},\boldsymbol{\lambda}\rangle}{q_2}x^{q_2}-\frac{\langle\mathbf{k},\boldsymbol{\mu}\rangle}{q_3}x^{q_3}} x^{\langle\mathbf{k},\mathbf{a}\rangle} y_{\mathbf{k}}(x), \quad C \in \mathbb{C}^n.$$

In section 5.5 it will be shown that a ‘partial’ formal transformation $\hat{T}_1(x,u)$ can be lifted to a holomorphic expression $T_1(x,u)$, using multisummability, and on the manifold defined by $y = T_1(x,u)$ the original difference equation is then transformed into the semi-canonical form $u'(x) + \Lambda(x)u(x) = 0$. Also the correspondence with solutions $y(x)$ that behave like $\hat{y}_0(x)$ as $x \to \infty$ will be studied: given such a solution on a sub-sector $S' \subset S$, then under some additional assumptions there exists a unique convergent transseries such that y equals the sum of the transseries (compare theorem 5.5.4 and the more general case of r levels in theorem 5.6.1).

5.2 Formal Reduction to a Normal Form

If we require that the transformation $y = \hat{T}(x,z) = \hat{y}_0(x) + \sum_{\mathbf{k}\in\mathbb{N}_1^n} \hat{y}_{\mathbf{k}} z^{\mathbf{k}}$ reduces (5.1.1) to the normal form (5.1.5), then $\hat{y}_{\mathbf{k}}$, $\mathbf{k} \in \mathbb{N}_1^n$, has to be a formal solution of the following linear equation

$$y'(x) + \left\{\Lambda_1(x) - \langle\mathbf{k},\boldsymbol{\omega}\rangle x^{q_1-1} - \langle\mathbf{k},\boldsymbol{\lambda}\rangle x^{q_2-1} - \langle\mathbf{k},\boldsymbol{\mu}\rangle x^{q_3-1} + \langle\mathbf{k},\mathbf{a}\rangle x^{-1}\right\} y(x) + t_{\mathbf{k}}(x) = 0,$$

where we used the convention (5.1.3). Here Λ_1 and $t_{\mathbf{k}}$ are defined in (2.2.3) and (2.2.5) respectively. Moreover $\mathbf{a}$, $\boldsymbol{\mu}$, $\boldsymbol{\lambda}$ and $\boldsymbol{\omega}$ are the n-vectors with components respectively a_j, μ_j, λ_j and ω_j, $j = 1, 2, \ldots, n$.

Remark 5.2.1 The expressions Λ_1 and $t_{\mathbf{k}}$ at first have to be interpreted as formal series. However, we will give them an analytic meaning in the proof of proposition 5.3.6.

Similarly to the equivalence of (5.1.1) and (5.1.4), the equation for $\hat{y}_{\mathbf{k}}$, $\mathbf{k} \in \mathbb{N}_1^n$, is equivalent to

$$\Big(\bigoplus_{j=1}^{3} x^{-q_j}\,\mathtt{I}_{\mathbf{k},j}\Big)x\frac{\mathrm{d}y}{\mathrm{d}x} + \Big(\bigoplus_{j=1}^{3} A_{\mathbf{k},j}\Big)y + A_{\mathbf{k},+}(x)\,y + \tilde{t}_{\mathbf{k}}(x) = 0, \tag{5.2.1}$$

where $\tilde{t}_{\mathbf{k}}(x) = (\oplus_{j=1}^{3} x^{1-q_j}\,\mathtt{I}_{\mathbf{k},j})t_{\mathbf{k}}(x)$ and where $A_{\mathbf{k},+}(x) \in x^{-1}\mathbb{C}^{n\times n}[[x^{-1}]]$. The matrices $A_{\mathbf{k},+}$ and $A_{\mathbf{k},j}$, $j = 1, 2, 3$, are defined by the equation

$$\begin{aligned}\big(\oplus_{j=1}^{3} x^{-q_j}\,\mathtt{I}_{\mathbf{k},j}\big)\big\{x\Lambda_1(x) - \langle\mathbf{k},\boldsymbol{\omega}\rangle x^{q_1} - \langle\mathbf{k},\boldsymbol{\lambda}\rangle x^{q_2} - \langle\mathbf{k},\boldsymbol{\mu}\rangle x^{q_3} + \langle\mathbf{k},\mathbf{a}\rangle\big\} =\\ \oplus_{j=1}^{3} A_{\mathbf{k},j} + A_{\mathbf{k},+}(x).\end{aligned} \tag{5.2.2}$$

The size of the matrices $\mathtt{I}_{\mathbf{k},j}$ and $A_{\mathbf{k},j}$ depends on the multi-index $\mathbf{k} \in \mathbb{N}_1^n$. If for $j \in \{1, 2, 3\}$ we associate with $\mathbf{k}$ the multi-indices $\mathbf{k}_j := \sum_{h\in\mathcal{J}_j}\langle\mathbf{k},\mathbf{e}_h\rangle\mathbf{e}_h$ (compare (4.2.1)), then one can distinguish the following three cases.

I *The case* $\mathbf{k}_1 \neq 0$.
The matrices $\mathtt{I}_{\mathbf{k},2}$, $\mathtt{I}_{\mathbf{k},3}$, $A_{\mathbf{k},2}$ and $A_{\mathbf{k},3}$ are 0×0-matrices, $\mathtt{I}_{\mathbf{k},1} = \mathtt{I}_n$, $A_{\mathbf{k},1}$ is an $n \times n$-matrix of the form

$$A_{\mathbf{k},1} = \mathrm{diag}\{\omega_1 - \langle\mathbf{k},\boldsymbol{\omega}\rangle, \dots, \omega_n - \langle\mathbf{k},\boldsymbol{\omega}\rangle\}.$$

II *The case* $\mathbf{k}_1 = 0$*, but* $\mathbf{k}_2 \neq 0$.
The matrices $\mathtt{I}_{\mathbf{k},3}$ and $A_{\mathbf{k},3}$ are 0×0-matrices, $\mathtt{I}_{\mathbf{k},1} = \mathtt{I}_{n_1}$, $\mathtt{I}_{\mathbf{k},2} = \mathtt{I}_{n_2+n_3}$,

$$A_{\mathbf{k},1} = \mathrm{diag}\{\omega_1, \dots, \omega_{n_1}\}, \quad A_{\mathbf{k},2} = \mathrm{diag}\{\lambda_{n_1+1} - \langle\mathbf{k},\boldsymbol{\lambda}\rangle, \dots, \lambda_n - \langle\mathbf{k},\boldsymbol{\lambda}\rangle\}.$$

III *The case* $\mathbf{k}_1 = \mathbf{k}_2 = 0$.
For $j = 1, 2, 3$ we now have $\mathtt{I}_{\mathbf{k},j} = \mathtt{I}_{n_j}$, while

$$A_{\mathbf{k},1} = \mathrm{diag}\{\omega_1, \dots, \omega_{n_1}\}, \quad A_{\mathbf{k},2} = \mathrm{diag}\{\lambda_{n_1+1}, \dots, \lambda_{n_1+n_2}\}$$

$$A_{\mathbf{k},3} = \mathrm{diag}\{\mu_{n_1+n_2+1} - \langle\mathbf{k},\boldsymbol{\mu}\rangle, \dots, \mu_n - \langle\mathbf{k},\boldsymbol{\mu}\rangle\}.$$

Proposition 5.2.2 *For each* $\mathbf{k} \succ 0$ *the differential equation (5.2.1) possesses a formal solution* $\hat{y}_{\mathbf{k}} \in \mathbb{C}^n[[x^{-1}]]$*. For* $j \in \{1, 2, \dots, n\}$ *the series* $\hat{y}_{\mathbf{e}_j}$ *is unique if we prescribe the constant term in* $\hat{y}_{\mathbf{e}_j}$ *to be equal to* $\mathbf{e}_j$*. Then also the series* $\hat{y}_{\mathbf{k}}$ *for* $|\mathbf{k}| > 1$ *are unique.*

Proof. The proof is given with induction on the length of $\mathbf{k}$. Moreover, for each $\mathbf{k} \in \mathbb{N}_1^n$ taken into consideration, we will use the notation $A := \oplus_{j=1}^{3} A_{\mathbf{k},j}$ and $B(x) := A_{\mathbf{k},+}(x)$, so A and B may have different meanings in different places.

If $\mathbf{k}$ is a multi-index of length 1, then one has to distinguish between the cases $\mathbf{k} = \mathbf{e}_m$, with $m \in \mathcal{J}_1$, $m \in \mathcal{J}_2$ or $m \in \mathcal{J}_3$. As an example of $m \in \mathcal{J}_1$ we take $m = 1$ and in that case (5.2.1) reads

$$x^{1-q_1}y'(x) + A\,y(x) + B(x)\,y(x) = 0, \tag{5.2.3}$$

where $A = \text{diag}\{\omega_j - \omega_1\}_{j=1}^n$ and where

$$B(x) = \text{diag}\big\{(\lambda_j - \lambda_1)x^{q_2-q_1} + (\mu_j - \mu_1)x^{q_3-q_1} - (a_j - a_1)x^{-q_1}\big\}_{j=1}^n + x^{1-q_1}\tilde{\Lambda}(x).$$

Here $\tilde{\Lambda}(x) = \text{col}\{d_{\mathbf{e}_1}(x), d_{\mathbf{e}_2}(x), \dots, d_{\mathbf{e}_n}(x)\} = O(x^{-2})$ as $x \to \infty$ (cf. (2.2.3)). Hence, B can be expanded as $B(x) = \sum_{m=q_1-q_2}^{\infty} B_m x^{-m}$ with B_m, $m = q_1 - q_2, \dots, q_1$, a diagonal matrix with the first diagonal element equal to 0. Substituting a series $\hat{y}(x) = \sum_{m=0}^{\infty} y_m x^{-m}$ in (5.2.3) and comparing coefficients of x^{-m}, $m \geqslant 0$, one obtains the following recursive relation:

$$\begin{cases} 0 &= A\, y_m + \sum_{l=0}^{m-q_1+q_2} B_{m-l}\, y_l, \qquad m \leqslant q_1 \\ (m-q_1) y_{m-q_1} &= A\, y_m + \sum_{l=0}^{m-q_1+q_2} B_{m-l}\, y_l, \qquad m \geqslant q_1 + 1. \end{cases}$$

For $m \leqslant q_1$ we easily deduce that the last $n-1$ components of y_m have to be equal to 0, while the first component of y_m is undetermined yet. The equations corresponding to $m \geqslant q_1 + 1$ can be written as

$$(m-q_1) y_{m-q_1} = A\, y_m + \sum_{l=0}^{m-q_1-1} B_{m-l}\, y_l + \sum_{l=m-q_1}^{m-q_1+q_2} B_{m-l}\, y_l, \quad m \geqslant q_1 + 1 \tag{5.2.4}$$

and the latter sum has first component equal to 0 since the first diagonal element of B_m, $m = q_1 - q_2, \dots, q_1$, is equal to 0. Now, let us prescribe $y_0 := \mathbf{e}_1$, then the equation corresponding $m = q_1 + 1$ uniquely determines the first component of y_1 and the last $n-1$ components of y_{q_1+1}. In general, (5.2.4) with $m \geqslant q_1 + 1$ recursively gives the first component of y_{m-q_1} and and the last $n-1$ components of y_m. Hence, (5.2.1), with $\mathbf{k} = \mathbf{e}_1$, possesses a unique formal solution in $\mathbb{C}^n[[x^{-1}]]$ if we prescribe the constant term to be equal to $\mathbf{e}_1$. Analogously, if $m \in \mathcal{J}_1 \backslash \{1\}$ then (5.2.1), with $\mathbf{k} = \mathbf{e}_m$, also has a unique formal solution if we prescribe the constant term to be equal to $\mathbf{e}_m$. In a similar way one may treat the equations for $\hat{y}_{\mathbf{e}_m}$, $m \in \mathcal{J}_2$ and $m \in \mathcal{J}_3$.

Next suppose ℓ to be an integer larger than 1 and assume that $\hat{y}_{\mathbf{k}'}$ has been found for all $\mathbf{k}' \in \mathbb{N}^n$, $|\mathbf{k}'| \leqslant \ell - 1$. Then, if $\mathbf{k}$ is a multi-index of length ℓ, we have

$$\Big(\bigoplus_{j=1}^{3} x^{1-q_j}\, \mathrm{I}_{\mathbf{k},j}\Big) y'(x) + A\, y + B(x)\, y(x) + \tilde{t}_{\mathbf{k}}(x) = 0. \tag{5.2.5}$$

As $\tilde{t}_{\mathbf{k}}$ only depends on $\hat{y}_{\mathbf{k}'}$ with $\mathbf{k}' \prec \mathbf{k}$, it has a known expansion. However, contrary to the situation in the preceding chapters, the matrix A is not always invertible. For example, if $\mathbf{k} = \mathbf{e}_1 + \mathbf{e}_{n_1+1}$, then $|\mathbf{k}| \geqslant 2$, but as $\mathbf{k}_1 = (1, 0, 0, \dots, 0)$ we are in case I and $A = \text{diag}\{0, \omega_2 - \omega_1, \dots, \omega_{n_1} - \omega_1, -\omega_1, \dots, -\omega_1\}$. So again we have to distinguish between several cases.

If $|\mathbf{k}_1| \geqslant 2$, then we are in case I and the matrix $A = A_{\mathbf{k},1}$ is invertible due to the fact that the set $\{\omega_m,\, m \in \mathcal{J}_1\}$ is linearly independent over $\mathbb{Z}$. In this case it is easily seen that (5.2.5) possesses a unique formal solution.

If $|\mathbf{k}_1| = 1$, for example $\mathbf{k}_1 = (1, 0, 0, \ldots, 0)$, then we still are in case I, but the matrix $A = \text{diag}\{\omega_j - \omega_1\}_{j=1}^n$ is no longer invertible. Now we distinguish between two cases, namely $|\mathbf{k}_2| \geqslant 1$ and $\mathbf{k}_2 = 0$. If $|\mathbf{k}_2| \geqslant 1$, then the matrix $B_{q_1-q_2}$ in the expansion of B equals $\text{diag}\{\lambda_j - \langle \mathbf{k}, \boldsymbol{\lambda}\rangle\}_{j=1}^n$ and thus is invertible. Substituting a series $\hat{y}(x) = \sum_{m=0}^\infty y_m x^{-m}$ in (5.2.5) and expanding $\tilde{t}_{\mathbf{k}}$ as $\tilde{t}_{\mathbf{k}}(x) = \sum_{m=q_1-1}^\infty \tilde{t}_{\mathbf{k},m}\, x^{-m}$ we obtain by comparing coefficients of x^{-m}, $m \geqslant 1$

$$(m - q_1) y_{m-q_1} = A\, y_m + \sum_{l=0}^{m-q_1+q_2} B_{m-l}\, y_l + \tilde{t}_{\mathbf{k},m}, \quad m \in \mathbb{N},$$

with the convention that $y_{m-q_1} = 0$ if $m < q_1$. Moreover, by definition $\tilde{t}_{\mathbf{k},m} = 0$ for $m < q_1 - 1$. If $m < q_1 - q_2$ this equation reduces to $Ay_m = 0$, which implies that the last $n-1$ components of y_m are equal to 0. If $m \geqslant q_1 - q_2$ we may rewrite the equation for y_m as

$$(m - q_1) y_{m-q_1} = A\, y_m + B_{q_1-q_2}\, y_{m-q_1+q_2} + \sum_{l=0}^{m-q_1+q_2-1} B_{m-l}\, y_l + \tilde{t}_{\mathbf{k},m}.$$

This equation recursively gives that last $n-1$ components of y_m and the first component of $y_{m-q_1+q_2}$, $m \geqslant q_1 - q_2$. If $\mathbf{k}_2 = 0$, then $|\mathbf{k}_3| \geqslant 1$, and in that case the first diagonal element of both A and B_m, $m < q_1 - q_3$, are equal to 0. However, $B_{q_1-q_3}$ is equal to $\text{diag}\{\mu_j - \langle \mathbf{k}, \boldsymbol{\mu}\rangle\}_{j=1}^n$ and thus invertible. Substituting a series $\sum_{m=0}^\infty y_m x^{-m}$ in the differential equation corresponding to this case and comparing coefficients of x^{-m}, $m \in \mathbb{N}$, gives in a similar manner as above that the last $n-1$ components of y_m are equal to 0 for $m < q_1 - q_3$, while for $m \geqslant q_1 - q_3$ the first component of $y_{m-q_1+q_3}$ and the last $n-1$ components of y_m can be determined recursively.

If $\mathbf{k}_1 = 0$, then $|\mathbf{k}_2| \geqslant 2$ or $|\mathbf{k}_2| = 1$ or $\mathbf{k}_2 = 0$. If $|\mathbf{k}_2| \geqslant 2$, then we are in case II and the matrix A is invertible. Hence, there exists a unique formal solution. If $\mathbf{k}_2 = 0$, then $|\mathbf{k}_3| \geqslant 2$, which again implies that A is invertible. So it remains to consider the case $\mathbf{k}_1 = 0$, together with $|\mathbf{k}_2| = 1$ and for convenience we assume that $\mathbf{k}_2 = (1, 0, 0, \ldots, 0)$. In this case $A = \text{diag}\{\omega_j\}_{j=1}^{n_1} \oplus \text{diag}\{\lambda_j - \lambda_{n_1+1}\}_{j=n_1+1}^n$, which is a diagonal matrix with only the $(n_1+1)^{\text{th}}$ diagonal element equal to 0. Moreover, B may be written as $B_1 \oplus B_2$, with

$$\left\{\begin{array}{lcll} B_1(x) & = & \sum_{m=q_1-q_2}^\infty B_{m,1}\, x^{-m}, & B_{q_1-q_2,1} = \text{diag}\{\lambda_j - \lambda_{n_1+1}\}_{j=1}^{n_1} \\ B_2(x) & = & \sum_{m=q_2-q_3}^\infty B_{m,2}\, x^{-m}, & B_{q_2-q_3,2} = \text{diag}\{\mu_j - \langle \mathbf{k}, \boldsymbol{\mu}\rangle\}_{j=n_1+1}^n. \end{array}\right.$$

Hence, both $B_{q_1-q_2,1}$ and $B_{q_2-q_3,2}$ are invertible. Substituting a series $\hat{y}(x) = \sum_{m=0}^\infty y_m x^{-m}$, comparing coefficients of x^{-m} for $m \in \mathbb{N}$ and using a similar splitting after the first n_1 components of y_m we deduce that each y_m satisfies

$$\begin{aligned}(m - q_1) y_{m-q_1,1} \oplus (m - q_2) y_{m-q_2,2} = \\ A_1\, y_{m,1} \oplus A_2\, y_{m,2} \;+ \sum_{l=0}^{m-q_1+q_2} B_{m-l,1}\, y_{l,1} \oplus \sum_{l=0}^{m-q_2+q_3} B_{m-l,2}\, y_{l,2} \;+\; \tilde{t}_{\mathbf{k},m},\end{aligned}$$

where $A_1 = \operatorname{diag}\{\omega_j\}_{j=1}^{n_1}$, $A_2 = \operatorname{diag}\{\lambda_j - \lambda_{n_1+1}\}_{j=n_1+1}^{n}$ and where $\tilde{t}_{\mathbf{k},m}$ is the coefficient of x^{-m} in the expansion of $\tilde{t}_{\mathbf{k}}$. In this case $\tilde{t}_{\mathbf{k},m} = \tilde{t}_{\mathbf{k},m,1} \oplus \tilde{t}_{\mathbf{k},m,2}$ with $\tilde{t}_{\mathbf{k},m,1} = 0$ for $m < q_1 - 1$ and $\tilde{t}_{\mathbf{k},m,2} = 0$ for $m < q_2 - 1$. For $m < q_2 - q_3$ all components of y_m are equal to 0, except for the $(n_1+1)^{\text{th}}$ component. Looking at the equation corresponding to $m = q_2 - q_3$ uniquely gives the $(n_1+1)^{\text{th}}$ component of y_0 as well as the components of y_m with index not equal to $n_1 + 1$. We can continue in this way and recursively determine all coefficients in $\hat{y}$. ∎

Remark 5.2.3 If we only take into consideration those $\mathbf{k} \in \mathbb{N}_1^n$ with $\mathbf{k}_2 = \mathbf{k}_3 = 0$, then we automatically are in case I. Hence, $\tilde{t}_{\mathbf{k}}(x) = x^{1-q_1} t_{\mathbf{k}}(x)$, and $t_{\mathbf{k}}$ only depends on $\hat{y}_{\mathbf{k}'}$, with multi-indices $\mathbf{k}' \prec \mathbf{k}$ satisfying the same property (if $\mathbf{k}$ is a multi-index with $\mathbf{k}_2 = \mathbf{k}_3 = 0$ and $\mathbf{k}' \prec \mathbf{k}$, then also $\mathbf{k}'_2 = \mathbf{k}'_3 = 0$). For such multi-indices $\mathbf{k}$ we then have $\hat{y}_{\mathbf{k}}(\infty) = \mathbf{k}$ if $|\mathbf{k}| = 1$, and $y_{\mathbf{k}}(\infty) = 0$ if $|\mathbf{k}| \geqslant 2$.

5.3 Multisummability of $\hat{y}_{\mathbf{k}}$

In this section we will show that each $\hat{y}_{\mathbf{k}}$ is (q_1, q_2, q_3)-summable. We begin by proving this for $\mathbf{k} = 0$, so for the formal solution of the original differential equation (5.1.1). Remember that this equation is equivalent to (5.1.4).

Definition 5.3.1 *Let $j \in \{1, 2, 3\}$. Then t is called a singular point of level q_j of (5.1.4) and the direction of t is called a singular direction of level q_j of (5.1.4) if $q_j t^{q_j}$ is an eigenvalue of $A_{0,j}$. The set of singular directions of level q_j of (5.1.4) will be denoted by $\mathcal{SD}_{0,j}$. A direction θ will be called a Stokes direction of (5.1.4) if there exists a $\sigma \in \mathcal{SD}_{0,j}$ for some $j \in \{1, 2, 3\}$ such that $\theta = -\sigma \pm \pi/(2q_j)$.*

Remark 5.3.2 Some authors use the term anti-Stokes direction instead. Those authors then refer to the singular direction as the Stokes direction.

The following results are proved by Braaksma in [Bra92].

Proposition 5.3.3 *For $j \in \{1, 2, 3\}$ take $\theta_j \notin \mathcal{SD}_{0,j}$ and take $\alpha_j > 0$ such that $S(\theta_j, \alpha_j)$ does not contain any singular direction of level q_j of (5.1.4) and assume that $S_1 \subset S_2 \subset S_3$, where $S_j = S(-\theta_j, \alpha_j + \pi/q_j)$, $j = 1, 2, 3$. Then $\hat{y}_0$ is (q_1, q_2, q_3)-summable in the multi-direction $-\boldsymbol{\theta}$, where $\boldsymbol{\theta} = (\theta_1, \theta_2, \theta_3)$. Its multi-sum y_0 is a holomorphic solution of (5.1.4) in a neighbourhood of ∞ in S_1 and $y_0(x) \sim \hat{y}_0(x)$ as $x \to \infty$ in S_1.*

In fact, the formal series $\hat{y}_0$ is (q_1, q_2, q_3)-summable. Its multi-sum exists on any sector which does not contain a pair of Stokes directions $\theta_\pm$ with $\theta_\pm = -\sigma \pm \pi/(2q_j)$, $\sigma \in \mathcal{SD}_{0,j}$, $j = 1, 2, 3$.

The last statement in this proposition says that if S_0 is a sector which does not contain a pair of Stokes directions $\theta_\pm = -\sigma \pm \pi/(2q_j)$, $\sigma \in \mathcal{SD}_{0,j}$, $j = 1, 2, 3$, then we may choose the sectors S_1, S_2 and S_3 above in such a way that the corresponding (q_1, q_2, q_3)-sum of $\hat{y}_0$ exists in a neighbourhood of ∞ in S_0, is a solution of (5.1.4) and asymptotically equals $\hat{y}_0(x)$ as $x \to \infty$ in S_0 (compare also the end of section 2 in [Bra91]).

Definition 5.3.4 *Let $j \in \{1,2,3\}$. Then t is a called singular point of level q_j of (5.2.1) and the direction of t is called a singular direction of level q_j of (5.2.1) if $q_j t^{q_j}$ is an eigenvalue of $A_{\mathbf{k}',j}$ for some $\mathbf{k}' \in \mathbb{N}^n$ with $\mathbf{k}' \preceq \mathbf{k}$. The set of singular directions of level q_j of (5.2.1) will be denoted by $\mathcal{SD}_{\mathbf{k},j}$. A direction θ will be called a Stokes direction of (5.2.1) if there exists a $\sigma \in \mathcal{SD}_{\mathbf{k},j}$ for some $j \in \{1,2,3\}$ such that $\theta = -\sigma \pm \pi/(2q_j)$.*

Remark 5.3.5 If t is a singular point of (5.1.4) or (5.2.1) we will also say that t is a singular point of the corresponding $\hat{y}_{\mathbf{k}}$. Similarly, a singular direction (Stokes direction) of (5.1.4) or (5.2.1) is also called a singular direction (Stokes direction) of the corresponding $\hat{y}_{\mathbf{k}}$.

Proposition 5.3.6 *For $j \in \{1,2,3\}$ take $\theta_j \notin \mathcal{SD}_{\mathbf{k},j}$ and take $\alpha_j > 0$ such that $S(\theta_j, \alpha_j)$ does not contain any singular direction of level q_j of (5.2.1) and assume that $S_1 \subset S_2 \subset S_3$, where $S_j = S(-\theta_j, \alpha_j + \pi/q_j)$, $j = 1,2,3$. Then $\hat{y}_{\mathbf{k}}$ is (q_1,q_2,q_3)-summable in the multi-direction $-\boldsymbol{\theta}$, where $\boldsymbol{\theta} = (\theta_1,\theta_2,\theta_3)$. Its multi-sum $y_{\mathbf{k}}$ is a holomorphic solution of (5.2.1) in a neighbourhood of ∞ in S_1 and $y_{\mathbf{k}}(x) \sim \hat{y}_{\mathbf{k}}(x)$ as $x \to \infty$ in S_1.*

In fact, the formal series $\hat{y}_{\mathbf{k}}$ is (q_1,q_2,q_3)-summable. Its multi-sum exists on any sector which does not contain a pair of Stokes directions $\theta_\pm$ with $\theta_\pm = -\sigma \pm \pi/(2q_j)$, $\sigma \in \mathcal{SD}_{\mathbf{k},j}$, $j = 1,2,3$.

Proof. According to proposition 5.3.3 the formal series $\hat{y}_0 \in x^{-2}\mathbb{C}^n[[x^{-1}]]$ is (q_1,q_2,q_3)-summable. Moreover, for each $\mathbf{j} \in \mathbb{N}^n$ the function $(x,y) \mapsto \sum_{\mathbf{l}\succeq\mathbf{j}} \binom{\mathbf{l}}{\mathbf{j}} g_{\mathbf{l}}(x) y^{\mathbf{l}-\mathbf{j}}$ is holomorphic for x in a neighbourhood of ∞ and y in a neighbourhood of $0 \in \mathbb{C}^n$. These two observations together imply that $d_{\mathbf{j}}$ (defined in (2.2.2)) is (q_1,q_2,q_3)-summable in every direction in which $\hat{y}_0$ is (q_1,q_2,q_3)-summable (cf. lemma 1.4.2). So $A_{\mathbf{k},+}$ is (q_1,q_2,q_3)-summable in every direction in which $\hat{y}_0$ is (q_1,q_2,q_3)-summable.

To prove the proposition we use induction on the length of $\mathbf{k} \in \mathbb{N}_1^n$. If $\mathbf{k}$ has length 1, then $t_{\mathbf{k}} \equiv 0$ and the result follows from [Bra91]. Next take $\ell > 1$ and assume the proposition to be true for all $\mathbf{k}' \in \mathbb{N}_1^n$ with $|\mathbf{k}'| \leqslant \ell - 1$. If $\mathbf{k}$ is a multi-index of length ℓ, then $t_{\mathbf{k}}(x) = \sum_{2\leqslant|\mathbf{j}|\leqslant|\mathbf{k}|} d_{\mathbf{j}}(x) \sum_{(\mathbf{i}_{mp};\mathbf{k})} \prod_{m=1}^{n} \prod_{p=1}^{j_m} (\hat{y}_{\mathbf{i}_{mp}})_m(x)$ is (q_1,q_2,q_3)-summable in every direction in which the series $\hat{y}_{\mathbf{k}'}$, $0 \preceq \mathbf{k}' \prec \mathbf{k}$, are (q_1,q_2,q_3)-summable. Again we apply [Bra91] to complete the proof of the proposition. ■

5.4 Exponential Estimates

In the preceding chapters we considered and derived estimates for the Borel transform of $x^{-|\mathbf{k}|}\hat{y}_{\mathbf{k}}(x)$, in order to ensure convergence of (a part of) the formal transformation that reduced the original system to a semi-canonical form. We will do something similar here. However, instead of looking at $x^{-|\mathbf{k}|}\hat{y}_{\mathbf{k}}(x)$ we now define $\hat{w}_{\mathbf{k}}(x) := x^{-q_1q_2q_3|\mathbf{k}|}\hat{y}_{\mathbf{k}}(x)$.

A short calculation shows that $\hat{w}_{\mathbf{k}}$, $\mathbf{k} \in \mathbb{N}_1^n$, satisfies

$$w'(x) + \left\{\Lambda_1(x) - \langle \mathbf{k}, \boldsymbol{\omega}\rangle x^{q_1-1} - \langle \mathbf{k}, \boldsymbol{\lambda}\rangle x^{q_2-1} - \langle \mathbf{k}, \boldsymbol{\mu}\rangle x^{q_3-1} + \langle \mathbf{k}, \mathbf{b}\rangle x^{-1}\right\} w(x) + u_{\mathbf{k}}(x) = 0,$$

where $\mathbf{b} = \mathbf{a} + q_1 q_2 q_3 (1, 1, \ldots, 1)$ and where $u_{\mathbf{k}}(x) = x^{-q_1 q_2 q_3 |\mathbf{k}|} t_{\mathbf{k}}(x)$. This equation can be rewritten as

$$\Big(\bigoplus_{j=1}^{3} x^{-q_j}\,\mathrm{I}_{\mathbf{k},j}\Big) x \frac{\mathrm{d}w}{\mathrm{d}x} + \Big(\bigoplus_{j=1}^{3} A_{\mathbf{k},j}\Big) w + \tilde{A}_{\mathbf{k},+}(x)\, w + \tilde{u}_{\mathbf{k}}(x) = 0, \tag{5.4.1}$$

where $\tilde{u}_{\mathbf{k}}(x) = (\oplus_{j=1}^{3} x^{1-q_j}\,\mathrm{I}_{\mathbf{k},j}) u_{\mathbf{k}}(x)$ and where $\tilde{A}_{\mathbf{k},+}$ is now defined by the equation

$$\begin{gathered}\big(\oplus_{j=1}^{3} x^{-q_j}\,\mathrm{I}_{\mathbf{k},j}\big)\big\{x\Lambda_1(x) - \langle \mathbf{k}, \boldsymbol{\omega}\rangle x^{q_1} - \langle \mathbf{k}, \boldsymbol{\lambda}\rangle x^{q_2} - \langle \mathbf{k}, \boldsymbol{\mu}\rangle x^{q_3} + \langle \mathbf{k}, \mathbf{b}\rangle\big\} = \\ \oplus_{j=1}^{3} A_{\mathbf{k},j} + \tilde{A}_{\mathbf{k},+}(x).\end{gathered} \tag{5.4.2}$$

Note that the matrices $A_{\mathbf{k},1}$, $A_{\mathbf{k},2}$ and $A_{\mathbf{k},3}$ remain unchanged and that $\tilde{A}_{\mathbf{k},+}$ is at least $O(x^{-1})$ as $x \to \infty$. We know that the formal solution $\hat{w}_{\mathbf{k}}$ is (q_1, q_2, q_3)-summable in every multi-direction $-\boldsymbol{\theta} \in \mathbb{R}^3$ in which $\hat{y}_{\mathbf{k}}$ is (q_1, q_2, q_3)-summable.

If we define

$$\kappa_j := (q_j^{-1} - q_{j-1}^{-1})^{-1}, \quad j = 1, 2, 3, \tag{5.4.3}$$

then the definition of multisummability (see definition 1.4.1) implies that $\hat{\mathcal{B}}_{q_3} \hat{w}_{\mathbf{k}}$ is convergent in some disc around the origin and its sum, denoted by $\Psi_{\mathbf{k},3}$, can be analytically continued in a sector $\tilde{S}_3$ and this continuation is of exponential growth of order at most κ_3 there. Hence, $\Psi_{\mathbf{k},2} := \mathcal{A}_{q_2,q_3} \Psi_{\mathbf{k},3}$ is defined and we know that it can be analytically continued in a sector $\tilde{S}_2$ and this continuation is of exponential growth of order $\leqslant \kappa_2$ there. Thus $\Psi_{\mathbf{k},1} := \mathcal{A}_{q_1,q_2} \Psi_{\mathbf{k},3}$ is defined, can be continued in a sector $\tilde{S}_1$ and this continuation is of exponential growth of order $\leqslant \kappa_1$. Finally the (q_1, q_2, q_3)-sum of $\hat{w}_{\mathbf{k}}$ is defined by $\mathcal{L}_{q_1} \Psi_{\mathbf{k},1}$ (note that $\kappa_1 = (q_1^{-1} - q_0^{-1})^{-1} = q_1$, so this Laplace integral is well defined) and this sum exists on any sector which does not contain a pair of Stokes directions $\theta_\pm$ with $\theta_\pm = -\sigma \pm \pi/(2q_j)$, $\sigma \in \mathcal{SD}_{\mathbf{k},j}$, $j = 1, 2, 3$.

As in the preceding chapters we show the existence of an analytic reduction to the (semi-)canonical form $z'(x) + \Lambda(x) z(x) = 0$, and this analytic reduction formula has to be the (q_1, q_2, q_3)-sum of the corresponding formal formula. One way to reach this goal is by using particular estimates on $\Psi_{\mathbf{k},j}$, similarly to the ones given in proposition 2.3.5 and proposition 4.3.3. Therefore we will make restrictions on the multi-indices we take into consideration.

5.4.1 Preparations

In the following we only take into consideration the multi-indices $\mathbf{k} \in \mathbb{N}_1^n$ with $\mathbf{k}_2 = \mathbf{k}_3 = 0$. With this restriction one can be sure to be in case I (compare section 5.2). Moreover, for such a multi-index $\mathbf{k}$ the inhomogeneous term $t_{\mathbf{k}}$ (or $u_{\mathbf{k}}$) only depends on $\hat{y}_{\mathbf{k}'}$ (or $\hat{w}_{\mathbf{k}'}$) with $\mathbf{k}' \prec \mathbf{k}$ satisfying the same requirement. Hence, we only have to estimate $\Psi_{\mathbf{k},j}$, $j = 3, 2, 1$, for $\mathbf{k} \in \mathbb{N}_1^n$ with $\mathbf{k}_2 = \mathbf{k}_3 = 0$. The reason to introduce such a restriction is the fact that if $\mathbf{k} \in \mathbb{N}_1^n$ satisfies case II, then the matrix $A_{\mathbf{k},1}^{-1}$ cannot be bounded by a constant times $|\mathbf{k}|^{-1}$. Such a bound is useful in order to obtain the desirable estimation of the corresponding $\Psi_{\mathbf{k},j}$,

$j = 3, 2, 1$ (compare sections 5.4.3, 5.4.4 and 5.4.5). With a similar argument case III is not taken into consideration.

With this restriction (5.4.1) reduces to[1]

$$x^{1-q_1} w'(x) + A_{\mathbf{k},1}\, w + \tilde{A}_{\mathbf{k},+}(x)\; w + x^{1-q_1} u_{\mathbf{k}}(x) = 0, \quad \mathbf{k} \in \mathbb{N}_1^{n_1}, \tag{5.4.4}$$

where $A_{\mathbf{k},1} = \mathrm{diag}\{\omega_1 - \langle \mathbf{k}, \boldsymbol{\omega}\rangle, \ldots, \omega_n - \langle \mathbf{k}, \boldsymbol{\omega}\rangle\}$ and where

$$\begin{aligned}\tilde{A}_{\mathbf{k},+}(x) \;&=\; x^{q_2-q_1}\mathrm{diag}\{\lambda_j - \langle \mathbf{k}, \boldsymbol{\lambda}\rangle\}_{j=1}^n + x^{q_3-q_1}\mathrm{diag}\{\mu_j - \langle \mathbf{k}, \boldsymbol{\mu}\rangle\}_{j=1}^n + \\ &\quad\; x^{-q_1}\mathrm{diag}\{\langle \mathbf{k}, \mathbf{b}\rangle - a_j\}_{j=1}^n + x^{1-q_1}\tilde{\Lambda}(x).\end{aligned}$$

Here $\tilde{\Lambda}(x) = \mathrm{col}\{d_{\mathbf{e}_1}(x), d_{\mathbf{e}_2}(x), \ldots, d_{\mathbf{e}_n}(x)\}$ (compare (2.2.3)). In this particular case the set of singular directions is given by

$$\mathcal{SD}_{\mathbf{k},1} = \left\{ \arg t \;\middle|\; \text{or} \;\begin{matrix} q_1 t^{q_1} = -\langle \mathbf{k}', \boldsymbol{\omega}\rangle, \text{ for some } \mathbf{k}' \text{ with } 0 \prec \mathbf{k}' \preceq \mathbf{k} \\ q_1 t^{q_1} = \omega_m - \langle \mathbf{k}', \boldsymbol{\omega}\rangle, \text{ for some } \mathbf{k}' \text{ with } 0 \preceq \mathbf{k}' \preceq \mathbf{k},\; m = 1, \ldots, n_1 \end{matrix} \right\},$$

$$\mathcal{SD}_{\mathbf{k},2} = \mathcal{SD}_{0,2} \quad \text{and} \quad \mathcal{SD}_{\mathbf{k},3} = \mathcal{SD}_{0,3}.$$

Let $\theta \in \mathbb{R}$ and define $i(\theta) := \big\{ j \in \{1, 2, \ldots, n_1\} \;\big|\; \Re(\omega_j e^{-i\theta}) > 0 \big\}$. Moreover, define $I(\theta) := \{\mathbf{k} \in \mathbb{N}^{n_1} \mid k_j = 0 \text{ if } j \notin i(\theta)\}$. With these definitions there are only finitely many singular points of $\hat{y}_{\mathbf{k}}$ (and thus of $\hat{w}_{\mathbf{k}}$), $\mathbf{k} \in I(\theta)$, in $H(\theta) := \{t \in \mathbb{C} \mid \Re(t^{q_1} e^{-i\theta}) > 0\}$ and it makes sense to consider two consecutive singular directions of this set of singular points in $H(\theta)$.

With a direction $\theta \in \mathbb{R}$ we associate two such consecutive singular directions θ_- and θ_+ in $H(\theta)$. If we define $\theta_0 := \frac{1}{2}(\theta_- + \theta_+)$ and $\alpha := \theta_+ - \theta_-$, then $S_0 := S(-\theta_0, \alpha + \pi/q_1)$ does not contain any pair of Stokes directions $-\sigma \pm \pi/(2q_j)$, $\sigma \in \mathcal{SD}_{\mathbf{k},j}$, $j = 1, 2, 3$, which can be seen as follows. Suppose that both $-\sigma - \pi/(2q_j)$ and $-\sigma + \pi/(2q_j)$ belong to S_0. Since $q_1 \geqslant q_j$, this implies that both $-\sigma - \pi/(2q_1)$ and $-\sigma + \pi/(2q_1)$ belong to S_0, and thus σ is a singular direction between θ_- and θ_+. This is a contradiction.

Given this S_0 we want to show that there exist S_1, S_2 and S_3 as in proposition 5.3.6 such that the corresponding (q_1, q_2, q_3)-sum of $\hat{y}_{\mathbf{k}}$, $\mathbf{k} \in I(\theta)$, exists in a neighbourhood of ∞ in S_0, is a solution of (5.2.1) and asymptotically equals $\hat{y}_{\mathbf{k}}(x)$ as $x \to \infty$ in S_0. Since $0 = q_4 < q_3 < q_2 < q_1 < q_0 = \infty$ we have $\pi/q_{h-1} < \alpha + \pi/q_1 \leqslant \pi/q_h$ for some $h \in \{1, 2, 3, 4\}$. Now, if $\alpha + \pi/q_1 = \pi/q_h$ we fix $0 < \alpha_1 < \alpha$ in such a way that $\pi/q_{h-1} < \alpha_1 + \pi/q_1 < \pi/q_h$ and otherwise we take $\alpha_1 = \alpha$. We define $S_1 := S(-\theta_0, \alpha_1 + \pi/q_1) \subset S_0$ and $S_j := S_1$ for $j \in \{2, \ldots, h-1\}$. As S_0 does not contain any pair of Stokes directions, the same holds for S_j, $j = 1, 2, \ldots, h-1$. Therefore $S(\theta_0, \alpha_1 + \pi/q_1 - \pi/q_j)$ does not contain a singular direction of level q_j if $j \leqslant h-1$. For $j \geqslant h$ we define $S_j := S(-\theta_0, \alpha_1 + \pi/q_j)$. As the opening of S_1 is less than π/q_j for all $j \geqslant h$ we have $S_1 \subset S_2 \subset S_3$. Moreover, if $j \geqslant h$ then $S(\theta_0, \alpha_1 + \pi/q_j - \pi/q_j) \subset \{t \in \mathbb{C}^* \mid \arg t \in (\theta_-, \theta_+)\}$, which does not contain any singular direction.

[1]Here we use a similar identification as in (2.3.1).

For convenience we will write $S_j =: S(-\theta_0, \alpha_j + \pi/q_j)$, so if $j \in \{1, 2, \ldots, h-1\}$ we have $\alpha_j = \alpha_1 + \pi(q_1^{-1} - q_j^{-1})$ and if $j \geqslant h$ we have $\alpha_j = \alpha_1$. Now let $\overline{S_j'}$ denote an arbitrary closed sub-sector of $S(\theta_0, \alpha_j)$. Moreover, let ρ denote a positive number which is smaller than the absolute value of each singular point of $\hat{y}_{\mathbf{k}}$, $\mathbf{k} \in I(\theta)$, and define $V_j = \overline{S_j'}$ for $j = 1, 2$, while $V_3 := \overline{S_3'} \cup \overline{\Delta}(0, \rho)$ (compare lemma 4.3.2).

Proposition 5.4.1 *For $j \in \{1, 2, 3\}$ let $\Psi_{\mathbf{k},j}$ be as in the introduction of this section and let V_j be as above. Then there exist positive constants K and R such that for all $\mathbf{k} \in I(\theta) \backslash \{0\}$ and $j \in \{1, 2, 3\}$ we have*

$$|t^{q_j-1}\Psi_{\mathbf{k},j}(t)| \;\leqslant\; K^{|\mathbf{k}|}\,\frac{|t|^{q_1q_2q_3|\mathbf{k}|-1}}{\Gamma\left(\frac{q_1q_2q_3|\mathbf{k}|}{q_j}\right)}\,e^{R|t|^{\kappa_j}}, \tag{5.4.5}$$

for all $t \in V_j$.

We will prove this proposition in the following subsections.

5.4.2 Some Auxiliary Lemmas

Lemma 5.4.2 *Let $a, b > 0$, $c \geqslant 0$, $\kappa \geqslant 1$ and $0 \leqslant a + b - 1 \leqslant a\kappa$. Then there exists a positive constant C such that for all $R \geqslant c + 1$ and all positive p we have*

$$p^{a-1}e^{cp^\kappa} * p^{b-1}e^{Rp^\kappa} \leqslant C\,R^{-(a+b-1)/\kappa}\,e^{Rp^\kappa}. \tag{5.4.6}$$

The proof of this lemma is given by Faber in [Fab98], chapter 5, (5.4.24).

Lemma 5.4.3 *If g is holomorphic in a full neighbourhood of ∞ and $q \in \mathbb{N}_+$, then the function $t \mapsto t^{q-1}(\mathcal{B}_q g)(t)$ is entire and of exponential growth of order $\leqslant q$.*

Proof. Given the Taylor expansion $\sum_{m=0}^\infty g_m x^{-m}$ of g, we know the existence of positive constants K and c such that $|g_m| \leqslant Kc^m$. Now $(\mathcal{B}_q g)(t) = \sum_{m=1}^\infty g_m \frac{t^{m-q}}{\Gamma(m/q)}$. Hence,

$$t^{q-1}(\mathcal{B}_q g)(t) = \sum_{m=0}^{\infty} g_{m+1}\frac{t^m}{\Gamma(\frac{m+1}{q})} = \sum_{j=0}^{q-1} t^j \sum_{m=0}^{\infty} g_{mq+j+1}\frac{t^{qm}}{\Gamma(m+\frac{j+1}{q})}.$$

Now for $m \to \infty$ we have $\frac{\Gamma(m+\frac{j+1}{q})}{\Gamma(m)} \sim m^{(j+1)/q}$, hence there exists a constant $\delta > 0$ such that for all $m \geqslant 1$ we have $\Gamma(m + \frac{j+1}{q}) \geqslant \delta\,\Gamma(m)$. Using this, together with the estimate for $|g_{mq+j+1}|$, it is easy to complete the proof of the lemma. ■

Lemma 5.4.4 *Let f be a $\mathbb{C}^n$-valued holomorphic function defined on a sector S and assume the existence of constants $q \in \mathbb{N}_+$, $K, c > 0$ and $\kappa \geqslant q$ such that $|t^{q-1}f(t)| \leqslant K\,e^{c|t|^\kappa}$. Then for $\mathbf{l} \in \mathbb{N}^n$, $|\mathbf{l}| \geqslant 1$, we have*

$$\left|t^{q-1}f^{*_q\mathbf{l}}(t)\right| \leqslant \frac{(\Gamma(\frac{1}{q})K)^{|\mathbf{l}|}}{\Gamma(\frac{|\mathbf{l}|}{q})}\,|t|^{|\mathbf{l}|-1}\,e^{c|t|^\kappa}, \quad \forall\, t \in S.$$

Here $f^{*_q\mathbf{l}}$ means the $\mathbf{l}$-fold q-convolution of f.

PROOF. We give the proof with induction on the length of $\mathbf{l}$. Trivially, the estimate is true if $|\mathbf{l}| = 1$. Now take a multi-index $\mathbf{l}$ of length > 1, and assume that the i^{th} component of $\mathbf{l}$ belongs to $\mathbb{N}_+$. Then write $\mathbf{l} = \mathbf{e}_i + \mathbf{l}'$ and assume the estimate to be true with $\mathbf{l}$ replaced by $\mathbf{l}'$. As $f^{*_q\mathbf{l}} = f_i *_q f^{*_q\mathbf{l}'}$, we obtain

$$\begin{aligned} \left|f^{*_q\mathbf{l}}(t)\right| &\leqslant \frac{\Gamma(\frac{1}{q})^{|\mathbf{l}|-1}K^{|\mathbf{l}|}}{\Gamma(\frac{|\mathbf{l}|-1}{q})}\int_0^{|t|^q}(|t|^q-\sigma)^{\frac{1-q}{q}}e^{c(|t|^q-\sigma)^{\frac{\kappa}{q}}}\sigma^{\frac{|\mathbf{l}|-1-q}{q}}e^{c\sigma^{\frac{\kappa}{q}}}\mathrm{d}\sigma \\ &= \frac{\Gamma(\frac{1}{q})^{|\mathbf{l}|-1}K^{|\mathbf{l}|}}{\Gamma(\frac{|\mathbf{l}|-1}{q})}|t|^{|\mathbf{l}|-q}\int_0^1(1-\sigma)^{\frac{1-q}{q}}\sigma^{\frac{|\mathbf{l}|-1-q}{q}}e^{c|t|^\kappa[(1-\sigma)^{\kappa/q}+\sigma^{\kappa/q}]}\mathrm{d}\sigma. \end{aligned}$$

For $\mu \geqslant 1$, $(1-\sigma)^\mu + \sigma^\mu \leqslant 1$ for all $\sigma \in [0,1]$. Moreover, $\int_0^1(1-\sigma)^{\frac{1-q}{q}}\sigma^{\frac{|\mathbf{l}|-1-q}{q}}\mathrm{d}\sigma$ equals $\Gamma(\frac{1}{q})\Gamma(\frac{|\mathbf{l}|-1}{q})/\Gamma(\frac{|\mathbf{l}|}{q})$. Using these results it is easy to complete the proof of the lemma. ■

Let $g = g(x,y)$ be as in (5.1.1). If we expand g as $g(x,y) = \sum_{\mathbf{l}\in\mathbb{N}^n} g_{\mathbf{l}}(x)y^{\mathbf{l}}$ and each $g_{\mathbf{l}}$ as $g_{\mathbf{l}}(x) = \sum_{m=0}^{\infty} g_{\mathbf{l},m}x^{-m}$, then we may assume the existence of positive constants K and c such that $|g_{\mathbf{l},m}| \leqslant Kc^{|\mathbf{l}|+m}$ (compare (2.3.4)).

Lemma 5.4.5 *Let $q \in \mathbb{N}_+$, $\kappa \geqslant q$ and assume Y to be a holomorphic $\mathbb{C}^n$-valued function in some sector S, satisfying the estimate $|t^{q-1}Y(t)| \leqslant K_1 e^{c_1|t|^\kappa}$, $t \in S$, for some positive constants K_1 and c_1. For $\mathbf{j} \in \mathbb{N}^n$ define*

$$D_{\mathbf{j}}^{(q)}(Y) := \mathcal{B}_q g_{\mathbf{j}} + \sum_{\mathbf{l}\succ\mathbf{j}}\binom{\mathbf{l}}{\mathbf{j}}\big(g_{\mathbf{l},0} + \mathcal{B}_q g_{\mathbf{l}} *_q\big)Y^{*_q(\mathbf{l}-\mathbf{j})}. \tag{5.4.7}$$

Then there exist positive constants K_2 and c_2 such that

$$\left|t^{q-1}D_{\mathbf{j}}^{(q)}(Y)(t)\right| \leqslant K_2^{|\mathbf{j}|+1}e^{c_2|t|^\kappa}$$

for all $t \in S$.

PROOF. First one should observe that a function of exponential growth of order $\leqslant q$ automatically is of exponential growth of order $\leqslant \kappa$. This, together with lemma 5.4.3 implies that $|t^{q-1}(\mathcal{B}_q g_{\mathbf{l}})(t)| \leqslant const. \cdot c^{|\mathbf{l}|}e^{c_0|t|^\kappa}$, $t \in \mathbb{C}$, for some positive constant c_0. Hence, it is sufficient to prove the estimate for the difference $D_{\mathbf{j}}^{(q)}(Y) - \mathcal{B}_q g_{\mathbf{j}}$. Using lemma 5.4.4 we easily deduce that $t^{q-1}\sum_{\mathbf{l}\succ\mathbf{j}}\binom{\mathbf{l}}{\mathbf{j}}g_{\mathbf{l},0}Y^{*_q(\mathbf{l}-\mathbf{j})}$ is of exponential growth of order $\leqslant \kappa$ and the definition of q-convolution, together with lemma 5.4.4, yields

$$\left|t^{q-1}\big(\mathcal{B}_q g_{\mathbf{l}} *_q Y^{*_q(\mathbf{l}-\mathbf{j})}\big)(t)\right| \leqslant const. \cdot c^{|\mathbf{l}|}\frac{(\Gamma(\frac{1}{q})K_1)^{|\mathbf{l}|-|\mathbf{j}|+1}}{\Gamma(\frac{|\mathbf{l}|-|\mathbf{j}|+1}{q})}|t|^{|\mathbf{l}|-|\mathbf{j}|}e^{\tilde{c}|t|^\kappa},$$

with $\tilde{c} = \max\{c_0, c_1\}$. Now it is easy to complete the proof of the lemma. ■

5.4.3 Estimate for $\Psi_{\mathbf{k},3}$

Remember that $\hat{w}_{\mathbf{k}}$, $\mathbf{k} \in I(\theta)\backslash\{0\}$, satisfies equation (5.4.4). In general, if $\Psi = \hat{\mathcal{B}}_q w$, then $\hat{\mathcal{B}}_q[x^{1-q} w'] = -q\, t^q\, \Psi$. Hence, if we denote $\Psi_{\mathbf{k},3} := \hat{\mathcal{B}}_{q_3} \hat{w}_{\mathbf{k}}$, then by applying $\hat{\mathcal{B}}_{q_3}$ to (5.4.4) this equation formally transforms into

$$A_{\mathbf{k},1}\Psi_{\mathbf{k},3} - \frac{t^{q_1-2q_3}}{\Gamma(-1+q_1/q_3)} *_{q_3} q_3 t^{q_3}\Psi_{\mathbf{k},3} + (\hat{\mathcal{B}}_{q_3}\tilde{A}_{\mathbf{k},+}) *_{q_3} \Psi_{\mathbf{k},3} + \hat{\mathcal{B}}_{q_3}\big[x^{1-q_1}u_{\mathbf{k}}\big] = 0. \quad (5.4.8)$$

With $q := q_3$, (5.4.8) may more conveniently be written as $\Psi_{\mathbf{k},3} = \mathcal{T}_{\mathbf{k},3}\Psi_{\mathbf{k},3} + \gamma_{\mathbf{k},3}$, where

$$\mathcal{T}_{\mathbf{k},3}\Psi = A_{\mathbf{k},1}^{-1}\Big\{\frac{t^{q_1-2q}}{\Gamma(-1+q_1/q)} *_q q t^q \Psi - (\hat{\mathcal{B}}_q\tilde{A}_{\mathbf{k},+}) *_q \Psi\Big\}, \quad \gamma_{\mathbf{k},3} = -A_{\mathbf{k},1}^{-1}\hat{\mathcal{B}}_q\big[x^{1-q_1}u_{\mathbf{k}}\big].$$

To get the estimate for $\Psi_{\mathbf{k},3}$, as described in proposition 5.4.1, we use the method that we also used to prove proposition 4.3.3 (see also [Bra01]).

For convenience we put $\kappa := \kappa_3$ (compare 5.4.3) and we define $p := q_1 q_2 q_3$. Let V_3 be as defined at the end of section 5.4.1. For $\ell \in \mathbb{N}_+$ and $R > 0$ we define a Banach space $\mathcal{V}_{\ell,R}$ consisting of continuous functions $\Psi : V_3 \to \mathbb{C}^n$, holomorphic in the interior of V_3, such that $\|\Psi\|_{\ell,R} := \sup_{t\in V_3} |t^{q-1}\Psi(t)|/\zeta_{\ell,R}(t) < \infty$, where $\zeta_{\ell,R}(t) := \frac{|t|^{p\ell-1}}{\Gamma(p\ell/q)}\, e^{R|t|^\kappa}$. If $|\mathbf{k}| = 1$, then the (q_1, q_2, q_3)-summability of $\hat{w}_{\mathbf{k}}$ implies the existence of a positive M such that $\Psi_{\mathbf{k},3} \in \mathcal{V}_{1,M}$.

If $\Psi \in \mathcal{V}_{\ell,R}$ and $r > 0$ and $s \geqslant 0$ are chosen in such a way that $r + qs \leqslant r\frac{\kappa}{q}$, then we have for $t \in V_3\backslash\{0\}$

$$\begin{aligned} |t^{r-q} *_q t^{sq}\Psi| &\leqslant \frac{\|\Psi\|_{\ell,R}}{\Gamma(p\ell/q)} \int_0^t |t^q - \sigma^q|^{r/q-1}|\sigma|^{sq}|\sigma|^{p\ell-q} e^{R|\sigma|^\kappa}|\mathrm{d}(\sigma^q)| \\ &\leqslant \frac{\|\Psi\|_{\ell,R}}{\Gamma(p\ell/q)}\, |t|^{p\ell-q} \int_0^{|t|^q} (|t|^q - \sigma)^{r/q-1}\sigma^s\, e^{R\sigma^{\kappa/q}}\mathrm{d}\sigma \\ &\leqslant \frac{C\|\Psi\|_{\ell,R}}{R^{(r+sq)/\kappa}}\, \frac{|t|^{p\ell-q}}{\Gamma(p\ell/q)}\, e^{R|t|^\kappa}, \end{aligned}$$

where in the last estimate we used (5.4.6). Hence,

$$\big\|t^{r-q} *_q t^{sq}\Psi\big\|_{\ell,R} \leqslant \frac{C}{R^{(r+sq)/\kappa}}\|\Psi\|_{\ell,R}, \quad (5.4.9)$$

where C is some positive constant independent of t, $\mathbf{k}$ and R, but which may be dependent on r and s.

For $\mathbf{k} \in I(\theta)$, $|\mathbf{k}| > 1$, we have $|A_{\mathbf{k},1}^{-1}| \leqslant K_0\, |\mathbf{k}|^{-1}$, for some constant K_0 independent of $\mathbf{k}$ (compare the proof of proposition 4.3.3). Moreover, for all $R \geqslant 1$, all integers $\ell \geqslant 2$ and all $\Psi \in \mathcal{V}_{\ell,R}$ we have by (5.4.9) with $r = q_1 - q$ and $s = 1$

$$\big\|t^{q_1-2q} *_q t^q\Psi\big\|_{\ell,R} \leqslant \frac{C}{R^{q_1/\kappa}}\|\Psi\|_{\ell,R},$$

where one should notice that $q_1 \leqslant (q_1 - q) \cdot \kappa/q$, so that (5.4.9) indeed can be applied. From the definition of $\tilde{A}_{\mathbf{k},+}$ we deduce that $\hat{\mathcal{B}}_q \tilde{A}_{\mathbf{k},+}$ equals

$$\frac{t^{q_1-q_2-q}}{\Gamma((q_1-q_2)/q)} \operatorname{diag}\{\lambda_1 - \langle \mathbf{k}, \boldsymbol{\lambda}\rangle, \ldots, \lambda_n - \langle \mathbf{k}, \boldsymbol{\lambda}\rangle\} \quad + \\ \frac{t^{q_1-2q}}{\Gamma(-1+q_1/q)} \operatorname{diag}\{\mu_1 - \langle \mathbf{k}, \boldsymbol{\mu}\rangle, \ldots, \mu_n - \langle \mathbf{k}, \boldsymbol{\mu}\rangle\} \quad + \\ \frac{t^{q_1-q}}{\Gamma(q_1/q)} \operatorname{diag}\{\langle \mathbf{k}, \mathbf{b}\rangle - a_1, \ldots, \langle \mathbf{k}, \mathbf{b}\rangle - a_n\} \quad + \quad \frac{t^{q_1-1-q}}{\Gamma((q_1-1)/q)} *_q (\mathcal{B}_q \tilde{\Lambda})(t).$$

Utilising (5.4.9) we obtain for all $R \geqslant 1$, all integers $\ell \geqslant 2$ and all $\Psi \in \mathcal{V}_{\ell,R}$

$$\left\| t^{q_1-q_2-q} *_q \Psi \right\|_{\ell,R} \leqslant \frac{C}{R^{(q_1-q_2)/\kappa}} \|\Psi\|_{\ell,R}\,, \quad \left\| t^{q_1-2q} *_q \Psi \right\|_{\ell,R} \leqslant \frac{C}{R^{(q_1-q)/\kappa}} \|\Psi\|_{\ell,R}\,,$$

$$\left\| t^{q_1-q} *_q \Psi \right\|_{\ell,R} \leqslant \frac{C}{R^{q_1/\kappa}} \|\Psi\|_{\ell,R}.$$

Since $\hat{y}_0$ is (q_1, q_2, q_3)-summable we know that $\hat{\mathcal{B}}_q \hat{y}_0$ converges in a punctured neighbourhood of the origin and its sum $\Psi_{0,3}$ can be analytically continued in $S(\theta_0, \alpha_3)$ and is of exponential growth of order $\leqslant \kappa$ there. From lemma 5.4.5 we then deduce the existence of two positive constants K_2 and c_2 such that $|t^{q-1}(\mathcal{B}_q\tilde{\Lambda})(t)| \leqslant K_2\, e^{c_2|t|^\kappa}$, $t \in V_3$. Hence, using (5.4.6), we get $|t^{q_1-1-q} *_q \mathcal{B}_q \tilde{\Lambda}| \leqslant \tilde{K}_2\, e^{c_2|t|^\kappa}$, $t \in V_3$, for some positive constant $\tilde{K}_2$. So for all $R \geqslant c_2+1$, all integers $\ell \geqslant 2$ and all $\Psi \in \mathcal{V}_{\ell,R}$ we obtain (again by using (5.4.6))

$$\left\| t^{q_1-1-q} *_q \mathcal{B}_q \tilde{\Lambda} *_q \Psi \right\|_{\ell,R} \leqslant \frac{\tilde{K}_2 C}{R^{q/\kappa}} \|\Psi\|_{\ell,R}.$$

Hence, there exists a positive $R_1 \geqslant \max\{c_2+1, M\}$ such that for all $R \geqslant R_1$, all integers $\ell \geqslant 2$ and all $\Psi \in \mathcal{V}_{\ell,R}$ we have

$$\|\mathcal{T}_{\mathbf{k},3}\Psi\|_{\ell,R} \leqslant \frac{1}{2} \|\Psi\|_{\ell,R}.$$

In the following we choose $R \geqslant R_1$ and we consider the equations for $\Psi_{\mathbf{k},3}$ in $\mathcal{V}_{|\mathbf{k}|,R}$. Now fix $\mathbf{k} \in I(\theta)$, $|\mathbf{k}| \geqslant 2$, and let us assume that we have found solutions $\Psi_{\mathbf{k}',3} \in \mathcal{V}_{|\mathbf{k}'|,R}$ for all $\mathbf{k}' \in I(\theta)\backslash\{0\}$ with $|\mathbf{k}'| < |\mathbf{k}|$. Using the definition of q-convolution one may show, in a similar way as we proved lemma 5.4.4, that given two multi-indices $\mathbf{l}, \mathbf{l}' \in I(\theta)\backslash\{0\}$ of length less than $|\mathbf{k}|$ we have, for $t \in V_3$,

$$|t^{q-1}(\Psi_{\mathbf{l},3} *_q \Psi_{\mathbf{l}',3})(t)| \leqslant \frac{\|\Psi_{\mathbf{l},3}\|_{|\mathbf{l}|,R} \|\Psi_{\mathbf{l}',3}\|_{|\mathbf{l}'|,R}}{\Gamma(\frac{p(|\mathbf{l}|+|\mathbf{l}'|)}{q})} |t|^{p(|\mathbf{l}|+|\mathbf{l}'|)-1}\, e^{R|t|^\kappa}. \tag{5.4.10}$$

Now, $u_{\mathbf{k}}(x) = \sum_{2 \leqslant |\mathbf{j}| \leqslant |\mathbf{k}|} d_{\mathbf{j}}(x) \sum_{(\mathbf{i}_{mp};\mathbf{k})} \prod_{m=1}^n \prod_{p=1}^{j_m} \hat{w}_{\mathbf{i}_{mp}}(x)$, so

$$\hat{\mathcal{B}}_q u_{\mathbf{k}} = \sum_{2 \leqslant |\mathbf{j}| \leqslant |\mathbf{k}|} \left(g_{\mathbf{j},0} + D_{\mathbf{j}}^{(q)}(\Psi_{0,3}) *\right) \sum_{(\mathbf{i}_{mp};\mathbf{k})} \prod_{m=1}^{n}{}^{*} \prod_{p=1}^{j_m}{}^{*} (\Psi_{\mathbf{i}_{mp},3})_m,$$

where $*$ means $*_q$. Using (5.4.10) we deduce that $t^{q-1}\prod_{m=1}^{*n}\prod_{p=1}^{*j_m}(\Psi_{\mathbf{i}_{mp},3})_m(t)$ can be estimated by $\prod_{m=1}^{n}\prod_{p=1}^{j_m}\|\Psi_{\mathbf{i}_{mp},3}\|_{|\mathbf{i}_{mp}|,R}\,\frac{|t|^{p|\mathbf{k}|-1}}{\Gamma(p|\mathbf{k}|/q)}\,e^{R|t|^\kappa}$, $t\in V_3$. According to lemma 5.4.5 there exist positive constants K_2 and c_2 such that $|t^{q-1}D_{\mathbf{j}}^{(q)}(\Psi_{0,3})(t)|\leqslant K_2^{|\mathbf{j}|+1}e^{c_2|t|^\kappa}$. Using these two results, together with (5.4.6), we have for $t\in V_3$

$$|t^{q-1}(\hat{\mathcal{B}}_q u_{\mathbf{k}})(t)|\leqslant \frac{|t|^{p|\mathbf{k}|-1}}{\Gamma(p|\mathbf{k}|/q)}\,e^{R|t|^\kappa}\cdot(1+C)\sum_{2\leqslant|\mathbf{j}|\leqslant|\mathbf{k}|}K_2^{|\mathbf{j}|+1}\sum_{(\mathbf{i}_{mp};\mathbf{k})}\prod_{m=1}^{n}\prod_{p=1}^{j_m}\|\Psi_{\mathbf{i}_{mp},3}\|_{|\mathbf{i}_{mp}|,R},$$

with possibly enlarged K_2. As the number of multi-indices $\mathbf{j}\in\mathbb{N}^n$ with $|\mathbf{j}|=h$ can be majorized by 2^{n+h-1} (cf. the proof of lemma 2.5.4), the estimate for $t^{q-1}(\hat{\mathcal{B}}_q u_{\mathbf{k}})(t)$ may be replaced by

$$|t^{q-1}(\hat{\mathcal{B}}_q u_{\mathbf{k}})(t)|\leqslant \frac{|t|^{p|\mathbf{k}|-1}}{\Gamma(p|\mathbf{k}|/q)}\,e^{R|t|^\kappa}\cdot 2^{n-1}(1+C)K_2\sum_{h=2}^{|\mathbf{k}|}(2K_2)^h\sum_{(\mathbf{i}_m;\mathbf{k})}\prod_{m=1}^{h}\|\Psi_{\mathbf{i}_m,3}\|_{|\mathbf{i}_m|,R},$$

where $\sum_{(\mathbf{i}_m;\mathbf{k})}$ denotes the sum over all $\mathbf{i}_m\in I(\theta)$ with $\mathbf{i}_m\succ 0$ and $\sum_{m=1}^{h}\mathbf{i}_m=\mathbf{k}$. From this one may deduce that $\gamma_{\mathbf{k},3}\in\mathcal{V}_{|\mathbf{k}|,R}$ and that its norm can be estimated by

$$\|\gamma_{\mathbf{k},3}\|_{|\mathbf{k}|,R}\leqslant\frac{\tilde{C}}{R^{(q_1-1)/\kappa}}\sum_{h=2}^{|\mathbf{k}|}(2K_2)^h\sum_{(\mathbf{i}_m;\mathbf{k})}\prod_{m=1}^{h}\|\Psi_{\mathbf{i}_m,3}\|_{|\mathbf{i}_m|,R},$$

where $\tilde{C}$ and K_2 are positive constants independent of $\mathbf{k}$ and R. Therefore there exists a unique $\Psi_{\mathbf{k},3}\in\mathcal{V}_{|\mathbf{k}|,R}$ such that $\Psi_{\mathbf{k},3}=\mathcal{T}_{\mathbf{k},3}\Psi_{\mathbf{k},3}+\gamma_{\mathbf{k},3}$.

To prove the estimate for $\Psi_{\mathbf{k},3}$, as formulated in proposition 5.4.1, we define, for a multi-index $\mathbf{k}\in I(\theta)$, with $|\mathbf{k}|=1$, $f_{\mathbf{k}}:=\|\Psi_{\mathbf{k},3}\|_{1,R}$. Now suppose that for some integer $\ell>1$ we have $\|\Psi_{\mathbf{k},3}\|_{|\mathbf{k}|,R}\leqslant f_{\mathbf{k}}$ for all $\mathbf{k}\in I(\theta)\backslash\{0\}$ with $|\mathbf{k}|<\ell$, then we want to show that the same holds for a multi-index $\mathbf{k}\in I(\theta)$ of length ℓ and some $f_{\mathbf{k}}$ which turns out to be defined recursively. From the results just obtained we get for such a multi-index $\mathbf{k}\in I(\theta)$ of length ℓ

$$\|\Psi_{\mathbf{k},3}\|_{|\mathbf{k}|,R}\leqslant 2\|\gamma_{\mathbf{k},3}\|_{|\mathbf{k}|,R}\leqslant 2\tilde{C}\sum_{h=2}^{|\mathbf{k}|}(2K_2)^h\sum_{(\mathbf{i}_m;\mathbf{k})}\prod_{m=1}^{h}f_{\mathbf{i}_m}=:f_{\mathbf{k}}.$$

As in the proof of proposition 2.5.11 one can show the existence of a positive constant K such that $f_{\mathbf{k}}<K^{|\mathbf{k}|}$. This proves (5.4.5) for $j=3$.

5.4.4 Estimate for $\Psi_{\mathbf{k},2}$

Let us first remark that the constants we use in this section are not related in any way to those used in the preceding section.

From [Bra91] one deduces that $\Psi_{\mathbf{k},2} = \mathcal{A}_{q_2,q_3}\Psi_{\mathbf{k},3}$ satisfies the equation obtained from (5.4.4) by applying $\mathcal{B}_{q_2}$ to this equation with the convention that $\mathcal{B}_{q_2}w$, $\mathcal{B}_{q_2}\tilde{\Lambda}$ and $\mathcal{B}_{q_2}u_{\mathbf{k}}$ have to be interpreted as $\mathcal{A}_{q_2,q_3}\mathcal{B}_{q_3}w$, $\mathcal{A}_{q_2,q_3}\mathcal{B}_{q_3}\tilde{\Lambda}$ and $\mathcal{A}_{q_2,q_3}\mathcal{B}_{q_3}u_{\mathbf{k}}$ respectively. Hence, $\Psi_{\mathbf{k},2}$ satisfies

$$A_{\mathbf{k},1}\Psi_{\mathbf{k},2} - \frac{t^{q_1-2q_2}}{\Gamma(-1+q_1/q_2)} *_{q_2} q_2 t^{q_2}\Psi_{\mathbf{k},2} + (\mathcal{B}_{q_2}\tilde{A}_{\mathbf{k},+}) *_{q_2} \Psi_{\mathbf{k},2} + \mathcal{B}_{q_2}\big[x^{1-q_1}u_{\mathbf{k}}\big] = 0.$$

This equation can be rewritten as $\Psi_{\mathbf{k},2} = \mathcal{T}_{\mathbf{k},2}\Psi_{\mathbf{k},2} + \gamma_{\mathbf{k},2}$, where

$$\mathcal{T}_{\mathbf{k},2}\Psi = A_{\mathbf{k},1}^{-1}\Big\{\frac{t^{q_1-2q_2}}{\Gamma(-1+q_1/q_2)} *_{q_2} q_2 t^{q_2}\Psi - (\mathcal{B}_{q_2}\tilde{A}_{\mathbf{k},+}) *_{q_2} \Psi\Big\}, \quad \gamma_{\mathbf{k},2} = -A_{\mathbf{k},1}^{-1}\mathcal{B}_{q_2}\big[x^{1-q_1}u_{\mathbf{k}}\big].$$

Let V_2 be as defined at the end of section 5.4.1. For $\ell \in \mathbb{N}_+$ and $R > 0$ we define a Banach space $\mathcal{V}_{\ell,R}$ consisting of functions $\Psi : V_2 \to \mathbb{C}^n$, holomorphic in the interior of V_2, such that $\|\Psi\|_{\ell,R} := \sup_{t\in V_2} |t^{q_2-1}\Psi(t)|/\zeta_{\ell,R}(t) < \infty$, where $\zeta_{\ell,R}$ is now defined by $\zeta_{\ell,R}(t) := \frac{|t|^{p\ell-1}}{\Gamma(p\ell/q_2)}\, e^{R|t|^{\kappa_2}}$.

One should observe that for $\mathbf{j} \in \mathbb{N}_1^n$ we have

$$\mathcal{B}_{q_2}d_{\mathbf{j}} = \mathcal{A}_{q_2,q_3}\mathcal{B}_{q_3}d_{\mathbf{j}} = \mathcal{B}_{q_2}g_{\mathbf{j}} + \sum_{\mathbf{l}\succ\mathbf{j}} \binom{\mathbf{l}}{\mathbf{j}} \big(g_{1,0} + \mathcal{B}_{q_2}g_{\mathbf{l}}*\big)\Psi_{0,2}^{*(\mathbf{l}-\mathbf{j})},$$

where $* = *_{q_2}$. From the multisummability property of $\hat{y}_0$ we know that there exist positive constants K_1 and c_1 such that $|t^{q_2-1}\Psi_{0,2}(t)| \leqslant K_1 e^{c_1|t|^{\kappa_2}}$ for all $t \in V_2$. Hence, lemma 5.4.5 gives $|t^{q_2-1}D_{\mathbf{j}}^{(q_2)}(\Psi_{0,2})(t)| \leqslant K_2^{|\mathbf{j}|+1}e^{c_2|t|^{\kappa_2}}$, $t \in V_2$, for some positive constants K_2 and c_2.

Using this observation, together with (5.4.6) one deduces in a similar way as in the preceding section the existence of positive constants K and R such that (5.4.5) holds for $j = 2$.

5.4.5 Estimate for $\Psi_{\mathbf{k},1}$

Again the constants we use in this section to estimate $\Psi_{\mathbf{k},1}$ are not related to those in the preceding two sections.

From [Bra91] one deduces that $\Psi_{\mathbf{k},1} = \mathcal{A}_{q_1,q_2}\Psi_{\mathbf{k},2}$ satisfies the equation obtained from (5.4.4) by applying $\mathcal{B}_{q_1}$ to this equation with the convention that $\mathcal{B}_{q_1}w$, $\mathcal{B}_{q_1}\tilde{\Lambda}$ and $\mathcal{B}_{q_1}u_{\mathbf{k}}$ have to be interpreted as $\mathcal{A}_{q_1,q_2}\mathcal{A}_{q_2,q_3}\mathcal{B}_{q_3}w$, $\mathcal{A}_{q_1,q_2}\mathcal{A}_{q_2,q_3}\mathcal{B}_{q_3}\tilde{\Lambda}$ and $\mathcal{A}_{q_1,q_2}\mathcal{A}_{q_2,q_3}\mathcal{B}_{q_3}u_{\mathbf{k}}$ respectively. Hence, $\Psi_{\mathbf{k},1}$ satisfies

$$\big(A_{\mathbf{k},1} - q_1 t^{q_1}\big)\Psi_{\mathbf{k},1} + (\mathcal{B}_{q_1}\tilde{A}_{\mathbf{k},+}) *_{q_1} \Psi_{\mathbf{k},1} + \mathcal{B}_{q_1}\big[x^{1-q_1}u_{\mathbf{k}}\big] = 0.$$

This equation can be rewritten as $\Psi_{\mathbf{k},1} = \mathcal{T}_{\mathbf{k},1}\Psi_{\mathbf{k},1} + \gamma_{\mathbf{k},1}$, where

$$\mathcal{T}_{\mathbf{k},1}\Psi = (q_1 t^{q_1} - A_{\mathbf{k},1})^{-1}\big((\mathcal{B}_{q_1}\tilde{A}_{\mathbf{k},+}) *_{q_1} \Psi\big), \quad \gamma_{\mathbf{k},1} = (q_1 t^{q_1} - A_{\mathbf{k},1})^{-1}\mathcal{B}_{q_1}\big[x^{1-q_1}u_{\mathbf{k}}\big].$$

Let V_1 be as defined at the end of section 5.4.1. For $\ell \in \mathbb{N}_+$ and $R > 0$ we define a Banach space $\mathcal{V}_{\ell,R}$ consisting of functions $\Psi : V_1 \to \mathbb{C}^n$, holomorphic in the interior of V_1, such that $\|\Psi\|_{\ell,R} := \sup_{t\in V_1} |t^{q_1-1}\Psi(t)|/\zeta_{\ell,R}(t) < \infty$, where $\zeta_{\ell,R}(t) := \frac{|t|^{p\ell-1}}{\Gamma(p\ell/q_1)}\, e^{R|t|^{\kappa_1}}$.

From the definition of V_1 it follows that there exists a positive constant K_0 such that

$$\left|(q_1 t^{q_1} - A_{\mathbf{k},1})^{-1}\right| \leqslant K_0 |\mathbf{k}|^{-1},$$

for all $t \in V_1$ and all $\mathbf{k} \in I(\theta)$, $|\mathbf{k}| \geqslant 2$ (compare the proof of proposition 4.3.3). Moreover, from lemma 5.4.5 we deduce the existence of two positive constants K_2 and c_2 such that for all $\mathbf{j} \in \mathbb{N}_1^n$ and all $t \in V_1$ we have $|t^{q_1-1} D_{\mathbf{j}}^{(q_1)}(\Psi_{0,1})(t)| \leqslant K_2^{|\mathbf{j}|+1} e^{c_2|t|^{\kappa_1}}$. Using this observation and (5.4.6) we obtain (5.4.5), with $j=1$, in a similar way as in the preceding two sections.

5.5 Analytic Reduction on a Manifold

Like in chapter 4 we write

$$y = y_0(x) + P(x,z) := y_0(x) + P_1(x,u) + v, \quad P_1(x,u) := \sum_{\mathbf{k}\in I(\theta)\setminus\{0\}} y_{\mathbf{k}}(x) u^{\mathbf{k}},$$

where $y_{\mathbf{k}}$, $\mathbf{k} \in I(\theta)$, now are (q_1,q_2,q_3)-sums of $\hat{y}_{\mathbf{k}}$ and where

$$z = \sum_{m=1}^{n} z_m \mathbf{e}_m, \quad u := \sum_{m \in i(\theta)} z_m \mathbf{e}_m \quad \text{and} \quad v := z - u.$$

As in the preceding section we fix $\theta \in \mathbb{R}$ and we define θ_- and θ_+ to be two consecutive singular directions in $H(\theta)$ of the set of all $\hat{y}_{\mathbf{k}}$, $\mathbf{k} \in I(\theta)$. For multi-indices $\mathbf{k} \in I(\theta)\setminus\{0\}$, we define $\Phi_{\mathbf{k},3} := \hat{\mathcal{B}}_{q_3}[x^{-q_1q_2q_3}\hat{y}_{\mathbf{k}}]$, $\Phi_{\mathbf{k},2} := \mathcal{A}_{q_2,q_3}\Phi_{\mathbf{k},3}$ and $\Phi_{\mathbf{k},1} := \mathcal{A}_{q_1,q_2}\Phi_{\mathbf{k},2}$. Since $\hat{y}_{\mathbf{k}}$ is (q_1,q_2,q_3)-summable, we know that each $\Phi_{\mathbf{k},j}$ exists and is of exponential growth of order $\leqslant \kappa_j$ on $S(\theta_0,\alpha_j)$, where $\theta_0 = \frac{1}{2}(\theta_- + \theta_+)$ and where α_j is defined at the end of section 5.4.1. Note that if $|\mathbf{k}| = 1$ then $\Phi_{\mathbf{k},j} = \Psi_{\mathbf{k},j}$, $j = 1,2,3$.

Fix $0 < \varepsilon_3 < \varepsilon_2 < \varepsilon_1$ such that for $j \in \{1,2,3\}$ we have $\varepsilon_j < \alpha_j/2$ (this for example can be achieved by choosing ε_1 smaller than $\alpha_j/2$ for all j). Define $\tilde{V}_j = \overline{S}(\theta_0, \alpha_j - 2\varepsilon_j)$ for $j = 1,2$ and $\tilde{V}_3 = \overline{S}(\theta_0, \alpha_3 - 2\varepsilon_3) \cup \overline{\Delta}(0,\rho/2)$. Moreover, if $V_j = \overline{S}(\theta_0, \alpha_j - \varepsilon_j)$ for $j = 1,2$ and $V_3 = \overline{S}(\theta_0, \alpha_3 - \varepsilon_3) \cup \overline{\Delta}(0,\rho)$, then $\tilde{V}_j \subset V_j$. According to proposition 5.4.1 there exist positive constants K and R such that

$$|t^{q_j-1}\Psi_{\mathbf{k},j}(t)| \;\leqslant\; K^{|\mathbf{k}|} \frac{|t|^{q_1q_2q_3|\mathbf{k}|-1}}{\Gamma\left(\frac{q_1q_2q_3|\mathbf{k}|}{q_j}\right)} e^{R|t|^{\kappa_j}},$$

for all $t \in V_j$.

Lemma 5.5.1 *There exist positive constants $\tilde{K}$ and $\tilde{R}$ such that for all $\mathbf{k} \in I(\theta)$, $|\mathbf{k}| \geqslant 1$, and $j \in \{1,2,3\}$ we have*

$$|\Phi_{\mathbf{k},j}(t)| \;\leqslant\; \tilde{K}^{|\mathbf{k}|} e^{\tilde{R}|t|^{\kappa_j}},$$

for all $t \in \tilde{V}_j$.

PROOF. Since the proof is similar for each $j \in \{1,2,3\}$, we only prove the proposition for $j = 1$. Moreover, we will omit the index 1 in α_1, q_1, κ_1, ε_1, V_1, $\tilde{V}_1$, $\Psi_{\mathbf{k},1}$ and $\Phi_{\mathbf{k},1}$. Let us recall that $|t^{q-1}\Psi_{\mathbf{k}}(t)| \leqslant K^{|\mathbf{k}|}\frac{|t|^{qq_2q_3|\mathbf{k}|-1}}{\Gamma(q_2q_3|\mathbf{k}|)}e^{R|t|^\kappa}$ for all $t \in V$. If $|\mathbf{k} = 1$, then $\Phi_{\mathbf{k}} = \Psi_{\mathbf{k}}$ and the lemma follows from the estimate for $\Psi_{\mathbf{k}}$ just given.

Now for $\mathbf{k} \in I(\theta)$, $|\mathbf{k}| \geqslant 2$, we have $\Psi_{\mathbf{k}} = \mathcal{B}_q[x^{-qq_2q_3(|\mathbf{k}|-1)}] *_q \Phi_{\mathbf{k}}$ and using the definition of q-convolution, we deduce that for those multi-indices $\mathbf{k}$ we have

$$\Psi_{\mathbf{k}}(t^{1/q}) = \Big(\frac{t^{q_2q_3(|\mathbf{k}|-1)-1}}{\Gamma\big(q_2q_3(|\mathbf{k}|-1)\big)} * \Phi_{\mathbf{k}}(t^{1/q})\Big)(t),$$

for every $t \in \mathbb{C}^*$ such that $t^{1/q}$ belongs to the domain in which $\Psi_{\mathbf{k}}$ is holomorphic. Hence, for every $\mathbf{k} \in I(\theta)\backslash\{0\}$ and every such t we have $\Phi_{\mathbf{k}}(t^{1/q}) = (\frac{\mathrm{d}}{\mathrm{d}t})^{q_2q_3(|\mathbf{k}|-1)}\Psi_{\mathbf{k}}(t^{1/q})$.

If $t \in \mathbb{C}^*$ is such that $t^{1/q}$ belongs to $\tilde{V}$ then we can write

$$\Phi_{\mathbf{k}}(t^{1/q}) = \frac{\big(q_2q_3(|\mathbf{k}|-1)\big)!}{2\pi i}\oint \frac{\Psi_{\mathbf{k}}(s^{1/q})}{(s-t)^{q_2q_3(|\mathbf{k}|-1)+1}}\,\mathrm{d}s,$$

where we take as path of integration the circle $s = t + |t|\eta e^{i\varphi}$, $0 \leqslant \varphi < 2\pi$, with $0 < \eta < 1$ so small that $t^{1/q} \in \tilde{V}$ implies $s^{1/q} \in V$. Then

$$|\Phi_{\mathbf{k}}(t^{1/q})| \leqslant \big(K(1+\eta^{-1})^{q_2q_3}\big)^{|\mathbf{k}|}|t|^{q_2q_3-1}e^{R(1+\eta)^{\kappa/q}|t|^{\kappa/q}} \leqslant \tilde{K}^{|\mathbf{k}|}e^{\tilde{R}|t|^{\kappa/q}}.$$

This proves the lemma. ■

In the following let $S := \{x \in \mathbb{C}^* \mid -\pi/(2q_1) - \theta_+ + \varepsilon < \arg x < \pi/(2q_1) - \theta_- - \varepsilon\}$, where $0 < \varepsilon < \frac{1}{2}(\theta_+ - \theta_-)$. With the sector S we associate $\tilde{S} := \{t \in \mathbb{C}^* \mid \theta_- + \varepsilon < \arg t < \theta_+ - \varepsilon\}$ and by choosing ε small enough we can assume without loss of generality that $\tilde{S} \subset S(\theta_0, \alpha_1)$. Moreover, we fix $0 < \varepsilon_3 < \varepsilon_2 < \varepsilon_1$ such that $0 < \varepsilon_j < \alpha_j/2$ and $\tilde{S} \subset \tilde{V}_1$. For $j \in \{1,2,3\}$ we define $S_j := S(-\theta_0, \alpha_j - 2\varepsilon_j + \pi/q_j)$.

Moreover, let us recall that $I(\theta) = \{\mathbf{k} \in \mathbb{N}^{n_1} \mid k_j = 0 \text{ if } j \notin i(\theta)\}$, where $i(\theta)$ is defined by $i(\theta) = \big\{j \in \{1,2,\ldots,n_1\} \bigm| \Re(\omega_j e^{-i\theta}) > 0\big\}$.

Theorem 5.5.2 *There exist positive constants δ and $\tilde{\rho}$ such that $P_1(x,u)$ converges uniformly for $|u| \leqslant \delta$ if $x \in S$, $|x| \geqslant \tilde{\rho}$, and $\sum_{\mathbf{k}\in I(\theta)\backslash\{0\}} \hat{y}_{\mathbf{k}}(x)u^{\mathbf{k}}$ is (q_1,q_2,q_3)-summable on (S_1,S_2,S_3) with multi-sum $P_1(x,u)$.*

Moreover, by means of $y = y_0(x) + P(x,z)$ the differential equation (5.1.1) is, in a neighbourhood of ∞ in S, transformed into

$$z'(x) + \Lambda(x)z(x) = \Lambda_2\big(x, z(x)\big)v(x), \tag{5.5.1}$$

where $\Lambda_2(x,z) = O(x^{-2}) + O(|z|)$ as $x \to \infty$ in S and $z \to 0$.

PROOF. Let us denote $\hat{P}_1(x,u) := \sum_{\mathbf{k}\in I(\theta)\backslash\{0\}} \hat{y}_{\mathbf{k}}(x)u^{\mathbf{k}}$. To prove the summability of $\hat{P}_1(x,u)$ with respect to x on S it is enough to prove that $x^{-q_1q_2q_3}\hat{P}_1(x,u)$ is (q_1,q_2,q_3)-summable on S, for $|u| \leqslant \delta$. Obviously $\hat{\mathcal{B}}_{q_3}\big[x^{-q_1q_2q_3}\hat{P}_1(x,u)\big] = \sum_{\mathbf{k}\in I(\theta)\backslash\{0\}} \Phi_{\mathbf{k},3}(t)u^{\mathbf{k}}$. Using the lemma above, this series converges uniformly for $|u| \leqslant \delta$ and $t \in \tilde{V}_3$, provided that δ is small enough, and is of exponential growth of order $\leqslant \kappa_3$ there. Hence, we can apply the acceleration operator $\mathcal{A}_{q_2,q_3}$ to this series and we obtain $\sum_{\mathbf{k}\in I(\theta)\setminus\{0\}} \Phi_{\mathbf{k},2}(t)u^{\mathbf{k}}$, which at first glance is defined in a neighbourhood of 0 in $S(\theta_0, \alpha_3 + \pi/\kappa_3 - 2\varepsilon_3)$. However, by definition each $\Phi_{\mathbf{k},2}$ can be analytically continued in $S(\theta_0, \alpha_2)$ and the lemma above tells us that this continuation is bounded by $\tilde{K}^{|\mathbf{k}|}e^{\tilde{R}|t|^{\kappa_2}}$ on $\tilde{V}_2$. Hence, $\sum_{\mathbf{k}\in I(\theta)\setminus\{0\}} \Phi_{\mathbf{k},2}(t)u^{\mathbf{k}}$ converges uniformly for $|u| \leqslant \delta$ and $t \in \tilde{V}_2$ and is of exponential growth of order $\leqslant \kappa_2$ there. In a similar way we see that $\sum_{\mathbf{k}\in I(\theta)\setminus\{0\}} \Phi_{\mathbf{k},1}(t)u^{\mathbf{k}}$ converges uniformly for $|u| \leqslant \delta$ and $t \in \tilde{V}_1$ and is of exponential growth of order $\leqslant \kappa_1 = q_1$ there. This proves the summability property of $\hat{P}_1(x,u)$.

To prove convergence of $P_1(x,u)$ we first recall that

$$|\Psi_{\mathbf{k},1}(t)| \leqslant K^{|\mathbf{k}|}\frac{|t|^{q_1q_2q_3|\mathbf{k}|-q_1}}{(q_2q_3|\mathbf{k}|-1)!}\,e^{R|t|^{q_1}},$$

for all $t \in V_1 = \overline{S}(\theta_0, \alpha_1 - \varepsilon_1)$. Now fix $\theta \in \big(\theta_0 - (\alpha_1 - \varepsilon_1)/2, \theta_0 + (\alpha_1 - \varepsilon_1)/2\big)$ and let $\eta = \sin(q_1\varepsilon_1/2)$. Then for $x \in \mathbb{C}^*$ with $-\theta - \pi/(2q_1) + \varepsilon_1/2 < \arg x < -\theta + \pi/(2q_1) - \varepsilon_1/2$ and $|x| \geqslant (2R/\eta)^{1/q_1}$ we have $\Re(x^{q_1}e^{iq_1\theta}) - R \geqslant \frac{\eta}{2}|x|^{q_1}$ and for these values of x the function $w_{\mathbf{k}}(x) := (\mathcal{L}_{q_1}\Psi_{\mathbf{k},1})(x)$ can be estimated by

$$|w_{\mathbf{k}}(x)| \;\leqslant\; \frac{K^{|\mathbf{k}|}}{(q_2q_3|\mathbf{k}|-1)!}\int_0^\infty \sigma^{q_2q_3|\mathbf{k}|-1}e^{-\frac{\eta}{2}|x|^{q_1}\sigma}\mathrm{d}\sigma \;\leqslant\; \big(K(2\eta^{-1})^{q_2q_3}\big)^{|\mathbf{k}|}x^{-q_1q_2q_3|\mathbf{k}|}. \quad (5.5.2)$$

By varying $\theta \in \big(\theta_0 - (\alpha_1 - \varepsilon_1)/2, \theta_0 + (\alpha_1 - \varepsilon_1)/2\big)$, we deduce that the estimate for $w_{\mathbf{k}}$ holds for all $x \in S(-\theta_0, \alpha_1 + \pi/q_1 - 2\varepsilon_1)$ and $|x| \geqslant \tilde{\rho} := (2R/\eta)^{1/q_1}$. In particular (5.5.2) is valid for all $x \in S$, $|x| \geqslant \tilde{\rho}$. Moreover, the properties of multisummability imply that $y_{\mathbf{k}}(x) = x^{q_1q_2q_3|\mathbf{k}|}w_{\mathbf{k}}(x)$, so $|y_{\mathbf{k}}(x)| \leqslant \big(K(2\eta^{-1})^{q_2q_3}\big)^{|\mathbf{k}|}$, for all $x \in S$, $|x| \geqslant \tilde{\rho}$. From this the first part of the theorem follows by choosing δ small enough.

The proof of the last part of the theorem runs in a similar way as the proof of the corresponding statement in theorem 4.3.4. ■

Corollary 5.5.3 *Let $\mathcal{M}$ be the manifold defined by $y = y_0(x) + P_1(x,u)$, with x in a neighbourhood of ∞ in S and u in a neighbourhood of 0. Then on $\mathcal{M}$ the differential equation (5.1.1) is transformed into $u'(x) + \Lambda(x)u(x) = 0$.*

Theorem 5.5.4 *Let S' be a sub-sector of S containing the direction $-\theta/q_1$ and assume that y is a solution of (5.1.1) such that $y(x) = O(x^{-2})$ as $x \to \infty$ on S'. Moreover, assume that $\Re(\lambda_m e^{-iq_2\theta/q_1}) \leqslant 0$ for all $m \in \mathcal{J}_2$ and $\Re(\mu_m e^{-iq_3\theta/q_1}) \leqslant 0$ for all $m \in \mathcal{J}_3$. Then there exists a unique vector $C \in \mathbb{C}^{n_1}$ such that*

$$y(x) = \sum_{\mathbf{k}\in I(\theta)} C^{\mathbf{k}} e^{-\frac{\langle\mathbf{k},\boldsymbol{\omega}\rangle}{q_1}x^{q_1} - \frac{\langle\mathbf{k},\boldsymbol{\lambda}\rangle}{q_2}x^{q_2} - \frac{\langle\mathbf{k},\boldsymbol{\mu}\rangle}{q_3}x^{q_3}} x^{\langle\mathbf{k},\mathbf{a}\rangle}y_{\mathbf{k}}(x), \quad x \in S'. \quad (5.5.3)$$

Here C is such that $C_m = 0$ unless $e^{-\omega_m x^{q_1}} \to 0$ as $x \to \infty$ in S'.

Conversely, if $C \in \mathbb{C}^{n_1}$ is such that $C_m \neq 0$ only if $e^{-\omega_m x^{q_1}} \to 0$ in S' then (5.5.3) converges in a neighbourhood of ∞ in S', defines a solution of (5.1.1) which is holomorphic in this neighbourhood and has the property $y(x) \sim \hat{y}_0(x)$ as $x \to \infty$ in S'.

Proof of theorem 5.5.4. To prove this statement we put $w(x) := y(x) - y_0(x)$ and we let z, u and v be associated with these solutions as in the introduction of section 5.5. As $D_z P(\infty, z)|_{z=0} = \mathtt{I}$, the implicit function theorem implies that the function $z \mapsto P(x,z)$ is invertible for x in a neighbourhood of ∞ in S' and z in a neighbourhood of the origin. So we can conclude that $z(x)$, and thus $u(x)$ and $v(x)$, are $O(x^{-2})$ as $x \to \infty$ on S'. From (5.5.1) we deduce a linear equation for v that involves the known function u. In fact, v satisfies $v'(x) + (\Lambda(x) + O(x^{-2}))v(x) = 0$ and it is well known that if S'' is a sub-sector of S' of opening less than π/q_1 and containing the direction $-\theta/q_1$, then this equation possesses a fundamental system of solutions

$$v_m(x) = e^{-\frac{\omega_m}{q_1}x^{q_1} - \frac{\lambda_m}{q_2}x^{q_2} - \frac{\mu_m}{q_3}x^{q_3}} x^{a_m} \left(\mathbf{e}_m + O(x^{-1}) \right), \quad m \in \{1,2,\ldots,n\} \backslash i(\theta),$$

as $x \to \infty$ in S'', with the understanding that $\omega_m = 0$ for $m \in \mathcal{J}_2 \cup \mathcal{J}_3$ and $\lambda_m = 0$ for $m \in \mathcal{J}_3$. Hence, there exist constants C_m, $m \in \{1,2,\ldots,n\} \backslash i(\theta)$, such that

$$v(x) = \sum_{\substack{m=1,2,\ldots,n \\ m \notin i(\theta)}} C_m e^{-\frac{\omega_m}{q_1}x^{q_1} - \frac{\lambda_m}{q_2}x^{q_2} - \frac{\mu_m}{q_3}x^{q_3}} x^{a_m} \left(\mathbf{e}_m + O(x^{-1}) \right),$$

as $x \to \infty$ in S''. However, $v(x) = O(x^{-2})$ as $x \to \infty$ in S'', which contradicts the exponential growth of v in a part of S'' if $C_m \neq 0$ for some $m \notin i(\theta)$. This proves that $v \equiv 0$ on S'' and thus on S'. ■

5.6 The Case of r Levels

Let us come back to the case of r levels, where $r \in \mathbb{N}_+$, i.e. let us consider the equation

$$y'(x) + \Lambda(x)y(x) + g(x, y(x)) = 0, \quad \text{with } \Lambda(x) = \bigoplus_{j=1}^{r} \left(x^{q_j - 1}\mathtt{M}_j + x^{q_{j+1}-1}\tilde{\mathtt{M}}_j(x) \right), \quad (5.6.1)$$

where $r \in \mathbb{N}_+$, $0 := q_{r+1} < q_r < q_{r-1} < \cdots < q_1$, $q_j \in \mathbb{N}$, $n = n_1 + n_2 + \cdots + n_r$, $n_j \in \mathbb{N}$, with the matrices $\mathtt{M}_j$ and $\tilde{\mathtt{M}}_j(x)$ defined in (5.1.2).

For $\theta \in \mathbb{R}$ we define $i(\theta) := \left\{ j \in \{1,2,\ldots,n_1\} \;\middle|\; \Re(\mu_j^{(1)} e^{-i\theta}) > 0 \right\}$ and let $I(\theta)$ and $H(\theta)$ be defined as in section 5.4.1. Let θ_- and θ_+ be two consecutive singular directions in $H(\theta)$ and define $S := \{x \in \mathbb{C}^* \mid -\pi/(2q_1) - \theta_+ + \varepsilon < \arg x < \pi/(2q_1) - \theta_- - \varepsilon\}$, where $0 < \varepsilon < \frac{1}{2}(\theta_+ - \theta_-)$. Let $P(x,z)$ and $P_1(x,u)$ be as in the introduction of section 5.5.

In a similar way as we proved theorem 5.5.2 and theorem 5.5.4 one may show the following theorem.

Theorem 5.6.1 *There exist positive constants δ and $\tilde{\rho}$ such that $P_1(x,u)$ converges uniformly for $|u| \leqslant \delta$ if $x \in S$, $|x| \geqslant \tilde{\rho}$, and there exist sectors $S_1, S_2, \dots, S_r$ such that $\sum_{\mathbf{k}\in I(\theta)\setminus\{0\}} \hat{y}_{\mathbf{k}}(x)u^{\mathbf{k}}$ is $(q_1, q_2, \dots, q_r)$-summable on $(S_1, S_2, \dots, S_r)$ with multi-sum $P_1(x,u)$.*

By means of $y = y_0(x) + P(x,z)$ the differential equation (5.6.1) is, in a neighbourhood of ∞ in S, transformed into

$$z'(x) + \Lambda(x)z(x) = \Lambda_2\big(x, z(x)\big)v(x), \tag{5.6.2}$$

where $\Lambda_2(x,z) = O(x^{-2}) + O(|z|)$ as $x \to \infty$ in S and $z \to 0$. Hence, on the manifold $\mathcal{M}$ defined by $y = y_0(x) + P_1(x,u)$, with x in a neighbourhood of ∞ in S and u in a neighbourhood of 0, (5.6.1) is transformed into $u'(x) + \Lambda(x)u(x) = 0$.

Let S' be a sub-sector of S containing the direction $-\theta/q_1$ and assume that y is a solution of (5.6.1) such that $y(x) = O(x^{-2})$ as $x \to \infty$ on S'. Moreover, assume that $\Re(\mu_m^{(j)} e^{-iq_j\theta/q_1}) \leqslant 0$ for all $m \in \mathcal{J}_j$, $j = 2, 3, \dots, r$. Then there exists a unique vector $C \in \mathbb{C}^{n_1}$ such that

$$y(x) = \sum_{\mathbf{k}\in I(\theta)} C^{\mathbf{k}} e^{-\sum_{j=1}^{r} \frac{\langle \mathbf{k}, \boldsymbol{\mu}^{(j)}\rangle}{q_j} x^{q_j}} x^{\langle \mathbf{k}, \mathbf{a}\rangle} y_{\mathbf{k}}(x). \tag{5.6.3}$$

Here $\boldsymbol{\mu}^{(j)} = \sum_{m=1}^{n} \mu_m^{(j)} \mathbf{e}_m$ (where $\mu_m^{(j)} := 0$ if $m \notin \mathcal{J}_1 \cup \cdots \cup \mathcal{J}_j$) and C is such that $C_m = 0$ unless $e^{-\mu_m^{(1)} x^{q_1}} \to 0$ as $x \to \infty$ in the sector S'.

Conversely, if $C \in \mathbb{C}^{n_1}$ is such that $C_m \neq 0$ only if $e^{-\mu_m^{(1)} x^{q_1}} \to 0$ in S' then (5.6.3) converges in a neighbourhood of ∞ in S', defines a solution of (5.6.1) which is holomorphic in this neighbourhood and has the property $y(x) \sim \hat{y}_0(x)$ as $x \to \infty$ in S'.

Appendix A

Staircase Distributions

A.1 Introduction

In this appendix we want to expose the theory on staircase distributions. Most of the proofs come from Costin's paper [Cos98]. However, we will give some additional results.

For an open set $\Omega \subseteq \mathbb{R}$ we define $\mathcal{D}(\Omega)$ to be the space of $\mathcal{C}^\infty$-functions $\varphi : \Omega \to \mathbb{R}$ with compact support in Ω. Such functions are called *test functions* on Ω. We will follow the theory of distributions as pointed out by Laurent Schwartz in [Sch66]. In fact, a distribution on an open set Ω is a continuous linear functional on $\mathcal{D}(\Omega)$. The set of all distributions on Ω will be denoted by $\mathcal{D}'(\Omega)$. We will not go into details, but refer to the book of Schwartz for a deep exposition of this theory.

Let $\mathcal{H}$ be the Heaviside one step function (at 0) defined by

$$\mathcal{H}(t) = \begin{cases} 1 & \text{if } t \geqslant 0, \\ 0 & \text{if } t < 0 \end{cases}$$

and let $\mathcal{P}$ be the operator defined on distributions $T \in \mathcal{D}'(\Omega)$ by $\mathcal{P}T = \mathcal{H} * T$. One should observe that $\mathcal{H} * T$ only is defined for those distributions T with $\big(\mathrm{supp}(\mathcal{H}), \mathrm{supp}(T)\big)$ satisfying the convolution property[1]. The distributions we take into consideration have support in $[0, \infty)$, and it is easily seen that $([0,\infty), [0,\infty))$ satisfies the convolution property. Moreover, if $T \in \mathcal{D}'(\Omega)$ is so that $\mathcal{P}T$ is defined, then $\mathcal{P}T' = H * T' = H' * T = \delta * T = T$. Here $\delta = \delta(0)$ is the Dirac delta distribution concentrated at 0. We will need the following result in what comes.

Lemma A.1.1 *Let $m, n \in \{1, 2, 3, \ldots\}$ and let $T \in \mathcal{D}'(0, n)$. Then $T^{(m)} = 0$ if and only if $T : \mathcal{D}(0, n) \to \mathbb{C}$ is given by*

$$T\varphi = \int_0^n p(t)\varphi(t)\mathrm{d}t,$$

where p is some polynomial of degree $\leqslant m - 1$.

[1] If A and B are closed subsets of $\mathbb{R}$, then (A, B) satisfies the convolution property if for every compact $K \subset \mathbb{R}$ the set $\{(x, y) \in A \times B \mid x + y \in K\}$ again is compact.

PROOF. Let us assume that $T^{(m)} = 0$ (the other implication is trivial). The proof that T is a polynomial of degree $\leqslant m-1$ is given with induction on m. If $m = 1$, then the assumption implies that $T = 0$ on $\{\varphi' \mid \varphi \in \mathcal{D}(0,n)\}$. One easily proves that this set is equal to $\{\varphi \in \mathcal{D}(0,n) \mid \int_0^n \varphi(t)\mathrm{d}t = 0\}$, which exactly is the kernel of the operator $T_1 : \mathcal{D}(0,n) \to \mathbb{C}$ defined by $T_1\varphi = \int_0^n \varphi(t)\mathrm{d}t$. Hence, T and T_1 are two linear operators on $\mathcal{D}(0,n)$ with the property $\ker(T_1) \subset \ker(T)$ and this implies that $T = cT_1$ for some constant $c \in \mathbb{C}$. Hence, $p \equiv c$, which proves the case $m = 1$.

Next fix $m \geqslant 2$ and assume the lemma to be true for all $k \leqslant m-1$, then for $k = m$ we have $(T')^{(m-1)} = 0$. Hence, the induction hypothesis implies that $T' = p$, with p a polynomial of degree $\leqslant m-2$. If $q(t) := \int_0^t p(s)\mathrm{d}s$, then q is a polynomial of degree $\leqslant m-1$, $q' = p$ (in the classical sense) and thus also $(T-q)' = 0$. Hence, $T - q$ equals a constant and we conclude that T is a polynomial of degree $\leqslant m-1$. ■

A.2 The Definition

For $a, b \in \mathbb{R}_+$ we introduce the notation $\mathbb{1}_{[a,b]}$ for the indicator function on the line-segment $[a,b]$, i.e.

$$\mathbb{1}_{[a,b]}(t) = \begin{cases} 1 & \text{if } t \in [a,b], \\ 0 & \text{otherwise.} \end{cases}$$

Definition A.2.1 *A distribution $f \in \mathcal{D}'(0,\infty)$ is called a* staircase distribution *of order $m \in \mathbb{N}_{\geqslant 1}$ if for all $k \in \mathbb{N}$ there exists a function $F_k \in L^1(0,k+1)$ such that the restriction of $\mathcal{P}^{mk} f$ to $\mathcal{D}(0,k+1)$ equals F_k. We will denote the set of staircase distributions of order m by $\mathcal{D}'_m$.*

Obviously $F_{k+1} = \mathcal{P}^m F_k$ when restricted to $\mathcal{D}(0,k+1)$. If F is an L^1-function defined on $\mathbb{R}_+ := [0,\infty)$ we may and will agree that $F(t) = 0$ for negative t and in that case $(\mathcal{H} * F)(t) = \int_{-\infty}^{\infty} \mathcal{H}(t-s)F(s)\mathrm{d}s$ coincides with the usual convolution $(1 * F)(t) = \int_0^t F(s)\mathrm{d}s$. Hence, $F_{k+1}\mathbb{1}_{[0,k+1]} = (1^{*m} * F_k)\mathbb{1}_{[0,k+1]}$ in the sense of L^1-functions. In particular we have $F_k \mathbb{1}_{[0,1]} = (1^{*mk} * F_0)\mathbb{1}_{[0,1]}$, and thus $F_k^{(j)}(0) = 0$ for all $j \in \{0, 1, \ldots, mk-1\}$.

Lemma A.2.2 (Decomposition lemma) *Each $f \in \mathcal{D}'_m$ can be written as*

$$f = \sum_{k=0}^{\infty} \big(\Delta_k(f)\big)^{(mk)}, \tag{A.2.1}$$

for unique $\Delta_k(f) \in L^1(0,\infty)$, with each $\Delta_k(f)$ supported in $[k,k+1]$. Moreover,

$$F_n = \sum_{k=0}^{n} \big(\mathcal{P}^{m(n-k)} \Delta_k(f)\big) \mathbb{1}_{[0,n+1]}. \tag{A.2.2}$$

Conversely, any series of the form $\sum_{k=0}^{\infty} \Delta_k^{(mk)}$ with $\Delta_k \in L^1(0,\infty)$ supported in $[k,k+1]$ represents an element in $\mathcal{D}'_m$.

PROOF. We construct the functions $\Delta_k := \Delta_k(f)$ by induction. For $k = 0$ we take $\Delta_0 := F_0$. Assume that Δ_k has been constructed such that (A.2.2) holds for all $k \leqslant n-1$, where $n \geqslant 1$. Then we define Δ_n by

$$\Delta_n := \left\{ F_n - \sum_{j=0}^{n-1} \mathcal{P}^{m(n-j)} \Delta_j \right\} 1_{[0,n+1]}.$$

Since $\left(\sum_{j=0}^{n-1} \mathcal{P}^{m(n-j)} \Delta_j\right) 1_{[0,n]} = (\mathcal{P}^m F_{n-1}) 1_{[0,n]} = F_n 1_{[0,n]}$, the function Δ_n has support in $[n, n+1]$. By definition the function F_n belongs to $L^1(0, n+1)$. On the other hand, for arbitrary $j \in \{0, 1, \ldots, n-1\}$ we have $\mathcal{P}^{m(n-j)} \Delta_j = 1^{*m(n-j)} * \Delta_j$ and so $\Delta_n \in L^1(0, \infty)$. Thus (A.2.2) is satisfied by construction. For arbitrary $n \in \mathbb{N}$ the restriction of f to $\mathcal{D}(0, n+1)$ equals

$$F_n^{(mn)} = \left\{ \sum_{k=0}^{n} (\mathcal{P}^{m(n-k)} \Delta_k) 1_{[0,n+1]} \right\}^{(mn)} = \sum_{k=0}^{n} \Delta_k^{(mk)},$$

which proves (A.2.1). To show uniqueness, let us assume that there are two sequences $\{\Delta_k\}_k$ and $\{\tilde{\Delta}_k\}_k$, satisfying the assertions in the lemma. Since f restricted to $\mathcal{D}(0,1)$ equals F_0, we get $\Delta_0 = \tilde{\Delta}_0$. Now assume $\Delta_k = \tilde{\Delta}_k$ for all $k \leqslant n-1$. Restricting f to $\mathcal{D}(0, n+1)$ and using (A.2.1) we see that $\Delta_n^{(mn)} = \tilde{\Delta}_n^{(mn)}$. Lemma A.1.1 then implies that $\Delta_n(t) = \tilde{\Delta}_n(t) + p(t)$, where p is a polynomial of degree $\leqslant mn - 1$. As $\Delta_n(t) = \tilde{\Delta}_n(t) = 0$ for $t \in [0, n)$, we have $p(t) = 0$ for all $t \in [0, n)$ and thus $p \equiv 0$. The last statement in the lemma is trivial. ■

A.3 The Convolution Algebra $\mathcal{D}'_{m,\nu}$

Suppose f and $\tilde{f}$ are staircase distributions in $\mathcal{D}'_m$, then there are unique sequences $\{\Delta_k\}_k$ and $\{\tilde{\Delta}_k\}_k$ respectively, satisfying the properties in the decomposition lemma. In the sense of distributions we have

$$\Delta_k^{(mk)} * \tilde{\Delta}_l^{(ml)} = \left(\Delta_k * \tilde{\Delta}_l\right)^{(mk+ml)}$$

and in the sense of L^1-functions $\Delta_k * \tilde{\Delta}_l = \left(\Delta_k * \tilde{\Delta}_l\right) 1_{[k+l,k+l+2]}$. Now, to find the decomposition of the convolution of two staircase distributions we use the following lemma.

Lemma A.3.1 *Let $k \in \mathbb{N}$ and $f = F^{(mk)} \in \mathcal{D}'(\mathbb{R})$, where $F \in L^1(0,\infty)$ is supported in $[k, \infty)$. Then f belongs to $\mathcal{D}'_m$ and the decomposition of f has the following terms*

$$\Delta_0 = \Delta_1 = \cdots = \Delta_{k-1} = 0, \quad \Delta_k = F 1_{[k,k+1]} \tag{A.3.1}$$

and for $n \geqslant 1$

$$\Delta_{k+n} = G_n 1_{[k+n,k+n+1]}, \qquad \text{where } G_n = \mathcal{P}^m\big(G_{n-1} 1_{[k+n,\infty)}\big), \quad G_0 = F. \tag{A.3.2}$$

In particular, if $f \in L^1(0,\infty)$, then $f \in \mathcal{D}'_m$.

PROOF. Taking $F_0 = F_1 = \cdots = F_{k-1} = 0$, $F_k = F\mathbb{1}_{[k,k+1]}$ and $F_{k+n} = (\mathcal{P}^{mn}F)\mathbb{1}_{[k,k+n+1]}$ for $n > 0$ we see that f indeed is a staircase distribution of order m. From (A.2.2) we deduce that for $j \geqslant 0$:

$$\Delta_j = \left\{ F_j - \sum_{i=0}^{j-1} \mathcal{P}^{m(j-i)} \Delta_i \right\} \mathbb{1}_{[j,j+1]}. \tag{A.3.3}$$

This implies (A.3.1) and moreover, if $n \in \mathbb{N}$ and

$$G_n := \mathcal{P}^{mn}F - \sum_{j=0}^{n-1} \mathcal{P}^{m(n-j)} \Delta_{k+j},$$

then (A.3.3) implies $\Delta_{k+n} = G_n \mathbb{1}_{[k+n,k+n+1]}$. On the other hand it is easily seen that $G_0 = F$, while for $n \geqslant 1$ we have

$$G_n = \mathcal{P}^m \Big(\mathcal{P}^{m(n-1)}F - \sum_{j=0}^{n-1} \mathcal{P}^{m(n-1-j)} \Delta_{k+j} \Big) = \mathcal{P}^m (G_{n-1} - \Delta_{k+n-1}).$$

Since $\Delta_{k+n-1} = G_{n-1}\mathbb{1}_{[k+n-1,k+n]}$ it follows that $G_n = \mathcal{P}^m(G_{n-1}\mathbb{1}_{[k+n,\infty)})$. ■

The convolution of two staircase distributions $f = \sum_{k=0}^{\infty} \Delta_k^{(mk)}$ and $\tilde{f} = \sum_{k=0}^{\infty} \tilde{\Delta}_k^{(mk)}$ is given by

$$f * \tilde{f} = \sum_{k,l=0}^{\infty} (\Delta_k * \tilde{\Delta}_l)^{(mk+ml)} = \sum_{k=0}^{\infty} \Big(\sum_{h=0}^{k} \Delta_h * \tilde{\Delta}_{k-h} \Big)^{(mk)}.$$

Here $\operatorname{supp}\big(\Delta_h * \tilde{\Delta}_{k-h}\big) \subset [k, k+2]$ and so by lemma A.3.1 there exist $\Delta_{k,l} \in L^1(0,\infty)$, $l \geqslant k$, such that $\operatorname{supp}(\Delta_{k,l}) \subset [l, l+1]$ and $\sum_{h=0}^{k} \big(\Delta_h * \tilde{\Delta}_{k-h}\big)^{(mk)} = \sum_{l=k}^{\infty} \Delta_{k,l}^{(ml)}$. Hence,

$$f * \tilde{f} = \sum_{l=0}^{\infty} \Big(\sum_{k=0}^{l} \Delta_{k,l} \Big)^{(ml)} \tag{A.3.4}$$

again is a staircase distribution and thus $(\mathcal{D}'_m, +, *)$ is a commutative algebra with respect to convolution.

In the sequel ν will always be a nonnegative number.

Definition A.3.2 *The space L^1_ν is defined as the space of functions $f \in L^1_{loc}(0,\infty)$ such that $\|f\|_\nu := \int_0^\infty e^{-\nu t}|f(t)|\mathrm{d}t$ is finite.*

It is easy to check that L^1_ν is a commutative Banach algebra with respect to addition and convolution. Moreover, if $f, g \in L^1_\nu$, then $\|f * g\|_\nu \leqslant \|f\|_\nu \cdot \|g\|_\nu$. Observe that if $f \in L^1_{\nu_0}$ for some ν_0 then $f \in L^1_\nu$ for all $\nu \geqslant \nu_0$ and the dominated convergence theorem implies $\|f\|_\nu \searrow 0$ if $\nu \nearrow \infty$.

Remark A.3.3 If $f \in L^1_\nu$, then in fact $\|f\|_\nu = \mathcal{L}(|f|)(\nu)$. If moreover g belongs to L^1_ν, then $|f * g| \leqslant |f| * |g|$ and thus $\mathcal{L}(|f * g|)(\nu) \leqslant \mathcal{L}(|f| * |g|)(\nu) = \mathcal{L}(|f|)(\nu) \cdot \mathcal{L}(|g|)(\nu)$, which shows that $\|f * g\|_\nu \leqslant \|f\|_\nu \cdot \|g\|_\nu$.

Lemma A.3.4 *Let* $\nu_0 > m$ *and let* f *be as in lemma A.3.1. Then for all* $\nu \geqslant \nu_0$ *we have* $\|\Delta_{k+n}\|_\nu \leqslant \nu^{-mn}\|F\|_\nu$ *if* $n = 0, 1, 2$ *and for* $n \geqslant 3$

$$\|\Delta_{k+n}\|_\nu \leqslant e^{2\nu - n\nu}\frac{n^{nm-1}}{(nm-1)!}\|F\|_\nu.$$

Moreover, there exists a constant C_m*, only depending on* m *and* ν_0*, such that*

$$\sum_{n=0}^{\infty}\nu^{m(k+n)}\|\Delta_{k+n}\|_\nu \leqslant C_m\nu^{mk}\|F\|_\nu.$$

Proof. The following proof was furnished by O. Costin in a private communication.

First one should observe that $|\mathcal{P}^m F| \leqslant \mathcal{P}^m(|F|)$ for every nonnegative integer m. As the estimate for $\|\Delta_k\|_\nu$ is evident, we start with proving the estimate for $\|\Delta_{k+1}\|_\nu$. Using Fubini's theorem we obtain

$$\begin{aligned}\|\Delta_{k+1}\|_\nu &\leqslant \int_{k+1}^{k+2} e^{-\nu t}\mathcal{P}^m\big(|F|\mathbb{1}_{[k+1,\infty)}\big)(t)\,\mathrm{d}t = \int_{k+1}^{k+2} e^{-\nu t}\int_{k+1}^{t}\frac{(t-s)^{m-1}}{(m-1)!}|F(s)|\,\mathrm{d}s\,\mathrm{d}t\\ &\leqslant \int_{k+1}^{k+2}\int_s^\infty e^{-\nu t}\frac{(t-s)^{m-1}}{(m-1)!}\,\mathrm{d}t\,|F(s)|\,\mathrm{d}s = \nu^{-m}\int_{k+1}^{k+2} e^{-\nu s}|F(s)|\,\mathrm{d}s.\end{aligned}$$

For Δ_{k+2} we get in a similar way

$$\begin{aligned}\|\Delta_{k+2}\|_\nu &= \int_{k+2}^{k+3} e^{-\nu t}\int_{k+2}^{t}\frac{(t-s)^{m-1}}{(m-1)!}\int_{k+1}^{s}\frac{(s-\sigma)^{m-1}}{(m-1)!}|F(\sigma)|\,\mathrm{d}\sigma\,\mathrm{d}s\,\mathrm{d}t\\ &\leqslant \int_{k+1}^{k+3}\int_{\max\{k+2,\sigma\}}^{\infty}\int_s^\infty e^{-\nu t}\frac{(t-s)^{m-1}}{(m-1)!}\frac{(s-\sigma)^{m-1}}{(m-1)!}|F(\sigma)|\,\mathrm{d}t\,\mathrm{d}s\,\mathrm{d}\sigma.\end{aligned}$$

As $\sigma \leqslant \max\{k+2,\sigma\}$ it is easy to deduce the desired estimate for the norm of Δ_{k+2}. To prove the estimate for the norm of Δ_{k+n}, $n \geqslant 3$, we first estimate G_n (cf. lemma A.3.1): for $n \geqslant 1$ we will show that

$$|G_n(t)| \leqslant e^{\nu(k+2)}\frac{(t-k-1)^{nm-1}}{(nm-1)!}\|F\|_\nu. \tag{A.3.5}$$

The proof of (A.3.5) can be given with induction on n. For $n = 1$ we have

$$|G_1(t)| \leqslant \int_{k+1}^{t}\frac{(t-s)^{m-1}}{(m-1)!}|F(s)|\,\mathrm{d}s$$

and using the inequality $|F(s)| = |F(s)|\mathbb{1}_{[k,k+2]} \leqslant e^{\nu(k+2)}\,e^{-\nu s}\,|F(s)|$ we obtain (A.3.5) for $n = 1$. Now assume (A.3.5) to be true for $n-1$, then

$$\begin{aligned}|G_n(t)| &\leqslant \int_{k+n}^{t}\frac{(t-s)^{m-1}}{(m-1)!}|G_{n-1}(s)|\mathrm{d}s\\ &\leqslant e^{\nu(k+2)}\|F\|_\nu\int_{k+n}^{t}\frac{(t-s)^{m-1}(s-k-1)^{(n-1)m-1}}{(m-1)!\big((n-1)m-1\big)!}\,\mathrm{d}s.\end{aligned}$$

Replacing $k+n$ in the lower bound of the integral by $k+1$ we obtain

$$\begin{aligned} |G_n(t)| &\leqslant e^{\nu(k+2)}\|F\|_\nu \int_0^{t-k-1} \frac{(t-k-1-\sigma)^{m-1}\sigma^{(n-1)m-1}}{(m-1)!\big((n-1)m-1\big)!}\,\mathrm{d}\sigma \\ &= e^{\nu(k+2)}\|F\|_\nu \frac{(t-k-1)^{mn-1}}{(nm-1)!}. \end{aligned}$$

So (A.3.5) holds for all $n \geqslant 1$. Hence, for $n \geqslant 3$ we have

$$\|\Delta_{k+n}\|_\nu \leqslant e^{\nu(k+2)}\,\|F\|_\nu \frac{1}{(nm-1)!}\int_{k+n}^{k+n+1} e^{-\nu t}(t-k-1)^{nm-1}\,\mathrm{d}t \leqslant e^{\nu(2-n)}\frac{n^{nm-1}}{(nm-1)!}\,\|F\|_\nu,$$

which proves the estimate for $\|\Delta_{k+n}\|_\nu$. Finally,

$$\sum_{n=0}^{\infty}\nu^{m(k+n)}\|\Delta_{k+n}\|_\nu \leqslant \nu^{mk}\|F\|_\nu\left\{3+\sum_{n=3}^{\infty}\nu^{nm}e^{2\nu-n\nu}\frac{n^{nm-1}}{(nm-1)!}\right\}$$

and using Stirling's formula[2] we get

$$\sum_{n=3}^{\infty}\nu^{nm}e^{-\nu(n-2)}\frac{n^{nm-1}}{(nm-1)!} \leqslant const.\cdot\frac{e^{2m}\nu^{2m}}{m^{2m-1/2}}\sum_{n=1}^{\infty}\left(\frac{e^{m-\nu}\nu^m}{m^m}\right)^n.$$

As $x \mapsto \frac{e^{m-x}x^m}{m^m}$ is decreasing on $[m,\infty)$, the last sum converges for $\nu \geqslant \nu_0 > m$. ■

Definition A.3.5 *On the set of staircase distributions $\mathcal{D}'_m$ we define a family of norms $\|\cdot\|_{m,\nu}$, depending on the parameter ν, by*

$$\|f\|_{m,\nu} := C_m\sum_{k=0}^{\infty}\nu^{mk}\|\Delta_k(f)\|_\nu, \tag{A.3.6}$$

where C_m is the positive constant found in lemma A.3.4. We define $\mathcal{D}'_{m,\nu}$ to be the set of distributions $f \in \mathcal{D}'_m$ with $\|f\|_{m,\nu} < \infty$.

The motivation to define the norm in this particular way is given in proposition A.3.8, where we will prove that $\mathcal{D}'_{m,\nu}$ is closed under convolution and $\|f * \tilde{f}\|_{m,\nu} \leqslant \|f\|_{m,\nu}\cdot\|\tilde{f}\|_{m,\nu}$ for $f,\ \tilde{f} \in \mathcal{D}'_{m,\nu}$.

Proposition A.3.6 *The pair $(\mathcal{D}'_{m,\nu}, \|\cdot\|_{m,\nu})$ defines a Banach space.*

Proof. Take a Cauchy sequence $\{f_n\}_n$ in $\mathcal{D}'_{m,\nu}$ and suppose that f_n has decomposition $\{\Delta_{k,n}\}_k$. Then for each $k \in \mathbb{N}$ the sequence $\{\Delta_{k,n}\}_n$ is a Cauchy sequence in L^1_ν. Now using the fact that on $L^1(k,k+1)$ the norm $\|\cdot\|_\nu$ is equivalent to the usual L^1-norm we see that $\{\Delta_{k,n}\}_n$ is a Cauchy sequence in L^1, and thus converges to say $\Delta_k = \Delta_k 1_{[k,k+1]}$ in L^1. Hence, $\{\Delta_{k,n}\}_n$ converges to Δ_k in L^1_ν. Taking $f = \sum_{k=0}^{\infty}\Delta_k^{(mk)}$ we see that $f \in \mathcal{D}'_{m,\nu}$ and $\{f_n\}_n$ converges to f in $\|\cdot\|_{m,\nu}$. ■

[2]Stirling's formula: $\Gamma(n) = \sqrt{2\pi}\,n^{n-1/2}e^{-n}[1+O(n^{-1})]$ as $n \to \infty$.

Proposition A.3.7 *Let $\nu_0 > m$. If $f \in \mathcal{D}'_{m,\nu_0}$, then $f \in \mathcal{D}'_{m,\nu}$ for all $\nu \geqslant \nu_0$ and we have $\|f\|_{m,\nu} \searrow 0$ as $\nu \nearrow \infty$.*

PROOF. First observe that $x \mapsto x^m e^{-x}$ is decreasing on $[m, \infty)$. Now, if $f \in \mathcal{D}'_{m,\nu_0}$, then for $\nu \geqslant \nu_0$ we have

$$\nu^{mk} \int_k^{k+1} e^{-\nu s} |\Delta_k(s)| \mathrm{d}s = \big(\nu^m e^{-\nu}\big)^k \int_0^1 e^{-\nu s} |\Delta_k(s+k)| \mathrm{d}s,$$

which is decreasing in ν, provided that $\nu \geqslant \nu_0$. The proposition now is an easy consequence of the dominated convergence theorem for sums. ∎

Proposition A.3.8 *Let $\nu_0 > m$, then for all $\nu \geqslant \nu_0$ the space $\mathcal{D}'_{m,\nu}$ is an algebra with respect to addition and convolution.*

PROOF. We only need to check whether $\mathcal{D}'_{m,\nu}$ is closed under convolution. Well, if f and $\tilde{f}$ are two staircase distributions with decomposition $\{\Delta_k\}_k$ and $\{\tilde{\Delta}_k\}_k$ respectively, then

$$f * \tilde{f} = \sum_{k,l=0}^{\infty} \big(\Delta_k * \tilde{\Delta}_l\big)^{(m(k+l))},$$

so lemma A.3.4 then implies

$$\|f * \tilde{f}\|_{m,\nu} \leqslant \sum_{k,l=0}^{\infty} \big\| \big(\Delta_k * \tilde{\Delta}_l\big)^{(m(k+l))} \big\|_{m,\nu} \leqslant \sum_{k,l=0}^{\infty} C_m^2 \nu^{m(k+l)} \|\Delta_k * \tilde{\Delta}_l\|_{\nu}.$$

Since $\|\Delta_k * \tilde{\Delta}_l\|_\nu \leqslant \|\Delta_k\|_\nu \cdot \|\tilde{\Delta}_l\|_\nu$, we conclude that $\|f * \tilde{f}\|_{m,\nu} \leqslant \|f\|_{m,\nu} \cdot \|\tilde{f}\|_{m,\nu}$. ∎

A.4 Embedding of $L^1_{\nu_0}$ in $\mathcal{D}'_{m,\nu}$ for $\nu > \nu_0$.

Proposition A.4.1 *Take $\nu > \nu_0 > m$ fixed and let $f \in L^1_{\nu_0}$, then $f \in \mathcal{D}'_{m,\nu}$ and there exists a constant C, which may depend on ν and ν_0, such that $\|f\|_{m,\nu} \leqslant C\|f\|_{\nu_0}$. Moreover, there exists a positive $\nu_1 > 2m$ such that for all $\nu \geqslant \nu_1$ we have*

$$\|f\|_{m,\nu} \leqslant 2\, C_m \|f\|_{\nu/2}. \tag{A.4.1}$$

PROOF. In a similar way as we proved lemma A.3.1 we can show that $f \in L^1_{\nu_0}$ implies $f \in \mathcal{D}'_m$ and has the following decomposition

$$\Delta_n = G_n \mathbb{1}_{[n,n+1]}, \quad G_n = \mathcal{P}^m\big(G_{n-1}\mathbb{1}_{[n,\infty)}\big), \quad G_0 = F_0 = f.$$

Now $\|\Delta_0\|_\nu \leqslant \|f\|_{\nu_0}$. If $n \geqslant 1$, then $|\Delta_n(t)| \leqslant |G_n(t)| = |(\mathcal{P}^m[G_{n-1}\mathbf{1}_{[n,\infty)}])(t)|$. This latter expression is bounded by $\mathcal{P}^m(|G_{n-1}|)(t)$, which in turn can be majorized by $\mathcal{P}^{2m}(|G_{n-2}|)(t)$ and so on. Hence, $|\Delta_n(t)| \leqslant \mathcal{P}^{mn}(|f|)(t)$ and this implies

$$\|\Delta_n\|_\nu \leqslant \int_n^{n+1} e^{-\nu t}\mathcal{P}^{mn}(|f|)(t)\mathrm{d}t \leqslant e^{-(\nu-\nu_0)n}\|\mathcal{P}^{mn}(|f|)\|_{\nu_0} \leqslant e^{-(\nu-\nu_0)n}\nu_0^{-mn}\|f\|_{\nu_0}.$$

If we put $\delta := \nu - \nu_0$, then $\|f\|_{m,\nu} \leqslant C_m\|f\|_{\nu_0}\sum_{n=0}^\infty(\nu/\nu_0)^{mn}e^{-\delta n}$ and the latter sum converges because $\delta - m\log((\nu_0+\delta)/\nu_0) > \delta - \nu_0\log((\nu_0+\delta)/\nu_0) > 0$. This shows that $L^1_{\nu_0} \subset \mathcal{D}'_{m,\nu}$, provided that $\nu > \nu_0 > m$.

In the special case where $\nu = 2\nu_0$, $\nu_0 > m$, the norm of f in the sense of distributions can be estimated by

$$\|f\|_{m,\nu} \;\leqslant\; C_m\|f\|_{\nu/2}\sum_{n=0}^\infty 2^{mn}e^{-\frac{\nu}{2}n} \;=\; \frac{C_m}{1-2^m e^{-\nu/2}}\,\|f\|_{\nu/2},$$

provided that $2^m e^{-\nu/2} < 1$. Since $1 - 2^m e^{-\nu/2} \geqslant 1/2$ for ν large enough, we obtain the required estimate (A.4.1). ■

Remark A.4.2

1. If $f \in \mathcal{D}'_{m,\nu}$, $\nu > m$, then also $\mathcal{P}f = \mathcal{H} * f$ is a staircase distribution (because $\mathcal{H} \in L^1_{\nu_0}$ for all $\nu_0 > 0$) and

$$\|\mathcal{P}f\|_{m,\nu} \leqslant \|\mathcal{H}\|_{m,\nu}\|f\|_{m,\nu} \leqslant 2C_m\,\|1\|_{\nu/2}\cdot\|f\|_{m,\nu} = \frac{4C_m}{\nu}\,\|f\|_{m,\nu},$$

   provided that ν is large enough.

2. If f and $\tilde{f}$ belong to $L^1_{\nu_0}$ for some positive $\nu_0 > m$, then they belong to $\mathcal{D}'_{m,\nu}$ for all $\nu > \nu_0$ and the convolution of f and $\tilde{f}$ in the sense of distributions coincides with the usual convolution of f and $\tilde{f}$ if we agree that $f(t) = \tilde{f}(t) = 0$ for negative t.

Proposition A.4.3 *The set $\mathcal{D}(\mathbb{R}_+\backslash\mathbb{N})$ is dense in $\mathcal{D}'_{m,\nu}$.*

PROOF. Let $f \in \mathcal{D}'_{m,\nu}$ and choose $\varepsilon > 0$. If f has the decomposition $\{\Delta_k\}_k$, then there exists $n_\varepsilon \in \mathbb{N}$ such that $\sum_{k=n_\varepsilon}^\infty \nu^{mk}\|\Delta_k\|_\nu \leqslant \varepsilon$. Moreover, since $\mathcal{D}(k,k+1)$ is dense in $L^1(k,k+1)$, we deduce that for all $k \in \{0,1,\ldots,n_\varepsilon-1\}$ there exists a $\varphi_k \in \mathcal{D}(k,k+1)$ such that $\|\Delta_k - \varphi_k\|_\nu \leqslant \varepsilon\,2^{-k}\nu^{-mk}$. If $g = \sum_{k=0}^{n_\varepsilon-1}\varphi_k^{(mk)}$, then g belongs to $\mathcal{D}(\mathbb{R}_+\backslash\mathbb{N})$ and, as $\mathrm{supp}(g) \subset [0,n_\varepsilon]$, we have $g \in L^1(0,\infty)$. Hence, $g \in \mathcal{D}'_{m,\nu}$ and

$$\|f-g\|_{m,\nu} \;=\; \sum_{k=0}^{n_\varepsilon-1}\nu^{mk}\|\Delta_k-\varphi_k\|_\nu + \sum_{k=n_\varepsilon}^\infty \nu^{mk}\|\Delta_k\|_\nu \;\leqslant\; 3\varepsilon,$$

which completes the proof of the proposition. ■

Proposition A.4.4 *Let $\nu > m$, $\alpha \geqslant 1$ and suppose $f \in \mathcal{D}'_{m,\nu}$. Then for every positive integer l the expression $\big(f(t-\alpha l)\mathbf{1}_{[\alpha l,\infty)}\big)^{(ml)}$ belongs to $\mathcal{D}'_{m,\nu}$ and there exists a constant C, independent of α, l and ν, such that*

$$\big\|\big(f(t-\alpha l)\mathbf{1}_{[\alpha l,\infty)}\big)^{(ml)}\big\|_{m,\nu} \leqslant C\big(K(\nu)\big)^l \|f\|_{m,\nu},$$

where $K(\nu) \to 0$ as $\nu \to \infty$.

Proof. Let $\{\Delta_k\}_k$ be the decomposition of f and suppose $\alpha l \in [l+r, l+r+1]$ for some integer $r \geqslant 0$. Writing $\big(f(t-\alpha l)\mathbf{1}_{[\alpha l,\infty)}\big)^{(ml)} = \mathcal{P}^{mr}\big[\big(f(t-\alpha l)\mathbf{1}_{[\alpha l,\infty)}\big)^{(ml+mr)}\big]$ and using the remark above it is sufficient to prove that $\big(f(t-\alpha l)\mathbf{1}_{[\alpha l,\infty)}\big)^{(ml+mr)}$ belongs to $\mathcal{D}'_{m,\nu}$. Now

$$\big(f(t-\alpha l)\mathbf{1}_{[\alpha l,\infty)}\big)^{(ml+mr)} = \sum_{k=0}^{\infty}\big(\Delta_k(t-\alpha l)\mathbf{1}_{[k+\alpha l,k+\alpha l+1]}\big)^{(m(k+l+r))}.$$

As $\tilde{\Delta}_{k+r+l}(t) := \Delta_k(t-\alpha l)\mathbf{1}_{[k+\alpha l,k+\alpha l+1]}$, $k \in \mathbb{N}$, belongs to $L^1(0,\infty)$ and has support in the interval $[k+l+r, k+l+r+2]$, lemma A.3.1, together with lemma A.3.4, implies that $\tilde{\Delta}_{k+l+r}^{m(k+l+r)}$ is a staircase distribution in $\mathcal{D}'_{m,\nu}$ with norm smaller than $C_m^2\,\nu^{m(k+l+r)}\|\tilde{\Delta}_{k+l+r}\|_\nu$. Hence, $\big(f(t-\alpha l)\mathbf{1}_{[\alpha l,\infty)}\big)^{(ml+mr)}$ is a staircase distribution and its norm can is less than

$$C_m^2\sum_{k=0}^{\infty}\nu^{m(k+l+r)}\|\tilde{\Delta}_{k+l+r}\|_\nu \;\leqslant\; C_m\,e^{-\nu\alpha l}\nu^{m(l+r)}C_m\sum_{k=0}^{\infty}\nu^{mk}\|\Delta_k\|_\nu.$$

Using proposition A.4.1, together with the fact that $\|1^{*mr}\|_{\nu/2} = \big(\frac{2}{\nu}\big)^{mr}$, we obtain

$$\begin{aligned}\big\|\big(f(t-\alpha l)\mathbf{1}_{[\alpha l,\infty)}\big)^{(ml)}\big\|_{m,\nu} &\leqslant 2C_m\,\|1^{*mr}\|_{\nu/2}\cdot\big\|\big(f(t-\alpha l)\mathbf{1}_{[\alpha l,\infty)}\big)^{(ml+mr)}\big\|_{m,\nu}\\ &\leqslant 2^{mr+1}C_m^2\,e^{-\nu\alpha l}\nu^{ml}\|f\|_{m,\nu}.\end{aligned}$$

Since $l+r \leqslant \alpha l$, we have $r \leqslant (\alpha-1)l$, and thus $2^{mr}e^{-\nu\alpha l}\nu^{ml}$ is bounded by $K(\nu)^l$ for some function K that tends to 0 as $\nu \to \infty$. ∎

A.5 Laplace Transforms in $\mathcal{D}'_{m,\nu}$

If $f \in \mathcal{D}'_{m,\nu}$, we can write it uniquely as $f = \sum_{k=0}^{\infty}\Delta_k^{(mk)}$ for a sequence of certain L^1-functions $\{\Delta_k\}_k$ with $\mathrm{supp}(\Delta_k) \subset [k, k+1]$. In this section we want to define the Laplace transform of a staircase distribution using this decomposition. Moreover, we will derive some elementary properties of this generalised Laplace transform.

Definition A.5.1 *The Laplace transform of a staircase distribution $f \in \mathcal{D}'_{m,\nu}$ is defined by*

$$(\mathcal{L}f)(x) := \sum_{k=0}^{\infty}x^{mk}\int_0^{\infty}\Delta_k(t)e^{-xt}\mathrm{d}t.$$

According to proposition A.5.3 this Laplace transform is an extension of the classical one.

Theorem A.5.2 *Let $f \in \mathcal{D}'_{m,\nu}$. Then the function $x \mapsto (\mathcal{L}f)(x)$ is holomorphic in some neighbourhood U of ∞ in $S(0,\pi)$ depending on m and ν. Moreover, U may be chosen such that for $x \in U$ we have $|(\mathcal{L}f)(x)| \leqslant \|f\|_{m,\nu}$. The Laplace transform is a continuous and injective operator on $\mathcal{D}'_{m,\nu}$.*

PROOF. Choose $x \in S(0,\pi)$, then for x so large that $\Re x > \nu$ we have

$$\begin{aligned}\sum_{k=0}^{\infty} |x|^{mk} \int_0^{\infty} |\Delta_k(t)||e^{-xt}|\mathrm{d}t &= \sum_{k=0}^{\infty} |x|^{mk} \int_k^{k+1} e^{-(\Re x-\nu)t} e^{-\nu t} |\Delta_k(t)|\mathrm{d}t \\ &\leqslant \sum_{k=0}^{\infty} \Big(\frac{|x|}{\nu}\Big)^{mk} e^{-(\Re x-\nu)k} \nu^{mk} \|\Delta_k\|_{\nu}.\end{aligned}$$

We define the neighbourhood U as the set of $x \in S(0,\pi)$ satisfying

$$m\log|x| - m\log\nu - \Re x + \nu = m\log|x| - m\log\nu - |x|\cos(\arg x) + \nu < 0,$$

i.e. $(|x|/\nu)^m e^{-\Re x+\nu} < 1$. This ensures absolute convergence of the series above in U and we have $|(\mathcal{L}f)(x)| \leqslant \|f\|_{m,\nu}$ if $x \in U$. Obviously $\mathcal{L}f$ is holomorphic in U. Hence, we only have to prove the injectivity of $\mathcal{L}$. Suppose $\mathcal{L}f \equiv 0$ for some $f \in \mathcal{D}'_{m,\nu}$, then there exists a positive r such that $(\mathcal{L}f)(x) = 0$ on (for example) $S := \big\{x \in \mathbb{C}^* \;\big|\; |x| \geqslant r,\ |\arg x| \leqslant \pi/4\big\}$. By just writing out the definition we deduce that

$$0 = (\mathcal{L}f)(x) = x^m e^{-x} \sum_{k=1}^{\infty} x^{m(k-1)} e^x \int_k^{k+1} \Delta_k(t) e^{-xt}\mathrm{d}t \;+\; (\mathcal{L}\Delta_0)(x).$$

In a similar way as we did above, the infinite sum in this last expression can be estimated by

$$\left|\sum_{k=1}^{\infty} x^{m(k-1)} e^x \int_k^{k+1} \Delta_k(t) e^{-xt}\mathrm{d}t\right| \leqslant \frac{e^{\nu}}{\nu^m} \sum_{k=1}^{\infty} \Big(\frac{|x|}{\nu}\Big)^{m(k-1)} e^{-(\Re x-\nu)(k-1)} \nu^{mk} \|\Delta_k\|_{\nu}, \quad \text{(A.5.1)}$$

which converges for $x \in S$ if we take r large enough. Moreover, the sum with k running from 2 to ∞ tends to 0 as $|x| \to \infty$ in S, so by enlarging r again, we can ensure that the right-hand side of (A.5.1) is smaller than $C_1 := e^{\nu}\|\Delta_1\|_{\nu} + 1$, so $|(\mathcal{L}\Delta_0)(x)| \leqslant C_1|x^m e^{-x}|$ on S.

Let us now define $g(x) := \int_0^1 e^{xt}\Delta_0(1-t)\mathrm{d}t$, then $g(x) = e^x(\mathcal{L}\Delta_0)(x)$. Obviously g is entire and can be estimated by $|g(x)| \leqslant C_1|x|^m$ in S. Moreover, in $\mathbb{C}$ the function g can be estimated by $C_2 e^{|x|}$, where C_2 is the L^1-norm of Δ_0. Now consider the function $g^{(m)}$, which also is an entire function. As $g^{(m)}(x) = \int_0^1 t^m e^{xt}\Delta_0(1-t)\mathrm{d}t$ it can be estimated by $C_2 e^{|x|}$ in $\mathbb{C}$. Moreover, if $\varphi \in [\pi/2, 3\pi/2]$, then $|g^{(m)}(se^{i\varphi})| \leqslant C_2$ for all $s \geqslant 0$. On the other hand we have

$$g^{(m)}(x) = \frac{m!}{2\pi i} \oint_{\gamma} \frac{g(\zeta)}{(\zeta - x)^{m+1}}\mathrm{d}\zeta,$$

where γ can be taken to be a positively orientated circle with centre x. For $x \in (2r, \infty)$ we take $\zeta = x + \frac{x}{2}e^{i\sigma}$, $0 \leqslant \sigma < 2\pi$, then $\gamma \subset S$ and $|g^{(m)}(x)| \leqslant C_1 m! 3^m$. For $x \in [0, 2r]$ the function $g^{(m)}$ obviously is bounded. Hence, $g^{(m)}$ is bounded on the positive real line and on $\{x \in \mathbb{C}^* \mid \arg x = \frac{3\pi}{4}, \frac{5\pi}{4}\}$. Three times an application of the Phragmén-Lindelöf theorem[3] then implies that $g^{(m)}$ is bounded in $\mathbb{C}$ and from Liouville's theorem we deduce that $g^{(m)}$ is a constant. Hence, g is a polynomial, which decays exponentially in the left half plane. So $g \equiv 0$ and thus $\mathcal{L}\Delta_0 \equiv 0$. This finally implies that $\Delta_0 = 0$.

Now assume that $\Delta_0 = \Delta_1 = \cdots = \Delta_{n-2} = 0$. To prove that $\Delta_{n-1} = 0$ we write $\mathcal{L}f$ as

$$(\mathcal{L}f)(x) = x^{mn} e^{-nx} \sum_{k=n}^{\infty} x^{m(k-n)} e^{nx} \int_k^{k+1} \Delta_k(t) e^{-xt} \mathrm{d}t \; + \; x^{m(n-1)} (\mathcal{L}\Delta_{n-1})(x).$$

and we proceed as in the initial case by defining $g(x) = \int_0^1 e^{xt} \Delta_{n-1}(n-t)\mathrm{d}t$. ■

Proposition A.5.3 *Let $\nu > \nu_0 > m$ and let $f \in L^1_{\nu_0}$. Then $f \in \mathcal{D}'_{m,\nu}$ and its Laplace transform in the sense of distributions coincides with $\int_0^\infty f(t)e^{-xt}\mathrm{d}t$.*

Proof. Lemma A.3.1 implies that $f = \sum_{n=0}^{\infty} \Delta_n^{(mn)}$, where $\Delta_n = G_n \mathbb{1}_{[n,n+1]}$, with $G_n = \mathcal{P}^m\big(G_{n-1}\mathbb{1}_{[n,\infty)}\big)$ and $G_0 = f$. Using partial integration one may show that for arbitrary $n \in \mathbb{N}_+$ the integral $x^{mn}\int_n^{n+1} \Delta_n(t)e^{-xt}\mathrm{d}t$ can be written as

$$x^{m(n-1)} \int_n^{n+1} G_{n-1}(t) e^{-xt}\mathrm{d}t - \sum_{j=0}^{m-1} x^{mn-j-1} e^{-xt} \big(\mathcal{P}^{m-j}\big(G_{n-1}\mathbb{1}_{[n,\infty)}\big)\big) \Big|_{t=n}^{n+1} .$$

Proceeding in this way we infer that $x^{mn}\int_n^{n+1} \Delta_n(t)e^{-xt}\mathrm{d}t$ equals

$$\begin{aligned} & \int_n^{n+1} f(t)e^{-xt}\mathrm{d}t - \sum_{j=0}^{m-1}\sum_{h=0}^{n-1} x^{m(n-h)-j-1} e^{-xt}\big(\mathcal{P}^{m-j}\big(G_{n-h-1}\mathbb{1}_{[n-h,\infty)}\big)\big) \Big|_{t=n}^{n+1} \\ = \; & \int_n^{n+1} f(t)e^{-xt}\mathrm{d}t - \sum_{j=0}^{m-1}\sum_{k=0}^{n-1} x^{m(k+1)-j-1} e^{-xt}\big(\mathcal{P}^{m-j}\big(G_{k}\mathbb{1}_{[k+1,\infty)}\big)\big) \Big|_{t=n}^{n+1} . \end{aligned}$$

Hence, the Laplace transform of f in the sense of distributions equals

$$\int_0^\infty f(t)e^{-xt}\mathrm{d}t \; - \; \sum_{n=1}^{\infty}\sum_{j=0}^{m-1}\sum_{k=0}^{n-1} x^{m(k+1)-j-1} e^{-xt}\big(\mathcal{P}^{m-j}\big(G_{k}\mathbb{1}_{[k+1,\infty)}\big)\big) \Big|_{t=n}^{n+1} .$$

Thus we have proved the proposition if we can show that the infinite sum in the expression above equals zero. This infinite sum can be rewritten as $\sum_{n=1}^{\infty} c_n e^{-xn}$ with

$$c_n = \sum_{j=0}^{m-1}\left[\sum_{k=0}^{n-2} x^{m(k+1)-j-1}\mathcal{P}^{m-j}\big(G_k\mathbb{1}_{[k+1,\infty)}\big)(n) - \sum_{k=0}^{n-1} x^{m(k+1)-j-1}\mathcal{P}^{m-j}\big(G_k\mathbb{1}_{[k+1,\infty)}\big)(n)\right]$$

[3][Tit32] Let $\alpha \geqslant 1$, $\beta > 1$ and let f be a holomorphic function in the region D between two straight lines making an angle π/β at the origin and on the lines themselves. Suppose that $|f(x)| \leqslant M$ on the lines and that $f(x) = O(e^{\alpha|x|})$, as $x \to \infty$ in D. Then $|f(x)| \leqslant M$ throughout the region D.

and $c_n = -\sum_{j=0}^{m-1} x^{mn-j-1}\mathcal{P}^{m-j}\big(G_{n-1}\mathbb{1}_{[n,\infty)}\big)(n) = 0.$ ■

We saw that the Laplace transform extends in quite a natural way to staircase distributions. The following proposition shows that the extension also holds for the most well known properties of the Laplace transform.

Proposition A.5.4 *Suppose $f, g \in \mathcal{D}'_{m,\nu}$, then we have*

(i) $\mathcal{L}(f * g) = \mathcal{L}f \cdot \mathcal{L}g$,

(ii) $\mathcal{P}f \in \mathcal{D}'_{m,\nu}$ *and* $\mathcal{L}(\mathcal{P}f) = x^{-1}\mathcal{L}f$,

(iii) *The operator e^{-t} maps $\mathcal{D}'_{m,\nu}$ into itself and $\mathcal{L}(e^{-t}f)(x) = (\mathcal{L}f)(x+1)$,*

(iv) *For $\alpha > 0$ we have $f(t-\alpha)\mathbb{1}_{[\alpha,\infty)} \in \mathcal{D}'_{m,\nu}$ and $\mathcal{L}[f(t-\alpha)\mathbb{1}_{[\alpha,\infty)}] = e^{-\alpha x}\mathcal{L}f$.*

(v) *If $f(t)$ asymptotically equals $\sum_{m=0}^{\infty} \frac{f_m}{\Gamma(m+r+1)} t^{m+r}$, $\Re r > -1$, as $t \to 0$, $t \in \mathbb{R}_+$, then $(\mathcal{L}f)(x) \sim \sum_{m=0}^{\infty} f_m x^{-m-r-1}$ as $x \to \infty$ in a neighbourhood of ∞ in $S(0,\pi)$.*

Proof. The first statement holds for functions $f, g \in \mathcal{D}(\mathbb{R}_+\backslash\mathbb{N})$. Next, we take staircase distributions $f, g \in \mathcal{D}'_{m,\nu}$ and, as $\mathcal{D}(\mathbb{R}_+\backslash\mathbb{N})$ is dense in $\mathcal{D}'_{m,\nu}$, we can construct sequences $\{f_n\}_n$ and $\{g_n\}_n$ in $\mathcal{D}(\mathbb{R}_+\backslash\mathbb{N})$ which converge to f and g in $\|\cdot\|_{m,\nu}$-norm. Then $f_n * g_n$ converges to $f * g$ as well and theorem A.5.2 implies the first statement for staircase distributions.

The fact that $\mathcal{P}$ maps $\mathcal{D}'_{m,\nu}$ into itself is shown in remark A.4.2. Let $\{f_n\}_n$ be a sequence in $\mathcal{D}(\mathbb{R}_+\backslash\mathbb{N})$ converging to f, then $\{\mathcal{P}f_n\}_n$ converges to $\mathcal{P}f$. But, as the f_n are L^1-functions, $\mathcal{P}f_n$ coincides with $1 * f_n$. We know that $\mathcal{L}(1 * f_n) = x^{-1}\,\mathcal{L}f_n$, so the second property also follows by density.

To prove the first part of the third statement we refer to lemma 2.5.7. The last part again follows by a density argument. The proof that for $\alpha > 0$ the distribution $f(t-\alpha)\mathbb{1}_{[\alpha,\infty)}$ belongs to $\mathcal{D}'_{m,\nu}$ may be given in the same way as in proposition A.4.4. It also follows from the proof of this proposition that the norm of $f(t-\alpha)\mathbb{1}_{[\alpha,\infty)}$ can be estimated by a constant times the norm of f. As $\mathcal{L}[f(t-\alpha)\mathbb{1}_{[\alpha,\infty)}] = e^{-\alpha x}\mathcal{L}f$ holds for functions $f \in \mathcal{D}(\mathbb{R}_+\backslash\mathbb{N})$, the proof of the fourth statement can again be completed using density arguments.

The last statement can be proven without using density arguments. Let $\{\Delta_k\}_k$ be the decomposition of f, then obviously $\Delta_0(t) \sim \sum_{m=0}^{\infty} \frac{f_m}{\Gamma(m+r+1)} t^{m+r}$ as $t \to 0$. As $\mathcal{L}\Delta_0$ in the sense of distributions coincides with the usual Laplace transform of Δ_0, well known theory on asymptotics (cf. [Bal00]) implies $(\mathcal{L}\Delta_0)(x) \sim \sum_{m=0}^{\infty} f_m x^{-m-r-1}$ as $x \to \infty$ in $S(0,\pi)$. Moreover, the expression $\sum_{k=1}^{\infty} x^{mk} \int_0^{\infty} \Delta_k(t) e^{-xt}\mathrm{d}t$ can be bounded by a constant times $|x|^{-N}$ for arbitrary $N \in \mathbb{N}$ on each closed sub-sector of $S(0,\pi)$, provided that $|x|$ is large enough. This may be shown similarly as the first part of theorem A.5.2. ■

A.6 Generalisations of $\mathcal{D}'_{m,\nu}$

Remember that the staircase distributions were introduced to solve (2.3.5) on singular rays, but such a singular ray will in general not be equal to the positive real axis. Hence, we have to define staircase distributions on half lines $\{te^{i\theta} \mid t \in \mathbb{R}_+\}$ with $\theta \in \mathbb{R}$. This is done in the same way as in the real case. So a distribution $f \in \mathcal{D}'(0, \infty e^{i\theta})$ is called a staircase distribution on the half line $\{te^{i\theta} \mid t \in \mathbb{R}_+\}$, $\theta \in \mathbb{R}$, of order $m \in \mathbb{N}_{\geqslant 1}$ if for all $k \in \mathbb{N}$ there exists a function $F_k \in L^1(0, (k+1)e^{i\theta})$ such that the restriction of $\mathcal{P}_\theta^{mk} f$ to $\mathcal{D}(0, (k+1)e^{i\theta})$ equals F_k. We denote this set of distributions by $\mathcal{D}'_m(\theta)$. Here $\mathcal{P}_\theta f$ is defined by $\mathcal{H}_\theta * f$, where the (generalised) Heaviside one step function is defined by

$$\mathcal{H}_\theta(t) = \begin{cases} 1 & \text{if } t \in \mathbb{R}_+ e^{i\theta}, \\ 0 & \text{otherwise.} \end{cases}$$

These staircase distributions have similar properties as those defined on the positive real line and the same proofs (with obvious modifications) hold. For example, each $f \in \mathcal{D}'_m(\theta)$ can be decomposed as in the decomposition lemma where in this case each Δ_k belongs to $L^1(0, \infty e^{i\theta})$ and is supported in $[ke^{i\theta}, (k+1)e^{i\theta}]$.

To define a family of norms on these generalised staircase distributions, we first introduce the space $L^1_\nu(\theta)$, consisting of functions $f \in L^1_{loc}(0, \infty e^{i\theta})$ such that

$$\|f\|_{\nu,\theta} := \int_0^\infty e^{-\nu t} |f(te^{i\theta})| \mathrm{d}t$$

is finite. Obviously, $L^1_\nu(\theta)$ satisfies the same properties as L^1_ν. On $\mathcal{D}'_m(\theta)$ we now define a family of norms $\|\cdot\|_{m,\nu,\theta}$ by

$$\|f\|_{m,\nu,\theta} := C_m \sum_{k=0}^{\infty} \nu^{mk} \|\Delta_k\|_{\nu,\theta}$$

and we use the notation $\mathcal{D}'_{m,\nu}(\theta)$ for the set of distributions f in $\mathcal{D}'_m(\theta)$ with $\|f\|_{m,\nu,\theta} < \infty$.

The Laplace transform of $f \in \mathcal{D}'_{m,\nu}(\theta)$ is defined by

$$(\mathcal{L}f)(x) := \sum_{k=0}^{\infty} x^{mk} \int_0^{\infty e^{i\theta}} \Delta_k(t) e^{-xt} \mathrm{d}t, \tag{A.6.1}$$

with integration along the half line $\arg t = \theta$. The function $x \mapsto (\mathcal{L}f)(x)$ is holomorphic in a neighbourhood of ∞ in $S(-\theta, \pi)$ and $\mathcal{L}$ satisfies properties similar to those in proposition A.5.4. Moreover, if $f \in L^1_{\nu_0}(\theta)$ for any $\nu_0 > m$, then f can be regarded as a staircase distribution in $\mathcal{D}'_{m,\nu}(\theta)$ for arbitrary $\nu > \nu_0$ and in that case the Laplace transform of f in the sense of distributions coincides with the usual Laplace transform of f in the direction θ.

If we choose a different interval length $\ell > 0$ instead of $\ell = 1$ in the partition associated with definition A.2.1 we will write $\mathcal{D}'_m(\ell, \theta)$ instead of $\mathcal{D}'_m(\theta)$. So $f \in \mathcal{D}'(0, \infty e^{i\theta})$ belongs to $\mathcal{D}'_m(\ell,\theta)$ if for all $k \in \mathbb{N}$ there exists a function $F_k \in L^1(0,(k+1)\ell e^{i\theta})$ such that the restriction of $\mathcal{P}_\theta^{mk} f$ to $\mathcal{D}(0,(k+1)\ell e^{i\theta})$ equals F_k. As in the decomposition lemma each $f \in \mathcal{D}'_m(\ell,\theta)$ can be decomposed as $f = \sum_{k=0}^{\infty} \Delta_k^{(mk)}$ for unique $\Delta_k \in L^1(0,\infty e^{i\theta})$, with each Δ_k supported in $[k\ell e^{i\theta},(k+1)\ell e^{i\theta}]$. The analogue of $\mathcal{D}'_{m,\nu}(\theta)$ will be denoted by $\mathcal{D}'_{m,\nu}(\ell,\theta)$ and the corresponding norm will be denoted by $\|\cdot\|_{m,\nu,\theta,\ell}$. With this notation one easily infers that

$$f \in \mathcal{D}'_{m,\nu}(\ell,\theta) \Longleftrightarrow f_\ell \in \mathcal{D}'_{m,\nu\ell}(\theta),$$

where $f_\ell = \sum_{k=0}^{\infty} \big[\ell^{-mk}(\Delta_k)_\ell\big]^{(mk)}$, $(\Delta_k)_\ell(t) := \Delta_k(\ell t)$. Hence, there is an obvious isomorphism between these two spaces and thus it is clear that for $\ell > 0$ the space $\mathcal{D}'_{m,\nu}(\ell,\theta)$ defines a Banach space with similar properties as $\mathcal{D}'_{m,\nu}(\theta)$.

We will end this chapter with the analogue of proposition A.4.4 in $\mathcal{D}'_{m,\nu}(\ell,\theta)$, which will be frequently used in chapter 2.

Proposition A.6.1 *Let $\nu > m$, $\alpha \geqslant \ell$ and suppose $f \in \mathcal{D}'_{m,\nu}(\ell,\theta)$. Then for every positive integer l the expression $\big(f(t - l\alpha e^{i\theta})\mathbb{1}_{[l\alpha e^{i\theta},\infty e^{i\theta})}\big)^{(ml)}$ belongs to $\mathcal{D}'_{m,\nu}(\ell,\theta)$ and there exists a constant C, independent of α, l and ν, such that*

$$\Big\|\big(f(t - l\alpha e^{i\theta})\mathbb{1}_{[l\alpha e^{i\theta},\infty e^{i\theta})}\big)^{(ml)}\Big\|_{m,\nu,\theta,\ell} \leqslant C\big(K(\nu)\big)^l \|f\|_{m,\nu,\theta,\ell},$$

where $K(\nu) \to 0$ as $\nu \to \infty$.

Appendix B

An Existence Theorem for Linear Difference Equations

In this appendix we give the existence theorem that we used in chapter 3 and 4. This is a slight extension of a result of Immink in [Imm84] (cf. also [Bra01]).

Let $S := \{x \in \mathbb{C}^* \mid \varphi_- < \arg x < \varphi_+\}$, where $\varphi_- < 0 < \varphi_+ < \varphi_- + \pi$ and let us consider the equation

$$y(x+1) = e^{-\mu}(1+x^{-1})^{\Lambda}\, y(x) + f(x), \tag{B.0.1}$$

where μ is a complex number and where Λ is a matrix of the form $\Lambda = a\mathtt{I} + \mathtt{N}$, with $a \in \mathbb{C}$, $\mathtt{I}$ the $n \times n$-identity matrix and $\mathtt{N}$ a nilpotent matrix of the form $\mathtt{N} = (b_{k,l})_{1 \leqslant k,l \leqslant n}$, $b_{k,k+1} \in \{0,1\}$ for $1 \leqslant k \leqslant n-1$ and $b_{k,l} = 0$ otherwise.

Moreover, assume that f is a $\mathbb{C}^n$-valued function, holomorphic in a neighbourhood of ∞ in S and $f(x) = O(x^{-2})$ as $x \to \infty$ in S. Let $Y_0(x) = e^{-\mu x}x^{\Lambda}$ be a fundamental matrix solution of the corresponding homogeneous equation, then a particular solution of (B.0.1) can be obtained as follows:

$$\begin{aligned} y(x) \;&=\; e^{\mu}(1+x^{-1})^{-\Lambda}\left[-f(x) + \int_{\Gamma_+(x)} Y_0(x+1)\,Y_0^{-1}(\xi+1)\,\frac{e^{2\pi i m_+(\xi-x)}}{e^{2\pi i(\xi-x)}-1}\,f(\xi)\,\mathrm{d}\xi\right] \\ &-\; e^{\mu}(1+x^{-1})^{-\Lambda}\int_{\Gamma_-(x)} Y_0(x+1)\,Y_0^{-1}(\xi+1)\,\frac{e^{2\pi i m_-(\xi-x)}}{e^{2\pi i(\xi-x)}-1}\,f(\xi)\,\mathrm{d}\xi \\ &+\; e^{\mu}(1+x^{-1})^{-\Lambda}\sum_{j=m_-}^{m_+-1}\int_{c_j}^{x+\frac{1}{2}} Y_0(x+1)\,Y_0^{-1}(\xi+1)\,e^{2\pi i j(\xi-x)}\,f(\xi)\,\mathrm{d}\xi, \end{aligned} \tag{B.0.2}$$

where $\Gamma_\pm(x)$ is the half line from $\xi = x + \frac{1}{2}$ to $\xi = \infty e^{i\alpha_\pm}$, with $\varphi_- < \alpha_- < 0 < \alpha_+ < \varphi_+$ and $\alpha_\pm$ sufficiently close to $\varphi_\pm$. Moreover, $m_\pm$ are integers such that the first two integrals in (B.0.2) converge, i.e. $m_- < 1 + \frac{1}{2\pi}(\Re\mu\cot\varphi_- - \Im\mu)$ and $m_+ > \frac{1}{2\pi}(\Re\mu\cot\varphi_+ - \Im\mu)$ (this will be explained below). However, these two inequalities do not determine unique values of $m_\pm$, but this can be achieved as follows. We choose m_- optimal in the sense that $\frac{1}{2\pi}(\Re\mu\cot\varphi_- - \Im\mu) \leqslant m_- < 1 + \frac{1}{2\pi}(\Re\mu\cot\varphi_- - \Im\mu)$. Now, if $m_- > \frac{1}{2\pi}(\Re\mu\cot\varphi_+ - \Im\mu)$

we take $m_+ := m_-$. If, in the other case, $m_- \leqslant \frac{1}{2\pi}(\Re\mu \cot\varphi_+ - \Im\mu)$ we take m_+ such that $\frac{1}{2\pi}(\Re\mu\cot\varphi_+ - \Im\mu) < m_+ \leqslant \frac{1}{2\pi}(\Re\mu\cot\varphi_+ - \Im\mu) + 1$. The constants c_j are suitably chosen numbers (possibly infinite) in the given neighbourhood of ∞ in S.

To prove the requirements on $m_\pm$ implying the first to integrals in (B.0.2) to converge, we first remark that if $\xi \in \Gamma_+(x)$, then $|1 - e^{2\pi i(\xi - x)}| \geqslant 1 - e^{-2\pi\Im(\xi-x)}$ which tends to 1 as $\xi \to \infty$ along $\Gamma_+(x)$. Hence, the integral along $\Gamma_+(x)$ converges if $\Re[(2\pi i m_+ + \mu)\xi] < 0$, that is $2\pi m_+ > \Re\mu\cot\alpha_+ - \Im\mu$. If $2\pi m_+ > \Re\mu\cot\varphi_+ - \Im\mu$, this obviously is satisfied if $\varphi_+ - \alpha_+$ is small enough. On the other hand, if $\xi \in \Gamma_-(x)$, then $|1 - e^{2\pi i(\xi-x)}| \geqslant \frac{1}{2}\,|e^{2\pi i(\xi - x)}|$, if $\xi \to \infty$. Hence, the integral along $\Gamma_-(x)$ converges if $\Re[(2\pi i(m_- - 1) + \mu)\xi] < 0$, which can be satisfied by choosing α_- close enough φ_-.

The fact that the expression for y solves (B.0.1) follows by studying the difference $y(x+1) - e^{-\mu}(1 + x^{-1})^\Lambda\, y(x)$. A straightforward calculation shows that this difference can be written as

$$\begin{aligned} & f(x) - e^{\mu}\Big(1 + \frac{1}{x+1}\Big)^{-\Lambda} f(x+1) + \\ + \;\; & Y_0(x+1)\Big[\int_{x+3/2}^{\infty e^{i\alpha_+}} - \int_{x+1/2}^{\infty e^{i\alpha_+}}\Big] Y_0^{-1}(\xi+1)\, \frac{e^{2\pi i m_+(\xi - x)}}{e^{2\pi i(\xi-x)} - 1}\, f(\xi)\, \mathrm{d}\xi + \\ - \;\; & Y_0(x+1)\Big[\int_{x+3/2}^{\infty e^{i\alpha_-}} - \int_{x+1/2}^{\infty e^{i\alpha_-}}\Big] Y_0^{-1}(\xi+1)\, \frac{e^{2\pi i m_-(\xi - x)}}{e^{2\pi i(\xi-x)} - 1}\, f(\xi)\, \mathrm{d}\xi + \\ + \;\; & Y_0(x+1) \sum_{j=m_-}^{m_+-1} \int_{x+1/2}^{x+3/2} Y_0^{-1}(\xi+1)\, e^{2\pi i j(\xi-x)}\, f(\xi)\, \mathrm{d}\xi. \end{aligned}$$

Now, if $m_- = m_+ =: m$ this expression reduces to

$$f(x) - e^{\mu}\Big(1 + \frac{1}{x+1}\Big)^{-\Lambda} f(x+1) + Y_0(x+1) \oint_{x+1} Y_0^{-1}(\xi+1)\, \frac{e^{2\pi i m(\xi-x)}}{e^{2\pi i(\xi - x)} - 1}\, f(\xi)\, \mathrm{d}\xi,$$

where $\oint_{x+1}$ denotes an integral with as path of integration a loop around the point $x+1$. Using that

$$2\pi i \cdot \mathrm{Res}_{\xi = x+1}\Big[Y_0(x+1)\, Y_0^{-1}(\xi+1) \frac{e^{2\pi i m(\xi - x)}}{e^{2\pi i(\xi-x)} - 1} f(\xi)\Big] = e^{\mu}\Big(1 + \frac{1}{x+1}\Big)^{-\Lambda} f(x+1),$$

it is concluded that y solves $y(x+1) - e^{-\mu}(1 + x^{-1})^\Lambda\, y(x) = f(x)$. In case $m_+ > m_-$ we write

$$\frac{e^{2\pi i m_+(\xi - x)}}{e^{2\pi i(\xi-x)} - 1} = \frac{e^{2\pi i m_-(\xi - x)}}{e^{2\pi i(\xi-x)} - 1} + \sum_{j=m_-}^{m_+-1} e^{2\pi i j(\xi - x)}$$

and the proof can be completed in a similar way. From (B.0.2) one may derive in a similar way as in [Imm84]:

Theorem B.0.2 *Let S be a sector as above, Λ as in (4.1.2) and A a holomorphic $n \times n$-matrix defined in a neighbourhood of ∞ in S with $A(x) = O(x^{-2})$ as $x \to \infty$ in S. Then*

$$y(x+1) = \big(\Lambda(x) + A(x)\big)y(x) \tag{B.0.3}$$

possesses a fundamental system of solutions $V(x) = \big(\mathtt{I} + O(x^{-1})\big)\big(\oplus_{j=1}^{r} e^{-\mu_j x} x^{\Lambda_j}\big)$ in a neighbourhood of ∞ in S.

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Samenvatting

In dit proefschrift worden niet-lineaire stelsels van differentiaal- en differentievergelijkingen bestudeerd in een omgeving van het punt ∞ in het complexe vlak $\mathbb{C}$. Om de gedachten te bepalen kijken we eerst naar een 1-dimensionale differentievergelijking van de vorm

$$y(x+1) = e^{-\mu}(1+x^{-1})^a y(x) + g(x, y(x)),$$

waarin $y = y(x)$ de onbekende functie van x is die we zoeken. We veronderstellen dat a en μ complexe getallen zijn met $\mu \not\equiv 0 \bmod 2\pi i$ (dat wil zeggen: $e^{-\mu} \neq 1$). Verder nemen we aan dat g een functie is die holomorf afhangt van de variabelen x^{-1} en y met enkele extra voorwaarden. Deze vergelijking kan op unieke wijze 'formeel' worden opgelost: substitutie van een machtreeks $\hat{y}_0(x) := \sum_{m=1}^{\infty} \alpha_m x^{-m}$, met $\alpha_m \in \mathbb{C}$ voor elke $m \in \{1, 2, 3, \dots, \}$, en het vergelijken van coëfficiënten voor overeenkomstige machten van x^{-1}, levert een recurrente betrekking voor de coëfficiënten die kan worden opgelost. De term 'formeel' wordt hier gebruikt omdat de zo gevonden formele reeks $\hat{y}_0$ in het algemeen niet convergeert. Met andere woorden: de formele oplossing bepaalt niet direct een holomorfe oplossing van bovenstaande differentievergelijking. Echter, met behulp van deze formele oplossing is het mogelijk aan te tonen dat er op bepaalde sectoren in het complexe vlak een 'echte' holomorfe oplossing y_0 van de differentievergelijking bestaat, die de formele oplossing als asymptotische ontwikkeling heeft wanneer de variabele x naar ∞ loopt in deze sectoren. We maken hierbij gebruik van de zogenaamde *Borel-sommatie* methode.

Door een Borel-transformatie (in bepaalde zin een inverse Laplace-transformatie) toe te passen op de differentievergelijking wordt een integraalvergelijking verkregen in het 'Borel-vlak'. Deze integraalvergelijking kan middels het contractieprincipe worden opgelost in gebieden die begrensd worden door zogenaamde *singuliere lijnen* (i.e. halflijnen in het complexe vlak waarop één of meer singuliere punten van de oplossing liggen). In zo'n gebied zal in het generieke geval de oplossing van de integraalvergelijking Laplace-transformeerbaar zijn en z'n Laplace-getransformeerde y_0 lost de oorspronkelijke differentievergelijking op. We zeggen dan dat de formele reeksoplossing $\hat{y}_0(x) := \sum_{m=1}^{\infty} \alpha_m x^{-m}$ Borel-sommeerbaar is met (Borel-)som y_0 en deze som bestaat juist op die sectoren die worden begrensd door twee halflijnen uit de oorsprong met richtingen $-\pi/2 - \theta_+$ en $\pi/2 - \theta_-$, waarin θ_- en θ_+ twee opeenvolgende singuliere richtingen zijn (i.e. richtingen van singuliere halflijnen). Deze sectoren bevatten dus gesloten halfvlakken.

Naast deze oplossing y_0 zoeken we naar alle oplossingen met hetzelfde asymptotische gedrag als y_0 (zogenaamde 'kleine' oplossingen), maar op sectoren met opening kleiner dan π (zo'n sector is dus bevat in een halfvlak). We doen dit op de volgende manier. Wanneer we de niet-lineaire term $g(x,y)$ weglaten uit bovenstaande vergelijking krijgen we een zogenaamde *normaalvorm*

$$z(x+1) = e^{-\mu}(1+x^{-1})^a z(x)$$

(een eenvoudige lineaire differentievergelijking) behorend bij de niet-lineaire vergelijking. Het is eenvoudig na te gaan dat de algemene oplossing van deze normaalvorm (in de ruimte van holomorfe functies op het Riemann oppervlak van de logaritme) geschreven kan worden als $z(x) = c(x)e^{-\mu x}x^a$, waarin c een willekeurige 1-periodieke holomorfe functie is. Het doel is nu het construeren van een (formele) transformatie die de niet-lineaire vergelijking transformeert naar de eenvoudige lineaire vergelijking. We eisen dat de formele transformatie $y = \hat{T}(x,z)$ de gedaante heeft

$$y = \hat{T}(x,z) := \sum_{k=0}^{\infty} \hat{y}_k(x) z^k,$$

waarin $\hat{y}_0$ de eerder geconstrueerde formele oplossing van de niet-lineaire vergelijking is en waarin $\hat{y}_k$, $k \in \mathbb{N}\backslash\{0\}$, in eerste instantie onbekende uitdrukkingen zijn die tot de verzameling van formele machtreeksen behoren. Wanneer we nu in $\hat{T}(x,z)$ voor z de algemene oplossing van de normaalvorm substitueren, verkrijgen we een zogenaamde *formele integraal*, welke een reeks is van exponentiële functies met als coëfficiënten formele machtreeksen vermenigvuldigd met periodieke functies

$$\hat{y}(x) = \sum_{k=0}^{\infty} c^k(x) e^{-k\mu x} x^{ka} \hat{y}_k(x).$$

Een dergelijke reeks is een voorbeeld van een *transseries*. Wanneer bovenstaande transformatie inderdaad bestaat, is de bijbehorende formele integraal per constructie een formele oplossing van de niet-lineaire differentievergelijking.

Het blijkt dat elke $\hat{y}_k$, $k > 0$, moet voldoen aan een lineaire differentievergelijking die in formele zin kan worden opgelost. Tevens blijkt elke $\hat{y}_k$ Borel-sommeerbaar te zijn, met som y_k die afhankelijk is van de keuze van een sector S_1, met opening groter dan π, waarop $\hat{y}_k$ Borel-gesommeerd kan worden. Hieruit kan worden afgeleid dat de formele transformatie $\hat{T} = \hat{T}(x,z)$ Borel-sommeerbaar is met betrekking tot $x \in S_1$, uniform in z mits $|z|$ klein genoeg. Ook blijkt dat elke oplossing y, die 'klein' is op een gegeven sector S_2 met opening kleiner dan π, geschreven kan worden als een unieke convergente transseries op S_2. Daarbij zijn de y_k's in de convergente transseries de Borel-sommen van de $\hat{y}_k$'s op een sector $S_1 \supset S_2$. Echter, er is afhankelijkheid van de sector S_1 waarop de Borel-sommen van de $\hat{y}_k$'s worden bepaald.

Na een introductie in hoofdstuk 1 wordt in hoofdstuk 2 het volgende stelsel niet-lineaire differentievergelijkingen bestudeerd:

$$y(x+1) = A_0(1+x^{-1})^{A_1} y(x) + g(x, y(x)).$$

Hierin is $y(x) = (y_1(x), y_2(x), \ldots, y_n(x))^t$ de onbekende vectorwaardige functie van x die we zoeken. De symbolen A_0 en A_1 representeren $n \times n$-matrices met coëfficiënten in $\mathbb{C}$ en g is een vectorwaardige functie die holomorf afhangt van de variabelen $x^{-1}, y_1, y_2, \ldots, y_n$ met enkele extra voorwaarden. In hoofdstuk 2 wordt verondersteld dat zowel A_0 als A_1 diagonaalmatrices zijn en dat A_0 geen eigenwaarden 0 of 1 bezit.

Ook in dit algemenere geval blijkt het mogelijk te zijn een formele transformatie van de vorm $y = \hat{T}(x,z) := \sum_{\mathbf{k}\in\mathbb{N}^n} \hat{y}_{\mathbf{k}}(x) z^{\mathbf{k}}$ te construeren die het bovenstaande niet-lineaire stelsel differentievergelijkingen transformeert naar de bijbehorende normaalvorm

$$z(x+1) = A_0(1+x^{-1})^{A_1} z(x)$$

en waarin elke $\hat{y}_{\mathbf{k}}$ Borel-sommeerbaar is met som $y_{\mathbf{k}}$ die afhankelijk is van de keuze van een sector S_1, met opening groter dan π, waarop $\hat{y}_{\mathbf{k}}$ Borel-gesommeerd kan worden. Analoog aan het bovenstaande bestaat ook in dit meer ingewikkelde geval het probleem onder andere uit het bepalen van een convergente transseries bij een gegeven 'kleine' oplossing y van de niet-lineaire vergelijking op een gegeven sector S_2. Dit wordt in het eerste gedeelte van hoofdstuk 2 bestudeerd.

Daarnaast wordt in het tweede hoofdstuk gekeken naar eigenschappen van de Borel-getransformeerden $Y_{\mathbf{k}}$ van de $\hat{y}_{\mathbf{k}}$'s op singuliere lijnen. Hier blijken zogenaamde *resurgentierelaties* op te treden. Zo'n resurgentierelatie bepaalt het gedrag van bijvoorbeeld Y_0 in een gegeven singulier punt door het gedrag van de $Y_{\mathbf{k}}$'s bij de oorsprong. Een speciaal geval van deze resurgentierelaties geeft ons tevens het asymptotisch gedrag van de coëfficiënten α_m in de formele oplossing $\hat{y}_0$ voor $m \to \infty$. Ook wordt de zogenaamde *Stokes-transitie* besproken. Hierbij gaan we uit van een 'kleine' oplossing y op een sector S en beschouwen twee representaties van y als convergente transseries die verschillen in de Borel-sommen van $\hat{y}_{\mathbf{k}}$; de laatste kunnen we namelijk op verschillende sectoren S_1 en S_1' bepalen, mits $S \subset (S_1 \cap S_1')$. De Stokes-transitie betreft nu juist de overgang van de ene op de andere convergente transseries voor dezelfde kleine oplossing y op S. Tot slot wordt in dit hoofdstuk gekeken naar zogenaamde *balanced averages*. Als het niet-lineaire stelsel reëel is (dat wil zeggen: A_0 en A_1 reëelwaardig en g reëel-analytisch), dan bestaan er oplossingen van dit stelsel in de vorm van convergente transseries die reëel zijn op de reële as. Deze oplossingen worden verkregen met behulp van balanced averages.

In hoofdstuk 3 wordt onder bepaalde aannames omtrent de matrices A_0 en A_1 bewezen dat er singulariteiten van oplossingen van het niet-lineaire stelsel differentievergelijkingen ontstaan nabij zogenaamde *Stokes-lijnen*. Deze Stokes-lijnen worden gekarakteriseerd door halflijnen vanuit de oorsprong met richtingen $\pm\pi/2 - \theta$, waarbij θ een singuliere richting is. De singuliere punten die ontstaan nabij deze Stokes-lijnen blijken in zekere zin gegroepeerd te zijn in 'bijna-periodieke rijen'. Tevens wordt er in dit hoofdstuk in meer detail gekeken naar het asymptotisch gedrag van de oplossingen y die we via de sommatie van transseries hebben verkregen.

In hoofdstuk 4 worden de resultaten met betrekking tot de formele reductie naar de normaalvorm, het toekennen van een holomorfe transformatie T aan de formele transformatie $\hat{T}$ en de correspondentie met 'kleine' oplossingen afgeleid in het geval dat de matrix

A_1 niet noodzakelijkerwijs diagonaliseerbaar is. Tevens wordt er een eenvoudige resurgentierelatie afgeleid. Naast een niet-lineair stelsel differentievergelijkingen, zoals eerder beschreven, wordt ook het analogon in het geval van differentiaalvergelijkingen behandeld. Het analoge stelsel ziet er dan uit als

$$y'(x) + \Lambda(x)y(x) + g(x, y(x)) = 0, \quad \text{met} \quad \Lambda(x) = A_0 - A_1 x^{-1},$$

waarin A_0 een diagonaalmatrix is met eventueel meervoudige, doch niet aan nul gelijk zijnde, eigenwaarden, maar waarin A_1 niet noodzakelijkerwijs diagonaliseerbaar is.

In hoofdstuk 5 wordt eveneens gekeken naar normaliserende transformaties van niet-lineaire stelsels differentiaalvergelijkingen. Echter, in dit hoofdstuk veronderstellen we dat de differentiaalvergelijking de vorm $y'(x) + \Lambda(x)y(x) + g(x, y(x)) = 0$ heeft, waarin Λ een diagonaalmatrix is waarvan de elementen tevens onbegrensde functies van x kunnen zijn. Preciezer gezegd nemen we aan dat een diagonaalelement van Λ geschreven kan worden als $ax^{-1} + p(x)$ met $a \in \mathbb{C}$ en p een polynoom in x. Het aantonen van het bestaan van een formele transformatie $\hat{T}$ geeft geen aanleiding tot nieuwe problemen. Echter, het sommeren van de transformatie $\hat{T}$ is tamelijk gecompliceerd. Het eerder beschreven sommatieproces schiet te kort in die zin dat dit proces ons nu niet leidt naar holomorfe oplossingen. In hoofdstuk 5 worden, met behulp van een meer ingewikkelde sommatieprocedure (*multisommatie*), uitspraken gedaan over het holomorfe analogon van $\hat{T}$ en daarmee over holomorfe oplossingen van de differentiaalvergelijking.

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