Geometric Quantization and Equivariant Cohomology

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Introduction

Let G be a real Lie group acting on a C^∞ even dimensional oriented manifold M. In many cases it is possible to associate to a G-equivariant Hermitian vector bundle $\mathcal{E}$ over M equipped with a G-invariant Hermitian connection $\mathbb{A}$ a *canonical* virtual unitary representation $Q(M, \mathcal{E}, \mathbb{A})$ of G in a virtual Hilbert space $H(M, \mathcal{E}, \mathbb{A})$. The space $H(M, \mathcal{E}, \mathbb{A})$ will be referred as the quantized space of $(M, \mathcal{E}, \mathbb{A})$. The meaning of canonical is the following. Although the geometric model for the quantized space $H(M, \mathcal{E}, \mathbb{A})$ may be evasive, we can give a *canonical character formula* for $Q(M, \mathcal{E}, \mathbb{A})$: there exists an *admissible bouquet of equivariant cohomology classes* $\mathrm{bch}(\mathcal{E}, \mathbb{A})$ on M such that

$$(F): \qquad \mathrm{Tr}(Q(M, \mathcal{E}, \mathbb{A})) = i^{-\frac{\dim M}{2}} \int_b \mathrm{bch}(\mathcal{E}, \mathbb{A})$$

as an equality of generalized functions on the group G. The notion of admissible bouquet and the notion of integration $\int_b$ of such bouquets will be described in this article. The bouquet $\mathrm{bch}(\mathcal{E}, \mathbb{A})$ will be called the bouquet of Chern characters of the bundle $\mathcal{E}$ with connection $\mathbb{A}$ (in fact, we will have to modify the notion of G-equivariant vector bundle to the notion of G-equivariant quantum bundle in order for $\mathrm{bch}(\mathcal{E}, \mathbb{A})$ to be an admissible bouquet).

The conjecture presented here is in common with M. Duflo and extends the earlier conjectures of Duflo-Heckmann-Vergne [25] and of Berline-Vergne

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[47] [14]. It also relates very strongly to my earlier work with Nicole Berline on equivariant cohomology and index formulas [12].

Our formalism is an extension of the notion of direct images in equivariant K-theory ([3], [5], [4], [6], [7]) . But we will compute, as direct images of geometric objects, *trace-class representations* of G instead of finite dimensional representations of G. The quantization assignment $(M, \mathcal{E}, \mathbb{A}) \to Q(M, \mathcal{E}, \mathbb{A})$ is also strongly inspired by the formalism of geometric quantization as initiated by Kostant-Souriau [39] , [45] which deals with symplectic manifolds M. However, I believe it is important to quantize vector bundles with connections or more generally superbundles with superconnections over general manifolds.

The formula (F) will be called the universal formula as it extends the universal formula for characters conjectured by Kirillov [37]. Let me immediately confess that it is not as universal as wished. However it is sufficiently general to give some new applications like formulas for the character of Zuckerman representations of real semi-simple connected Lie groups and, when allowing superconnections, index formulas for transversally elliptic operators [48].

The plan of this article is as follows.

In part 1, I will first motivate the map Q and the character formula (F) on elementary examples. Then, when G and M are compact, I will define the map Q in terms of the equivariant index of Dirac operators and express the trace of Q in function of the equivariant cohomology of M. Finally I will state a first version of the universal formula as an equality of generalized functions on a neighborhood of 0 in the Lie algebra of G. I will give examples of applications.

Part 1 is written for a large audience, thus I will here define the basic notions of characters, connections and equivariant cohomology used to state the first version of the formula (F).

In part 2, I will show the relation of metalinear structures with orientations of fixed points submanifolds. Then I will introduce the notions of G-equivariant quantum bundles and of admissible bouquets of equivariant differential forms. They are the "good" objects to consider for defining direct images. I will indeed define under some assumptions on M the notion of integration of admissible bouquets. Finally I will state the universal formula (F) and I will give examples of its validity. In particular, the formula (F) gives a formula for the character of Zuckerman representations of connected real semi-simple Lie groups. The formula (F) is also valid for the Weil representation of the metaplectic group.

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1 Motivations

1.1 Characters of representations

Let G be a Lie group. We denote by $\mathfrak{g}$ the Lie algebra of G. If $g \in G$, we denote by $G(g)$ the centralizer of g. If G acts on a set M, we denote by $M(g)$ the subset of fixed points of the action of g in M. If $m \in M$, we denote by $G(m)$ the stabilizer of m.

Recall first what is the character of a representation T of G. If the group G acts on a finite dimensional complex vector space E, each element $g \in G$ is represented by a matrix $T(g) \in \mathrm{GL}(E)$ and the character of the representation T of G in E is by definition the function $g \to \operatorname{Tr} T(g)$ on G. For $g = e$, the identity element of the group G, then $\operatorname{Tr} T(e)$ is the dimension of the vector space E.

If E is an infinite dimensional Hilbert space and T is an unitary representation of G in E, we may also be able to define the character of the representation T of G in E by the same formula

$$\operatorname{Tr} T(g) = \sum_k (g \cdot e_k, e_k)$$

whenever the sum of all the diagonal coefficients $(g \cdot e_k, e_k)$ of the matrix $T(g)$ written in any orthonormal basis e_k exists in the space of generalized functions on G . We will then say that the representation T is trace-class.

If E^+, E^- are two Hilbert spaces with trace-class representations T^+ and T^-, then for $[T] = [T^+] - [T^-]$, we write:

$$\operatorname{Tr} T(g) = \operatorname{Tr} T^+(g) - \operatorname{Tr} T^-(g).$$

We will say that $\operatorname{Tr} T^+(g) - \operatorname{Tr} T^-(g)$ is a virtual character of G.

Let us give some examples of characters of representations .

Example 1. Consider a finite group G acting on itself by left translations. Let

$$L^2(G) = \oplus_{g \in G} \mathbb{C}\delta_g$$

be the space of functions on G. The element δ_g is the point-mass function $\delta_g(g') = \delta_g^{g'}$. Consider the action of G on the space of functions on G given by $(L(g_0)\phi)(g) = \phi(g_0^{-1}g)$ for g_0, g in G and $\phi \in L^2(G)$.

Let $e \in G$ be the identity. If $g_0 \neq e$, the action of g_0 on G given by $g \to g_0 g$ moves all the points $g \in G$. If $g_0 = e$, on the contrary, the action of e leaves fixed all the points $g \in G$. Thus we see on the formula

$$L(g_0)\delta_g = \delta_{g_0 g}$$

for $L(g_0)$ acting on the basis δ_g that $\operatorname{Tr} L(g_0) = 0$ if $g_0 \neq e$ while $\operatorname{Tr} L(e) = |G|$ so that

$$\operatorname{Tr} L(g) = |G|\delta_e(g)$$

where $|G|$ denotes the cardinal of G.

Example 2. Let $T = \{e^{i\theta}; \theta \in \mathbb{R}/2\pi\mathbb{Z}\}$ be the 1-dimensional rotation group. We denote by δ_1 the (generalized) δ-function at the identity 1 of the group T.

The group T acts on $S^1 = \{z \in \mathbb{C}; |z| = 1\}$ by $e^{i\theta} \cdot z = e^{i\theta} z$. Thus T acts on $L^2(S^1)$ by $(L(e^{i\theta})f)(z) = f(e^{-i\theta}z)$. Take $e_k = z^k$ as orthonormal basis of $L^2(S^1)$. The action of $e^{i\theta}$ is diagonal on this basis: $L(e^{i\theta}) \cdot z^k = e^{-ik\theta} z^k$, so that we obtain

$$\operatorname{Tr} L(e^{i\theta}) = \sum_{k \in \mathbb{Z}} e^{-ik\theta}.$$

But the sum of functions $\sum_{k\in\mathbb{Z}} e^{-ik\theta}$ on T converges to the generalized function $2\pi\delta_1$:

$$\sum_{k \in \mathbb{Z}} e^{ik\theta} = 2\pi\delta_1(e^{i\theta}). \tag{1}$$

Let us note also here for later use the following similar formula for the generalized function δ_0 on $\mathbb{R}$:

$$\int_{\mathbb{R}} e^{i\theta\xi} d\xi = 2\pi\delta_0(\theta). \tag{2}$$

Thus we have

$$\operatorname{Tr} L(g) = 2\pi\delta_1(g).$$

Remark here again that an element $g \neq 1$ of T is acting on S^1 without fixed points and that the character of the associated representation L of T on $L^2(S^1)$ is zero when $g \neq 1$.

1.2 Connections

If M is a manifold , we denote by TM its tangent bundle. If $N \subset M$ is a submanifold of M, denote by $T(M/N) = (TM|N)/TN$ the normal bundle of N in M. If $\mathcal{E} \to M$ is a real or complex vector bundle over M, we denote by $\Gamma(M, \mathcal{E})$ the space of its smooth sections.

Let $\mathcal{A}^{\bullet}(M) = \sum_k \mathcal{A}^{[k]}(M)$ be the graded algebra of differential forms. If $N \subset M$ is a submanifold and if $\alpha \in \mathcal{A}(M)$, we denote by $\alpha|N$ the restriction of α to N. If M is oriented and if α is a differential form with compact support, we write $\int_M \alpha$ for the integral of the top dimensional term of α.

Let d be the exterior differential. Let $\mathcal{A}(M, \mathcal{E}) = \Gamma(M, \Lambda T^* M ø \mathcal{E})$ be the space of $\mathcal{E}$-valued differential forms. A connection $\mathbb{A}$ on $\mathcal{E}$ is an operator

$$\mathbb{A} : \mathcal{A}^{\bullet}(M, \mathcal{E}) \to \mathcal{A}^{\bullet+1}(M, \mathcal{E})$$

satisfying Leibniz'rule

$$\mathbb{A}(\alpha\nu) = (d\alpha) \wedge \nu + (-1)^k \alpha \wedge \mathbb{A}(\nu)$$

if $\alpha \in \mathcal{A}^{[k]}(M)$ and $\nu \in \mathcal{A}(M, \mathcal{E})$.

If $\mathcal{E} = M \times E$ is a trivial bundle, then $\mathcal{A}(M, \mathcal{E}) = \mathcal{A}(M) ø E$. Any connection $\mathbb{A}$ on E is an operator on the form

$$\mathbb{A} = d ø I + \sum_a \omega_a ø X_a$$

where $\omega_a \in \mathcal{A}^{[1]}(M)$ acts on $\mathcal{A}(M)$ by left exterior multiplication and $X_a \in \mathrm{End}(E)$. We write $\omega = \sum_a \omega_a ø X_a$.

Let us now assume that a Lie group G acts on $\mathcal{E} \to M$. We will say that $\mathbb{A}$ is a G-invariant connection if the operator $\mathbb{A}$ on $\mathcal{A}(M, \mathcal{E})$ commutes with the natural action of each element $g \in G$ on $\mathcal{A}(M, \mathcal{E})$. Furthermore, we will assume that $\mathcal{E}$ has a Hermitian structure. We will say that $\mathbb{A}$ is a Hermitian connection if it preserves the Hermitian structure on $\mathcal{E}$. We denote by $\mathcal{Q}_G(M)$ the set of G-equivariant Hermitian vector bundles with G-invariant Hermitian connections (up to isomorphism). (This notion will have to be slightly modified in part 2).

Assume that M is even dimensional and oriented. We would like to associate to an element $(\mathcal{E}, \mathbb{A}) \in \mathcal{Q}_G(M)$ a quantized virtual space $H(M, \mathcal{E}, \mathbb{A})$ with a G-action $Q(M, \mathcal{E}, \mathbb{A})$. Furthermore we would like to compute the trace of the representation of G in $H(M, \mathcal{E}, \mathbb{A})$ in function of the geometric object $(\mathcal{E}, \mathbb{A})$.

Let us start by some examples.

1.2.1 Points

The first example is when $M = \bullet$ is a point. Then $\mathcal{E} = E$ is a finite dimensional representation space for G.

Another very simple example but already significant is when M is a finite set with an action of G. Thus $\mathcal{E}$ is just a collection of Hermitian spaces E_x indexed by $x \in M$. Then we associate to $\mathcal{E} \to M$ the space $H(M, \mathcal{E})$ of sections of $\mathcal{E}$ with its natural G-action. In other words

$$H(M, \mathcal{E}) = \oplus_{x \in M} E_x$$

is the "integral" of $\mathcal{E}$ over M.

As the action of $g \in G$ moves the space E_x to the space $E_{g \cdot x}$, the character of the natural action $Q(M, \mathcal{E})$ of G on $H(M, \mathcal{E})$ is given by the fixed point formula:

$$\operatorname{Tr} Q(M, \mathcal{E})(g) = \sum_{x \in M(g)} \operatorname{Tr}_{E_x} g \tag{3}$$

where $M(g)$ is the subset of M fixed by the action of $g \in G$.

1.2.2 Cotangent bundles

Let $M = T^*B$ be the cotangent bundle to a manifold B. Let α be the canonical 1-form on T^*B. In local coordinates, q_1, q_2, $\ldots$, q_n, of the base, $\alpha = p_1 dq_1 + p_2 dq_2 + \cdots + p_n dq_n$. Let $\mathcal{L} = M \times \mathbb{C}$ be the trivial line bundle on M with connection $\mathbb{A} = d + i\alpha$. Let G be a real Lie group acting on B. Then G acts on M and $(\mathcal{L}, \mathbb{A})$ is an element of $\mathcal{Q}_G(M)$. The quantized representation space is undoubtedly to be the natural representation L of G in

$$H(M, \mathcal{L}, \mathbb{A}) = L^2(B, dx)$$

if G leaves invariant a positive measure dx on B, or more generally the Hilbert space of $\frac{1}{2}$-densities on B. The representation L is trace-class if B is compact and homogeneous.

1.2.3 Symplectic vector spaces

Let (V, B) be a symplectic vector space with symplectic coordinates p_1, q_1, $\ldots$, p_n, q_n. Let $\mathcal{L}$ be the trivial line bundle over V with connection $\mathbb{A} = d +$

$iB(v,dv)/2 = d + \frac{i}{2}(p_1dq_1 - q_1dp_1 + \cdots + p_ndq_n - q_ndp_n)$. Then we associate to $(V, \mathcal{L}, \mathbb{A})$ the Weil representation W of the metaplectic group $\mathrm{Mp}(2n, \mathbb{R})$. This is the archetype of the quantization map.

1.2.4 Hamiltonian spaces

Let M be a G-Hamiltonian manifold with symplectic form Ω and moment map $\mu : M \to \mathfrak{g}^*$. Here $\mathfrak{g}$ is the Lie algebra of G and $\mathfrak{g}^*$ is the dual vector space of $\mathfrak{g}$. Assume as in Kostant-Souriau framework [39] that M is prequantized, i.e. there is a G-equivariant Hermitian line bundle $(\mathcal{L}, \mathbb{A})$ with a G-invariant Hermitian connection $\mathbb{A}$ of curvature $i\Omega$. (The notion of prequantization will be slightly modified in part 2). In many cases, we know how to associate to a prequantized Hamiltonian space $(M, \mathcal{L}, \mathbb{A})$ a unitary representation $Q(M, \mathcal{L}, \mathbb{A})$ of G in a Hilbert space $H(M, \mathcal{L}, \mathbb{A})$. If M is a prequantized orbit of the coadjoint representation of G in $\mathfrak{g}^*$, then due to the work of Harish-Chandra [31] [32], Rossmann [43], Kirillov [36] [37], Auslander-Kostant [8], Pukanszky [41], Duflo [21] [23], Khalgui [35],... the representation $Q(M, \mathcal{L}, \mathbb{A})$ is constructed and is of trace-class at least when G is sufficiently algebraic and M is closed and of maximal dimension. Moreover the trace of $Q(M, \mathcal{L}, \mathbb{A})$ is given in a neighborhood of the identity by the universal character formula conjectured by Kirillov [37]. We will review this construction in paragraph 1.6.4.

1.3 Quantization and Dirac operators

Let us denote by $\mathrm{Rep}^{\pm}(G)$ the set of virtual unitary representations of G (up to isomorphism). Our aim is to find a canonical map

$$Q : \mathcal{Q}_G(M) \to \mathrm{Rep}^{\pm}(G).$$

If G and M are compacts, let $K_G(M)$ be the Grothendieck group of G-equivariant vector bundles over M. An element $(\mathcal{E}, \mathbb{A}) \in \mathcal{Q}_G(M)$ gives us an element in $K_G(M)$ just forgetting the choice of the G-invariant connection $\mathbb{A}$ on $\mathcal{E}$. Thus there is a natural (surjective) map $\mathcal{Q}_G(M) \to K_G(M)$. Assume that M has a G-invariant spin structure. Then Atiyah-Hirzebruch [3] and Atiyah-Segal-Singer [5] [4] [6] [7] have defined an "integration" map in K-theory

$$K_G(M) \to \mathrm{Rep}^{\pm}(G)$$

which associates to a G-equivariant vector bundle over a compact even dimensional spin manifold M a virtual finite dimensional representation of G (we will modify the notion of vector bundle to the notion of quantum bundle in order to consider more generally manifolds without spin structure). It is this assignment

$$Q : \mathcal{Q}_G(M) \to K_G(M) \to \mathrm{Rep}^{\pm}(G)$$

that we want to extend.

Let us recall the construction of $Q(M, \mathcal{E}, \mathbb{A}) = Q(M, \mathcal{E})$ when G and M are compacts. Choose a G-invariant Riemannian structure on M. Let $\mathcal{S}^{\pm}$ be the half-spin bundles over M determined by the orientation of M and the spin structure. Choose a G-invariant connection $\mathbb{A}$ on $\mathcal{E}$. (These choices can be made as G is compact). Consider the twisted Dirac operator (see [9], ch. 3)

$$D^+_{\mathcal{E},\mathbb{A}} : \Gamma(M, \mathcal{S}^+ ø \mathcal{E}) \to \Gamma(M, \mathcal{S}^- ø \mathcal{E}).$$

The operator $D^+_{\mathcal{E},\mathbb{A}}$ is a G-invariant elliptic operator, so that its kernel and its cokernel are finite-dimensional representation spaces for G. Consider the index space of solutions of $D^+_{\mathcal{E},\mathbb{A}}$, with a change of signs that we will explain below,

$$H(M, \mathcal{E}, \mathbb{A}) = (-1)^{\dim M/2}([\mathrm{Ker}\, D^+_{\mathcal{E},\mathbb{A}}] - [\mathrm{Coker}\, D^+_{\mathcal{E},\mathbb{A}}]).$$

The virtual representation $Q(M, \mathcal{E}, \mathbb{A})$ of G so obtained is independent of the choice of the connection $\mathbb{A}$ and of the choice of the Riemannian structure on M. Thus we also denote $Q(M, \mathcal{E}, \mathbb{A})$ by $Q(M, \mathcal{E})$.

It may be more concrete to reformulate the map $\mathcal{E} \to Q(M, \mathcal{E})$ in the case where M is a compact complex manifold of complex dimension n and $\mathcal{E}$ is a holomorphic Hermitian bundle over M. In this case the map Q associates to $\mathcal{E}$ the direct image of the sheaf of holomorphic sections of the vector bundle $\mathcal{E}$ with a slight change of labels and of signs. We thus assume that G is a group of holomorphic transformations of $\mathcal{E} \to M$. By our assumption on the existence of a spin structure, there is a line bundle ρ over M which is the square root of the line bundle of $(n, 0)$-forms. Then it is not difficult to see that $Q(M, \mathcal{E})$ is, up to sign, the natural representation of G in the finite dimensional graded cohomology space of the sheaf of holomorphic sections of $\mathcal{E} ø \rho^*$

$$H(M, \mathcal{E}) = (-1)^{\dim M/2} \left(\oplus_{k=0}^{n} (-1)^k [H^k(M, \mathcal{O}(\mathcal{E} ø \rho^*))] \right).$$

Here, in the definition of $H(M, \mathcal{E})$, the manifold M has the orientation given by its complex structure.

Let $\mathcal{L}$ be a positive line bundle on M. It provides a non-degenerate symplectic form and thus an orientation $o_{\mathcal{L}}$ which differs from the complex orientation by the factor $(-1)^{\dim M/2}$. Our convention is such that for the orientation $o_{\mathcal{L}}$, and for $\mathcal{L}$ sufficiently positive the quantized space $H(M, \mathcal{L})$ is the space of holomorphic sections of the line bundle $\mathcal{L}$.

The character of $Q(M, \mathcal{E})$ is given by Atiyah-Segal-Singer formula. The knowledge of the function $\operatorname{Tr} Q(M, \mathcal{E})(g)$ determines up to isomorphism the virtual representation $Q(M, \mathcal{E})$. We first state a special case of this formula in the case where the element $g \in G$ acts on M with just a finite number of fixed points. In this case, we have the simple fixed point formula for $\operatorname{Tr} Q(M, \mathcal{E})(g)$ due to Atiyah-Bott [1] [2]:

$$\operatorname{Tr} Q(M, \mathcal{E})(g) = \sum_{x \in M(g)} \operatorname{sign}(g, x) \frac{\operatorname{Tr}_{\mathcal{E}_x} g}{|\det^{1/2}_{T_x M}(1 - g)|} \tag{4}$$

where $\operatorname{sign}(g, x) = \pm 1$ is a sign determined by the spin structure and the orientation of M (see paragraph 2.1). The similarity of this formula with the formula (3) of paragraph 1.2.1 may be observed.

1.4 De Rham cohomology and Atiyah-Segal-Singer formula

In order to state Atiyah-Segal-Singer formula we need to recall the construction of Chern-Weil of some characteristic forms in de Rham cohomology.

Let M be a C^∞-manifold. Let $\mathcal{E} \to M$ be a bundle over M and $\mathbb{A} : \mathcal{A}(M, \mathcal{E}) \to \mathcal{A}(M, \mathcal{E})$ a connection on $\mathcal{E}$. Although $\mathbb{A}$ is a first order differential operator, the operator $F = \mathbb{A}^2$ is a differential operator of order 0. This is easily deduced from the fact that $d^2 = 0$ on $\mathcal{A}(M)$. Thus the operator F is given by the action on $\mathcal{A}(M, \mathcal{E})$ of an element $\mathcal{A}(M, \operatorname{End}(\mathcal{E}))$ still denoted by F and called the curvature of $\mathbb{A}$. In a local frame $e_i, 1 \leq i \leq N$ of $\mathcal{E}$ where $\mathbb{A} = d + \omega$, then $F = d\omega + \omega^2$ is a matrix of 2-forms $F = F_{i,j}, 1 \leq i, j \leq N$. The Chern character form of the bundle $\mathcal{E}$ with connection $\mathbb{A}$ is the closed differential form on M given by

$$\operatorname{ch}(\mathcal{E}, \mathbb{A}) = \operatorname{Tr}(e^F) \in \mathcal{A}(M). \tag{5}$$

The convention taken here differs from the convention in [9].

Let V be a real vector space. Define for $Y \in \mathrm{End}(V)$

$$J_V(Y) = \det\left(\frac{e^{Y/2} - e^{-Y/2}}{Y}\right). \tag{6}$$

Let $\mathcal{V} \to M$ be a real vector bundle. Consider a connection ∇ on $\mathcal{V}$. Let $R = \nabla^2$ be the curvature of ∇. Define

$$J(M, \mathcal{V}, \nabla) = \det\left(\frac{e^{R/2} - e^{-R/2}}{R}\right) = 1 + \frac{1}{24}\mathrm{Tr}(R^2) + \cdots \tag{7}$$

Then $J(M, \mathcal{V}, \nabla)$ is a de Rham closed form on M. If ∇ is understood, we will write it simply by $J(M, \mathcal{V})$. Anyway the cohomology class $J(M, \mathcal{V})$ of $J(M, \mathcal{V}, \nabla)$ is independent of the choice of ∇. We will say that $J(M, \mathcal{V})$ is the J-genus of $\mathcal{V}$. If $TM \to M$ is the tangent bundle, we denote $J(M, TM)$ simply by $J(M)$. If M is given a Riemannian structure, we can choose the Levi-Civita connection on TM to define $J(M)$. The cohomology class of the form $J^{-1/2}(M) = 1 - \frac{1}{48}\mathrm{Tr}(R^2) + \cdots$ is, apart from normalization factors of 2π, equal to the $\hat{A}$-genus of M [5] [6].

We can now state:

Theorem 1 (Atiyah-Singer)[5] [6] *Let $\mathcal{E} \to M$ be a vector bundle over a compact oriented spin manifold M. Let $\mathbb{A}$ be a connection on $\mathcal{E}$. Consider the twisted Dirac operator $D^+_{\mathcal{E},\mathbb{A}}$ associated to $(\mathcal{E}, \mathbb{A})$. Then*

$$(-1)^{\dim M/2}(\dim \mathrm{Ker}\, D^+_{\mathcal{E},\mathbb{A}} - \dim \mathrm{Coker}\, D^+_{\mathcal{E},\mathbb{A}}) = \int_M (2i\pi)^{-\dim M/2}\,\mathrm{ch}(\mathcal{E}, \mathbb{A}) J^{-1/2}(M).$$

A beautiful proof of this theorem has been given by E. Getzler [30] (see [9], ch. 4).

Let G be a compact Lie group acting on $\mathcal{E} \to M$. In the next paragraph, we will generalize this formula to a formula for the trace of the representation $Q(M, \mathcal{E})$ in terms of the equivariant cohomology of M. The above formula corresponds to the case where G is reduced to the identity transformation.

1.5 Equivariant cohomology

Let G be a real Lie group acting on M. Unless otherwise indicated, we do not assume G nor M to be compact. Let us recall H. Cartan [19] [20] model for the equivariant cohomology of M (see [9], ch.7).

Let $\mathfrak{g}$ be the Lie algebra of G. We denote by $C[\mathfrak{g}]$ the space of polynomial functions on $\mathfrak{g}$, by $C^{\mathrm{hol}}(\mathfrak{g}_{\mathbb{C}})$ the space of holomorphic functions on the complexification $\mathfrak{g}_{\mathbb{C}}$ of $\mathfrak{g}$. We will also use the spaces $C^{\infty}(\mathfrak{g})$ of C^{∞}-functions on $\mathfrak{g}$ and the space $C^{-\infty}(\mathfrak{g})$ of generalized functions on $\mathfrak{g}$. If G acts on a vector space E, we denote by E^G the subspace of invariants.

Let $X \in \mathfrak{g}$. We denote by X_M the vector field produced by the action of $\exp(-tX)$ on M. Let $\iota(X_M) : \mathcal{A}^{\bullet}(M) \to \mathcal{A}^{\bullet-1}(M)$ be the contraction by X_M.

On the space $C[\mathfrak{g}]\text{ø}\mathcal{A}(M)$ of polynomial maps from $\mathfrak{g}$ to $\mathcal{A}(M)$, we introduce a total $\mathbb{Z}$-grading: for $P \in C[\mathfrak{g}]$ an homogeneous polynomial and $\alpha \in \mathcal{A}^{[k]}(M)$ a form on M of exterior degree k:

$$\deg(P\text{ø}\alpha) = 2\deg(P) + k. \tag{8}$$

Consider the space

$$\mathcal{A}_G(\mathfrak{g}, M) = (C[\mathfrak{g}]\text{ø}\mathcal{A}(M))^G$$

of G-invariant polynomial maps from $\mathfrak{g}$ to $\mathcal{A}(M)$. An element $\alpha \in \mathcal{A}_G(\mathfrak{g}, M)$ will be called an equivariant form on M with polynomial coefficients. Thus, for $X \in \mathfrak{g}$ and $\alpha \in \mathcal{A}_G(\mathfrak{g}, M)$, $\alpha(X) \in \mathcal{A}(M)$ is a form on M depending polynomially on $X \in \mathfrak{g}$.

We consider also the spaces:

$$\mathcal{A}_G^{\mathrm{hol}}(\mathfrak{g}, M) = C^{\mathrm{hol}}(\mathfrak{g}_{\mathbb{C}}, \mathcal{A}(M))^G, \qquad \mathcal{A}_G^{\infty}(\mathfrak{g}, M) = C^{\infty}(\mathfrak{g}, \mathcal{A}(M))^G.$$

An element $\alpha \in \mathcal{A}_G^{\mathrm{hol}}(\mathfrak{g}, \mathcal{A}(M))$ will be written as a series $\alpha = \sum_{k=0}^{\infty} \alpha_k$ where the term $\alpha_k \in \mathcal{A}_G(\mathfrak{g}, M)$ is homogeneous of degree k for the total grading (8) of $\mathcal{A}_G(\mathfrak{g}, M)$.

Define $d_{\mathfrak{g}} : \mathcal{A}_G(\mathfrak{g}, M) \to \mathcal{A}_G(\mathfrak{g}, M)$ by

$$(d_{\mathfrak{g}}\alpha)(X) = d(\alpha(X)) - \iota(X_M)(\alpha(X)).$$

The operator $d_{\mathfrak{g}}$ is of degree 1 for the total grading of $\mathcal{A}_G(\mathfrak{g}, M)$ and satisfies $d_{\mathfrak{g}}^2 = 0$. We say that an equivariant form α is closed if $d_{\mathfrak{g}}\alpha = 0$, exact if $\alpha = d_{\mathfrak{g}}\beta$ for some equivariant form β. We denote by $\mathcal{H}_G(\mathfrak{g}, M)$ the cohomology space of $d_{\mathfrak{g}} : \mathcal{A}_G(\mathfrak{g}, M) \to \mathcal{A}_G(\mathfrak{g}, M)$. If G is reduced to the identity transformation, then $d_{\mathfrak{g}} = d$ and the complex $\mathcal{A}_G(\mathfrak{g}, M)$ is just de Rham complex. The operator $d_{\mathfrak{g}}$ extends to $\mathcal{A}_G^{\mathrm{hol}}(\mathfrak{g}, M)$, $\mathcal{A}_G^{\infty}(\mathfrak{g}, M)$, etc... in an operator such that $d_{\mathfrak{g}}^2 = 0$. We denote by $\mathcal{H}_G^{\infty}(\mathfrak{g}, M)$, $\mathcal{H}_G^{\mathrm{hol}}(\mathfrak{g}, M)$ the spaces

$\operatorname{Ker} d_{\mathfrak{g}} / \operatorname{Im} d_{\mathfrak{g}}$ in this various spaces. The spaces $\mathcal{H}_G^\infty(\mathfrak{g}, M), \mathcal{H}_G^{\mathrm{hol}}(\mathfrak{g}, M)$ are only $\mathbb{Z}/2\mathbb{Z}$ graded. These spaces are modules for $C[\mathfrak{g}]^G$. If $M = \bullet$ is a point

$$\mathcal{H}_G(\mathfrak{g}, \bullet) = C[\mathfrak{g}]^G, \quad \mathcal{H}_G^\infty(\mathfrak{g}, \bullet) = C^\infty(\mathfrak{g})^G, \quad \mathcal{H}_G^{\mathrm{hol}}(\mathfrak{g}, \bullet) = C^{\mathrm{hol}}(\mathfrak{g}_{\mathbb{C}})^G.$$

More generally we say that $M = G/H$ is a reductive homogeneous space if there exists an H-invariant decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{q}$. In this case [28]

(9)
$$\mathcal{H}_G(\mathfrak{g}, M) = C[\mathfrak{h}]^H, \quad \mathcal{H}_G^\infty(\mathfrak{g}, M) = C^\infty(\mathfrak{h})^H, \quad \mathcal{H}_G^{\mathrm{hol}}(\mathfrak{g}, M) = C^{\mathrm{hol}}(\mathfrak{h}_{\mathbb{C}})^H.$$

It is easy to see that if $\mathcal{H}_G(\mathfrak{g}, M)$ is free and finitely generated over $C[\mathfrak{g}]^G$, then

$$\mathcal{H}_G^{\mathrm{hol}}(\mathfrak{g}, M) = C^{\mathrm{hol}}(\mathfrak{g}_{\mathbb{C}})^G \emptyset_{C[\mathfrak{g}]^G} \mathcal{H}_G(\mathfrak{g}, M). \tag{10}$$

This is the case for example if $M = G \cdot \lambda$ is a *closed* orbit of the coadjoint representation of a real reductive group G. Indeed the stabilizer H of $\lambda \in \mathfrak{g}^*$ is a reductive subgroup of G and the property 10 follows from Chevalley theorem and formula 9 above.

If M is *compact* and oriented the integration over M of equivariant forms is defined by $(\int_M \alpha)(X) = \int_M \alpha(X)$. The integration map sends $\mathcal{A}_G(\mathfrak{g}, M)$ to $C[\mathfrak{g}]^G$ and $\mathcal{A}_G^\infty(\mathfrak{g}, M)$ to $C^\infty(\mathfrak{g})^G$. Furthermore, if α is closed, the integral of α is a function on $\mathfrak{g}$ depending only of the cohomology class of α.

If M is non compact, we may sometimes be able to define the integral of $\alpha \in \mathcal{A}_G^\infty(\mathfrak{g}, M)$ in a generalized sense.

Definition 2 *Let $\alpha \in \mathcal{A}_G^\infty(\mathfrak{g}, M)$. We will say that α is weakly integrable if, for every test function ϕ on $\mathfrak{g}$, $\int_{\mathfrak{g}} \alpha(X)\phi(X)dX$ is a form on M which is integrable and if the map $\phi \to \int_M (\int_{\mathfrak{g}} \alpha(X)\phi(X)dX)$ defines a generalized function on $\mathfrak{g}$.*

Thus if α is weakly integrable, we denote by $\int_M \alpha$ the generalized function on $\mathfrak{g}$ such that for every test function ϕ on $\mathfrak{g}$

$$\int_{\mathfrak{g}} (\int_M \alpha(X))\phi(X)dX = \int_M (\int_{\mathfrak{g}} \alpha(X)\phi(X)dX).$$

Let us now define, following [11], some equivariant closed forms (see [9] ch.7). We begin with the equivariant Chern character. Let $\mathbb{A}$ be a G-invariant connection on a G-equivariant bundle $\mathcal{E} \to M$. Consider for $X \in \mathfrak{g}$ the operator $F(X)$ on $\mathcal{A}(M, \mathcal{E})$ given by

$$F(X) = (\mathbb{A} - \iota(X_M))^2 + \mathcal{L}^{\mathcal{E}}(X)$$

where $\mathcal{L}^{\mathcal{E}}(X)$ is the first-order differential operator given by the Lie derivative of the action of G on $\mathcal{A}(M, \mathcal{E})$. The operator $F(X)$ is given by the action on $\mathcal{A}(M, \mathcal{E})$ of an element of $\mathcal{A}(M, \mathrm{End}(\mathcal{E}))$ that we still denote by $F(X)$ and that we call the equivariant curvature of $\mathbb{A}$. If $F = \mathbb{A}^2$ is the ordinary curvature of $\mathbb{A}$, then

$$F(X) = F + \mu(X)$$

where $\mu(X) \in \Gamma(M, \mathrm{End}(\mathcal{E}))$ is the vertical action of X determined by the connection $\mathbb{A}$: for $m \in M$, $v \in \mathcal{E}_m$, $(\mu(X)v)_m$ is the vertical projection on $\mathcal{E}_m$ (determined by $\mathbb{A}$) of the vector $(-X_{\mathcal{E}})_{(m,v)}$.

Definition 3 *Let $(\mathcal{E}, \mathbb{A})$ be a G-equivariant vector bundle with a G-invariant connection $\mathbb{A}$. Let $F(X)$ be the equivariant curvature of $\mathbb{A}$. The equivariant Chern character* $\mathrm{ch}(\mathcal{E}, \mathbb{A}) \in \mathcal{A}_G^\infty(\mathfrak{g}, M)$ *is defined, for $X \in \mathfrak{g}$, by:*

$$\mathrm{ch}(\mathcal{E}, \mathbb{A})(X) = \mathrm{Tr}(e^{F(X)}).$$

The equivariant Chern character is a closed equivariant form. Our convention here differs from the convention in [9]. If $M = \bullet$ and $\mathcal{E} = E$ is a representation space for G, then $\mathrm{ch}(\mathcal{E})(X) = \mathrm{Tr}_E\, e^X$. Of course, the Chern character form in de Rham cohomology (formula 5 in paragraph 1.4) is the evaluation at $X = 0$ of the equivariant Chern character.

Definition 4 *Let $\mathcal{V} \to M$ be a G-equivariant real vector bundle over M. Let us suppose that $\mathcal{V} \to M$ has a G-invariant connection ∇ with equivariant curvature $R(X)$, then $J(M, \mathcal{V}, \nabla) \in \mathcal{A}_G^{\mathrm{hol}}(\mathfrak{g}, M)$ is defined , for $X \in \mathfrak{g}$, by:*

$$J(M, \mathcal{V}, \nabla)(X) = \det\left(\frac{e^{R(X)/2} - e^{-R(X)/2}}{R(X)}\right).$$

The equivariant form $J(M, \mathcal{V}, \nabla)$ is a closed form; we will often write it simply as $J(M, \mathcal{V})$, the connection ∇ being implicit. Anyway the cohomology class $J(M, \mathcal{V})$ in $\mathcal{H}_G^{\mathrm{hol}}(\mathfrak{g}, M)$ is independent of the choice of the G-invariant connection on $\mathcal{V}$. We will say that $J(M, \mathcal{V})$ is the (equivariant) J-genus of $\mathcal{V}$. For reasons which will appear later on, the J-genus will be considered as a limit of polynomial cohomology classes. This is the reason why we consider its cohomology classes in $\mathcal{H}_G^{\mathrm{hol}}(\mathfrak{g}, M)$ rather than in $\mathcal{H}_G^{\infty}(\mathfrak{g}, M)$.

If $TM \to M$ is the tangent bundle, we denote $J(M, TM)$ simply by $J(M)$. If M admits a G-invariant Riemannian structure, we can choose the Levi-Civita connection on TM to define $J(M)$. When $X = 0$, then $J(M)(0) = 1 + \frac{1}{24}\operatorname{Tr} R^2 + \cdots$ is an invertible form. Thus we can define for X in a neighborhood of 0 the form $J^{-1/2}(M)(X)$ by choosing $J^{-1/2}(M)(0) = 1 - \frac{1}{48}\operatorname{Tr} R^2 + \cdots$, at least when M is compact.

Let G be a compact Lie group acting on a compact oriented even dimensional spin manifold M. Let $\mathcal{E} \to M$ be a G-equivariant vector bundle with G-invariant connection $\mathbb{A}$. Consider the twisted Dirac operator $D^+_{\mathcal{E},\mathbb{A}}$. We can now state a formula for

$$\operatorname{Tr} Q(M, \mathcal{E})(g) = (-1)^{\dim M/2}(\operatorname{Tr}_{\operatorname{Ker} D^+_{\mathcal{E},\mathbb{A}}} g - \operatorname{Tr}_{\operatorname{Coker} D^+_{\mathcal{E},\mathbb{A}}} g).$$

Theorem 5 *[12] Let G be a compact Lie group acting on a compact oriented even dimensional spin manifold M. Let $\mathcal{E} \to M$ be a G-equivariant vector bundle with G-invariant connection $\mathbb{A}$. The character of the representation $Q(M, \mathcal{E})$ is given in a neighborhood of the identity of G by the formula: for $X \in \mathfrak{g}$ sufficiently near 0,*

$$\operatorname{Tr} Q(M, \mathcal{E})(\exp X) = \int_M (2i\pi)^{-\dim M/2} \operatorname{ch}(\mathcal{E}, \mathbb{A})(X) J^{-1/2}(M)(X).$$

The independence of $Q(M, \mathcal{E})$ of the choice of $\mathbb{A}$ and the Riemannian structure on M is reflected in the fact that the equivariant cohomology class of $\operatorname{ch}(\mathcal{E}, \mathbb{A})$ is independent of $\mathbb{A}$ and that the class $J(M)$ is independent of the connection on TM.

Using the localization theorem [10], this formula is just a reformulation of Atiyah-Segal-Singer [4] formula (see [9] ch.8). Let us also mention the direct proof of this theorem (in its refined local form) given by J.M. Bismut [15] (see [9] ch.8).

The preceding formula gives only a formula for $g \in G$ in the neighborhood of the identity element. There is also a formula at any element $g \in G$ that we will give in paragraph 2.3. In this formula, signs depending on the spin structure on M will appear.

1.6 The universal formula near the identity

Let G be a Lie group acting on an oriented manifold M. We do not necessarily assume that G and M are compacts. We assume in this paragraph that M is an even dimensional oriented manifold provided with a G-invariant metalinear structure (see definition 12 in paragraph 2.1). The assumption on existence of metalinear structure will be removed in part 2.

Let $(\mathcal{E}, \mathbb{A}) \in \mathcal{Q}_G(M)$ be a G-equivariant Hermitian bundle with G-invariant Hermitian connection $\mathbb{A}$. Recall that our aim is to find a map

$$Q : \mathcal{Q}_G(M) \to \mathrm{Rep}^{\pm}(G).$$

This map should be the analogue of the integration map in K-theory

$$Q : \mathcal{Q}_G(M) \to K_G(M) \to \mathrm{Rep}^{\pm}(G)$$

defined when G and M are compacts and M spinorial. Thus dictated by the above form of the character formula for the representation $Q(M, \mathcal{E}, \mathbb{A})$, we state a first version of our conjecture.

Conjecture: Let $(\mathcal{E}, \mathbb{A}) \in \mathcal{Q}_G(M)$. There exists a virtual unitary representation $Q(M, \mathcal{E}, \mathbb{A})$ associated to $(\mathcal{E}, \mathbb{A})$. Furthermore, if the representation $Q(M, \mathcal{E}, \mathbb{A})$ is trace-class, its character is given in a neighborhood of 1 in G by the following formula. For X sufficiently near 0 in $\mathfrak{g}$

$$\mathrm{Tr}\, Q(M, \mathcal{E}, \mathbb{A})(\exp X) = \int_M (2i\pi)^{-\dim M/2} \mathrm{ch}(\mathcal{E}, \mathbb{A})(X) J^{-1/2}(M)(X).$$

I also believe that the same conjecture is valid when allowing superconnections instead of connections. I will explain the corresponding formalism and applications to index formulas for transversally elliptic operators in another article. In order to give a meaning to the right hand side of the formula (F), there are two problems:

(a) when G is not compact, there might not exist a G-invariant connection on TM and the class $J(M)(X)$ might not be defined.

(b) when M is not compact, it is not clear that the right hand side defines a function (even generalized) on $\mathfrak{g}$. In fact in some of the examples we need to treat, although $\mathrm{ch}(\mathcal{E}, \mathbb{A})$ is weakly integrable, the form $\mathrm{ch}(\mathcal{E}, \mathbb{A}) J^{-1/2}(M)$ is not weakly integrable. The right hand side of the equality will be given a meaning by methods similar to oscillatory integrals.

In part 2, for G-manifolds without metalinear structure, we will slightly modify the definition of $\mathcal{Q}_G(M)$. Also, we will refine this conjecture in order to understand the character $\operatorname{Tr} Q(M, \mathcal{E}, \mathbb{A})(g)$ in a neighborhood of any point of $g \in G$. However let us first review in terms of this conjecture some of our basic cases.

1.6.1 Points

In the case of a point $M = \bullet$ and $\mathcal{E} = E$ a representation space for G, the formula is tautological as for $X \in \mathfrak{g}$

$$\operatorname{ch}(\mathcal{E}, \mathbb{A})(X) = \operatorname{Tr}_E e^X.$$

1.6.2 Cotangent bundles

Let us consider the cotangent bundle $M = T^*B$ of a compact oriented manifold B. The manifold M has a canonical symplectic form $\Omega = d\alpha$ and so is canonically oriented. Let G be a compact Lie group acting on B and preserving the orientation. Then M has a G-invariant metalinear structure (see lemma 14 in paragraph 2.1). Let $\mathcal{L} = M \times \mathbb{C}$ with connection $\mathbb{A} = d + i\alpha$. Then $(\mathcal{L}, \mathbb{A}) \in \mathcal{Q}_G(M)$.

For $X \in \mathfrak{g}$, let $\mu(X) \in C^\infty(M)$ be the symbol of the vector field $-X_B$. As $-\alpha(X_M) = \mu(X)$, the equivariant curvature of $\mathcal{L} = M \times \mathbb{C}$ with connection $\mathbb{A} = d + i\alpha$ is $i(\mu(X) + \Omega)$ and the equivariant Chern character of $(\mathcal{L}, \mathbb{A})$ is given, for $X \in \mathfrak{g}$, by

$$\operatorname{ch}(\mathcal{L}, \mathbb{A})(X) = e^{i(\mu(X)+\Omega)}.$$

We write more explicitly what is the universal formula (F) in this case.

As G is compact, there is a G-invariant connection ∇ on $TB \to B$. We can then define the equivariant form $J(B) = J(B, \nabla)$. As B is compact, if X is sufficiently small, $J(B)(X)$ is invertible. The connection ∇ provides a connection (still denoted by ∇) on $TM \to M$, as (using the connection ∇) the bundle $TM \to M$ is isomorphic to the inverse image of the bundle $TB \oplus T^*B$ over B. The form $J^{1/2}(M) = J^{1/2}(M, \nabla)$ is the lift from B to M of the form $J(B)$ on the base B (still noted $J(B)$).

The universal formula (F) takes the form

$$\operatorname{Tr} Q(M, \mathcal{L}, \mathbb{A})(\exp X) = \int_{T^*B} (2i\pi)^{-\dim B} \operatorname{ch}(\mathcal{L}, \mathbb{A})(X) J^{-1}(B)(X).$$

We have asserted that the quantized representation $Q(M,\mathcal{L},\mathbb{A})$ is the natural representation L of G in $L^2(B)$. Assume that B is homogeneous under G. Then the representation L has a trace. The following proposition justify the assignment $Q(M,\mathcal{L},\mathbb{A})=L$.

Proposition 6 *For every smooth function ϕ on $\mathfrak{g}$ with support in a sufficiently small neigborhood of* 0*, we have the equality*

$$\mathrm{Tr}\int_{\mathfrak{g}}\phi(X)L(\exp X)dX=(2i\pi)^{-\dim B}\int_M\left(\int_{\mathfrak{g}}\mathrm{ch}(\mathcal{L},\mathbb{A})(X)J^{-1}(B)(X)\phi(X)dX\right).$$

This is easy to verify [14] and is essentially equivalent to the integral formula (2) for the δ-function on a vector space. Let us do this verification for the character of the representation of the group $T=\{e^{i\theta}\}$ in $L^2(S^1)$ already considered in paragraph 1.1. We write $S^1=\mathbb{R}/2\pi\mathbb{Z}$. Let

$$M=T^*S^1=\{(x,\xi);x\in\mathbb{R}/2\pi\mathbb{Z},\xi\in\mathbb{R}\}.$$

In these coordinates, the canonical form is $\alpha=\xi dx$. The group T has Lie algebra

$$\mathfrak{g}=\{\theta V;\theta\in\mathbb{R}\},$$

where V gives by infinitesimal action on S^1 the vector field $V_{S^1}=-\frac{\partial}{\partial x}$.

Let $\mathbb{A}=d+i\alpha$. Then the equivariant curvature $F(X)$ of $\mathbb{A}$ is, for $X=\theta V\in\mathfrak{g}$

$$F(\theta)=i(\theta\xi+d\xi\wedge dx)=i(d_{\mathfrak{g}}\alpha)(X).$$

For every C^∞-function $\phi(\theta)$ with compact support on $\mathfrak{g}$, then

$$\int_{\mathfrak{g}}e^{i\theta\xi}e^{id\xi\wedge dx}\phi(\theta)d\theta=\left(\int_{\mathbb{R}}e^{i\theta\xi}\phi(\theta)d\theta\right)(1+id\xi\wedge dx)$$

is a form on $S^1\times\mathbb{R}$ which is rapidly decreasing in the ξ variable as Fourier transforms of test functions are Schwartz functions.

It is easy to see that the equivariant J-form of $TS^1=S^1\times\mathbb{R}$ is identically 1. The second member of the universal formula (F) is

$$\begin{aligned}(2i\pi)^{-1}\int_{S^1\times\mathbb{R}}e^{i(\theta\xi+d\xi\wedge dx)} &= (2i\pi)^{-1}\int_{S^1\times\mathbb{R}}e^{i\theta\xi}(id\xi\wedge dx)\\ &= \int_{\mathbb{R}}e^{i\theta\xi}d\xi\\ &= (2\pi)\delta_0(\theta).\end{aligned}$$

This is equal indeed in a neighborhood of 0 to $\mathrm{Tr}\, L(e^{i\theta}) = 2\pi\delta_1(e^{i\theta})$ as shown in paragraph 1.1 (example 2).

Remark 1.1 *In contrast with the compact case, we see that for $M = T^*B$ the universal formula (F) for $Q(M,\mathcal{L},\mathbb{A})$ depends not only on $\mathcal{L}$ but also on $\mathbb{A}$. In fact, as $\mathcal{L}$ is trivial, we could take the trivial connection on $T^*B \times \mathbb{C}$ but the formula (F) would not be meaningful. Here the prequantization rule of Kostant-Souriau is essential. We have to take as connection $\mathbb{A}$ on $\mathcal{L}$ a connection of curvature $i\Omega$, where Ω is the canonical form on M. Thus we are led to choose the non trivial connection $\mathbb{A} = d + i\alpha$ on the trivial line bundle $\mathcal{L}$. This choice implies that the universal formula is meaningful as a generalized function and gives the right answer.*

Since there is a trivial connection, according to the general theory, the equivariant Chern character $e^{F(X)}$ of $(\mathcal{L},\mathbb{A})$ is $d_{\mathfrak{g}}$-equivalent in $\mathcal{A}^{\infty}_G(\mathfrak{g}, M)$ to 1. For instance, in this case,

$$e^{i(\xi\theta + d\xi dx)} - 1 = d_{\mathfrak{g}} \left(\frac{e^{i\theta\xi} - 1}{\theta} dx \right).$$

Thus the highest term $ie^{i\xi\theta} d\xi dx$ is d-exact. However, its integral (in the generalized sense) over M is not 0, and $e^{F(X)} - 1$ is not the differential of a weakly integrable form.

1.6.3 Symplectic vector spaces

Let (V, B) be a $2n$-dimensional symplectic space with symplectic coordinates $p_1, q_1, \ldots, p_n, q_n$. Let $\alpha = B(v, dv)/2$. Let $\mathcal{L} = V \times \mathbb{C}$ with connection $\mathbb{A} = d + i\alpha = d + iB(v, dv)/2$. Let G be the metaplectic group. It is a two-fold cover of the symplectic group. Let $\mathfrak{g} \subset \mathrm{End}(V)$ be the Lie algebra of G. Then $(\mathcal{L},\mathbb{A}) \in \mathcal{Q}_G(V)$. Let $\Omega = B(dv, dv)/2 = dp_1 \wedge dq_1 + \cdots + dp_n \wedge dq_n$. Then, for $X \in \mathfrak{g}$,

$$\mathrm{ch}(\mathcal{L},\mathbb{A})(X) = e^{iB(v,Xv)/2} e^{i\Omega}.$$

Let $\beta_V = (2\pi)^{-n} dp_1 \wedge dq_1 \wedge \cdots \wedge dp_n \wedge dq_n$ be the canonical Liouville form on V.

As $TV = V \times V$ with diagonal action of G, the trivial connection is G-invariant and the corresponding equivariant J-genus of TV is given by $J_V(X) 1_V$ where 1_V is the constant function 1 on V. Then as the top dimensional term of $(2i\pi)^{-n} e^{i\Omega}$ is β_V, the universal formula (F) takes the form: for

$X \in \mathfrak{g}$ sufficiently small

$$\operatorname{Tr} Q(V,\mathcal{L},\mathbb{A})(\exp X) = \int_V (2i\pi)^{-n} \operatorname{ch}(\mathcal{L},\mathbb{A})(X) J_V^{-1/2}(X) = J_V^{-1/2}(X) \int_V e^{iB(v,Xv)/2} d\beta_V.$$

We asserted that the quantized representation $Q(V,\mathcal{L},\mathbb{A})$ is the Weil representation. This is justified by the following proposition.

Proposition 7 *The character of the Weil representation W satisfies the formula: for $X \in \mathfrak{g}$, in a neighborhood of 0,*

$$J_V^{1/2}(X) \operatorname{Tr} W(\exp X) = \int_V e^{iB(v,Xv)/2} d\beta_V.$$

This proposition follows easily from the character formula for W given for example in [46] or from ([33],16.3).

Here again, as in remark 1.1, the prequantization rule is crucial in order to give the right meaning to the universal formula.

1.6.4 Hamiltonian spaces

Let M be a symplectic manifold with symplectic form Ω. Let G be a Lie group acting on M by an Hamiltonian action. This means that there is a G-invariant moment map $\mu : M \to \mathfrak{g}^*$ such that for each $X \in \mathfrak{g}$ the function $\mu(X)(m) = (\mu(m), X)$ satisfies

$$d\mu(X) = \iota(X_M)\Omega,$$

that is such that $\Omega(X) = \mu(X) + \Omega$ is a closed equivariant form on M.

Important examples of Hamiltonian spaces are orbits $M \subset \mathfrak{g}^*$ of the coadjoint representation of G. In this case the moment map is the injection $M \to \mathfrak{g}^*$.

Another important example is the situation $M = T^*B$ of a cotangent bundle considered in example 1.6.2. In this case, the moment map $\mu(X)$ is the symbol of the vector field $-X_B$.

Still another nice example is the situation $M = V$ of example 1.6.3. In this case, $\mu(X)(v) = \frac{1}{2}B(v, Xv)$.

In this paragraph we assume that M has a G-invariant metaplectic structure. Assume that there is a line bundle $(\mathcal{L},\mathbb{A}) \in \mathcal{Q}_G(M)$ over M such that the equivariant curvature of $(\mathcal{L},\mathbb{A})$ is $i\Omega(X) = i(\mu(X) + \Omega)$. For example, let $M \subset \mathfrak{g}^*$ be the coadjoint orbit of $\lambda \in \mathfrak{g}^*$. Then the orbit M can be prequantized if and only if there exists an unitary 1-dimensional representation $\tau : G(\lambda) \to T$ such that $\tau(\exp X) = e^{i(\lambda,X)}$ for all $X \in \mathfrak{g}(\lambda)$. Indeed in this case the line bundle $\mathcal{L}_\tau = G \times_{G(\lambda)} \mathbb{C}_\tau$ can be provided with a unique G-invariant connection $\mathbb{A}$ with equivariant curvature $i\Omega(X)$ [39].

Let $(\mathcal{L}, \mathbb{A})$ be a quantum line bundle for (M, Ω). Then the Chern character $\mathrm{ch}(\mathcal{L}, \mathbb{A})$ is given, for $X \in \mathfrak{g}$, by the formula

$$\mathrm{ch}(\mathcal{L}, \mathbb{A})(X) = e^{i\Omega(X)}.$$

Thus the formula for the quantized representation $Q(M, \mathcal{L}, \mathbb{A})$ near the identity of the group should be:

$$\mathrm{Tr}\, Q(M, \mathcal{L}, \mathbb{A})(\exp X) = \int_M (2i\pi)^{-\dim M/2}\, \mathrm{ch}(\mathcal{L}, \mathbb{A})(X) J^{-1/2}(M)(X).$$

Let $\dim M = 2d$ and let $\beta_M = (2\pi)^{-d}(d!)^{-1}\Omega^d$ be the Liouville form on M.

We will first assume that $\mathrm{ch}(\mathcal{L}, \mathbb{A})$ is weakly integrable. This means that

$$F_M(X) = \int_M (2i\pi)^{-\dim M/2}\, \mathrm{ch}(\mathcal{L}, \mathbb{A})(X) = \int_M e^{i(\mu(m),X)} d\beta_M(m)$$

is well defined as a generalized function on $\mathfrak{g}$. The function $F_M(X)$ is the Fourier transform of the image $\mu_*(\beta_M)$ of the Liouville measure of M under the moment map. It exists if $\mu_*(\beta_M)$ is a tempered measure on $\mathfrak{g}^*$.

In particular, if M is a closed orbit of the coadjoint representation of a real algebraic group, then $F_M(X)$ is well defined since the canonical measure of a real algebraic closed coadjoint orbit is a tempered measure on $\mathfrak{g}^*$.

We now try to define the class $J(M)$. Let us assume that our Hamiltonian space M is embedded in a linear representation space V of G. This is the case for $M = V$ a symplectic vector space as well as for orbits $M \subset \mathfrak{g}^*$ of the coadjoint representation.

Consider the normal bundle $T(V/M)$ of the embedding $M \subset V$. We have the exact sequence of vector bundles:

$$0 \to TM \to M \times V \to T(V/M) \to 0.$$

We write $J(V/M)$ for $J(T(V/M))$. As J-genera are multiplicative, it is natural to reformulate the universal formula for the character of the quantized representation as in [13] [47]:

Conjecture:

for $X \in \mathfrak{g}$ in a neighborhood of 0, then

$$J_V^{1/2}(X)\,\mathrm{Tr}\,Q(M,\mathcal{L},\mathbb{A})(\exp X) = \int_M (2i\pi)^{-\dim M/2}\,\mathrm{ch}(\mathcal{L},\mathbb{A})(X)J^{1/2}(V/M)(X).$$

To define the class $J^{1/2}(V/M)$ might be easier than to define $J^{1/2}(M)$. Indeed, let us assume that the normal bundle $T(V/M)$ admits a G-invariant connection. This is not unreasonable: if f is a G-invariant function defined on a neighborhood of M such that $f = 0$ on M, $df|M$ is an invariant section of the conormal bundle. Thus if $M \subset V$ can be defined as the zero set of $(\dim V - \dim M)$ G-invariant equations , the normal bundle $T(V/M)$ of the embedding of M in V is a trivial G-equivariant bundle: there exists an isomorphism $T(V/M) \cong M \times \mathbb{R}^N$ (where N is the codimension of M) with action of G given by its action of the first factor M and trivial action on $\mathbb{R}^N$. In this case, the J-class of $T(V/M)$ is the constant function 1_M. In particular, for $M = V$ a symplectic vector space, the conjectural formula above coincide with the formula given in proposition 7 of paragraph 1.6.3 for the character of the Weil representation.

More generally, if $T(V/M)$ admits a G-invariant connection, we are able to define $J(V/M)$. In some cases we will be able to give a meaning to the right hand side by partial integration on M.

1.6.4.1 *Orbits of maximal dimension*

Let $\lambda \in \mathfrak{g}^*$ and let $M \subset \mathfrak{g}^*$ be the orbit of λ. We assume that M has a metaplectic structure and that M can be prequantized with a quantum line bundle $(\mathcal{L},\mathbb{A})$ (uniquely determined by a 1-dimensional representation τ of $G(\lambda)$). We will make less restrictive assumptions in part 2.

We have

$$J_{\mathfrak{g}^*}(X) = J_{\mathfrak{g}}(X) = \det{}_{\mathfrak{g}}\left(\frac{e^{\mathrm{ad}\,X/2} - e^{-\mathrm{ad}\,X/2}}{X}\right).$$

Assume that M is of maximal dimension, then the normal bundle $T(\mathfrak{g}^*/M)$ is a trivial bundle [26]. In this generic case, the universal formula (F) becomes the universal formula for characters that Kirillov conjectured

Theorem 8 *Let M be a prequantized orbit of maximal dimension of G in $\mathfrak{g}^*$ with quantum line bundle $(\mathcal{L},\mathbb{A})$ and G-invariant metaplectic structure. Then there exists a quantized representation $Q(M,\mathcal{L},\mathbb{A})$. Assume moreover* *that β_M is a tempered measure on $\mathfrak{g}^*$. Then $Q(M,\mathcal{L},\mathbb{A})$ is trace-class and its character is given in a neigborhood of the identity by the formula: for $X \in \mathfrak{g}$ in a neighborhood of* 0

$$J_{\mathfrak{g}}^{1/2}(X)\operatorname{Tr} Q(M,\mathcal{L},\mathbb{A})(\exp X) = F_M(X).$$

References for the proof of this theorem were given in paragraph 1.2.4.

1.6.4.2 *Closed orbits of reductive Lie groups*

Let $M = G\cdot\lambda \subset \mathfrak{g}^*$ be a closed coadjoint orbit of a real semi-simple Lie group. The Killing form is non degenerate on T_xM and the orthonormal decomposition $\mathfrak{g}^* = T_xM \oplus \mathcal{N}_x$ determines G-invariant connections on TM and on $T(V/M)$. Furthermore the bundles TM and $T(\mathfrak{g}^*/M)$ have pseudo Euclidean structures. It follows that $J^{1/2}(\mathfrak{g}^*/M)(X)$ can be defined for all $X \in \mathfrak{g}$, and $J^{1/2}(\mathfrak{g}^*/M)$ defines an element of $\mathcal{H}_G^{\mathrm{hol}}(\mathfrak{g},M)$.

Again we assume that M has a metaplectic structure and that there exists a one dimensional representation τ of $G(\lambda)$ such that for $X \in \mathfrak{g}(\lambda)$, $\tau(\exp X) = e^{i(\lambda,X)}$. Let $(\mathcal{L}_\tau,\mathbb{A})$ be the (unique) quantum line bundle on M determined by τ. Consider the weakly integrable form $\mathrm{ch}(\mathcal{L}_\tau,\mathbb{A})$. If $\alpha \in \mathcal{A}_G(\mathfrak{g},M)$ is an equivariant form with polynomial coefficients, it is easy to see that $\mathrm{ch}(\mathcal{L}_\tau,\mathbb{A})\alpha$ is also weakly integrable, thus

$$\int_M \mathrm{ch}(\mathcal{L}_\tau,\mathbb{A})(X)\alpha(X)$$

defines a generalized function on $\mathfrak{g}$. Furthermore if α is closed, then the generalized function $\int_M \mathrm{ch}(\mathcal{L}_\tau,\mathbb{A})\alpha$ depends only of the cohomology class of α. Consider now a closed form $\alpha = \sum_{j=0}^{\infty}\alpha_j$ in $\mathcal{A}_G^{\mathrm{hol}}(\mathfrak{g}_{\mathbb{C}},M)$. Define

$$\int_M \mathrm{ch}(\mathcal{L}_\tau,\mathbb{A})[\alpha] = \lim_{k\to\infty}\sum_{j=0}^{k}\int_M \mathrm{ch}(\mathcal{L}_\tau,\mathbb{A})\alpha_j. \tag{11}$$

This limit exists: recall that $\mathcal{H}_G^{\mathrm{hol}}(\mathfrak{g},M) = C^{\mathrm{hol}}(\mathfrak{g}_{\mathbb{C}})^G \o_{C[\mathfrak{g}]^G}\mathcal{H}_G(\mathfrak{g},M)$ (formula 10 of paragraph 1.5) and write the class of α as congruent modulo $d_{\mathfrak{g}}$ to a finite sum $\sum_a F_a\o\nu_a$, with $F_a \in C^{\mathrm{hol}}(\mathfrak{g}_{\mathbb{C}})^G$ and $\alpha_a \in \mathcal{H}_G(\mathfrak{g},M)$, then we see that the above limit exists and is equal to $\sum_a F_a(X)\int_M \mathrm{ch}(\mathcal{L}_\tau,\mathbb{A})(X)\nu_a(X)$.

We can thus define the G-invariant generalized function on $\mathfrak{g}$

$$\int_M \mathrm{ch}(\mathcal{L}_\tau,\mathbb{A})(X)[J^{1/2}(\mathfrak{g}^*/M)](X).$$

With the help of Zuckerman functor and usual parabolic induction, it is possible to associate to the data (λ, τ) at least a virtual $(\mathfrak{g}, K)$-module $Z(\lambda, \tau)$ (see [49]) with a G-invariant Hermitian form [27]. If the orbit M admits a real polarization, then $Z(\lambda, \tau)$ is the unitary representation of G induced by a one dimensional representation of a parabolic subgroup (not necessarily minimal). In general when λ is sufficiently generic, then $Z(\lambda, \tau)$ is a unitary irreducible representation of G [49] [50]. The assignment

$$Q(M, \mathcal{L}_\tau, \mathbb{A}) = Z(\lambda, \tau)$$

for the quantization of the orbit $M = G \cdot \lambda$ with quantum line bundle $\mathcal{L}_\tau$ is justified by the following proposition [29].

Proposition 9 *Let $M = G \cdot \lambda$ be a closed orbit of a real semi-simple connected Lie group G. We assume here that M has a G-invariant metaplectic structure and that there exists a one dimensional representation τ of $G(\lambda)$ such that for $X \in \mathfrak{g}(\lambda)$, $\tau(\exp X) = e^{i(\lambda, X)}$. Let $(\mathcal{L}_\tau, \mathbb{A})$ be the quantum line bundle with connection associated to τ and let $Z(\lambda, \tau)$ be the virtual $(\mathfrak{g}, K)$-module associated to (λ, τ). Then for $X \in \mathfrak{g}$ in a neighborhood of 0, we have the equality of generalized functions:*

$$J_{\mathfrak{g}}^{1/2}(X) \operatorname{Tr} Z(\lambda, \tau)(\exp X) = \int_M (2i\pi)^{-\dim M/2} \operatorname{ch}(\mathcal{L}_\tau, \mathbb{A})(X)[J^{1/2}(\mathfrak{g}^*/M)](X).$$

1.7 Geometric constructions

Finally let us make some comments on the construction of the quantized representation $Q(M, \mathcal{E}, \mathbb{A}) \in \operatorname{Rep}^{\pm}(G)$.

We have already discussed several possible ways to associate to $(\mathcal{E}, \mathbb{A}) \in \mathcal{Q}_G(M)$ a representation.

If M has a G-invariant Riemannian structure, then extending the prescription of Atiyah-Hirzebruch-Segal-Singer, it is natural to realize $Q(M, \mathcal{E}, \mathbb{A})$ in the index space of L^2-solutions of the twisted Dirac operator $D^+_{\mathcal{E},\mathbb{A}}$. This leads for example to a construction of the discrete series of representations of a semi-simple Lie group G [40].

If M has a G-invariant Kähler structure and $\mathcal{E}$ is an holomorphic vector bundle, we consider the L^2-cohomology space of the $\overline{\partial}$ -complex on the holomorphic bundle $\mathcal{E} ø \rho_M^*$ where ρ_M is the square root of the line bundle of $(n, 0)$-forms on M. This leads to another construction of the discrete series of representations of a semi-simple Lie group G [44].

If M is a cotangent bundle T^*B of an homogeneous space B, we will associate to M the induced representation of G in $L^2(B)$. A similar construction for coadjoint orbits with real polarizations leads to induced representations.

If $M = G \cdot \lambda$ is a closed orbit of maximal dimension of the coadjoint representation, the combination of the above methods indeed construct the representation $T_{\lambda,\tau} = Q(M, \mathcal{L}_\tau, \mathbb{A})$ associated to $(M, \mathcal{L}_\tau, \mathbb{A})$.

If $M = G \cdot \lambda$ is a closed orbit of a real semi-simple Lie group, the construction of $Q(M, \mathcal{L}_\tau, \mathbb{A})$ is similar. Indeed, if λ is elliptic, then $Z(\lambda, \tau)$ is obtained in studying the index space of the $\overline{\partial}$-operator. However, Zuckerman considered $\overline{\partial}$ as acting on the Taylor expansions of sections of the holomorphic line bundle $\mathcal{L}_\tau \emptyset \rho_M^*$ defined on a tubular neighborhood of $K \cdot \lambda$ in $M = G \cdot \lambda$. Here M may not have a G-invariant Riemannian metric (the Killing form gives only a structure of pseudo-Riemannian manifold on M). Thus the unitary structure on $H(M, \mathcal{L}, \mathbb{A})$ is not evident [27], [50].

Remark that there might be many different choices of these extra structures on $(M, \mathcal{E}, \mathbb{A})$ (Riemannian metrics, polarizations,...) leading to several models for $Q(M, \mathcal{E}, \mathbb{A})$. It is remarkable however that the universal formula tells us in advance that all these models are isomorphic. However if V is a symplectic vector space and G the metaplectic group, there is no G-invariant metric nor G-invariant polarization to help us to construct a "concrete" model for the Weil representation W. Thus W is a mysterious representation with a canonical character formula but no canonical model.

Finally let us say that we are unable to treat the case of general orbits of the coadjoint representation, for example we are not able to propose a character formula based on this scheme for general unipotent representations of a semi-simple Lie group which are believed to be "attached" to nilpotent orbits (even if we can do it for some particular representation like the Weil representation).

When G and M are compacts, the quantized representation $Q(M, \mathcal{E}, \mathbb{A})$ is independent of the choice of $\mathbb{A}$. Clearly this is not anymore the case when M is not compact. In the example of characters of induced representations from a subgroup H of G, the choice of the non trivial connection $d + i\alpha$ on the trivial line bundle $\mathcal{L}$ on $T^*(G/H)$ is fundamental in order to obtain a meaningful formula (see remark 1.1 of paragraph 1.6.2). Using superconnections instead of connections, we can give, based on the same scheme, character formulas for the index of transversally elliptic operators on a compact manifold B [48]. Here the [0]-exterior degree term of the superconnection as well as the 1-form α of T^*B is of fundamental importance. Thus, it seems that the fundamental objects of quantization are the connections or superconnections on bundles modulo some equivalence relations. However, it is not clear what are the homotopies to allow on $\mathbb{A}$ in order that the representation $Q(M, \mathcal{E}, \mathbb{A})$ remains the same.

2 Quantum bundles and descent

2.1 On orientations of fixed submanifolds

Let G be a real Lie group with Lie algebra $\mathfrak{g}$. An element $s \in G$ is called elliptic if it is contained in a compact subgroup of G. We denote by G_{ell} the set of elliptic elements of G. Of course, if G is compact then $G_{ell} = G$. An element $S \in \mathfrak{g}$ is called elliptic if $\exp tS$ is relatively compact. We denote by $\mathfrak{g}_{ell}$ the set of elliptic elements of $\mathfrak{g}$.

Let V be a real finite dimensional vector space. Consider for $s \in \mathrm{GL}(V)_{ell}$ and $S \in \mathfrak{gl}(V)_{ell}$ the spaces

$$V(s) = \{v \in V, s \cdot v = v\}, \quad V(S) = \{v \in V, S \cdot v = 0\}.$$

These spaces have canonical supplementary subspaces

$$V = V(s) \oplus V_1(s), \quad V = V(S) \oplus V_1(S) \tag{12}$$

with $V_1(s) = (1 - s)(V)$ and $V_1(S) = S(V)$.

If $S \in \mathfrak{gl}(V)_{ell}$, then $\det_V S \geq 0$. If o is an orientation of V, there exists a canonical square root $\det_{V,o}^{1/2} S$ of $\det_V S$. The convention will be taken as follows. If $\det_V S$ is non zero the dimension of V is even and we can choose an oriented basis e_i such that

$$Se_{2i-1} = \lambda_i e_{2i}, \quad Se_{2i} = -\lambda_i e_{2i-1}.$$

Then we define $\det_{V,o}^{1/2} S = \lambda_1 \cdots \lambda_{n/2}$.

Definition 10 *Let $S \in \mathfrak{gl}(V)_{ell}$. Then the space $V/V(S)$ is even dimensional. We call the orientation o_S defined by $\det^{1/2}_{V/V(S),o_S}(S) > 0$ the canonical orientation.*

Let $s \in \mathrm{GL}(V)_{ell}$. Then the only possible real eigenvalues of s are ± 1 so that $\det_V(1 - s) \geq 0$. We have $\det_V s = \pm 1$. If $det_V(s) = 1$ then the dimension of $V/V(s)$ is even. In contrast to the case of $S \in \mathfrak{gl}(V)_{ell}$, the map s does not provide an orientation o_s on $V/V(s)$. However, it is the case if s belongs to the double cover of $\mathrm{SL}(V)$. This fact does not seem to be as well known as it should be. Let us explain this. Let $\mathrm{GL}^+(V)$ be the group of invertible linear transformations of V with positive determinant. There is a canonical two-fold cover

$$j : \mathrm{DL}(V) \to \mathrm{GL}^+(V).$$

If $\dim V > 2$, this is the universal cover, thus $\mathrm{DL}(V)$ is simply connected. If Q is a positive definite symmetric bilinear form on V, then the inverse image of the group $\mathrm{SO}(V, Q)$ in $\mathrm{DL}(V)$ is the spin group. Similarly if B is a non-degenerate skew-symmetric bilinear form on V, the inverse image of $\mathrm{Sp}(V, B)$ is the metaplectic group $\mathrm{Mp}(V, B)$. We denote by $\{e, \epsilon\}$ the kernel of j where e is the identity of G.

Let $s \in \mathrm{DL}(V)_{ell}$. We assert that there is a canonical choice of an orientation o_s of $V/V(s)$ (we write $V(s) = V(j(s))$). If $\dim V > 1$ the group $\mathrm{DL}(V)$ is connected and there exists $S \in \mathfrak{gl}(V)_{ell}$ such that $s = \exp S$. We write

$$V = V(s) \oplus V_1(s).$$

Then S induces elliptic invertible transformations of $V(s)/V(S)$ and of $V_1(s)$. Thus $\dim(V(s)/V(S)) = 2p$ and $\dim V_1(s) = 2q$. We choose a basis $e_1, e_2, \dots, e_{2p-1}, e_{2p}$ of $V(s)/V(S)$, a basis $f_1, f_2, \dots, f_{2q-1}, f_{2q}$ of $V_1(s)$ and real numbers α_i, λ_j such that

$$\begin{aligned} Se_{2i-1} &= \lambda_i e_{2i}, \\ Se_{2i} &= -\lambda_i e_{2i-1}, \\ Sf_{2j-1} &= \alpha_j f_{2j}, \\ Sf_{2j} &= -\alpha_j f_{2j-1}. \end{aligned}$$

As $V(s) = V(e^S)$, the real numbers λ_i belong to $2\pi\mathbb{Z}$, so that $\cos(\lambda_i/2) = \pm 1$, while the α_j do not belong to $2\pi\mathbb{Z}$. Let o be an orientation of $V/V(s) = V_1(s)$. Assume that $f_1, f_2, \dots, f_{2q}$ is of orientation o. Define

$$c(S, o) = \prod_{i=1}^{p} \cos(\lambda_i/2) \prod_{j=1}^{q} \sin(-\alpha_j/2).$$

Then $c(S,o)^2 = \det_{V/V(s)}(1-e^S)$. Furthermore, as $e^S = 1$ in $\mathrm{DL}(V)$ if and only if $V_1(s) = 0$ and $\sum \lambda_i \in 4\pi\mathbb{Z}$, the number $c(S,o)$ depends only of $s = e^S$. Thus we define for $s \in \mathrm{DL}(V)$:

$$D^{1/2}(s,o) = c(S,o). \tag{13}$$

Remark that $D^{1/2}(s\epsilon, o) = -D^{1/2}(s,o)$ so that $D^{1/2}(s,o)$ cannot be defined for $s \in \mathrm{GL}^+(V)_{ell}$. We then obtain

Lemma 11 *Let $s \in \mathrm{DL}(V)_{ell}$. Then the space $V/V(s)$ has a canonical orientation o_s defined by $D^{1/2}(s, o_s) > 0$.*

Let us also remark that for $s \in \mathrm{DL}(V)$ and $g \in \mathrm{GL}^+(V)$ the element gsg^{-1} is well defined. If s is elliptic, $o_{gsg^{-1}} = g \cdot o_s$.

If $\mathcal{V} \to M$ is an oriented vector bundle with a G-action, we say that $\mathcal{V}$ is G-oriented if G preserves the orientations of the fibers. If $\mathcal{V}$ is G-oriented, the frame bundle $GL^+(\mathcal{V})$ is a G-equivariant $\mathrm{GL}^+(V)$-principal bundle.

Definition 12 *Let $\mathcal{V} \to M$ be a G-oriented vector bundle over M with frame bundle $GL^+(\mathcal{V})$. We will say that $\mathcal{V}$ admits a G-invariant metalinear structure if there is a G-equivariant two-fold cover P of $GL^+(\mathcal{V})$ which is a principal bundle with structure group* $\mathrm{DL}(V)$.

In particular a spin structure or a metaplectic structure on $\mathcal{V}$ provides a metalinear structure.

If $s \in G_{ell}$, we denote by $\mathcal{V}(s)$ the set of fixed points of the action of s on $\mathcal{V}$. As s is elliptic, this is a vector bundle over the submanifold $M(s)$ of M.

Proposition 13 *Let $\mathcal{V} \to M$ be a G-oriented vector bundle with a metalinear structure P. Let $s \in G_{ell}$. Then the bundle $\mathcal{V}(s) \to M(s)$ is $G(s)$-oriented.*

Proof. Let $m \in M(s)$. Let $p : V \to \mathcal{V}_m$ be a frame of $\mathcal{V}_m$. Let $\tilde{p}$ be an element of P above p. Then $s \cdot \tilde{p} = \tilde{p} s_D$ with $s_D \in \mathrm{DL}(V)$. Let $\tilde{s} = p s_D p^{-1} \in \mathrm{DL}(\mathcal{V}_m)$. By the remark following lemma 11 the element $\tilde{s}$ is well defined. It determines an orientation on $\mathcal{V}_m/\mathcal{V}_m(s)$ independent of the choice of p. As $\mathcal{V}$ itself is oriented, we obtain in this way an orientation on $\mathcal{V}(s) \to M(s)$. ∎

Our main interest will be the tangent bundle TM. Then $T(M(s)) = (TM)(s)$. Thus we see that if a group G acts on M and leaves invariant a metalinear structure (for example a spin structure) then fixed point submanifolds of elliptic transformations are canonically oriented. Many more properties are true, as is pointed out in the proof by R. Bott and C. Taubes [17] of Witten's rigidity theorem [51].

Let V_1 be a real vector space. Let $V = V_1 \oplus V_1^*$. Consider the symplectic form on V given by

$$B(x_1 + f_1, x_2 + f_2) = f_2(x_1) - f_1(x_2)$$

for $x_1, x_2 \in V_1,\ f_1, f_2 \in V_1^*$. Consider the homomorphism $I(g) = (g,^t g^{-1})$ of $\mathrm{GL}(V_1)$ into $\mathrm{Sp}(V, B)$. It is well known that the restriction of I to $\mathrm{SL}(V_1)$ lifts in a homomorphism $\tilde{I}$ from $\mathrm{SL}(V_1)$ to $\mathrm{Mp}(V, B) \subset \mathrm{DL}(V)$. However the homomorphism I itself does not lift. Thus we have the following lemma.

Lemma 14 *Let $\mathcal{V}_1 \to M$ be a G-equivariant vector bundle. Consider the vector bundle $\mathcal{V} = \mathcal{V}_1 \oplus \mathcal{V}_1^*$ with its natural G-action. Then $\mathcal{V}$ admits a G-invariant metalinear structure if and only the bundle $\mathcal{V}_1$ admits a G-invariant orientation.*

In fact, we do not want to assume the existence of an invariant metalinear structure. In this case, we have to modify the notion of G-equivariant vector bundle into the notion of G-equivariant quantum bundle. Our modification is a generalization of the notion of quantum line bundles introduced by J. Rawnsley and P. Robinson [42].

Let W be a Hermitian space. Let $U(W)$ be the group of unitary transformations of W. We denote by $-I$ the transformation $w \to -w$ of W. We embed $\mathbb{Z}/2\mathbb{Z}$ as a central subgroup Z in $\mathrm{DL}(V) \times U(W)$ by sending $(-1) \in \mathbb{Z}/2\mathbb{Z}$ to $(\epsilon, -I) \in \mathrm{DL}(V) \times U(W)$. Let

$$\mathrm{DL}^W(V) = (\mathrm{DL}(V) \times U(W))/Z$$

be the quotient group. We refer to this group as the metalinear group with coefficients in W. By definition there is a canonical homomorphism still denoted by

$$j : \mathrm{DL}(V) \times U(W) \to \mathrm{DL}^W(V).$$

We have canonical homomorphisms

$$f : \mathrm{DL}^W(V) \to \mathrm{GL}^+(V)$$

and
$$u : \mathrm{DL}^W(V) \to U(W)/\pm I$$
obtained respectively by projecting an element of $(DL(V) \times U(W))/Z$ to its first and second components.

Let us return to the example where V is the symplectic space $V = V_1 \oplus V_1^*$. Let $I(g) = (g, {}^t g^{-1})$ the homomorphism of $\mathrm{GL}(V_1)$ into $\mathrm{Sp}(V, B)$. If $s \in \mathrm{GL}(V_1)_{ell}$, it follows from formula 12 of paragraph 2.1 that $V(s) = V(I(s))$ is canonically isomorphic to $V_1(s) \oplus V_1(s)^*$. Thus $V(s)$ has a canonical orientation given by its symplectic structure. We denote by o_B the quotient orientation on $V/V(s)$. Let $\mathrm{DL}^{\mathbb{C}}(V)$ be the metalinear group with coefficients in $\mathbb{C}$. Choose any element $s^D \in \mathrm{DL}(V)$ above $I(s)$. Then the element $j(s^D, \mathrm{sign}\, D^{1/2}(s^D, o_B) i^{\dim(V_1/V_1(s))})$ of $\mathrm{DL}^{\mathbb{C}}(V)$ depends only of s. We denote it by $\ell(s)$.

Let us state for later use the following lemma.

Lemma 15 *There exists a unique homomorphism $h : \mathrm{GL}(V_1) \to \mathrm{DL}^{\mathbb{C}}(V)$ such that*
$$h(g) = j(\tilde{I}(g), 1) \qquad \text{for } g \in \mathrm{SL}(V_1)$$
and
$$h(s) = \ell(s) \qquad \text{for } s \in \mathrm{GL}(V_1)_{ell}.$$

Let $P \to M$ be a principal bundle with structure group $\mathrm{DL}^W(V)$. The homomorphism f defines an associated principal bundle P^f with structure group $\mathrm{GL}^+(V)$. The homomorphism u defines an associated principal bundle P^u with structure group $U(W)/\pm I$.

Definition 16 *Let $\mathcal{V} \to M$ be a G-oriented real vector bundle. A G-equivariant quantum bundle for $\mathcal{V}$ is a G-equivariant principal bundle $\tau : P \to M$ with structure group $\mathrm{DL}^W(V)$ such that the associated bundle P^f with structure group $\mathrm{GL}^+(V)$ is the frame bundle $GL^+(\mathcal{V})$ of $\mathcal{V}$.*

Let $\mathcal{V}$ be the tangent bundle to M. In this case we will say that $\tau : P \to M$ is a G-equivariant quantum bundle, or just a quantum bundle if G is understood. The space W will be called the fiber of τ. In particular, if $\dim W = 1$ we will say that τ is a quantum line bundle.

Definition 17 *We denote by $K_G^t(M)$ the set of G-equivariant quantum bundles (up to isomorphism).*

The letter t indicates the tangent bundle. The definition above is related to the groups $K_G^{TM}(M)$ introduced by Karoubi [34] when G is compact.

Lemma 18 *If $TM \to M$ has a G-invariant metalinear structure $\tilde{P}$, the set $K_G^t(M)$ of G-equivariant quantum bundles is in one-to-one correspondence with the set of G-equivariant Hermitian vector bundles over M.*

If $\mathcal{W}$ is a Hermitian bundle with typical fiber W and frame bundle $U(\mathcal{W})$, we define $P = (\tilde{P} \times_M U(\mathcal{W}))/Z$.

Let $M = G/H$ be an homogeneous space of a real Lie group G with a G-invariant orientation. We can construct the two fold cover

$$\tilde{H}_M = \{(h,g) \in H \times \mathrm{DL}(\mathfrak{g}/\mathfrak{h}); Ad_{\mathfrak{g}/\mathfrak{h}} h = j(g)\} \tag{14}$$

of H. We still denote by ε the element $(1,\epsilon)$ of $\tilde{H}_M$. A representation τ of $\tilde{H}_M$ in a space W is said to be genuine if $\tau(\epsilon) = -I$.

Definition 19 *We denote by $K^M(H)$ the set of genuine finite dimensional unitary representations of $\tilde{H}_M$ (up to isomorphism).*

A representation $\tau \in K^M(H)$ determines a G-equivariant quantum bundle still denoted by τ. Indeed let W be the representation space of τ. For $h \in H$ choose $s(h) \in \mathrm{DL}(\mathfrak{g}/\mathfrak{h})$ above the transformation $Ad_{\mathfrak{g}/\mathfrak{h}}(h)$. Then the map $h \to j(s(h), \tau(h, s(h)))$ gives us an homomorphism of H in $\mathrm{DL}(\mathfrak{g}/\mathfrak{h}, W)$. The principal bundle $P(G/H, \tau) = G \times_H \mathrm{DL}^W(V)$ is a G-equivariant quantum bundle over M. This construction induces an isomorphism

$$K^M(H) \cong K_G^t(G/H).$$

Let $\lambda \in \mathfrak{g}^*$ with stabilizer H. Let $M = G \cdot \lambda = G/H$. Then the group $\tilde{H}_M$ is the two-fold cover of the group $G(\lambda)$ introduced by Duflo [22].

The lemma 15 implies the following complement to lemma 14.

Lemma 20 *Let $\mathcal{V}_1 \to M$ be a G-equivariant vector bundle. Consider the vector bundle $\mathcal{V} = \mathcal{V}_1 \oplus \mathcal{V}_1^*$ with its natural G-action. Then $\mathcal{V}$ admits a canonical G-equivariant quantum line bundle.*

Let $\tau : P \to M$ be a G-equivariant quantum bundle. If the associated principal bundle P^u with structure group $U(W)/\pm 1$ admits a G-invariant connection, we will say that τ is a G-equivariant quantum bundle with G-invariant Hermitian connection.

Definition 21 *We denote by $\mathcal{Q}_G(M)$ the set of G-equivariant quantum bundles with Hermitian connection (up to isomorphism).*

2.2 Equivariant cohomology and descent

The notion of equivariant quantum bundles is strongly related to the notion of descent and of admissible bouquets of equivariant differential forms. Let us introduce now some definitions. This is a slightly simplified version of the notions given in [28]. Similar notions are introduced for compact groups in [16].

Let G be a real algebraic group acting on a manifold M. If $s \in G_{ell}$, the set $M(s)$ is a submanifold of M, as s is contained in a compact subgroup of G. If $S \in \mathfrak{g}$ is elliptic, we denote by $M(S) = \{m \in M; (S_M)_m = 0\}$ the manifold of zeros of the vector field S_M.

Definition 22 *A bouquet of equivariant differential forms on M is a family $(\alpha_s)_{s\in G_{ell}}$ where each $\alpha_s \in \mathcal{A}^{\infty}_{G(s)}(\mathfrak{g}(s), M(s))$ is a closed $G(s)$-equivariant form. Furthermore the family α_s satisfies the following conditions:*

1. *Invariance:*
$$\alpha_{gsg^{-1}} = g \cdot \alpha_s$$
for all $g \in G$ and $s \in G_{ell}$.

2. *Compatibility: Let $s \in G_{ell}$, then for all $S \in \mathfrak{g}(s)$* elliptic *and* sufficiently small
$$\alpha_{se^S}(Y) = \alpha_s(S+Y)|M(se^S)$$
for all $Y \in \mathfrak{g}(se^S)$.

Remark 2.1 *If $S \in \mathfrak{g}(s)$ is sufficiently small then $M(se^S) = M(s) \cap M(S)$ and $\mathfrak{g}(se^S) = \mathfrak{g}(s) \cap \mathfrak{g}(S)$ so that the right hand-side of the equality (2) has a meaning.*

Definition 23 *We denote by $\mathcal{Z}_G(M)$ the space of bouquets of equivariant forms.*

When G and M are compacts, the quotient space of $\mathcal{Z}_G(M)$ by the subspace of exact forms is equal to the equivariant cyclic homology of M [16].

If G is a real algebraic group and M a point $\bullet$, then $\mathcal{Z}_G(\bullet) = C^\infty(G)^G$. This is seen as follows. To a C^∞-function ϕ on G, we associate the family $\alpha(\phi)_s$ of functions on $\mathfrak{g}(s)$ given for $X \in \mathfrak{g}(s)$ by

$$\alpha(\phi)_s(X) = \phi(se^X).$$

As by polar decomposition, any element g of a real algebraic group G can be written $g = s \exp H$, with s elliptic and $H \in \mathfrak{g}(s)$ (and hyperbolic), it follows easily that the map $\phi \to \alpha(\phi)$ is an isomorphism.

Let us give an important example of bouquet. Let $\mathcal{E}$ be a G-equivariant vector bundle with a G-invariant connection $\mathbb{A}$. Let $F(X)$ be the equivariant curvature of $\mathbb{A}$. Over $M(s)$ the action $s^{\mathcal{E}}$ of s on $\mathcal{E}|M(s)$ is a transformation that we still denote by $s^{\mathcal{E}}$ (or simply s) acting fiberwise. Then $\mathrm{bch}(\mathcal{E}, \mathbb{A}) = (\mathrm{ch}_s(\mathcal{E}, \mathbb{A}))_{s \in G_{ell}}$ where

$$\mathrm{ch}_s(\mathcal{E}, \mathbb{A})(X) = \mathrm{Tr}(s^{\mathcal{E}} e^{F(X)|M(s)}) \quad \text{pour } X \in \mathfrak{g}(s) \tag{15}$$

is a bouquet of equivariant forms which we call the bouquet of Chern characters.

We will need to integrate over the submanifolds $M(s)$. Thus we need to produce densities on $M(s)$ rather than differential forms. We will now see that the Chern character of equivariant quantum bundles produces such families.

If M is a manifold, we introduce the two-fold cover $M_{or} = \{(m, o)\}$ of M, where $m \in M$ and o is an orientation of $T_m M$. If o is a local orientation of M, a differential form α on M_{or} gives us a local differential form α_o on M. We say that α is a *folded differential form* on M if α is a differential form on M_{or} such that $\alpha_o = -\alpha_{-o}$. The term of maximum exterior degree of a folded differential form α is a density on M. We can then define $\int_M \alpha$. If M is orientable with orientation o, then $\int_M \alpha = \int_{M,o} \alpha_o$.

We introduce now the notion of *admissible bouquets* of equivariant differential forms on M. Let $s \in G_{ell}$. Let $S \in \mathfrak{g}(s)_{ell}$. Let $m \in M(s) \cap M(S)$. The normal space $N_m = T_m M(s)/T_m(M(s) \cap M(S))$ is an even dimensional space and has a canonical orientation o_S (definition 10). If (o, o') are orientations of the tangent bundle $TM(s)$ and of $T(M(s) \cap M(S))$ at $m \in M(s) \cap M(S)$, we write $\mathrm{sign}(S, o, o') = \pm 1$ depending on whether the orientations o, o', o_S are compatible or not.

Definition 24 *An admissible bouquet of equivariant differential forms is a family $(\alpha_s)_{s \in G_{ell}}$ where $\alpha_s \in \mathcal{A}^{\infty}_{G(s)}(\mathfrak{g}(s), M(s)_{or})$ is a folded closed $G(s)$-equivariant form on $M(s)$. Furthermore we assume that the family α_s satisfy:*

1. *Invariance:*

$$\alpha_{gsg^{-1}} = g \cdot \alpha_s$$

for all $g \in G$, $s \in G_{ell}$.

2. *Compatibility: Let $s \in G_{ell}$, then for all $S \in \mathfrak{g}(s)$ elliptic and sufficiently small*

$$\alpha_{se^S,o'}(Y) = \mathrm{sign}(-S,o,o')\alpha_{s,o}(S+Y)|M(se^S)$$

for all $Y \in \mathfrak{g}(se^S)$, o, o' local orientations of $TM(s), TM(se^S)$.

We denote by $\mathcal{Z}^t_G(M)$ the space of admissible bouquets.

Assume that M is G-oriented, with orientation o_M. Let $\tau : P \to M$ be a G-equivariant quantum bundle over M with fiber W and G-invariant Hermitian connection $\mathbb{A}$. If $s \in G_{ell}$, $m \in M(s)$ and p is an element of P above m, we denote by $g(p,s)$ the element $g(p,s) \in \mathrm{DL}^W(V)$ such that $sp = pg(p,s)$. Let $(s^D, s^W) \in \mathrm{DL}(V) \times U(W)$ such that $j(s^D,s^W) = g(p,s)$. Let o be a local orientation of $M(s)$. Recall (lemma 11) that s^D determines an orientation on $V/V(s)$. We write $\mathrm{sign}(s^D, o_M, o) = \pm 1$ depending on whether the orientations o_{s^D}, o_M, o are compatible or not. Define for $X \in \mathfrak{g}(s)$

$$\mathrm{Tr}_{o,W}(s; e^{F(X)}) = \mathrm{sign}(s^D, o_M, o)\,\mathrm{Tr}(s^W e^{F(X)|M(s)}).$$

Here we have identified locally P^u to $M \times (U(W)/\pm I)$. The Lie algebra of $(U(W)/\pm I)$ is $\mathfrak{su}(W) \subset End(W)$ so that in local coordinates $F(X)$ is (as in the case of vector bundles) a matrix of differential forms. The function $\mathrm{Tr}_{o,W}(s; e^{F(X)})$ is a differential form on $M(s)_{or}$.

Definition 25 *The Chern character* $\mathrm{bch}(\tau, \mathbb{A})$ *of the equivariant quantum bundle τ with G-invariant connection $\mathbb{A}$ is the family of folded equivariant differential forms*

$$\mathrm{ch}_{s,o}(\tau,\mathbb{A})(X) = \mathrm{Tr}_{o,W}(s; e^{F(X)|M(s)}).$$

Proposition 26 *The Chern character* $\mathrm{bch}(\tau, \mathbb{A})$ *of a G-equivariant quantum bundle with G-invariant connection $\mathbb{A}$ is an admissible bouquet.*

Remark 2.2 *If (M, o_M) is an oriented manifold with a metalinear structure, then there is an isomorphism $I : \mathcal{Z}_G(M) \to \mathcal{Z}^t_G(M)$ such that $I(\alpha)_{s,o} =$* $\mathrm{sign}(s, o_M, o)\alpha_s$ *for $\alpha \in \mathcal{Z}_G(M)$.*

2.3 Integration of admissible families

Let G be a real algebraic group acting on a manifold M. When G and M are compacts, we have defined in [28] a direct image (or integration) map

$$\int_b : \mathcal{Z}_G^t(M) \to C^\infty(G)^G.$$

To formulate the conjecture (F) when G and M are not necessarily compact, we need to construct an integration map

$$\int_b : \mathcal{Z}_G^t(M) \to C^{-\infty}(G)^G.$$

We are able to extend the notion of integration $\int_b$ only in some cases of G-manifolds. Let us recall some definitions:

Definition 27 *Let $\mathcal{V} \to M$ be a G-equivariant real vector bundle over M with a G-equivariant connection of curvature $R(X)$. Let $s \in G_{ell}$. Assume that s acts trivially on M (however s acts on $\mathcal{V}$). Define :*

$$D_s(\mathcal{V}, \nabla)(X) = \det\left(1 - s^{\mathcal{V}} e^{R(X)}\right)$$

for $X \in \mathfrak{g}(s)$.

The $G(s)$-equivariant form $D_s(\mathcal{V}, \nabla)$ is a closed equivariant form on M. We say that $s \in G_{ell}$ is an elliptic non-degenerate transformation of $\mathcal{V}$ if $\mathcal{V}(s) = 0$. Remark that $D_s(\mathcal{V}, \nabla)^{[0]}(0)$ is > 0 if s is non-degenerate.

We will have to take square roots of the forms $J(\mathcal{V}, \nabla)$ and $D_s(\mathcal{V}, \nabla)$. For example if $\mathcal{V}$ can be provided with a G-invariant Euclidean or pseudo Euclidean structure and with a G-invariant Euclidean or pseudo Euclidean connection ∇, then we can define the form $J^{1/2}(\mathcal{V}, \nabla) \in \mathcal{A}_G^{\mathrm{hol}}(\mathfrak{g}, M)$. We normalize it by $J^{1/2}(\mathcal{V})^{[0]}(0) = 1$. Similarly, if s acts trivially on M and produces a non-degenerate transformation of $\mathcal{V}$, there exists a square root $D_s^{1/2}(\mathcal{V}, \nabla) \in \mathcal{A}_{G(s)}^{\mathrm{hol}}(\mathfrak{g}(s), M)$ and we normalize it such that $D_s^{1/2}(\mathcal{V}, \nabla)^{[0]}(0) > 0$.

Let G be a compact Lie group acting on a compact manifold M. Then we can choose a G-invariant Euclidean structure and G-invariant Euclidean connection ∇ on the tangent bundle $TM \to M$. Let $s \in G$. The connection ∇ induces connections on $T(M/M(s)) \to M(s)$ and $TM(s) \to M(s)$. The action of s on $T(M/M(s)) \to M(s)$ is non degenerate and allows us to construct the equivariant form $D_s^{1/2}(M/M(s)) = D_s^{1/2}(T(M/M(s)), \nabla)$. This form is invertible if $X \in \mathfrak{g}$ is sufficiently small.

Theorem 28 *Let G be a compact Lie group acting on a compact manifold M. Let $\alpha = (\alpha_s)_{s\in G} \in \mathcal{Z}^t_G(M)$. There exists a unique G-invariant C^∞ function $\Theta(\alpha)$ on G such that, for all $s \in G$ and all $X \in \mathfrak{g}(s)$ sufficiently small,*

$$\Theta(\alpha)(se^X) = \int_{M(s)} (2\pi)^{-\dim M(s)/2} \frac{\alpha_s(X)}{D_s^{1/2}(M/M(s))(X) J^{1/2}(M(s))(X)}.$$

Note that α_s is a folded form on $M(s)$ and so the integral is well defined even if $M(s)$ is not orientable.

Remark 2.3 *The importance of admissible bouquets appears clearly in this theorem. Indeed this is our main motivation for introducing the notion of admissible bouquets: if we try to define a global function $\Theta(\alpha)$ on G by the set of formulas above, we obtain two formulas for $\Theta(se^{S+X}) = \Theta((se^S)e^X)$ if $S \in \mathfrak{g}(s)_{ell}$ and if $X \in \mathfrak{g}(s) \cap \mathfrak{g}(S)$, one given by integration over $M(s)$ and the second by integration over $M(s) \cap M(S)$. The localization formula [10] prescribes the condition (2) in the definition 24 of admissible bouquets for the two formulas to be compatible.*

The function $\Theta(\alpha)$ is determined by its pointwise evaluation

$$\Theta(\alpha)(s) = \int_{M(s)} (2\pi)^{-\dim M(s)/2} \frac{\alpha_s(0)}{D_s^{1/2}(M/M(s))(0) J^{1/2}(M(s))(0)}.$$

However, it seems difficult to see a priori on this pointwise formula that $\Theta(\alpha)(s)$ depends smoothly on s as the dependence of $M(s)$ on s can be quite chaotic.

We denote by $\int_b \alpha$ the function $\Theta(\alpha)$. Then we have defined a map

$$\int_b : \mathcal{Z}^t_G(M) \to C^\infty(G)^G.$$

The definition of this integration map is modeled on Atiyah-Hirzebruch "integration" map in K-theory. The next theorem [28] is an easy consequence of Atiyah-Segal-Singer theorem.

Theorem 29 *If $\mathcal{E}$ is a G-equivariant vector bundle over a spin manifold M, then*

$$\operatorname{Tr} Q(M, \mathcal{E}) = i^{-\dim M/2} \int_b \operatorname{bch}(\mathcal{E}, \mathbb{A}).$$

In the above formula, we have identified $\operatorname{bch}(\mathcal{E}) \in \mathcal{Z}_G(M)$ with an element of $\mathcal{Z}^t_G(M)$ with the help of the spin structure (remark 2.2). Atiyah-Segal-Singer formula is the fixed point formula above for $X = 0$:

$$(16) \qquad \operatorname{Tr} Q(M,\mathcal{E})(s) = i^{-\dim M/2} \int_{M(s)} \frac{(2\pi)^{-\dim M(s)/2} \operatorname{ch}_s(\mathcal{E},\mathbb{A})(0)}{D_s^{1/2}(M/M(s))(0) J^{1/2}(M(s))(0)}.$$

In the case of non degenerate fixed points, it coincides with the fixed point formula (4) given in 1.3.

In the particular case of the bouquet of Chern characters $\operatorname{bch}(\mathcal{E},\mathbb{A})$, we know that the result of integration $\int_b \operatorname{bch}(\mathcal{E},\mathbb{A})$ is a global C^∞ function on G as it is the trace of the finite dimensional representation $Q(M,\mathcal{E})$ of G. However this is not apparent on the fixed point formula (16). It is a consequence of the fact that the equivariant forms $\operatorname{ch}_s(\mathcal{E},\mathbb{A})$ on $M(s)$ satisfy the compatibility condition (2) of the definition 24.

Let us now consider a Lie group G acting on a manifold M. Let $(\tau,\mathbb{A})$ be a quantum bundle over M with a G-invariant Hermitian connection $\mathbb{A}$. If the associated quantized representation $Q(M,\tau,\mathbb{A})$ has a trace, the trace of $Q(M,\tau,\mathbb{A})$ is a generalized function on G. Thus to define it, we cannot define its value at a point $g \in G$ but we can define its restriction to an open neighborhood of g. We will here assume that $G \subset \mathrm{GL}(V)$ is a real algebraic group. Then using polar decompositions, we will cover the group G by well adapted neighborhoods in order to give such formulas.

Let $a \in \mathbb{R}$ be a strictly positive real number and let $\mathfrak{g}_a$ be the set of $X \in \mathfrak{g}$ such that the imaginary part $Im\lambda$ of any eigenvalue λ of the transformation $X \in \operatorname{End}(V)$ satisfies $|Im\lambda| < a$. Recall that the exponential map is a diffeomorphism of $\mathfrak{g}_a$ on an neighborhood of the identity in G if a is small. If G is compact, the open sets $\mathfrak{g}_a, a \in \mathbb{R}$, form a system of neighborhoods of 0. At the opposite if $G \subset \mathrm{GL}(V)$ is unipotent, then $\mathfrak{g}_a = \mathfrak{g}$ for every $a > 0$.

Consider for a small real number a the set $\mathcal{W}_{s,a} = \{u(s\exp X)u^{-1}; u \in G, X \in \mathfrak{g}(s)_a\}$. It is a G-invariant open set in G. If Θ is a G-invariant generalized function on G, then $\Theta(u(s\exp X)u^{-1})$ is constant on u so that we can define the restriction $X \to \Theta(s\exp X)$ as a generalized function of $X \in \mathfrak{g}(s)_a$.

Every element $g \in G$ has a polar decomposition $g = s\exp H$ where s is elliptic and $H \in \mathfrak{g}(s)$ has only real eigenvalues. Thus $H \in \mathfrak{g}(s)_a$ for all positive real numbers a. It follows that for any choice of $a(s) > 0$ we have

$$G = \bigcup_{s \in G_{ell}} \mathcal{W}_{s,a(s)}.$$

For some "good bouquets" $\alpha \in \mathcal{Z}_G^t(M)$, we will also be able to define $\int_b \alpha \in C^{-\infty}(G)^G$ by the formula of theorem 28: for $X \in \mathfrak{g}(s)_a$

(17)
$$(\int_b \alpha)(s \exp X) = \int_{M(s)} (2\pi)^{-\dim M(s)/2} \frac{\alpha_s(X)}{D_s^{1/2}(M/M(s))(X) J^{1/2}(M(s))(X)}.$$

As in paragraph 1.6, there are two problems:

(a) when G is not compact, there might not exist a G-invariant connection on TM.

(b) it is not clear how the right hand side defines a generalized function on a neighborhood of 0 in $\mathfrak{g}(s)$.

There is an additional problem. In order to give a global formula on $G = \cup \mathcal{W}_{s,a(s)}$, we need to give a formula valid on a neighborhood of 0 in $\mathfrak{g}(s)$ of the form $\mathfrak{g}(s)_a$ (recall that these neighborhoods of 0 are rather big if G is not compact).

Let $M \subset V$ be a closed G-invariant real algebraic submanifold of the linear representation space V of G.

For $s \in G_{ell}$, let $V = V_0 \oplus V_1$ with $V_0 = V(s)$ and $V_1 = (1-s)V$. Let $M(s) = M_0$. We have $M_0 \subset V_0$. Denote by $\mathcal{M}_1$ the normal bundle $T(M/M_0)$. It is a subbundle of the bundle $M_0 \times V_1$. Let us denote by $\mathcal{V}_1/\mathcal{M}_1$ the quotient bundle:

$$0 \to \mathcal{M}_1 \to M_0 \times V_1 \to \mathcal{V}_1/\mathcal{M}_1 \to 0.$$

Let $\alpha \in \mathcal{Z}_G^t(M)$. As in paragraph 1.6, it is natural to consider instead of the formula (17) for $\int_b \alpha$ the formula: for $X \in \mathfrak{g}(s)_a$

$$\det\nolimits_{V_1}^{1/2}(1 - se^X) J_{V_0}^{1/2}(X)(\int_b \alpha)(s \exp X) =$$
$$\int_{M(s)} (2\pi)^{-\dim M(s)/2} \alpha_s(X) D_s^{1/2}(V_1/M_1)(X) J^{1/2}(V_0/M_0)(X)$$

where $D_s(V_1/M_1) = D_s(\mathcal{V}_1/\mathcal{M}_1)$ and $J(V_0/M_0) = J(T(V_0/M_0))$. Remark now that, for a sufficiently small, and if $X \in \mathfrak{g}(s)_a$, then $\det_{V_1}(1 - se^X) > 0$

and $J_{V_0}(X) > 0$. Thus, if we can define the right hand side as a generalized function on $\mathfrak{g}(s)$, we obtain a formula for $(\int_b \alpha)(s \exp X)$ for $X \in \mathfrak{g}(s)_a$.

Consider first the case where $T(V/M)$ is a trivial G-equivariant bundle $M \times \mathbb{R}^N$.

Lemma 30 *If $T(V/M)$ is a trivial G-equivariant bundle, then $T(M/M_0) = M_0 \times V_1$ and $T(V_0/M_0)$ is a trivial $G(s)$-equivariant vector bundle.*

Proof. Over a point $m \in M(s)$, the decomposition of $T(V/M)$ with respect to the action of s shows that $T(V/M)|M_0 = T(V_0/M_0)$ and that $T(M/M_0) = M_0 \times V_1$.

In this case $\mathcal{V}_1/\mathcal{M}_1 = 0$ and the above conjectural formula becomes much simpler.

Conjecture:

Assume M is a real algebraic closed G-invariant submanifold of V. Assume $T(V/M)$ admits $(\dim V - \dim M)$ G-invariant sections. Let $\alpha \in \mathcal{Z}^t_G(M)$ be a admissible bouquet of weakly integrable forms . Then there is a G-invariant generalized function $\int_b \alpha \in C^{-\infty}(G)^G$ such that for every $s \in G_{ell}$ and $X \in \mathfrak{g}(s)_a$ (a small)

$$\det{}^{1/2}_{V_1}(1 - se^X) J^{1/2}_{V_0}(X)(\int_b \alpha)(s \exp X) = \int_{M(s)} (2\pi)^{-\dim M(s)/2} \alpha_s(X).$$

As $G = \bigcup_{s \in G_{ell}} \mathcal{W}_{s,a(s)}$, the (generalized) function $\int_b \alpha$ if it exists is unique. The existence of the global function $\int_b \alpha$ is equivalent to a conjectural localization formula for the non compact group G (however with respect to elliptic elements of $\mathfrak{g}$). If G is compact and M is a vector bundle over a compact base, then the above conjecture holds. If $M = G/T$ where G is a real semi-simple Lie group G and T a Cartan subgroup of G, these are the descent formulas of Harish-Chandra. Then, with the help of the powerful results of Harish-Chandra on invariant eigen-distributions, it is possible to prove this conjecture for many weakly integrable bouquets α as we will explain in paragraph 2.4.4

Let us return to the general case. Assume that the normal bundle $T(V/M)$ of M in V admits a G-invariant connection. In this case, the bundles $\mathcal{V}_1/\mathcal{M}_1$ and $T(V_0/M_0)$ have $G(s)$-invariant connections. We also assume that the forms $J^{1/2}(V_0/M_0)$, $D^{1/2}_s(V_1/M_1)$ exist in $\mathcal{A}^{\mathrm{hol}}_{G(s)}(\mathfrak{g}(s)_{\mathbb{C}}, M(s))$. Then we define $N_s(X) = D^{1/2}_s(V_1/M_1)(X) J^{1/2}(V_0/M_0)(X)$ and we may be able to treat the integral of the form $\alpha_s(X)N_s(X)$ over $M(s)$ by the procedure of formula (11). Namely we define

$$\int_{M(s)} \alpha_s(X)[N_s](X)$$

as the limit when $k \to \infty$ of

$$\sum_{j=0}^{k} \int_{M(s)} \alpha_s(X)N_{s,j}(X)$$

where $N_{s,j}$ is the homogeneous component of N_s for the total equivariant grading (8). We will give examples in paragraph 2.4.4 where indeed we can treat integration of some admissible bouquets by this pharmacopoeia.

2.4 The universal formula

In the preceding paragraph, for some "good" admissible bouquets $\alpha \in \mathcal{Z}_G^t(M)$, we were able to give a meaning to $\int_b \alpha$. Furthermore we constructed a map $\mathrm{bch} : \mathcal{Q}_G(M) \to \mathcal{Z}_G^t(M)$ by taking bouquets of Chern characters. In view of the theorem 29, we then conjecture:

Conjecture:

Let G be a real algebraic group acting on an even dimensional oriented real algebraic manifold M. Let $(\tau, \mathbb{A}) \in \mathcal{Q}_G(M)$ be a G-equivariant quantum bundle with connection. Then there exists a quantized representation $Q(M, \tau, \mathbb{A}) \in \mathrm{Rep}^{\pm}(G)$. If $Q(M, \tau, \mathbb{A})$ is trace-class, then

$$\operatorname{Tr} Q(M, \tau, \mathbb{A}) = i^{-\dim M/2} \int_b \mathrm{bch}(\tau, \mathbb{A}). \tag{18}$$

This formula is a fixed point formula: it gives a formula for the character of $Q(M, \tau, \mathbb{A})$ near a point g as an integral formula over the fixed submanifold $M(s)$ under the elliptic part s of $g = s \exp H$. However the formula given near purely hyperbolic elements $\exp H$ involves integration on all M and the result is usually not equal to zero even if $\exp H$ acts freely on M.

I now return to the examples and indicate the meaning of the universal formula in these cases.

2.4.1 Points

Let $M = \bullet$ and E be a representation space of G. The formula is tautological: the bouquet $\mathrm{bch}(E)$ is the family of functions on $\mathfrak{g}(s)$ given by

$$\mathrm{ch}_s(X) = \mathrm{Tr}_E(s \exp X).$$

The integral of this bouquet is clearly the function $g \to \mathrm{Tr}_E(g)$.

2.4.2 Cotangent bundles

Let us generalize proposition 6. Let G be a compact Lie group acting on a compact manifold B. Let $M = T^*B$. We consider (lemma 20) the canonical G-equivariant quantum line bundle with connection $(\mathcal{L}, \mathbb{A})$. If B is G-oriented, then TM has a G-invariant metalinear structure (14) and $(\mathcal{L}, \mathbb{A})$ corresponds to the trivial line bundle 14 (lemma 18).

If $s \in G$, the fixed point set $M(s)$ of the action of s on M is canonically isomorphic to $T^*B(s)$. Let α_s be the canonical 1-form on $T^*B(s)$. As $T^*B(s)$ is canonically oriented by its symplectic form, we identify folded differential forms on $T^*B(s)$ with ordinary differential forms. The bouquet of Chern characters of the quantum line bundle $(\mathcal{L}, \mathbb{A})$ is the family

$$\mathrm{ch}_s(\mathcal{L}, \mathbb{A}) = i^{(\dim B - \dim B(s))} e^{i d_{\mathfrak{g}(s)} \alpha_s}.$$

As explained in 1.5, example 2, for every element $s \in G$, the form $J^{1/2}(M(s))$ is defined and is the lift of the form $J(B(s))$. In particular, $J(B(s))(X)$ is invertible for $X \in \mathfrak{g}(s)$ small. Similarly $D_s^{1/2}(M/M(s)) = D_s^{1/2}(T(M/M(s)))$ is equal to the lift to $M(s)$ of the form $D_s(B/B(s)) = D_s(T(B/B(s)))$ on $B(s)$.

Assume that B is homogeneous under the action of G. Then for each test function ϕ on $\mathfrak{g}(s)$, $\int_{\mathfrak{g}(s)} \mathrm{ch}_s(\mathcal{L}, \mathbb{A})(X)\phi(X)dX$ is a form on $T^*B(s)$ which is rapidly decreasing in the fiber direction and

$$\int_{M(s)} \left(\int_{\mathfrak{g}(s)} \frac{\mathrm{ch}_s(\mathcal{L}, \mathbb{A})(X)}{D_s(B/B(s))(X) J(B(s))(X)} \phi(X) dX \right)$$

is well defined if ϕ is supported in a sufficiently small neighborhood of 0.

Proposition 31 *Assume B homogeneous under G. There exists a unique G-invariant $C^{-\infty}$ function Θ on G such that for all $s \in G$ and all $X \in \mathfrak{g}(s)$*

sufficiently small,

$$\Theta(se^X) = \int_{M(s)} (2i\pi)^{-\dim B(s)} \frac{e^{id_{\mathfrak{g}(s)}\alpha_s(X)}}{D_s(B/B(s))(X)J(B(s))(X)}.$$

Here $M(s) = T^*B(s)$ has its canonical orientation.

Thus we have

$$\Theta = i^{-\dim B} \int_b \mathrm{bch}(\mathcal{L}, \mathbb{A})$$

for the generalized function on G so obtained.

Theorem 32 *[14] Assume that B is homogeneous under the action of the compact Lie group G. Then the character of the representation L of G in $L^2(B)$ is given by*

$$\mathrm{Tr}\, L = i^{-\dim B} \int_b \mathrm{bch}(\mathcal{L}, \mathbb{A}).$$

2.4.3 The character of the Weil representation

The universal formula applies to the archetype of the quantization map: the Weil representation. Let (V, B) be a symplectic vector space and let G be the metaplectic group. It is the inverse image of the group $\mathrm{Sp}(B)$ in $\mathrm{DL}(V)$. The trivial bundle $\tau = V \times \mathrm{DL}(V)$ with G-action $g_0 \cdot (v, g) = (g_0 \cdot v, g_0 g)$ is a quantum bundle (with right action of $\mathrm{DL}(V)$ on the second factor) The associated line bundle $\mathcal{L}$ is the trivial line bundle $V \times \mathbb{C}$. We thus write informally $(\tau, \mathbb{A}) = (\mathcal{L}, \mathbb{A})$ with $\mathbb{A} = d + i\alpha = d + iB(v, dv)/2$. Let $s \in G_{ell}$. Let $V = V_0 \oplus V_1$ with $V_0 = V(s)$, $V_1 = V_1(s)$. The symplectic form B induces an orientation o_B on V_1. Recall that $s \in G$ induces also an orientation on V_1. We write $\mathrm{sign}(o_s, o_B) = \pm 1$ according on whether the orientations o_s and o_B coincide or not. We identify folded differential forms on $V(s)$ with ordinary differential forms by the symplectic orientation. Considering the formula 15, we obtain that the bouquet of Chern characters of $(\mathcal{L}, \mathbb{A})$ is given by

$$\mathrm{ch}_s(\mathcal{L}, \mathbb{A}) = \mathrm{sign}(o_s, o_B) e^{id_{\mathfrak{g}(s)}\alpha|V(s)}.$$

In view of the conjectural formula 18 it is natural to consider the generalized function Θ_s on $\mathfrak{g}(s)_a$ determined by

$$
\begin{aligned}
&\det\nolimits_{V_1}^{1/2}(1-se^X)J_{V_0}^{1/2}(X)\Theta_s(X)\\
&= i^{-\dim V/2}\operatorname{sign}(o_s,o_B)\int_{V_0}(2\pi)^{-\dim V_0/2}e^{iB(v,Xv)/2+iB(dv,dv)/2}\\
&= i^{-\dim V_1/2}\operatorname{sign}(o_s,o_B)\int_{V_0}e^{iB(v,Xv)/2}d\beta_{V_0}.
\end{aligned}
$$

Theorem 33 *Let W be the Weil representation of the metaplectic group. Then for $X \in \mathfrak{g}(s)_a$, a small,*

$$\operatorname{Tr} W(s\exp X) = \Theta_s(X).$$

This formula is easily verified using for example the method of [24]. It would be however useful to have a direct proof of this simple formula.

The trace of the representation W is a locally sommable function, analytic in the open set $\det_V(1-g) \neq 0$. In particular the formula of theorem 33 describes this function.

Proposition 34 *Let $g \in G$ such that $\det_V(1-g) \neq 0$. Let $g = s\exp H$ be the polar decomposition of g. Let $V = V_0 \oplus V_1$ the canonical decomposition of V produced by s. Then*

$$\operatorname{Tr} W(g) = i^{-\dim V_1/2}\operatorname{sign}(o_s,o_B)|\det\nolimits_V(1-g)|^{-1/2}.$$

2.4.4 The orbit method

Let M be a symplectic manifold. Let Ω be the symplectic form on M. Let G be a Lie group acting on M by an Hamiltonian action with moment map μ. Let $\Omega(X) = \mu(X) + \Omega$ be the equivariant symplectic form.

Let W be a Hermitian vector space. We denote by $E = iI_W \in \mathfrak{su}(W)$.

Definition 35 *Let $(\tau, \mathbb{A})$ be a G-equivariant quantum bundle over M with G-invariant Hermitian connection. We say that τ is an admissible bundle for (Ω, μ) if the equivariant curvature $F(X)$ of $\mathbb{A}$ is equal to $i(\mu(X) + \Omega)E$. We denote by $\mathcal{Q}_G(M, \Omega, \mu)$ the space of G-equivariant quantum bundles admissible for (Ω, μ).*

If M has a G-invariant metaplectic structure then there is a one to one correspondence between quantum line bundles admissible for (Ω, μ) and Kostant-Souriau prequantization data: G-equivariant Hermitian line bundles over M with connection of equivariant curvature $i\Omega(X)$. Our definition of quantum bundles admissible for (Ω, μ) is a slight generalization of the notion of quantum line bundles introduced by J. Rawnsley and P. Robinson [42]. However we admit quantum bundles with arbitrary fiber dimension. Each element $(\tau, \mathbb{A}) \in \mathcal{Q}_G(M, \Omega, \mu)$ should give rise to a possible way to quantize the classical Hamiltonian space (M, Ω, μ) in a representation $Q(M, \tau, \mathbb{A})$ of G in a Hilbert space $H(M, \tau, \mathbb{A})$.

Let $\lambda \in \mathfrak{g}^*$. Let $H = G(\lambda)$ and let $M = G/H$. In paragraph 2.1, we have identified $K_G^t(M)$ with $K^M(H)$. The map $g \to g \cdot \lambda$ gives us a Hamiltonian structure μ on M (depending on $\lambda \in \mathfrak{g}^*$).

Definition 36 *Let*

$$X(\lambda) = \{\tau \in K^M(H); \tau(\exp X) = e^{i\lambda(X)} I, \text{for } X \in \mathfrak{h}\}.$$

Recall that in Duflo terminology [23] an orbit $G\cdot\lambda$ is admissible if $X(\lambda)$ is non empty. If $\tau \in X(\lambda)$, then τ determines a quantum bundle still denoted by τ on M. It is remarkable [39] that indeed this quantum bundle has a *unique* G-invariant connection $\mathbb{A}$ of equivariant curvature $F(X) = i(\mu(X) + \Omega)$. Thus for $M = G \cdot \lambda$ the set $X(\lambda)$ is isomorphic with the set $\mathcal{Q}_G(M, \Omega, \mu)$.

In the rest of this paragraph, I consider the case where $M = G \cdot \lambda$ is an orbit of the coadjoint representation. If $s \in G_{ell}$, then $M(s)$ is a symplectic submanifold of M. Let us write $\mathfrak{z} = \mathfrak{g}(s)$ and $\mathfrak{q} = (1 - s)\mathfrak{g}$. We have a direct sum decomposition

$$\mathfrak{g} = \mathfrak{z} \oplus \mathfrak{q}.$$

The submanifold $M(s)$ is a finite union of coadjoint orbits of $G(s)$. We denote by $\beta_{M(s)}$ its Liouville form.

Let $\tau \in X(\lambda)$. Let us first describe a scalar valued function c_s on $M(s)$ (depending on τ). Let $f \in M(s)$. Let $g \in G$ such that $g\cdot\lambda = f$, then $g^{-1}sg \in H$. Let $(g^{-1}sg, u) \in \tilde{H}_M$ covering $g^{-1}sg$. The element u is in $DL(V)$ where V is the vector space $\mathfrak{g}/\mathfrak{g}(\lambda)$. The symplectic form $B_\lambda(X, Y) = -\lambda([X, Y])$ on $\mathfrak{g}/\mathfrak{g}(\lambda) = V$ is the canonical symplectic form on $T_\lambda M$. It gives a canonical orientation to V (this convention differs from the one of [25]). Similarly the space $V/V(u)$ is a symplectic space and $V/V(u)$ has a canonical orientation

o_λ. The element $u \in DL(V)$ determines the orientation o_u on $V/V(u)$. We define a locally constant function c_s on $M(s)$ by

$$c_s(f) = \mathrm{sign}(o_u, o_\lambda)\,\mathrm{Tr}(\tau(u)).$$

This is independent of $g \in G$ such that $g \cdot \lambda = f$ and of u above $g^{-1}sg$.

We denote also by τ the element of $\mathcal{Q}_G(M, \Omega, \mu)$ determined by τ. As $\mathbb{A}$ is unique, we write $\mathrm{bch}(\tau)$ for the bouquet of Chern characters $\mathrm{bch}(\tau, \mathbb{A})$. It is an admissible bouquet. We give to the submanifolds $M(s)$ their orientation by their symplectic structure. Thus $\mathrm{bch}(\tau)$ is a family of differential forms on $M(s)$. Clearly we have for $X \in \mathfrak{g}(s)$

$$\mathrm{ch}_s(\tau)(X) = c_s e^{i\Omega(X)|M(s)}.$$

Consider $M \subset \mathfrak{g}^*$. We have $M(s) \subset \mathfrak{z}^*$. We denote $J(\mathfrak{z}^*/M(s)) = J(T(\mathfrak{z}^*/M(s)))$. The normal bundle $T(M/M(s))$ is a subbundle of $M(s) \times \mathfrak{q}^*$. We denote by $T(\mathfrak{q}^*/M_1)$ the quotient bundle

$$T(\mathfrak{q}^*/M_1) = (M(s) \times \mathfrak{q}^*)/T(M/M(s))$$

and by $D_s(\mathfrak{q}^*/M_1) = D_s(T(\mathfrak{q}^*/M_1))$. The classes $J(\mathfrak{z}^*/M(s))$ and $D_s(\mathfrak{q}^*/M_1)$ can at present be defined only under additional assumptions.

In view of the conjectural formula for $\int_b$ the conjectural formula for $\mathrm{Tr}\, Q(M, \tau)$ becomes: for $X \in \mathfrak{z}_a$ and a small

$$|J_{\mathfrak{z}}^{1/2}(X) \det{}_{\mathfrak{q}}^{1/2}(1 - se^X)|\,\mathrm{Tr}\, Q(M, \tau)(s \exp X) =$$
$$i^{-\dim M/2} \int_{M(s)} (2\pi)^{-\dim M(s)/2}\, \mathrm{bch}(\tau)_s(X)[J^{1/2}(\mathfrak{z}^*/M(s))(X) D_s^{1/2}(\mathfrak{q}^*/M_1)(X)].$$

In particular, if $M = G \cdot \lambda$ is a coadjoint orbit of G in $\mathfrak{g}^*$ of maximal dimension, the normal bundle $T(\mathfrak{g}^*/M)$ of the embedding of M in $\mathfrak{g}^*$ is a trivial G-equivariant bundle. Recall (lemma 30) that this implies that $T(\mathfrak{z}^*/M(s))$ is a trivial $G(s)$-equivariant bundle and that $T(\mathfrak{q}^*/M_1) = 0$.

Let $\tau \in X(\lambda)$. The universal formula (F) becomes the conjectured formula of [25] for the character of the representation $\mathrm{Tr}\, T_{\lambda,\tau}$:

Duflo-Heckmann-Vergne conjecture:

Let $M = G \cdot \lambda$ be a closed coadjoint orbit of maximal dimension. Let $\tau \in X(\lambda)$. Then for every $s \in G_{ell}$, we have for $X \in \mathfrak{z}_a$ and a small,

$$|J_{\mathfrak{z}}^{1/2}(X)\det{}_{\mathfrak{q}}^{1/2}(1-se^X)|\operatorname{Tr} T_{\lambda,\tau}(s\exp X) \\ = i^{(\dim M(s)-\dim M)/2}\int_{M(s)} c_s(f)e^{if(X)}d\beta_{M_s}(f).$$

This conjecture has been proved in [25], [18] for closed orbits of maximal dimension of a real reductive group. Thus the universal formula (F) holds at least for the reduced dual of a semi-simple Lie group G.

Let $M = G\cdot\lambda$ be a closed coadjoint orbit of a real reductive group G. Let $H = G(\lambda)$. Then H is a reductive subgroup of G. Let $\tau \in X(\lambda)$. Then, with the help of Zuckerman functor, we can associate to τ a $(\mathfrak{g}, K)$-module $Z(\lambda,\tau)$. If λ is sufficiently large (among the set of λ with $G(\lambda) = H$), then $Z(\lambda,\tau)$ is an unitary irreducible representation of G (see [50]). Consider the quantum bundle $(\tau,\mathbb{A})$ associated to τ and the bouquet of Chern characters $\operatorname{bch}(\tau)$. For each equivariant form $\alpha \in \mathcal{A}_{G(s)}(\mathfrak{g}(s), M(s))$ with polynomial coefficients, $\operatorname{ch}_s(\tau)\alpha$ is a weakly integrable form on $M(s)$. The space $M(s)$ is a finite union of closed coadjoints orbits under $G(s)$. Thus

$$\mathcal{H}^{\mathrm{hol}}_{G(s)}(\mathfrak{g}(s), M(s)) = C^{\mathrm{hol}}(\mathfrak{g}(s)_{\mathbb{C}})^{G(s)}\emptyset_{C[\mathfrak{g}(s)]^{G(s)}}\mathcal{H}_{G(s)}(\mathfrak{g}(s), M(s))$$

and for each $\alpha \in \mathcal{H}^{\mathrm{hol}}_{G(s)}(\mathfrak{g}(s), M(s)) = C^{\mathrm{hol}}(\mathfrak{g}(s)_{\mathbb{C}})^{G(s)}$ we can define $\int_{M(s)}\operatorname{ch}(\tau)_s[\alpha]$ by the formula 11.

We say that an element λ is elliptic if $G\cdot\lambda$ admits a G-invariant complex structure.

Theorem 37 *[29]. Let $M = G\cdot\lambda$ be an elliptic orbit of a connected real semi-simple Lie group G. Let $\tau \in X(\lambda)$. Let $Z(\lambda,\tau)$ be the virtual $(\mathfrak{g}, K)$-module associated to λ. Let for $s \in G_{ell}$, $\mathfrak{z} = \mathfrak{g}(s)$ and $\mathfrak{q} = (1-s)\mathfrak{g}$. Then the character of the $(\mathfrak{g}, K)$-module $Z(\lambda,\tau)$ is entirely determined by the descent formulas: for $X \in \mathfrak{z}_a$ and a small*

$$|J_{\mathfrak{z}}^{1/2}(X)\det{}_{\mathfrak{q}}^{1/2}(1-se^X)|\operatorname{Tr} Z(\lambda,\tau)(s\exp X) = \\ i^{(\dim M(s)-\dim M)/2}\int_{M(s)}(2\pi)^{-\dim M(s)/2}\operatorname{ch}(\tau)_s(X)[J^{1/2}(\mathfrak{z}^*/M(s))(X)D_s^{1/2}(\mathfrak{q}^*/M_1)(X)].$$

The same formula for the character of $Z(\lambda,\tau)$ should hold whenever G is connected or not.

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