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Projectively Dual Varieties

Preface

During several centuries various reincarnations of projective duality have inspired research in algebraic and differential geometry, classical mechanics, invariant theory, combinatorics, etc. On the other hand, projective duality is simply the systematic way of recovering the projective variety from the set of its tangent hyperplanes. In this survey we have tried to collect together different aspects of projective duality and points of view on it. We hope, that the exposition is quite informal and requires only a standard knowledge of algebraic geometry and algebraic (or Lie) groups theory. Some chapters are, however, more difficult and use the modern intersection theory and homology algebra. But even in these cases we have tried to give simple examples and avoid technical difficulties.

An interesting feature of projective duality is given by the observation that most important examples carry the natural action of the Lie group. This is especially true for projective varieties that have extremal properties from the point of view of projective geometry. We have tried to stress this phenomenon in this survey and to discuss many variants of it. However, one aspect is completely omitted – we are not discussing the dual varieties of toric varieties and the corresponding theory of A-discriminants. This theory is presented in the beautiful book [GKZ2] and we feel no need to reproduce it.

Parts of this survey were written during my visits to the Erwin Shroedinger Institute in Vienna and Mathematic Institute in Basel. I would like to thank my hosts for the warm hospitality. I have discussed the contents of this book with many people, including E. Vinberg, V. Popov, A. Kuznetsov, S. Keel, H. Kraft, D. Timashev, D. Saltman, P. Katsylo, and learned a lot from them. I am especially grateful to F. Zak for providing a lot of information on projective duality and other aspects of projective geometry.

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Table of Contents

1. Dual Varieties 1
- 1.1 Definitions and First Properties 1
- 1.2 Reflexivity Theorem 4
  - 1.2.A The Conormal Variety 4
  - 1.2.B Applications of the Reflexivity Theorem 6
- 1.3 Dual Plane Curves 8
  - 1.3.A Parametric Representation of the Dual Plane Curve . . 8
  - 1.3.B The Legendre Transformation and Caustics 9
  - 1.3.C Correspondence of Branches. Plücker Formulas. 10
- 1.4 Projections and Linear Normality 12
  - 1.4.A Projections 12
  - 1.4.B Linear Normality 14
- 1.5 Dual Varieties of Smooth Divisors 17

2. Dual Varieties of Algebraic Group Orbits 21
- 2.1 Polarized Flag Varieties 21
  - 2.1.A Definitions and Notations 21
  - 2.1.B Basic Examples 23
- 2.2 The Pyasetskii Pairing 30
  - 2.2.A Actions With Finitely Many Orbits 30
  - 2.2.B The Multisegment Duality. 32
- 2.3 Parabolic Subgroups With Abelian Unipotent Radical 34

3. The Cayley Method for Studying Discriminants 41
- 3.1 Jet Bundles and Koszul Complexes 41
- 3.2 Cayley Determinants of Exact Complexes 43
- 3.3 Discriminant Complexes 46

4. Resultants and Schemes of Zeros 51
- 4.1 Ample Vector Bundles 51
- 4.2 Resultants 52
- 4.3 Zeros of Generic Global Sections 54
- 4.4 Moore–Penrose Inverse and Applications 61

5. Fulton–Hansen Theorem and Applications 71
5.1 Fulton–Hansen Connectedness Theorem 71
5.2 Secant and Tangential Varieties 74
5.3 Zak Theorems 76

6. Dual Varieties and Projective Differential Geometry 81
6.1 The Katz Dimension Formula 81
6.2 Product Theorem and Applications 84
6.2.A Product Theorem 84
6.2.B Hyperdeterminants 87
6.2.C Associated Hypersurfaces 88
6.3 Ein Theorems 89
6.4 The Projective Second Fundamental Form 96
6.4.A Gauss Map 96
6.4.B Moving Frames 98
6.4.C Fundamental Forms of Projective Homogeneous Spaces 101

7. The Degree of a Dual Variety 105
7.1 Katz–Kleiman–Holme Formula 105
7.2 Degrees of Discriminants of Polarized Flag Varieties 108
7.2.A The Degree of the Dual Variety to G/B 108
7.2.B Degrees of Hyperdeterminants 110
7.2.C One Calculation 111
7.3 Degree of the Discriminant as a Function in $\mathcal{L}$ 116
7.3.A General Positivity Theorem 117
7.3.B Applications to Polarized Flag Varieties 118
7.4 Gelfand–Kapranov Formula 119

8. Milnor Classes and Multiplicities of Discriminants 121
8.1 Class Formula 121
8.2 Milnor Classes 122
8.3 Applications to Multiplicities of Discriminants 124
8.4 Multiplicities of the Dual Variety of a Surface 126
8.5 Further Applications to Dual Varieties 128

9. Mori Theory and Dual Varieties 133
9.1 Some Results From Mori Theory 133
9.2 Mori Theory and Dual Varieties 138
9.2.A The Nef Value and the Defect 138
9.2.B The Defect of Fibers of the Nef Value Morphism 140
9.2.C Varieties With Small Dual Varieties 142
9.3 Polarized Flag Varieties With Positive Defect 144
9.3.A Nef Value of Polarized Flag Varieties 144
9.3.B Classification 147

10. Some Applications of the Duality 151
10.1 Discriminants and Automorphisms 151
10.1.A Matsumura–Monsky Theorem 151
10.1.B Quasiderivations of Commutative Algebras 153
10.2 Discriminants of Anticommutative Algebras 157
10.3 Self–dual Varieties 168
10.3.A Self–dual Polarized Flag Varieties 168
10.3.B Around Hartshorne Conjecture 170
10.3.C Self–dual Nilpotent Orbits 172
10.4 Linear Systems of Quadrics of Constant Rank 172

References 178

Index 187

1. Dual Varieties

Preliminaries

The projective duality gives a remarkably simple method to recover any projective variety from the set of its tangent hyperplanes. In this chapter we recall this classical notion, give the proof of the Reflexivity Theorem and its consequences, provide a number of examples and motivations, and fix the notation to be used throughout the book. The exposition is fairly standard and classical. In the proof of the Reflexivity Theorem we follow [GKZ2] and deduce this result from the classical theorem of symplectic geometry saying that any conical Lagrangian subvariety of the cotangent bundle is equal to some conormal variety.

1.1 Definitions and First Properties

For any finite-dimensional complex vector space V we denote by $\mathbb{P}(V)$ its projectivization, that is, the set of 1-dimensional subspaces. For example, if $V = \mathbb{C}^{n+1}$ is a standard complex vector space then $\mathbb{P}^n = \mathbb{P}(\mathbb{C}^{n+1})$ is a standard complex projective space. A point of $\mathbb{P}^n$ is defined by $(n+1)$ homogeneous coordinates $(x_0 : \ldots : x_1)$, $x_i \in \mathbb{C}$, which are not all equal to 0 and are considered up to a scalar multiple.

If $U \subset V$ is a non-trivial linear subspace then $\mathbb{P}(U)$ is a subset of $\mathbb{P}(V)$, subsets of this form are called projective subspaces. Projective subspaces of dimension 1, 2, or of codimension 1 are called lines, planes, and hyperplanes.

For any vector space V we denote by V^* the dual vector space, the vector space of linear forms on V. Points of the dual projective space $\mathbb{P}(V)^* = \mathbb{P}(V^*)$ correspond to hyperplanes in $\mathbb{P}(V)$. Conversely, to any point p of $\mathbb{P}(V)$, we can associate a hyperplane in $\mathbb{P}(V)^*$, namely the set of all hyperplanes in $\mathbb{P}(V)$ passing through p. Therefore, $\mathbb{P}(V)^{**}$ is naturally identified with $\mathbb{P}(V)$. Of course, this reflects nothing else but a usual canonical isomorphism $V^{**} = V$.

To any vector subspace $U \subset V$ we associate its annihilator $\mathrm{Ann}(U) \subset V^*$. Namely, $\mathrm{Ann}(U) = \{f \in V^* \mid f(U) = 0\}$. We have $\mathrm{Ann}(\mathrm{Ann}(U)) = U$. This corresponds to the projective duality between projective subspaces in $\mathbb{P}(V)$ and $\mathbb{P}(V)^*$: for any projective subspace $L \subset \mathbb{P}(V)$ we denote by $L^* \subset \mathbb{P}(V)^*$ its dual projective subspace, parametrizing all hyperplanes that contain L.

Remarkably, the projective duality between projective subspaces in $\mathbb{P}^n$ and $\mathbb{P}^{n*}$ can be extended to the involutive correspondence between irreducible algebraic subvarieties in $\mathbb{P}^n$ and $\mathbb{P}^{n*}$.

First, suppose that $X \subset \mathbb{P}^n$ is a smooth irreducible algebraic subvariety. For any $x \in X$, we denote by $\hat{T}_x X \subset \mathbb{P}^n$ an embedded projective tangent space. More precisely, if $X \in \mathbb{P}(V)$ is any projective variety then we define the cone $\mathrm{Cone}(X) \subset V$ over it as a conical variety formed by all lines l such that $\mathbb{P}(l) \in X$. If $x \in X$ is a smooth point then any non-zero point v of the corresponding line is a smooth point of $\mathrm{Cone}(X)$ and $\hat{T}_x(X)$ is defined as $\mathbb{P}(T_v \mathrm{Cone}(X))$, where $T_v \mathrm{Cone}(X)$ is a tangent space of $\mathrm{Cone}(X)$ at v considered as a linear subspace of V (it does not depend on a choice of v). For any hyperplane $H \subset \mathbb{P}^n$, we say that H is tangent to X at x if H contains $\hat{T}_x X$. We define the dual variety $X^* \subset \mathbb{P}^{n*}$ as the set of all hyperplanes tangent to X.

In other words, a hyperplane H belongs to X^* if and only if the intersection $X \cap H$ (regarded as a scheme) is singular (is not a smooth algebraic variety). In most parts of this book we shall be interested only in dual varieties of smooth varieties, so this description of X^* will be sufficient. However, with this definition we can not expect the duality $X^{**} = X$ because X^* can be singular (and in most interesting cases it is actually singular). So we should define X^* for a singular X as well. There are in fact two possibilities: first, we can imitate the previous definition and consider embedded tangent spaces at all points, not necessarily smooth. But it turns out that the dual variety defined in this fashion does not have good properties, for example it can be reducible. The better way is to pick only the 'main' component of the dual variety.

Definition 1.1 Let $X \subset \mathbb{P}^n$ be an irreducible projective variety. A hyperplane $H \subset \mathbb{P}^n$ is called tangent to X if it contains an embedded tangent space $\hat{T}_x X$ at some smooth point $x \in X$. The closure of the set of all tangent hyperplanes is called the dual variety $X^* \subset \mathbb{P}^{n*}$.

We shall discuss the Reflexivity Theorem $X^{**} = X$ and its consequences in the next section. First we shall establish some simple properties of dual varieties and give further definitions.

Definition 1.2 Let $X \subset \mathbb{P}^n$ be an irreducible projective variety with the smooth locus X_{sm}. Consider the set $I_X^0 \subset \mathbb{P}^n \times \mathbb{P}^{n*}$ of pairs (x, H) such that $x \in X_{sm}$ and H is the hyperplane tangent to X at x. The Zariski closure I_X of I_X^0 is called the conormal variety of X.

The projection $\mathrm{pr}_1 : I_X^0 \to X_{sm}$ makes I_X^0 into a bundle over X_{sm} whose fibers are projective subspaces of dimension $n - \dim X - 1$. Therefore, I_X^0 and I_X are irreducible varieties of dimension $n - 1$. By definition, X^* is the image of the projection $\mathrm{pr}_2 : I_X \to \mathbb{P}^{n*}$. Therefore, we have the following proposition:

Proposition 1.3 *X^* is an irreducible variety.*

Moreover, since $\dim I_X = n-1$, we can expect that in 'typical' cases X^* is a hypersurface. Having this in mind, we give the following definition:

Definition 1.4 The number $\operatorname{codim}_{\mathbb{P}^{n*}} X^* - 1$ is called the defect of X, denoted by $\operatorname{def} X$.

Typically, $\operatorname{def} X = 0$. In this case X^* is defined by an irreducible homogeneous polynomial Δ_X.

Definition 1.5 Δ_X is called the discriminant of X.

If $\operatorname{def} X > 0$ then, for convenience, we set $\Delta_X = 1$. Clearly, Δ_X is defined only up to a scalar multiple. Roughly speaking, the study of dual varieties and discriminants includes the following 3 steps:

– To define some nice natural class of projective varieties X.
– To find all exceptional cases when $\operatorname{def} X > 0$.
– In the remaining cases, to say something about Δ_X or X^*.

In the last step the minimal program is to determine the degree of Δ_X, the maximal program is to determine Δ_X as a polynomial, or at least to describe its monomials. Another interesting problem is to describe singularities of X^*.

Example 1.6 The most familiar example of a discriminant Δ_X is, of course, the discriminant of a binary form. In order to show that it actually coincides with some Δ_X we need first to give an equivalent definition of Δ_X. Suppose that $x_1, \ldots, x_k$ are some local coordinates on $\operatorname{Cone}(X) \subset V$. Any $f \in V^*$, a linear form on V, being restricted to $\operatorname{Cone}(X)$ becomes an algebraic function in $x_1, \ldots, x_k$. Then Δ_X is just an irreducible polynomial, which vanishes at $f \in V^*$ whenever the function $f(x_1, \ldots, x_k)$ has a multiple root, that is, vanishes at some $v \in \operatorname{Cone}(X)$, $v \neq 0$, together with all first derivatives $\partial f / \partial x_i$.

Consider now the d-dimensional projective space $\mathbb{P}^d = \mathbb{P}(V)$ with homogeneous coordinates $z_0, \ldots, z_d$, and let $X \subset \mathbb{P}^d$ be the Veronese curve

$$(x^d : x^{d-1}y : x^{d-2}y^2 : \ldots : xy^{d-1} : y^d),\ x, y \in \mathbb{C},\ (x, y) \neq (0, 0)$$

(the image of the Veronese embedding $\mathbb{P}^1 \subset \mathbb{P}^d$). Any linear form $f(z) = \sum a_i z_i$ is uniquely determined by its restriction to the cone $\operatorname{Cone}(X)$, which is a binary form $f(x, y) = \sum a_i x^{d-i} y^i$. Therefore, $f \in \operatorname{Cone}(X^*)$ if and only if $f(x, y)$ vanishes at some point $(x_0, y_0) \neq (0, 0)$ (so $(x_0 : y_0)$ is a root of $f(x, y)$) with its first derivatives (so $(x_0 : y_0)$ is a multiple root of $f(x, y)$). It follows that Δ_X is the classical discriminant of a binary form.

1.2 Reflexivity Theorem

In this section we shall prove the Reflexivity Theorem. We follow the exposition in [GKZ2], which shows that there exists a deep connection between projective duality and symplectic geometry. Other proofs, including the investigation of a prime characteristic case, could be found, e.g., in [Se], [M], [Wa].

Theorem 1.7

(a) *For any irreducible projective variety $X \subset \mathbb{P}^n$, we have $X^{**} = X$.*
(b) *More precisely, If z is a smooth point of X and H is a smooth point of X^*, then H is tangent to X at z if and only if z, regarded as a hyperplane in $\mathbb{P}^{n*}$, is tangent to X^* at H.*

The proof will be given in the next section.

1.2.A The Conormal Variety

We shall need some standard definitions.

Definition 1.8 If X is a smooth algebraic variety, then TX denotes the tangent bundle of X. If $Y \subset X$ is a smooth algebraic subvariety then TY is a subbundle in $TX|_Y$. The quotient $TX|_Y/TY$ is called the normal bundle of Y in X, denoted by $N_Y X$. By taking dual bundles we obtain the cotangent bundle T^*X and the conormal bundle $N_Y^* X$. The conormal bundle can be naturally regarded as a subvariety of T^*X.

Recall that $I_X \subset \mathbb{P}^n \times \mathbb{P}^{n*}$ is the conormal variety and the dual variety X^* coincides with $\mathrm{pr}_2(I_X)$. The projection $\mathrm{pr}_1 : I_X^0 \to X_{sm}$ is a projective bundle, where pr_1, pr_2 denote the projections of $\mathbb{P}^n \times \mathbb{P}^{n*}$ to its factors. More precisely, pr_1 identifies I_X^0 with the projectivization $\mathbb{P}(N^*_{X_{sm}}\mathbb{P}^n)$ of the conormal bundle $N^*_{X_{sm}}\mathbb{P}^n$. Indeed, the choice of a hyperplane $H \subset \mathbb{P}^n$ tangent to X_{sm} at x is equivalent to the choice of a hyperplane T_xH in the tangent space $T_x\mathbb{P}^n$, which contains T_xX. The equation of this hyperplane is an element of $N^*_{X_{sm}}\mathbb{P}^n$ at x. The Reflexivity Theorem can be reformulated as follows:

$$I_X = I_{X^*}. \tag{1.1}$$

It is more convenient to prove (1.1) by working with vector spaces instead of projective spaces.

We assume that $\mathbb{P}^n = \mathbb{P}(V)$ and $\mathbb{P}^{n*} = \mathbb{P}(V^*)$. Then we have affine cones $Y = \mathrm{Cone}(X) \subset V$ and $Y^* = \mathrm{Cone}(X^*) \subset V^*$. We denote by $\mathrm{Lag}(Y)$ the closure of the conormal bundle $N^*_{Y_{sm}}V$ in the cotangent bundle T^*V.

The space T^*V is canonically identified with $V \times V^*$. Denote by pr_1, pr_2 the projections of this product to its factors. Then Y^* coincides with $\mathrm{pr}_2(\mathrm{Lag}(Y))$. Therefore, (1.1) can be reformulated as follows:

$$\mathrm{Lag}(Y) = \mathrm{Lag}(Y^*), \tag{1.2}$$

where we identify T^*V and T^*V^* with $V \times V^*$.

Recall that a smooth algebraic variety M is called the symplectic variety, if M admits a symplectic structure, that is, a differential 2-form ω with the following properties:

– ω is closed, $d\omega = 0$.
– ω is non-degenerate, for any $p \in M$ the restriction of ω on T_pM is a non-degenerate skew product.

In this case $\dim M$ is necessarily even. An irreducible closed subvariety $\Lambda \subset M$ is called Lagrangian if $\dim \Lambda = \dim M/2$ and the restriction of ω to the smooth locus Λ_{sm} vanishes as a 2-form (is totally isotropic).

The most important and the most typical (having in mind the Darboux Theorem) example of a symplectic variety is the cotangent bundle T^*X of a smooth algebraic variety X. T^*X carries a canonical symplectic structure defined as follows. Let $(x_1, \dots, x_n)$ be a local coordinate system in X. Let ξ_i be the fiberwise linear function on T^*X given by the pairing of a 1-form with the vector field $\partial/\partial x_i$. Then $(x_1, \dots, x_n, \xi_1, \dots, \xi_n)$ forms a local coordinate system in T^*X. The form ω is defined by

$$\omega = \sum_{i=1}^{n} d\xi_i \wedge dx_i. \tag{1.3}$$

It easy to give an equivalent definition of ω without any coordinate systems. We shall define a canonical 1-form ν on T^*X and then we shall take $\omega = d\nu$. Let $\pi : T^*X \to X$ be the canonical projection. Let $p \in T^*X$ and $v \in T_p(T^*X)$ be a vector tangent to T^*X at p. Then $\nu(v) = p(\pi_* v)$.

An important (and typical, as we shall see) example of a Lagrangian subvariety in T^*X is obtained as follows. Let $Y \subset X$ be any irreducible subvariety with smooth locus Y_{sm} and let

$$\mathrm{Lag}(Y) = \overline{N^*_{Y_{sm}} X}$$

be the closure of the conormal bundle of Z_{sm} (in T^*X). Clearly, $\mathrm{Lag}(Y)$ is a conical subvariety (invariant under dilations of fibers of T^*X).

Theorem 1.9

(a) $\mathrm{Lag}(Y)$ *is a Lagrangian subvariety.*
(b) *Any conical Lagrangian subvariety has the form* $\mathrm{Lag}(Y)$ *for some irreducible subvariety* $Y \subset X$.

Proof. Let us show that $\mathrm{Lag}(Y)$ is Lagrangian. Clearly, $\dim \mathrm{Lag}(Y) = \dim Y + (\dim X - \dim Y) = \dim T^*X/2$. So we need only to verify that $\omega|_{\mathrm{Lag}(Y)} = 0$. It is sufficient to consider the smooth locus of Y only. Let

$x_1, \ldots, x_n$ be a local coordinate system on X such that Y is locally defined by equations $x_1 = \ldots = x_r = 0$. Then the fibers of the conormal bundle over points of Y are generated by 1-forms $dx_1, \ldots, dx_r$. Hence $\xi_{r+1} = \ldots = \xi_n = 0$ on $N^*_{Y_{sm}}X$ and by (1.3) we see that $\omega = 0$ on $N^*_{Y_{sm}}X$. Therefore, $\mathrm{Lag}(Y)$ is indeed Lagrangian.

Suppose now that $\Lambda \subset T^*X$ is a conical Lagrangian subvariety. We take $Y = \mathrm{pr}(\Lambda)$, where $\mathrm{pr} : T^*X \to X$ is the projection, and claim that $\Lambda = \mathrm{Lag}(Y)$. It suffices to show that $\Lambda \subset \mathrm{Lag}(Y)$, because Λ and $\mathrm{Lag}(Y)$ are irreducible varieties of the same dimension. In turn, to prove that $\Lambda \subset \mathrm{Lag}(Y)$ it suffices to check that for any smooth point $y \in Y$ the fiber $\mathrm{pr}^{-1}(y) \cap \Lambda$ is contained in the conormal space $(N^*_Y X)_y$. Let ξ be any covector from $\mathrm{pr}^{-1}(y) \cap \Lambda$. Since $T^*_y X$ is a vector space, we can regard ξ as a "vertical" tangent vector to T^*X at a point $y \subset X \subset T^*X$, where we identify X with the zero section of pr. Since Λ is conical, $\xi \in T_\xi \Lambda$. Therefore, since Λ is Lagrangian, ξ is orthogonal with respect to ω to any tangent vector from $T_\xi \Lambda$ and, hence, to any tangent vector $v \in T_y Y$. But by (1.3) it is easy to see that this is equivalent to $\xi \in (N^*_Y X)_y$. □

Now we can prove the Reflexivity Theorem. We shall prove (1.2). The identification $T^*V = V \times V^* = T^*V^*$ takes the canonical symplectic structure on T^*V to minus the canonical symplectic structure on T^*V^*. Therefore $\mathrm{Lag}(Y)$ regarded as a subvariety of T^*V^* is still Lagrangian. Moreover, clearly $\mathrm{Lag}(Y) \subset V \times V^*$ is invariant under dilations of V and V^*, therefore, $\mathrm{Lag}(Y)$ is a conical Lagrangian variety of T^*V^*. Therefore, by Theorem 1.9 $\mathrm{Lag}(Y) = \mathrm{Lag}(Z)$, where Z is the projection of $\mathrm{Lag}(Y)$ on V^*. But this projection coincides with Y^*. Therefore, $\mathrm{Lag}(Y) = \mathrm{Lag}(Y^*)$. The Reflexivity Theorem is proved.

1.2.B Applications of the Reflexivity Theorem

Suppose that $X \subset \mathbb{P}^n$ and $X^* \subset \mathbb{P}^{n*}$ are projectively dual varieties, $I_X = I_{X^*} \subset \mathbb{P}^n \times \mathbb{P}^{n*}$ is the conormal variety. We have the diagram of projections

$$X \xleftarrow{\mathrm{pr}_1} I_X \xrightarrow{\mathrm{pr}_2} X^*.$$

Theorem 1.10

(a) *If X is smooth then I_X is smooth.*
(b) *If X^* is a hypersurface then pr_2 is a birational isomorphism.*
(c) *If X is smooth and X^* is a hypersurface then pr_2 is a resolution of singularities.*

Proof. The map pr_1 is a projective bundle over a smooth locus of X. Therefore, if X is smooth then I_X is smooth. If X^* is a hypersurface then $\dim X^* = \dim I_X = n - 1$. Since pr_2 is generically a projective bundle, it is birational. Finally, (c) follows from (a) and (b). □

Typically the dual variety $X^* \subset \mathbb{P}^{n*}$ is a hypersurface. Namely, we shall see that if X^* is not a hypersurface then X is ruled in projective subspaces.

Definition 1.11 We say that X is ruled in projective subspaces of dimension r if for any $x \in X$ there exists a projective subspace L of dimension r such that $x \in L \subset X$. By a standard closedness argument it is sufficient to check this property only for points x from some Zariski open dense subset $U \subset X$.

Recall, that the number $\operatorname{def} X = \operatorname{codim}_{\mathbb{P}^{n*}} X^* - 1$ is called the defect of X.

Theorem 1.12 *Suppose that* $\operatorname{def} X = r \geq 1$. *Then*

(1) *X is ruled in projective subspaces of dimension r.*
(2) *If X is smooth then for any $H \in X^*{}_{sm}$ the contact locus* $\operatorname{Sing}(H \cap X)$ *is a projective subspace of dimension r and the union of these projective subspaces is dense in X.*

Proof. By Reflexivity Theorem 1.7, (a) is equivalent to the following: if $\operatorname{codim} X = r + 1$ then X^* is ruled in projective subspaces of dimension r. The condition for a hyperplane H to be tangent to X at a smooth point x is that $\hat{T}_x H$ contains $\hat{T}_x X$. For a given x, all H with this property form a projective subspace of dimension r. But the set of hyperplanes of X^* tangent to X at some smooth point obviously contains a Zariski open subset of X^*.

(b) is proved by the same argument involving Reflexivity Theorem. □

Example 1.13 Suppose that $X \subset \mathbb{P}^n$ is a non-linear curve. Then X^* is a hypersurface. Indeed, X obviously could not contain a projective subspace $\mathbb{P}^k$ for $k > 0$.

The following Theorem is also an easy corollary of Reflexivity Theorem. It allows to find singular points of hyperplane sections of smooth projective varieties.

Theorem 1.14 *Suppose that $X \subset \mathbb{P}^{n-1}$ is smooth and $X^* \subset \mathbb{P}^{n-1*}$ is a hypersurface. Let $z_1, \ldots, z_n$ be homogeneous coordinates on $\mathbb{P}^{n-1}$ and $a_1, \ldots, a_n$ the dual homogeneous coordinates on $\mathbb{P}^{n-1*}$. Suppose that $f = (a_1, \ldots, a_n)$ is a smooth point of X^*. Then the hyperplane section $\{f = 0\}$ of X has a unique singular point with coordinates given by $(\frac{\partial \Delta_X}{\partial a_1}(f) : \ldots : \frac{\partial \Delta_X}{\partial a_n}(f))$.*

Proof. Let $H \subset \mathbb{P}^{n-1}$ be the hyperplane corresponding to f. By the Reflexivity Theorem, H is tangent to X at z if and only if the hyperplane in $\mathbb{P}^{n-1}$ corresponding to z is tangent to X^* at f. Since X^* is smooth at f, such a point z is unique and is given by $z_i = \frac{\partial \Delta_X}{\partial a_i}(f)$. □

1.3 Dual Plane Curves

1.3.A Parametric Representation of the Dual Plane Curve

Perhaps the most classical example of a dual variety is the dual curve C^* of a non-linear plane curve $C \subset \mathbb{P}^2$. By definition, generic points of C^* are the tangents to C at smooth points. In this case the Reflexivity Theorem has a fairly intuitive meaning. The tangent line $\hat{T}_p \in \mathbb{P}^{2^*}$ at a smooth point $p \in C$ is the limit of secants $\overline{pq}$ for $q \in C$, $q \to p$. Similarly, the point in $\mathbb{P}^2$ that corresponds to the tangent to $C^* \subset \mathbb{P}^{2^*}$ at a non-singular point $\hat{T}_p$ is the limit of the intersection points of the tangents $\hat{T}_p$ and $\hat{T}_q$ as $q \to p$. Of course, this point is p.

In fact, it is quite easy to write down a parametric representation of C^* using a given parametric representation of C. Let x, y, z be homogeneous coordinates on $\mathbb{P}^2$ and p, q, r the dual homogeneous coordinates on $\mathbb{P}^{2^*}$. We choose the affine chart $\mathbb{C}^2 = \{z \neq 0\} \subset \mathbb{P}^2$ with affine coordinates x, y, so the third homogeneous coordinate z is set to be 1. The dual chart $\mathbb{C}^{2^*} \subset \mathbb{P}^{2^*}$ with coordinates p, q is obtained by setting the third homogeneous coordinate r in $\mathbb{P}^{2^*}$ to be -1. Then $\mathbb{C}^{2^*}$ consists of lines in $\mathbb{P}^2$ not passing through the point $(0,0) \in \mathbb{C}^2 \subset \mathbb{P}^2$. Every such line that meets $\mathbb{C}^2$ is given by the affine equation $px + qy = 1$, the line in $\mathbb{P}^2$ with coordinates $p = q = 0$ is the line "at infinity". Suppose that a local parametric equation of C has the form $x = x(t)$, $y = y(t)$, where t is a local coordinate on C, and $x(t)$, $y(t)$ are analytic functions. By definition, the dual curve C^* has the parametrization $p = p(t)$, $q = q(t)$, where $p(t)x + q(t)y = 1$ is the affine equation of the tangent line to C at the point $(x(t), y(t))$. Therefore, we have

$$p(t) = \frac{-y'(t)}{x'(t)y(t) - x(t)y'(t)}, \quad q(t) = \frac{x'(t)}{x'(t)y(t) - x(t)y'(t)}. \tag{1.4}$$

Applying this formula two times we obtain Reflexivity Theorem once again.

Example 1.15 In analysis, there is a well-known duality between Banach spaces L_p and L_q for $\frac{1}{p} + \frac{1}{q} = 1$, see, e.g., [Ru]. Let us give an algebraic version of this duality.

Consider the curve X in $\mathbb{C}^2$ given by

$$x^a + y^a = 1, \quad a > 1, \ a \in \mathbb{Q}.$$

This curve, and its closure in $\mathbb{P}^2$, is usually called the Fermat curve (especially if $a \in \mathbb{Z}$). If $a \notin \mathbb{Z}$, this equation involves multivalued fractional power functions, but it is possible to put this equation into the polynomial form. In order to find the dual curve, we can use a parametric representation of X of the form

$$x = t, \ y = \sqrt[a]{1 - t^a}.$$

Using (1.4) we get the parametric representation of the dual curve as follows:

$$p = t^{a-1},\ q = (1 - t^a)^{\frac{a-1}{a}}.$$

The relation between p and q has the form

$$p^b + q^b = 1, \text{ where } \frac{1}{a} + \frac{1}{b} = 1.$$

1.3.B The Legendre Transformation and Caustics

The projective duality is closely related to the Legendre transformation of classical mechanics, and, therefore, is in some sense analogous to the duality of the Lagrange and the Hamilton pictures of classical mechanics. To illustrate this analogy it will be sufficient to recall the classical definition of the Legendre transformation of real functions in one variable. Details can be found in [Ar].

Suppose that $y = f(x)$ is a smooth convex real function, $f''(x) > 0$. The Legendre transformation of the function f is a new function g of a new variable p, which is constructed in the following way. Consider the line $y = px$. We take the point $x = x(p)$ at which the graph of $y = f(x)$ has a slope p (so $f'(x(p)) = p$), and define $g(p)$ as $g(p) = px(p) - f(x(p))$. Equivalently, we define $x(p)$ as a unique point, where the function $F(p, x) = px - f(x)$ has a maximum with respect to x and define $g(p) = F(p, x(p))$.

The Legendre transformation is easily seen to be involutive. To see how it is related to the dual curve, let us notice that the straight line $y = G(x, p) = xp - g(p)$ is nothing else but a tangent line to the graph of f with slope p.

To link projective duality and Legendre transformation we need a notion of a caustic curve. To introduce it, let us express the projective duality entirely in terms of the projective plane $\mathbb{P}^2$. By definition, a tangent line to a curve C at some point x is the line that contains x and which is infinitesimally close to the curve C near x. A point of $\mathbb{P}^{2^*}$ is a line $l \subset \mathbb{P}^2$. A curve in $\mathbb{P}^{2^*}$ is a 1-parameter family of lines in $\mathbb{P}^2$. For example, a line in $\mathbb{P}^{2^*}$ is a pencil x^* of all lines in $\mathbb{P}^2$ passing through a given point $x \in \mathbb{P}^2$. The dual curve C^* is a 1-parameter family of tangent lines to C. Suppose now that $C' \subset \mathbb{P}^{2^*}$ is some curve (some 1-parameter family of lines in $\mathbb{P}^2$). Let us find a geometric interpretation of the dual curve $C'^* \subset \mathbb{P}^2$. Take some line $l \in C'$. The condition that x^* is tangent to C' at l means that the line $l \in \mathbb{P}^{2^*}$ is a member of a family C' and other lines from C' near l are infinitesimally close to the pencil of lines x^*. This is usually expressed by saying that x is a caustic point for the family of lines C'. The set of all caustic points of the family of lines C' is usually called the caustic (or the envelope) of C'. This is nothing else but the projectively dual curve C'^*. Now the Reflexivity Theorem means that any curve $C \subset \mathbb{P}^2$ coincides with the caustic curve for the family of its tangent lines. The "dual" form of this theorem is less intuitively obvious, it means that any 1-parameter family of lines in $\mathbb{P}^2$ consists of tangent lines to some curve C and this curve is a caustic curve for this family of lines. For

(real) families of lines the caustic could be found (locally) with the help of the Legendre transformation:

Theorem 1.16 *Consider a family of real straight lines $y = px - g(p)$. Then its caustic curve has the equation $y = f(x)$, where f is the Legendre transformation of g.*

This Theorem is, in fact, a version of the Reflexivity Theorem, see [Ar] for the proof.

1.3.C Correspondence of Branches. Plücker Formulas.

Even if a plane curve $C \subset \mathbb{P}^2$ is smooth, the dual curve $C^* \subset \mathbb{P}^{2^*}$ almost always has singularities. We have a natural map $C \to C^*$, sending a point $p \in C$ to a tangent line l to C at p. This map is clearly a resolution of singularities. In general, curves C and C^* are birationally equivalent. Indeed, consider the conormal variety $I_C \subset \mathbb{P}^2 \times \mathbb{P}^{2^*}$, the closure of the set of pairs (p, l), $p \in C_{sm}$, $l \in {C^*}_{sm}$, l is tangent to C at p. Then, by Reflexivity Theorem, I projects birationally both on C and C^*. Therefore, C and C^* are birationally equivalent, in particular, they have the same geometric genus g.

A line l which is tangent to C in at least two points, is a singular point of C^*. It is known as the multiple tangent. If a multiple tangent l has exactly two tangency points on C and the intersection multiplicity at each of them is equal exactly to 2, then l is called the bitangent. A bitangent corresponds to an ordinary double point of C^*.

If the tangent $l = \hat{T}_p$ at a non-singular point $p \in C$ intersects C at p with multiplicity ≥ 3, it is again a singular point on C^*. If the intersection multiplicity is precisely 3, and l is not tangent to C at any other point, then p is called an inflection point (or flex) of C. Then l is a cuspidal point (or cusp) of C^*.

Now we may introduce a class of "generic" curves with singularities, which is preserved by the projective duality. Namely, we say that a curve C is generic if both C and C^* have only double points and cusps as their singularities. Suppose that C is generic in this sense. Let d, g, κ, δ, b, f be the degree, the geometric genus, the number of cusps, the number of double points, the number of bitangents, and the number of flexes of C. Let d^*, g^*, κ^*, δ^*, b^*, f^* be the corresponding numbers for C^* (d^* is also sometimes called the class of C). Then by Reflexivity Theorem we have the following

Proposition 1.17 $g = g^*$, $\kappa = f^*$, $\delta = b^*$, $b = \delta^*$, $f = \kappa^*$.

It turns out that there is another remarkable set of equations linking these numbers, that was discovered by Plücker and Clebsch. The proof can be found in [GH].

Theorem 1.18

$$g = \frac{1}{2}(d^* - 1)(d^* - 2) - b - f,$$
$$g = \frac{1}{2}(d - 1)(d - 2) - \delta - \kappa,$$
$$d = d^*(d^* - 1) - 2b - 3f,$$
$$d^* = d(d - 1) - 2\delta - 3\kappa.$$

Example 1.19 Let $C \subset \mathbb{P}^2$ be a smooth conic. In homogeneous coordinates x_1, x_2, x_3, the curve C is given by

$$(Ax, x) = \sum_{i,j=1}^{3} a_{ij} x_i x_j = 0,$$

where $A = ||a_{ij}||$ is a non-degenerate symmetric 3×3-matrix. It is clear that the tangent line to C at a point $x_0 \in C$ is given by the equation $(Ax_0, x) = 0$. Hence the point $\xi \in \mathbb{P}^{2^*}$ corresponding to this tangent line has homogeneous coordinates Ax_0, which implies $(A^{-1}\xi, \xi) = 0$. Therefore, $C^* \subset \mathbb{P}^{2^*}$ is also a smooth conic defined by the inverse matrix A^{-1}.

Example 1.20 Let $C \subset \mathbb{P}^2$ be a smooth cubic curve. By Bezout Theorem, C is automatically generic and Plücker formulas are always applicable. C has no bitangents and has exactly 9 flexes. The dual curve C^* is a very special curve of degree 6 with 9 cusps and no double points. Schläffli has found a beautiful determinantal formula for C^*. Let x_1, x_2, x_3 be homogeneous coordinates in $\mathbb{P}^2$ and p_1, p_2, p_3 the dual coordinates in $\mathbb{P}^{2^*}$. Let $f(x_1, x_2, x_3) = 0$ be the homogeneous equation of C and $F(p_1, p_2, p_3) = 0$ be the homogeneous equation of C^*. Consider the polynomial

$$V(p, x) = \begin{vmatrix} 0 & p_1 & p_2 & p_3 \\ p_1 & \frac{\partial^2 f}{\partial x_1 \partial x_1} & \frac{\partial^2 f}{\partial x_1 \partial x_2} & \frac{\partial^2 f}{\partial x_1 \partial x_3} \\ p_2 & \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2 \partial x_2} & \frac{\partial^2 f}{\partial x_2 \partial x_3} \\ p_3 & \frac{\partial^2 f}{\partial x_3 \partial x_1} & \frac{\partial^2 f}{\partial x_3 \partial x_2} & \frac{\partial^2 f}{\partial x_3 \partial x_3} \end{vmatrix}.$$

Clearly $V(p, x)$ has degree 2 in x. Then Schläffli's formula is as follows:

Theorem 1.21

$$F(p_1, p_2, p_3) = \begin{vmatrix} 0 & p_1 & p_2 & p_3 \\ p_1 & \frac{\partial^2 V}{\partial x_1 \partial x_1} & \frac{\partial^2 V}{\partial x_1 \partial x_2} & \frac{\partial^2 V}{\partial x_1 \partial x_3} \\ p_2 & \frac{\partial^2 V}{\partial x_2 \partial x_1} & \frac{\partial^2 V}{\partial x_2 \partial x_2} & \frac{\partial^2 V}{\partial x_2 \partial x_3} \\ p_3 & \frac{\partial^2 V}{\partial x_3 \partial x_1} & \frac{\partial^2 V}{\partial x_3 \partial x_2} & \frac{\partial^2 V}{\partial x_3 \partial x_3} \end{vmatrix}.$$

The proof can be found in [GKZ2].

Example 1.22 Suppose that $C \subset \mathbb{P}^2$ is a generic smooth quartic curve. By applying Plücker formulas we see that C has genus 3, 24 flexes, and 28 bitangents. There are two known visual descriptions of these bitangents. The first classical approach is to realize the quartic curve as a 'shade' of a cubic surface in $\mathbb{P}^3$. Namely, suppose that $S \subset \mathbb{P}^3$ is a generic cubic surface, $x \in S$ is a generic point. Consider the projective plane $\mathbb{P}^2$ of lines in $\mathbb{P}^3$ passing through x. Then lines $l \in \mathbb{P}^2$ that are tangent to S form a quartic curve C. The 28 bitangents then coincide with projections of 27 lines on S plus one extra line: the projectivization of a tangent space $T_x S$.

Another classical description is less known but it is more in style of these notes. Consider a generic 3-dimensional linear system L of quadrics in $\mathbb{P}^3$ (an element of $\mathrm{Gr}(3, S^2(\mathbb{C}^4)^*)$. Then singular members of this linear system form a quartic curve in $\mathbb{P}(L) = \mathbb{P}^2$. It can be shown that any generic quartic curve can be obtained this way (for example by calculating the differential at a generic point of the map from linear systems to quartic forms). By the Bezout theorem L has $2^3 = 8$ base points in $\mathbb{P}^3$. For any two base points p, q let $l_{pq} \subset \mathbb{P}(L)$ be the line formed by all quadrics that contain not only points p and q, but also a line $[pq]$ connecting them. Therefore, we have 28 lines $l_{pq} \subset \mathbb{P}(L)$. Let us show that these lines are bitangents to C.

Fix a basis $\{e_1, e_2, e_3, e_4\}$ in $\mathbb{C}^4$ such that e_1 belongs to p and e_2 belongs to q. Then quadrics from l_{pq} have the form

$$G = \begin{pmatrix} 0 & A^T \\ A & B \end{pmatrix},$$

therefore,

$$\det G = (\det A)^2.$$

This exactly means that l_{pq} intersects C in two double points, i.e. l_{pq} is a bitangent to C.

1.4 Projections and Linear Normality

1.4.A Projections

Let $P = \mathbb{P}(V)$ be an n-dimensional projective space and $L \subset P$ a projective subspace of dimension k, $L = \mathbb{P}(U)$, $U \subset V$. The quotient projective space $P/L = \mathbb{P}(V/U)$ has, as points, $(k+1)$-dimensional projective subspaces in P containing L. The projection with center L is the map $\pi_L : P \setminus L \to P/L$ which takes any point $x \in P \setminus L$ to the $(k+1)$-dimensional projective subspace spanned by x and L. In other words, it corresponds to the projection $V \to V/U$. Classically, both P/L and π_L can be visualized inside the initial projective space P. Namely, P/L can be identified with any $(n-k-1)$-dimensional projective subspace $H \subset P$ not intersecting L. Then π_L sends

any point $x \in P \setminus L$ to the unique intersection point of H and $(k+1)$-dimensional projective subspace spanned by x and L. Clearly, the dual projective space $(P/L)^*$ is canonically embedded in P^* as the set of hyperplanes containing L, so it coincides with L^*, the dual variety of L.

Theorem 1.23

(a) *Let $X \subset P = \mathbb{P}(V)$ be an algebraic subvariety not intersecting the projective subspace $L = \mathbb{P}(U)$ and such that $\dim X < \dim P/L$. Then*

$$(\pi_L(X)^* \subset L^* \cap X^*.$$

The discriminant $\Delta_{\pi_L(X)}$ (as a polynomial on $(V/U)^* \subset V^*$) is a factor of the restriction of Δ_X to $(V/U)^*$.

(b) *Suppose further that $\pi_L : X \to \pi_L(X)$ is an isomorphism. Then*

$$(\pi_L(X)^* = L^* \cap X^*, \text{ and } \Delta_{\pi_L(X)} = \Delta_X|_{(V/U)^*}.$$

Proof. A hyperplane in P/L is just a hyperplane in P containing L. It is clear that if a hyperplane $H \subset P/L$ is tangent to $\pi_L(X)$ at some smooth point $y = \pi_L(x)$, where $x \in X$ is also smooth, then H as a hyperplane in P is tangent to X at x. This proves (a).

Suppose now that π_L is an isomorphism. The same argument as above also shows that if $X_0^* \subset X^*$ (resp. $\pi_L(X)_0^* \subset \pi_L(X)^*$) is a dense subset of hyperplanes tangent to X (resp. $\pi_L(X)$) at some smooth point then $\pi_L(X)_0^* = L^* \cap X_0^*$, since π_L induces an isomorphism of smooth loci $X_{sm} \to \pi_L(X)_{sm}$. Moreover, it is clear that at a generic point of $\pi_L(X)_0^*$ varieties L^* and X_0^* intersect transversally. Since $X^* = \overline{X_0^*}$ and $\pi_L(X)^* = \overline{\pi_L(X)_0^*}$, in order to prove (b) it suffices to show that $L^* \cap X^* = \overline{L^* \cap X_0^*}$. Here the assumption that π_L is an isomorphism is crucial, because otherwise critical points of π_L may produce extra components of $L^* \cap X^*$.

We need to show that if $H(t)$ is a 1-parameter family of hyperplanes such that for $t \neq 0$ the hyperplane $H(t)$ is tangent to X at some smooth point $x(t)$ and $H(0)$ contains L, then there exists another 1-parameter family $H'(t)$ such that $H'(0) = H(0)$ and for $t \neq 0$ the hyperplane $H'(t)$ is tangent to X at $x(t)$ and contains L. Since X is projective, hence compact, we can suppose that the limit $x(0) = \lim_{t\to 0} x(t)$ exists. However, $x(0)$ may be a singular point of X. Let $\hat{T}_{x(0)}X$ be an embedded Zariski projective tangent space to X at $x(0)$. Since π_L is an isomorphism, it induces an isomorphism of Zariski tangent spaces. Therefore $\hat{T}_{x(0)}X$ does not intersect L. Let $T_0 \subset \hat{T}_{x(0)}X$ be a limit position of embedded tangent Zariski spaces $T_t = \hat{T}_{x(t)}X$ as $t \to 0$. Then for any t the projective subspace T_t does not intersect L. Therefore we may consider a family T_t' of $(\dim L + \dim X + 1)$-dimensional projective subspaces such that for any t we have $L \subset T_t'$ and for $t \neq 0$ the subspace T_t' is tangent to X at $x(t)$. Namely, T_t' is a projective subspace spanned by L and T_t. Since $\dim X < \dim P/L$, we can embed T_t' into a family of hyperplanes H_t' with same properties. $\square$

1.4.B Linear Normality

Definition 1.24 The projective variety $X \subset \mathbb{P}^n$ is called non-degenerate if X is not contained in any hyperplane H.

The next theorem shows that we may restrict ourselves to non-degenerate varieties while studying dual varieties and discriminants. First we shall need the following definition. We recall from a previous section that if $\mathbb{P}^k = L \subset \mathbb{P}^n$ is a projective subspace then a dual projective space $\mathbb{P}^{k^*}$ is not canonically embedded in $\mathbb{P}^{n*}$, but (by some abuse of notation) is canonically isomorphic to the quotient projective space $\mathbb{P}^{n*}/L^*$, where L^* is a projective subspace of $\mathbb{P}^{n*}$ projectively dual to L. However, if $Y \subset \mathbb{P}^{k^*}$ is any subvariety then we can canonically define a cone over Y with vertex L^* as the closure $\pi_{L^*}^{-1}(Y)$.

Theorem 1.25 *Let $X \subset \mathbb{P}^n$ be an irreducible subvariety.*

(a) *Assume that X is contained in a hyperplane $H = \mathbb{P}^{n-1}$. If $X^{*\prime}$ is the dual variety of X, when we consider X as a subvariety of $\mathbb{P}^{n-1}$, then X^* is the cone over $X^{*\prime}$ with vertex p corresponding to H.*
(2) *Conversely, if X^* is a cone with vertex p, then X is contained in the corresponding hyperplane H.*

Proof. If $H' \neq H$ is a tangent hyperplane of X then $H \cap H'$ is a tangent hyperplane of X in $\mathbb{P}^{n-1}$. Conversely, if T is a tangent hyperplane of X in $\mathbb{P}^{n-1}$ then each hyperplane H' in $\mathbb{P}^n$ containing T is tangent to X. Therefore, X^* is the cone over $X^{*\prime}$. This proves (a).

By Reflexivity Theorem, we also have (b). Namely, each hyperplane which is tangent to X^* at a smooth point necessarily contains p. Therefore $X = X^{**}$ is contained in the hyperplane corresponding to p. □

In terms of discriminants, Theorem 1.25 can be reformulated as follows. Consider a surjection $\pi : V \to U$ of vector spaces and consider $\mathbb{P}(V^*)$, $\mathbb{P}(U^*)$. Then we have an embedding $i : \mathbb{P}(U^*) \hookrightarrow \mathbb{P}(V^*)$. Let $X \subset \mathbb{P}(U^*)$. Then Δ_X is a polynomial function on U. If we consider X as a subvariety in $\mathbb{P}(V^*)$ then $\Delta_{i(X)}$ is a function on V. By Theorem 1.25 these polynomial functions are related as follows:

$$\Delta_{i(X)}(f) = \Delta_X(\pi(f)).$$

In other words, $\Delta_{i(X)}$ does not depend on some of the arguments and forgetting these arguments gives Δ_X.

Theorems 1.23 and 1.25 show that in order to study dual varieties and discriminants we need to consider only projective varieties X that are non-degenerate and not equal to a non-trivial projection. These projective varieties are called linearly normal. To give a more intrinsic definition of linearly normal varieties we need to recall the correspondence between invertible sheaves, linear systems, and projective embeddings.

An invertible sheaf on an irreducible algebraic variety X is simply the sheaf of sections of some algebraic line bundle. For example, the structure

sheaf of regular functions $\mathcal{O}_X$ corresponds to the trivial line bundle. Usually we shall not distinguish notationally between invertible sheaves and line bundles. Invertible sheaves form the group $\mathrm{Pic}(X)$ with respect to the tensor product.

A Cartier divisor on X is a family (U_i, g_i), $i \in I$, where U_i are open subsets of X covering X, and g_i are rational functions on U_i such that g_i/g_j is regular on each intersection $U_i \cap U_j$. The functions g_i are called local equations of the divisor. More precisely, a Cartier divisor is an equivalence class of such data. Two collections (U_i, g_i) and (U_i', g_i') are equivalent if their union is still a divisor. Cartier divisors can be added, by multiplying their local equations. Thus they form a group, denoted by $\mathrm{Div}(X)$.

If each local equation g_i is regular on U_i, then we say that the divisor D is effective, and we write $D \geq 0$. The subschemes $\{g_i = 0\}$ of the U_i can then be glued together into a subscheme of X, also denoted by D. Therefore, effective Cartier divisors can be identified with subschemes of X that are locally given by one equation. Any non-zero rational function $f \in \mathbb{C}(X)$ determines a Cartier divisor (f) which is said to be principal. Principal divisors form a subgroup of $\mathrm{Div}(X)$.

Let $\mathcal{K}_X$ denote the sheaf of rational functions on X, $\mathcal{K}_X(U) = \mathbb{C}(U)$. To every Cartier divisor $D = (U_i, g_i)_{i \in I}$ we can attach a subsheaf $\mathcal{O}_X(D) \subset \mathcal{K}_X$. Namely, on U_i it is defined as $g_i^{-1}\mathcal{O}_{U_i}$. On the intersection $U_i \cap U_j$ the sheaves $g_i^{-1}\mathcal{O}_{U_i}$ and $g_j^{-1}\mathcal{O}_{U_j}$ coincide, since g_i/g_j is invertible. Therefore these sheaves can be pasted together into a sheaf $\mathcal{O}_X(D) \subset \mathcal{K}_X$. For instance, $\mathcal{O}_X(0) = \mathcal{O}_X$ and $\mathcal{O}_X(D_1 + D_2) = \mathcal{O}_X(D_1) \otimes \mathcal{O}_X(D_2)$. The sheaves $\mathcal{O}_X(D)$ are invertible. In fact, multiplication by g_i defines an isomorphism ("trivialization") $\mathcal{O}_X(D)|_{U_i} \simeq \mathcal{O}_{U_i}$. Therefore, we have a homomorphism $\mathrm{Div}(X) \to \mathrm{Pic}(X)$. This homomorphism is surjective and its kernel consists of principal divisors.

A non-zero section of $\mathcal{O}_X(D)$ is a rational function f on X such that the functions fg_i are regular on the U_i, in other words, such that the divisor $(f) + D$ is effective. If D itself is effective, the sheaf $\mathcal{O}_X(D)$ has a canonical section s_D, which corresponds to the constant function 1. By contrast, the sheaf $\mathcal{O}_X(-D)$, for D effective, is an ideal sheaf in $\mathcal{O}_X$. It defines D as a subscheme. The sections of invertible sheaves define some divisors. Let $s \in H^0(X, \mathcal{L})$ be a non-trivial global section of an invertible sheaf $\mathcal{L}$. After choosing some trivializations $\phi_i : \mathcal{L}_{U_i} \simeq \mathcal{O}_{U_i}$ on a covering (U_i), we obtain an effective divisor $(U_i, \phi_i(s_i))$, which we denote by $\mathrm{div}(s, \mathcal{L})$. For instance, if D is effective, the canonical section s_D of the sheaf $\mathcal{O}(D)$ defines D. Thus we have established the possibility to define any effective divisor by one equation $s = 0$, keeping in mind that s is not a function, but a section of an invertible sheaf. If s' is another non-zero global section of $\mathcal{L}$ then the divisors $\mathrm{div}(s', \mathcal{L})$ and $\mathrm{div}(s, \mathcal{L})$ differ by the divisor of a rational function s/s'. One also says that they are linearly equivalent.

Example 1.26 Let us recall the construction of invertible sheaves on projective spaces $\mathbb{P}(V)$. All these sheaves have the form $\mathcal{O}(d)$, where $\mathcal{O}(d)$ is

the sheaf of degree d homogeneous functions on $\mathbb{P}(V)$. More precisely, let $\pi : V \setminus \{0\} \to \mathbb{P}(V)$ be the canonical projection. If $U \subset \mathbb{P}(V)$ is a Zariski open set, then the sections of $\mathcal{O}(d)$ over U are, by definition, regular functions f on $\pi^{-1}(U) \subset V$, which are homogeneous of degree d, $f(\lambda v) = \lambda^d f(v)$. It is well-known that these sheaves are invertible and any invertible sheaf has the form $\mathcal{O}(d)$ for some d. For example, $\mathcal{O}(-1)$, as a line bundle, is the tautological line bundle, the fiber of $\mathcal{O}(-1)$, as a line bundle, over a point of $\mathbb{P}(V)$ represented by a 1-dimensional subspace l is l itself. If $d < 0$ then $H^0(\mathbb{P}, \mathcal{O}(d)) = 0$. If $d \geq 0$ then $H^0(\mathbb{P}, \mathcal{O}(d)) = S^d V^*$, homogeneous polynomials of degree d. For any non-zero $s \in H^0(\mathbb{P}, \mathcal{O}(d))$ the corresponding effective divisor $\operatorname{div}(s, \mathcal{O}(d))$ is just the hypersurface defined by the polynomial s. In particular, hyperplanes in $\mathbb{P}$ correspond to global sections of $\mathcal{O}(1)$.

Suppose now that X is a projective variety in $\mathbb{P}^n$. Then any sheaf $\mathcal{O}(d)$ can be restricted on X, thus we get a sheaf $\mathcal{O}_X(d)$ for any d. In particular, we have a restriction homomorphism of global sections:

$$V^* = H^0(\mathbb{P}^n, \mathcal{O}(1)) \stackrel{\text{res}}{\to} H^0(X, \mathcal{O}_X(1)).$$

Clearly this map is not injective if and only if X is degenerate. So suppose now that X is non-degenerate. In general, map res is not surjective. Its image is a vector subspace $W \subset H^0(X, \mathcal{O}_X(1))$ with the following obvious property: for any $x \in X$ there exist some section $s \in W$ such that $s(x) \neq 0$. Clearly divisors of the form $\operatorname{div}(s, \mathcal{O}_X(1))$, $s \in W$ are just hyperplane sections $X \cap H$ for various hyperplanes H.

In general, if X is any variety with invertible sheaf $\mathcal{L}$ then any family of divisors $|W|$ of the form $\operatorname{div}(s, \Lambda)$, where $s \in W \subset H^0(X, \mathcal{L})$, $\dim W < \infty$, is called a linear systems of divisors. If X is projective (or just complete) then $H^0(X, \mathcal{L})$ is finite-dimensional and we can take $W = H^0(X, \mathcal{L})$. The corresponding linear system is called complete. It is denoted by $|\mathcal{L}|$, or by $|D|$ if $\mathcal{L} = \mathcal{O}(D)$. Every effective divisor that is linearly equivalent to D appears in $|D|$, and exactly once.

A linear system W is called base-point free if the intersection of all its divisors is empty, or, equivalently, if for any $x \in X$ there exists some $s \in W$ such that $s(x) \neq 0$. As we have seen, if $X \subset \mathbb{P}(V)$ is a projective variety then the image of res : $V^* \to H^0(X, \mathcal{O}_X(1))$ defines a base-point free linear system. It turns out that any base-point free linear system $|W|$ defines a morphism into a projective space. Namely, for any $x \in X$ we have a hyperplane $\Gamma(x) \subset W$ consisting of all sections vanishing at x. Therefore, we have a regular map $\Gamma : X \to \mathbb{P}(W^*)$. If Γ is an embedding then we see that W is identified with the image of a restriction map $W \to H^0(X, \mathcal{O}_X(1))$ and $\mathcal{O}_X(1)$ is identified with $\mathcal{L}$. So, we need a final definition.

Definition 1.27 A linear system $|W|$ is called very ample if it defines an embedding in a projective space. An invertible sheaf $\mathcal{L}$ is called very ample if the complete linear system $|\mathcal{L}|$ is very ample. $\mathcal{L}$ is called ample if some tensor power $\mathcal{L}^{\otimes m}$, $m > 1$, is very ample.

In particular, we see that non-degenerate projective embeddings of X are in a $1-1$ correspondence with very ample linear systems of divisors.

Now we can describe linearly normal varieties.

Theorem 1.28 *A projective variety $X \subset \mathbb{P}(V)$ is linearly normal if and only if the restriction homomorphism* $\mathrm{res} : V^* \to H^0(X, \mathcal{O}_X(1))$ *is an isomorphism if and only if the embedding $X \subset \mathbb{P}(V)$ is given by a complete linear system corresponding to some very ample invertible sheaf $\mathcal{L}$ on X.*

Proof. Clearly res is injective if and only if X is non-degenerate. Suppose that res is not surjective. Then the linear system $|\mathcal{O}_X(1)|$ gives a non-degenerate embedding of X to a higher-dimensional vector space $H^0(X, \mathcal{O}_X(1))$. The initial embedding is obtained from this one by the projection with center $\mathbb{P}(\mathrm{Ann}(\mathrm{Im}(\mathrm{res})))$. Conversely, if X can be obtained as a non-trivial isomorphic projection of a non-degenerate variety $\tilde{X}$ in a larger projective space $\mathbb{P}(U)$ then we have a proper embedding $V^* \subset U^*$. The map $U^* \to H^0(X, \mathcal{O}_{\tilde{X}}(1))$, given by restricting linear functions from $\mathbb{P}(U)$ to $\tilde{X}$, is an injection, since $\tilde{X}$ is non-degenerate. Thus the map res is not surjective, being the composition $V^* \hookrightarrow U^* \to H^0(X, \mathcal{O}_X(1))$. □

We see that in order to study dual varieties and discriminants we can restrict ourselves to linearly normal varieties.

Definition 1.29 A pair $(X, \mathcal{L})$ of a projective variety and a very ample invertible sheaf on it is called the polarized variety.

Any polarized variety admits a canonical embedding in a projective space with a linearly normal image, namely, the embedding given by the complete linear system $|\mathcal{L}|$. Therefore we may speak about dual varieties, defect, discriminants, etc. of polarized varieties without any confusion.

1.5 Dual Varieties of Smooth Divisors

Suppose that $X \subset \mathbb{P}^N$ is a smooth projective variety. The following theorem allows to find the defect of smooth hyperplane or hypersurface sections of X.

Theorem 1.30 ([E2, HK, Ho1])

(a) *Assume that $Y = X \cap H$ is a smooth hyperplane section of X. Then*

$$\operatorname{def} Y = \max\{0, \operatorname{def} X - 1\}.$$

Moreover, if X^* is not a hypersurface, then the dual variety Y^* is the cone over X^* with H as a vertex.

(b) *Assume that Y is a smooth divisor corresponding to a section of $\mathcal{O}_X(d)$ for $d \geq 2$. Then* $\operatorname{def} Y = 0$.

Proof. Consider the conormal variety

$$I_X = \mathbb{P}(N_X^* \mathbb{P}^N(1)) \subset \mathbb{P}^N \times \mathbb{P}^{N^*}$$

and the following diagram of projections

$$\mathbb{P}^N \supset X \overset{\pi_1}{\leftarrow} I_X \overset{\pi_2}{\rightarrow} X^* \subset \mathbb{P}^{N^*}.$$

We use notation $\mathcal{O}_{I_X}(a,b)$ for the line bundle $\pi_1^*\mathcal{O}_X(a) \otimes \pi_2^*\mathcal{O}_{X^*}(b)$. We will need the notion of the tautological line bundle on a projective bundle. Let E be a vector bundle on X. We consider the variety $X_E = \mathbb{P}(E^*)$, the projectivization of the bundle E^*. There is a projection $p:\ E_X \to X$ whose fibers are projectivizations of fibers of E^*, and a projection $\pi:\ E^* \setminus X \to X_E$, where X is embedded into the total space of E^* as the zero section. We denote by $\xi(E)$ the tautological line bundle on X_E defined as follows. For open $U \subset X_E$, a section of $\xi(E)$ over U is a regular function on $\pi^{-1}(U)$ which is homogeneous of degree 1 with respect to dilations of E^*. The restriction of $\xi(E)$ to every fiber $p^{-1}(x) = \mathbb{P}(E_x^*)$ is the tautological line bundle $\mathcal{O}(1)$ of the projective space $\mathbb{P}(E_x^*)$. For example, $\mathcal{O}_{I_X}(0,1)$ is the tautological line bundle of $\mathbb{P}(N_X^*\mathbb{P}^N(1))$.

Suppose that Y is a smooth member of a linear system $|\mathcal{O}_X(d)|$. Clearly we have $N_Y X = \mathcal{O}_Y(d)$. Consider the following exact sequence

$$0 \to \mathcal{O}_Y(d-1) \to N_Y\mathbb{P}^N(-1) \to N_X\mathbb{P}^N(-1)|_Y \to 0.$$

Let $D = \mathbb{P}(N_X^*\mathbb{P}^N(1)|_Y)$. Then there is a natural embedding of D as a divisor in $I_Y = \mathbb{P}(N_Y^*\mathbb{P}^N(1))$. We see that $\mathcal{O}_{I_Y}(D) \simeq \mathcal{O}_{I_Y}(1-d,1)$.

We shall study the following diagram

$$\begin{array}{ccccccccc}
\mathbb{P}^{N^*} & \supset & Y^* & \supset & \pi_2(D) & \subset & X^* & \subset & \mathbb{P}^{N^*} \\
 & & \uparrow{\scriptstyle \pi_2} & & \uparrow{\scriptstyle \pi_2} & & \uparrow{\scriptstyle \pi_2} & & \\
 & & I_Y & \supset & D & \subset & I_X & & \\
 & & \downarrow{\scriptstyle \pi_1} & & \downarrow{\scriptstyle \pi_1} & & \downarrow{\scriptstyle \pi_1} & & \\
\mathbb{P}^N & \supset & Y & = & Y & \subset & X & \subset & \mathbb{P}^N
\end{array}$$

Consider the morphism g given by the complete linear system $|\mathcal{O}_{I_Y}(0,1)|$. Since $\pi_2^*\mathcal{O}_{Y^*}(1) = \mathcal{O}_{I_Y}(0,1)$, we can regard Y^* as the linear projection of $g(I_Y)$. Thus $\dim Y^* = \dim g(I_Y)$.

(a) Assume that $d = 1$. Then $\mathcal{O}_{I_Y}(0,1) = \mathcal{O}_{I_Y}(D)$. Therefore $g(D)$ is a hyperplane section of $g(I_Y)$ and

$$\dim g(D) = \dim \pi_2(D) = \dim Y^* - 1.$$

Therefore it is sufficient to prove that

(A) If $\operatorname{def} X = k > 0$, then $\dim \pi_2(D) = \dim X^*$.
(B) If $\operatorname{def} X = 0$ then $\dim \pi_2(D) = \dim X^* - 1$.

For any $q \in X^*$ the preimage $\pi_2^{-1}(q)$ is isomorphic to the contact locus of the hyperplane $q = 0$ and X. Therefore, $\dim \pi_2^{-1}(q) \geq \operatorname{def} X$ and for generic $q \in X^*$ we have an equality. Therefore, if $\operatorname{def} X = k > 0$ then $\dim D \cap \pi_2^{-1}(q) > 0$ and (A) follows. Moreover, we see that any hyperplane tangent to X is also tangent to Y, therefore, Y^* is a cone over X^* with the vertex H. Suppose now that $\operatorname{def} X = 0$. Then π_2 is birational, and, by the same argument as above, $\dim \pi_2(D) \leq \dim X^* - 1$. Moreover, if Y passes through a point $x \in X$ such that there exists a hyperplane $H \in {X^*}_{sm}$ tangent to X at x then, in fact, $\dim \pi_2(D) = \dim X^* - 1$. This proves (B) for generic hyperplane sections and, hence, for arbitrary smooth hyperplane sections by a standard argument.

(b) Assume $d \geq 2$. We claim that $g|_{I_Y \setminus D}$ is one to one, and, therefore, $\dim Y^* = \dim g(I_Y) = \dim I_Y = N - 1$ and Y^* is a hypersurface. Indeed, let y_1 and y_2 be two distinct points in $I_Y \setminus D$. We need to show that the linear system $|\mathcal{O}_{I_Y}(0,1)|$ separates y_1 and y_2. If $\pi_1(y) = \pi_1(y_2)$ then $\pi_2(y_1) \neq \pi_2(y_2)$ because π_2 maps $\pi_1^{-1}(\pi_1(y_1))$ isomorphically to a projective subspace in Y^*. So we may assume that $\pi_1(y_1) \neq \pi_1(y_2)$. Then we can find a hypersurface S of degree $d-1$ in $\mathbb{P}^N$ which will contain $\pi_1(y_1)$ but not $\pi_1(y_2)$. Recall that $\mathcal{O}_{I_Y}(D) \simeq \mathcal{O}_{I_Y}(1-d,1)$. Therefore, $D + \pi_1^{-1}(S) \in |\mathcal{O}_{I_Y}(0,1)|$. But $y_1 \in D + \pi_1^{-1}(S)$ and $y_2 \notin D + \pi_1^{-1}(S)$. $\square$

2. Dual Varieties of Algebraic Group Orbits

Preliminaries

It is quite difficult to find the explanation of the following phenomenon: most interesting discriminants and dual varieties appear when the projective variety X is either homogeneous or a closure of a homogeneous variety. There are three known big classes of examples: projective embeddings of flag varieties, algebraic groups acting on projective spaces with finitely many orbits, and projective embeddings of torical varieties. The last case leads to the theory of A-discriminants studied in detail in the beautiful book [GKZ2] and omitted here.

2.1 Polarized Flag Varieties

2.1.A Definitions and Notations

Let G be a connected simply-connected semisimple complex algebraic group with a Borel subgroup B and a maximal torus $T \subset B$. Let $\mathcal{P}$ be the character group of T (the weight lattice). Let $\Delta \subset \mathcal{P}$ be the set of roots of G relative to T. To every root $\alpha \in \Delta$ we can assign the 1-dimensional unipotent subgroup $U_\alpha \subset G$. We define the negative roots Δ^- as those roots α such that $U_\alpha \subset B$. The positive roots are $\Delta^+ = \Delta \setminus \Delta^- = -\Delta^-$. Let $\Pi \subset \Delta^+$ be simple roots, $\Pi = \{\alpha_1, \dots, \alpha_n\}$, where $n = \operatorname{rank} G = \dim T$. Any root $\alpha \in \Delta$ is an integral combination $\sum_i n_i \alpha_i$ with nonnegative n_i (for $\alpha \in \Delta^+$) or nonpositive n_i (for $\alpha \in \Delta^-$). If G is simple then we use the Bourbaki numbering [Bo] of simple roots.

The weight lattice $\mathcal{P}$ is generated as a $\mathbb{Z}$-module by the fundamental weights $\omega_1, \dots, \omega_n$ dual to the simple roots under the Killing form

$$\langle \omega_i | \alpha_j \rangle = \frac{2(\omega_i, \alpha_j)}{(\alpha_j, \alpha_j)} = \delta_{ij}.$$

A weight $\lambda = \sum_i n_i \omega_i$ is called dominant if all $n_i \geq 0$. We denote the dominant weights by $\mathcal{P}^+$. Dominant weights parametrize finite-dimensional irreducible G-modules: to any $\lambda \in \mathcal{P}^+$ we assign an irreducible G-module V_λ

with highest weight λ. A weight $\lambda = \sum_i n_i\omega_i$ is called strictly dominant if all $n_i > 0$. We denote the strictly dominant weights by $\mathcal{P}^{++}$. There is a partial order on $\mathcal{P}$: $\lambda > \mu$ if $\lambda - \mu$ is an integral combination of simple roots with nonnegative coefficients. If $\lambda \in \mathcal{P}^+$ then we denote by $\lambda^* \in \mathcal{P}^+$ the highest weight of the dual G-module V_λ^*. Let W be the Weil group of G relative to T. If $w_0 \in W$ is the longest element then $\lambda^* = -w_0(\lambda)$.

The character group of B is identified with a character group of T. Therefore, for any $\lambda \in \mathcal{P}$ we can assign a 1-dimensional B-module $\mathbb{C}_\lambda$, where B acts on $\mathbb{C}_\lambda$ by a character λ. Now we can define the twisted product $G \times_B \mathbb{C}_\lambda$ to be the quotient of $G \times \mathbb{C}_\lambda$ by the diagonal action B:

$$b \cdot (g, z) = (gb^{-1}, \lambda(b)z).$$

Projection onto the first factor induces the map $G \times_B \mathbb{C}_\lambda \to G/B$, which realizes the twisted product as an equivariant line bundle $\mathcal{L}_\lambda$ on G/B with fiber $\mathbb{C}_\lambda$. It is well-known that the correspondence $\lambda \to \mathcal{L}_\lambda$ is an isomorphism of $\mathcal{P}$ and $\mathrm{Pic}(G/B)$. $\mathcal{L}_\lambda$ is ample if and only if $\mathcal{L}_\lambda$ is very ample if and only if $\lambda \in \mathcal{P}^{++}$. By the Borel–Weil–Bott theorem [Bot], for strictly dominant λ the vector space of global sections $H^0(G/B, \mathcal{L}_\lambda)$ is isomorphic as a G-module to V_λ, the irreducible G-module with highest weight λ. The embedding $G/B \subset \mathbb{P}(V_{\lambda^*})$ identifies G/B with the projectivization of the cone of highest weight vectors. The dual variety $(G/B)^*$ therefore lies in $\mathbb{P}(V_\lambda)$ and parametrises global sections $s \in H^0(G/B, \mathcal{L}_\lambda)$ such that the scheme of zeros $Z(s)$ is a singular divisor.

More generally, consider any flag variety of the form G/P, where $P \subset G$ is an arbitrary parabolic subgroup. The subgroup $P \subset G$ is called parabolic if G/P is a projective variety. P is parabolic if and only if it contains some Borel subgroup. Up to conjugacy, me may assume that P contains B. The combinatorial description is as follows. Let $\Pi_P \subset \Pi$ be some subset of simple roots. Let $\Delta_P^+ \subset \Delta^+$ denote the positive roots that are linear combinations of the roots in Π_P. Then P is generated by B and by the root groups U_α for $\alpha \in \Delta_P^+$. We denote $\Pi \setminus \Pi_P$ by $\Pi_{G/P}$ and $\Delta^+ \setminus \Delta_P^+$ by $\Delta_{G/P}^+$. A parabolic subgroup is maximal if and only if $\Pi_{G/P}$ is a single simple root.

The fundamental weights $\omega_{i_1}, \ldots \omega_{i_k}$ dual to the simple roots in $\Pi_{G/P}$ generate the sublattice $\mathcal{P}_{G/P}$ of $\mathcal{P}$. We denote $\mathcal{P}^+ \cap \mathcal{P}_{G/P}$ by $\mathcal{P}_{G/P}^+$. The subset $\mathcal{P}_{G/P}^{++} \subset \mathcal{P}_{G/P}^+$ consists of all weights $\lambda = \sum n_k\omega_{i_k}$ such that all $n_k > 0$. Any weight $\lambda \in \mathcal{P}_{G/P}$ defines a character of P, and therefore a line bundle $\mathcal{L}_\lambda$ on G/P. Then the following is well-known.

- The correspondence $\lambda \to \mathcal{L}_\lambda$ is an isomorphism of $\mathcal{P}_{G/P}$ and $\mathrm{Pic}(G/P)$. In particular, $\mathrm{Pic}(G/P) = \mathbb{Z}$ if and only if P is maximal.
- $\mathcal{L}_\lambda$ is ample if and only if $\mathcal{L}_\lambda$ is very ample if and only if $\lambda \in \mathcal{P}_{G/P}^{++}$.
- If $\lambda \in \mathcal{P}_{G/P}^+$ then the linear system corresponding to $\mathcal{L}_\lambda$ is base–point free. The corresponding map given by sections is a factorization $G/P \to G/Q$,

where Q is a parabolic subgroup such that Π_Q is a union of Π_P and all simple roots in $\Pi_{G/P}$ orthogonal to λ.

– For strictly dominant $\lambda \in \mathcal{P}^{++}_{G/P}$ the vector space of global sections $H^0(G/P, \mathcal{L}_\lambda)$ is isomorphic as a G-module to V_λ, the irreducible G-module with highest weight λ.

The embedding $G/P \subset \mathbb{P}(V_{\lambda^*})$ identifies G/P with the projectivization of the cone of highest weight vectors. The dual variety $(G/P)^*$ therefore lies in $\mathbb{P}(V_\lambda)$ and parametrises global sections $s \in H^0(G/P, \mathcal{L}_\lambda)$ such that the scheme of zeros $Z(s)$ is a singular divisor.

2.1.B Basic Examples

Example 2.1 (Projective spaces and Grassmanians) Let $V = \mathbb{C}^n$ be a finite-dimensional vector space, $G = \mathrm{SL}(V)$. The projective space $\mathbb{P}(V)$ is identified with a flag variety G/P, where P is the normalizer in G of a line in V. P is a maximal parabolic subgroup, therefore $\mathrm{Pic}(\mathbb{P}(V)) = \mathbb{Z}$. All line bundles on $\mathbb{P}(V)$ have a form $\mathcal{O}(d)$ for $d \in \mathbb{Z}$. The line bundle $\mathcal{O}(d)$ is ample if and only if it is very ample if and only if $d > 0$. In this case $H^0(\mathbb{P}(V), \mathcal{O}(d))$ is canonically isomorphic to $S^d V^*$. The embedding $\mathbb{P}(V) \subset \mathbb{P}(S^d V)$ is called the Veronese embedding. It assigns to a line spanned by $v \in V$ the line in $S^d V$ spanned by v^d. The zero scheme $Z(s)$ of the global section $s \in S^d(V^*)$ are just the hypersurface in $\mathbb{P}(V)$ of degree d given by $s = 0$. Therefore, the dual variety in this case parametrises singular hypersurfaces. In particular, the corresponding discriminant is the classical discriminant of a homogeneous form of degree d in n variables.

The Grassmanian $\mathrm{Gr}(k, V)$ of k-dimensional vector subspaces of V is a flag variety G/P, where P is the normalizer in $\mathrm{SL}(V)$ of a k-dimensional subspace in V. The k-dimensional subspaces in V are in the bijective correspondence with $(k-1)$-dimensional subspaces in $\mathbb{P}(V)$. Having this in mind we shall sometimes write $\mathrm{Gr}(k-1, \mathbb{P}(V))$ instead of $\mathrm{Gr}(k, V)$. The Grassmanian $\mathrm{Gr}(k, V)$ carries a natural equivariant vector bundle $\mathcal{S}$, called the tautological vector bundle. The fiber of $\mathcal{S}$ over a k-dimensional linear subspace $U \subset V$ is U itself. Therefore, $\mathcal{S}$ is a subbundle of the trivial bundle $\mathrm{Gr}(k, V) \times V$ with fiber V at each point.

P is a maximal parabolic subgroup corresponding to the fundamental weight ω_k, therefore $\mathrm{Pic}(\mathrm{Gr}(k, V)) = \mathbb{Z}$. The ample generator of $\mathrm{Pic}(\mathrm{Gr}(k, V))$ is the line bundle $\Lambda^k \mathcal{S}^*$. The corresponding embedding is called the Plücker embedding, $\mathrm{Gr}(k, V) \subset \mathbb{P}(\Lambda^k V)$. Namely, if $U \in \mathrm{Gr}(k, V)$, $u_1, \ldots, u_k$ is any basis of U then $U \mapsto [u_1 \wedge \ldots \wedge u_k]$. The discriminant of the Plücker embedding was studied by Lascoux [Las]. Clearly, all minimal flag varieties of $\mathrm{SL}(V)$ are isomorphic to Grassmanians.

The tangent space to $\mathrm{Gr}(k, V)$ at the point U is a vector space with an additional structure. Namely, it is easy to verify the natural isomorphism

$$T_U \operatorname{Gr}(k,V) = \operatorname{Hom}(U, V/U) = U^* \otimes V/U.$$

In particular, the tangent bundle $T\operatorname{Gr}(k,V)$ is canonically isomorphic to $\mathcal{S}^* \otimes (V/\mathcal{S})$. Notice also that $\operatorname{Gr}(k,V) \simeq \operatorname{Gr}(n-k,V^*)$. Namely, any subspace $U \in \operatorname{Gr}(k,V)$ maps to its annihilator $\operatorname{Ann} U \in \operatorname{Gr}(n-k,V^*)$.

Example 2.2 (Spinor varieties) Let $V = \mathbb{C}^n$ be a vector space equipped with a non-degenerate symmetric scalar product Q or a non-degenerate symplectic form ω (n should be even in this case). Let $G = \operatorname{SO}(V)$ or $G = \operatorname{Sp}(V)$, respectively. For any $k \le n/2$ let $\operatorname{Gr}_Q(k,V)$ (resp. $\operatorname{Gr}_\omega(k,V)$) denote the isotropic Grassmanian of k-dimensional isotropic subspaces. Recall that a subspace $U \subset V$ is called isotropic if for any $v_1, v_2 \in U$ we have $Q(v_1,v_2) = 0$ (resp. $\omega(v_1,v_2) = 0$). Then $\operatorname{Gr}_Q(k,V)$ and $\operatorname{Gr}_\omega(k,V)$ are projective G-equivariant varieties. It easily follows from the Witt theorem that $\operatorname{Gr}_Q(k,V)$ and $\operatorname{Gr}_\omega(k,V)$ are irreducible homogeneous varieties (hence flag varieties) in all cases except the symmetric case when $n = 2k$. In the latter case $\operatorname{Gr}_Q(k,\mathbb{C}^{2k})$ consists of two homogeneous components $\operatorname{Gr}_Q^+(k,\mathbb{C}^{2k})$ and $\operatorname{Gr}_Q^-(k,\mathbb{C}^{2k})$. These two varieties are in fact isomorphic to each other (as algebraic varieties, not as algebraic varieties with the group G action). Indeed, let s be any orthogonal reflection. Then $s(\operatorname{Gr}_Q^\pm(k,\mathbb{C}^{2k})) = \operatorname{Gr}_Q^\mp(k,\mathbb{C}^{2k})$. Moreover, $\operatorname{Gr}_Q^\pm(k,\mathbb{C}^{2k})$ is isomorphic to $\operatorname{Gr}_Q(k-1,\mathbb{C}^{2k-1})$. To see this let $H = \mathbb{C}^{2k-1} \subset \mathbb{C}^{2k}$ be any hyperplane such that the restriction of Q on H is non-degenerate. Then we have obvious maps

$$\operatorname{Gr}_Q^\pm(k,\mathbb{C}^{2k}) \to \operatorname{Gr}_Q(k-1,\mathbb{C}^{2k-1}), \quad U \mapsto U \cap H.$$

It is easy to see that both maps are isomorphisms. The variety $\operatorname{Gr}_Q^\pm(k,\mathbb{C}^{2k})$ is called the spinor variety, denoted by $\mathbb{S}_k$.

The introduced varieties represent the complete list of minimal flag varieties of groups $\operatorname{SO}(V)$ and $\operatorname{Sp}(V)$. Namely, $\operatorname{Gr}_\omega(k,V)$, $k = 1,\dots,n/2$, is the minimal flag variety of $\operatorname{Sp}(V)$ corresponding to the fundamental weight ω_k (for example, $\operatorname{Gr}_\omega(1,V) = \mathbb{P}(V)$). $\operatorname{Gr}_Q(k,V)$, $0 < k < n/2$, is the minimal flag variety of $\operatorname{SO}(V)$ corresponding to the fundamental weight ω_k (for example, $\operatorname{Gr}_Q(1,V)$ is a quadric hypersurface in $\mathbb{P}(V)$). Finally, $\operatorname{Gr}_Q^\pm(k,\mathbb{C}^{2k})$ is the minimal flag variety of $\operatorname{SO}(\mathbb{C}^{2k})$ corresponding to the fundamental weights ω_{k-1} and ω_k.

In particular, the Picard group of all these varieties is isomorphic to $\mathbb{Z}$. Let us describe the embedding corresponding to the ample generator of $\mathbb{Z}$.

Consider the symplectic case $G = \operatorname{Sp}(V)$ first. Then we have the Plücker embedding $\operatorname{Gr}_\omega(k,V) \subset \mathbb{P}(\Lambda^k V)$. However, $\Lambda^k V$ is reducible as $\operatorname{Sp}(V)$-module, it contains the submodule $\omega \wedge \Lambda^{k-2} V$. The complement $\Lambda_0^k V$ of this submodule is an irreducible module with highest weight ω_k. The Plücker embedding in fact gives the embedding of $\operatorname{Gr}_\omega(k,V)$ in $\mathbb{P}(\Lambda_0^k V)$ as the projectivization of the cone of highest weight vectors. This embedding corresponds to the ample generator of the Pickard group.

Now consider the orthogonal case $G = \mathrm{SO}(V)$. Then we have the Plücker embedding $\mathrm{Gr}_Q(k, V) \subset \mathbb{P}(\Lambda^k V)$. Suppose first that $k < m$ if $n = 2m + 1$ or $k < m - 1$ if $n = 2m$. Then $\Lambda^k V$ is irreducible as G-module and has the highest weight ω_k. Therefore, this embedding corresponds to the ample generator of the Pickard group.

Finally, the embedding of the spinor variety $\mathbb{S}_m$ corresponding to the ample generator of the Pickard group can be described as follows. Let W be the spinor representation of $\mathrm{Spin}(2m-1)$ or the half-spinor representation of $\mathrm{Spin}(2m)$ (they are isomorphic with respect to the embedding $\mathrm{Spin}(2m-1) \subset \mathrm{Spin}(2m)$). Then $\mathbb{S}_m$ embeds into $\mathbb{P}(W)$ as the projectivization of the highest weight vector orbit.

To give a more precise model, let us recall the definition of the half-spinor representation. Let $V = \mathbb{C}^{2m}$ be an even-dimensional vector space with a non-degenerate symmetric scalar product Q. Let $\mathrm{Cl}(V)$ be the Clifford algebra of V. Recall that $\mathrm{Cl}(V)$ is in fact a superalgebra, $\mathrm{Cl}(V) = \mathrm{Cl}^0(V) \oplus \mathrm{Cl}^1(V)$, where $\mathrm{Cl}^0(V)$ (resp. $\mathrm{Cl}^1(V)$) is a linear span of elements of the form $v_1 \cdot \ldots \cdot v_r$, $v_i \in V$, r is even (resp. r is odd). Then we have

$$\mathrm{Spin}(V) = \{a \in \mathrm{Cl}^0(V) \,|\, a = v_1 \cdot \ldots \cdot v_r,\ v_i \in V,\ Q(v_i, v_i) = 1\}.$$

$\mathrm{Spin}(V)$ is a connected simply-connected algebraic group.

Given $a \in \mathrm{Cl}(V)$, $a = v_1 \cdot \ldots \cdot v_r$, $v_i \in V$, let $\overline{a} = (-1)^r v_r \cdot \ldots \cdot v_1$. This is a well-defined involution of $\mathrm{Cl}(V)$. Using it, we may define an action R of $\mathrm{Spin}(V)$ on V by formula $R(a)v = a \cdot v \cdot \overline{a}$. Then R is a double covering $\mathrm{Spin}(V) \to \mathrm{SO}(V)$.

Let $U \subset V$ be a maximal isotropic subspace, hence $\dim U = m$. Take also any maximal isotropic subspace U' such that $U \oplus U' = V$. For any $v \in U$ we define an operator $\rho(v) \in \mathrm{End}(\Lambda^* U)$ by the formula

$$\rho(v) \cdot u_1 \wedge \ldots \wedge u_r = v \wedge u_1 \wedge \ldots u_r.$$

For any $v \in U'$ we define an operator $\rho(v) \in \mathrm{End}(\Lambda^* U)$ by the formula

$$\rho(v) \cdot u_1 \wedge \ldots \wedge u_r = \sum_{i=1}^{r} (-1)^{i-1} Q(v, u_i) \wedge u_1 \wedge \ldots u_{i-1} \wedge u_{i+1} \wedge \ldots \wedge u_r.$$

These operators are well-defined and therefore by linearity we have a linear map $\rho : V \to \mathrm{End}(\Lambda^* U)$. It is easy to check that for any $v \in V$ we have $\rho(v)^2 = Q(v, v)\,\mathrm{Id}$. Therefore we have a homomorphism of associative algebras $\mathrm{Cl}(V) \to \mathrm{End}(\Lambda^* U)$, i.e., $\Lambda^* U$ is a $\mathrm{Cl}(V)$-module, easily seen to be irreducible. Therefore, $\Lambda^* U$ is also a $\mathrm{Spin}(V)$-module, now reducible. However, the even part $\Lambda^{ev} U$ is an irreducible $\mathrm{Spin}(V)$-module, called the half-spinor module $\mathcal{S}^+$. Now let us describe the embedding of the spinor variety $\mathbb{S}_m \subset \mathbb{P}(\mathcal{S}^+)$. This embedding will correspond to the ample generator of $\mathrm{Pic}(\mathbb{S}_m)$. By definition, $\mathbb{S}_m$ is an $\mathrm{SO}(V)$-orbit of a fixed maximal isotropic subspace, say U'. We claim that the normalizer of U' in $\mathrm{Spin}(V)$ (acting

on V via the representation R) is equal to the normalizer of $1 \in \Lambda^{ev}U$ in $\mathrm{Spin}(V)$ (acting on Λ^{ev} via the representation ρ). This will show that $\mathbb{S}_m$ is naturally isomorphic to the projectivization of the orbit of 1 in the half-spinor representation. Consider the map

$$\Psi : V \times \Lambda^{ev}U \to \Lambda^{odd}U, \quad (v, w) \mapsto \rho(v)w.$$

$\mathrm{Spin}\, V$ acts on V via the representation R and on Λ^{ev} and Λ^{odd} via the representation ρ. We claim that Ψ is $\mathrm{Spin}(V)$-equivariant. Indeed, for any $g \in \mathrm{Spin}(V)$ we have

$$\begin{aligned}\Psi(R(g)v, \rho(g)w) &= \rho(R(g)v)(\rho(g)w) = \rho(gv\overline{g})(\rho(g)w)\\ &= \rho(gv\overline{g}g)w = \rho(g)(\rho(v)w) = \rho(g)\Psi(v, w),\end{aligned}$$

since $g\overline{g} = 1$. It is easy to check that $\Psi(U', 1) = 0$ and if for any $v \in U'$ we have $\Psi(v, w) = 0$ then $w = \lambda 1$ for some $\lambda \in \mathbb{C}$. Suppose now that $g \in \mathrm{Spin}(V)$ normalizes $1 \in \Lambda^{ev}U$. Then for any $v \in U'$ we have

$$\Psi(R(g)v, \rho(g)1) = \rho(g)\Psi(v, 1) = 0,$$

therefore g normalizes U'. On the contrary, if g normalizes U' then for any $v \in U'$

$$0 = \Psi(R(g)(R(g^{-1})v), \rho(g)1) = \Psi(v, \rho(g)1),$$

therefore g normalizes $1 \in \Lambda^{ev}U$.

Example 2.3 (Severi varieties) Let $\mathbb{D}_{\mathbb{R}}$ denote one of four division algebras over $\mathbb{R}$: real numbers $\mathbb{R}$, complex numbers $\mathbb{C}$, quaternions $\mathbb{H}$, or octonions $\mathbb{O}$. Let $\mathbb{D} = \mathbb{D}_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$ be its complexification, therefore $\mathbb{D}$ is either $\mathbb{C}$, $\mathbb{C} \oplus \mathbb{C}$, $\mathrm{Mat}_2(\mathbb{C})$, or $\mathbb{C}a$ (the algebra of complex Cayley numbers). All these algebras have the standard involution. Let $\mathcal{H}_3(\mathbb{D})$ denote the 3×3 Hermitian matrices over $\mathbb{D}$. If $x \in \mathcal{H}_3(\mathbb{D})$ then we may write

$$x = \begin{pmatrix} c_1 & u_1 & u_2 \\ \overline{u_1} & c_2 & u_3 \\ \overline{u_2} & \overline{u_3} & c_3 \end{pmatrix}, \quad c_i \in \mathbb{C},\ u_i \in \mathbb{D}.$$

$\mathcal{H}_3(\mathbb{D})$ is a Jordan algebra with respect to the operation $x \circ y = \dfrac{xy + yx}{2}$, that is, for any a, b we have

$$a \circ b = b \circ a \quad \text{and} \quad ((a \circ a) \circ b) \circ a = (a \circ a) \circ (b \circ b),$$

see [J]. In case $\mathbb{D} = \mathbb{C}a$ this algebra is called the Albert algebra. The group of automorphisms $G = \mathrm{Aut}(\mathcal{H}_3)$ is a semisimple algebraic group of type SO_3, SL_3, Sp_6, or F_4, respectively. The action of G on $\mathcal{H}_3$ has two irreducible components: scalar matrices and traceless matrices. It is possible to enlarge

the group of automorphisms by considering the group of norm similarities. Namely, $\mathcal{H}_3(\mathbb{D})$ has a cubic form det defined by the formula

$$\det(x) = \frac{1}{6}\left((\operatorname{Tr} x)^3 + 2\operatorname{Tr}(x^3) - 3(\operatorname{Tr} x)\operatorname{Tr}(x^2)\right).$$

The group of norm similarities $\tilde{G}$ is defined as a subgroup in $\mathrm{SL}(\mathcal{H}_3)$ preserving det. $\tilde{G}$ is a semisimple algebraic group of type SL_3, $\mathrm{SL}_3 \times \mathrm{SL}_3$, SL_6, or E_6, respectively. The representation of $\tilde{G}$ on $\mathcal{H}_3$ is irreducible. The projectivization of the highest weight vector orbit is called the Severi variety corresponding to $\mathbb{D}$. One may think of Severi varieties as of complexifications of projective planes over $\mathbb{D}_{\mathbb{R}}$.

The Severi variety corresponding to $\mathbb{C}a$ is the minimal flag variety of E_6 (corresponding to the fundamental weight ω_1) embedded into $\mathbb{P}^{26}$ by the ample generator of its Picard group. The corresponding elements of the Albert algebra are known as elements of rank 1, see [J]. Other Severi varieties have the following description.

If $\mathbb{D}_{\mathbb{R}} = \mathbb{R}$ then $\mathcal{H}_3$ is equal to the Jordan algebra of symmetric 3×3 complex matrices and det is the standard determinant. $\tilde{G} = \mathrm{SL}_3$ acts on symmetric matrices as on quadratic forms, the Severi variety is the projectivization of the cone of rank 1 symmetric matrices. Therefore, this Severi variety is isomorphic to $\mathbb{P}^2$ in the second Veronese embedding.

If $\mathbb{D}_{\mathbb{R}} = \mathbb{C}$ then $\mathcal{H}_3$ is equal to the Jordan algebra of 3×3 complex matrices and det is the standard determinant. $\tilde{G} = \mathrm{SL}_3 \times \mathrm{SL}_3$ acts on matrices by left and right multiplication, the Severi variety is the projectivization of the cone of rank 1 matrices. Therefore, this Severi variety is isomorphic to $\mathbb{P}^2 \times \mathbb{P}^2$ in the Segre embedding.

If $\mathbb{D}_{\mathbb{R}} = \mathbb{H}$ then $\mathcal{H}_3$ is equal to the Jordan algebra $\Lambda^2\mathbb{C}^6$ with the multiplication given by $\alpha_1 \circ \alpha_2 = \alpha_1 \wedge \alpha_2$, where we identify $\Lambda^2\mathbb{C}^6$ and $\Lambda^4\mathbb{C}^6$ by means of a standard symplectic form ω (it is easy to see that $\Lambda^4\mathbb{C}^6 = \omega \wedge \Lambda^2\mathbb{C}^6$). The function det is defined as $\det(\alpha) = \alpha \wedge \alpha \wedge \alpha$ (in the matrix form det is equal to the Pfaffian of the skew-symmetric matrix corresponding to α). $\tilde{G} = \mathrm{SL}_6$ acts on $\Lambda^2\mathbb{C}^6$ traditionally. The Severi variety is the projectivization of the variety of decomposable bivectors of the form $u \wedge v$, $u, v \in \mathbb{C}^6$. Therefore, this Severi variety is isomorphic to $\mathrm{Gr}(2, \mathbb{C}^6)$ in the Plücker embedding.

For $\mathbb{D}_{\mathbb{R}}$ equal to $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$ it is possible to define higher dimensional Severi varieties in the same way (isomorphic to $\mathbb{P}^n$ in the second Veronese embedding, $\mathbb{P}^n \times \mathbb{P}^n$ in the Segre embedding, or $\mathrm{Gr}(2, \mathbb{C}^{2n+2})$ in the Plücker embedding). However, for $\mathbb{D}_{\mathbb{R}} = \mathbb{O}$ only the case of 3×3 matrices makes sense.

Example 2.4 (Adjoint varieties) For the adjoint representation of $\mathrm{SL}_n = \mathrm{SL}(V)$ there is a natural notion of the discriminant defined as follows. For any operator $A \in \mathfrak{sl}(V)$ let $P_A = \det(t\,\mathrm{Id} - A)$ be the characteristic polynomial. Then its discriminant, $D(A) = D(P_A)$ is a homogeneous form on $\mathfrak{sl}(V)$ of

degree $n^2 - n$. Clearly $D(A) \neq 0$ if and only if all eigenvalues of A are distinct, that is, if A is a regular semisimple operator. This notion can be carried over any simple Lie algebra. Before doing that, notice that $D(A)$ can be also defined as follows. Consider the characteristic polynomial

$$Q_A = \det(t\,\mathrm{Id} - \mathrm{ad}(A)) = \sum_{i=0}^{n^2-1} t^i D_i(A)$$

of the adjoint operator $\mathrm{ad}(A)$. If $\lambda_1, \ldots, \lambda_r$ are eigenvalues of A (counted with multiplicities) then the set of eigenvalues of $\mathrm{ad}(A)$ consists of $n-1$ zeros and $\lambda_i - \lambda_j$, $i, j = 1, \ldots, n$, $i \neq j$. Therefore, $D_0(A) = \ldots = D_{n-2}(A) = 0$ and $D_{n-1}(A)$ coincides with $D(A)$ up to a non-zero scalar.

Suppose now that $\mathfrak{g}$ is a simple Lie algebra of rank r. Let $\dim \mathfrak{g} = n$. For any $x \in \mathfrak{g}$ let

$$Q_x = \det(t\,\mathrm{Id} - \mathrm{ad}(x)) = \sum_{i=0}^{n} t^i D_i(x)$$

be the characteristic polynomial of the adjoint operator. Then $D(x) = D_r(x)$ is called the discriminant of x. Clearly, D is a homogeneous Ad-invariant polynomial on $\mathfrak{g}$ of degree $n - r$. Since the dimension of the centralizer $\mathfrak{g}_x$ of any element $x \in \mathfrak{g}$ is greater or equal to r, it follows that $D_0 = \ldots = D_{r-1} = 0$, therefore $D(x) = 0$ if and only if $\mathrm{ad}(x)$ has the eigenvalue 0 with multiplicity $> r$. We claim that actually $D(x) \neq 0$ if and only if x is regular semisimple (recall that x is called regular if $\dim \mathfrak{g}_x = r$). Indeed, if x is semisimple then $\mathrm{ad}(x)$ is a semisimple operator, therefore $D(x) = 0$ if and only if the dimension of the centralizer $\mathfrak{g}_x$ is greater than r, i.e. x is not regular. If x is not semisimple then we take the Jordan decomposition $x = x_s + x_n$, where x_s is semisimple, x_n is nilpotent, and $[x_s, x_n] = 0$. Then x_s is automatically not regular, therefore since $Q_x = Q_{x_s}$ we have $D(x) = D(x_s) = 0$ (of course, this last equality also follows from the Ad-invariance of D and the fact that the Ad-orbit of x contains the Ad-orbit of x_s in its closure).

To study $D(x)$ further, we can use the Chevalley restriction theorem (see [PV]) $\mathbb{C}[\mathfrak{g}]^G = \mathbb{C}[\mathfrak{t}]^W$, where $\mathfrak{t} \subset \mathfrak{g}$ is any Cartan subalgebra and W is the Weil group. Let $\Delta \subset \mathfrak{t}^*$ be the root system, $\#\Delta = n - r$. Then, clearly, for any $x \in \mathfrak{t}$ $D(x) = 0$ if and only if x is not regular if and only if $\alpha(x) = 0$ for some $\alpha \in \Delta$. Since $\deg D = n - r$ and $D|_{\mathfrak{t}}$ is W-invariant, it easily follows that

$$D|_{\mathfrak{t}} = \prod_{\alpha \in \Delta} \alpha.$$

The Weyl group acts transitively on the set of roots of the same length. Therefore, $D|_{\mathfrak{t}}$, and hence D, is irreducible if and only if all roots in Δ have the same length, i.e. Δ is of type A, D, or E. If Δ is of type B, C, F, or G, we have $\Delta = \Delta_s \cup \Delta_l$, where Δ_s is the set of short roots and Δ_l is the set

of long roots. Then we have $D = D_l D_s$, where D_l and D_s are irreducible polynomials and

$$D_l|_{\mathfrak{t}} = \prod_{\alpha \in \Delta_l} \alpha, \quad D_s|_{\mathfrak{t}} = \prod_{\alpha \in \Delta_s} \alpha.$$

In the $A - D - E$ case we also set $D_l = D$ to simplify notations.

We are going to show that D_l is also the discriminant in our regular sense. The adjoint representation $\mathrm{Ad} : G \to \mathrm{GL}(\mathfrak{g})$ is irreducible. In $A - D - E$ case let $\mathcal{O} = \mathcal{O}_l$ be the Ad-orbit of any root vector, In $B - C - F - G$ case let $\mathcal{O}_l \subset \mathfrak{g}$ (resp. $\mathcal{O}_s \subset \mathfrak{g}$) be the Ad-orbit of any long root vector (resp. any short root vector). Then $\mathcal{O}_l$ is the orbit of the highest weight vector. Its projectivization is called the adjoint variety. Both orbits $\mathcal{O}_l$ and $\mathcal{O}_s$ are conical. Let $X_l = \mathbb{P}(\mathcal{O}_l)$, $X_s = \overline{\mathbb{P}(\mathcal{O}_s)}$.

Theorem 2.5 *D_l is the discriminant of X_l. D_s is the discriminant of X_s.*

Proof. We identify $\mathfrak{g}$ and $\mathfrak{g}^*$ via the Killing form. Let $\alpha \in \Delta$ be any root, $e_\alpha \in \mathfrak{g}$ the corresponding root vector. Since $[\mathfrak{g}, e_\alpha]^\perp = \mathfrak{g}_{e_\alpha}$, we have

$$\left(\overline{\mathbb{P}(\mathrm{Ad}(G) \cdot e_\alpha)}\right)^* = \overline{\mathbb{P}(\mathrm{Ad}(G) \cdot \mathfrak{g}_{e_\alpha})}.$$

Let

$$\mathfrak{t}_\alpha = \{x \in \mathfrak{t} \,|\, \alpha(x) = 0\}, \quad \hat{\mathfrak{t}}_\alpha = \mathfrak{t}_\alpha + e_\alpha.$$

Then, clearly, $\hat{\mathfrak{t}}_\alpha \in \mathfrak{g}_{e_\alpha}$. Moreover, if α is long then $D_l|_{\hat{\mathfrak{t}}_\alpha} = D_l|_{\mathfrak{t}_\alpha} = 0$. If α is short then $D_s|_{\hat{\mathfrak{t}}_\alpha} = D_s|_{\mathfrak{t}_\alpha} = 0$. Therefore, in order to prove Theorem it suffices to check that

$$\dim \mathrm{Ad}(G) \cdot \hat{\mathfrak{t}}_\alpha = n - 1. \tag{2.1}$$

Clearly, for generic $x \in \hat{\mathfrak{t}}_\alpha$ we have

$$\dim \mathrm{Ad}(G) \cdot \hat{\mathfrak{t}}_\alpha = \dim G + \dim \hat{\mathfrak{t}}_\alpha - \dim \mathrm{Tran}(x, \hat{\mathfrak{t}}_\alpha), \tag{2.2}$$

where

$$\mathrm{Tran}(x, \hat{\mathfrak{t}}_\alpha) = \{g \in G \,|\, \mathrm{Ad}(g)x \in \hat{\mathfrak{t}}_\alpha\}.$$

Let $x = y + e_\alpha \in \hat{\mathfrak{t}}_\alpha$, where $y \in \mathfrak{t}_\alpha$ is such that for any $\beta \in \Delta \setminus \{\pm\alpha\}$ we have $\beta(y) \neq 0$. Suppose that $\mathrm{Ad}(g)x \in \hat{\mathfrak{t}}_\alpha$. Since $[e_\alpha, \mathfrak{t}_\alpha] = 0$, it easily follows that $\mathrm{Ad}(g)y \in \mathfrak{t}_\alpha$ and $\mathrm{Ad}(g)e_\alpha = e_\alpha$. Under our assumptions on y this means that

$$\mathrm{Tran}(x, \hat{\mathfrak{t}}_\alpha) = (N_G(\mathfrak{t}_\alpha))_{e_\alpha},$$

where $N_G(\mathfrak{t}_\alpha)$ is the normalizer of t_α in G. Therefore, in order to prove (2.1) using (2.2), it suffices to check that

$$\dim(\mathfrak{g}_{\mathfrak{t}_\alpha})_{e_\alpha} = r, \tag{2.3}$$

where $\mathfrak{g}_{\mathfrak{t}_\alpha}$ is the centralizer of t_α in $\mathfrak{g}$. Now, $\mathfrak{g}_{\mathfrak{t}_\alpha}$ is a Levi subalgebra equal to $\mathfrak{t} + \mathbb{C}e_\alpha + \mathbb{C}e_{-\alpha}$. Therefore, $(\mathfrak{g}_{\mathfrak{t}_\alpha})_{e_\alpha} = \mathfrak{t}_\alpha + \mathbb{C}e_\alpha$ and (2.3) follows. $\square$

2.2 The Pyasetskii Pairing

2.2.A Actions With Finitely Many Orbits

In [Py] Pyasetskii has shown that if a connected algebraic group acts linearly on a vector space with a finite number of orbits then the dual representation has the same property and, moreover, the number of orbits is the same. In this section we deduce this result from the projective duality and give some examples and applications.

Theorem 2.6 *Suppose that a connected algebraic group G acts on a projective space $\mathbb{P}^n$ with a finite number of orbits. Then the dual action $G : \mathbb{P}^{n*}$ has the same number of orbits. Let $\mathbb{P}^n = \bigsqcup_{i=1}^{N} \mathcal{O}_i$ and $\mathbb{P}^{n*} = \bigsqcup_{i=1}^{N} \mathcal{O}'_i$ be the orbit decompositions. Let $\mathcal{O}_0 = \mathcal{O}'_0 = \emptyset$. Then the bijection is defined as follows: $\mathcal{O}_i$ corresponds to $\mathcal{O}'_j$ if and only if $\overline{\mathcal{O}_i}$ is projectively dual to $\overline{\mathcal{O}'_j}$.*

Proof. Indeed, take any G-orbit $\mathcal{O} \subset \mathbb{P}^{n*}$. If $\overline{\mathcal{O}} \neq \mathbb{P}^{n*}$ then the projectively dual variety $\overline{\mathcal{O}}^*$ is the closure of some orbit $\mathcal{O}' \subset \mathbb{P}^n$, since it is G-invariant. Therefore, by the Reflexivity Theorem we have $\overline{\mathcal{O}} = \overline{\mathcal{O}'}^*$. □

Remark 2.7 If the number of orbits for the action $G : \mathbb{P}^n$ is infinite then, in general, there is no natural bijection between orbits in $\mathbb{P}^n$ and $\mathbb{P}^{n*}$. However, it can be shown (see e.g. [Py]) that the modalities of these actions are equal (modality is equal to the maximal number of parameters that the family of orbits can depend on). This also easily follows from the arguments involving projective duality.

The linear action of an algebraic group on a vector space V is called conical if any G-orbit $\mathcal{O}$ is conical, i.e. $\mathcal{O}$ is preserved by homotheties. The action is conical if and only if for any $v \in V$, v is contained in the tangent space $\mathfrak{g} \cdot v$ to the orbit Gv. For example, if there are only finitely many orbits then the action is conical. If the action is conical then its non-zero orbits are in a 1-1 correspondence with orbits of the projectivization $G : \mathbb{P}(V)$.

Proposition 2.8 *If the linear action $G : V$ is conical then the dual action $G : V^*$ is conical as well.*

Proof. We can embed $V = \mathbb{C}^n$ and $V^* = (\mathbb{C}^n)^*$ in projective spaces $\mathbb{P}^n$ and $\mathbb{P}^{n*}$ as the affine charts. These embeddings can be done equivariantly. Points of V^* correspond to hyperplanes in $\mathbb{P}^n$ not passing through the origin $0 \in V \subset \mathbb{P}^n$. Suppose that $\mathcal{O} \subset V^*$ is a non-conical orbit. Then the dual variety $\overline{\mathcal{O}}^* \subset \mathbb{P}^n$ intersects with $\mathbb{C}^n$ non-trivially. Therefore, $\overline{\mathcal{O}}^* \subset \mathbb{P}^n$ is the closure of a conical variety in $\mathbb{C}^n$. Therefore, its dual variety $\overline{\mathcal{O}}^{**} \subset \mathbb{P}^{n*}$ does not intersect $(\mathbb{C}^n)^*$. But this contradicts the Reflexivity Theorem. □

Now the Pyasetskii Theorem is an easy corollary of Theorem 2.6 and Proposition 2.8.

Corollary 2.9 ([Py]) *Suppose that a connected algebraic group G acts linearly on a vector space V with a finite number of orbits. Then the dual action $G : V^*$ has the same number of orbits. Let $V = \bigsqcup_{i=1}^{N} \mathcal{O}_i$ and $V^* = \bigsqcup_{i=1}^{N} \mathcal{O}'_i$ be the orbit decompositions. Then the bijection is defined as follows: $\mathcal{O}_i$ corresponds to $\mathcal{O}'_j$ if and only if $\mathbb{P}(\overline{\mathcal{O}_i})$ is projectively dual to $\mathbb{P}(\overline{\mathcal{O}'_j})$.*

Recall that the Reflexivity Theorem is equivalent to the formula (1.2) that $\mathrm{Lag}(\mathcal{O}_i) = \mathrm{Lag}(\mathcal{O}'_j)$, where for any conical variety $Y \subset V$ we denote by $\mathrm{Lag}(Y)$ the closure of the conormal bundle $N^*_{Y_{sm}} V$ in the cotangent bundle $T^*V = V \oplus V^*$. The varieties $\mathrm{Lag}(\mathcal{O}_i)$, $i = 1, \ldots, N$ have a nice interpretation. Namely, the action of $\mathfrak{g}$ (the Lie algebra of G) on V is given by the element of

$$\mathrm{Hom}(\mathfrak{g}, \mathrm{End}(V)) = \mathfrak{g}^* \otimes V^* \otimes V = \mathrm{Hom}(V^* \otimes V, \mathfrak{g}^*).$$

The corresponding bilinear map $\mu : V \oplus V^* \to \mathfrak{g}^*$ is called the moment map. Let $L = \mu^{-1}(0)$ be its zero fiber. Then it is easy to see that $L = \cup_i \mathrm{Lag}(\mathcal{O}_i)$ is the decomposition of L into irreducible components.

The large class of linear actions of reductive groups with finitely many orbits is provided by graded semisimple Lie algebras. Suppose that $\mathfrak{g} = \oplus_{k \in \mathbb{Z}} \mathfrak{g}_k$ is a graded semi-simple Lie algebra of the connected semi-simple group G. Then there exists a unique semisimple element $\xi \in \mathfrak{g}_0$ such that

$$g_k = \{x \in \mathfrak{g} \mid [\xi, x] = kx\}.$$

The connected component H of the centralizer $G_\xi \subset G$ is a reductive subgroup of G called the Levi subgroup (because it is the Levi part of the parabolic subgroup). $\mathfrak{g}_0$ is the Lie algebra of H. H acts on each graded component $\mathfrak{g}_k$.

Theorem 2.10 ([Ri, Vi]) *H acts on $\mathfrak{g}_k$ with finitely many orbits.*

Proof. Clearly all elements of $\mathfrak{g}_k$ are nilpotent in $\mathfrak{g}$. Since G has finitely many nilpotent orbits, it is sufficient to prove that for any nilpotent G-orbit $\mathcal{O}$ the intersection $\mathcal{O} \cap \mathfrak{g}_k$ consists of finitely many H-orbits. By the general Richardson Lemma (see [Ri, PV]), it is enough to check that for any $x \in \mathcal{O} \cap \mathfrak{g}_k$ the tangent space $[\mathfrak{g}_0, x]$ to the H-orbit $\mathrm{Ad}(H)x$ is equal to the intersection of $\mathfrak{g}_k$ with the tangent space $[\mathfrak{g}, x]$ to the G-orbit $\mathrm{Ad}(G)x$. But this is clear:

$$[\mathfrak{g}, x] \cap \mathfrak{g}_k = [\oplus_i \mathfrak{g}_i, x] \cap \mathfrak{g}_k = \oplus_i [\mathfrak{g}_i, x] \cap \mathfrak{g}_k = [\mathfrak{g}_0, x].$$

Theorem is proved. □

Notice that the dual H-module $(\mathfrak{g}_k)^*$ is isomorphic to $\mathfrak{g}_{-k}$ under the Killing form and the moment map $\mathfrak{g}_k \times \mathfrak{g}_{-k} \to \mathfrak{g}_0$ is given by the Lie bracket.

2.2.B The Multisegment Duality.

Apart from actions associated with $\mathbb{Z}$-graded semisimple Lie algebras, another class of actions with finitely many orbits is provided by the theory of representations of quivers (see [Ga]). These two classes overlap: the representations of quivers of type A give the same class of actions as the standard gradings of of $\mathrm{SL}(V)$. The Pyasetskii pairing in this case was studied in a series of papers under the name of the multisegment duality, or the Zelevinsky involution. Here we give an overview of some of these results.

We fix a positive integer r and consider the set $S = S_r$ of pairs of integers (i, j) such that $1 \le i \le j \le r$. Let $\mathbb{Z}_+^S$ denote the semigroup of families $m = (m_{ij})_{(i,j)\in S}$ of non-negative integers indexed by S. The set S and the semigroup $\mathbb{Z}_+^S$ have several interpretations. First, S_r is naturally identified with the set of positive roots of type A_r. Namely, if $\alpha_1, \ldots, \alpha_r$ denote simple roots then each $(i, j) \in S$ corresponds to a positive root $\alpha_i + \alpha_{i+1} + \ldots + \alpha_j$. Another useful interpretation is to regard a pair $(i, j) \in S$ as a segment $[i, j] = \{i, i + 1, \ldots, j\}$ in $\mathbb{Z}$. A family $m = m_{ij} \in \mathbb{Z}_+^S$ can be regarded as a collection of segments, containing m_{ij} copies of each $[i, j]$. Thus, elements of $\mathbb{Z}_+^S$ can be called multisegments. The weight $|m|$ of a multisegment m is defined as a sequence $\gamma = \{d_1, \ldots, d_r) \in \mathbb{Z}_+^r$ given by

$$d_i = \sum_{i\in[k,l]} m_{kl} \quad \text{for } i = 1, \ldots, r.$$

In other words, $|m|$ records how many segments of m contain any given number $i \in [1, r]$. For any $\gamma \in \mathbb{Z}_+^r$ we set $\mathbb{Z}_+^S(\gamma) = \{m \in \mathbb{Z}_+^S \mid |m| = \gamma\}$. In the language of positive roots, $\mathbb{Z}_+^r$ is the semigroup generated by simple roots. Under this identification, the elements $m \in \mathbb{Z}_S^+(\gamma)$ become the partitions of γ into a sum of positive roots. The cardinality of $\mathbb{Z}_S^+(\gamma)$ is known as a Kostant partition function (of γ).

Another important interpretation of $\mathbb{Z}_S^+(\gamma)$ is that it parametrizes isomorphism classes of representations of quivers of type A. Let A_r be the quiver equal to the Dynkin diagram of type A_r, where all edges are oriented from the left to the right. Let A_r^* be a dual quiver with all orientations reversed. The representation of A_r with the dimension vector $\gamma = \{d_1, \ldots, d_r) \in \mathbb{Z}_+^r$ is the collection of vector spaces $\mathbb{C}^{d_1}, \ldots, \mathbb{C}^{d_r}$ and linear maps

$$\varphi_1 : \mathbb{C}^{d_1} \to \mathbb{C}^{d_2}, \ldots, \varphi_{r-1} : \mathbb{C}^{d_{r-1}} \to \mathbb{C}^{d_r}.$$

The representation of A_r^* with the dimension vector $\gamma = \{d_1, \ldots, d_r) \in \mathbb{Z}_+^r$ is the collection of vector spaces $\mathbb{C}^{d_1}, \ldots, \mathbb{C}^{d_r}$ and linear maps

$$\varphi_1 : \mathbb{C}^{d_2} \to \mathbb{C}^{d_1}, \ldots, \varphi_{r-1} : \mathbb{C}^{d_r} \to \mathbb{C}^{d_{r-1}}.$$

Therefore, representations of A_r with dimension vector γ are parametrized by points of a vector space $V(\gamma) = \bigoplus_{i=1}^{r-1} \mathrm{Hom}(\mathbb{C}^{d_i}, \mathbb{C}^{d_{i+1}})$. Representations

of A_r^* with dimension vector γ are parametrized by points of a vector space $V(\gamma)^* = \bigoplus_{i=1}^{r-1} \mathrm{Hom}(\mathbb{C}^{d_{i+1}}, \mathbb{C}^{d_i})$. Notice that vector spaces $V(\gamma)$ and $V(\gamma)^*$ are naturally dual to each other. Moreover, $V(\gamma)$ and $V(\gamma)^*$ are dual modules of the group $G(\gamma) = \mathrm{GL}_{d_1} \times \ldots \times \mathrm{GL}_{d_r}$ with respect to the natural action. These actions could also be described in terms of graded Lie algebras. Namely, consider the Lie algebra $\mathfrak{g} = \mathfrak{gl}_{d_1+\ldots+d_r}$. Elements of $\mathfrak{g}$ can be represented as block matrices with diagonal blocks of shapes $d_1 \times d_1$, ..., $d_r \times d_r$. This determines $\mathbb{Z}$-grading: block diagonal matrices have grade 0, etc. Then $\mathfrak{g}_0$ is a Lie algebra of $G(\gamma)$ and the action of $G(\gamma)$ on $V(\gamma)$ and $V(\gamma)^*$ is equivalent to the action of $G(\gamma)$ on $\mathfrak{g}_{-1}$ and $\mathfrak{g}_1$.

The orbits of $G(\gamma)$ on $V(\gamma)$ (resp. $V(\gamma)^*$) correspond to isoclasses of representations of A_r (resp. A_r^*) with dimension vector γ. These orbits are parametrized by elements of $\mathbb{Z}_S^+(\gamma)$. Elements of S parametrize indecomposable A_r-modules (or A_r^*-modules). Namely, each $(i,j) \in S$ corresponds to an indecomposable module R_{ij} with dimension vector

$$|(i,j)| = (0^{i-1}, 1^{j-i+1}, 0^{r-j})$$

and all maps are isomorphisms, if it is possible, or zero maps otherwise. Then any family $(m_{ij}) \in \mathbb{Z}_S^+$ corresponds to an A_r (or A_r^*) module $\oplus_S R_{ij}^{m_{ij}}$.

By Pyasetskii Theorem 2.9, there is a natural bijection of $G(\gamma)$-orbits in $V(\gamma)$ and $V(\gamma)^*$. Therefore, there exists a natural involution ζ of $\mathbb{Z}_+^S(\gamma)$, which can be extended to a weight-preserving involution of $\mathbb{Z}_+^S$ called the multisegment duality.

An explicit description of ζ was found in [KZ] using the Poljak Theorem [Po] from the combinatorial theory of networks. To formulate it, we need the following definition. For any multisegment $m \in \mathbb{Z}_+^S$ the ranks $r_{ij}(m)$ are given by

$$r_{ij}(m) = \sum_{[i,j] \subset [k,l]} m_{kl}.$$

It is easy to see that the multisegment m can be recovered from its ranks by formula

$$m_{ij} = r_{ij}(m) - r_{i-1,j}(m) + r_{i-1,j+1}(m).$$

If the multisegment corresponds to the representation of A_r given by

$$(\varphi_1, \ldots, \varphi_{r-1}) \in V(\gamma)$$

then r_{ij} is equal to the rank of the map $\varphi_{j-1} \circ \ldots \circ \varphi_{i+1} \circ \varphi_i$. In particular, $r_{ii} = d_i$.

For any $(i,j) \in S$ let T_{ij} denote the set of all maps $\nu : [1,i] \times [j,r] \to [i,j]$ such that $\nu(k,l) \leq \nu(k',l')$ whenever $k \leq k'$, $l \leq l'$ (in other words, ν is a morphism of partially ordered sets, where $[1,i] \times [j,r]$ is supplied with the product order).

Theorem 2.11 ([KZ]) *For every $m = (m_{ij}) \in \mathbb{Z}_+^S$ we have*

$$r_{ij}(\zeta(m)) = \min_{\nu \in T_{ij}} \sum_{(k,l) \in [1,i] \times [j,r]} m_{\nu(k,l)+k-i, \nu(k,l)+l-j}.$$

The inductive description of ζ was given in [MW]. Let $m = (m_{ij})$ be a multisegment of weight $\gamma = (d_1, \ldots, d_r)$. We set $i_1 = \min\{i \mid d_i \neq 0\}$ and define the sequence of indices $j_1, \ldots, j_p$ as follows:

$$j_1 = \min\{j \mid m_{i_1 j} \neq 0\}, \ldots, j_{t+1} = \min\{j \mid j > j_t,\ m_{i_1+t,j} \neq 0\},$$

where $t = 1, \ldots, p-1$. The sequence terminates when j_{p+1} does not exist. Let $i_t = i_1 + t - 1$ for $t = 1, \ldots, p+1$. We associate to m the multisegment m' given by

$$m' = m - (i_1, j_1) - (i_2, j_2) - \ldots - (i_p, j_p) + (i_2, j_1) + (i_3, j_2) + \ldots + (i_{p+1}, j_p),$$

where we use the convention that $(i, j) = 0$ unless $1 \leq i \leq j \leq r$.

Theorem 2.12 ([MW]) *If the multisegment m' is associated to m then*

$$\zeta(m) = \zeta(m') + (i_1, i_p).$$

The involution ζ can also be described in terms of irreducible finite-dimensional representations of affine Hecke algebras and in terms of canonical bases for quantum groups, see [KZ].

2.3 Parabolic Subgroups With Abelian Unipotent Radical

Let L be a simple algebraic group and $P \subset L$ a parabolic subgroup with abelian unipotent radical. In this case $\mathfrak{l} = \operatorname{Lie} L$ admits a $\mathbb{Z}$-grading with only three non-zero parts:

$$\mathfrak{l} = \mathfrak{l}_{-1} \oplus \mathfrak{l}_0 \oplus \mathfrak{l}_1.$$

Such a grading is said to be short. Here $\mathfrak{l}_0 \oplus \mathfrak{l}_1 = \operatorname{Lie} P$ and $\exp(\mathfrak{l}_1)$ is the abelian unipotent radical of P.

There exists a unique semisimple element $\xi \in \mathfrak{l}_0$ such that

$$l_k = \{x \in \mathfrak{l} \mid [\xi, x] = kx\}.$$

We denote by G the connected component of the centralizer $L_\xi \subset L$. Then $\mathfrak{l}_0 = \mathfrak{g} = \operatorname{Lie} G$ and by Theorem 2.10 G acts on $\mathfrak{l}_{\pm 1}$ with a finite number of orbits. In this section we describe explicitly these G-orbits and the Pyasetskii pairing. These results are well-known, we follow mainly [Pan] and [MRS].

We denote by T the maximal torus in G (and hence in L), $\mathfrak{t} = \operatorname{Lie} T$, Δ is the root system of $(\mathfrak{l}, \mathfrak{t})$,

$$\Delta = \Delta_{-1} \cup \Delta_0 \cup \Delta_1$$

is the partition corresponding to the short grading. We fix a Borel subgroup $B \subset G$ containing T. This choice determines a set of positive roots

$$\Delta_0^+ \subset \Delta_0 \quad \text{and} \quad \Delta^+ = \Delta_0^+ \cup \Delta^1 \subset \Delta.$$

We denote by γ the highest root in Δ^+, in fact in Δ_1. Π is the set of simple roots in Δ^+, then $\Pi_0 = \Pi \cap \Delta_0$ is the set of simple roots in Δ_0^+. Let W (resp. W_0) be the Weil group of $\mathfrak{l}$ (resp. of $\mathfrak{g}$) with respect to $\mathfrak{t}$. For $\alpha \in \Delta$, we let w_α denote the corresponding reflection in W, $\alpha^\vee = \dfrac{2\alpha}{(\alpha, \alpha)}$ the corresponding coroot, and e_α a non-zero root vector.

Proposition 2.13 *The representation of $\mathfrak{g}$ on $\mathfrak{l}_1$ is faithful and irreducible.*

Proof. Indeed, $\mathfrak{k} = \{x \in \mathfrak{g} \mid \operatorname{ad}(x)\mathfrak{l}_1 = 0\}$ is an ideal in $\mathfrak{g}$. Since $\mathfrak{l}_1$ and $\mathfrak{l}_{-1}$ are dual $\mathfrak{g}$-modules, $[\mathfrak{k}, \mathfrak{l}_{-1}] = 0$. It follows that $\mathfrak{k}$ is an ideal in $\mathfrak{l}$. Therefore, $\mathfrak{k} = 0$ since $\mathfrak{l}$ is simple.

Suppose now that $\mathfrak{l}_1$ is not irreducible, $\mathfrak{l}_1 = \mathfrak{l}_1^1 \oplus \mathfrak{l}_1^2$ is the $\mathfrak{g}$-module decomposition. Since $\mathfrak{l}_{-1}$ is dual to $\mathfrak{l}_1$, we have $\mathfrak{l}_{-1} = \mathfrak{l}_{-1}^1 \oplus \mathfrak{l}_{-1}^2$, where $\mathfrak{l}_{-1}^i = (\mathfrak{l}_1^j)^\perp$ for $i \neq j$. Then $[\mathfrak{l}_{-1}^i, \mathfrak{l}_1^j] = 0$ for $i \neq j$. Indeed, for any $x \in \mathfrak{l}_{-1}^i$, $y \in \mathfrak{l}_0$, $z \in \mathfrak{l}_1^j$ we have $(y, [x, z]) = -([x, y], z) = 0$, therefore $[x, z] = 0$. It follows that $\mathfrak{l}_{-1}^i \oplus [\mathfrak{l}_{-1}^i, \mathfrak{l}_1^i] \oplus \mathfrak{l}_1^i$ is an ideal in l. But $\mathfrak{l}$ is simple. Contradiction. $\square$

Therefore, the center of $\mathfrak{g}$ is one-dimensional, spanned by ξ, P is a maximal parabolic subgroup, and $\#\Pi_0 = \#\Pi - 1$. Thus, there is a unique simple root in Δ_1. Call it β.

Proposition 2.14

(a) *β is long.*
(b) *The β-height of any root $\alpha_0 \in \Delta_1$, i.e. the coefficient n_β in the sum $\alpha_0 = n_\beta \beta + \sum_{\alpha \in \Pi_0} n_\alpha \alpha$, is equal to 1.*
(c) *For any $\alpha, \alpha' \in \Delta_1$, $(\alpha, \alpha') \geq 0$.*
(d) *W_0 acts transitively on long roots in Δ_1.*

Proof. (a) Indeed, since $\mathfrak{l}_1$ is an irreducible $\mathfrak{g}$-module, β is the unique lowest weight of the $\mathfrak{g}$-module $\mathfrak{l}_1$. The longest element in W_0 takes β to the highest weight of $\mathfrak{l}_1$, i.e. to γ. Hence β is long.

(b) Since $\mathfrak{l}_1 = U\mathfrak{g} \cdot \beta$, the β-height of any root in Δ_1 is equal to 1.

(c) Suppose that $\alpha, \alpha' \in \Delta_1$. Since $[e_\alpha, e_{\alpha'}] = 0$, it follows that $\alpha + \alpha'$ is not a root, therefore, $(\alpha, \alpha') \geq 0$.

(d) Suppose that $\alpha_0 \in \Delta_1$ is a long root, $\alpha_0 = \beta + \sum_{\alpha \in \Pi_0} n_\alpha \alpha$. If we have $(\alpha_0, \alpha) < 0$ for some $\alpha \in \Pi_0$ then $w_\alpha(\alpha_0) > \alpha_0$ and we may finish by

induction. Suppose that $(\alpha_0, \alpha) \geq 0$ for all $\alpha \in \Pi_0$. Since $(\alpha_0, \beta) \geq 0$ by (c), α_0 is a dominant weight for $\mathfrak{l}$. Since α_0 is long, it follows that $\alpha_0 = \delta$. Therefore, all long roots in Δ_1 are W_0-conjugate to δ. □

On the contrary, if $\mathfrak{l}$ is an arbitrary simple Lie algebra, $\Delta \supset \Pi$ is the root system and the simple roots, $\beta \in \Pi$, and the β-height of the highest root is equal to 1, then, clearly, the maximal parabolic subgroup corresponding to β has an abelian unipotent radical. Using this observation, it is easy to find all parabolic subgroups with abelian unipotent radical in simple algebraic groups. If $\beta = \alpha_k$ in the Bourbaki numbering of simple roots then we shall denote the corresponding parabolic subgroup by P_k.

– A_r: any maximal parabolic subgroup P has an abelian unipotent radical.
– B_r: the only possibility is P_1.
– C_r: the only parabolic subgroup with abelian unipotent radical is P_r.
– D_r: there are three possibilities: P_1, P_{r-1}, and P_r
– Finally, there are two exceptional cases: P_1 (or P_6) of E_6 and P_7 of E_7.

Most results about shortly graded simple Lie algebras can be proved by induction using the following procedure. Let

$$\Delta' = \{\alpha \in \Delta \mid (\alpha, \beta) = 0\}.$$

Then $\Delta' = \Delta'_{-1} \cup \Delta'_0 \cup \Delta'_1$ is the root system of the graded semisimple subalgebra $\mathfrak{l}' \subset \mathfrak{l}$.

Proposition 2.15 *Suppose that $\Delta'_1 \neq \emptyset$. Then*

(a) $\delta \in \Delta'_1$.
(b) $\mathfrak{l}'_0 = \mathfrak{l}''_0 \oplus \mathfrak{l}'''_0$, *where $\mathfrak{l}'_{-1} \oplus \mathfrak{l}''_0 \oplus \mathfrak{l}'_1$ is a shortly graded simple Lie algebra.*
(c) Δ'_1 *contains a unique minimal root, say β'. β' is long.*

Proof. Indeed, suppose that $(\beta, \alpha_0) = 0$ for some $\alpha_0 \in \Delta_1$. Then

$$(\beta, \delta) = (\beta, \alpha_0) + \left(\beta, \sum_{\alpha \in \Pi_0} n_\alpha \alpha\right),$$

where all $n_\alpha \geq 0$. Since $(\beta, \alpha) \leq 0$ for any $\alpha \in \Pi_0$, we get $(\beta, \delta) \leq 0$. Since $(\beta, \delta) \geq 0$ by Proposition 2.14, we see that $\delta \in \Delta'_1$. It follows that δ is the unique highest weight in $\mathfrak{l}'_0$-module $\mathfrak{l}'_1$. Therefore, $\mathfrak{l}'_1$ is an irreducible $\mathfrak{l}'_0$-module and we obtain (b). Now we can apply Proposition 2.14 to the shortly graded simple Lie algebra $\mathfrak{l}'_{-1} \oplus \mathfrak{l}''_0 \oplus \mathfrak{l}'_1$ and get (c). □

It is possible to define two canonical strings of pairwise orthogonal long roots in Δ_1. The construction is originally due to Harish Chandra. The lower canonical string $\beta_1, \ldots, \beta_r$ is the cascade up from $\beta_1 = \beta$ within Δ_1: at each stage β_{i+1} is the minimal root in

$$\Delta_1^{(i)} = \{\alpha \in \Delta_1 \mid (\alpha, \beta_1) = \ldots = (\alpha, \beta_i) = 0\}.$$

The process terminates when Δ_1^i is empty. The previous proposition shows that the lower canonical string $\beta_1, \ldots, \beta_r$ is well-defined, i.e. on each step Δ_1^i contains a unique minimal element. Clearly, all the roots β_i are long.

The upper canonical string $\gamma_1, \ldots, \gamma_{r'}$ is the cascade down from $\gamma_1 = \gamma$ within Δ_1. At each stage, γ_{i+1} is the unique maximal element in

$$^{(i)}\Delta_1 = \{\alpha \in \Delta_1 \,|\, (\alpha, \gamma_1) = \ldots = (\alpha, \gamma_i) = 0\}.$$

The process terminates when $^{(i)}\Delta_1$ becomes empty. Obviously, the longest element in W_0 takes the lower canonical string to the upper canonical string. Hence, the upper canonical string is well-defined and both strings have the same cardinality. In fact, we have the following proposition

Proposition 2.16 ([RRS]) *Suppose that $\beta_1', \ldots, \beta_{r'}'$ is a string of pairwise orthogonal long roots in Δ_1. Then $r' \leq r$ and there exists $w \in W_0$ such that $\beta_i' = w(\beta_i)$ for $i \leq r'$.*

Proof. By Proposition 2.14 (d) we may suppose that $\beta_1' = \beta_1$. Now the proof goes by induction: both $\beta_2', \ldots, \beta_{r'}'$ and $\beta_2, \ldots, \beta_r$ belong to Δ_1' and, hence, there exists $w \in W_0'$ (the Weil group of $\mathfrak{l}_0'$) such that $\beta_i' = w(\beta_i)$ for $i \leq r'$. It remains to notice that $w\beta_1 = \beta_1$. □

Now we can describe the orbit decomposition and the Pyasetskii pairing.

Theorem 2.17 *Let $\alpha_1, \ldots, \alpha_r \in \Delta_1$ be any maximal sequence of pairwise orthogonal long roots. Set $e_k = e_{\alpha_1} + \ldots + e_{\alpha_k}$ for $k = 0, \ldots, r$ (so $e_0 = 0$). Denote the G-orbit of e_k by $\mathcal{O}_k \subset \mathfrak{l}_1$. Set $f_k = e_{-\alpha_1} + \ldots + e_{-\alpha_k}$ for $k = 0, \ldots, r$ (so $f_0 = 0$). Denote the G-orbit of f_k by $\mathcal{O}_k' \subset \mathfrak{l}_{-1}$. Then*

(a) $\mathfrak{l}_1 = \bigsqcup_{i=0}^{r} \mathcal{O}_i$, $\mathfrak{l}_{-1} = \bigsqcup_{i=0}^{r} \mathcal{O}_i'$.
(b) *$O_i \subset \overline{O_j}$ if and only if $i \leq j$ if and only if $O_i' \subset \overline{O_j'}$.*
(c) *$\mathcal{O}_k$ corresponds to $\mathcal{O}_{r-k}'$ in the Pyasetskii pairing.*
(d) *If $h_k = \alpha_1^\vee + \ldots + \alpha_k^\vee$ then $\langle e_k, h_k, f_k \rangle$ is a homogeneous $\mathfrak{sl}_2$-subalgebra.*

Proof. (d) This is obvious.

(a) By Proposition 2.16 the choice of a maximal string is irrelevant. We take $\alpha_i = \beta_i$, where $\beta_1, \ldots, \beta_r$ is the lower canonical string. First, we need to show that any element in $\mathfrak{l}_1$ is G-conjugate to one of e_k. We argue by the induction on r. Let $x \in \mathfrak{l}_1$, $x \neq 0$. We can write x as

$$x = \sum_{\alpha \in \Delta_1} x_\alpha e_\alpha.$$

Since $\mathfrak{l}_1$ is an irreducible G-module, after conjugating x we may assume that $x_\beta \neq 0$. Since the G-orbit of x is conical, we may assume that $x_\beta = 1$. Let $\hat{\Delta}_0 = \{\alpha \in \Delta_0 \,|\, (\alpha, \beta) < 0$. Clearly, $\hat{\Delta}_0 \subset \Delta_0^+$. Then

$$\mathfrak{u} = \langle e_\alpha \,|\, \alpha \in \hat{\Delta}_0 \rangle \subset \mathfrak{g}$$

is a unipotent subalgebra. Let $U \subset G$ be an unipotent subgroup with $\mathfrak{u} = \operatorname{Lie} U$. Then U acts on $\mathfrak{l}_1$. Clearly, for any roots $\alpha \in \hat{\Delta}_0$, $\alpha' \in \Delta'_1$ we have $\alpha + \alpha' \notin \Delta$ by Proposition 2.14 (c). Therefore, U acts trivially on $\mathfrak{l}'_1$. Consider the action of U on $\mathfrak{l}_1/\mathfrak{l}'_1$. Clearly, the function x_β is U-invariant, therefore each U-orbit in $\mathfrak{l}_1/\mathfrak{l}'_1$ lies on a hyperplane $x_\beta = \text{const}$. For any root $\alpha \in \Delta_1$ such that $(\alpha, \beta) > 0$ we have $\alpha = \beta + \hat{\alpha}$, where $\hat{\alpha} \in \hat{\Delta}_0$. Therefore, the tangent space $\mathfrak{u} \cdot (x \mod \mathfrak{l}'_1)$ is equal to the hyperplane $x_\beta = 0$. Since any orbit of a linear unipotent group is closed [OV], it follows that the U-orbit of $x \mod \mathfrak{l}'_1$ is the affine hyperplane $x_\beta = 1$. In particular, x is U-conjugate to an element $e_\beta + x'$, where $x' \in \mathfrak{l}'_1$. By induction, x' is G'-conjugate to one of $e'_0, \ldots, e'_{r-1}$, where $\operatorname{Lie} G' = \mathfrak{l}'_0$. Since $G' \cdot e_\beta = e_\beta$, we see that any nonzero element of $\mathfrak{l}_1$ is G-conjugate to $e_\beta + e'_k = e_{k+1}$ for some $k = 0, \ldots, r-1$.

To finish the proof of (a) we need to show that all orbits $\mathcal{O}_i$ are distinct. To prove this it suffices to check that $\dim \mathfrak{g} \cdot e_i = \dim \mathcal{O}_i < \dim \mathcal{O}_j = \dim \mathfrak{g} \cdot e_j$ for $i < j$. This will follow, in turn, from the inequality

$$\dim \operatorname{Ann}(\mathfrak{g} \cdot e_i) = \dim \mathfrak{l}_{-1}^{e_i} > \dim \mathfrak{l}_{-1}^{e_j} = \dim \operatorname{Ann}(\mathfrak{g} \cdot e_i),$$

where $\mathfrak{l}_{-1}^{e_i} = \{x \in \mathfrak{l}_{-1} \mid [x, e_i] = 0\}$. Since $[f_i, \mathfrak{l}_{-1}] = 0$, it follows from the $\mathfrak{sl}_2$-theory that $\mathfrak{l}_{-1}^{e_i} = \mathfrak{l}_{-1}^{h_i}$. But

$$\mathfrak{l}_{-1}^{h_i} = \langle e_{-\alpha} \mid \alpha \in \Delta^{(i)} \rangle,$$

therefore, $\dim \mathfrak{l}_{-1}^{e_i} = \#\Delta^{(i)}$. Since $\#\Delta^{(i)} > \#\Delta^{(j)}$, we see that all orbits $\mathcal{O}_i$ are distinct.

It is clear from the above that $\operatorname{Ann}(\mathfrak{g} \cdot e_i) \cap G f_{r-i}$ is open in $\operatorname{Ann}(\mathfrak{g} \cdot e_i)$, therefore, we have (c).

Finally, to prove (b) we need only to show that $\mathcal{O}_i \subset \overline{\mathcal{O}_j}$ for $i < j$. But

$$\lim_{t \to -\infty} \exp t(\alpha_{i+1}^\vee + \ldots + \alpha_j^\vee) e_j = e_i.$$

□

After these case-by-case-independent considerations it is worthy to sort out all examples.

Example 2.18 Consider the short gradings of $\mathfrak{l} = \mathfrak{sl}_{n+m}$. Then

$$G = \{(A, B) \in \mathrm{GL}_n \times \mathrm{GL}_m \mid \det(A) \det(B) = 1\}.$$

$\mathfrak{l}_1$ can be identified with $\mathbb{C}^n \times \mathbb{C}^m$. There are $r = \min(n, m)$ non-zero G-orbits. Namely, $\mathcal{O}_i$, $i = 1, \ldots, r$, is the variety of $m \times n$-matrices of rank i. The projectivization of $\mathcal{O}_1$ is identified with $X = \mathbb{P}^{n-1} \times \mathbb{P}^{m-1}$ in the Segre embedding. Therefore, the dual variety X^* is equal to the projectivization of the closure $\overline{\mathcal{O}_{r-1}}$, the variety of matrices of rank less than or equal to $r-1$. X^* is a hypersurface if and only if $n = m$, in which case Δ_X is the ordinary determinant of a square matrix. Another interesting case is $n = 2$, $m \geq 2$: we see that the Segre embedding of $\mathbb{P}^1 \times \mathbb{P}^k$ is self-dual.

Example 2.19 Consider the short grading of $\mathfrak{l} = \mathfrak{so}_{n+2}$ that corresponds to β being the first simple root. Then $G = \mathbb{C}^* \times \mathrm{SO}_n$ and $l_1 = C^n$ with a simplest action. There are two non-zero orbits: the dense one and the self-dual quadric hypersurface $Q \subset \mathbb{C}^n$ preserved by G.

Example 2.20 Consider the short grading of $\mathfrak{l} = \mathfrak{so}_{2n} = D_n$ that corresponds to $\beta = \alpha_n$ or $\beta = \alpha_{n-1}$. Then $G = \mathbb{C}^* \times \mathrm{SL}_n$ acts naturally on $\mathfrak{l}_1 = \Lambda^2\mathbb{C}^n$. There are $r = [n/2]$ non-zero orbits, where $\mathcal{O}_i$, $i = 1, \ldots, r$, is the variety of skew–symmetric matrices of rank $2r$. The projectivization of $\mathcal{O}_1$ is identified with $X = \mathrm{Gr}(2, n)$ in the Plücker embedding. Therefore, the dual variety X^* is equal to the projectivization of the closure $\overline{\mathcal{O}_{r-1}}$, the variety of matrices of rank less than or equal to $2r-2$. X^* is a hypersurface if and only if n is even, in which case Δ_X is the Pfaffian of a skew-symmetric matrix. If n is odd then $\operatorname{def} X = \operatorname{codim} \mathcal{O}_{r-1} = 2$.

Example 2.21 The short grading of E_6 gives the following action. $G = \mathbb{C}^* \times \mathrm{SO}_{10}$ and $\mathfrak{l}_1$ is the half-spinor representation. Except for the zero orbit and the dense orbit there is only one orbit $\mathcal{O}$. In particular, the projectivization of $\mathcal{O}$ is smooth and self-dual. It is a spinor variety $\mathbb{S}_5$. It could also be described via Cayley numbers. Let $\mathbb{C}\mathrm{a}$ be the algebra of split Cayley numbers (therefore $\mathbb{C}\mathrm{a} = \mathbb{O} \otimes \mathbb{C}$, where $\mathcal{O}$ is the real division algebra of octonions). Let $u \mapsto \overline{u}$ be the canonical involution in $\mathbb{C}\mathrm{a}$. Let $\mathbb{C}^{16} = \mathbb{C}\mathrm{a} \oplus \mathbb{C}\mathrm{a}$ have octonionic coordinates u, v. Then the spinor variety $\mathcal{S}_5 \subset \mathbb{P}(\mathbb{C}^{16})$ is defined by homogeneous equations

$$u\overline{u} = 0,\ v\overline{v} = 0, u\overline{v} = 0,$$

where the last equation is equivalent to 8 complex equations.

Example 2.22 The final example appears from the short grading of E_7. Here $G = \mathbb{C}^* \times E_6$, and $\mathfrak{l}_1 = \mathbb{C}^{27}$ can be identified with the exceptional simple Jordan algebra (the Albert algebra), see [J]. Then E_6 is the group of norm similarities and $\mathbb{C}^*$ acts by homotheties. There are 3 non-zero orbits: the dense one, the cubic hypersurface (defined by the norm in the Jordan algebra), and the closed conical variety with smooth projectivization consisting of elements of rank one, i.e. the exceptional Severi variety (the model of the Cayley projective plane).

3. The Cayley Method for Studying Discriminants

Preliminaries

The remarkable observation due to Cayley [Ca] gives an expression of the discriminant as the determinant of a certain complex of finite-dimensional vector spaces. Of course, complexes were unknown at his time and he was studying resultants rather than discriminants (though he also noticed that resultants and discriminants are, in fact, the equivalent notions). This approach was clarified and developed in the beautiful serie of papers by Gelfand, Kapranov, and Zelevinsky [GKZ1, GKZ2].

3.1 Jet Bundles and Koszul Complexes

Let X be a smooth irreducible algebraic variety with an algebraic line bundle $\mathcal{L}$. We consider the bundle $J(\mathcal{L})$ of first jets of sections of $\mathcal{L}$. By definition, the fiber of $J(\mathcal{L})$ at a point $x \in X$ is the quotient of the space of all sections of $\mathcal{L}$ near x by the subspace of sections which vanish at x with their first derivatives. In other words, $J(\mathcal{L})_x = \mathcal{L}/I_x^2\mathcal{L}$, where I_x is the ideal of functions vanishing at x. Thus $J(\mathcal{L})$ is a vector bundle of rank $\dim X + 1$.

To any section f of $\mathcal{L}$, we associate a section $j(f)$ of $J(\mathcal{L})$, called the first jet of f. Namely, the value of $j(f)$ at x is the class of f modulo $I_x^2\mathcal{L}$. The correspondence $f \mapsto j(f)$ is $\mathbb{C}$-linear, but being a differential operator, is not $\mathcal{O}_X$-linear.

Let us summarize here without proofs some well-known properties of jet bundles.

Theorem 3.1 ([GKZ2, KS])

(a) *For any two line bundles $\mathcal{L}$, $\mathcal{M}$ on X, there exists a canonical isomorphism $J(\mathcal{L} \otimes \mathcal{M}) \simeq J(\mathcal{L}) \otimes \mathcal{M}$.*

(b) *Let Ω_X^1 be the sheaf of regular differential 1-forms on X. Then there exists a canonical exact sequence of vector bundles*

$$0 \to \Omega_X^1 \otimes \mathcal{L} \to J(\mathcal{L}) \to \mathcal{L} \to 0. \tag{3.1}$$

(c) *Let $X = \mathbb{P}(V)$ be a projective space. Then the jet bundle $J(\mathcal{O}_X(1))$ is a trivial vector bundle naturally identified with $\mathcal{O}_X \otimes V^*$*

The relevance of jets to dual varieties is as follows. Suppose that $X \subset \mathbb{P}(V)$ is an irreducible projective variety. We take $\mathcal{L} = \mathcal{O}_X(1)$. Then any $f \in V^*$ is a linear function on V and, hence, can be regarded as a global section of $\mathcal{L}$. The following result is an immediate consequence of definitions:

Proposition 3.2 *$f \in V^*$ represents a point in the dual variety X^* if and only if the section $j(f)$ of a jet bundle $J(\mathcal{L})$ vanishes at some point $x \in X$.*

Let E be an algebraic vector bundle of rank r on an irreducible algebraic variety X. For any global section s of E consider the following complexes of sheaves on X, called Koszul complexes.

$$\mathcal{K}_+(E,s) = \left\{0 \to \mathcal{O}_X \xrightarrow{s} E \xrightarrow{\wedge s} \Lambda^2 E \xrightarrow{\wedge s} \dots \xrightarrow{\wedge s} \Lambda^r E \to 0\right\},$$

$$\mathcal{K}_-(E,s) = \left\{0 \to \Lambda^r E^* \xrightarrow{i_s} \dots \xrightarrow{i_s} \Lambda^2 E^* \xrightarrow{i_s} E^* \xrightarrow{i_s} \mathcal{O}_X \to 0\right\}.$$

The differential in $\mathcal{K}_+$ is given by the exterior multiplication with s and the differential in $\mathcal{K}_-$ is given by the contraction with s, that is, by the map $\Lambda^j E^* \to \Lambda^{j-1} E^*$ dual to the map $\Lambda^{j-1} E \to \Lambda^j E$ given by $\wedge s$. We fix gradings in $\mathcal{K}_+$ and $\mathcal{K}_-$ by assigning the degree j to $\Lambda^j E$ in $\mathcal{K}_+$ and by assigning the degree $-j$ to $\Lambda^j E^*$ in $\mathcal{K}_-$. Notice that we have an isomorphism of complexes

$$\mathcal{K}_-(E,s) \simeq \mathcal{K}_+(E,s) \otimes \Lambda^r E^*[r],$$

where r in brackets means the shift in the grading by r. The following Theorem shows that the cohomology of Koszul complexes 'represents' the vanishing set of s.

Theorem 3.3

(a) *Koszul complexes $\mathcal{K}_+(E,s)$ or $\mathcal{K}_-(E,s)$ are exact if and only if s vanishes nowhere on X (the set of zeros $Z(s)$ is empty).*

(b) *Suppose that X is smooth and that s vanishes along a smooth subvariety $Z(s)$ of codimension $r = \operatorname{rank} E$, and, moreover, s is transverse to the zero section. Then $\mathcal{K}_-(E,s)$ has only one non-trivial cohomology sheaf, namely $\mathcal{O}_{Z(s)}$ (regarded as a sheaf on X) in highest degree 0. The complex $\mathcal{K}_+(E,s)$ in this situation has the only non-trivial cohomology sheaf in the highest degree r, and this sheaf is the restriction $\det E = \Lambda^r E|_{Z(s)}$ regarded as a sheaf on X.*

Sketch of the proof. We shall prove only (a), the proof of (b) follows the same ideas but is more technically involved, see e.g. [Fu1, GH1]. If V is a vector space and $v \in V$ is a non-zero vector then the differential in the exterior algebra $\Lambda^* V$ given by $\wedge v$ is exact. Indeed, we can include v in some basis as a first vector, then both the kernel and the image of the differential are spanned by monomials containing v. Therefore, the dual differential i_v in $\Lambda^* V^*$ is also exact. Applying this to our situation we see that if the section

s does not vanish anywhere then both Koszul complexes are exact fiberwise and, therefore, are exact as complexes of sheaves. However, if $Z(s)$ is not empty then the cokernel of the last differential in Koszul complexes is not trivial. Indeed, if $x \in Z(s)$ then, trivializing E near x, we represent s as a collection of functions $(f_1, \dots, f_r) \in \mathcal{O}^r$. Then the last differential in both complexes has a form $\mathcal{O}^r \to \mathcal{O}$, $(u_1, \dots, u_r) \mapsto \sum u_i f_i$. If s vanishes at x then all $f_i = 0$ and this map is not surjective. □

Let now $X \subset \mathbb{P}(V)$ be a smooth projective variety. Then we can apply Theorem 3.3 to the jet bundle $E = J(\mathcal{L})$, where $\mathcal{L} = \mathcal{O}_X(1)$. Combining Proposition 3.2 and Theorem 3.3 we get the following

Theorem 3.4 ([GKZ2]) *For any $f \in V^*$ let $j = j(f)$ denote its first jet. Then a vector $f \in V^*$ represents a point in X^* if and only if any of the following complexes of sheaves on X is not exact:*

$$\mathcal{K}_+(J(\mathcal{L}), j) = \left\{0 \to \mathcal{O}_X \xrightarrow{j} J(\mathcal{L}) \xrightarrow{\wedge j} \Lambda^2 J(\mathcal{L}) \xrightarrow{\wedge j} \dots \xrightarrow{\wedge j} \Lambda^r J(\mathcal{L}) \to 0\right\},$$

$$\mathcal{K}_-(J(\mathcal{L}), j) = \left\{0 \to \Lambda^r J(\mathcal{L})^* \xrightarrow{i_j} \dots \xrightarrow{i_j} \Lambda^2 J(\mathcal{L})^* \xrightarrow{i_j} J(\mathcal{L})^* \xrightarrow{i_j} \mathcal{O}_X \to 0\right\}.$$

3.2 Cayley Determinants of Exact Complexes

The determinants of exact complexes (in the implicit form) were first introduced by Cayley in his paper [Ca] on resultants. A systematic early treatment of this subject was undertaken by Fisher [Fi] whose aim was to give a rigorous proof of Cayley results. In topology determinants of complexes were introduced in 1935 by Reidermeister and Franz [Fra]. They used the word 'torsion' for determinant-type invariants constructed. In this section we give only definitions that are necessary for the formulation of results related to dual varieties. More details could be found in [GKZ2, KnMu, Del, Q, RS].

The base field k can be arbitrary. Suppose that V is a finite-dimensional vector space. Then the top-degree component of the exterior algebra $\Lambda^{\dim V} V$ is called the determinant of V, denoted by $\operatorname{Det} V$. If $V = 0$ then we set $\operatorname{Det} V = k$. It is easy to see that for any exact triple $0 \to U \to V \to W \to 0$ we have a natural isomorphism $\operatorname{Det} V \simeq \operatorname{Det} U \otimes \operatorname{Det} W$.

Suppose now that $V = V_0 \oplus V_1$ is a finite-dimensional supervector space. Then, by definition, $\operatorname{Det} V$ is set to be $\operatorname{Det} V_0 \otimes (\operatorname{Det} V_1)^*$. Once again, for any exact triple of supervector spaces $0 \to U \to V \to W \to 0$ we have $\operatorname{Det} V \simeq \operatorname{Det} U \otimes \operatorname{Det} W$. For any supervector space V we denote by $\tilde{V}$ the new supervector space given by $\tilde{V}_0 = V_1$, $\tilde{V}_1 = V_0$. Clearly, we have a natural isomorphism

$$\operatorname{Det} \tilde{V} = (\operatorname{Det} V)^*. \tag{3.2}$$

Now let (V, ∂) be a finite-dimensional supervector space with a differential ∂ such that $\partial V_0 \subset V_1$, $\partial V_1 \subset V_0$, $\partial^2 = 0$. Then $\operatorname{Ker} \partial$, $\operatorname{Im} \partial$, and

the cohomology space $H(V) = \operatorname{Ker}\partial/\operatorname{Im}\partial$ are again supervector spaces. We claim that there exists a natural isomorphism

$$\operatorname{Det} V \simeq \operatorname{Det} H(V). \tag{3.3}$$

Indeed, from the exact sequence

$$0 \to \operatorname{Ker}\partial \to V \xrightarrow{\partial} \widetilde{\operatorname{Im}\partial} \to 0$$

we see that $\operatorname{Det} V \simeq \operatorname{Det}(\operatorname{Ker}\partial) \otimes \operatorname{Det}(\operatorname{Im}\partial)^*$. From the exact sequence

$$0 \to \operatorname{Im}\partial \to \operatorname{Ker}\partial \to H(V) \to 0$$

we get that $\operatorname{Det} H(V) \simeq \operatorname{Det}(\operatorname{Ker}\partial) \otimes \operatorname{Det}(\operatorname{Im}\partial)^*$. Therefore, we have (3.3). In particular, if ∂ is exact, $H(V) = 0$, then we have a natural isomorphism

$$\operatorname{Det} V \simeq k. \tag{3.4}$$

Let us fix some bases $\{e_1, \ldots, e_{\dim V_0}\}$ in V_0 and $\{e'_1, \ldots, e'_{\dim V_1}\}$ in V_1. Let $\{f'_1, \ldots, f'_{\dim V_1}\}$ be a dual basis in V_1^*. Then we have a basis vector

$$e_1 \wedge \ldots \wedge e_{\dim V_0} \otimes f'_1 \wedge \ldots \wedge f'_{\dim V_1} \in \operatorname{Det} V.$$

Therefore, if (V, ∂, e) is a based supervector space with an exact differential then by (3.4) we get a number $\det(V, \partial, e)$ called the Cayley determinant of a based supervector space with an exact differential. If we fix other bases $\{\tilde{e}_1, \ldots, \tilde{e}_{\dim V_0}\}$ in V_0 and $\{\tilde{e}'_1, \ldots, \tilde{e}'_{\dim V_1}\}$ in V_1 then, clearly,

$$\det(V, \partial, \tilde{e}) = \det A_0 (\det A_1)^{-1} \det(V, \partial, e), \tag{3.5}$$

where $(A_0, A_1) \in \operatorname{GL}(V_0) \times \operatorname{GL}(V_1)$ are transition matrices from bases e to bases $\tilde{e}$. One upshot of this is the fact that if bases e and $\tilde{e}$ are equivalent over some subfield $k_0 \subset k$ then the Cayley determinants with respect to these bases are equal up to a non-zero multiple from k_0. Since all isomorphisms are natural so far, the homogeneity condition (3.5) can be reformulated as follows. For any based supervector space (V, ∂, e) with an exact differential and any $(A_0, A_1) \in \operatorname{GL}(V_0) \times \operatorname{GL}(V_1)$ we can consider a new based supervector space $(V, A \cdot \partial, e)$ with the exact differential $A \cdot \partial$ given by

$$A \cdot \partial|_{V_0} = A_1 \cdot \partial \cdot A_0^{-1}, \quad A \cdot \partial|_{V_1} = A_0 \cdot \partial \cdot A_1^{-1}.$$

Then we have

$$\det(V, A \cdot \partial, e) = (\det A_0)^{-1} \det A_1 \det(V, \partial, e).$$

In particular, it is easy to see that the following formula is valid:

$$\det(V, \lambda\partial, e) = \lambda^{\dim(\operatorname{Ker}\partial)_1 - \dim(\operatorname{Ker}\partial)_0} \det(V, \partial, e). \tag{3.6}$$

In the matrix form the Cayley determinant can be calculated as follows. Suppose that (V, ∂, e) is a based supervector space with an exact differential,

$$\dim V_0 = \dim \operatorname{Ker} \partial_0 + \dim \operatorname{Im} \partial_0 = \dim \operatorname{Im} \partial_1 + \dim \operatorname{Ker} \partial_1 = \dim V_1 = n.$$

Therefore, both ∂_0 and ∂_1 are represented by $(n \times n)$-matrices D_0 and D_1. For any subsets $I, J \subset B = \{1, \dots, n\}$ we denote by D_0^{IJ} and D_1^{IJ} the submatrices of D_0 and D_1 formed by columns indexed by I and rows indexed by J. Then it is easy to see that there exist subsets $I_0, I_1 \subset B$ such that submatrices $D_0^{B \setminus I_0, I_1}$, $D_1^{B \setminus I_1, I_0}$ are invertible. In particular,

$$\#I_1 = \operatorname{rk} \partial_0, \quad \#I_0 = \operatorname{rk} \partial_1 = n - \#I_1.$$

The formula (3.5) implies that

$$\det(V, \partial, e) = \det D_0^{B \setminus I_0, I_1} (\det D_1^{B \setminus I_1, I_0})^{-1}. \tag{3.7}$$

This formula gives an explicit matrix description of the Cayley determinant.

Now we consider a finite complex of finite-dimensional vector spaces

$$\dots \overset{\partial_{i-1}}{\to} V^i \overset{\partial_i}{\to} V^{i+1} \overset{\partial_{i+1}}{\to} \dots.$$

Then we can define the finite-dimensional supervector space $V = V_0 \oplus V_1$,

$$V_0 = \bigoplus_{i \equiv 0 \bmod 2} V^i, \quad V_1 = \bigoplus_{i \equiv 1 \bmod 2} V^i,$$

with an induced differential ∂. In particular, all previous considerations are valid. Therefore, if the complex $(V^\bullet, \partial)$ is exact and there are some fixed bases $\{e_1^i, \dots, e_{\dim V^i}^i\}$ in each component V^i then we have the corresponding Cayley determinant $\det(V^\bullet, \partial, e) \in k^*$. For example, if L and M are based vector spaces and $A : L \to M$ is an invertible operator then the complex $0 \to L \overset{A}{\to} M \to 0$ is exact and the corresponding Cayley determinant is equal to $\det A$ (if L is located in the even degree of the complex).

In fact, Cayley in [Ca] has found some matrix representation of his determinant. In order to give his formula we shall write our complex in the explicit coordinate form

$$V^\bullet = \{0 \to k^{B_0} \overset{D_0}{\to} k^{B_1} \overset{D_1}{\to} \to \dots \overset{D_{r-1}}{\to} k^{B_r} \to 0\}, \tag{3.8}$$

where for simplicity we assume that the non-zero terms of the complex are located in degrees between 0 and $r > 0$. In general, if $V^\bullet[m]$ is the same complex as $V^\bullet$ but with a grading shifted by m, then by (3.2) we have $\det(V^\bullet[m], \partial, e) = (\det(V^\bullet[m], \partial, e))^{(-1)^m}$ In the formula (3.8) we may suppose that all B_i are some finite sets. Therefore, D_i is a matrix with columns indexed by B_i and rows indexed by B_{i+1}. For any subsets $X \subset B_i$, $Y \subset B_{i+1}$

we denote by $(D_i)_{XY}$ the submatrix in D_i with columns from X and rows from Y.

A collection of subsets $I_i \subset B_i$ is called admissible if $I_0 = \emptyset$, $I_r = B_r$, for any $i = 0, \dots, r-1$ we have $\#(B_i \setminus I_i) = \#I_{i+1}$, and the submatrix $(D_i)_{B_i \setminus I_i, I_{i+1}}$ is invertible. The following theorem easily follows from (3.7).

Theorem 3.5 ([GKZ2])

(a) *Admissible collections exist. For any admissible collection we have*

$$\#I_i = \sum_{j=0}^{i-1} (-1)^{i-1-j} \#B_j.$$

(b) *Let $\{I_i\}$ be an admissible collection. Denote by Δ_i the determinant of the matrix $(D_i)_{B_i \setminus I_i, I_{i+1}}$. Then*

$$\det(V^\bullet, D, B) = \prod_{i=0}^{r-1} \Delta_i^{(-1)^i}.$$

In particular, this Theorem (or formula (3.6)) implies

Corollary 3.6 *For any based exact complex $(V^\bullet, d, e)$ we have*

$$\det(V^\bullet, \lambda d, e) = \lambda^{\sum_i (-1)^{i+1} \cdot i \cdot \dim V^i} \det(V^\bullet, d, e).$$

3.3 Discriminant Complexes

Let $X \subset \mathbb{P}(V)$ be a smooth projective variety. We denote $\mathcal{L} = \mathcal{O}_X(1)$, so any $f \in V^*$ can be regarded as a section of $\mathcal{L}$. Let $\mathcal{M}$ be another line bundle on X. We define discriminant complexes $C_+^*(X, \mathcal{M})$ and $C_-^*(X, \mathcal{M})$ as complexes of global sections of Koszul complexes $\mathcal{K}_+(J(\mathcal{L}), j(f))$ and $\mathcal{K}_-(J(\mathcal{L}), j(f))$ tensored by $\mathcal{M}$. More precisely,

$$C_+^i(X, \mathcal{M}) = H^0(X, \Lambda^i J(\mathcal{L}) \otimes \mathcal{M}),$$

$$C_-^i(X, \mathcal{M}) = H^0(X, \Lambda^{-i} J(\mathcal{L})^* \otimes \mathcal{M}).$$

Thus the terms of discriminant complexes are fixed and the differentials depend on $f \in V^*$. We shall denote this differential by ∂_f. By Theorem 3.4 a vector $f \in V^*$ represents a point in X^* if and only if any of two Koszul complexes is not exact. We want to get the same condition but for discriminant complexes, which are complexes of finite-dimensional vector spaces instead of complexes of sheaves. However, the exactness of a complex of sheaves does not necessarily imply the exactness of the corresponding complex of global sections. The obstruction to this is given by the higher cohomology of the sheaves of the complex.

Definition 3.7 *The discriminant complexes are called stably twisted if all terms of the corresponding Koszul complexes have no higher cohomology.*

For example, suppose that $\mathcal{M}$ is any ample line bundle on X. Ampleness is equivalent to the fact that for any coherent sheaf $\mathcal{F}$ on X, the sheaves $\mathcal{F} \otimes \mathcal{M}^{\otimes n}$ have no higher cohomology for $n \gg 0$. Therefore, for sufficiently big n the discriminant complexes $C_+^*(X, \mathcal{M}^{\otimes n})$ and $C_-^*(X, \mathcal{M}^{\otimes n})$ will be stably twisted. In this situation we have the following Theorem that immediately follows from Theorem 3.4 and 'abstract de Rham theorem' (see e.g.[GH1]).

Theorem 3.8 ([GKZ2]) *Suppose that the discriminant complex $C_+^*(X, \mathcal{M})$ (resp. $C_-^*(X, \mathcal{M})$) is stably twisted. Let $f \in V^*$ be such that its projectivization does not belong to the dual variety $X^* \subset \mathbb{P}(V^*)$. Then the complex $(C_+^*(X, \mathcal{M}), \partial_f)$ (resp. $(C_-^*(X, \mathcal{M}), \partial_f)$) is exact.*

Remarkably, it turns out that much more is true. Suppose that the discriminant complex $C_+^*(X, \mathcal{M})$ (or $C_-^*(X, \mathcal{M})$) is stably twisted. Then for a generic $f \in V^*$ the complex $C_+^*(X, \mathcal{M})$ (or $C_-^*(X, \mathcal{M})$) is exact by the previous Theorem. Let $f_1, \ldots, f_n$ be a basis of V^*. Let $\mathbb{C}(x_1, \ldots, x_n) = \mathbb{C}(V^*)$ be a field of rational functions on V^*. Consider the complex $C_+^*(X, \mathcal{M}) \otimes \mathbb{C}(V^*)$ (or $C_-^*(X, \mathcal{M}) \otimes \mathbb{C}(V^*)$) with the differential given by $\partial = \sum x_i \partial_{f_i}$. Then this complex is exact, since its generic specialization is exact. We fix some bases over $\mathbb{C}$ in each component $C_+^i(X, \mathcal{M})$ (or $C_-^{-i}(X, \mathcal{M})$) and consider them as bases over $\mathbb{C}(V^*)$ in each $C_+^i(X, \mathcal{M}) \otimes \mathbb{C}(V^*)$ (or $C_-^{-i}(X, \mathcal{M}) \otimes \mathbb{C}(V^*)$). Then we can calculate the Cayley determinant $\Delta_{X,\mathcal{M}}^+$ (or $\Delta_{X,\mathcal{M}}^-$) of the complex $C_+^*(X, \mathcal{M}) \otimes \mathbb{C}(V^*)$ (or $C_-^*(X, \mathcal{M}) \otimes \mathbb{C}(V^*)$), which will be the non-zero element of the field of rational functions $\mathbb{C}(V^*)$. Clearly the Cayley determinants depend only on the discriminant complexes (up to a scalar multiple) and do not depend on the choice of bases. The following Theorem shows that the Cayley determinant coincides with the discriminant.

Theorem 3.9 ([GKZ2]) *If the discriminant complex $C_-^*(X, \mathcal{M})$ is stably twisted then, up to a non-zero scalar, we have*

$$\Delta_{X,\mathcal{M}}^- = \Delta_X,$$

where Δ_X is the discriminant of X. If $C_+^*(X, \mathcal{M})$ is stably twisted then, up to a non-zero scalar, we have

$$(\Delta_{X,\mathcal{M}}^+)^{(-1)^{\dim X+1}} = \Delta_X.$$

The proof of this Theorem involves derived categories and Cayley determinants for complexes over arbitrary Noetherian integral domains, see [GKZ2]. Let us consider several examples.

Sylvester Formula. Let $X = \mathbb{P}^1 = \mathbb{P}(\mathbb{C}^2)$ be a projective line embedded into $\mathbb{P}(S^d\mathbb{C}^2)$ via the Veronese embedding. The space $V^* = (S^d\mathbb{C}^2)^*$ is the space of binary forms

$$f(x_0, x_1) = a_0 x_0^d + a_1 x_0^{d-1} x_1 + \ldots + a_d x_1^d$$

and the discriminant $\Delta_X(f)$ is the classical discriminant of this binary form. It is quite easy to implement Theorem 3.9 in this case. We take a twisting bundle $\mathcal{M} = \mathcal{O}_{\mathbb{P}^1}(2d-3)$. The discriminant complex $C^*_-(X, \mathcal{M})$ has only two non-zero terms and, therefore, the Cayley determinant is reduced to a determinant of a square matrix. More precisely, the Cayley determinant in this case is equal to the determinant of the linear map

$$\partial_f : S^{d-2}\mathbb{C}^2 \oplus S^{d-2}\mathbb{C}^2 \to S^{2d-3}\mathbb{C}^2,$$

given by

$$\partial_f(u, v) = \frac{\partial f}{\partial x_0} u + \frac{\partial f}{\partial x_1} v.$$

The final formula coincides with the classical Sylvester formula for the discriminant of a binary form (up to a scalar):

$$\Delta_X(f) = \frac{(-1)^{d-1}}{d^{d-2}} D,$$

where D is the determinant of the following matrix

$$\begin{vmatrix} a_1 & 2a_2 & \ldots & (d-1)a_{d-1} & da_d & \ldots & 0 \\ 0 & a_1 & \ldots & (d-2)a_{d-2} & (d-1)a_{d-1} & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & a_1 & 2a_2 & \ldots & da_d \\ da_0 & (d-1)a_1 & \ldots & 2a_{d-2} & a_{d-1} & \ldots & 0 \\ 0 & da_0 & \ldots & 3a_{d-3} & 2a_{d-2} & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & da_0 & (d-1)a_1 & \ldots & a_{d-1} \end{vmatrix}.$$

The Dual of an Algebraic Curve. Let $X \subset \mathbb{P}(V)$ be a smooth algebraic curve of degree d and genus g. We take the twisting bundle $\mathcal{M}$ to be a generic line bundle on X of degree $2d+3g-3$. The discriminant complex $C^*_-(X, \mathcal{M})$ has only two non-zero terms and, therefore, the Cayley determinant is given by the determinant of a square matrix. The size of this matrix (and hence the degree of the dual variety X^*) is equal to $\dim H^0(X, \mathcal{M})$. The latter is equal to $2d + 2g - 2$ by the Riemann-Roch theorem.

Discriminant Spectral Sequences. One disadvantage of Theorem 3.9 is the restricting condition on the twisting line bundle $\mathcal{M}$. It is not always convenient to make the discriminant complexes $C^\bullet_\pm$ stably twisted. Another

possibility is to take into account the higher cohomology as well. The standard tool for this is the spectral sequence. Recall [GH1] that if $\mathcal{F}^\bullet$ is a finite complex of sheaves on a topological space X, then one can define the hypercohomology groups $\mathbf{H}^i(X, \mathcal{F}^\bullet)$. To define them, consider the complexes of Abelian groups $C_j^\bullet$ calculating the cohomology of every individual $\mathcal{F}^j$ (for example, the Čech complexes with respect to an appropriate open covering of X). The differentials $\mathcal{F}^i \to \mathcal{F}^{i+1}$ make this collection of complexes into a double complex $C^{\bullet\bullet}$. The hypercohomology groups $\mathbf{H}^i(X, \mathcal{F}^\bullet)$ are the cohomology groups of its total complex. In particular, we have the spectral sequence of the double complex $C^{\bullet\bullet}$

$$E_1^{pq} = H^q(X, \mathcal{F}^p) \Rightarrow \mathbf{H}^{p+q}(X, \mathcal{F}^\bullet).$$

The first differential d_1 is induced by the differential in $\mathcal{F}^\bullet$. In particular, the complex of global sections

$$\ldots \to H^0(X, \mathcal{F}^i) \to H^0(X, \mathcal{F}^{i+1}) \to \ldots$$

is just the bottom row of the term E_1.

If the complex of sheaves $\mathcal{F}^\bullet$ is exact then all hypercohomology groups $\mathbf{H}^i(X, \mathcal{F}^\bullet)$ vanish. Indeed, the hypercohomology can be calculated using another spectral sequence 'dual' to the first:

$$'E_2^{pq} = H^q(X, \mathcal{H}^p(\mathcal{F}^\bullet)) \Rightarrow \mathbf{H}^{p+q}(X, \mathcal{F}^\bullet),$$

where $\mathcal{H}^p(\mathcal{F}^\bullet)$ is the p-th cohomology sheaf of $\mathcal{F}^\bullet$

$$\mathcal{H}^p(\mathcal{F}^\bullet) = \frac{\mathrm{Ker}\{\mathcal{F}^p \to^d \mathcal{F}^{p+1}\}}{\mathrm{Im}\{\mathcal{F}^{p-1} \to^d \mathcal{F}^p\}}.$$

Now let $X \subset \mathbb{P}(V)$ be a smooth projective variety, $\mathcal{L} = \mathcal{O}_X(1)$. Let $\mathcal{M}$ be any line bundle on X. We define discriminant spectral sequences $C_{r,\pm}^{pq}(X, \mathcal{M}, f)$ to be the spectral sequences of complexes $\mathcal{K}_\pm(J(\mathcal{L}), j(f))$ tensored by $\mathcal{M}$. Then the following theorem is a consequence of the discussion above and Theorem 3.4.

Theorem 3.10 ([GKZ2]) *Suppose that the projectivization of $f \in V^*$ does not belong to the dual variety X^*. Then the discriminant spectral sequences $C_{r,\pm}^{pq}(X, \mathcal{M}, f)$ are exact, i.e. they converge to zero.*

Moreover, it is possible to define the Cayley determinants of exact spectral sequences and to prove an analogue of Theorem 3.9, see [GKZ2] for details.

Weyman's Bigraded Complexes The calculation of the determinant of a spectral sequence with many non-trivial terms may be very involved. J. Weyman [We] has suggested a procedure that replaces the discriminant spectral sequence by a bigraded complex which incorporates all higher differentials at

once. Bigraded complex is, by definition, a complex $(C^\bullet, \partial)$ of vector spaces in which each term C^i is equipped with additional grading $C^i = \oplus_{p+q=i} C^{pq}$. The underlying complex $(C^\bullet, \partial)$ may then be called a total complex of the bigraded complex $C^{\bullet\bullet}$. The differential ∂ in $C^\bullet$ is decomposed into a sum $\partial = \sum_r \partial_r$, where ∂_r is bihomogeneous of degree $(r, 1-r)$. We shall consider only bigraded complexes with $\partial_r = 0$ for $r < 0$.

Now let $X \subset \mathbb{P}(V)$ be a smooth projective variety, $\mathcal{L} = \mathcal{O}_X(1)$. Let $\mathcal{M}$ be any line bundle on X.

Theorem 3.11 ([We]) *There exists a bigraded complex $(C^{\bullet\bullet}, \partial_f)$ with*

$$C^{pq} = H^q\left(X, \Lambda^{-p} J(\mathcal{L})^* \otimes \mathcal{M}\right)$$

and the differential ∂_f depends polynomially on $f \in V^*$. This complex has the following properties:

- *The differential ∂_f has the form $\partial_f = \sum_{r \geq 1} \partial_{r,f}$, where $\partial_{r,f}$ has bidegree $(r, 1-r)$ and its matrix elements are homogeneous polynomials of degree r in coefficients of f. Moreover, $\partial_{1,f}$ is induced by the differential in the Koszul complex $\mathcal{K}_-(J(\mathcal{L}), j(f))$.*
- *The Cayley determinant of the total complex $(C^\bullet, \partial_f)$ of $C^{\bullet\bullet}$ equals, up to a scalar factor, the discriminant Δ_X.*

4. Resultants and Schemes of Zeros

Preliminaries

Classically, discriminants were studied together with resultants. Resultants and their generalizations appear while looking at vector bundles: they detect their global sections that have "correct" schemes of zeros. Some results in this direction are reviewed in this chapter.

4.1 Ample Vector Bundles

Classically, a line bundle L over an algebraic variety X is said to be ample if the map $X \to \mathbb{P}^N$ associated to the sections of some power L^n gives an imbedding of X. This notion has been extended to vector bundles as well by Hartshorne [Ha3]. Let E be a vector bundle on X. We consider the variety $X_E = \mathbb{P}(E^*)$, the projectivization of the bundle E^*. There is a projection $p : E_X \to X$ whose fibers are projectivizations of fibers of E^*, and a natural projection $\pi : E^* \setminus X \to X_E$, where X is embedded into the total space of E^* as the zero section. We denote by $\xi(E)$ the tautological line bundle on X_E defined as follows. For open $U \subset X_E$, a section of $\xi(E)$ over U is a regular function on $\pi^{-1}(U)$ which is homogeneous of degree 1 with respect to dilations of E^*. The restriction of $\xi(E)$ to every fiber $p^{-1}(X) = \mathbb{P}(E_x^*)$ is the tautological line bundle $\mathcal{O}(1)$ of the projective space $\mathbb{P}(E_x^*)$.

Definition 4.1 A vector bundle E on X is called ample if $\xi(E)$ is an ample line bundle on X_E.

If E is itself a line bundle then $X_E = X$, $\xi(E) = E$, and this definition gives nothing new. Basic properties and criteria for ampleness were obtained in [Ha3]. For example, the Grothendieck definition of ample line bundles can be transported to vector bundles as well. Namely, E is ample on X if and only if for every coherent sheaf F, there is an integer $n_0 > 0$, such that for every $n \geq n_0$, the sheaf $F \otimes S^n(E)$ (where $S^n(E)$ is the n-th symmetric power of E) is generated as an $\mathcal{O}_X$-module by its global sections. Using this description it is easy to see, for example, that the direct sum of ample vector bundles is ample, etc.

The following theorem is an easy application of ample vector bundles properties.

Theorem 4.2 ([E1]) *Assume that X is a nonlinear smooth projective variety in $\mathbb{P}^N$. If X is a complete intersection then X^* is a hypersurface.*

Proof. We may assume that X is nondegenerate. Let $\dim X = n$. Then X is scheme-theoretic intersection of $N-n$ hypersurfaces $H_1, \ldots, H_{N-n}$ such that $\deg H_k = d_k \geq 2$. Therefore the normal bundle $N_X\mathbb{P}^N$ is equal to $\mathcal{O}_X(d_1) \oplus \ldots \oplus \mathcal{O}_X(d_{N-n})$. It follows that

$$N_X\mathbb{P}^N(-1) = \mathcal{O}_X(d_1 - 1) \oplus \ldots \oplus \mathcal{O}_X(d_{N-n} - 1)$$

is an ample vector bundle. Let $I_X = \mathbb{P}(N_X^*\mathbb{P}^N(1)) \subset \mathbb{P}^N \times (\mathbb{P}^N)^*$ be the conormal variety and let $\mathrm{pr} : I_X \to X^*$ be the projection. Then $\mathrm{pr}^*\,\mathcal{O}_{X^*}(1)$ is the tautological line bundle on $\mathbb{P}(N_X^*\mathbb{P}^N(1))$. Since $N_X\mathbb{P}^N(-1)$ is ample, it follows that pr is a finite morphism. Therefore,

$$\dim X^* = \dim I_X = N - 1.$$

□

Sommese [S1] generalized the notion of ampleness to the following.

Definition 4.3 *A vector bundle E is called k-ample if $\xi(E)^n$ is spanned by global sections for some $n > 0$ and the induced map $X_E \to \mathbb{P}^N$ has at most k-dimensional fibers.*

It can be verified that the number k is independent of the integer n for which $\xi(E)^n$ is spanned and a vector bundle is 0-ample if and only if it is ample. This number k is called the ampleness of E, denoted by $a(E)$. It can be shown that $0 \leq a(E) \leq \dim X$. If $\xi(E)^n$ is not spanned for any $n > 0$, we define $a(E) = \dim X$.

The ampleness of homogeneous vector bundles on flag varieties was described combinatorially by Snow [Sn1] inspired by an earlier work of Goldstein [Go] who determined the ampleness for tangent bundles on flag varieties.

4.2 Resultants

Classically, discriminants were studied in conjunction with resultants. Let X be a smooth irreducible projective variety and let E be a vector bundle on X of rank $k = \dim X + 1$. Set $V = H^0(X, E)$. We shall assume that E is very ample, i.e. $\xi(E)$ is a very ample line bundle on X_E. In particular, $\xi(E)$ (and hence E) is generated by global sections. Notice that $H^0(X, E) = H^0(X_E, \xi(E))$, therefore $\xi(E)$ embeds X_E in $\mathbb{P}(V^*)$.

Definition 4.4 The resultant variety $\nabla \subset \mathbb{P}(V)$ is the set of all sections vanishing at some point $x \in X$.

Example 4.5 Suppose that $X = \mathbb{P}^{k-1} = \mathbb{P}(\mathbb{C}^k)$ and

$$E = \mathcal{O}(d_1) \oplus \ldots \oplus \mathcal{O}(d_k).$$

Then $V = S^{d_1}(\mathbb{C}^k)^* \oplus \ldots \oplus S^{d_k}(\mathbb{C}^k)^*$ and ∇ is the classical resultant variety parametrizing k-tuples of homogeneous forms on $\mathbb{C}^k$ of degrees $d_1, \ldots, d_k$ having a common non-zero root.

The assertion (b) of the following Theorem is sometimes called the Cayley trick who first noticed that the resultant can be written as a discriminant.

Theorem 4.6 ([GKZ2])

(a) ∇ *is an irreducible hypersurface of degree* $\int_X c_{k-1}(E)$.
(b) ∇ *is projectively dual to* X_E.

Proof. $\xi(E)$ embeds X_E in $\mathbb{P}(V^*)$ in such a way that all fibers become projective subspaces of dimension $k-1$. Geometrically, ∇ parametrizes hyperplanes in $\mathbb{P}(V^*)$ that contain at least one fiber of X_E. Since the embedded tangent space to any point of X_E contains the fiber through it, it follows that $\nabla \supset {X_E}^*$. On the contrary, suppose that $v \in \nabla$. Then the corresponding section $s \in H^0(X, E)$ vanishes at some point $x \in X$. Therefore, s defines a linear map $ds : T_xX \to T_xE = T_xX \oplus E_x$. Let $\mathrm{pr} : T_xE \to E_x$ be the projection. Since $\dim X = \dim E - 1$, $\mathrm{pr} \circ ds(T_xX)$ is contained in some hyperplane $H \subset E_x$. An easy local calculation shows that the hyperplane in $\mathbb{P}(V^*)$ corresponding to v contains the embedded tangent space to X_E at the point corresponding to H. Therefore, $\nabla = {X_E}^*$.

Now let us prove that ∇ is a hypersurface. An easy dimension count shows that it is sufficient to prove that a generic section s of $H^0(X, E)$ vanishing at $x \in X$ does not vanish at other points. In other words, a generic hyperplane in $P = \mathbb{P}(V^*)$ containing some fiber F of E_X does not contain other fibers. Consider the linear projection $\pi_F : P \setminus F \to P/F$. If F' is any fiber of X_E distinct from F then $\pi_F(F')$ is a projective subspace of dimension $\dim F' = k-1$. Therefore, images of fibers of X_E distinct from F form at most $(k-1)$-dimensional family of projective subspaces of dimension $k-1$. An easy dimension count shows that there exists a hyperplane $H \subset P/F$ that does not contain any projective subspace from this family. The preimage of this hyperplane in P contains only one fiber F.

It remains to calculate the degree of ∇. To find it, we should take two generic sections s_1 and s_2 of V and find the number of values $\lambda \in \mathbb{C}$ such that $s_1 + \lambda s_2$ vanishes at some $x \in X$. In other words, we need to find the number of points (with multiplicities) of a zero-dimensional cycle $Z \subset X$ consisting of all points $x \in X$ where s_1 and s_2 are linearly dependent. But this is exactly the geometric definition of $c_{k-1}(E)$, see [Fu1]. □

The same argument also proves the following result from the folklore. Suppose that $X \subset \mathbb{P}^N$ is a scroll, i.e. a projectivization of a vector bundle E

over a smooth variety Y such that all fibers are embedded linearly. Assume that $\dim E > \dim Y$. Then

$$\operatorname{def} X = \dim E - \dim Y - 1$$

and X^* parametrizes hyperplanes in $\mathbb{P}^N$ that contain at least one fiber of the projective bundle $X \to Y$.

The proof of the following theorem can be found in [E2]:

Theorem 4.7 ([E2]) *If X is a smooth n-dimensional projective variety in $\mathbb{P}^N$ and $\operatorname{def} X = k \geq n/2$, then X is a projective bundle $X \simeq \mathbb{P}_Y(F)$, where F is a vector bundle of rank $(n+k+2)/2$ on a smooth $(n-k)/2$-dimensional variety Y and the fibers are embedded linearly.*

4.3 Zeros of Generic Global Sections

Both resultants and discriminants are related to the following general construction. Suppose that X is a smooth projective variety and E is a vector bundle on X generated by global sections. Let $Z(s)$ denote the scheme of zeros of any global section $s \in H^0(X, E)$. One might expect that for generic s the scheme $Z(s)$ is a smooth variety of codimension $\dim E$. Then we can define the degeneration variety $D \subset H^0(X, E)$ parametrizing all global sections s such that $Z(s)$ is not smooth of expected codimension. For example, if E is a very ample line bundle then D is a cone over the dual variety. If E is a very ample vector bundle and $\dim E = \dim X + 1$ then D is the resultant variety. If E is a very ample vector bundle and $\dim E = \dim X$ then the corresponding homogeneous polynomial can be called Bezoutian. Indeed, if $X = \mathbb{P}^n$ and $E = \mathcal{O}(d_1) \oplus \ldots \oplus \mathcal{O}(d_n)$ then D parametrizes sets of homogeneous forms of degrees $d_1, \ldots, d_n$ such that the Bezout theorem is not applicable.

In general, in order to make the theory consistent, it is necessary to impose very strong conditions on the vector bundle E. These conditions are not always satisfied even in the case of homogeneous vector bundles on flag varieties. Even the question whether or not a generic zero scheme is non-empty can be quite difficult.

Assume that G is a connected reductive group, T is a fixed maximal torus, B is a fixed Borel subgroup, $T \subset B \subset G$, B_- is the opposite Borel subgroup, P is a parabolic subgroup containing B_-, $\mathbf{X}(T)$ is the lattice of characters of T, and $\lambda \in \mathbf{X}(T)$ is the dominant weight. Consider the homogeneous vector bundle $\mathcal{L}_\lambda = G \times_P U_\lambda$ over G/P, where U_λ is the irreducible P-module with highest weight λ. See Section 2.1 for further details. By the Borel–Weil–Bott theorem (see [Bot]), $V_\lambda = H^0(G/P, \mathcal{L}_\lambda)$ is an irreducible G-module with highest weight λ.

Theorem 4.8 *Let $s \in V_\lambda$ be a generic global section. Then*

(a) *If* $\dim U_\lambda > \dim G/P$*, the scheme of zeros* Z_s *is empty.*
(b) *If* $\dim U_\lambda \leq \dim G/P$*, either* Z_s *is empty or* s *intersects the zero section of* $\mathcal{L}_\lambda$ *transversally and* Z_s *is a smooth unmixed subvariety of expected codimension* $\dim U_\lambda$*.*
(c) *If* $\dim U_\lambda = \dim G/P$*, the geometric number of points in* Z_s *is equal to the top Chern class of* $\mathcal{L}_\lambda$*.*

Proof. (a) is obvious and follows by an easy dimension count. (b) and (c) follow from the fact that $\mathcal{L}_\lambda$ is generated by global sections. Let us recall this argument here. Let $\dim U_\lambda \leq \dim G/P$, and assume that every global section has a zero. We have to prove that a generic global section s intersects the zero section of $\mathcal{L}_\lambda$ transversally. This will imply, in particular, that Z_s is a smooth unmixed subvariety of codimension $\dim U_\lambda$. For simplicity we suppress the index λ. Consider the incidence variety

$$Z \subset G/P \times V, \quad Z = \{(x, s) \,|\, x \in (Z_s)_{red}\}.$$

Since G/P is homogeneous and Z is invariant, it follows that Z is obtained by spreading the fiber $Z_e = \{s \in V \,|\, s(eP) = 0\}$ by the group G. Since U is irreducible, we have $\dim Z_e = \dim V - \dim U$. Hence, Z is a smooth irreducible subvariety of dimension $\dim V + \dim G/P - \dim U$.

Let $\pi : Z \to V$ be the restriction to Z of the projection of $G/P \times V$ on the second summand. By assumption, π is a surjection. By Sard's lemma for algebraic varieties (see [Mum1]), for a generic point $s \in V$ and any point (x, s) in $\pi^{-1}s$ the differential $d\pi_{(x,s)}$ is surjective. We claim that s has the transversal intersection with the zero section. Indeed, Z can be regarded as a subbundle of the trivial bundle $G/P \times V$. Then $\mathcal{L} = (G/P \times V)/Z$ and the zero section of $\mathcal{L}$ is identified with $Z \mod Z$. The section s is identified with $(G/P \times \{s\}) \mod Z$. Hence, it is sufficient to prove that $G/P \times \{s\}$ is transversal to Z, which is equivalent to the following claim: $d\pi_{(x,s)}$ is surjective for all $(x, s) \in Z$. Now statement (c) of the theorem follows from the standard intersection theory (see [Fu1]). □

Example 4.9 It should be noted that Theorem 4.8 cannot be strengthened to the point where the non-emptiness and the irreducibility of the scheme of zeros in Theorem 4.8 could be established apriori, as the following example shows. Consider the vector bundle $S^2\mathbb{S}^*$ on $\mathrm{Gr}(k, 2n)$. The dimension of a fiber does not exceed the dimension of the Grassmanian as $k \leq \dfrac{4n-1}{3}$, but a generic section (that is, a non-degenerate quadratic form in $\mathbb{C}^{2n}$) has a zero (that is, a k-dimensional isotropic subspace) only if $k \leq n$. For $k = n$ the scheme of zeros is a reducible variety of dimension $\dfrac{n(n-1)}{2}$ with two irreducible components (spinor varieties) that correspond to two families of maximal isotropic subspaces on an even-dimensional quadric.

Consider the P-submodule $M_\lambda = \{s \in H^0(G/P, \mathcal{L}_\lambda) \mid s(P) = 0\}$ in V_λ. Here $s(P)$ is a germ of s in the point of G/P corresponding to P. It is easy to see that M_λ can be characterized as the unique maximal proper P-submodule of V_λ. Clearly $V_\lambda/M_\lambda \simeq U_\lambda$ as P-modules. Consider the map $\Psi : G \times M_\lambda \to V_\lambda$, $\Psi(g, v) = gv$. Then generic global sections of $\mathcal{L}$ have zeros iff Ψ is dominant iff the differential of Ψ at a generic point is surjective. We write $\mathfrak{g}$ and $\mathfrak{p}$ for Lie algebras of G and P, respectively. The natural "orbital" map $\mathfrak{g} \times V_\lambda \to V_\lambda$ defines the P-equivariant map $\psi : \mathfrak{g}/\mathfrak{p} \times M_\lambda \to U_\lambda$. Then Ψ is dominant iff $\psi(\cdot, x)$ is surjective at a generic point $x \in M_\lambda$. Let $Z_\lambda = \psi^{-1}(0)_{red} \subset \mathfrak{g}/\mathfrak{p} \times M_\lambda$ be the incidence variety. Then the following proposition follows from an easy dimension count:

Proposition 4.10 *Suppose that*

$$\dim Z_\lambda = \dim \mathfrak{g}/\mathfrak{p} + \dim M_\lambda - \dim U_\lambda.$$

Then generic global sections of $\mathcal{L}_\lambda$ have zeros.

Denote by π the restriction on Z_λ of the projection of $\mathfrak{g}/\mathfrak{p} \times M_\lambda$ to the first factor. The variety Z_λ could be rather complicated. Say, it is not irreducible in general. We shall use the fact that fibers $\pi^{-1}(x)$ are linear spaces. So consider the function $l_\lambda(x) = \dim \pi^{-1}(x)$, $x \in \mathfrak{g}/\mathfrak{p}$.

Conjecture 4.11 *There exists an algebraic stratification $\mathfrak{g}/\mathfrak{p} = \bigsqcup_{i=1}^{r} X_i$ such that for any λ the function $l_\lambda(x)$ is constant along each X_i:*

$$l_\lambda(x) = l_\lambda^i, \quad x \in X_i.$$

If such stratification exists then

$$\dim Z_\lambda = \max_i (\dim X_i + l_\lambda^i). \tag{4.1}$$

If P acts on $\mathfrak{g}/\mathfrak{p}$ with a finite number of orbits then we can take these orbits as X_i (since ψ is P-equivariant). Unfortunately, this class of parabolic subgroups is very small. The dual P-module $(\mathfrak{g}/\mathfrak{p})^*$ is isomorphic to the representation of P on the unipotent radical $\mathfrak{p}_n$ of $\mathfrak{p}$, hence by Pyasetskii Theorem 2.9 the action $P : \mathfrak{g}/\mathfrak{p}$ has a finite number of orbits iff the action $P : \mathfrak{p}_n$ satisfies this property. Such actions were studied in [PR], the complete classification for classical groups was obtained in [HRo]. But even in these cases the problem of the complete description of the orbital decomposition seems very messy. In the next section we shall introduce another class of parabolic subgroups that satisfy the Conjecture 4.11.

The most wonderful class of parabolic subgroups that will fit all our needs is the class of parabolic subgroups P with abelian unipotent radical $\mathfrak{p}_n$, see Section 2.3. So from now on P will be a parabolic subgroup with *aura*. Let P^- be an opposite parabolic subgroup, let $L = P \cap P^-$ be a Levi subgroup

with Lie algebra $\mathfrak{l}$, let $\mathfrak{p}_n^-$ be a unipotent radical of $\mathfrak{p}^-$. Then $\mathfrak{p}_n$ is abelian iff the decomposition $\mathfrak{g} = \mathfrak{p}_n^- \oplus \mathfrak{l} \oplus \mathfrak{p}_n$ is a $\mathbb{Z}$-grading: $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$. Notice that $\mathfrak{g}/\mathfrak{p}$ is isomorphic to $\mathfrak{p}_n^-$ as a L-module. It is well-known that the action of L on $\mathfrak{p}_n^-$ has a finite number of orbits in this case. Take the orbital decomposition $\mathfrak{p}_n^- = \bigcup_{i=1}^{r} Lf_i$. For its detailed description see Theorem 2.17. Then the equality (4.1) takes a form

$$\dim Z_\lambda = \max_i(\dim Lf_i + l_\lambda(f_i)). \tag{4.2}$$

It remains to calculate $l_\lambda(f_i)$. If $f_i = 0$ then $l_\lambda(f_i) = \dim M_\lambda$. In the opposite case we can include f_i in a $\mathfrak{sl}_2$-triple $\langle f_i, h_i, e_i\rangle$. Evidently we can assume that $h_i \in \mathfrak{l}$, $e_i \in \mathfrak{p}_n$. These h_i's were also written down explicitly in Theorem 2.17. We shall use the following easy lemma from the $\mathfrak{sl}_2$-theory:

Lemma 4.12 *Let V be a finite-dimensional $\mathfrak{sl}_2$-module, $\mathfrak{sl}_2 = \langle f, h, e\rangle$. Let $\mathfrak{b} = \langle h, e\rangle$. Suppose that $M \subset V$ is a $\mathfrak{b}$-submodule such that $eV \subset M$. Then*

$$\dim \operatorname{Coker}(M \subset V \xrightarrow{f} V \to V/M) = \dim(V/M)^h,$$

where $(V/M)^h = \{v \in V/M \mid hv = 0\}$.

Applying this Lemma to the representation of $\mathfrak{sl}_2$ in V_λ we get

$$\begin{aligned} l_\lambda(f_i) = \dim \operatorname{Ker} \psi(f_i, \cdot) \quad &= \dim M_\lambda - \dim U_\lambda + \dim \operatorname{Coker} \psi(f_i, \cdot) = \\ &= \dim M_\lambda - \dim U_\lambda + \dim U_\lambda^{h_i}. \end{aligned}$$

Notice that this formula remains valid for $f_i = 0$ if we assume that in this case $h_i = 0$. Joining this formula with (4.2) and applying the Lemma above we get the following result.

Theorem 4.13 ([T3]) *Suppose that for any i*

$$\dim U_\lambda^{h_i} \le \operatorname{codim}_{\mathfrak{g}/\mathfrak{p}} Lf_i.$$

Then generic global sections of the bundle $\mathcal{L}_\lambda$ have zeros.

All ingredients of the formula (4.13) can be easily computed in many particular cases. We will apply Theorem 4.13 for the proof of the following

Theorem 4.14 ([T3]) *Let $w \in S^dV^*$, resp. $w \in \Lambda^dV^*$, be a generic form. Then V contains a k-dimensional isotropic subspace w.r.t. w if and only if*

$$n \ge \frac{\binom{d+k-1}{d}}{k} + k, \ \textit{resp.}\ n \ge \frac{\binom{k}{d}}{k} + k, \tag{4.3}$$

with the following exceptions (a form is supposed to be generic). V contains a k-dimensional isotropic subspace if and only if

– $n \geq 2k$ *for* $w \in S^2V^*$ *or* $w \in \Lambda^2 V^*$.
– $k \leq n-2$ *for* $w \in \Lambda^{n-2}V^*$, n *is even.*
– $k \leq 4$ *for* $w \in \Lambda^3 V^*$, $n = 7$.

It is interesting that the variety of 4-dimensional isotropic subspaces of a generic skewsymmetric 3-form in $\mathbb{C}^7$ is a smooth 8-dimensional Fano variety. Moreover, this variety is a compactification of the unique symmetric space of the simple algebraic group G_2.

Proof. We choose a basis $\{e_1, \ldots, e_n\}$ in V and identify GL_n with the group of non-singular matrices. Let T, B, B^- be the subgroups of diagonal, upper- and lower-triangular matrices. We fix an integer k. Consider the parabolic subgroup

$$P = \begin{pmatrix} A & 0 \\ * & B \end{pmatrix},$$

where B is a $k \times k$-matrix. Then G/P is the Grassmanian $\mathrm{Gr}(k, V)$. Consider the vector bundle $\mathcal{L} = S^d\mathcal{S}^*$ (resp. $\mathcal{L} = \Lambda^d\mathcal{S}^*$, but only in the case $k \geq d$) on G/P, where $\mathcal{S}$ is the tautological bundle. Then $\mathcal{L} = \mathcal{L}_\lambda$, where λ is the highest weight of the GL_n-module S^dV^* (resp. Λ^dV^*). Let $w \in S^dV^*$ (resp. $w \in \Lambda^dV^*$), let s_w be the corresponding global section. It is easy to see that $(Z_{s_w})_{red}$ coincides with the variety of k-dimensional isotropic subspaces w.r.t. w. Notice that inequalities (4.3) are equivalent to the condition $\dim U_\lambda \leq \dim G/P$. In the sequel we suppose that these inequalities hold.

We take

$$f_i = E_{1,n} + E_{2,n-1} + \ldots + E_{i,n+1-i}, \quad i = 0, \ldots, r = \min(k, n-k).$$

The unipotent radical $\mathfrak{p}_n^-$ can be identified with matrices of shape $k \times (n-k)$ then the orbit Lf_i is identified with the variety of matrices of rank i, therefore

$$\mathrm{codim}_{\mathfrak{g}/\mathfrak{p}}\, Lf_i = (k-i)(n-k-i).$$

We have

$$h_i = (E_{n,n} - E_{1,1}) + (E_{n-1,n-1} - E_{2,2}) + \ldots + (E_{n+1-i,n+1-i} - E_{i,i}),$$

hence

$$\dim U_\lambda^{h_i} = \binom{d+k-i-1}{d} \text{ if } V_\lambda = S^dV^*;$$
$$\dim U_\lambda^{h_i} = \binom{k-i}{d} \text{ if } V_\lambda = \Lambda^dV^*.$$

Applying Theorem4.13 we get

Proposition 4.15

(a) *Suppose that for any* $i = 0, \ldots, r$ *we have*

$$\binom{d+k-i-1}{d} \leq (k-i)(n-k-i).$$

Then a generic form $w \in S^d V^*$ *has a* k*-dimensional isotropic subspace.*

(b) *Suppose that for any* $i = 0, \ldots, r$ *we have*

$$\binom{k-i}{d} \leq (k-i)(n-k-i).$$

Then a generic form $w \in \Lambda^d V^*$ *has a* k*-dimensional isotropic subspace.*

It remains to clarify when the conditions of Proposition 4.15 follow from formulas (4.3).

A. Symmetric Case.

Clearly if $i = k$ then the conditions of Proposition 4.15 are satisfied. Therefore it suffices to find out when the inequality

$$n \geq \frac{\binom{d+k-i-1}{d}}{k-i} + k + i$$

follows from the inequality

$$n \geq \frac{\binom{d+k-1}{d}}{k} + k$$

as $i = 1, \ldots, \min(k-1, n-k)$. If $d = 1$ then Theorem 4.14 is obvious, if $d = 2$ it reduces to a well-known result about isotropic subspaces of quadratic form. So assume that $d \geq 3$. We use a following Lemma that can be easily verified by induction:

Lemma 4.16 *Let* $d \geq 3$, $\alpha \geq 2$. *Then* $\frac{\binom{d+\alpha-1}{d}}{\alpha} \geq \frac{\binom{d+\alpha-2}{d}}{\alpha-1} + 1$.

It follows from Lemma 4.16 that

$$n \geq \frac{\binom{d+k-1}{d}}{k} + k \geq \frac{\binom{d+k-2}{d}}{k-1} + k + 1$$
$$\geq \frac{\binom{d+k-3}{d}}{k-2} + k + 2 \geq \ldots \geq \frac{\binom{d+k-1-(k-1)}{d}}{k-(k-1)} + k + k - 1.$$

QED.

B. Skew-symmetric Case.

Clearly if $i > k - d$ then the conditions of Proposition 4.15 are satisfied. Therefore it suffices to check when the inequality

$$n \geq \frac{\binom{k-i}{d}}{k-i} + k + i$$

follows from the inequality

$$n \geq \frac{\binom{k}{d}}{k} + k$$

as $i = 1, \ldots, \min(k-d, n-k)$. If $d = 1$ then Theorem 4.14 is obvious, if $d = 2$ it reduces to a well-known result about isotropic subspaces of a 2-form. So assume that $d \geq 3$. We use a following Lemma that can be easily verified by induction:

Lemma 4.17 *Let* $3 \leq d \leq \alpha - 2$. *Then* $\frac{\binom{\alpha}{d}}{\alpha} \geq \frac{\binom{\alpha-1}{d}}{\alpha-1} + 1$.

It follows from Lemma 4.17 that

$$n \geq \frac{\binom{k}{d}}{k} + k \geq \frac{\binom{k-1}{d}}{k-1} + k + 1$$
$$\geq \frac{\binom{k-2}{d}}{k-2} + k + 2 \geq \ldots \geq \frac{\binom{k-(k-3-d)}{d}}{k-(k-3-d)} + k + (k-3-d).$$

It remains to consider 3 cases: $i = k - d - 2$, $i = k - d - 1$, $i = k - d$ as $i \leq n - k$.

Let $i = k - d - 2$, $n \geq 2k - d - 2$. We want to deduce the inequality $n \geq \frac{\binom{d+2}{d}}{d+2} + 2k - d - 2 = 2k - \frac{d-1}{2} - 2$ from $n \geq \frac{\binom{k}{d}}{k} + k$. It suffices to check $\binom{k}{d} \geq k(k - \frac{d-1}{2} - 2)$. Since $d \geq 3$ and $k - d = i + 2 \geq 3$ we have $\binom{k}{d} \geq \binom{k}{3} \geq k(k-3) \geq k(k - \frac{d-1}{2} - 2)$.

Let $i = k - d - 1$, $n \geq 2k - d - 1$. We want to deduce the inequality $n \geq \frac{\binom{d+1}{d}}{d+1} + 2k - d - 1 = 2k - d$ from $n \geq \frac{\binom{d+1}{d}}{d+1} + 2k - d - 1 = 2k - d$. It suffices to check $\binom{k}{d} \geq k(k-d)$. Since $d \geq 3$ and $k - d = i + 1 \geq 2$ we have $\binom{k}{d} \geq \binom{k}{3} \geq k(k-3) \geq k(k-d)$ as $k - d \geq 3$. If $k - d = 2$ then $\binom{k}{d} = \frac{(d+2)(d+1)}{2} \geq 2(d+2) \geq k(k-d)$.

Let $i = k - d$, $n \geq 2k - d$. We need to deduce the inequality $n \geq \frac{\binom{d}{d}}{d} + 2k - d = 2k - d + \frac{1}{d}$ from $n \geq \frac{\binom{k}{d}}{k} + k$. All exceptions are defined by the following system of equalities and inequalities:

$$n = 2k - d, \quad \binom{k}{d} \leq k(k-d), \quad k - d \geq 1.$$

There exist only two possibilities: either $k = d+1$, $n = d+2$ or $d = 3$, $k = 5$, $n = 7$.

In the first case we need to clarify whether a generic form $w \in \Lambda^{n-2}V^*$ has a $(n-1)$-dimensional isotropic subspace or not. This holds iff a generic 2-form in V^* has a non-zero kernel. It is well-known that this is true only for odd n.

To complete the proof it remains to find out whether a generic form $w \in \Lambda^3(\mathbb{C}^7)^*$ has a 5-dimensional isotropic subspace. In suitable coordinates this isotropic subspace will coincide with the linear span $\langle e_1, \dots, e_5 \rangle$. Therefore $w = \sum_{1 \le i < j < k \le 7} \alpha_{ijk} x_i \wedge x_j \wedge x_k$, where $\alpha_{ijk} = 0$ as $k \le 5$. Consider the one-parameter subgroup

$$H(t) = \mathrm{diag}(t^2, t^2, t^2, t^2, t^2, t^{-5}, t^{-5})$$

in SL_7. Clearly, $\lim_{t \to 0} H(t)w = 0$. Therefore w belongs to the Null-cone of the action $\mathrm{SL}_7 : \Lambda^3(\mathbb{C}^7)^*$. It follows that any non-constant homogeneous invariant of this action vanishes at w. But it is known that this action has non-constant invariants (see [PV]), for example, the discriminant is well-defined in this case. Therefore generic forms do not admit 5-dimensional isotropic subspaces.

Moreover, it can be shown that a 3-form in the 7-dimensional vector space admits a 5-dimensional isotropic subspace if and only if its discriminant is equal to zero. □

4.4 Moore–Penrose Inverse and Applications

The nice notion of a generalized inverse of an arbitrary matrix (possibly singular or even non-square) has been discovered independently by Moore [Mo] and Penrose [Pe]. The following definition belongs to Penrose (Moore's definition is different but equivalent):

Definition 4.18 A matrix A^+ is called *a MP-inverse* of a matrix A if

$$AA^+A = A, \quad A^+AA^+ = A^+,$$

and AA^+, A^+A are Hermitian matrices.

It is quite surprising but a MP-inverse always exists and is unique. Since the definition is symmetric with respect to A and A^+ it follows that $(A^+)^+ = A$. If A is a non-singular square matrix then A^+ coincides with an ordinary inverse matrix A^{-1}. The theory of MP-inverses and their numerous modifications becomes now a separate subfield of Linear Algebra [CM] with various applications. Here we show that this notion quite naturally arises in the theory of shortly graded simple Lie algebras and give applications. To explain this connection let us first give another definition of a MP-inverse.

Let $A \in \mathrm{Mat}_{n,m}(\mathbb{C})$. Then it is easy to see that a matrix $A^+ \in \mathrm{Mat}_{m,n}(\mathbb{C})$ is a MP-inverse of A if and only if there exist Hermitian matrices

$$B_1 \in \mathrm{Mat}_{n,n}(\mathbb{C}) \text{ and } B_2 \in \mathrm{Mat}_{m,m}(\mathbb{C})$$

such that the following matrices form an $\mathfrak{sl}_2$-triple in $\mathfrak{sl}_{n+m}(\mathbb{C})$:

$$E = \begin{pmatrix} 0 & A \\ 0 & 0 \end{pmatrix}, \quad H = \begin{pmatrix} B_1 & 0 \\ 0 & B_2 \end{pmatrix}, \quad F = \begin{pmatrix} 0 & 0 \\ A^+ & 0 \end{pmatrix}.$$

By an $\mathfrak{sl}_2$-triple $\langle e, h, f \rangle$ in a Lie algebra $\mathfrak{g}$ we mean a collection of (possibly zero) vectors such that

$$[e, f] = h, \quad [h, e] = 2e, \quad [h, f] = -2f.$$

In other words, an $\mathfrak{sl}_2$-triple is a homomorphic image of canonical generators of $\mathfrak{sl}_2$ with respect to some homomorphism of Lie algebras $\mathfrak{sl}_2 \to \mathfrak{g}$.

This definition admits an immediate generalization. Suppose that $\mathfrak{g}$ is a simple complex Lie algebra, G is a corresponding simple simply-connected Lie group. Suppose further that P is a parabolic subgroup of G with abelian unipotent radical (with *aura*). Then $\mathfrak{g}$ admits a short grading

$$\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$$

with only three nonzero parts. Here $\mathfrak{p} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ is a Lie algebra of P and $\exp \mathfrak{g}_1$ is the abelian unipotent radical of P. Let $\mathfrak{k}_0$ be a compact real form of $\mathfrak{g}_0$. In this section we shall permanently consider compact real forms of reductive subalgebras of simple Lie algebras. These subalgebras will always be Lie algebras of algebraic reductive subgroups of a corresponding simple complex algebraic group. Their compact real forms will always be understood as Lie algebras of compact real forms of corresponding algebraic groups. For example, a Lie algebra of an algebraic torus has a unique compact real form.

Suppose now that $e \in \mathfrak{g}_1$. It is well-known that there exists a *homogeneous* $\mathfrak{sl}_2$-triple $\langle e, h, f \rangle$ such that $h \in \mathfrak{g}_0$ and $f \in \mathfrak{g}_{-1}$.

Definition 4.19 An element $f \in \mathfrak{g}_{-1}$ is called *a MP-inverse* of $e \in \mathfrak{g}_1$ if there exists a homogeneous $\mathfrak{sl}_2$-triple $\langle e, h, f \rangle$ with $h \in i\mathfrak{k}_0$.

MP-inverses of elements $f \in \mathfrak{g}_{-1}$ are defined in the same way. It is clear that if f is a MP-inverse of e then e is a MP-inverse of f.

Example 4.20 Suppose that $G = \mathrm{SL}_{n+m}$ and $P \subset G$ is a maximal parabolic subgroup of block triangular matrices of the form

$$\begin{pmatrix} B_1 & A \\ 0 & B_2 \end{pmatrix}, \quad \text{where} \quad B_1 \in \mathrm{Mat}_{n,n},\ A \in \mathrm{Mat}_{n,m},\ B_2 \in \mathrm{Mat}_{m,m}\,.$$

The graded components of the correspondent grading consist of matrices of the following form:

$$\mathfrak{g}_{-1} = \begin{pmatrix} 0 & 0 \\ A' & 0 \end{pmatrix}, \quad \mathfrak{g}_0 = \begin{pmatrix} B_1 & 0 \\ 0 & B_2 \end{pmatrix}, \quad \mathfrak{g}_1 = \begin{pmatrix} 0 & A \\ 0 & 0 \end{pmatrix},$$

where $A' \in \mathrm{Mat}_{m,n}$, $B_1 \in \mathrm{Mat}_{n,n}$, $B_2 \in \mathrm{Mat}_{m,m}$, and $A \in \mathrm{Mat}_{n,m}$. One can take $\mathfrak{k}_0$ to be the real Lie algebra of block diagonal skew-Hermitian matrices with zero trace. Then $i\mathfrak{k}_0$ is a vector space of block diagonal Hermitian matrices with zero trace. Therefore in this case we return to a previous definition of a Moore–Penrose inverse.

Theorem 4.21 ([T4]) *For any $e \in \mathfrak{g}_1$ there exists a unique Moore–Penrose inverse $f \in \mathfrak{g}_{-1}$.*

The proof is similar to the proof of Theorem 4.23 below.

It obviously follows that for any non-zero $f \in \mathfrak{g}_{-1}$ there exists a unique MP-inverse $e \in \mathfrak{g}_1$. So taking a MP-inverse is a well-defined involutive operation. In general, it is not equivariant with respect to a Levi subgroup $L \subset P$ with Lie algebra $\mathfrak{g}_0$, but only with respect to its maximal compact subgroup $K_0 \subset L$.

Let us give an intrinsic description of the Moore-Penrose inverse in all cases arising arising from short gradings of classical simple Lie algebras. Exceptional cases may be found in [T4].

Linear Maps.

This is, of course, the classical Moore–Penrose inverse. Let us recall its intrinsic description. Suppose that $\mathbb{C}^n$ and $\mathbb{C}^m$ are vector spaces equipped with standard Hermitian scalar products. For any linear map $F : \mathbb{C}^n \to \mathbb{C}^m$ its Moore-Penrose inverse is a linear map $F^+ : \mathbb{C}^m \to \mathbb{C}^n$ defined as follows. Let $\mathrm{Ker}\, F \subset \mathbb{C}^n$ and $\mathrm{Im}\, F \subset \mathbb{C}^m$ be a kernel and an image of F. Let $\mathrm{Ker}^\perp F \subset \mathbb{C}^n$ and $\mathrm{Im}^\perp F \subset \mathbb{C}^m$ be their orthogonal complements with respect to the Hermitian scalar products. Then F defines via restriction a bijective linear map $\tilde{F} : \mathrm{Ker}^\perp F \to \mathrm{Im}\, F$. Then $F^+ : \mathbb{C}^m \to \mathbb{C}^n$ is a unique linear map such that $F^+|_{\mathrm{Im}^\perp F} = 0$ and $F^+|_{\mathrm{Im}\, F} = \tilde{F}^{-1}$. This MP-inverse corresponds to short gradings of $\mathfrak{sl}_{n+m}$.

Symmetric and Skew-symmetric Bilinear Forms.

Let $V = \mathbb{C}^n$ be a vector space equipped with a standard Hermitian scalar product. For any symmetric (resp. skew-symmetric) bilinear form ω on V its Moore-Penrose inverse is a symmetric (resp. skew-symmetric) bilinear form ω^+ on V^* defined as follows. Let $\mathrm{Ker}\ \omega \subset V$ be the kernel of ω. Then ω induces a non-degenerate bilinear form $\tilde{\omega}$ on $V/\mathrm{Ker}\ \omega$. Let $\mathrm{Ann}(\mathrm{Ker}\ \omega) \subset V^*$ be an annihilator of $\mathrm{Ker}\ \omega$. Then $\mathrm{Ann}(\mathrm{Ker}\ \omega)$ is canonically isomorphic to the dual of $V/\mathrm{Ker}\ \omega$. Therefore the form $\tilde{\omega}^{-1}$ on $\mathrm{Ann}(\mathrm{Ker}\ \omega)$ is well-defined. The form ω^+ is defined as a unique form such that its restriction on on $\mathrm{Ann}(\mathrm{Ker}\ \omega)$ coincides with $\tilde{\omega}^{-1}$ and its kernel is $\mathrm{Ann}(\mathrm{Ker}\ \omega)^\perp$, the orthogonal complement with respect to a standard Hermitian scalar product on V^*. This MP-inverse corresponds to the short grading of $\mathfrak{sp}_{2n+2}$ (resp. $\mathfrak{so}_{2n+2}$).

Vectors in a Vector Space With the Scalar Product.

Suppose that $V = \mathbb{C}^n$ is a vector space with the standard bilinear scalar product $(\cdot,\cdot)$. For any vector $v \in V$ its Moore-Penrose inverse $v^\vee$ is again a vector in V defined as follows:

$$v^\vee = \begin{cases} \dfrac{2v}{(v,v)}, & \text{if } (v,v) \neq 0 \\ \dfrac{\overline{v}}{(\overline{v},v)}, & \text{if } (v,v) = 0,\ v \neq 0 \\ 0, & \text{if } v = 0. \end{cases}$$

This MP-inverse corresponds to the short grading of $\mathfrak{so}_{n+2}$.

It is quite natural to ask whether it is possible to extend the notion of the Moore–Penrose inverse from parabolic subgroups with aura to arbitrary parabolic subgroups. It is also interesting to consider the "non-graded" situation. Let us start with it. Suppose G is a simple connected simply-connected Lie group with Lie algebra $\mathfrak{g}$. We fix a compact real form $\mathfrak{k} \subset \mathfrak{g}$.

Definition 4.22 A nilpotent orbit $\mathcal{O} \subset \mathfrak{g}$ is called *a Moore–Penrose orbit* if for any $e \in \mathcal{O}$ there exists an $\mathfrak{sl}_2$-triple $\langle e, h, f\rangle$ such that $h \in i\mathfrak{k}$.

It turns out that it is quite easy to find all Moore-Penrose orbits. Recall that the height $\mathrm{ht}(\mathcal{O})$ of a nilpotent orbit $\mathcal{O} = \mathrm{Ad}(G)e$ is equal to the maximal integer k such that $\mathrm{ad}(e)^k \neq 0$. Clearly $\mathrm{ht}(\mathcal{O}) \geq 2$.

Theorem 4.23 *$\mathcal{O}$ is a Moore–Penrose orbit if and only if* $\mathrm{ht}(\mathcal{O}) = 2$. *In this case for any $e \in \mathcal{O}$ there exists a unique $\mathfrak{sl}_2$-triple $\langle e, h, f\rangle$ such that $h \in i\mathfrak{k}$.*

Proof. Let $x \to \overline{x}$ denotes a complex conjugation in $\mathfrak{g}$ with respect to the compact form $\mathfrak{k}$. Therefore $x = \overline{x}$ iff $x \in \mathfrak{k}$ and $x = -\overline{x}$ iff $x \in i\mathfrak{k}$. Let $B(x,y) = \mathrm{Tr}\ \mathrm{ad}(x)\,\mathrm{ad}(y)$ be the Killing form of $\mathfrak{g}$. Finally, let $H(x,y) = -B(x,\overline{y})$ be a positive-definite Hermitian form on $\mathfrak{g}$.

Lemma 4.24 *We fix a nilpotent element $e \in \mathfrak{g}$. Suppose that $\langle e, h, f\rangle$ is an $\mathfrak{sl}_2$-triple in $\mathfrak{g}$ such that $h \in i\mathfrak{k}$. Then for any other $\mathfrak{sl}_2$-triple $\langle e, h', f'\rangle$ we have $H(h,h) < H(h',h')$. In particular, if there exists an $\mathfrak{sl}_2$-triple $\langle e, h, f\rangle$ with $h \in i\mathfrak{k}$ then the $\mathfrak{sl}_2$-triple with this property is unique.*

Proof. Recall that if $\langle e, h, f\rangle$ is an $\mathfrak{sl}_2$-triple then h is called *a characteristic* of e. Consider the subset $\mathcal{H} \subset \mathfrak{g}$ consisting of all possible characteristics of e. It is well-known that $\mathcal{H}$ is an affine subspace in $\mathfrak{g}$ such that the corresponding linear subspace is precisely the unipotent radical $\mathfrak{z}^u_{\mathfrak{g}}(e)$ of the centralizer $\mathfrak{z}_{\mathfrak{g}}(e)$ in $\mathfrak{g}$ of the element e. Since $H(h',h')$ is a strongly convex function on $\mathcal{H}$, there exists a unique element $h_0 \in \mathcal{H}$ such that $H(h_0,h_0) < H(h',h')$ for any $h' \in \mathcal{H}$, $h' \neq h_0$. We need to show that $h_0 = h$. It is clear that an element $h_0 \in \mathcal{H}$ minimizes $H(h,h)$ on $\mathcal{H}$ iff $H(h_0, \mathfrak{z}^u_{\mathfrak{g}}(e)) = 0$ iff $B(\overline{h}_0, \mathfrak{z}^u_{\mathfrak{g}}(e)) = 0$. If $h \in \mathcal{H} \cap i\mathfrak{k}_0$ then $\overline{h} = -h$ and we have

$$B(\overline{h}, \mathfrak{z}^u_{\mathfrak{g}}(e)) = -B(h, \mathfrak{z}^u_{\mathfrak{g}}(e)) = -B([e,f], \mathfrak{z}^u_{\mathfrak{g}}(e)) = B(f, [e, \mathfrak{z}^u_{\mathfrak{g}}(e)]) = 0.$$

Therefore $h = h_0$. □

Suppose that $\langle e, h, f\rangle$ is an $\mathfrak{sl}_2$-triple in $\mathfrak{g}$. Consider the grading $\mathfrak{g} = \bigoplus_k \mathfrak{g}_k$ such that $x \in \mathfrak{g}_k$ iff $[h, x] = kx$. Let $\mathfrak{n}_+ = \bigoplus_{k>0} \mathfrak{g}_k$, $\mathfrak{n}_- = \bigoplus_{k<0} \mathfrak{g}_k$. It is well known that $\mathfrak{z}^u_{\mathfrak{g}}(e) \subset \mathfrak{n}_+$.

Lemma 4.25 *Suppose that* $\mathfrak{z}^u_{\mathfrak{g}}(e) = \mathfrak{n}_+$. *Then* $\mathcal{O} = \mathrm{Ad}(G)e$ *is a Moore–Penrose orbit.*

Proof. We need to prove that for any element $e' \in \mathcal{O}$ there exists an $\mathfrak{sl}_2$-triple $\langle e', h', f'\rangle$ such that $h' \in i\mathfrak{k}$, where $\mathfrak{k}$ is a *fixed* compact real form of $\mathfrak{g}$. Clearly it is sufficient to prove that for an *arbitrary* compact real form $\mathfrak{k}$ there exists an $\mathfrak{sl}_2$-triple $\langle e, h, f\rangle$ with $h \in i\mathfrak{k}$. According to the proof of the previous Lemma we should choose h to be a unique characteristic such that $B(\overline{h}, \mathfrak{z}^u_{\mathfrak{g}}(e)) = 0$, where $x \to \overline{x}$ denotes a complex conjugation in $\mathfrak{g}$ with respect to the compact form $\mathfrak{k}$. It remains to prove that $h \in i\mathfrak{k}$. Since B is a non-degenerate ad-invariant scalar product on $\mathfrak{g}$ and $\mathfrak{z}^u_{\mathfrak{g}}(e) = \mathfrak{n}_+$ it follows that $\overline{h} \in \mathfrak{p}$, where $\mathfrak{p} = \mathfrak{g}_0 \oplus \mathfrak{n}_+$. Let $\mathfrak{l}$ be some "standard" compact real form of $\mathfrak{g}$ such that $h \in i\mathfrak{l}$ and $\tilde{\mathfrak{n}}_\pm = \mathfrak{n}_\mp$, where $x \to \tilde{x}$ denotes a complex conjugation in $\mathfrak{g}$ with respect to the compact form $\mathfrak{l}$. Let $P \subset G$ be a parabolic subgroup of G with the Lie algebra $\mathfrak{p}$, let $H \subset P$ be its Levi subgroup with the Lie algebra $\mathfrak{g}_0$. Since all compact real forms of a semisimple complex Lie algebra are conjugated by elements of any fixed Borel subgroup it follows that there exists $g \in P$ such that $\mathrm{Ad}(g)\mathfrak{k} = \mathfrak{l}$. Therefore

$$\widetilde{\mathrm{Ad}(g)h} = \mathrm{Ad}(g)\overline{h} \subset \mathrm{Ad}(g)(\mathfrak{p}) = \mathfrak{p}.$$

We can express g as a product uz, where $u \in \exp(\mathfrak{n}_+)$, $\mathrm{Ad}(z)h = h$. Then $\widetilde{\mathrm{Ad}(g)h} = \widetilde{\mathrm{Ad}(u)h}$. If u is not the identity element of G then $\mathrm{Ad}(u)h = h + \xi$, where $\xi \in \mathfrak{n}_+$ and $\xi \neq 0$. Therefore $\widetilde{\mathrm{Ad}(u)h} = -h + \tilde{\xi}$. But $\tilde{\xi} \in \mathfrak{n}_-$ and hence $\widetilde{\mathrm{Ad}(u)h} \notin \mathfrak{p}$, contradiction. Therefore u is trivial and since $\mathrm{Ad}(z)h = h$ we finally get

$$\overline{h} = \tilde{h} = -h. \square$$

□

Now we shall try to reverse this argument.

Lemma 4.26 *Suppose that* $\mathcal{O} = \mathrm{Ad}(G)e$ *is a Moore–Penrose orbit. Then* $\mathfrak{z}^u_{\mathfrak{g}}(e) = \mathfrak{n}_+$.

Proof. We choose a standard compact real form $\mathfrak{l}$ as in the proof of the previous Lemma. Clearly, $\mathfrak{z}^u_{\mathfrak{g}}(e)$ is a graded subalgebra of $\mathfrak{n}_+ = \bigoplus_{k>0} \mathfrak{g}_k$. Suppose, on the contrary, that $\mathfrak{z}^u_{\mathfrak{g}}(e) \neq \mathfrak{n}_+$. Let $\xi \in \mathfrak{g}_p$, $p > 0$, be a homogeneous element

that does not belong to $\mathfrak{z}^u_{\mathfrak{g}}(e)$. Let $u = \exp(\xi)$. Let $e' = \mathrm{Ad}(u)e$. We claim that all characteristics of e' do not belong to $i\mathfrak{l}$. Indeed, all characteristics of e' have a form $\mathrm{Ad}(u)h + \mathrm{Ad}(u)x$, where $x \in \mathfrak{z}^u_{\mathfrak{g}}(e)$. Suppose that for some x we have $\mathrm{Ad}(u)h + \mathrm{Ad}(u)x \in i\mathfrak{l}$. Since $h \in i\mathfrak{l}$, $\tilde{\mathfrak{n}}_{\pm} = \mathfrak{n}_{\mp}$, and $\mathrm{Ad}(u)(h+x) - h \in \mathfrak{n}_+$ it follows that $\mathrm{Ad}(u)(h+x) = h$. In $\mathfrak{n}_+$ modulo $\bigoplus_{k>p} \mathfrak{g}_k$ we obtain the equation $[\xi, h] + x = 0$, but $[h, \xi] = p\xi$ and therefore $\xi \in \mathfrak{z}^u_{\mathfrak{g}}(e)$. Contradiction. □

Now we can finish the proof of Theorem 4.23. Combining previous lemmas we see that $\mathcal{O}$ is a Moore–Penrose orbit if and only if $\mathfrak{z}^u_{\mathfrak{g}}(e) = \mathfrak{n}_+$. It follows from the $\mathfrak{sl}_2$-theory that $\dim \mathfrak{z}^u_{\mathfrak{g}}(e) = \dim \mathfrak{g}_1 + \dim \mathfrak{g}_2$. Therefore $\mathfrak{z}^u_{\mathfrak{g}}(e) = \mathfrak{n}_+$ if and only if $\mathfrak{g}_p = 0$ for $p > 2$. Clearly, this is precisely equivalent to $\mathrm{ht}(\mathcal{O}) = 2$. In this case for any $e \in \mathcal{O}$ there exists a unique $\mathfrak{sl}_2$-triple $\langle e, h, f\rangle$ such that $h \in i\mathfrak{k}$ by Lemma 4.24. □

Now let us turn to the graded situation. Suppose that $\mathfrak{g}$ is a $\mathbb{Z}$-graded simple Lie algebra, $\mathfrak{g} = \bigoplus_{k\in\mathbb{Z}} \mathfrak{g}_k$. Let $P \subset G$ be a parabolic subgroup with Lie algebra $\mathfrak{p} = \bigoplus_{k\geq 0} \mathfrak{g}_k$. Let $L \subset P$ be a Levi subgroup with Lie algebra $\mathfrak{g}_0$. We choose a compact real form $\mathfrak{k}_0$ of $\mathfrak{g}_0$. Suppose now that $e \in \mathfrak{g}_k$. It is well-known that there exists a homogeneous $\mathfrak{sl}_2$-triple $\langle e, h, f\rangle$ with $h \in \mathfrak{g}_0$ and $f \in \mathfrak{g}_{-k}$.

Definition 4.27 Take any $k > 0$ and any L-orbit $\mathcal{O} \subset \mathfrak{g}_k$. Then $\mathcal{O}$ is called *a Moore-Penrose orbit* if for any $e \in \mathcal{O}$ there exists a homogeneous $\mathfrak{sl}_2$-triple $\langle e, h, f\rangle$ such that $h \in i\mathfrak{k}_0$. In this case f is called a MP-inverse of e. A grading is called *a Moore–Penrose grading in degree* $k > 0$ if all L-orbits in $\mathfrak{g}_k$ are Moore-Penrose. A grading is called *a Moore–Penrose grading* if it is a Moore–Penrose grading in any positive degree. A parabolic subgroup $P \subset G$ is called *a Moore–Penrose parabolic subgroup* if there exists a Moore–Penrose grading $\mathfrak{g} = \bigoplus_{k\in\mathbb{Z}} \mathfrak{g}_k$ such that $\mathfrak{p} = \bigoplus_{k\geq 0} \mathfrak{g}_k$ is a Lie algebra of P.

One should be careful comparing graded and non-graded situation: if $\mathcal{O} \subset \mathfrak{g}_k$ is a Moore–Penrose L-orbit then $\mathrm{Ad}(G)\mathcal{O} \subset \mathfrak{g}$ is not necessarily a Moore–Penrose G-orbit. Let us give a criterion for an L-orbit to be Moore–Penrose. Suppose $\mathcal{O} = \mathrm{Ad}(L)e \subset \mathfrak{g}_k$. Take any homogeneous $\mathfrak{sl}_2$-triple $\langle e, h, f\rangle$. Then h defines a grading $\mathfrak{g}_0 = \bigoplus_{n\in\mathbb{Z}} \mathfrak{g}_0^n$, such that $\mathrm{ad}(h)|_{\mathfrak{g}_0^n} = n \cdot \mathrm{Id}$.

Theorem 4.28 ([T4]) *$\mathcal{O}$ is a Moore–Penrose orbit if and only if* $\mathrm{ad}(e)g_0^n = 0$ *for any $n > 0$. In this case for any $e' \in \mathcal{O}$ there exists a unique homogeneous $\mathfrak{sl}_2$-triple $\langle e', h', f'\rangle$ such that $h' \in i\mathfrak{k}_0$.*

The proof is similar to the proof of Theorem 4.23.

Example 4.29 Suppose that G is a simple group of type G_2. We fix a root decomposition. There are two simple roots α_1 and α_2 such that α_1 is short and α_2 is long. There are 3 proper parabolic subgroups: Borel subgroup B and two maximal parabolic subgroups P_1 and P_2 such that a root vector of

α_i belongs to a Levi subgroup of P_i. Then the following is an easy application of Theorem 4.28. B is a Moore–Penrose parabolic subgroup (actually Borel subgroups in all simple groups are Moore–Penrose parabolic subgroups with respect to any grading). P_1 is not Moore–Penrose, but it is a Moore–Penrose parabolic subgroup in degree 2 (with respect to the natural grading of height 2). P_2 is a Moore–Penrose parabolic subgroup.

Example 4.30 Suppose $G = \mathrm{SL}_n$. We fix positive integers $d_1, \ldots, d_k$ such that $n = d_1 + \ldots + d_k$. We consider the parabolic subgroup $P(d_1, \ldots, d_k) \subset \mathrm{SL}_n$ that consists of all upper-triangular block matrices with sizes of blocks equal to $d_1, \ldots, d_k$. We take a standard grading. Then $\mathfrak{g}_1$ is identified with the linear space of all tuples of linear maps $\{f_1, \ldots, f_k\}$,

$$\mathbb{C}^{d_1} \xleftarrow{f_1} \mathbb{C}^{d_2} \xleftarrow{f_2} \ldots \xleftarrow{f_{k-1}} \mathbb{C}^{d_k},$$

$\mathfrak{g}_{-1}$ is identified with the linear space of all tuples of linear maps $\{g_1, \ldots, g_k\}$,

$$\mathbb{C}^{d_1} \xrightarrow{g_1} \mathbb{C}^{d_2} \xrightarrow{g_2} \ldots \xrightarrow{g_{k-1}} \mathbb{C}^{d_k},$$

and Levi subgroup $L(d_1, \ldots, d_k)$ is just the group

$$(A_1, \ldots, A_k) \in \mathrm{GL}_{d_1} \times \ldots \times \mathrm{GL}_{d_k} \quad \text{such that} \quad \det(A_1) \cdot \ldots \cdot \det(A_k) = 1.$$

It acts on these spaces of linear maps in an obvious way. The most important among L-orbits are varieties of complexes. To define them, let us fix in addition non-negative integers $m_1, \ldots, m_{k-1}$ such that $m_{i-1} + m_i \leq d_i$ (we set $m_0 = m_k = 0$), and consider the subvariety of all tuples $\{f_1 \ldots, f_{k-1}\}$ as above such that $\mathrm{rk}\, f_i = m_i$ and $f_{i-1} \circ f_i = 0$ for any i. These tuples form a single L-orbit $\mathcal{O}$ called a variety of complexes. For any tuple $\{f_1, \ldots, f_{k-1}\} \in \mathcal{O}$ consider the tuple $\{f_1^+, \ldots, f_{k-1}^+\} \in \mathfrak{g}_{-1}$, where f_i^+ is a classical "matrix" Moore–Penrose inverse of f_i. It is easy to see that this new tuple is again a complex, moreover, this complex is a Moore–Penrose inverse (in our latest meaning of this word) of an original complex. In particular, orbits of complexes are Moore–Penrose orbits.

From the first glance only few parabolic subgroups are Moore–Penrose. But this is scarcely true. For example, we have the following Theorem:

Theorem 4.31 ([T4]) *Any parabolic subgroup in SL_n is Moore–Penrose.*

To explain our interest in Moore–Penrose parabolic subgroups let us recall Conjecture 4.11 from the previous section. Suppose once again that G is a simple connected simply-connected Lie group, P is its parabolic subgroup, $\mathfrak{p} \subset \mathfrak{g}$ are their Lie algebras. We take any irreducible G-module V. There exists a unique proper P-submodule M_V of V. We have the inclusion $i : M_V \to V$, the projection $\pi : V \to V/M_V$ and the map $R_V : \mathfrak{g} \to \mathrm{End}(V)$ defining the

representation. Therefore we have a linear map $\tilde{R}_V : \mathfrak{g} \to \mathrm{Hom}(M_V, V/M_V)$, namely $\tilde{R}_V(x) = \pi \circ R_V(x) \circ i$. Clearly $\mathfrak{p} \subset \mathrm{Ker}\, \tilde{R}_V$. Finally, we have a linear map $\Psi_V : \mathfrak{g}/\mathfrak{p} \to \mathrm{Hom}(M_V, V/M_V)$. The Conjecture 4.11 states that there exists an algebraic stratification $\mathfrak{g}/\mathfrak{p} = \bigcup_{i=1}^{n} X_i$ such that for any V the function rk Ψ_V is constant along each X_i.

The following theorem shows the connection of this problem with the Moore–Penrose inverse.

Theorem 4.32 ([T4]) *Suppose that a grading $\mathfrak{g} = \bigoplus_{k\in\mathbb{Z}} \mathfrak{g}_k$ is a Moore–Penrose grading in all positive degrees except at most one. Then the Conjecture is true for the corresponding parabolic subgroup P.*

Proof. Suppose that G is a connected reductive group with a Lie algebra $\mathfrak{g}$. For any elements $x_1, \dots, x_r \in \mathfrak{g}$ let $\langle x_1, \dots, x_r\rangle_{alg}$ denote the minimal algebraic Lie subalgebra of $\mathfrak{g}$ that contains $x_1, \dots, x_r$ (algebraic subalgebras are the Lie algebras of algebraic subgroups). By a theorem of Richardson [Ri1] $\langle x_1, \dots, x_r\rangle_{alg}$ is reductive if and only if an orbit of the r-tuple $(x_1, \dots, x_r)$ in $\mathfrak{g}^r$ is closed with respect to the diagonal action of G. Suppose now that $h_1, \dots, h_r$ are semi-simple elements of $\mathfrak{g}$. Consider the closed subvariety $\hat{\mathcal{O}} = (\mathrm{Ad}(G)h_1, \dots, \mathrm{Ad}(G)h_r) \subset \mathfrak{g}^r$. For any closed G-orbit $\mathcal{O} \subset \hat{\mathcal{O}}$ let us denote by $G(\mathcal{O})$ the conjugacy class of the reductive subalgebra $\langle x_1, \dots, x_r\rangle_{alg}$ for $(x_1, \dots, x_r) \in \mathcal{O}$.

Lemma 4.33 *There are only finitely many conjugacy classes $G(\mathcal{O})$.*

Proof. We shall use induction on $\dim \mathfrak{g}$. Suppose that the claim of the Lemma is true for all reductive groups H with $\dim H < \dim G$. Let $\mathfrak{z} \subset \mathfrak{g}$ be the center of $\mathfrak{g}$, $\mathfrak{g}' \subset \mathfrak{g}$ be its derived algebra. Consider two canonical homomorphisms

$$\mathfrak{g} \xrightarrow{\pi} \mathfrak{g}' \quad \text{and} \quad \mathfrak{g} \xrightarrow{\pi'} \mathfrak{z}.$$

We take any closed G-orbit $\mathcal{O} \subset \hat{\mathcal{O}}$. Let $(x_1, \dots, x_r) \in \mathcal{O}$, $y_i = \pi(x_i)$ for $i = 1, \dots, r$. Then $\langle y_1, \dots, y_r\rangle_{alg} = \pi(\langle x_1, \dots, x_r\rangle_{alg})$ and, therefore, is reductive. Let us consider two cases.

Suppose first, that $\langle y_1, \dots, y_r\rangle_{alg} = \mathfrak{g}'$. Then $\mathfrak{g}'$ is a derived algebra of $\langle x_1, \dots, x_r\rangle_{alg}$ and, therefore, $\langle x_1, \dots, x_r\rangle_{alg} = \langle \pi'(h_1), \dots, \pi'(h_r)\rangle \oplus \mathfrak{g}'$. In this case we get one conjugacy class.

Suppose now, that $\langle y_1, \dots, y_r\rangle_{alg} \neq \mathfrak{g}'$. Then $\langle y_1, \dots, y_r\rangle_{alg}$ is contained in some maximal reductive Lie subalgebra of $\mathfrak{g}'$. It is well-known (and not difficult to prove) that in a semisimple Lie algebra there are only finitely many conjugacy classes of maximal reductive subalgebras. Let $\mathfrak{h}'$ be one of them, $\mathfrak{h} = \mathfrak{z} \oplus \mathfrak{h}' \subset \mathfrak{g}$. Let H be a corresponding reductive subgroup of G. It is sufficient to prove that for any closed G-orbit $\mathcal{O}$ of $\hat{\mathcal{O}}$ that meets $\mathfrak{h}^r$ there are only finitely many possibilities for $G(\mathcal{O})$. It easily follows from Richardson's Lemma [Ri] that for any i the intersection $\mathrm{Ad}(G)h_i \cap \mathfrak{h}$ is a union of finitely

many closed H-orbits, say $\mathrm{Ad}(H)h_i^1, \ldots, \mathrm{Ad}(H)h_i^{s_i}$. It remains to prove that if for some r-tuple $(x_1, \ldots, x_r) \in \mathrm{Ad}(H)h_1^{k_1} \times \ldots \mathrm{Ad}(H)h_r^{k_r}$ the corresponding subalgebra $\langle x_1, \ldots, x_r \rangle_{alg}$ is reductive then there are only finitely many possibilities for its conjugacy class. But this is precisely the claim of Lemma for the group H, which is true by the induction hypothesis. □

Suppose that $\mathfrak{k}$ is a compact real form of $\mathfrak{g}$.

Lemma 4.34 *If r-tuple $(x_1, \ldots, x_r)$ belongs to $(i\mathfrak{k})^r$, then its G-orbit is closed in $\mathfrak{g}^r$.*

Proof. Indeed, let B be a non-degenerate ad-invariant scalar product on $\mathfrak{g}$, which is negative-definite on $\mathfrak{k}$. Let $H(x) = -B(\overline{x}, x)$ be a positive-definite $\mathfrak{k}$-invariant Hermitian quadratic form on $\mathfrak{g}$, where the complex conjugation is taken with respect to $\mathfrak{k}$. Let H^r be a corresponding Hermitian quadratic form on $\mathfrak{g}^r$. More precisely, $H^r(x_1, \ldots, x_r) = H(x_1) + \ldots + H(x_r)$. By a Kempf–Ness criterion [PV] in order to prove that the G-orbit of $(x_1, \ldots, x_r)$ is closed it is sufficient to prove that the real function $H^r(\cdot)$ has a critical point on this orbit. Let us show that $(x_1, \ldots, x_r)$ is this critical point. Indeed, for any $g \in \mathfrak{g}$

$$-B(\overline{x}_1, [g, x_1]) - \ldots - B(\overline{x}_r, [g, x_r]) = B(x_1, [g, x_1]) + \ldots + B(x_r, [g, x_r]) = 0.$$

□

Now let G be a simple simply-connected Lie group, let $\mathfrak{g}$ be its Lie algebra with a $\mathbb{Z}$-grading $\mathfrak{g} = \bigoplus_{k \in \mathbb{Z}} \mathfrak{g}_k$. Let r be a maximal integer such that $\mathfrak{g}_r \neq 0$. We denote the non-positive part of the grading $\bigoplus_{k \leq 0} \mathfrak{g}_k$ by $\mathfrak{p}$. Let $P \subset G$ be a parabolic subgroup with the Lie algebra $\mathfrak{p}$. We shall identify $\mathfrak{g}/\mathfrak{p}$ with $\bigoplus_{k>0} \mathfrak{g}_k$. Let $L \subset G$ be a connected reductive subgroup with Lie algebra $\mathfrak{g}_0$. Let V be an irreducible G-module. If we choose a Cartan subalgebra $\mathfrak{t} \subset \mathfrak{g}_0$ then the grading of $\mathfrak{g}$ originates from some $\mathbb{Z}$-grading on $\mathfrak{t}^*$. Therefore, there exists a $\mathbb{Z}$-grading $V = \bigoplus_{k \in \mathbb{Z}} V_k$ such that $\mathfrak{g}_i V_j \subset V_{i+j}$. Let R be a maximal integer such that $V_R \neq 0$. It is easy to see that $M_V = \bigoplus_{k<R} V_k$ (notice that V_R is an irreducible L-module).

Now we can prove the Theorem. It is sufficient to prove that there exists a finite set of points $\{x_1, \ldots, x_N\} \subset \mathfrak{g}/\mathfrak{p}$ such that for any $x \in \mathfrak{g}/\mathfrak{p}$ and for any V we have $\mathrm{rk}\ \Psi_V(x) = \mathrm{rk}\ \Psi_V(x_i)$ for some i. Recall that L has finitely many orbits on each $\mathfrak{g}_k$, see Theorem 2.10. We pick some L-orbit $\mathcal{O}_i$ in each $\mathfrak{g}_i$. Then it is sufficient to find a finite set of points as above only for points $x \in \mathfrak{g}/\mathfrak{p}$ of a form $x = x_1 + \ldots + x_r$, where $x_i \in \mathcal{O}_i$. For any orbit $\mathcal{O}_i$ let $\mathcal{H}_i$ denote the set of all possible homogeneous characteristics of all elements from $\mathcal{O}_i$. Clearly $\mathcal{H}_i$ is a closed $\mathrm{Ad}(L)$-orbit. Let $\hat{\mathcal{O}} = \mathcal{H}_1 \times \ldots \times \mathcal{H}_r \subset \mathfrak{g}_0^r$. Then by the first Lemma the set of conjugacy classes of subgroups $G(\mathcal{O})$ for closed L-orbits $\mathcal{O}$ in $\hat{\mathcal{O}}$ is finite. Let us show that for any r-tuple $(x_1, \ldots, x_r) \in$

$\mathcal{O}_1 \times \ldots \times \mathcal{O}_r$ there exists an r-tuple $(h_1, \ldots, h_r) \in \hat{\mathcal{O}}$ such that h_i is a homogeneous characteristic of x_i and an L-orbit $\mathrm{Ad}(L)(h_1, \ldots, h_r)$ is closed. Indeed, after simultaneous conjugation of elements x_i by some element $g \in L$ we may suppose that any x_i has a homogeneous characteristic $h_i \in i\mathfrak{k}_0$ (in all degrees except at most one no conjugation is needed because of Moore–Penrose property, for one degree this is obvious). Then by the second Lemma an orbit $\mathrm{Ad}(L)(h_1, \ldots, h_r)$ is closed. Since all functions rk Ψ_V are L-invariant, we may restrict ourselves to the points $x = \sum_i x_i \in \mathfrak{g}/\mathfrak{p}$ such that $x_i \in \mathcal{O}_i$, any x_i has a homogeneous characteristic $h_i \in i\mathfrak{k}_0$, and a conjugacy class of $\langle h_1, \ldots, h_r \rangle_{alg}$ is fixed. We claim that any function rk Ψ_V is constant along the set of these points. Moreover, we shall prove that

$$\mathrm{rk}\ \Psi_V(x) = \dim V_R - \dim V_R^{\langle h_1, \ldots, h_r \rangle_{alg}}. \tag{4.4}$$

Indeed,

$$\mathrm{rk}\ \Psi_V(x) = \dim \sum_i \mathrm{Im}\left(\mathrm{ad}(x_i)|_{V_{R-i}}\right).$$

Clearly V_R is $\mathrm{ad}(h_i)$-invariant and is killed by $\mathrm{ad}(e_i)$, therefore from the $\mathfrak{sl}_2$-theory we get that $V_R = \bigoplus_{k \geq 0} V_R^k$, where $\mathrm{ad}(h_i)|_{V_R^k} = k \cdot \mathrm{Id}$. Moreover,

$$\mathrm{Im}\left(\mathrm{ad}(x_i)|_{V_{R-i}}\right) = \bigoplus_{k>0} V_R^k.$$

Let H be a contravariant Hermitian form on V_R with respect to the compact form $\mathfrak{k}_0$ of $\mathfrak{g}_0$. Since $h_i \in i\mathfrak{k}_0$ and H is a contravariant form we get that $\bigoplus_{k>0} V_R^k = (V_R^0)^\perp$. Therefore,

$$\sum_i \mathrm{Im}\left(\mathrm{ad}(x_i)|_{V_{R-i}}\right) = \left(\cap_i V_R^{h_i}\right)^\perp = \left(V_R^{\langle h_1, \ldots, h_r \rangle_{alg}}\right)^\perp.$$

The formula (4.4) follows. □

For example, combining Theorem 4.32 and Theorem 4.31 we get the following corollary:

Corollary 4.35 *The conjecture is true for any parabolic subgroup in* SL_n.

Though the conditions of Theorem 4.32 are not always satisfied, it seems that one can prove the conjecture for any simple group in this direction.

5. Fulton–Hansen Theorem and Applications

Preliminaries

Fulton–Hansen connectedness Theorem has numerous applications to the projective geometry including famous Zak Theorem on tangencies. Aside from dual varieties, projective geometry also studies secant and tangential varieties, joins, etc. In this chapter we review some basic results related to them.

5.1 Fulton–Hansen Connectedness Theorem

Theorem 5.1 ([FH]) *Let $f : X \to \mathbb{P}^N$ be a finite morphism from an irreducible complete variety X to the projective space $\mathbb{P}^N$. Then for any projective subspace $L_0 \subset \mathbb{P}^N$ such that $\dim X > \operatorname{codim}_{\mathbb{P}^N} L_0$ the inverse image $f^{-1}(L_0)$ is connected.*

Sketch of the proof. Let $k = \dim L$. Let $U \subset \mathrm{Gr}(k, \mathbb{P}^N)$ be the open set of all subspaces that meet $f(X)$ properly, and let $U_0 \subset \mathrm{Gr}(k, \mathbb{P}^N)$ be the set of all subspaces L such that $f^{-1}(L)$ is irreducible. Then U_0 is non-empty by Bertini's theorem (see [Ha1] or [Wei]). It follows from Zariski's principle of degeneration (see [Fu1] or [Za]) that for all $L \in U$ the preimage $f^{-1}(L)$ is connected, non-empty, of codimension $\dim X + k - N$. For each $L \in U$, there is a positive cycle $f^*(L)$ on X whose support is $f^{-1}(L)$. This determines a morphism from U to the Chow variety Z of cycles on X. Let

$$\Gamma \subset \mathrm{Gr}(k, \mathbb{P}^N) \times Z$$

be the closure of the graph of this morphism, $\pi : \Gamma \to \mathrm{Gr}(k, \mathbb{P}^N)$ the projection. Since Z is complete, π is proper as well as birational. Zariski's principle of degeneration implies that for any $\gamma \in \Gamma$ the corresponding cycle C_γ is connected (i.e. its support $|C_\gamma|$ is connected). Zariski's main theorem implies that $\pi^{-1}(L_0)$ is connected. It follows that the union of the $|C_\gamma|$ for $\gamma \in \pi^{-1}(L_0)$ is connected. It remains to prove that

$$f^{-1}(L_0) = \bigcup_{\gamma \in \pi^{-1}(L_0)} |C_\gamma|.$$

The inclusion $\supset$ follows from the continuity of limit cycles. So it remains to prove the inclusion $\subset$. Let $x \in f^{-1}(L_0)$ and let $z = f(x)$. Consider the subvariety G of $\mathrm{Gr}(k, \mathbb{P}^N)$ consisting of all subspaces passing through z (therefore G is isomorphic to $\mathrm{Gr}(k-1, \mathbb{P}^{N-1})$. An easy dimension count shows that G meets U. Let $\Gamma' \subset \Gamma$ be an irreducible subvariety that maps birationally to G. For each $L \subset G \cap U$, the cycle $f^*(L)$ contains x, therefore, for any $\gamma \in \Gamma'$ the cycle C_γ contains x. In particular, there exists the limit cycle C_γ for $\gamma \in \Gamma' \cap \pi^{-1}(L_0)$ that contains x. □

Now we shall derive a number of corollaries.

Corollary 5.2 ([FH]) *Let $f : X \to \mathbb{P}^N$ be a morphism from an irreducible complete variety X. Then for any projective subspace $L \subset \mathbb{P}^N$ such that $\dim f(X) > \mathrm{codim}_{\mathbb{P}^N} L$ the inverse image $f^{-1}(L)$ is connected.*

Proof. This follows from Theorem 5.1 using the Stein factorization (see e.g. [Ha1]). Indeed, for a non-finite f, let $f = g \circ h$ be its Stein factorization, i.e. $h : X \to X'$ has connected fibers, and $g : X' \to \mathbb{P}^N$ is finite. Then $\dim X' = \dim f(X)$, therefore $g^{-1}(L)$ is connected by Theorem 5.1, therefore $f^{-1}(L)$ is also connected. □

Theorem 5.3 ([FH]) *Let $\mathbb{P} = \mathbb{P}^N \times \ldots \times \mathbb{P}^N$ (r copies), and let Δ be the diagonal. If X is an irreducible variety, $f : X \to \mathbb{P}$ a morphism, with*

$$\dim f(X) > \mathrm{codim}_{\mathbb{P}} \Delta = (r-1)N,$$

then $f^{-1}(\Delta)$ is connected.

Proof. Identify $\mathbb{P}$ as the set of ordered r-tuples of points of $\mathbb{P}^N$, so that a point in $\mathbb{P}$ has r factors in $\mathbb{P}^N$; the diagonal Δ consists of those points whose factors are all equal.

For simplicity suppose that f is a finite morphism, a general case is handled with a help of the Stein factorization as in the proof of the previous Corollary. Suppose that $f^{-1}(\Delta)$ is not connected, choose points x and x' in different connected components of $f^{-1}(\Delta)$. Choose a hyperplane H in $\mathbb{P}^N$ not containing any of the r factors of $f(x)$ or $f(x')$. Let $\mathbb{P}_0$ be the open set of $\mathbb{P}$ consisting of those points none of whose factors lie in H. Then $\mathbb{P}_0$ is a product of r copies of $\mathbb{C}^N$, which is identified with $\mathbb{C}^{rN}$ and therefore with $\mathbb{P}^{rN} \setminus \{z_0 = 0\}$, where $z_0, \ldots, z_{rN}$ are the homogeneous coordinates on $\mathbb{P}^{rN}$. Let $W \subset \mathbb{P} \times \mathbb{P}^{rN}$ be the closure of the graph of this birational correspondence. If coordinates are chosen on $\mathbb{P}^N$ such that H is the hyperplane $x_0 = 0$, and the homogeneous coordinates on the k-th copy of $\mathbb{P}^N$ are $x_0^k, \ldots, x_N^k$, then a point $(x^1) \times \ldots \times (x^r) \times (z)$ in $\mathbb{P} \times \mathbb{P}^{rN}$ belongs to W if and only if there are constants $\lambda_1, \ldots, \lambda_r$ such that

$$(z_0, z_{(k-1)N+1}, \ldots, z_{kN}) = \lambda_k (x_0^k, \ldots, x_N^k)$$

for all $k = 1, \ldots, r$. So W is defined by the equations

$$\begin{cases} x_j^k z_0 = x_0^k z_{(k-1)N+j} & 1 \le k \le r,\ 1 \le j \le N \\ x_i^k z_{(k-1)N+j} = x_j^k z_{(k-1)N+i} & 1 \le k \le r,\ 1 \le i < j \le N. \end{cases}$$

Let $\alpha : W \to \mathbb{P}$, $\beta : W \to \mathbb{P}^{rN}$ be the two birational projections. Let L be the N-dimensional linear subspace of $\mathbb{P}^{rN}$ defined by the equations

$$z_{(k-1)N+j} = z_{kN+j}, \quad 1 \le k \le r-1,\ 1 \le j \le N.$$

Then it is easy to check that

$$\alpha^{-1}(\Delta) \supset \beta^{-1}(L) \text{ and } \alpha^{-1}(\Delta \cap \mathbb{P}_0) \subset \beta^{-1}(L). \tag{5.1}$$

Let $\tilde{X}$ be the irreducible component of the fiber product $X \times_{\mathbb{P}} W$ that maps onto X. Then there is a diagram

$$\begin{tikzcd} \tilde{X} \arrow[r, "g"] \arrow[d, "\pi"'] & W \arrow[r, "\beta"] \arrow[d, "\alpha"] & \mathbb{P}^{rN} \\ X \arrow[r, "f"'] & \mathbb{P} & \end{tikzcd}$$

where the square commutes, and π is surjective. Then equations (5.1) imply that

$$f^{-1}(\Delta) \supset \pi(g^{-1}\beta^{-1}(L)), \text{ and } f^{-1}(\Delta \cap \mathbb{P}_0) \subset \pi(g^{-1}\beta^{-1}(L)). \tag{5.2}$$

Now $\dim \beta g(\tilde{X}) = \dim f(X)$ since $f(X)$ meets the open set $\mathbb{P}_0$ which corresponds isomorphically to an open set in $\mathbb{P}^{rN}$. Then by Theorem 5.1 we know that $(\beta g)^{-1}(L)$ is connected. Therefore $\pi((\beta g)^{-1}(L))$ is connected. But by (5.2) $\pi((\beta g)^{-1}(L))$ is then a connected subset of $f^{-1}(\Delta)$ which contains both x and x'. Contradiction. □

Corollary 5.4 ([FH])

(a) *Let Y be an irreducible subvariety of the projective space $\mathbb{P}^N$ and let $f : X \to \mathbb{P}^N$ be a morphism from an irreducible variety X to $\mathbb{P}^N$ with $\dim f(X) > \operatorname{codim}_{\mathbb{P}^N} Y$. Then $f^{-1}(Y)$ is connected.*

(b) *If X and Y are irreducible subvarieties of $\mathbb{P}^N$ with $\dim X + \dim Y > N$ then $X \cap Y$ is connected.*

Proof. (a) We apply Theorem 5.3 for the map $F : X \times Y \to \mathbb{P}^N \times \mathbb{P}^N$ given by $F = (f, \mathrm{Id})$.

(b) Follows from (a). □

Corollary 5.5 ([FH])

(a) *Let X be an irreducible variety of dimension n, $f : X \to \mathbb{P}^N$ an unramified morphism, with $N < 2n$. Then f is a closed embedding.*

(b) *Let X be an irreducible n-dimensional subvariety of $\mathbb{P}^N$ with $N < 2n$. Then the algebraic fundamental group of X is trivial* (*i.e. X has no non-trivial étale coverings*).

Proof. (a) If $f : X \to \mathbb{P}^N$ is unramified, consider the product mapping $f \times f : X \times X \to \mathbb{P}^N \times \mathbb{P}^N$. The unramified assumption is equivalent to the assertion that the diagonal Δ_X in $X \times X$ is open as well as closed in $(f \times f)^{-1}(\Delta_{\mathbb{P}^N})$. Since f is automatically finite, $\dim(f \times f)(X \times X) = 2n$, so $(f \times f)^{-1}(\Delta_{\mathbb{P}^N})$ is connected by the Theorem 5.3. Hence

$$\Delta_X = (f \times f)^{-1}(\Delta_{\mathbb{P}^N}),$$

so f is one-to-one, and therefore a closed embedding.

(b) Follows from (a). □

5.2 Secant and Tangential Varieties

Definition 5.6 Suppose that $X \subset \mathbb{P}^N$ is a smooth n-dimensional projective variety. The tangential variety $\mathrm{Tan}(X)$ is the union of all (embedded) tangent spaces to X

$$\mathrm{Tan}(X) = \bigcup_{x \in X} \hat{T}_x X.$$

The secant variety $\mathrm{Sec}(X)$ is the closure of the union of all secant lines to X

$$\mathrm{Sec}(X) = \overline{\bigcup_{x,y \in X} \mathbb{P}^1_{xy}},$$

where $\mathbb{P}^1_{xy}$ is a line connecting x and y.

Clearly, $\mathrm{Tan}(X)$ and $\mathrm{Sec}(X)$ are closed irreducible subvarieties of $\mathbb{P}^N$, with $\mathrm{Tan}(X) \subset \mathrm{Sec}(X)$, and

$$\dim \mathrm{Tan}(X) \le 2n, \quad \dim \mathrm{Sec}(X) \le 2n + 1.$$

Example 5.7 Let $V = \mathbb{C}^{k+1} \otimes \mathbb{C}^{l+1}$ denote the space of $(k+1) \times (l+1)$ matrices. Let $X \subset \mathbb{P}(V)$ be the projectivization of the variety of rank 1 matrices. Then X is isomorphic to $\mathbb{P}^k \times \mathbb{P}^l$ embedded into $\mathbb{P}(V)$ via the Segre embedding. It is easy to see that $\mathrm{Sec}(X)$ is equal to the projectivization of the variety of matrices of rank at most 2.

Theorem 5.8 ([FH], [Z2]) *Suppose that $X \subset \mathbb{P}^N$ is a smooth projective variety. Then either*

$$\dim \mathrm{Tan}(X) = 2n, \quad \dim \mathrm{Sec}(X) = 2n + 1$$

or

$$\mathrm{Tan}(X) = \mathrm{Sec}(X).$$

Proof. Indeed, suppose that $\mathrm{Tan}(X) \neq \mathrm{Sec}(X)$. Since $\mathrm{Sec}(X)$ is irreducible, it is sufficient to prove that in this case $\dim \mathrm{Tan}(X) = 2n$. Suppose, on the contrary, that $\dim \mathrm{Tan}(X) < 2n$. Let $L \subset \mathbb{P}^N$ be a linear subspace such that $L \cap \mathrm{Tan}(X) = \emptyset$ and $\dim L = N - 1 - \dim \mathrm{Tan}(X)$ (since $\mathrm{Tan}(X) \neq \mathrm{Sec}(X)$ we obviously have $\dim \mathrm{Tan}(X) \leq N - 1$). Let π be the linear projection $\mathbb{P}^N \to \mathbb{P}^{\dim \mathrm{Tan}(X)}$ with center L. Since $L \cap \mathrm{Tan}(X) = \emptyset$, the restriction of π on X is an unramified morphism. Since $\dim \mathrm{Tan}(X) < 2n$, it follows by Theorem 5.5 that the restriction of π on X is a closed embedding. Therefore, $L \cap \mathrm{Sec}(X) = \emptyset$. Since $\dim L + \dim \mathrm{Sec}(X) \geq N$, this is a contradiction. $\square$

It is possible to generalize this theorem in the following direction. Suppose that $Y \subset X \subset \mathbb{P}^N$ are arbitrary irreducible projective varieties. Then we can define relative secant and tangential varieties as follows. The relative secant variety $\mathrm{Sec}(Y, X)$ is the closure of the union of all secants $\mathbb{P}^1_{x,y}$, where $x \in X$, $y \in Y$. The relative tangential variety $\mathrm{Tan}(Y, X)$ is the union of tangent stars $T^\star_y X$ for $y \in Y$, where the tangent star $T^\star_x X$ for $x \in X$ is the union of limit positions of secants $\mathbb{P}^1_{x',x''}$, where $x', x'' \in X$ and $x', x'' \to x$.

Example 5.9 Let $V = \mathbb{C}^{k+1} \otimes \mathbb{C}^{l+1}$ denote the space of $(k+1) \times (l+1)$ matrices. Let $X^r \subset \mathbb{P}(V)$ be the projectivization of the variety of matrices of rank at most r. Then $X^r \subset X^{r+1}$ and for $r < s$ we have

$$\mathrm{Sec}(X^r, X^s) = \begin{cases} X^{r+s} & \text{if } r + s \leq \min(k+1, l+1) \\ \mathbb{P}(V) & \text{if } r + s > \min(k+1, l+1). \end{cases}$$

The proof of the following theorem is analogous to the proof of Theorem 5.8.

Theorem 5.10 ([Z2]) *Suppose that $Y \subset X \subset \mathbb{P}^N$ are irreducible projective varieties. Then either*

$$\dim \mathrm{Tan}(Y, X) = \dim X + \dim Y, \quad \dim \mathrm{Sec}(Y, X) = \dim X + \dim Y + 1$$

or

$$\mathrm{Tan}(Y, X) = \mathrm{Sec}(Y, X).$$

One can go even further and define the join $S(X, Y)$ of any two subvarieties $X, Y \subset \mathbb{P}(V)$,

$$S(X, Y) = \overline{\bigcup_{x \in X,\ y \in Y} \mathbb{P}^1_{xy}},$$

where the closure is not necessary if the two varieties do not intersect. For example, if X and Y are projective subspaces then

$$S(X, Y) = \mathbb{P}(\mathrm{Cone}(X) + \mathrm{Cone}(Y)).$$

An important result about joins is the following Terracini lemma.

Lemma 5.11 ([Te]) *Let $X, Y \subset \mathbb{P}(V)$ be irreducible varieties and let*

$$x \in X, \quad y \in Y, \quad z \in \mathbb{P}^1_{xy}.$$

Then

$$\hat{T}_z S(X,Y) \supset S(\hat{T}_x X, \hat{T}_y Y).$$

Moreover, if z is a generic point of $S(X,Y)$ then equality holds.

If an algebraic group G acts on $\mathbb{P}^N$ with finitely many orbits, then for any two orbits $\mathcal{O}_1$ and $\mathcal{O}_2$ the join of their closures $S(\overline{\mathcal{O}_1}, \overline{\mathcal{O}_2})$ is G-invariant therefore coincides with some orbit closure $\overline{\mathcal{O}}$. This gives an interesting associative product on the set of G-orbits.

Example 5.12 Let L be a simple algebraic group and P a parabolic subgroup with abelian unipotent radical. In this case $\mathfrak{l} = \operatorname{Lie} L$ admits a short $\mathbb{Z}$-grading with only three non-zero parts:

$$\mathfrak{l} = \mathfrak{l}_{-1} \oplus \mathfrak{l}_0 \oplus \mathfrak{l}_1.$$

Here $\mathfrak{l}_0 \oplus \mathfrak{l}_1 = \operatorname{Lie} P$ and $\exp(\mathfrak{l}_1)$ is the abelian unipotent radical of P. Let $G \subset L$ be a reductive subgroup with Lie algebra $\mathfrak{l}_0$. Recall that by Theorem 2.17 G has finitely many orbits in $\mathfrak{l}_1$ naturally labeled by integers from the segment $[0, r]$. Let $X_i = \overline{\mathbb{P}(\mathcal{O}_i)}$, $i = 0, \ldots, r$. Then it is easy to see that

$$S(X_i, X_j) = X_{\min(i+j,r)}.$$

Indeed, by associativity and induction it suffices to prove that $S(X_i, X_1) = X_{\min(i+1,r)}$ and this easily follows from the results of Section 2.3.

5.3 Zak Theorems

The following theorem is called Zak Theorem on tangencies.

Theorem 5.13 ([Z2], [FL])

(a) *Suppose that $X \subset \mathbb{P}^N$ is a smooth nondegenerate projective variety, $\dim X = n$. If L is a k-plane in $\mathbb{P}^N$ and $k \geq n$ then*

$$\dim \operatorname{Sing}(L \cap X) \leq k - n.$$

(b) *If $X \subset \mathbb{P}^N$ is a non-linear smooth projective variety, then $\dim X^* \geq X^*$. Moreover, if X^* is smooth, then $\dim X = \dim X^*$.*

Proof. (a) Indeed, let $Y = \operatorname{Sing} L \cap X$. Then $x \in Y$ if and only if $\hat{T}_x X \subset L$. Therefore, $\operatorname{Tan}(Y, X) \subset L$. In particular, $\dim \operatorname{Tan}(Y, X) \leq k$. On the over hand, since X is non-degenerate, $X \not\subset L$. Therefore $\operatorname{Sec}(Y, X) \not\subset L$.

In particular, $\mathrm{Tan}(Y,X) \neq \mathrm{Sec}(Y,X)$. By Theorem 5.10 it follows that $\dim \mathrm{Tan}(Y,X) = \dim Y + n$. Finally, we have $\dim Y \leq k - n$.

(b) Let $H \subset X^*{}_{sm}$. Then the contact locus $\mathrm{Sing}\, X \cap H$ is the projective subspace of dimension $\mathrm{def}\, X$. Therefore, by (a) we have

$$\mathrm{def}\, X \leq \dim H - \dim X = \mathrm{codim}\, X - 1.$$

It follows that $\dim X \leq \dim X^*$. If X^* is also smooth then using the reflexivity theorem and the same argument as above we get $\dim X^* \leq \dim X$, therefore $\dim X = \dim X^*$. □

To remove the condition of smoothness in Theorem 5.13 (a) we need the notion of Gauss images. Let $X \subset \mathbb{P}^N$ be an irreducible n-dimensional projective variety.

Definition 5.14 *For any $k \geq n$ the variety*

$$\gamma_k(X) = \overline{\cup_{x \in X_{sm}} \{L \subset \mathrm{Gr}(k, \mathbb{P}^N) \,|\, L \supset \hat{T}_x X\}} \subset \mathrm{Gr}(k, \mathbb{P}^N)$$

is called the k-th Gauss image of X.

It is easy to see that Gauss images are irreducible varieties. For example, $\gamma_{N-1}(X) \subset \mathrm{Gr}(N-1, \mathbb{P}^N) = \mathbb{P}^{N*}$ is equal to the dual variety X^*.

Example 5.15 One might expect that usually $\dim X = \dim \gamma_n(X)$. Indeed, suppose that $\dim X > \dim \gamma_n(X)$. Then a generic fiber F of the map $X_{sm} \to \gamma_n(X)$, $x \mapsto \hat{T}_x X$ has positive dimension $\dim X - \dim \gamma_n(X)$. Therefore, if the hyperplane H is tangent to X_{sm} at some point $x \in F$ then H is tangent to X at all points of F. It easily follows that $\mathrm{def}\, X > \dim F = \dim X - \dim \gamma_n(X)$. So X has a positive defect in this case.

The proof of the following theorem is analogous to the proof of Theorem 5.13 (a).

Theorem 5.16 ([Z2]) *Suppose that $X \subset \mathbb{P}^N$ is a nondegenerate projective variety. If $L \subset \gamma_k(X)$ then*

$$\dim\{x \in X \,\hat{T}_x X \subset L\} \leq k - n + (\dim \mathrm{Sing}\, X + 1),$$

where we set $\dim \mathrm{Sing}\, X = -1$ *if X is smooth.*

The following theorem is called Zak's Theorem on linear normality.

Theorem 5.17 ([Z4]) *Suppose that $X \subset \mathbb{P}^N$ is a smooth non-degenerate projective variety,* $\dim X = n$. *If*

$$\mathrm{codim}\, X < \frac{N+4}{3}$$

then $\mathrm{Sec}(X) = \mathbb{P}^N$.

Proof. By Theorem 5.8 either $\mathrm{Tan}(X) = \mathrm{Sec}(X)$ or $\dim \mathrm{Sec}(X) = 2n+1$. Suppose that $\mathrm{Sec}(X) \neq \mathbb{P}^N$ and $\dim \mathrm{Sec}(X) = 2n+1$. Then $2n+1 < N$, which contradicts $\operatorname{codim} X < \frac{N+4}{3}$.

Suppose now that $\mathrm{Tan}(X) = \mathrm{Sec}(X)$ and $\mathrm{Sec}(X) \neq \mathbb{P}^N$. Let $z \in \mathrm{Sec}(X)$ be a generic point, $L = \hat{T}_z \mathrm{Sec}(X)$, and $Q = \{x \in X \mid z \in \hat{T}_x X\}$. Then Q is closed and an easy dimension count shows that $\dim Q = 2n - \dim \mathrm{Sec}(X)$. For any point $x \in Q$ we have $z \in \hat{T}_x X \subset \mathrm{Sec}(X)$ therefore $\hat{T}_x X \subset \hat{T}_z \mathrm{Sec}(X) = L$. It follows that $\mathrm{Tan}(Q, X) \subset L$. However, $\mathrm{Sec}(Q, X) \not\subset L$ because L is a proper subspace and X is not degenerate. By Theorem 5.10 we have

$$\dim \mathrm{Sec}(Q, X) = \dim Q + \dim X + 1 = 3n - \dim \mathrm{Sec}(X) + 1.$$

Since $\mathrm{Sec}(Q, X) \subset \mathrm{Sec}(X)$, we have $\dim \mathrm{Sec}(Q, X) \leq \dim \mathrm{Sec}(X)$. Then

$$2 \dim \mathrm{Sec}\, X \geq 3n + 1$$

Hence $2N > 3n+1$. But this contradicts $\operatorname{codim} X < \frac{N+4}{3}$. □

Theorem 5.17 is called the theorem on linear normality due to the following corollary first conjectured in [Ha2].

Corollary 5.18 *Suppose that $X \subset \mathbb{P}^N$ is a smooth non-degenerate projective variety. If*

$$\operatorname{codim} X < \frac{N+2}{3}$$

then X is linearly normal.

Proof. Recall from Section 1.4 that if X is not linearly normal then there exists a subvariety $X' \subset \mathbb{P}^{N+1}$ and a point $p \in \mathbb{P}^{N+1}$ such that X is isomorphic to X' via the linear projection with center p. In particular, $\mathrm{Sec}(X') \neq \mathbb{P}^{N+1}$. But this contradicts Theorem 5.17. □

Zak [Z2] has also classified the varieties in the borderline case

$$\operatorname{codim} X = \frac{N+4}{3}.$$

Theorem 5.19 ([Z4]) *Suppose that $X \subset \mathbb{P}^N$ is a smooth non-degenerate n-dimensional projective variety such that* $\operatorname{codim} X = \frac{N+4}{3}$. *Then* $\mathrm{Sec}(X) = \mathbb{P}^N$ *except the* 4 *following cases:*

- $n = 2$, $X = \mathbb{P}^2$, $X \subset \mathbb{P}^5$ *is the Veronese embedding.*
- $n = 4$, $X = \mathbb{P}^2 \times \mathbb{P}^2$, $X \subset \mathbb{P}^8$ *is the Segre embedding.*
- $n = 8$, $X = \mathrm{Gr}(2, \mathbb{C}^6)$, $X \subset \mathbb{P}^{14}$ *is the Plücker embedding.*
- $n = 16$, $X \subset \mathbb{P}^{26}$ *is the projectivization of the highest weight vector orbit in the* 27*-dimensional irreducible representation of* E_6.

The varieties listed in Theorem 5.19 are called Severi varieties after Severi who proved that the unique 2-dimensional Severi variety is the Veronese surface. Scorza and Fujita–Roberts [FR] have shown that the unique 4-dimensional Severi variety is the Segre embedding of $\mathbb{P}^2 \times \mathbb{P}^2$. It is also shown in [FR] that $n \equiv 0 \mod 16$ as $n > 8$. Finally, Tango [Tan] proved that if $n > 16$ then either $n = 2^a$ or $n = 3 \cdot 2^b$, where $a \geq 7$ and $b \geq 5$. The 16-dimensional Severi variety was discovered by Lazarsfeld [La]. In fact, all Severi varieties (1)–(4) arise from Example 5.12. Another description of Severi varieties can be found in Example 2.3.

6. Dual Varieties and Projective Differential Geometry

Preliminaries

In sections 6.1 and 6.2 we review applications of the Katz dimension formula expressing the codimension of the dual variety in terms of the rank of a certain Hessian matrix. The section 6.3 contains results of Ein about the contact locus. In the last section 6.4 we give a useful formalism for dealing with dual varieties "under the microscope".

6.1 The Katz Dimension Formula

Let $X \subset \mathbb{P}^n = \mathbb{P}(V)$ be an irreducible k-dimensional projective variety and $X^* \subset \mathbb{P}^{n*} = \mathbb{P}(V^*)$ be the projectively dual variety. The Katz dimension formula expresses the defect $\operatorname{def}(X) = \operatorname{codim} X^* - 1$ in terms of the rank of a certain Hessian matrix.

For $x \in \mathbb{P}(V)$ let $x^\perp \subset V^*$ denote the annihilator of the line in V corresponding to x. Let x_0 be a smooth point of X. Then one can choose linear functionals

$$T_0 \in V^* \setminus x_0^\perp, \quad T_1, \ldots, T_k \in x_0^\perp$$

so that the functions

$$t_1 = T_1/T_0,\ t_2 = T_2/T_0, \ldots, t_k = T_k/T_0$$

are local coordinates on X in the neighborhood of x_0. For every $U \in x_0^\perp$ the function $u = U/T^0$ on X near x_0 is an analytic function of $t_1, \ldots, t_k$ such that $u(0, \ldots, 0) = 0$. Consider the Hessian matrix

$$\operatorname{Hes}(u) = \operatorname{Hes}(U; T_0, T_1, \ldots, T_k; x_0) = \left(\frac{\partial^2 u}{\partial t_i \partial t_j}(0, \ldots, 0) \right)_{i,j=1,\ldots,k}.$$

Theorem 6.1 ([Ka]) *Let* $\operatorname{def}(X) = \operatorname{codim} X^* - 1$ *be the defect of* X. *Then*

(a) *We have*

$$\operatorname{def}(X) = \min \operatorname{corank} \operatorname{Hes}(u),$$

the minimum over all possible choices of x_0 *and* U.

(b) *X^* is a hypersurface if and only if for some choice of x_0 and u the Hessian matrix* $\mathrm{Hes}(u)$ *is invertible.*

Proof. Let $I_X^0 \subset \mathbb{P}(V) \times \mathbb{P}(V^*)$ be a "smooth part" of the conormal variety consisting of all pairs (x, H) such that $x \in X$ is smooth and H is tangent to X at x. Let $\pi : I_X^0 \to \mathbb{P}(V^*)$ be the second projection. Then X^* is the closure of $\pi(I_X^0)$. We shall compute the Jacobian matrix of π in appropriate local coordinates and relate it to the Hessian matrices appearing in the Theorem. Then we shall use an obvious fact that for any regular map $\pi : Z \to S$ of algebraic varieties the dimension of the image is equal to the maximal rank of the Jacobian matrix of π at smooth points of Z.

Let $(x_0, H_0) \in I_X^0$. Then one can choose

$$T_0 \in V^* \setminus x_0^\perp, \quad T_1, \ldots, T_k \in x_0^\perp$$

so that the functions

$$t_1 = T_1/T_0,\ t_2 = T_2/T_0,\ \ldots,\ t_k = T_k/T_0$$

are local coordinates on X in the neighborhood of x_0. Extend $T_1, \ldots, T_k$ to a basis

$$\{T_1, \ldots, T_k, U_1, \ldots, U_{n-k}\}$$

of $x_0^\perp$. Then each of functions $u_i = U_i/T_0$ on X is an analytic function of $t_1, \ldots, t_k$ near $x_0 = (0, \ldots, 0)$, and we have

$$u_i(0, \ldots, 0) = 0 \text{ for } i = 1, \ldots, n-k.$$

Let $x \in X$ be a point close to x_0 with local coordinates $(t_1, \ldots, t_k)$, so in homogeneous coordinates

$$x = (1 : t_1 : \ldots : t_k : u_1 : \ldots : u_{n-k}).$$

Let $H \subset \mathbb{P}(V^*)$ be the hyperplane defined by an equation

$$\sum_{j=0}^{k} \tau_j T_j + \sum_{i=1}^{n-k} \eta_i U_i = 0.$$

Then H is tangent to X at x if and only if the function

$$\tau_0 + \sum_{j=1}^{k} \tau_j t_j + \sum_{i=1}^{n-k} \eta_i u_i$$

vanishes at x together with all its first derivatives. Therefore, for a given x, the hyperplanes tangent to X at x form a projective space of dimension $n-k-1$ with homogeneous coordinates $(\eta_1 : \ldots : \eta_{n-k})$, and the remaining coordinates are given by

$$\tau_j = -\sum_{i=1}^{n-k} \eta_i \frac{\partial u_i}{\partial t_j}, \quad j = 1, \ldots, k,$$

$$\tau_0 = -\left(\sum_{j=1}^{k} \tau_j t_j + \sum_{i=1}^{n-k} \eta_i u_i\right) = -\sum_{i=1}^{n-k} \eta_i u_i + \sum_{j=1}^{k}\sum_{i=1}^{n-k} \eta_i t_j \frac{\partial u_i}{\partial t_j}.$$

Without loss of generality we may assume that the hyperplane H_0 has coordinates

$$(\eta_1 : \ldots : \eta_{n-k}) = (0 : \ldots : 0 : 1).$$

It follows that we can set $\eta_{n-k} = 1$ in above formulas, and use

$$(t_1, \ldots, t_k, \eta_1, \ldots, \eta_{n-k-1})$$

as local coordinates on I_X^0 near (x_0, H_0). In these coordinates the projection $\pi : I_X^0 \to \mathbb{P}(V^*)$ has a form

$$(t_1, \ldots, t_k, \eta_1, \ldots, \eta_{n-k-1}) \mapsto (\tau_0, \tau_1, \ldots, \tau_k, \eta_1, \ldots, \eta_{n-k-1}),$$

where τ_i are given by formulas above with $\eta_{n-k} = 1$. Therefore, the Jacobian matrix of π at the origin is the following matrix:

$$\frac{\partial(\tau_0, \tau_1, \ldots, \tau_k, \eta_1, \ldots, \eta_{n-k-1})}{\partial(t_1, \ldots, t_k, \eta_1, \ldots, \eta_{n-k-1})}(0, \ldots, 0) =$$

$$= \begin{pmatrix} 0 & 0 \\ -\operatorname{Hes}(u_{n-k}) & -\left(\frac{\partial(u_1, \ldots, u_{n-k})}{\partial(t_1, \ldots, t_k)}(0, \ldots, 0)\right)^t \\ 0 & \operatorname{Id}_{n-k-1} \end{pmatrix}.$$

Here vertical sizes of the blocks are $1, k, n-k-1$, and horizontal sizes are k, $n-k-1$. Clearly, this matrix has rank

$$n - k - 1 + \operatorname{rk} \operatorname{Hes}(u_{n-k}).$$

It follows that

$$\dim X^* = n - k - 1 + \max \operatorname{rk} \operatorname{Hes}(u),$$

and the Theorem is proved. □

If $X \subset \mathbb{P}^n$ is a hypersurface (or a complete intersection) then it is possible to rewrite Hessian matrices in homogeneous coordinates. In case of hypersurfaces the corresponding result was first formulated by B. Segre [Se]:

Theorem 6.2 *Let $f(x_0, \ldots, x_n)$ be an irreducible homogeneous polynomial and let $X \subset \mathbb{P}^n$ be the hypersurface with the equation $f = 0$. Let m be the largest number with the following property: there exists $(m \times m)$-minor of the Hessian matrix $\left(\partial^2 f / \partial x_i \partial x_j\right)$ that is not divisible by f. Then $\dim X^* = m - 2$.*

Example 6.3 An irreducible surface in $\mathbb{P}^3$ with equation $f(x_1, \ldots, x_3) = 0$ is projectively dual to a space curve if and only if the Hessian $\det \left|\partial^2 f/\partial x_i \partial x_j\right|$ is divisible by f.

The Segre Theorem is applicable in the most common situation: the projectively dual variety is usually a hypersurface, therefore by taking duals of hypersurfaces we can get a considerable amount of varieties that are not hypersurfaces.

Example 6.4 The following remark belongs to Zak (unpublished). In notations of Theorem 6.2, suppose that the Hessian $H(f) = \left|\partial^2 f/\partial x_i \partial x_j\right|$ is not trivial.. Then $H(f)$ is divisible by f^{n+1-m} (this result can be easily verified by induction using the well-known fact that if any $(p \times p)$-minor of a $(p+1) \times (p+1)$-matrix A over the UFD D is divisible by f^l, where f is prime, then $\det A$ is divisible by f^{l+1}). Let $\deg f = d$. We can compare degrees of $H(f)$ and f^{n+1-m} and obtain the formula $(n+1)(d-2) \geq d(n+1-m)$, i.e. $d \geq \dfrac{2(n+1)}{m}$. In other words, if Y is a projective variety in $\mathbb{P}^n$ such that Y^* is a hypersurface with the non-trivial Hessian, then

$$\deg Y^* \geq \frac{2(n+1)}{\dim Y + 2}.$$

Zak conjectures that the non-triviality of the Hessian can be substituted for the non-degeneracy of Y and, moreover, that this inequality turns to the equality if and only if Y is the projectivization of the variety of rank 1 elements in a simple Jordan algebra.

6.2 Product Theorem and Applications

6.2.A Product Theorem

Let $X_1 \subset \mathbb{P}^{n_1}$ and $X_2 \subset \mathbb{P}^{n_2}$ be two irreducible projective varieties. The product $X_1 \times X_2$ is then naturally embedded in $\mathbb{P}^{n_1} \times \mathbb{P}^{n_2}$ and the latter is embedded in $\mathbb{P}^n$ via the Segre embedding, where

$$n + 1 = (n_1 + 1)(n_2 + 1).$$

Therefore we have an embedding $X_1 \times X_2 \subset \mathbb{P}^n$. Let $(X_1 \times X_2)^* \subset \mathbb{P}^{n*}$ be the dual variety. It turns out that it is quite easy to calculate $\operatorname{def} X_1 \times X_2$.

Theorem 6.5 ([WZ])

$$\operatorname{def} X_1 \times X_2 = \max(0, \operatorname{def} X_1 - \dim X_2, \operatorname{def} X_2 - \dim X_1).$$

Proof. We shall use Theorem 6.1. Let $k_1 = \dim X_1$, $k_2 = \dim X_2$. Let $x_0 = (x_{01}, x_{02})$ be a smooth point of $X_1 \times X_2$. We choose local coordinates

$$t_{11}, t_{21}, \ldots, t_{k_1 1}$$

of X_1 near x_{01} and

$$t_{12}, t_{22}, \ldots, t_{k_2 2}$$

of X_2 near x_{02}. So for $\nu = 1, 2$, a point x_ν close to $x_{0\nu}$ has homogeneous coordinates

$$x_\nu = (1 : t_{1\nu} : \ldots : t_{k_\nu \nu} : u_{1\nu} : \ldots : u_{n_\nu - k_\nu, \nu}),$$

where each $u_{1\nu}$ is an analytic function of

$$t_{1\nu}, \ldots, t_{k_\nu \nu}$$

vanishing at the origin.

By definition of the Segre embedding, a point $(x_1, x_2) \in X_1 \times X_2$ has as homogeneous coordinates all pairwise products of homogeneous coordinates of x_1 and x_2. Therefore, the set of homogeneous coordinates (near (x_{01}, x_{02})) is given by

$$(1 : t_1 : t_2 : u_1 : u_2 : tt : tu : ut : uu),$$

where symbols

$$t_1, t_2, u_1, u_2, tt, tu, ut, uu$$

stand for the following sets of variables:

$$t_1 = \{t_{j1}\},\ t_2 = \{t_{j2}\},\ u_1 = \{u_{i1}\},\ u_2 = \{u_{i2}\},$$

$$tt = \{t_{j_1 1} t_{j_2 2}\},\ tu = \{t_{j1} u_{i2}\},\ ut = \{u_{i1} t_{j2}\},\ uu = \{u_{i_1 1} u_{i_2 2}\}.$$

The coordinates in sets t_1 and t_2 can be chosen as local coordinates on $X_1 \times X_2$ near (x_{01}, x_{02}). The coordinates from remaining sets are analytic functions of these local coordinates vanishing at the origin.

To apply Theorem 6.1, we have to consider the Hessian matrix $\mathrm{Hes}(u)$, where u is a linear combination of all the coordinates from the sets

$$t_1, t_2, u_1, u_2, tt, tu, ut, uu$$

regarded as a function of local coordinates from t_1 and t_2. We can write $\mathrm{Hes}(u)$ as

$$\mathrm{Hes}(u) = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix},$$

where

$$A_{\alpha\beta} = \left(\frac{\partial u}{\partial t_{i\alpha} \partial t_{j\beta}}(0, \ldots, 0) \right)_{1 \le i \le k_\alpha,\ 1 \le j \le k_\beta}.$$

Then it is easy to see that $A_{11} = \mathrm{Hes}(u_1)$, $A_{22} = \mathrm{Hes}(u_2)$, and

$$\mathrm{Hes}(u) = \begin{pmatrix} \mathrm{Hes}(u_1) & A \\ A^{\perp} & \mathrm{Hes}(u_2) \end{pmatrix},$$

where A can be any $(k_1 \times k_2)$-matrix. Therefore, by Theorem 6.1 it is sufficient to use the following fact from linear algebra:

Lemma 6.6 *For any integers $k_1 \geq c_1 \geq 0$, $k_2 \geq c_2 \geq 0$, let $\mathrm{Bil}(c_1, k_1; c_2, k_2)$ denote the set of all bilinear forms on $\mathbb{C}^{k_1} \oplus \mathbb{C}^{k_2}$ such that their restrictions on $\mathbb{C}^{k_1}$ and $\mathbb{C}^{k_2}$ have coranks c_1 and c_2. Then a generic form $\varphi \in \mathrm{Bil}(c_1, k_1; c_2, k_2)$ has corank*

$$\mathrm{corank}(\varphi) = \max(0, c_1 - k_2, c_2 - k_1).$$

Let us prove this lemma. The corank of a bilinear form is equal to the dimension of its kernel. Let $\varphi \in \mathrm{Bil}(c_1, k_1; c_2, k_2)$, $U = \mathrm{Ker}\,\varphi|_{\mathbb{C}^{k_1}}$, $\dim U = c_1$. The form φ induces a linear map $\psi : U \to (\mathbb{C}^{k_2})^*$. Let $U_0 = \mathrm{Ker}\,\psi$. Then it is clear that $U_0 \subset \mathrm{Ker}\,\varphi$ and $\dim U_0 \geq \dim U - \dim(\mathbb{C}^{k_2})^* = c_1 - k_2$. This argument shows that

$$\mathrm{corank}(\varphi) \geq \max(0, c_1 - k_2, c_2 - k_1).$$

It remains to find a form such that this inequality turns to an equality. We shall proceed by induction. Suppose first that c_1 and c_2 are positive. Take a form $\varphi' \in \mathrm{Bil}(c_1 - 1, k_1 - 1; c_2 - 1, k_2 - 1)$ such that

$$\mathrm{corank}(\varphi') = \max(0, c_1 - k_2, c_2 - k_1).$$

We can define the form φ on

$$\mathbb{C}^{k_1} \oplus \mathbb{C}^{k_2} = \big(\mathbb{C}^{k_1 - 1} \oplus \langle e \rangle\big) \oplus \big(\mathbb{C}^{k_2 - 1} \oplus \langle f \rangle\big)$$

such that its restriction on $\mathbb{C}^{k_1-1} \oplus \mathbb{C}^{k_2-1}$ coincides with φ', subspaces $\mathbb{C}^{k_1-1} \oplus \mathbb{C}^{k_2-1}$ and $\langle e, f \rangle$ are orthogonal, and $\varphi(e, e) = \varphi(f, f) = 0$, $\varphi(e, f) = 1$. Then it is clear that $\varphi \in \mathrm{Bil}(c_1, k_1; c_2, k_2)$ and

$$\mathrm{corank}(\varphi) = \mathrm{corank}(\varphi') = \max(0, c_1 - k_2, c_2 - k_1).$$

Now we may assume that $c_1 = 0$. Suppose that c_2 and k_1 are both positive. Take a form $\varphi' \in \mathrm{Bil}(0, k_1 - 1; c_2 - 1, k_2 - 1)$ such that

$$\mathrm{corank}(\varphi') = \max(0, c_2 - k_1).$$

We define the form φ on

$$\mathbb{C}^{k_1} \oplus \mathbb{C}^{k_2} = \big(\mathbb{C}^{k_1 - 1} \oplus \langle e \rangle\big) \oplus \big(\mathbb{C}^{k_2 - 1} \oplus \langle f \rangle\big)$$

such that the restriction of φ on $\mathbb{C}^{k_1-1} \oplus \mathbb{C}^{k_2-1}$ coincides with φ', subspaces $\mathbb{C}^{k_1-1} \oplus \mathbb{C}^{k_2-1}$ and $\langle e, f\rangle$ are orthogonal, and $\varphi(e,e) = \varphi(e,f) = 1$, $\varphi(f,f) = 0$. It is clear then that $\varphi \in \mathrm{Bil}(0, k_1; c_2, k_2)$ and

$$\mathrm{corank}(\varphi) = \mathrm{corank}(\varphi') = \max(0, c_2 - k_1).$$

It remains to consider only cases when either $k_1 = 0$ or $c_1 = c_2 = 0$. In both cases lemma is obvious. □

It follows immediately from Theorem 6.5 that the same result holds for the product of any number of factors. Let $X_k \subset \mathbb{P}^{n_k}$, $k = 1 \ldots, r$ be irreducible projective varieties. The product $X = X_1 \times \ldots \times X_r$ is then naturally embedded in $\mathbb{P}^{n_1} \times \ldots \times P^{n_r}$ and the latter is embedded in $\mathbb{P}^n$ via the Segre embedding, where $n + 1 = (n_1 + 1) \times \ldots \times (n_r + 1)$. Therefore we have an embedding $X \subset \mathbb{P}^n$. Let $X^* \subset \mathbb{P}^{n*}$ be the dual variety.

Theorem 6.7

$$\operatorname{def} X + \dim X = \max(\dim X, \operatorname{def} X_1 + \dim X_1, \ldots, \operatorname{def} X_r + \dim X_r).$$

In particular, X^* is a hypersurface if and only if

$$\operatorname{def} X_k + \dim X_k \le \dim X, \quad k = 1 \ldots, r.$$

6.2.B Hyperdeterminants

For $i = 1, \ldots, r$ consider a projective space $\mathbb{P}^{l_i} = \mathbb{P}(\mathbb{C}^{l_i+1})$. The Segre embedding $\mathbb{P}^{l_1} \times \ldots \times \mathbb{P}^{l_r} \subset \mathbb{P}^l$, $\mathbb{P}^l = \mathbb{P}(\mathbb{C}^{l_i+1} \otimes \ldots \otimes \mathbb{C}^{l_r+1})$, identifies the affine cone $\mathrm{Cone}(\mathbb{P}^{l_1} \times \ldots \times \mathbb{P}^{l_r})$ with a variety of decomposable tensors in $\mathbb{C}^{l_i+1} \otimes \ldots \otimes \mathbb{C}^{l_r+1}$. The corresponding discriminant (if exists) is called a hyperdeterminant for the matrix format $(l_1 + 1) \times \ldots \times (l_r + 1)$, cf. Example 2.18. Theorem 6.5 allows to get a simple criterion for the existence of hyperdeterminants:

Theorem 6.8 *The dual variety $(\mathbb{P}^{l_1} \times \ldots \times \mathbb{P}^{l_r})^*$ is a hypersurface (and, hence, defines a hyperdeterminant) if and only if*

$$2l_k \le l_1 + \ldots + l_r$$

for $k = 1, \ldots, r$.

Proof. Since the dual variety of $\mathbb{P}^{n*}$ is an empty set, $\operatorname{def} \mathbb{P}^n = n$. Now everything follows from Theorem 6.5. □

6.2.C Associated Hypersurfaces

Let $X \subset \mathbb{P}(V)$ be an irreducible subvariety, $\mathbb{P}^l = \mathbb{P}(U)$. By Theorem 6.5, the dual variety $(X \times \mathbb{P}^l)^*$ is a hypersurface in $\mathbb{P}(V \otimes U)^*$ for $\operatorname{def} X \le l \le \dim X$. Since we always have $\operatorname{def} X \le \dim X$, we obtain a system of hypersurfaces naturally associated to X. Their interpretation was found in [WZ].

We can identify $\mathbb{P}(V \otimes U)^*$ with the projectivization of the space of linear maps $\operatorname{Hom}(U, V^*)$. By some abuse of notation we shall denote a non-zero linear map and its projectivization by the same letter. Let $\mathbb{P}(V \otimes U)^*_0$ be an open subset consisting of operators having maximal possible rank $l+1$. For any $f \in \mathbb{P}(V \otimes U)^*_0$ we can associate two projective subspaces

$$U(f) = \mathbb{P}(\operatorname{Im}(f)) \subset \mathbb{P}(V)^*$$

and the orthogonal subspace $U(f)^\perp \subset \mathbb{P}(V)$. Recall that $I^0_X \subset \mathbb{P}(V) \times \mathbb{P}(V)^*$ is an open part of the conormal variety consisting of pairs (x, H) such that $x \in X_{sm}$ and H is the hyperplane tangent to X at x.

Theorem 6.9 ([WZ]) *The dual variety $(X \times \mathbb{P}^l)^*$ is the closure of the set of points $f \in \mathbb{P}(V \otimes U)^*_0$ such that*

$$\big(U(f)^\perp \times U(f)\big) \cap I^0_X \neq \emptyset.$$

In two extreme cases $l = \dim X$ or $l = \operatorname{def} X$ this result can be made more precise. To go further, we need a notion of an associated hypersurface. Let $X \subset \mathbb{P}^n$ be a k-dimensional irreducible subvariety. Consider the set $\mathcal{Z}(X)$ of all $(n-k-1)$-dimensional projective subspaces in $\mathbb{P}^n$ that intersect X. This is a subvariety in the Grassmanian $\operatorname{Gr}(n-k, n+1)$ parametrizing all $(n-k-1)$-dimensional projective subspaces in $\mathbb{P}^n$.

Theorem 6.10 *The subvariety $\mathcal{Z}(X)$ is an irreducible hypersurface, called an associated hypersurface of X.*

Proof. Let $B(X) \subset X \times \mathcal{Z}(X)$ be an incidence variety consisting of pairs (x, L) such that $x \in X$, $L \in \operatorname{Gr}(n-k, n+1)$, and $x \in L$. We have two projections $p: B(X) \to X$ and $q: B(X) \to \mathcal{Z}(X)$. Then p is a Grassmannian fibration, any fiber of p is isomorphic to $\operatorname{Gr}(n-k-1, n)$. In particular, $B(X)$ (and hence $\mathcal{Z}(X)$) is irreducible. On the other hand, q is birational, because a generic $(n-k-1)$-dimensional projective subspace intersecting X meets X in exactly one point. Now an easy dimension count shows that $\mathcal{Z}(X)$ is a hypersurface. □

Now we can relate associated hypersurfaces and dual varieties. We continue to use the notation of Theorem 6.9. The following theorem is sometimes called a Cayley trick:

Theorem 6.11 ([WZ]) *Let $k = \dim X$, $k^* = \dim X^*$. Then*

(a) *Let* $l = \dim X$, *and* $p_1 : \mathbb{P}(V \otimes U)_0^* \to \mathrm{Gr}(n-k-1, n+1)$ *be the map* $f \mapsto U(f)^\perp$. *Then*
$$(X \times \mathbb{P}^l)^* = \overline{p_1^{-1}(\mathcal{Z}(X))}.$$
(b) *Let* $l = \operatorname{def} X$, *and* $p_2 : \mathbb{P}(V \otimes U)_0^* \to \mathrm{Gr}(n-k^*-1, n+1)$ *be the map* $f \mapsto U(f)$. *Then*
$$(X \times \mathbb{P}^l)^* = \overline{p_2^{-1}(\mathcal{Z}(X^*))}.$$

We refer the reader to [GKZ2] for further results on associated varieties and Chow forms.

6.3 Ein Theorems

Suppose that $X \subset \mathbb{P}^N$ is a non-degenerate smooth projective variety, $\dim X = n$. For any hyperplane $H \subset \mathbb{P}^N$ the contact locus $\operatorname{Sing} X \cap H$ is a subvariety of X consisting of all points $x \in X$ such that the embedded tangent space $\hat{T}_x X$ is contained in H. One can use the Jacobian ideal of $X \cap H$ to define a scheme structure on the contact locus, however this scheme could be not reduced. Clearly the contact locus is non-empty if and only if H belongs to the dual variety X^*. By Theorem 1.12 if $\operatorname{def} X = d$ then for any $H \in {X^*}_{sm}$ the contact locus $\operatorname{Sing}(H \cap X)$ is a projective subspace of dimension d and a union of this projective subspaces is dense in X. In this section we study further properties of the contact locus, most results here belong to L. Ein.

Theorem 6.12 ([E1]) *Suppose that* q *is a generic point of* X *and* H *is generic tangent hyperplane of* X *at* q, $L = \mathbb{P}^d$ *is the contact locus of* H *with* X. *Then*

(a) *if* p *is a point in* $\mathbb{P}^d$, *then the tangent cone of the hyperplane section* $H \cap X$ *at* p *is a quadric hypersurface of rank* $n-d$ *in* $\hat{T}_p(X)$.
(b) *Let* s_h *be the section of* $\mathcal{O}_X \otimes \mathcal{O}_X(1)$ *defining* $H \cap X$. *Then* s_h *factors through* I_L^2, *where* $I_L \subset \mathcal{O}_X$ *is the ideal sheaf of* L *in* X.
(c) *Let* t_h *be the section of* $I_L^2/I_L^3 \otimes \mathcal{O}_L(1) \cong S^2(N_L^* X) \otimes \mathcal{O}_L(1)$ *induced by* s_h. *Then* t_h *defines a nonsingular quadric hypersurface in* $\mathbb{P}(N_L^* X|_p)$.

Proof. (a) This follows from the proof of Theorem 6.1 using Sard's Lemma for algebraic varieties.

(b) We choose a local coordinate system $\{x_1, \ldots, x_n\}$ of X near p. We may assume that I_L is generated by $x_1, \ldots, x_{n-d}$. Since $L \subset H \cap X$, we can write the power series of s_h in the following form:
$$s_h = x_1 f_1 + \ldots + x_{n-d} f_{n-d} + \sum_{i=1}^{n-d} \sum_{j=1}^{n-d} x_i x_j g_{ij},$$

where $f_1, \ldots, f_{n-d}$ are power series in variables $x_{n-d+1}, \ldots, x_n$ only. But $\operatorname{Sing} H \cap X = L$, therefore $\left.\dfrac{\partial s_h}{\partial x_i}\right|_L = 0$ for $i = 1, \ldots, n-d$. Hence

$$f_1 = \ldots, f_{n-d} = 0$$

and

$$s_h = \sum_{i=1}^{n-d} \sum_{j=1}^{n-d} x_i x_j g_{ij}.$$

Thus s_h factors through I_L^2.

(c) We can write $g_{ij} = a_{ij} + h_{ij}$, where a_{ij}'s are constants and h_{ij}'s are power series without the constant term. Now

$$s_h = \sum_{i=1}^{n-d} \sum_{j=1}^{n-d} x_i x_j (a_{ij} + h_{ij}).$$

By (a), $\sum\limits_{i=1}^{n-d} \sum\limits_{j=1}^{n-d} x_i x_j a_{ij}$ is a quadratic form of rank $n-d$. But this is also the equation for the quadric hypersurface in $\mathbb{P}(I_L/I_L^2)$ induced by s_h. □

Recall that any vector bundle on $\mathbb{P}^1$ has a form $\oplus_i \mathcal{O}(a_i)$ for some integers a_i (see e.g. [Ha1]).

Definition 6.13 *A vector bundle E on a projective space $\mathbb{P}^N$ is called uniform if for any line $T \subset \mathbb{P}^N$ the restriction $E|_T$ is a fixed vector bundle $\oplus_i \mathcal{O}(a_i)$.*

The following theorem is the main result of this section.

Theorem 6.14 ([E1]) *Suppose that X is a nonlinear smooth projective variety in $\mathbb{P}^N$, $\operatorname{def} X = d > 0$, q is a generic point of X and H is generic tangent hyperplane of X at q, $L = \mathbb{P}^d$ is the contact locus of H with X. Then*

(a) $N_L X \cong (N_L X)^*(1)$.

(b) *If $T = \mathbb{P}^1$ is a line in $L = \mathbb{P}^d$, then*

$$N_L X|_T \cong \mathcal{O}_T^{\oplus(n-d)/2} \oplus \mathcal{O}_T(1)^{\oplus(n-d)/2}$$

(*in particular $N_L X$ is a uniform vector bundle*) *and*

$$N_T X \cong \mathcal{O}_T^{\oplus(n-d)/2} \oplus \mathcal{O}_T(1)^{\oplus(n+d-2)/2}.$$

(c) *There is an irreducible $\dfrac{3n+d-4}{2}$-dimensional family of lines in X. If p is a generic point in X, then there is an $\dfrac{n+d-2}{2}$-dimensional family of lines in X through p.*

Proof. (a) We continue to use the notation of Theorem 6.12. By this theorem, the section s_h gives a section of

$$I_L^2/I_L^3(1) = S^2(N_L^*X)(1) \subset \mathrm{Hom}(N_LX, N_L^*X(1)).$$

Therefore s_h induces a map from $N_L(X)$ to $N_LX(1)$. This map is an isomorphism by Theorem 6.12 (c).

(b) Consider the exact sequence

$$0 \to N_LX|_T \to N_L\mathbb{P}^N|_T \to N_X\mathbb{P}^N|_T \to 0,$$

where $N_L\mathbb{P}^N|_T = (N-d)\mathcal{O}_T(1)$. Suppose that

$$N_LX|_T = \bigoplus_{i=1}^{n-d} \mathcal{O}_T(a_i).$$

Then exactness implies that all $a_i \le 1$. Using the isomorphism between N_LX and $N_L^*X(1)$ we observe that each $a_i \ge 0$. Therefore,

$$N_LX|_T \cong \mathcal{O}_T^{\oplus(n-d)/2} \oplus \mathcal{O}_T(1)^{\oplus(n-d)/2}$$

and N_LX is a uniform vector bundle. This implies that

$$N_TX \cong \mathcal{O}_T^{\oplus(n-d)/2} \oplus \mathcal{O}_T(1)^{\oplus(n+d-2)/2}.$$

The statement (c) is a consequence of the standard deformation theory. Indeed, let $p \in L$ and let T_0 be a line in L through p. Since

$$N_{T_0}X \cong \mathcal{O}_{T_0}^{\oplus(n-d)/2} \oplus \mathcal{O}_{T_0}(1)^{\oplus(n+d-2)/2},$$

we have

$$\dim H^0(T_0, N_{T_0}X) = \frac{n-d}{2} + 2\frac{n+d-2}{2} = \frac{3n+d-4}{2}$$

and

$$H^1(T_0, N_{T_0}X) = 0.$$

Therefore, the Hilbert scheme of lines in X is smooth at the point T_0 and there is a unique irreducible component of the Hilbert scheme containing the point T_0. This component has dimension $(3n+d-4)/2$. The proof of the second statement of (c) is similar. □

As a quite formal consequence of Theorem 6.14 (b) we get the following parity theorem that was first proved by A. Landman, using the Picard–Lefschetz theory (unpublished):

Theorem 6.15 ([E1]) *If* $\operatorname{def} X > 0$ *then* $\dim X \equiv \operatorname{def} X \mod 2$.

Example 6.16 ([GH]) Suppose that $X \subset \mathbb{P}^n$ is a smooth non-linear surface. Then X^* is a hypersurface. Indeed, if $\operatorname{def} X > 0$ then $\operatorname{def} X$ is even by Theorem 6.15. But X obviously can not contain an even-dimensional projective subspace $\mathbb{P}^{\operatorname{def} X}$.

Example 6.17 Suppose that $X \subset \mathbb{P}^N$ is a smooth projective nonlinear variety, $\dim X \geq 2$. Then $\operatorname{def} X \leq \dim X - 2$. Indeed, it is clear that $\operatorname{def} X \leq \dim X - 1$. Therefore, by Theorem 6.15 we have $\operatorname{def} X \leq \dim X - 2$. Moreover, it can be shown [E1] that $\operatorname{def} X = \dim X - 2$ if and only if X is a projective bundle over a curve C, $X = \mathbb{P}_C(F)$, where F is a rank n vector bundle on C and all fibers are embedded into $\mathbb{P}^N$ linearly (c.f. Theorem 4.6 and remarks after it).

Example 6.18 Suppose that $X \subset \mathbb{P}^N$ is a smooth projective nonlinear variety, $\dim X \geq 3$, $N = 2n - 1$, and $\dim X = \dim X^*$. Since $\operatorname{def} X = n - 2$ in this case, by Example 6.17 we have that X is a projective bundle over a curve C, $X = \mathbb{P}_C(F)$, where F is a rank n vector bundle on C and all fibers are embedded into $\mathbb{P}^N$ linearly. Moreover, by a theorem of S. Kleiman [Kl5] in this case X is the Segre embedding of $\mathbb{P}^1 \times \mathbb{P}^{n-1}$.

It was first observed by Griffiths and Harris [GH] that any smooth projective variety with positive defect has the negative Kodaira dimension, $H^0(X, K_X^{\otimes m}) = 0$ for $m > 0$, where K_X is the canonical line bundle. This result will have important consequences in Chapter 9. Here is the detailed story:

Theorem 6.19 ([E1]) *Suppose that X is a nonlinear smooth projective variety in $\mathbb{P}^N$, $\operatorname{def} X = d > 0$, q is a generic point of X and H is a generic tangent hyperplane of X at q, $L = \mathbb{P}^d$ is the contact locus of H with X. Then*

(a) $K_X|_L = \mathcal{O}_L\left(\dfrac{-n-d-2}{2}\right)$.

(b) *The Kodaira dimension of X is negative.*

(c) *If $K_X = \mathcal{O}_X(a)$, then $a = \dfrac{-n-d-2}{2}$.*

(d) *If $n > \dfrac{N}{2} + 1$, then $K_X = \mathcal{O}_X\left(\dfrac{-n-d-2}{2}\right)$.*

Proof. (a) By Theorem 6.14 (b), if $T = \mathbb{P}^1$ is a line in $L = \mathbb{P}^d$, then

$$N_L X|_T \cong \mathcal{O}_T^{\oplus(n-d)/2} \oplus \mathcal{O}_T(1)^{\oplus(n-d)/2}.$$

It follows that $\Lambda^{n-d} N_L X|_T = \mathcal{O}_T\left(\dfrac{n-d}{2}\right)$ and, hence, $\Lambda^{n-d} N_L X = \mathcal{O}_L\left(\dfrac{n-d}{2}\right)$. Therefore $K_X|_L = \mathcal{O}_L\left(\dfrac{-n-d-2}{2}\right)$ by the adjunction formula.

(b) Since there is such a plane L through a generic point $p \in X$, the linear system $|K_X^m|$ is empty for $m \geq 0$.

(c) follows from (a).

(d) follows from (c). Indeed, if $\dim X > \dfrac{N}{2} + 1$, then $\operatorname{Pic} X = \mathbb{Z}$ is generated by $\mathcal{O}_X(1)$ by the Barth Theorem [Ba]. □

Example 6.20 ([E1]) Suppose that X is a nonlinear smooth projective variety in $\mathbb{P}^N$, $\operatorname{codim} X = 2$. Then X^* is a hypersurface, unless X is the Segre embedding of $\mathbb{P}^1 \times \mathbb{P}^2$ in $\mathbb{P}^5$. Indeed, assume first that $\dim X \geq 4$ and $\operatorname{def} X = d > 0$. Then by Theorem 6.19 (d) the canonical class $K_X = \mathcal{O}_X(\dfrac{-n-d-2}{2})$. However, it follows from results of [BC] that in this case X is automatically a complete intersection. This contradicts Theorem 4.2. Now, if $\dim X = 1$ or $\dim X = 2$ then X^* is a hypersurface by Example 1.13 and Example 6.16. The case $\dim X = 3$ is ruled out by Example 6.18.

In case of not generic contact loci the following version of Theorem 6.19 holds:

Theorem 6.21 ([BFS1]) *Suppose that X is a smooth n-dimensional projective variety in $\mathbb{P}^N$, $\operatorname{def} X = d > 0$. Let $H \subset X^*$. Then $\operatorname{Sing} H \cap X$ is the union of projective subspaces $\mathbb{P}^d$ with $K_X|_{\mathbb{P}^d} = \mathcal{O}_{\mathbb{P}^d}\left(\dfrac{-n-d-2}{2}\right)$.*

Proof. Let $I_X \subset X \times \mathbb{P}^{N^*}$ be the conormal variety, $\pi : I_X \to \mathbb{P}^{N^*}$ be the projection. Then $\pi(I_X) = X^*$. Let $p : I_X \to X$ be the first projection, then p is a projective bundle. Clearly, I_X is smooth and irreducible. The general fibers F of π are isomorphic to the contact loci of X with generic tangent hyperplanes, therefore, they are isomorphic to the projective subspaces $\mathbb{P}^d$ and by Theorem 6.19 we have $p^*(K_X + ((n+d)/2+1)L)_F \simeq \mathcal{O}_F$. It thus follows that every fiber of π is a union of projective subspaces $\mathbb{P}^d$ with $p^*(K_X + ((n+d)/2+1)L)_{\mathbb{P}^d} \simeq \mathcal{O}_{\mathbb{P}^d}$. □

Let X be a non-linear n-dimensional smooth projective variety in $\mathbb{P}^N$ such that $\operatorname{def} X = d > 0$. Let H be a generic tangent hyperplane of X. Then the contact locus $\operatorname{Sing} H \cap X$ of X with H is a d-dimensional projective subspace L. Let $\tilde{X}$ be a blowing up of X along L. Let $p : \tilde{X} \to X$ be the corresponding map. Denote by E the exceptional divisor and denote by F the proper transform of $H \cap X$. Therefore, we have a following diagram

$$\begin{array}{ccccccc} \mathbb{P}(N_L X) & = & E & \subset & \tilde{X} & \supset & F \\ & & \downarrow & & \downarrow & & \downarrow \\ & & L & \subset & X & \supset & H \cap X. \end{array}$$

We shall denote by $\mathcal{O}_{\tilde{X}}(a,b)$ the line bundle $p^*\mathcal{O}_X(a) \otimes \mathcal{O}_{\tilde{X}}(-bE)$. For example, $\mathcal{O}_{\tilde{X}}(F) = \mathcal{O}_{\tilde{X}}(1,2)$.

The following vanishing theorem holds.

Theorem 6.22 ([E1]) *Assume that $K_X = \mathcal{O}_X(b)$ for some $b \in \mathbb{Z}$. Then*

(a) $K_X = \mathcal{O}_X\left(\dfrac{-n-d-2}{2}\right)$.

(b) $K_{\tilde{X}} = \mathcal{O}_{\tilde{X}}\left(\dfrac{-n-d-2}{2}, -n+d+1\right)$.

(c) $H^i(\mathcal{O}_{\tilde{X}}(a,1)) = 0$, *if* $i > 0$ *and* $a \geq \dfrac{n-3d-2}{2}$.

(d) $H^i(\mathcal{O}_{\tilde{X}}(a,2)) = 0$, *if* $i > 0$ *and* $a \geq \dfrac{n-3d}{2}$.

Assume further that $\dfrac{n-3d-2}{2} \leq 0$. *Then*

(e) $H^0(N_L^*X(a)) = 0$ *for* $a \leq 0$.
(f) $H^k(N_L^*X(a)) = 0$ *for* $a \geq -d$.

(g) $H^i(N_L^*X(a)) = 0$ *if* $0 < i < d$ *and* $a \geq \dfrac{n-3d}{2}$.

(h) $H^i(N_L^*X(a)) = 0$ *if* $0 < i < d$ *and* $a \leq \dfrac{d-n}{2}$.

Proof. (a) follows from Theorem 6.19 (a).

(b) follows from (a) and the fact that $\tilde{X}$ is the blowing up of X along L.

(c) Let $f : \tilde{X} \to \mathbb{P}^{N-1-d}$ be the projection with center L. Let $Y = f(\tilde{X})$. The hyperplane section $H \cap X$ will correspond to a hyperplane section D of Y. Then $f^{-1}(D) = E + F$ and $f^*\mathcal{O}_Y(1) = \mathcal{O}_{\tilde{X}}(1,1)$. Let $y \in Y \setminus D$, $Z = f^{-1}(y)$. Suppose that $\dim Z \geq 1$. Since $Z \cap (E \cup F) = \emptyset$, p maps Z isomorphically to a variety in X. Therefore, $\mathcal{O}_{\tilde{X}}(0,1)|_Z$ is non-trivial. But $\mathcal{O}_{\tilde{X}}(1,1)|_Z = f^*\mathcal{O}_Y(1)|_Z$ is trivial. So $\mathcal{O}_{\tilde{X}}(0,1)$ is non-trivial. Hence $Z \cap E \neq \emptyset$, which is a contradiction. Therefore, all positive-dimensional fibers of f belong to $E \cap F$, in particular $\dim Y = \dim X$. Now,

$$\mathcal{O}_{\tilde{X}}(a,1) = K_{\tilde{X}} \otimes f^*\mathcal{O}_Y(n-d) \otimes \mathcal{O}_{\tilde{X}}\left(a - \frac{n-3d-2}{2}, 0\right).$$

It follows from the Grauert–Rimenschneider vanishing theorem [GR] that $H^i(\mathcal{O}_{\tilde{X}}(a,1)) = 0$ if $i > 0$ and $a \geq \dfrac{n-3d-2}{2}$.

(d) The proof is similar to (c).

(e) Consider the exact sequence

$$0 = H^0(\mathcal{O}_{\tilde{X}}(0,1)) \to H^0(\mathcal{O}_E(0,1)) \to H^1(\mathcal{O}_{\tilde{X}}(0,2)).$$

Now $H^1(\mathcal{O}_{\tilde{X}}(0,2)) = 0$ by (d). So

$$H^0(\mathcal{O}_E(0,1)) \simeq H^0(N_L^*X) = 0.$$

Hence $H^0(N_L^*X(a)) = 0$ for $a \leq 0$.

(f) Recall that $N_LX = N_L^*X(1)$. So (f) follows from (e) and Serre duality.

(g) Consider the exact sequence

$$H^i(\mathcal{O}_{\tilde{X}}(a,1)) \to H^i(\mathcal{O}_E(a,1)) \to H^{i+1}(\mathcal{O}_{\tilde{X}}(a,2)).$$

By (c) and (d), we conclude that

$$H^i(\mathcal{O}_E(a,1)) \simeq H^i(N_L^*X(a)) = 0$$

for $a \geq \dfrac{n-3d}{2}$.

(h) follows from (g) and Serre duality. □

Theorem 6.23 ([E1]) *Suppose that X is a nonlinear smooth projective variety in $\mathbb{P}^N$, $\dim X = n \geq 3$. Assume that $K_X = \mathcal{O}_X(b)$ for some $b \in \mathbb{Z}$. Then*

(a) $\operatorname{def} X \leq \dfrac{n-2}{2}$.

(b) *If* $\dim X = 4m+2$ *and* $\operatorname{def} X = 2m > 0$ *then*

$$N_L^*X = H^m\left(N_L^*X(-m)\right) \otimes \Omega_L^m(m),$$

and $m \leq 2$.

Proof. Consider the Beilinson spectral sequence [OSS] with

$$E_1^{pq} = H^q\left(N_L^*X(p)\right) \otimes \Omega_L^{-p}(-p),$$

which converges to

$$E^i = \begin{cases} N_L^*X & \text{if } i = 0 \\ 0 & \text{otherwise (i.e. } E_\infty^{pq} = 0 \text{ if } p+q \neq 0). \end{cases}$$

(a) If $k = \operatorname{def} X \geq \dfrac{n-1}{2}$, then $\dfrac{n-3k}{2} - 1 \leq \dfrac{k-n}{2}$. It follows from Theorem 6.22 that $H^q(N_L^*X(p)) = 0$ for $-k \leq p \leq 0$. It follows that $N_L^*X = 0$. This is a contradiction.

(b) In this case $H^q(N_L^*X(p)) = 0$ for $-2m \leq p \leq 0$ unless $p = -m$. This implies that $N_L^*X = H^m(N_L^*X(-m)) \otimes \Omega_L^m(m)$. So

$$2m+2 = \operatorname{rank} N_L^*X \geq \operatorname{rank} \Omega_L^m(m) = \binom{2m}{m}.$$

It follows that $m \leq 2$. □

We conclude this section with the following simple but very useful result first appeared in [LS] (where it was attributed to the referee). It could be called the monotonicity theorem.

Theorem 6.24 ([LS]) *Let $X \subset \mathbb{P}^N$ be a smooth projective n-dimensional variety. Suppose that through its generic point there passes a smooth subvariety Y of dimension h and defect θ. Then* $\operatorname{def} X \geq \theta - k + h$. *In other words,*

$$\dim X + \operatorname{def} X \geq \dim Y + \operatorname{def} Y.$$

Proof. Suppose that $x \in X$ is a generic point and H is a generic hyperplane tangent to X at x. Let Y be an h-dimensional submanifold passing through x such that $\operatorname{def} Y = \theta$. Since the defect is the dimension of a generic contact locus, which is a projective subspace, there exists a θ-dimensional projective subspace $Z \subset Y$ containing x such that H is tangent to Y along Z (though H is not a generic tangent hyperplane to Y). Let $f = 0$ be a local equation for $H \cap X$ at x. In a neighborhood U of x, the differential df annihilates the tangent spaces $T_x X$ and $T_z Y$ for every $z \in Z \cap U$. Hence df defines on $Z \cap U$ a section of the conormal bundle $N_Y^* X|_Z$ vanishing at x. Since $N_Y^* X$ is a $(n-h)$-dimensional vector bundle, it follows that this section in fact vanishes on a subvariety $Z' \subset Z \cap U$ of codimension less then or equal to $n - h$. Therefore, H is tangent to X along Z'. But $\operatorname{Sing} X \cap H$ is $(\operatorname{def} X)$-dimensional, therefore $\operatorname{def} X \geq \theta - (n - h)$. □

6.4 The Projective Second Fundamental Form

6.4.A Gauss Map

Suppose that $X \subset \mathbb{P}(V)$ is a smooth irreducible n-dimensional variety, not necessarily closed. Therefore, for any $x \in X$ we have an embedded tangent space $\hat{T}_x X$. A natural way to keep track of the motion of $\hat{T}_x X$ (while x moves in X) is the Gauss map.

Definition 6.25 The Gauss map γ is defined as follows.

$$\gamma : X \to \operatorname{Gr}(n, \mathbb{P}(V)), \quad x \mapsto \hat{T}_x X,$$

where $\operatorname{Gr}(n, \mathbb{P}(V)) = \operatorname{Gr}(n+1, V)$ is the Grassmanian of n-dimensional projective subspaces in $\mathbb{P}^N$.

To measure how $\hat{T}_x X$ moves to the first order, one calculates the differential of γ

$$(d\gamma)_x : T_x X \to T_{\hat{T}_x X} \operatorname{Gr}(n, \mathbb{P}(V)) = \operatorname{Hom}(\operatorname{Cone}(T_x X), V/\operatorname{Cone}(T_x X)).$$

It is easy to see that $\operatorname{Cone}(x) \subset \operatorname{Ker}(d\gamma)_x(v)$ for any $v \in T_x X$, therefore $d(\gamma)_x$ factors to a map

$$\mathbb{FF}_x^2 : T_x X \to \operatorname{Hom}(T_x X, N_X \mathbb{P}(V)|_x).$$

Moreover, $\mathbb{FF}_x^2$ is symmetric, essentially because the Gauss map is already the derivative of a map and mixed partial derivatives commute. Finally, we get the following definition.

Definition 6.26 The section $\mathbb{FF}^2$ of the vector bundle $S^2T^*X \otimes N_X\mathbb{P}(V)$ constructed above is called the projective second fundamental form of X.

If $X \subset \mathbb{P}(V)$ is a singular variety then $\mathbb{FF}^2$ is still defined over the smooth locus X_{sm}.

It is convenient to consider $\mathbb{FF}^2$ as a map $\mathbb{FF}^2 : N_X^*\mathbb{P}(V) \to S^2T^*X$ and to set $|\mathbb{FF}^2| = \mathbb{P}(\mathbb{FF}^2(N_X^*\mathbb{P}(V)))$. One can think of $|\mathbb{FF}^2|_x$ as a linear family of quadric hypersurfaces in $\mathbb{P}(T_xX)$. The space $\mathbb{P}(N_X^*\mathbb{P}(V)|_x)$ has the geometric interpretation as the space of hyperplanes tangent to X at x, i.e., the hyperplanes H such that $X \cap H$ is singular at x. Then $\mathbb{FF}_x^2$ can be viewed as the map that sends a hyperplane to the quadratic part of the singularity of $X \cap H$ at x.

Let $\operatorname{Base}|\mathbb{FF}^2|_x \subset \mathbb{P}(T_xX)$ denote the variety of directions tangent to the lines that osculate to order 2 at x (that is, X appears to contain these lines to a second order). More precisely,

$$\begin{aligned}\operatorname{Base}|\mathbb{FF}^2|_x &= \mathbb{P}\{v \in T_xX \mid \mathbb{FF}_x^2(v,v) = 0\} \\ &= \{p \in \mathbb{P}(T_xX) \mid p \in Q,\ \forall Q \in |\mathbb{FF}^2|\}. \end{aligned} \tag{6.1}$$

Let $\operatorname{Sing}|\mathbb{FF}^2|_x$ denote the set of tangent directions such that the embedded tangent space does not move to first order in these directions.

$$\begin{aligned}\operatorname{Sing}|\mathbb{FF}^2|_x &= \mathbb{P}\{v \in T_xX \mid \mathbb{FF}^2(v,w) = 0\ \forall w \in T_xX\} \\ &= \{p \in \mathbb{P}(T_xX) \mid p \in Q_{sing}\ \forall Q \in |\mathbb{FF}^2|\}.\end{aligned}$$

The following theorem may be called the smoothness principle.

Theorem 6.27 ([GH]) *Let $X \subset \mathbb{P}^N$ be an irreducible projective variety. If X is smooth then* $\operatorname{Sing}|\mathbb{FF}^2|_x = \emptyset$ *for generic $x \in X$.*

In other words, if X is smooth then the Gauss map γ is generically finite. In fact, much more is true.

Theorem 6.28 ([Z3, E4, Ra3]) *If X is smooth then the Gauss map is finite and birational.*

These theorem do not imply that generic quadrics from the second fundamental form are not degenerate. Indeed, the following theorem is just a reformulation of Theorem 6.1:

Theorem 6.29 *If X is any projective variety and $x \in X_{sm}$ then the projective second fundamental form $|\mathbb{FF}^2|_x$ of X at x is a system of quadrics of rank bounded above by* $\dim X - \operatorname{def} X$. *Moreover, if $x \in X_{sm}$ is a generic point and $Q \subset |\mathbb{FF}^2|_x$ is a generic quadric then* $\operatorname{rank} Q = \dim X - \operatorname{def} X$.

6.4.B Moving Frames

The most convenient formalism for dealing with second (and higher) fundamental forms is the language of moving frames. We shall recall it briefly here, more details and applications can be found in [GH].

In $\mathbb{P}^N$ a frame is denoted by $\{A_0, \ldots, A_N\}$. It is given by a basis $A_0, \ldots, A_N$ for $\mathbb{C}^{N+1}$. The set of all frames forms an algebraic variety $\mathcal{F}(\mathbb{P}^n)$ the principal homogeneous space of the general linear group GL_{N+1}. Each of the vectors A_i may be viewed as a mapping $v : \mathcal{F}(\mathbb{P}^N) \to \mathbb{C}^{N+1}$. Expressing the exterior derivative dv in terms of the basis $\{A_i\}$ gives

$$dA_i = \sum_j \omega_{ij} A_j.$$

The $(N+1)^2$ differential forms ω_{ij} are the Maurer–Cartan forms on the group GL_{N+1}, and by taking their exterior derivatives we obtain the Maurer–Cartan equations

$$d\omega_{ij} = \sum_k \omega_{ik} \wedge \omega_{kj}.$$

Geometrically we may think of a frame $\{A_0, \ldots, A_N\}$ as defining a coordinate simplex in $\mathbb{P}^N$, and then ω_{ij} gives the rotation matrix when this coordinate simplex is infinitesimally displaced.

There is a fibering $\pi : \mathcal{F}(\mathbb{P}^N) \to \mathbb{P}^N$ given by

$$\pi\{A_0, \ldots, A_N\} \to A_0.$$

If we set $\omega_i = \omega_{0i}$ then these 1-forms are horizontal for the fibering π, i.e., they vanish at the fibers $\pi^{-1}(p)$, $p \in \mathbb{P}^N$. From $d\omega_i = \sum_j \omega_j \wedge \omega_{ji}$ we see that the forms $\{\omega_i\}$ satisfy the Frobenius integrability condition. Thus we may think of the fibration π as defined by the foliation

$$\omega_1 = \ldots = \omega_N = 0.$$

The equation

$$dA_0 = \sum_{i=0}^{N} \omega_i A_i = \sum_{i=1}^{N} \omega_i A_i \mod A_0$$

has the following geometric interpretation. For each choice of the frame $\{A_0, \ldots, A_N\}$ lying over $p \in \mathbb{P}^N$ the horizontal 1-forms $\omega_1, \ldots, \omega_N$ give a basis for the cotangent space $T_p^*\mathbb{P}^N$. The corresponding basis $v_1, \ldots, v_N \in T_p\mathbb{P}^N$ for the tangent space has the property that v_i is tangent to the line $\overline{A_0 A_i}$.

Assume now that we are given a connected n-dimensional smooth subvariety $M \subset \mathbb{P}^N$, not necessarily closed. Associated to M is the submanifold $\mathcal{F}(M) \subset \mathcal{F}(\mathbb{P}^N)$ of Darboux frames

$$\{A_0; A_1, \ldots, A_n; A_{n+1}, \ldots, A_N\}$$

defined by the conditions that A_0 lies over some $p \in M$ and $A_0, A_1, \ldots, A_n$ span $\hat{T}_p M$, the embedded tangent space of M at p. On $\mathcal{F}(M)$ we have the condition that differential 1-forms $\omega_1, \ldots, \omega_n$ give a basis of $T_p^* M$ and $\omega_i = 0$ for $i > n$. Therefore it is natural to think of the $\omega_1, \ldots, \omega_n$ as homogeneous coordinates in the projectivized tangent spaces $\mathbb{P}(T_p M)$. For example, a quadric in $\mathbb{P}(T_p M)$ is defined by an equation $\sum_{i,j=1}^n q_{ij}\omega_i\omega_j = 0$, $q_{ij} = q_{ji}$.

Since $\omega_i = 0$ for $i > n$, we also have $d\omega_i = 0$ on $\mathcal{F}(M)$. On the over hand, by the structure equations we have $d\omega_i = \sum_{j=1}^N \omega_j \wedge \omega_{ji}$. Therefore, for any $i > n$ we have

$$\sum_{j=1}^n \omega_j \wedge \omega_{ji} = 0.$$

Since $\omega_1, \ldots, \omega_n$ are linearly independent, an easy calculation (called Cartan Lemma) shows that for any $\mu > n$, for any $i = 1, \ldots, n$ we have

$$\omega_{i\mu} = \sum_{j=1}^n q_{ij\mu}\omega_j, \quad q_{ij\mu} = q_{ji\mu}.$$

We set for any $\mu = n+1, \ldots, N$

$$Q_\mu = \sum_{i,j=1}^n q_{ij\mu}\omega_i\omega_j.$$

The equation $Q_\mu = 0$ defines a quadric hypersurface in $\mathbb{P}(T_p M)$.

Then an easy local calculation (see [GH1]) shows that we have

Proposition 6.30 *The linear system of quadrics in $\mathbb{P}(T_p M)$ spanned by Q_μ, $\mu = n+1, \ldots, N$ is the projective second fundamental form $|\mathbb{FF}^2|$.*

Thus, if we think of $\mathbb{FF}^2$ as of the map $S^2 T_p M \to (N_M \mathbb{P}^N)_p$ and identify $(N_M \mathbb{P}^N)_p$ with $\mathbb{C}^{N+1}/\operatorname{Cone}(\hat{T}_p M)$ then for any vector $v \in T_p M$ the vector $\mathbb{FF}^2(v)$ is given in coordinates by

$$\mathbb{FF}^2(v) = \sum_{i,j=1}^n \sum_{\mu=n+1}^N \omega_i(v)\omega_j(v) A_\mu,$$

where $v = \sum_{i=1}^n \omega_i(v) v_i \in T_p M$.

Another geometric interpretation of the second fundamental form is via the classical Meusnier–Euler Theorem.

Definition 6.31 *If $\{p(t)\}$ is any holomorphic arc in $\mathbb{P}^N$ described by a vector–valued function $A_0(t)$, then the osculating sequence is the sequence of linear spaces spanned by the following collection of vectors*

$$\{A_0(t)\},\ \{A_0(t), A_0'(t)\},\ \{A_0(t), A_0'(t), A_0''(t)\},\ \ldots$$

We then have the Meusnier–Euler Theorem

Theorem 6.32 ([GH1]) *For a tangent vector $v \in T_pM$, the normal vector $\mathbb{FF}^2(v,v) \in (N_M\mathbb{P}^N)_p$ gives the projection in $\mathbb{C}^{n+1}/\operatorname{Cone}(\hat{T}_pM)$ of the second osculating space to any curve $p(t)$, $p(0) = p$ with tangent v at $t = 0$*

Proof. We choose an arbitrary field of Darboux frames $\{A_i(t)\}$ along $p(t)$ and write

$$\frac{dA_0}{dt} = \sum_{i=1}^{n} \left(\frac{\omega_i}{dt}\right) A_i \mod A_0.$$

Recall that ω_i is a 1-form on the frame manifold, so the expression $\frac{\omega_i}{dt}$ makes sense and is equal to the value of this 1-form on the tangent vector to the curve $\{A_i(t)\}$ on the frame manifold. If we interpret $\{\omega_1, \ldots, \omega_n\}$ as a basis in the cotangent space T_p^*M then this equation shows that $\frac{\omega_i}{dt}$ is equal to the i-th coordinate of $v \in T_pM$ in the dual basis $\{v_1, \ldots, v_n\}$ of T_pM (in fact, $v_i = A_i \mod A_0$).

Differentiating further, we get

$$\begin{aligned}\frac{d^2A_0}{dt^2} &= \sum_{i=1}^{n} \left(\frac{\omega_i}{dt}\right)\left(\frac{dA_i}{dt}\right) \mod A_0, \ldots, A_n \\ &= \sum_{i,j=1}^{n} \sum_{\mu=n+1}^{N} q_{ij\mu} \left(\frac{\omega_i}{dt}\right)\left(\frac{\omega_j}{dt}\right) \mod \operatorname{Cone}(\hat{T}_pM).\end{aligned}$$

But we also have

$$\mathbb{FF}^2(v,v) = \sum_{i,j=1}^{n} \sum_{\mu=n+1}^{N} q_{ij\mu} \left(\frac{\omega_i}{dt}\right)\left(\frac{\omega_j}{dt}\right).$$

This finishes the proof. □

Finally, let us give a useful alternative coordinate description of the second fundamental form of M in $\mathbb{P}^N$. This one will be absolutely identical to the description implicitly used in the proofs of Theorems 6.1 and 6.12. For $p \in M$, we choose a homogeneous coordinate system $X_0, \ldots, X_N$ for $\mathbb{P}^N$ so that $p = (1 : 0 : \ldots : 0)$ and $\hat{T}_pM$ is spanned by the first $n+1$ coordinate vectors. If $x_1 = X_1/X_0$, …, $x_N = X_N/X_0$ is the corresponding affine coordinate system for the affine chart of $\mathbb{P}^N$ and $(z_1, \ldots, z_n)$ is any holomorphic coordinate system for M centered at p, then for $\mu = n+1, \ldots, N$ we have in the neighborhood of p that $x_\mu|_M = x_\mu(z)$ vanishes to second order at $z = 0$. Therefore,

$$x_\mu(z) = \sum_{i,j=1}^{n} q_{ij\mu} z_i z_j + (\text{higher order terms}).$$

The second fundamental form is the linear system of quadrics generated by $\sum_{i,j=1}^{n} q_{ij\nu} dz_i dz_j$.

6.4.C Fundamental Forms of Projective Homogeneous Spaces

It is interesting to explicitly compute the second fundamental form of some familiar homogeneous spaces.

Example 6.33 ([GH1]) Suppose that $\mathbb{P}^m \times \mathbb{P}^n \subset \mathbb{P}^{nm+n+m}$ is the Segre embedding. Recall that if $\mathbb{P}^m = \mathbb{P}(V)$, $\mathbb{P}^n = \mathbb{P}(W)$ then the image M of the Segre variety is the image of the natural inclusion $\mathbb{P}(V) \times \mathbb{P}(W) \subset \mathbb{P}(V \otimes W)$. Choosing coordinates $\{X_\alpha\}$ in V and $\{Y_\mu\}$ in W, this inclusion is given by mapping $(\{X_\alpha\}, \{Y_\mu\}) \to \{X_\alpha Y_\mu\}$. We denote by $\{A_0, \ldots, A_m\}$ and $\{B_0, \ldots, B_n\}$ frames for $\mathbb{P}(V)$ and $\mathbb{P}(W)$. We shall use the range of indices $1 \le i, j \le m$, $1 \le \alpha, \beta \le n$. If A_0 lies over $v_0 \in \mathbb{P}(V)$ and B_0 over $w_0 \in \mathbb{P}(W)$, then the frame

$$\{A_0 \otimes B_0, A_0 \otimes B_\alpha, A_i \otimes B_0, A_i \otimes B_\alpha\} \tag{6.2}$$

in $\mathbb{P}(V \otimes W)$ lies over $v_0 \otimes w_0$. If we write

$$dA_0 = \sum_i \phi_i A_i \mod A_0, \quad dB_0 = \sum_\alpha \psi_\alpha B_\alpha \mod B_0,$$

then

$$d(A_0 \otimes B_0) = \sum_\alpha \psi_\alpha A_0 \otimes B_\alpha + \sum_i \phi_i A_i \otimes B_0 \mod A_0 \otimes B_0.$$

Therefore, (6.2) is a Darboux frame for M and

$$\hat{T}_{v_0 \otimes w_0} M = \mathbb{P}(V \otimes w_0 + v_0 \otimes W),$$

so that

$$(N_M)_{v_0 \otimes w_0} = (V \otimes W)/(V \otimes w_0 + v_0 \otimes W).$$

Differentiating further, we get

$$d^2(A_0 \otimes B_0) = 2 \sum_{i,\alpha} \phi_i \psi_\alpha A_i \otimes B_\alpha \mod \{A_0 \otimes B_0, A_i \otimes B_0, A_0 \otimes B_\alpha\}.$$

Therefore, the second fundamental form is given by

$$\mathbb{FF}^2(v_0 \otimes w + v \otimes w_0, v_0 \otimes w + v \otimes w_0) = 2v \otimes w \mod V \otimes w_0 + v_0 \otimes W.$$

In terms of homogeneous coordinates $\{\phi_i\}$ for $T_{v_0}\mathbb{P}^m$ and $\{\psi_\alpha\}$ for $T_{w_0}\mathbb{P}^n$, $|\mathbb{FF}^2|$ is the linear system of quadrics

$$\sum_{i,\alpha} q_{i\alpha} \phi_i \psi_\alpha$$

for any matrix $q_{i\alpha}$. In other words, matrices of quadrics from this linear system have the form

$$\begin{pmatrix} 0 & Q \\ Q^t & 0 \end{pmatrix},$$

where Q is an arbitrary matrix.

Summarizing, we see that the projectivization of the tangent space at any point $x \in M$ is $\mathbb{P}^{n+m-1}$ and $|\mathbb{FF}^2|_x$ is the complete linear system of quadrics having as a base locus the union $\mathbb{P}^{m-1} \cap \mathbb{P}^{n-1}$ of two skew linear subspaces.

Example 6.34 ([GH1]) Suppose that $\mathbb{P}^n = \mathbb{P}(V) \subset \mathbb{P}(S^d V)$ is the d-th Veronese embedding. In terms of homogeneous coordinates $\{X_0, \ldots, X_n\}$ for $\mathbb{P}(V)$ the embedding is given by

$$(\ldots : X_\alpha : \ldots) \mapsto (X_0^d : X_0^{d-1} X_1 : \ldots : X_0^{d-1} X_n : \ldots : F_\lambda(X) : \ldots),$$

where F_λ varies over a basis for the homogeneous forms of degree d. Given a point $v_0 \in \mathbb{P}^n$, e.g., $v_0 = (1 : \ldots : 0)$, v_0 maps to $(1 : \ldots : 0)$; the projectivization of the tangent space at v_0 is $\mathbb{P}^{n-1}$ spanned by first n coordinate vectors in $\mathbb{P}(S^d V)$. The affine coordinate system in the affine chart containing v_0 is given by $F_\lambda(X)/X_0^d = F_\lambda(X_1/X_0, \ldots, X_n/X_0)$. Therefore, $|\mathbb{FF}^2|_{v_0}$ is the linear system of all quadrics.

Example 6.35 ([GH1]) Let $\mathrm{Gr}(k, V) \subset \mathbb{P}(\Lambda^k V)$ be the Grassmanian in the Plücker embedding, $\dim V = n$. Then the projectivization of the tangent space at any point x is $\mathbb{P}(\mathrm{Mat}_{k,n-k})$ and an easy calculation (see [GH1]) shows that $|\mathbb{FF}^2|_x$ generated by (2×2)-minors of this matrix. Using this description, it is quite straightforward to find the defect of the Grassmanian in the Plücker embedding. All we need to do is to find the maximal possible rank of a quadric generated by (2×2)-minors of a rectangular matrix. The answer is as follows. We may suppose that $k \leq n/2$. Then the dual variety is always a hypersurface except the case $k = 1$ (when the dual variety is empty) and the case $k = 2$, n is odd (when the defect is equal to 2).

Example 6.36 ([Lan4]) Let $\mathbb{S}_m \subset \mathbb{P}(\Lambda^{ev}\mathbb{C}^m)$ be the spinor variety (c.f. Example 2.2). Then calculating the tangent space at the point $v_0 = 1$ it is easy to see that $T_{v_0}\mathbb{S}_m = \Lambda^2\mathbb{C}^m$. Let $\mathrm{Gr}(2, \mathbb{C}^m) \subset \mathbb{P}(\Lambda^2\mathbb{C}^m)$ be the Plücker embedding then the linear system $|\mathbb{FF}^2|$ consists of all quadrics containing $\mathrm{Gr}(2, \mathbb{C}^m)$.

Example 6.37 ([Lan4]) Consider the Severi variety, c.f. Example 2.3. Let $\mathbb{D}_\mathbb{R}$ denote one of four division algebras over $\mathbb{R}$: real numbers $\mathbb{R}$, complex numbers $\mathbb{C}$, quaternions $\mathbb{H}$, or octonions $\mathbb{O}$. Let $\mathbb{D} = \mathbb{D}_\mathbb{R} \otimes_\mathbb{R} \mathbb{C}$ denotes its complexification, therefore $\mathbb{D}$ is either $\mathbb{C}$, $\mathbb{C} \oplus \mathbb{C}$, $\mathrm{Mat}_2(\mathbb{C})$, or $\mathbb{C}\mathrm{a}$ (the algebra of Cayley numbers). All these algebras have the standard involution. Let $\mathcal{H}_3(\mathbb{D})$ denote the 3×3 Hermitian matrices over $\mathbb{D}$. If $x \in \mathcal{H}_3(\mathbb{D})$ then we may write

$$x = \begin{pmatrix} c_1 & u_1 & u_2 \\ \overline{u_1} & c_2 & u_3 \\ \overline{u_2} & \overline{u_3} & c_3 \end{pmatrix}, \quad c_i \in \mathbb{C},\ u_i \in \mathbb{D}.$$

The projectivization of the highest weight vector orbit with respect to the group of norm similarities is called the Severi variety corresponding to $\mathcal{D}$. One may think of Severi varieties as of complexifications of projective planes over $\mathbb{D}_{\mathbb{R}}$. Take a point p of the Severi variety of the form

$$p = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Choose the affine coordinates $u_1, u_2, u_3 \in \mathbb{D}$, $c_2, c_3 \in \mathbb{C}$ of $\mathcal{H}_3(\mathbb{D})$ centered at p. Then the tangent space at p is $\{u_1, u_2\}$ and the second fundamental form is generated by quadrics

$$u_1\overline{u_1} = 0,\ u_2\overline{u_2} = 0,\ \overline{u_2}u_1 = 0.$$

For example, if $\mathbb{D} = \mathbb{C}a$ then the base locus of this linear system is the spinor variety $\mathbb{S}_5$ (c.f. Example 2.21).

The higher fundamental forms are defined similar to the second fundamental form $\mathbb{FF}^2$. The most geometric way is to utilize the Euler–Meusnier Theorem 6.32 as the definition.

Recall that if $\{p(t)\}$ is any holomorphic arc in $\mathbb{P}^N$ described by a vector–valued function $A_0(t)$, then the osculating sequence at $p = p(0)$ is the sequence of linear spaces spanned by the following collection of vectors

$$\{A_0(t)\},\ \{A_0(t), A_0'(t)\},\ \{A_0(t), A_0'(t), A_0''(t)\},\ \ldots$$

Suppose now that $M \subset \mathbb{P}^N = \mathbb{P}(V)$ is a smooth variety, $p \in M$, $\tilde{T}_pM \subset V$ is the cone over the embedded tangent space $\hat{T}_pM$. Then the sequence of osculated spaces $T_p^{(0)}M = \mathbb{C}p$, $T_p^{(1)}M = \tilde{T}_pM$, $T_p^{(2)}M$, $\ldots$ is defined as follows. The osculating space $T_p^{(k)}M$ is the linear span of k-th osculating spaces for smooth curves $p(t) \subset M$, $p(0) = p$. The k-th normal space $N_p^{(k)}M$ is defined as $V/T_p^{(k)}M$, so we have $N_p^{(1)}M = (N_M\mathbb{P}^N)_p$. Suppose that $v \in T_pM = T_p^{(1)}M/T_p^{(0)}M$ is any tangent vector, $\{p(t)\}$ is any holomorphic arc in $\mathbb{P}^N$ with tangent vector v at $t = 0$. Let $A(t) \subset V$ be a curve projected to $p(t)$. Then it is easy to see that the vector

$$\mathbb{FF}^k(v) = \frac{d^kA(t)}{dt^k} \mod T_p^{(k-1)}M \in N_p^{(k-1)}M$$

depends only on v. This construction gives a map

$$\mathbb{FF}^k : S^kT_pM \to N_p^{(k-1)}M$$

called the k-th projective fundamental form. Of course, it can be also described via moving frames, in coordinates, by differentiating Gauss map, etc., see e.g. [GH1, Lan4]. The sequence of dimensions of osculating spaces $T_p^{(k)}M$ can, of course, depend on $p \in M$. This will not be very essential, first, because we can restrict ourselves to an open subset of M where these dimensions are fixed. Secondly, we shall be interested only in projective homogeneous varieties in equivariant embeddings, in which case all points have the same rights.

Example 6.38 It is easy to see that if $M \subset \mathbb{P}^N$ is a hypersurface of degree d, $p \in M$ is a smooth point, and $l \subset \mathbb{P}(T_p^{(d)}M)$ is a line passing through p then $l \subset M$. More generally, if $M \subset \mathbb{P}^N$ is a projective variety whose homogeneous ideal is generated by elements of degree at most d, $p \in M$ is a smooth point, and $l \subset \mathbb{P}(T_p^{(d)}M)$ is a line passing through p then $l \subset M$. If V is an irreducible module of a simple group G and $M = G/P \subset \mathbb{P}(V)$ is the projectivization of the orbit of the highest weight then its ideal is generated in degree two (see e.g. [Li]). In particular, for any $p \in M$ $\mathrm{Base}\,|\mathbb{FF}^2|_p$ is the set of tangent directions to lines on M through p. Using this idea, in [LM] the thorough study of varieties of lines and linear subspaces on minimal equivariant projective embeddings of minimal flag varieties was undertaken.

Let G be a connected semisimple complex Lie group, $\mathfrak{g}$ its Lie algebra, $U\mathfrak{g}$ the universal enveloping algebra of $\mathfrak{g}$. $U\mathfrak{g}$ has a natural filtration $U\mathfrak{g}^{(k)}$ such that the associated graded algebra is the symmetric algebra $S\mathfrak{g}$ of $\mathfrak{g}$.

Let V_λ be an irreducible G-module with highest weight λ, and $v_\lambda \in V_\lambda$ a highest weight vector. The action of $U\mathfrak{g}$ on V_λ induces a filtration of V_λ whose k-th term is $V_\lambda^{(k)} = U\mathfrak{g}^{(k)}v_\lambda$.

Let $x = [v_\lambda] \in \mathbb{P}(V_\lambda)$ be the line in V_λ generated by v_λ, and let $X = G/P \subset \mathbb{P}(V_\lambda)$ be the projectivization of the orbit of the highest weight vector. Here $P \subset G$ is a parabolic subgroup, namely, the stabilizer of x. Let $\mathfrak{p}$ be the Lie algebra of P. The tangent bundle TX is a homogeneous vector bundle and we can canonically identify T_xX with $\mathfrak{g}/\mathfrak{p}$ as a P-module. Then it is easy to verify that the osculating spaces and the fundamental forms of X have a following simple representation-theoretic interpretation.

Proposition 6.39 *Let $X = G/P \subset \mathbb{P}(V_\lambda)$ be a polarized flag variety and let $x = eP \in X$. Then the k-th osculating space $T_x^{(k)}X = V_\lambda^{(k)}$, therefore $N_x^{(k)}X = V_\lambda/V_\lambda^{(k)}$. Moreover, there is a commutative diagram*

$$\begin{array}{ccccc} S^k\mathfrak{g} & = & U\mathfrak{g}^{(k)}/U\mathfrak{g}^{(k-1)} & & \\ {\scriptstyle\pi_1}\downarrow & & \downarrow{\scriptstyle\pi_2} & & \\ S^k(T_xX) = S^k(\mathfrak{g}/\mathfrak{p}) & \xrightarrow[\mathbb{FF}^k]{} & V_\lambda/V_\lambda^{(k)} & = & N_x^{(k-1)}X \end{array}$$

where π_1 is induced by the projection $\mathfrak{g} \to \mathfrak{g}/\mathfrak{p}$ and π_2 is the map given by $u \mapsto u \cdot v_\lambda$.

7. The Degree of a Dual Variety

Preliminaries

Let $X \subset \mathbb{P}(V)$ be a smooth projective variety, $\mathcal{L} = \mathcal{O}_X(1)$. In this chapter we review several approaches for finding the degree of the dual variety X^*. In Section 7.1 we give several applications of the Katz–Kleiman Holme formula. In Section 7.2 we collect various calculations of degrees of discriminants of polarized flag varieties. In final sections we study the degree of the discriminant of a polarized variety as a function at polarization and express the degree in terms of Hilbert polynomials.

7.1 Katz–Kleiman–Holme Formula

For any vector bundle E on X we denote by $c_i(E)$ its i-th Chern class (see e.g. [Fu1]). For example, $c_1(\mathcal{L}) = H$, the hyperplane section divisor of X in $\mathbb{P}(V)$. For any 0-dimensional cycle Z on X we denote its degree ('the number of points' in Z) by $\int_X Z$. Let $J(\mathcal{L})$ be the first jet bundle of $\mathcal{L}$. Let $\dim X = n$.

Theorem 7.1 ([BFS1])

(a) $\deg \Delta_X = \int_X c_n(J(\mathcal{L}))$. *In particular,* X^* *is a hypersurface if and only if* $c_n(J(\mathcal{L})) \neq 0$.

(b) *Moreover,* $\operatorname{def} X = k$ *if and only if* $c_r(J(\mathcal{L})) = 0$ *for* $r \geq n-k+1$, *and* $c_{n-k}(J(\mathcal{L})) \neq 0$. *In this case*

$$\deg X^* = \int_X c_{n-k}(J(\mathcal{L})) \cdot H^k.$$

Sketch of the proof. Suppose that $\operatorname{def} X = k$. Notice that jets $j(f)$, $f \in V^*$, span $J(\mathcal{L})$. We choose generic elements $f_1, \ldots, f_v \in V^*$, where $v = \dim V^*$. Using the classical representation of a Chern class of a spanned vector bundle (see e.g. [Fu1]), we have that $c_{n-k+1}(J(\mathcal{L}))$ is represented by the set

$$\{x \in X \mid j(f_1)(x), \ldots, j(f_{k+1})(x) \text{ are linearly dependent}\}.$$

Since $\operatorname{codim}_{\mathbb{P}(V^*)} X^* = k+1$, generic k-dimensional projective subspace in $\mathbb{P}(V^*)$ does not meet X^*. Therefore, any non-trivial linear combination of $f_1, \ldots, f_{k+1}$ does not belong to the $\operatorname{Cone}(X^*)$. But this is equivalent to

$$\dim \operatorname{Span}\{j(f_1)(x), \ldots, j(f_{k+1})(x)\} = k+1$$

for any $x \in X$. This implies by the above representation that $c_{n-k+1}(J(\mathcal{L}) = 0$ and moreover that $c_r(J(\mathcal{L})) = 0$ for $r \geq n-k+2$. Now, the Chern class $c_{n-k}(J(\mathcal{L}))$ is represented by the set

$$\{x \in X \mid j(f_1)(x), \ldots, j(f_{k+2})(x) \text{ are linearly dependent}\}.$$

This set is a union of the singular loci of the hyperplane sections $H \cap X$, where $H \in X^* \cap L$ for a generic $(k+1)$-dimensional subspace $L \subset \mathbb{P}(V^*)$. Notice that the number of these hyperplane sections is equal to the degree of the dual variety $\deg X^*$. Moreover, by dimension reasons we may suppose that $X^* \cap L \subset X^*{}_{sm}$. By Theorem 1.12 these singular loci are projective subspaces of dimension k. Therefore $c_{n-k}(J(\mathcal{L}))$ is represented by $\deg X^*$ k-dimensional projective subspaces. In particular, $c_{n-k}(J(\mathcal{L})) \neq 0$ and $\deg X^* = \int_X c_{n-k}(J(\mathcal{L})) \cdot H^k$. □

Using the exact sequence (3.1) it is possible to substitute Chern classes of the jet bundle by Chern classes of the cotangent bundle $\Omega^1_X = T^*_X$. The resulting formula was found by N. Katz and S. Kleiman [Ka, Kl1] in the case where X^* is a hypersurface and by A. Holme [Ho1, Ho2] in the general case. Consider the Chern polynomial of X with respect to the given projective embedding

$$c_X(q) = \sum_{i=0}^{n} q^{i+1} \int_X c_{n-i}(\Omega^1_X) \cdot H^i.$$

Theorem 7.2

(a) $\deg \Delta_X = c'_X(1) = \sum_{i=0}^{n}(i+1) \int_X c_{n-i}(\Omega^1_X) \cdot H^i$.
(b) *The codimension of X^* equals the order of the zero at $q = 1$ of the polynomial $c_X(q) - c_X(1)$. If this order is μ then* $\deg X^* = c_X^{(\mu)}(1)/\mu!$.

The formula in (a) can be rewritten in a numerous number of ways, sometimes more convenient for calculations. The proofs of the following formulas can be found in [Kl1].

Example 7.3 ([Kl1])

$$\deg \Delta_X = \int_X s_n(N_X \mathbb{P}^N(-1)),$$

where s_n is the n-th Segre class (see e.g. [Fu1]).

Example 7.4 ([Kl1])

$$\deg \Delta_X = \int_X \frac{1}{c(N^*_X \mathbb{P}^N(1))}.$$

For example, if X is a hypersurface of degree d in $\mathbb{P}^N$, then $N_X^*\mathbb{P}^N(1) = \mathcal{O}_X(1-d)$ and

$$\deg \Delta_X = \int_X \frac{1}{1-(d-1)H} = \int_X (d-1)^n H^n = d(d-1)^n.$$

In particular, X^* is a hypersurface (if X is not a hyperplane). In the same way one can show that X^* is a hypersurface if X is a non-degenerate smooth complete intersection (c.f. Theorem 4.2).

Example 7.5 ([Kl1, Lam]) The following non-trivial consequence of Theorem 7.2 is called the class formula.

$$\deg \Delta_X = (-1)^n \left[\chi(X) - 2\chi(X \cap H) + \chi(X \cap H \cap H')\right],$$

where H and H' are generic hyperplanes (such that $X \cap H$ and $X \cap H \cap H'$ are smooth) and χ is a topological Euler characteristic. This formula is the main ingredient in the proof of the following Landman formula (see [Lan, Kl3]):

$$\begin{aligned}\deg \Delta_X = \big(b_n(X) - b_{n-2}(X)\big) + 2\big(b_{n-1}(X \cap H) - b_{n-1}(X)\big)\\ +\big(b_{n-2}(X \cap H \cap H') - b_{n-2}(X)\big),\end{aligned}$$

where b_i is the i-th Betti number. All three summands are nonnegative numbers due to the strong and weak Lefschetz theorems. Hence the characteristic condition for the positive defect is

$$b_n(X) = b_{n-2}(X),\ b_{n-1}(X \cap H) = b_{n-1}(X),\ b_{n-2}(X \cap H \cap H') = b_{n-2}(X).$$

Example 7.6 ([Kl1]) Let $X \subset \mathbb{P}^2$ be a smooth plane curve of degree d. Consider the Veronese embedding $\mathbb{P}^2 \subset \mathbb{P}^N$, $N+1 = \binom{r+2}{2}$, and the corresponding embedding $X \subset \mathbb{P}^N$. Then the dual variety $X^* \subset \mathbb{P}^{N*}$ parametrizes plane curves of degree r tangent to X at some point. It is easy to see that

$$\deg \Delta_X = d(d+2r-3).$$

For example, the variety of conics tangent to a given conic is a hypersurface in $\mathbb{P}^5$ of degree 6.

Example 7.7 Suppose that $X \subset \mathbb{P}^N$ is a smooth projective curve of genus g and degree d. Let $H \subset X$ be a hyperplane section. Then X^* is a hypersurface by Example 1.13, and its degree is given by

$$\deg \Delta_X = \deg K_X + 2\deg H = 2g - 2 + 2d,$$

where K_X denotes the canonical divisor.

Example 7.8 Let $X \subset \mathbb{P}^N$ be a smooth projective surface, $H \subset X$ be a smooth hyperplane section. Then X^* is a hypersurface by Example 6.16, and its degree is given by

$$\deg \Delta_X = \deg c_2(\Omega^1_X) + 2K_X \cdot H + 3H \cdot H = \chi(X) - 2\chi(H) + H \cdot H,$$

where χ is the topological Euler characteristic.

Example 7.9 Suppose that $X = \mathbb{P}^n = \mathbb{P}(\mathbb{C}^{n+1})$ is embedded into $\mathbb{P}(S^d\mathbb{C}^{n+1})$ via the Veronese embedding. Then elements of X^* parametrize hypersurfaces with singularities of degree d in $\mathbb{P}^n$, therefore Δ_X is the classical discriminant of homogeneous forms of degree d in $n+1$ variables. Let us find its degree. The Chow ring (or the cohomology ring) of X is equal to $\mathbb{Z}(t)/t^{n+1}$, the element t being the class of the hyperplane. The first Chern class of the line bundle corresponding to the Veronese embedding is equal to dt. The total Chern class $\sum_i c_i(\Omega^1_{\mathbb{P}^n})$ equals $(1-t)^{n+1}$, therefore $c_i(\Omega^1_{\mathbb{P}^n}) = (-1)^i\binom{n+1}{i}t^i$. Clearly $\int_X t^n = 1$. So we have

$$c_X(q) = \sum_{i=0}^{n} q^{i+1}(-1)^{n-i}\binom{n+1}{i+1}d^i = \frac{(qd-1)^{n+1} - (-1)^{n+1}}{d}.$$

Therefore, $\deg \Delta_X = c'_X(1) = (n+1)(d-1)^n$. This formula is known as a Boole formula.

A little bit more explicit form of Theorem 7.2 can be given as follows.

Definition 7.10 *Let X be a n-dimensional smooth projective variety in $\mathbb{P}^N$, $\mathcal{L} = \mathcal{O}_X(1)$. Then the s-th rank of X with respect to the projective embedding is defined as*

$$\delta_s = \sum_{i=s}^{n} \binom{i+1}{s+1} e_{n-i},$$

where e_j denotes the degree of $c_j(T^*X)$ with respect to the given projective embedding.

Then Theorem 7.2 can be reformulated as follows

Corollary 7.11 *The defect* $\operatorname{def} X$ *is the smallest integer $r \geq 0$ such that $\delta_r \neq 0$, and for $r = \operatorname{def} X$ the degree of the dual variety X^* equals δ_r.*

7.2 Degrees of Discriminants of Polarized Flag Varieties

7.2.A The Degree of the Dual Variety to G/B

We follow the notation of Section 2.1. Let G be a simple algebraic group of rank r with Borel subgroup B. The minimal equivariant projective embedding

of G/B corresponds to the line bundle $\mathcal{L}_\rho$, where $\rho = \omega_1 + \ldots + \omega_r$ is a half sum of positive roots. We shall follow [CW] here and calculate the degree of the corresponding discriminant (in fact, in most cases the dual variety will be a hypersurface, see precise statements in Section 9.3).

Let $\{\beta_1, \ldots, \beta_N\} = \Delta_+$ be the collection of all positive roots and let $\{\beta_1^\vee, \ldots, \beta_N^\vee\} = \Delta_+^\vee$ be the collection of all positive coroots. We consider the matrix

$$M = \left(\frac{(\beta_i^\vee, \beta_j)}{(\beta_i^\vee, \rho)} \right)_{i,j=1,\ldots,N}.$$

Let $P_s(M)$ be the sum of all permanents of $s \times s$ submatrices of M. Recall that the permanent of a $n \times n$ matrix (a_{ij}) is equal to $\sum\limits_{\sigma \in S_n} \prod\limits_{i=1}^{n} a_{i,\sigma(i)}$.

Theorem 7.12 ([CW])

$$\deg \Delta_{(G/B, \mathcal{L}_\rho)} = \sum_{s=0}^{N} (s+1)! P_{N-s}(M).$$

Proof. Recall that by the Kleiman formula 7.2 we have

$$\deg \Delta_{(G/B, \mathcal{L}_\rho)} = \sum_{i=0}^{N} (i+1) \int_{G/B} c_{n-i}(\Omega^1_{G/B}) \cdot c_1(\mathcal{L}_\rho)^i.$$

Therefore we need to calculate these integrals. The correspondence $\lambda \mapsto \mathcal{L}_\lambda$ identifies $\mathcal{P}_\mathbb{Q}$ (the character lattice tensored by $\mathbb{Q}$) with $\mathrm{Pic}(G/B) \otimes \mathbb{Q}$. Therefore, we have a homomorphism of commutative algebras

$$c : S^\bullet(\mathcal{P}_\mathbb{Q}) \to A^\bullet(G/B),$$

where $S^\bullet(\mathcal{P}_\mathbb{Q})$ is the symmetric algebra of $\mathcal{P}_\mathbb{Q}$ and $A^\bullet(G/B)$ is the rational Chow ring. It is well-known (see e.g. [BGG]) that this homomorphism is surjective and its kernel is generated as an ideal by W-invariants $(S^\bullet(\mathcal{P}_\mathbb{Q}))^W_+$ of positive degree, where W is the Weyl group. For example,

$$c_1(\mathcal{L}_\rho) = c(\rho).$$

Using an appropriate filtration of the cotangent bundle $\Omega^1_{G/B}$ and 'the splitting principle' it is easy to see that

$$c_{n-i}(\Omega^1_{G/B}) = c(X_i), \quad X_i = \sum_{\substack{\Gamma \subset \Delta_+ \\ \#\Gamma = n-i}} \prod_{\alpha \in \Gamma} \alpha.$$

Therefore, it is sufficient to compute $c(\rho^i X_i)$.

For any coroot $\alpha^\vee \in \mathcal{P}^*_\mathbb{Q}$ we define a linear function

$$D_{\alpha^\vee}(\lambda) = \langle \alpha^\vee, \lambda \rangle.$$

This linear function extends by a Leibniz rule to the differential operator on $S^\bullet \mathcal{P}_{\mathbb{Q}}$. We consider now the differential operator D on $S^\bullet \mathcal{P}_{\mathbb{Q}}$ given by

$$D = \frac{\prod_{\alpha>0} D_{\alpha^\vee}}{\prod_{\alpha>0} (\alpha^\vee, \rho)}.$$

The operator D decreases the degree by N, so in particular we get a linear form

$$D : S^N \mathcal{P}_{\mathbb{Q}} \to \mathbb{Q}.$$

One can show (see [CW]) that for any $x \in S^N \mathcal{P}_{\mathbb{Q}}$ we have

$$D(x) = \int_{G/B} c(x).$$

Now the claim of the theorem follows by an easy calculation. □

7.2.B Degrees of Hyperdeterminants

Consider the flag variety $\mathbb{P}^{k_1} \times \ldots \times \mathbb{P}^{k_r}$ of the group $\mathrm{SL}_{k_1} \times \ldots \times \mathrm{SL}_{k_r}$. The projectively dual variety of its 'minimal' equivariant projective embedding corresponds to a nice theory of hyperdeterminants that was initiated by Cayley and Schläffli [Ca1, Ca2, Schl].

Let $r \geq 2$ be an integer, and $A = (a_{i_1 \ldots i_r})$, $0 \leq i_j \leq k_j$ be an r-dimensional complex matrix of format $(k_1+1) \times \ldots \times (k_r+1)$. The hyperdeterminant of A is defined as follows, c.f. 6.2.B. Consider the product $X = \mathbb{P}^{k_1} \times \ldots \times \mathbb{P}^{k_r}$ of several projective spaces embedded into the projective space $\mathbb{P}^{(k_1+1)\times\ldots\times(k_r+1)-1}$ via the Segre embedding. Let X^* be the projectively dual variety. If X^* is a hypersurface then it is defined by a corresponding discriminant Δ_X, which in this case is called the hyperdeterminant (of format $(k_1+1) \times \ldots \times (k_r+1)$), denoted by Det. Clearly $\mathrm{Det}(A)$ is a polynomial function in matrix entries of A invariant under the action of he group $\mathrm{SL}_{k_1+1} \times \ldots \times \mathrm{SL}_{k_r+1}$. If X^* is not a hypersurface then we set $\mathrm{Det} = 1$. Theorem 6.8 shows that X^* is a hypersurface (and, hence, defines a hyperdeterminant) if and only if $2k_j \leq k_1 + \ldots + k_r$ for $j = 1, \ldots, r$. If for one of j we have an equality $2k_j = k_1 + \ldots + k_r$ then the format is called boundary. Let $N(k_1, \ldots, k_r)$ be the degree of the hyperdeterminant of format $(k_1+1) \times \ldots \times (k_r+1)$. The proof of the following theorem can be found in [GKZ2] or [GKZ3].

Theorem 7.13

(a) *The generating function for the degrees $N(k_1, \ldots, k_r)$ is given by*

$$\sum_{k_1,\ldots,k_r \geq 0} N(k_1, \ldots, k_r) z_1^{k_1} \ldots z_r^{k_r} = \frac{1}{\left(1 - \sum_{i=2}^{r} (i-1) e_i(z_1, \ldots, z_r)\right)^2},$$

where $e_i(z_1, \ldots, z_r)$ is the i-th elementary symmetric polynomial.

(b) *The degree* $N(k_1, \ldots, k_r)$ *of the boundary format is* (*assuming that* $k_1 = k_2 + \ldots + k_r$)

$$N(k_2 + \ldots + k_r, k_2, \ldots, k_r) = \frac{(k_2 + \ldots + k_r + 1)!}{k_2! \ldots k_r!}.$$

(c) *The degree of the hyperdeterminant of the cubic format is given by*

$$N(k, k, k) = \sum_{0 \le j \le k/2} \frac{(j + k + 1)!}{(j!)^3 (k - 2j)!} \cdot 2^{k-2j}.$$

(d) *The exponential generating function for the degree* N_r *of the hyperdeterminant of format* $2 \times 2 \times \ldots \times 2$ (r *times*) *is given by*

$$\sum_{r \ge 0} N_r \frac{z^r}{r!} = \frac{e^{-2z}}{(1 - z)^2}.$$

7.2.C One Calculation

In most circumstances, known formulas for the degree of the discriminant depend on the certain set of discrete parameters. There are two possibilities for these parameters. First, we may fix a projective variety and vary its polarizations. The degree of the discriminant as a function in polarization is studied in the next section. Second possibility is to change a variety and to fix a polarization (in a certain sense). For example, consider the irreducible representation of SL_{n_0} with the highest weight λ Then this weight can be considered as a highest weight of SL_n for any $n \ge n_0$ with respect to the natural embedding $\mathrm{SL}_n \subset \mathrm{SL}_{n+1} \subset \ldots$. As a result, we shall obtain a tower of flag varieties with "the same" polarization. The degree of the corresponding discriminants will be a function in n. This function can be very complicated. For example, in [Las] a very involved combinatorial formula for this degree was found in the case λ is a fundamental weight (i.e. for the degree of a dual variety of the Grassmanian $\mathrm{Gr}(k, n)$ in the Plücker embedding). However, sometimes this function has a closed expression. For example, the Boole formula 7.9 is a formula of this kind. Another simple example is the formula 2.20 for the degree of the dual variety to $\mathrm{Gr}(2, n)$ in the Plücker embedding. The following theorem is the mixture of these two cases. We are not aware of any other similar formulas.

Theorem 7.14 ([T1]) *Let* V *be an irreducible* SL_n*-module with the highest weight* $(a - 1)\varphi_1 + \varphi_2$, $a \ge 2$. *Then the variety* $X^* \subset \mathbb{P}(V^*)$ *projectively dual to the projectivization* X *of the orbit of the highest vector is a hypersurface of degree*

$$\frac{(n^2 - n)a^{n+1} - (n^2 + n)a^{n-1} - 2n(-1)^n}{(a + 1)^2}.$$

Proof. We have to calculate the degree of Δ_X. We use Kleiman's formula 7.2 for the degree of the dual variety. In the present case $X = G/P$, where $G = \mathrm{SL}_n$ and $P \subset G$ is the parabolic subgroup of matrices

$$\begin{pmatrix} * & 0 & 0 & \dots & 0 \\ * & * & 0 & \dots & 0 \\ * & * & * & \dots & * \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ * & * & * & \dots & * \end{pmatrix}. \tag{7.1}$$

Assume that $T \subset G$ is the diagonal torus, B is the Borel subgroup of lower-triangular matrices, $x_1, \dots, x_n$ are the weights of the tautological representation, $X(T)$ is the lattice of characters of T, S is the symmetric algebra of $X(T)$ (over $\mathbb{Q}$), $W \simeq S_n$ is the Weil group of G, and $W_P \simeq S_{n-2}$ is the Weil group of P. It is well known (see [BGG]) that the map $c\colon X(T) \to \mathrm{Pic}(G/B)$ that assigns to λ the first Chern class of the invertible sheaf $\mathcal{L}_\lambda$ can be extended to a surjective homomorphism $c\colon S \to A^*(G/B)$ in the (rational) Chow ring, and its kernel coincides with $S_+^W S$. The projection $\alpha : G/B \to G/P$ induces an embedding $\alpha^* : A^*(G/P) \to A^*(G/B)$. The image coincides with the subalgebra of W_P-invariants. Hence, $A^*(G/P) = S^{W_P}/S_+^W S^{W_P}$. We denote the homomorphism $S^{W_P} \to A^*(G/P)$ by the same letter c.

To apply Kleiman's formula we need $c_1(\mathcal{L})$ (which is equal to $c(ax_1 + x_2)$) and the total Chern class of Ω_Z^1, which is equal to

$$c(\Omega_Z^1) = c\left((1 - x_1 + x_2) \prod_{i=3,\dots,n} (1 - x_1 + x_i) \prod_{i=3,\dots,n} (1 - x_2 + x_i) \right).$$

(This can be shown by standard arguments using the filtration of Ω_Z^1 and splitting principle.)

Let $\alpha_1, \dots, \alpha_{n-2}$ be the elementary symmetric polynomials in $x_3, \dots, x_n$. Then $S^{W_P} = \mathbb{Q}[x_1, x_2, \alpha_1, \dots, \alpha_{n-2}]$, and the ideal $S_+^W S^{W_P}$ is generated by

$$\begin{gathered} x_1 + x_2 + \alpha_1, \quad x_1x_2 + x_1\alpha_1 + x_2\alpha_1 + \alpha_2, \\ x_1x_2\alpha_1 + x_1\alpha_2 + x_2\alpha_2 + \alpha_3, \ \dots, \ x_1x_2\alpha_{n-4} + x_1\alpha_{n-3} + x_2\alpha_{n-3} + \alpha_{n-2}, \\ x_1x_2\alpha_{n-3} + x_1\alpha_{n-2} + x_2\alpha_{n-2}, \quad x_1x_2\alpha_{n-2}. \end{gathered}$$

Hence, $\alpha_i = (-1)^i(x_1^i + x_1^{i-1}x_2 + \dots + x_2^i) \mod S_+^W S^{W_P}$, and $A^*(G/P)$ is isomorphic to the quotient ring $\mathbb{Q}[x_1, x_2]/\langle f_1, f_2\rangle$, where $f_1 = x_1^{n-1} + x_1^{n-2}x_2 + \dots + x_2^{n-1}$ and $f_2 = x_1^n$. Note that f_1, f_2 is the Gröbner basis of the ideal $\langle f_1, f_2\rangle$ with respect to the ordering $x_2 > x_1$ (see [Berg]). Hence, the set of X^iY^j, where $X = x_1 \mod \langle f_1, f_2\rangle$, $Y = x_2 \mod \langle f_1, f_2\rangle$, $i = 1, \dots, n-1$, $j = 1, \dots, n-2$, is a basis of the quotient algebra.

To calculate the degree of the discriminant by Kleiman's formula, we have to calculate $\int_Z c(X^{n-1}Y^{n-2})$. Let $\tilde{w}_0$ be the longest element in W, and let w_0 be the shortest element in $\tilde{w}_0 W_P$ with the reduced factorization

$$w_0 = \begin{pmatrix} 1 & 2 & 3 & 4 & \dots & n \\ n & n-1 & 1 & 2 & \dots & n-2 \end{pmatrix}$$
$$= (n-1,n)(n-2,n-1)\dots(12)\cdot(n-1,n)(n-2,n-1)\dots(23).$$

Let

$$A_{w_0} = A_{(n-1,n)}A_{(n-2,n-1)}\dots A_{(12)}A_{(n-1,n)}A_{(n-2,n-1)}\dots A_{(23)}$$

be the corresponding endomorphism of degree $-(2n-3)$ in S, where

$$A_{(ij)} = \frac{id - s_{(ij)}}{x_i - x_j},$$

and $s_{(ij)}$ is the reflection that transposes x_i and x_j. Then

$$\int_Z c(X^{n-1}Y^{n-2}) = A_{w_0}(x_1^{n-1}x_2^{n-2})$$

(see [BGG]). It is obvious that

$$A_{w_0}(x_1^{n-1}x_2^{n-2}) = A_{\rho_1}A_{\rho_2}(x_1^{n-1}x_2^{n-2})$$
$$= A_{\rho_1}(x_1^{n-1}A_{\rho_2}(x_2^{n-2})) = A_{\rho_1}(x_1)A_{\rho_2}(x_2),$$

where $\rho_k = (n-1,n)(n-2,n-1)\dots(k,k+1)$. These factors are both equal to 1, since they are equal to $\int_{\mathbb{P}^{n-1}} c_1(\mathcal{O}(1))^{n-1}$ and $\int_{\mathbb{P}^{n-2}} c_1(\mathcal{O}(1))^{n-2}$, respectively, and it is obvious that these integrals are equal to 1. We finally obtain that $\int_Z c(X^{n-1}Y^{n-2}) = 1$.

It remains to calculate the polynomial

$$\sum_{i=0}^{2n-3}(i+1)c_{2n-3-i}(aX+Y)^i, \tag{7.2}$$

in the ring $\mathbb{Q}[X,Y]$ (with the basis X^iY^j, $i = 1,\dots,n-1$, $j = 1,\dots,n-2$, and relations $X^{n-1}+X^{n-2}Y+\dots+Y^{n-1} = 0$, $Y^n = 0$, and $X^n = 0$, which follows from the preceding relations), where c_k is the kth homogeneous component of the polynomial

$$(1-X+Y)\prod_{i=3}^{n}(1-X+x_i)\prod_{i=3}^{n}(1-Y+x_i),$$

in which the ith symmetric function of $x_3,\dots,x_n$ must be replaced by $(-1)^i(X^i + X^{i-1}Y + \dots + Y^i)$. By the previous discussion, the result of this calculation will be $\deg(\Delta_X)X^{n-1}Y^{n-2}$.

Note that the polynomial 7.2 is equal to

$$F'(T)|_{T=2X+Y} = (F_1F_2F_3F_4)'(T)|_{T=2X+Y},$$

where

$$F_1 = T, \quad F_2 = T - X + Y,$$
$$F_3 = \prod_{i=3}^{n}(T - X + x_i) = \sum_{i=0}^{n-2}(T - X)^{n-2-i}(-1)^i(X^i + \ldots + Y^i),$$
$$F_4 = \prod_{i=3}^{n}(T - Y + x_i) = \sum_{i=0}^{n-2}(T - Y)^{n-2-i}(-1)^i(X^i + \ldots + Y^i).$$

We have, further,

$$F_3(T) = \left(\sum_{i=0}^{n-2}(T - X)^{n-2-i}(-1)^i \frac{X^{i+1} - Y^{i+1}}{X - Y}\right) =$$
$$\frac{X(T - X)^{n-2}}{X - Y}\left(\sum_{i=0}^{n-2}(T - X)^{-i}(-1)^i X^i\right) -$$
$$\frac{Y(T - X)^{n-2}}{X - Y}\left(\sum_{i=0}^{n-2}(T - X)^{-i}(-1)^i Y^i\right) =$$
$$\frac{X\left((T - X)^{n-1} - (-X)^{n-1}\right)}{T(X - Y)} - \frac{Y\left((T - X)^{n-1} - (-Y)^{n-1}\right)}{(T - X + Y)(X - Y)}.$$

We obtain, likewise, that

$$F_4(T) = \left(\sum_{i=0}^{n-2}(T - Y)^{n-2-i}(-1)^i \frac{X^{i+1} - Y^{i+1}}{X - Y}\right) =$$
$$\frac{X(T - Y)^{n-2}}{X - Y}\left(\sum_{i=0}^{n-2}(T - Y)^{-i}(-1)^i X^i\right) -$$
$$\frac{Y(T - Y)^{n-2}}{X - Y}\left(\sum_{i=0}^{n-2}(T - Y)^{-i}(-1)^i Y^i\right) =$$
$$\frac{X\left((T - Y)^{n-1} - (-X)^{n-1}\right)}{(T + X - Y)(X - Y)} - \frac{Y\left((T - Y)^{n-1} - (-Y)^{n-1}\right)}{T(X - Y)}.$$

We deduce from the latest formula that

$$F'(T) = \frac{n(X - Y)(T - X)^{n-1} + (-X)^n - (-Y)^n}{X - Y} \times$$
$$\frac{(X - Y)(T - Y)^n + T(-X)^n - (T + X - Y)(-Y)^n}{T(T + X - Y)(X - Y)} +$$
$$\frac{(X - Y)(T - X)^n + (T - X + Y)(-X)^n - T(-Y)^n}{T(X - Y)} \times$$
$$\frac{\Big(n(T + X - Y) - (T - Y)\Big)(T - Y)^{n-1} + (-X)^n}{(T + X - Y)^2} -$$
$$\frac{(X - Y)(T - X)^n + (T - X + Y)(-X)^n - T(-Y)^n}{T(X - Y)} \times$$

$$\frac{(X-Y)(T-Y)^n+T(-X)^n-(T+X-Y)(-Y)^n}{T(T+X-Y)(X-Y)}.$$

Using the elementary formulae

$$\frac{\alpha^n(\gamma-\beta)+\beta^n(\alpha-\gamma)+\gamma^n(\beta-\alpha)}{(\alpha-\beta)(\beta-\gamma)(\gamma-\alpha)}=\sum_{i+j+k=n-2}\alpha^i\beta^j\gamma^k,$$

$$\frac{\alpha^n(\gamma-\beta)+\beta^n(\alpha-\gamma)+\gamma^n(\beta-\alpha)}{(\alpha-\beta)(\gamma-\beta)}=\sum_{i=0}^{n-2}\beta^i(\alpha^{n-1-i}-\gamma^{n-1-i}),$$

$$\frac{(n(\alpha-\beta)-\alpha)\alpha^{n-1}+\beta^n}{(\alpha-\beta)^2}=\sum_{i=0}^{n-2}(n-1-i)\alpha^{n-2-i}\beta^i,$$

we obtain

$$\begin{aligned}F'(T)=&\left(n(T-X)^{n-1}+(-1)^n\sum_{i=0}^{n-1}X^{n-1-i}Y^i\right)\\&\times\left(\sum_{i+j+k=n-2}(T-Y)^i(-X)^j(-Y)^k\right)+\\&\left(\sum_{i=0}^{n-2}(-X)^i((T-X)^{n-1-i}-(-Y)^{n-1-i})\right)\\&\times\left(\sum_{i=0}^{n-2}(n-1-i)(T-Y)^{n-2-i}(-X)^i\right)-\\&\left(\sum_{i=0}^{n-2}(-X)^i((T-X)^{n-1-i}-(-Y)^{n-1-i})\right)\\&\times\left(\sum_{i+j+k=n-2}(T-Y)^i(-X)^j(-Y)^k\right).\end{aligned}$$

Putting $T=aX+bY$ in the latest formula, we obtain

$$\begin{aligned}&\left(\sum_{i=0}^{n-1}\Big(1+(-1)^n n(a-1)^{n-1-i}b^i\binom{n-1}{i}\Big)X^{n-1-i}Y^i\right)\\&\qquad\times\left(\sum_{i=0}^{n-2}\begin{bmatrix}n-2\\i\end{bmatrix}_{a,b-1}X^iY^{n-2-i}\right)-\\&\left(\sum_{i=0}^{n-1}\Big(\begin{bmatrix}n-1\\i\end{bmatrix}_{a-1,b}-\begin{bmatrix}n-2\\i-1\end{bmatrix}_{a-1,b}-1\Big)X^{n-1-i}Y^i\right)\\&\qquad\times\left(\sum_{i=0}^{n-2}\Big(\begin{Bmatrix}n-2\\i\end{Bmatrix}_{a,b-1}-\begin{Bmatrix}n-3\\i\end{Bmatrix}_{a,b-1}\Big)X^{n-2-i}Y^i\right)+\\&\left(\sum_{i=0}^{n-1}\Big(\begin{bmatrix}n-1\\i\end{bmatrix}_{a-1,b}-\begin{bmatrix}n-2\\i-1\end{bmatrix}_{a-1,b}-1\Big)X^iY^{n-1-i}\right)\end{aligned}$$

$$\times\left(\sum_{i=0}^{n-2}\begin{bmatrix} n-2 \\ i \end{bmatrix}_{a,b-1} X^i Y^{n-2-i}\right),$$

where

$$\begin{bmatrix} n \\ k \end{bmatrix}_{xy} = \sum_{p=0}^{k}\sum_{q=0}^{n-k}(-1)^{p+q}x^p y^q \binom{p+q}{p},$$

$$\begin{Bmatrix} n \\ k \end{Bmatrix}_{xy} = \sum_{p=0}^{k}\sum_{q=0}^{n-k}(p+q+1)(-1)^{p+q}x^p y^q \binom{p+q}{p}.$$

It remains to calculate the degree of the discriminant, that is, the difference between the coefficients of $X^{n-1}Y^{n-2}$ and $X^{n-2}Y^{n-1}$ in the expression for $F'(aX+bY)$. After some transformations we obtain

$$\sum_{i=0}^{n-1}\left((-1)^n n(a-1)^{n-1-i}b^i\binom{n-1}{i} + W_i\right)\times \left(\begin{bmatrix} n-2 \\ i \end{bmatrix}_{a,b-1} - \begin{bmatrix} n-2 \\ i-1 \end{bmatrix}_{a,b-1}\right) - \sum_{i=0}^{n-1}(W_i - 1)\times \left(\begin{Bmatrix} n-2 \\ i \end{Bmatrix}_{a,b-1} - \begin{Bmatrix} n-3 \\ i \end{Bmatrix}_{a,b-1} - \begin{Bmatrix} n-2 \\ i-1 \end{Bmatrix}_{a,b-1} + \begin{Bmatrix} n-3 \\ i-1 \end{Bmatrix}_{a,b-1}\right),$$

where

$$W_i = \begin{bmatrix} n-1 \\ n-1-i \end{bmatrix}_{a-1,b} - \begin{bmatrix} n-2 \\ n-1-i \end{bmatrix}_{a-1,b}.$$

This is the degree of the discriminant of the irreducible SL_n-module with the highest weight $(a-b)\varphi_1 + b\varphi_2$.

Substituting $b=1$ in the last formula, we obtain

$$\frac{(n^2-n)a^{n+1} - (n^2+n)a^{n-1} - 2n(-1)^n}{(a+1)^2}.$$

□

7.3 Degree of the Discriminant as a Function in $\mathcal{L}$

Suppose that X is a smooth projective variety. If $\mathcal{L}$ is a very ample line bundle on X then we call the pair $(X,\mathcal{L})$ a polarized variety. Any polarized variety admits a canonical embedding $X \subset \mathbb{P}(H^0(X,\mathcal{L})^*)$. Let $(X,\mathcal{L})^* \subset \mathbb{P}(H^0(X,\mathcal{L}))$ be the corresponding dual variety, which parametrises singular divisors in the linear system $|\mathcal{L}|$. If $(X,L)^*$ is a hypersurface then the discriminant $\Delta_{(X,\mathcal{L})}$ is a unique (up to a scalar) polynomial defining $(X,\mathcal{L})^*$. If $\mathrm{def}(X,\mathcal{L}) > 0$ then we set $\Delta_{(X,\mathcal{L})} = 1$.

7.3.A General Positivity Theorem

We start with the following simple but very useful observation

Theorem 7.15 . *Suppose that $\mathcal{L}$, $\mathcal{M}$ are very ample line bundles on X. Then* $\operatorname{def}(X, \mathcal{L} \otimes \mathcal{M}) = 0$.

Proof. Indeed, we have $\deg \Delta_{(X,\mathcal{L}\otimes\mathcal{M})} = \int_X c_n(J(\mathcal{L} \otimes \mathcal{M})$ by Theorem 7.1. Therefore,

$$\deg \Delta_{(X,\mathcal{L}\otimes\mathcal{M})} = \int_X c_n(J(\mathcal{L}) \otimes \mathcal{M}) = \sum_{i=0}^{n} (i+1) \int_X c_1(\mathcal{M})^i \cdot c_{n-i}(J(\mathcal{L})).$$

Since $J(\mathcal{L})$ is spanned, all summands are non-negative. Since $\int_X c_1(\mathcal{M})^n > 0$ (being equal to the degree of X in the embedding determined by $\mathcal{M}$), the whole sum is positive. □

For example, if X is embedded in $\mathbb{P}(V)$ and then re-embedded in $\mathbb{P}(S^d V)$ via the Veronese embedding then the dual variety X^* of this re-embedding is a hypersurface.

Suppose now that $\mathcal{L}_1, \ldots, \mathcal{L}_r$ are line bundles on X such that the corresponding linear systems $|\mathcal{L}_i|$ have no base points and such that for each i there is a representative of the linear system $|\mathcal{L}_i|$ which is a smooth divisor on X. Suppose further that any line bundle $\mathcal{L}$ of the form $\mathcal{L} = \mathcal{L}_1^{\otimes n_1} \otimes \ldots \otimes \mathcal{L}_r^{\otimes n_r}$ is very ample for positive n_i. Then the degree $\deg \Delta_{(X,\mathcal{L})}$ is a function in $n_1, \ldots, n_r$. Let us introduce new non-negative integers $m_i = n_i - 1$. Then $\deg \Delta_{(X,\mathcal{L})} = f(m_1, \ldots, m_r)$.

Theorem 7.16 ([CW])

(a) *The function f is a non-trivial polynomial with non-negative coefficients.*
(b) *If each $n_i \geq 2$ then $(X, \mathcal{L})^*$ is a hypersurface.*

Proof. Let $n = \dim X$. Recall that by Kleiman–Katz–Holme formula 7.2 we have

$$\deg \Delta_{(X,\mathcal{L})} = \sum_{i=0}^{n} (i+1) \int_X c_{n-i}(\Omega_X^1) \cdot \mathcal{L}^i = P'(\mathcal{L}),$$

where

$$P(t) = \sum_{i=0}^{n} c_{n-i}(\Omega_X^1) t^{i+1}.$$

This clearly implies that f is a polynomial in n_i and, hence, in m_i. This polynomial is non-trivial by Theorem 7.15. Therefore, we need to check positivity only. Using the Taylor expansion, it suffices to check the following claim. For each $j = 1, \ldots, n$ and for each collection of nonnegative integers $u_1, \ldots, u_r$ such that $u_1 + \ldots + u_r = j - 1$, we have

$$\int_X P^{(j)}(\mathcal{L}) \cdot \mathcal{L}_1^{u_1} \ldots \mathcal{L}_r^{u_r} \geq 0.$$

The conditions imposed on $\mathcal{L}_i$ imply that any intersection of the form $\mathcal{L}_1^{u_1} \dots \mathcal{L}_r^{u_r}$ (as an element of a Chow ring) is represented by a smooth subvariety of codimension $j-1$. Therefore, it suffices to check that for any such subvariety $Y \subset X$ of codimension $j-1$ we have

$$\int_X P^{(j)}(\mathcal{L}) \cdot [Y] \geq 0.$$

Using exact sequence 3.1 it is easy to see that

$$P^{(j)}(\mathcal{L}) = j! c_{n+1-j}(J(\mathcal{L})).$$

Therefore it remains to show that

$$\int_X c_{n+1-j}(J(\mathcal{L})) \cdot [Y] \geq 0.$$

This follows from the fact that $J(\mathcal{L})$ is spanned by its global sections (see [Fu1]). □

7.3.B Applications to Polarized Flag Varieties

The conditions of Theorem 7.16 are perfectly well satisfied if X is a flag variety G/P. We shall use notations from 2.1. The fundamental weights $\omega_1, \dots \omega_k$ dual to the simple roots in $\Pi_{G/P}$ generate the lattice $\mathcal{P}_{G/P}$ isomorphic to $\mathrm{Pic}(G/P)$. We shall denote the line bundle L_{ω_i} by $\mathcal{L}_i$. Then any line bundle on G/P have a form $\mathcal{L} = \mathcal{L}_1^{\otimes n_1} \otimes \dots \otimes \mathcal{L}_r^{\otimes n_r}$. $\mathcal{L}$ is very ample if and only if all $n_i > 0$. Then the degree $\deg \Delta_{(X,\mathcal{L})}$ is a function in $n_1, \dots, n_r$. Let us introduce new non-negative integers $m_i = n_i - 1$. Then $\deg \Delta_{(X,\mathcal{L})} = f(m_1, \dots, m_r)$. All conditions of Theorem 7.16 are clearly satisfied, therefore f is a non-trivial polynomial with non-negative coefficients.

Example 7.17 Let $V = \mathbb{C}^r$, $G = \mathrm{SL}(V)$ and let P be the stabiliser of the line in V. Then $G/P = \mathbb{P}(V)$. Take $\lambda = n\omega_1$, $n > 0$. Then $\mathcal{L}_1 = \mathcal{O}(n)$. The degree of $\Delta_{(G/P,\lambda)}$ is the degree of the classical discriminant of a homogeneous form of degree n in r variables. Therefore by Example 7.9 we have $\deg \Delta_{(G/P,\lambda)} = r(n-1)^{r-1}$. Therefore the polynomial f in this case has a form $f = rm^{r-1}$, so it is a polynomial with nonnegative coefficients.

Example 7.18 Let $G = SL_3$ and let $P = B$ be the Borel subgroup. The weight λ can be written as $\lambda = m_1\omega_1 + m_2\omega_2$. One can check using the Kleiman formula that

$$\deg \Delta_{(G/B,\lambda)} = 12(m_1^2 m_2 + m_1 m_2^2) - 6(m_1^2 + 4m_1 m_2 + m_2^2) + 12(m_1 + m_2) - 6.$$

Substituting $n_1 = m_1 - 1$, $n_2 = m_2 - 1$ we see that

$$\deg \Delta_{(G/B,\lambda)} = 12(n_1^2 n_2 + n_1 n_2^2) + 6(n_1^2 + 4n_1 n_2 + n_2^2) + 12(n_1 + n_2) + 6.$$

Again, we get a polynomial with nonnegative coefficients.

Example 7.19 Assume that the weight λ satisfies the conditions $m_i \geq 2$ (is very strictly dominant). Then the results of this section clearly imply that the dual variety $(G/P, \lambda)^*$ is a hypersurface.

7.4 Gelfand–Kapranov Formula

Theorem 3.9 identifies Δ_X with the determinant of any of the discriminant complexes $C^*_\pm(X, \mathcal{M})$, provided the complex is stably twisted. By Corollary 3.6 this implies a formula for $\deg \Delta_X$.

Theorem 7.20 ([GKZ2]) *If the complex $C^*_-(X, \mathcal{M})$ is stably twisted then*

$$\deg \Delta_X = \sum_{i=-\dim X-1}^{0} (-1)^i \cdot i \cdot \dim C^i_-(X, \mathcal{M}).$$

If the complex $C^*_+(X, \mathcal{M})$ is stably twisted then

$$\deg \Delta_X = (-1)^{\dim X+1} \sum_{i=0}^{\dim X+1} (-1)^i \cdot i \cdot \dim C^i_+(X, \mathcal{M}).$$

Thus $\deg \Delta_X$ can be expressed through the dimensions of vector spaces

$$H^0\left(X, \Lambda^{-i} J(\mathcal{L})^* \otimes \mathcal{M}\right) \text{ or } H^0\left(X, \Lambda^i J(\mathcal{L}) \otimes \mathcal{M}\right).$$

It is possible to rewrite these dimensions using simpler quantities associated with X, at least in case $\mathcal{M} = \mathcal{O}_X(l)$, $l \gg 0$. For any coherent sheaf $\mathcal{F}$ on X, we write

$$\mathcal{F}(l) = \mathcal{F} \otimes \mathcal{O}_X(l),\ h^i(\mathcal{F}) = \dim H^i(X, \mathcal{F}),\ \chi(\mathcal{F}) = \sum (-1)^i h^i(\mathcal{F}).$$

The number $\chi(\mathcal{F})$ is called the Euler characteristic of $\mathcal{F}$. It follows from the Riemann-Roch-Hirzebruch theorem [Hir] that $\chi(\mathcal{F}(l))$ is a polynomial in l. This polynomial is called the Hilbert polynomial of $\mathcal{F}$ and denoted by $h_{\mathcal{F}}(l)$. For $l \gg 0$, the higher cohomology of $\mathcal{F}(l)$ vanish and we have

$$h_{\mathcal{F}}(l) = \chi(\mathcal{F}(l) = h^0(\mathcal{F}(l)).$$

This is a more usual definition of Hilbert polynomials, see [Ha1]. Simple calculations show that Theorem 7.20 can be reformulated as follows. Recall that T_X is the tangent bundle of X and $\Omega^i_X = \Lambda^i T^*_X$ is the bundle of differential i-forms on X.

Theorem 7.21 ([GKZ2]) *For any $l \in \mathbb{Z}$ we have*

$$\deg \Delta_X = \sum_{i=-\dim X-1}^{0} (-1)^i \cdot i \cdot \left(h_{\Lambda^{-i} T_X}(i+l) + h_{\Lambda^{-i-1} T_X}(i+l)\right),$$

$$\deg \Delta_X = (-1)^{\dim X+1} \sum_{i=0}^{\dim X+1} (-1)^i \cdot i \cdot \left(h_{\Omega_X^i}(i+l) + h_{\Omega_X^{i-1}}(i+l) \right).$$

This theorem was generalized in [GK] to handle the case of the positive def X. For any $l \in \mathbb{Z}$ consider the polynomial $f_l(q)$ in a formal variable q

$$f_l(q) = \sum_{i=0}^{\dim X+1} (-1)^i \left(h_{\Lambda^i T_X}(i+l) + h_{\Lambda^{i-1} T_X}(i+l) \right) q^i,$$

where all polynomials that have no sense are supposed to be equal to 0. Clearly $f_l(q)$ is also a polynomial in l.

Theorem 7.22 ([GK, GKZ2]) *Let $l \in \mathbb{Z}_+$ be any non-negative integer. The codimension of X^* equals the order of the zero of $f_l(q)$ at $q = 1$. Let this order be μ and let $f_l(q) = a_\mu(l)(1-q)^\mu + O((1-q)^{\mu+1})$. Then $a_\mu(l) = \binom{\mu+l-1}{\mu-1} \cdot \deg X^*$.*

8. Milnor Classes and Multiplicities of Discriminants

Preliminaries

The notion of the Milnor class was introduced in [Al] generalizing Milnor numbers of isolated singularities [Mi] and Parusiǹski μ-numbers [Pa1]. We review some applications of this technique to dual varieties, in particular for the description of their singularities. We also give other results in this direction. In this chapter we assume that the reader is more familiar with modern intersection theory than is necessary for other chapters of this book. But to illustrate how the topology can be used for study of dual varieties, we give an elementary proof of the class formula in the first section (cf. Example 7.5).

8.1 Class Formula

Suppose that $X \subset \mathbb{P}^N$ is a smooth projective n-dimensional variety with the dual variety $X^* \subset (\mathbb{P}^N)^*$. Consider a generic line $L \subset (\mathbb{P}^N)^*$ such that the intersection $L \cap X^*$ consists of d smooth on X^* points if $\operatorname{def} X = 0$, where d is the degree of X^*. If $\operatorname{def} X > 0$ then this intersection is empty and we set $d = 0$.

L can be considered as a pencil (one-dimensional linear system) of divisors on X, therefore it defines a rational map $F : X \to \mathbb{P}^1$. More precisely, if L is spanned by hyperplanes H_1 and H_2 with equations $f_1 = 0$ and $f_2 = 0$ then F has a form $x \mapsto (f_1(x) : f_2(x))$ for any $x \in X$. Thus F is not defined along the subvariety $X \cap H_1 \cap H_2$ of codimension 2. We shall blow it up and get the variety $\tilde{X}$ and the regular morphism $\tilde{F} : \tilde{X} \to \mathbb{P}^1$. Let $D \subset \tilde{X}$ be a preimage of $X \cap H_1 \cap H_2$.

Now let us calculate the topological Euler characteristic $\chi(\tilde{X})$ in two ways. First,

$$\chi(\tilde{X}) = \chi(\tilde{X} \setminus D) + \chi(D) = \chi(X \setminus X \cap H_1 \cap H_2) + \chi(D) = \\ \chi(X \setminus X \cap H_1 \cap H_2) + 2\chi(X \cap H_1 \cap H_2) = \chi(X) + \chi(X \cap H_1 \cap H_2).$$

Here we use the fact that D is a $\mathbb{P}^1$-bundle over $X \cap H_1 \cap H_2$, therefore $\chi(D) = 2\chi(X \cap H_1 \cap H_2)$. Notice that the blowing up is an isomorphism on the complement of the exceptional divisor, hence $\chi(\tilde{X} \setminus D) = \chi(X \setminus X \cap H_1 \cap H_2)$.

On the other hand, we may use $\tilde{F}$ to calculate $\chi(\tilde{X})$. For $x \in \mathbb{P}^1$, we have $\chi(\tilde{F}^{-1}(x)) = \chi(X \cap H_x)$, where H_x is a hyperplane corresponding to x. Let $x_1, \ldots, x_d \in \mathbb{P}^1$ be points that correspond to the intersection of L with X^*. Then for any $x \in \mathbb{P}^1 \setminus \{x_1, \ldots, x_d\}$, $X \cap H_x$ is a smooth divisor. For any $x \in \{x_1, \ldots, x_d\}$, $X \cap H_x$ has a simple quadratic singularity (see Theorem 6.12). It is clear then that $\chi(X \cap H_x) = \chi(X \cap H_y) = \chi(X \cap H)$, where $x, y \in \mathbb{P}^1 \setminus \{x_1, \ldots, x_d\}$ and H is a generic hyperplane. For $x \in \{x_1, \ldots, x_d\}$, near the simple quadratic singularity of $X \cap H_x$, the family of divisors $X \cap H_y$, $y \to x$, looks like the family of smooth $(n-1)$-dimensional quadrics Q_ε with equations $T_1^2 + \ldots + T_{n-1}^2 = \varepsilon T_n^2$ near the unique singular point $(0 : \ldots : 0 : 1)$ of the quadric Q_0 with the equation $T_1^2 + \ldots + T_{n-1}^2 = 0$. Therefore,

$$\chi(\tilde{X}) = \chi(\mathbb{P}^1)\chi(X \cap H) + d(\chi(Q_0) - \chi(Q_1)) = 2\chi(X \cap H) + d(\chi(Q_0) - \chi(Q_1)).$$

Combining two formulas for $\chi(\tilde{X})$, we get

$$\chi(X) + \chi(X \cap H_1 \cap H_2) = 2\chi(X \cap H) + d(\chi(Q_0) - \chi(Q_1)).$$

The Euler charachteristic of k-dimensional smooth quadric is equal to $k+2$ for k even and to $k+1$ for k odd. The Euler charachteristic of a k-dimensional quadric with a unique singularity is equal to $k+1$ for k even and to $k+2$ for k odd, since it is a cone over $(k-1)$-dimensional quadric. Finally, we obtain the class formula

$$d = (-1)^n \left[\chi(X) - 2\chi(X \cap H) + \chi(X \cap H \cap H')\right].$$

This formula has a large number of applications, see Example 7.5, 7.6, 7.7, 7.8, etc. In this chapter we consider various attempts to generalise this formula in different directions.

8.2 Milnor Classes

Suppose that M is a smooth n-dimensional algebraic variety, $\mathcal{L}$ is a line bundle on M, and X is the zero-scheme of a section of $\mathcal{L}$. The singular scheme of X, $\operatorname{Sing} X$, is the scheme supported on the singular locus of X, and defined locally by the Jacobian ideal $\left(\frac{\partial F}{x_1}, \ldots, \frac{\partial F}{x_n}\right)$, where $x_1, \ldots, x_n$ are local parameters for M, and F is the section of $\mathcal{L}$ defining X.

Definition 8.1 *Let Y be the singular scheme of a section of a line bundle $\mathcal{L}$ on a smooth variety M. The Milnor class of Y with respect to $\mathcal{L}$ is the class*

$$\mu_{\mathcal{L}}(Y) = c(T^*M \otimes \mathcal{L}) \cap s(Y, M)$$

in the Chow group A_*Y of Y.

Here c denotes the total Chern class and s is the Segre class (in the sense of [Fu1]). The following theorem summarizes the main properties of the Milnor class. The reader may consult [Al] for proofs and further details.

Theorem 8.2 ([Al])

(a) *The Milnor class $\mu_{\mathcal{L}}(Y)$ only depends on Y and $\mathcal{L}|_Y$. That is, the definition of the Milnor class is independent on the choice of a smooth ambient variety M in which Y is realized as the singular scheme of a section of a line bundle restricting to $\mathcal{L}|_Y$.*

(b) *Suppose that $\mathcal{L}$ is generated by global sections and let X_g be a generic section of $\mathcal{L}$. Then*

$$\mu_{\mathcal{L}}(Y \cap X_g) = c_1(\mathcal{L}) \cap \mu_{\mathcal{L}}(Y).$$

(c) *Suppose that Y is non-singular. Then*

$$\mu_{\mathcal{L}}(Y) = c(T^*Y \otimes \mathcal{L}) \cap [Y].$$

Example 8.3 If Y is supported on a point P, then $\mu_{\mathcal{L}}(Y) = m[P]$, where m is the Milnor number of X at P. Indeed, in this case $\mu_{\mathcal{L}}(Y) = s(Y, M)$, so m is the coefficient of $[P]$ in $s(Y, M)$, which agrees with the definition of the Milnor number (see [Fu1]).

Example 8.4 Even if $\dim Y > 0$, let $|Y|$ be the support variety of Y, assume that Y is complete. Then in [Pa1] there is introduced an invariant, the μ-number of X at $|Y|$, agreeing with the ordinary Milnor number in case $|Y|$ is a point. We claim that in fact this number equals the degree of the 0-dimensional component of $\mu_{\mathcal{L}}(Y)$. Namely, the differential of the local equations of X in M determines a section of $(T^*M \otimes \mathcal{L})|_X$. This can be extended to a holomorphic section s_X of $T^*M \otimes \mathcal{L}$ over the whole of M. Then the μ-number is defined in [Pa1] as the contribution of $|Y|$ to the intersection of s_X with the zero section of $T^*M \otimes \mathcal{L}$. In the neighborhood of $|Y|$ we have the fiber diagram

$$\begin{array}{ccc} Y & \longrightarrow & M \\ \downarrow & & \downarrow s_X \\ M & \xrightarrow{s_0} & T^*M \otimes \mathcal{L} \end{array}$$

that is, Y is the scheme-theoretic intersection of the two sections, so the contribution of $|Y|$ to the intersection number of the sections is equal (see [Fu1]) the degree of the 0-dimensional component of

$$c(T^*M \otimes \mathcal{L}) \cap s(Y, M),$$

which is the claim. In fact, in view of Theorem 8.3, the numerical information carried by the Milnor class of Y (when Y is complete) is essentially equivalent to μ-numbers of $|Y|$ and its generic sections $|Y \cap X_g|$, $|X \cap X_{g_1} \cap X_{g_2}|$, ...

8.3 Applications to Multiplicities of Discriminants

We shall now apply the machinery of the previous section to our usual situation. Suppose that $X \subset \mathbb{P}^N$ is a smooth projective variety, $\mathcal{L} = \mathcal{O}_X(1)$, $X^* \subset \mathbb{P}^{N^*}$ is the dual variety parametrizing hyperplanes H such that the singular scheme (or the contact scheme in this case) $\operatorname{Sing} X \cap H$ is not empty. Then according to the previous section we have the Milnor class $\mu_{\mathcal{L}}(\operatorname{Sing} X \cap H) \in A_*(\operatorname{Sing} X \cap H)$. The problem that we address here is how to calculate the multiplicity of X^* at H in terms of the Milnor class.

Theorem 8.5 ([Al, AC, Di1, N, Pa2]) *Let $X \cap H$ be any singular hyperplane section of X. Then*

(a) *The defect of the dual variety X^* is the smallest integer $r \geq 0$ such that*

$$\int c_1(\mathcal{L})^r c(\mathcal{L}) \cap \mu_{\mathcal{L}}(\operatorname{Sing} X \cap H) \neq 0;$$

and for $r = \operatorname{def} X$ this number equals the multiplicity $m_H X^*$ of X^* at H.

(b) *If $\operatorname{def} X = r$ then $m_H X^*$ is equal to*

$$\int (-1)^r \mu_{\mathcal{L}}(\operatorname{Sing} X \cap H) + c_1(\mathcal{L})^{r+1} \cap \mu_{\mathcal{L}}(\operatorname{Sing} X \cap H).$$

Proof. (a) Induction on r. If $r = 0$ then the integral is equal to

$$\begin{aligned}\int c(\mathcal{L}) \cap \mu_{\mathcal{L}}(\operatorname{Sing} X \cap H) &= \int c(\mathcal{L}) c(T^*X \otimes \mathcal{L}) \cap s(\operatorname{Sing} X \cap H, X) \\ &= \int c((T^*X \oplus \mathcal{O}) \otimes \mathcal{L}) \cap s(\operatorname{Sing} X \cap H, X).\end{aligned}$$

Thinking of H as a point of X^*, we have $s(H, X^*) = (m_H X^*) X^*$, thus

$$m_H X^* = \deg s(H, X^*).$$

The dual variety is the projection from the conormal variety I_X, $X^* = \pi(I_X)$. Since $r = 0$, the degree of this projection is equal to 1. The fiber Y of π over H is identified with the contact locus $\operatorname{Sing} H \cap X$ (as a scheme). Therefore, we have $s(H, X^*) = \pi_*(Y, I_X)$. Taking degrees, we get

$$m_H X^* = \deg s(Y, I_X).$$

Now using the second projection $I_X \to X$, which is the projective bundle, it is quite easy to see that

$$s(Y, I_X) = c(J(\mathcal{L})) \cap s(\operatorname{Sing} X \cap H, X),$$

where $J(\mathcal{L})$ is the jet bundle of $\mathcal{L}$. It remains to notice that $c(J(\mathcal{L})) = c((T^*X \oplus \mathcal{O}) \otimes \mathcal{L})$ by the exact sequence (3.1).

Now let $r > 0$. By the similar argument as above, we have $\int c(\mathcal{L}) \cap \mu_{\mathcal{L}}(\operatorname{Sing} X \cap H) = 0$. By Theorem 1.30, if X^* is not a hypersurface then the dual variety X'^* of a generic hyperplane section $X' = X \cap H_g$ is the cone over X^* with H_g as a vertex. The multiplicity of X^* at H then evidently equals the multiplicity of X'^* at H; but $\operatorname{def} X' = \operatorname{def} X - 1$, so the statement follows by induction because

$$\begin{aligned} &\int c_1(\mathcal{L})^r c(\mathcal{L}) \cap \mu_{\mathcal{L}}(\operatorname{Sing} X \cap H) \\ &= \int c_1(\mathcal{L})^{r-1} c(\mathcal{L}) \cap (c_1(\mathcal{L}) \cap \mu_{\mathcal{L}}(\operatorname{Sing} X \cap H)) \\ &= \int c_1(\mathcal{L})^{r-1} c(\mathcal{L}) \cap \mu_{\mathcal{L}}(\operatorname{Sing} X' \cap H), \end{aligned}$$

where we used Theorem 8.2 in the last equality.

(b) By (a) we see that $\operatorname{def} X \geq r$ if and only if the components of dimension i, $0 \leq i < r$ of $c(\mathcal{L}) \cap \mu_{\mathcal{L}}(\operatorname{Sing} X \cap H)$ vanish, for all hyperplane sections $X \cap H$. That is, if $\operatorname{def} X \geq r$, then for all hyperplane sections $X \cap H$

$$c(\mathcal{L}) \cap \mu_{\mathcal{L}}(\operatorname{Sing} X \cap H) = A_{\dim \operatorname{Sing} X \cap H} + \ldots + A_r$$

with A_j a class in dimension j, $j = r, \ldots, \dim \operatorname{Sing} X \cap H$ (depending on H). Therefore,

$$\mu_{\mathcal{L}}(\operatorname{Sing} X \cap H) = c(\mathcal{L})^{-1} \cap (A_{\dim \operatorname{Sing} X \cap H} + \ldots + A_r)$$

(with $A_r \neq 0$), and

$$\int (-1)^r \mu_{\mathcal{L}}(\operatorname{Sing} X \cap H) = \sum_{i=r}^{\dim \operatorname{Sing} X \cap H} (-1)^{i-r} c_1(\mathcal{L})^i \cap A_i$$

while

$$\int c_1(\mathcal{L})^{r+1} \mu_{\mathcal{L}}(\operatorname{Sing} X \cap H) = \sum_{i=r+1}^{\dim \operatorname{Sing} X \cap H} (-1)^{i-r+1} c_1(\mathcal{L})^i \cap A_i.$$

Hence,

$$\int (-1)^r \mu_{\mathcal{L}}(\operatorname{Sing} X \cap H) + c(\mathcal{L})^{r+1} \cap \mu_{\mathcal{L}}(\operatorname{Sing} X \cap H) = \int c_1(\mathcal{L})^r \cap A_r,$$

which coincides with the formula in (a). □

In case $\operatorname{def} X = 0$ and $\operatorname{Sing} X \cap H$ is finite, the theorem takes the most simple form:

Corollary 8.6 ([Di1, N]) *If* $\operatorname{def} X = 0$, $H \in X^*$, *and* $\operatorname{Sing} X \cap H$ *is finite we have*

$$m_H X^* = \sum_{p \in \operatorname{Sing} X \cap H} \mu(X \cap H, p),$$

the sum over ordinary Milnor numbers. In particular, if $\operatorname{Sing} X \cap H$ *is finite then* H *is smooth on* X^* *if and only if* $\operatorname{Sing} X \cap H = \{p_0\}$ *and the Hessian of the function* f *defining* $X \cap H$ *is non-zero at* p_0.

Remark 8.7 An easy Chern class computation shows that ranks (see Section 7.1) could be expressed via Milnor classes as follows:

$$\delta_i = \int_X c_1(\mathcal{L})^i c(\mathcal{L}) \cap \mu_{\mathcal{L}}(X).$$

Moreover, Theorem 7.11 can be viewed as a particular case of Theorem 8.5 applied to the zero-section of $H^0(X, \mathcal{L})$.

8.4 Multiplicities of the Dual Variety of a Surface

Let X be a smooth projective algebraic surface, $\mathcal{L}$ a very ample line bundle on X, $X^* \subset |\mathcal{L}|$ the dual variety, consisting of singular members of $|\mathcal{L}|$. By Example 6.16, X^* is a hypersurface. We want to determine, for each $D \in |\mathcal{L}|$, the multiplicity $m_D X^*$ of the hypersurface X^* at the point D. Suppose that the divisor

$$D = \sum_{i=1}^{r} n_i D_i,$$

where any divisor D_i is reduced and irreducible. Then we have $D_{red} = \sum D_i$.

Theorem 8.8 ([Al]) *With notation as above, we have*

$$m_D X^* = (D - D_{red}) \cdot (K_X + 2D + D_{red}) + \sum_{p \in D_{red}} \mu(D_{red}, p),$$

where K_X *is the canonical divisor of* X *and* μ *stands for the ordinary Milnor number.*

Proof. Take a general $H \in |\mathcal{L}|$ (i.e. a H intersecting D_{red} transversally) and denote by $L \subset |\mathcal{L}|$ the pencil containing D and H. Then

$$m_D X^* = \deg X^* - s, \tag{8.1}$$

where s is the number of singular members of L different from D (each of this singular members has one node as a singular locus). In order to determine s we shall blow up X to construct a family parametrized by L and use Lefschetz's formula [GH1].

For each i $(1 \leq i \leq r)$ denote by p_{ij} $(1 \leq j \leq H \cdot D_i)$ the points of intersection of H and D_i. Let $\hat{X}$ denote the surface obtained from X by blowing up each point p_{ij} n_i times (in the direction of H). The induced pencil $\hat{L}$ on $\hat{X}$ is base point free and gives a map $f : \hat{X} \to \mathbb{P}^1$. If E_{ij}^k, $k = 1, \ldots, n_i$ are the exceptional divisors at p_{ij} then the fiber of f at the point 0 (corresponding to D) is

$$f^*(0) = D + \sum_{i,j} \sum_{1 \leq k \leq n_i - 1} (n_i - k) E_{ij}^k.$$

In other words, the special fiber is isomorphic to D with strings of $\mathbb{P}^1$'s (each $\mathbb{P}^1$ with a certain multiplicity) attached at the points p_{ij}; each string has $n_i - 1$ components.

We now denote

$$D' = (f^*(0))_{red} = D_{red} + \sum_{i,j} T_{ij}$$

the reduced fiber of f at 0, where $T_{ij} = \sum_{1 \leq k \leq n_i - 1} E_{ij}^k$ is the reduced string attached at p_{ij}. Then the standard argument involving Lefschetz's formula (see [GH1]) gives

$$\chi(X) = 2\chi(H) - H \cdot H + (\chi(D') - \chi(H)) + \sum_{1 \leq \lambda \leq s} (\chi(H_\lambda) - \chi(H)),$$

where χ denotes the topological Euler characteristic and H_λ are the singular fibers of f for $\lambda \neq 0$. Since $\chi(H_\lambda) - \chi(H) = 1$ (see (8.4) below) and $\deg X^* = \chi(X) - 2\chi(H) + H \cdot H$ (Example 7.8), we can rewrite (8.1) as

$$m_D X^* = \chi(D') - \chi(H). \tag{8.2}$$

In order to compute $\chi(D')$, denoting $T = \cap_{i,j} T_{ij}$ we have

$$\chi(D') = \chi(D_{red} \cap T) = \chi(D_{red}) + \chi(T) - \chi(D_{red} \cap T) \tag{8.3}$$

$$= \chi(D_{red}) + \sum_{i,j} \chi(T_{ij}) - \sum_{i,j} \chi(\{p_{ij}\})$$

$$= \chi(D_{red}) + \sum_{i,j} n_i - \sum_{i,j} 1 = \chi(D_{red}) + D \cdot (D - D_{red}).$$

Now we compute $\chi(D_{red})$. Let $Z = \sum_{1 \leq i \leq r} Z_i$ be a reduced connected curve with normalization

$$\rho : \tilde{Z} = \bigsqcup_{1 \leq i \leq r} \tilde{Z}_i \to Z.$$

If $p \in Z$ is a singular point, denote $B(p) = \rho^{-1}(p)$ the set of branches at p. Topologically, Z is obtained from the smooth real surface $\tilde{Z}$ by identifying each of the sets $B(p)$ to a point p. Therefore, we have

$$\chi(\tilde{Z}) - \chi(Z) = \sum_{p \in Z} (b(Z,p) - 1), \tag{8.4}$$

where $b(Z,p)$ is the number of branches of Z at p. Also, from the exact sequence of sheaves

$$0 \to \mathcal{O}_Z \to \rho_* \mathcal{O}_{\tilde{Z}} \to \rho_* \mathcal{O}_{\tilde{Z}} / \mathcal{O}_Z \to 0$$

we obtain

$$1 - r + \sum_{p \in Z} \delta(Z,p) - h^1(\mathcal{O}_Z) + h^1(\mathcal{O}_{\tilde{Z}}) = 0,$$

where we set $\delta(Z,p) = \mathrm{length}_p(\rho_* \mathcal{O}_{\tilde{Z}} / \mathcal{O}_Z)$, Combining this with (8.4) we obtain

$$\chi(Z) = 2 - 2p_a(Z) + \sum_{p \in Z} \mu(Z,p), \tag{8.5}$$

where

$$\mu(Z,p) = 2\delta(Z,p) - b(Z,p) + 1$$

is the Milnor number of (Z,p). Now writing together (8.2), (8.3), and (8.5) we get

$$\begin{aligned} m_D X^* &= \chi(D_{red}) - \chi(H) + D \times (D - D_{red}) \\ &= (2 - 2p_a(D_{red})) - (2 - 2p_a(H) + D \cdot (D - D_{red}) + \sum_{p \in D_{red}} \mu(D_{red}, p) \\ &= -(K_X + D_{red}) \cdot D_{red} + (K_X + H) \cdot H \\ &\quad + D \cdot (D - D_{red}) + \sum_{p \in D_{red}} \mu(D_{red}, p). \end{aligned}$$

After rearranging we finally obtain the claim of the Theorem. □

Example 8.9 Suppose that $X = \mathbb{P}^2$, $\mathcal{L} = \mathcal{O}_X(d)$. Then X^* parametrizes plane singular curves of degree d. If C is such curve then applying Theorem 8.8 we get

$$m_C X^* = (3(d-1) - d')\, d' + \mu,$$

where d' is the degree of $C - C_{red}$ and μ is the sum of the Milnor numbers of the singularities of C_{red}. For example, the multiplicity of the discriminant of plane conics at a double line is 2.

8.5 Further Applications to Dual Varieties

Theorem 6.15 shows that if $\operatorname{def} X > 0$ then the defect and the dimension have the same parity. Since the defect equals the dimension of a contact locus with a generic tangent hyperplane, this can be reformulated by saying

that if the generic contact locus is positive-dimensional then it has the same parity as $\dim X$. It is possible to generalize this result for not necessarily generic contact loci using the machinery of Milnor classes. The proof of the following theorem can be found in [Al].

Theorem 8.10 ([Al]) *Let $X \subset \mathbb{P}^N$ be a smooth projective variety. Let $H \in X^*$ be such that the contact locus $Y = \operatorname{Sing} X \cap H$ is pure-dimensional and non-singular in a neighborhood of some complete curve $C \subset Y$ of odd degree. Then*

$$\dim Y \equiv \dim X \mod 2.$$

For example, if Y is a generic contact locus then $Y = \mathbb{P}^{\operatorname{def} X}$. Therefore, if Y is not a point then it contains a line C and we get Theorem 6.15.

The following result follows quite immediately from Theorem 8.5:

Theorem 8.11 ([Al]) *Let $X \subset \mathbb{P}^N$ be a smooth projective variety. Suppose that the contact scheme $\operatorname{Sing} X \cap H$ of a hyperplane H with X is a projective space $\mathbb{P}^r$. Then the defect $\operatorname{def} X$ is equal to r and the dual variety X^* is smooth at H.*

Recall that by the contact scheme we mean the contact locus endowed of the scheme structure defined by the Jacobian ideal. The surprising element of this result is that we are not assuming the hyperplane to be generic. The dimension of the contact scheme of the hyperplane need not be equal the dimension of the contact locus of the general hyperplane. The claim, however, is that it is so if the contact scheme is a projective subspace as a scheme. Moreover, it can be shown (also using Theorem 8.5) that if the contact scheme of a variety with a given hyperplane is smooth then the defect of the variety equals the defect of the contact scheme. This 'explains' the previous theorem, because the defect of a projective subspace equals its dimension.

In case $\operatorname{def} X = 0$ Theorem 8.11 can be formulated a little bit more explicitly. Let $X \subset \mathbb{P}^n$ be a smooth variety with dual variety X^* being a hypersurface. Let $I_X \subset \mathbb{P}^n \times \mathbb{P}^{n*}$ be an conormal variety. We denote by π the projection $I_X \to X^*$. Let $U \subset X^*$ be the set of all points, where the projection π is unramified. Clearly, U is non-empty and open in X^*. In other words, $H \subset U$ if the contact locus $\operatorname{Sing} X \cap H$ consists of a single point p scheme-theoretically (the Hessian of the function f defining $X \cap H$ is non-zero at p). Then we have the following

Theorem 8.12 ([Ka]) *Suppose that $X \subset \mathbb{P}^N$ is smooth and X^* is a hypersurface. U consists of smooth points of X^* and is the biggest open set in X^* for which the projection $\pi : \pi^{-1}(U) \to U$ is an isomorphism.*

In fact, one may expect that $U = X_{sm}$. For example, this is so if $X^* \setminus U$ has codimension 1.

Proposition 8.13 *Suppose that $X \subset \mathbb{P}^N$ is smooth and X^* is a hypersurface. If $Z = X^* \setminus U$ is a hypersurface in X^* (possibly reducible) then $Z = X^*{}_{sing}$.*

Proof. By a previous theorem $X^*{}_{sing} \subset Z$. We need to prove an opposite inclusion. Let $Y \subset X^*$ be the variety of all points $x \in X^*$ such that $\dim \pi^{-1}(x) > 0$. Then $Y \cap U = \emptyset$, therefore $\operatorname{codim}_{X^*} Y \geq 2$. Therefore, if $Z_0 \subset Z$ is an irreducible component then for generic point $H \in Z_0$ the contact locus $\operatorname{Sing} H \cap X$ consists of finitely many points. Therefore, $H \in X^*{}_{sing}$ by Corollary 8.6. □

The description of the singular locus of the dual variety X^* is known only in some special cases. The most detailed study was performed for hyperdeterminants in [WZ]. To formulate this result, we'll need the following notations. Suppose first that $X \subset \mathbb{P}^N$ is a smooth variety. We shall define two subvarieties $X^*{}_{cusp}$ and $X^*{}_{node}$ in X^*. The variety $X^*{}_{node}$ is the closure of the set of hyperplanes H such that H is tangent to X at two distinct points. In other words, $X^*{}_{node} = \overline{\operatorname{pr}_1(I_X^2)}$, where

$$I_X^2 = \{(H, x, y) \in X^* \times X \times X \mid x \neq y,\ \hat{T}_x X, \hat{T}_y X \in H\}.$$

The variety $X^*{}_{cusp}$ consists of all hyperplanes H such that there exists a point $x \in \operatorname{Sing} X \cap H$ that is not a simple quadratic singularity. That is, the Hessian of the function f defining $X \cap H$ is equal to zero at x. Suppose now that $X = \mathbb{P}^{k_1} \times \ldots \mathbb{P}^{k_r}$ in the Segre embedding. By Theorem 6.8 X^* is a hypersurface if and only if

$$k_j \leq \sum_{i \neq j} k_i, \quad j = 1, \ldots, r.$$

In this case the defining equation of X^* is called the hyperdeterminant of the matrix format $(k_1 + 1) \times \ldots \times (k_r + 1)$. Without loss of generality we may suppose that $k_1 \geq k_2 \geq \ldots \geq k_r$. Then in fact we have only one inequality, namely $k_1 \leq k_2 + \ldots + k_r$. We call the format $(k_1 + 1) \times \ldots \times (k_r + 1)$ boundary if $k_1 = k_2 + \ldots + k_r$ and interior if $k_1 < k_2 + \ldots + k_r$. For the hyperdeterminants, the variety $X^*{}_{node}$ can be decomposed further. Namely, for any subset $J \subset \{1, \ldots, r\}$ we set $X^*{}_{node}(J) = \overline{\operatorname{pr}_1(I_X^2(J))}$, where

$$I_X^2(J) = \{(H, x, y) \in I_X^2 \mid x^{(j)} = y^{(j)} \text{ if and only if } j \in J\}.$$

The proof of the following theorem can be found in [WZ].

Theorem 8.14 ([WZ]) *Suppose that $k_1 \leq k_2 + \ldots + k_r$, $k_1 \geq \ldots \geq k_r$.*

(a) *If the format is boundary then $X^*{}_{sing}$ is an irreducible hypersurface in X^*. Furthermore, we have $X^*{}_{sing} = X^*{}_{node}(\emptyset)$ if the format is different from $4 \times 2 \times 2$. In this exceptional case $X^*{}_{sing} = X^*{}_{node}(1)$.*

(b) *If the format is interior and does not belong to the following list of exceptions, then* ${X^*}_{sing}$ *has two irreducible components* ${X^*}_{cusp}$ *and* ${X^*}_{node}(\emptyset)$*, both having codimension* 1 *in* X^**. The list of exceptional cases consists of the following* 3*- and* 4*-dimensional formats:*

(1) *For* $2 \times 2 \times 2$ *matrices* ${X^*}_{sing} = {X^*}_{cusp}$ *and has three irreducible components, all of codimension* 2 *in* X^**.*

(2) *For* $3 \times 2 \times 2$ *matrices* ${X^*}_{sing} = {X^*}_{cusp}$ *and is irreducible.*

(3) *For* $3 \times 3 \times 3$ *matrices* ${X^*}_{sing}$ *has five irreducible components* ${X^*}_{cusp}$*,* ${X^*}_{node}(\emptyset)$*,* ${X^*}_{node}(1)$*,* ${X^*}_{node}(2)$*,* ${X^*}_{node}(3)$*.*

(4) *For* $m \times m \times 3$ *matrices with* $m > 3$ ${X^*}_{sing}$ *has three irreducible components* ${X^*}_{cusp}$*,* ${X^*}_{node}(\emptyset)$*, and* ${X^*}_{node}(3)$*.*

(5) *For* $2 \times 2 \times 2 \times 2$ *matrices* ${X^*}_{sing}$ *has eight irreducible components* ${X^*}_{cusp}$*,* ${X^*}_{node}(\emptyset)$*, and* ${X^*}_{node}(i,j)$ *for* $1 \leq i < j \leq 4$*.*

(6) *For* $m \times m \times 2 \times 2$ *matrices with* $m > 2$ ${X^*}_{sing}$ *has three irreducible components* ${X^*}_{cusp}$*,* ${X^*}_{node}(\emptyset)$*, and* ${X^*}_{node}(3,4)$*.*

In each of the cases (2) *to* (6) *all irreducible components of* ${X^*}_{sing}$ *have codimension* 1*.*

9. Mori Theory and Dual Varieties

Preliminaries

One of the basic ideas of the minimal model program (see e.g. [CKM, KMM, Ko, Wi]) is to study projective varieties and their morphisms by understanding the behaviour of rational curves on these varieties. Since projective varieties with positive defect are covered by projective lines, it is not very surprising that the machinery of the minimal model program can be used to study them. This approach was implemented in a remarkable serie of papers by several authors, starting from [BSW]. In this chapter we give a survey of these results. In section 9.1 we give all necessary definitions and facts from the minimal model program. In section 9.2 we explain how to apply this machinery to study projective varieties with positive defect. Most results of this section are due to [BSW, BFS1, Sn]. We also give a list of smooth projective varieties X with positive defect such that $\dim X \leq 10$. The study of these varieties was initiated in [E1, E2], continued in [LS] and finished in [BFS1], with many contributions of other people. Though the natural framework for this theory is the class of smooth projective varieties, recently [LPS] many of these results where extended to projective varieties with mild singularities, say with smooth normalization.

Polarized flag varieties with positive defect were first classified in [KM] using Theorem 6.1. Later a much more quicker approach was suggested in [Sn]. The idea is to use the machinery of the minimal model program. In Section 9.3 we give the classification of polarized flag varieties with positive defect.

9.1 Some Results From Mori Theory

Divisors. Let X be a smooth projective variety. If H is a very ample Cartier divisor on X (hyperplane section of some projective embedding) then a pair (X, H) will be called a polarized variety. A divisor D on X is called semi-ample if the linear system $|mD|$ is base-point-free for some $m \gg 0$. Let K_X denote a canonical divisor.

We denote by $\mathrm{Div}(X)$ a free abelian group of Cartier divisors on X. If two divisors are linearly equivalent (differ by a principal divisor) it will be

denoted by $D_1 \sim D_2$. $\mathrm{Pic}(X)$ is the group of algebraic linear bundles on X. It is naturally isomorphic to a factorgroup of $\mathrm{Div}(X)$ modulo principal divisors. For any $D \in \mathrm{Div}(X)$ the corresponding line bundle will be denoted by $\mathcal{O}(D)$.

1-cycles. A 1-cycle on X is a finite sum

$$Z = \sum n_i C_i$$

of irreducible curves $C_i \subset X$ with integer multiplicities n_i. A 1-cycle Z is called effective if all $n_i \geq 0$.

The intersection product. If D is a Cartier divisor on X and $C \subset X$ is an irreducible curve then we define an intersection product $D \cdot C \in \mathbb{Z}$ as $\deg g^*\mathcal{O}(D)|_C$, where $g : \bar{C} \to C$ is a normalisation. This product can be extended by linearity to a $\mathbb{Z}$-valued pairing of free abelian groups $\mathrm{Div}(X)$ and a group of 1-cycles. This pairing is degenerate. A 1-cycle C (resp. a Cartier divisor D) is called numerically trivial if $D \cdot C = 0$ for any Cartier divisor D (resp. for any 1-cycle C). Two 1-cycles are numerically equivalent, $Z_1 \equiv Z_2$, if $Z_1 - Z_2$ is numerically trivial. Dually, two Cartier divisors are numerically equivalent, $D_1 \equiv D_2$, if $D_1 - D_2$ is numerically trivial. The group $N^1_{\mathbb{Z}}(X)$ of Cartier divisors modulo numerical equivalence is a factorgroup of the Neron–Severi group $NS(X)$ of Cartier divisors modulo algebraic equivalence, which is in turn isomorphic to a subgroup of $H^2(X, \mathbb{Z})$ by a standard argument involving the exponential exact sequence of sheaves

$$0 \to \mathbb{Z} \to \mathcal{O}_X \to \mathcal{O}_X^* \to 0.$$

Since $H^2(X, \mathbb{Z})$ is a finitely generated abelian group it follows that $N^1_{\mathbb{Z}}(X)$ is a free abelian group of finite rank $\rho(X)$, this rank is called a Picard number of X. Therefore, the group $N_1^{\mathbb{Z}}(X)$ of 1-cycles modulo numerical equivalence is also a free abelian group of rank $\rho(X)$.

The cone of effective 1-cycles. We consider two real vector spaces of dimension $\rho(X)$:

$$N_1(X) = N_1^{\mathbb{Z}}(X) \otimes \mathbb{R}, \quad N^1(X) = N^1_{\mathbb{Z}}(X) \otimes \mathbb{R}.$$

These vector spaces are naturally dual to each other by means of the intersection product. The main combinatorial object of the minimal model program is defined as follows:

Definition 9.1

- *The cone of effective* 1*-cycles* $NE_1(X) \subset N_1(X)$ *is a cone generated by classes of effective* 1*-cycles.*
- $\overline{NE}_1(X)$ *is its closure in* $N_1(X)$*.*
- *The positive part* $NE_+(X)$ *of* $\overline{NE}_1(X)$ *is defined as*

$$NE_+(X) = \{Z \in \overline{NE}_1(X) \mid K_X \cdot Z \geq 0\}.$$

Nef divisors. For $D \in N^1(X)$, we say that D is nef (or numerically effective) if $D \cdot Z \geq 0$ for any $Z \in NE_1(X)$ (or equivalently for any $Z \in \overline{NE}_1(X)$). Therefore, nef vectors form a closed cone $NE^1(X)$ in $N^1(X)$, dual to the cone $\overline{NE}_1(X)$. For example, if D is an ample divisor then for any curve $C \subset X$ the intersection number $C \cdot D > 0$. Therefore, D is nef. In fact, we have the following Kleiman criterion:

Theorem 9.2 ([Kl6]) *D is ample iff $D \cdot Z > 0$ for any $Z \in \overline{NE}_1(X)$, $Z \neq 0$.*

Therefore, ample divisors correspond to the interior of the cone $NE^1(X)$.

Clearly, all introduced cones are non-degenerate (are not contained in any hyperplane).

$\mathbb{Q}$-divisors. We define $\mathbb{Q}$-Cartier divisors as elements of a $\mathbb{Q}$-vector space $\mathrm{Div}(X) \otimes \mathbb{Q}$. The $\mathbb{Q}$-Cartier divisor is called nef if the corresponding element in $N^1(X)$ is nef. The $\mathbb{Q}$-Cartier divisor is called ample if some multiple of it is ample in a usual sense. In the sequel we shall call $\mathbb{Q}$-Cartier divisors simply divisors. This will not lead to any confusion.

Extremal rays. An extremal ray is a half line R in $\overline{NE}_1(X)$ such that

- $K_X \cdot R < 0$ ($K_X \cdot Z < 0$ for any non-zero $Z \in R$);
- R is a face of $\overline{NE}_1(X)$, i.e. for any $Z_1, Z_2 \in \overline{NE}_1(X)$ if $Z_1 + Z_2 \in R$ then Z_1 and Z_2 are in R.

An extremal rational curve C is a rational curve on X such that $\mathbb{R}_+[C]$ is an extremal ray and $-K_X \cdot C \leq \dim X + 1$. The following is a fundamental result of Mori Theory, usually referred to as the Mori Cone Theorem:

Theorem 9.3 ([Mor]) *The cone $\overline{NE}_1(X)$ is the smallest convex cone containing $NE_+(X)$ and all the extremal rays. For any open convex cone U containing $NE_+(X) \setminus \{0\}$ there are finitely many extremal rays that do not lie in $U \cup \{0\}$. Every extremal ray is spanned by an extremal rational curve.*

Another fundamental result in Mori theory is the Kawamata–Shokurov Contraction Theorem:

Theorem 9.4 ([Kaw]) *Let D be a numerically effective divisor on X. If, for some $a \geq 1$, the divisor $aD - K_X$ is nef and big (i.e. $(aD - K_X)^n > 0$), then D is semi-ample.*

One of the most useful forms of the Contraction Theorem is the following

Theorem 9.5 *Let R be an extremal ray in $\overline{NE}_1(X)$. Then there exists a normal projective variety Y and a surjective morphism with connected fibers $\mathrm{contr}_R : X \to Y$ with connected fibers such that for any curve C on X, contr_R contracts C to a point if and only if $[C] \in R$.*

Proof. Indeed, let $D \in NE^1(X)$ be such that $D \cdot R = 0$ and $D \cdot Z > 0$ for any $Z \in \overline{NE}_1(X) \setminus R$. It follows from the Mori Cone Theorem that we may assume D is a divisor. (In fact, usually this theorem is proved as a part of the proof of the Mori Cone Theorem. Then the fact that D can be chosen to be a divisor is deduced from the Rationality Theorem, see below.) The Kleiman Criterion applies that for $a \gg 0$ the divisor $aD - K_X$ is ample. Therefore, D is semi-ample by the Contraction Theorem. □

The nef value and the nef morphism. Suppose that (X, H) is a smooth polarized projective variety. Assume that K_X is not nef.

Definition 9.6

$$\tau = \min\{t \in \mathbb{R} \mid K_X + tH \text{ is nef}\}$$

is called a nef value of (X, H).

If X is fixed then τ will be called a nef value of H, or a nef value of L, where L is a line bundle corresponding to H. Clearly $0 < t < \infty$. Notice also that τ is a nef value of (X, H) if and only if $K_X + \tau H$ is nef, but not ample. The following theorem is known as the Kawamata Rationality Theorem:

Theorem 9.7 ([Kaw]) *The nef value τ is a rational number.*

The next theorem is a particular case of the Kawamata–Shokurov Contraction Theorem

Theorem 9.8 *Suppose that τ is a nef value of (X, H). The linear system $|m(K_X + \tau H)|$ is base point free for $m \gg 0$. It defines a regular morphism $\Phi : X \to Y$ onto a normal variety Y with connected fibers. Φ is called a nef value morphism.*

Length of extremal rays. If R is an extremal ray, then its length $l(R)$ is defined as follows

$$l(R) = \min\{-K_X \cdot C \mid C \text{ is a rational curve with } [C] \in R\}.$$

The motivation for this definition is to give an estimate of the dimension of locus of extremal rational curves in an extremal ray. The idea of the proof of the following theorem is, basically, due to Mori [Mor1]. In the sequel we shall need the similar argument at least 4 times (!), but the details will be left to the reader.

Theorem 9.9 ([Wis]) *Let C be a rational curve with its class in an extremal ray R, such that $-K_X \cdot C = l(R)$. If Q is a smooth point on C then the dimension of locus of points of curves that belong to R and pass through the point Q is at least $l(R) - 1$.*

Proof. By the local deformation theory the space of deformations of the morphism $f : \mathbb{P}^1 \to X$, $f(\mathbb{P}^1) = C$, has dimension at least

$$h^0(C, f^*T_X) - h^1(C, f^*T_X) = -C \cdot K_X + (1 - g(C)) \dim X = \dim X - C \cdot K_X.$$

Since we want to fix the point Q, we have $\dim X$ more restrictions. Therefore, the space of deformations of the morphism $f : \mathbb{P}^1 \to X$ fixing the point Q has dimension at least

$$-K_X \cdot C = l(R).$$

The group of automorphisms of $\mathbb{P}^1$ fixing a point $x \in \mathbb{P}^1$ has dimension 2, therefore, there exists a family of rational curves on X passing through Q of dimension at least $l(R) - 2$. All these curves are deformations of C, therefore their numerical classes belong to R. To finish the proof it remains to verify that for a generic point P in the locus of curves from our family, there exists only finitely many curves from the family passing through P. Suppose, on the contrary, that there exists a family of rational curves passing through P and Q, parametrized by some affine curve D with smooth compactification $\overline{D}$. Passing to infinite points $\overline{D} \setminus D$, we get limit positions of these curves, that are irreducible and reduced, since otherwise we shall obtain a contradiction with minimality of $-K_X \cdot C$ for rational curves from the ray R.

Therefore, there exists a ruled surface S and a morphism $F : S \to X$ such that $C_1 = F^{-1}(P)$ and $C_2 = F^{-1}(Q)$ are one-dimensional sections of this ruled surface over its base. Moreover, $\dim F(S) = 2$, otherwise all curves in our pencil represent the same 1-cycle. It follows that

$$C_1^2 < 0, \ C_2^2 < 0, \ \text{and } C_1 \cdot C_2 = 0, \tag{9.1}$$

the last equality is satisfied because C_1 and C_2 do not intersect. But this is impossible, (9.1) can not hold on a ruled surface, see e.g. [Ha1]. □

Fano varieties. A smooth variety X is called a Fano variety if its anti-canonical divisor $-K_X$ is ample. Its index is defined as follows:

$$r(X) = \max\{m > 0 \,|\, mH \sim -K_X \text{ for some } H \in \mathrm{Pic}(X)\}.$$

It is well-known that $r(X) \leq \dim X + 1$. Indeed, suppose that $K_X = mH$, where $-H$ is ample, let $\dim X = n$. The Poincare polynomial $\chi(vH)$ has at most n zeros, therefore $\chi(vH) \neq 0$ for some $1 \leq v \leq n+1$. But $\chi(vH) = \pm h^n(X, vH)$ by the Kodaira vanishing theorem and $h^n(X, vH) = h^0(X, K_X - vH)$ by the Serre duality. Therefore, $m \leq n+1$. The following theorem was conjectured by Mukai and proved in [Wis1] using the Mori theory.

Theorem 9.10 *[Wis1] Let X be a n-dimensional Fano variety of index $r(X)$. If $r(X) > \frac{1}{2} \dim X + 1$ then $\mathrm{Pic}(X) = \mathbb{Z}$.*

9.2 Mori Theory and Dual Varieties

9.2.A The Nef Value and the Defect

Suppose that (X, H) is a smooth polarized variety. Then the linear system $|H|$ defines an embedding $X \subset \mathbb{P}^N$. Recall that $X^* \subset (\mathbb{P}^N)^*$ is the dual variety and

$$\operatorname{def}(X, H) = \operatorname{codim} X^* - 1$$

is the defect of (X, H).

Theorem 9.11 *If* $\operatorname{def}(X, H) > 0$ *then* K_X *is not nef.*

Proof. By Theorem 1.12 X is covered by lines. It is well-known that if a smooth projective variety is covered by rational curves then K_X is not nef. In our case it also follows directly from Theorem 6.19 (a). □

It follows that if $\operatorname{def}(X, H)$ is positive, we can apply the technique of the previous section. It turns out that the nef value and the defect of (X, H) are connected in a very simple way.

Theorem 9.12 ([BSW, BFS1]) *Assume that* $\operatorname{def}(X) > 0$. *Then*

(a) *The nef value* τ *of* X *is equal to*

$$\tau = \frac{\dim X + \operatorname{def}(X, H)}{2} + 1.$$

(b) *If* $\operatorname{Sing} X \cap H$, $H \in X^*$, *is a generic contact locus* (*hence a projective subspace*) *and* $l \in \operatorname{Sing} X \cap H$ *is any line then* $R = \mathbb{R}_+[l]$ *is an extremal ray in* $\overline{NE}_1(X)$ *and* contr_R *coincides with the nef value morphism* Φ. *In particular,* Φ *contracts* $\operatorname{Sing} X \cap H$ *to a point.*

(c) *If* F *is a generic fiber of the nef value morphism* Φ *then* $\operatorname{Pic} F = \mathbb{Z}$.

(d)

$$\dim \Phi(X) \leq \frac{\dim X - \operatorname{def}(X, H)}{2}.$$

Notice that since

$$\dim X \equiv \operatorname{def}(X, H) \mod 2$$

by Theorem 6.15, the nef value in this case is an integer. Theorem 9.12 can be used to show that the defect of a polarized variety is equal to 0. One needs to calculate the nef value first and then to obtain a contradiction using this Theorem. Some examples will be discussed in next sections.

Proof. We denote $n = \dim X$ and $k = \operatorname{def}(X)$.

Step 1. Let $\mathbb{P}^k = \operatorname{Sing} X \cap H$, $H \in X^*$, be a generic contact locus and $l \in \operatorname{Sing} X \cap H$ be any line. Since $(-K_X) \cdot l = \dfrac{n+k}{2} + 1$ by Theorem 6.19 (a), it follows that

$$\tau \geq \frac{n+k}{2} + 1. \tag{9.2}$$

Step 2. Let Φ be the nef value morphism. Since Φ is not ample, it follows from the Kleiman Criterion and the Mori Cone Theorem that there exists an extremal rational curve C contracted by Φ, $(K_X + \tau H) \cdot C = 0$. We claim that $C \cdot H = 1$, i.e. C is a projective line. Indeed, if $C \cdot H \geq 2$ then

$$\tau \leq \frac{1}{2}\tau H \cdot C = \frac{1}{2}(-K_X) \cdot C \leq \frac{1}{2}(n+1).$$

This contradicts (9.2). Therefore,

$$\tau = (-K_X) \cdot C = l(R),$$

where $R = \mathbb{R}_+[C]$ is an extremal ray. Applying Theorem 9.9, wee see that if F is a positive-dimensional fiber of Φ then

$$\dim F \geq \tau - 1. \tag{9.3}$$

Step 3. Suppose that $\tau > \dfrac{n+k}{2} + 1$. Let F be a positive dimensional fiber of Φ, $x \in F$. By Theorem 6.21 there exists a line $l \subset X$ passing through x and such that $(-K_X) \cdot l = \dfrac{n+k}{2} + 1$. Therefore, Φ does not contract l, hence $l \not\subset F$. Let $y \in l$, $y \notin F$. Arguing as in the proof of Theorem 9.9, we see that there exists a $\dfrac{n+k}{2}$-dimensional subvariety $Y \subset X$ covered by rational curves passing through y that are deformations of l. Applying the similar argument with ruled surfaces once again, it is easy to see that $\dim Y \cap F = 0$. However, using (9.3) we get that

$$\dim Y \cap F \geq \dim Y + \dim F - \dim X \geq \frac{n+k}{2} + \frac{n+k}{2} - n > 0.$$

Contradiction. This proves (a).

Step 4. Clearly, Φ contracts any generic contact locus $\operatorname{Sing} X \cap H$, $H \in X^*$. Moreover, arguing as above (or using Theorem 6.14 (d)), we see that any fiber of Φ is at least $\dfrac{n+k}{2}$-dimensional, therefore $\dim Y \leq n - \dfrac{n+k}{2} = \dfrac{n-k}{2}$. This proves (d). Let F be a generic fiber. Then $K_F = K_X|_F = -\tau H|_F$. Therefore, F is a Fano variety of index at least

$$\tau = \frac{n+k}{2} + 1 > \frac{\dim F}{2} + 1.$$

Therefore, $\operatorname{Pic} F = \mathbb{Z}$ by Theorem 9.10. This proves (c).

Step 5. It remains to prove that Φ is a contraction of an extremal ray $\mathbb{R}_+[l]$. Suppose, on the contrary, that there exists an extremal ray R such that $l \notin R$ and Φ contracts any curve from R. Let C be an extremal rational curve from R. Arguing as in Step 2, we see that $l(R) = (-K_X) \cdot C = \tau$. Consider the morphism $\Psi = \mathrm{contr}_R$. By Theorem 9.9, there exists a fiber F of Ψ such that $\dim F \geq \tau - 1$, let $x \in F$. By Theorem 6.21 there exists a line $l \subset X$ passing through x and such that $(-K_X) \cdot l = \tau$. This line is either a line in a generic contact locus or its limit position, therefore Ψ does not contract l. Hence $l \not\subset F$. Let $y \in l$, $y \notin F$. Arguing as in the proof of Theorem 9.9, we see that there exists a $(\tau - 1)$-dimensional subvariety $Y \subset X$ covered by rational curves passing through y that are deformations of l. Applying the similar argument once again, it is easy to see that $\dim Y \cap F = 0$. But

$$\dim Y \cap F \geq \dim Y + \dim F - \dim X \geq 2\tau - 2 - n = k > 0.$$

Contradiction. □

9.2.B The Defect of Fibers of the Nef Value Morphism

Suppose now that (X, H) is a smooth polarized variety and K_X is not nef. Let $\Phi : X \to Y$ denote a nef value morphism. Then its generic fiber F is a smooth irreducible variety. It has a canonical polarization (F, H_F), where H_F is a hyperplane section of F in the embedding $F \subset X \subset \mathbb{P}^N$.

Theorem 9.13 ([BFS1]) *The following conditions are equivalent:*

(i) $\mathrm{def}(X, H) > 0$;
(ii) $\mathrm{def}(F, H_F) > \dim Y$;
(iii) $\mathrm{def}(X, H) = \mathrm{def}(F, H_F) - \dim Y > 0$.

Proof. It follows from Theorem 6.24 that

$$\dim X + \mathrm{def}(X, H) \geq \dim F + \mathrm{def}(F, H_F).$$

Therefore, (i) follows from (ii). Obviously (ii) follows from (iii). Therefore, it is sufficient to prove that (iii) follows from (i).

Since $\mathrm{def}(X, H) > 0$ we have

$$\tau(X, H) = \frac{\dim X + \mathrm{def}(X, H)}{2} + 1$$

by Theorem 9.12. Since F is a generic fiber of Φ we have

$$\mathcal{O}(K_F + \tau H_F) = \mathcal{O}(K_X + \tau H)|_F$$

is a trivial bundle. Therefore, K_F is not nef and $\tau(F, H_F) = \tau(X, H)$. If $\mathrm{def}(F, H_F) > 0$ then again by Theorem 9.12

$$\operatorname{def}(F, H_F) = \frac{\dim F + \operatorname{def}(F, H_F)}{2} + 1$$

and, therefore,

$$0 < \operatorname{def}(X, H) = -\dim X + \dim F + \operatorname{def}(F, H_F) = \operatorname{def}(F, H_F) - \dim Y.$$

It remains to show that if $\operatorname{def}(X, H) > 0$ then $\operatorname{def}(F, H_F) > 0$ as well. Let $L = \mathcal{O}(H)$, $L_F = \mathcal{O}(H_F)$. Recall that for any line bundle L on a smooth variety X we denote by $J_1(X, L)$ the vector bundle of 1-jets of L; for any vector bundle V we denote by $c_r(V)$ its r Chern class. By Theorem 7.1 we need to show that $c_{\dim F}(J_1(F, L_F)) = 0$. From the exact sequence

$$0 \to T_F^*(X) \otimes L_F \to J_1(X, L)|_F \to J_1(F, L_F) \to 0$$

we see that the total Chern class

$$c(J_1(X, L)|_F) = c(T_F^*(X) \otimes L_F) \cdot c(J_1(F, L_F)).$$

Since F is a generic fiber of Φ, $T_F^*(X) \otimes L_F = L_F + \ldots + L_F$ ($\dim Y$ copies). Therefore, $c(T_F^*(X) \otimes L_F) = (1 + H_F)^{\dim Y}$. In particular,

$$c_{\dim F}(J_1(X, L)|_F) = \\ c_{\dim F}(J_1(F, L_F)) + \sum_{i=1}^{\dim Y} \binom{\dim Y}{i} c_{\dim F - i}(J_1(F, L_F)) \cdot H_F^i.$$

All summands on the right are nonnegative since H_F is very ample and $J_1(F, L_F)$ is spanned. Therefore in order to prove that $c_{\dim F}(J_1(F, L_F)) = 0$ it remains to prove that $c_{\dim F}(J_1(X, L)|_F) = 0$.

It will be easier to prove a more general statement that $c_k(J_1(X, L)|_F) = 0$ for $k \geq \dim F - \operatorname{def}(X, H) + 1$. By the classical representation of a Chern class of a spanned vector bundle [Fu1] we see that $c_k(J_1(X, L)) = 0$ is represented by a set $C_k \subset X$ of all points where $\dim X - k + 2$ generic sections of $H^0(X, J_1(X, L))$ are not linearly independent. We may suppose that these sections have a form $j_1(s_1), \ldots, j_1(s_{\dim X - k + 2})$, where $s_1, \ldots, s_{\dim X - k + 2}$ are generic sections of $H^0(X, L)$ and j_1 is a natural map sending a germ of a section to its 1-jet. It suffices to prove that for $k \geq \dim F - \operatorname{def}(X, H) + 1$ we have $C_k \cap F = \emptyset$. Since $C_k \supset C_{k+1}$, we only need to prove that $C \cap F = \emptyset$, where $C = C_{\dim F - \operatorname{def}(X,H) + 1}$. Suppose that this intersection is non-empty. Therefore, the restriction map $\Phi_C : C \to Y$ is surjective. Since $\dim C = \dim Y + \operatorname{def}(X, H) - 1$, it follows that a generic fiber of Φ_C has dimension $\operatorname{def}(X, H) - 1$.

Suppose that $x \in C \cap F$. There exist λ_i for $1 \leq i \leq \dim Y + \operatorname{def}(X, H) - 1$, not all zero, such that $\sum_i \lambda_i j_1(s_i(x)) = 0$. Therefore $s = \sum_i \lambda_i s_i$ is a non-trivial global section of L vanishing at x with its 1-jet. Hence the divisor H_s is singular at x. By Theorem 6.21 there exists a linear subspace $\mathbb{P}^{\operatorname{def}(X,H)} \subset X$ containing x, contained in $\operatorname{Sing} H_s$, and such that $\mathcal{O}(K_X + \tau(X, H)H)|_{\mathbb{P}^{\operatorname{def}(X,H)}}$

is a trivial bundle. Since this is a trivial bundle, $\mathbb{P}^{\mathrm{def}(X,H)}$ is contracted by Φ to a point and, therefore, $\mathbb{P}^{\mathrm{def}(X,H)} \subset F$. On the other hand, since $\mathbb{P}^{\mathrm{def}(X,H)}$ is contained in $\operatorname{Sing} H_s$ it is also contained in C. Therefore, a generic fiber of Φ_C has dimension greater or equal to $\mathrm{def}(X,H)$. Contradiction. $\square$

9.2.C Varieties With Small Dual Varieties

In this section we describe without proofs smooth polarized varieties (X,H) with positive defect for $\dim X \leq 10$. First we need a couple of definitions.

Definition 9.14 *A smooth polarized variety (X,H) is called a $\mathbb{P}^d$-bundle over a normal variety Y if there exists a surjective morphism $p: X \to Y$ such that all the fibers F of p are projective subspaces $\mathbb{P}^d$ of $\mathbb{P}^N$, where $X \subset \mathbb{P}^N$ is an embedding corresponding to H.*

Definition 9.15 *A smooth polarized variety (X,H) is called a scroll (resp. a quadric, Del Pezzo, or Mukai fibration) over a normal variety Y if there exists a surjective morphism $p: X \to Y$ with connected fibers such that $K_X + (\dim X - \dim Y + 1)H \sim p^*H'$ (resp. $K_X + (\dim X - \dim Y)H \sim p^*H'$, $K_X + (\dim X - \dim Y - 1)H \sim p^*H'$, or $K_X + (\dim X - \dim Y - 2)H \sim p^*H'$) for some ample Cartier divisor H' on Y.*

To adjust this definition we need to note that it easily follows from the Kobayashi–Ochiai characterisation of projective spaces and quadrics [KO] that if (X,H) is a scroll over Y then a generic fiber F of p is a projective subspace $\mathbb{P}^d$ of $\mathbb{P}^N$, where $X \subset \mathbb{P}^N$ is an embedding corresponding to H. If (X,H) is a quadric fibration then a generic fiber F is isomorphic to a quadric hypersurface Q in some projective space and the line bundle corresponding to this embedding is $\mathcal{O}(H)|_F$. In fact, quadric fibrations will not appear in the classification because if (X,H) is a quadric fibration then $\mathrm{def}(X,H) = 0$ by [BFS1].

$\dim X \leq 2$. If $\dim X \leq 2$ then $\mathrm{def}(X,H) = 0$.

$\dim X = 3$. Suppose that $\dim X = 3$. Then $\mathrm{def}(X,H) > 0$ if an only if (X,H) is a $\mathbb{P}^2$-bundle over a smooth curve. In this case $\mathrm{def}(X,H) = 1$.

$\dim X = 4$. Suppose that $\dim X = 4$. Then $\mathrm{def}(X,H) > 0$ if an only if (X,H) is a $\mathbb{P}^3$-bundle over a smooth curve. In this case $\mathrm{def}(X,H) = 2$.

$\dim X = 5$. Let $\dim X = 5$ and $\mathrm{def}(X,H) > 0$. Then we have the following possibilities:

(a) X is a hyperplane section of the Plücker embedding of the Grassmanian $\mathrm{Gr}(2,5)$ in $\mathbb{P}^9$, $\mathrm{def}(X,H) = 1$.
(b) (X,H) is a $\mathbb{P}^4$-bundle over a smooth curve, $\mathrm{def}(X,H) = 3$.
(c) (X,H) is a $\mathbb{P}^3$-bundle over a smooth surface, $\mathrm{def}(X,H) = 1$.

$\dim X = 6$. Let $\dim X = 6$ and $\operatorname{def}(X, H) > 0$. Then either:

(a) (X, H) is the Plücker embedding of the Grassmanian $\mathrm{Gr}(2, 5)$ in $\mathbb{P}^9$, $\operatorname{def}(X, H) = 2$.
(b) (X, H) is a $\mathbb{P}^5$-bundle over a smooth curve, $\operatorname{def}(X, H) = 4$.
(c) (X, H) is a $\mathbb{P}^4$-bundle over a smooth surface, $\operatorname{def}(X, H) = 2$.

$\dim X = 7$. Let $\dim X = 7$ and $\operatorname{def}(X, H) > 0$. Then we have the following possibilities:

(a) (X, H) is a $\mathbb{P}^6$-bundle over a smooth curve, $\operatorname{def}(X, H) = 5$.
(b) (X, H) is a $\mathbb{P}^5$-bundle over a smooth surface, $\operatorname{def}(X, H) = 3$.
(c) X is a 7-dimensional Fano variety with $\operatorname{Pic} X = \mathbb{Z}$ and $K_X \sim -5H$. In this case $\operatorname{def}(X, H) = 1$. An example of such variety is the section of 10-dimensional spinor variety S in $\mathbb{P}^{15}$ by three generic hyperplanes [Mu, LS].
(d) (X, H) is a Del Pezzo fibration over a smooth curve such that the general fiber F is isomorphic to $\mathrm{Gr}(2, 5)$ and the embedding corresponding to $\mathcal{O}(H)|_F$ is the Plücker embedding in $\mathbb{P}^9$. In this case $\operatorname{def}(X, H) = 1$.
(e) (X, H) is a scroll over a 3-dimensional variety, $\operatorname{def}(X, H) = 1$.

$\dim X = 8$. Let $\dim X = 8$ and $\operatorname{def}(X, H) > 0$. Then either:

(a) (X, H) is a $\mathbb{P}^7$-bundle over a smooth curve, $\operatorname{def}(X, H) = 6$.
(b) (X, H) is a $\mathbb{P}^6$-bundle over a smooth surface, $\operatorname{def}(X, H) = 4$.
(c) X is is a 8-dimensional Fano variety with $\operatorname{Pic} X = \mathbb{Z}$ and $K_X \sim -6H$. In this case $\operatorname{def}(X, H) = 2$.
(d) (X, H) is a scroll over a 3-dimensional variety, $\operatorname{def}(X, H) = 2$.

$\dim X = 9$. Let $\dim X = 9$ and $\operatorname{def}(X, H) > 0$. Then either:

(a) (X, H) is a $\mathbb{P}^8$-bundle over a smooth curve, $\operatorname{def}(X, H) = 7$.
(b) (X, H) is a $\mathbb{P}^7$-bundle over a smooth surface, $\operatorname{def}(X, H) = 5$.
(c) X is a 9-dimensional Fano variety with $\operatorname{Pic} X = \mathbb{Z}$ and $K_X \sim -7H$. In this case $\operatorname{def}(X, H) = 3$.
(d) (X, H) is a scroll over a 3-dimensional variety, $\operatorname{def}(X, H) = 3$.
(e) X is a Fano 9-dimensional variety with $K_X \sim -6H$ and and $\operatorname{Pic} X = \mathbb{Z}$, $\operatorname{def}(X, H) = 1$.
(f) (X, H) is a Mukai fibration over a smooth curve with $\operatorname{Pic} F = \mathbb{Z}$ for a generic fiber F, $\operatorname{def}(X, H) = 1$.
(g) (X, H) is a scroll over a 4-dimensional variety, $\operatorname{def}(X, H) = 1$.

$\dim X = 10$. Let $\dim X = 10$ and $\operatorname{def}(X, H) > 0$. Then we have the following possibilities:

(a) (X, H) is a $\mathbb{P}^9$-bundle over a smooth curve, $\operatorname{def}(X, H) = 8$.
(b) (X, H) is a $\mathbb{P}^8$-bundle over a smooth surface, $\operatorname{def}(X, H) = 6$.
(c) X is a 10-dimensional Fano variety with $\operatorname{Pic} X = \mathbb{Z}$ and $K_X \sim -8H$. In this case $\operatorname{def}(X, H) = 4$.

(d) (X, H) is a scroll over a 3-dimensional variety, $\operatorname{def}(X, H) = 4$.
(e) X is a Fano 10-dimensional variety with $K_X \sim -7H$ and $\operatorname{Pic} X = \mathbb{Z}$, $\operatorname{def}(X, H) = 2$;
(f) (X, H) is a Mukai fibration over a smooth curve with $\operatorname{Pic} F = \mathbb{Z}$ for a generic fiber F, $\operatorname{def}(X, H) = 2$.
(g) (X, H) is a scroll over a 4-dimensional variety, $\operatorname{def}(X, H) = 2$.

9.3 Polarized Flag Varieties With Positive Defect

9.3.A Nef Value of Polarized Flag Varieties

We continue to use the notation from Section 2.1. Let G be a connected simply-connected semisimple complex algebraic group with a Borel subgroup B and a maximal torus $T \subset B$. Let $\mathcal{P}$ be a character group of T (the weight lattice). Let $\Delta \subset \mathcal{P}$ be a set of roots of G relative to T. To every root $\alpha \in \Delta$ we can assign the 1-dimensional unipotent subgroup $U_\alpha \subset G$. We define the negative roots Δ^- as those roots α such that $U_\alpha \subset B$. Let $\Pi \subset \Delta^+$ be simple roots, $\Pi = \{\alpha_1, \ldots, \alpha_n\}$, where $n = \operatorname{rank} G = \dim T$. The root lattice $\mathcal{Q} \subset \mathcal{P}$ is a sublattice spanned by Δ. For any simple root $\alpha_i \in \Pi$ we can associate a smooth curve $C_{\alpha_i} \subset G/B$, namely $C_{\alpha_i} = \overline{\pi(U_{\alpha_i})}$, where $\pi : G \to G/B$ is a factorization map. By linearity to any element $\alpha = \sum n_i \alpha_i \in \mathcal{Q}$ we assign a 1-cycle $C_\alpha = \sum n_i C_{\alpha_i}$ in G/B. Clearly this cycle is effective if $\alpha \in \mathcal{Q}^+$, the semigroup generated by simple roots.

$\mathcal{P}$ is generated as a $\mathbb{Z}$-module by fundamental weights $\omega_1, \ldots, \omega_n$ dual to the simple roots under the Killing form. We denote the dominant weights by $\mathcal{P}^+$ and the strictly dominant weights by $\mathcal{P}^{++}$.

The character group of B is identified with a character group of T. Therefore, for any $\lambda \in \mathcal{P}$ we can assign a 1-dimensional B-module $\mathbb{C}_\lambda$, where B acts on $\mathbb{C}_\lambda$ by a character λ. Now we can define the twisted product $G \times_B \mathbb{C}_\lambda$ to be the quotient of $G \times E$ by the diagonal action B:

$$b \cdot (g, z) = (gb^{-1}, b \cdot z).$$

Projection onto the first factor induces a map $G \times_B \mathbb{C}_\lambda \to G/B$, which realizes the twisted product as an equivariant line bundle $\mathcal{L}_\lambda$ on G/B with fiber $\mathbb{C}_\lambda$.

The following theorem is well-known:

Theorem 9.16

(1) *The correspondence $\alpha \to C_\alpha$ is an isomorphism of $\mathcal{Q}$ and $N_1^{\mathbb{Z}}(G/B)$, the group of 1-cycles modulo numerical equivalence.*
(2) *The correspondence $\lambda \to L_\lambda$ is an isomorphism of $\mathcal{P}$ and $\operatorname{Pic}(G/B)$. The latter group is, in turn, isomorphic to $N_Z^1(G/B)$, the group of Cartier divisors modulo numerical equivalence.*

(3) *The pairing* $\mathcal{P} \times \mathcal{Q} \to \mathbb{Z}$ *given by duality corresponds to the intersection product on* $N^1_Z(G/B) \times N^Z_1(G/B)$.
(4) L_λ *is nef if and only if* $\lambda \in \mathcal{P}^+$. L_λ *is ample if and only if* L_λ *is very ample if and only if* $\lambda \in \mathcal{P}^{++}$.

From this theorem it is easy to derive the corresponding description for flag varieties G/P, where $P \subset G$ is an arbitrary parabolic subgroup. Let $\Pi_P \subset \Pi$ be some subset of simple roots. Let $\Delta^+_P \subset \Delta^+$ denote the positive roots that are linear combinations of the roots in Π_P. Then we may suppose that P is generated by B and by the root groups U_α for $\alpha \in \Delta^+_P$. We denote $\Pi \setminus \Pi_P$ by $\Pi_{G/P}$ and $\Delta^+ \setminus \Delta^+_P$ by $\Delta^+_{G/P}$. A parabolic subgroup is maximal if and only if $\Pi_{G/P}$ is a single simple root. We denote by $\mathcal{Q}_{G/P}$ the sublattice of $\mathcal{Q}$ generated by $\Pi_{G/P}$ and by $\mathcal{Q}^+_{G/P}$ the corresponding subsemigroup. For any $\alpha_i \in \Pi_{G/P}$ we can associate a smooth curve $C_{\alpha_i} \subset G/P$, namely $C_{\alpha_i} = \overline{\pi(U_{\alpha_i})}$, where $\pi : G \to G/P$ is a factorization map. By linearity to any element $\alpha = \sum n_i \alpha_i \in \mathcal{Q}_{G/P}$ we assign a 1-cycle $C_\alpha = \sum n_i C_{\alpha_i}$ in G/P. Clearly this cycle is effective if $\alpha \in \mathcal{Q}^+_{G/P}$.

The fundamental weights $\omega_{i_1}, \dots \omega_{i_k}$ dual to the simple roots in $\Pi_{G/P}$ generate the sublattice $\mathcal{P}_{G/P}$ of $\mathcal{P}$. We denote $\mathcal{P}^+ \cap \mathcal{P}_{G/P}$ by $\mathcal{P}^+_{G/P}$. The subset $\mathcal{P}^{++}_{G/P} \subset \mathcal{P}^+_{G/P}$ consists of all weights $\lambda = \sum n_k \omega_{i_k}$ such that all $n_k > 0$. Any weight $\lambda \in \mathcal{P}_{G/P}$ defines a character of P, and, therefore, a line bundle L_λ on G/P.

Theorem 9.17

(1) *The correspondence* $\alpha \to C_\alpha$ *is an isomorphism of* $\mathcal{Q}_{G/P}$ *and* $N^{\mathbb{Z}}_1(G/P)$.
(2) *The correspondence* $\lambda \to L_\alpha$ *is an isomorphism of* $\mathcal{P}_{G/P}$ *and* $\mathrm{Pic}(G/P)$. *The latter group is, in turn, isomorphic to* $N^1_Z(G/P)$. *In particular,* $\mathrm{Pic}(G/P) = \mathbb{Z}$ *if and only if* P *is maximal.*
(3) *The pairing* $\mathcal{P}_{G/P} \times \mathcal{Q}_{G/P} \to \mathbb{Z}$ *given by duality corresponds to the intersection product on* $N^1_Z(G/P) \times N^Z_1(G/P)$.
(4) L_λ *is nef if and only if* $\lambda \in \mathcal{P}^+_{G/P}$. L_λ *is ample if and only if* L_λ *is very ample if and only if* $\lambda \in \mathcal{P}^{++}_{G/P}$.
(5) *If* $\lambda \in \mathcal{P}^+_{G/P}$ *then the linear system corresponding to* L_λ *is base–point free. The corresponding map given by sections is a factorization* $G/P \to G/Q$, *where* Q *is a parabolic subgroup such that* Π_Q *is a union of* Π_P *and all simple roots in* $\Pi_{G/P}$ *orthogonal to* λ.

Now it is quite straightforward to find nef values of ample line bundles.

Theorem 9.18

(1) *The anticanonical bundle* $K^*_{G/P}$ *corresponds to the dominant weight* $\rho_{G/P} = \sum_{\alpha \in \Delta^+_{G/P}} \alpha$, $\rho_{G/P} \in \mathcal{P}^{++}_{G/P}$. *Canonical line bundle* $K_{G/P}$ *is not nef.*

(2) *The nef value of an ample line bundle L_λ given by the weight $\lambda \in \mathcal{P}^{++}_{G/P}$ is equal to*

$$\tau = \max_{\alpha \in \Pi_{G/P}} \frac{(\rho_{G/P}, \alpha)}{(\lambda, \alpha)}.$$

(3) *The nef value morphism Φ is a homogeneous fiber bundle $\Phi : G/P \to G/Q$, where Q is the parabolic subgroup defined by $\Pi_Q = \Pi_P \cup \Pi'$, where $\Pi' \subset \Pi_{G/P}$ is a set of simple roots for which the above maximum occurs.*

Proof. The tangent bundle $T(G/P)$ is an equivariant vector bundle such that its fiber over the identity coset is isomorphic to the quotient of Lie algebras $\operatorname{Lie} G / \operatorname{Lie} P$ on which P acts via an adjoint representation on $\operatorname{Lie} G$ projected to the quotient. The T-weights of this representation consist of $\Delta^+_{G/P}$. Therefore the anticanonical bundle $K^*_{G/P} = \Lambda^{\dim G/P} T(G/P)$ corresponds to the weight $\rho_{G/P}$. Notice that for any $\alpha \in \Pi_{G/P}$

$$(\rho_{G/P}, \alpha) = \left(\sum_{\beta \in \Delta^+} \beta, \alpha \right) - \left(\sum_{\beta \in \Delta^+_P} \beta, \alpha \right).$$

The first product is equal to 2, the second one is non-positive, because any root from Δ^+_P is a linear combination of simple roots from Π_P with non-negative coefficients and the scalar product of two distinct simple roots is non-positive. Therefore, $\rho_{G/P} \in \mathcal{P}^{++}_{G/P}$ and $K_{G/P}$ is not nef.

Let

$$\rho_{G/P} = \sum_{\alpha_i \in \Pi_{G/P}} n_i \omega_i, \quad \lambda_{G/P} = \sum_{\alpha_i \in \Pi_{G/P}} m_i \omega_i.$$

Then, clearly, the nef value τ of L is given by

$$\tau = \max_{\alpha_i \in \Pi_{G/P}} \frac{n_i}{m_i} = \max_{\alpha \in \Pi_{G/P}} \frac{(\rho_{G/P}, \alpha)}{(\lambda, \alpha)}.$$

Part (3) follows from Theorem 9.17, part (5). □

Using this theorem, it is straightforward to calculate nef values in particular cases. Suppose that G is a simple group, P is a maximal parabolic subgroup such that $\Pi_{G/P} = \{\alpha_i\}$ is a single simple root. Suppose further that L is a generator of $\operatorname{Pic} G/P$, that is, L corresponds to the fundamental weight ω_i. In the following tables we give for each G and i (in the Bourbaki numbering of fundamental weights) the dimension $\dim G/P$ and the nef value τ of L.

$G = A_l$

i	$1, \ldots, l$
$\dim G/P$	$i(l+1-i)$
τ	$l+1$

$G = B_l$

i	$1, \ldots, l-1$	l
$\dim G/P$	$i(4l+1-3i)/2$	$l(l+1)/2$
τ	$2l-i$	$2l$

$G = C_l$

i	$1, \ldots, l$
$\dim G/P$	$i(4l+1-3i)/2$
τ	$2l-i+1$

$G = D_l$

i	$1, \ldots, l-2$	$l-1, l$
$\dim G/P$	$i(4l-1-3i)/2$	$l(l-1)/2$
τ	$2l-i-1$	$2l-2$

$G = E_6$

i	$1,5$	$2,4$	3	6
$\dim G/P$	16	25	29	21
τ	12	9	7	11

$G = E_7$

i	1	2	3	4	5	6	7
$\dim G/P$	27	42	50	53	47	33	42
τ	18	13	10	8	11	17	14

$G = E_8$

i	1	2	3	4	5	6	7	8
$\dim G/P$	57	83	97	104	106	98	78	92
τ	29	19	14	11	9	13	23	17

$G = F_4$

i	1	2	3	4
$\dim G/P$	15	20	20	15
τ	8	5	7	11

$G = G_2$

i	1	2
$\dim G/P$	5	5
τ	5	3

9.3.B Classification

In this section we use the calculations from the previous section and Theorems 9.12, 9.13 to classify all polarized flag varieties with positive defect. We continue to use notations from the previous section.

Suppose first that G is a simple group, $P \subset G$ is a maximal parabolic subgroup corresponding to a simple root α_i. Then all very ample linear bundles on G/P have a form $L^{\otimes k}$ for $k > 0$, where L corresponds to the fundamental weight ω_i. If $k > 1$ then by Theorem 7.15 we have $\operatorname{def}(G/P, L^{\otimes k}) = 0$. Therefore it suffices to find $\operatorname{def}(G/P, L)$. By Theorem 9.12 if $\operatorname{def}(G/P, L) > 0$ the $\operatorname{def}(G/P, L) = 2\tau(G/P, L) - 2 - \dim G/P$. Therefore we may look through the table of the previous section and pick all entries such that $d_P = 2\tau(G/P, L) - 2 - \dim G/P > 0$. The following table contains all these

entries. Therefore all polarized flag varieties with positive defect (such that P is a maximal parabolic subgroup) are contained in this table.

No.	G	i	$\tau(G/P, L)$	$\dim G/P$	d_P
1	A_l	$1, l$	$l+1$	l	l
2	A_l, l even	$2, l-1$	$l+1$	$2l-2$	2
3	C_l	1	$2l$	$2l-1$	$2l-1$
4	B_2	2	4	3	3
5	B_4	4	8	10	4
6	D_5	$4, 5$	8	10	4
7	A_l, l odd	$2, l-1$	$l+1$	$2l-2$	2
8	A_5	3	6	9	1
9	B_l	1	$2l-1$	$2l-1$	$2l-3$
10	B_3	3	6	6	4
11	D_4	$3, 4$	6	6	4
12	B_5	5	10	15	3
13	D_6	$5, 6$	10	15	3
14	B_6	6	12	21	1
15	D_7	$6, 7$	12	21	1
16	C_l	2	$2l-1$	$4l-5$	1
17	D_l	1	$2l-2$	$2l-2$	$2l-4$
18	E_6	$1, 5$	12	16	6
19	E_7	1	18	27	7
20	F_4	4	11	15	5
21	G_2	1	5	5	3

In fact, only first 6 entries of this table correspond to polarized flag varieties with positive defect, and entry 4 is a particular case of entry 3. This can be done case-by-case (see [Sn]). Therefore, we have the following theorem

Theorem 9.19 *Suppose that P_i is a maximal parabolic subgroup of a simple algebraic group corresponding to the simple root α_i. Let L be a very ample line bundle on G/P_i corresponding to the fundamental weight ω_i. Then*

(1) $\operatorname{def}(G/P_i, L^{\otimes k}) = 0$ *for* $k > 1$.
(2) $\operatorname{def}(G/P_i, L) > 0$ *if and only if* G/P_i *is one of the following:*
- A_l/P_1, A_l/P_l. $\operatorname{def}(G/P) = \dim(G/P) = l$.
- A_l/P_2, A_l/P_{l-1}, $l \geq 4$ *even.* $\operatorname{def}(G/P) = 2$, $\dim(G/P) = 2l-2$.
- C_l/P_1. $\operatorname{def}(G/P) = \dim(G/P) = 2l-1$.
- B_4/P_4, D_5/P_4, D_5/B_5. $\operatorname{def}(G/P) = 4$, $\dim(G/P) = 10$.

Notice that we do not follow the usual tradition and consider isomorphic polarized varieties with different group actions as distinct varieties. The next step is to consider flag varieties corresponding to arbitrary parabolic subgroups in simple group.

Theorem 9.20 *Suppose that $(G/P, L)$ is a polarized flag variety of a simple algebraic group G, where $P \subset G$ is a non-maximal parabolic subgroup. Then* $\operatorname{def}(G/P, L) = 0$.

Proof. Suppose that $\operatorname{def}(G/P, L) > 0$. By Theorem 9.18 the nef value morphism is a homogeneous fiber bundle $\Phi : G/P \to G/Q$, where $Q \subset G$ is another parabolic subgroup. The fiber F of Φ is isomorphic to Q/P.

By Theorem 9.12 we have $\operatorname{Pic} F = \mathbb{Z}$, therefore F is isomorphic to a quotient of a simple group by a maximal parabolic subgroup and Q is a proper subgroup of G. By Theorem 9.13 $\operatorname{def}(F, L) > \dim G/Q > 0$. Therefore, the polarized flag variety (F, L) is one of the varieties from Theorem 9.19. Let us consider 3 cases.

If F is equal to A_l/P_2 or A_l/P_{l-1}, $l \geq 4$ even, then $\operatorname{def}(F, L) = 2$. Therefore, $\dim G/Q = 1$. It follows that $G = \mathrm{SL}_2$, Q is a Borel subgroup, and there are no possibilities for P.

If F is equal to B_4/P_4, D_5/P_4, or D_5/B_5 then $\operatorname{def}(F, L) = 4$. Therefore, $\dim G/Q$ is equal to 1, 2, or 3. It follows that G is equal to SL_2, SL_3, SL_4, or Sp_4. Again none of these groups contain B_4 or D_5 as a simple component of a Levi subgroup.

Finally, let F be A_l/P_1, A_l/P_l, or $C_{\frac{l+1}{2}}/P_1$, so F is isomorphic to $\mathbb{P}^l$. We want to show that $\dim G/Q > k$, which will be a contradiction. The set of simple roots Π_Q contains a subset Ψ with Dynkin diagram of type A_l or $C_{\frac{l+1}{2}}$. Moreover, there exists a simple root $\alpha \in \Pi$ such that $\alpha \notin \Pi_Q$ and the Dynkin diagram of $\Psi' = \Psi \cup \{\alpha\}$ is connected. Then Ψ' is a set of simple roots of a simple subgroup $G' \subset G$ and $\dim G/Q \geq \dim G'/Q'$, where $Q' = G' \cap Q$ is a maximal parabolic subgroup defined by Ψ. It is easy to check that in all arising cases $\dim G'/Q' > k$. □

Now we can handle the case of an arbitrary semisimple group and its arbitrary parabolic subgroup.

Theorem 9.21 *Suppose that $(G/P, L)$ is a polarized flag variety of a complex semisimple algebraic group G. Then* $\operatorname{def}(G/P, L) > 0$ *if and only if either $(G/P, L)$ is one of polarized varieties described in Theorem 9.19 or the following conditions hold:*

- $G = G_1 \times G_2$, $P = P_1 \times P_2$, *where* $P_1 \subset G_1$, $P_2 \subset G_2$.
- $L = \mathrm{pr}_1^* L_1 \otimes \mathrm{pr}_2^* L_2$, *where* $\mathrm{pr}_i : G/P \to G_i/P_i$ *are projections and* L_i *are very ample line bundles on* G_i/P_i.
- $(G_1/P_1, L_1)$ *is one of polarized varieties described in Theorem 9.19.*
- $\operatorname{def}(G_1/P_1, L_1) > \dim G_2/P_2$.

In this case $\operatorname{def}(G/P, L) = \operatorname{def}(G_1/P_1, L_1) - \dim G_2/P_2$.

Proof. This Theorem easily follows from Theorem 6.5. □

10. Some Applications of the Duality

Preliminaries

In this chapter we collect several applications, variations, and illustrations of the projective duality.

10.1 Discriminants and Automorphisms

10.1.A Matsumura–Monsky Theorem

Let $D \subset \mathbb{P}^n$ be a smooth hypersurface of degree d. It was first proved in [MM] that the group of projective automorphisms preserving D is finite if $d > 2$. In fact, it was also proved that the group of biregular automorphisms of D is finite if $d > 2$ (except the cases $d = 3$, $n = 2$ and $d = 4$, $n = 3$). Though this generalization looks much stronger, actually it is an easy consequence of the "projective" version and the Bart Theorem [Ba].

More generally, let G/P be a flag variety of a simple Lie group and $D \subset G/P$ be a smooth ample divisor. Let $L_\lambda = \mathcal{O}(D)$ be the corresponding ample line bundle, where $\lambda \in \mathcal{P}^+$ is a dominant weight. Then one might expect that the normalizer $N_G(D)$ of D in G is finite if λ is big enough. Indeed, if $N_G(D)$ (or actually any linear algebraic group of transformations of D) contains a one-parameter subgroup of automorphisms of D then D is covered by rational curves. However, if λ is big enough then the canonical class K_D is nef by the adjunction formula, and, therefore, D can not be covered by rational curves. Unfortunately, this transparent approach does not give strong estimates on λ. Much better estimates can be obtained using an original proof of Matsumura-Monsky theorem.

The problem can be reformulated as follows. Suppose that V_λ is an irreducible G-module with highest weight λ. Let $\mathcal{D} \subset V_\lambda$ be the discriminant variety (the dual variety to the orbit of the highest weight vector in V_λ^*, see Section 9.3 for further details). We shall show that if λ is big enough then any point $x \in V_\lambda \setminus \mathcal{D}$ has a finite stabilizer G_x and the orbit of x is closed, $Gx = \overline{Gx}$ (therefore x is a stable point of V_λ in the sense of Geometric Invariant Theory, see [MFK]). This result can be compared with the results of [AVE], where all irreducible modules of simple algebraic groups with infinite

stabilizers of generic points were found (this classification was extended later in [El1] and [El2] to handle irreducible representations of semisimple groups and any representations of simple groups as well).

If $\mathcal{D}$ is not a hypersurface then an easy inspection using Theorem 9.21 shows that the stabilizer of any point is infinite. So from now on we shall assume that $\mathcal{D}$ is a hypersurface defined by vanishing of the discriminant Δ.

Recall that a dominant weight λ is called self-dual if V_λ is isomorphic to V_λ^* as a G-module. Let $\mathcal{P}_S^+ \subset \mathcal{P}^+$ be the subcone of self-dual dominant weights. Let γ be the highest root.

Theorem 10.1 *Let V_λ be an irreducible representation of a simple algebraic group G with highest weight λ such that $\mathcal{D}$ is a hypersurface. Suppose that $(\lambda - \gamma, \mu) > 0$ for any $\mu \in \mathcal{P}_S^+$. Let $x \in V_\lambda \setminus \mathcal{D}$. Then G_x is finite and $Gx = \overline{Gx}$. Moreover, $G_{[x]}$ is also finite, where $[x]$ is the line spanned by x.*

Proof. If G_x is finite and $G_{[x]}$ is infinite then $[x] \setminus \{0\} \subset Gx$. Therefore, $0 \in \overline{Gx}$ and $\Delta(x) = \Delta(0) = 0$, hence, $x \in \mathcal{D}$. The same argument shows that if $x \in V_\lambda \setminus \mathcal{D}$ then $\overline{Gx} \subset V_\lambda \setminus \mathcal{D}$. If Gx is not closed then G_y is infinite for any point $y \in \overline{Gx} \setminus Gx$. Therefore, in order to prove the Theorem it suffices to prove that for any $x \in V_\lambda \setminus \mathcal{D}$ the stabilizer G_x is finite.

Suppose that G_x is infinite. Then G_y is infinite and reductive (see e.g. [PV]) for any point y from the closed orbit in $\overline{Gx}$. Therefore, it suffices to prove that if $S \subset G$ is a one-dimensional torus and $Sx = x$ then $x \in \mathcal{D}$.

Without loss of generality we may assume that $S \subset T$, where T is the fixed maximal torus, and $\mathfrak{t} = \operatorname{Lie} T$ is the Cartan subalgebra. Let $x = \sum_{\pi \in \mathcal{P}} x_\pi$ be the weight decomposition of x. Let $\operatorname{Supp}(x) = \{\pi \in \mathcal{P} \mid x_\pi \neq 0\}$ be the support of x. For any $\mu \in \mathcal{P}$ let H_μ denote the hyperplane of weights perpendicular to μ. Then there exists $\mu \in \mathcal{P}$ such that $\operatorname{Supp}(x) \subset H_\mu$. Using the action of the Weil group we may assume that $\mu \in \mathcal{P}^+$.

For any $\lambda \in \mathcal{P}^+$ let λ^* be the highest weight of the dual module V_λ^*. Then $\lambda^* = -w_0(\lambda)$, where w_0 is the longest element of the Weil group.

Suppose that $(\mu, \lambda^* - \gamma) > 0$. Then for any positive root α we have $(\mu, \lambda^* - \alpha) > 0$. Therefore, x is perpendicular to $[\mathfrak{g}, v_{\lambda^*}]$, where v_{λ^*} is the highest weight vector of V_λ^*. It follows that $x \in \mathcal{D}$.

Suppose that $(\mu^*, \lambda^* - \gamma) > 0$. Then for any positive root α we have $(\mu^*, \lambda^* - \alpha) > 0$. Therefore, $w_0(x)$ is perpendicular to $[\mathfrak{g}, v_{\lambda^*}]$. It follows that $w_0(x) \in \mathcal{D}$. Therefore, It follows that $x \in \mathcal{D}$ as well.

Suppose now that $(\mu, \lambda^* - \gamma) \leq 0$ and $(\mu^*, \lambda^* - \gamma) \leq 0$. Then

$$(\mu + \mu^*, \lambda^* - \gamma) \leq 0.$$

But $\mu + \mu^*$ is a self-dual weight, hence this contradicts assumptions of the Theorem. $\square$

Example 10.2 If $G = \mathrm{SL}_n$ and $\lambda = \sum n_i\omega_i$, where $\omega_1, \ldots, \omega_n$ are the fundamental weights, then the assumptions of the Theorem are satisfied if and only if $\sum n_i > 2$, for example if $\lambda = n\omega_1$, $n > 2$. In particular, we recover the original Matsumura–Monsky Theorem.

10.1.B Quasiderivations of Commutative Algebras

Commutative algebras without identities. Let $V = \mathbb{C}^n$. Consider the vector space $\mathcal{A} = S^2V^* \otimes V$ parametrizing bilinear commutative multiplications in V. In the sequel we identify points of $\mathcal{A}$ with the corresponding commutative algebras.

Definition 10.3 Let $A \in \mathcal{A}$. A non-zero element $v \in A$ is called a quadratic nilpotent if $v^2 = 0$. Let $\mathcal{D}_1 \subset \mathcal{A}$ be a subset of all algebras containing quadratic nilpotents. A one-dimensional subalgebra $U \subset A$ is called singular if there exists linear independent vectors $u \in U$ and $v \in A$ such that

$$u^2 = \alpha u, \quad uv = \frac{\alpha}{2}v, \quad \text{where } \alpha \in \mathbb{C}.$$

Let $\mathcal{D}_2 \subset \mathcal{A}$ be a subset of all algebras containing singular subalgebras.

Then the following theorem holds.

Theorem 10.4 ([T6])

(1) *$\mathcal{D}_1$ and $\mathcal{D}_2$ are irreducible hypersurfaces.*
(2) *Let $A \in \mathcal{A}$. Then A contains a one-dimensional subalgebra.*
(3) *Let $A \in \mathcal{A} \setminus (\mathcal{D}_1 \cup \mathcal{D}_2)$. Then A contains exactly $2^n - 1$ one-dimensional subalgebras; all these subalgebras are spanned by idempotents.*

Proof. Clearly $\mathcal{A}$ can be identified with a set of at most n-dimensional linear systems of quadrics in V. Namely, any linear function $f \in V^*$ defines a homogeneous quadratic function

$$v \to f(v^2).$$

Then $\mathcal{D}_1$ corresponds to the set of linear systems of quadrics with zero resultant. This proves that $\mathcal{D}_1$ is an irreducible hypersurface.

There exists, however, another useful identification. Any algebra A determines the n-dimensional linear system of *affine* quadrics in V. Namely, any linear form $f \in V^*$ defines an affine quadratic form

$$v \to f(v - v^2).$$

We can embed V into a projective space $\mathbb{P}$ as an affine chart. Then this linear system is naturally identified with a n-dimensional linear system of quadrics in $\mathbb{P}$. Clearly, the base points of this linear system that do not lie on the infinity coincide with idempotents of A. Infinite base points are the

projectivizations of lines spanned by quadratic nilpotents. It is easy to see that quadrics from our linear system intersect transversally at 0. Moreover, quadrics intersect non-transversally at some point $v \neq 0$ if and only if the subalgebra spanned by v (if v is finite) or the subalgebra with the projectivization v (if v is infinite) is singular. Therefore, (3) follows from the Bezout theorem.

$\mathcal{D}_2$ is irreducible since $\mathcal{D}_2 = \mathrm{GL}_n \cdot \mathcal{D}_2'$, where $D_2' \subset \mathcal{A}$ is the linear subset of all algebras that have a fixed singular subalgebra and a fixed line spanned by the vector v from the definition of a singular subalgebra. It is also quite easy to check that D_2 is actually a closed hypersurface.

Since quadrics of our linear system intersect transversally at 0, it follows that there exist other base points, i.e. there exists at least one 1-dimensional subalgebra. □

The algebras $A \in \mathcal{A}$ that do not belong to discriminant varieties $\mathcal{D}_1$ and $\mathcal{D}_2$ will be called regular. Of course, both hypersurfaces $\mathcal{D}_1$ and $\mathcal{D}_2$ can be interpreted as ordinary discriminants. First, we can enlarge the symmetry group and consider $S^2(\mathbb{C}^n)^* \otimes \mathbb{C}^n$ as $\mathrm{SL}_n \times \mathrm{SL}_n$-module. Then this module is irreducible and its discriminant variety (the dual variety of the projectivization of the highest weight vector orbit) coincides with $\mathcal{D}_1$. Now, consider $\mathcal{A} = S^2V^* \otimes V$ as $\mathrm{SL}(V)$-module. Then this module is reducible, $\mathcal{A} = \mathcal{A}_0 + \tilde{\mathcal{A}}$, where $\mathcal{A}_0$ is a set of algebras with zero trace and $\tilde{\mathcal{A}}$ is isomorphic to V^* as $\mathrm{SL}(V)$-module. Consider the discriminant of $\mathcal{A}_0$ as a function on $\mathcal{A}$ (forgetting other coordinates). Then the corresponding hypersurface is exactly $\mathcal{D}_2$. If we consider the set of linear operators $\mathrm{Hom}(V, V) = V^* \otimes V$ instead of $\mathcal{A}$, then this construction will give us the determinant and the discriminant of the linear operator (see Example 2.18, Theorem 2.5).

Quasiderivations. Let $\mathfrak{g}$ be a Lie algebra with representation $\rho : \mathfrak{g} \to \mathrm{End}(V)$. Consider any $v \in V$. Then subalgebra

$$\mathfrak{g}_v = \{g \in \mathfrak{g} \mid \rho(g)v = 0\}$$

is called the annihilator of v. The subset

$$Q\mathfrak{g}_v = \{g \in \mathfrak{g} \mid \rho(g)^2 v = 0\}$$

is called the quasi-annihilator of v. Clearly, $\mathfrak{g}_v \subset Q\mathfrak{g}_v$. Of course, the quasi-annihilator is not a linear subspace in general, However, we have the following version of a Jordan decomposition:

Lemma 10.5 *Suppose that ρ is the differential of the representation of an algebraic group. Let $g \in Q\mathfrak{g}_v$. Consider the Jordan decomposition in $\mathfrak{g}$, $g = g_s + g_n$, where g_s is semisimple and g_n is nilpotent, $[g_s, g_n] = 0$. Then $g_s \in \mathfrak{g}_v$ and $g_n \in Q\mathfrak{g}_v$.*

The proof is obvious.

For example, suppose that $\mathfrak{g} = \mathfrak{gl}_n$ and ρ is the natural representation in the vector space $V^* \otimes V^* \otimes V$ that parametrizes bilinear multiplications in V. Let A be any algebra. Then the annihilator $\mathfrak{g}_A$ is identified with the Lie algebra of derivations $\mathrm{Der}(A)$. Operators $D \in Q\mathfrak{g}_A$ are called quasiderivations. Of course, it is possible to write down explicit equations that determine $Q\,\mathrm{Der}(A) = Q\mathfrak{g}_A$ in $\mathrm{End}(A)$, but this formula is quite useless (see [Vi1]). We shall use its particular case that is quite easy to verify:

Lemma 10.6 *Let A be an algebra, $D \in \mathrm{End}(A)$, $D^2 = 0$. Then $D \in Q\,\mathrm{Der}(A)$ if and only if for any $x, y \in A$ we have*

$$D(x)D(y) = D(D(x)y) + D(xD(y)). \tag{10.1}$$

Example 10.7 It was conjectured in [Vi1] that all quasiderivations of the algebra of matrices Mat_n have the form $D(x) = ax+xb$, where $(a+b)^2 = [a, b]$. Let us give an example of a different quasiderivation. Consider the linear operator $D(x) = exe$, where $e^2 = 0$, $e \neq 0$. Then it is to check using (10.1) that D is a quasiderivation. However, D of course can not be written in the form $ax + xb$.

Quasiderivations can be used to define naive deformations. Namely, suppose that A is any algebra with the multiplication $u{\cdot}v$ and D is its quasiderivation. Consider the algebra A_D with the same underlying vector space and with the multiplication given by $u \star v = u \cdot D(v) + D(u) \cdot v - D(u \cdot v)$. Then if A satisfies any polynomial identity then A_D satisfies this identity as well. More generally, let $\rho : \mathfrak{g} \to \mathrm{End}(V)$ be the differential of a representation of an algebraic group G, $v \in V$, $D \in Q\mathfrak{g}_v$. Suppose that $H \subset V$ is a closed conical G-equivariant hypersurface (the vanishing set of a homogeneous G-semiinvariant) and $v \in H$. Then $\rho(D)v$ also belongs to H. Indeed, since H is equivariant, $\exp(\lambda\rho(D))v$ belongs to H for any $\lambda \in \mathbb{C}$. Since D is a quasiderivation, $\exp(\lambda\rho(D))v = v + \lambda\rho(D)v$. Since H is conical, $v/\lambda + \rho(D)v$ belongs to H. Since H is closed, $\rho(D)v$ also belongs to H.

Now suppose that A is a regular commutative algebra. Then we claim that $Q\,\mathrm{Der}(A) = 0$.

Theorem 10.8 *Let $A \in \mathcal{A}$, $A \notin \mathcal{D}_1 \cup \mathcal{D}_2$. Then $Q\,\mathrm{Der}(A) = 0$.*

Proof. By Lemma 10.5 it is sufficient to check that A has no semisimple derivations and no nilpotent quasiderivations. Suppose that $\mathrm{Der}(A)$ contains a non-zero semisimple element. Then the group $\mathrm{Aut}(A)$ contains a one-dimensional algebraic torus T. Let $\mathfrak{t}$ be its Lie algebra, $H \in \mathfrak{t}$, $H \neq 0$. We way assume that the spectrum of H in A is integer-valued. Let $A_n \subset A$ be a weight space of weight n. Then $A = \oplus A_n$ is a $\mathbb{Z}$-grading. Let $v \in A$ be a homogeneous element of a maximal positive (or minimal negative) degree. Then $v^2 = 0$, hence $A \in \mathcal{D}_1$.

Suppose now that E is a non-zero nilpotent quasiderivation. We can embed E in an $\mathfrak{sl}_2$-triple $\langle F, H, E\rangle \subset \mathrm{End}(A)$. Let

$$A = \oplus m_d R_d$$

be an $\mathfrak{sl}_2$-module decomposition, here R_d is an irreducible $(d+1)$-dimensional module. Consider also the weight decomposition

$$A = \oplus A^n \quad \text{and} \quad \mathcal{A} = \oplus \mathcal{A}^n$$

with respect to H. Let

$$J^n = A^n + A^{n+1} + A^{n+2} + \dots.$$

To avoid the abuse of notations denote by $\alpha \in \mathcal{A}$ the point corresponding to the algebra A. Let Supp α be the support of α (i.e. all $n \in \mathbb{Z}$ such that $\alpha_n \neq 0$, where $\alpha = \sum \alpha_n$, $\alpha_n \in \mathcal{A}^n$). Since $E^2\alpha = 0$, the vector $E\alpha$ is a linear combination of highest weight vectors, therefore Supp $\alpha \subset \{-1, 0, 1, 2, \dots\}$. This is equivalent to $J^n J^m \subset J^{n+m-1}$. In particular, if $v \in A$ is a weight vector of the weight n then $v^2 = 0$ as $n > 1$. Since A is regular, it follows that

$$A = m_0 R_0 \oplus m_1 R_1 = A^{-1} \oplus A^0 \oplus A^1.$$

Since $A^1 = J^1$, A^1 is a subalgebra in A. By Theorem 10.4 (2), A^1 has a one-dimensional subalgebra U spanned by an idempotent u (since A has no quadratic nilpotents). Let $v \in A^{-1}$ be a unique vector such that $Ev = u$. Since $E^2 = 0$, we can apply formula (10.1) with $x = y = v$. We get $E(v)^2 = 2E(E(v)v)$, hence $u = 2E(uv)$. Therefore, $uv - \frac{1}{2}v \in J^0$. Notice that the operator of left multiplication by u preserves J^0. Since this operator has an eigenvector in A/J^0 with eigenvector $1/2$, it has such an eigenvector in A. Therefore, U is a singular subalgebra in A. □

The following Corollary was first proved in [An] using combinatorial methods.

Corollary 10.9 ([An]) *Let A be an n-dimensional semisimple commutative algebra, i.e. A is a direct sum of n copies of $\mathbb{C}$, that is A is the algebra of diagonal $n \times n$ matrices. Then A has no nonzero quasiderivations.*

Proof. It is sufficient to check that $A \notin \mathcal{D}_1 \cup \mathcal{D}_2$. Clearly, A has no nilpotents. Suppose that $U \subset A$ is a one-dimensional subalgebra spanned by an idempotent $e \in U$. Since the spectrum of the operator of left multiplication by e is integer-valued (actually consists of 0 and 1), it follows that U is not singular. □

10.2 Discriminants of Anticommutative Algebras

If the set of certain objects is parametrized by an algebraic variety X (for example, by a vector space) then it makes sense to speak about generic objects. Namely, we say that a generic object satisfies some property if there exists a dense Zariski-open subset $X_0 \subset X$ such that all objects parametrized by points from X_0 share this property. However, sometimes it is possible to find some discriminant-type closed subvariety $Y \subset X$ and then to study properties of 'regular' objects parametrized by points from $X \setminus Y$. For example, instead of studying generic hypersurfaces it is usually worthy to study smooth hypersurfaces. In this section we implement this program for the study of some quite non-classical object, namely anticommutative algebras, with the multiplication depending on a number of arguments.

Generic anticommutative algebras. Let $V = \mathbb{C}^n$. We fix an integer k, $1 < k < n-1$. Let $\mathcal{A}_{n,k} = \Lambda^k V^* \otimes V$ be the vector space of k-linear anticommutative maps from V to V. We identify the points of $\mathcal{A}_{n,k}$ with the corresponding algebras, that is, we assume that $A \in \mathcal{A}_{n,k}$ is the space V equipped with the structure of a k-argument anticommutative algebra. Subalgebras in generic algebras with $k = 2$ were studied in [T2]. The following theorem is a generalization of these results.

Theorem 10.10 ([T1]) *Let $A \in \mathcal{A}_{n,k}$ be a generic algebra. Then*

(i) *Every m-dimensional subspace is a subalgebra if $m < k$.*
(ii) *A contains no m-dimensional subalgebras with $k+1 < m < n$.*
(iii) *The set of k-dimensional subalgebras is a smooth irreducible $(k-1)(n-k)$-dimensional subvariety in the Grassmanian $\mathrm{Gr}(k, A)$.*
(iv) *There are finitely many $(k+1)$-dimensional subalgebras, and their number is*

$$\sum_{\substack{n-k-1\geq\mu_1\geq\ldots\geq\mu_{k+1}\geq 0\\ n-k-1\geq\lambda_1\geq\ldots\geq\lambda_{k+1}\geq 0\\ \mu_1\leq\lambda_1,\ldots,\mu_{k+1}\leq\lambda_{k+1}}} (-1)^{|\mu|}\frac{(\lambda_1+k)!(\lambda_2+k-1)!\ldots\lambda_{k+1}!}{(\mu_1+k)!(\mu_2+k-1)!\ldots\mu_{k+1}!}(|\lambda|-|\mu|)!\times$$
$$\times\left|\frac{1}{(i-j+\lambda_j-\mu_i)!}\right|^2_{i,j=1,\ldots,k+1},$$

where $|\lambda| = \lambda_1 + \ldots + \lambda_{k+1}$, $|\mu| = \mu_1 + \ldots + \mu_{k+1}$, $1/N! = 0$ if $N < 0$.
(v) *A contains a $(k+1)$-dimensional subalgebra.*
(vi) *If $k = n-2$, then the number of $(k+1)$-dimensional subalgebras is equal to*

$$\frac{2^n-(-1)^n}{3}.$$

Proof. The GL_n-module $\mathcal{A}_{n,k}$ is a sum of two irreducible submodules:

$$\mathcal{A}_{n,k} = \mathcal{A}^0_{n,k} \oplus \tilde{\mathcal{A}}_{n,k}. \tag{10.2}$$

Here $\tilde{\mathcal{A}}_{n,k}$ is isomorphic to $\Lambda^{k-1}V^*$: we assign to every $(k-1)$-form ω the algebra with multiplication

$$[v_1, \dots, v_k] = \sum_{i=1}^{k} (-1)^{i-1} \omega(v_1, \dots, \hat{v}_i, \dots, v_k) v_i.$$

Note that every subspace of this algebra is a subalgebra. Hence, the lattice of subalgebras of $A \in \mathcal{A}_{n,k}$ coincides with the lattice of subalgebras of A^0, where $A \mapsto A^0$ is the GL_n-equivariant projector on the first summand in (10.2). Algebras in $\mathcal{A}^0_{n,k}$ will be called *zero trace algebras*, since $A \in \mathcal{A}^0_{n,k}$ if and only if the $(k-1)$-form $\mathrm{Tr}[v_1, \dots, v_{k-1}, \cdot]$ is equal to zero. Hence, the theorem will be proved once we have proved it for generic algebras in $\mathcal{A}^0_{n,k}$.

We choose a basis $\{e_1, \dots, e_n\}$ in V, identify GL_n with the group of matrices, consider the standard diagonal maximal torus T, and take for B and B_- the subgroups of upper- and lower-triangular matrices. We fix an $m \geq k$. Consider the parabolic subgroup of matrices

$$P = \begin{pmatrix} A & 0 \\ * & B \end{pmatrix},$$

where B is an $m \times m$-matrix and A is an $(n-m) \times (n-m)$-matrix. Then G/P coincides with $\mathrm{Gr}(m, V)$. Consider the vector bundle

$$\mathcal{L} = \Lambda^k \mathcal{S}^* \otimes V/\mathcal{S}$$

on G/P, where $\mathcal{S}$ is the tautological bundle and $V/\mathcal{S}$ is the factor-tautological bundle. Then the assumptions of Theorem 4.8 are fulfilled, since $\mathcal{L} = \mathcal{L}_\lambda$, where λ is the highest weight of $\mathcal{A}^0_{n,k}$. Therefore,

$$\mathcal{A}^0_{n,k} = H^0(\mathrm{Gr}(m, V), \Lambda^k \mathcal{S}^* \otimes V/\mathcal{S}).$$

Let $A \in \mathcal{A}^0_{n,k}$, and let s_A be the corresponding global section. Then the scheme of zeros $(Z_{s_A})_{red}$ coincides with the variety of m-dimensional subalgebras of A.

Let us return to the theorem. Statement (i) is obvious. (ii) follows from Theorem 4.8 (i). Theorem 4.8 (ii) implies that if every k-argument anticommutative algebra A contains a k-dimensional subalgebra, then the variety of k-dimensional subalgebras of a generic algebra is a smooth unmixed $(k-1)(n-k)$-dimensional subvariety in $\mathrm{Gr}(k, A)$. We claim that any $(k-1)$-dimensional subspace U can be included in a k-dimensional subalgebra. The multiplication in the algebra defines a linear map from V/U to V/U. Let $v + U$ be a non-zero eigenvector. It is obvious that $\mathbb{C}v \oplus U$ is a k-dimensional subalgebra.

We have only to prove that the variety of k-dimensional subalgebras is irreducible. Assume that a section s of the bundle $\mathcal{L} = \Lambda^k \mathcal{S}^* \otimes V/\mathcal{S}$ over $\mathrm{Gr}(k, V)$ corresponding to A has transversal intersection with the zero section. Then the Koszul complex

$$0 \to \Lambda^{n-k}\mathcal{L}^* \xrightarrow{s} \ldots \xrightarrow{s} \Lambda^2\mathcal{L}^* \xrightarrow{s} \mathcal{L}^* \xrightarrow{s} \mathcal{O} \to \mathcal{O}_{Z(s)} \to 0$$

is exact by Theorem 3.3.

Note that $\Lambda^p\mathcal{L}^*$ is isomorphic to the bundle $S^p\Lambda^k\mathcal{S} \otimes \Lambda^p(V/\mathcal{S})^*$. This is a homogeneous bundle over G/P of the form $\mathcal{L}_\mu$, where

$$\mu = -\varepsilon_1 - \ldots - \varepsilon_p + \varepsilon_{n-k+1} + \ldots + \varepsilon_n$$

and ε_i are the weights of the diagonal torus in the tautological representation. Note that the weight $\mu + \rho$ (where ρ is the half-sum of the positive roots) is singular (belongs to the wall of the Weil chamber) for any p, $1 \le p \le n-k$. By the Borel–Weil–Bott theorem, $H^*(\mathrm{Gr}(k, V), \Lambda^p\mathcal{L}^*) = 0$ for $1 \le p \le n-k$. Hence, $H^0(Z(s), \mathcal{O}_{Z(s)}) = H^0(\mathrm{Gr}(k, V), \mathcal{O}) = \mathbb{C}$, as was to be shown.

We postpone the proof of assertion (v) till the end of this section and consider assertions (iv) and (vi), that is, we shall calculate the highest Chern class of the bundle $\Lambda^k\mathcal{S}^* \otimes V/\mathcal{S}$ over $\mathrm{Gr}(k+1, V)$.

We use the standard notation, facts, and formulae from the Schubert calculus (cf. [Fu1]. The letters λ and μ always denote Young diagrams in the rectangle with $k+1$ rows and $n-k-1$ columns. It is well known that such diagrams parametrize the basis of the Chow ring of $\mathrm{Gr}(k+1, V)$. The cycle corresponding to λ is denoted by σ_λ, $\sigma_\lambda \in A^{|\lambda|}(\mathrm{Gr}(k+1, V))$. (We grade the Chow ring by the codimension, $|\lambda| = \lambda_1 + \ldots + \lambda_{k+1}$, where λ_i is the length of the ith row of λ.) We need the total Chern class of the bundles $\mathcal{S}$ and $V/\mathcal{S}$:

$$\begin{aligned} c(\mathcal{S}) &= 1 - \sigma_1 + \sigma_{1,1} - \ldots + (-1)^{k+1}\sigma_{1,\ldots,1}, \\ c(V/\mathcal{S}) &= 1 + \sigma_1 + \sigma_2 + \ldots + \sigma_{n-k-1}. \end{aligned}$$

We begin by calculating the total Chern class of $\mathcal{S} \otimes V/\mathcal{S}$. The standard formula for the total Chern class of the tensor products of two bundles implies that

$$c(\mathcal{S} \otimes V/\mathcal{S}) = \sum_{\mu \subset \lambda} d_{\lambda\mu} \Delta_{\tilde{\mu}}(c(\mathcal{S})) \Delta_{\lambda'}(c(V/\mathcal{S})), \tag{10.3}$$

where

$$d_{\lambda\mu} = \left| \binom{\lambda_i + k + 1 - i}{\mu_j + k + 1 - j} \right|_{1 \le i,j \le k+1}, \quad \Delta_\lambda(c) = |c_{\lambda_i + j - i}|,$$

$\lambda' = (n-k-1-\lambda_{k+1}, n-k-1-\lambda_k, \ldots, n-k-1-\lambda_1)$, and $\tilde{\mu}$ is the diagram obtained from μ by transposition. We shall use the fact that $\Delta_{\tilde{\mu}}(c(\mathcal{S})) = \Delta_\mu(s(\mathcal{S}))$, where $s(E)$ is the Segre class of E. Another fact is that $\Delta_\lambda(c(V/\mathcal{S})) = \sigma_\lambda$, $\Delta_\lambda(s(\mathcal{S})) = (-1)^{|\lambda|}\sigma_\lambda$ (since $s(\mathcal{S}) = 1 - \sigma_1 + \sigma_2 + \ldots + (-1)^{n-k-1}\sigma_{n-k-1}$). Therefore, (10.3) can be written as

$$c(\mathcal{S}\otimes V/\mathcal{S}) = \sum_{\mu\subset\lambda} d_{\lambda\mu}(-1)^{|\mu|}\sigma_\mu\sigma_{\lambda'}. \tag{10.4}$$

To calculate the highest Chern class of $\mathcal{L} = \Lambda^k\mathcal{S}^*\otimes V/\mathcal{S}$, we use the formula $\mathcal{L} = \Lambda^{k+1}\mathcal{S}^*\otimes(\mathcal{S}\otimes V/\mathcal{S})$. The total Chern class of the first factor is equal to $c(\Lambda^{k+1}\mathcal{S}^*) = 1+\sigma_1$. The total Chern class of the second is given by (10.4). Hence, the highest Chern class of $\mathcal{L}$ is

$$c_{top}(\mathcal{L}) = \sum_{\mu\subset\lambda} d_{\lambda\mu}(-1)^{|\mu|}\sigma_\mu\sigma_{\lambda'}\sigma_1^{|\lambda|-|\mu|}. \tag{10.5}$$

The last result of the Schubert calculus that we need is the exact formula for the degree of a product of two cycles. In our case this can be written as

$$\sigma_\mu\sigma_{\lambda'}\sigma_1^{|\lambda|-|\mu|} = \deg(\sigma_\mu\sigma_{\lambda'}) = (|\lambda|-|\mu|)!\left|\frac{1}{(i-j+\lambda_j-\mu_i)!}\right|_{i,j=1,\dots,n}, \tag{10.6}$$

where $1/N! = 0$ if $N<0$. The formula in assertion (iv) of the Theorem can be obtained from (10.5) and (10.6) by a slight modification of the determinant in the formula for $d_{\lambda\mu}$.

It remains to verify the formula in the assertion (vi) of the Theorem. Let $k = n-2$. Then the formula in the assertion (iv) can be written as

$$\sum_{0\le i\le j\le k+1}(-1)^i\frac{(k+1)!k!\dots(k-j+2)!(k-j)!\dots 0!}{(k+1)!k!\dots(k-i+2)!(k-i)!\dots 0!}(j-i)!\det{}^2 A, \tag{10.7}$$

where

$$A = \begin{pmatrix} X & 0 & 0\\ * & Y & 0\\ * & * & Z\end{pmatrix},$$

X is an $i\times i$-matrix, Y is a $(j-i)\times(j-i)$-matrix, and the format of Z is a $(k+1-j)\times(k+1-j)$. X and Z are lower-triangular matrices with 1s on the diagonal and Y is given by

$$Y = \begin{pmatrix} 1 & 1 & 0 & 0 & \cdots & 0\\ 1/2! & 1 & 1 & 0 & \cdots & 0\\ 1/3! & 1/2! & 1 & 1 & \cdots & 0\\ 1/4! & 1/3! & 1/2! & 1 & \cdots & 0\\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots\\ 1/(j-i)! & 1/(j-i-1)! & 1/(j-i-2)! & 1/(j-3)! & \cdots & 1\end{pmatrix}.$$

It is easy to verify that $\det Y = 1/(j-i)!$, which enables us to rewrite (10.7) as

$$\sum_{i\le j}(-1)^i\frac{(1+(k-i))!\dots(1+(k-j+1))!}{(k-i)!\dots(k-j+1)!}/(j-i)! =$$

$$\sum_{i\le j}(-1)^i\binom{k+1-i}{j-i}=\sum_i(-1)^i2^{k+1-i}=\frac{2^{k+2}-(-1)^{k+2}}{3}=\frac{2^n-(-1)^n}{3},$$

as was to be shown.

It remains to prove assertion (v) of the Theorem. We have to prove that every $A\in\mathcal{A}_{n,k}$ has a $(k+1)$-dimensional subalgebra. We fix a basis $\{e_1,\dots,e_n\}$ in V and consider the subspace $U=\langle e_{n-k},\dots,e_n\rangle$. Let $M\subset\mathcal{A}_{n,k}$ be the subspace that consists of the algebras for which U is a $(k+1)$-dimensional subalgebra. We claim that $\mathcal{A}_{n,k}=\mathrm{GL}_n\cdot M$. It is sufficient to prove that the differential of the canonical morphism $\phi:\mathrm{GL}_n\times M\to\mathcal{A}_{n,k}$ is surjective at a point (e,A). Consider the algebra $A\in M$ in which $[e_{n-k},\dots,\hat{e}_i,\dots,e_n]=e_i$ for all $n-k\le i\le n$ and the other products are zero. We claim that $d\phi$ is surjective at (e,A). Consider the map $\pi:\mathcal{A}_{n,k}\to\mathcal{A}_{n,k}/M$. It is sufficient to verify that $\pi\circ d\phi_{(e,A)}(\mathfrak{gl}_n,0)=\mathcal{A}_{n,k}/M\simeq\Lambda^kU^*\otimes V/U$. But this is obvious, since multiplication in $d\phi_{(e,A)}(E_{ji},0)$ for $n-k\le i\le n$, $1\le j\le n-k-1$ (E_{ji} is the matrix identity) is given by $[e_{n-k},\dots,\hat{e}_i,\dots,e_n]=e_j$ with the other products equal to zero. This completes the proof of the theorem. □

***D*-regular algebras.** An essential drawback of Theorem 10.10 is the fact that it does not enable us to study the structure of subalgebras of any particular algebra. To correct this, we shall need to introduce some explicit class of 'regular' algebras instead of implicit class of generic algebras. The natural way to remove degeneracies is to consider discriminants.

Let $\mathcal{A}^{0\,*}_{n,k}$ be the GL_n-module dual to $\mathcal{A}^0_{n,k}$, and let S_D be the closure of the orbit of the highest vector, $S_D\subset\mathcal{A}^{0\,*}_{n,k}$. Let $PS_D\subset P\mathcal{A}^{0\,*}_{n,k}$ be its projectivization, let $P\mathcal{D}\subset P\mathcal{A}^0_{n,k}$ be the subvariety projectively dual to the subvariety PS_D, and let $\mathcal{D}\subset\mathcal{A}^0_{n,k}$ be the cone over it. We shall call $\mathcal{D}$ the *D-discriminant subvariety.* The algebras $A\in\mathcal{D}$ are said to be *D-singular.* The algebras $A\notin\mathcal{D}$ are said to be *D-regular.*

Theorem 10.11 ([T1])

(i) *$\mathcal{D}$ is a hypersurface.*
(ii) *Let A be a D-regular algebra. Then the set of k-dimensional subalgebras of A is a smooth irreducible $(k-1)(n-k)$-dimensional subvariety in* $\mathrm{Gr}(k,A)$.
(iii) *Let $k=n-2$. Then the degree of $\mathcal{D}$ is equal to*

$$\frac{(3n^2-5n)2^n-4n(-1)^n}{18.}\tag{10.8}$$

Hence, the D-singularity of A is determined by the vanishing of the SL_n-invariant polynomial D that defines $\mathcal{D}$. This polynomial is called the D-discriminant.

Proof. The fact that $\mathcal{D}$ is a hypersurface follows immediately from the results of Section 9.3. Assertion (ii) can be deduced from the corresponding assertion

of Theorem 10.10 by an easy calculation with differentials. Namely, let A be a D-regular algebra. We use the arguments in the proof of Theorem 4.8 (ii) and Theorem 10.10 (iii). According to these calculations, it is sufficient to verify that if U is a k-dimensional subalgebra of A, then the map

$$\psi : \mathfrak{gl}_n \to \Lambda^k U \otimes A/U,$$

$$\psi(g)(v_1 \wedge \ldots \wedge v_k) = g[v_1, \ldots, v_k] - [gv_1, \ldots, v_k] - \ldots - [v_1, v_2, \ldots, gv_k] + U$$

is surjective. Assume the contrary. Then there is a hyperplane $H \supset U$ such that the image of ψ lies in $\Lambda^k U \otimes H/U$. Consider a non-zero algebra $\tilde{A}$ in $S_D \subset \mathcal{A}^{0\,*}_{n,k}$ such that $[U^\perp, V^*, \ldots, V^*] = 0$, $[V^*, \ldots, V^*] \subset H^\perp$, where $U^\perp$ and $H^\perp$ are the annihilators of U and H in $\mathcal{A}^{0\,*}_{n,k}$. (Such an algebra is unique up to a scalar.) Then $\tilde{A}$ annihilates $[\mathfrak{gl}_n, A]$, which is equivalent to the fact that A annihilates $[\mathfrak{gl}_n, \tilde{A}]$, that is, the tangent space to S_D at $\tilde{A}$. This means that A lies in $\mathcal{D}$, that is, it is a D-singular algebra.

Finally, assertion (iii) follows from Theorem 7.14. □

***E*-regular algebras.** We define the E-discriminant and E-regularity only for $(n-2)$-argument n-dimensional anticommutative algebras. Let $\mathcal{A} = \mathcal{A}^0_{n,n-2}$. Consider the projection $\pi : \mathrm{Gr}(n-1, V) \times P\mathcal{A} \to P\mathcal{A}$ on the second summand and the incidence subvariety $Z \subset \mathrm{Gr}(n-1, V) \times P\mathcal{A}$ that consists of pairs $S \subset PA$, where S is a subalgebra in A. Let $\tilde{\pi} = \pi|_Z$. By Theorem 10.10, we have $\tilde{\pi}(Z) = \mathcal{A}$. Let $\tilde{\mathcal{E}} \subset Z$ be the set of critical points of $\tilde{\pi}$, let $P\mathcal{E} = \tilde{\pi}(\tilde{\mathcal{E}})$ be the set of critical values of $\tilde{\pi}$, and let $\mathcal{E} \subset \mathcal{A}$ be the cone over $P\mathcal{E}$. Then $\mathcal{E}$ is called the *E-discriminant subvariety.* The algebras $A \in \mathcal{E}$ are said to be *E-singular.* The algebras $A \notin \mathcal{E}$ are said to be *E-regular.*

Theorem 10.12 ([T1])

(i) *$\mathcal{E}$ is an irreducible hypersurface.*
(ii) *Let A be an E-regular algebra. Then A has precisely*

$$\frac{2^n - (-1)^n}{3}$$

$(n-1)$-dimensional subalgebras.
(iii) *The map $\tilde{\pi} : \tilde{\mathcal{E}} \to P\mathcal{E}$ is birational.*

Hence, the E-singularity of A is determined by the vanishing of the SL_n-invariant polynomial that defines $\mathcal{E}$. This polynomial is called the *E-discriminant.* Assertion (iii) can be formulated as follows: a generic E-singular algebra has precisely one "critical" $(n-1)$-dimensional subalgebra.

Proof. The proof of assertion (ii) is similar to the proof of Theorem 10.10 (iv). We claim that statement (i) follows from statement (iii). We choose in V^* a basis $\{f_1, \ldots, f_n\}$ dual to the basis $\{e_1, \ldots, e_n\}$ in V. Let $U \in \mathrm{Gr}(n-1, V)$ be the hyperplane $f_1 = 0$. It is clear that $\tilde{\mathcal{E}} = \mathrm{GL}_n \cdot M_0$, where $M_0 = \tilde{\mathcal{E}} \cap (U, P\mathcal{A})$.

Moreover, $M = Z \cap (U, P\mathcal{A})$ is the linear subspace of the algebras for which U is an $(n-1)$-dimensional subalgebra. Let P be the parabolic subgroup of matrices $\begin{pmatrix} A & 0 \\ * & B \end{pmatrix}$, and let $\mathfrak{u}$ be the Lie algebra of matrices $\begin{pmatrix} 0 & X \\ 0 & 0 \end{pmatrix}$, where B is an $(n-1)\times(n-1)$-matrix and X is an $1\times(n-1)$-matrix. Every algebra $A \in M$ defines a linear map $\mathfrak{u} \to \Lambda^{n-1}U^* \otimes V/U$. Since $A \in M_0$ if and only if this map is degenerate, we have $\operatorname{codim}_M M_0 = 1$. It is obvious that M_0 is irreducible, since M_0 is the spreading of the subspace

$$M_0^1 = \{A \in M_0 \,|\, \text{the algebra } E_{12}A \text{ has a subalgebra } U\}$$

by the group P. Hence, $\tilde{\mathcal{E}}$ is an irreducible divisor in Z, and assertion (i) follows from assertion (iii).

We prove (iii). We shall say that an $(n-1)$-dimensional subalgebra U' of $A \in \mathcal{E}$ is *critical* if (U', A) lies in $\tilde{\mathcal{E}}$. In this case there is an $(n-2)$-dimensional subspace $W' \subset U'$ such that if $V = U' \oplus \mathbb{C}e$ and $v \in \mathfrak{gl}_n$ is a non-zero linear operator such that $v(V) \subset \mathbb{C}e$ and $v(W') = 0$, then U' is a subalgebra of vA. To prove assertion (iii) it is sufficient to prove that in generic algebras in M_0^1 the subalgebra U is the unique critical subalgebra. Let $N \subset M_0^1$ be the subvariety of all algebras that have another critical subalgebra.

Note that M_0^1 is normalized by the parabolic subgroup

$$Q = \begin{pmatrix} * & 0 & 0 & \dots & 0 \\ * & * & 0 & \dots & 0 \\ * & * & * & \dots & * \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ * & * & * & \dots & * \end{pmatrix}.$$

Then N is the spreading of the subvarieties N_2 and N_3 by the group Q, where $A \in N_i$ if and only if the hyperplane $f_i = 0$ is a critical subalgebra. In turn, N_2 is the spreading of the vector spaces N_2^1 and N_2^3, and N_3 is the spreading of the subspaces N_3^1, N_3^2, and N_3^4, where $N_i^j \subset N_i$ is the subspace of all algebras such that the $(n-2)$-dimensional subspace W' (mentioned above) is given by $f_i = f_j = 0$. Let $Q_i^j \subset Q$ be the subgroup that normalizes the flag $f_i \subset \langle f_i, f_j \rangle$. It is easy to verify that

$$\operatorname{codim}_Q Q_2^1 = 1,\ \operatorname{codim}_Q Q_2^3 = n-1,\ \operatorname{codim}_Q Q_3^1 = n-1,$$

$$\operatorname{codim}_Q Q_3^2 = n,\ \operatorname{codim}_Q Q_3^4 = 2n-3.$$

On the other hand,

$$\operatorname{codim}_{M_0^1} N_2^1 = n,$$

$$\operatorname{codim}_{M_0^1} N_2^3 = \operatorname{codim}_{M_0^1} N_3^1 = \operatorname{codim}_{M_0^1} N_3^2 = \operatorname{codim}_{M_0^1} N_3^4 = 2n-2.$$

Hence, $\operatorname{codim}_{M_0^1} QN_i^j \geq 1$ in all cases, which completes the proof of the theorem. $\square$

Regular 4-dimensional anticommutative algebras. An $(n-2)$-argument n-dimensional anticommutative algebra is said to be *regular* if it is D-regular and E-regular. In this subsection we consider 2-argument 4-dimensional algebras. The corresponding generic algebras were studied in [T2].

Theorem 10.13 ([T1]) *Let A be a 4-dimensional regular anticommutative algebra. Then*

(i) *A has precisely five 3-dimensional subalgebras. The set of these subalgebras is a generic configuration of five hyperplanes. In particular, A has a pentahedral normal form, that is, it can be reduced by a transformation that belongs to GL_4 to an algebra such that the set of its five subalgebras is a Sylvester pentahedron $x_1 = 0$, $x_2 = 0$, $x_3 = 0$, $x_4 = 0$, $x_1 + x_2 + x_3 + x_4 = 0$.*

(ii) *A has neither one- nor two-dimensional ideals.*

(iii) *The set of two-dimensional subalgebras of A is a del Pezzo surface of degree 5 (a blowing up of $\mathbb{P}^2$ at four generic points).*

(iv) *A has precisely 10 fans, that is, flags $V_1 \subset V_3$ of 1-dimensional and 3-dimensional subspaces such that every intermediate subspace U, $V_1 \subset U \subset V_3$, is a two-dimensional subalgebra.*

Proof. We begin with assertion (i). We have to prove that if A is a 4-dimensional regular anticommutative algebra with zero trace, then the set of its three-dimensional subalgebras $S_1, S_2, \dots, S_5$ is a generic configuration of hyperplanes, that is, the intersection of any three of them is one-dimensional and the intersection of any four of them is zero-dimensional. Indeed, assume, for example, that $U = S_1 \cap S_2 \cap S_3$ is two-dimensional. Let $v \in U$ and $v \neq 0$. Then $[v, \cdot]$ induces a linear operator on A/U, since U is a subalgebra. S_1/U, S_2/U, and S_3/U are one-dimensional eigenspaces. Since $\dim A/U = 2$, the operator is a dilation. Since this is true for any $v \in U$, any three-dimensional subspace that contains U is a three-dimensional subalgebra, which contradicts the fact that there are precisely five such subalgebras.

Now assume that $U = S_1 \cap S_2 \cap S_3 \cap S_4$ is one-dimensional, and let $v \in U$, $v \neq 0$. Then the operator $[v, \cdot]$ induces an operator on A/U. This operator has four two-dimensional eigenspaces $S_1/U, \dots, S_4/U$ of which any three have zero intersection. Hence, this operator is a dilation. Let $W \supset U$ be an arbitrary two-dimensional subspace, and let $w \in W$ be a vector that is not proportional to v. Then the operator $[w, \cdot]$ induces a linear operator on A/W. Let z be a non-zero eigenvector. Then $\langle v, w, z \rangle$ is a three-dimensional subalgebra. Therefore, every vector can be included in a three-dimensional subalgebra, which contradicts the fact that there are only five such subalgebras.

This argument also shows that A has no one-dimensional ideals. Since every three-dimensional subspace that contains a two-dimensional ideal is a

subalgebra, there are no two-dimensional ideals, which completes the proof of assertion (ii).

To prove assertion (iii), we consider the subvariety X of two-dimensional subalgebras in A. Then $X \subset \mathrm{Gr}(2,4)$. Consider the Plücker embedding $\mathrm{Gr}(2,4) \subset \mathbb{P}^5 = P(\Lambda^2\mathbb{C}^4)$. First we claim that the embedding $X \subset \mathbb{P}^5$ is non-degenerate, that is, the image is contained in no hyperplane. Let S_1, S_2, S_3, S_4 be four three-dimensional subalgebras. Since it is a generic configuration, we can choose a basis $\{e_1, e_2, e_3, e_4\}$ in A such that $S_i = \langle e_1, \dots, \hat{e}_i, \dots, e_4\rangle$. Since the intersection of three-dimensional subalgebras is a two-dimensional subalgebra, A has six subalgebras $\langle e_i, e_j\rangle$, $i \neq j$. The set of corresponding bivectors $e_i \wedge e_j$ is a basis in $\Lambda^2\mathbb{C}^4$. Therefore, they can lie in no hyperplane. Simple calculation with Koszul complexes (cf. Theorem 3.3) shows that $H^0(X, \mathcal{O}_X(1))^* = \Lambda^2\mathbb{C}^4$.

To prove that X is a del Pezzo surface of degree five, we have only to verify that $\mathcal{O}_X(1)$ coincides with the anticanonical sheaf (see [Ma]). Since $Y = \mathrm{Gr}(2,4)$ is a quadric in $\mathbb{P}^5$, we have

$$\omega_Y = \mathcal{O}_Y(2-5-1) = \mathcal{O}_Y(-4).$$

The set X is a non-singular subvariety of codimension 2 in Y. Therefore, $\omega_X = \omega_Y \otimes \Lambda^2\mathcal{N}_{X/Y}$, where $\mathcal{N}_{X/Y}$ is the normal sheaf. Further, X is the scheme of zeros of a regular section of the vector bundle $\mathcal{L} = \Lambda^2\mathcal{S}^* \otimes V/\mathcal{S}$, whence $\mathcal{N}_{X/Y} = \mathcal{L}|_Y$ and $\Lambda^2\mathcal{N}_{X/Y} = \mathcal{O}_X(3)$, since $c_1(\mathcal{L}) = 3H$. We obtain that $\omega_X = \mathcal{O}_X(-4) \otimes \mathcal{O}_X(3) = \mathcal{O}_X(-1)$, as was to be shown.

It remains to prove assertion (iv). Since X is a del Pezzo surface of degree five, it contains ten straight lines. Since the embedding $X \subset P(\Lambda^2\mathbb{C}^4)$ is anticanonical, these straight lines are ordinary straight lines in $P(\Lambda^2\mathbb{C}^4)$ that lie in $\mathrm{Gr}(2,4)$. It remains to establish a bijection between these straight lines and fans. If $b \in \Lambda^2\mathbb{C}^4$, then b belongs to the cone over $\mathrm{Gr}(2,4)$ if and only if $b \wedge b = 0$. If b_1 and b_2 belong to this cone, then the straight line that joins them belongs to this cone if and only if $b_1 \wedge b_2 = 0$, which coincides with the fan condition. □

Dodecahedral section. Let us start with some definitions. Let X be an irreducible G-variety (a variety with an action of algebraic group G), $S \subset X$ be an irreducible subvariety. Then S is called a *section* of X if $\overline{G \cdot S} = X$. The section S is called a relative section if the following condition holds: there exists a dense Zariski-open subset $U \subset S$ such that if $x \in U$ and $gx \in S$ then $g \in H$, where $H = N_G(S) = \{g \in G \,|\, gS \subset S\}$ is the normalizer of S in G (see [PV]). In this case for any invariant function $f \in \mathbb{C}(X)^G$ the restriction $f|_S$ is well-defined and the map

$$\mathbb{C}(X)^G \to \mathbb{C}(S)^H, \quad f \mapsto f|_S,$$

is an isomorphism. Any relative section defines a G-equivariant rational map $\psi : X \to G/H$: if $g^{-1}x \in S$ then $x \mapsto gH$. Conversely, any G-equivariant

rational map $\psi : X \to G/H$ with irreducible fibers defines the relative section $\overline{\psi^{-1}(eH)}$.

We are going to apply Theorem 10.13 and to construct a relative section in the SL_4-module $\mathcal{A}_0$ (the module of 4-dimensional anticommutative algebras with zero trace). The action of SL_4 on 'Sylvester pentahedrons' is transitive with finite stabilizer H (which is the central extension of the permutation group S_5). In the sequel the Sylvester pentahedron will always mean the standard configuration formed by the hyperplanes

$$x_1 = 0,\ x_2 = 0,\ x_3 = 0,\ x_4 = 0,\ x_1 + x_2 + x_3 + x_4 = 0.$$

Let $S \subset \mathcal{A}_0$ be a linear subspace formed by all algebras such that the hyperplanes of Sylvester pentahedron are their subalgebras. Then Theorem 10.13 implies that S is a 5-dimensional linear relative section of SL_4-module $\mathcal{A}_0$.

It is easy to see that the multiplication in algebras from S is given by formulas

$$[e_i\,, e_j] = a_{ij}e_i + b_{ij}e_j \quad (1 \le i < j \le 4),$$

where a_{ij} and b_{ij} satisfy the certain set of linear conditions. Consider 6 algebras $A_1, \ldots, A_6$ with the following structure constants

	A_1	A_2	A_3	A_4	A_5	A_6
a_{12}	0	1	-1	1	0	-1
b_{12}	1	0	1	-1	-1	0
a_{13}	1	1	-1	0	-1	0
b_{13}	0	-1	0	1	1	-1
a_{14}	1	0	0	1	-1	-1
b_{14}	-1	1	-1	0	0	1
a_{23}	-1	-1	0	1	0	1
b_{23}	1	0	-1	0	1	-1
a_{24}	0	-1	-1	0	1	1
b_{24}	-1	1	0	1	-1	0
a_{34}	-1	1	1	-1	0	0
b_{34}	0	0	-1	1	-1	1

Then it is easy to see that $A_i \in S$ for any i. Moreover, algebras A_i satisfy the unique linear relation $A_1 + \ldots + A_6 = 0$. It follows that any $A \in S$ can be written uniquely in the form $\alpha_1 A_1 + \ldots + \alpha_6 A_6$, where $\alpha_1 + \ldots + \alpha_6 = 0$. The coordinates α_i are called dodecahedral coordinates and S is called the dodecahedral section (this name will be clear later).

The stabilizer of the standard Sylvester pentahedron in PGL_4 is isomorphic to $\mathbb{S}_5$ represented by permutations of its hyperplanes. The group S_5 is generated by the transposition (12) and the cycle (12345). The preimages of these elements in GL_4 are given by matrices

$$\sigma = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad \tau = \begin{pmatrix} -1 & -1 & -1 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

The preimage of S_5 in SL_4 is the group H of 480 elements. The representation of H in S induces the projective representation of S_5 in $\mathbb{P}^4$. We have the following

Proposition 10.14 *This projective representation is the projectivization of the* 5*-dimensional irreducible representation of* S_5 *(any of two possible).*

Proof. Recall certain 'folklore' facts about the representation theory of S_5. It is well-known that S_5 admits exactly two embeddings in S_6 up to conjugacy. One is standard via permutations of the first five elements of the six-element set permuted by S_6. The second one can be obtained from the first one by taking the composition with the unique (up to conjugacy) outer involution of S_6. S_5 has exactly two irreducible 5-dimensional representations, which have the same projectivizations. One of these representations has the following model. One takes the tautological 5-dimensional irreducible representation of S_6 and consider its composition with the non-standard embedding $S_5 \subset S_6$.

Now let us return to our projective representation. The action of σ and τ on algebras A_i is given by formulas

$$\begin{pmatrix} \sigma A_1 \\ \sigma A_2 \\ \sigma A_3 \\ \sigma A_4 \\ \sigma A_5 \\ \sigma A_6 \end{pmatrix} = \begin{pmatrix} -A_2 \\ -A_1 \\ -A_4 \\ -A_3 \\ -A_6 \\ -A_5 \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} \tau A_1 \\ \tau A_2 \\ \tau A_3 \\ \tau A_4 \\ \tau A_5 \\ \tau A_6 \end{pmatrix} = \begin{pmatrix} A_1 \\ A_6 \\ A_2 \\ A_3 \\ A_4 \\ A_5 \end{pmatrix}. \tag{10.9}$$

Therefore, S_5 permutes the lines spanned by A_i. Moreover, the induced embedding is clearly non-standard: transposition in S_5 maps to the composition of 3 independent transpositions in S_6. It is clear that that the corresponding projective representation of S_5 is isomorphic to the projectivization of the 5-dimensional irreducible representation in the model described above. □

Remark 10.15 The section S is called dodecahedral by the following reason. Though the surjection $H \to S_5$ does not split, the alternating group A_5 can be embedded in H. The induced representation of A_5 in S has the following description. $\mathbb{A}_5$ can be realised as a group of rotations of dodecahedron. Let $\{\Gamma_1, \ldots, \Gamma_6\}$ be the set of pairs of opposite faces of dodecahedron. Consider the vector space of functions

$$f : \{\Gamma_1, \ldots, \Gamma_6\} \to \mathbb{C}, \quad \sum_{i=1}^{6} f(\Gamma_i) = 0.$$

Then this vector space is an A_5-module. It is easy to see that this module is isomorphic to S via the identification $A_i \mapsto f_i$, where $f_i(\Gamma_i) = 5$, $f_i(\Gamma_j) = -1$, $j \neq i$.

The following proposition follows from the discussion above

Theorem 10.16 *The restriction of invariants induces an isomorphism of invariant fields*

$$\mathbb{C}(\mathcal{A}_0)^{GL_4} \simeq \mathbb{C}(\mathbb{C}^5)^{\mathbb{C}^* \times \mathbb{S}_5},$$

where $\mathbb{C}^*$ *acts on* $\mathbb{C}^5$ *by homotheties and* $\mathbb{S}_5$ *acts via any of two* 5*-dimensional irreducible representations.*

Remark 10.17 The Sylvester pentahedron also naturally arises in the theory of cubic surfaces. The SL_4-module of cubic forms $S^3(\mathbb{C}^4)^*$ admits the relative section (the so-called Sylvester section, or Sylvester normal form). Namely, a generic cubic form in a suitable system of homogeneous coordinates $x_1, \ldots, x_5$, $x_1 + \ldots + x_5 = 0$, can be written as a sum of 5 cubes $x_1^3 + \ldots + x_5^3$. The Sylvester pentahedron can be recovered from a generic cubic form f in a very interesting way: its 10 vertices coincide with 10 singular points of a quartic surface $\det \mathrm{Hes}(f)$. The Sylvester section has the same normalizer H as our dodecahedral section. It can be proved [Bek] that in this case the restriction of invariants induces an isomorphism

$$\mathbb{C}(S^3(\mathbb{C}^4)^*)^{GL_4} \simeq \mathbb{C}(\mathbb{C}^5)^{\mathbb{C}^* \times \mathbb{S}_5},$$

where $\mathbb{C}^*$ acts via homotheties and $\mathbb{S}_5$ via permutations of coordinates (i.e. via the *reducible* 5-dimensional representation). Other applications of the Sylvester pentahedron to moduli varieties can be found in [Bar].

These results were used in [T2] in order to prove that the field of invariant functions of the 5-dimensional irreducible representation of S_5 is rational (is isomorphic to the field of invariant functions of a vector space). From this result it is easy to deduce that in fact the field of invariant functions of any representation of S_5 is rational (see [T5]).

10.3 Self–dual Varieties

10.3.A Self–dual Polarized Flag Varieties

A projective variety $X \subset \mathbb{P}^n$ is called self-dual if X is isomorphic to X^* as an embedded projective variety, i.e. there exists an isomorphism $f : \mathbb{P}^N \to \mathbb{P}^{n*}$ such that $f(X) = X^*$. A lot of examples is provided by the Pyasetskii pairing 2.9. Indeed, if G is a connected reductive algebraic group acting on a vector space V then the dual action $G : V^*$ can be described as follows.

Having passed to a finite covering of G, one may assume that there is an involution $\theta \in \mathrm{Aut}(G)$ (a so-called involution of maximal rank) such that $\theta(t) = t^{-1}$ for any $t \in T$, where T is the fixed maximal torus in G. Along with the action of G on V we may consider the twisted action given by $g \star v = \omega(g)v$. Then it is easy to see that the twisted action is isomorphic to the dual action $G : V^*$. Therefore, we may canonically identify G-orbits in V and in V^*. In particular, if G acts on V with finitely many orbits then we may consider the Pyasetskii pairing (hence the projective duality of projectivizations) as pairing between G-orbits in V.

For example, let L be a simple algebraic group and P a parabolic subgroup with Abelian unipotent radical. In this case $\mathfrak{l} = \mathrm{Lie}\, L$ admits a short $\mathbb{Z}$-grading with only three non-zero parts:

$$\mathfrak{l} = \mathfrak{l}_{-1} \oplus \mathfrak{l}_0 \oplus \mathfrak{l}_1.$$

Here $\mathfrak{l}_0 \oplus \mathfrak{l}_1 = \mathrm{Lie}\, P$ and $\exp(\mathfrak{l}_1)$ is the Abelian unipotent radical of P. Let $G \subset L$ be a reductive subgroup with Lie algebra $\mathfrak{l}_0$. Recall that by Theorem 2.17 G has finitely many orbits in $\mathfrak{l}_1$ naturally labelled by integers from the segment $[0, r]$ such that $\mathcal{O}_k$ corresponds to $\mathcal{O}_{r-k}$ via Pyasetskii pairing (here we use the above identifications of G-orbits in $\mathfrak{l}_1$ and $\mathfrak{l}_{-1}$). Therefore, if r is even then the closure of the projectivization of $\mathcal{O}_{\frac{r}{2}}$ is a self-dual projective variety.

However, it is well-known that the closure of the projectivization of $\mathcal{O}_i$ is smooth if and only if $i = 1$ or $i = r$ (in the last case $\overline{\mathbb{P}(\mathcal{O}_r)} = \mathbb{P}(\mathfrak{l}_1)$), see e.g. [Pan] for the equivariant resolution of singularities of $\overline{\mathcal{O}_i}$. Therefore, the construction above gives a smooth self-dual projective variety if and only if $r = 2$. It is worthy to write down all arising cases. We use the notation from Section 2.3.

Example 10.18 Consider the short gradings of $\mathfrak{l} = \mathfrak{sl}_{n+2}$. Then $\mathcal{O}_1$ is the variety of $n \times 2$-matrices of rank 1. The projectivization of $\mathcal{O}_1$ is identified with $X = \mathbb{P}^{n-1} \times \mathbb{P}^1$ in the Segre embedding.

Example 10.19 Consider the short grading of $\mathfrak{l} = \mathfrak{so}_{n+2}$ that corresponds to β being the first simple root. The projectivization of $\mathcal{O}_1$ is identified with the quadric hypersurface in $\mathbb{P}^{n-1}$. The quadric hypersurface in $\mathbb{P}^5$ also arises from the short grading of $\mathfrak{l} = \mathfrak{so}_8 = D_4$ that corresponds to $\beta = \alpha_1$, $\beta = \alpha_3$, or $\beta = \alpha_4$ (Plücker quadric).

Example 10.20 Consider the short grading of $\mathfrak{l} = \mathfrak{so}_{10} = D_5$ that corresponds to $\beta = \alpha_4$ or $\beta = \alpha_5$. The projectivization of $\mathcal{O}_1$ is identified with $X = \mathrm{Gr}(2, 5)$ in the Plücker embedding.

Example 10.21 The short grading of E_6 gives the spinor variety $\mathbb{S}_5$.

Remarkably, there are no other known examples of smooth non-linear self-dual varieties. Moreover, it is widely expected that this list is complete. At least, this is true for polarized flag varieties:

Theorem 10.22 ([Sn]) *Let $X = G/P \subset \mathbb{P}^N$ be a non-linear polarized flag variety. If $\dim X = \dim X^*$ the X is one of the following*

- *A quadric hypersurface.*
- *The Segre embedding of $\mathbb{P}^n \times \mathbb{P}^1$.*
- *The Plücker embedding of $\mathrm{Gr}(2,5)$.*
- *The 10-dimensional spinor variety $\mathbb{S}_5$.*

Proof. If $\dim X = \dim X^*$ then either X is a hypersurface or $\operatorname{def} X > 0$. In the first case X is necessarily a quadric hypersurface. Indeed, let $Y \subset \mathbb{P}^{N^*}$ be the projectivization of the highest weight vector orbit of the dual representation. Then Y is isomorphic to X as an embedded projective variety. The intersection $Y^* \cap X$ is non-empty and G-invariant, therefore, $Y^* = X$ and X is a self-dual smooth hypersurface. Then X is a quadric hypersurface by Example 7.4. An alternative proof follows from the fact that the closure of the highest weight vector orbit is always cut out by quadrics [Li].

If $\operatorname{def} X > 0$ then the claim easily follows from Theorem 9.21. □

Theorem 10.22 was used in [Sn] to recover the classification of homogeneous real hypersurface in a complex projective space due to [Ta].

Theorem 10.23 ([Ta, Sn]) *Let M be a homogeneous complete real hypersurface embedded equivariantly in $\mathbb{P}^N$. Then M is a tube over a linear projective space or one of the 4 self-dual homogeneous spaces $X \subset \mathbb{P}^N$ listed in Theorem 10.22.*

Sketch of the proof. Let $M = K/L$, where K is a compact Lie group. The main idea is to use the fact that M is necessarily a tube over a *complex* submanifold $X \subset \mathbb{P}^N$ called a focal submanifold, see e.g. [CR]. Then X is easily seen to be homogeneous, therefore X is a flag variety, $X = G/P$, where G is the complexification of K. Moreover, G acts transitively on the normal directions to X. It follows that the conormal variety $\mathbb{P}(N_X\mathbb{P}^N)$ is also homogeneous, therefore the dual variety X^* is also homogeneous being the image of the conormal variety. Hence X^* is smooth and therefore $\dim X = \dim X^*$ by Theorem 5.13. Now we can apply Theorem 10.22. □

10.3.B Around Hartshorne Conjecture

In [Ha2] R. Hartshorne has suggested a number of conjectures related to the geometry of projective varieties of small codimension. This work has stimulated a serie of remarkable researches, for example the Zak's proofs of Theorem 5.17 on linear normality and Theorem 5.19 on Severi varieties. Undoubtfully, the most famous conjecture from this paper is the so-called Hartshorne conjecture on complete intersections:

Conjecture 10.24 ([Ha2]) *If X is a smooth n-dimensional projective variety in $\mathbb{P}^N$ and* $\operatorname{codim} X < N/3$*, then X is a complete intersection.*

This conjecture is still very far from being solved. Only partial results are known, for example the following Landsberg's theorem on complete intersections. The proof is based on the technique of Section 6.4.

Theorem 10.25 ([Lan1]) *Let $X \subset \mathbb{P}(V) = \mathbb{P}^N$ be an irreducible projective variety cut out by quadrics, that is, the homogeneous ideal $I_{\mathrm{Cone}(X)}$ in $\mathbb{C}[V]$ is generated by its second homogeneous component $I^2_{\mathrm{Cone}(X)}$. Let $b = \dim \operatorname{Sing} X$ (set $b = -1$ if X is smooth). If* $\operatorname{codim} X < \dfrac{N-(b+1)+3}{4}$*, then X is a complete intersection.*

If Hartshorne conjecture is true then any smooth projective variety X in $\mathbb{P}^N$ such that $\operatorname{codim} X < N/3$ is a complete intersection. Therefore, X^* should be a hypersurface by Theorem 4.2. It is not known whether it is true or not. In particular, if X is a smooth self-dual variety then, up to Hartshorne Conjecture, either X is a quadric hypersurface or $\operatorname{codim} X \geq N/3$. In particular, the following theorem should give a complete list of smooth self-dual varieties:

Theorem 10.26 ([E1]) *Let X be a nonlinear smooth projective variety in $\mathbb{P}^N$. Assume that* $\operatorname{codim} X \geq N/3$*. Suppose that* $\dim X = \dim X^*$*. Then X is one of the following varieties:*

(i) *X is a hypersurface in $\mathbb{P}^2$ or $\mathbb{P}^3$.*
(ii) *X is the Segre embedding of $\mathbb{P}^1 \times \mathbb{P}^{n-1}$ in $\mathbb{P}^{2n-1}$.*
(iii) *X is the Plücker embedding of* $\mathrm{Gr}(2,5)$ *in $\mathbb{P}^9$.*
(iv) *X is the* 10*-dimensional spinor variety $\mathbb{S}_5$ in $\mathbb{P}^{15}$.*

Sketch of the proof. Let $\dim X = n$. We may assume that $n \geq 3$ by Example 1.13 and Example 6.16. Now $\operatorname{def} X = N - 1 - n$. Since $\operatorname{def} X \leq n-2$ by Example 6.17, we conclude that $n \geq \dfrac{N+1}{2}$. If $n = \dfrac{N+1}{2}$, then $\operatorname{def} X = n-2$ and, therefore, we have case (ii) by Example 6.18. So we may assume that $n \geq N/2+1$. Then $K_X = \mathcal{O}_X(\dfrac{-N-1}{2})$ by Theorem 6.19 (d). We conclude that $\operatorname{def} X = N-1-n \leq \dfrac{n-2}{2}$ by Theorem 6.23 (a). Hence $n \geq 2N/3$. By our assumption $n \leq 2N/3$. Therefore, $n = 2N/3$. Now $\operatorname{def} X = N-1-n = \frac{1}{2}n - 1$. Therefore n is even. Since $\operatorname{def} X \equiv n \mod 2$ by Theorem 6.15, we conclude that $n = 4m+2$ and $\operatorname{def} X = 2m$. It follows that $m \leq 2$ by Theorem 6.23 (b). This leaves two cases: X is either a 6-dimensional variety in $\mathbb{P}^9$ or a 10-dimensional variety in $\mathbb{P}^{15}$. A very delicate treatment of both cases can be found in [E1]. □

10.3.C Self–dual Nilpotent Orbits

If $X \subset \mathbb{P}^n$ is a smooth projective variety then X is almost never self-dual. Moreover, up to Hartshorne conjecture Theorem 10.26 provides a complete list of them. However, there are a lot of non-smooth self-dual varieties. Many equivariant self-dual varieties are provided by Pyasetskii Theorem 2.9. Perhaps, the most interesting examples of self-dual varieties are the Kummer surface in $\mathbb{P}^3$ [GH1] and the Coble quartic in $\mathbb{P}^7$ [Pau].

Another interesting examples were found in [Pop]. Let G be a connected reductive group acting linearly on a vector space V. Suppose that there exists a G-invariant non-degenerate scalar product $(\cdot,\cdot)$ on V, in particular we have a natural isomorphism $V \simeq V^*$. Finally, suppose that G acts on the null-cone

$$\mathcal{R}(V) = \{v \in V \mid \overline{G \cdot v} \ni 0\}$$

with finitely many orbits. In particular, any G-orbit $\mathcal{O} \subset \mathcal{R}(V)$ is conical.

Theorem 10.27 ([Pop]) *Let $\mathcal{O} = G \cdot v \subset \mathcal{R}(V)$ be a non-zero orbit, $X = \overline{\mathbb{P}(\mathcal{O})}$ be the closure of its projectivization. Then $X = X^*$ if and only if $(\mathfrak{g} \cdot v)^\perp \subset \mathcal{R}(V)$.*

Proof. Since $\mathcal{O}$ is conical, $v \in \mathfrak{g} \cdot v$. Therefore, $\mathfrak{g} \cdot v$ is the affine cone $\mathrm{Cone}(\hat{T}_{[v]}X)$ over the embedded tangent space $\hat{T}_{[v]}X$. Hence, X^* is equal to $\overline{\mathbb{P}(G \cdot (\mathfrak{g} \cdot v)^\perp)}$. Since $(\cdot,\cdot)$ is G-invariant, we have $v \in (\mathfrak{g} \cdot v)^\perp$, therefore, $X \subset X^*$. If $X = X^*$ then, obviously, $(\mathfrak{g} \cdot v)^\perp \subset \mathcal{R}(V)$. Suppose that $(\mathfrak{g} \cdot v)^\perp \subset \mathcal{R}(V)$. Then $X^* \subset \mathbb{P}(\mathcal{R}(V))$. Therefore, $X^* = \overline{\mathbb{P}(\mathcal{O}')}$ for some orbit $\mathcal{O}' \subset \mathcal{R}(V)$. Applying the same arguments we get that $X^* \subset X^{**}$. By Reflexivity Theorem, we finally obtain $X = X^*$. $\square$

Let G be semisimple, $V = \mathfrak{g}$ be the adjoint representation. Then the Killing form $(\cdot,\cdot)$ is G-invariant, $\mathcal{R}(\mathfrak{g})$ is the cone of nilpotent elements. Since there are finitely many nilpotent orbits, we can apply the previous Theorem. For any $x \in \mathfrak{g}$ the orthogonal complement $(\mathrm{ad}(\mathfrak{g})x)^\perp$ is identified with the centralizer $\mathfrak{g}_x$ of x in $\mathfrak{g}$. Therefore, a nilpotent orbit $\mathcal{O} = \mathrm{Ad}(G)x$ has a self-dual projectivization if and only if the centralizer $\mathfrak{g}_x$ belongs to $\mathcal{R}(\mathfrak{g})$, i.e. has no semi-simple elements. These orbits are called the distinguished nilpotent orbits. For example, the regular nilpotent orbit (the orbit dense in $\mathcal{R}(\mathfrak{g})$) is distinguished, hence the null-cone $\mathcal{R}(\mathfrak{g})$ itself has a self-dual projectivization.

The distinguished nilpotent orbits are, in a sense, the building blocks for the set of all nilpotent orbits. Namely, due to the Bala–Carter correspondence [BCa1, BCa2] the set of all nilpotent orbits in a semisimple Lie algebra $\mathfrak{g}$ is in the natural bijection with the set of isoclasses of pairs $(\mathfrak{l}, \mathcal{O})$, where $\mathfrak{l} \subset \mathfrak{g}$ is a Levi subalgebra and $\mathcal{O} \subset \mathfrak{l}$ is a distinguished nilpotent orbit in $[\mathfrak{l}, \mathfrak{l}]$.

10.4 Linear Systems of Quadrics of Constant Rank

Theorem 10.28 ([IL]) *Let $X \subset \mathbb{P}^N$ be a smooth n-dimensional variety. If $H \in X^*{}_{sm}$ then the projective second fundamental form $|II|_H$ of X^* at H is a system of quadrics of projective dimension* $\operatorname{def} X$ *and constant rank* $n - \operatorname{def} X$.

This Theorem (analogous to Theorem 6.12) leads to the examination of linear systems of quadrics of constant rank.

More generally, let $V = \mathbb{C}^m$, $W = \mathbb{C}^n$ and let $A \subset \operatorname{Hom}(V, W)$ (resp. $A \subset S^2V^*$, resp. $A \subset \Lambda^2 V^*$) be a linear subspace.

Definition 10.29 A is said to be of bounded rank r if for all $f \in A$, $\operatorname{rank} f \leq r$. A is said to be of constant rank r if for all non-zero $f \in A$, $\operatorname{rank} f = r$. A is said to be of rank bounded below by r if for all non-zero $f \in A$, $\operatorname{rank} f \geq r$.

We define the numbers

$$l(r, m, n) = \max\{\dim A \mid A \subset \operatorname{Hom}(V, W) \text{ is of constant rank } r\}$$

$$\underline{l}(r, m, n) = \max\{\dim A \mid A \subset \operatorname{Hom}(V, W) \text{ is of bounded rank } r\}$$

$$\bar{l}(r, m, n) = \max\{\dim A \mid A \subset \operatorname{Hom}(V, W) \text{ is of rank bounded below by } r\}$$

Similarly, define $c(r, m)$, $\underline{c}(r, m)$, and $\bar{c}(r, m)$ in the symmetric case and $\lambda(r, m)$, $\underline{\lambda}(r, m)$, and $\bar{\lambda}(r, m)$ in the skew-symmetric case. Recall that any skew-symmetric matrix has even rank.

Proposition 10.30 ([IL]) *The upper bounds for dimensions of linear subspaces of matrices of rank bounded below by r are as follows.*

(a) $\bar{l}(r, m, n) = (m - r)(n - r)$;
(b) $\bar{c}(r, m) = \binom{m-r+1}{2}$;
(c) $\bar{\lambda}(r, m) = \binom{m-r}{2}$, *$r$ even.*

Proof. Let

$$X_r = \{f \in \mathbb{P}(\operatorname{Hom}(V, W)) \mid \operatorname{rank} f \leq r\}$$

and similarly for $X_r^c \subset \mathbb{P}(S^2V)$ and $X_r^\lambda \subset \mathbb{P}(\Lambda^2 V)$. Clearly, $\bar{l}(r, m, n) = \operatorname{codim} X_r$, $\bar{c}(r, m) = \operatorname{codim} X_r^c$, and $\bar{\lambda}(r, m) = \operatorname{codim} X_r^\lambda$. Now the theorem follows by an easy dimension count. □

Given a linear subspace $A \subset \operatorname{Hom}(V, W)$ we can also consider A as a linear subspace A^c of $S^2(V^* \oplus W)$ or a linear subspace A^λ of $\Lambda^2(V^* \oplus W)$. In matrix form, for any matrix $a \in A$ we associate a symmetric matrix $a^c \in A^c$ or skew-symmetric matrix $a^\lambda \in A^\lambda$ of the form

$$a^c = \begin{pmatrix} 0 & a \\ a^t & 0 \end{pmatrix}, \quad a^\lambda = \begin{pmatrix} 0 & a \\ -a^t & 0 \end{pmatrix}.$$

If A has constant rank r (resp. bounded rank r, resp. rank bounded below by r) then A^c and A^λ have constant rank $2r$ (resp. bounded rank $2r$, resp. rank bounded below by $2r$). A^c and A^λ are called doublings of A.

Proposition 10.31 ([IL]) *The lower bounds for dimensions of linear subspaces of matrices of constant rank are as follows.*

(a) *If* $0 < r \le m \le n$ *then* $l(r,m,n) \ge n - r + 1$.
(b) *If* $r \ge 2$ *is even then* $c(r,m) \ge m - r + 1$ *and* $\lambda(r,m) \ge m - r + 1$.

Proof. $X_{r-1} \subset \mathbb{P}(\mathrm{Hom}(\mathbb{C}^r, \mathbb{C}^n)$ has codimension $n-r+1$. Thus we can find an $(n-r+1)$-dimensional subspace $A \subset \mathrm{Hom}(\mathbb{C}^r, \mathbb{C}^n)$ of constant rank r and (a) follows. The doublings of A produce $(n-r+1)$-dimensional subspaces $A^c \subset S^2\mathbb{C}^{r+n}$ and $A^\lambda \subset \Lambda^2\mathbb{C}^{r+n}$ of constant rank $2r$. This implies (b). □

Any linear map $\psi : A \to \mathrm{Hom}(V,W)$ can be viewed as an element of

$$\begin{aligned} A^* \otimes \mathrm{Hom}(V,W) &= H^0(\mathbb{P}(A), \mathcal{O}_{\mathbb{P}(A)}(1)) \otimes \mathrm{Hom}(V,W) \\ &= H^0(\mathbb{P}(A), \mathcal{O}_{\mathbb{P}(A)}(1) \otimes \mathrm{Hom}(V,W)). \end{aligned}$$

Thus to give a linear map $A \to \mathrm{Hom}(V,W)$ (say, embedding) is the same as to give a vector bundle map

$$\overline{\psi} : V \otimes \mathcal{O}_{\mathbb{P}(A)} \to W \otimes \mathcal{O}_{\mathbb{P}(A)}(1)$$

on $\mathbb{P}(A)$. Let K, N, and E be the kernel, the cokernel, and the image of $\overline{\psi}$. Then it is easy to see that ψ is an embedding of constant rank r if and only if all K, N, and E are vector bundles of ranks $\dim V - r$, $\dim W - r$, and r if and only if any of K, N, and E is a vector bundle of rank as above. Therefore, all embeddings $A \subset \mathrm{Hom}(V,W)$ of constant rank are in one-one correspondence with exact sequences of vector bundles

$$0 \to K \to \mathcal{O}_{\mathbb{P}(A)} \otimes V \to E \to 0 \text{ and } 0 \to E \to \mathcal{O}_{\mathbb{P}(A)}(1) \otimes W \to N \to 0.$$

Similarly, all embeddings $A \subset \mathcal{S}^2 V$ (resp. $A \subset \Lambda^2 V$) of constant rank are in one-one correspondence with exact sequences of vector bundles

$$0 \to K \to \mathcal{O}_{\mathbb{P}(A)} \otimes V^* \to E \to 0,$$

where E is a rank r vector bundle such that $E \simeq E^*(1)$ and this isomorphism is symmetric (resp. skew-symmetric).

Theorem 10.32 ([IL])

(a) *Let* $A \subset S^2V$ *or* $A \subset \Lambda^2 V$ *be a constant rank* r *subspace,* $\dim A \ge 2$. *Then* E *is a uniform vector bundle of splitting type* $\mathcal{O}_{\mathbb{P}^1}^{r/2} \oplus \mathcal{O}_{\mathbb{P}^1}^{r/2}(1)$. *In particular,* r *is even.*
(b) *If* r *is odd then* $c(r,m) = \lambda(r,m) = 1$.

Proof. (a) The proof is by the same argument as in the proof of Theorem 6.14 (b).

(b) This follows from (a). However, in the symmetric case it was also known classically as a consequence of the Kronecker–Weierstrass theory giving a normal form for pencils of symmetric matrices of bounded rank, see [Gan, HP, Me]. □

Theorem 10.33 ([E1, IL]) *A smooth n-dimensional projective variety $X \subset \mathbb{P}^N$ with positive defect* $\operatorname{def} X$ *determines a* $(\operatorname{def} X + 1)$*-dimensional linear subspace $A \subset S^2\mathbb{C}^{N-1-\operatorname{def} X}$ with associated short exact sequence*

$$0 \to N_X^* H(1)|_L \to \mathcal{O}_L \otimes \mathbb{C}^{N-1-\operatorname{def} X} \to N_L X \to 0,$$

where $H \in X^*{}_{sm}$ and $L = \operatorname{Sing} X \cap H$ is the contact locus.

Proof. Recall that L is a projective subspace of dimension $\operatorname{def} X$. We have the following exact sequence

$$0 \to N_L X \to N_L H \to N_X H|_L \to 0$$

which holds because H is tangent to X along L. Dualising and twisting by $\mathcal{O}_L(1)$ we get

$$0 \to N_X^* H(1)|_L \to N_L^* H(1) \to N_L^* X(1) \to 0.$$

It remains to use the fact that $N_L H \simeq \mathcal{O}_L(1) \otimes \mathbb{C}^{N-1-\operatorname{def} X}$ and, most importantly, that $N_L X \simeq N_L^* X(1)$ by Theorem 6.14 (a). □

The following theorem is a consequence of the general theory of uniform vector bundles [Sa].

Theorem 10.34 ([E1, IL])

(a) *Suppose that $A \subset S^2 V$ is a linear $(l+1)$-dimensional subspace of constant rank r. Let $0 \to K \to \mathcal{O}_{\mathbb{P}(A)} \otimes V^* \to E \to 0$ be the corresponding exact sequence of vector bundles. If $r \le l$ then $E \simeq \mathcal{O}_{\mathbb{P}(A)}^{r/2} \oplus \mathcal{O}_{\mathbb{P}(A)}^{r/2}(1)$ unless $r = l = 2$ and $E = T\mathbb{P}^2(-1)$.*

(b) *Suppose that $X \subset \mathbb{P}^N$ is a smooth projective variety. Let L be a contact locus with a generic tangent hyperplane. If* $\operatorname{def} X \ge N/2$*, then*

$$N_L X = \mathcal{O}_L^{(\dim X - \operatorname{def} X)/2} \oplus \mathcal{O}_L^{(\dim X - \operatorname{def} X)/2}(1).$$

Proof. (a) Indeed, E is uniform by Theorem 10.32 (a). Therefore, if $r \le l$ then E either splits into a direct sum of line bundles or E is isomorphic to $T\mathbb{P}^l(a)$ or $T^*\mathbb{P}^l(a)$ by [Sa]. If E splits then $E \simeq \mathcal{O}_{\mathbb{P}(A)}^{r/2} \oplus \mathcal{O}_{\mathbb{P}(A)}^{r/2}(1)$ by Theorem 10.32 (a). Suppose now that E is isomorphic to $T\mathbb{P}^l(a)$ or $T^*\mathbb{P}^l(a)$. Since $E \simeq E^*(1)$, the first Chern class $c_1(E)$ is equal to $(l/2)H$, where H is the class of the hyperplane section. Taking first Chern classes we conclude that E is isomorphic either to $T\mathbb{P}^2(-1)$ or to $T^*\mathbb{P}^2(2)$, but these last two bundles are isomorphic.

(b) Let $\dim X = n$ and $\operatorname{def} X = k$. By (a) and by Theorem10.33 we see that $N_L X$ is isomorphic to either $\mathcal{O}_L^{(n-k)/2} \oplus \mathcal{O}_L^{(n-k)/2}(1)$ or $T\mathbb{P}^2(-1)$. However, in the last case $n = 4$, $k = 2$, therefore X is a projective bundle $\mathbb{P}_C(F)$ over a curve C by Example 6.17. Then L is embedded as a 2-plane in a fibre f of $\mathbb{P}_C(F)$. Consider the exact sequence

$$0 \to N_L f \to N_L X \to N_f X|_L \to 0.$$

Since $N_L f = \mathcal{O}_L(1)$ and $N_f X = \mathcal{O}_f$, this exact sequence reduces to

$$0 \to \mathcal{O}_L(1) \to N_L X \to \mathcal{O}_L \to 0.$$

It follows that $N_L X = \mathcal{O}_L \oplus \mathcal{O}_L(1)$. □

The following theorem was proved in [W]. Weaker results (but similar arguments) first appeared in [Sy].

Theorem 10.35 ([W]) *Suppose $2 \le r \le m \le n$. Then*

(a) $l(r, m, n) \le m + n - 2r + 1$.
(b) $l(r, m, n) = n - r + 1$ *if* $n - r + 1$ *does not divide* $(m-1)!/(r-1)!$.
(c) $l(r, r+1, 2r-1) = r + 1$.

Sketch of the proof. Let $A \subset \mathrm{Hom}(V, W)$ be an l-dimensional linear subspace of constant rank r, $\dim V = m$, $\dim W = n$, $m \ge n$. Let

$$0 \to K \to \mathcal{O}_{\mathbb{P}(A)} \otimes V \to E \to 0 \text{ and } 0 \to E \to \mathcal{O}_{\mathbb{P}(A)}(1) \otimes W \to N \to 0$$

be the corresponding exact sequences of vector bundles. Then we have the following equations of Chern classes $c(K)c(E) = 1$ and $c(E)c(N) = (1+H)^n$, where H is the class of the hyperplane section. Thus, $c(K)(1+H)^n = c(N)$. If $n - r + 1 \le i \le l - 1$ then $c_i(N) = 0$ and looking at the coefficient of H^i we get

$$\sum_{j=0}^{m-r} \binom{n}{i-j} k_j = 0,$$

where $c_j(K) = k_j H^j$ and we use the convention that $\binom{n}{i} = 0$ if $i < 0$ or $i > n$. The coefficient matrix of this system of linear equations is

$$\left(\binom{n}{i-j}\right)_{0 \le j \le m-r,\ n-r+1 \le i \le l-1}.$$

If $l = m + n - 2r + 2$ then this is square invertible matrix with determinant $\prod_{j=0}^{m-r} j!$. Thus $k_0 = 0$ which is a contradiction since $k_0 = 1$. This proves (a).

Proofs of (b) and (c) can be found in [W]. Notice that by Proposition 10.31 (a) we have $l(r, m, n) \ge n - r + 1$, so only the opposite inequality should be proved. □

Another possible way of proving Theorem 10.35 (a) is to use the following theorem of R. Lazarsfeld (its application is immediate):

Theorem 10.36 ([La1]) *Let X be l-dimensional projective variety. Let E and F be vector bundles of ranks m and n, respectively. Suppose that $E^* \otimes F$ is ample and there is a constant rank r vector bundle map $E \to F$. Then $l \le m + n - 2r$.*

The (partial) symmetric analogue of this theorem was found in [IL]. Theorems 10.36 and 10.37 (and their proofs) are analogous to results related to the problem of the non-emptiness and the connectedness of degeneracy loci, see [La1, FL1, HT, HT1, Tu, IL].

Theorem 10.37 ([IL]) *Let X be a smooth simply connected m-dimensional variety. Let E be a rank n vector bundle on X, and L a line bundle on X. Suppose that $S^2E^* \otimes L$ is an ample vector bundle and that there is a constant even rank r symmetric bundle map $E \to E^* \otimes L$. Then $m \leq n - r$.*

Proof. Consider the projective bundle $\pi : \mathbb{P}(E) \to X$. The symmetric bundle map $E \to E^* \otimes L$ defines a section $s \in H^0(X, S^2E^* \otimes L)$ which pulls back to give a section $\pi^*s \in H^0(\mathbb{P}(E), \pi^*(S^2E^* \otimes L))$. There is a natural map $\pi^*\pi_*\mathcal{O}_{\mathbb{P}(E)}(2) \to \mathcal{O}_{\mathbb{P}(E)}(2)$. Since $\pi_*\mathcal{O}_{\mathbb{P}(E)}(2) \simeq S^2E^*$ this gives after tensoring with π^*L a map $u : \pi^*(S^2E^* \otimes L) \to \mathcal{O}_{\mathbb{P}(E)}(2) \otimes \pi^*L$. Let $t = u \circ \pi^*s \in H^0(\mathbb{P}(E), \mathcal{O}_{\mathbb{P}(E)}(2) \otimes \pi^*L)$. Let $Y \subset \mathbb{P}(E)$ be the zero locus of t.

Let $x \in X$. The section t at $[v] \in \mathbb{P}(E(x))$ is the linear map $\lambda \to s(x)(v,v)\lambda$ and so vanishes if and only if $s(x)(v,v) = 0$. But by the hypothesis, this defines a rank r quadric hypersurface in $\mathbb{P}(E(x))$. Therefore, the fiber of $\pi_Y : Y \to X$ over each $x \in X$ is a rank r quadric hypersurface in $\mathbb{P}(E(x))$.

We claim that $\mathcal{O}_{\mathbb{P}(E)}(2) \otimes \pi^*(L)$ is ample on $\mathbb{P}(E)$. Indeed, since $S^2E^* \otimes L$ is ample on X, $\mathcal{O}_{\mathbb{P}(S^2E^* \otimes L)}(1)$ is ample on $\mathbb{P}(S^2E^* \otimes L)$. Clearly, $\mathbb{P}(S^2E^* \otimes L) \simeq \mathbb{P}(S^2E^*)$ and $\mathcal{O}_{\mathbb{P}(S^2E^* \otimes L)}(1) \simeq \mathcal{O}_{\mathbb{P}(S^2E^*)}(1) \otimes \sigma^*L$, where $\sigma : \mathbb{P}(S^2E^*) \to X$ is the natural projection. Therefore, $\mathcal{O}_{\mathbb{P}(S^2E^*)}(1) \otimes \sigma^*L$ is ample on $\mathbb{P}(S^2E^*)$. The second Veronese map gives an inclusion $i : \mathbb{P}(E) \subset \mathbb{P}(S^2E^*)$ such that $\pi = i \circ \sigma$. Then $i^*\mathcal{O}_{\mathbb{P}(S^2E^*)}(1) = \mathcal{O}_{\mathbb{P}(E)}(2)$. So $i^*(\mathcal{O}_{\mathbb{P}(S^2E^*)}(1) \otimes \sigma^*L) \simeq \mathcal{O}_{\mathbb{P}(E)}(2) \otimes \pi^*L$ is ample on $\mathbb{P}(E)$ as required. In particular, $\mathbb{P}(E) \setminus Y$ is an affine variety and its homology vanishes above its middle dimension: $H_i(\mathbb{P}(E) \setminus Y, \mathbb{C}) = 0$ for $i \geq m + n$.

Let K denote the kernel and F denote the image of the map $E \to E^* \otimes L$. Then since the map has constant rank r, K and F are vector bundles on X of ranks $n-r$ and r respectively. Since the map is symmetric, there is a symmetric isomorphism $F \simeq F^* \otimes L$. We have a natural map $p : \mathbb{P}(E) \setminus \mathbb{P}(K) \to \mathbb{P}(F)$ given on the fibers by the linear projection centered at $\mathbb{P}(K(x)) \subset \mathbb{P}(E(x))$. Then p is a $\mathbb{C}^{n-r}$-fiber bundle.

The isomorphism $F \simeq F \otimes L$ determines a hypersurface $Z \subset \mathbb{P}(F)$ such that if $\rho : \mathbb{P}(F) \to X$ is the natural projection then the fiber of $\rho|_Z : Z \to X$ over each $x \in X$ is a smooth quadric in $\mathbb{P}(F(x))$. Now $\mathbb{P}(K) \subset Y$ and so p restricts to a $\mathbb{C}^{n-r}$-fiber bundle $\mathbb{P}(E) \setminus Y \to \mathbb{P}(F) \setminus Z$. Thus, $H_i(\mathbb{P}(E) \setminus Y, \mathbb{C}) \simeq H_i(\mathbb{P}(F) \setminus Z, \mathbb{C})$ and by Lefschetz duality this is isomorphic to $H^{2(m+r-1)-i}(\mathbb{P}(F), Z, \mathbb{C})$. Hence $H^i(\mathbb{P}(F), Z, \mathbb{C}) = 0$ for $i \leq 2r + m - n - 2$ and then using the long exact sequence of the pair $(\mathbb{P}(F), Z)$ we conclude that $H^i(Z, \mathbb{C}) \simeq H^i(\mathbb{P}(F), \mathbb{C})$ for $i \leq 2r + m - n - 3$.

Both bundles $\rho : \mathbb{P}(F) \to X$ and $\rho|_Z : Z \to X$ have smooth fibers and are defined over smooth simply-connected variety. Therefore, by Deligne' theorem [GH1], the Leray spectral sequence for ρ and $\rho|_Z$ degenerates at the E_2 term. It follows that $H^*(\mathbb{P}(F), \mathbb{C}) = H^*(X, \mathbb{C}) \otimes H^*(\mathbb{P}^{r-1}, \mathbb{C})$ and $H^*(Z, \mathbb{C}) = H^*(X, \mathbb{C}) \otimes H^*(Q, \mathbb{C})$, where $Q \subset \mathbb{P}^{r-1}$ is a smooth quadric. Let $b_j(B) = \dim H^j(B, \mathbb{C}$ be the Betti numbers. Then it is well-known that

$$b_i(\mathbb{P}^{r-1}) = \begin{cases} 1 & \text{for even } i,\ 0 \le i \le 2(r-1) \\ 0 & \text{otherwise.} \end{cases}$$

Since r is even, we have

$$b_i(Q) = \begin{cases} 1 & \text{for even } i,\ 0 \le i \le 2(r-2) \\ 2 & i = r-2 \\ 0 & \text{otherwise.} \end{cases}$$

It follows that $b_{r-2}(\mathbb{P}(F) \neq b_{r-2}(Z)$. Since $b_i(\mathbb{P}(F) = b_i(Z)$ for $i \le 2r + m - n - 3$, we finally obtain $r - 2 > 2r + m - n - 3$. Equivalently, $m \le n - r$. $\square$

As an easy application of Theorem 10.37 we have the following theorem.

Theorem 10.38 ([IL]) *Let V be an m-dimensional vector space, let $r > 0$ be an even number*

$$\max\{\dim A \mid A \subset \mathbb{P}(S^2 V^*) \text{ is of constant rank } r\} = m - r.$$

Example 10.39 Theorems 10.28 and 10.38 combined furnish a new proof of the formula $\dim X^* \ge \dim X$ (Theorem 5.13 (b)).

References

[Al] P. Aluffi, *Singular schemes of hypersurfaces*, Duke Math. J., **80** (1995), 325–351

[AC] P. Aluffi, F. Cukierman, *Multiplicities of discriminants*, Manuscripta Math., **78** (1993), 245–258

[An] A.Z. Anan'in, *Quasiderivations of the algebra of diagonal matrices*, Tret'ya Sibirskaya shkola po algebre i analizu, Irkutsk University, 1990, 8–15 (in Russian)

[AVE] E.M. Andreev, E.B. Vinberg, A.G. Elashvili, *Orbits of greatest dimension in semisimple linear Lie groups*, Functional Anal. Appl., **1** (1967), 257–261

[ACGH] E. Arbarello, M. Cornalba, P.A. Griffiths, J. Harris, *Geometry of Algebraic Curves, Vol. 1*, Springer–Verlag, New York, 1985

[Ar] V.I. Arnold, *Mathematical Methods of Classical Mechanics*, Springer–Verlag, New York–Heidelberg–Berlin, 1978

[At] M.D. Atkinson, *Primitive spaces of matrices of bounded rank, II*, Jour. Australian Math. Soc., **34** (1983), 306–315

[AW] M.D. Atkinson, R. Westwik, *Spaces of linear transformations of equal rank*, Linear and Multilinear Algebra, **13** (1983), 231–239

[BCa1] P. Bala, R.W. Carter, *Classes of unipotent elements in simple algebras. I*, Math. Proc. Camb. Phil. Soc., **79** (1976), 401–425

[BCa2] P. Bala, R.W. Carter, *Classes of unipotent elements in simple algebras. II*, Math. Proc. Camb. Phil. Soc., **80** (1976), 1–18

[BC] E. Ballico, L. Chiantini, *On smooth subcanonical varieties of codimension* 2 *in* $\mathbb{P}^n$, $n \geq 4$, Ann. Mat. Pure Appl., **135:4** (1983), 99-117

[Bar] F. Bardelli, *The Sylvester pentahedron and some applications to moduli spaces*, preprint

[Ba] W. Barth, *Transplanting cohomology class in complex projective space*, Amer. J. Math, **92** (1970), 951–961

[Be] L.B. Beasley, *Null spaces of matrices of bounded rank*, Current Trends in Matrix Theory (R. Grone and F. Uhlig, Eds.), North–Holland, Amsterdam, 1987, 45–50

[Bea] A. Beauville, *Fano contact manifolds and nilpotent orbits*, Comm. Math. Helv., **73:4** (1998), 566–583

[Bek] N.D. Beklemishev, *Classification of quaternary cubic forms not in general position*, In: Voprosy teorii grupp i gomologicheskoy algebry, Yaroslavl, 1981 (in Russian)

[BFS1] M.C. Beltrametti, M.L. Fania, A.J. Sommese, *On the discriminant variety of a projective manifold*, Forum Math, **4** (1992), 529–547

[BFS2] M.C. Beltrametti, M.L. Fania, A.J. Sommese, *On the adjunction theoretic classification of projective varieties*, Math. Ann., **290** (1991), 31–62

[BS] M.C. Beltrametti, A.J. Sommese, *The adjunction theory of complex projective varieties*, Expositions in Math., **16** (1995), W. de Gruyter, Berlin

[BSW] M.C. Beltrametti, A.J. Sommese, J.A. Wiśniewski, *Results on varieties with many lines and their applications to adjunction theory*, Proceedings of the Bayreuth Conference "Complex Algebraic Varieties", April 1990. Lecture Notes in Math. 1507, Springer, 1992, 16–38

[Berg] G.M. Bergman, *The diamond lemma for ring theory*, Adv. in Math., **29/2** (1978), 178–218

[BGG] I.N. Bernstein, I.M. Gelfand, S.I. Gelfand, *Schubert cells and the cohomology of G/P*, Uspechi Matematicheskih Nauk, **28/3** (1973), pp. 3–26

[Ber] E. Bertini, *Introduzione alla geometria proiettriva degli iperspazi*, Enrico Spaerri, Piza, 1907

[Bot] R. Bott, *Homogeneous vector bundles*, Ann. Math., **66/2** (1957), pp. 203–248

[Bo] N. Bourbaki, *Groupes et algébres de Lie*, Hermann, Paris 1968

[Br] M. Brion, *Représentations exceptionnelles des groupes semi-simples*, Ann. Sci. Ec. Norm. Sup., **18** (1985), 345–387

[Ca] A. Cayley, *On the theory of elimination*, Collected papers Vol. 1, Cambridge Univ. Press, 1889, pp. 370–374

[Ca1] A. Cayley, *On the theory of linear transformations*, Collected papers Vol. 1, Cambridge Univ. Press, 1889, pp. 80–94

[Ca2] A. Cayley, *On linear transformations*, Collected papers Vol. 1, Cambridge Univ. Press, 1889, pp. 95–112

[CR] T. Cecil, P. Ryan, *Focal sets and real hypersurfaces in complex projective space*, Trans. Amer. Math. Soc., **269** (1982)

[CKM] H. Clemens, J. Kollar, S. Mori, *Higher dimensional complex geometry*, Asterisque, **166** (1988)

[CM] D. Collingwood, W. McGovern, *Nilpotent orbits in semisimple Lie algebras*, Van Nostrand Reinhold Company, 1993

[CW] C. De Concini, J. Weiman, *A formula with nonnegative terms for the degree of the dual variety of a homogeneous space*, Proc. AMS, **125:1** (1997), 1–8

[De] L. Degoli, *Due Nouvi Teoremi Sui Sistemi Lineari Di Quadriche A Jacobiana Identicamente Nulla*, Collect. Math., **35** (1984), 125–139

[Del] P. Deligne, *Le déterminant de la cohomologie*, In: Current trends in arithmetic algebraic geometry, Contemporary Math., No. 67,Amer. Math. Soc., 1987, pp. 93–177

[DK] P. Deligne, N.Katz, *Groupes de monodromie en geometrie algebrique*, Lecture Notes in Math. 340, Springer, 1973

[Di1] A. Dimca, *Milnor numbers and multiplicities of dual varieties*, Rev. Roumaine Math. Pures (1986), 535–538

[Di2] A. Dimca, *Topics on real and complex singularities*, Vieweg Advanced Lectures (1987)

[D] R. Donagi, *On the geometry of Grassmanians*, Duke Math. J., **44** (1977), 795–837

[E1] L. Ein, *Varieties with small dual varieties, I*, Invent. Math., **86** (1986), 63–74

[E2] L. Ein, *Varieties with small dual varieties, II*, Duke. Math. J., **52** (1985), 895–907

[E3] L. Ein, *Stable vector bundles on projective space in char $p > 0$*, Math. Ann., **254** (1980), 53–72

[E4] L. Ein, *The ramification divisors for branched coverings of $\mathbb{P}^n_k$*, Math. Ann., **261** (1982), 483–485

[ES] L. Ein, N. Shepherd–Barron, *Some special Cremona transformations*, Amer. J. Math., **111** (1989), 783–800

[EH] D. Eisenbud, J. Harris, *Vector spaces of matrices of low rank*, Adv. Math., **70** (1988), 135–155

[El1] A.G. Elashvili, *Canonical form and stationary subalgebras of points of general position for simple linear Lie groups*, Functional Anal. Appl., **6** (1972), 44–53

[El2] A.G. Elashvili, *Stationary subalgebras of points of general position for irreducible linear Lie groups*, Functional Anal. Appl., **6** (1972), 139–148

[EHS] G. Elencwajg, A. Hirschowitz, M. Schneider, *Les fibres uniformes de rang au plus n sur $\mathbb{P}_n(\mathbb{C})$*, Proceedings of the Nice Conference 1979 on Vector bundles and Differential equations

[Fa] B. Fantechi, *On the superadditivity of secant defects*, Bull. Soc. Math. France, **118** (1990), 85–100

[Fi] E. Fisher, *Über die Cayleyshe Eliminationsmethode*, Math. Zeit., **26** (1927), 497–550

[Fra] W. Franz, *Die torsion einer Überdeckung*, J. Reine u. angew. Math., **173** (1935), 245–254

[Fr] H. Freudenthal, *Beziehungen der E_7 und E_8 zur Oktavenebene I-XI*, Nederl. Akad. Wet., Proc., **57** (1954), 218–230, 363–368, **58** (1955), 151–157, 277–285, **62** (1959), 165–179, 180–191, 192–201, 447–465, 466–474, **66** (1963), 457–471, 472–487

[Fr1] H. Freudenthal, *Lie groups in the foundation of geometry*, Adv. Math., **1** (1964), 145–190

[Fub] G. Fubini, *Il problema della deformazione proiettiva delle ipersuperficie*, Rend. Acad. Naz. dei Lincei, **27** (1918), 147–155

[F] T. Fujita, *Projective threefolds with small secant varieties*, Sci. Papers College of General Education. Univ. Tokio, **32:1** (1982), 33–46

[FR] T. Fujita, J. Roberts, *Varieties with small secant varieties: the extremal case*, Amer. J. Math., **103** (1981), 953–976

[Fu1] W. Fulton, *Intersection Theory*, Ergebnisse der Mathematik 2, Springer, 1984

[Fu2] W. Fulton, *On the topology of algebraic varieties*, Proc. Symp. Pure Math., **46** (1987), 15–46

[FH] W. Fulton, J. Hansen, *A connectedness theorem for projective varieties, with applications*, Ann. Math., **110:1** (1979), 159–166

[FL] W. Fulton, R. Lazarsfeld, *Connectivity and its applications in algebraic geometry*, Proc. Midwest Algebraic Geometry Conf. Univ. Illinois, 1980. Lect. Notes Math. Springer–Verlag, Berlin–Heidelberg–New York, **862** (1981), 26–92

[FL1] W. Fulton, R. Lazarsfeld, *On the connectedness of degeneracy loci and special divisors*, Acta Math., **146** (1981), 271–283

[Ga] P. Gabriel, *Unzerlegbare Darstellungen, I*, Manuscripta Math., **6** (1972), 71–103

[Gan] F.R. Gantmacher, *The theory of matrices Vol.2*, Chelsea Publishing Company, New York, 1959

[GK] I.M. Gelfand, M.M. Kapranov, *On the dimension and degree of projectively dual varieties: a q-analogue of the Katz–Kleiman formula*, in: Gelfand Mathematical Seminars 1990–1992, Birkhaüser 1993, pp. 27–34

[GKZ1] I.M. Gelfand, M.M. Kapranov, A.V. Zelevinski, *A-discriminants and the Cayley–Koszul complexes*, Dokl. Akad. Nauk SSSR, **307** (1989), 1307–1311 (Russian)

[GKZ2] I.M. Gelfand, M.M. Kapranov, A.V. Zelevinski, *Discriminants, resultants, and multidimensional determinants*, Birkhauser, Boston, 1994

[GKZ3] I.M. Gelfand, M.M. Kapranov, A.V. Zelevinski, *Hyperdeterminants*, Adv. Math., **84** (1990), pp. 237–254

[Go] N. Goldstein, *Ampleness and connectedness in complex G/P*, Trans. AMS, **274** (1982), 361–374

[GR] H. Grauert, O. Rimenschneider, *Verschwindungssätze für Annalische Kogomologiegruppen auf Komplexen Räumen*, Inv. Math., **11** (1970), 263–292

[GH] P. Griffiths, J. Harris, *Algebraic geometry and local differential geometry*, Ann. Sci. Ec. Norm. Super., **12** (1979), 355–432

[GH1] P. Griffiths, J. Harris, *Principles of Algebraic Geometry*, Wiley–Interscience, London–New–York, 1979

[HT] J. Harris, L.W. Tu, *The connectedness of symmetric degeneracy loci: odd ranks*, *Topics in Algebra: Banach Center Publications, Vol. 26, Part 2*, PWN-Polish Scientific Publishers, Warsaw, 1990, pp. 249–256

[HT1] J. Harris, L.W. Tu, *On symmetric and skew-symmetric determinantal varieties*, Topology, **23** (1984), 71–84

[Ha1] R. Hartshorne, *Algebraic Geometry*, Graduate Texts in Mathematics, **52**, Springer–Verlag, New York–Heidelberg–Berlin, 1977

[Ha2] R. Hartshorne, *Varieties of low codimension in projective space*, Bull. Amer. Math. Soc., **80** (1974), 1017–1032

[Ha3] R. Hartshorne, *Ample vector bundles*, IHES Publ. Math., **29** (1966), 63–94

[HK] A. Hefez, S. Kleiman, *Notes on duality for projective varieties*

[HRo] L. Hille, G. Röhrle, *A classification of parabolic subgroups of classical groups with a finite number of orbits on the unipotent radical*, Transformation Groups, **4:1** (1999), 35–52

[Hir] F. Hirzebruch, *Topological Methods in Algebraic Geometry*, Grund. Math. Wiss. **131**, 1966, Springer–Verlag

[HP] W.V.D. Hodge, D. Pedoe, *Methods of Algebraic Geometry*, Cambridge University Press, Cambridge, 1994

[Ho1] A. Holme, *On the dual of the smooth variety*, Lect. Notes in Math., **732** (1979), Springer–Verlag, 144–156

[Ho2] A. Holme, *The geometric and numerical properties of duality in projective algebraic geometry*, Manuscripta Math., **61** (1988), 145–162

[HR] A. Holme, J. Roberts, *Pinch points and multiple locus for generic projections of singular varieties*, Adv. Math., **33** (1979), 212–256

[IL] B. Ilic, J.M. Landsberg, *On symmetric degeneracy loci, spaces of symmetric matrices of constant rank and dual varieties*, Math. Ann., **314** (1999), 159–174

[I] P. Ionescu, *An enumeration of all smooth projective varieties of degree* 5 *and* 6, Increst Preprint Series Math., **74** (1981)

[J] N. Jacobson, *Structure and representations of Jordan algebras*, Publ. Amer. Math. Soc., **39** (1968)

[KOY] H. Kaji, M. Ohno, O. Yasukura, *Adjoint varieties and their secant varieties*, Indag. Math., **10** (1999), 45–57

[Kan] I.L. Kantor, *Models of exceptional Lie algebras*, Soviet Math. Dokl., **14** (1973), 254–258

[Ka] N. Katz, *Pinceaux de Lefschetz; Theoreme d'existence*, in: SGA 7, Lecture Notes in Math., **340**, 212–253

[KMM] Y. Kawamata, K. Matsuda, K. Matsuki, *Introduction to the minimal model program*, Algebraic Geometry, Sendai 1985. Adv. Stud. Pure Math., **10** (1987), 283–360

[Kaw] Y. Kawamata, *The cone of curves of algebraic varieties*, Ann. Math, **119** (1984), 603–633

[Ke] G. Kempf, *On the collapsing of homogeneous bundles*, Invent. Math. **37** (1976), 229–239

[Kl1] S. Kleiman, *Enumerative theory of singularities*, Real and Complex Singularities (Proc. Ninth Nordic Summer School/NAVF Sympos. Math., Oslo, 1976), 297–396

[Kl2] S. Kleiman, *About the conormal scheme*, "Complete intersections", Proceedings of Acireale (1983), Lecture Notes in Math. 1092, Springer, 1984, 161–197

[Kl3] S. Kleiman, *Tangency and duality*, Proceedings of the 1984 Vancouver Conference in Algebraic Geometry, July 2–12, 1984, ed. by J. Carrell, A.V. Geramita, and R. Russell. Canadian Math. Soc. Conference Proceedings **6**, (1986), 163–225

[Kl4] S. Kleiman, *Concerning the dual variety*, Proc. 18th Scand. Congr. Math. Aarhus, 1981, Birkhäuser Verlag, Basel–Boston–Stuttgart, 386–396

[Kl5] S. Kleiman, *Plane forms and multiple point formulas*, Algebraic threefolds (Varenna, 1981), pp. 287–310, Lecture Notes in Math., **947** (1982)

[Kl6] S. Kleiman, *Toward a numerical theory of ampleness*, Ann. Math, **84** (1966), 293–344

[KZ] K. Knight, A. Zelevinsky, *Representations of quivers of type A and the multisegment duality*, Adv. Math, **117** (1996), 273–293

[KM] F. Knop, G. Mentzel, *Duale Varietäten von Fahnenvarietäten*, Comm. Math. Helv., **62** (1987), 38–61

[KnMu] F. Knudsen, D. Mumford, *Projectivity of moduli spaces of stable curves I. Preliminaries on* Det *and* Div, Math. Scand., **39** (1976), 19–55

[KO] S. Kobayashi, T. Ochiai *Characterization of complex projective spaces and hyperquadrics*, J. Math. Kyoto Univ., **13:1** (1973), 31–47

[Ko] Y. Kollar, *The structure of algebraic threefolds. An introduction to Mori's program*, Bull. AMS, **17:2** (1987), 211–273

[KS] A. Kumpera, D. Spencer, *Lie Equations, Vol. 1: General Theory*, Ann. of Math. Stud., **73** (1972), Princeton University Press

[Lam] I. Lamotke, *The topology of complex projective varieties after S. Lefschetz*, Topology, **20** (1981), 15–51

[Land] A. Landman, *Examples of varieties with small dual varieties. Picard–Lefschetz theory and dual varieties*, Two lectures at Aarhus Univ., (1976)

[Lan] J.M. Landsberg, *On degenerate secant and tangential varieties and local differential geometry*, Duke Math. J., **85** (1996), 605–634

[Lan1] J.M. Landsberg, *Differential–geometric characterization of complete intersections*, J. of Diff. Geom., **44** (1996), 32–73

[Lan2] J.M. Landsberg, *On second fundamental form of projective varieties*, Invent. Math., **117** (1994), 303–315

[Lan3] J.M. Landsberg, *On degenerate secant and tangential varieties and local differential geometry*, Duke Math. J., **85** (1996), 1–30

[Lan4] J.M. Landsberg, *Algebraic geometry and projective differential geometry*, preprint, (1998)

[LM] J.M. Landsberg, L. Manivel, *On the projective geometry of homogeneous varieties*, preprint, (2000)

[LP] A. Lanteri, M. Palleschi, *Projective manifolds containing many rational curves*, Indiana Math. J., **36** (1987), 857–865

[LPS] A. Lanteri, M. Palleschi, A.J. Sommese, *Discriminant loci of varieties with smooth normalization*, preprint (1998)

[LS] A. Lanteri, D. Struppa, *Projective* 7*-folds with positive defect*, Compositio Math., **61** (1987), 329–337

[L] M.E. Larsen, *On the topology of complex projective manifolds*, Invent. Math., **19** (1972), 251–260

[Las] A. Lascoux, *Degree of the dual Grassman variety*, Comm. Algebra, **9/11** (1981), 1215–1225

[La] R. Lazarsfeld, *An example of* 16*-dimensional projective variety with a* 25*-dimensional secant variety*, Math. Letters, **7** (1981), 1–4

[La1] R. Lazarsfeld, *Some applications of the theory of positive vector bundles*, LNM 1092, Complete Intersections (Acireale 1983), Springer–Verlag, Berlin, 1984, pp. 29–61

[LV] R. Lazarsfeld, A. Van de Ven, *Topics in the Geometry of Projective Space, Recent Work of F.L. Zak*, DMV Seminar, Birkhauser, 1984

[Li] W. Lichtenstein, *A system of quadrics describing the orbit of the highest weight vector*, Proc. A.M.S., **84** (1982), 605–608

[Ma] Yu.I. Manin, *Cubic forms. Algebra, geometry, arithmetic*, M., Nauka, 1972. English transl.: North–Holland, Amsterdam–London 1974

[MM] H. Matsumura, P. Monsky, *On the automorphisms of hypersurfaces*, J. Math. Kyoto Univ., **3:3** (1964), 347–361

[Me] R. Meshulam, *On two-parameter families of symmetric matrices*, Lin. Alg. Appl., **216** (1995), 93–96

[Mi] J. Milnor, *Singular points of complex hypersurfaces*, Ann. of Math. Studies **61**, Princeton Univ. Press (1968)

[MW] C. Moeglin, J.L. Waldspurger, *Sur l'involution de Zelevinski*, J. Reine Angew. Math, **372** (1986), 136–177

[M] B.G. Moishezon, *Algebraic homology classes of algebraic varieties*, Izvestiya AN SSSR, **31** (1967) (in Russian)

[Mo] E.H. Moore, *General Analysis – Part I*, Mem. Amer. Phil. Soc., **1** (1935)

[Mor] S. Mori, *Threefolds whose canonical bundle is not numerically effective*, Ann. Math., **116** (1982), 133–176

[Mor1] S. Mori, *Projective manifolds with ample tangent bundle*, Ann. Math., **110** (1979), 593–606

[Mu] S. Mukai, *New classification of Fano threefolds and Fano manifolds of coindex* 3, preprint (1988)

[MRS] I. Muller, H. Rubenthaler, G. Schiffmann, *Structure des espaces préhomogènes associés à certaines algébres de Lie graduées*, Math. Ann., **274** (1986), 95–123

[Mum] D. Mumford, *Some footnote of the work of C.P. Ramanujam*, in: C.P. Ramanujam, a Tribute, pp. 247–262, Springer, 1978

[Mum1] D. Mumford, *Complex projective varieties*, Springer–Verlag, Berlin–Heidelberg–New York, 1979

[MFK] D. Mumford, J. Fogarty, F. Kirwan, *Geometric Invariant Theory*, Ergebnisse der Mathematik und Grenzgebiete, vol. 34, Springer–Verlag, New York–Heidelberg–Berlin, 1994. (3rd edition)

[N] A. Némethi, *Lefschetz theory for complex hypersurfaces*, Rev. Roumaine Math. Pures (1988), 233–250

[OSS] C. Okonek, H. Spindler, M. Schneider, *Vector bundles on complex projective spaces*, Prog. Math. **3**, Basel–Boston, Birkhäuser, 1980

[OV] A.L. Onischik, E.B. Vinberg, *Lie groups and Lie algebras III*, Encyclopedia of Mathematical Sciences **41**, Springer–Verlag 1994

[Pan] D.I. Panyushev, *Parabolic subgroups with abelian unipotent radical as a testing site for invariant theory*, Canad. J. Math., **51:3** (1999), 616–635

[Pa1] A. Parusiǹski, *A generalization of the Milnor number*, Math. Ann., **281** (1988), 247–254

[Pa2] A. Parusiǹski, *Multiplicity of the dual variety*, Bull. London Math. Soc., **23** (1991), 428–436

[Pau] C. Pauly, *Self-duality of Coble's quartic hypersurface and applications*, preprint

[Pe] R.A. Penrose, *A generalized inverse for matrices*, Proc. Camb. Phil. Soc., **51** (1955)

[P] R. Piene, *Plücker formulas*, Doctoral thesis, M.I.T. Cambridge, MA, 1976

[Po] S. Poljak, *Maximum rank of powers of a matrix of a given pattern*, Proc. Amer. Math. Soc., **106:4** (1989), 1137–1144

[Pop] V. Popov, *Self-dual algebraic varieties and nilpotent orbits*, Proceedings of the International Colloquium "Algebra, Arithmetic, and Geometry", Tata Institute of fundamental Research, Bombay, 2000

[PR] V. Popov, G. Röhrle, *On the number of orbits of a parabolic subgroup on its unipotent radical*, Algebraic Groups and Lie Groups, Lehrer G.I., Ed., Austr. Math. Soc. Lecture Series, **9** (1997)

[PV] V.L. Popov, E.B. Vinberg, *Invariant Theory*, in: Algebraic Geometry IV, Encyclopedia of Mathematical Sciences, Springer–Verlag, **55** (1994), 123–278

[Py] V.S. Pyasetskii, *Linear Lie groups acting with finitely many orbits*, Funct. Anal. Appl. **9** (1975), 351–353

[Q] D. Quillen, *Determinants of Cauchy–Riemann operators on Riemann surfaces*, Funct. Anal. Appl., **19** (1985), 31–34

[RS] N. Ray, I. Singer, *R-torsion and the Laplacian for Riemannian manifolds*, Adv. Math., **7** (1971), 145–210

[Ra1] Z. Ran, *The (dimension+2)-secant Lemma*, Invent. Math., **106** (1991), 65–71

[Ra2] Z. Ran, *On projective varieties of codimension two*, Invent. Math., **73** (1983), 333-336

[Ra3] Z. Ran, *The structure of Gauss-like maps*, Compos. Math., **52** (1984), 171–177

[Ri] R.W. Richardson, *Finiteness theorems for orbits of algebraic groups*, Indag. Math., **88** (1985), 337–344

[Ri1] R.W. Richardson, *Conjugacy classes of n-tuples in Lie algebras and algebraic groups*, Duke Math. J., **57** (1988), 1–35

[RRS] R.W. Richardson, G. Röhrle, R. Steinberg, *Parabolic subgroups with abelian unipotent radical*, Invent. Math., **110** (1992), 649–671

[Rob] J. Roberts, *Generic projections of algebraic varieties*, Amer. J. Math., **93** (1971), 191–214

[R] T. Room, *A Synthesis of Clifford matrices and its generalization*, Amer. J. Math., **74** (1952), 967–984

[Ro] B. Rosenfeld, *Geometry of Lie groups*, Mathematics and its Applications **393**, Kluwer Academic Publishes 1997

[Ru] W. Rudin, *Real and Complex Analysis*, Mc Graw Hill, 1987

[Sa] E. Sato, *Uniform vector bundles on a projective space*, J. Math. Soc. Japan, **28** (1976), 123–132

[Schl] L. Schläffli, *Gesammelte Abhandlungen*, Band. 2, pp. 9–112L, Birkhäuser Verlag, Basel, 1953

[Se] B. Segre, *Bertini forms and hessian matrices*, L. London Math. Soc., **26** (1951), 164–176

[Sn] D. Snow, *The nef value and defect of homogeneous line bundles*, Trans. AMS, **340:1** (1993), 227–241

[Sn1] D. Snow, *On the ampleness of homogeneous vector bundles*, Trans. AMS, **294:2** (1986), 227–241

[S] A.J. Sommese, *Hyperplane section of projection surface I — the adjunction mapping*, Duke Math. J., **46** (1979), 377-401

[S1] A. Sommese, *Submanifolds of abelian varieties*, Math. Ann., **233** (1978), 229–256

[Sy] J. Sylvester, *On the dimension of spaces of linear transformations satisfying rank conditions*, Linear Algebra and its Applications, **76** (1986), 1–10

[Ta] R. Takagi, *On homogeneous real hypersurfaces in a complex projective space*, Osaka J. Math., **10** (1973), 495–506

[Tan] H. Tango, *Remark on varieties with small secant varieties*, Bul. Kyoto Univ. Ed. Ser. B, **60** (1982), 1–10

[Te] Terracini, *Alcune questioni sugli spazi tangenti e osculatori ad una varieta, I, II, III*, Societa dei Naturalisti e Matematici (1913), 214–247

[T1] E. Tevelev, *Subalgebras and discriminants of anticommutative algebras*, Izvestiya RAN, Ser. Mat., **63:3** (1999), 169–184

[T2] E. Tevelev, *Generic algebras*, Transformation Groups, **1** (1996), 127–151

[T3] E. Tevelev, *Isotropic subspaces of multilinear forms*, Matematicheskie Zametki, 2001

[T4] E. Tevelev, *Moore–Penrose inverse, parabolic subgroups, and Jordan pairs*, preprint, 2000

[T5] E. Tevelev, *Generic algebras*, PhD Thesis

[T6] E. Tevelev, *Discriminants and quasi-derivations of commutative algebras*, Uspechi Matematicheskih Nauk, **54:2** (1999), 189–190

[Tu] L.W. Tu, *The connectedness of symmetric and skew symmetric degeneracy loci: even ranks*, Trans. AMS, **313** (1989), 381–392

[V] A. Van de Ven, *On the* 2*-connectedness of the very ample divisor on a surface*, Duke Math. J., **46** (1979), 403–407

[Vi] E.B. Vinverg, *The Weyl group of a graded Lie algebra*, Math. USSR Izv., **10** (1976), 463–495

[Vi1] E.B. Vinberg, *Generalized derivations of algebras*, AMS Transl., **163** (1995), 185–188

[Wa] A.H. Wallace, *Tangency and duality over arbitrary fields*, Proc. London Math. Soc., **6:3** (1956), 321–342

[Wei] A. Weil, *Foundations of algebraic geometry*, A.M.S. Coloquium Publ., **29** (1962), Providence, R.I.

[W] R. Westwick, *Spaces of matrices of fixed rank*, Linear and Multilinear Algebra, **20** (1987), 171–174

[W1] R. Westwick, *Spaces of matrices of fixed rank, II*, Linear Algebra and its Applications, **235** (1996), 163–169

[We] J. Weyman, *Calculating discriminants by higher direct images*, Trans. Amer. Math. Soc., **343** (1994), 367–389

[WZ] J. Weyman, A. Zelevinsky, *Singularities of Hyperdeterminants*, Ann. Inst. Fourier, **46:3** (1996), 591–644

[WZ1] J. Weyman, A. Zelevinsky, *Multiplicative properties of projectively dual varieties*, Manuscripta Math., **82:2** (1994), 139–148

[Wi] P. Wilson, *Towards birational classification of algebraic varieties*, Bull. London Math. Soc., **19** (1987), 1–48

[Wis] J.A. Wiśniewski, *Length of extremal rays and generalized adjunction*, Math. Z., **200** (1989), 409–427

[Wis1] J.A. Wiśniewski, *On a conjecture of Mukai*, Manuscripta Math., **68** (1990), 135–141

[Z1] F.L. Zak, *Projection of algebraic varieties*, Mat. Sbornik, **116** (1981), 593–602 (in Russian). English Translation: Math. USSR Sbornik, **44** (1983), 535–544

[Z2] F.L. Zak, *Severi varieties*, Mat. Sbornik, **126** (1985), 115–132 (in Russian).

[Z3] F.L. Zak, *The structure of Gauss maps*, Funct. Anal. i Prilozh., **21** (1987), 39–50 (in Russian).

[Z4] F.L. Zak, *Tangents and secants of algebraic varieties*, Translations of Math. Monographs, **127**, AMS 1993

[Z5] F.L. Zak, *Linear systems of hyperplane sections of varieties of low codimension*, Funct. Anal. i Prilozh., **19** (1985), 165–173

[Za] O. Zariski, *Theory and applications of holomorphic functions on algebraic varieties over arbitrary ground fields*, Mem. A.M.S. No. 5 (1951), Providence, R.I.

Index

1-cycle 134
$\mathbb{P}^d$-bundle 142
$\mathbb{Q}$-divisors 135
μ-number 123

adjoint variety 29
admissible collection 46
Albert algebra 26, 39
ample 16
ample vector bundle 51
ampleness 52
annihilator 1, 154
associated hypersurfaces 88

base-point free 16
based exact complex 46
Beilinson spectral sequence 95
bigraded complex 50
bitangent 10
Borel subgroup 21
Borel–Weyl-Bott theorem 22
boundary format 110
Bourbaki numbering 21

Cartan Lemma 99
Cartier divisor 15
caustic 9
Cayley determinant 43
Cayley numbers 26
Cayley trick 53, 88
character group 21
characteristic of a nilpotent 64
Chern class 105
Chern polynomial 106
Chow forms 89
class formula 107, 122
class of a plane curve 10
classical discriminant 23, 48, 108, 118
classical resultant 53
Clifford algebra 25
cohomology of Koszul complexes 42
complete linear system 16
cone of effective 1-cycles 134
conical action 30
conical orbit 30
conormal bundle 4
conormal variety 2
contact locus 7, 89
contact scheme 129
Contraction Theorem 135
correspondence of branches 10
cotangent bundle 4
cusp 10
cuspidal point 10

Darboux frame 98
defect of a projective variety 3
defect of smooth divisors 17
degeneracy locus 176
degree of a cycle 105
Del Pezzo fibration 142
desingularization of a dual variety 6
determinant of a complex 45
determinant of a supervector space 43
determinant of a vector space 43
discriminant 3
discriminant complex 46
discriminant of a binary form 3
discriminant of a linear operator 27
discriminant of an element in a Lie algebra 28
discriminant spectral sequence 49
dominant weight 21
doublings of linear systems 173
dual of the curve 7
dual of the smooth surface 92
dual plane curve 8
dual projective subspace 1
dual quiver 32
dual variety 2
dual variety of a complete intersection 52

dual variety of a curve 48
dual variety of a smooth complete intersection 106
dual vector space 1

effective 1-cycle 134
effective divisor 15
Ein Theorems 89
embedded projective tangent space 2
Euler characteristic 119
extremal rational curve 135
extremal ray 135

Fano variety 137
Fermat curve 8
first jet 41
flex 10
Frobenius integrability condition 98
Fulton–Hansen connectedness theorem 71
fundamental form 97
fundamental forms of flag varieties 104
fundamental weights 21, 144

Gauss image 77
Gauss map 96
Grassmanian 23
Grassmanian fibration 88
group of norm similarities 39

half-spinor representation 25
Hartshorne conjecture 78, 170
height of a nilpotent orbit 64
Hessian matrix 81
higher fundamental forms 103
Hilbert polynomial 119
Hilbert scheme 91
homogeneous $\mathfrak{sl}_2$-triple 62
homogeneous coordinates 1
homogeneous real hypersurface 170
horizontal 1-form 98
hyperdeterminants 87, 110, 130
hyperdeterminants, existence of 87
hyperplane 1

index of a Fano variety 137
inflection point 10
intersection product 134
isotropic Grassmanian 24
isotropic subspace 24

Jacobian ideal 122
jet bundle 41

join 75
Jordan algebra 26
Jordan decomposition 154

Katz dimension formula 81
Katz–Kleiman–Holme formula 106
Killing form 21
Kleiman criterion 135
Kobayashi–Ochiai characterisation 142
Kodaira dimension 92
Kostant partition function 32
Koszul complex 42
Kronecker–Weierstrass theory 174

Lagrangian variety 5
Landman formula 107
Legendre transformation 9
length of extremal rays 136
Levi subgroup 31
linear system of divisors 16
linear systems of bounded rank 172
linear systems of constant rank 172
linear systems of quadrics 172
linear systems of rank bounded below 172
linearly equivalent divisors 15
linearly normal variety 14
lines on flag varieties 104
local equations of a divisor 15
lower canonical string 36

matrix format 130
Matsumura-Monsky theorem 151
Maurer–Cartan equations 98
Maurer–Cartan forms 98
maximal parabolic subgroup 22
maximal torus 21
Meusnier–Euler theorem 99
Milnor class 123
Milnor number 123
moment map 31
Monotonicity theorem 95
Moore–Penrose inverse 61
Mori Cone Theorem 135
moving frames 98
Mukai fibration 142
multiple tangent 10
multiplicities of the discriminant 124
multisegment 32
multisegment duality 32

nef divisors 135
nef value 136

nef value morphism 136
negative roots 21
Neron–Severi group 134
non-degenerate variety 14
norm similarities 26
normal bundle 4
numerically equivalent 134
numerically trivial 134

octonions 26
ordinary double point 10
osculating sequence 99
osculating space 99, 103

parabolic subgroup 22
parity theorem 91, 129
partial order on the weight lattice 22
pencils of symmetric matrices 174
permanent 109
Picard number 134
Plücker embedding 23, 102
Plücker formulas 10
polarized variety 17, 116, 133
positive roots 21
Product theorem 84
projection from a subspace 12
projective second fundamental form 97
projective subspace 1
properties of jet bundles 41
Pyasetskii Theorem 30

quadratic nilpotent 153
quadric fibration 142
quasi-annihilator 154
quasiderivation 155
quotient projective space 12

ranks of a projective variety 108
Rationality Theorem 136
Reflexivity Theorem 4
regular algebra 154
Reidermeister torsion 43
relative secant variety 75
relative tangential variety 75
representations of quivers 32
resultant variety 52
Richardson Lemma 31
root lattice 144
ruled variety 7

Schläffli formula 11
scroll 142
secant variety 74
Segre class 106
Segre embedding 74, 84, 101
Segre Theorem 83
self-dual variety 168
semi-ample divisor 133
Severi variety 27, 78, 79, 102
short grading 34
simple roots 21
singular locus of hyperdeterminants 130
singular locus of the dual variety 130
singular scheme 122
singular subalgebra 153
smoothness principle 97
spectral sequence of the double complex 49
spinor variety 24, 102
stable point 151
stably twisted discriminant complex 47
standard projective space 1
strictly dominant weight 21
supervector space 43
Sylvester formula 48
symplectic structure 5
symplectic structure on the cotangent bundle 5
symplectic variety 5

tangent bundle 4
tangent hyperplane 2
tangent star 75
tangential variety 74
Terracini lemma 75

uniform vector bundle 90, 174
upper canonical string 37

varieties with small dual varieties 142
variety of complexes 67
Veronese curve 3
Veronese embedding 23, 102
very ample 16
very ample vector bundle 52

weight lattice 21
weight of a multisegment 32
Weyman's bigraded complex 50

Zak's theorem on linear normality 77
Zak's theorem on Severi varieties 78
Zak's theorem on tangencies 76
Zelevinsky involution 32