Lecture Notes in Mathematics 1842

Springer
Berlin
Heidelberg
New York
Hong Kong
London
Milan
Paris
Tokyo

Trygve Johnsen
Andreas Leopold Knutsen

*K*3 Projective Models in Scrolls

Springer

Authors

Trygve Johnsen

Department of Mathematics
University of Bergen
Johs. Bruns gt. 12
5008 Bergen, Norway

e-mail: johnsen@mi.uib.no

Andreas Leopold Knutsen

Department of Mathematics
University of Oslo
Box 1053, Blindern
0316 Oslo, Norway

e-mail: andreas.knutsen@math.uio.no

Library of Congress Control Number: 2004103750

Mathematics Subject Classification (2000): 14J28, 14H51

ISSN 0075-8434
ISBN 3-540-21505-0 Springer-Verlag Berlin Heidelberg New York

Springer-Verlag is a part of Springer Science + Business Media

http://www.springeronline.com

Printed in Germany

Typesetting: Camera-ready TEX output by the author

SPIN: 10999523 41/3142/ du - 543210 - Printed on acid-free paper

Preface

The cover picture shows a smooth quartic surface in space, the simplest example of a projective model of a $K3$ surface. In the following pages we will encounter many more examples of models of such surfaces.

The purpose of this volume is to study and classify projective models of complex $K3$ surfaces polarized by a line bundle L such that all smooth curves in $|L|$ have non-general Clifford index. Such models are in a natural way contained in rational normal scrolls.

These models are *special* in moduli in the sense that they do not represent the general member in the countable union of 19-dimensional families of polarized $K3$ surfaces. However, they are of interest because they fill up the set of models in $\mathbf{P}^g$ for $g \leq 10$ not described as complete intersections in projective space or in a homogeneous space as described by Mukai, with a few classificable exceptions.

Thus our study enables us to classify and describe all projective models of $K3$ surfaces of genus $g \leq 10$, which is the main aim of the volume.

Acknowledgements. We thank Kristian Ranestad, who suggested to study certain projective models of $K3$ surfaces in scrolls that had shown up in connection with his work on varieties of sums of powers (see [I-R1], [I-R2] and [R-S]). This idea was the starting point of our work.

We are also grateful to M. Coppens, G. Fløystad, S. Ishii, S. Lekaus, R. Piene, J. Stevens, S. A. Strømme, B. Toen and J. E. Vatne for useful conversations, and to G. M. Hana for pointing out several mistakes in an earlier version of the manuscript.

We also thank Alessandra Sarti who made us the nice picture we have used on the cover.

Most of this book was written while the authors were visitors at the Department of Mathematics, University of Utah, and at the Max-Planck-Institut für Mathematik, Bonn, in the year 2000. We thank both institutions for their hospitality.

The second author was supported by a grant from the Research Council of Norway.

Bergen/Oslo, March 2004

Trygve Johnsen
Andreas Leopold Knutsen

Contents

1

Introduction

1.1 Background

A $K3$ surface is a smooth compact complex connected surface with trivial canonical bundle and vanishing first Betti number. The mysterious name $K3$ is explained by A. Weil in the comment on his *Final report on contract AF18(603)-57* (see [We] p 546):

> *Dans la seconde partie de mon rapport, il s'agit des variétés kähleriénnes dites $K3$, ainsi nommées en l'honneur de Kummer, Kodaira, Kähler et de la belle montagne $K2$ au Cachemire.*

It is well known that all $K3$ surfaces are diffeomorphic, and that there is a 20-dimensional family of analytical isomorphism classes of $K3$ surfaces. However, the general element in this family is not algebraic, in fact the algebraic ones form a countable union of 19-dimensional families. More precisely, for any $n > 0$ there is a 19-dimensional irreducible family of $K3$ surfaces equipped with a base point free line bundle of self-intersection n. Moreover, the family of $K3$ surfaces having $\geq k$ linearly independent divisors (i.e. the surfaces with *Picard number* $\geq k$, where the Picard number is by definition the rank of the Picard group) forms a dense countable union of subvarieties of dimension $20 - k$ in the family of all $K3$ surfaces. In particular, on the general algebraic $K3$ surface all divisors are linearly equivalent to some rational multiple of the hyperplane class (see [G-H, pp. 590–594]).

A pair (S, L) of a $K3$ surface S and a base point free line bundle L with $L^2 = 2g - 2$ will be called a *polarized* $K3$ surface of genus g. Note that $g = h^0(L) - 1$ and that g is the arithmetic genus of any member of $|L|$. The sections of L give a map φ_L of S to $\mathbf{P}^g$, and the image is called a projective model of S. When φ_L is birational, the image is a surface of degree $2g - 2$ in $\mathbf{P}^g$. It is also easy to see that a projective model of genus 2 is a $2:1$ map $S \to \mathbf{P}^2$ branched along a sextic curve.

A very central point in the theory of projective models of $K3$ surfaces is that by the adjunction formula every smooth hyperplane section of a projec-

T. Johnsen and A.L. Knutsen: LNM 1842, pp. 1–14, 2004.

tive model of S (these are the images by φ_L of the smooth members of $|L|$) are canonical curves, i.e. curves for which $\omega_C \simeq O_C(1)$.

The first examples of projective models of $K3$ surfaces are the ones which are complete intersections in projective space. Using the fact that for a complete intersection surface S of $n-2$ hypersurfaces in $\mathbf{P}^n$ of degrees $d_1, \ldots, d_{n-2}$ we have $\omega_S \simeq \mathcal{O}_S(\sum d_i - n - 1)$ and $h^1(\mathcal{O}_S) = 0$ (see e.g. [Hrts, Exercises II,8.4 and III,5,5]), we find that there are exactly three types of $K3$ complete intersections, namely a hyperquartic in $\mathbf{P}^3$, a complete intersection of a hyperquadric and a hypercubic in $\mathbf{P}^4$ and a complete intersection of three hyperquadrics in $\mathbf{P}^5$.

In fact one can show that any birational projective model of genus 3 is a quartic surface and of genus 4 a complete intersection of a quadric and a cubic hypersurface. But already for genus 5 the situation is not as simple: The general model is a complete intersection of three hyperquadrics, but there are models which are not. In fact, take a 3-dimensional smooth rational normal scroll X of degree 3 in $\mathbf{P}^5$, which can be seen as the union of $\mathbf{P}^2$s parametrized by $\mathbf{P}^1$, i.e. a $\mathbf{P}^2$-bundle over $\mathbf{P}^1$. Intersect this scroll by a (sufficiently general) cubic hypersurface $\mathcal{C}$ containing one of the $\mathbf{P}^2$-fibers, call it F, then the intersection is $\mathcal{C} \cap X = F \cup S$, where S is a smooth surface of degree 8 in $\mathbf{P}^5$, i.e. a $K3$ surface. The ideal of this surface cannot be generated only by quadrics, whence S is not a complete intersection of three hyperquadrics. Note that the intersection of $\mathcal{C}$ with a general $\mathbf{P}^2$-fiber is a smooth curve of degree 3 in $\mathbf{P}^2$, which is elliptic by the genus formula, so S contains a pencil of elliptic curves of degree 3. In particular, since such a curve cannot be linearly equivalent to a multiple of the hyperplane section, S contains two linearly independent divisors, whence these surfaces can at most fill up an 18-dimensional family (in fact we will show that they do fill up an 18-dimensional family). Another interesting point is that the elliptic pencil on the surface cuts out a g^1_3 (i.e. a linear system of dimension 1 and degree 3) on each hyperplane section of S. Conversely, by a classical theorem of Enriques-Petri, the homogeneous ideal of a canonical curve with a g^1_3 is generated by both quadrics and cubics, so any projective model in $\mathbf{P}^5$ of a $K3$ surface whose hyperplane sections have a g^1_3 cannot be the complete intersection of three hyperquadrics.

For $6 \leq g \leq 10$ and $g = 12$ it is shown by Mukai in [Mu1] and [Mu2] that the general projective models are complete intersections in certain homogeneous varieties contained in projective spaces of larger dimension than g. The ambient varieties are constructed using special divisors on the hyperplane sections, and the general models have the property that their hyperplane sections do not carry certain particular g^r_ds induced from divisors on the surface.

That the projective model of a $K3$ surface somehow has to do with special divisors carried by the curves in $|L|$ dates back to the classical paper [SD] of Saint-Donat, which has become the main reference for all later work on projective models of or curves on $K3$ surfaces.

As remarked in [SD] it is clear from Zariski's Main Theorem (see e.g. [Hrts, V, Thm. 5.2]) that

$$\varphi_L = u_L \circ \theta_L,$$

where u_L is a finite morphism and θ_L maps S birationally onto a normal surface by contracting finitely many curves to rational double points and is an isomorphism outside these curves (the contracted curves are the curves sent to a point, and these are precisely the curves Δ such that $L.\Delta = 0$).

One of the main results in [SD] describes exactly when the map u_L is an identity, in other words when φ_L is birational.

Theorem 1.1 (Saint-Donat [SD]). *Let L be a base point free line bundle with $L^2 > 0$ on a $K3$ surface S. The following conditions are equivalent:*

(a) φ_L is not birational.
(b) There is a smooth hyperelliptic curve in $|L|$.
(c) All the smooth curves in $|L|$ are hyperelliptic.
(d) $L^2 = 2$; or there is a smooth elliptic curve E on S satisfying $E.L = 2$; or $L \sim 2B$ for a smooth curve B with $B^2 = 2$ and $L \sim 2B$.

A linear system $|L|$ satisfying these properties is said to be hyperelliptic.

Furthermore, if L is not hyperelliptic, then the natural maps $S_n H^0(L) \to H^0(nL)$ are surjective for all n

(Recall that a smooth curve is said to be hyperelliptic if it carries a g^1_2.) This “lifts” the classical fact that the canonical morphism of a smooth curve is an embedding if and only if the curve is not hyperelliptic and also Noether's theorem, to the surface:

Theorem 1.2 (Noether [No]). *If C is not hyperelliptic, then the ring $\oplus H^0(C, n\omega_C)$ is the homogeneous coordinate ring of C in its canonical embedding in $\mathbf{P}^g$.*

Moreover, Saint-Donat's result tells that a g^1_2 on a smooth curve on a $K3$ surface “propagates” to the other smooth members of the linear system. In fact, except for the trivial case where all the curves have genus 2 (the case $L^2 = 2$) and are therefore trivially hyperelliptic, such a propagating g^1_2 is given by the pencils $\mathcal{O}_C(E)$ or $\frac{1}{2}\mathcal{O}_C(B)$ for any smooth curve $C \in |L|$, corresponding to the curves E and B in (d).

Another main result in [SD] describes the homogeneous ideal of the image $\varphi_L(S)$:

Theorem 1.3 (Saint-Donat [SD]). *Let L be a base point free non-hyperelliptic line bundle with $L^2 \geq 8$ on a $K3$ surface S. Denote by I the graded ideal defined as the kernel of the map $S_* H^0(L) \to \oplus H^0(nL)$. Then I is generated by quadrics and cubics. Moreover the following conditions are equivalent:*

(a) I is generated not only by quadrics.
(b) $|L|$ contains a smooth curve carrying a g^1_3 or a g^2_5.
(c) All the smooth curves in $|L|$ carry a g^1_3 or all carry a g^2_5.

(d) There is a smooth elliptic curve E on S satisfying $E.L = 3$; or $L \sim 2B + \Gamma$ for a smooth curve B with $B^2 = 2$ and Γ a smooth rational curve with $B.\Gamma = 1$ (and $\Gamma^2 = -2$, in particular $L^2 = 10$).

Again, this lifts the classical result of Petri from the curve to the surface:

Theorem 1.4 (Petri [Pe]). *The homogeneous ideal of a non-hyperelliptic canonical curve C is generated by quadrics, unless C has a g^1_3 or a g^2_5.*

In the cases $L^2 = 4$ or 6 all the smooth curves in $|L|$ have genus 3 or 4, so they necessarily carry a g^1_3 (i.e. they are *trigonal*). For higher genus the last result again tells that g^1_3s and g^2_5s "propagate" among the smooth curves in $|L|$. Indeed the linear systems $|E|$ and $|B|$ on S given in (d) cut out a g^1_3 and a g^2_5 respectively on all the members of $|L|$.

Moreover, Saint-Donat gives a thorough description of the projective models in the special cases where $|L|$ is hyperelliptic or I is generated not only by quadrics. The models happen to lie in rational normal scrolls.

To broaden our perspective, let us recall the definition of the *Clifford index* of a smooth curve C of genus g, introduced by H. H. Martens in [HMa]. This is denoted by $\operatorname{Cliff} C$ and is the minimal integer $\deg A - 2(h^0(A) - 1)$ for all line bundles A on C satisfying $h^0(A) \geq 2$ and $h^1(A) \geq 2$. (The latter requirements presuppose that $g \geq 4$; however one can give ad hoc definitions in the cases of genus 2 or 3, by setting $\operatorname{Cliff} C = 0$ for C of genus 2 or hyperelliptic of genus 3, and $\operatorname{Cliff} C = 1$ for C non-hyperelliptic of genus 3.) Clifford's theorem states that $\operatorname{Cliff} C \geq 0$ with equality if and only if C is hyperelliptic and $\operatorname{Cliff} C = 1$ if and only if C is trigonal or a smooth plane quintic. Moreover, we also have $\operatorname{Cliff} C \leq \lfloor \frac{g-1}{2} \rfloor$, with equality for the general curve (cf. [A-C-G-H, V]).

We can rephrase the two results above of Saint-Donat by saying that φ_L is birational if and only if $\operatorname{Cliff} C > 0$ for every smooth curve $C \in |L|$ and that in addition I is generated only by quadrics if and only if $\operatorname{Cliff} C > 1$ for every smooth curve $C \in |L|$.

Moreover, Saint-Donat's results yield that either all or none of the smooth curves in a complete linear system on a $K3$ surface have Clifford index 0 (resp. 1). It is then a natural question to ask whether this also holds for higher indices.

Around ten years after the appearance of Saint-Donat's paper, interesting new techniques were introduced in the study of projective varieties.

One tool was the introduction of *Koszul cohomology* in [Gr] in connection with the study of *syzygies* and the resulting famous conjecture of Green.

Consider a smooth variety X with a base point free line bundle L with $r := h^0(L) - 1$ on it and the graded ring $R := \oplus_{m \geq 0} H^0(X, mL)$. This is in a natural way a finitely generated module over $T := \operatorname{Sym} H^0(X, L)$, the coordinate ring of the projective space $\mathbf{P}(H^0(L))$, and so has a *minimal graded free resolution*

$$0 \longrightarrow M_{r-1} \longrightarrow \ldots \longrightarrow M_1 \longrightarrow M_0 \longrightarrow R \longrightarrow 0,$$

where each M_i is a direct sum of twists of T:

$$M_i = \oplus_j T(-j) \otimes M_{i,j} \simeq \oplus_j T(-j)^{\beta_{i,j}}.$$

The finite dimensional vector space $M_{i,j}$ is called the *syzygy* of order i and weigth j and the $\beta_{i,j} := \dim M_{i,j}$ are called the *graded Betti-numbers.* Now L is said to satisfy property N_p if

$$M_0 = T \quad \text{and} \quad M_i = T(-i-1)^{\beta_{i,i-1}} \quad \text{for all} \quad 1 \leq i \leq p.$$

To be more concrete, N_0 means that $\varphi_L(X)$ is projectively normal, N_1 that in addition its homogeneous ideal is generated by quadrics, and more generally N_p for $p \geq 2$ means that in addition the matrices in the minimal graded free resolution have linear entries from the second to the pth step.

Now if $X = C$ is a smooth curve Green conjectured the following:

Conjecture 1.5 (Green [Gr]). *The Clifford index of C is the least integer p for which property N_p fails for the canonical bundle.*

For Cliff $C = 0$ this is Noether's theorem and for Cliff $C = 1$ this is Petri's theorem.

A "Lefschetz theorem" as in [Gr, (3.b.7)] implies that the syzygies of a hyperplane section of a $K3$ surface are the same as the ones of the $K3$ surface, so that all linearly equivalent smooth curves on a $K3$ surface have the same syzygies. Therefore an immediate consequence of Green's conjecture would be that all the smooth curves in a linear system on a $K3$ surface have the same Clifford index (since all such are canonically embedded by φ_L, by the adjunction formula).

A second important tool was the vector bundle techniques introduced by Lazarsfeld [La2] and Tyurin [Ty] (and also by Reider [Rdr] in a slightly different context). Using these techniques Green and Lazarsfeld [G-L4] proved that all the smooth curves in a linear system on a $K3$ surface have the same Clifford index. Moreover, they proved that if non-general, i.e. if $< \lfloor \frac{g-1}{2} \rfloor$, the Clifford index is induced by a line bundle on the surface, similarly to the cases studied by Saint-Donat.

Theorem 1.6 (Green-Lazarsfeld [G-L4]). *Let L be a base point free line bundle on a $K3$ surface S with $L^2 > 0$. Then Cliff C is constant for all smooth irreducible $C \in |L|$, and if Cliff $C < \lfloor \frac{g-1}{2} \rfloor$, then there exists a line bundle M on S such that $M_C := M \otimes \mathcal{O}_C$ computes the Clifford index of C for all smooth irreducible $C \in |L|$.*

As an immediate consequence we see that in the general case, i.e. when Pic $S \simeq \mathbf{Z}L$, then there can exist no line bundle M as above, so on the general $K3$ surface all curves have the general Clifford index.

By the result of Green and Lazarsfeld it makes sense to define the Clifford index Cliff L of a base point free line bundle, or the Clifford index Cliff $_L(S)$

of a polarized $K3$ surface (S, L), as the Clifford index of the smooth curves in $|L|$.

The fact that the Clifford index somehow influences the projective model of S was also remarked in [Kn4], where the second author studies higher order embeddings of $K3$ surfaces. Roughly speaking the Clifford index determines the amount of $(k+1)$-secant $(k-1)$-planes of the projective model.

In this book we study the projective models of those polarized $K3$ surfaces of genus g of *non-general Clifford index*, i.e. with $\text{Cliff}_L(S) < \lfloor \frac{g-1}{2} \rfloor$. These surfaces are special in moduli, since they can only fill up at most 18-dimensional families (except in the particular cases where S has Picard number one and L is non-primitive, i.e. L is an integral multiple ≥ 2 of the generator of Pic S).

As in the cases of Clifford index 0 and 1 studied by Saint-Donat, these models lie in rational normal scrolls in a natural way.

The central point is that by the result of Green and Lazarsfeld there exists in these cases a linear system $|D|$ on S computing the Clifford index of L. We can moreover choose such a linear system which is base point free and such that the general member is a smooth curve. We call such a divisor (class) D a *free Clifford divisor* for L.

The images of the members of $|D|$ by φ_L span sublinear spaces inside $\mathbf{P}^g$. Each subpencil $\{D_\lambda\}$ within the complete linear system $|D|$ then gives rise to a pencil of sublinear spaces. For each fixed pencil the union of these spaces will be a rational normal scroll $\mathcal{T}$. These scrolls are the natural ambient spaces for non-Clifford general $K3$ surfaces. Our description is inspired by and uses methods developed by Schreyer in [Sc], where the authour studies scrolls containing canonical curves and uses this to prove Green's conjecture for $g \leq 8$. In the same spirit as Saint-Donat, we so to speak lift Schreyer's results from the curve to the surface.

In the cases of Clifford index 1 and 2 with $D^2 = 0$, the description of the projective models is particularly nice, since they are then complete intersections in their corresponding scrolls.

Another important tool, which was still not available at the time [SD] was written, are the results on lattices by Nikulin [Ni], which allows to construct families of $K3$ surfaces with prescribed lattices, and thus show the existence of several interesting families. Using this, the second author proved the following *Existence Theorem* in [Kn2]: For any pair of integers (g, c) such that $g \geq 2$ and $0 \leq c \leq \lfloor \frac{g-1}{2} \rfloor$, there exists an 18-dimensional family of polarized $K3$ surfaces of genus g and of Clifford index c. Similar techniques allow us to prove the existence of all the families we study in this book and also compute their number of moduli.

We also give a description of those projective models for $g \leq 10$ that are Clifford general, but still not general in the sense of Mukai (i.e. they are not complete intersections in homogeneous spaces). These models are also contained in scrolls, and can be analysed in a similar manner. Together with Mukai's results this then gives a complete picture of the birational projective models for $g \leq 10$. For $g = 11$ and $g \geq 13$ our description of non-Clifford general projective models is not supplemented by any description of general projective models at all. We hope, however, that our description of the non-general models may have some interest in themselves.

1.2 Related literature

$K3$ surfaces in scrolls have also been studied in [Br] and [Ste].

Saint-Donat's results on the propagation of g^1_2s and g^1_3s among the smooth curves in a linear system on a $K3$ surface were extended to other g^1_ds by Reid [Re3]. The general question of propagation of g^r_ds came out of work of Harris and Mumford [H-M]. In fact they conjectured (unpublished) that the gonality (i.e. the minimal degree of a pencil on a curve) should be constant among the smooth curves in a linear system. Subsequently, Donagi and Morrison [D-M] pointed out the following counterexample:

Example 1.7. [D-M, (2.2)] Let $\pi : S \to \mathbf{P}^2$ be a $K3$ surface of genus 2, i.e. a double cover of $\mathbf{P}^2$ branched along a smooth sextic, and let $L := \pi^*\mathcal{O}_{\mathbf{P}^2}(3)$. The arithmetic genus of the curves in $|L|$ is 10. We have $H^0(L) = \pi^* H^0\mathcal{O}_{\mathbf{P}^2}(3) \oplus W$, where W is the one-dimensional subspace of sections vanishing on the ramification locus. The smooth curves C in the first summand are double covers of cubics, whence tetragonal (they all carry a 1-parameter family of g^1_4s which is the pullback of the 1-parameter family of g^1_2s on $\pi(C)$). On the other hand, the general curve in $|L|$ is isomorphic to a smooth plane sextic and is therefore of gonality 5. (Note that, in full accordance with the theorem of Green and Lazarsfeld, all the curves have Clifford index 1.)

The question is still open whether there exist other counterexamples. Ciliberto and Pareschi [C-P] proved that this is indeed the only counterexample when L is ample.

Exceptional curves, i.e. curves for which the Clifford index is not computed by a pencil, so that Cliff $C <$ gon $C - 2$, were studied in [E-L-M-S], where a whole class of examples were constructed as curves on $K3$ surfaces.

As for other surfaces, the constancy of the Clifford index and gonality of the smooth curves in a linear system on a Del Pezzo surface was studied by Pareschi [Pa] and the second author [Kn1, Kn3], who also classifies the exceptional curves on Del Pezzo surfaces.

As for recent work on Green's conjecture we refer to the recent brilliant work of Voisin [Vo1, Vo2], who - most interestingly - uses curves on $K3$ surfaces.

1.3 How the book is organised

Chapter 2. We recall the definition and some basic facts about rational normal scrolls, and how to obtain such scrolls from surfaces with pencils on them.

Most of this stems from [Sc]. At the end we give some special results when the surface is $K3$.

Chapter 3. The Clifford index of a curve is defined and the result of Green and Lazarsfeld for curves on $K3$ surfaces is given. We define the Clifford index of a base point free line bundle L with $L^2 = 2g - 2$ (or the polarized surface (S, L)) to be the Clifford index of all the smooth curves in $|L|$. The divisor class D on S computing the Clifford index c of L, when this is less than $\lfloor \frac{g-1}{2} \rfloor$, is studied, and we show that we can always find one such satisfying $0 \leq D^2 \leq c + 2$ and such that $|D|$ is base point free and the general member of $|D|$ is a smooth curve. Such a divisor (class) will be called a *free Clifford divisor* for L (Definition 3.6). (The definition only depends on the class of D.)

The images of the members of $|D|$ by φ_L span sublinear spaces inside $\mathbf{P}^g$. Each subpencil $\{D_\lambda\}$ within the complete linear system $|D|$ then gives rise to a pencil of sublinear spaces. For each fixed pencil the union of these spaces will be a rational normal scroll $\mathcal{T}$.

Chapter 4. The main result from [Kn2], the above mentioned *Existence Theorem*, and its proof are recalled.

Chapter 5. We study in detail the singular locus of the projective model $\varphi_L(S)$ and the scroll $\mathcal{T}$ in which we choose to view this model as contained. We show (Theorem 5.7) that we can always find a free Clifford divisor D such that the singular locus of $\mathcal{T}$ is "spanned" by the images of the base points of the pencil $\{D_\lambda\}$ and the contractions of smooth rational curves across the members of the pencil. A free Clifford divisor with this extra property will be called a *perfect Clifford divisor* (Definition 5.9). The proofs use results about higher order embeddings of $K3$ surfaces as developed by the second author in [Kn4], which we briefly recall in Section 1.4 below. We also include a study of the projective model if $c = 0$ (the hyperelliptic case), which is Saint-Donat's classical result [SD]. Some proofs are postponed until the next chapter.

Chapter 6. Here some of the longer proofs of the results in the previous chapter are given.

Chapter 7. We study and find (up to certain invariants) a resolution of $\varphi_L(S)$ inside its scroll $\mathcal{T}$ when $\mathcal{T}$ is smooth. In this case a general hyperplane section of $\mathcal{T}$ is a scroll formed in a similar way from a pencil computing the gonality on a canonical curve C of genus g (the gonality is $c+2$). Such scrolls were studied in [Sc], and our results (Lemma 7.1 and Proposition 7.2) for $K3$ surfaces in smooth scrolls are quite parallel to those of [Sc].

Chapter 8. We treat the case when the scroll $\mathcal{T}$ is singular. The approach is to study the blow up $f : \tilde{S} \to S$ at the D^2 base points of the pencil $\{D_\lambda\}$ and the projective model $S'' := \varphi_H(\tilde{S})$ of $\tilde{S}$ by the base point free line bundle $H := f^*L + f^*D - E$, where E is the exceptional divisor. The pencil $|f^*D - E|$ defines a smooth rational normal scroll $\mathcal{T}_0$ that contains S'' and is a desingularization of $\mathcal{T}$.

We use Koszul cohomology and techniques inspired by Green and Lazarsfeld to compute some Betti-numbers of the $\varphi_L(D_\lambda)$ and we obtain that they all have the same Betti-numbers for low values of D^2 and this is a necessary and sufficient condition for “lifting” the resolutions of the fibers to one of the surface S'' in $\mathcal{T}_0$. We prove that S'' is normal, and use this to give more details about the resolution. We give conditions under which we can push down the resolution to one of $\varphi_L(S)$ in $\mathcal{T}$. Here we use results from [Sc]. We end the section by investigating some examples for low genera.

Chapter 9. We consider in more detail the projective models in smooth scrolls for $c = 1$, 2 and 3 ($< \lfloor \frac{g-1}{2} \rfloor$). The description is particularly nice for $c = 1$ and 2, since the projective models are complete intersections in their corresponding scrolls.

We study the sets of projective models in $(c+2)$-dimensional scrolls of given types. Since the scroll type is dependent on which rational curves that exist on S, and therefore on the Picard lattice, it is natural that the dimension of the set of models in question in a scroll as described is dependent on the scroll type. We study this interplay, and obtain a fairly clear picture for $c = 1$ and 2. Most of the information presented can also be obtained from combining material in [Re2], [Ste], and [Br]. For $c = 3$ we study a Pfaffian map of the resolution of $\varphi_L(S)$ in the scroll. In Remark 9.19 we predict the dimension of the set of projective $K3$ models inside a fixed smooth scroll of a given type, for arbitrary $c < \lfloor \frac{g-1}{2} \rfloor$. We state the special case $c = 3$ as Conjecture 9.15.

Chapter 10. We give the definition of BN general polarized $K3$ surfaces introduced by Mukai in [Mu2]: A polarized $K3$ surface (S, L) is said to be *Brill-Noether (BN) general* if for all non-trivial decompositions $L \sim M + N$ one has $h^0(M)h^0(N) < h^0(L)$. (One easily sees that this is for instance satisfied if any smooth curve $C \in |L|$ is Brill-Noether general, i.e. carries no line bundle $\mathcal{A}$ for which $\rho(A) := g - h^0(\mathcal{A})h^1(\mathcal{A}) < 0$.) In [Mu1] it is shown that all such projective models of BN general surfaces of genus $g \leq 10$ and $g = 12$ are complete intersections in certain homogeneous spaces, and that being BN general is also a necessary condition to have such a model (see Theorem 10.3 below).

We study the projective models for $g \leq 10$ that are Clifford general but not BN general. By the concrete description in [Mu2] of such surfaces it follows that their projective models are also contained in scrolls. We analyse them in a similar manner.

Chapter 11. We conclude by giving a complete list and descripton of *all* birational projective models of $K3$ surfaces for $g \leq 10$ (including both the general ones in the sense of Mukai and the remaining ones, that we give a detailed classification of here).

Chapter 12. Some related issues and applications of the ideas developed in this book are discussed, like rational curves in families of Calabi-Yau threefolds and scrolls containing Enriques surfaces.

1.4 Notation and conventions

We use standard notation from algebraic geometry, as in [Hrts].

The ground field is the field of complex numbers. All surfaces are reduced and irreducible *algebraic surfaces*.

By a $K3$ surface is meant a smooth surface S with trivial canonical bundle and such that $H^1(\mathcal{O}_S) = 0$. In particular $h^2(\mathcal{O}_S) = 1$ and $\chi(\mathcal{O}_S) = 2$.

By a *curve* is always meant a *reduced* and *irreducible curve* (possibly singular). The *adjunction formula* for a curve C on a surface S reads $\mathcal{O}_C(C + K_S) \simeq \omega_C$, where ω_C is the dualising sheaf of C, which is just the canonical bundle when C is smooth. In particular, the arithmetic genus p_a of C is given by $C.(C + K_S) = 2p_a - 2$.

On a smooth surface we use line bundles and divisors, as well as the multiplicative and additive notation, with little or no distinction. We denote by Pic S the *Picard group* of S, i.e. the group of linear equivalence classes of line bundles on S. The *Hodge index theorem* yields that if $H \in$ Pic S with $H^2 > 0$, then $D^2H^2 \leq (D.H)^2$ for any $D \in$ Pic S, with equality if and only if $(D.H)H \equiv H^2D$.

Linear equivalence of divisors is denoted by $\sim$, and numerical equivalence by $\equiv$. Note that on a $K3$ surface S linear and numerical equivalence is the same, so that Pic S is torsion free. The usual intersection product of line bundles (or divisors) on surfaces therefore makes the Picard group of a $K3$ surface into a lattice, the *Picard lattice* of S, which we also denote by Pic S.

For two divisors or line bundles M and N on a surface, we use the notation $M \geq N$ to mean $h^0(M-N) > 0$ and $M > N$, if in addition $M-N$ is nontrivial.

If L is any line bundle on a smooth surface, L is said to be *numerically effective*, or simply *nef*, if $L.C \geq 0$ for all curves C on S. In this case L is said to be *big* if $L^2 > 0$.

If $\mathcal{F}$ is any coherent sheaf on a variety V, we shall denote by $h^i(\mathcal{F})$ the complex dimension of $H^i(V, \mathcal{F})$, and by $\chi(\mathcal{F})$ the *Euler characteristic* $\sum(-1)^i h^i(\mathcal{F})$. In particular, if D is any divisor on a normal surface S, the *Riemann-Roch formula* for D is $\chi(\mathcal{O}_S(D)) = \frac{1}{2}D.(D-K_S)+\chi(\mathcal{O}_S)$. Moreover, if D is effective and nonzero and $\mathcal{L}$ is any line bundle on D, the Riemann-Roch formula for $\mathcal{L}$ on D is $\chi(\mathcal{L}) = \deg \mathcal{L} + 1 - p_a(D) = \deg \mathcal{L} - \frac{1}{2}D.(D + K_S)$.

We will make use of the following results of Saint-Donat on line bundles on $K3$ surfaces. The first result will be used repeatedly, without further mention.

Proposition 1.8. [SD, Cor. 3.2] *A complete linear system on a $K3$ surface has no base points outside of its fixed components.*

Proposition 1.9. [SD, Prop. 2.6] *Let L be a line bundle on a $K3$ surface S such that $|L| \neq \emptyset$ and such that $|L|$ has no fixed components. Then either*

(i) $L^2 > 0$ *and the general member of $|L|$ is a smooth curve of genus $L^2/2+1$. In this case $h^1(L) = 0$, or*
(ii) $L^2 = 0$, *then $L \simeq \mathcal{O}_S(kE)$, where k is an integer ≥ 1 and E is a smooth curve of arithmetic genus 1. In this case $h^1(L) = k-1$ and every member of $|L|$ can be written as a sum $E_1+\cdots+E_k$, where $E_i \in |E|$ for $i = 1, \ldots, k$.*

Lemma 1.10. [SD, 2.7] *Let L be a nef line bundle on a $K3$ surface S. Then $|L|$ is not base point free if and only if there exist smooth irreducible curves E, Γ and an integer $k \geq 2$ such that*

$$L \sim kE + \Gamma, \qquad E^2 = 0, \qquad \Gamma^2 = -2, \qquad E.\Gamma = 1.$$

In this case, every member of $|L|$ is of the form $E_1 + \cdots + E_k + \Gamma$, where $E_i \in |E|$ for all i.

To show the existence of $K3$ surfaces with certain divisors on it, a very useful result is the following by Nikulin [Ni] (the formulation we use is due to Morrison):

Proposition 1.11. [Mo, Cor. 2.9(i)] *Let $\rho \leq 10$ be an integer. Then every even lattice of signature $(1, \rho-1)$ occurs as the Picard group of some algebraic $K3$ surface.*

Consider now the group generated by the Picard-Lefschetz reflections

$$\begin{aligned} \phi_\Gamma : \text{Pic } S &\longrightarrow \text{Pic } S \\ D &\mapsto D + (D.\Gamma)\Gamma \end{aligned}$$

where $\Gamma \in \text{Pic } S$ satisfies $\Gamma^2 = -2$. Note that a reflection leaves the intersections between divisors invariant.

The following result will also be useful for our purposes:

Proposition 1.12. [B-P-V, VIII, Prop. 3.9] *A fundamental domain for this action, restricted to the positive cone, is the big-and-nef cone of S.*

This means that given a certain Picard lattice, we can perform Picard-Lefschetz reflections on it, and thus assume that some particular line bundle chosen (with positive self-intersection) is nef.

We will need some results about higher order embeddings of $K3$ surfaces from [Kn4], which we here recall:

Proposition 1.13. *Let $k \geq 0$ be an integer and L a big and nef line bundle on a $K3$ surface with $L^2 \geq 4k$. Assume $\mathcal{Z}$ is a 0-dimensional subscheme of S of length $h^0(\mathcal{O}_\mathcal{Z}) = k+1$ such that*

$$H^0(L) \longrightarrow H^0(L \otimes \mathcal{O}_\mathcal{Z})$$

is not onto, and for any proper subscheme $\mathcal{Z}'$ of $\mathcal{Z}$, the map

$$H^0(L) \longrightarrow H^0(L \otimes \mathcal{O}_{\mathcal{Z}'})$$

is onto.

Then there exists an effective divisor D passing through $\mathcal{Z}$ satisfying $D^2 \geq -2$, $h^1(D) = 0$ and the numerical conditions

$$\begin{gathered}2D^2 \overset{(i)}{\leq} L.D \leq D^2 + k + 1 \overset{(ii)}{\leq} 2k + 2 \\ \textit{with equality in (i) if and only if } L \sim 2D \textit{ and } L^2 \leq 4k + 4, \\ \textit{and equality in (ii) if and only if } L \sim 2D \textit{ and } L^2 = 4k + 4.\end{gathered} \tag{1.1}$$

Furthermore, either $L - 2D \geq 0$, *or* $L^2 = 4k$ *and* $h^0(\mathcal{O}_S(L - D) \otimes \mathcal{O}_{\mathcal{Z}}) > 0$.

Finally, if $L^2 = 4k + 4$ *and* $L \sim 2D$, *then* $h^0(\mathcal{O}_S(D) \otimes \mathcal{J}_{\mathcal{Z}}) = 2$, *and if* $L^2 = 4k + 2$ *and* $D^2 = k$, *then* $L \sim 2D + \Gamma$, *for a smooth rational curve* Γ *satisfying* $\Gamma.D = 1$ *and* $\Gamma \cap \mathcal{Z} \neq \emptyset$.

Proof. All the statements are implicitly contained in [Kn4], but we will go through the main steps in the proof for the sake of the reader.

Under the above hypotheses, it follows from the first part of the proof of [B-S, Thm. 2.1] or from [Ty, (1.12)] that there exists a rank 2 vector bundle E on S fitting into the exact sequence

$$0 \longrightarrow \mathcal{O}_S \longrightarrow E \longrightarrow L \otimes \mathcal{J}_{\mathcal{Z}} \longrightarrow 0, \tag{1.2}$$

and such that the coboundary map

$$\delta : H^1(L \otimes \mathcal{J}_{\mathcal{Z}}) \longrightarrow H^2(\mathcal{O}_S) \simeq \mathbf{C},$$

is an isomorphism. In particular $H^1(E) = H^2(E) = 0$ and we also have $\det E = L$ and $c_2(E) = \deg \mathcal{Z} = k + 1$.

Secondly, since $L^2 \geq 4k$, one computes by Riemann-Roch

$$\chi(E \otimes E^*) = c_1(E)^2 - 4c_2(E) + 4\chi(\mathcal{O}_S) \geq 4,$$

whence $h^0(E \otimes E^*) \geq 2$ by Serre duality. This means that E has nontrivial endomorphisms, and by standard arguments, as for instance in [D-M, Lemma 4.4], there are line bundles M and N on S and a zero-dimensional subscheme $A \subset S$ such that E fits in an exact sequence

$$0 \longrightarrow N \longrightarrow E \longrightarrow M \otimes \mathcal{J}_A \longrightarrow 0 \tag{1.3}$$

and either $N \geq M$, or $A = \emptyset$ and the sequence splits. In the latter case, we can and will assume by symmetry that $N.L \geq M.L$ (which is automatically fulfilled in the first case, by the nefness of L).

Combining (1.2) and (1.3) we find

$$\det E = L \sim M + N \quad \text{and} \quad c_2(E) = \deg \mathcal{Z} = k + 1 = M.N + \deg A. \tag{1.4}$$

It follows that $N.L \geq \frac{1}{2}(N.L + M.L) \geq \frac{1}{2}L^2 > 0$ and $N^2 = N.L - M.N \geq \frac{1}{2}L^2 - M.N \geq 2k - (k+1) > -2$, so $N > 0$ by Riemann-Roch. It folloms that $h^0(N^\vee) = 0$, so tensoring (1.2) and (1.3) by $N^\vee$ and taking cohomology, we get $h^0(M \otimes \mathcal{J}_{\mathcal{Z}}) \geq h^0(E \otimes N^\vee) > 0$, whence $M > 0$ as well and there is an effective divisor $D \in |M|$ such that $D \supseteq \mathcal{Z}$, as stated.

Moreover, since $h^1(E) = h^2(E) = h^2(N) = 0$, we get from (1.3) that $h^1(D) = h^1(M) = 0$, whence $D^2 \geq -2$ by Riemann-Roch.

From (1.4) we have $M.N \leq k+1$, in other words $L.D \leq D^2 + k + 1$, and by the Hodge index theorem

$$2(D.L)D^2 \leq D^2L^2 \leq (D.L)^2.$$

Hence $2D^2 \leq D.L$, with equality if and only if $2D.L = L^2$ and $L \sim 2D$, in which case we have $L^2 = 4D^2 \leq 4(k+1)$. It also follows that $D^2 \leq D.L - D^2 = D.M \leq k+1$, with equalities if and only if $L \sim 2D$ and $D^2 = k+1$, so that $L^2 = 4k+4$. This establishes the numerical criteria.

Now we want to show that, possibly after interchanging M and N, either $L - 2D \geq 0$, or $L^2 = 4k$ and $h^0(N \otimes \mathcal{J}_{\mathcal{Z}}) > 0$. So assume that $h^0(L-2D) = 0$. Then the sequence (1.3) splits, and by arguing as above with N and M interchanged, we find $h^0(N \otimes \mathcal{J}_{\mathcal{Z}}) > 0$. Since $(L-2D)^2 = L^2 - 4M.N \geq 4k - 4(k+1) = -4$ and $(L-2D).L \geq 0$, we see by Riemann-Roch and the Hodge index theorem that we must have $(L-2D)^2 = -2$ or -4. If $(L-2D)^2 = -2$, Riemann-Roch yields that $2D - L > 0$. By the nefness of L we must have $(L-2D).L = 0$, whence $M.L = N.L$ and $M^2 = N^2$, and we get the desired result after interchanging M and N. If $(L-2D)^2 = -4$, then $L^2 = 4k$, as stated.

We now prove the two last assertions.

If $L^2 = 4k+4$ and $L \sim 2D$, then from (1.4) we get $A = \emptyset$, so by tensoring (1.2) and (1.3) by $M^\vee$ and taking cohomology, we get $h^0(M \otimes \mathcal{J}_{\mathcal{Z}}) \geq h^0(E \otimes M^\vee) = 2$, as stated.

If $L^2 = 4k+2$ and $D^2 = k$, then clearly $L \not\sim 2D$, so by the numerical conditions above, we get $2k < L.D \leq 2k+1 < 2k+2$, whence $L.D = 2k+1$. Moreover, we find that $L \sim 2D + \Delta$, for a $\Delta > 0$ satisfying $\Delta^2 = -2$ and $\Delta.D = 1$. Since $\Delta.L = 0$, we have that $h^0(\Delta) = 1$ and Δ is supported only on smooth rational curves, and there has to exist a smooth rational curve Γ with $\Gamma.D > 0$. Since L is big and nef, we get by Riemann-Roch

$$\begin{aligned} h^0(L) &= \frac{1}{2}L^2 + 2 = \frac{1}{2}(2D+\Delta)^2 + 2 = \frac{1}{2}(4k+4-2) + 2 \\ &\leq \frac{1}{2}(2D+\Gamma)^2 + 2 = h^0(2D+\Gamma), \end{aligned}$$

and since L is not of the particular form in Lemma 1.10 above, L is base point free, so we must have $L \sim 2D + \Gamma$. So $N \sim D + \Gamma$, and Γ is fixed in N. Since $N^2 = D^2$ and $h^0(N) = h^0(D)$, it follows by Riemann-Roch that $h^1(N) = h^1(D) = 0$. Moreover, we see from (1.4) that $A = \emptyset$, and by tensoring (1.2) and (1.3) with $N^\vee$ and $M^\vee$ respectively, using $H^1(\Gamma) = H^1(N) = 0$, we get $h^0(M \otimes \mathcal{J}_{\mathcal{Z}}) = 1$ and $h^0(N \otimes \mathcal{J}_{\mathcal{Z}}) = 2$, respectively. This means that we can choose two distinct elements N_1 and N_2 in $|N|$ both containing $\mathcal{Z}$ (scheme-theoretically). But since Γ is a base component of $|N|$, we must have $N_1 = D_1 + \Gamma$ and $N_2 = D_2 + \Gamma$, for two distinct elements D_1 and D_2 of $|D|$.

If $\mathcal{Z}$ does not meet Γ, we would have both D_1 and D_2 containing $\mathcal{Z}$ (scheme-theoretically). But this contradicts the fact that $h^0(\mathcal{O}_X(D) \otimes \mathcal{J}_{\mathcal{Z}}) = 1$. So $\mathcal{Z}$ meets Γ and we are done.

Remark 1.14. If we replace the assumptions that L be big and nef with $L^2 > 0$ and $h^1(L) = 0$, one can check from the proof of [B-S, Thm. 2.1] or from [Ty, (1.12)] that we still have a rank 2 vector bundle E on S fitting into an exact sequence as in (1.2). Moreover, if the stronger condition $L^2 > 4k+4$ is fulfilled, then $c_1(E)^2 > 4c_2(E)$, and we can use Bogomolov's theorem (see [Bo] or [Re1]) to find an exact sequnce as (1.3) with the properties that $(N-M)^2 > 0$ and $(N-M).H > 0$ for any ample line bundle on S. These two numerical conditions yield with Riemann-Roch that $N > M$ and it follows almost automatically that $h^0(N^\vee) = 0$, so as in the proof of Proposition 1.13 we find that $h^0(M \otimes \mathcal{J}_{\mathcal{Z}}) > 0$, $h^1(M) = 0$ and $M^2 \geq -2$. Furthermore, (1.4) still holds.

Since L is not necessarily nef, we cannot assume that $N.L \geq M.L$, so we do not get the numerical conditions as in Proposition 1.13.

To sum up, under the assumptions of Proposition 1.13, with L being big and nef replaced by $h^1(L) = 0$, and $L^2 \geq 4k$ replaced by $L^2 > 4k+4$, we get the weaker result there is a nontrivial effective decomposition $L \sim D + N$ such that $N > D$, $N.D \leq k+1$, $h^1(D) = 0$, $D^2 \geq -2$ and D passes through $\mathcal{Z}$.

2

Surfaces in Scrolls

In this chapter we briefly review the definition of rational normal scrolls and some basic facts that can be found in [Sc]. The case when a scroll contains a surface will be of particular interest to us.

In Section 2.2 we gather some special results valid in the $K3$ case, which will be useful to us later.

2.1 Rational normal scrolls

Definition 2.1. *Let $\mathcal{E} = \mathcal{O}_{\mathbf{P}^1}(e_1) \oplus \cdots \oplus \mathcal{O}_{\mathbf{P}^1}(e_d)$, with $e_1 \geq \ldots \geq e_d \geq 0$ and $f = e_1 + \cdots + e_d \geq 2$. Consider the linear system $\mathcal{L} = \mathcal{O}_{\mathbf{P}(\mathcal{E})}(1)$ on the corresponding $\mathbf{P}^{d-1}$-bundle $\mathbf{P}(\mathcal{E})$ over $\mathbf{P}^1$. We map $\mathbf{P}(\mathcal{E})$ into $\mathbf{P}^r$ with the complete linear system $H^0(\mathcal{L})$, where $r = f + d - 1$. The image T is by definition a rational normal scroll of type* $\mathbf{e}$= $(e_1, \ldots, e_d)$. *The image is smooth, and isomorphic to $\mathbf{P}(\mathcal{E})$, if and only if $e_d \geq 1$.*

Remark 2.2. Some authors, like in [P-S], use the term *rational normal scroll* only if $e_d \geq 1$ (so that T is smooth), but for our purposes it will be more convenient to use the more liberal definition above. The definition of rational normal scrolls goes back at least to C. Segre, see [Se1] and [Se2].

Definition 2.3. *Let $\mathcal{T}$ be a rational normal scroll of type* $(e_1, \ldots, e_d)$. *We say that $\mathcal{T}$ is a scroll of maximally balanced type if $e_1 - e_d \leq 1$.*

Let L be a base point free and big line bundle on a smooth surface S. We denote by φ_L the natural morphism

$$\varphi_L : S \longrightarrow \mathbf{P}^{h^0(L)-1} := \mathbf{P}^g$$

defined by the complete linear system $|L|$.

Assume that L can be decomposed as

$$L \sim D + F, \quad \text{with} \quad h^0(D) \geq 2 \text{ and } h^0(F) \geq 2. \tag{2.1}$$

T. Johnsen and A.L. Knutsen: LNM 1842, pp. 15–18, 2004.

Choose a 2-dimensional subspace $W \subseteq H^0(S,D)$, which then defines a pencil

$$\{D_\lambda\}_{\lambda \in \mathbf{P}^1} \subseteq |D|.$$

Each $\varphi_L(D_\lambda)$ will span a $(h^0(L) - h^0(L-D) - 1)$-dimensional subspace of $\mathbf{P}^g$, which is called the linear span of $\varphi_L(D_\lambda)$ and denoted by $\overline{D_\lambda}$. The variety swept out by these linear spaces,

$$T = \cup_{\lambda \in \mathbf{P}^1} \overline{D_\lambda} \subseteq \mathbf{P}^g,$$

is a rational normal scroll:

Proposition 2.4. [Sc] *The multiplication map*

$$W \otimes H^0(S,F) \longrightarrow H^0(S,L)$$

yields a $2 \times h^0(F)$ *matrix with linear entries whose* 2×2 *minors vanish on* $\varphi_L(S)$. *The variety* T *defined by these minors contains* $\varphi_L(S)$ *and is a rational normal scroll of degree* $f := h^0(F)$ *and dimension* $d := h^0(L) - h^0(L-D)$. *In particular* $d + f = g + 1$.

Decomposing the pencil $\{D_\lambda\}$ into its moving part $\{D'_\lambda\}$ and fixed part Δ,

$$D_\lambda \sim D'_\lambda + \Delta,$$

the type $(e_1, \ldots, e_d)$ of the scroll T is given by

$$e_i = \#\{j \mid d_j \geq i\} - 1, \tag{2.2}$$

where

$$\begin{aligned} d = d_0 &:= h^0(L) - h^0(L-D), \\ d_1 &:= h^0(L-D) - h^0(L-2D'-\Delta), \\ &\vdots \\ d_i &:= h^0(L-iD'-\Delta) - h^0(L-(i+1)D'-\Delta), \\ &\vdots \end{aligned}$$

Remark 2.5. The d_i form a non-increasing sequence. This follows essentially as in the proof of Exercise B-4 in [A-C-G-H], using the socalled "base-point-free pencil trick".

Conversely, if $\varphi_L(S)$ is contained in a scroll T of degree f, the ruling of T will cut out on S a pencil of divisors (possibly with base points) $\{D_\lambda\} \subseteq |D|$ with $h^0(L-D) = f \geq 2$, whence inducing a decomposition as in (2.1).

For any decomposition $L \sim D + F$, with $h^0(D) \geq 2$ and $h^0(F) \geq 2$, denote by c the integer $D.F - 2$. We may assume $D.L \leq F.L$, or equivalently

$D^2 \leq F^2$. Then we have by the Hodge index theorem that D satisfies the numerical conditions below:

$$2D^2 \overset{(i)}{\leq} L.D = D^2 + c + 2 \overset{(ii)}{\leq} 2c+4$$

with equality in (i) or (ii) if and only if $L \equiv 2D$ and $L^2 = 4c+8$.

Indeed, the condition $D.L \leq F.L$ can be rephrased as $2D.L \leq L^2$, and by the Hodge index theorem $2D^2(D.L) \leq D^2L^2 \leq (D.L)^2$, with equalities if and only if $L \equiv 2D$.

If the set

$$\mathcal{A}(L) := \{D \in \text{Pic } S \ \mid \ h^0(D) \geq 2 \text{ and } h^0(L-D) \geq 2\}$$

is nonempty, define the integer $\mu(L)$ as

$$\begin{aligned} \mu(L) &:= \min\,\{D.F - 2 \ \mid \ L \sim D + F \text{ and } D, F \in \mathcal{A}(L)\} \\ &= \min\,\{D.L - D^2 - 2 \ \mid \ D \in \mathcal{A}(L)\} \end{aligned}$$

and set

$$\mathcal{A}^0(L) := \{D \in \mathcal{A}(L) \ \mid \ D.(L-D) = \mu(L) + 2\}$$

2.2 Specializing to $K3$ surfaces

For $K3$ surfaces we have the following result:

Proposition 2.6. *Let L be a base point free and big line bundle on a $K3$ surface S such that $\mathcal{A}(L) \neq \emptyset$. Then $\mu(L) \geq 0$ and any divisor D in $\mathcal{A}^0(L)$ will have the following properties:*

(i) the (possibly empty) base divisor Δ of D satisfies $L.\Delta = 0$,
(ii)$h^1(D) = 0$.

Proof. The first statement follows from the fact that any member of the complete linear system of a base point free and big line bundle on a $K3$ surface is numerically 2-connected (see [SD, (3.9.6)], or [Kn4, Thm. 1.1] for a more general statement).

We first show (i).

If D is nef but not base point free, then by Lemma 1.10, $D \sim kE + \Gamma$, for an integer $k \geq 2$ and divisors E and Γ satisfying $E^2 = 0$, $\Gamma^2 = -2$ and $E.\Gamma = 1$. Since L is base point free, we must have $E.L \geq 2$ (see [SD] or [Kn4, Thm. 1.1]), so $D.L - D^2 = (kE+\Gamma).L - (2k-2) \geq kE.L - 2(k-1) \geq E.L + 2(k-1) - 2(k-1) = E.L$, which implies $E.L = 2$, $\mu(L) = 0$, and as asserted $\Gamma.L = 0$.

If D is not nef, there exists a smooth rational curve Γ such that $\Gamma.D < 0$. Letting $D' := D - \Gamma$ and we have $D' \in \mathcal{A}(L)$ and $D'.(L - D') = D.(L - D) - L.\Gamma + 2\Gamma.D + 2 \leq D.(L - D)$, whence $L.\Gamma = 0$, $\Gamma.D = -1$, ${D'}^2 = D^2$, $L.D' = L.D$ and $D'(L - D') = D.(L - D) = c + 2$. Continuing inductively, we get that $\Delta.L = 0$, as desired.

Since $\Delta.L = 0$ and $(D - \Delta).L - (D - \Delta)^2 \geq D.L - D^2$, we must have $D^2 \geq (D - \Delta)^2 \geq 0$.

We now prove (ii).

If $h^1(D) \neq 0$, there exists by Ramanujam's lemma an effective decomposition $D \sim D_1 + D_2$ such that $D_1.D_2 \leq 0$. By the Hodge index theorem (and the fact that $D^2 \geq 0$) we can assume $D_1^2 \geq 0$ and $D_2^2 \leq 0$, with equalities occurring simultaneously. The divisor D_1 is in $\mathcal{A}(L)$, and writing $F := L - D$ we get $D_1.(F + D_2) = D.F + D_1.D_2 - D_2.F \geq F.D$, whence $D_2.F \leq D_1.D_2 \leq 0$. But L is nef, so $D_2.L = D_2.D + D_2.F \geq 0$, which implies $D_2.F = D_1.D_2 = D_1^2 = D_2^2 = 0$. Now the same argument works for D_1, so $D_1.F = 0$ and we get the contradiction $D.F = (D_1 + D_2).F = 0$. $\square$

Writing $L \sim D + F$, the above result is of course symmetric in D and F. It turns out that we can choose one of them to have an additional property. More precisely, we have :

Proposition 2.7. *Let L be a base point free line bundle on a $K3$ surface S such that $\mathcal{A}(L) \neq \emptyset$. We can find a divisor D in $\mathcal{A}^0(L)$ such that either $|D|$ or $|L - D|$ (but not necessarily both at the same time) is base point free and its general member is smooth and irreducible. If L is ample, then for any divisor D in $\mathcal{A}^0(L)$ the above conditions will be satisfied for both $|D|$ and $|L - D|$.*

Proof. Let $D \in \mathcal{A}^0(L)$. Denote its base locus by Δ and assume it is not zero. Then $L.\Delta = 0$ by the previous proposition.

If D is nef but not base point free, then $D \sim kE + \Gamma$ as above and the smooth curve E will satisfy the desired conditions.

If D is not nef, there exists a smooth rational curve Γ such that $\Gamma.D < 0$. Letting $D' := D - \Gamma$, we can argue inductively as above until we reach a divisor which is base point free or of the form $kE + \Gamma$.

This procedure can of course not be performed on both D and $L - D$ simultaneously, but if L is ample, they are both automatically base point free.

The fact that the general member of $|D|$ (or $|L - D|$) is a smooth curve now follows from Proposition 1.9, since $h^1(D) = 0$. $\square$

Remark 2.8. Note that by the proofs of the two previous propositions, if D is not nef, then ${D'}^2 = D^2$. This means that given a divisor $D \in \mathcal{A}(L)$, we can find a divisor $D_0 \in \mathcal{A}(L)$ satisfying the additional conditions in Proposition 2.7 and such that $D_0^2 \leq D^2$.

3

The Clifford index of smooth curves in $|L|$ and the definition of the scrolls $\mathcal{T}(c, D, \{D_\lambda\})$

We will see that in the $K3$ case natural decompositions of L into two moving classes, as in (2.1), occur when the smooth curves in $|L|$ do not have the general Clifford index, by the theorem of Green and Lazarsfeld mentioned in the introduction.

We recall the definition and basic properties of the Clifford index of a smooth curve in Section 3.1 and the result of Green and Lazarsfeld, saying that all the smooth curves in a complete linear system on a $K3$ surface have the same Clifford index, in Section 3.2, together with some of its corollaries.

The central point for us is that if the curves in $|L|$ have non-general Clifford index, i.e. $< \lfloor \frac{g-1}{2} \rfloor$, where g is the sectional genus of L, then there is a divisor D such that $L \sim D + (L - D)$ is a decomposition into two moving classes. We study these divisors D in Section 3.3 and the scrolls arising from the decomposition of L in Section 3.4.

3.1 Gonality and Clifford index of curves

We briefly recall the definition and some properties of gonality and Clifford index of curves.

Let C be a smooth irreducible curve of genus $g \geq 2$. We denote by g^r_d a linear system of dimension r and degree d and say that C is *k-gonal* (and that k is its *gonality*) if C posesses a g^1_k but no g^1_{k-1}. In particular, we call a 2-gonal curve *hyperelliptic* and a 3-gonal curve *trigonal*. We denote by gon C the gonality of C. Note that if C is k-gonal, all g^1_k's must necessarily be base point free and complete.

If A is a line bundle on C, then the *Clifford index* of A (introduced by H. H. Martens in [HMa]) is the integer

$$\operatorname{Cliff} A = \deg A - 2(h^0(A) - 1).$$

If $g \geq 4$, then the *Clifford index of C* itself is defined as

T. Johnsen and A.L. Knutsen: LNM 1842, pp. 19–29, 2004.

$$\text{Cliff } C = \min \{\text{Cliff } A \mid h^0(A) \geq 2, h^1(A) \geq 2\}.$$

Clifford's theorem then states that Cliff $C \geq 0$ with equality if and only if C is hyperelliptic and Cliff $C = 1$ if and only if C is trigonal or a smooth plane quintic.

At the other extreme, we get from Brill-Noether theory (cf. [A-C-G-H, V]) that the gonality of C satisfies gon $C \leq \lfloor \frac{g+3}{2} \rfloor$, whence Cliff $C \leq \lfloor \frac{g-1}{2} \rfloor$. For the general curve of genus g, we have Cliff $C = \lfloor \frac{g-1}{2} \rfloor$.

We say that a line bundle A on C *contributes to the Clifford index of* C if both $h^0(A) \geq 2$ and $h^1(A) \geq 2$ and that it *computes the Clifford index of* C if in addition Cliff C = Cliff A.

Note that Cliff A = Cliff $(\omega_C \otimes A^{-1})$.

The *Clifford dimension* of C is defined as

$$\min \{h^0(A) - 1 \mid A \text{ computes the Clifford index of } C\}.$$

A line bundle A which achieves the minimum and computes the Clifford index, is said to *compute* the Clifford dimension. A curve of Clifford index c is $(c+2)$-gonal if and only if it has Clifford dimension 1. For a general curve C, we have gon $C = c + 2$.

Following [G-L4] we give ad hoc definitions of Cliff C for C of genus 2 or 3: We set Cliff $C = 0$ for C of genus 2 or hyperelliptic of genus 3, and Cliff $C = 1$ for C non-hyperelliptic of genus 3. This convention will be used throughout the book, with no further mention.

Lemma 3.1 (Coppens-Martens [C-M]). *The gonality k of a smooth irreducible projective curve C of genus $g \geq 2$ satisfies*

$$\textit{Cliff } C + 2 \leq k \leq \textit{Cliff } C + 3.$$

The curves satisfying gon C = Cliff $C + 3$ are conjectured to be very rare and called *exceptional* (cf. [GMa, (4.1)]).

3.2 The result of Green and Lazarsfeld

Recall the following result of Green and Lazarsfeld already mentioned in the introduction:

Theorem 3.2 (Green-Lazarsfeld [G-L4]). *Let L be a base point free line bundle on a $K3$ surface S with $L^2 > 0$. Then Cliff C is constant for all smooth irreducible $C \in |L|$, and if Cliff $C < \lfloor \frac{g-1}{2} \rfloor$, then there exists a line bundle M on S such that $M_C := M \otimes \mathcal{O}_C$ computes the Clifford index of C for all smooth irreducible $C \in |L|$.*

Note that since $(L - M) \otimes \mathcal{O}_C \simeq \omega_C \otimes {M_C}^{-1}$, the result is symmetric in M and $L - M$.

With Theorem 3.2 in mind we make the following definition:

Definition 3.3. *Let L be a base point free and big line bundle on a $K3$ surface. We define the* Clifford index of L *to be the Clifford index of all the smooth curves in $|L|$ and denote it by Cliff L.*

Similarly, if (S, L) is a polarized $K3$ surface we will often call Cliff L the Clifford index of S *and denote it by Cliff $_L(S)$.*

Definition 3.4. *A polarized $K3$ surface (S, L) of genus g is called* Clifford general *if Cliff $L < \lfloor \frac{g-1}{2} \rfloor$.*

It turns out that we can choose the line bundle M appearing in Theorem 3.2 above so that it satisfies certain properties. We will need the following result in the sequel.

Lemma 3.5. [Kn4, Lemma 8.3] *Let L be a base point free line bundle on a $K3$ surface S with $L^2 = 2g - 2 \geq 2$ and Cliff $L = c$.*

If $c < \lfloor \frac{g-1}{2} \rfloor$, then there exists a smooth curve D on S satisfying $0 \leq D^2 \leq c+2$, $2D^2 \leq D.L$ (either of the latter two inequalities being an equality if and only if $L \sim 2D$) and

$$\mathit{Cliff}\ C = \mathit{Cliff}\ (\mathcal{O}_S(D) \otimes \mathcal{O}_C) = D.L - D^2 - 2$$

for any smooth curve $C \in |L|$.

It is also known (see e.g. [GMa]) that D satisfies $h^0(D \otimes \mathcal{O}_C) = h^0(D)$ and $h^0((L-D) \otimes \mathcal{O}_C) = h^0(L-D) = h^1(D \otimes \mathcal{O}_C)$ for any smooth curve $C \in |L|$.

From the results in the previous chapter, it is also clear that

$$\text{Cliff } C = \min \{\mu(L), \lfloor \frac{g-1}{2} \rfloor\}$$

for any smooth $C \in |L|$.

3.3 Clifford divisors

Summarizing the results of the previous section, and using Propositions 2.6 and 2.7, we have that if L is a base point free line bundle on a $K3$ surface S, of *sectional genus*

$$g = g(L) = \frac{1}{2}L^2 + 1,$$

and the smooth curves in $|L|$ have Clifford index

$$c < \lfloor \frac{g-1}{2} \rfloor = \lfloor \frac{L^2}{4} \rfloor,$$

(which in particular implies $L^2 \geq 4c + 4$), then there exists a divisor (class) D on S with the following properties (with $F := L - D$):

(C1) $c = D.L - D^2 - 2 = D.F - 2$,

(C2) $D.L \leq F.L$ (eqv. $D^2 \leq F^2$) and if equality occurs, then either $L \sim 2D$ or $h^0(2D - L) = 0$,
(C3) $h^1(D) = h^1(F) = 0$.

A divisor (class) D with the properties (C1) and (C2) above will be called a *Clifford divisor* for L. This means in other words that D and $L - D$ compute the Clifford index of all smooth curves in $|L|$. The property (C2) can be considered an ordering of D and $L-D$. Any Clifford divisor will automatically fulfill property (C3) and the (possibly empty) base loci Δ' of $|D|$ and Δ of $|L - D|$ will satisfy $L.\Delta = L.\Delta' = 0$ by Proposition 2.6.

By Proposition 2.7 we can find a Clifford divisor D satisfying the properties

(C4) the (possibly empty) base divisor Δ of F satisfies $L.\Delta = 0$,
(C5) $|D|$ is base point free and its general member is a smooth curve,

Definition 3.6. *A divisor D satisfying all properties (C1)-(C5) will be called a free Clifford divisor for L.*

Since this definition only depends on the class of D, we will by abuse of notation never distinguish between D and its divisor class. Hopefully, this will not cause any confusion. From now on, for a free Clifford divisor D, the term Δ will always denote the base divisor of $F = L - D$.

Note that any Clifford divisor D will satisfy the numerical conditions:

$$2D^2 \overset{(i)}{\leq} L.D = D^2 + c + 2 \overset{(ii)}{\leq} 2c + 4$$

$(*)$ with equality in (i) or (ii) if and only if $L \sim 2D$ and $L^2 = 4c + 8$.

In particular,

$$D^2 \leq c + 2, \text{with equality if and only if } L \sim 2D \text{ and } L^2 = 4c + 8, \tag{3.1}$$

and by the Hodge index theorem

$$D^2 L^2 \leq (L.D)^2 = (D^2 + c + 2)^2. \tag{3.2}$$

The special limit case $D^2 = c + 2$ (where by (3.1) we necessarily have $L \sim 2D$ and $L^2 = 4c + 8$) will henceforth be denoted by (Q).

We will now take a closer look at two particular kinds of free Clifford divisors, namely:

(a) $D^2 = c + 1$, or
(b) $D^2 = c$, $L \sim 2D + \Delta$, with $\Delta > 0$.

It turns out that these free Clifford divisors are of a particular form.

Proposition 3.7. *Let L be a base point free and big line bundle of Clifford index $c < \lfloor \frac{g-1}{2} \rfloor$ on a $K3$ surface, and let D be a free Clifford divisor.*

If D is as in (a) above, then $L^2 = 4c + 6$ and

(E0) $L \sim 2D + \Gamma$, where Γ is a smooth rational curve satisfying $\Gamma.D = 1$.

If D is as in (b) above, then $L^2 = 4c+4$ and (with all Γ_i denoting smooth rational curves) either

(E1) $L \sim 2D + \Gamma_1 + \Gamma_2$, $D^2 = c$, $D.\Gamma_1 = D.\Gamma_2 = 1$, $\Gamma_1.\Gamma_2 = 0$, *or*
(E2) $L \sim 2D+2\Gamma_0+2\Gamma_1+\cdots+2\Gamma_N+\Gamma_{N+1}+\Gamma_{N+2}$, $D^2 = c$, *and the following configuration:*

$$
\begin{tikzcd}
D \arrow[r, dash] & \Gamma_0 \arrow[r, dash, dashed] & \Gamma_N \arrow[r, dash] \arrow[d, dash] & \Gamma_{N+1} \\
 & & \Gamma_{N+2} &
\end{tikzcd}
$$

Furthermore, there can only exist Clifford divisors of one of the three types (E0)-(E2) (on the same surface), and all such are linearly equivalent.

Proof. If D is as in case (a), it follows that D is of the desired form in the same way as in the proof of the last statement of Proposition 1.13.

Now assume D is as in case (b). Then we have $L^2 = 2D.L+\Delta.L = 2(D^2+c+2) = 4c+4$. This gives $\Delta^2 = (L-2D)^2 = -4$ and $D.\Delta = \frac{1}{2}(L.\Delta-\Delta^2) = 2$. By Proposition 5.3 below, any smooth rational curve Γ component of Δ such that $\Gamma.D > 0$, satisfies $\Gamma.D = 1$, and any two such curves are disjoint. So we have to distinguish between two cases.

If there exist two distinct rational curves Γ_1 and Γ_2 in Δ such that $\Gamma_1.D = \Gamma_2.D = 1$ and $\Gamma_1.\Gamma_2 = 0$, write $L \sim 2D+\Gamma_1+\Gamma_2+\Delta'$, for some $\Delta' \geq 0$. Then $0 = \Gamma_i.L = 2-2+\Gamma_i.\Delta'$ gives $\Delta'.\Gamma_i = 0$, for $i = 1, 2$. Clearly $D.\Delta' = 0$, so

$$(2D + \Gamma_1 + \Gamma_2).\Delta' = 0,$$

whence $\Delta' = 0$, since L is numerically 2-connected, and we are in case (E1).

If there exists a rational curve Γ occurring with multiplicity 2 in Δ such that $\Gamma.D = 1$, write $L \sim 2D + 2\Gamma + \Delta'$, for some $\Delta' \geq 0$. Since $0 = \Gamma.L = 2-4+\Delta'.\Gamma$, we get $\Delta'.\Gamma = 2$. Iterating the process, we get case (E2).

Assume now D is given and B is any free Clifford divisor as in (E0)-(E2). We want to show that $B \sim D$.

If D is of type (E0), we must have $L^2 = 4c+6$, so B must also be of type (E0), which means that $L \sim 2B + \Gamma_0$, where Γ is a smooth rational curve such that $B.\Gamma_0 = 1$.

Since

$$0 = \Gamma_0.L = 2D.\Gamma_0 + \Gamma.\Gamma_0,$$

and D is nef, we get the two possibilities

$$\text{(i) } D.\Gamma_0 = \Gamma.\Gamma_0 = 0 \quad \text{or} \quad \text{(ii) } D.\Gamma_0 = 1, \Gamma = \Gamma_0.$$

If (i) were to happen, we would get

$$4D.B = (L-\Gamma)(L-\Gamma_0) = L^2 = 4c+6,$$

which is clearly impossible.

Hence $\Gamma = \Gamma_0$ and $D \sim B$.

If D is of type (E1) or (E2), we have $L^2 = 4c+4$, so B must also be of type (E1) or (E2) and will therefore satisfy either

(1) $L \sim 2B + \Gamma_1' + \Gamma_2'$, or
(2) $L \sim 2B + 2\Gamma_0' + \cdots + 2\Gamma_N' + \Gamma_{N+1}' + \Gamma_{N+2}'$,

where the Γ_i' are smooth rational curves with configurations as in the cases (E1) and (E2).

Assume now that D is of type (E1). The proof if D is of type (E2) is similar.

If (2) holds, we get from

$$0 = \Gamma_i'.L = 2D.\Gamma_i' + \Gamma_1.\Gamma_i' + \Gamma_2.\Gamma_i',$$

and the fact that D is nef, that

$$\Gamma_j.\Gamma_i' = 0 \quad \text{or} \quad \Gamma_j = \Gamma_i', \quad i = 0, \dots, N+2, \quad j = 1, 2.$$

This gives

$$4D.B = (L - \Gamma_1 - \Gamma_2)(L - 2\Gamma_0' - \cdots - 2\Gamma_N' + \Gamma_{N+1}' - \Gamma_{N+2}') \leq L^2 = 4(c+1),$$

whence $D.B \leq c+1$ and $(D-B)^2 \geq -2$, and by Riemann-Roch, if $D \not\sim B$, either $D - B$ or $B - D$ is effective. The argument below is symmetric in those two cases, so assume $D \sim B + \Sigma$, for Σ effective and $\Sigma^2 \geq -2$. Then $\Sigma.L = 0$ and $\Sigma^2 = -2$ by the Hodge index theorem. Furthermore,

$$L \sim 2D + \Gamma_1 + \Gamma_2 \sim 2B + \Gamma_1 + \Gamma_2 + 2\Sigma \sim 2B + 2\Gamma_0' + \cdots + 2\Gamma_N' + \Gamma_{N+1}' + \Gamma_{N+2}',$$

whence

$$2\Sigma \sim 2\Gamma_0' + \cdots + 2\Gamma_N' + \Gamma_{N+1}' + \Gamma_{N+2}' - \Gamma_1 - \Gamma_2,$$

and $2\Sigma.B = 2 - (\Gamma_1 + \Gamma_2).B \leq 2$ (since B is nef). By

$$0 = \Sigma.L = 2B.\Sigma + (\Gamma_1 + \Gamma_2).\Sigma + 2\Sigma^2,$$

we get $(\Gamma_1 + \Gamma_2).\Sigma \geq 2$, and

$$2\Sigma.D = (L - \Gamma_1 - \Gamma_2).\Sigma \leq -2,$$

contradicting the nefness of D. So we are in case (1) above and again from

$$0 = \Gamma_i'.L = 2D.\Gamma_i' + \Gamma_1.\Gamma_i' + \Gamma_2.\Gamma_i',$$

and the fact that D is nef, we get the three possibilities:

(i) $D.\Gamma_1' = 1$, $\Gamma_1' = \Gamma_1$, $D.\Gamma_2' = \Gamma_2.\Gamma_2' = 0$,
(ii) $D.\Gamma_i' = \Gamma_1.\Gamma_i' = \Gamma_2.\Gamma_i' = 0$, i=1,2,

(iii) $D.\Gamma_i' = 1$, $\Gamma_i' = \Gamma_i$, $i = 1, 2$.

In case (i) we get the absurdity $4D.B = (L - \Gamma_1 - \Gamma_2).(L - \Gamma_1' - \Gamma_2') = 4(c+1) - 2$.

In case (ii) we get $4D.B = 4(c+1)$, whence $D.B = c+1$. We calculate $(D-B)^2 \geq -2$, and by Riemann-Roch, if $D \not\sim B$, either $D - B$ or $B - D$ is effective. Writing $D \sim B + \Sigma$, for Σ effective and $\Sigma^2 = -2$, we get the same contradiction as above.

So we are in case (iii) and $B \sim D$. $\square$

The following proposition describes the case (E0) further.

Proposition 3.8. *Let L be a base point free and big line bundle on a $K3$ surface and let c be the Clifford index of all smooth curves in $|L|$. Then the following conditions are equivalent:*

(i) all smooth curves in $|L|$ are exceptional (i.e. have gonality $c+3$),
(ii) there is a free Clifford divisor of type (E0),
(iii) all free Clifford divisors are linearly equivalent and of type (E0).

Furthermore, if any of these conditions are satisfied, then all the smooth curves in $|L|$ have Clifford dimension $r = h^0(D) - 1 = \frac{1}{2}(c+3)$ and D_C computes the Clifford dimension of all smooth $C \in |L|$.

Proof. The equivalence between (i) and (iii) follows from the proof of [Kn4, Prop. 8.6]. We will however go through the whole proof for the sake of the reader.

We first prove that (i) implies (iii).

Assume, to get a contradiction, that all smooth curves in $|L|$ have gonality $c+3$, and that there is a free Clifford divisor D which is not of type (E0). We claim that in this case, the line bundle $F_{D'}$ is base point free, for any smooth $D' \in |D|$. This clearly holds in the case (Q), so we can assume that $D^2 \leq c$, whence for any smooth $D' \in |D|$ we have $\deg F_{D'} = c+2 \geq D_0^2 + 2 = 2g(D_0)$, so $F_{D'}$ is base point free. By [C-P, Lemma 2.2], we have that there exists a smooth curve in $|L|$ of gonality $F.D = c+2$, a contradiction.

Next we prove that (iii) implies (i).

By Lemma 3.1, we have $c + 2 \leq \operatorname{gon} C \leq c + 3$, for any smooth curve $C \in |L|$. Assume, to get a contradiction, that there is a smooth curve $C \in |L|$ with $\operatorname{gon} C = c+2$. Let $|B|$ be a g^1_{c+2} on C and pick any $Z \in |A|$ lying outside the finitely many rational curves Γ' on S satisfying $\Gamma'.L \leq 2c+4$ (we can find such a Z since B is base point free). By Riemann-Roch, one easily computes $h^1(\mathcal{O}_C(Z)) = h^0(\omega_C(-Z)) = g - 1 - c = h^0(L) - c - 2$. From the short exact sequence

$$0 \longrightarrow \mathcal{O}_S \longrightarrow L \otimes \mathcal{J}_Z \longrightarrow \omega_C(-Z) \longrightarrow 0$$

we then find $h^0(L \otimes \mathcal{J}_Z) = h^0(L) - c - 1$. In particular, the restriction map $H^0(L) \to H^0(L \otimes \mathcal{O}_Z)$ is not surjective. One easily sees that for any proper subscheme Z' of Z, the map $H^0(L) \longrightarrow H^0(L \otimes \mathcal{O}_{Z'})$ is surjective, since otherwise $h^0(\mathcal{O}_C(Z')) = 2$, and gon $C \leq \deg Z' < \deg Z$, a contradiction. So Z satisfies the conditions in Proposition 1.13 and there exists an effective divisor D_0 passing through $\mathcal{Z}$ satisfying the conditions in Proposition 1.13. Since $D_0.L \leq 2c+4$ and Z by assumption does not meet any smooth rational curve Γ' on S satisfying $\Gamma'.L \leq 2c+4$, we must have $D_0^2 \geq 0$. let $F_0 := L - D_0$. Since $L^2 = 4c+6$, we have that $h^0(F_0) \geq h^0(D_0) \geq 2$, so $D_0 \in \mathcal{A}(L)$, and since $D_0.F_0 \leq c+2$, we must have $D_0.F_0 = c+2$ and $D_0 \in A_0(L)$, i.e. D_0 is a Clifford divisor for L. By our assumptions, the moving part of $|D_0|$ is of type (E0), whence either $D_0 \sim D$ or $D_0 \sim D + \Gamma$, but the latter is ruled out since $L - 2D_0 \geq 0$. So $D_0 \sim D$, and by the last statement in Proposition 1.13, we have that Z meets Γ, but this is a contradiction on our choice of Z.

It is clear that (iii) implies (ii) and we now show that (ii) implies (iii) by showing that if D is a free Clifford divisor of type (E0) and D' is any other free Clifford divisor, then $D' \sim D$.

Let $B := D - D'$. Define $R' := L - 2D'$ and note that $R' \sim 2B + \Gamma$. We have

$$\begin{aligned} c+2 = D.(L-D) &= (D'+B).(D'+B+\Gamma) \\ &= D'^2 + (2B+\Gamma).D' + B.(B+\Gamma) = c+2+B.(B+\Gamma), \end{aligned}$$

whence $B^2 + B.\Gamma = 0$. Combined with $\Gamma.D = \Gamma.(D'+B) = 1$, and since $\Gamma.D' = 0$ or 1 by Lemma 6.3(c), we get

$$\Gamma.D' = 1, B^2 = 0, B.\Gamma = 0,$$

whence

$$B.R' = B.(2B+\Gamma) = 0 \quad \text{and} \quad R'.D' = 2B.D' + 1.$$

This gives

$$L.B = 2D'.B + R'.B = 2D'.B = R'.D' - 1.$$

But this implies

$$B.L - B^2 - 2 = R'.D' - 3 < D'^2 + R'.D' - 2 = c,$$

whence we must have $B \sim 0$, as desired.

It remains to prove the last statement. If $D^2 = 2$, then $h^0(D) = 3$, and clearly all the smooth curves in $|L|$ have Clifford dimension 2, so there is nothing more to prove. We therefore can assume $D^2 \geq 4$.

We first show that D cannot be decomposed into two moving classes, i.e. that we cannot have $D \sim D_1 + D_2$, with $h^0(D_i) \geq 2$ for $i = 1, 2$.

Indeed, if this were the case, then since $D.\Gamma = 1$, we can assume that $D_2.\Gamma \geq 1$, whence the contradiction

$$\begin{aligned} D_1.L - D_1^2 = (D-D_2).L - (D-D_2)^2 &= D.L - D^2 - D_2.L + 2D.D_2 - D_2^2 \\ &\leq D.L - D^2 - D_2(L-2D) = c+2 - D_2.\Gamma < c+2. \end{aligned}$$

It follows that D_C is very ample for any smooth $C \in |L|$. Indeed, if Z is a length two scheme that $|D|$ fails to separate, then by the results in [Kn4] (see Proposition 1.13) and the fact that $D^2 \geq 4$, we have that Z is contained in a divisor B satisfying $B^2 = -2$, $B.D = 0$, or $B^2 = 0$, $B.D = 2$, or $B^2 = 2$, $D \sim 2D$. One easily sees that the two last cases would induce a decomposition $D \sim B + (D - B)$ into two moving classes, which we have just seen is impossible. So $B^2 = -2$ and $B.D = 0$. Now $(D - B)^2 \geq 2$ and since

$$(D - B).L - (D - B)^2 = D.L - D^2 - \Gamma.L + 2,$$

we must have $B.L \leq 1$, by the condition (iii). This means that none of the smooth curves in $|L|$ contain Z, whence D_C is very ample for any smooth $C \in |L|$, as claimed.

Since $h^0(D - C) = h^1(D - C) = 0$, we have $r := h^0(D) - 1 = h^0(D_C) - 1$, which means that $|D|$ embeds C as a smooth curve of genus $g = 4r - 2$ and degree $d := g - 1$ in $\mathbf{P}^r$. To show that the Clifford dimension of C is r, it suffices by [E-L-M-S, Thm. 3.6 (Recognition Theorem)] to show that C (embedded by $|D|$) is not contained in any quadric of rank ≤ 4.

But if this were the case, the two rulings would induce a decomposition of D into two moving classes, which is impossible by the above.

So C has Clifford dimension r. $\square$

Remark 3.9. This result can be seen as a generalization of [SD, Rem. 7.13] and [E-L-M-S, Thm. 4.3]. In [E-L-M-S, Thm. 4.3] the authors prove essentially the same as above, but with the hypotheses that Pic $S \simeq \mathbf{Z}D \oplus \mathbf{Z}\Gamma$.

Moreover, note that for $r \geq 3$, given any of the equivalent conditions in Proposition 3.8, all the smooth curves in $|L|$ satisfy the conjecture in [E-L-M-S, p. 175]. Indeed, one immediately sees that it satisfies condition (1) in that conjecture, and in [E-L-M-S] it is also shown that any curve satisfying condition (1) also satisfies the remaining conditions (2)-(4) in that conjecture.

The following result shows that, except for one particular case, any free Clifford divisor is itself Clifford general. We will need this result later.

Proposition 3.10. *Let L be a base point free and big line bundle of Clifford index $c < \lfloor \frac{g-1}{2} \rfloor$ on a $K3$ surface, and let D be a free Clifford divisor with $D^2 \geq 2$.*

Then D is Clifford general (i.e. all the smooth curves in $|D|$ have Clifford index $\lfloor \frac{g(D)-1}{2} \rfloor$), except for the case (Q), with $c = 2$ (in particular $D^2 = 4$ and $L \sim 2D$), when there exists a smooth elliptic curve E such that $E.D = 2$ In this case D is hyperelliptic.

Proof. Assume D is not Clifford general. Then we can assume $D^2 \geq 4$, and there is an effective decomposition $D \sim A + B$, with $h^0(A) \geq 2$, $h^0(B) \geq 2$ and $A.B = \text{Cliff } D + 2 \leq \lfloor \frac{g(D)-3}{2} \rfloor + 2 \leq \lfloor \frac{1}{4} D^2 \rfloor + 1$.

By symmetry, we can assume $B.F \geq \frac{1}{2} D.F = \frac{1}{2}(c+2)$. Moreover, we must have $A.L - A^2 \geq c + 2$. Hence

$$\begin{aligned} c+2 \leq A.L - A^2 &= A.(B+F) = D.F - B.F + A.B \\ &= c+2 - B.F + A.B \leq c+2 - \frac{1}{2}(c+2) + \frac{1}{4}D^2 + 1, \end{aligned}$$

so $D^2 \geq 2c$. Combining with (3.1) we get $c = 2$, $D^2 = 4$ and $L \sim 2D$, as asserted.

Since D is hyperelliptic, there either exists a smooth curve E satisfying $E^2 = 0$ and $E.D = 2$, or a smooth curve B such that $B^2 = 2$ and $D \sim 2B$. However, in the second we get the contradiction

$$D.L - D^2 - 2 = 6 > 4 = B.L - B^2 - 2,$$

so we must be in the first case. □

3.4 Getting a scroll

Now we return to the theory of scrolls. Let D be a free Clifford divisor.

If $D^2 = 0$, then $|D| = \{D_\lambda\}_{\lambda \in \mathbf{P}^1}$ is a pencil, which defines in a natural way a scroll containing $\varphi_L(S)$.

If $D^2 > 0$, then $\dim |D| = \frac{1}{2}D^2 + 1 > 1$, and we choose a subpencil $\{D_\lambda\}_{\lambda \in \mathbf{P}^1} \subseteq |D|$ as follows: Pick any two smooth members D_1 and $D_2 \in |D|$ intersecting in D^2 distinct points and such that none of these points belong to the union of the finite set of curves

$$\{\Gamma \mid \Gamma \text{ is a smooth rational curve, } \Gamma.L \leq c+2\}. \tag{3.3}$$

Then

$$\{D_\lambda\}_{\lambda \in \mathbf{P}^1} := \text{the pencil generated by } D_1 \text{ and } D_2.$$

The pencil $\{D_\lambda\}$ will be without fixed components (but with D^2 base points) and define in a natural way a scroll containing $\varphi_L(S)$ and of type determined as in equation (2.2) by the integers

$$d_i = h^0(L - iD) - h^0(L - (i+1)D), \quad i \geq 0. \tag{3.4}$$

Since all choices of subpencils of $|D|$ will give scrolls of the same type, and scrolls of the same type are isomorphic, the scrolls arising are up to isomorphism only dependent on D. We denote these scrolls by $\mathcal{T} = \mathcal{T}(c, D, \{D_\lambda\})$. If $h^0(D) = 2$, we sometimes write only $\mathcal{T}(c, D)$.

Since $h^1(L) = 0$, we get by the conditions (C1) and (C5) that $\mathcal{T}$ has dimension

$$\dim \mathcal{T} = d_0 = h^0(L) - h^0(F) = c + 2 + \frac{1}{2}D^2, \tag{3.5}$$

and degree

$$\deg \mathcal{T} = h^0(F) = g - c - 1 - \frac{1}{2}D^2. \tag{3.6}$$

Furthermore, each $D_\lambda \in \{D_\lambda\}$ has linear span

$$\overline{D_\lambda} = \mathbf{P}^{c+1+\frac{1}{2}D^2}. \tag{3.7}$$

Remark 3.11. If $h^0(D) = r + 1 \geq 3$, then $|D|$ is parametrized by a $\mathbf{P}^r$. For each D_λ in $|D|$ we may take the linear span $\overline{D_\lambda} = \mathbf{P}^{c+1+\frac{1}{2}D^2}$. Taking the union of *all* these linear spaces, and not only of those corresponding to a subpencil of $|D|$, we obtain some sort of "ruled" variety, which perhaps is a more natural ambient variety for $S' := \varphi_L(S)$ than the scrolls described above (since it is independent of a choice of pencil). Such a variety is an image of a $\mathbf{P}^{c+1+\frac{1}{2}D^2}$-bundle over $\mathbf{P}^r$. The main reason why we choose to study the scrolls described above rather than these big "ruled" varieties, is that we know too little about the latter ones to be able to use them constructively. By using the scrolls above we are able to utilize the results in [Sc] and in many cases find the resolutions of $\mathcal{O}_{S'}$ as an $\mathcal{O}_{\mathcal{T}}$-module. A detailed explanation will be given in Chapter 8.

4

Two existence theorems

Given integers $g \geq 2$ and $0 \leq c \leq \lfloor \frac{g-1}{2} \rfloor$, one may ask whether there actually exists a pair (S, L), where S is a $K3$ surface, $L^2 = 2g - 2$ and all smooth curves in $|L|$ have Clifford index c.

Theorem 4.1 below gives a positive answer to this question. Theorem 4.4 below answers the same kind of question concerning the possible gonalities of a curve on a $K3$ surface.

The results in this chapter were first given in [Kn2]. We also include the material here, to obtain a complete exposition.

Theorem 4.1. *Let g and c be integers such that $g \geq 3$ and $0 \leq c \leq \lfloor \frac{g-1}{2} \rfloor$. Then there exists a polarized $K3$ surface of genus g and Clifford index c.*

The theorem is an immediate consequence of the following

Proposition 4.2. *Let d and g be integers such that $g \geq 3$ and $2 \leq d \leq \lfloor \frac{g-1}{2} \rfloor + 2$. Then there exists a $K3$ surface S with Pic $S = \mathbf{Z}L \oplus \mathbf{Z}E$, where $L^2 = 2(g-1)$, $E.L = d$ and $E^2 = 0$. Moreover L is base point free, and*

$$c := \text{Cliff } L = d - 2 \leq \lfloor \frac{g-1}{2} \rfloor$$

Furthermore, E is the only Clifford divisor for L (modulo equivalence class) if $d < \lfloor \frac{g-1}{2} + 2 \rfloor$.

To prove this proposition, we first need the following basic existence result:

Lemma 4.3. *Let $g \geq 3$ and $d \geq 2$ be integers. Then there exists a $K3$ surface S with Pic $S = \mathbf{Z}L \oplus \mathbf{Z}E$, such that L is base point free and E is a smooth curve, $L^2 = 2(g-1)$, $E.L = d$ and $E^2 = 0$.*

Proof. By Propositions 1.11 and 1.12, we can find a $K3$ surface S with Pic $S = \mathbf{Z}L \oplus \mathbf{Z}E$, with intersection matrix

$$\begin{bmatrix} L^2 & L.E \\ E.L & E^2 \end{bmatrix} = \begin{bmatrix} 2(g-1) & d \\ d & 0 \end{bmatrix}$$

T. Johnsen and A.L. Knutsen: LNM 1842, pp. 31–33, 2004.

and such that L is nef. If L is not base point free, there exists by Proposition 1.10 a curve B such that $B^2 = 0$ and $B.L = 1$. An easy calculation shows that this is impossible. By [Kn5, Proposition 4.4], we have that $|E|$ contains a smooth curve. $\square$

Proof of Proposition 4.2. Let S, L and E be as in Lemma 4.3, with $d \leq \lfloor \frac{g-1}{2} \rfloor + 2$. Note that since E is irreducible, we have $h^1(E) = 0$. By the cohomology of the short exact sequence

$$0 \longrightarrow \mathcal{O}_S(E-L) \longrightarrow \mathcal{O}_S(E) \longrightarrow \mathcal{O}_C(E) \longrightarrow 0,$$

where C is any smooth curve in $|L|$, we find that $h^0(\mathcal{O}_C(E)) \geq h^0(E) = 2$ and $h^1(\mathcal{O}_C(E)) = h^0(L-E) \geq 2$, so $\mathcal{O}_C(E)$ contributes to the Clifford index of C and

$$c \leq \text{Cliff } \mathcal{O}_C(E) \leq E.L - E^2 - 2 = d - 2 < \lfloor \frac{g-1}{2} \rfloor.$$

If $c = \lfloor \frac{g-1}{2} \rfloor$, then we are finished. If $c < \lfloor \frac{g-1}{2} \rfloor$, then there has to exist an effective divisor D on S satisfying

$$c = \text{Cliff } \mathcal{O}_C(D) = D.L - D^2 - 2.$$

Since both D and $L - D$ must be effective and E is nef, we must have

$$D.E \geq 0 \text{ and } (L-D).E \geq 0.$$

Writing $D \sim xL + yE$ this is equivalent to

$$dx \geq 0 \text{ and } d(1-x) \geq 0,$$

which gives $x = 0$ or 1. These two cases give, respectively, $D = yE$ or $L - D = -yE$. Since $h^1(D) = h^1(L-D) = 0$ by (C3), we must have $y = 1$ and $D \sim E$. This shows that $c = E.L - E^2 - 2 = d - 2$ and that there are no other Clifford divisors but E (modulo equivalence class). $\square$

This concludes the proof of Theorem 4.1.

The proof of this theorem also gives the following result, which is of its own interest:

Theorem 4.4. *Let g and k be integers such that $g \geq 2$ and $2 \leq k \leq \lfloor \frac{g+3}{2} \rfloor$. Then there exists a $K3$ surface containing a smooth curve of genus g and gonality k.*

The surfaces constructed in Proposition 4.2 all have the property that the only free Clifford divisor (modulo equivalence class) is a smooth elliptic curve E. One could also perform the same construction with lattices of the form

$$\begin{bmatrix} L^2 & L.D \\ D.L & D^2 \end{bmatrix}$$

with $D^2 > 0$ (and satisfying the constraints given by equations (3.1) and (3.2)), but for each pair (g, c) there might be values of D^2 that cannot occur. We will in Chapters 10 and 11 perform more such constructions, also with lattices of higher ranks. See Proposition 11.5 for a result concerning low values of c.

5

The singular locus of the surface S' and the scroll $\mathcal{T}$

The singular locus of $\varphi_L(S) = S'$ is well-known, and the results date back to Saint-Donat [SD], which we briefly present in the first section of this chapter.

In the second section we describe the singular locus of the rational normal scroll $\mathcal{T} = \mathcal{T}(c, D, \{D_\lambda\})$ constructed as in the previous chapter. Most of the results are due to the second author, in particular Propositions 5.3, 5.5 and 5.6 and Theorem 5.7. The latter is the main result of this chapter and states that we can always find a free Clifford divisor D such that Sing $\mathcal{T}$ is "spanned" by the images of the base points of the pencil $\{D_\lambda\}$ and the images of the contracted smooth rational curves across the members of the pencil, i.e. the curves Γ with $\Gamma.L = 0$ and $\Gamma.D > 0$ (in fact we will see that $\Gamma.D = 1$). We call such a Clifford divisor a *perfect Clifford divisor*. In the proofs we will need Proposition 1.13.

The proofs of Propositions 5.3, 5.5 and 5.6 and part of Theorem 5.7 are postponed until the next chapter.

5.1 The singular locus of $\varphi_L(S)$

We start this chapter by describing the image $S' := \varphi_L(S)$ by the complete linear system $|L|$ on the $K3$ surface S.

Proposition 5.1. *Let L be a base point free and big line bundle on a $K3$ surface S, and denote by φ_L the corresponding morphism and by c the Clifford index of the smooth curves in $|L|$.*

(i) If $c = 0$, φ_L is $2:1$ onto a surface of degree $\frac{1}{2}L^2$,
(ii) If $c > 0$, then φ_L is birational onto a surface of degree L^2 (in fact it is an isomorphism outside of finitely many contracted smooth rational curves), and $S' := \varphi_L(S)$ is normal and has only rational double points as singularities. In particular $K_{S'} \simeq \mathcal{O}_{S'}$, and $p_a(S') = 1$.

Proof. These are well-known results due to Saint-Donat [SD] (see also [Kn4] for further discussions). □

T. Johnsen and A.L. Knutsen: LNM 1842, pp. 35–45, 2004.

Let D be a free Clifford divisor and $\{D_\lambda\}$ a subpencil of $|D|$ chosen as described in the previous chapter.

Define the subset $\mathcal{D}$ of the pencil $\{D_\lambda\}$ by

$$\mathcal{D} := \{D_\lambda \in \{D_\lambda\} \mid \varphi_L \text{ does not contract any component of } D_\lambda\}.$$

We then have

Lemma 5.2. *If $c > 0$, then L_{D_λ} is very ample for all $D_\lambda \in \mathcal{D}$.*

Proof. By [C-F, Thm. 3.1] it is sufficient to show that for any effective subdivisor A of D_λ we have $L.A \geq A^2 + 3$.

If $A^2 \geq 0$, then we have $L.A \geq A^2 + c + 2 \geq A^2 + 3$ (which actually holds for any divisor A on S). If $A^2 \leq -2$, then $L.A \geq 1 \geq A^2 + 3$, unless $L.A = 0$, which proves the lemma. □

5.2 The singular locus of $\mathcal{T}$ and perfect Clifford divisors

In the rest of this chapter we focus on the singular locus of the rational normal scroll $\mathcal{T} = \mathcal{T}(c, D, \{D_\lambda\})$.

It is well-known that the singular locus of a rational normal scroll of type $(e_1, \ldots, e_d)$ is a projective space of dimension $r - 1$, where

$$r := \#\{e_i \mid e_i = 0\}. \tag{5.1}$$

From equation (2.2) we have

$$r = d_0 - d_1 = h^0(L) + h^0(L - 2D) - 2h^0(L - D).$$

By property (C2), when $L \not\sim 2D$, we have for $R := L - 2D$ by Riemann-Roch

$$h^0(R) = \frac{1}{2}R^2 + 2 + h^1(R).$$

Note that we have $h^0(R) > 0$ if $L^2 \geq 4c + 6$, and $h^0(R) = 0$ if and only if $L^2 = 4c + 4$ and $h^1(R) = 0$.

If $L \sim 2D$, we have $D^2 = c + 2$ and $h^0(L - 2D) = h^0(\mathcal{O}_S) = 1$. In general, using Riemann-Roch and the fact that $H^1(L) = H^1(L - D) = 0$, we get the following expression for r:

$$r = \begin{cases} D^2 + h^1(L - 2D) & \text{if } L \not\sim 2D \text{ (equiv. } D^2 \neq c + 2), \\ D^2 - 1 & \text{if } L \sim 2D \text{ (equiv. } D^2 = c + 2) \end{cases} \tag{5.2}$$

The next results will show that the term D^2 (or $D^2 - 1$) can be interpreted geometrically as follows: The pencil $\{D_\lambda\}$ has $n = D^2$ distinct base points, denote their images by φ_L by $x_1, \ldots, x_n$. The linear spaces $\overline{D_\lambda}$ that sweep out the scroll $\mathcal{T}$ will intersect in the linear space spanned by these points, which we denote by $< x_1, \ldots, x_n >$. This is a $\mathbf{P}^{n-1}$ when $L \not\sim 2D$ and a $\mathbf{P}^{n-2}$ when $L \sim 2D$.

Define the set

$$\mathcal{R}_{L,D} := \{\Gamma \mid \Gamma \text{ is a smooth rational curve, } \Gamma.L = 0 \text{ and } \Gamma.D > 0\}. \quad (5.3)$$

The members of $\varphi_L(\{D_\lambda\})$ will intersect in the points $\{\varphi_L(\Gamma)\}_{\Gamma \in \mathcal{R}_{L,D}}$ in addition to the images of the D^2 base points of $\{D_\lambda\}$. If these extra points pose new independent conditions, they will contribute to the singular locus of $\mathcal{T}$. We will show below that among all free Clifford divisors, we can choose one such that the term $h^1(L-2D)$ will correspond exactly to the singularities of the scroll arising from the contractions of the curves in $\mathcal{R}_{L,D}$.

The contraction of smooth rational curves Γ which are not in $\mathcal{R}_{L,D}$, will occur in some fiber. Indeed, since $D.\Gamma = 0$ one calculates $h^0(D-\Gamma) = h^0(D) - 1$, whence Γ will be a component of a unique reducible member of $\{D_\lambda\}$. Clearly, such contractions which occur in some fiber, and not transversally to the fibers, will not influence the singularities of $\mathcal{T}$.

The proofs of the next three propositions are rather long and tedious, and will therefore be postponed until the next chapter.

Proposition 5.3. *Let D be a free Clifford divisor for L and Γ a curve in $\mathcal{R}_{L,D}$.*

Then $D.\Gamma = 1$, $F.\Gamma = -1$ and Γ is contained in the base locus Δ of F. As a consequence, $\Delta.D = \#\mathcal{R}_{L,D}$, where the elements are counted with the multiplicity they have in Δ.

Furthermore, if γ is any reduced and connected effective divisor such that $\gamma.L = 0$ and $\gamma.D > 0$, then $D.\gamma = 1$.

In particular, the curves in $\mathcal{R}_{L,D}$ are disjoint.

We defined the cases (E0)-(E2) in Chapter 3. We also need to define the following two cases for $c = 0$:

(E3) $L \sim 3D + 2\Gamma_0 + \Gamma_1$, Γ_0 and Γ_1 are smooth rational curves, $c = D^2 = 0$, $L^2 = 6$, $D.\Gamma_0 = 1$, $D.\Gamma_1 = 0$, $\Gamma_0.\Gamma_1 = 1$.
(E4) $L \sim 4D + 2\Gamma$, Γ is a smooth rational curve, $c = D^2 = 0$, $L^2 = 8$, $D.\Gamma = 1$.

Note that in all cases (E0)-(E4) we have $h^1(L-2D) = \Delta.D - 1$. More precisely we have:

(E0) $\Delta = \Gamma$, $\Delta.D = 1$, $h^1(L-2D) = 0$,
(E1) $\Delta = \Gamma_1 + \Gamma_2$, $\Delta.D = 2$, $h^1(L-2D) = 1$,
(E2) $\Delta = 2\Gamma_0 + 2\Gamma_1 + \cdots + 2\Gamma_N + \Gamma_{N+1} + \Gamma_{N+2}$, $\Delta.D = 2$, $h^1(L-2D) = 1$,
(E3) $\Delta = 2\Gamma_0 + \Gamma_1$, $\Delta.D = 2$, $h^1(L-2D) = 1$,
(E4) $\Delta = 2\Gamma$, $\Delta.D = 2$, $h^1(L-2D) = 1$.

Remark 5.4. If D is any free Clifford divisor not of type (E0)-(E4), then it will follow from equation (6.14) in Chapter 6 that $h^1(L-2D) \geq \Delta.D$.

Proposition 5.5. *Among all free Clifford divisors for L there is one, call it D, with the following property (denoting by Δ the base locus of $F := L - D$): If D is not of type (E0)-(E4), then*

$$h^1(L - 2D) = \Delta.D.$$

We will also need the following:

Proposition 5.6. *We have for D a free Clifford divisor*

$$h^1(L - 2D) \leq \frac{1}{2}c + 1 - D^2,$$

except possibly for the case $L^2 \leq 4c+6$ and $\Delta = 0$, the cases (E0)-(E2) *above, and the case*

$$L^2 = 4c + 4, D.\Delta = 1, \Delta^2 = -2. \tag{5.4}$$

In this latter case, $D^2 < c$.

We now study the singular locus V of the scroll $\mathcal{T}$. By equation (5.2) we know its dimension $r - 1$, and in the following results we will see which points in $\varphi_L(S)$ that span V and how $\varphi_L(S)$ intersects V. We will divide the treatment into the two cases $c = 0$ and $c > 0$. We recall from Proposition 5.1 that these two cases are naturally different.

We will now treat the case $c > 0$. Since we choose the base points of the pencil $\{D_\lambda\}$ to be distinct and to lie outside of the finitely many curves in $\mathcal{R}_{L,D}$, the images by φ_L of these points will be $n = D^2$ distinct points in $\varphi_L(S)$, denote them by $x_1, \ldots, x_n$, and their preimages by $p_1, \ldots, p_n$. Let $m = D.\Delta$ and let

$$\mathcal{R}_{L,D} = \{\Gamma_1, \ldots, \Gamma_t\}, \tag{5.5}$$

and define

$$m_i := \text{ multiplicity of } \Gamma_i \text{ in } \Delta. \tag{5.6}$$

Then $m = \sum_{i=1}^t m_i$. Denote by $y_1, \ldots, y_t$ the images (distinct from $x_1, \ldots, x_n$) of the contractions of the curves in $\mathcal{R}_{L,D}$, and by $q_{1,\lambda}, \ldots, q_{t,\lambda}$ their corresponding preimages in each fiber. So $q_{i,\lambda} = \Gamma_i \cap D_\lambda$.

In the cases (E0)-(E2) of Proposition 5.5, we use the following notation:

(E0) $y = \varphi_L(\Gamma)$,
(E1) $y_1 = \varphi_L(\Gamma_1)$, $y_2 = \varphi_L(\Gamma_2)$,
(E2) $y_0 = \varphi_L(\Gamma_0)$.

We will denote by q_λ, $q_{1,\lambda}$, $q_{2,\lambda}$ and $q_{0,\lambda}$ their respective preimages in the fiber D_λ.

Also, recall from p. 22 that we denote the special case $L \sim 2D$ by (Q).

For each $D_\lambda \in \mathcal{D}$, we can identify D_λ with its image $D'_\lambda := \varphi_L(D_\lambda)$ on S' by Lemma 5.2. Moreover, we clearly have that the multiplicities of the points $p_1, \ldots, p_n, q_{1,\lambda}, \ldots, q_{t,\lambda}$ on each D_λ is one, hence these points are all smooth points of D_λ, and consequently all $x_1, \ldots, x_n, y_1, \ldots, y_t$ are smooth points of D'_λ.

For any $D_\lambda \in \mathcal{D}$, we define Z_λ to be the zero-dimensional subscheme of length $n+m$ of D_λ defined by

$$Z_\lambda := p_1 + \cdots + p_n + m_1 q_{1,\lambda} + \cdots + m_t q_{t,\lambda}. \tag{5.7}$$

In particular

$$\mathcal{O}_{D_\lambda}(Z_\lambda) \simeq \mathcal{O}_{D_\lambda}(D + \sum_{i=1}^{t} m_i \Gamma_i). \tag{5.8}$$

This zero-dimensional scheme can, by the isomorphism between D_λ and D'_λ, be identified with the following zero-dimensional subscheme of D'_λ, which we by abuse of notation denote by the same name:

$$Z_\lambda = x_1 + \cdots + x_n + m_1 y_{1,\lambda} + \cdots + m_t y_{t,\lambda}.$$

Note that in the case (Q) all the Z_λ are equal to $p_1 + \cdots + p_n$ and will be denoted by Z.

In the special cases (Q), (E0)-(E2) we will also define the following zero-dimensional subschemes of Z_λ (which we again will identify to their corresponding subschemes of D'_λ):

(Q) $Z^i := p_1 + \cdots + \hat{p}_i + \cdots + p_n$,
(E0) $Z_{0,\lambda} := p_1 + \cdots + p_n$,
(E1) $Z_{i,\lambda} := p_1 + \cdots + p_n + q_{i,\lambda}$, $i = 1, 2$,
(E2) $Z_{0,\lambda} := p_1 + \cdots + p_n + q_{0,\lambda}$.

By $< Z >$ we will mean the linear span of a zero-dimensional scheme Z on S'.

The following is the main result of this chapter:

Theorem 5.7. *Assume $c > 0$. Among all free Clifford divisors for L there is one, call it D, satisfying the property in Proposition 5.5 and with the following three additional properties:*

(a) If D is not of type (Q), (E0), (E1) or (E2), then for all $D_\lambda \in \mathcal{D}$ we have

$$V := \mathit{Sing}\ \mathcal{T} =< Z_\lambda > \simeq \mathbf{P}^{n+m-1},$$

and if D is of one of the particular types above, then:
(Q) $V =< Z^i > =< Z > \simeq \mathbf{P}^{n-2}$, all i.
(E0) $V =< Z_{0,\lambda} > =< Z_\lambda > \simeq \mathbf{P}^{n-1}$,
(E1) $V =< Z_{1,\lambda} > =< Z_{2,\lambda} > =< Z_\lambda > \simeq \mathbf{P}^{n}$,
(E2) $V =< Z_{0,\lambda} > =< Z_\lambda > \simeq \mathbf{P}^{n}$.

(b) V does not intersect S' (set-theoretically) outside the points in the support of Z_λ.

(c) For any irreducible D_λ, we have

$$V \cap D_\lambda = Z_\lambda.$$

In the theorem above, the following convention is used: $\mathbf{P}^{-1} = \emptyset$ (which happens if and only if $n = m = 0$ and implies that the scroll is smooth).

Remark 5.8. *If D is any free Clifford divisor, we have $V \supseteq < Z_\lambda > \simeq \mathbf{P}^{n+m-1}$, except in the cases (Q), (E0)-(E2), where the property (a) is automatically fulfilled.*

If D is not of type (E1) or (E2), the properties (b) and (c) automatically hold. If D is of type (E1) or (E2), then it might be that V intersects S' outside of the support of $< Z_\lambda >$.

The proof of Theorem 5.7 will be divided in the general case and in the special cases (Q), (E0)-(E2). We will only prove the two first properties. The last one will be left to the reader.

In this chapter, we give the proofs for the general case and the cases (Q) and (E0). The proof of the case (E1) is postponed until the next chapter, and the proof of the case (E2) is similar and therefore left to the reader.

We will write $\lambda \in \mathcal{D}$ for a λ such that $D_\lambda \in \mathcal{D}$.

Proof of Theorem 5.7 in the general case. Let $s := n + m$. To prove that $< Z_\lambda > \simeq \mathbf{P}^{n+m-1}$ it suffices to prove that the natural map

$$H^0(L) \longrightarrow H^0(L \otimes \mathcal{O}_{Z_\lambda})$$

is surjective for all $\lambda \in \mathcal{D}$.

So assume this map is not surjective for some λ. Then there exists a subscheme $Z' \subseteq Z_\lambda$ of length $s' \leq s$, for some integer $s' \geq 2$ (since L is base point free), such that the map $H^0(L) \to H^0(L \otimes \mathcal{O}_{Z'})$ is not surjective, but such that the map $H^0(L) \to H^0(L \otimes \mathcal{O}_{Z''})$ is surjective for all proper subschemes $Z'' \subsetneqq Z'$. We now use Propositions 5.5 and 5.6.

If $\Delta = 0$ and $L^2 \leq 4c + 6$, we have $n = D^2 \leq c$ and $m = 0$, so

$$L^2 \geq 4(c+1) \geq 4(n+1) = 4(s+1).$$

If we are in the case given by (5.4), we have

$$L^2 \geq 4(c+1) \geq 4(n+2) = 4(s+1).$$

In all other cases, we have $s = m + n \leq \lfloor \frac{1}{2}c \rfloor + 1$ by Propositions 5.5 and 5.6, so

$$L^2 \geq 4(c+1) \geq 4(\lfloor \frac{1}{2}c \rfloor + 2) \geq 4(s+1).$$

Therefore, by Proposition 1.13, there exists an effective divisor B passing through Z' and satisfying $B^2 \geq -2$, $h^1(B) = 0$ and the numerical conditions

$$2B^2 < B.L \leq B^2 + s' < 2s'.$$

If $B^2 \geq 0$, then B would induce a Clifford index $c_B \leq s' - 2 \leq n + m - 2$ on the smooth curves in $|L|$. If $\Delta = 0$ and $L^2 \leq 4c + 6$, we get the contradiction $c_B \leq n - 2 \leq c - 2$. If we are in the case given by (5.4), we get the contradiction $c_B \leq n - 1 \leq c - 2$. Finally, in all other cases, we have $c_B \leq n + m - 2 < \lfloor \frac{1}{2} c \rfloor$, again a contradiction.

Hence $B^2 = -2$ and B is supported on a union of smooth rational curves. Furthermore, $B.L \leq s' - 2$ and $B.D \geq s'$ (the last inequality follows since D_λ passes through Z_λ).

We now consider the effective decomposition

$$L \sim (D + B) + (F - B).$$

Firstly note that $L.(D+B) \leq n + s' + c$ and $(D+B)^2 \geq n + 2s' - 2$, whence $(F - B)^2 = (L - D - B)^2 \geq 2c - n + 2 \geq c + 2 > 0$, so that $h^0(F - B) \geq 2$.

Secondly, $L.(D + B) - (D + B)^2 - 2 \leq c - s' < c$, a contradiction.

For the second statement, it suffices to show that there is no point $x_0 \in S' - \{x_1, \ldots, x_n, y_1, \ldots, y_t\}$ such that S' has an $(s+1)$-secant $(s-1)$-plane through Z_λ and x_0 for all λ.

Assume, to get a contradiction, that there is such a point x_0. Choose any preimage p_0 of x_0, and denote by X_λ the zero-dimensional scheme defined as the union of Z_λ and p_0. Fix any λ such that D_λ is irreducible.

In these terms we have that the natural map

$$H^0(L) \longrightarrow H^0(L \otimes \mathcal{O}_{X_\lambda})$$

is not surjective.

Then there exists a subscheme $X' \subseteq X_\lambda$ of length $s' + 1 \leq s + 1$, for some integer $s' \geq 1$, such that the map $H^0(L) \to H^0(L \otimes \mathcal{O}_{X'})$ is not surjective, but such that the map $H^0(L) \to H^0(L \otimes \mathcal{O}_{X''})$ is surjective for all proper subschemes $X'' \subsetneqq X'$.

Since $L^2 \geq 4(s+1)$ by the above, there exists by Proposition 1.13 again an effective divisor B passing through X' and satisfying $B^2 \geq -2$, $h^1(B) = 0$ and the numerical conditions

$$2B^2 \leq B.L \leq B^2 + s' + 1 \leq 2s' + 2.$$

As above, if $B^2 \geq 0$, we would get a contradiction on the Clifford index c. Hence $B^2 = -2$ and B is supported on a union of smooth rational curves. Furthermore, $B.L \leq s' - 1$ and $B.D \geq s'$ (the last inequality follows since D_λ is irreducible).

As above, the effective decomposition

$$L \sim (D + B) + (F - B)$$

induces a Clifford index $< c$ on the smooth curves in $|L|$, unless $s' = 1$, $B.L = 0$ and $B.D = 1$. This means that p_0 lies in some divisor which is contracted to one of the points $y_1, \ldots, y_t$. Hence x_0 is one of these points, a contradiction. $\square$

Proof of Theorem 5.7 in the case $L \sim 2D$. It suffices to prove that if there is a point $x_0 \in S' - \{x_1, \dots, \hat{x_i}, \dots, x_n\}$ for some i, such that S' has an n-secant $(n-2)$-plane through x_0 and Z^i, then $x_0 = x_i$.

Choose any preimage p_0 of x_0, and denote by X_i the zero-dimensional scheme defined by p_0 and Z^i. We will show that if the natural map

$$H^0(L) \longrightarrow H^0(L \otimes \mathcal{O}_{X_i})$$

is not surjective, then $p_0 = p_i$.

Let $X' \subseteq X_i$ be a subscheme of length $n' \leq n$, for some integer $n' \geq 2$, such that the map $H^0(L) \to H^0(L \otimes \mathcal{O}_{X'})$ is not surjective, but such that the map $H^0(L) \to H^0(L \otimes \mathcal{O}_{X''})$ is surjective for all proper subschemes $X'' \subsetneqq X'$.

By assumption, we have $n = D^2 = c + 2$ and $L^2 = 4c + 8 = 4n$. Hence, by Proposition 1.13, there exists an effective divisor B passing through X' and satisfying $B^2 \geq -2$ and the numerical conditions

$$2B^2 \overset{(a)}{\leq} L.B \leq B^2 + n' \overset{(b)}{\leq} 2n',$$

with equality in (a) or (b) implying $L \sim 2B$.

Since B passes through X', we have $B.D \geq n' - 1$, whence $B.L \geq 2n' - 2$. From the inequalities above, we get $B^2 \geq n' - 2 \geq 0$, so we have $n' = n$ and $B.L = B^2 + n$, since otherwise B would induce a Clifford index $< n - 2 = c$ on the smooth members of $|L|$. This leaves us with the two possibilities:

$$(i) \quad B^2 = n \text{ and } L \sim 2B, \qquad \text{or} \qquad (ii) \quad B^2 = n - 1.$$

But in the second case, by Proposition 1.13, we have $L \sim 2B + \Gamma$, for Γ a smooth rational curve, which is impossible, since $L \sim 2D$. So we are in case (i), and $B \in |D|$. By the last assertion in Proposition 1.13 we have $h^0(B \otimes \mathcal{J}_{X'}) = h^0(E - B) = 2$, so there is a pencil P of divisors in $|D|$ passing through X'.

We claim that any divisor $D_0 \in |D|$ passing through $n - 1$ of the points $p_1, \dots, p_n$, will also pass through the last one. Indeed, by the surjectivity of the map $H^0(D) \to H^0(\mathcal{O}_{D_0}(D))$, we reduce to the same statement for $\mathcal{O}_{D_0}(D)$. By Riemann-Roch, this is equivalent to $h^0(\mathcal{O}_{D_0}(Z)) \geq 2$ and $\mathcal{O}_{D_0}(Z)$ base point free, which are both satisfied since $\mathcal{O}_{D_0}(Z) \simeq \mathcal{O}_{D_0}(D)$, and $\mathcal{O}_S(D)$ is base point free.

Since Z contains the points $p_0, \dots, \hat{p_i}, \dots, p_n$, we have that all the members in P contain all the points $p_0, \dots, p_n$. Therefore, P is the pencil $\{D_\lambda\}$, whose general member is smooth and irreducible. Since all the members intersect in n points, we have $p_0 = p_i$, as asserted. $\square$

Proof of Theorem 5.7 in the case (E0). We first prove that $< Z_{0,\lambda} > \simeq \mathbf{P}^{n-1}$ for all λ. If this were not true, the natural map

$$H^0(L) \longrightarrow H^0(L \otimes \mathcal{O}_{Z_{0,\lambda}})$$

would not be surjective for some λ.

As usual let $Z' \subseteq Z_{0,\lambda}$ be a subscheme of length $n' \leq n$, for some integer $n' \geq 2$, such that the map $H^0(L) \to H^0(L \otimes \mathcal{O}_{Z'})$ is not surjective, but such that the map $H^0(L) \to H^0(L \otimes \mathcal{O}_{Z''})$ is surjective for all proper subschemes $Z'' \subsetneqq Z'$.

Since $L^2 = 4n + 2 = 4(n-1) + 6$, we get by Proposition 1.13 that there exists an effective divisor B passing through Z' such that $B^2 \geq -2$, $h^1(B) = 0$ and

$$2B^2 < B.L \leq B^2 + n' < 2n'.$$

If $B^2 \geq 0$, we would get that B induces a Clifford index $c_B \leq n' - 2 \leq c - 1$ on the smooth curves in $|L|$, a contradiction.

So $B^2 = -2$, and B is necessarily supported on a union of smooth rational curves, since $h^1(B) = 0$. But $B.L \leq n' - 2 \leq n - 2 = c - 1$ and Z' consists of base points of $\{D_\lambda\}$. This means that B passes through some of these base points, which contadicts the fact that we have chosen these base points to lie outside of smooth rational curves of degree $\leq c + 2$ with respect to L.

So $< Z_{0,\lambda} > \simeq \mathbf{P}^{n-1}$, and by equation (5.2) and Proposition 5.5 we know that $V \simeq \mathbf{P}^{n-1}$, so the point y does not pose any additional conditions.

To prove the last assertion, assume to get a contradiction that there exists a point $x_0 \in S' - \{x_1, \ldots, x_n, y\}$ such that S' has an $(n+1)$-secant $(n-1)$-plane through x_0 and $Z_{0,\lambda}$. Choose any preimage p_0 of x_0 and denote by X_λ the zero-dimensional scheme defined by p_0 and $Z_{0,\lambda}$. We then have that the natural map

$$H^0(L) \longrightarrow H^0(L \otimes \mathcal{O}_{X_\lambda})$$

is not surjective. Fix a λ.

Again let $X' \subseteq X_\lambda$ be a subscheme of length $n' + 1 \leq n + 1$, for some integer $n' \geq 1$, such that the map $H^0(L) \to H^0(L \otimes \mathcal{O}_{X'})$ is not surjective, but such that the map $H^0(L) \to H^0(L \otimes \mathcal{O}_{X''})$ is surjective for all subschemes $X'' \subseteq X'$.

Since $L^2 = 4n + 2$ and L is not divisible by assumption, we get by Proposition 1.13 again that there exists an effective divisor B passing through X' satisfying $B^2 \geq -2$, $h^1(B) = 0$ and the numerical conditions

$$2B^2 < B.L \leq B^2 + n' + 1 < 2n' + 2.$$

If $B^2 = -2$ then, since Z' has length ≥ 2, we must have that B passes through some of the base points of $\{D_\lambda\}$, a contradiction as above.

So $B^2 \geq 0$, $n = n'$ and $X' = X_\lambda$. Since $h^0(L - B) \geq h^0(B) \geq 2$ by Proposition 1.13 again, we have that B is a Clifford divisor, and by Proposition 3.8, we have $D \sim B$. By the last statement in Proposition 1.13, we have $\Gamma \cap X_\lambda \neq \emptyset$, whence we conclude that $p_0 \in \Gamma$. This gives the desired contradiction $x_0 = y$. $\square$

It will be convenient to make the following definition:

Definition 5.9. *A free Clifford divisor satisfying the properties described in Proposition 5.5 and Theorem 5.7 will be called a* perfect *Clifford divisor.*

In the next chapter we will prove Proposition 5.5 and Theorem 5.7, thus proving that we can find a perfect Clifford divisor.

The main advantage of choosing a perfect Clifford divisor is that we then get a nice description of the singular locus of $\mathcal{T}$ and how it intersects S' as in Theorem 5.7 above. This theorem states that Sing $\mathcal{T}$ is "spanned" by the images of the base points of the chosen subpencil of $|D|$ and the contracted curves, and moreover that it intersects S' in only these points. If D is not perfect, then Sing $\mathcal{T} \supsetneq Z_\lambda$, as seen in Remarks 5.4 and 5.8. In Proposition 8.39 below we will see an example where this occurs.

It will also be practical, for classification purposes, to restrict the attention to perfect Clifford divisors, as we will do in Chapter 11.

Apart from this, any free Clifford divisor will be equally fit for our purposes.

We include an additional description of the case (Q):

Proposition 5.10. *Assume D is a free Clifford divisor of type (Q) and $c \geq 2$. Then $\varphi_L(S)$ is the 2-uple embedding of $\varphi_D(S)$, except in the special case described in Proposition 3.10 (where $c = 2$ and there exists a smooth elliptic curve E such that $E.D = 2$, in which case D is hyperelliptic).*

Proof. By [SD, Thm. 6.1] φ_L is the 2-uple embedding of $\varphi_D(S)$, when D is not hyperelliptic.

Conversely, if D is hyperelliptic, then φ_D is not birational, so φ_L cannot be the 2-uple embedding of $\varphi_D(S)$.

Since we assume $c \geq 2$, we have $D^2 \geq 4$, and we can use Proposition 3.10 to conclude the proof. □

The special case appearing in the proposition will be thouroughly described in Proposition 8.39 below.

If $c = 0$, there exist two kinds of (free) Clifford divisors for L, namely:

1. $D^2 = 0$, $D.L = 2$ and
2. $D^2 = 2$, $L \sim 2D$.

In both these cases $\varphi_L(S)$ is $2:1$ on each fiber.

In the case $c = 0$ we have the following result:

Proposition 5.11. *Assume $c = 0$. Let D be a free Clifford divisor for L. Then $D^2 = 0$ and $V = \emptyset$ except in the following cases:*

(Q) $L \sim 2D$, $D^2 = 2$, $V = \{x\}$, *where x is the common image of the two base points of the chosen pencil $\{D_\lambda\}$,*
(E1) $D^2 = 0$, $L \sim 2D + \Gamma_1 + \Gamma_2$, $V = \{\varphi_L(\Gamma_1)\} = \{\varphi_L(\Gamma_2)\}$,
(E2) $D^2 = 0$, $L \sim 2D + 2\Gamma_0 + 2\Gamma_1 + \cdots + 2\Gamma_N + \Gamma_{N+1} + \Gamma_{N+2}$, $V = \{\varphi_L(\Gamma_0)\}$,
(E3) $D^2 = 0$, $L \sim 3D + 2\Gamma_0 + \Gamma_1$, $V = \{\varphi_L(\Gamma_0)\}$,
(E4) $D^2 = 0$, $L \sim 4D + 2\Gamma$, $V = \{\varphi_L(\Gamma)\}$.

Proof. For $D^2 = 0$, this follows from the fact that except for the cases (E1)-(E4), the base locus Δ of $L - D$ is zero, which is shown in the proof of [SD, Prop. 5.7]. In the other case, it follows from the equation (5.1). $\square$

All these cases have been completely described in [SD, Prop. 5.6 and 5.7].

When $V = \emptyset$, then $\varphi_L(S)$ is a rational ruled surface.

The cases where there are contractions across the fibers, are the cases (E1)-(E4). In these cases $\varphi_L(S)$ is a cone.

In the case (Q), $\varphi_L(S)$ is the Veronese surface in $\mathbf{P}^5$.

6

Postponed proofs

In this chapter we will give the proofs omitted in the previous chapter.

Throughout this chapter L will be a base point free and big line bundle of non-general Clifford index c. In particular, this implies $L^2 \geq 4c+4$.

Also we write $F := L - D$ and $R := L - 2D = F - D$, and denote the (possibly zero) base divisor of $|F|$ by Δ. Recall that $L.\Delta = 0$ and that we have $h^0(R) = 0$ if and only if $L^2 = 4c+4$ and $h^1(R) = 0$. In particular, $h^1(R) > 0$ implies that $R > 0$.

Furthermore, we have

Lemma 6.1. *If* $h^0(R) = 0$*, then* $\Delta = 0$*.*

Proof. We have $L^2 = 4c+4$, so we cannot be in the cases (Q) or (E1), whence $D^2 \leq c$. Choose any smooth curve $D_0 \in |D|$ and let $F_{D_0} := F \otimes \mathcal{O}_{D_0}$. Then $\deg F_{D_0} = c+2 \geq D^2+2 = 2g(D_0)$, whence F_{D_0} is base point free.

We first will show that this implies that F is nef.

Taking cohomology of the short exact sequence

$$0 \longrightarrow R \longrightarrow F \longrightarrow F_{D_0} \longrightarrow 0,$$

and of the same sequence tensored with $-\Delta$, we get the following two exact sequences (using $h^0(R) = h^0(R-\Delta) = h^1(R) = 0$)

$$\begin{tikzcd}
0 \arrow[r] & H^0(F) \arrow[r] \arrow[d, equal] & H^0(F_{D_0}) \arrow[r] & 0 \\
0 \arrow[r] & H^0(F-\Delta) \arrow[r] & H^0((F-\Delta)_{D_0}). &
\end{tikzcd}$$

This gives $h^0((F-\Delta)_{D_0}) \geq h^0(F_{D_0})$, whence $\Delta.D = 0$, since F_{D_0} is base point free. This means that for any smooth rational curve Γ in the support of Δ, we have $\Gamma.D = \Gamma.F = 0$. Hence F is nef.

By Lemma 1.10 it now suffices to show that F is not of the type $F \sim kE+\Gamma$, for E a smooth elliptic curve and Γ a smooth rational curve satisfying

T. Johnsen and A.L. Knutsen: LNM 1842, pp. 47–57, 2004.

$E.\varGamma = 1$ and an integer $k \geq 2$. But if this were the case, we would have $E.L = 2 + c/k$. If $c \neq 0$, this would mean that E induces a lower Clifford index than c on the smooth curves in $|L|$, a contradiction. If $c = 0$, we get $D.F = 2$ and $D^2 = 0$. But this would give $R^2 = (F - D)^2 \geq -2$ and by Riemann-Roch, we would then get the contradiction $h^0(F - D) \geq 1$. □

By this lemma, if $h^0(R) = 0$, the Propositions 5.3, 5.5 and 5.6 will automatically be satisfied. So for the rest of this chapter, we will assume $R > 0$.

Let F_0 be the moving component of $|F|$. Since $R > 0$, we can write $F_0 \sim D + A$ for some divisor $A \geq 0$. Thus we have

$$F \sim D + R \sim F_0 + \varDelta \sim D + A + \varDelta, \tag{6.1}$$

and

$$L \sim 2D + A + \varDelta. \tag{6.2}$$

We will first study the divisors above more closely.

Lemma 6.2. *Except for the cases (E3) and (E4), the general member of $|F_0|$ is smooth and irreducible.*

Proof. Since $|F_0|$ is base point free, by Proposition 1.9 we only need to show that $F_0 \not\sim kE$, for E a smooth elliptic curve and an integer $k \geq 2$.

Assume, to get a contradiction, that $F_0 \sim kE$, then by (6.1), we have $D \sim E$ and $A \sim (k-1)E$. Let $d := c + 2 = E.L \geq 2$.

Since $L \sim (k+1)E + \varDelta$, we get $L^2 = d(k+1)$, so $h^0(L) = d(k+1)/2 + 2$, and $h^0(F) = h^0(kE) = k + 1$. On the other hand, by equation (3.6), we have $h^0(F) = h^0(L) - d$.

Combining the last three equations, we get

$$k + 1 = d(k-1)/2 + 2,$$

which is only possible if $d = 2$, i.e. $c = 0$. A case by case study as in the proof of [SD, Prop. 5.7] establishes the lemma in this latter case. □

We gather some basic properties of R.

Lemma 6.3. *(a) If $R = R_1 + R_2$ is an effective decomposition, then $R_1.R_2 \geq 0$.*
(b) If γ is an effective divisor satisfying $\gamma^2 = -2$ and $\gamma.R < 0$, then $\gamma.R = -1$ or -2.
(c) If γ is an effective divisor satisfying $\gamma^2 = -2$ and $\gamma.L = 0$, then either $\gamma.D = \gamma.F = \gamma.R = 0$ or $\gamma.D = 1$, $\gamma.F = -1$ and $\gamma.R = -2$.
(d) If $\varGamma$ is a smooth rational curve, then $\varGamma \in \mathcal{R}_{L,D}$ if and only if $\varGamma.R = -2$ and $\varGamma.L = 0$.

Proof. To prove (a), one immediately sees that if $R_1.R_2 < 0$, then the effective decomposition $L \sim (D + R_1) + (D + R_2)$ would induce a Clifford index $< c$.

The other assertions are immediate consequences of (a). □

This concludes the proof of Proposition 5.3.

Lemma 6.4. *Except for the cases (E3) and (E4), the following holds:* $\Delta^2 = -2D.\Delta$ *and* $\Delta.A = 0$.

Proof. By Lemma 6.2, we have $h^1(F_0) = 0$. From $0 = \Delta.L = 2\Delta.D + \Delta.A + \Delta^2$, we get

$$\Delta^2 = -2\Delta.D - \Delta.A. \tag{6.3}$$

Furthermore, we also have

$$h^0(F_0) = h^0(F) = \frac{1}{2}F_0^2 + F_0.\Delta + \frac{1}{2}\Delta^2 + 2 = h^0(F_0) + (D+A).\Delta + \frac{1}{2}\Delta^2,$$

which implies

$$\Delta^2 = -2\Delta.D - 2\Delta.A. \tag{6.4}$$

Combining equations (6.3) and (6.4), we get $\Delta.A = 0$ and $\Delta^2 = -2D.\Delta$. □

We have seen in Proposition 3.8, that if there exists a free Clifford divisor of type (E0), then all free Clifford divisors are linearly equivalent and of type (E0).

We now take a closer look at the types (E1) and (E2).

Proposition 6.5. *Let* L *be a base point free and big line bundle of non-general Clifford index* c *on a* $K3$ *surface and let* D *be a free Clifford divisor of type (E1) or (E2).*

If $D' \not\sim D$ *is any other free Clifford divisor, then* $B := D - D' > 0$ *and*

$$\Delta.D' = 0, \Delta.B = 2, B^2 = -2. \tag{6.5}$$

Proof. Let $R' := L - 2D'$ as usual, and note that $R' \sim 2B + \Delta$.

Since ${R'}^2 = L^2 - 4(c+2) = \Delta^2 = -4$, we get $B^2 + B.\Delta = 0$. Combined with $\Delta.D = \Delta.(D' + B) = 2$, we get the two possibilities

(a) $\Delta.D' \geq 2$, $\Delta.B \leq 0$, $B^2 \geq 0$,
(b) $\Delta.D' = 0$, $\Delta.B = 2$, $B^2 = -2$.

Using ${D'}^2 \leq c$, we calculate

$$\begin{aligned} B.L &= \frac{1}{2}(R' - \Delta).L = \frac{1}{2}R'.L = \frac{1}{2}(L - 2D').L \\ &= \frac{1}{2}(L^2 - 2({D'}^2 + c + 2)) \geq 2c + 2 - c - c - 2 = 0. \end{aligned} \tag{6.6}$$

In case (a) we then must have $B.L > 0$ by the Hodge index theorem, so $B > 0$ by Riemann-Roch. We get

$$B.R' = B.(2B + \Delta) = B^2 \geq 0 \quad \text{and} \quad R'.D' = 2B.D' + \Delta.D' \geq 2B.D' + 2,$$

which gives

$$L.B = 2D'.B + R'.B = 2D'.B + B^2 \leq R'.D' + B^2 - 2.$$

But this implies

$$B.L - B^2 - 2 \leq R'.D' - 4 < {D'}^2 + R'.D' - 2 = c,$$

whence we must have $B \sim 0$.

So we must be in case (b), and by Riemann-Roch we have either $B > 0$ or $-B > 0$. We see from (6.6) that $B.L > 0$ unless ${D'}^2 = D^2 = c$. But if the latter holds, since both D' and D are assumed to be free Clifford divisors (so that $h^1(D) = h^1(D') = 0$), we have $h^0(D) = h^0(D')$, whence $D \sim D'$ and $B \sim 0$, a contradiction. Hence $D.L > 0$, so $B > 0$ and we are done. $\square$

As seen below, we will distinguish between inclusions $D' < D$ as in Proposition 6.5 with $\Delta' = 0$ and $\Delta' \neq 0$ (where Δ' is the base divisor of $|L - D'|$).

By Propositions 3.7 and 3.8 it is clear that we can choose a free Clifford divisor D with the two additional properties (recall that Δ as usual denotes the base divisor of $|L - D|$):

(C6) If D' is any other free Clifford divisor such that $D' > D$, then $\Delta \neq 0$ and D' is of type (E1) or (E2).
(C7) If D is of type (E1) or (E2) above, and D' is any other free Clifford divisor satisfying (C6), then $D' \sim D$.

Property (C6) is a *maximality condition*: it means that we choose a free Clifford divisor which is not contained in any other free Clifford divisor, unless possibly when $\Delta \neq 0$ and it is contained in some free Clifford divisor of type (E1) or (E2).

Property (C7) means that if we can, we will choose among all free Clifford divisors satisfying (C6), one that is not of type (E1) or (E2).

It turns out, as we will show in this chapter, that free Clifford divisors satisfying the additional properties (C6) and (C7) will be perfect, i.e. they will satisfy Propositions 5.5 and 5.7.

Now assume $R = R_1 + R_2$ is an effective decomposition such that $R_1.R_2 = 0$. Then $L \sim (D + R_1) + (D + R_2)$ is an effective decomposition satisfying

$$(D + R_1).(D + R_2) = D^2 + D.(R_1 + R_2) = D.F = c + 2,$$

so this decomposition induces the same Clifford index c. This means that either $D + R_1$ or $D + R_2$ is a Clifford divisor. This enables us to prove the following:

Proposition 6.6. *Assume D is not of type (E3) or (E4) and satisfies (C6) and (C7). Assume furthermore that there exists an effective decomposition $R = R_1 + R_2$ such that $R_1.R_2 = 0$ and such that $D + R_1$ is a Clifford divisor.*

Then either $\Delta \neq 0$ and $D + R_1$ is of type (E1) or (E2), or there exists a smooth rational curve Γ satisfying either

(I) $\Gamma.D = \Gamma.F = \Gamma.L = 0$, $\Gamma.R_1 = -1$, $\Gamma.R_2 = 1$, *or*
(II) $\Gamma.D = 1$, $\Gamma.F = -1$, $\Gamma.L = 0$, $\Gamma.R_1 = -2$, $\Gamma.R_2 = 0$.

Proof. Let $D_1 := D + R_1$ and $D_2 := D + R_2$. Since D_1 is a Clifford divisor containing D, we have by condition (C6) that either D_1 is not a free Clifford divisor, or $\Delta \neq 0$ and D_1 is of type (E1) or (E2).

So we can assume D_1 is not a free Clifford divisor, which means that D_1 is not base point free.

If D_1 is nef, then by Lemma 1.10 it is of the form

$$D_1 \sim lE + \Gamma_0,$$

for some smooth elliptic curve E and smooth rational curve Γ_0 satisfying $E.\Gamma_0 = 1$, and some integer $l \geq 2$. This gives

$$R_1 \sim (l-1)E + \Gamma_0 \quad \text{and} \quad D \sim E.$$

Write

$$D_2 = D + R_2 \sim D + M + B,$$

where $B \geq 0$ is the base divisor of D_2 and $M \geq 0$. Note that $M + B \sim R_2$.

We have

$$\begin{aligned} 0 = R_1.R_2 &= ((l-1)D + \Gamma_0).(M + B) \\ &= (l-1)D.M + (l-1)D.B + \Gamma_0.M + \Gamma_0.B. \end{aligned} \tag{6.7}$$

Also, we have an effective decomposition

$$R \sim ((l-1)D + M + B) + \Gamma_0,$$

such that, using (6.7),

$$((l-1)D + M + B).\Gamma_0 = l - 1 + \Gamma_0.M + \Gamma_0.B = (l-1)(1 - D.M - D.B). \tag{6.8}$$

By [SD, Lemma 3.7], if $M \neq 0$, either $M \sim kD$, with $D^2 = 0$, for some integer $k \geq 1$, or $D.M \geq 2$. In this latter case, the latter product in (6.8) would be negative, contradicting Lemma 6.3. So we must have $M \sim kD$, for some integer $k \geq 0$ and $D.B = 0$ or 1.

So $R \sim R_1 + R_2 \sim (l-1)D + \Gamma_0 + M + B \sim (k + l - 1)D + \Gamma_0 + B$ and

$$c + 2 = D.F = D.(D + R) = ((k + l)D + \Gamma_0 + B).D \leq 2,$$

which gives $c = 0$ and $B.D = 1$. A short analysis as in part (b) of the proof of [SD, Lemma 5.7.2] shows that D is then of type (E3) or (E4).

So D_1 is not nef, which means that there exists a smooth rational curve Γ such that $\Gamma.D_1 < 0$, whence Γ is fixed in $|D_1|$ and $\Gamma.L = 0$, by Proposition 2.6. Combining $\Gamma.D_1 = \Gamma.D + \Gamma.R_1 \leq -1$ and $0 = \Gamma.L = 2\Gamma.D + \Gamma.R_1 + \Gamma.R_2$, we get

$$1 - \varGamma.R_2 \leq \varGamma.D \leq -1 - \varGamma.R_1. \tag{6.9}$$

Furthermore, by Lemma 6.3(b), we have

$$\varGamma.R = \varGamma.R_1 + \varGamma.R_2 \geq -2. \tag{6.10}$$

If $R_1 = \varGamma$, we are done by Lemma 6.3(c), so we can assume that $R_1 - \varGamma > 0$. Then by Lemma 6.3(a) we have $(R_1 - \varGamma).(R_2 + \varGamma) = R_1.R_2 + \varGamma.R_1 - \varGamma.R_2 + 2 \geq 0$, which implies

$$\varGamma.R_1 - \varGamma.R_2 \geq -2. \tag{6.11}$$

Combining (6.10) and (6.11), we get

$$-2 - \varGamma.R_1 \leq \varGamma.R_2 \leq 2 + \varGamma.R_1 \quad \text{and} \quad \varGamma.R_1 \geq -2. \tag{6.12}$$

Combining (6.12) with (6.9) and Lemma 6.3(c), we end up with the two possibilities given by (I) and (II) above. □

We now need a basic lemma about A.

Lemma 6.7. *If $A = 0$, then D is of one of the types (E0)-(E2).*
If $A^2 \leq -2$, then one of the following holds:

(a) $A^2 = -4$, $\varDelta = 0$, $L^2 = 4c + 4$,
(b) $A^2 = -2$, $\varDelta = 0$, $L^2 = 4c + 6$,
(c) $A^2 = -2$, $\varDelta^2 = -2$, $D.\varDelta = 1$, $L^2 = 4c + 4$.

Moreover, in case (c) we have $D^2 < c$.

Proof. If $A = 0$, we must have $-4 \leq \varDelta^2 = R^2 \leq -2$, whence $\varDelta^2 = -4$ or -2, $D.\varDelta = 2$ or 1 respectively (by Lemma 6.4), and $L^2 = 4c + 4$ or $4c + 6$ respectively. An analysis as in Proposition 3.7 now gives that D is as in one of the cases (E0)-(E2).

If $A^2 \leq -2$, we have by $R^2 = A^2 + \varDelta^2 = L^2 - 4(c + 2)$ (where we have used Lemma 6.4) that either $\varDelta = 0$ and we are in case (a) or (b) above, or that $A^2 = -2$, $\varDelta^2 = -2$, $D.\varDelta = 1$ (by Lemma 6.4) and $L^2 = 4c + 4$, i.e. case (c).

In this latter case, we have

$$c + 2 = D.F = D^2 + D.A + D.\varDelta = D^2 + D.A + 1,$$

whence $D^2 = c + 1 - D.A$. Since $D + A \sim F_0$ is base point free, we have $D.A \geq 2$ by [SD, (3.9.6)], whence $D^2 < c$. □

We can now prove Proposition 5.6.

First note that the Proposition is true for the cases (E3) and (E4), so we will from now on assume that we are not in any of these two cases.

When we are not in the exceptional cases of the proposition (which are the cases (E0)-(E2) and the cases (a)-(c) of the last lemma), we have $A \neq 0$

and $A^2 \geq 0$. In particular $h^0(A) \geq 2$. Moreover $h^0(L-A) \geq h^0(2D) \geq 3$. From the standard exact sequence for any $C \in |L|$

$$0 \longrightarrow A - L \longrightarrow A \longrightarrow A_C \longrightarrow 0,$$

we see that A_C contributes to the Clifford index of C, and moreover that $h^0(A_C) \geq h^0(A)$.

We first claim that

$$h^1(A) = D^2 - c - 2 + D.A + h^1(R). \tag{6.13}$$

Indeed, we have by Lemma 6.2 that $h^1(F_0) = 0$, whence

$$\begin{aligned} h^0(F) = h^0(F_0) = h^0(D+A) &= \frac{1}{2}D^2 + D.A + \frac{1}{2}A^2 + 2 \\ &= \frac{1}{2}D^2 + D.A + h^0(A) - h^1(A), \end{aligned}$$

which gives

$$\begin{aligned} h^0(A) = h^0(R) &= h^0(F) - L.D + \frac{3}{2}D^2 + h^1(R) \\ &= (\frac{1}{2}D^2 + D.A + h^0(A) - h^1(A)) - L.D + \frac{3}{2}D^2 + h^1(R) \\ &= h^0(A) + D^2 - c - 2 + D.A + h^1(R) - h^1(A), \end{aligned}$$

whence (6.13) follows.

Now we get

$$\begin{aligned} \text{Cliff } A_C &= \deg A_C - 2(h^0(A_C) - 1) \\ &\leq L.A - 2(\frac{1}{2}A^2 + 1 + h^1(A)) \\ &= L.A - A^2 - 2 - 2h^1(A) \\ &= 2D.A - 2 - 2(D^2 - c - 2 + D.A + h^1(R)) \\ &= 2(c + 1 - D^2 - h^1(R)). \end{aligned}$$

But since A_C contributes to the Clifford index of C, we must have Cliff $A_C \geq c$, whence Proposition 5.6 follows.

Before proving the next result, we will need the following easy lemma.

Lemma 6.8. *Assume D is not as in (E3) or (E4). If $A \sim A_1 + A_2$ is an effective decomposition such that $A_1.A_2 \leq 0$, then*

$$A_1.A_2 = A_1.\Delta = A_2.\Delta = 0,$$

and either $D + A_1$ or $D + A_2$ is a Clifford divisor.

Proof. By Lemma 6.4 we have $\Delta.A = 0$, so we can assume (possibly after interchanging A_1 and A_2) that $A_1.\Delta \leq 0$ and $A_2.\Delta \geq 0$. Then $R \sim A_1 + (A_2 + \Delta)$ is an effective decomposition of R such that

$$A_1.(A_2 + \Delta) = A_1.A_2 + A_1.\Delta \leq 0.$$

By Lemma 6.3(a) we must have equality, whence $A_1.A_2 = A_1.\Delta = A_2.\Delta = 0$.

If $A_1.L > A_2.L$ (resp. $A_2.L > A_1.L$), then clearly $D + A_1$ (resp. $D + A_2$) is a Clifford divisor by condition (C2).

If $A_1.L = A_2.L$, then $D + A_i$ is not a Clifford divisor if and only if $h^0((D + A_i) - (D + A_{3-i} + \Delta)) = h^0(A_i - A_{3-i} - \Delta) > 0$. Clearly this condition cannot hold for both $i = 1$ and 2. So we are done. $\square$

The next result is the crucial one to prove Proposition 5.5.

Proposition 6.9. *If D satisfies (C6) and (C7), then $H^1(A) = 0$ except for the case (E4).*

Proof. The result is trivial if $A = 0$. So we will assume $A > 0$. Also, the result is fulfilled in the case (E3), so we can assume D is not as in (E3) or (E4). In particular, we can use the Lemmas 6.4 and 6.8.

If $h^1(-A) = h^1(A) > 0$, then A cannot be numerically 1-connected, whence there exists a nontrivial effective decomposition $A \sim A_1 + A_2$ such that $A_1.A_2 \leq 0$. By the previous lemma, we have $A_1.A_2 = A_1.\Delta = A_2.\Delta = 0$, and (possibly after interchanging A_1 and A_2) we can assume that $D + A_1$ is a Clifford divisor.

Assume first that D and $D + A_1$ are as in the special case where $\Delta \neq 0$ and $D' := D + A_1$ is a free Clifford divisor of type (E1) or (E2), so the base divisor Δ' of

$$F' := L - D' \sim D + A_2 + \Delta \sim D + A_1 + \Delta'$$

satisfies ${\Delta'}^2 = -4$. Furthermore, by Proposition 6.5, $\Delta'.D = 0$, $A_1^2 = -2$ and $A_1.\Delta' = 2$. Also, since $\Delta'.L = 0$ and $D'L = F'.L$, we must have $A_1.L = A_2.L$. Also note that $A_1 \not\sim A_2$, since $A_1.A_2 = 0$.

Since $(A_1 - A_2).L = 0$, we must by the Hodge index theorem have $A^2 = (A_1 - A_2)^2 \leq -2$. By Lemma 6.7 and the fact that $\Delta \neq 0$, this gives us

$$A_1^2 = -2, \quad A_2^2 = 0, \quad \Delta^2 = -2,$$

We then get from $L^2 = 4c + 4 = (2D + A + \Delta).L = 2D.L + A.L = 2D^2 + 2c + 4 + A.L$, that

$$A.L = 2(c - D^2).$$

Since $A_1.L = A_2.L$, we have

$$A_1.L = A_2.L = c - D^2.$$

So A_2 would induce a Clifford index $\leq L.A_2 - A_2^2 - 2 = c - D^2 - 2 < c$ on the smooth curves in $|L|$, a contradiction.

So we can now use Proposition 6.6 and find a smooth rational curve satisfying one of the two conditions:

(I) $\Gamma.D = 0$, $\Gamma.A_1 = -1$, $\Gamma.(A_2 + \Delta) = 1$,
(II)$\Gamma.D = 1$, $\Gamma.A_1 = -2$, $\Gamma.(A_2 + \Delta) = 0$.

In case (I) we get $\Gamma.A = \Gamma.F_0 - \Gamma.D \geq 0$, whence $\Gamma.A_2 \geq 1$. Since $\Gamma.A_1 = -1$, we have $A_1 - \Gamma > 0$, and we get an effective decomposition $A \sim (A_1 - \Gamma) + (A_2 + \Gamma)$ such that

$$(A_1 - \Gamma).(A_2 + \Gamma) = A_1.A_2 - \Gamma.A_2 + \Gamma.A_1 - \Gamma^2 \leq 0,$$

so by Lemma 6.8, we must have $\Gamma.A_2 = 1$ and $(A_1 - \Gamma).(A_2 + \Gamma) = 0$. Obviously, $D + A_1 - \Gamma$ is a Clifford divisor, and we can now repeat the process with A_1 and A_2 replaced by $A_1 - \Gamma$ and $A_2 + \Gamma$. This will eventually bring us in case (II) after a finite number of steps.

So we can assume that A_1 and A_2 are as in case (II). Again, by $\Gamma.A = \Gamma.F_0 - \Gamma.D \geq -1$, we have $\Gamma.A_2 \geq 1$, whence $\Gamma \neq A_1$ and $A_1 - \Gamma > 0$. Since

$$(A_1 - \Gamma).(A_2 + \Gamma) = A_1.A_2 - \Gamma.A_2 + \Gamma.A_1 - \Gamma^2 \leq -1,$$

we have a contradiction by Lemma 6.8.

This concludes the proof of the proposition. $\square$

We can now prove Proposition 5.5.

By Lemma 6.4, we can assume $A.\Delta = 0$ and $\Delta^2 = -2D.\Delta$.

One easily sees that the base divisor of R must contain Δ, so $h^0(A) = h^0(R) = h^0(A + \Delta)$.

If $A > 0$, we have

$$\begin{aligned} h^0(A) = h^0(A + \Delta) &= \frac{1}{2}A^2 + 2 + \frac{1}{2}\Delta^2 + h^1(R) \\ &= h^0(A) - h^1(A) + D.\Delta + h^1(R), \end{aligned} \tag{6.14}$$

whence $h^1(R) = D.\Delta + h^1(A)$. If we choose D such that it satisfies (C6) and (C7), then $h^1(A) = 0$ by Proposition 6.9.

If $A = 0$, then $R = \Delta$ and D is of one of the types (E0)-(E2) by Lemma 6.7. This concludes the proof of Proposition 5.5.

Note that in the case $A = 0$, we have

$$1 = h^0(R) = \frac{1}{2}R^2 + 2 + h^1(R) = -D.\Delta + 2 + h^1(R), \tag{6.15}$$

whence $h^1(R) = D.\Delta - 1$, as we have already noted.

We now give the proof of Theorem 5.7 in the case (E1), which was left out in the previous chapter. The proof in the case (E2) is similar, and therefore left to the reader.

Proof of Theorem 5.7 in the case (E1). We first show that $< Z_{i,\lambda} > \simeq \mathbf{P}^n$ for $i = 1, 2$ and any λ. (Recall the definition of $Z_{i,\lambda}$ on p.Í39. In particular, $\deg Z_{i,\lambda} = n+1$.) If this were not true, the natural map

$$H^0(L) \longrightarrow H^0(L \otimes \mathcal{O}_{Z_{i,\lambda}})$$

would not be surjective.

Let $Z' \subseteq Z_{i,\lambda}$ be a subscheme of length $n'+1 \leq n+1$, for some integer $n' \geq 1$, such that the map $H^0(L) \to H^0(L \otimes \mathcal{O}_{Z'})$ is not surjective, but such that the map $H^0(L) \to H^0(L \otimes \mathcal{O}_{Z''})$ is surjective for all proper subschemes $Z'' \subsetneqq Z'$.

Since $L^2 = 4c+4 = 4(n+1)$, we have by Proposition 1.13 that there exists an effective divisor B passing through Z' such that $B^2 \geq -2$, $h^1(B) = 0$ and

$$2B^2 \leq B.L \leq B^2 + n' + 1 \leq 2n' + 2.$$

If $B^2 \geq 0$, we would get that B induces a Clifford index $c_B \leq n'-1 \leq c-1$ on the smooth curves in $|L|$, a contradiction.

So $B^2 = -2$, and B is necessarily supported on a union of smooth rational curves, since $h^1(B) = 0$. But $B.L \leq n'-1 \leq n-1 = c-1$ and Z' has length ≥ 2, so B passes through some of the base points of $\{D_\lambda\}$. This contradicts the fact that we have chosen these base points to lie outside of smooth rational curves of degree $\leq c+2$ with respect to L.

To prove the second assertion, we will show that if there is a point $x_0 \in S' - \{x_1, \ldots, x_n, y_1\}$ such that S' has an $(n+2)$-secant n-plane through x_0 and $Z_{1,\lambda}$, then $x_0 = y_2$. By symmetry, this will suffice.

As usual choose any preimage p_0 of x_0 and denote by $X_{1,\lambda}$ the zero-dimensional scheme defined by p_0 and $Z_{1,\lambda}$. We then have that the natural map

$$H^0(L) \longrightarrow H^0(L \otimes \mathcal{O}_{X_{1,\lambda}})$$

is not surjective for any λ.

As usual let Let $X'_{1,\lambda} \subseteq X_{1,\lambda}$ be a subscheme of length $n'_{1,\lambda} + 2 \leq n+2$, for some integer $n'_{1,\lambda} \geq 0$, such that the map $H^0(L) \to H^0(L \otimes \mathcal{O}_{X'_{1,\lambda}})$ is not surjective, but such that the map $H^0(L) \to H^0(L \otimes \mathcal{O}_{X''_{1,\lambda}})$ is surjective for all proper subschemes $X''_{1,\lambda} \subsetneqq X'_{1,\lambda}$.

Since $n = D^2 = c \geq 2$ and $L^2 = 4c+4 = 4(n+1)$, we again have by Proposition 1.13 that there for each λ exists an effective divisor $B_{1,\lambda}$ passing through $X_{1,\lambda}$ and satisfying $B_{1,\lambda}^2 \geq -2$, $h^1(B_{1,\lambda}) = 0$ and the numerical conditions

$$2B_{1,\lambda}^2 \overset{(a)}{\leq} L.B_{1,\lambda} \leq B_{1,\lambda}^2 + n'_{1,\lambda} + 2 \overset{(b)}{\leq} 2n'_{1,\lambda} + 4,$$

with equality in (a) or (b) implying $L \sim 2B_{1,\lambda}$.

Assume first that $B_{1,\lambda}^2 = -2$ for some λ. We then get the same contradiction on the choice of the base points of $\{D_\lambda\}$, since $B_{1,\lambda}.L \leq n'_{1,\lambda} \leq n = c$.

So we must have $B_{1,\lambda}^2 \geq 0$ for all λ. Then $n'_{1,\lambda} = n$, $X'_{1,\lambda} = X_{1,\lambda}$, $L.B_{1,\lambda} = B_{1,\lambda}^2 + n + 2$, and $B_{1,\lambda}$ is a Clifford divisor. The moving part $B'_{1,\lambda}$ of $|B_{1,\lambda}|$ is then a free Clifford divisor, so by condition (C7) we have that either $B'_{1,\lambda} \sim D$ or there exists a free Clifford divisor $P_{1,\lambda}$ such that $B'_{1,\lambda} \leq P_{1,\lambda} < D$ with the last inclusion as described in Proposition 6.5, with the additional property that $|L - P_{1,\lambda}|$ has no fixed divisor, by the conditions (C6) and (C7).

We will show that this latter case cannot occur.

We have that $B_{1,\lambda}$ passes through $X_{1,\lambda}$. Now a (possible) base divisor in $|B_{i,\lambda}|$ cannot pass through any of the points $p_0, \dots, p_n$, since these points lie outside all the rational curves contracted by L. So we must have $B'_{i,\lambda}.D \geq n$.

In addition, by Proposition 6.5 we must have

$$D \sim B'_{1,\lambda} + \gamma_{1,\lambda},$$

for some $\gamma_{1,\lambda} > 0$ satisfying $\gamma_{1,\lambda}^2 = -2$, and $B'_{1,\lambda}.\Delta = 0$. Hence

$$B'_{1,\lambda}.L = B'_{1,\lambda}.(L - \Delta) = 2B'_{1,\lambda}.D \geq 2n,$$

so that

$$B'^{\,2}_{1,\lambda} \geq n - 2 = D^2 - 2.$$

Since $h^1(B'_{1,\lambda}) = h^1(P_{1,\lambda}) = h^1(D) = 0$, we have $h^0(B'_{1,\lambda}) \geq \frac{1}{2}D^2 + 1 \geq h^0(P_{1,\lambda})$, so $B'_{1,\lambda} \sim P_{1,\lambda}$.

Since $|L - P_{1,\lambda}| = |L - B_{1,\lambda}|$ has no fixed divisor, we have $B'_{1,\lambda} \sim B_{1,\lambda}$, so $B_{1,\lambda} < D$ and $B_{1,\lambda}$ is a free Clifford divisor. Since $h^0(B_{1,\lambda} \otimes \mathcal{J}_{X_{1,\lambda}}) > 0$, there must exist an element of $|D|$ of the form $B_{1,\lambda} + A_{1,\lambda}$ passing through $Z_{1,\lambda}$, for $A_{i,\lambda} > 0$. But since there is only one element of $|D|$ passing through $p_1, \dots, p_n, q_{1,\lambda}$, which we called D_λ and which is smooth and irreducible, we have $B_{1,\lambda} = D_\lambda$, a contradiction.

So we must have $B'_{1,\lambda} \sim D$. By Proposition 1.13 either $L - B_{1,\lambda} \geq B_{1,\lambda}$, or both $h^0(B_{i,\lambda} \otimes \mathcal{J}_{X_{1,\lambda}}) \neq 0$ and $h^0((L - B_{1,\lambda}) \otimes \mathcal{J}_{X_{1,\lambda}}) \neq 0$. This gives us the two possibilities:

1. $B_{1,\lambda} \in |D|$,
2. $B_{1,\lambda} \in |D| + \Gamma_{j(\lambda)}$, for $j(\lambda) = 1$ or 2, and there exists an $F_{1,\lambda} \in |D| + \Gamma_{3-j(\lambda)}$ passing through $X_{1,\lambda}$.

In case 1., since there is only one member of $|D|$ containing $p_1, \dots, p_n, q_{1,\lambda}$, which we called D_λ, we have $B_{1,\lambda} = D_\lambda$. But this would mean that $p_0 \in D_\lambda$ for all λ, a contradiction.

In case 2. one easily sees that the only option is $p_0 \in \Gamma_2$, which means that $x_0 = y_2$, as desired. $\square$

The proof of Theorem 5.7 in the case (E2) is similar, and therefore left to the reader.

Since we have seen that the crucial point in proving Propositon 5.5 is to prove that $h^1(A) = 0$, we get the following result (by checking that the proof of Theorem 5.7 goes through):

Lemma 6.10. *Let D be a free Clifford divisor, not of type (E1) or (E2). If $h^1(A) = 0$, then D is perfect.*

7

Projective models in smooth scrolls

Let D be a free Clifford divisor on a non-Clifford general polarized $K3$ surface S. Assume that $\mathcal{T} = \mathcal{T}(c, D) = \mathcal{T}(c, D, \{D_\lambda\})$ is smooth. This is equivalent to the conditions $D^2 = 0$ and $\mathcal{R}_{L,D} = \emptyset$ when D is perfect. In any case these two conditions are necessary to have $\mathcal{T}$ smooth, so $|D|$ has projective dimension 1 and the pencil D_λ is uniquely determined. We recall that $\varphi_L(S)$ is denoted by S'.

Since $\mathcal{T}$ is smooth, it can be identified with the $\mathbf{P}^1$-bundle $\mathbf{P}(\mathcal{E})$, where $\mathcal{E} = \mathcal{O}_{\mathbf{P}^1}(e_1) \oplus \mathcal{O}_{\mathbf{P}^1}(e_2) \oplus \cdots \oplus \mathcal{O}_{\mathbf{P}^1}(e_{c+2})$, and $(e_1, e_2, \ldots, e_{c+2})$ is the type of the scroll.

We will construct a resolution of the structure sheaf $\mathcal{O}_{S'}$ as an $\mathcal{O}_{\mathcal{T}}$-module.

The contents in this chapter will be very similar to that in [Sc], where canonical curves of genus g are treated. This is quite natural, since a general hyperplane section of S' is indeed such a canonical curve.

The following are well-known facts about $\mathcal{T}$ in $\mathbf{P}^g$ (see [Har] and [E-H]):

(1) $\deg \mathcal{T} = g - c - 1$.
(2) $\dim \mathcal{T} = c + 2$.
(3) The Chow ring of $\mathcal{T}$ is $\mathbf{Z}[\mathcal{H}, \mathcal{F}]/(\mathcal{F}^2, \mathcal{H}^{c+3}, \mathcal{H}^{c+2}\mathcal{F}, \mathcal{H}^{c+2} - (g-c-1)\mathcal{H}^{c+1}\mathcal{F})$, where $\mathcal{H}$ is the hyperplane section, and $\mathcal{F}$ is the class of the ruling.
(4) The canonical class of $\mathcal{T}$ is $-(c+2)\mathcal{H} + (g-c-3)\mathcal{F}$.
(5) The class of S' in the Chow ring of $\mathcal{T}$ is $(c+2)\mathcal{H}^c + (c^2 + 3c - cg)\mathcal{H}^{c-1}\mathcal{F}$.

We will need the Betti-numbers of the $\varphi_L(D_\lambda)$ in $\mathbf{P}^{c+1}$. These can be found also when $\mathcal{T}$ is singular, and will be needed in this case later on.

Lemma 7.1. *Let (S, L) be a polarized $K3$ surface of genus g and of non-general Clifford index $c > 0$. Let D be a free Clifford divisor satisfying $D^2 = 0$. For $c \geq 2$, all the $\varphi_L(D_\lambda)$ in $\mathbf{P}^{c+1}$ have minimal resolutions*

$$\begin{aligned} 0 &\longrightarrow \mathcal{O}_{\mathbf{P}^{c+1}}(-(c+2)) \longrightarrow \mathcal{O}_{\mathbf{P}^{c+1}}(-c)^{\beta_{c-1}} \longrightarrow \mathcal{O}_{\mathbf{P}^{c+1}}(-(c-1))^{\beta_{c-2}} \longrightarrow \\ &\cdots \longrightarrow \mathcal{O}_{\mathbf{P}^{c+1}}(-3)^{\beta_2} \longrightarrow \mathcal{O}_{\mathbf{P}^{c+1}}(-2)^{\beta_1} \to \mathcal{O}_{\mathbf{P}^{c+1}} \to \mathcal{O}_{\varphi_L(D_\lambda)} \longrightarrow 0, \end{aligned}$$

T. Johnsen and A.L. Knutsen: LNM 1842, pp. 59–61, 2004.

where

$$\beta_i = i\binom{c+1}{i+1} - \binom{c}{i-1}.$$

For $c = 1$ all the $\varphi_L(D_\lambda)$ in $\mathbf{P}^2$ have the resolution

$$0 \longrightarrow \mathcal{O}_{\mathbf{P}^2}(-3) \longrightarrow \mathcal{O}_{\mathbf{P}^2} \longrightarrow \mathcal{O}_{\varphi_L(D_\lambda)} \longrightarrow 0.$$

Proof. Pick any $D'_\lambda := \varphi_L(D_\lambda)$. We will show in Proposition 8.8 below that any such D'_λ is arithmetically normal, whence projectively Cohen-Macaulay, since the D_λ have pure dimension one. Then its Betti-numbers (see Section 8.3 below for the definition) are equal to those of a general hyperplane section of it. It is sufficient that the linear term defining the hyperplane is not a zero divisor in its coordinate ring R_λ. This is essentially [Na, Theorem 27.1].

Now choose a sufficiently general hyperplane H_λ in $\mathbf{P}^g$ so that $C_\lambda := H_\lambda \cap S'$ is a smooth canonical curve, H_λ does not contain any of the linear spaces $\overline{D_\lambda}$, and the hyperplane section $A_\lambda := H_\lambda \cap D_\lambda$ is not a zero divisor of R_λ.

We can identify C_λ with an element in $|L|$, and by abuse of notation write $\mathcal{O}_{C_\lambda}(A_\lambda) = \mathcal{O}_{C_\lambda}(D_\lambda) = \mathcal{O}_{C_\lambda}(D)$. This linear system is complete and base point free (in fact it is a pencil computing the gonality) of degree $c+2$ on C_λ. By [Sc, Lemma p.119] (where there is a misprint) and [Sc, Proposition 4.3] the zero-dimensional scheme A_λ then has the Betti-numbers $\beta_{i,i+1} = \beta_i = i\binom{c+1}{i+1} - \binom{c}{i-1}$.

In particular, these numbers are independent of λ. $\square$

The following result is analogous to [Sc, Corollary (4.4)].

Proposition 7.2. *Let S be a polarized $K3$ surface of non-general Clifford-index $c > 0$, whose associated scroll $\mathcal{T}$ as above is smooth.*

(a) $\mathcal{O}_{S'}$ has a unique $\mathcal{O}_\mathcal{T}$-resolution F_* (up to isomorphism). If $c = 1$, the resolution is:

$$0 \longrightarrow \mathcal{O}_\mathcal{T}(-3\mathcal{H} + (g-4)\mathcal{F}) \longrightarrow \mathcal{O}_\mathcal{T} \longrightarrow \mathcal{O}_{S'} \longrightarrow 0,$$

If $c \geq 2$, the resolution is of the following type:

$$0 \longrightarrow \mathcal{O}_\mathcal{T}(-(c+2)\mathcal{H} + (g-c-3)\mathcal{F}) \longrightarrow \oplus_{k=1}^{\beta_{c-1}} \mathcal{O}_\mathcal{T}(-c\mathcal{H} + b_{c-1}^k\mathcal{F}) \longrightarrow$$
$$\cdots \longrightarrow \oplus_{k=1}^{\beta_1} \mathcal{O}_\mathcal{T}(-2\mathcal{H} + b_1^k\mathcal{F}) \longrightarrow \mathcal{O}_\mathcal{T} \longrightarrow \mathcal{O}_{S'} \longrightarrow 0,$$

where $\beta_i = i\binom{c+1}{i+1} - \binom{c}{i-1}$.

(b) F_* is self-dual: $\mathcal{H}om(F_*, \mathcal{O}_\mathcal{T}(-(c+2)\mathcal{H} + (g-c-3)\mathcal{F})) \simeq F_*$.

(c) If all $b_i^k \geq -1$, then an iterated mapping cone

$$[[\mathcal{C}^{g-c-3}(-(c+2)) \longrightarrow \oplus_{k=1}^{\beta_{c-1}} \mathcal{C}^{b_c^k}(-c)]\ldots] \longrightarrow \mathcal{C}^0$$

is a (not necessarily minimal) resolution of $\mathcal{O}_{S'}$ as an $\mathcal{O}_{\mathbf{P}^g}$-module.

(d) The b_i^k satisfy the following polynomial equation in n if $c \geq 2$:

$$\binom{n+c+1}{c+1}(\frac{n(g-c-1)}{c+2}+1) - n^2(g-1) - 2 =$$

$$\sum_{i=1}^{c-1}((-1)^{i+1}\binom{n-i+c}{c+1}(\frac{((n-i-1)(g-c-1)+(c+2))\beta_i}{c+2} + \sum_{k=1}^{\beta_i} b_i^k) +$$

$$(-1)^{c+1}\binom{n-1}{c+1}(\frac{(n-c-2)(g-c-1)}{c+2} + g-c-2).$$

Proof. We start by proving (a). We have $\overline{D_\lambda} \simeq \mathbf{P}^{c+1}$ by (3.7). The $\varphi_L(D_\lambda)$ have Betti-numbers $\beta_{i,j}^\lambda = \dim(Tor_i^{R_\lambda}(R,k)_j)$, where R is the homogeneous coordinate ring of $\mathbf{P}^{c+1}$, and R_λ the coordinate ring R/I_λ of $\varphi_L(D_\lambda)$. Following [Sc], for $c \geq 2$ it is enough to prove:

(1) For fixed i, j the $\beta_{i,j}^\lambda$ are the same for all λ.
(2) If $c \geq 2$, then $\beta_{i,j}^\lambda = 0$, unless $j = i+1$ and $i \leq c-1$, or $(i,j) = (c, c+2)$.
(3) The common value $\beta_{i,i+1} = \beta_{i,i+1}^\lambda$ is $\beta_i = i\binom{c+1}{i+1} - \binom{c}{i-1}$ for $i \leq c-1$, and $\beta_{c,c+2} = 1$.

This follows immediately from the lemma above.

The easier case $c = 1$ is dealt with in an analogous manner.

The proof of (b) is almost identical to that of [Sc, Corollary 4.4(ii)]. In our case we have $\mathcal{E}xt^i_{\mathcal{T}}(\mathcal{O}_{S'}, \omega_{\mathcal{T}}) = \omega_{S'}$ if $i = c$, and zero otherwise, $\omega_{S'} = \mathcal{O}_{S'}$, and $\omega_{\mathcal{T}} = \mathcal{O}_{\mathcal{T}}(-(c+1)\mathcal{H} + (g-c-3)\mathcal{F})$.

The proof of (c) is identical to that of [Sc, Corollary 4.4(iii)].

Denote the term i places to the left of $\mathcal{O}_{\mathcal{T}}$ in the resolution F_* by F_i. The proof of (d) then follows from the identity

$$\chi(\mathcal{O}_{\mathcal{T}}(n\mathcal{H})) - \chi(\mathcal{O}_{S'}(n\mathcal{H})) = \sum_i (-1)^i \chi(F_i(n\mathcal{H})).$$

The contribution from the F_c-term is written out separately. Moreover it is clear that for all large n, we have $\chi(F_i(n\mathcal{H})) = h^0(F_i(n\mathcal{H})$, for all i, and $\chi(\mathcal{O}_{\mathcal{T}}(n\mathcal{H})) = h^0(\mathcal{O}_{\mathcal{T}}(n\mathcal{H}))$ since $\mathcal{H}$ is (very) ample on $\mathcal{T}$. Then one uses the following well-known fact for $a \geq 0$:

$$h^0(\mathbf{P}(\mathcal{E}), \mathcal{O}_{\mathbf{P}(\mathcal{E})}(a\mathcal{H} + b\mathcal{F})) = h^0(\mathbf{P}^1, \mathrm{Sym}\,^a(\mathcal{E}) \otimes \mathcal{O}_{\mathbf{P}^1}(b)). \tag{7.1}$$

□

Remark 7.3. Part (d) of the proposition only gives us the sums of the b_i^k for each fixed i. The values $n = 2, 3, \ldots, c$ give enough equations to determine these sums. The duality of part (c) gives $\beta_i = \beta_{c-i}$, for $i = 1, \ldots, c-1$, and $i \neq c/2$, and after a possible renumeration of the b_i^k, for $k = 1, \ldots, \beta_i$, we also have $b_{c-i}^k = g-c-3-b_i^k$ for these k. In particular this enables us to identify the sums of the b_i^k with those of the $g-c-3-b_{c-i}^k$. To obtain more information about the individual b_i^k a more refined study is necessary.

8

Projective models in singular scrolls

Let D be a free Clifford divisor on a non-Clifford general polarized $K3$ surface (S, L). In this chapter we will make a thourough study of the case where the scroll $\mathcal{T} = \mathcal{T}(c, D, \{D_\lambda\})$ is singular. A useful tool will be the blowing-up $f : \tilde{S} \to S$ of S at the D^2 base points of the pencil $\{D_\lambda\}$. We study this blowing-up in Section 8.1 (If $D^2 = 0$, then $\tilde{S} = S$). Moreover we will show in Section 8.1 that the projective model $S'' := \varphi_H(\tilde{S})$ is normal, where $H := f^*L + f^*D - E$ is base point free and E is the exceptional divisor.

In Section 8.2 we show that the pencil $|f^*D - E|$ defines a smooth rational normal scroll $\mathcal{T}_0$ that contains S'' and is a desingularization of $\mathcal{T}$. The real interest in the varieties $\mathcal{T}_0, \tilde{S}, S''$ is of course to use them to understand S' and $\mathcal{T}$. In Proposition 8.7 we describe the class of S'' in the Chow ring of $\mathcal{T}_0$, and associated to this, the class of S' in the Chow group of $\mathcal{T}$.

In Section 8.3 we use results from [Gr] to study the resolutions of the $\varphi_L(D_\lambda)$ in their linear spans in $\mathbf{P}^g$, for each D_λ (from the chosen subpencil of $|D|$). We show that all $\varphi_L(D_\lambda)$ are arithmetically normal in the spaces they span, and in Proposition 8.17 we find that for convenient pencils and low values of D^2 all members of the pencil have the same Betti-numbers (appearing in their minimal resolutions). The results in Proposition 8.17 are sufficient to calculate these Betti-numbers explicitly in all cases we wish to study.

In Section 8.4 we start with describing resolutions of the $\varphi_L(D_\lambda)$ for some concrete values of c, D^2. We then in general describe how one can obtain resolutions of the S'' in their associated scrolls $\mathcal{T}_0$. Such resolutions are given in Proposition 8.23. Moreover, in Proposition 8.29 we give sufficient conditions under which we can push down the resolution to one of $\varphi_L(S)$ in $\mathcal{T}$. Here we use results from [Sc].

In Section 8.5 we recall the definition of socalled rolling factors coordinates, which we apply both here and later to give more details about resolutions of S' in $\mathcal{T}$ in various cases.

In Section 8.6 we apply much of our technical machinery to study more examples of such resolutions, with special emphasis on the right ends of these.

T. Johnsen and A.L. Knutsen: LNM 1842, pp. 63–98, 2004.

The examples enable us to obtain a detailed picture for $g \leq 10$, and even a fairly complete one, when it comes to description of generators of the ideal of S' in $\mathcal{T}$.

Remark 8.1. As seen above, the scroll $\mathcal{T} = \mathcal{T}(c, D, \{D_\lambda\})$ is singular if $D^2 > 0$ or the set $\mathcal{R}_{L,D}$ is non-empty. Moreover $\mathcal{T}$ is singular if and only if one of these two conditions holds, if D is perfect.

We will always assume $c > 0$, so that $\varphi_L : S \to S'$ is birational.

The type $(e_1, \dots, e_d)$ of the scroll, where $d = \frac{1}{2}D^2 + c + 2$, is such that the last r of the e_i are zero, where r is defined as in equation (5.1) and can be computed as in equation (5.2).

As we have seen, when D is perfect we have

$$r = \begin{cases} D^2 - 1 & \text{if } D \text{ is of type (Q)}, \\ D^2 + D.\Delta - 1 & \text{if } D \text{ is of one of the types (E0)-(E2)}, \\ D^2 + D.\Delta & \text{otherwise.} \end{cases} \tag{8.1}$$

We will however not assume that D is perfect, unless explicitly stated.

8.1 Blowing up S

Let $n := D^2$ and denote by $p_1, \dots, p_n$ the n base points of the pencil $\{D_\lambda\}$. Let

$$\tilde{S} \xrightarrow{f} S$$

be the blow up of S at $p_1, \dots, p_n$. Denote by E_i the exceptional line over p_i and let

$$E := \sum_{i=1}^{n} E_i$$

denote the exceptional divisor. Define

$$H := f^*L + f^*D - E.$$

The first observation is:

Lemma 8.2. *H is generated by its global sections, $h^1(H) = 0$ and φ_H is birational; in fact φ_H is an isomorphism outside of finitely many contracted smooth rational (-2)-curves.*

Moreover, a smooth rational curve γ is contracted by H if and only if $\gamma = f^*\Gamma$, for some smooth rational curve Γ on S such that $\Gamma.L = \Gamma.D = 0$.

Proof. Since $H - E \sim (f^*L - E) + (f^*D - E)$ is clearly nef and $(H - E)^2 \geq 10$, we have $h^1(H) = 0$. Furthermore, since $|f^*D - E|$ is a base point free pencil and f^*L is base point free, H is base point free as well.

The morphism given by $|f^*L|$ is clearly an isomorphism outside of the n exceptional curves and the strict transforms of the finitely many smooth rational curves on S which are contracted by $|L|$. By our choice of pencil (see 3.3), these curves do not intersect the n blown up points.

Since $E_i.H = 1$ for all i, every exceptional curve E_i is mapped by φ_H isomorphically to a line, so φ_H is an isomorphism along the exceptional curves. Moreover if $\gamma = f^*\Gamma$ for some smooth rational curve Γ on S such that $\Gamma.L = 0$ and $\Gamma.D > 0$ then $\Gamma.D = 1$ by Lemma 6.3(c), so γ is mapped isomorphically to a line by φ_H and φ_H is an isomorphism along these curves as well.

Hence φ_H is an isomorphism outside of finitely many contracted smooth rational (-2)-curves, which are precisely the ones of the form $f^*\Gamma$, for some smooth rational curve Γ on S such that $\Gamma.L = \Gamma.D = 0$. $\square$

We have $h^0(H) = \frac{1}{2}H.(H-E)+2 = \frac{1}{2}L^2+\frac{1}{2}D^2+c+4 = g+\frac{1}{2}D^2+c+2 = g+d+1$. Denote by S'' the surface $\varphi_H(\tilde{S})$ in $\mathbf{P}^{g+d}$.

One easily obtains $\deg S'' = 2g + 2c + 2 + 2D^2$.

Proposition 8.3. *The surface S'' is normal, $p_a(S'') = 1$, and $K_{S''} \simeq \mathcal{O}_{S''}(E')$, where E' is the sum of D^2 lines that are (-1)-curves on S''.*

Proof. The two last assertions are immediate consequences of S'' being normal, by [Ar].

Consider the blow-up $f : \tilde{S} \to S$ described above.

Denote by $\mathcal{E}_H$ the set of irreducible curves $\tilde{\Gamma}$ on $\tilde{S}$ such that $\tilde{\Gamma}.H = 0$. From the Hodge Index theorem it follows that such a curve has negative self-intersection. Moreover, by Lemma 8.2

$$\tilde{\Gamma} = f^*\Gamma,$$

for some smooth rational curve Γ on S such that $\Gamma.L = \Gamma.D = 0$. Thus we can write

$$\mathcal{E}_H = f^*(\mathcal{E}_L - \mathcal{R}_{L,D}).$$

Now let $\tilde{\delta}$ be the fundamental cycle of a connected component of $\mathcal{E}_H$, p the image of $\tilde{\delta}$ on S'' and U the inverse image of an affine open neighborhood of p. To prove the normality of p it will be sufficient to prove the surjectivity of

$$H^0(U, \mathcal{O}_U(H - \tilde{\delta})) \longrightarrow H^0(\tilde{\delta}, \mathcal{O}_{\tilde{\delta}}(H - \tilde{\delta})),$$

hence of

$$H^0(\tilde{S}, \mathcal{O}_{\tilde{S}}(H - \tilde{\delta})) \longrightarrow H^0(\tilde{\delta}, \mathcal{O}_{\tilde{\delta}}(H - \tilde{\delta})).$$

To show the latter, it will suffice to show

$$H^1(\tilde{S}, \mathcal{O}_{\tilde{S}}(H - 2\tilde{\delta})) = 0.$$

By the degeneration of the Leray spectral sequence

$$0 \longrightarrow H^1(S, f_*(H - 2\tilde{\delta})) \longrightarrow H^1(\tilde{S}, H - 2\tilde{\delta}) \longrightarrow H^0(S, R^1 f_*(H - 2\tilde{\delta}))$$

it will suffice to show that

$$h^1(S, f_*(H - 2\tilde{\delta})) = h^0(S, R^1 f_*(H - 2\tilde{\delta})) = 0.$$

Denote by δ the divisor on S such that $f^*\delta = \tilde{\delta}$. Then δ is connected and $\delta^2 = -2$ (δ is in fact a fundamental cycle for a connected component of $\mathcal{E}_L$ minus a curve Γ that is a tail of δ and is such that $\Gamma.D = 1$. The fact that $\delta^2 = -2$ can be checked by inspection for each of the five platonic configurations [Ar]). We then have

$$f_*(H - 2\tilde{\delta}) = (L + D - 2\delta) \otimes \mathcal{J}_Z,$$

where Z is the zero-dimensional scheme corresponding to the n blown up points, and

$$R^1 f_*(H - 2\tilde{\delta}) = R^1 f_*(-E) \otimes (L + D - 2\delta).$$

Since $f_*\mathcal{O}_E \simeq \mathcal{O}_Z$, we have $R^1 f_*(-E) = 0$, whence we are reduced to proving the vanishing of $H^1((L + D - 2\delta) \otimes \mathcal{J}_Z)$. This will be proved in Lemma 8.4 below. □

Lemma 8.4. *With the notation as above,* $H^1((L + D - 2\delta) \otimes \mathcal{J}_Z) = 0$.

Proof. We will first need the following fact:

$$h^1(L + D - 2\delta) = 0.$$

The proof for this is rather long and tedious, but does not involve any new ideas and is similar in principle to the proof of [Co, Lemma 5.3.5]. We therefore leave it to the reader.

Note that if $D^2 = 0$, then $Z = \emptyset$, and we are done. So we will from now on assume that $n = D^2 > 0$.

Because of the vanishing of $H^1(L+D-2\delta)$, the vanishing of $H^1((L+D-2\delta) \otimes \mathcal{J}_Z)$ is equivalent to the surjectivity of the map

$$H^0(L + D - 2\delta) \longrightarrow H^0((L + D - 2\delta) \otimes \mathcal{O}_Z).$$

Assume, to get a contradiction, that this map is not surjective. Let $Z' \subseteq Z$ be a subscheme of length $l + 1 \leq n = D^2$, for some integer $l \geq 1$, such that $H^0(L + D - 2\delta) \longrightarrow H^0((L + D - 2\delta) \otimes \mathcal{O}_{Z'})$ is not surjective, but $H^0(L+D-2\delta) \longrightarrow H^0((L+D-2\delta)\otimes \mathcal{O}_{Z''})$ is for all proper subschemes Z''.

Since $(L + D - 2\delta)^2 > 4l + 4$ and $h^1(L + D - 2\delta) = 0$, we get by Remark 1.14 that there is an effective decomposition $L + D - 2\delta \sim A + B$ such that $A > B$, $A.B \leq l + 1$, $h^1(B) = 0$, $B^2 \geq -2$ and B passes through Z'.

If $B^2 = -2$ (so that B is necessarily supported on a union of smooth rational curves), then we use the fact that we have chosen Z to lie outside of any rational curve Γ such that $\Gamma.L \leq c + 2$ by (3.3) and

$$L.B \leq (L + D).B \leq l - 1 + 2\delta.B \leq D^2 - 2 + 2\delta.B \leq c + 2\delta.B,$$

to conclude that we must have $\delta.B \geq 2$. Hence $(\delta + B)^2 \geq 0$.

This yields that we in all cases have

$$h^0(\delta + B) \geq 2.$$

We now want to show that also

$$h^0(\delta + A - D) \geq 2.$$

We can write

$$F \sim A + B + 2\delta - 2D \sim (A + \delta - D) + (B + \delta - D) := F_1 + F_2.$$

This is not necessarily an effective decomposition, but we have $F_1 > F_2$, since $A > B$.

We can easily calculate

$$F_1.F_2 = A.B - D^2 - c \leq -c < 0,$$

and since $F^2 = {F_1}^2 + {F_2}^2 + 2F_1.F_2 \geq D^2 \geq 2$, we must have ${F_1}^2 \geq 2$ or ${F_2}^2 \geq 2$.

If ${F_1}^2 \geq 2$, then either $h^0(F_1) \geq 2$ or $h^0(-F_1) \geq 2$ by Riemann-Roch. Since $L.F_1 = L.A - L.D > \frac{1}{2}(L^2 + L.D) - L.D = \frac{1}{2}(c + 2 + F^2) > 0$, we must have $h^0(F_1) \geq 2$, and we are done.

If ${F_2}^2 \geq 2$, then either $h^0(F_2) \geq 2$ or $h^0(-F_2) \geq 2$. In the first case, we get $h^0(F_1) \geq h^0(F_2) \geq 2$. In the second, we get $F_1 \sim F - F_2 > F$, since $-F_2$ is effective, whence $h^0(F_1) \geq h^0(F) \geq 2$ again.

So we have an effective decomposition of L as

$$L \sim (B + \delta) + (A + \delta - D),$$

such that both $h^0(B + \delta)$ and $h^0(A + \delta - D) \geq 2$ and such that

$$(B + \delta).(A + \delta - D) = A.B - D.B + 2 \leq l - D.B + 3.$$

Since $l + 1 \leq D^2 \leq c + 2$, and $D.B \geq 2$, since D is base point free and $h^0(B + \delta) \geq 2$, we must have $D.B = 2$ and $l + 1 = n = D^2 = c + 2$. But since B passes through Z, we must have $D.B \geq n$, whence the contradiction $c = 0$.

This concludes the proof of the lemma and hence of Proposition 8.3. □

8.2 The smooth scroll $\mathcal{T}_0$

Define the following line bundle on $\tilde{S}$:

$$\tilde{D} := f^*D - E.$$

The members of $|\tilde{D}|$ are in one-to-one correspondence with the members of the pencil $\{D_\lambda\}$. One computes $\tilde{D}^2 = 0$, so $|\tilde{D}|$ is a pencil of disjoint members. Furthermore

$$h^0(H - \tilde{D}) = h^0(f^*L) > 2,$$

so $|\tilde{D}|$ defines a rational normal scroll $\mathcal{T}_0$ containing S''.

Proposition 8.5. *$\mathcal{T}_0$ has dimension d and degree $g+1$ and is smooth of type $(e_1+1,\dots,e_d+1)$.*

Proof. The two first assertions are easily checked.

We have to calculate the numbers $h^0(\tilde{S},H-i\tilde{D}) = h^0(\tilde{S},f^*(L-(i-1)D)+(i-1)E)$ for all $i \geq 0$.

One easily sees that $(i-1)E$ is a fixed divisor in $|f^*(L-(i-1)D)+(i-1)E|$ for all $i \geq 1$, so we get for all $i \geq 1$:

$$\begin{aligned} h^0(\tilde{S},f^*(L-(i-1)D+(i-1)E) &= h^0(\tilde{S},f^*(L-(i-1)D) \\ &= h^0(S,L-(i-1)D). \end{aligned} \tag{8.2}$$

We also have

$$h^0(H)-h^0(H-\tilde{D}) = d. \tag{8.3}$$

Defining $d'_i := h^0(\tilde{S},H-i\tilde{D})-h^0(\tilde{S},H-(i+1)\tilde{D})$, we get by combining (8.2) and (8.3) that

$$d'_0 = d_0 \quad \text{and} \quad d'_i = d_{i-1} \quad \text{for} \quad i \geq 1.$$

It follows immediately that the type of $\mathcal{T}_0$ is as claimed. $\square$

Since $\mathcal{T}_0$ is smooth, we have $\mathcal{T}_0 \simeq \mathbf{P}(\mathcal{E})$, where $\mathcal{E} = \oplus_{i=1}^d \mathcal{O}_{\mathbf{P}^1}(e_i+1)$. Also, we have the maps

$$\begin{tikzcd} \mathbf{P}(\mathcal{E}) \arrow[r, "j"] \arrow[d, "\pi"] & \mathcal{T}_0 \subseteq \mathbf{P}^{g+d} \\ \mathbf{P}^1 & \end{tikzcd}$$

where j is an isomorphism. Then the Picard group of $\mathbf{P}(\mathcal{E})$ satisfies

$$\text{Pic } \mathbf{P}(\mathcal{E}) \simeq \mathbf{Z}\mathcal{H}_0 \oplus \mathbf{Z}\mathcal{F},$$

where $\mathcal{H}_0 := j^*\mathcal{O}_{\mathbf{P}^{g+d}}(1)$ and $\mathcal{F} := \pi^*\mathcal{O}_{\mathbf{P}^1}(1)$.

Furthermore, the Chow ring of $\mathbf{P}(\mathcal{E})$ is

$$\mathbf{Z}[\mathcal{H}_0,\mathcal{F}]/(\mathcal{F}^2,\mathcal{H}_0^{s+2},\mathcal{H}_0^{s+1}\mathcal{F},\mathcal{H}_0^{s+1}-(g+1)\mathcal{H}_0^s\mathcal{F}), \tag{8.4}$$

where we set $s := c+1+\frac{1}{2}D^2$.

Consider now the morphism i given by the base point free line bundle $\mathcal{H} := \mathcal{H}_0 - \mathcal{F}$, where $\mathcal{H}_0 = \mathcal{H}+\mathcal{F}$:

$$i : \mathbf{P}(\mathcal{E}) \longrightarrow \mathbf{P}^g.$$

One easily sees that i maps $\mathbf{P}(\mathcal{E})$ onto a rational normal scroll of dimension d and type $(e_1,\dots,e_d)$, whence isomorphic to $\mathcal{T}$. So we can assume that i maps $\mathbf{P}(\mathcal{E})$ onto $\mathcal{T}$. By abuse of notation we write

$$i : \mathcal{T}_0 \longrightarrow \mathcal{T},$$

and this is a rational resolution of singularities of $\mathcal{T}$ (in the sense that $\mathcal{T}_0$ is smooth and $R^1 i_* \mathcal{O}_{\mathcal{T}_0} = 0$). Furthermore one easily sees that by construction i restricts to a map

$$g : S'' \longrightarrow S'$$

which is a resolution of some singularities of S' (precisely the singularities of S' arising from the contractions of rational curves across the fibers in S, i.e. the curves in $\mathcal{R}_{L,D}$) and a blow up at the images of the base points of $\{D_\lambda\}$.

We get the following commutative diagram:

$$\begin{tikzcd}
\mathbf{P}(\mathcal{E}) \arrow[r, "j"] & \mathcal{T}_0 \arrow[r, "i"] & \mathcal{T} \arrow[r] & \mathbf{P}^g \\
& S'' \arrow[u] \arrow[r, "g"] & S' \arrow[u] & \\
& \tilde{S} \arrow[u, "\varphi_H"] \arrow[r, "f"] & S. \arrow[u, "\varphi_L"'] &
\end{tikzcd}$$

By construction, one has $g \circ \varphi_H = \varphi_{f^*L}$.

Proposition 8.6. *Let $\mathcal{J}_{S''/\mathcal{T}_0}$ denote the ideal sheaf of S'' in $\mathcal{T}_0$ and $\mathcal{J}_{S'/\mathcal{T}}$ the ideal sheaf of S' in $\mathcal{T}$.*

We have $\mathcal{J}_{S'/\mathcal{T}} = i_* \mathcal{J}_{S''/\mathcal{T}_0}$.

Proof. This follows since $i_* \mathcal{O}_{\mathcal{T}_0} = \mathcal{O}_\mathcal{T}$ and $i_* \mathcal{O}_{S''} = \mathcal{O}_S$. The latter fact is a consequence of g being a birational map of normal surfaces. $\square$

We recall that the Chow ring of $\mathcal{T}_0$ is given by (8.4). Define $\mathcal{H}_\mathcal{T}$ and $\mathcal{F}_\mathcal{T}$ to be the push-down of cycles by i of $\mathcal{H}$ and $\mathcal{F}$ respectively.

We have the following description:

Proposition 8.7. *(a) The class of S'' in the Chow ring of $\mathcal{T}_0$ is*

$$(D^2 + c + 2)\mathcal{H}_0^{d-2} + (c - cg - D^2(g-1))\mathcal{H}_0^{d-3}\mathcal{F}.$$

(b) The class of S' in the Chow group of $\mathcal{T}$ is

$$(D^2 + c + 2)(\mathcal{H}_\mathcal{T})^{d-2} + (D^2(d-1-g) - 4 - cg - c + cd + 2d)(\mathcal{H}_\mathcal{T})^{d-3}\mathcal{F}_\mathcal{T}.$$

Proof. The class of S'' is of the type $m\mathcal{H}_0^{d-2} + n\mathcal{H}_0^{d-3}\mathcal{F}$, for two integers m and n. To determine m and n one has the equations $S''\mathcal{H}_0^2 = \deg S'' = 2g + 2c + 2 + 2D^2$ and $S''\mathcal{H}_0\mathcal{F} = \deg(\varphi_L(D)) = c + 2 + D^2$.

Statement (b) is an immediate consequence of i being birational by using the cap product map $A^*(\mathbf{P}^g) \otimes A_*(\mathcal{T}) \to A_*(\mathcal{T})$. $\square$

8.3 Techniques for finding Betti-numbers of the $\varphi_L(D_\lambda)$

We would like to study the resolution of S'' in $\mathbf{P}(\mathcal{E}) \simeq \mathcal{T}_0$. We say that S'' has constant Betti-numbers $\beta_{ij} = \beta_{ij}(\lambda)$ over $\mathbf{P}^1$ if the one-dimensional schemes obtained by intersecting S'' by the linear spaces F_λ in the pencil of fibres of $\mathcal{T}_0$ have Betti-numbers in $\mathbf{P}^{c+1+\frac{1}{2}D^2}$ that are independent of λ. By [Sc], if S'' has constant Betti-numbers over $\mathbf{P}^1$, we can (at least in principle) find a resolution of $\mathcal{O}_{S''}$ by free $\mathcal{O}_{\mathbf{P}(\mathcal{E})}$-modules which restricts to the minimal resolution of $\mathcal{O}_{S''_\lambda}$ on each fiber $\mathbf{P}(\mathcal{E})_\lambda \simeq \mathbf{P}^{c+1+\frac{1}{2}D^2}$.

Clearly, since the map i is the identity on each fiber, the Betti-numbers of S''_λ are the same as the Betti-numbers of $\varphi_L(D_\lambda)$.

Recall that a projective scheme V is called arithmetically normal if the natural map

$$S_k H^0(V, \mathcal{O}_V(1)) \longrightarrow H^0(V, \mathcal{O}_V(k))$$

is surjective for all $k \geq 0$.

We start by showing that the $\varphi_L(D_\lambda)$ are all arithmetically normal.

Proposition 8.8. *All the $\varphi_L(D_\lambda)$ are arithmetically normal in $\overline{D_\lambda} = \mathbf{P}^{c+1+\frac{1}{2}D^2}$.*

Proof. We can easily show that

$$h^1(\mathcal{O}_S(qL - D)) = 0 \text{ for all } q. \tag{8.5}$$

Furthermore, by [SD, Thm. 6.1], we have that

$$S_k H^0(S, L) \longrightarrow H^0(S, kL) \text{ is surjective for all } k \geq 0. \tag{8.6}$$

We have a commutative diagram

$$\begin{array}{ccc} H^0(\mathcal{O}_{\mathbf{P}^g}(q)) & \longrightarrow & H^0(\mathcal{O}_{\overline{D_\lambda}}(qL)) \\ \downarrow{\scriptstyle \alpha_1} & & \downarrow{\scriptstyle \alpha_3} \\ H^0(\mathcal{O}_S(qL)) & \xrightarrow{\alpha_2} & H^0(\mathcal{O}_{D_\lambda}(qL)). \end{array}$$

Now α_2 is surjective by (8.5) and α_1 is surjective by (8.6). Hence α_3 is surjective and $\varphi_L(D_\lambda)$ is arithmetically normal. □

For each $\lambda \in \mathbf{P}^1$ define

$$B^\lambda := \oplus_{q \in \mathbf{Z}} H^0(D_\lambda, qL) \quad \text{and} \quad V^\lambda := H^0(D_\lambda, L).$$

The symmetric algebra $S(V^\lambda)$ of V^λ satisfies

$$S(V^\lambda) \simeq R_\lambda,$$

where R_λ is the homogeneous coordinate ring of $\mathbf{P}(H^0((D_\lambda, L) \simeq \mathbf{P}^{c+1+\frac{1}{2}D^2}$, and B^λ is a graded R_λ-module. Since all the R_λ are isomorphic, we will sometimes suppress the λ, hoping to cause no confusion.

We have the *Koszul complex*

$$\cdots \longrightarrow \wedge^{i+1} V^\lambda \otimes B^\lambda_{j-1} \xrightarrow{d^\lambda_{i+1,j-1}} \wedge^i V^\lambda \otimes B^\lambda_j \xrightarrow{d^\lambda_{i,j}} \cdots$$

with the *Koszul cohomology groups* defined by

$$\mathcal{K}^\lambda_{i,j} := \mathcal{K}_{i,j}(B^\lambda, V^\lambda) := \frac{\ker\, d^\lambda_{i,j}}{\operatorname{im}\, d^\lambda_{i+1,j-1}}.$$

For each λ we have a minimal free resolution of B^λ as an R_λ-module:

$$\begin{aligned} \cdots \longrightarrow \oplus_j R_\lambda(-j)^{\beta^\lambda_{i,j}} \longrightarrow \cdots \longrightarrow \oplus_j R_\lambda(-j)^{\beta^\lambda_{1,j}} \\ \longrightarrow \oplus_i R_\lambda(-j)^{\beta^\lambda_{0,j}} \longrightarrow B^\lambda \longrightarrow 0, \end{aligned}$$

and the $\beta^\lambda_{i,j}$ are the *(graded) Betti-numbers* for $\varphi_L(D_\lambda)$ (since $\varphi_L(D_\lambda)$ is arithmetically normal).

By the well-known *Syzygy Theorem* [Gr, Thm. (1.b.4)], we have

$$\beta^\lambda_{i,i+j} = \dim \mathcal{K}^\lambda_{i,j}$$

(where the dimension is as vector space over $\mathbf{C}$).

Example 8.9. As an example we look at the case where $D^2 = 0$ and $\mathcal{T}$ is singular (i.e. D is not perfect or $\mathcal{R}_{L,D}$ is non-empty). In this case the scroll $\mathcal{T}_0$ can be analyzed with the techniques of Section 7. Proposition 8.3 gives that the canonical sheaf on S'' is trivial. Lemma 7.1 gives us the Betti-numbers of all the D_λ. Hence the analogue of Proposition 7.2 goes through completely (we need the triviality of the canonical sheaf to prove the analogue of part (b)) to give a resolution of $\mathcal{O}_{S''}$ as an $\mathcal{O}_{\mathcal{T}_0}$-module. Set $g_0 = g + d = g + c + 2$. Since $\mathcal{T}_0$ has degree $g_0 - c - 1$, dimension $c + 2$, and spans $\mathbf{P}^{g_0}$, we only need to replace g by g_0 in Theorem 7.2.

Unfortunately, finding the Betti-numbers β^λ_{ij} for the $\varphi_L(D_\lambda)$ when $D^2 > 0$ is not as easy as in the case $D^2 = 0$. In fact, we are not able to compute all of them, nor to show that they are constant over $\mathbf{P}^1$, in general, but we will manage for the cases $D^2 = 2$ and 4, which are the cases we need for the classification of projective models of genus $g \leq 10$.

By our choice the general element in the pencil $\{D_\lambda\}$ is smooth and irreducible, whence by Lemma 5.2 also the general $\varphi_L(D_\lambda)$ is a smooth irreducible curve. To compute its Betti-numbers in $\overline{D_\lambda} = \mathbf{P}^{c+1+\frac{1}{2}D^2}$, we can use several results of Green and Lazarsfeld, and it will turn out that these results are sufficient to determine its Betti-numbers uniquely for $D^2 \leq 4$. However, there might be singular, reducible or even nonreduced elements in the pencil

$\{\varphi_L(D_\lambda)\}$, and one then has to check that the results of Green and Lazarsfeld can still be applied to these cases. Roughly speaking, since the Betti-numbers do not change when taking general hyperplane sections (since all the $\varphi_L(D_\lambda)$ are arithmetically normal whence projectively Cohen-Macaulay), we can avoid the isolated singularities, so the biggest problems arise from nonreduced fibers. It is therefore convenient to choose a pencil $\{D_\lambda\}$ with as few such cases as possible. Also note that the existence of a reducible element in $\{D_\lambda\}$, will require the existence of some effective divisors linearly independent of L and D, so in the general case of every family we study, all elements in $\{D_\lambda\}$ will be reduced and irreducible.

It will be of use to us that we can choose a pencil $\{D_\lambda\}$ subject to the following additional condition when $D^2 > 0$:

$$\text{Any member of } \{\varphi_L(D_\lambda)\} \text{ is one of the following:} \tag{8.7}$$

- A smooth irreducible curve of genus $p_a(D)$.
- A singular irreducible curve of arithmetic genus $p_a(D)$ or $p_a(D)+1$ with exactly one node or one cusp.
- $E_1 + E_2$, where E_1 and E_2 are distinct smooth elliptic curves intersecting in $\frac{1}{2}D^2$ points or in one point (the latter happening if and only if we are in the special case of Proposition 3.10, where $D^2 = 4$, $L \sim 2D$ and D is hyperelliptic).
- $\overline{D} + \Omega$ with $\overline{D}$ a smooth irreducible curve of genus $p_a(D) - 1$ and Ω of degree 1 or 2
- $\overline{D} + \Omega$ with $\overline{D}$ an irreducible curve of genus $p_a(D)$ with exactly one node or cusp and Ω of degree 1 or 2.
  (Note that Ω is either a conic, a union of two distinct lines, a double line or a line. In particular a nonreduced component of a member of $\{\varphi_L(D_\lambda)\}$ has to be a double line.)

Lemma 8.10. *Let D be a free Clifford divisor with $D^2 > 0$. Then we can choose a pencil $\{D_\lambda\}$ such that (8.7) is satisfied.*

Proof. Any irreducible element of $|D|$ is mapped isomorphically by φ_L by Lemma 5.2. Since the codimension of the set of irreducible elements in $|D|$ having more than one node or cusp as singularity is well-known to be > 1, we can find a pencil so that all irreducible elements are mapped to irreducible curves which are either smooth of genus $p_a(D)$ or has at most one node or cusp and therefore have arithmetic genus $p_a(D)+1$.

Now we have to consider reducible elements of $|D|$ living in codimension one.

Assume that an element of $|D|$ has two components of arithmetic genus ≥ 1. This means that $D \sim A + B$ with $h^0(A) \geq 2$ and $h^0(B) \geq 2$. A quick analysis as in the proof of Proposition 3.10 shows that $A^2 = B^2 = 0$ (otherwise either A or B would induce a Clifford index $< c$ on L). So $D \sim E_1 + E_2 + \Sigma$ for E_1 and E_1 smooth elliptic curves and an effective Σ which is either zero or only supported on smooth rational curves. In the first case, since the general elements in both $|E_1|$ and $|E_2|$ are smooth elliptic curves, we can choose a pencil containing at most the union of two smooth elliptic curves E_1 and E_2. Such a $D_0 = E_1 + E_2$ is mapped isomorphically by φ_L by Lemma 5.2. In the second, as in the rest of the proof, we are reduced to studying the cases where B is an effective divisor on S only supported on smooth rational curves such that $h^0(D \otimes \mathcal{J}_B) = h^0(D - B) = h^0(D) - 1 \geq 2$.

By adding base divisors to B, we can assume that $|D - B|$ is base point free. Hence, by Proposition 1.9 either $h^1(D - B) = 0$, or $D - B \sim kE$, for an integer $k \geq 2$ and a smooth elliptic curve E.

In the first case we have $h^0(D-B) = \frac{1}{2}D^2 - D.B + \frac{1}{2}B^2 + 2 = h^0(D) - D.B + \frac{1}{2}B^2$. If $B^2 > 0$, then by the Hodge index theorem and the fact that $B^2 < D^2$ (since $h^0(B) < h^0(D)$) we get $(D.B)^2 \geq D^2B^2 > (B^2)^2$, so $D.B > B^2$, and in particular $D.B \geq 3$, whence $h^0(D - B) < h^0(D) - \frac{1}{2}D.B \leq h^0(D) - 2$, a contradiction. If $B^2 = 0$, then $B.D \geq 2$, since D is base point free (by [SD, (3.9.6)] or [Kn4, Thm. 1.1]), so again $h^0(D - B) \leq h^0(D) - 2$.

So the only possibility remaining is $B^2 \leq -2$, and we see that $h^0(D-B) = h^0(D) - 1$ if and only if $B^2 = -2$ and $D.B = 0$. Since $h^0(D) \geq 3$, we have that $L \sim (D - B) + (F + B)$ is a decomposition into two moving classes with $(D - B).(F + B) = D.F + 2 - B.L = c + 2 - (B.L - 2)$, so we must have $B.L \leq 2$.

This means that there is a codimension one subset of $|D|$ whose elements are of the form $D' + B$, with D' base point free with $p_a(D') = p_a(D) - 1$, $h^1(D') = 0$ and B only supported on smooth rational curves and satisfying $B^2 = -2$, $B.D' = 2$ and $B.L \leq 2$. Clearly, since the general element in $|D'|$ is a smooth irreducible curve, we can choose a pencil in $|D|$ such that elements of this form are of the form $D' + B$ with D' a smooth curve of genus $p_a(D) - 1$. Now the contracted part B_0 of B satisfies $B_0.D' \leq B.D' = 2$, whence D' is mapped by φ_L to a curve with at worst one point of multiplicity two, i.e. either a node or a cusp. If $\varphi_L(D')$ is smooth then it has genus $p_a(D) - 1$, if not it has arithmetic genus $p_a(D)$. The divisor B is either zero or is mapped to a point or to an effective divisor Ω on S' of degree $B.L \leq 2$, whence a line, a conic, a union of two distict lines, or a double line.

In the second case we have $h^0(D - B) = k + 1 = h^0(D) - 1 = \frac{1}{2}D^2 + 1$, whence $D^2 = 2k \geq 4$, so $2k = D^2 = (kE + B)^2 = 2kE.B + B^2 = 2kE.D + B^2$. At the same time, by the base point freeness of D, we have $B.D = B.(kE + B) = kE.B + B^2 = kE.D + B^2 \geq 0$ and $E.D \geq 2$, so the only possibility is $E.B = E.D = 2$, $B.D = 0$ and $B^2 = -2k \leq -4$. In particular D is hyperelliptic, so $c = 2$, $D^2 = 4$ and $L \sim 2D$ by Proposition 3.10, which means that $k = 2$. Since $D.L = 8$, and $D \sim 2E + B$, we must have $E.L = 4$ and $B.L = 0$. Now there is a codimension one subset of $|D|$ whose elements are of the form $E_1 + E_2 + B$ where E_1 and E_2 are smooth elliptic curves in $|E|$. Since $B.E_1 = B.E_2 = 1$ and B is contracted by φ_L, the elements are mapped to a union of two smooth elliptic curves intersecting in one point. $\square$

Remark 8.11. We see from the proof above that in the cases where there exists a reducible fibre $\varphi_L(D_\lambda)$, then we are either in the case with $D \sim A+B$ into two moving classes or $D \sim D' + B$ with D' either irreducible or twice an elliptic pencil and B supported on rational curves with $B^2 = -2$ and $B.L \leq 2$. In the first case we find that A and B are Clifford divisors for L and in the second that D' is a free Clifford divisor. In particular we see that we can always find a free Clifford divisor D satisfying either $D^2 = 0$ or that $|D|$ contains a subpencil $\{D_\lambda\}$ such that all the members of $\{\varphi_L(D_\lambda)\}$ are irreducible.

Such a D need however not be perfect.

Note that the property (8.7) also yields that the singular locus of any $\varphi_L(D_\lambda)$ is either a finite number of points or at most a finite number of points and a double line, so it has an open set of regular points. The same applies for any D_λ.

Moreover, again by the property (8.7), a general hyperplane section of any $\varphi_L(D_\lambda)$ is of a scheme of length $L.D = D^2 + c + 2$ which either consists of distinct points (outside of Sing $\varphi_L(D_\lambda)$) or of a union of $L.D - 2 = D^2 + c$ distinct points (outside of Sing $\varphi_L(D_\lambda)$) and a scheme of length two situated in one point, namely the intersection with the double line, or equivalently, the image by φ_L of the unique element in $|\mathcal{O}_C(2\Gamma)|$ for a general $C \in |L|$.

We will from now on always work with a pencil satisfying (8.7).

We will need the following *general position statement*:

Lemma 8.12. *Assume $c > 0$ and $D^2 > 0$ and let $D' \subseteq \mathbf{P}^{c+1+\frac{1}{2}D^2}$ be any member of $\{\varphi_L(D_\lambda)\}$. Then a general hyperplane section Z is a scheme of length $L.D = D^2 + c + 2$ in general position, i.e. any subscheme of length $c + \frac{1}{2}D^2$ spans $\mathbf{P}^{c+1+\frac{1}{2}D^2}$.*

Proof. A general hyperplane section Z consists either of $D^2 + c + 2$ distinct points, or of D^2+c+1 distinct points, where one carries an additional tangent direction.

Set $r := c + 1 + \frac{1}{2}D^2$, then $r \geq 3$. The proof now follows the lines of the proof of the well-known General Position Theorem on p. 109 in [A-C-G-H]. We leave it to the reader to verify that the steps (i)-(iii) in that proof go through and that we can reduce to showing (correspondingly to the lemma on p. 109 in [A-C-G-H]) that a general hyperplane section of D' contains no subscheme of length 3 spanning only a $\mathbf{P}^1$.

So assume there is a general hyperplane section Z of D' containing a subscheme Z_0 of length three spanning a $\mathbf{P}^1$. Since we assume Z is general, we can avoid it to touch the singular points of S'. So we can consider Z and Z_0 as subschemes of S and we get that the natural map $H^0(L) \longrightarrow H^0(L \otimes \mathcal{O}_{Z_0})$ fails to be surjective. Since $c \geq 1$, we must have $L^2 \geq 4c + 4 \geq 8$, and we can use Proposition 1.13 to conclude that there is an effective divisor B passing though Z_0 satisfying either $B^2 = -2$ and $B.L \leq 1$, $B^2 = 0$ and $B.L \leq 3$ or $B^2 = 2$ and $B.L \leq 5$.

In the first case we have $B.L = 1$ and B irreducible, since we assume that Z lies outside the singular locus of S'. So Z_0 lies on a line, and a general hyperplane will only meet this line in one point, a contradiction.

In the two other cases we see that B induces the Clifford index one on L and we must have $(B^2, B.L) = (0,3)$ or $(2,5)$. Since we assume $D^2 \geq 2$, we must have $D^2 = c+1 = 2$, which means that we are in the case (E0), where $L \sim 2D + \Gamma$ for a smooth rational curve Γ satisfying $\Gamma.D = 1$. Since D is base point free, any other free Clifford divisor D' must satisfy $D'.D \geq 2$, whence $D'.L \geq 4$. Now the moving part of B is a free Clifford divisor, whence we must have $(B^2, B.L) = (2,5)$. It follows from Proposition 1.13 that $Z_0 \cap \Gamma \neq \emptyset$, and since $\Gamma.L = 0$, it follows that Z_0 meets the singularities of S', a contradiction. □

We will make use of the following lemma, which is well-known if D_0 is a smooth curve (see e.g. [G-L3, Lemma 3.1] or [A-C-G-H, Exc. K-2 p. 152] for (a)):

Lemma 8.13. *Let $D_0 \in |D|$.*

(a) If $x_1, \ldots, x_n$ are $n := h^0(L_{D_0}) - 2 = \frac{1}{2}D^2 + c$ distinct general points of D_0, outside of Sing D_0, then $L_{D_0} - x_1 - \ldots - x_n$ is base point free, $h^0(L_{D_0} - x_1 - \ldots - x_n) = 2$ and $h^1(L_{D_0} - x_1 - \cdots - x_n) = 0$.

(b) If $x_1, \ldots, x_k$ are $k \geq p_a(D)$ distinct general points of D_0, outside of Sing D_0, then $h^1(\mathcal{O}_{D_0}(x_1 + \cdots + x_k)) = 0$.

Proof. Since $n = L.D - \frac{1}{2}D^2 - 2 \leq L.D - 3$ the statement (a) immediately follows from the previous lemma.

As for (b), by Serre duality we have $h^1(\mathcal{O}_{D_0}(x_1 + \cdots + x_k)) = h^0(\mathcal{O}_{D_0}(D)(-x_1 - \cdots - x_k))$. Denoting the ideal defined by the points $x_1, \ldots, x_k$ by Z, we have an exact sequence

$$0 \longrightarrow \mathcal{O}_S \longrightarrow \mathcal{O}_S(D) \otimes \mathcal{J}_Z \longrightarrow \mathcal{O}_{D_0}(D)(-Z) \longrightarrow 0, \tag{8.8}$$

so $h^1(\mathcal{O}_{D_0}(x_1 + \cdots + x_k)) = 0$ if and only if $h^0(\mathcal{O}_S(D) \otimes \mathcal{J}_Z) = 1$. Clearly we can assume that $k = p_a(D)$. Then $h^0(\mathcal{O}_S(D) \otimes \mathcal{J}_Z) = 1$ if and only if the k points pose independent conditions on $|D|$. Proceeding inductively, we only have to show that for k' distinct points on D_0, with $1 \leq k' \leq k$, posing independent conditions on $|D|$, then a general point $p \in D_0$ away from Sing D_0 poses one more additional condition. Let Σ be the base divisor of $|D \otimes \mathcal{J}_{Z'}|$, where Z' is the scheme defined by the k' distinct points. Then we are done, unless all the regular points of D_0 are contained in Σ. However, by the property (8.7), it would then follow that $h^0(\Sigma) \geq h^0(D) - 1$. But then the moving part of $|D \otimes \mathcal{J}_{Z'}|$ has dimension zero, i.e. it consists only of D_0 itself, so $h^0(D \otimes \mathcal{J}_{Z'}) = 1$, and it follows that $k = k'$ and we are done. □

We write $L_\lambda := L_{D_\lambda}$.

We first define a vector bundle $\mathcal{E}_\lambda$ on every D_λ, as follows. If B is an effective divisor on S and $\mathcal{A}$ is any globally generated invertible sheaf on B, then the evaluation map $H^0(\mathcal{A}) \otimes \mathcal{O}_B \to \mathcal{A}$ is surjective, and the kernel is a vector bundle on B:

$$0 \longrightarrow \mathcal{E}_{\mathcal{A}} \longrightarrow H^0(\mathcal{A}) \otimes_{\mathbf{C}} \mathcal{O}_B \longrightarrow \mathcal{A} \longrightarrow 0. \tag{8.9}$$

Note that det $\mathcal{E}_{\mathcal{A}} = \mathcal{A}^\vee$ and rank $\mathcal{E}_{\mathcal{A}} = h^0(\mathcal{A})$, so that $\mathcal{E}_{\mathcal{A}} = \mathcal{A}^\vee$ when $h^0(\mathcal{A}) = 2$.

For every λ we set $\mathcal{E}_\lambda := \mathcal{E}_{L_\lambda}$.

Taking exterior powers in (8.9) and twisting by suitable powers of L, we get for any $i \geq 0$ and any $j \geq 0$

$$0 \longrightarrow \wedge^i \mathcal{E}_\lambda \otimes L_\lambda^{\otimes j} \longrightarrow \wedge^i H^0(L_\lambda) \otimes_{\mathbf{C}} L_\lambda^{\otimes j} \longrightarrow \wedge^{i-1} \mathcal{E}_\lambda \otimes L_\lambda^{\otimes (j+1)} \longrightarrow 0. \tag{8.10}$$

Moreover, we get and we see that $d_{i,j}^\lambda = H^0(f_{i,j}^\lambda)$ for all $i, j \geq 0$.

$$\begin{tikzcd}
 & 0 \arrow[d] & & & \\
 & \wedge^{i+1}\mathcal{E}_\lambda \otimes L_\lambda^{\otimes(j-1)} \arrow[d] & & & \\
 & \wedge^{i+1}H^0(L_\lambda) \otimes_{\mathbf{C}} L_\lambda^{\otimes(j-1)} \arrow[d] \arrow[dr, "f_{i+1,j-1}^\lambda"] & & 0 \arrow[d] & \\
0 \arrow[r] & \wedge^{i}\mathcal{E}_\lambda \otimes L_\lambda^{\otimes j} \arrow[r] \arrow[d] & \wedge^{i}H^0(L_\lambda) \otimes_{\mathbf{C}} L_\lambda^{\otimes j} \arrow[r] \arrow[dr, "f_{i,j}^\lambda"] & \wedge^{i-1}\mathcal{E}_\lambda \otimes L_\lambda^{\otimes(j+1)} \arrow[r] \arrow[d] & 0 \\
 & 0 & & \wedge^{i-1}H^0(L_\lambda) \otimes_{\mathbf{C}} L_\lambda^{\otimes(j+1)}, &
\end{tikzcd}$$

Chasing the diagram, and using that $h^1(L_\lambda^{\otimes k}) = 0$ for all $k \geq 1$, together with $h^0(L_\lambda) = c + 2 + \frac{1}{2}D^2$, we easily get that the Koszul cohomology groups $\mathcal{K}_{i,j}^\lambda$ satisfy

$$\mathcal{K}_{i,j}^\lambda = 0 \text{ for all } j \geq 3. \tag{8.11}$$

$$\dim \mathcal{K}_{i,2}^\lambda = h^1(\wedge^{i+1}\mathcal{E}_\lambda \otimes L_\lambda). \tag{8.12}$$

$$\dim \mathcal{K}_{i,1}^\lambda = h^1(\wedge^{i+1}\mathcal{E}_\lambda) - \binom{c + 2 + \frac{1}{2}D^2}{i+1}(\frac{1}{2}D^2 + 1) + h^1(\wedge^i \mathcal{E}_\lambda \otimes L_\lambda) \tag{8.13}$$

Of course we also have

$$\mathcal{K}_{i,j}^\lambda = 0 \text{ for } \quad i \geq h^0(L_{D_\lambda}) - 1 = c + 1 + \frac{1}{2}D^2. \tag{8.14}$$

We now want to show that $h^1(\wedge^{i+1}\mathcal{E}_\lambda)$ is independent of λ for $i \leq h^0(L_{D_\lambda}) - 2 = c + \frac{1}{2}D^2$.

By Lemma 8.13(a), if $x_1, \ldots, x_n$ are $n := h^0(L_\lambda) - 2 = c + \frac{1}{2}D^2$ general distinct points of D_λ, outside of the singular points of D_λ, then $L_\lambda - x_1 - \ldots - x_n$ is generated by its global sections and $h^1(L_\lambda - x_1 - \ldots - x_n) = h^1(L_\lambda) = 0$, so from [G-L1] or [La1, Lemma 1.4.1] we have an exact sequence

$$0 \longrightarrow \mathcal{E}_{L_\lambda - x_1 - \ldots - x_n} \longrightarrow \mathcal{E}_{L_\lambda} \longrightarrow \Sigma \longrightarrow 0, \tag{8.15}$$

where $\Sigma := \oplus_{i=1}^n \mathcal{O}_{D_\lambda}(-x_i)$. (We leave it to the reader to check that this also holds in our case when D_λ is singular or possibly reducible). Set $\mathcal{B} := \mathcal{O}_{D_\lambda}(x_1 + \cdots + x_n)$. Since $h^0(L_\lambda - \mathcal{B}) = 2$, we have $\mathcal{E}_{L_\lambda - x_1 - \ldots - x_n} = \mathcal{B} - L_\lambda$. Taking exterior products yields

$$0 \longrightarrow \wedge^i \Sigma \otimes (\mathcal{B} - L_\lambda) \longrightarrow \wedge^{i+1}\mathcal{E}_{L_\lambda} \longrightarrow \wedge^{i+1}\Sigma \longrightarrow 0. \tag{8.16}$$

The term on the right is a direct sum of $\binom{n}{i+1}$ line bundles of the form $\mathcal{O}_{D_\lambda}(-x_{k_1} - \cdots - x_{k_{i+1}})$, whence for all $i \geq 0$ we have $h^0(\wedge^{i+1}\Sigma) = 0$ and by Riemann-Roch $h^1(\wedge^{i+1}\Sigma) = \binom{n}{i+1}(\frac{1}{2}D^2 + 1 + i)$.

The term on the left is a direct sum of $\binom{n}{i}$ line bundles of the form $\mathcal{O}_{D_\lambda}(x_{k_1} + \cdots + x_{k_{n-i}}) \otimes L^\vee$. Now by Serre duality $h^0(\mathcal{O}_{D_\lambda}(x_{k_1} + \cdots + x_{k_{n-i}}) \otimes L^\vee = h^1(\mathcal{O}_{D_\lambda}(L + D)(-x_{k_1} - \cdots - x_{k_{n-i}})$. By the sequence

$$0 \longrightarrow L \longrightarrow \mathcal{O}_S(L + D) \otimes \mathcal{J}_Z \longrightarrow \mathcal{O}_{D_\lambda}(L + D)(-Z) \longrightarrow 0, \tag{8.17}$$

with Z the ideal defined by $x_{k_1}, \ldots, x_{k_{n-i}}$, $h^1(\mathcal{O}_{D_\lambda}(L + D)(-x_{k_1} - \cdots - x_{k_{n-i}})) = h^1(\mathcal{O}_S(L + D) \otimes \mathcal{J}_Z)$. Since $h^1(L + D) = 0$, this is equivalent to saying that $x_{k_1}, \ldots, x_{k_{n-i}}$ pose independent conditions on $L + D$. but since $n - i \leq n$, the points pose independent conditions on L by Lemma 8.13(a), whence also on $L + D$, since D is base point free. So $h^0(\wedge^{i+1}\Sigma) = 0$ and by Riemann-Roch $h^1(\wedge^{i+1}\Sigma) = \binom{n}{i}(D^2 + 2 + i)$. Inserting for n, it follows that

$$h^1(\wedge^{i+1}\mathcal{E}_\lambda) = \binom{c + \frac{1}{2}D^2}{i}(D^2 + 2 + i) + \binom{c + \frac{1}{2}D^2}{i + 1}(\frac{1}{2}D^2 + 1 + i). \tag{8.18}$$

This improves (8.13):

$$\begin{aligned} \dim \mathcal{K}_{i,1}^\lambda = &\binom{c + \frac{1}{2}D^2}{i}(D^2 + 2 + i) + \binom{c + \frac{1}{2}D^2}{i + 1}(\frac{1}{2}D^2 + 1 + i) \\ &-\binom{c + 2 + \frac{1}{2}D^2}{i + 1}(\frac{1}{2}D^2 + 1) + h^1(\wedge^i\mathcal{E}_\lambda \otimes L_\lambda). \end{aligned} \tag{8.19}$$

In particular, we see that

$$\begin{aligned} \dim \mathcal{K}_{i,1}^\lambda - \dim \mathcal{K}_{i-1,2}^\lambda = &\binom{c + \frac{1}{2}D^2}{i}(D^2 + 2 + i) \\ &+ \binom{c + \frac{1}{2}D^2}{i + 1}(\frac{1}{2}D^2 + 1 + i) - \binom{c + 2 + \frac{1}{2}D^2}{i + 1}(\frac{1}{2}D^2 + 1) \end{aligned} \tag{8.20}$$

is independent of λ.

Recall that the line bundle L_λ on D_λ is said to satisfy property N_p if the Betti-numbers satisfy the following:

$$\beta^\lambda_{0,j} = \begin{cases} 1 & \text{if } j = 0, \\ 0 & \text{if } j \neq 0 \end{cases} \quad \text{and} \quad \beta^\lambda_{i,j} \neq 0 \text{ if and only if } j = i+1, \text{ for } 0 < i \leq p. \tag{8.21}$$

This means that B^λ has a resolution of the form

$$\begin{aligned} \cdots \longrightarrow R_\lambda(-p-1)^{\beta_{p,p+1}} \longrightarrow \cdots \longrightarrow R_\lambda(-3)^{\beta_{2,3}} \\ \longrightarrow R_\lambda(-2)^{\beta_{1,2}} \longrightarrow R_\lambda \longrightarrow B^\lambda \longrightarrow 0. \end{aligned}$$

In our case, we have

Proposition 8.14. *Assume $c > 0$. Then L_λ satisfies property N_{c-1} but not N_c.*

Proof. If D_λ is smooth, then the second statement is immediate, since we have by [Kn4] and the conditions $(*)$ that L_λ fails to be $(c+1)$-very ample, and the result follows from [G-L3, Thm. 2]. By semicontinuity, N_c fails for all L_λ.

The first statement is also immediate if D_λ is smooth: Indeed, it follows from [Gr, Thm. (4.a.1)], since $\deg L_{D_\lambda} = 2g(D_\lambda) + c$ and $h^1(L_{D_\lambda}) = 0$.

We have to argue that the result still holds for the singular and reducible D_λ, in other words we have to show that $\mathcal{K}^\lambda_{i,2} = 0$ for all $i \leq c-1$ and all λ.

By (8.16) we have to show that $h^1(\wedge^{i+1}\mathcal{E}_\lambda \otimes L_\lambda) = 0$ for all $i \leq c-1$.

Choose as above $n := c + \frac{1}{2}D^2$ general points $x_1, \ldots, x_n$ of D_λ. We then get a sequence as (8.16), and tensoring this sequence with L_λ yields

$$0 \longrightarrow \wedge^i \Sigma \otimes \mathcal{B} \longrightarrow \wedge^{i+1}\mathcal{E}_{L_\lambda} \otimes L_\lambda \longrightarrow \wedge^{i+1}\Sigma \otimes L_\lambda \longrightarrow 0. \tag{8.22}$$

The term on the right is a direct sum of $\binom{n}{i+1}$ line bundles of the form $L_\lambda(-x_{k_1} - \cdots - x_{k_{i+1}})$. By Lemma 8.13(a) it follows that for all $i \geq 0$ we have $h^1(\wedge^{i+1}\Sigma \otimes L_\lambda) = 0$.

The term on the left is a direct sum of $\binom{n}{i}$ line bundles of the form $\mathcal{O}_{D_\lambda}(x_{k_1} + \cdots + x_{k_{n-i}})$, whence of degrees $n - i \geq \frac{1}{2}D^2 + 1 = p_a(D)$. By Lemma 8.13(b) it follows that $h^1(\wedge^i \Sigma \otimes \mathcal{B}) = 0$.

It follows that $h^1(\wedge^{i+1}\mathcal{E}_\lambda \otimes L_\lambda) = 0$ for all $i \leq c-1$.

An alternative proof of the fact that $\mathcal{K}^\lambda_{i,2} = 0$ for all $i \leq c-1$ and all λ goes as follows: Since $\varphi_L(D_\lambda)$ is arithmetically normal and D_λ is of pure dimension one, the Betti-numbers of $\varphi_L(D_\lambda)$ are equal to the Betti-numbers of a general hyperplane section of it. This is a scheme X of length $L.D = D^2 + c + 2$ in general position by Lemma 8.12. We now argue as in the proof of [G-L3, Thm. 2.1] to show that the scheme X satisfies N_{c-1}. Recall that X either consists of distinct points or at worst of a union of $L.D - 2 = D^2 + c$ distinct points and a scheme of length two supported in one point, call it Z. The case of distinct points is exactly the statement in [G-L3, Thm. 2.1], so we have to show that the proof goes through in the other case. We leave it to the reader to verify that everything works as long as one writes the scheme X as a disjoint union $X = X_1 \cup X_2$ as in the proof of [G-L3, Thm. 2.1], taking care that $Z \subseteq X_2$. This is possible, since X_1 should consist of $\frac{1}{2}D^2 + c + 1$ distinct points, which yields length $X_2 = \frac{1}{2}D^2 + 1 \geq 2$. □

From this proposition we therefore get

$$\mathcal{K}^\lambda_{i,2} = 0 \text{ for all } i \leq c - 1. \tag{8.23}$$

$$\mathcal{K}^\lambda_{c,2} \neq 0. \tag{8.24}$$

Also, by the Theorem in [G-L5], we have that for $D^2 > 0$:

$$\beta^\lambda_{c,c+1} \neq 0 \text{ for all smooth irreducible } D_\lambda.$$

Indeed, $L_{D_\lambda} \simeq F_{D_\lambda} + \omega_{D_\lambda}$, and $D^2 \leq 2c$ for $c > 0$, and we calculate

$$h^0(F_{D_\lambda}) = h^0(F) - \chi(F - D) = c + 2 - \frac{1}{2}D^2 \geq 2,$$

$$h^0(\omega_{D_\lambda}) = \frac{1}{2}D^2 + 1 \geq 1$$

and

$$h^0(F_{D_\lambda}) + h^0(\omega_{D_\lambda}) - 3 = c.$$

By semicontinuity it follows that

$$\mathcal{K}^\lambda_{c,1} \neq 0 \text{ for all } \lambda. \tag{8.25}$$

Finally, recall that the line bundle L_λ on D_λ is said to satisfy property M_q if $\mathcal{K}^\lambda_{i,j} = 0$ for all $i \geq h^0(L_\lambda) - 1 - q = \frac{1}{2}D^2 + c + 1 - q$ and $j \neq 2$.

We have

Proposition 8.15. *(a) If $c > 0$, then L_λ satisfies M_1.*

(b) If $D^2 \geq 4$ and $c \geq 3$, then L_λ satisfies M_2.

Proof. The main ingredient in this proof is the proof of Green's $\mathcal{K}_{p,1}$ theorem [Gr, (3.c.1)].

Set $r := h^0(L_\lambda) = \frac{1}{2}D^2 + c + 1$.

To show (a), we argue as in the proof of statement (2) in [Gr, (3.c.1)], and assume that $\mathcal{K}^\lambda_{p,1} \neq 0$, for $p = \frac{1}{2}D^2 + c = r - 1$. Taking a general hyperplane section Z of $\varphi_L(D_\lambda)$ we get that $\mathcal{K}^\lambda_{p,1} \neq 0$ for $Z \subseteq \mathbf{P}^{r-1}$. By Lemma 8.12 Z is in general position, so if it consists of distinct points, then it follows from Green's *Strong Castelnuovo Lemma* [Gr, (3.c.6)] that Z lies on a rational normal curve, whence the contradiction $D^2 + c + 2 = L.D = \deg \varphi_L(D_\lambda) \leq r = \frac{1}{2}D^2 + c + 1$.

If Z consists of $L.D - 2$ distinct points and a scheme of length two with support in one point we have to show that Green's Strong Castelnuovo Lemma still can be used. The key point is where Green uses that any $r + 2$ distinct points in general position in $\mathbf{P}^{r-1}$ lie on a unique rational normal curve. This still holds true if we have $r+1$ distinct points with one additional tangent direction at one of them, when the whole scheme is in general position.

We leave it to the reader to verify that the Strong Castelnuovo Lemma holds in our case and that we can conclude as above that Z lies on a rational normal curve, and get the same contradiction.

Now we prove (b). Once we have checked that the Strong Castelnuovo Lemma holds in our case, we can argue as in the proof of (3) in [Gr, (3.c.1)], and find that either $D^2+c+2 = L.D = \deg \varphi_L(D_\lambda) \leq r+1 = \frac{1}{2}D^2+c+2$, which is not our case, or that $\varphi_L(D_\lambda)$ lies on a surface of minimal degree, i.e. the Veronese surface in $\mathbf{P}^5$, a ruled surface or a cone over a rational normal curve.

In the first case we must have $r=5$, whence $c=2$ and $D^2=4$.

In the two other cases, then if $\varphi_L(D_\lambda)$ does not pass through the vertex of the cone the ruling restricts to a Cartier divisor on $\varphi_L(D_\lambda)$ and it cuts out a g^1_2 on $\varphi_L(D_\lambda)$ which we can pull back by φ_L to S. Then every element Z in this g^1_2 on S is a 0-dimensional scheme of length 2 failing to pose independent conditions on $|D|$. Therefore $|D|$ must be hyperelliptic and by Proposition 3.10 we have $c=2$ and $D^2=4$.

We have left to treat the case where $\varphi_L(D_\lambda)$ lies in a cone and passes through its vertex. Since $\varphi_L(D_\lambda)$ cannot be a union of lines by (8.10) the ruling cuts out a g^1_1 on the component of $\varphi_L(D_\lambda)$ obtained by removing the components which are lines of the ruling, if any. So this component is an irreducible curve birational to $\mathbf{P}^1$. By (8.10) this curve is either smooth of genus $p_a(D)$ or $p_a(D)-1$ or has only one node or cusp and arithmetic genus $p_a(D)$ or $p_a(D)+1$. In all these cases we get that the curve has geometric genus $\geq p_a(D)-1 \geq 2$, since we assume $D^2 \geq 4$, a contradiction. $\square$

The following lemma settles the remaining case $D^2=4$ and $c=2$, where in fact L_λ does not satisfy M_2:

Lemma 8.16. *Let* $(c, D^2) = (2,4)$. *Then* $\dim \mathcal{K}^\lambda_{3,1} = 3$ *for all* λ.

Proof. We are in the case (Q) with $L \sim 2D$. By Proposition 5.10 either φ_L is the 2-uple embedding of $\varphi_D(S)$, or there is an elliptic pencil $|E|$ such that $E.D=2$. We will treat these two cases separately.

In the first case $\varphi_L(D_\lambda)$ is the 2-uple embedding of $\varphi_D(D_\lambda)$, for all λ. Now φ_D maps D_λ into $\mathbf{P}^2$, so $\varphi_L(D_\lambda)$ lies on the Veronese surface V in $\mathbf{P}^5$, i.e. the 2-uple embedding of $\mathbf{P}^2$.

We have Pic $V \sim \mathbf{Z}l$, where $l^2=1$. The hyperplane class H_V satisfies $H_V \sim 2l$, and since $\varphi_L(D_\lambda)$ has degree $L.D=8$, we have $\varphi_L(D_\lambda) \sim 4l \sim 2H_V$. By [Gr, (3.b.4)] we have

$$\mathcal{K}^\lambda_{3,1} = \mathcal{K}^\lambda_{3,1}(V, H_V) \oplus \mathcal{K}^\lambda_{2,0}(V, H_V).$$

Both the latter are well-known, since V is a variety of minimal degree (see e.g. [Sc, Lemma 5.2]). In fact $\dim \mathcal{K}^\lambda_{3,1}(V, H_V) = 3$ and $\mathcal{K}^\lambda_{2,0}(V, H_V) = 0$. Hence $\dim \mathcal{K}^\lambda_{3,1} = 3$, as asserted.

In the second case, any D_λ has a g^1_2 given by $\mathcal{O}_{D_\lambda}(E)$. Compare the two morphisms $f_E : D_\lambda \to \mathbf{P}^1$ given by $|\mathcal{O}_{D_\lambda}(E)|$ and $f_D : D_\lambda \to \mathbf{P}^2$ given by $|\mathcal{O}_{D_\lambda}(D)|$. Since $h^1(E-D) = h^1(\mathcal{O}_S) = 0$, these are the restrictions of φ_E and φ_D respectively. Since they both collapse every member of the g^1_2, we see that $f_D = g \circ f_E$, where $g : \mathbf{P}^1 \to \mathbf{P}^2$ is the 2-uple embedding. It follows that $\mathcal{O}_{D_\lambda}(D) = \mathcal{O}_{D_\lambda}(D)^{\otimes 2}$ and consequently $L_{D_\lambda} = 4E_{D_\lambda}$.

The members of the g^1_2 sweep out a scrollar surface S_0 containing $\varphi_L(D_\lambda)$. As before, we can compute its scroll type (e_1, e_2) by first computing the "dual scrollar invariants"

$$\begin{aligned} d_i &= h^0(L_{D_\lambda} - iE_{D_\lambda}) - h^0(L_{D_\lambda} - (i+1)E_{D_\lambda}) \\ &= h^0((4-i)E_{D_\lambda}) - h^0((3-i)E_{D_\lambda}). \end{aligned}$$

We easily get $d_0 = 2$, $d_1 = d_2 = d_3 = d_4 = 1$ and $d_{\geq 5} = 0$. Recalling that $e_i = \#\{j \mid d_j \geq i\} - 1$ we get $(e_1, e_2) = (4, 0)$, whence S_0 is a cone over a rational normal quartic.

Note that since the g^1_2 is base point free $\varphi_L(D_\lambda)$ does not intersect the vertex of S_0, so we can work with the desingularization S_0, which we by abuse of notation also denote by S_0.

We have $\mathrm{Num} S_0 \simeq \mathbf{Z}H_0 \oplus \mathbf{Z}L_0$, where H_0 is the hyperplane class and L_0 is the class of the ruling, whence $H_0.L_0 = 1$, $H_0^2 = 4$ and $L_0^2 = 0$. Since $\varphi_L(D_\lambda).H_0 = \deg \varphi_L(D_\lambda) = 8$ and $\varphi_L(D_\lambda).L_0 = 2$, we find $D_0 \sim 2H_0$. Moreover we have $H^0(S_0, H_0 - D_0) = H^1(S_0, qH_0 - D_0) = H^1(S_0, qD_0) = 0$ for all $q \geq 0$, so by [Gr, (3.b.4)] we have

$$\mathcal{K}^\lambda_{3,1} = \mathcal{K}^\lambda_{3,1}(S_0, H_0) \oplus \mathcal{K}^\lambda_{2,0}(S_0, H_0).$$

Again it is well-known that $\dim \mathcal{K}^\lambda_{3,1}(S_0, H_0) = 3$ and $\mathcal{K}^\lambda_{2,0}(S_0, H_0) = 0$ (see e.g. [Sc, Lemma 5.2]), so $\dim \mathcal{K}^\lambda_{3,1} = 3$, as asserted. $\square$

Summing up, we have

Proposition 8.17. *Let D be a free Clifford divisor on a polarized $K3$ surface (S, L) of non-general Clifford index $c > 0$ satisfying $D^2 > 0$. Then the Betti-numbers of the $\varphi_L(D_\lambda)$ satisfy:*

(a) $\beta^\lambda_{0,j} = \begin{cases} 1 & \text{if } j = 0, \\ 0 & \text{if } j \neq 0 \end{cases}.$

(b) For $0 < i \leq c-1$, $\beta^\lambda_{i,j} \neq 0$ if and only if $j = i+1$,

(c) $\beta^\lambda_{ij} = 0$ for $i \geq c + 1 + \frac{1}{2}D^2$.

(d) $\beta^\lambda_{ij} = 0$ for $j \geq i + 3$.

(e) $\beta^\lambda_{i,i+1} - \beta^\lambda_{i-1,i+1} = \binom{c+\frac{1}{2}D^2}{i}(D^2 + 2 + i) + \binom{c+\frac{1}{2}D^2}{i+1}(\frac{1}{2}D^2 + 1 + i) - \binom{c+2+\frac{1}{2}D^2}{i+1}(\frac{1}{2}D^2 + 1)$, *for* $i > 0$.

(f) $\beta^\lambda_{\frac{1}{2}D^2+c, \frac{1}{2}D^2+c+1} = 0$.

(g) $\beta^\lambda_{\frac{1}{2}D^2+c-1, \frac{1}{2}D^2+c} = 0$ *for* $D^2 \geq 4$ *and* $c \geq 3$.

(h) $\beta^{\lambda}_{3,4} = 3$ *if* $(c, D^2) = (2, 4)$
(i) $\beta^{\lambda}_{c,c+1} \neq 0$ *for* $D^2 > 0$.
(j) $\beta^{\lambda}_{c,c+2} \neq 0$.

So for $D^2 > 0$, the $\varphi_L(D_\lambda)$ all have a resolution of the form:

$$0 \longrightarrow R_\lambda(-\frac{1}{2}D^2 - c - 2)^{\beta^{\lambda}_{\frac{1}{2}D^2+c,\frac{1}{2}D^2+c+2}} \longrightarrow \cdots \longrightarrow$$
$$R_\lambda(-\frac{1}{2}D^2 - c)^{\beta^{\lambda}_{\frac{1}{2}D^2+c-1,\frac{1}{2}D^2+c}} \oplus R_\lambda(-\frac{1}{2}D^2 - c - 1)^{\beta^{\lambda}_{\frac{1}{2}D^2+c-1,\frac{1}{2}D^2+c+1}} \longrightarrow$$
$$\cdots \longrightarrow R_\lambda(-c-2)^{\beta^{\lambda}_{c+1,c+2}} \oplus R_\lambda(-c-3)^{\beta^{\lambda}_{c+1,c+3}} \longrightarrow$$
$$R_\lambda(-c-1)^{\beta^{\lambda}_{c,c+1}} \oplus R_\lambda(-c-2)^{\beta^{\lambda}_{c,c+2}} \longrightarrow R_\lambda(-c)^{\beta^{\lambda}_{c-1,c}} \longrightarrow$$
$$\cdots \longrightarrow R_\lambda(-3)^{\beta^{\lambda}_{2,3}} \longrightarrow R_\lambda(-2)^{\beta^{\lambda}_{1,2}} \longrightarrow R_\lambda \longrightarrow B^\lambda \longrightarrow 0.$$

It is easy to see that all the Betti-numbers for $D^2 = 2$ and $D^2 = 4$ are uniquely determined by the information above. Combining with Example 8.9, we get:

Corollary 8.18. *For* $0 \leq D^2 \leq 4$ *the Betti-numbers of the* $\varphi_L(D_\lambda)$ *are the same for all* λ *and uniquely given by the results above.*

We will compute some concrete examples in the next section.

8.4 Resolutions for projective models

In this section we will study various examples of projective models of $K3$ surfaces contained in singular scrolls $\mathcal{T}$. We will use the results in the previous section to obtain minimal resolutions of the $\varphi_L(D_\lambda)$ in the projective spaces they span. We will also give results (Proposition 8.23 and 8.29) showing how one can lift these resolutions to resolve $\mathcal{O}_{S''}$ and $\mathcal{O}_{S'}$ as $\mathcal{O}_{\mathcal{T}_0}$- and $\mathcal{O}_{\mathcal{T}}$-modules, respectively.

Example 8.19. As our first example we study the case (E0) with $c = 1$, $D^2 = 2$ and $g = 6$. By Proposition 8.17 all the $\varphi_L(D_\lambda) \subseteq \mathbf{P}^3$ have minimal resolutions

$$0 \longrightarrow R(-4)^2 \longrightarrow R(-2) \oplus R(-3)^2 \longrightarrow R \longrightarrow B \longrightarrow 0.$$

This is the well-known resolution of a smooth curve of genus 2 in $\mathbf{P}^3$ (see e.g. [Si]).

Example 8.20. As another example we study the case when $D^2 = 2$ and $c \geq 2$, where $\varphi_L(D_\lambda) \subseteq \mathbf{P}^{c+2}$.

For $c = 2$ a minimal resolution is of the following form:

$$0 \longrightarrow R(-5)^2 \longrightarrow R(-3)^2 \oplus R(-4)^3$$
$$\longrightarrow R(-2)^4 \longrightarrow R \longrightarrow B^\lambda \longrightarrow 0,$$

For $c = 3$ a minimal resolution is of the following form:

$$\begin{aligned}0 \longrightarrow R(-6)^2 \longrightarrow R(-5)^4 \oplus R(-4)^3 \longrightarrow R(-3)^{12}\\ \longrightarrow R(-2)^8 \longrightarrow\!\!\longrightarrow B^\lambda \longrightarrow 0,\end{aligned}$$

For $c = 4$ a minimal resolution is of the following form:

$$\begin{aligned}0 \longrightarrow R(-7)^2 \longrightarrow R(-6)^5 \oplus R(-5)^4 \longrightarrow R(-4)^{25}\\ \longrightarrow R(-3)^{30} \longrightarrow R(-2)^{13} \longrightarrow R \longrightarrow B^\lambda \longrightarrow 0,\end{aligned}$$

Example 8.21. As yet another example we study the case when $D^2 = 4$ and $c \geq 2$, where $\varphi_L(D_\lambda) \subseteq \mathbf{P}^{c+3}$.

For $c = 2$ a minimal resolution is of the following form:

$$\begin{aligned}0 \longrightarrow R(-6)^3 \longrightarrow R(-5)^8 \oplus R(-4)^3 \longrightarrow\\ R(-4)^6 \oplus R(-3)^8 \longrightarrow R(-2)^7 \longrightarrow R \longrightarrow B^\lambda \longrightarrow 0.\end{aligned}$$

For $c = 3$ a minimal resolution is of the following form:

$$\begin{aligned}0 \longrightarrow R(-7)^3 \longrightarrow R(-6)^{10} \longrightarrow R(-5)^6 \oplus R(-4)^{15}\\ \longrightarrow R(-3)^{25} \longrightarrow R(-2)^{12} \longrightarrow R \longrightarrow B^\lambda \longrightarrow 0.\end{aligned}$$

Remark 8.22. If we twist the resolution following Proposition 8.17 with n and use the additivity of the Euler characteristic, we obtain the following polynomial identity in the variable n:

$$\begin{aligned}(c+2+D^2)n - \frac{1}{2}D^2 = \binom{n+c+1+\frac{1}{2}D^2}{c+1+\frac{1}{2}D^2} +\\ \sum_{j=2}^{\frac{1}{2}D^2+c+2} (-1)^j \binom{n+c+\frac{1}{2}D^2+1-j}{c+1+\frac{1}{2}D^2}(\beta_{j-2,j} - \beta_{j-1,j}).\end{aligned}$$

It is easy to see that this identity alone determines the $\beta_{j-2,j} - \beta_{j-1,j}$ uniquely. Since $\beta_{j-2,j} = 0$, for $j \leq c+1$, the $\beta_{j-1,j}$, for $j = 2, \ldots, c+1$ are determined uniquely. On the other hand this observation gives nothing that is not already contained in the statement of Proposition 8.17.

We now return to the general resolution, following Proposition 8.17. From Corollary 8.18, Example 8.9 for the case $D^2 = 0$ and [Sc, Thm. (3.2)] in general, we obtain the following:

Proposition 8.23. *If $D^2 = 0$ and $c = 1$, then the $\mathcal{O}_{\mathcal{T}_0}$-resolution F_* of $\mathcal{O}_{S''}$ is*

$$0 \longrightarrow \mathcal{O}_{\mathcal{T}_0}(-3\mathcal{H}_0 + (g-1)\mathcal{F}) \longrightarrow \mathcal{O}_{\mathcal{T}_0} \longrightarrow \mathcal{O}_{S''} \longrightarrow 0.$$

If $D^2 = 0$ and $c \geq 2$, the resolution is of the following type:

$$0 \longrightarrow \mathcal{O}_{\mathcal{T}_0}(-(c+2)\mathcal{H}_0 + (g-1)\mathcal{F}) \longrightarrow \oplus_{k=1}^{\beta_{c-1}} \mathcal{O}_{\mathcal{T}_0}(-c\mathcal{H}_0 + b_{c-1}^k \mathcal{F}) \longrightarrow$$
$$\cdots \longrightarrow \oplus_{k=1}^{\beta_1} \mathcal{O}_{\mathcal{T}_0}(-2\mathcal{H}_0 + b_1^k \mathcal{F}) \longrightarrow \mathcal{O}_{\mathcal{T}_0} \longrightarrow \mathcal{O}_{S''} \longrightarrow 0,$$

where $\beta_i = i\binom{c+1}{i+1} - \binom{c}{i-1}$.

If $D^2 = 2$ *or* 4, *or more generally if the Betti-numbers of all the* $\varphi_L(D_\lambda)$ *are the same for all* λ, *then* $\mathcal{O}_{S''}$ *has a* $\mathcal{O}_{\mathcal{T}_0}$*-resolution* F_* *of the following type:*

$$0 \longrightarrow F_{\frac{1}{2}D^2+c} \cdots \longrightarrow F_{c+1} \longrightarrow F_c$$
$$\longrightarrow \oplus_{k=1}^{\beta_{c-1}} \mathcal{O}_{\mathcal{T}_0}(-c\mathcal{H}_0 + b_{c-1}^k \mathcal{F}) \longrightarrow \cdots \longrightarrow \oplus_{k=1}^{\beta_2} \mathcal{O}_{\mathcal{T}_0}(-3\mathcal{H}_0 + b_2^k \mathcal{F})$$
$$\longrightarrow \oplus_{k=1}^{\beta_1} \mathcal{O}_{\mathcal{T}_0}(-2\mathcal{H}_0 + b_1^k \mathcal{F}) \longrightarrow \mathcal{O}_{\mathcal{T}_0} \longrightarrow \mathcal{O}_{S''} \longrightarrow 0.$$

Here $\beta_i = \beta_{i,i+1}$, *for* $i = 1, \ldots, c$, *and* F_c *is an extension of the non-zero term*

$$\oplus_{k=1}^{\beta_{c,c+2}} \mathcal{O}_{\mathcal{T}_0}(-(c+2)\mathcal{H}_0 + b_{c,c+2}^k \mathcal{F})$$

by the non-zero term

$$\oplus_{k=1}^{\beta_{c,c+1}} \mathcal{O}_{\mathcal{T}_0}(-(c+1)\mathcal{H}_0 + b_{c,c+1}^k \mathcal{F}).$$

Moreover F_i *is an extension of the term*

$$\oplus_{k=1}^{\beta_{i,i+2}} \mathcal{O}_{\mathcal{T}_0}(-(i+2)\mathcal{H}_0 + b_{i,i+2}^k \mathcal{F})$$

by the term

$$\oplus_{k=1}^{\beta_{i,i+1}} \mathcal{O}_{\mathcal{T}_0}(-(i+1)\mathcal{H}_0 + b_{i,i+1}^k \mathcal{F})$$

for $i = c+1, \ldots, \frac{1}{2}D^2 + c$.

Proof. Since the Betti-numbers are the same for all λ if $D^2 = 0$ by Example 8.9, the case $D^2 = 0$ is a direct application of [Sc, Thm. (3.2)]. The case $D^2 \geq 2$ follows from [Sc, Thm. (3.2)] and Corollary 8.18. □

We recall the definition $\beta_i = \beta_{i,i+1}$, for $i = 1, \ldots, c-1$, and $d = c + 2 + \frac{1}{2}D^2 = \dim \mathcal{T}$.

Proposition 8.24. *The* $b_{i,j}^k$ *and* b_i^k *in Proposition 8.23 satisfy the following polynomial equation in* n *(set* $b_{i,i+2}^k = \beta_{i,i+2}^k = 0$ *for all* i *and* k *if* $D^2 = 0$, *and set* $b_{i,i+1}^k = b_i^k$, *for* $i = 1, \ldots, c-1$ *for all values of* D^2*):*

$$\binom{n+d-1}{d-1}\left(\frac{n(g+1)}{d} + 1\right) - n^2(g+1+c+D^2) + \frac{1}{2}nD^2 - 2 =$$
$$\sum_{i=1}^{c+\frac{1}{2}D^2} (-1)^{i+1} \binom{n+d-i-2}{d-1} \left(\frac{((n-i-1)(g+1)+d)\beta_{i,i+1}}{d} + \sum_{k=1}^{\beta_{i,i+1}} b_{i,i+1}^k\right) +$$
$$\sum_{i=c}^{c+\frac{1}{2}D^2} (-1)^{i+1} \binom{n+d-i-3}{d-1} \left(\frac{((n-i-2)(g+1)+d)\beta_{i,i+2}}{d} + \sum_{k=1}^{\beta_{i,i+2}} b_{i,i+2}^k\right).$$

Proof. Denote the term i places to the left of $\mathcal{O}_{\mathcal{T}_0}$ in the resolution F_* by F_i. The result follows, similarly as in the proof of [Sc, Prop. 4.4(c)] from the identity

$$\chi(\mathcal{O}_{\mathcal{T}_0}(n\mathcal{H}_0)) - \chi(\mathcal{O}_{S''}(n\mathcal{H}_0)) = \sum_i (-1)^{i+1}\chi(F_i(n\mathcal{H}_0)).$$

To calculate $\chi(\mathcal{O}_{S''}(n\mathcal{H}_0))$ one uses Riemann-Roch on S'' and $\deg S'' = 2g + 2c + 2 + 2D^2$.

Moreover it is clear that for all large n, we have $\chi(F_i(n\mathcal{H})) = h^0(F_i(n\mathcal{H}))$, for all i, and $\chi(\mathcal{O}_{\mathcal{T}_0}(n\mathcal{H}_0)) = h^0(\mathcal{O}_{\mathcal{T}_0}(n\mathcal{H}_0))$ since $\mathcal{H}_0$ is (very) ample on $\mathcal{T}_0$. Then one uses (7.1) again. □

Remark 8.25. From the last result it is clear that the sums $\sum_{k=1}^{\beta_{i,j}} b_{i,j}^k$ are uniquely determined, but this does not necessarily apply to the $b_{i,j}^k$ individually. If $D^2 > 0$, it is not even a priori clear that the $b_{i,j}^k$ are independent of the choice of pencil inside $|D|$ giving rise to $\mathcal{T}(c, D, \{D_\lambda\})$.

Corollary 8.26. *(a) We have*

$$\sum_{k=1}^{\beta_{1,2}} b_{1,2}^k = (\frac{1}{2}D^2 + c - 1)g + (1 - c - D^2).$$

(b) If $D^2 > 0$, then

$$\sum_{k=1}^{\beta_{c+\frac{1}{2}D^2, c+\frac{1}{2}D^2+2}} b_{c+\frac{1}{2}D^2, c+\frac{1}{2}D^2+2}^k = g(\frac{1}{2}D^2 + 1) + 1.$$

Proof. We insert $n = 2$ in Proposition 8.24. That gives part (a) directly. Then we insert $n = 0$ in Proposition 8.24. That gives

$$-1 = g\beta_{c+\frac{1}{2}D^2, c+\frac{1}{2}D^2+2} - \sum_{k=1}^{\beta_{c+\frac{1}{2}D^2, c+\frac{1}{2}D^2+2}} b_{c+\frac{1}{2}D^2, c+\frac{1}{2}D^2+2}^k.$$

This immediately gives the statement of part (b), since it follows from Proposition 8.17 that

$$\beta_{c+\frac{1}{2}D^2, c+\frac{1}{2}D^2+2} = \frac{1}{2}D^2 + 1.$$

□

Example 8.27. We return to the situation studied in Example 8.20, with $D^2 = c = 2$.

From that example and Corollary 8.26 we see that $\beta_{3,5} = 2$ and

$$\sum_{k=1}^{\beta_{3,5}} b_{3,5}^k = 2g - 1.$$

We now apply Proposition 8.24:

Setting $n = 1$, we get nothing, but setting $n = 2$ we obtain

$$\sum_{k=1}^{\beta_1} b_1^k = 2g + 1 - \beta_{1,2} = 2g - 3.$$

Setting $n = 3$ we obtain

$$\sum_{k=1}^{\beta_{2,3}} b_{2,3}^k = 2g - 3 - \beta_{2,3} = 2g - 5.$$

Continuing this way, we find the difference of the $b_{3,4}^k$ and the $b_{2,4}^k$ in terms of $\beta_1, \beta_{2,3}, \beta_{2,4}, \beta_{3,4}$, by setting $n = 4$. This gives

$$\sum_{k=1}^{\beta_{3,4}} b_{3,4}^k - \sum_{k=1}^{\beta_{2,4}} b_{2,4}^k = (\beta_{2,3} - 4)g + (\beta_{2,3} + \beta_{3,4} - \beta_{2,4} - 6) = -2g - 7.$$

8.4.1 Pushing down resolutions

We will now "push down" results for S'' in $\mathcal{T}_0$ to results for S' in $\mathcal{T}$.

Definition 8.28. *We define, for integers a and b,*

$$\mathcal{O}_{\mathcal{T}}(a\mathcal{H} + b\mathcal{F}) := i_* \mathcal{O}_{\mathcal{T}_0}(a\mathcal{H} + b\mathcal{F}).$$

In particular, by the projection formula,

$$i_* \mathcal{O}_{\mathcal{T}_0}(a\mathcal{H}_0 + b\mathcal{F}) = i_* \mathcal{O}_{\mathcal{T}_0}(a\mathcal{H} + (a+b)\mathcal{F}) = \mathcal{O}_{\mathcal{T}}(a) \otimes i_*((a+b)\mathcal{F}).$$

We now return to the general situation. As a consequence of Proposition 8.23 we have the following result:

Proposition 8.29. *If $D^2 = 0$ and $c = 1$, then the $\mathcal{O}_{\mathcal{T}}$-resolution F_* of $\mathcal{O}_{S'}$ is*

$$0 \longrightarrow \mathcal{O}_{\mathcal{T}}(-3\mathcal{H} + (g-4)\mathcal{F}) \longrightarrow \mathcal{O}_{\mathcal{T}} \longrightarrow \mathcal{O}_{S'} \longrightarrow 0.$$

In all other cases we assume $b_i^k \geq i$, for $i = 1, \ldots, c-1$ and all k and $b_{i,j}^k \geq j-1$ for $j = i+1, i+2$, $i = c, \ldots, \frac{1}{2}D^2 + c$, and all k.

If $D^2 = 0$ and $c \geq 2$, then

$$\begin{aligned} 0 &\longrightarrow \mathcal{O}_{\mathcal{T}}(-(c+2)\mathcal{H} + (g-c-3)\mathcal{F}) \longrightarrow \oplus_{k=1}^{\beta_{c-1}} \mathcal{O}_{\mathcal{T}}(-c\mathcal{H} + (b_{c-1}^k - c)\mathcal{F}) \\ &\longrightarrow \cdots \longrightarrow \oplus_{k=1}^{\beta_1} \mathcal{O}_{\mathcal{T}}(-2\mathcal{H} + (b_1^k - 2)\mathcal{F}) \longrightarrow \mathcal{O}_{\mathcal{T}} \longrightarrow \mathcal{O}_{S'} \longrightarrow 0, \end{aligned}$$

is an $\mathcal{O}_{\mathcal{T}}$-resolution of $\mathcal{O}_{S'}$.

If $D^2 \geq 2$, and if there exists a resolution as described in Proposition 8.23, in particular if the Betti-numbers are the same for all $\{\varphi_L(D_\lambda)\}$, then $\mathcal{O}_{S'}$ has a $\mathcal{O}_{\mathcal{T}}$-resolution F'_* of the following type:

$$
\begin{aligned}
0 \longrightarrow F'_{\frac{1}{2}D^2+c} \longrightarrow \cdots \longrightarrow F'_{c+1} \longrightarrow F'_c \\
\longrightarrow \oplus_{k=1}^{\beta_{c-1}} \mathcal{O}_{\mathcal{T}}(-c\mathcal{H} + (b^k_{c-1} - c)\mathcal{F}) \longrightarrow \cdots \longrightarrow \oplus_{k=1}^{\beta_2} \mathcal{O}_{\mathcal{T}}(-3\mathcal{H} + (b^k_2 - 3)\mathcal{F}) \\
\longrightarrow \oplus_{k=1}^{\beta_1} \mathcal{O}_{\mathcal{T}}(-2\mathcal{H} + (b^k_1 - 2)\mathcal{F}) \longrightarrow \mathcal{O}_{\mathcal{T}} \longrightarrow \mathcal{O}_{S'} \longrightarrow 0.
\end{aligned}
$$

Here $F'_i = i_*(F_i)$, *for all* i.

Proof. See [Sc, p.117]. The essential fact is that the map $i : \mathcal{T}_0 \simeq \mathbf{P}(\mathcal{E}) \to \mathcal{T}$ is a rational resolution of singularities, and that we therefore have $R^1 i_* \mathcal{O}_{\mathcal{T}_0} = 0$. Moreover $i_* \mathcal{O}_{S''} = \mathcal{O}_{S'}$, and $i_* \mathcal{O}_{\mathcal{T}_0} = \mathcal{O}_{\mathcal{T}}$. The condition on the b_i and the $b_{i,j}$ gives that each term (except $\mathcal{O}_{S''}$) in the resolution of $\mathcal{O}_{S''}$ in Proposition 8.23 is an extension of terms of the form $\mathcal{O}_{\mathcal{T}_0}(a\mathcal{H} + b\mathcal{F})$, with $b \geq -1$. As in [Sc, (3.5)] we then conclude that the resolution therefore remains exact after pushing down. $\square$

Remark 8.30. By Proposition 8.6 we already know that the ideal of S' in $\mathcal{T}$ is the push-down by i of the ideal of S'' in $\mathcal{T}_0$.

If $b^k_2 \geq 2$ for all k when $c \geq 3$ or $D^2 = 0$ (resp. $b^k_{2,3} \geq 2$ and $b^k_{2,4} \geq 3$ when $c \leq 2$ and $D^2 > 0$), then it automatically follows that $R^1 i_* F_2 = R^1 i_* F_1 = 0$, so that we get an exact pushed-down right end

$$i_* F_1 \longrightarrow \mathcal{O}_{\mathcal{T}} \longrightarrow \mathcal{O}_{S'} \longrightarrow 0.$$

This means that the ideal of S' in $\mathcal{T}$ is generated by the push-down by i of the generators of the ideal of S'' in $\mathcal{T}_0$.

The next two results give the first examples of applications of the proposition.

Corollary 8.31. *Assume* $D^2 = 0$.

(a) If $c = 1$ *then* $\mathcal{O}_{S'}$ *has the following* $\mathcal{O}_{\mathcal{T}}$*-resolution:*

$$0 \longrightarrow \mathcal{O}_{\mathcal{T}}(-3\mathcal{H} + (g-4)\mathcal{F}) \longrightarrow \mathcal{O}_{\mathcal{T}} \longrightarrow \mathcal{O}_{S'} \longrightarrow 0. \qquad (8.26)$$

(b) If $c = 2$, *then* $\mathcal{O}_{S'}$ *has the following* $\mathcal{O}_{\mathcal{T}}$*-resolution:*

$$
\begin{aligned}
0 \longrightarrow \mathcal{O}_{\mathcal{T}}(-4\mathcal{H} + (g-5)\mathcal{F}) \longrightarrow \\
\mathcal{O}_{\mathcal{T}}(-2\mathcal{H} + a_1\mathcal{F}) \oplus \mathcal{O}_{\mathcal{T}}(-2\mathcal{H} + a_2\mathcal{F}) \longrightarrow \mathcal{O}_{\mathcal{T}} \longrightarrow \mathcal{O}_{S'} \longrightarrow 0,
\end{aligned}
$$

for two integers a_1 *and* a_2 *such that* $a_1 \geq a_2 \geq 0$ *and* $a_1 + a_2 = g - 5$.

Proof. Set $g_0 = g + c + 2$.

If $c = 1$, then this is a part of Proposition 8.29. The essence is as follows: By Proposition 8.9 a resolution of $\mathcal{O}_{S''}$ as an $\mathcal{O}_{\mathcal{T}_0}$-module is

$$0 \longrightarrow \mathcal{O}_{\mathcal{T}_0}(-3\mathcal{H}_0 + (g_0 - 4)\mathcal{F}) \longrightarrow \mathcal{O}_{\mathcal{T}_0} \longrightarrow \mathcal{O}_{S''} \longrightarrow 0.$$

Here $g_0 - 4 \geq 5$, whence (a) follows.

If $c=2$, we have $g_0 \geq 10$ and Proposition 7.2(a), gives that a resolution of $\mathcal{O}_{S''}$ as an $\mathcal{O}_{\mathcal{T}_0}$-module is:

$$0 \longrightarrow \mathcal{O}_{\mathcal{T}_0}(-4\mathcal{H}_0 + (g_0-5)\mathcal{F}) \longrightarrow$$
$$\mathcal{O}_{\mathcal{T}_0}(-2\mathcal{H}_0 + b_1\mathcal{F}) \oplus \mathcal{O}_{\mathcal{T}_0}(-2\mathcal{H}_0 + b_2\mathcal{F}) \longrightarrow \mathcal{O}_{\mathcal{T}_0} \longrightarrow \mathcal{O}_{S'} \longrightarrow 0,$$

for two integers b_1 and b_2 such that $b_1 \geq b_2 \geq 0$ and $b_1+b_2 = g_0-5$. From [Br, Thm. 5.1] we have that S'' is singular along a curve if $b_1 > \frac{2(e_1+e_2+e_3)}{3}$, where (e_1,e_2,e_3) denotes the type of $\mathcal{T}_0$. This is equivalent to $b_2 < \frac{g_0-9+2e_4}{3}$. Hence $b_2 \geq 1$ and (b) follows. □

Corollary 8.32. *Let $c=1$, $D^2=2$ and $g=6$ as in Example 8.19. Then $\mathcal{O}_{S''}$ has the following $\mathcal{O}_{\mathcal{T}_0}$-resolution:*

$$\begin{aligned} 0 &\longrightarrow \mathcal{O}_{\mathcal{T}_0}(-4\mathcal{H}_0+6\mathcal{F}) \oplus \mathcal{O}_{\mathcal{T}_0}(-4\mathcal{H}_0+5\mathcal{F}) \\ &\longrightarrow \mathcal{O}_{\mathcal{T}_0}(-2\mathcal{H}_0+4\mathcal{F}) \oplus \mathcal{O}_{\mathcal{T}_0}(-3\mathcal{H}_0+4\mathcal{F}) \oplus \mathcal{O}_{\mathcal{T}_0}(-3\mathcal{H}_0+3\mathcal{F}) \\ &\longrightarrow \mathcal{O}_{\mathcal{T}_0} \longrightarrow \mathcal{O}_{S''} \longrightarrow 0. \end{aligned}$$

In particular, $\mathcal{O}_{S'}$ has the following $\mathcal{O}_{\mathcal{T}}$-resolution:

$$\begin{aligned} 0 &\longrightarrow \mathcal{O}_{\mathcal{T}}(-4\mathcal{H}+2\mathcal{F}) \oplus \mathcal{O}_{\mathcal{T}}(-4\mathcal{H}+\mathcal{F}) \\ &\longrightarrow \mathcal{O}_{\mathcal{T}}(-2\mathcal{H}+2\mathcal{F}) \oplus \mathcal{O}_{\mathcal{T}}(-3\mathcal{H}+\mathcal{F}) \oplus \mathcal{O}_{\mathcal{T}}(-3\mathcal{H}) \\ &\longrightarrow \mathcal{O}_{\mathcal{T}} \longrightarrow \mathcal{O}_{S'} \longrightarrow 0. \end{aligned}$$

Proof. In Example 8.19 the minimal resolutions of all the $\varphi_L(D_\lambda)$ are given. Corollary 8.26 gives $b^1_{1,2}=4$ and $b^1_{2,4}+b^2_{2,4}=11$, while inserting $n=3$ in Proposition 8.24 gives $b^1_{1,3}+b^2_{1,3}=7$. Then Proposition 8.23 gives the following resolution:

$$\begin{aligned} 0 \longrightarrow \mathcal{O}_{\mathcal{T}_0}(-4\mathcal{H}_0+b^1_{2,4}\mathcal{F}) \oplus \mathcal{O}_{\mathcal{T}_0}(-4\mathcal{H}_0+(11-b^1_{2,4})\mathcal{F}) &\longrightarrow F_1 \\ \longrightarrow \mathcal{O}_{\mathcal{T}_0} \longrightarrow \mathcal{O}_{S''} &\longrightarrow 0, \end{aligned} \tag{8.27}$$

where F_1 is an extension

$$\begin{aligned} 0 &\longrightarrow \mathcal{O}_{\mathcal{T}_0}(-2\mathcal{H}_0+4\mathcal{F}) \longrightarrow F_1 \\ &\longrightarrow \mathcal{O}_{\mathcal{T}_0}(-3\mathcal{H}_0+b^1_{1,3}\mathcal{F}) \oplus \mathcal{O}_{\mathcal{T}_0}(-3\mathcal{H}_0+(7-b^1_{1,3})\mathcal{F}) \longrightarrow 0. \end{aligned} \tag{8.28}$$

Without loss of generality we assume $b := b^1_{1,3} \geq 4$, and $a := b^1_{2,4} \geq 6$. The type of $\mathcal{T}_0$ is $(3,2,1,1)$.

Look at the composite morphism given by (8.27) and (8.28)

$$\begin{aligned} \alpha : \mathcal{O}_{\mathcal{T}_0}(-4\mathcal{H}_0+a\mathcal{F}) &\oplus \mathcal{O}_{\mathcal{T}_0}(-4\mathcal{H}_0+(11-a)\mathcal{F}) \\ &\longrightarrow \mathcal{O}_{\mathcal{T}_0}(-3\mathcal{H}_0+b\mathcal{F}) \oplus \mathcal{O}_{\mathcal{T}_0}(-3\mathcal{H}_0+(7-b)\mathcal{F}). \end{aligned}$$

Now α can be expressed by a matrix

$$\begin{bmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_4 \end{bmatrix},$$

with

$$\begin{aligned}
&\alpha_1 \in H^0(\mathcal{H}_0 + (b-a)\mathcal{F}) \\
&\alpha_2 \in H^0(\mathcal{H}_0 + (a+b-11)\mathcal{F}) \\
&\alpha_3 \in H^0(\mathcal{H}_0 + (7-a-b)\mathcal{F}) \\
&\alpha_4 \in H^0(\mathcal{H}_0 + (a-b-4)\mathcal{F}),
\end{aligned}$$

whose determinant gives a section $g \in H^0(2\mathcal{H}_0 - 4\mathcal{F})$ whose zero scheme contains S''.

If $(a,b) \neq (6,4)$, we have

$$\begin{aligned}
&H^0(\mathcal{H}_0 + (7-a-b)\mathcal{F}) = \\
&H^0(\mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1}(10-a-b) \oplus \mathcal{O}_{\mathbf{P}^1}(9-a-b) \oplus \mathcal{O}_{\mathbf{P}^1}(8-a-b)^2) = 0,
\end{aligned}$$

whence $\alpha_3 = 0$ and g is a product of two sections of $\mathcal{H}_0 + (b-a)\mathcal{F}$ and $\mathcal{H}_0 + (a-b-4)\mathcal{F}$ respectively. But then S'' would have degenerate fibers S''_λ, contradicting Proposition 8.17(b).

So $(a,b) = (6,4)$ and we compute

$$\begin{aligned}
\mathrm{Ext}^1(\mathcal{O}_{\mathcal{T}_0}(-3\mathcal{H}_0 + 4\mathcal{F}) \oplus \mathcal{O}_{\mathcal{T}_0}(-3\mathcal{H}_0 + 3\mathcal{F}), \mathcal{O}_{\mathcal{T}_0}(-2\mathcal{H}_0 + 4\mathcal{F}) &= \\
H^1(\mathcal{H}_0) \oplus H^1(\mathcal{H}_0 + \mathcal{F}) &= \\
H^1(\mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1}(3) \oplus \mathcal{O}_{\mathbf{P}^1}(2) \oplus \mathcal{O}_{\mathbf{P}^1}(1)^2) &\oplus \\
H^1(\mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1}(4) \oplus \mathcal{O}_{\mathbf{P}^1}(3) \oplus \mathcal{O}_{\mathbf{P}^1}(2)^2) &= 0,
\end{aligned}$$

whence the sequence (8.28) splits and the first assertion follows.

The second is then an immediate consequence of Proposition 8.29. □

Note that by this result, S'' is cut out in $\mathcal{T}_0$ by three sections q, c_1 and c_2 of $\mathcal{O}_{\mathcal{T}_0}(2\mathcal{H}_0 - 4\mathcal{F})$, $\mathcal{O}_{\mathcal{T}_0}(3\mathcal{H}_0 - 4\mathcal{F})$ and $\mathcal{O}_{\mathcal{T}_0}(3\mathcal{H}_0 - 3\mathcal{F})$ respectively.

Now look at the three dimensional subvariety V of $\mathcal{T}_0$ defined by $q \in \mathcal{O}_{\mathcal{T}_0}(2\mathcal{H}_0 - 4\mathcal{F})$. Arguing as in the proof of Proposition 8.7 we find that the class of $i(V)$ in the Chow group of $\mathcal{T}$ is $2\mathcal{H}_\mathcal{T} - 2\mathcal{F}_\mathcal{T}$, whence $i(V)$ has degree 4 and dimension 3 in $\mathbf{P}^6$. As in [SD, (7.12)] we have that $i(V)$ is a cone over the Veronese surface (whose vertex is the image of Γ) and that this variety is the (reduced) intersection of all quadrics containing S'.

8.5 Rolling factors coordinates

A very useful result is the following, involving so called "rolling factors" coordinates (see for example [Har, p.59], [Ste, p.3] or [Re2]):

Lemma 8.33. *The sections of $a\mathcal{H} - b\mathcal{F}$ on a smooth rational normal scroll of type $(e_1, \ldots, e_d)$ can be identified with weighted-homogeneous polynomials of the form*

$$P = \sum P_{i_1,\ldots,i_d}(t,u) Z_1^{i_1} \ldots Z_d^{i_d},$$

where $i_1 + \cdots + i_d = a$, and $P_{i_1,\ldots,i_d}(t,u)$ is a homogeneous polynomial of degree $-b + (i_1 e_1 + \cdots + i_d e_d)$.

If we multiply P by a homogeneous polynomial of degree b in t, u, then we get a defining equation of the zero scheme of the section, in term of homogeneous coordinates of the projective space, within which the scroll is embedded. Here $X_{k,j} = t^j u^{e_k - j} Z_k$, for $k = 1, \ldots, d$, and $j = 0, \ldots, e_k$, are coordinates for this space. The equation is uniquely determined modulo the homogeneous ideal of the scroll.

As a first application, we prove the analogue of Corollary 8.31 for $c = 3$.

Corollary 8.34. *Let $D^2 = 0$ and $c = 3$. Then a resolution of $\mathcal{O}_{S'}$ as an $\mathcal{O}_{\mathcal{T}}$-module is:*

$$0 \longrightarrow \mathcal{O}_{\mathcal{T}}(-5\mathcal{H} + (g-6)\mathcal{F}) \longrightarrow \oplus_{i=1}^5 \mathcal{O}_{\mathcal{T}}(-3\mathcal{H} + a_i\mathcal{F}) \longrightarrow$$
$$\oplus_{i=1}^5 \mathcal{O}_{\mathcal{T}}(-2\mathcal{H} + b_i\mathcal{F}) \longrightarrow \mathcal{O}_{\mathcal{T}} \longrightarrow \mathcal{O}_{S'} \longrightarrow 0,$$

where all the a_i and $b_i \geq -1$ and satisfy $\sum_{i=1}^5 b_i = 2g - 12$ and $a_i = g - 6 - b_i$.

Proof. First recall that $g \geq 9$.

As a special case of Proposition 8.23 we obtain that the resolution of $\mathcal{O}_{S''}$ as an $\mathcal{O}_{\mathcal{T}_0}$-module is:

$$0 \longrightarrow \mathcal{O}_{\mathcal{T}_0}(-5\mathcal{H}_0 + (g-1)\mathcal{F}) \longrightarrow \oplus_{k=1}^5 \mathcal{O}_{\mathcal{T}_0}(-3\mathcal{H}_0 + b_2^k\mathcal{F}) \longrightarrow$$
$$\oplus_{k=1}^5 \mathcal{O}_{\mathcal{T}_0}(-2\mathcal{H}_0 + b_1^k\mathcal{F}) \longrightarrow \mathcal{O}_{\mathcal{T}_0} \longrightarrow \mathcal{O}_{S''} \longrightarrow 0.$$

From Corollary 8.26 we get $\sum_{i=1}^5 b_1^k = 2g - 2$. The self-duality of the resolution in this particular case gives $b_2^k = g - 1 - b_1^k$, if we for example order the b_1^k in a non-increasing way, and the b_2^k in a non-decreasing way.

We will show that for all $g \geq 9$ all the $b_1^k \geq 1$ and all the $b_2^k \geq 2$, so that we can push down the resolution to one of $\mathcal{O}_{S'}$ as an $\mathcal{O}_{\mathcal{T}}$-module.

Look at the map

$$\Phi : \oplus_{k=1}^5 \mathcal{O}_{\mathcal{T}_0}(-3\mathcal{H}_0 + b_2^k\mathcal{F}) \longrightarrow \oplus_{k=1}^5 \mathcal{O}_{\mathcal{T}_0}(-2\mathcal{H}_0 + b_1^k\mathcal{F}).$$

Just like in the analysis of pentagonal curves in [Sc], it follows from [B-E] that the map Φ is skew-symmetrical and that its Pfaffians generate the ideal of S'' in $\mathcal{T}_0$. See also [Wa].

Let the type of $\mathcal{T}_0$ be $(e_1, \ldots, e_5)$, where $e_5 = 1$ and $\sum_{i=1}^4 e_i = g$.

A key observation is the following: $b_1^1 \leq e_1 + e_3$ and $b_1^1 \leq 2e_2$. The first inequality holds, since otherwise we would have a quadratic relation of the form $f(t, u, Z_1, Z_2) = 0$ in each fiber. Hence the general fiber would be reducible, a contradiction. The second inequality follows since its negation implies that Z_1 is factor in one quadratic relation satisfied by the points of S'', a contradiction. This gives $b_1^1 \leq \frac{2(e_1+e_2+e_3)}{3} = \frac{2(g-e_4)}{3}$. Hence

$$b_2^1 = g-1-b_1^1 \geq g-1-\frac{2g}{3}+\frac{2e_4}{3} = \frac{g-3+2e_4}{3} \geq 2,$$

since $g \geq 9$. Hence $b_2^k \geq b_2^1 \geq 3$ for all k.

Another key observation is the following: $b_1^2 \leq e_1+e_4$ and $b_1^2 \leq e_2+e_3$. The first inequality holds, since otherwise the two-step projection of the general fiber D'' of S'' into the Z_1, Z_2, Z_3-plane from $P = (0,0,0,0,1)$ and $Q = (0,0,0,1,0)$ would be contained in 2 quadrics. This in only possible if the projected image is a line, and in that case the general fiber would be degenerate (contained in the $\mathbf{P}^3$ spanned by this line and P and Q). This is impossible. The second inequality holds, since otherwise there would be two independent relations of the form

$$Z_1 f(t,u,Z_1,\ldots,Z_5) + aZ_2^2 = 0$$

for each fiber. In that case we could eliminate the Z_2^2-term and obtain one relation with Z_1 as a factor, a contradiction.

These two inequalities for b_1^2 imply

$$b_1^2 \leq \frac{e_1+e_2+e_3+e_4}{2} = \frac{g}{2}.$$

Now we assume for contradiction that $b_1^5 \leq 0$. Then we get

$$b_2^5 - b_1^k \geq b_2^5 - b_1^2 = (g-1-b_1^5) - b_1^2 \geq g-1-\frac{g}{2} = \frac{g}{2}-1,$$

for $k = 2,3,4$. In the matrix description of the map Φ there is one submaximal minor with one column consisting of zero and sections of $\mathcal{H}_0 - (b_2^5 - b_1^k)\mathcal{F}$, for $k = 2,3,4$. If all entries of this column have Z_1 as a factor, that would lead to a contradiction, since the minor is the square of one of the generators of the (Pfaffian) ideal of S'' on $\mathcal{T}_0$. To avoid that Z_1 is a factor in each such entry, we must have $e_2 \geq \frac{g}{2} - 1$. This gives $e_1 + e_2 \geq g-2$, and $e_3 + e_4 \leq g - (e_1+e_2) \leq g-(g-2) = 2$. Hence $e_3 = e_4 = e_5 = 1$. But this implies that $D^2 + h^1(R) \geq 3$, contradicting Proposition 5.6.

Hence the assumption $b_1^5 \leq 0$ leads to a contradiction, and $b_1^k \geq 1$, for all k. Hence the entire resolution can be pushed down to one of $\mathcal{O}_{S'}$ as an $\mathcal{O}_{\mathcal{T}}$-module and the result follows. □

Remark 8.35. Assume that we are in the situation of Proposition 8.23 (i.e. the Betti-numbers of all the $\varphi_L(D_\lambda)$ are the same for all λ, for instance if $D^2 \leq 4$), so that a finite set of sections of line bundles of type $a\mathcal{H}_0 - b\mathcal{F}$ generate the ideal of the surface S'' on the smooth rational normal scroll $\mathcal{T}_0$ of type $(e_1,\ldots,e_{d-r-1},e_{d-r},\ldots,e_d)$, where $e_{d-r} = \cdots = e_d = 1$, $e_{d-r-1} \geq 2$, and $V = \operatorname{Sing} \mathcal{T} \simeq \mathbf{P}^{r-1}$ for some $r \geq 0$. Let $W = i^{-1}(V)$. This is a subscroll of $\mathcal{T}_0$ of type $(1, \ldots, 1)$, that is $\mathbf{P}^r \times \mathbf{P}^1$. The ideal generators can be classified into 3 types:

(a) Those that are sections of $a\mathcal{H}_0 - b\mathcal{F} = a\mathcal{H} - (b-a)\mathcal{F}$, with $b > a$.
(b) Those that are sections of $a\mathcal{H}_0 - b\mathcal{F} = a\mathcal{H} - (b-a)\mathcal{F}$, with $b = a$.
(c) Those that are sections of $a\mathcal{H}_0 - b\mathcal{F} = a\mathcal{H} - (b-a)\mathcal{F}$, with $b < a$.

For those of type (a) it is clear from Lemma 8.33 that their zero scheme contains W. Likewise one sees that those of type (b) can be written as

$$f(t, u, Z_1, \ldots, Z_d) + g(Z_{s+1}, \ldots, Z_d),$$

where Z_1 or Z_2 or ... or Z_s is a factor in every monomial of $f(t, u, Z_1, \ldots, Z_d)$, while g is homogeneous of degree $a = b$. Those of type (c) can be written as

$$f(t, u, Z_1, \ldots, Z_d) + h(t, u, Z_{s+1}, \ldots, Z_d),$$

where f is as described for type (b), while h is bihomogeneous, of degree $a - b > 0$ in t, u and degree a in $Z_{s+1}, \ldots, Z_d$.

There is one fundamental difference between the sections of types (a) and (b) on one hand, and those of type (c) on the other. Those of types (a) and (b) are “constant” on the fibers of i, their zero scheme contains either the whole fiber, or no point on the fiber, for each $\mathbf{P}^1$, which is a fiber of i. For the sections of type (c) this is only true if h is the zero polynomial, and then its zero scheme contains all of W.

We therefore see (referring to the notation of Proposition 8.23) that if $b_1^k \geq 2$, for $k = 1, \ldots, \beta_1$ (we must use the formulation $b_{1,j}^k \geq j$, for $j = 2, 3$ and $k = 1, \ldots, \beta_{1,j}^k$ in the special case (E0) in Corollary 8.32 with $c = 1$, and $D^2 = 2$) in the resolution of $\mathcal{O}_{S''}$ as an $\mathcal{O}_{\mathcal{T}_0}$-module, then the ideal of S'' is generated by “fiber constant” equations, and if Q is a point on $\mathcal{T}_0$ not on S'', then there is a fiber constant section of the type described, which does not contain Q in its zero scheme. In short, fiber constant equations cutting out S'' in $\mathcal{T}_0$ are also equations of S' in $\mathcal{T}$.

In Chapter 11 we will classify the possible projective models for $g \leq 10$, and in particular the singular scrolls $\mathcal{T}$ appearing as $\mathcal{T}(c, D, \{D_\lambda\})$ in the various cases, and we will also show that projective models giving scrolls of all these types exist.

8.6 Some examples

In Corollaries 8.31, 8.32 and 8.34 we showed that in some particular cases we can push down the entire resolution of $\mathcal{O}_{S''}$ to one of $\mathcal{O}_{S'}$. In the rest of this chapter, through a series of additional examples, we take a closer look at the rest of the singular scrolls appearing for $g \leq 10$, and using Lemma 8.33 we will find restrictions on the b_1^k.

Recall the b_1^k and β_i described in Proposition 8.23 (see also Remark 8.25). To make the notation simpler in the examples below, we give the following:

Definition 8.36. *Let b_k denote b_1^k, for $i = 1, \cdots \ \beta_1$.*

In all the examples below we have $D^2 \leq 4$, so by Corollary 8.18 the Betti-numbers of the $\varphi_L(D_\lambda)$ are independent of the λ.

Example 8.37. We return to the situation studied in Example 8.20 and Example 8.27, with $c = 2$, $D^2 = 2$ and $g \geq 7$. From Example 8.27 we see that the ideal of S'' in $\mathcal{O}_{\mathcal{T}_0}$ is generated by four sections of the type $2\mathcal{H}_0 - b_k\mathcal{F}$, where $\sum_{k=1}^4 b_k = 2g - 3$.

The type of $\mathcal{T}_0$ is $(e_1{+}1, e_2{+}1, e_3{+}1, 1, 1)$, where all $e_i \geq 0$ and $e_1{+}e_2{+}e_3 = g-4$. Let Q be the subscroll of $\mathcal{T}_0$ formed by the two last directrices, so Q is the inverse image by i of the line in $\mathbf{P}^g$ spanned by the images by φ_L of the basepoints of D. We see that Q is a quadric surface in $\mathbf{P}^3$. All the four sections of type $2\mathcal{H}_0 - b_k\mathcal{F}$ must intersect Q in, and therefore contain, the two lines that form the inverse image by i in $\mathcal{T}_0$ of the images by φ_L of the basepoints of D . But this is simply the two last directrices. The intersection with Q for one such section is obtained by using Lemma 8.33 to express each of the sections, and set $Z_1 = Z_2 = Z_3 = 0$. What remains must be a term of the type $P_{2-b_k}(t,u)Z_4Z_5$, where P_{2-b_k} is zero if $b_k \geq 3$, and a polynomial of degree $2 - b_k$ otherwise.

We order the b_k as $b_1 \geq b_2 \geq b_3 \geq b_4$. We see that if $b_4 \leq 1$, in particular if $b_4 \leq 0$, then $b_3 \leq 2$, since otherwise the total intersection of Q with the four sections will consist of $2 - b_4$ lines transversal to the two directrices in addition to the two directrices.

If $g = 7$, it is clear that $\mathcal{T}_0$ has type $(2,2,2,1,1)$ or $(3,2,1,1,1)$. Then Z_1 is a factor in all sections of $2\mathcal{H}_0 - b\mathcal{F}$ for $b \geq 5$ for both scroll types. Hence $b_1 \leq 4$. If $b_1 = 3$, then the only possible combination is $(b_1, \ldots, b_4) = (3,3,3,2)$, since $\sum b_k = 11$. If $b_1 = 4$, then the only a priori possibilities are $(4,4,2,1)$ and $(4,3,2,2)$. But $(4,4,2,1)$ is impossible for type $(2,2,2,1,1)$, since we then have two quadratic relations between Z_1, Z_2, Z_3 only. To see that this is impossible, let D'' be any smooth curve in $|f^*D - E|$, which can be identified with its image under φ_H. The variables Z_1, Z_2, Z_3 restricted to D'' correspond to sections of $(H-E)_{D_0}$. Since this line bundle has degree $2g(D'') = 4$, it is base point free and its sections map D'' into $\mathbf{P}^2$ by a one-to-one or two-to-one map. This means that there is at most one quadratic relation between Z_1, Z_2, Z_3, whence $b_2 \leq 3$. So for type $(2,2,2,1,1)$ the only possibilities for $(b_1, \ldots, b_4)$ are

$$(3,3,3,2) \quad \text{and} \quad (4,3,2,2).$$

For the type $(3,2,1,1,1)$, for each fiber of $\mathcal{T}_0$, the equations with $b_k \leq 2$, restricted to the subscroll $Z_1 = Z_2 = 0$ with plane fibers, must cut out a subscheme of length 4 (such that each subscheme of length 3 spans a $\mathbf{P}^2$) (these are the cases (E1) and (E2)). It takes 2 equations to do this. Hence

$b_3 \le 2$. Moreover $b_2 \le 3$. Assume $b_2 \ge 4$. Then we would have two independent equations of type

$$Z_1 f(t,u,Z_1,Z_2,Z_3,Z_4,Z_5) + Z_2^2$$

for general (t,u). From these equations we can eliminate the Z_2^2-term, and derive one equation with Z_1 as a factor. This is a contradiction.

Hence the only possibility for $(b_1,\dots,b_4)$ for the type $(3,2,1,1,1)$ is $(4,3,2,2)$.

If $g=8$, then $\mathcal{T}_0$ has type $(3,2,2,1,1)$. A similar argument as for $g=7$, gives that the only possible combinations for $(b_1,\dots,b_4)$ are

$$(5,4,2,2), \quad (5,3,3,2) \quad \text{and} \quad (4,4,3,2).$$

For $g=9$ the type of $\mathcal{T}_0$ is a priori either $(3,3,2,1,1)$ or $(4,2,2,1,1)$. In Chapter 11 the type $(4,2,2,1,1)$ is ruled out when D is perfect. For the type $(3,3,2,1,1)$ we see that $b_1 \ge 6$ is impossible, since $2\mathcal{H}_0 - 6\mathcal{F}$ only has sections of the form $f(Z_1,Z_2)$. Moreover $b_4 \le 2$, since we need to cut out the exceptional fibers. Hence $b_1 = 5$, otherwise the sum of the b_i would be at most 14, and it is 15. Any section of $2\mathcal{H}_0 - 5\mathcal{F}$ can be written in terms of t,u,Z_1,Z_2,Z_3 only. As for one case with $g=7$ we see that we cannot have two quadratic relations between Z_1,Z_2,Z_3 for general fixed (t,u), so $b_2 \le 4$. We then see that the only possible combination is $(b_1,b_2,b_3,b_4) = (5,4,4,2)$.

For $g=10$ the type of $\mathcal{T}_0$ is a priori $(3,3,3,1,1)$, $(4,3,2,1,1)$ or $(5,2,2,1,1)$. In Chapter 11 it is shown that only the type $(4,3,2,1,1)$ occurs when D is perfect. In this case a more detailed analysis gives that the only possible combinations for $(b_1,\dots,b_4)$ are

$$(6,5,4,2) \quad \text{and} \quad (5,5,5,2).$$

Example 8.38. Let us study the case $c=2$, $D^2=4$ and $g=9$. This gives a non-primitive projective model, with $L \simeq 2D$ (it is the case (Q) described in the text above Theorem 5.7). The scroll $\mathcal{T}$ necessarily has type $(2,1,1,0,0,0)$ and $\mathcal{T}_0$ has type $(3,2,2,1,1,1)$.

We will now find the b_k.

Lemma 8.33 gives $b_k \le 4$, for all k, since Z_1 is factor in every section of $2\mathcal{H}_0 - b\mathcal{F}$, for $b \ge 5$. The complete intersection $Z_1 = Z_2 = Z_3 = 0$ in $\mathcal{T}_0$ is a subscroll N of type $(1,1,1)$ with a plane in each fiber. The 7 equations cutting out S'' in $\mathcal{T}_0$ must together cut out four points in each plane fiber, such that no three of these points are collinear, by Theorem 5.7. It is clear that such a configuration of points is contained in exactly two quadrics in each plane. All sections of $2\mathcal{H}_0 - b\mathcal{F}$ with $b \ge 3$ vanish on N, so we must have at least two of the b_k less than 3. Moreover, every section of $2\mathcal{H}_0 - 4\mathcal{F}$ can be written

$$Z_1 f(t,u,Z_1,\dots,Z_5) + aZ_2^2 + bZ_2Z_3 + cZ_3^2.$$

If $b_4 \ge 4$, there are four independent equations of this kind, so we could eliminate the three last terms and obtain one relation with Z_1 as a factor. This is a contradiction, so $b_4 \le 3$.

This leaves the unique possibility $(b_1,\dots,b_7) = (4,4,4,3,3,2,2)$.

The following proposition describes the particular case in Proposition 5.10, that is we have $L \sim 2D$, $D^2 = 4$ and $c = 2$ and D is hyperelliptic, which is also a particular case of the last example. In this case there is a smooth curve E (which is a perfect Clifford divisor for D) satisfying $E^2 = 0$ and $E.D = 2$. Since $(D-E)^2 = 0$ and $(D-E).L = 4$, we have $D > E$, so E does not satisfy the conditions (C6) and (C7). We will also see that E is not always a perfect Clifford divisor for L.

Proposition 8.39. *Let L and D be as in the particular case of Proposition 5.10 (where $S' \subseteq \mathbf{P}^9$ is not the 2-uple embedding of $\varphi_D(S)$).*

Then we are in one of the following three cases:

(i) $\mathcal{R}_{L,E} = \emptyset$ and $D \sim E + E'$, where E' is a smooth elliptic curve such that $E.E' = 2$.
(ii) $\mathcal{R}_{L,E} = \{\Gamma_1, \Gamma_2\}$ and $D \sim 2E + \Gamma_1 + \Gamma_2$.
(iii) $\mathcal{R}_{L,E} = \{\Gamma_0\}$ and $D \sim 2E + \Delta_0$, where Δ_0 has a configuration with respect to E as in (E2).

Let $\mathcal{T} = \mathcal{T}(2, E)$ be the scroll defined by $|E|$.

In case (i), $\mathcal{T}$ is of type $(2, 2, 2, 0)$ and $\mathcal{O}_{S'}$ has the following $\mathcal{O}_{\mathcal{T}}$-resolution:

$$0 \longrightarrow \mathcal{O}_{\mathcal{T}}(-4\mathcal{H} + 4\mathcal{F}) \longrightarrow \mathcal{O}_{\mathcal{T}}(-2\mathcal{H} + 4\mathcal{F}) \oplus \mathcal{O}_{\mathcal{T}}(-2\mathcal{H}) \\ \longrightarrow \mathcal{O}_{\mathcal{T}} \longrightarrow \mathcal{O}_{S'} \longrightarrow 0.$$

In this case Sing $\mathcal{T} \cap S' = \emptyset$, so $S' \simeq S''$ where S'' sits in the smooth scroll $\mathcal{T}_0$ of type $(3, 3, 3, 1)$.

In the cases (ii) and (iii), $\mathcal{T}$ is of type $(4, 2, 0, 0)$ and its singular locus is spanned by $< Z_\lambda >$ (using the same notation as in Theorem 5.7). Moreover $\mathcal{O}_{S'}$ has the following $\mathcal{O}_{\mathcal{T}}$-resolution:

$$0 \longrightarrow \mathcal{O}_{\mathcal{T}}(-4\mathcal{H}+4\mathcal{F}) \longrightarrow \mathcal{O}_{\mathcal{T}}(-2\mathcal{H}+4\mathcal{F}) \oplus \mathcal{O}_{\mathcal{T}}(-2\mathcal{H}) \longrightarrow \mathcal{O}_{\mathcal{T}} \longrightarrow \mathcal{O}_{S'} \longrightarrow 0. \tag{8.29}$$

Proof. The three cases follow from Proposition 5.11, by noting that we clearly have $\mathcal{R}_{L,E} = \mathcal{R}_{D,E}$.

We have $h^0(L) = 10$ and $h^0(L-E) = 6$. We leave it to the reader to verify that in case (i) we have

$$h^0(L - 2E) = 3 \quad \text{and} \quad h^0(L - 3E) = 0,$$

and that we in the cases (ii) and (iii) have

$$h^0(L-2E) = 3, \quad h^0(L-3E) = 2, \quad h^0(L-4E) = 1 \quad \text{and} \quad h^0(L-5E) = 0.$$

This yields the two scroll types $(2, 2, 2, 0)$ and $(4, 2, 0, 0)$ respectively.

In the cases (ii) and (iii), one can show as in the proof of Theorem 5.7 that Sing $\mathcal{T} =< Z_\lambda >$.

The statement about the resolution in part (ii) and (iii) follows from Proposition 8.23 and the upper (large) table in Section 9.2.2 below. In case (i) the corresponding statement follows in part from these results. Proposition 8.23 and the table give that the resolution of $\mathcal{O}_{S''}$ in case (i) is

$$0 \longrightarrow \mathcal{O}_{\mathcal{T}_0}(-4\mathcal{H}_0 + 8\mathcal{F}) \longrightarrow \mathcal{O}_{\mathcal{T}_0}(-2\mathcal{H}_0 + 4\mathcal{F})^2 \longrightarrow \mathcal{O}_{\mathcal{T}_0} \longrightarrow \mathcal{O}_{S''} \longrightarrow 0,$$

or

$$\begin{aligned} 0 \longrightarrow \mathcal{O}_{\mathcal{T}_0}(-4\mathcal{H}_0 + 8\mathcal{F}) \longrightarrow \mathcal{O}_{\mathcal{T}_0}&(-2\mathcal{H}_0 + 6\mathcal{F}) \oplus \mathcal{O}_{\mathcal{T}_0}(-2\mathcal{H}_0 + 2\mathcal{F}) \\ &\longrightarrow \mathcal{O}_{\mathcal{T}_0} \longrightarrow \mathcal{O}_{S''} \longrightarrow 0. \end{aligned}$$

On the other hand it is clear that there are no contractions across the fibres in this case. Assume we have the upper of these two resolutions. From Lemma 8.33 we then get that S'' is cut out in $\mathcal{T}_0$ by two equations of the form:

$$P_1(t,u)Z_1^2 + P_2(t,u)Z_1Z_2 + P_3(t,u)Z_1Z_3 + P_4(t,u)Z_2^2 +$$

$$P_5(t,u)Z_2Z_3 + P_6(t,u)Z_3^2 + c_1Z + 1Z_4 + c_2Z_2Z_4 + c_3Z_3Z_4 = 0.$$

Here all the $P_i(t,u)$ are quadratic in t,u, and the c_j are constants. But both these quations contain the directrix line $(Z_1, Z_2, Z_3, Z_4) = (0,0,0,1)$ of $\mathcal{T}_0$. This is precisely the inverse image of Sing $\mathcal{T}$. Hence the inverse image of this line on S is contracted, a contradiction. Hence we are left with the lower of the two resolutions. From Corollary 8.31 we then get the given resolution of $\mathcal{O}_{S'}$. The last details in the description of case (i) follow from Remark 9.14. □

Remark 8.40. We see that in case (i) above, E is not perfect, since Sing $\mathcal{T}$ is a point, but there are no contractions across the fibers.

In the cases (ii) and (iii), E is however perfect.

These two cases are therefore included in the table on p. 148 (under scroll type $(4,2,0,0)$). However, also D is a perfect Clifford divisor, so these cases can also be described as the case with scroll type $(2,1,1,0,0,0)$ in the same table.

Example 8.41. Let us study the case $c = 3$, $D^2 = 2$ and $g = 9$. In Chapter 11 we will show that projective models with such invariants occur, and that the scroll type of $\mathcal{T}$ is either $(1,1,1,1,0,0)$ or $(2,1,1,0,0,0)$ when D is perfect. By Proposition 8.23 we have $\beta_{1,3} = 0$, and by Corollary 8.26 we have $\beta_{1,2} = 8$ and $\sum_{k=1}^{8} b_k = 3g - 4 = 23$.

Assume first that the type is $(1,1,1,1,0,0)$, which implies that $\mathcal{T}_0$ has type $(2,2,2,2,1,1)$. Lemma 8.33 gives $b_k \leq 4$, for all k, since $h^0(2\mathcal{H}_0 - b\mathcal{F}) = 0$ for $b \geq 5$. The complete intersection $Z_1 = Z_2 = Z_3 = Z_4 = 0$ in $\mathcal{T}_0$ is a subscroll Q of type $(1,1)$ with a line in each fiber. The 8 equations cutting out S'' in $\mathcal{T}_0$ must together cut out two points in each fiber of Q (the inverse image in S'' of Sing $\mathcal{T} \cap S'$). For a general fiber, call these points P_1 and P_2. Order the

b_k in a non-increasing way. To cut out the two points we must have $b_8 \leq 2$, since all sections of $2\mathcal{H}_0 - b\mathcal{F}$ vanish on Q for $b \geq 3$.

For general (t, u), where the fiber D'' of S'' is smooth, D'' is a smooth curve of degree 7 and genus 2, which can be identified with a smooth curve in $|f^*D - E|$. The complete linear system $|(H-E)_{D''}|$ is of degree $2g(D'')+1 = 5$ and in particular very ample. Now Z_1, Z_2, Z_3, Z_4 (restricted to D'') span $H^0((H-E)_{D''})$, which embeds D'' as a curve of degree 5 and genus 2 in $\mathbf{P}^3$. As in Example 8.19 we conclude from [Si] that this curve is contained in only one quadric surface. On the other hand all sections of $2\mathcal{H}_0 - 4\mathcal{F}$ can be expressed in terms of Z_1, Z_2, Z_3, Z_4 only. Hence no more than one of the b_k can be 4. This leaves us with only two possible cases:

$$(b_1, \ldots, b_8) = (4,3,3,3,3,3,2,2) \quad \text{or} \quad (3,3,3,3,3,3,3,2).$$

Assume now that the type is $(2,1,1,0,0,0)$, which implies that $\mathcal{T}_0$ has type $(3,2,2,1,1,1)$. Lemma 8.33 gives $b_k \leq 4$, for all k, since all sections of $2\mathcal{H}_0 - b\mathcal{F}$ have Z_1 as factor if $b \geq 5$. The complete intersection $Z_1 = Z_2 = Z_3 = 0$ in $\mathcal{T}_0$ is a subscroll N of type $(1,1,1)$ with a plane in each fiber. The 8 equations cutting out S'' in $\mathcal{T}_0$ must together cut out three independent points in each fiber of N (the inverse image in S'' of $\operatorname{Sing} \mathcal{T} \cap S'$).

Order the b_k as above. To cut out the three points we must have $b_6 \leq 2$, since all sections of $2\mathcal{H}_0 - b\mathcal{F}$ vanish on N for $b \geq 3$, and a net of three quadrics is needed to cut out three independent points in a plane. On the other hand every section of $2\mathcal{H}_0 - 4\mathcal{F}$ can be written

$$Z_1 f(t, u, Z_1, \ldots, Z_5) + aZ_2^2 + bZ_2Z_3 + cZ_3^2.$$

This gives $b_4 \leq 3$ as in Example 8.38. This leaves

$$(b_1, \ldots, b_8) = (4,4,4,3,2,2,2,2) \quad \text{or} \quad (4,4,3,3,3,2,2,2),$$

as the only possibilities.

Example 8.42. The case $c = 3$, $D^2 = 2$ and $g = 10$ is very similar to the analogous one for $g = 9$, treated in Example 8.41 and one can show in a similar way that

$$(b_1, \ldots, b_8) = (4,4,4,3,3,3,3,2).$$

We show in Chapter 11 that the only possible scroll type for $\mathcal{T}$ is $(2,1,1,1,0,0)$ when D is perfect, corresponding to the type $(3,2,2,2,1,1)$ for $\mathcal{T}_0$.

Example 8.43. In the case $c = 3$, $D^2 = 4$ and $g = 10$, it follows from Proposition 8.17 that the Betti-numbers of the $\varphi_L(D_\lambda)$ are independent of the λ. Now we have a projective model of type type (E0) (with $\beta_{1,2} = 12$ and $\sum b_k = 4g - 6 = 34$) One can show that

$$(b_1, \ldots, b_{12}) = (4,4,4,3,3,3,3,2,2,2,2,2).$$

In Chapter 11 it is shown that the only scroll type occurring for $\mathcal{T}$ is $(2,1,1,0,0,0,0)$, which means $(3,2,2,1,1,1,1)$ for $\mathcal{T}_0$.

Example 8.44. Let $D^2 = 0$ and $c = 3$, as in Corollary 8.34.

We will show in Chapter 11 that for $g = 9$ the only smooth scroll occurring as $\mathcal{T} = \mathcal{T}(3, D)$ is of type $(1,1,1,1,1)$, and the singular scrolls occurring are of types $(2,1,1,1,0)$, $(2,2,1,0,0)$ and $(3,1,1,0,0)$, corresponding to the smooth types $(3,2,2,2,1)$, $(3,3,2,1,1)$ and $(4,2,2,1,1)$ for $\mathcal{T}_0$. We also show that all these occur. By using similar techniques as in the previous examples, one can show that $(b_1, b_2, b_3, b_4) = (2,1,1,1)$ for the type $(1,1,1,1,1)$, $(b_1, b_2, b_3, b_4) = (4,3,3,3,3)$, $(4,4,3,3,2)$ or $(4,4,4,2,2)$ for the type $(3,2,2,2,1)$, and $(b_1, b_2, b_3, b_4) = (4,4,3,3,2)$ or $(4,4,4,2,2)$ for the types $(3,3,2,1,1)$ and $(4,2,2,1,1)$.

For $g = 10$ we will show in Chapter 11 that the only smooth scroll occurring as $\mathcal{T} = \mathcal{T}(3, D)$ is of type $(2,1,1,1,1)$, and the singular scrolls occurring are of types $(2,2,1,1,0)$, $(2,2,2,0,0)$, $(3,2,1,0,0)$, corresponding to the types $(3,3,2,2,1)$, $(3,3,3,1,1)$ and $(4,3,2,1,1)$ for $\mathcal{T}_0$. We also show that all these occur. Again one can show that $(b_1, b_2, b_3, b_4) = (2,2,2,1,1)$ for the type $(2,1,1,1,1)$, $(b_1, b_2, b_3, b_4) = (4,4,4,3,3)$ or $(4,4,4,4,2)$ for the scroll type $(3,3,3,1,1)$, and $(b_1, b_2, b_3, b_4) = (5,5,4,2,2)$, $(5,5,3,3,2)$, $(5,4,4,3,2)$, $(4,4,4,3,3)$ or $(4,4,4,4,2)$ for the scroll types $(3,3,2,2,1)$ and $(4,3,2,1,1)$.

9

More on projective models in smooth scrolls of $K3$ surfaces of low Clifford-indices

In this chapter we will have a closer look at the situation described in Chapter 7 for $c = 1, 2$ and 3. We recall that D is a free Clifford divisor on a non-Clifford general polarized $K3$ surface S, and that $\mathcal{T} = \mathcal{T}(c, D)$ is smooth, which is equivalent to the conditions $D^2 = 0$ and $\mathcal{R}_{L,D} = \emptyset$ when D is perfect. In any case these two conditions are necessary to have $\mathcal{T}$ smooth, and the pencil D_λ is uniquely determined. The resolution of $\mathcal{O}_{S'}$ as an $\mathcal{O}_{\mathcal{T}}$-module was given in Proposition 7.2.

By Corollaries 8.31 and 8.34 such resolutions exist also if $\mathcal{T}$ is singular if $D^2 = 0$. We will use this to take a closer look also at the situation for singular $\mathcal{T}(c, D)$ when $D^2 = 0$ and $c = 1, 2, 3$. We end the chapter with a statement valid for general c.

From the proof of Theorem 4.1 it is clear that for each of the possible combinations of c and g there is an 18-dimensional family of isomorphism classes of polarized $K3$ surfaces with smooth scroll $\mathcal{T}(c, D, \{D_\lambda\})$. Moreover it follows that there will be an $18 + \dim(\text{Aut } (\mathbf{P}^g)) = 17 + (g+1)^2$-dimensional family of such projective models of $K3$ surfaces. This is true, simply because there is only a finite number of linear automorphisms that leaves a smooth polarized $K3$ surface invariant.

For each value of c (and g) one can pose several questions about the set (or subscheme of the Hilbert scheme) of projective models S' of $K3$ surfaces S with elliptic free Clifford divisor D and such that $\mathcal{T}$ is smooth.

All scrolls of the same type are projectively equivalent, and hence the configuration of projective models of $K3$ surfaces in one such scroll is a projectively equivalent copy of that in another. Some questions one can pose, are: How many scrolls are there of a given type? How many projective models S' are there within each scroll? In how many scrolls of a given type is a given S' included?

The answer to the first question is well-known, the remaining ones we will study more closely.

We start with the following well-known result from [Har]:

T. Johnsen and A.L. Knutsen: LNM 1842, pp. 99–120, 2004.

Proposition 9.1. *The dimension of the set of scrolls of type* $\mathbf{e}$ *and dimension* d *in* $\mathbf{P}^g$ *is*

$$\dim(Aut\ (\mathbf{P}^g)) - \dim(Aut\ (\mathbf{P}(\mathcal{E}))) = (g+1)^2 - 3 - d^2 - \delta_1,$$

where $\delta_1 := \sum_{i,j} max\ (0, e_i - e_j - 1)$.

If $D^2 = 0$, we recall that $d = c + 2$ for the scroll $\mathcal{T}(c, D, \{D_\lambda\})$.

9.1 Projective models with $c = 1$

We have $g \geq 5$. Let the projective model S' and the smooth scroll $\mathcal{T} = \mathcal{T}(1, D)$ be given. As is seen from Proposition 7.2 the surface S' corresponds to the divisor class $3\mathcal{H} - (g-4)\mathcal{F}$ on $\mathcal{T}$. Moreover, part (c) of the proposition can be applied so we can obtain a resolution in $\mathbf{P}^g$. This is even minimal, by the comment in [Sc, Example 3.6].

Assume $\mathcal{T}$ has scroll type (e_1, e_2, e_3), with $e_1 \geq e_2 \geq e_3$

Proposition 9.2. *The (projective) dimension of the set of sections of divisor type* $3\mathcal{H} - (g-4)\mathcal{F}$ *in* $\mathcal{T}$ *is equal to* $29 + \delta_2$, *where* $\delta_2 := \sum max\ (0, g - 5 - \sum_{i=1}^{3} a_i e_i)$. *Here the first summation is taken over those triples* (a_1, a_2, a_3) *such that* $a_i \geq 0$, *for* $i = 1, 2, 3$, *and* $\sum_{i=1}^{3} a_i = 3$. *If* S' *is smooth, a general section is a smooth projective model of a* $K3$ *surface.*

Proof. We use the formula $h^0(\mathbf{P}(\mathcal{E}), a\mathcal{H} + b\mathcal{F}) = h^0(\mathbf{P}^1, \mathrm{Sym}\ ^a(\mathcal{E}) \otimes \mathcal{O}_{\mathbf{P}^1}(b))$, with $a = 3$ and $b = g - 4$. This gives $30 + \delta_2$. Being a smooth model of a $K3$ surface is an open condition on the set of sections of $3\mathcal{H} - (g-4)\mathcal{F}$, and since one section, the one giving S', is smooth, a general section of the linear system is so, too. □

We also have:

Proposition 9.3. *Each projective model* S' *of a* $K3$ *surface* S *of Clifford index 1 in* $\mathbf{P}^g$ *for* $g \geq 5$, *with* $\mathcal{T}(c, D, \{D_\lambda\})$ *smooth, is contained in only one smooth rational normal scroll of dimension 3.*

Proof. In [SD, Part (7.12)] it is shown that the scroll $\mathcal{T}$ is the intersection of all quadric hypersurfaces containing S'. Moreover, any other smooth scroll $\mathcal{T}_2$ containing S' is an intersection of quadric hypersurfaces ([SD, Prop. 1.5(ii)]), each of course containing S'. Hence $\mathcal{T}_2$ contains $\mathcal{T}$. Since the two scrolls have the same dimension and degree, they must be equal. □

From this we conclude:

Corollary 9.4. *Let a scroll type* (e_1, e_2, e_3) *be given, and let* δ_1 *and* δ_2 *be defined as above. Then* $\delta_1 \geq \delta_2$, *and there is a set of dimension*

$$(g+1)^2 + 17 + \delta_2 - \delta_1 = \dim(Aut\ (\mathbf{P}^g)) + 18 + \delta_2 - \delta_1,$$

parametrizing projective models of $K3$ *surfaces in smooth scrolls of the given type.*

Proof. From Proposition 9.1 we see that there is a $((g+1)^2 - 12 - \delta_1)$-dimensional set of scrolls of the same type as $\mathcal{T}$ in $\mathbf{P}^g$. We know that each S' in each such scroll is contained in only one scroll. In each scroll there is a $(29+\delta_2)$-dimensional set of projective models of $K3$ surfaces as described. We have $\delta_1 \geq \delta_2$, since otherwise there would be too many models with Clifford index 1. $\square$

Remark 9.5. For $c = 1$ we have

$$\delta_1 = max\ (0, e_1 - e_2 - 1) + max\ (0, e_1 - e_3 - 1) + max\ (0, e_2 - e_3 - 1),$$

and

$$\delta_2 = max\ (0, e_1 - e_2 - 3) + max\ (0, e_1 - e_3 - 3) + max\ (0, e_2 - e_3 - 3) + \\ max\ (0, e_1 + e_2 - 2e_3 - 3) + max\ (0, e_1 - 2e_2 + e_3 - 3).$$

Moreover $\delta_1 = 0$ if and only if the scroll type is maximally balanced, and $\delta_2 = 0$ if the scroll type is "reasonably well balanced". It is clear that $\delta_1 = 0$ implies $\delta_2 = 0$. We also see that if $5 \leq g \leq 8$, then $\delta_2 = 0$. Hence the cases $(e_1, e_2, e_3) = (3, 1, 1)$ or $(3, 2, 1)$ are cases where the number $(g+1)^2 + 17 + \delta_2 - \delta_1$ is strictly less than $(g+1)^2 + 17 = \dim(\mathrm{Aut}\ (\mathbf{P}^g)) + 18$, and it is clear that scrolls of these types cannot represent the general projective models with $c = 1$ and fixed g since by the construction as in Proposition 4.1 we get an 18-dimensional family of such models.

The inequality $\delta_1 \geq \delta_2$ does not follow directly from the formulas in Remark 9.5, for example since $\delta_1 < \delta_2$, for scroll types $(g-4, 1, 1)$, when $g \geq 11$. This enables us to conclude that these and other scroll types with $\delta_1 < \delta_2$ do not occur for the scrolls formed from projective models of $K3$ surfaces as described. This statement will be strengthened to apply for $g \geq 8$ below. On the other hand the mentioned type $(3, 1, 1)$ does occur for $g = 7$. This can be seen by using Lemma 8.33.

For $g = 7$ and type $(3, 1, 1)$ one then gets a polynomial P of the form:

$$P_{6,1}(t,u)Z_1^3 + P_{4,1}(t,u)Z_1^2Z_2 + P_{4,2}Z_1^2Z_3 + P_{2,1}(t,u)Z_1Z_2^2 + \\ P_{2,2}(t,u)Z_1Z_2Z_3 + P_{2,3}(t,u)Z_1Z_3^2 + c_1Z_2^3 + c_2Z_2^2Z_3 + c_3Z_2Z_3^2 + c_4Z_3^3,$$

where the $P_{i,j}$ are homogeneous of degree i, and the c_k are constants. For any fixed (t,u) and any fixed point in the $\mathbf{P}^2$ thus obtained, we see that we can avoid that point lying on the zero scheme of P by choosing the $P_{i,j}$ and c_k properly, so we conclude that the linear system $|3\mathcal{H} - (g-4)\mathcal{F}| = |3\mathcal{H} - 3\mathcal{F}|$ is base point free, and hence its general section is smooth, by Bertini. Irreducibility also follows by a similar argument.

Using Lemma 8.33, we see that for $g = 8$ any section of $3\mathcal{H} - (g-4)\mathcal{F} = 3\mathcal{H} - 4\mathcal{F}$ on a scroll of type $(3, 2, 1)$ can be identified with a P of the form

$$P_{5,1}(t,u)Z_1^3 + P_{4,1}(t,u)Z_1^2Z_2 + P_{3,1}(t,u)Z_1^2Z_3 + P_{3,2}(t,u)Z_1Z_2^2 + \\ P_{2,1}(t,u)Z_1Z_2Z_3 + P_{1,1}(t,u)Z_1Z_3^2 + P_{2,2}(t,u)Z_2^3 + P_{1,2}(t,u)Z_2^2Z_3 + c_1Z_2Z_3^2.$$

So here there is no Z_3^3-term, and from that we conclude that any section of $3\mathcal{H} - 4\mathcal{F}$ on a scroll of type $(3,2,1)$ must have the directrix line, say Γ, corresponding to $e_3 = 1$ as a part of its zero scheme. The fact that $\delta_2 < \delta_1$ indicates that if we can form smooth projective models of $K3$ surfaces this way, the surface must have a Picard lattice of higher rank than two. We may check this. If L, E and Γ sit inside a lattice of rank 2, then we can write $\Gamma = aL + bE$, for rational numbers a, b. In addition we must have $\Gamma^2 = -2$ and $L.\Gamma = E.\Gamma = 1$. It is easy to check that this is impossible.

It is clear that the set of base points of the linear system $3\mathcal{H} - 4\mathcal{F}$ is just the directrix line. This is true since for a fixed value of (t, u) (each fixed fiber) and a point Q outside $(0,0,1)$, we can just change the $P_{5,1}(t,u)Z_1^3$-term or the $P_{2,2}(t,u)Z_2^3$-term, if we want to avoid Q. If we choose $c_1 \neq 0$, then we obtain that the zero scheme of the corresponding section intersects all fibers in curves, smooth at $(0,0,1)$. (Here we set $Z_3 = 1$ in order to write the equation of the curve in affine coordinates around $(0,0,1)$. The existence of the non-zero linear term c_1Z_2 gives smoothness at this point.) Hence the zero scheme of a general section is smooth on all of $\mathcal{T}$. We have basically used the identities $3e_2 \geq g - 4$ and $e_2 + 2e_3 = g - 4$, to conclude as we do. See Remark 9.8 for references to other authors who have already used this kind of reasoning.

Using Lemma 8.33 again we see that if $g \geq 8$, then any section of $3\mathcal{H} - (g-4)\mathcal{F}$ on a scroll of type $(g-4,1,1)$ corresponds to a polynomial P with Z_1 as a factor, which means that its zero scheme must be reducible as a sum of sections $\mathcal{H} - (g-4)\mathcal{F}$ (a subscroll of type $(1,1)$) and $2\mathcal{H}$. Hence these scroll types cannot occur for $\mathcal{T}(c, D, \{D_\lambda\})$. In a similar way one can draw conclusions about sections on other scroll types. The observation above also has an interesting consequence for the types of singular scrolls $\mathcal{T}(c, D, \{D_\lambda\})$ arising for the case $c = 1$, $D^2 = 0$, $g \geq 5$.

Corollary 9.6. *The type of $\mathcal{T}(c, D, \{D_\lambda\})$ is never $(g-2,0,0)$ for $c = 1$ and $D^2 = 0$.*

Proof. Assume the type of $\mathcal{T}(c, D, \{D_\lambda\})$ is $(g-2,0,0)$. Then the type of the associated scroll $\mathcal{T}_0$ is $(g-1,1,1) = (g_0 - 4, 1, 1)$, and the divisor type of S'' in $\mathcal{T}_0$ is $3\mathcal{H}_0 - (g_0 - 4)\mathcal{F}$ (see Example 8.9 and the proof of Corollary 8.31). But we just observed that this is impossible. □

In general we conclude in the same way:

Proposition 9.7. *If a type (a,b,c) is impossible for a smooth scroll $\mathcal{T}(c, D, \{D_\lambda\})$ in $\mathbf{P}^g$ with $c = 1$ and $D^2 = 0$, then the type $(a-1, b-1, c-1)$ is impossible for any scroll $\mathcal{T}(c, D, \{D_\lambda\})$ in $\mathbf{P}^{g-3}$ with $c = 1$ and $D^2 = 0$.*

We will make a list including all possible scroll types for smooth $\mathcal{T}(c, D, \{D_\lambda\})$, for $g \leq 13$, with $c = 1$ and $D^2 = 0$. By the previous lemma, this will give a list including all scroll types of $\mathcal{T}(c, D, \{D_\lambda\})$, smooth or singular, for $g \leq 10$ and $c = 1$ and $D^2 = 0$. In the column with headline "# mod." we give the value of $18 - \delta_1 + \delta_2$.

The information in the list is essentially contained in [Re2] and [Ste, p.8-10]. We include it for completeness, and for the benefit of the reader we also include, in Remark 9.8 below, a few words about how the information can be obtained.

g	scroll type	# mod.	g	scroll type	# mod.	g	scroll type	# mod.
5	$(1,1,1)$	18	9	$(3,2,2)$	18	12	$(5,3,2)$	16
6	$(2,1,1)$	18	10	$(5,2,1)$	16	12	$(4,4,2)$	17
7	$(3,1,1)$	16	10	$(4,3,1)$	17	12	$(4,3,3)$	18
7	$(2,2,1)$	18	10	$(4,2,2)$	16	13	$(7,3,1)$	18
8	$(3,2,1)$	17	10	$(3,3,2)$	18	13	$(6,3,2)$	16
8	$(2,2,2)$	18	11	$(5,3,1)$	17	13	$(5,4,2)$	17
9	$(4,2,1)$	16	11	$(4,3,2)$	17	13	$(5,3,3)$	16
9	$(3,3,1)$	17	11	$(3,3,3)$	18	13	$(4,4,3)$	18
			12	$(6,3,1)$	17			

This gives the following possibilities for singular types:

g	singular scroll types
5	$(2,1,0)$
6	$(3,1,0)$, $(2,2,0)$
7	$(4,1,0)$, $(3,2,0)$
8	$(4,2,0)$
9	$(5,2,0)$
10	$(6,2,0)$

The dimensions of the families on the singular scrolls of type (e_1, e_2, e_3) in $\mathbf{P}^g$ are equal to those of type (e_1+1, e_2+1, e_3+1) on the corresponding smooth scrolls in $\mathbf{P}^{g+3}$.

Remark 9.8. Among the smooth scroll types listed above, we may immediately conclude that a general section of $3\mathcal{H} - (g-4)\mathcal{F}$ is smooth, and hence a smooth projective model of a $K3$ surface, for the types

$$(1,1,1), (2,1,1), (3,1,1), (2,2,1), (2,2,2), (3,2,2),$$
$$(4,2,2), (3,3,2), (3,3,3), (4,3,3), (5,3,3), (4,4,3).$$

These are the ones with $3e_3 \geq g-4$. The last inequality implies that the complete linear system $3\mathcal{H} - (g-4)\mathcal{F}$ has no base points, and hence a Bertini argument gives smoothness of the general section. The remaining types have the third directrix (of degree e_3) as base locus. We have seen above that for the type $(3,2,1)$ the zero scheme of the general section of $3\mathcal{H} - (g-4)F$ is smooth, since $3e_2 \geq g-4$ and $e_2 + 2e_3 = g-4$. The same identities hold for the types $(4,3,2)$, $(4,4,2)$ and $(5,4,2)$ also, so the zero scheme of a general section of $3\mathcal{H} - (g-4)\mathcal{F}$ is smooth for these types too.

A similar argument can be made for the types $(3,3,1)$, $(4,3,1)$, $(5,3,1)$, $(6,3,1)$ and $(7,3,1)$. Here the identity $3e_3 < g-4$ gives that the third directrix curve (a line) consists of base points for the linear system. The identity $3e_2 \geq g-4$ gives that there are no other base points. The identity $e_1 + 2e_3 = g-4$ gives that in each fiber the curve that arises as the intersection of that fiber and the zero scheme of a section of $3\mathcal{H} - (g-4)\mathcal{F}$ is smooth at $(0,0,1)$, provided we choose the section with a non-zero $cZ_1Z_3^2$-term. The total zero scheme is also smooth then.

The remaining possible smooth scroll types for $g \leq 13$ on the list above are different, in the sense that for a general section of $3\mathcal{H}-(g-4)\mathcal{F}$, the zero scheme of the section is singular at finitely many points. It turns out that for these types, which are $(4,2,1)$, $(5,2,1)$, $(5,3,2)$ and $(6,3,2)$, the general zero schemes are singular at exactly one point each.

The reason is the following: Since $3e_3 < g-4$, the third directrix curve consists of base points for the linear system. Since $3e_2 \geq g-4$ for these types, there are no other base points. Since $e_2 + 2e_3 < g-4$, there is no $Z_2Z_3^2$-term for any section. Since $e_1 + 2e_3 > g-4$, in fact $e_1 + 2e_3 = (g-4)+1$ for all these types, there is no $cZ_1Z_3^2$-term with c a constant, but there is an $L(t,u)Z_1Z_3^2$-term with $L(t,u)$ a linear expression in t and u. If the section is chosen general, $L(t,u)$ is not identically equal to zero. For all fixed (t,u) where $L(t,u) \neq 0$, the zero scheme of the section of $3\mathcal{H} - (g-4)\mathcal{F}$ is then smooth. For the single zero of $L(t,u)$, the zero scheme is however singular.

A comment about the types not appearing on the list above: The smooth scroll types $(4,4,1), (5,4,1), (6,4,1), (5,5,1)$ are eliminated the following way: A section of $3\mathcal{H}-(g-4)\mathcal{F}$ can have no term containing a $Z_3^2Z_i$-term for these scroll types. Hence all plane cubics in the pencil are singular where they meet the linear directrix. But this is a contradiction, since the general element in the pencil $|D|$ is smooth. Here we have used the identity $e_1 + 2e_3 < g-4$ for these types. The other types are eliminated because Z_1 must be a factor in each relevant section, a contradiction. These are the types with $3e_2 < g-4$.

The necessary and sufficient condition "$e_1 + 2e_3 < g-4$ or $3e_2 < g-4$" for eliminating scroll types is given in [Re2], as quoted in [Ste, Lemma 1.8.]. In [Ste, p.9] one also describes on which scroll types a general section of $3\mathcal{H} - (g-4)\mathcal{F}$ is singular in a finite number of points.

In Chapter 11 we will show that all the types listed above for $g \leq 10$ actually occur.

Remark 9.9. One does not have to use the resolution from Proposition 7.2 to see that a projective model S' of a $K3$ surface of Clifford index one in a smooth scroll $\mathcal{T}$ as above must be of divisor type $3\mathcal{H} - (g-4)\mathcal{F}$ in $\mathcal{T}$. Define the vector space $W = H^0(\mathcal{J}_{S'}(3))/H^0(\mathcal{J}_{\mathcal{T}}(3))$. In a natural way W represents the space of cubic functions on $\mathcal{T}$ that vanish on S'.

Study the exact sequences:

$$0 \longrightarrow H^0(\mathcal{J}_{S'}(3)) \longrightarrow H^0(\mathcal{O}_{\mathbf{P}^g}(3)) \longrightarrow H^0(\mathcal{O}_{S'}(3)) \longrightarrow H^1(\mathcal{J}_{S'}(3)) \longrightarrow 0$$

and

$$0 \longrightarrow H^0(\mathcal{J}_{\mathcal{T}}(3)) \longrightarrow H^0(\mathcal{O}_{\mathbf{P}^g}(3)) \longrightarrow H^0(\mathcal{O}_{\mathcal{T}}(3)) \longrightarrow H^1(\mathcal{J}_{\mathcal{T}}(3)) \longrightarrow 0.$$

One obtains $\dim W = h^0(\mathcal{J}_{S'}(3)) - h^0(\mathcal{J}_{\mathcal{T}}(3)) = g-3$, since $h^0(\mathcal{O}_{\mathcal{T}}(3)) - h^0(\mathcal{O}_{S'}(3)) = g-3$ and $h^1(\mathcal{J}_{S'}(3)) = h^1(\mathcal{J}_{\mathcal{T}}(3)) = 0$ (see [SD, Prop 1.5(i)]). Take $g-3$ arbitrary fibers F of the ruling on $\mathcal{T}$, that is $g-3$ planes. For each plane it is one linear condition on the elements in W to contain it (since this is equivalent to contain an point in the plane outside S'). Hence containing all the $g-3$ planes imposes $g-3$ conditions. These conditions must be independent, since otherwise there would be a cubic in $\mathbf{P}^g$, not containing $\mathcal{T}$, and containing the union of S' and $g-3$ planes. This union has degree $(2g-2)+(g-3) = 3g-5$. But by Bezout's theorem the cubic and $\mathcal{T}$ intersect in a surface of degree $3g-6$. Hence, in particular, any choice of $g-4$ planes gives independent conditions, and there is one, and only one, hypercubic (modulo the ideal of $\mathcal{T}$), which contains S' and $g-4$ planes in the pencil. By Bezout's theorem, it does not contain more. Hence S' in a natural way is a section of $3\mathcal{H} - (g-4)\mathcal{F}$.

9.2 Projective models with $c = 2$

Let $\mathcal{T} = \mathcal{T}(2, D)$ with $D^2 = 0$. We have $g \geq 7$. Denote the type of $\mathcal{T}$ by (e_1, e_2, e_3, e_4). Proposition 7.2(a) (or Corollary 8.31 if $\mathcal{T}$ is not smooth) gives that S' is a complete intersection in $\mathcal{T}$ of two divisors of type $2\mathcal{H} - b_1\mathcal{F}$ and $2\mathcal{H} - b_2\mathcal{F}$. By convention, we set $b_1 \geq b_2$. Part (d) of the same proposition gives the well-known fact that $b_1 + b_2 = g-5$. Such a situation has been thoroughly investigated in [Br]. As already mentioned in the proof of Corollary 8.34, it follows from [Br, Thm. 5.1] that S' is singular along a curve if $b_1 > \frac{2(e_1+e_2+e_3)}{3}$, or equivalently $b_2 < \frac{g-9+2e_4}{3}$. Hence $b_2 \geq 0$ for $g \geq 7$, for complete intersections with only finitely many singularities. Since in particular $b_2 \geq -1$, it is clear that part (b) and (c) of Proposition 7.2 can be used to give a resolution of S' in $\mathbf{P}^g$. This resolution is minimal because of the comment in [Sc, Example 3.6]. The fact that $b_2 \geq 0$ means that S' can be viewed geometrically as a complete intersection of one hyperquadric containing b_1 three-planes in the pencil, and another containing b_2 three-planes (throwing away the three-planes and taking the closure of what remains).

Let us study projective models in smooth scrolls for $c = 2$ and $D^2 = 0$ in general. We see from Proposition 9.1 that the set parametrizing the scrolls having the same type as $\mathcal{T}$ has dimension $(g+1)^2 - 19 - \delta_1 = \dim(\mathrm{Aut}\ (\mathbf{P}^g)) - 18 - \delta_1$, where $\delta_1 := \sum_{i,j} \max\ (0, e_i - e_j - 1)$. Therefore one expects the set of projective models of smooth $K3$ surfaces in each scroll to have dimension 36 if the scroll type is reasonably well balanced, to get a set of total dimension $\dim(\mathrm{Aut}\ (\mathbf{P}^g)) + 18$. This "expectation" is based on the natural assumption that a set of total dimension $\dim(\mathrm{Aut}\ (\mathbf{P}^g)) + 18$ arises from S' that sit inside maximally balanced scrolls. In this case there are two different sources of imbalance; that of the e_i, and that of the b_k. We will look more closely at this.

Set $\delta_2 := \max\,(0, b_1 - b_2 - 1)$. Assume first $b_1 > b_2$. We calculate

$$\dim |\mathcal{O}_{\mathcal{T}}(2\mathcal{H} - b_1\mathcal{F})| = 5g - 6 - 10b_1 + \delta_3,$$

where $\delta_3 := h^1(\mathrm{Sym}\ ^2\mathcal{E} \otimes \mathcal{O}_{\mathbf{P}^1}(-b_1)) = 0$ if and only if $e_4 \geq \frac{b_1-1}{2}$.

By the same sort of calculation we of course get

$$\dim |\mathcal{O}_{\mathcal{T}}(2\mathcal{H} - b_2\mathcal{F})| = 5g - 6 - 10b_2 + \delta_4,$$

where $\delta_4 := h^1(\mathrm{Sym}\ ^2\mathcal{E} \otimes \mathcal{O}_{\mathbf{P}^1}(-b_2)) = 0$ if and only if $e_4 \geq \frac{b_2-1}{2}$.

Now fix a zero scheme Y of a section s of $2\mathcal{H} - b_1\mathcal{F}$, and study the exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathcal{T}}((b_1 - b_2)\mathcal{F}) \longrightarrow \mathcal{O}_{\mathcal{T}}(2\mathcal{H} - b_2\mathcal{F}) \longrightarrow \mathcal{O}_Y(2\mathcal{H} - b_2\mathcal{F}) \longrightarrow 0.$$

This induces a sequence

$$\begin{aligned} 0 \longrightarrow H^0(\mathcal{O}_{\mathcal{T}}((b_1 - b_2)\mathcal{F})) &\longrightarrow H^0(\mathcal{O}_{\mathcal{T}}(2\mathcal{H} - b_2\mathcal{F})) \\ \longrightarrow H^0(\mathcal{O}_Y(2\mathcal{H} - b_2\mathcal{F})) &\longrightarrow H^1(\mathcal{O}_{\mathcal{T}}((b_1 - b_2)\mathcal{F})). \end{aligned}$$

Since $h^0(\mathcal{O}_{\mathcal{T}}((b_1-b_2)\mathcal{F})) = b_1 - b_2 + 1$ and $h^1(\mathcal{O}_{\mathcal{T}}((b_1-b_2)\mathcal{F})) = 0$, we obtain

$$\begin{aligned} \dim |\mathcal{O}_Y(2\mathcal{H} - b_2\mathcal{F})| &= \dim |\mathcal{O}_{\mathcal{T}}(2\mathcal{H} - b_2\mathcal{F})| - (b_1 - b_2 + 1) \\ &= 5g - 6 - 10b_2 + \delta_4 - (b_1 - b_2 + 1) \\ &= 5g - 8 - 10b_2 + \delta_4 - \delta_2. \end{aligned}$$

Summing up, we obtain

$$\begin{aligned} &\dim \mathcal{O}_{\mathcal{T}}(2\mathcal{H} - b_1\mathcal{F}) + \dim \mathcal{O}_Y(2\mathcal{H} - b_2\mathcal{F}) \\ &= 5g - 6 - 10b_1 + \delta_3 + 5g - 8 - 10b_2 + \delta_4 - \delta_2 \\ &= 10g - 14 - 10(b_1 + b_2) - \delta_2 + \delta_3 + \delta_4 \\ &= 36 - \delta_2 + \delta_3 + \delta_4. \end{aligned}$$

In particular, if g is even, $b_1 - b_2 = 1$ and $e_4 \geq \frac{b_1-1}{2} = \frac{g-6}{4}$, we get 36. We remark that $\frac{g-6}{4}$ is just $\frac{3}{4}$ less than the average values of the e_i. In general there is thus a $(36 - \delta_2 + \delta_3 + \delta_4)$-dimensional set of complete intersections of type $(2\mathcal{H} - b_1\mathcal{F}, 2\mathcal{H} - b_2\mathcal{F})$, provided the section s is uniquely determined. The latter fact follows, for example by the same kind of argument as in [Sc, (6.2)], , where scrolls arising from tetragonal curves are treated (see also the proof of Proposition 9.12 below).

We now assume $b_1 = b_2 (= b = \frac{g-5}{2})$, which can only occur if g is odd. Then $h^0(2\mathcal{H} - b\mathcal{F}) = h^0(\mathbf{P}^1, \mathrm{Sym}\ ^2(\mathcal{E}) \otimes \mathcal{O}_{\mathbf{P}^1}(b))$. This number is of the form $20 + \delta_3 = 20 + \delta_4$, where δ_3 and δ_4 are defined as in the case above. We see that

$\delta_3 = \delta_4 = 0$ if and only if $2e_4 - b \geq -1$, or equivalently $e_4 \geq \frac{g-7}{4}$. The average value of the e_i is $\frac{g-3}{4}$, which is just one more. The set of complete intersections corresponds to an open set in the Grassmannian $G(2, h^0(2\mathcal{H} - b\mathcal{F}))$, which has dimension $36 + 2\delta_3$. Hence we get the expected number if $b_1 = b_2$, and the e_i are well balanced. Whether $b_1 = b_2$ or not, we have now proved: Let a scroll type $e_1, e_2, e_3, e_4)$ be given, and let $\delta_2, \delta_3, \delta_4$ be defined as above in this section, and let δ_1 be defined as in Proposition 9.1

Proposition 9.10. *The set of complete intersections of type* $(2\mathcal{H} - b_1\mathcal{F}, 2\mathcal{H} - (g-5-b_1)\mathcal{F})$ *on a smooth rational normal scroll of dimension* 4 *in* $\mathbf{P}^g$ *of type* (e_1, e_2, e_3, e_4) *is either empty or of dimension* $36 - \delta_2 + \delta_3 + \delta_4$.

Corollary 9.11. *The set of complete intersections of type* $(2\mathcal{H} - b_1\mathcal{F}, 2\mathcal{H} - (g-5-b_1)\mathcal{F})$ *with no or finitely many singularities on a smooth rational normal scroll of dimension* 4 *in* $\mathbf{P}^g$ *of type* (e_1, e_2, e_3, e_4)*, is either empty or of dimension at least* 36 *if* $\delta_1 \geq 1$.

Proof. Set $s = b_1 - \frac{g-5}{2}$. If $s = 0$, then $\delta_2 = 0$, and there is nothing to prove. If $s \geq \frac{1}{2}$, then $\delta_2 = 2s - 1$. We split into 4 subcases: $e_1 + e_2 + e_3 + e_4 = 4e + h$, where $h = 0, 1, 2, 3$. If $h = 0, 1$, we have if $\delta_1 \geq 1$: $2e_4 \leq 2(\lfloor \frac{g-3}{4} \rfloor - 1) \leq \frac{g-7}{2}$, and $e_3 + e_4 \leq \frac{g-3}{2} - 1 = \frac{g-5}{2}$. This gives $b_1 - 2e_4 - 1 \geq \frac{g-5}{2} + s - \frac{g-7}{2} - 1 = s$, and $b_1 - (e_3 + e_4) - 1 \geq \frac{g-5}{2} + s - \frac{g-1}{2} - 1 = s - 1$. Hence

$$\delta_3 = h^1(\mathbf{P}^1, \mathrm{Sym}^{\,2}(\mathcal{E}) \otimes \mathcal{O}_{\mathbf{P}^1}(b_1)) \geq s + (s-1) = 2s - 1 = \delta_2,$$

and then the result follows from Proposition 9.10.

If $h = 3$, essentially the same method works (look at the three terms $b_1 - 2e_3 - 1$, $b_1 - e_3 - e_4 - 1$ and $b_1 - 2e_4 - 1$).

If $h = 2$, then essentially the same method works (look at the six terms of the form $b_1 - e_i - e_j - 1$, for $i, j = 2, 3, 4$), except in the case $s = 1$, and $(e_1, e_2, e_3, e_4) = (e+2, e, e, e)$. But in that case $b_1 = \frac{g-1}{2} + 1 = 2e + 1$. Using Lemma 8.33 we see that that Z_1 is a factor in every section of $2\mathcal{H} - b_1\mathcal{F}$, so this case simply does not occur. See also the appendix of [Br]. □

If we add the assumption that there exists at least one smooth model S' of a given intersection type, giving rise to a scroll of a given type, then we can conclude that there is a set of dimension $(36 - \delta_2 + \delta_3 + \delta_4)$ parametrizing smooth projective models S' in a scroll of the given type. We see that if the scroll type or (b_1, b_2)-type is unbalanced, the dimension of the set of complete intersections (smooth or singular) can a priori go up, or it can go down from its "expected" value 36. From the last corollary, however, we see that the dimension goes down only if the scroll type is maximally balanced, and the intersection type is not.

If we just start with an arbitrary scroll type (e_1, e_2, e_3, e_4), with $e_4 \geq 1$, $\sum_{i-1}^{c+2} e_i = g - 3$ and "intersection type" $b_1 \geq b_2 \geq 0$, with $b_1 + b_2 = g - 5$, it is an intricate question to decide whether there are any that are smooth projective models of $K3$ surface, or any that are not singular along a curve. This problem is studied in detail in [Br], and we will study the cases for low g in Section 9.2.2.

The question whether a projective model S' can be included in several scrolls of the same type simultaneously is not as simple to answer as in the case $c = 1$. In the case $c \geq 2$ the scroll $\mathcal{T}$ is no longer the intersection of the hyperquadrics containing S'; in fact S' itself is that intersection [SD]. The question is essentially how many divisor classes of elliptic curves E with $E.L = c + 2$ there are on S.

On the other hand it is clear that a projective model cannot be contained in a positive dimensional set of scrolls of the type in question. There is a discrete family of divisor classes on S, so the dimension of the family of smooth projective models of $K3$ surfaces on scrolls of the type in question is now, as for $c = 1$, equal to the sum of the dimension of the set of scrolls of a given type and the dimension of the set of smooth projective models of that type. This sum is

$$\begin{aligned}&(g+1)^2 - 19 - \delta_1 + (36 - \delta_2 + \delta_3 + \delta_4)\\ &= \dim(\text{Aut }(\mathbf{P}^g)) + 18 - \delta_1 - \delta_2 + \delta_3 + \delta_4.\end{aligned}$$

To obtain the dimension of the set of projective equivalence classes, since only a finite number of automorphisms of $\mathbf{P}^g$ fixes a $K3$ surface, we subtract the number $\dim(\text{Aut }(\mathbf{P}^g))$ and get

$$18 - \delta_1 - \delta_2 + \delta_3 + \delta_4.$$

By Theorem 4.1 this number is equal to 18 for at least one scroll type, where there exists smooth complete intersections of that type, and where the fibers of the complete intersection represent a free Clifford divisor on S with $c = 2$. It is clear that if we choose the most balanced scroll type for a fixed g, then $\delta_1 = 0$. If in addition we choose the most balanced (b_1, b_2)-type, then $\delta_2 = \delta_3 = \delta_4 = 0$. Using Lemma 8.33 it is also easy to prove that for all g the most balanced scroll and intersection type (the unique combination for fixed g with $\delta_1 = \delta_2 = 0$) then a general complete intersection will be a smooth projective model of a $K3$ surface.

9.2.1 An interpretation of b_1 and b_2

We will briefly study the case $c = 2$ in an analogous manner as the case $c = 1$ was studied in Remark 9.9 when we showed that a projective model of a $K3$ surface of Clifford index one with smooth asssociated scroll $\mathcal{T}$ must be of divisor type $3\mathcal{H} + (g-4)\mathcal{F}$ in $\mathcal{T}$ without using the resolution from Proposition 7.2.

Define the vector space $W = H^0(\mathcal{J}_{S'}(2))/H^0(\mathcal{J}_{\mathcal{T}}(2))$. In a natural way W represents the space of quadric functions on $\mathcal{T}$ that vanish on S'.

As in Remark 9.9 one obtains $\dim W = h^0(\mathcal{J}_{S'}(2)) - h^0(\mathcal{J}_{\mathcal{T}}(2)) = g - 3$, since $h^0(\mathcal{O}_{\mathcal{T}}(2)) - h^0(\mathcal{O}_{S'}(2)) = g - 3$ and $h^1(\mathcal{J}_{S'}(2)) = h^1(\mathcal{J}_{\mathcal{T}}(2)) = 0$ (see [SD, Prop. 1.5(i) and Theorem 6.1(ii)]. Assume g is odd. Take $b = \frac{g-5}{2}$ arbitrary fibers F of the ruling on $\mathcal{T}$, that is three-planes. For each three-plane we have two independent linear conditions on the quadric hypersurfaces to contain it. (First, take one point in the three-plane, not on S'. There is only one quadric surface in the threespace containing this point and the intersection with S'. Then take another point outside this quadric surface. To contain these two points and S' is equivalent to containing the threespace and S'.) So, one naively expects there to be $2b = g - 5$ conditions to contain all the b three-planes. Hence there should be a pencil, and only a pencil, of elements of W doing so. Intersecting the elements of the pencil, one would expect to get the projective model of the $K3$ surface. If it really were so simple, however, all intersection types (b_1, b_2) would be completely balanced. This is not always true, and one reason is that two different elements of W may intersect $\mathcal{T}$ in a common threedimensional component dominating $\mathbf{P}^1$ in the fibration on $\mathcal{T}$.

We are therefore not able to imitate the reasoning of Remark 9.9, and thereby establish the fact that the ideal of S' in $\mathcal{T}$ is generated as it is, without using Proposition 7.2. On the other hand we may use the knowledge that we have from Proposition 7.2, that S' is indeed of intersection type $(2\mathcal{H} - b_1\mathcal{F}, 2\mathcal{H} - b_2\mathcal{F})$ in its scroll. Make no assumption on the parity of g.

Proposition 9.12. *The invariant b_1 is equal to the largest number k, such that there exists a non-zero element Q of W (a hyperquadric in $\mathbf{P}^g$ containing S', but not $\mathcal{T}$) containing k three-planes in the pencil.*

Moreover, b_2 is the largest number m, such that there exists a non-zero element of W containing m three-planes in the pencil, and intersecting $\mathcal{T}$ in a different 3-dimensional dominant component than Q does.

Proof. Define two elements in W to be congruent if they have the same dominating three-dimensional component (but possibly differ in which three-planes they contain). Define the index of an element of W as the number of three-planes it contains (if necessary, counted with multiplicity). It is clear that if two elements of W are congruent, then they have the same index (they correspond to a well defined divisor class $2\mathcal{H} - i\mathcal{F}$, where i is the index). It follows from a Bezout argument that if the sum of the indices of two elements is larger than $2b = g - 5$, then they must be congruent. This shows both assertions. It also shows that the element in $2\mathcal{H} - b_1\mathcal{F}$ which any element in W with index larger than b gives rise to, is the same. (This element is nothing but the congruence class of the element in W.) □

9.2.2 Possible scroll types for $c = 2$

Almost all the information in this subsection can also be found in [Br] and [Ste], taken together, but we include it for completeness, and present it in our own way, for the sake of the reader.

Also in the case $c = 2$ it is possible to use Lemma 8.33 to obtain useful conclusions for many concrete scroll types. First we will look at possible smooth scroll types for $g \leq 10$.

For $g = 7$ and $g = 8$, the only possible smooth scroll types are $(1, 1, 1, 1)$ and $(2, 1, 1, 1)$, respectively. For $g = 7$, Lemma 8.33 immediately gives $b_1 \leq 2$. Here $b_1 + b_2 = 2$, so $b_2 \geq 0$. For $g = 8$ we see that Z_1 is a factor in every section of $2\mathcal{H} - b_1\mathcal{F}$, for $b_1 \geq 3$. Since a reducible section which is the sum a section $\mathcal{H} - 2\mathcal{F}$ and a section $\mathcal{H} - (b - 2)\mathcal{F}$ would intersect another section of type $2\mathcal{H} - b_2\mathcal{F}$ in something reducible, alternatively since S' is non-degenerate, we must have $b_1 = 2$ and $b_2 = 1$.

For $g = 9$ we have $b_1 + b_2 = 4$ and two smooth scroll types $(2, 2, 1, 1)$ and $(3, 1, 1, 1)$. For the latter type Lemma 8.33 gives $b_1 = b_2 = 2$, since any section of $2\mathcal{H} - b\mathcal{F}$, with $b \geq 3$ must be a product of a section $\mathcal{H} - b\mathcal{F}$ and a section of $\mathcal{H}$. For the type $(2, 2, 1, 1)$ any section of $2\mathcal{H} - 3\mathcal{F}$ has a zero scheme containing the subscroll generated by the two linear directrices (a quadric surface Q). If $b_1 = 3$, then $b_2 = 1$, and the section $2\mathcal{H} - b_2\mathcal{F}$ intersects Q as a curve of type $(1, 2)$ on Q. This is a rational twisted cubic Γ. This corresponds to the fact that $\delta_1 = \delta_3 = \delta_4 = 0$ and $\delta_2 = 1$, so that the set of complete intersections inside $\mathcal{T}$ has dimension at most $36 - \delta_2 + \delta_3 + \delta_4 = 35$, and taking the union over all $\mathcal{T}$ of the same type we get dimension at most $\dim(\mathrm{Aut}\ (\mathbf{P}^g)) + 17$.

Any section of $2\mathcal{H} - 4\mathcal{F}$ has a zero scheme, which restricts to two lines in each fiber of $\mathcal{T}$. This is impossible if this scheme shall contain a (necessarily non-degenerate) model S'. We also have $h^0(2\mathcal{H} - b\mathcal{F}) = 0$ for $b \geq 5$. Hence b_1 is 2 or 3.

For $g = 10$, we have $b_1 + b_2 = 5$ and a priori three possible scroll types $(4, 1, 1, 1)$, $(3, 2, 1, 1)$ and $(2, 2, 2, 1)$. But the first cannot occur, since any section of $2\mathcal{H} - b_1\mathcal{F}$ must have total weight $-b_1 \leq -3$, and then Z_1 must be a factor, using Lemma 8.33. This is impossible. Likewise, if $\mathcal{T}$ has type $(3, 2, 1, 1)$ and $b_1 \geq 5$, we conclude that Z_1 must be a factor, again impossible. The cases $b_1 = 4$ and $b_1 = 3$ are however possible.

If $\mathcal{T}$ has type $(2, 2, 2, 1)$, then $b_1 \leq 4$, since $h^0(2\mathcal{H} - b\mathcal{F}) = 0$ if $b \geq 5$. If $b_1 = 4$, then the zero scheme of any section of $2\mathcal{H} - b_1\mathcal{F} = 2\mathcal{H} - 4\mathcal{F}$ contains the linear directrix of $\mathcal{T}$ twice (its equation is a homogeneous quadric in Z_1, Z_2, Z_3 involving neither Z_4, t nor u). If we intersect with a section of $2\mathcal{H} - b_2\mathcal{F} = 2\mathcal{H} - \mathcal{F}$, and interpret it as the intersection with a quadric containing a fiber, and throw away the fiber, the residual intersection with $\mathcal{T}$ must contain one point of its linear directrix. This must then be a singular point of S'. Hence only $b_1 = 3$, $b_2 = 2$ gives a smooth S' for this scroll type. For these invariants $\delta_i = 0$, for $i = 1, 2, 3, 4$, so we get a family of total dimension $\dim(\mathrm{Aut}\ (\mathbf{P}^g)) + 18$.

So far we have studied smooth scroll types for $7 \leq g \leq 10$. At this point we could either proceed with smooth scroll types for $g \geq 11$, or look at singular scroll types for $g \geq 7$. These topics are closely related. Assume we have a singular scroll $\mathcal{T}(c, D, \{D_\lambda\})$ for $g \geq 7$, $D^2 = 0$, $c = 2$. Then the associated smooth scroll $\mathcal{T}_0$ is contained in $\mathbf{P}^{g+4}$, with a resolution as in Proposition 8.23:

$$0 \longrightarrow \mathcal{O}_{\mathcal{T}_0}(-4\mathcal{H}_0 + (g-1)\mathcal{F}) \longrightarrow \oplus_{k=1}^{2}\mathcal{O}_{\mathcal{T}_0}(-2\mathcal{H}_0 + b_k\mathcal{F}) \\ \longrightarrow \mathcal{O}_{\mathcal{T}_0} \longrightarrow \mathcal{O}_{S''} \longrightarrow 0.$$

So, at this point we can use the method of rolling factors to check what scroll types in $\mathbf{P}^{g+4}$ that may contain a surface like S''. Scroll types $(e_1+1, \ldots, e_4+1)$ for $\mathcal{T}_0$ correspond to types $(e_1, \ldots, e_4)$ for $\mathcal{T}$.

Let us study complete intersection surfaces in smooth scroll types for $g = 11$ with this dual viewpoint. Now $b_1 + b_2 = 6$ and $\deg \mathcal{T} = 8$ and there are a priori 5 different possible scroll types:

$$(5,1,1,1), \quad (4,2,1,1), \quad (3,2,2,1), \quad (3,3,1,1) \quad \text{and} \quad (2,2,2,2).$$

The type $(5,1,1,1)$ cannot occur, for the same reason that $(4,1,1,1)$ cannot occur for $g = 10$.

For $(4,2,1,1)$ and $(3,3,1,1)$ we can conclude that Z_1 or Z_2 is a factor in every term of every section of $2\mathcal{H}-3\mathcal{F}$. This gives that the subscroll formed by the two linear directrices is contained in S' (or S''). This is clearly impossible. Hence $b_1 \geq 4$ for these types.

If $b_1 \geq 5$, then Z_1 is a factor in every section of $2\mathcal{H} - b_1\mathcal{F}$, for each of the types $(4,2,1,1), (3,2,2,1)$ and $(2,2,2,2)$, which gives a contradiction. For the type $(3,3,1,1)$ we argue as follows: If $b_1 \geq 5$, then no term of the form Z_iZ_3 or Z_iZ_4 can occur as factor in a monomial of a section of $2\mathcal{H} - b_1\mathcal{F}$, so for each fixed value of (t, u) we get a quadric in Z_1, Z_2 only. This defines a union of two planes in each $\mathbf{P}^3$ which is a fiber of $\mathcal{T}$ (or $\mathcal{T}_0$). Hence each fiber of S' (or S'') is degenerate, a contradiction. So $b_1 = 4$.

We make the same kind of considerations for all $g \leq 14$. We end up with the following a priori possible combinations of smooth scroll type and b_1, for $7 \leq g \leq 14$ (of course $b_2 = g - 5$). For each scroll type and intersection type (b_1, b_2) we indicate whether the general zero scheme of a complete intersection of type $(2\mathcal{H} - b_1\mathcal{F}, 2\mathcal{H} - b_2\mathcal{F})$ is smooth or singular. See also Remark 9.13. In the column with headline " # mod." we give the value of $18 - \delta_1 - \delta_2 + \delta_3 + \delta_4$. This table contains information that can also be found in [Br] and [Ste], taken together. In [Br] all possible scroll and intersection types for smooth scrolls for all $g \geq 7$ are listed, and in [Ste] the information on the moduli of the corresponding families are given. We include the list for $g \leq 14$, since it will be useful in the study of projective models on singular scrolls, and for the lattice-theoretical considerations in Chapter 11.

g	scroll type	b_1	comp. int.	# mod.	g	scroll type	b_1	comp. int.	# mod.
7	$(1,1,1,1)$	1	Smooth	18	12	$(4,3,1,1)$	5	Smooth	16
7	$(1,1,1,1)$	2	Smooth	17	12	$(4,2,2,1)$	4	Singular	15
8	$(2,1,1,1)$	2	Smooth	18	13	$(3,3,2,2)$	4	Smooth	18
9	$(2,2,1,1)$	2	Smooth	18	13	$(3,3,2,2)$	5	Smooth	17
9	$(2,2,1,1)$	3	Smooth	17	13	$(3,3,3,1)$	4	Smooth	17
9	$(3,1,1,1)$	2	Smooth	15	13	$(3,3,3,1)$	6	Smooth	18
10	$(2,2,2,1)$	3	Smooth	18	13	$(4,2,2,2)$	4	Smooth	15
10	$(2,2,2,1)$	4	Singular	17	13	$(4,3,2,1)$	4	Singular	16
10	$(3,2,1,1)$	3	Smooth	16	13	$(4,3,2,1)$	5	Smooth	16
10	$(3,2,1,1)$	4	Singular	17	13	$(4,3,2,1)$	6	Smooth	18
11	$(2,2,2,2)$	3	Smooth	18	13	$(5,3,1,1)$	6	Smooth	17
11	$(2,2,2,2)$	4	Smooth	17	14	$(3,3,3,2)$	5	Smooth	18
11	$(3,2,2,1)$	3	Smooth	17	14	$(3,3,3,2)$	6	Singular	17
11	$(3,2,2,1)$	4	Smooth	17	14	$(4,3,2,2)$	5	Smooth	16
11	$(4,2,1,1)$	4	Singular	15	14	$(4,3,2,2)$	6	Singular	17
11	$(3,3,1,1)$	4	Smooth	16	14	$(4,3,3,1)$	5	Smooth	17
12	$(3,2,2,2)$	4	Smooth	18	14	$(4,4,2,1)$	5	Singular	16
12	$(3,3,2,1)$	4	Smooth	17	14	$(5,3,2,1)$	5	Singular	15
12	$(3,3,2,1)$	5	Smooth	17	14	$(5,3,2,1)$	6	Singular	16

For perfect Clifford divisors D with $D^2 = 0$ and singular scrolls $\mathcal{T} = \mathcal{T}(2, D)$ we get the following list of a priori possible cases in $\mathbf{P}^g$, for $7 \le g \le 10$ (subtracting 2 from all values of b_i for S'' in $\mathcal{T}_0$ in $\mathbf{P}^{g+4}$, for $i = 1, 2$):

g	sing. scroll type	b_1	$(S'')_{virt}$	g	sing. scroll type	b_1	$(S'')_{virt}$
7	$(2,1,1,0)$	$1,2$	Smooth	9	$(4,2,0,0)$	4	Smooth
7	$(2,2,0,0)$	2	Smooth	9	$(3,2,1,0)$	$2,3$	Smooth only for $b_1 = 3$
7	$(3,1,0,0)$	2	Singular	9	$(2,2,2,0)$	2	Smooth
8	$(3,2,0,0)$	3	Smooth	10	$(4,2,1,0)$	$3,4$	Singular
8	$(3,1,1,0)$	2	Singular	10	$(3,3,1,0)$	3	Singular
8	$(2,2,1,0)$	2	Smooth	10	$(3,2,2,0)$	3	Smooth

Remark 9.13. For each smooth scroll $\mathcal{T}$ and intersection type where the general element S' is smooth (on the upper list of types for $7 \le g \le 14$) it is clear that we have a smooth projective model of a $K3$ surface. For the remaining cases (on that list) it is natural to interpret them as projective models S' of $K3$ surfaces by non-ample linear systems. The types on the upper list are the only ones for $g \le 14$ where a general complete intersection $(2\mathcal{H} - b_1\mathcal{F}, 2\mathcal{H} - b_2\mathcal{F})$, with $b_1 + b_2 = g - 5$, is either smooth, or singular in a finite number of points. In Chapter 11, moreover, we describe all projective models for low g, including those with $c = 2$. All scroll types listed above

(smooth as in the upper list or singular as in the lower list) for $g \leq 10$ reappear in the description in Chapter 11.

To decide which intersections that are in general smooth, which intersections that are in general singular in finitely many points, and which intersections that are in general singular along a curve (or even reducible) one uses Lemma 8.33, similarly as in [Br] and [Ste]. In particular we have checked with the Appendix in [Br], which gives a list of smooth complete intersection $K3$ surfaces in 4-dimensional smooth rational normal scrolls and also a list of relevant intersections with only finitely many singularities.

On the lower list, concerning singular scrolls for ≤ 10, we have listed all scroll and intersection types which might a priori appear as "images" by the map i of scrolls $\mathcal{T}_0$ and surfaces S'' on the upper list, provided that the Clifford divisor D is perfect.

In the columns with heading $(S'')_{virt}$ we have indicated whether a general complete intersection of type in question on $\mathcal{T}_0$, which contains the exceptional divisor of the map i from $\mathcal{T}_0$ to $\mathcal{T}$, is smooth or singular. We call such a complete intersection $(S'')_{virt}$, since we do not a priori know that it is an S''. If D is perfect and the scroll $\mathcal{T}$ is singular, each occurring projective model S' on $\mathcal{T}$ is of course also singular, but S'' smooth means that all singularities of S' are due to contractions across the fibers; there are no contractions in the individual fibers.

We will illustrate that the issues whether a smooth scroll $\mathcal{T}(2, D)$ and an associated intersection type for a model S' appears on the upper list, is different from the issue whether the scroll and intersection type appears for a $\mathcal{T}_0$ and a S''. If D is perfect, it is a priori possible that all complete intersections, or a general one, represents a model S', but not an S''. As an example, look at the type $(3, 3, 3, 1)$, with $(b_1, b_2) = (6, 2)$. Then S' consists of the common zeroes of two sections of the form

$$c_1 Z_1^2 + c_2 Z_1 Z_2 + c_3 Z_1 Z_3 + c_4 Z_2^2 + c_5 Z_2 Z_3 + c_6 Z_3^2$$

and

$$f(t, u, Z_1, Z_2, Z_3, Z_4) + c_7 Z_4^2,$$

where $f(t, u, Z_1, Z_2, Z_3, Z_4)$ is contained in the ideal generated by Z_1, Z_2, Z_3. The general such intersection is smooth, and does not intersect the last directrix ($Z_1 = Z_2 = Z_3 = 0$) at all. But in order to be a surface of the form S'', associated to a perfect Clifford divisor D, the intersection must contain the last directrix. This forces c_7 to be zero. In that case the intersection is no longer smooth, in fact it contains the directrix in its singular locus, and hence it cannot be an S''. Hence scroll type $(3, 3, 3, 1)$ with $b_1 = 6$ appears on the upper list, but the corresponding "pushed down" type $(2, 2, 2, 0)$ does not appear in combination with (the revised) $b_1 = 4$.

A similar, but slightly different case, is the scroll type $(3, 2, 2, 1)$ and intersection type $(4, 2)$. Then a surface of the form S' would consist of the common zeroes of two sections of the form:

$$P_{2,1}(t,u)Z_1^2 + P_{1,1}(t,u)Z_1Z_2 + P_{1,2}(t,u)Z_1Z_3 + \\ c_1Z_2^2 + c_2Z_2Z_3 + c_3Z_3^2 + c_4Z_1Z_4$$

and

$$f(t,u,Z_1,Z_2,Z_3) + P_{2,2}(t,u)Z_1Z_4 + \\ P_{1,3}(t,u)Z_2Z_4 + P_{1,4}(t,u)Z_3Z_4 + c_5Z_4^2.$$

If this is an S'' for a perfect Clifford divisor D, then it contains the last directrix, which means $c_5 = 0$. Even if $c_5 = 0$, the intersection will in general be smooth if $c_4 \neq 0$, and $P_{1,3}$ and $P_{1,4}$ have no common roots. In this example only a subfamily of positive codimension of the $(\dim \operatorname{Aut}(\mathbf{P}^g) + 18 - \delta_1 - \delta_2 + \delta_3 + \delta_4)$-dimensional family of all complete intersections of that type are of the form S''.

Remark 9.14. If we only assume that D is free (and not perfect), we get the following additional a priori possible cases:

g	sing. scroll type	b_1	$(S'')_{virt}$
8	$(2,2,1,0)$	3	Smooth
9	$(2,2,2,0)$	4	Smooth
9	$(3,2,1,0)$	4	Singular

As proven in Remark 9.13 above, if $\mathcal{T}(2,D)$ has type $(2,2,2,0)$ with $b_1 = 4$, then S'' cannot contain the inverse image by $i : \mathcal{T}_0 \to \mathcal{T}$ of the point singular locus of $\mathcal{T}$. Therefore S' cannot contain the point singular locus of $\mathcal{T}(2,D)$, and $S'' \simeq S'$, and D is not perfect. This completes the proof of Proposition 8.39. A similar conclusion can be drawn about the two other cases in the last table, if they occur.

9.3 Projective models with $c = 3$

Assume $\mathcal{T} = \mathcal{T}(3,D)$ for a free Clifford divisor D with $D^2 = 0$. If $\mathcal{T}$ is smooth, we get from Proposition 7.2(a) that $\mathcal{O}_{S'}$ has a resolution (as an $\mathcal{O}_{\mathcal{T}}$-module) of the following form:

$$0 \to \mathcal{O}_{\mathcal{T}}(-5\mathcal{H} + (g-6)\mathcal{F}) \to \oplus_{k=1}^{5}\mathcal{O}_{\mathcal{T}}(-3\mathcal{H} + b_k\mathcal{F}) \to$$

$$\oplus_{k=1}^{5}\mathcal{O}_{\mathcal{T}}(-2\mathcal{H} + a_k\mathcal{F}) \to \mathcal{O}_{\mathcal{T}} \to \mathcal{O}_{S'} \to 0.$$

From Corollary 8.34 we conclude that we have such a resolution even if $\mathcal{T}$ is non-smooth. We see from [Sc] that we are in a situation very similar to that of a pentagonal canonical curve, which is natural, since a general hyperplane section of S' is such a curve. We do not intend to say as much about this situation as about the cases $c = 1$ and 2. Study the skew-symmetrical map Φ in the resolution above, already introduced in Corollary 8.34:

$$\Phi : \oplus_{k=1}^{5}\mathcal{O}_{\mathcal{T}}(-3\mathcal{H} + b_k\mathcal{F}) \to \oplus_{k=1}^{5}\mathcal{O}_{\mathcal{T}}(-2\mathcal{H} + a_k\mathcal{F}).$$

Recall that the Pfaffians of this map generate the ideal of S' in $\mathcal{T}$. Clearly $\mathcal{T}$ is a rational normal scroll of degree $g-4$ in $\mathbf{P}^g$. Let its type be $\mathbf{e}= (e_1, \ldots, e_5)$.

From Proposition 9.1 the dimension of the set of scrolls of type $\mathbf{e}$ in $\mathbf{P}^g$ is equal to $(g+1)^2 - 28 - \delta_1 = \dim(\mathrm{Aut}\ (\mathbf{P}^g) - 27 - \delta_1$, where $\delta_1 := \sum_{i,j} \max\ (0, e_i - e_j - 1)$. To obtain the number $18 + \dim(\mathrm{Aut}\ (\mathbf{P}^g)$ for the dimension of the set of projective models of $K3$ surfaces in scrolls of some type, one expects a 45-dimensional set of such models in a given scroll, provided the scroll type is reasonably well balanced. We will look into this issue, but we will not give a rigorous proof that we can find such a 45-dimensional set.

A given projective model S' is characterized by the ten above-diagonal entries of a five-by-five matrix description of the map Φ. These entries are sections of:

$$\begin{aligned}
&\mathcal{H} - (b_2 - a_1)\mathcal{F}, \mathcal{H} - (b_3 - a_1)\mathcal{F}, \mathcal{H} - (b_4 - a_1)\mathcal{F}, \mathcal{H} - (b_5 - a_1)\mathcal{F},\\
&\qquad\qquad \mathcal{H} - (b_3 - a_2)\mathcal{F}, \mathcal{H} - (b_4 - a_2)\mathcal{F}, \mathcal{H} - (b_5 - a_2)\mathcal{F},\\
&\qquad\qquad \mathcal{H} - (b_4 - a_3)\mathcal{F}, \mathcal{H} - (b_5 - a_3)\mathcal{F}, \mathcal{H} - (b_5 - a_4)\mathcal{F}.
\end{aligned}$$

We have $h^0(\mathcal{T}, \mathcal{H} - (b_i - a_j)\mathcal{F}) = g + 1 - 5(b_i - a_j) + \delta_{2,i,j}$, where $\delta_{2,i,j} := h^1(\mathbf{P}^1, \mathcal{E} \otimes \mathcal{O}_{\mathbf{P}^1}(a_j - b_i))$ and is zero if and only if $e_5 - (b_i - a_j) \geq -1$. In all, there set of choices of the ten linear terms has dimension

$$10(g+1) - \sum_{i>j}(5(b_i - a_j) + \delta_{2,i,j}).$$

Moreover, we have

$$\begin{aligned}
\sum_{i>j}(b_i - a_j) &= b_2 + 2b_3 + 3b_4 + 4b_5 - 4a_1 - 3a_2 - 2a_3 - a_4\\
&= b_2 + 2b_3 + 3b_4 + 4b_5 - 4(g-6-b_1) - 3(g-6-b_2)\\
&\quad -2(g-6-b_3) - (g-6-b_4)\\
&= 4(b_1 + \cdots + b_5) - 10(g-6)\\
&= 2g - 12,
\end{aligned}$$

where we have used the self-duality of the resolution (Proposition 7.2(b)) which gives $a_i = g - 6 - b_i$, for $i = 1, \ldots, 5$

We see from $a_i = g - 6 - b_i$, for $i = 1, \ldots, 5$, that $\sum_{i=1}^{5} a_i = 2g - 12$, so the average value of the $b_i - a_j$ is $\frac{g-6}{5}$. The average value of the e_i is $\frac{g-4}{5}$, so if both the e_i and the b_j (and therefore the a_j) are maximally balanced, we will in fact have $e_5 - (b_i - a_j) \geq -1$, so $\delta_{2,i,j} = 0$, for each $i > j$.

To obtain the desired value 45 in the maximally balanced situation, one needs to argue that it is correct to subtract 25, in the sence that there is typically a 25-dimensional family of matrix decriptions giving rise to each projective model of a $K3$ surface as described. We do not know how to do this in a rigorous way, but the problem is related to the one mentioned in [B-E, p. 457], where one treats matrix descriptions of maps between two free modules of rank 5 over a ring (see also [Be]). Translating the discussion in [B-E] into our situation, the issue is: Do two matrices A' and A have the same Pfaffian ideal if and only if there is a matrix B, such that $A' = BAB^t$? In an extremely simple case, take $g = 11$ and $a_i = 2$ for all i (and consequently $b_j = 3$ for all j), so that all entries in the matrix representation A of Φ are sections of the same line bundle on $\mathcal{T}$ (in this case $\mathcal{H} - \mathcal{F}$). One can imagine the set of five-by-five matrices acting on the matrix A representing Φ as $A \to BAB^t$, for all B in $GL(5)$. In a situation where the a_i are less balanced, one can imagine an analogous matrix B with entries in suitably manufactured line bundles, so that the "shape" of A is preserved under a similar action. By this we mean that if entry $A_{i,j}$ of A is a section of a line bundle $L_{i,j}$, then entry $A'_{i,j}$ of BAB^t is also a section of $L_{i,j}$. One must then count the sections in the entries of B, control the stabilizers of the action, and show that all A with the same Pfaffian ideal are in the same orbit by the action.

A natural candidate for such a matrix B is one where the entry B_{ij} is chosen as a general section of $\mathcal{O}_{\mathcal{T}}((a_j - a_i)\mathcal{F}) = \mathcal{O}_{\mathcal{T}}((b_i - b_j)\mathcal{F})$, for all (i, j). Since $h^0(\mathcal{O}_{\mathcal{T}}((a_j - a_i)\mathcal{F})) + h^0(\mathcal{O}_{\mathcal{T}}((a_i - a_j)\mathcal{F}) = 2 + \max\,(0, |a_i - a_j| - 1)$, we see that $\sum_{i,j} h^0(\mathcal{O}_{\mathcal{T}}((a_j - a_i)\mathcal{F}) = 25$ if and only if the a_i are chosen in a maximally balanced way. Set $\delta_3 = \sum_{i>j} \max\,(0, |a_i - a_j| - 1)$. Then the dimension of the set of choices of matrix B as described is $25 + \delta_3$ (we see that det B is a constant, and we look at the closed subset of those B with non-zero determinant). One checks that BAB^t is antisymmetric, and has entries that are sections in the same line bundles as the corresponding ones for A. This leads to the following:

Conjecture 9.15. *Let $\mathcal{T}$ be a fixed rational normal scroll of maximally balanced type and dimension* 5 *in $\mathbf{P}^g$, for $g \geq 9$. Let $\mathcal{M}(\mathcal{T}, c)$ be the set of projective models of $K3$ surfaces S of Clifford index* 3*, with a perfect, elliptic Clifford divisor D, such that $\mathcal{T} = \mathcal{T}(c, D)$. Then* $\dim M = 45$. *For an arbitrary scroll type (not necessarily smooth), and given combination* $(a_1, \ldots, a_5)$ *the corresponding set $\mathcal{M}(\mathcal{T}, c)$ is empty, or it has dimension* $45 + \delta_2 - \delta_3$. *We have $\delta_2 \geq \delta_3$ if $\delta_1 \geq 1$.*

Remark 9.16. The first statement of the conjecture will be proved in Proposition 9.18 below. For the second statement, see the discussion above. The last statement ($\delta_2 \geq \delta_3$ if $\delta_1 \geq 1$) of the conjecture does not follow directly from purely numerical considerations. As an example, take the case $g = 11$, scroll type $(3, 1, 1, 1, 1)$ and $(a_1, \ldots, a_5) = (1, 2, 2, 2, 3)$, which gives $(b_1, \ldots, b_5) = (4, 3, 3, 3, 2)$. Here $\delta_1 = 4$, and $\delta_3 = 1$. For all (i, j) with $i > j$, we have $e_5 - (b_i - a_j) \geq -1$, so $\delta_2 = 0$.

On the other hand the entries outside the diagonal in the first row of a matrix description of Φ are sections of $\mathcal{H} - (b_2 - a_1)\mathcal{F}, \mathcal{H} - (b_3 - a_1)\mathcal{F}, \mathcal{H} - (b_4 - a_1)\mathcal{F}$ and $\mathcal{H} - (b_5 - a_1)\mathcal{F}$, which here are $\mathcal{H} - 2\mathcal{F}, \mathcal{H} - 2\mathcal{F}, \mathcal{H} - 2\mathcal{F}$ and $\mathcal{H} - \mathcal{F}$. Taking the submaximal minor where we disregard the term $\mathcal{H} - \mathcal{F}$, we see that Z_1 is a factor, since Z_1 is a factor in every section of $\mathcal{H} - 2\mathcal{F}$. This is a contradiction, and hence the case does not occur.

Remark 9.17. We have now seen (as a special case) that one way to prove the (well known) formula $\dim(\mathrm{Aut}\ (\mathbf{P}^g)) + 18$ for the dimension of the set of projective models on rational normal scrolls of maximally balanced types in $\mathbf{P}^g$ (with elliptic Clifford divisor D), at least in each of the cases $c = 1, 2, 3$, is to first compute the dimension of the set of scrolls, and then add the dimension of the set of projective models in each scroll. Using the same method, one deduces the well-known fact that the set of k-gonal curves in $\mathbf{P}^g$ on rational scrollar surfaces of maximally balanced types is empty or has dimension

$$\dim(\mathrm{Aut}\ (\mathbf{P}^{g-1})) + 2g + 2k - 5$$

in each of the cases $k = 3, 4, 5$. But the set is not empty, as is shown for example in [Ba], where one shows that for all k, the general canonical k-gonal curve has maximally balanced scroll type (for its gonality scroll). For canonical curves, the scroll type is determined by the dual scrollar invariants $h^0(K - rD)$, in other words by $h^0(rD)$, for $r = 1, 2, \ldots$ for the gonality divisor D. One sees that for $k = 3, 4, 5$ one can find the dimension of the sets of k-gonal curves with fixed scrollar invariants (if non-empty) in $\mathbf{P}^{g-1}$, corresponding to sets of curves with prescribed values of $h^0(rD)$, for $r = 1, 2, \ldots$, by using similar methods as in the subsections above.

9.4 Higher values of c

From Proposition 9.1 we see that the dimension of the set of scrolls of a given type in $\mathbf{P}^g$ is $(g+1)^2 - 3 - (c+2)^2 - \delta_1$, where δ_1 is a non-negative number, which is zero if and only if the scroll type is maximally balanced. We recall the exact value:

$$\delta_1 = \sum_{i,j} \max\ (0, e_i - e_j - 1).$$

Since we know that for all c in the range in question there exists a set of dimension $\dim(\mathrm{Aut}\ (\mathbf{P}^g)) + 18 = (g+1)^2 + 17$ parametrizing projective model of $K3$ surfaces in $\mathbf{P}^g$ with Clifford-index c fibered by elliptic curves on a scroll of some type, we know that for this type, the set of projective models of $K3$

surfaces of Clifford index c, with smooth associated scrolls $\mathcal{T}$, has dimension at least

$$(g+1)^2 + 17 - ((g+1)^2 - 3 - (c+2)^2) = (c+2)^2 + 20.$$

A scroll type with $\delta_1 = 0$ is then a natural candidate. We have:

Proposition 9.18. *Let $g \geq 5$ and $1 \leq c < \lfloor \frac{g-1}{2} \rfloor$. Let $\mathcal{T}$ be a fixed rational normal scroll of maximally balanced type of dimension $c+2$ in $\mathbf{P}^g$. Let $\mathcal{M}(\mathcal{T}, c)$ be the set of projective models of $K3$ surfaces S of Clifford index c, with a perfect, elliptic Clifford divisor D, such that $\mathcal{T} = \mathcal{T}(c, D)$. Then*

$$\dim \mathcal{M}(\mathcal{T}, c) = (c+2)^2 + 20.$$

For (not necessarily smooth) scrolls $\mathcal{T}$ with types with $\delta_1 > 0$, the corresponding set $\mathcal{M}(\mathcal{T}, c)$ is empty, or

$$\dim \mathcal{M}(\mathcal{T}, c) \leq (c+2)^2 + 20 + \delta_1.$$

Proof. Let S be a $K3$ surface with Picard group as in Lemma 4.3, that is such that $\operatorname{Pic} S \simeq \mathbf{Z}L + \mathbf{Z}D$, with $L^2 = 2g-2$, $D^2 = 0$ and $LD = c+2$. Let us study the scroll $\mathcal{T}(c, D)$. By Proposition 4.2, we have that D is a free Clifford divisor and the "dual scrollar invariants" d_r (see Chapter 2) have the form:

$$d_r = h^0(L - rD) - h^0(L - (r+1)D).$$

Assume that S contains a smooth rational curve Γ. Then $\Gamma = aL + bD$, for integers a and b. This gives $a^2(2g-2) + 2ab(c+2) = -2$, which gives $a(a(g-1) + b(c+2)) = -1$. This, together with $D.\Gamma \geq 0$ gives $a = 1$ and $b = \frac{-g}{c+2}$. Hence S contains a rational curve Γ if and only if $(c+2)|g$, in which case $\Gamma \sim L - nD$, for $n := \frac{g}{c+2}$.

We will show that the scroll $\mathcal{T}(c, D)$ will be of maximally balanced type. From the way the scrollar invariants $e_1, \dots, e_{c+2}$ are formed from the dual scrollar invariants $d_1, d_2, \dots$ we see that the scroll type is maximally balanced if and only if

$$h^0(L - rD) = (g+1) - r(c+2),$$

for all $r \geq 0$, such that $L - rD$ is effective. By Riemann-Roch we see that this happens if and only if $h^1(L - rD) = 0$ for these r.

Set $B_r := L - rD$.

Assume first that B_r is not nef. Then $|B_r|$ has a fixed component Σ supported on a union of smooth rational curves. But we have just seen that the only such curve is of the form $\Gamma \sim L - nD$, with $n := \frac{g}{c+2} \in \mathbf{Z}$. So we can write $\Sigma = m\Gamma$, for an integer $m \geq 1$, and denoting the (possibly zero) moving part of $|B_r|$ by B_r^0, we have

$$B_r \sim B_r^0 + m\Gamma.$$

Furthermore, by our assumptions that B_r is not nef, we have $B_r.\Gamma < 0$.

We have

$$B_r^0 \sim B_r - m\Gamma \sim L - rD - m\Gamma \sim (1-m)\Gamma + (n-r)D. \tag{9.1}$$

Since D is nef, we have $\Gamma.D \geq 0$ and $B_r^0.D = (1-m)\Gamma.D \geq 0$, whence $m = 1$.

By (9.1) this implies that $B_r^0 \sim (n-r)D = (\frac{g}{c+2} - r)D$, whence

$$\Gamma.B_r^0 = (\frac{g}{c+2} - r)\Gamma.D = (\frac{g}{c+2} - r)L.D = (\frac{g}{c+2} - r)(c+2) = g - r(c+2) \geq 0,$$

and since $\Gamma.B_r = \Gamma.B_r^0 - 2 < 0$, we must have

$$g - r(c+2) = 0 \quad \text{or} \quad 1.$$

In the first case, we get $r = \frac{g}{c+2} = n$, whence $B_r^0 = 0$ and $B_r = \Gamma$. In the second case we get the contradiction

$$n = \frac{g}{c+2} = r + \frac{1}{c+2}.$$

So if B_r is not nef, then $B_r = \Gamma$ and $h^1(B_r) = h^1(\Gamma) = 0$.

Now assume B_r is nef.

By Proposition 1.9 and Lemma 1.10, we have that $h^1(B_r) > 0$ if and only if $B_r \sim mE$ for an integer $m \geq 2$ and a smooth elliptic curve E. By $B_r^2 = (L - rD)^2 = 2g - 2 - 2r(c+2) = 0$, we get

$$r = \frac{g-1}{c+2}.$$

Furthermore $L^2 = 2g - 2 = 2rmD.E = \frac{2m(g-1)}{c+2}D.E > 0$, whence $D.E > 0$ and $c + 2 = mD.E$. But this gives

$$0 < |\text{disc } (D, E)| = (D.E)^2 = \frac{(c+2)^2}{m^2} < (c+2)^2 = |\text{disc } (L, D)|,$$

a contradiction, since L and D generate Pic S.

This shows that $h^1(L - rD) = 0$ for all r such that $L - rD \geq 0$.

From Lemma 4.3 we then have an abstract 18-dimensional family of $K3$ surfaces. From the argument above we know that these $K3$ surfaces give rise to projective models with balanced $(c+2)$-dimensional scrolls, i.e. an (Aut $(\mathbf{P}^g)+$ 18)-dimensional set of projective models of such surfaces. Hence the first part of the statement of the proposition follows, since there is an (Aut $(\mathbf{P}^g) - 2 - (c+2)^2$)-dimensional family of $(c+2)$-dimensional rational normal scrolls of maximally balanced type in $\mathbf{P}^g$, and all projective models are contained in finitely many such scrolls, and all scrolls of the same type are projectively equivalent. We see that we can construct a concrete family of dimension $(c+2)^2 + 20$ in each scroll of maximally balanced type, by using the surfaces from Lemma 4.3.

Assume the scroll $\mathcal{T}$ is not maximally balanced, that is $\delta_1 > 0$. Then the statement

$$\dim \mathcal{M}(\mathcal{T}, c) \leq (c+2)^2 + 20 + \delta_1$$

follows from the fact that there is no abstract 19-dimensional family of $K3$ surfaces in $\mathbf{P}^g$ with Clifford index c, and perfect elliptic Clifford divisor. Assume $\dim \mathcal{M}(\mathcal{T}, c) \geq (c+2)^2 + 21 + \delta_1$. Then, by taking the union over the (Aut $(\mathbf{P}^g) - 2 - (c+2)^2 - \delta_1$)-dimensional family of rational normal scrolls in $\mathbf{P}^g$ of the same type as $\mathcal{T}$, we obtain an Aut $(\mathbf{P}^g) + 19$-dimensional set of projective models of $K3$ surfaces in question. Here we use again that all projective models are contained in finitely many such scrolls, and all scrolls of the same type are projectively equivalent. □

Remark 9.19. We also conjecture that $\mathcal{M}(\mathcal{T}, c)$ (defined as above) is empty or:

$$(c+2)^2 + 20 \leq \dim \mathcal{M}(\mathcal{T}, c)$$

even if the scroll type of $\mathcal{T}$ is not maximally balanced. This conjecture is inspired by Proposition 9.2, Corollary 9.11 and Remark 9.16. (In many examples for $c = 1, 2$ with non-zero δ_1 a strict inequality is impossible.)

Set $M' =$ (the largest component of) Hilb $_{\mathcal{T}}^{(g-1)x^2+2}$. Then it is clear that

$$(c+2)^2 + 20 \leq \dim M'.$$

This is true because we can define the relative Hilbert scheme

$$\mathcal{M}'_H = \text{Hilb}\,_{\mathcal{T}_H}^{(g-1)x^2+2},$$

where H is the (parameter) Hilbert scheme of rational curves of degree $g - c - 1$ in $G(c+1, g)$, that is: The parameter space of rational normal $(c+2)$-dimenional scrolls in $\mathbf{P}^g$. Here $\mathcal{T}_H$ is the "universal scroll", such that the fibre $\mathcal{T}_{[t]}$ is $\mathcal{T}$ if $[t]$ is the parameter point in H corresponding to $\mathcal{T}$. It is well known, and follows from for example [Str], [R-R-W], and [Har, p. 62], that H is irreducible, and that the maximally balanced scrolls correspond to an open dense stratum of H. Since the fibre $M'_{[t]}$ of $\mathcal{M}'_H$ has dimension at least $\dim \mathcal{M}(\mathcal{T}, c) = (c+2)^2 + 20$ for all $[t]$ corresponding to scrolls of maximally balanced type, we have $\dim M'_{[t]} \geq= (c+2)^2 + 20$ for the $[t]$ corresponding to scrolls of less balanced types.

In order to prove the conjecture, we have to pass from M' to $\mathcal{M}(\mathcal{T}, c)$. It is not entirely clear to us how to do this. If the conjecture is true, we get

$$(c+2)^2 + 20 \leq \dim \mathcal{M}(\mathcal{T}, c) \leq c+2)^2 + 20 + \delta_1.$$

Moreover the cases $c = 1$, scroll type $(7, 3, 1)$, and $c = 2$, scroll types $(3, 3, 3, 1)$ with $b_1 = 6$, and $(4, 3, 2, 1)$ with $b_1 = 6$, reveal that a strict inequality $\dim \mathcal{M}(\mathcal{T}, c) < (c+2)^2 + 20 + \delta_1$ is not always correct, even if $\delta_1 > 0$. In Chapter 11 one sees that these cases indeed occur with the fiber D a perfect Clifford divisor. In these cases both the most balanced scroll/intersection type and the mentioned non-balanced types give families of dimension dim(Aut $(\mathbf{P}^g)) + 18$.

10

BN general and Clifford general $K3$ surfaces

In this chapter we will first, in Section 10.1, recall some results by Mukai describing projective models of Brill-Noether (BN) general polarized $K3$ surfaces of low genera. Mukai describes the models as "complete intersections" in various homogeneous varieties. The definition of a BN general polarized $K3$ surface, due to Mukai and given in Section 10.1 below, is valid in any genus, and we easily see that for example any polarized $K3$ surface (S, L) with Pic $S \simeq \mathbf{Z}L$ is Brill-Noether general.

In Section 10.2 we compare the concepts of BN generality and Clifford generality (described in Chapter 3). We show that BN general $K3$ polarized surfaces are Clifford general. We also give two more technical results concerning $K3$ surfaces of low genus. It turns out that the two concepts coincide for all genera below 11, except 8 and 10. In sections 10.3 and 10.4 we describe the projective models of polarized $K3$ surfaces og genera 8 and 10, respectively, that are Clifford general but not BN general.

10.1 The results of Mukai

It is shown in [Mu1] that a projective model of a general $K3$ surface in $\mathbf{P}^g$, for $g = 6$, 7, 8, 9 and 10, is a complete intersection in a homogeneous spaces described below.

We recall the following definition of Mukai:

Definition 10.1 (Mukai [Mu2]). *A polarized $K3$ surface (S, L) of genus g is said to be* Brill-Noether (BN) general *if the inequality $h^0(M)h^0(N) < h^0(L) = g + 1$ holds for any pair (M, N) of non-trivial line bundles such that $M \otimes N \simeq L$.*

Remark 10.2. One easily sees that this is for instance satisfied if any smooth curve $C \in |L|$ is Brill-Noether general, i.e. carries no line bundle $\mathcal{A}$ for which $\rho(A) := g - h^0(\mathcal{A})h^1(\mathcal{A}) < 0$. This is because any nontrivial decomposition $L \sim$

T. Johnsen and A.L. Knutsen: LNM 1842, pp. 121–128, 2004.

$M+N$ with $h^0(M)h^0(N) \geq g+1$ yields $h^0(M_C)h^1(M_C) \geq h^0(M)h^1(N) > g$. It is an open question whether the converse is true.

Clearly the polarized $K3$ surfaces which are BN general form a 19-dimensional Zariski open subset in the moduli space of polarized $K3$ surfaces of a fixed genus g.

The following theorem is due to Mukai. We use the following convention: For a vector space V^i of dimension i, we write $G(r, V^i)$ (resp. $G(V^i, r)$) for the Grassmann variety of r-dimensional subspaces (resp. quotient spaces) of V.

The variety $\Sigma_{12}^{10} \subseteq \mathbf{P}^{15}$ is a 10-dimensional spinor variety of degree 12. Let V^{10} be a 10-dimensional vector space with a nondegenerate second symmetric tensor λ. Then Σ_{12}^{10} is one of the two components of the subset of $G(V^{10}, 5)$ consisting of 5-dimensional totally isotropic quotient spaces [1].

The variety $\Sigma_{16}^{6} \subseteq \mathbf{P}^{13}$ is the Grassmann variety of 3-dimensional totally isotropic quotient spaces of a 6-dimensional vector space V^6 with a nondegenerate second skew-symmetric tensor σ. It has dimension 6 and degree 16.

Also, $\Sigma_{18}^{5} = G/P \subseteq \mathbf{P}^{13}$, where G is the automorphism group of the Cayley algebra over $\mathbf{C}$ and P is a maximal parabolic subgroup. The variety has dimension 5 and degree 18.

Finally, in the case $g = 12$, let V^7 be a 7-dimensional vector space and $N \subseteq \wedge^2 V^{\vee}$ a 3-dimensional vector space of skew-symmetric bilinear forms, with basis $\{m_1, m_2, m_3\}$. We denote by Grass $(3, V^7, m_i)$ the subset of Grass $(3, V^7)$ consisting of 3-dimensional subspaces u of V such that the restriction of m_i to $U \times U$ is zero. Then $\Sigma_{12}^{3} = \text{Grass } (3, V^7, N) := \cap \text{Grass } (3, V^7, m_i)$. It has dimension 3 and degree 12.

Theorem 10.3 (Mukai [Mu2]). *The projective models of BN general polarized $K3$ surfaces of small genus are as follows:*

genus	*projective model of BN general polarized $K3$ surface*
2	$S_2 \longrightarrow \mathbf{P}^2$ *double covering with branch sextic*
3	$(4) \subseteq \mathbf{P}^3$
4	$(2,3) \subseteq \mathbf{P}^4$
5	$(2,2,2) \subseteq \mathbf{P}^5$
6	$(1,1,1,2) \cap G(2, V^5) \subseteq \mathbf{P}^6$
7	$(1^8) \cap \Sigma_{12}^{10} \subseteq \mathbf{P}^7$
8	$(1^6) \cap G(V^6, 2) \subseteq \mathbf{P}^8$
9	$(1^4) \cap \Sigma_{16}^{6} \subseteq \mathbf{P}^9$
10	$(1^3) \cap \Sigma_{18}^{5} \subseteq \mathbf{P}^{10}$
12	$S_{12} = (1) \subseteq \Sigma_{12}^{3}$

[1] A quotient $f : V \to V'$ is totally isotropic with respect to λ if $(f \otimes f)(\lambda)$ is zero on $V' \otimes V'$.

10.2 Notions of generality

In this section we will compare the notion of BN generality with our notion of Clifford generality as given in Chapter 3 . We only treat the cases $g \leq 10$.

It is an easy computation to check that a BN general $K3$ surface is also Clifford general:

Proposition 10.4. *Let (S, L) be a polarized $K3$ surface of genus g. If (S, L) is BN general, then it is Clifford general.*

Proof. Assume that (S, L) is not Clifford general, and let $c = \operatorname{Cliff} L < \lfloor \frac{g-1}{2} \rfloor$ and D any Clifford divisor with $F := L - D$. Using (C1) and (C3) together with Riemann-Roch one easily computes

$$\begin{aligned} h^0(D) + h^0(F) &= \frac{1}{2}D^2 + 2 + \frac{1}{2}F^2 + 2 \\ &= \frac{1}{2}L^2 + 2 - D.F + 2 = g + 1 - c \geq \frac{g+5}{2}. \end{aligned} \tag{10.1}$$

Since $h^0(F) \geq h^0(D) \geq 2$ and for fixed $d \geq 2$ the function $f_d(x) = x(d-x)$ obtains its maximal value in $[2, d]$ at $x = 2$, we get

$$h^0(D)h^0(F) \geq 2(h^0(D) + h^0(F) - 2) \geq 2(\frac{g+5}{2} - 2) = g + 1 = h^0(L).$$

Hence (S, L) is not BN general. $\square$

For low genera we have:

Proposition 10.5. *Let (S, L) be a polarized $K3$ surface of genus $g = 2, 3 \ldots, 7$ or 9. Then (S, L) is BN general if and only if it is Clifford general.*

If $g = 8$ resp. 10, then (S, L) is Clifford general but not BN general if and only if there is an effective divisor D satisfying $D^2 = 2$ and $D.L = 7$ resp. 8, and there are no divisors satisfying the conditions $(*)$ for $c < 3$ resp. 4.

Proof. We must investigate the condition that there exists an effective decomposition $L \sim D + F$ such that $h^0(F)h^0(D) \geq g + 1$, but $\operatorname{Cliff} \mathcal{O}_C(D) \geq \lfloor \frac{g-1}{2} \rfloor$ for any smooth curve $C \in |L|$.

By Riemann-Roch, we have $\operatorname{Cliff} \mathcal{O}_C(D) = g+1-h^0(\mathcal{O}_C(D))-h^1(\mathcal{O}_C(D)) \leq g + 1 - h^0(D) - h^0(F)$, so we easily see that we must be in one of the two cases above. $\square$

Since the divisor D in the proposition satisfies $\operatorname{Cliff} L = \operatorname{Cliff} \mathcal{O}_C(D)$, we have $D \in \mathcal{A}^0(L)$, so we get the following from Propositions 2.6 and 2.7:

Corollary 10.6. *Any divisor D as in Proposition 10.5 must satisfy $h^1(D) = h^1(L - D) = 0$, and among all such divisors we can find one satisfying the conditions (C1)-(C5).*

By arguing as in the proof of Proposition 4.2, with the lattice $\mathbf{Z}L \oplus \mathbf{Z}D$, with

$$\begin{bmatrix} L^2 & L.D \\ D.L & D^2 \end{bmatrix} = \begin{bmatrix} 2(g-1) & c+4 \\ c+4 & 2 \end{bmatrix}$$

for $g = 8$ and 10 and $c = \lfloor \frac{g-1}{2} \rfloor = 3$ and 4 respectively, we see that there exists an 18-dimensional family of isomorphism classes of polarized $K3$ surfaces that are Clifford general but not BN general for both $g = 8$ and $g = 10$.

We will in the next two sections investigate these two cases. A choice of a subpencil $\{D_\lambda\}$ of $|D|$ gives as before a rational normal scroll $\mathcal{T}$ within which $\varphi_L(S) = S'$ is contained. Unfortunately, as we will see, we no longer have such a nice result about $V = \operatorname{Sing} \mathcal{T}$ as Theorem 5.7, since the Clifford index c is now the general one. We will however be able to describe these particular cases in a similar manner, too.

10.3 The case $g = 8$

Let us first consider the case $g = 8$, where $c = 3$. We have $D^2 = 2$, $D.L = 7$, $h^0(L) = 9$ and $h^0(L - D) = 3$. Since $(L - 2D)^2 = -6$ and $(L - 2D).L = 0$, we have $h^0(L - 2D) = 0$ or 1 and $h^0(L - 3D) = 0$.

Recall that the type $(e_1, \ldots, e_d)$ of the scroll $\mathcal{T}$, with $d = d_0 = \dim \mathcal{T}$, is given by

$$e_i = \#\{j \mid d_j \geq i\} - 1, \tag{10.2}$$

where

$$d_i := h^0(L - iD) - h^0(L - (i+1)D).$$

We have $d_{\geq 3} = 0$ and $(d_0, d_1, d_2) = (6, 3 - h^0(L - 2D), h^0(L - 2D))$ and the two possible scroll types

$$(e_1, \ldots, e_6) = \begin{cases} (1,1,1,0,0,0) & \text{if} \quad h^0(L - 2D) = 0 \\ (2,1,0,0,0,0) & \text{if} \quad h^0(L - 2D) = 1 \end{cases}$$

We first study the case $h^0(L - 2D) = 0$, that is $h^1(L - 2D) = 1$. We have $V = \operatorname{Sing} \mathcal{T} = \mathbf{P}^2$. Here we already see that Theorem 5.7 will not apply, since it is clear by the examples given by the lattice above that there are such cases with no contractions across the fibers. Denote the two base points of the pencil $\{D_\lambda\}$ by p_1 and p_2 and their images under φ_L by x_1 and x_2. We have the following result:

Lemma 10.7. *Either*

(i) $\mathcal{R}_{L,D} = \emptyset$, *or*

(ii) $\mathcal{R}_{L,D} = \{\Gamma\}$ *and* V *intersects* S' *in* x_1, x_2 *and* $y := \varphi_L(\Gamma)$, *and* $V = < x_1, x_2, y >$.

Proof. We first show that $\mathcal{R}_{L,D}$ is either empty or contains at most one curve.

Choose any smooth $D_0 \in |D|$. Set $F := L - D$ as usual. Since $\deg F_{D_0} = c + 2 = 5 = 2p_a(D) + 1$, one has that F_{D_0} is very ample, and by arguing as in Lemma 6.1, we get that $D.\Delta = 0$ or 1. This shows the assertion.

By arguing as in the proof of Theorem 5.7, we get that V intersects S' in at most three points (two of which must of course be x_1 and x_2) and that these three points are independent. $\square$

By this lemma, there are only two cases occurring for $h^0(L - 2D) = 0$, which we denote by (CG1) and (CG2), since they are Clifford general:

(CG1) $\mathcal{R}_{L,D} = \emptyset$,
(CG2) $\mathcal{R}_{L,D} = \{\Gamma\}$.

If $h^0(L - 2D) = 1$, then since $F^2 = 2$ we have

$$L \sim 2D + \Delta,$$

where Δ is the base divisor of $|F|$ and satisfies $\Delta^2 = -6$, $\Delta.L = 0$ and $\Delta.D = 3$. By arguing as in the proof of Proposition 3.7, we find that L is as in one of the five following cases (where all the Γ and Γ_i are smooth rational curves):

(CG3) $L \sim 2D + \Gamma_1 + \Gamma_2 + \Gamma_3$, with the following configuration:

$$\begin{tikzcd} D \arrow[r, dash] \arrow[d, dash] \arrow[dr, dash] & \Gamma_1 \\ \Gamma_2 & \Gamma_3 \end{tikzcd}$$

and $\mathcal{R}_{L,D} = \{\Gamma_1, \Gamma_2, \Gamma_3\}$,

(CG4) $L \sim 2D + \Gamma + 2\Gamma_0 + 2\Gamma_1 + \cdots 2\Gamma_N + \Gamma_{N+1} + \Gamma_{N+2}$, with the following configuration:

$$\begin{tikzcd} D \arrow[r, dash] \arrow[d, dash] & \Gamma_0 \arrow[r, dashed, dash] & \Gamma_N \arrow[r, dash] \arrow[d, dash] & \Gamma_{N+2} \\ \Gamma & & \Gamma_{N+1} & \end{tikzcd}$$

and $\mathcal{R}_{L,D} = \{\Gamma, \Gamma_0\}$,

(CG5) $L \sim 2D + 3\Gamma_1 + 2\Gamma_2 + 2\Gamma_3 + \Gamma_4 + \Gamma_5$, with the following configuration:

$$\begin{tikzcd} D \arrow[r, dash] & \Gamma_1 \arrow[r, dash] \arrow[dr, dash] & \Gamma_2 \arrow[r, dash] & \Gamma_4 \\ & & \Gamma_3 \arrow[r, dash] & \Gamma_5 \end{tikzcd}$$

and $\mathcal{R}_{L,D} = \{\Gamma_1\}$.

(CG6) $L \sim 2D + 3\Gamma_0 + 4\Gamma_1 + 2\Gamma_2 + 3\Gamma_3 + 2\Gamma_4 + \Gamma_5$, with the following configuration:

$$\begin{tikzcd}
D \arrow[r, dash] & \Gamma_0 \arrow[r, dash] & \Gamma_1 \arrow[r, dash] \arrow[dr, dash] & \Gamma_2 & & \\
 & & & \Gamma_3 \arrow[r, dash] & \Gamma_4 \arrow[r, dash] & \Gamma_5
\end{tikzcd}$$

and $\mathcal{R}_{L,D} = \{\Gamma_0\}$.

(CG7) $L \sim 2D + 3\Gamma_0 + 4\Gamma_1 + 5\Gamma_2 + 6\Gamma_3 + 4\Gamma_4 + 2\Gamma_5 + 3\Gamma_6$, with the following configuration:

$$\begin{tikzcd}
D \arrow[r, dash] & \Gamma_0 \arrow[r, dash] & \Gamma_1 \arrow[r, dash] & \Gamma_2 \arrow[r, dash] & \Gamma_3 \arrow[r, dash] \arrow[dr, dash] & \Gamma_4 \arrow[r, dash] & \Gamma_5 \\
 & & & & & \Gamma_6 &
\end{tikzcd}$$

and $\mathcal{R}_{L,D} = \{\Gamma_0\}$.

Defining $\mathcal{Z}_\lambda$ as in (5.5)-(5.8), we see that length $\mathcal{Z}_\lambda = 5$ and by arguing as in the proof of Theorem 5.7 in these five cases we get that for any $D \in \mathcal{D}$:

$$V =< \mathcal{Z}_\lambda >= \mathbf{P}^3,$$

any subscheme of length 4 spans a $\mathbf{P}^3$ and $V \cap S'$ has support only on this scheme.

For the cases (CG1)-(CG7) we can now argue as in Chapter 8. In particular, we get a commutative diagram as on page 69, and Proposition 8.17, Corollary 8.18, Propositions 8.23 and 8.24 and Corollary 8.26 still apply. All the $\varphi_L(D_\lambda)$ have the same Betti-numbers and their resolutions are given in Example 8.20.

In the cases (CG1) and (CG2) the type of $\mathcal{T}_0$ is $(2,2,2,1,1,1)$. We leave it to the reader to use Lemma 8.33 to show that the only possible combinations of the b_i's (defined in Definition 8.36) are

$$(b_1, \dots, b_8) = (4,3,3,3,2,2,2,1), \quad (4,3,3,2,2,2,2,2), \quad (3,3,3,3,2,2,2,2).$$

In the cases (CG3)-(CG7) the type of $\mathcal{T}_0$ is $(3,2,1,1,1,1)$. We again leave it to the reader to show that $(b_1, \dots, b_8) = (4,3,3,2,2,2,2,2)$ is the only possibility.

We conclude this section by showing that all the cases (CG1)-(CG7) actually exist, by arguing with the help of Propositions 1.11 and 1.12.

The case (CG1) can be realized by the lattice just below Corollary 10.6 and therefore has number of moduli 18.

We now show that the case (CG2) can be realized by the lattice $\mathbf{Z}D \oplus \mathbf{Z}F \oplus \mathbf{Z}\Gamma$, with intersection matrix:

$$\begin{bmatrix} D^2 & D.F & D.\Gamma \\ F.D & F^2 & F.\Gamma \\ \Gamma.D & \Gamma.F & \Gamma^2 \end{bmatrix} = \begin{bmatrix} 2 & 5 & 1 \\ 5 & 2 & -1 \\ 1 & -1 & -2 \end{bmatrix}$$

One easily checks that this matrix has signature $(1, 2)$, so by Proposition 1.11 there is an algebraic $K3$ surface with this lattice as its Picard lattice.

Set $L := D + F$. By Proposition 1.12 we can assume that L is nef, whence by Riemann-Roch D, $F > 0$.

We first show that L is base point free and that Cliff $L = 3$. Since $D.L - D^2 - 2 = 3$, we only need to show that there is no effective divisor B on S satisfying either

$$B^2 = 0,\ B.L = 1, 2, 3, 4, \quad \text{or}$$
$$B^2 = 2, \qquad B.L = 6.$$

Setting $B \sim xD + yF + z\Gamma$, one finds

$$B.L = 7(x + y),$$

which is not equal to any of the values above. Furthermore, D forces (S, L) to be non-BN general. Since one easily sees that we cannot be in any of the cases (CG1), (CG3)-(CG7), we must be in case (CG2).

We can argue in the same way for the cases (CG3)-(CG7), with the obvious lattices. The number of moduli of these cases are 16, 15, 14, 13 and 12, respectively. We leave these cases to the reader.

10.4 The case $g = 10$

The case $g = 10$ is very similar to the previous case. We have $c = 4$, $D.L = 8$, $h^0(L) = 11$ and $h^0(L - D) = 4$. Since $(L - 2D)^2 = -6$ and $(L - 2D).L = 2$, we have $h^0(L - 2D) = 0$ or 1 and $h^0(L - 3D) = 0$. This gives as before $d_{\geq 3} = 0$ and $(d_0, d_1, d_2) = (7, 4 - h^0(L - 2D), h^0(L - 2D))$ and the two possible scroll types

$$(e_1, \ldots, e_6) = \begin{cases} (1, 1, 1, 1, 0, 0, 0) & \text{if} \quad h^0(L - 2D) = 0 \\ (2, 1, 1, 0, 0, 0, 0) & \text{if} \quad h^0(L - 2D) = 1 \end{cases}$$

We now get exactly analogous cases (CG1)' and (CG2)' as for $g = 8$, corresponding to the scroll type $(1, 1, 1, 1, 0, 0, 0)$. If $h^0(L - 2D) = 1$, write as usual $F := L - D$ and denote by Δ the base divisor of $|F|$, so that we have

$$L \sim 2D + A + \Delta.$$

for some $A > 0$ satisfying $A.L = (L - 2D).L = 2$ and $A.\Delta = 0$. We can now show that $2 = h^1(R) = \Delta.D$, so that $A^2 = -2$ and $A.D = 2$. By arguing as in the proof of Proposition 3.7 again, we find that L is as in one of the two following cases (where all the Γ_i are smooth rational curves such that $\Gamma_i.A = 0$):

(CG3)' $L \sim 2D + A + \Gamma_1 + \Gamma_2$, with $\Gamma_1.D = \Gamma_2.D = 1$, $\Gamma_1.\Gamma_2 = 0$ and $\mathcal{R}_{L,D} = \{\Gamma_1, \Gamma_2\}$,

(CG4)' $L \sim 2D + A + 2\Gamma_0 + 2\Gamma_1 + \cdots + 2\Gamma_N + \Gamma_{N+1} + \Gamma_{N+2}$, with all the Γ_i having a configuration as in (E2), $\Gamma_i.A = 0$, $\mathcal{R}_{L,D} = \{\Gamma_0\}$,

Defining $\mathcal{Z}_\lambda$ as in (5.5)-(5.8), we see that length $\mathcal{Z}_\lambda = 4$. By arguing as in the proof of Theorem 5.7 in these two cases we get that for any $D \in \mathcal{D}$:

$$V =< \mathcal{Z}_\lambda >= \mathbf{P}^3$$

and $V \cap S'$ has support only on this scheme.

As above, we can argue as in Chapter 8, and find that for the cases (CG1')-(CG4') the $\varphi_L(D_\lambda)$ have the same Betti-numbers and their resolutions are given in Example 8.20.

For the cases (CG1)' and (CG2)' the type of $\mathcal{T}_0$ is $(2,2,2,2,1,1,1)$. Again one can use Lemma 8.33 to show that the only possible combinations of the b_i's (defined in Definition 8.36) are

$$(4,3,3,3,3,3,3,3,2,2,2,2,2) \quad \text{and} \quad (3,3,3,3,3,3,3,3,3,2,2,2,2)$$

for the case (CG1)', and

$$(4,3,3,3,3,3,3,3,2,2,2,2,2), \quad (3,3,3,3,3,3,3,3,3,2,2,2,2) \quad \text{and}$$

$$(4,3,3,3,3,3,3,3,3,2,2,2,1)$$

for the case (CG2)'.

The type of $\mathcal{T}_0$ for the cases (CG3)' and (CG4)' is $(3,2,2,1,1,1,1)$. The only possible cases for the b_i's are found to be

$$(4,4,4,3,3,3,2,2,2,2,2,2,2) \quad \text{and} \quad (4,4,3,3,3,3,3,2,2,2,2,2,2),$$

In the same way as for the cases (CG1)-(CG7), we can show the existence of each of the types (CG1)'-(CG4)' for $g = 10$ and show that their number of moduli 18, 17, 16 and 15 respectively.

These results will all be summarized in the next chapter, together with all non-Clifford general projective models for $g \leq 10$.

11

Projective models of $K3$ surfaces of low genus

In this chapter we will use the results obtained in the previous ones to classify all projective models of non-BN-general $K3$ surfaces of genus at most 10. Together with Mukai's description of the general models we are then able to give a complete classification and characterization for these genera. The central part of the chapter is Section 11.5 where we give tables summing up the essential information concerning the various projective models appearing of non-BN general $K3$ surfaces for $5 \leq g \leq 10$.

An important intermediate step is performed in Section 11.2 where we describe the possible perfect Clifford divisors for $c = 1$, 2 and 3 and also in some more detail the cases where $h^1(L - 2D) > 0$, since this last number determines the singular locus of the scroll $\mathcal{T}$ by (5.2). The description is valid for all genera, not only the small ones, but for $g \geq 11$ cases with $c \geq 4$ appear, even for (S, L) non-Clifford general. For $g \leq 10$ we always have $c \leq 3$ for the non-Clifford general models.

The reason why we concentrate on perfect Clifford divisors is to make the classification in Section 11.5 simpler. If we did not restrict to perfect Clifford divisors, we would get more projective models, but the extra projective models would also have been possible to describe with a perfect Clifford divisor, whence they would belong to our list.

Section 11.1 is purely technical, and devoted to a new decomposition of the divisor $R = L - 2D$ for each free Clifford divisor D. The new decomposition, with the added property (11.4) is necessary to make the description in Section 11.2 work.

In Section 11.3 we show how one can calculate the scroll types of the relevant ambient scrolls appearing in the various cases.

The exposition in Section 11.5 contains detailed information about the Picard lattice of S, and the singularity type of $S' = \varphi_L(S)$ in many subcases. In Section 11.4 we show how this information can be obtained in some typical cases, and leave the arguments in the remaining ones to the reader.

T. Johnsen and A.L. Knutsen: LNM 1842, pp. 129–154, 2004.

11.1 A new decomposition of R

Assume that D is a free Clifford divisor. We recall from Chapter 6 that $R = L - 2D$, and that $\Delta = 0$ if $H^0(R) = 0$ by Lemma 6.1. If $R > 0$, we have $L = 2D + A + \Delta$, where $D + A$ is the moving part of $F := L - D$, and Δ is the base divisor of F. So $R \sim A + \Delta$ is an effective decomposition of R. Recall from Lemma 6.4 that $\Delta.A = 0$, except for the cases (E3) and (E4). To make the classification simpler, we would like to find a new effective decomposition of R, say $R \sim A' + \Delta'$, with a stronger property than the one in Lemma 6.4, namely that $\Delta''.A' = 0$ for every effective $\Delta'' \leq \Delta'$. At the same time we would like A' and Δ' to enjoy the same intersection properties and cohomological properties as A and Δ, so that the results in Chapter 6 are still valid. (We are grateful to Gert M. Hana for pointing out the need for such a new decomposition)

Proposition 11.1. *Let (S, L) be a polarized $K3$ surface of non-general Clifford index, with free Clifford index D not as in (E3) or (E4), and such that $R := L - 2D > 0$. Let A and Δ be defined as above. Then there exists an effective decomposition $R = A' + \Delta'$ such that the following properties hold:*

$$\Delta' \leq \Delta \quad \textit{and} \quad A' \geq A \tag{11.1}$$

$$\begin{bmatrix} D^2 & D.A & D.\Delta \\ D.A & A^2 & A.\Delta \\ D.\Delta & A.\Delta & \Delta^2 \end{bmatrix} = \begin{bmatrix} D^2 & D.A' & D.\Delta' \\ D.A' & A'^2 & A'.\Delta' \\ D.\Delta' & A'.\Delta' & \Delta'^2 \end{bmatrix} \tag{11.2}$$

$$h^i(A') = h^i(A) \quad \textit{and} \quad h^i(\Delta') = h^i(\Delta) \quad \textit{for } i = 0, 1, 2. \tag{11.3}$$

$$\Delta''.A' = 0 \quad \textit{for every effective} \quad \Delta'' \leq \Delta'. \tag{11.4}$$

Remark 11.2. Note that $R \sim A + \Delta$ always satisfies (11.1)-(11.3), so that property (11.4) is the reason why we want to find a new decomposition. Moreover note that (11.1)-(11.4) ensure that all the important results in Chapters 5 and 6 for A and Δ are still valid for A' and Δ'. To be more precise, Proposition 5.3, Remark 5.4, Proposition 5.5, Lemmas 6.1, 6.4, 6.7, 6.8 and Proposition 6.9 are valid with A and Δ replaced by A' and Δ'.

We will give an algorithmic proof of Proposition 11.1. First we will state and prove the following result.

Lemma 11.3. *Assume we are neither in case (E3) nor (E4), and that we have an effective decomposition $R \sim A_i + \Delta_i$ such that (11.1)-(11.3) hold. If there exists a smooth rational curve $\Gamma \leq \Delta_i$ such that $\Gamma.A_i > 0$, then $\Gamma.A_i = 1$, and $\Gamma.D = 0$.*

Proof. Remember that $F_0 \sim D + A$ is the moving part of F. We write $F_i := D + A_i$. Then $F_0 \leq F_i \leq F$. Hence we have $h^0(F_i) = h^0(F_0)$. Since (A_i, Δ_i) satisfies (11.2), we have $F_0^2 = F_i^2$. Riemann-Roch then gives $h^1(F_i) = h^1(F_0) = 0$ by Lemma 6.2. Here we use that we are not in any of the cases (E3) or (E4).

Let now $\Gamma \leq \Delta_i$ be a smooth rational curve such that $\Gamma.A_i > 0$.

Using Riemann-Roch yet another time gives

$$h^0(F_i + \Gamma) - h^0(F_i) = F_i.\Gamma - 1 + h^1(F_i + \Gamma) = 0.$$

Hence $F_i.\Gamma \leq 1$.

Since $D.\Gamma \geq 0$ we get $\Gamma.A_i \leq 1$. So if $\Gamma.A_i > 0$, then $\Gamma.A_i = 1$ and $\Gamma.D = 0$. □

Proof of Proposition 11.1 . Write $\Delta_0 := \Delta$ and $A_0 := A$. Given an effective decomposition $R \sim A_i + \Delta_i$ satisfying (11.1)-(11.3), assume that there exists a smooth rational curve $\Gamma \leq \Delta_i$ such that $\Gamma.A_i > 0$. Write $A_{i+1} := A_i + \Gamma$ and $\Delta_{i+1} := \Delta_i - \Gamma$. Then $R \sim A_{i+1} + \Delta_{i+1}$ satisfies (11.1)-(11.2) by the previous lemma. Clearly $h^2(A_{i+1}) = h^2(\Delta_{i+1}) = 0$, and since $A_{i+1}^2 = A_i^2$ and $\Delta_{i+1}^2 = \Delta_i^2$, it suffices to show that $h^0(A_{i+1}) = h^0(A_i)$ and $h^0(\Delta_{i+1}) = h^0(\Delta_i)$ to show that $R \sim A_{i+1} + \Delta_{i+1}$ satisfies (11.3). It is obvious that $h^0(\Delta_{i+1}) = h^0(\Delta_i) = 1$ since $\Delta_{i+1} \leq \Delta_i$. Furthermore $h^0(A_{i+1}) = h^0(A_i)$ since Γ is fixed in $A_i + \Gamma$, as $\Gamma.(A_i + \Gamma) = -1$. Hence $R \sim A_{i+1} + \Delta_{i+1}$ satisfies (11.3).

We repeat this process if necessary, and it is obvious that the procedure will stop after finitely many steps, say for $i = n \geq 0$, since $\Delta_0 > \Delta_1 > \ldots > \Delta_n$. For the effective decomposition $R \sim A_n + \Delta_n$ there exists no smooth rational curve $\Gamma \leq \Delta_n$ such that $\Gamma.A_n > 0$, whence the decomposition satisfies (11.4) as well. □

Lemma 11.4. *Assume we are neither in case (E3) nor (E4), and that for every $\Gamma \in \mathcal{R}_{L,D}$ we have $\Gamma.A = 0$. Then $R \sim A + \Delta$ satisfies (11.1)-(11.4).*

Proof. If an effective divisor $B \leq \Delta$ satisfies $A.B \neq 0$, then some smooth rational curve $\Gamma \leq \Delta$ (possibly equal to B), must satisfy $A.\Gamma < 0$. But $(D + A).\Gamma = 0$ or 1. Hence $\Gamma \in \mathcal{R}_{L,D}$. But then $A.\Gamma = 0$ by the assumptions, a contradiction. □

11.2 Perfect Clifford divisors for low *c*

From now on (A', Δ') will be a pair of divisors satisfying (11.1)-(11.4).

Furthermore, in the list below we have:

- Γ is a smooth rational curve such that $\Gamma.D = 1$ and $\Gamma.A' = 0$.
- Γ_1 and Γ_2 are smooth rational curves such that $\Gamma_1.D = \Gamma_2.D = 1$ and $\Gamma_1.A' = \Gamma_2.A' = \Gamma_1.\Gamma_2 = 0$.
- $\Delta_0 := 2\Gamma_0 + 2\Gamma_1 + \cdots + 2\Gamma_N + \Gamma_{N+1} + \Gamma_{N+2}$, for $N \geq 0$, with a configuration with respect to D as in (E2) and such that $A'.\Gamma_i = 0$ for $i = 0, \ldots, N + 2$.

Also we denote the different cases by $\{c, D^2\}$.

Here is the list of all possible perfect Clifford divisors for $c = 1$, 2 and 3, and the cases where $h^1(R) > 0$:

$$\mathbf{c = 1, L^2 \geq 8}$$

$\{1,0\}$ $D^2 = 0$, $D.L = 3$, $\dim \mathcal{T} = 3$.
$\{1,2\}$ $D^2 = 2$, $L^2 = 10$, $L \sim 2D + \Gamma$ as in (E0), $\dim \mathcal{T} = 4$.

Moreover, $h^1(R) \neq 0$ if and only if L is as in the following case:

$\{1,0\}^a$ $L \sim 2D + A' + \Gamma$, $A'^2 \geq -2$, $D.A' = 2$, $L^2 = A'^2 + 10 \leq 18$ with equality if and only if $L \sim 6D + 3\Gamma$, $h^1(R) = 1$, $\mathcal{R}_{L,D} = \{\Gamma\}$.

$\mathbf{c = 2, L^2 \geq 12}$

$\{2,0\}$ $D^2 = 0$, $D.L = 4$, $\dim \mathcal{T} = 4$.
$\{2,2\}$ $D^2 = 2$, $D.L = 6$, $L^2 \leq 18$ with equality if and only if $L \sim 3D$, $\dim \mathcal{T} = 5$.
$\{2,4\}$ $D^2 = 4$, $L^2 = 16$, $L \sim 2D$ as in (Q), $\dim \mathcal{T} = 6$.

Moreover, $h^1(R) \neq 0$ if and only if L is as in one of the following cases:

$\{2,0\}^a$ $L \sim 2D + A' + \Gamma$, $A'^2 \geq -2$, $D.A' = 3$, $L^2 = A'^2 + 14 \leq 32$ with equality if and only if $L \sim 8D + 4\Gamma$, $h^1(R) = 1$, $\mathcal{R}_{L,D} = \{\Gamma\}$.
$\{2,0\}^b$ $L \sim 2D + A' + \Gamma_1 + \Gamma_2$, $A'^2 \geq 0$, $D.A' = 2$, $L^2 = A'^2 + 12 \leq 16$ with equality if and only if $L \sim 4D + 2\Gamma_1 + 2\Gamma_2$, $h^1(R) = 2$, $\mathcal{R}_{L,D} = \{\Gamma_1, \Gamma_2\}$ [1].
$\{2,0\}^c$ $L \sim 2D + A' + \Delta_0$, $A'^2 \geq 0$, $D.A' = 2$, $L^2 = A'^2 + 12 \leq 16$ with equality if and only if $L \sim 4D + 2\Delta_0$, $h^1(R) = 2$, $\mathcal{R}_{L,D} = \{\Gamma_0\}$ [2].
$\{2,2\}^a$ $L \sim 2D + \Gamma_1 + \Gamma_2$ as in (E1), $L^2 = 12$, $h^1(R) = 1$, $\mathcal{R}_{L,D} = \{\Gamma_1, \Gamma_2\}$.
$\{2,2\}^b$ $L \sim 2D + \Delta_0$ as in (E2), $L^2 = 12$, $h^1(R) = 1$, $\mathcal{R}_{L,D} = \{\Gamma_0\}$.

$\mathbf{c = 3, L^2 \geq 16}$

$\{3,0\}$ $D^2 = 0$, $D.L = 5$, $\dim \mathcal{T} = 5$.
$\{3,2\}$ $D^2 = 2$, $D.L = 7$, $L^2 \leq 22$, $\dim \mathcal{T} = 6$.
$\{3,4\}$ $D^2 = 4$, $L^2 = 18$, $L \sim 2D + \Gamma$ as in (E0), $\dim \mathcal{T} = 7$.

Moreover, $h^1(R) \neq 0$ if and only if L is as in one of the following cases:

$\{3,0\}^a$ $L \sim 2D + A' + \Gamma$, $A'^2 \geq -2$, $D.A' = 4$, $L^2 = A'^2 + 18 \leq 50$ with equality if and only if $L \sim 10D + 5\Gamma$, $h^1(R) = 1$, $\mathcal{R}_{L,D} = \{\Gamma\}$.
$\{3,0\}^b$ $L \sim 2D + A' + \Gamma_1 + \Gamma_2$, $A'^2 \geq 0$, $D.A' = 3$, $L^2 = A'^2 + 16 \leq 24$, $h^1(R) = 2$, $\mathcal{R}_{L,D} = \{\Gamma_1, \Gamma_2\}$.
$\{3,0\}^c$ $L \sim 2D + A' + \Delta'$, $A'^2 \geq 0$, $D.A' = 3$, $L^2 = A'^2 + 16 \leq 24$, $h^1(R) = 2$, $\mathcal{R}_{L,D} = \{\Gamma_0\}$.
$\{3,2\}^a$ $L \sim 2D + A'' + \Gamma$, $A'^2 = -2$, $D.A' = 2$, $L^2 = 16$, $h^1(R) = 1$, $\mathcal{R}_{L,D} = \{\Gamma\}$.

[1] If $L^2 = 14$, then the moving part of A' is a perfect Clifford divisor of type $\{2,2\}$ containing D, and if $L^2 = 16$, then A' is a perfect Clifford divisor of type $\{2,4\}$ containing D.
[2] Same comment as above.

This list is obtained by using the relations $(*)$ and (3.2) in Chapter 3 together with Propositions 5.3, 5.5 and 5.6. We now show how it works for $c = 3$.

The three cases $\{3,0\}$, $\{3,2\}$ and $\{3,4\}$ follow directly from the relations $(*)$. If $D^2 > 0$, then by (3.2) we must have $L^2 \leq 24$. Assume $L^2 = 24$ and consider the divisor $E := L - 3D$. This satisfies $E^2 = 0$ and $E.L = 3$, thus inducing a Clifford index 1 on L, a contradiction. So $L^2 \leq 22$.

Now assume we are in case $\{3,0\}$ and $h^1(R) > 0$. By Propositions 5.5, 5.6 and 11.1 we have $1 \leq D.\Delta' = D.\Delta \leq 2$. Since

$$5 = D.L = A'.D + \Delta'.D,$$

we have the two possibilities:

(a) $\Delta'.D = 1$ and $D.A' = 4$,
(b) $\Delta'.D = 2$ and $D.A' = 3$.

In case (a), there has to exist a smooth rational curve Γ in the support of Δ' such that $\Gamma.D = 1$ and $\Gamma.A' = 0$ (the last equality follows from (11.4) of Proposition 11.1). Write

$$L \sim 2D + A' + \Gamma + \Delta''.$$

Clearly $D.\Delta'' = A'.\Delta'' = 0$, and by $0 = \Gamma.L = 2 - 2 + \Gamma.\Delta''$, we also get $\Gamma.\Delta'' = 0$, whence

$$(2D + A' + \Gamma).\Delta'' = 0,$$

and we must have $\Delta'' = 0$ since L is numerically 2-connected. This establishes case $\{3,0\}^a$. From the Hodge index theorem on L and A' it follows that $L^2 \leq 50$ with equality if and only if $4L \sim 5A'$.

In case (b), there either exist two (and only two) disjoint smooth rational curves Γ_1 and Γ_2 in the support of Δ' such that $\Gamma_1.D = \Gamma_2.D = 1$ and $\Gamma_1.A' = \Gamma_2.A' = 0$, or there exists one and only one smooth rational curve Γ_0 in the support of Δ' (necessarily with multiplicity 2) such that $\Gamma_0.D = 1$ and $\Gamma_0.A' = 0$. Arguing as in the proof of Proposition 3.7, these two cases give the cases $\{3,0\}^b$ and $\{3,0\}^c$ respectively. Again it follows from the Hodge index theorem on L and A' that $L^2 \leq 24$.

Assume we are in case $\{3,2\}$ and $h^1(R) > 0$. By Propositions 5.5, 5.6, and 11.1 we have $L^2 = 16$, $D.\Delta' = 1$ and $\Delta'^2 = -2$. There has to exist a smooth rational curve Γ in the support of Δ' such that $\Gamma.D = 1$ and $\Gamma.A' = 0$. Arguing as above, we easily find that $L \sim 2D + A' + \Gamma$. Since $7 = D.L = 2D^2 + A'.D + \Delta'.D$, we have $A'.D = 2$, and since $16 = L^2 = 18 + {A'}^2$, we must have ${A'}^2 = -2$. This is case $\{3,2\}^a$.

We leave the easier cases $c = 1$ and 2 to the reader, but make a comment on the cases $\{2,0\}^b$ and $\{2,0\}^c$.

From the Hodge index theorem on L and A' we get that $L^2 \leq 16$ with equality if and only if $L \sim 2A'$. If ${A'}^2 = 2$ or 4, one calculates

$$A'.L - A^2 - 2 = 2,$$

$(A'-D)^2 \geq -2$ and $(A'-D).D = 2$, whence by Riemann-Roch $A' \geq D$, so D does not satisfy the condition (C6). However, since A' computes the Clifford index of L, we have $h^1(A) = h^1(A') = 0$ by Proposition 11.1, whence D is perfect by Lemma 6.10. If $L \sim 2A'$, one easily sees that A' is base point free, whence perfect.

These cases are particularly interesting, since S' is contained in two scrolls of different types.

Note that for $g \leq 10$ (equivalently $L^2 \leq 18$) a polarized $K3$ surface of non-general Clifford index must have $c \leq 3$, so the above cases are sufficient to consider these surfaces. We know that the general $K3$ surface has general Clifford index. The following proposition considers the dimension of the families in the list above.

Proposition 11.5. *The number of moduli of polarized $K3$ surfaces of genus g, with $5 \leq g \leq 10$, and non-general Clifford index $c > 0$ of each of the types $\{1,0\}$, $\{1,2\}$, $\{2,0\}$, $\{2,2\}$ with $g \leq 9$, $\{3,0\}$, $\{3,2\}$ and $\{3,4\}$ is 18, and of each of the types $\{2,2\}$ with $g = 10$ and $\{2,4\}$ is 19.*

Furthermore the general projective model of each of these types satisfies $h^1(L-2D) = 0$, and the general projective model of each of these types except for the types $\{1,2\}$ and $\{3,4\}$ is smooth.

The number of moduli of each of these types with $h^1(L-2D) > 0$ is ≤ 17, except for the type $\{1,0\}^a$ for $g = 10$, whose number is 18.

Proof. In the cases $\{2,2\}$ with $g = 10$ and $\{2,4\}$ we have $L \sim 3D$ and $L \sim 2D$ respectively, so it is clear that those cases can be realized with a Picard group of rank 1 and hence live in 19-dimensional families.

In the other cases, one easily sees that L and D are linearly independent, and we will show that these cases can all be realized with a Picard group of rank 2. Arguing as in the proof of Proposition 4.2 we easily see that there is a $K3$ surface S such that Pic $S \simeq \mathbf{Z}L \oplus \mathbf{Z}D$ such that L^2, $L.D$ and D^2 have the values corresponding to the different cases in question and such that D is a perfect Clifford divisor for L. This has already been done for $D^2 = 0$ in the proof of Proposition 4.2, and a case by case study establishes the proof in the other cases.

Recall now that $h^1(L-2D) > 0$ if and only if there exists a smooth rational curve Γ such that $\Gamma.L = 0$ and $\Gamma.D = 1$, and (since $c > 0$) $\varphi_L(S)$ is singular if and only if there exists a smooth rational curve Γ such that $\Gamma.L = 0$ and $\Gamma.D = 0$ or 1.

Assuming that the rank of the Picard group is two, we can write $\Gamma = aL + bD$, for $a, b \in \mathbf{Q}$. The conditions $\Gamma^2 = -2$, $\Gamma.L = 0$ and $\Gamma.D = 0$ or 1 give the equations:

$$a^2(g-1) + ab(c+2+D^2) + \frac{b^2D^2}{2} = -1,$$

$$a(2g-2)+b(c+2+D^2)=0 \text{ and}$$
$$a(c+2+D^2)+bD^2=0 \text{ or } 1.$$

A case by case check reveals that we have a solution only when $\Gamma.D=1$ and then in the following cases:

(a) $\{1,2\}$, with $\Gamma \sim L-2D$,
(b) $\{3,4\}$, with $\Gamma \sim L-2D$,
(c) $\{1,0\}$ for $g=10$, with $L \sim 6D+3\Gamma$.

One can easily show that case (c) can be realized with a lattice of the form $\mathbf{Z}D \oplus \mathbf{Z}\Gamma$, with $D^2=0$, $D.\Gamma=1$ and $\Gamma^2=-2$.

This concludes the proof of the Proposition. □

11.3 The possible scroll types

We now would like to study which scroll types are possible for each value of (g,c,D^2) with $5 \leq g \leq 10$ and $1 \leq c \leq 3$. Recall that the type $(e_1,\dots,e_d)$ of the scroll $\mathcal{T}$, with $d=\dim \mathcal{T}$, is given by

$$e_i = \#\{j \mid d_j \geq i\} - 1, \tag{11.5}$$

where

$$\begin{aligned} d = d_0 &:= h^0(L)-h^0(L-D) = c+2+\frac{1}{2}D^2, \\ d_1 &:= h^0(L-D)-h^0(L-2D) = d_0 - r, \\ &\vdots \\ d_i &:= h^0(L-iD)-h^0(L-(i+1)D), \\ &\vdots \end{aligned}$$

with

$$r = \begin{cases} D^2+h^1(L-2D) & \text{if } L \not\sim 2D \text{ (equiv. } D^2 \neq c+2), \\ D^2-1 & \text{if } L \sim 2D \text{ (equiv. } D^2 = c+2) \end{cases} \tag{11.6}$$

In the cases $\{1,2\}$ and $\{3,4\}$, which are both of type (E0), and the case $\{2,4\}$, which is of type (Q), we have $h^0(L-2D)=1$ and $h^0(L-iD)=0$ for all $i \geq 3$, so the scroll types are immediately given.

In the case $\{2,2\}$ with $g=10$, we have $L \sim 3D$, so $h^0(L-2D)=h^0(D)=2$, $h^0(L-3D)=1$ and $h^0(L-iD)=0$ for all $i \geq 3$.

We will now consider one by one the remaining cases and gather the result in the tables in Section 11.5 below.

If $c = 1$ or 2 and $D^2 = 0$ the possible scroll types are given in Chapter 9. We now briefly review these cases.

Let us first consider the case $c = 1$ and $D^2 = 0$ (case $\{1, 0\}$).

For $g = 5$ the two possible scroll types are $(1, 1, 1)$ and $(2, 1, 0)$. One easily sees that the first case corresponds to $(d_0, d_1, d_2) = (3, 3, 0)$, whence $h^0(L - 2D) = h^1(L - 2D) = 0$ and the second corresponds to $(d_0, d_1, d_2) = (3, 2, 1)$, whence $h^1(L - 2D) = 1$ and we are in case $\{1, 0\}^a$.

For $g = 6$ we have three possible scroll types: $(2, 1, 1)$, $(2, 2, 0)$ and $(3, 1, 0)$. Comparing with the possible values of the d_i, one finds that the first case corresponds to $h^0(L - 2D) = 1$ (and $h^1(L - 2D) = 0$). Moreover, the two last cases corresponds to the case $\{1, 0\}^a$ with $A' \not> D$ and $A' > D$ respectively.

For $g = 7$ there are four possible scroll types: $(2, 2, 1)$, $(3, 1, 1)$, $(3, 2, 0)$ and $(4, 1, 0)$. We see that the two first cases correspond to $h^1(L - 2D) = 0$, with $h^0(L - 3D) = 0$ and 1 respectively. The two last cases have $h^1(L - 2D) = 1$ and therefore correspond to $\{1, 0\}^a$ with $A' \not> 2D$ and $A' > 2D$ respectively.

We now leave the cases $g = 8$, 9 and 10 to the reader.

If $c = 2$ and $D^2 = 0$ (case $\{2, 0\}$) then $12 \leq L^2 \leq 18$.

We leave the easiest case $g = 7$ to the reader.

If $g = 8$ we have seen that the four possible scroll types are $(2, 1, 1, 1)$, $(2, 2, 1, 0)$, $(3, 1, 1, 0)$ and $(3, 2, 0, 0)$. The scroll $(2, 1, 1, 1)$ corresponds to $h^1(R) = 0$, whereas the scrolls $(2, 2, 1, 0)$ and $(3, 1, 1, 0)$ correspond to the case $\{1, 0\}^a$ with $A' \not> D$ and $A' > D$ respectively. The type $(3, 2, 0, 0)$ corresponds to a polarized surface that also has a different perfect Clifford divisor, and is hence contained in another scroll as well, by the footnote on page 132.

If $g = 9$ we have seen that the five possible scroll types are $(2, 2, 1, 1)$, $(3, 1, 1, 1)$, $(2, 2, 2, 0)$, $(3, 2, 1, 0)$ and $(4, 2, 0, 0)$. The types $(2, 2, 1, 1)$ and $(3, 1, 1, 1)$ correspond to $h^1(R) = 0$ with $h^0(L - 3D) = 0$ and 1 respectively. (One easily sees that the scroll type $(3, 1, 1, 1)$ can be realized by a $K3$ surface S with Picard group $\operatorname{Pic} S \simeq \mathbf{Z}D \oplus \mathbf{Z}\Gamma$, for a smooth rational curve Γ satisfying $\Gamma.D = 2$, and with $L \sim 3D + \Gamma$. Therefore, it has number of moduli 18.) The scroll types $(2, 2, 2, 0)$ and $(3, 2, 1, 0)$ correspond to the case $\{1, 0\}^a$ with $A' \not> D$ and $A' > D$ respectively. The type $(4, 2, 0, 0)$ corresponds to a polarized surface that also has a different perfect Clifford divisor, and is hence contained in another scroll as well, by the footnote on page 132.

If $g = 10$ there are again five possible scroll types: $(2, 2, 2, 1)$, $(3, 2, 1, 1)$, $(3, 2, 2, 0)$, $(3, 3, 1, 0)$ and $(4, 2, 1, 0)$. Again the two first correspond to $h^1(R) = 0$ with $h^0(L - 3D) = 0$ and 1 respectively. The three last cases correspond to the case $\{1, 0\}^a$ with $h^0(A' - D) = 1$ and 2 respectively, but $A' \not> 2D$ for the two first cases, and $A' > 2D$ for the last case.

If $c = 2$ and $D^2 = 2$ (case $\{2, 2\}$), then $12 \leq L^2 \leq 16$ (the case $L^2 = 18$ being already treated). We have

$$(L - 3D).L = L^2 - 18 < 0.$$

By the nefness of L we must have $h^0(L - 3D) = 0$. Since $(L - 2D)^2 = L^2 - 16$, we get by Riemann-Roch $h^0(L - 2D) = \frac{1}{2}L^2 - 6 + h^1(R)$. This gives $d_{\geq 3} = 0$

and the two possibilities $(d_0, d_1, d_2) = (5, 3, \frac{1}{2}L^2 - 6)$ or $(5, 2, 1)$, the latter occurring if and only if $L^2 = 12$ and L is of type (E1) or (E2) (the special cases $\{2,2\}^a$ and $\{2,2\}^b$). The corresponding scroll types in the first situation are then $(1,1,1,0,0)$ for $g = 7$, $(2,1,1,0,0)$ for $g = 8$ and $(2,2,1,0,0)$ for $g = 9$. For $g = 7$ and L of type (E1) or (E2) the scroll type is $(2,1,0,0,0)$.

If $c = 3$ and $D^2 = 0$ (case $\{3,0\}$) then $L^2 = 16$ or 18. We have

$$(L - 3D).L = L^2 - 15 \leq 3$$

and

$$(L - 4D).L = L^2 - 20 < 0.$$

This gives immediately $h^0(L - iD) = 0$ for all $i \geq 3$, whence $d_{\geq 4} = 0$. Also, since $c = 3$, we must have $h^0(L - 3D) \leq 1$. We also have by Riemann-Roch $h^0(L - 2D) = \frac{1}{2}L^2 - 8 + h^1(R)$.

Let us first consider the case $g = 9$. Then we have $(d_0, d_1, d_2, d_3) = (5, 5 - h^1(R), h^1(R) - h^0(L - 3D), h^0(L - 3D))$. If $h^1(R) = 0$, then $h^0(L - 2D) = h^0(L - 3D) = 0$ and $(d_0, d_1, d_2, d_3) = (5,5,0,0)$. The corresponding scroll type is $(1,1,1,1,1)$. The cases with $h^0(R) = h^1(R) > 0$ are $\{3,0\}^a$, $\{3,0\}^b$ and $\{3,0\}^c$. In the first we have $h^0(R) = h^1(R) = 1$, whence $h^0(L - 3D) = 0$ and $(d_0, d_1, d_2, d_3) = (5,4,1,0)$. The corresponding scroll type is $(2,1,1,1,0)$. In the cases $\{3,0\}^b$ and $\{3,0\}^c$, we have $h^0(R) = h^1(R) = 2$. If $h^0(L - 3D) = 0$ (eqv. $A' \not> D$), we get $(d_0, d_1, d_2, d_3) = (5,3,2,0)$ and the scroll type is $(2,2,1,0,0)$. If $h^0(L - 3D) = 1$ (eqv. $A' > D$), we get $(d_0, d_1, d_2, d_3) = (5,3,1,1)$ and the scroll type is $(3,1,1,0,0)$.

If $g = 10$, we have $(d_0, d_1, d_2, d_3) = (5, 5 - h^1(R), 1 + h^1(R) - h^0(L - 3D), h^0(L - 3D))$. If $h^1(R) = 0$, then $h^0(L - 2D) = 1$ and $h^0(L - 3D) = 0$ and $(d_0, d_1, d_2, d_3) = (5,5,1,0)$. The corresponding scroll type is $(2,1,1,1,1)$. The cases with $h^1(R) > 0$ are $\{3,0\}^a$, $\{3,0\}^b$ and $\{3,0\}^c$ as in the case $g = 9$. Arguing as in that case, we get $(d_0, d_1, d_2, d_3) = (5,4,2,0)$ and scroll type $(2,2,1,1,0)$ in the case $\{3,0\}^a$ (where $h^1(R) = 1$), and we get $(d_0, d_1, d_2, d_3) = (5,3,3,0)$ and scroll type $(2,2,2,0,0)$ if $h^0(L - 3D) = 0$ (eqv. $A' \not> D$), and $(d_0, d_1, d_2, d_3) = (5,3,2,1)$ and scroll type $(3,2,1,0,0)$ if $h^0(L - 3D) = 1$ (eqv. $A' > D$) in the two latter cases (where $h^1(R) = 2$).

If $c = 3$ and $D^2 = 2$ (case $\{3,2\}$), then $L^2 = 16$ or 18. We have

$$(L - 3D).L = L^2 - 21 < 0,$$

whence $h^0(L - iD) = 0$ for all $i \geq 3$, whence $d_{\geq 3} = 0$. By Riemann-Roch, $h^0(L - 2D) = \frac{1}{2}L^2 - 8 + h^1(R)$ and we have $(d_0, d_1, d_2) = (6, 4 - h^1(R), \frac{1}{2}L^2 - 8 + h^1(R), 0)$.

If $g = 9$ and $h^1(R) = 0$, then $(d_0, d_1, d_2) = (6,4,0)$ and the corresponding scroll type is $(1,1,1,1,0,0)$. The case with $h^1(R) > 0$ is given by $\{3,2\}^a$. In this case we have $(d_0, d_1, d_2) = (6,3,1)$ and the corresponding scroll type is $(2,1,1,0,0,0)$.

If $g = 10$, then we automatically have $h^1(R) = 0$, whence $(d_0, d_1, d_2) = (6,4,1)$ and the corresponding scroll type is $(2,1,1,1,0,0)$.

We will summarize these results below.

Furthermore, we can prove, by arguing with lattices that all the cases mentioned above exist, and calculate the number of their moduli. In many cases, we can also explicitly find an expression for L in terms of D and some smooth rational curves on the surface. Also, by studying the Picard lattices, we can find the curves that are contracted by L, and hence find the singularities of the generic surfaces in question.

All these informations are also summarized below, in section 11.5.

11.4 Some concrete examples

In this section, we focus on some concrete examples, to give the reader an idea of the proofs. We then leave all the other cases to the reader, and conclude the chapter by giving the list of all projective models of genus ≤ 10 in section 11.5.

Example 11.6. We start with an easy case: $g = 6$, $c = 1$, $D^2 = 0$ and the scroll type $(3, 1, 0)$. This occurs if L is of type $\{1, 0\}^a$ with $A' > D$ (and also $\mathcal{R}_{L,D} = \{\Gamma\}$). By arguing as in the proof of Proposition 3.7 we find that $L \sim 3D + 2\Gamma + \Gamma_0 + \Gamma_1$, where Γ, Γ_0 and Γ_1 are smooth rational curves, with the following configuration:

$$\begin{tikzcd} D \arrow[r, dash] \arrow[d, dash] & \Gamma \arrow[r, dash] & \Gamma_1 \\ \Gamma_0. & & \end{tikzcd}$$

By Propositions 1.11 and 1.12 there is an algebraic $K3$ surface S with Picard group $\operatorname{Pic} S = \mathbf{Z}D \oplus \mathbf{Z}\Gamma \oplus \mathbf{Z}\Gamma_0 \oplus \mathbf{Z}\Gamma_1$ and intersection matrix

$$\begin{bmatrix} D^2 & D.\Gamma & D.\Gamma_0 & D.\Gamma_1 \\ \Gamma.D & \Gamma^2 & \Gamma.\Gamma_0 & \Gamma.\Gamma_1 \\ \Gamma_0.D & \Gamma_0.\Gamma & \Gamma_0^2 & \Gamma_0.\Gamma_1 \\ \Gamma_1.D & \Gamma_1.\Gamma & \Gamma_1.\Gamma_0 & \Gamma_1^2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & -2 & 0 & 1 \\ 1 & 0 & -2 & 0 \\ 0 & 1 & 0 & -2 \end{bmatrix},$$

and such that $L := 3D + 2\Gamma + \Gamma_0 + \Gamma_1$ is nef (whence by Riemann-Roch D and $\Gamma_0 > 0$).

We have $D.L - D^2 - 2 = 1$. To show that L is base point free and of Clifford index 1, it suffices to show that there is no effective divisor E such that $E^2 = 0$ and $E.L = 1$ or 2.

Set $E \sim xD + y\Gamma + z\Gamma_0 + w\Gamma_1$. Since we can assume $E \in \mathcal{A}^0(L)$, and E base point free, we easily see that

$$E.\Gamma_0 = x - 2z = 0 \quad \text{or} \quad 1,$$

whence

$$E.L = 3x + z = 7z \quad \text{or} \quad 7z + 3,$$

which can never be equal to 1 or 2.

By Riemann-Roch either $\Gamma > 0$ or $-\Gamma > 0$. If the latter is the case, write $\Gamma = -\gamma$, and we then have $D = D_0 + \gamma$ with $D_0 > 0$, since $D.\gamma = -1$. Therefore, we can write

$$L \sim 3(D_0 + \gamma) - 2\gamma + \Gamma_0 + \Gamma_1 = 3D_0 + 2\gamma + \Gamma_0 + \Gamma_1.$$

We can use the same argument if $-\Gamma_1 > 0$, so possibly after a change of basis, we can assume D, Γ, Γ_0 and $\Gamma_1 > 0$. It is then easy to check that D is nef, whence a perfect Clifford divisor.

Example 11.7. Let us consider the case $g = 9$, $c = 2$, $D^2 = 0$ and the scroll type $(3, 2, 1, 0)$. This occurs if L is of type $\{2, 0\}^a$ with $A' > D$ (and also $\mathcal{R}_{L,D} = \{\Gamma\}$). An analysis as in the proof of Proposition 3.7 shows that L is one of the following three types:

(a) $L \sim 3D + 2\Gamma + \Gamma_1 + \Gamma_2$, with the following configuration:

$$\begin{tikzcd}
D \arrow[r, dash] \arrow[d, dash] \arrow[dr, dash] & \Gamma \arrow[d, dash] \\
\Gamma_2 & \Gamma_1
\end{tikzcd}$$

(b) $L \sim 3D + 2\Gamma + \Gamma_1 + \Gamma_2$, with the following configuration:

$$\begin{tikzcd}
D \arrow[r, dash] \arrow[d, Rightarrow, no head] & \Gamma \arrow[r, dash] & \Gamma_1 \\
\Gamma_2 & &
\end{tikzcd}$$

(c) $L \sim 3D + 2\Gamma + \Gamma_1 + \Gamma_2 + \cdots + \Gamma_{n+3}$, for $n \geq 0$ (in general $n = 0$) with the following configuration:

$$\begin{tikzcd}
 & D \arrow[r, dash] \arrow[d, dash] \arrow[dr, dash] & \Gamma_2 \arrow[d, dashed, no head] \\
\Gamma_1 \arrow[r, dash] & \Gamma & \Gamma_{n+3}.
\end{tikzcd}$$

(Actually case (b) can be looked at as a special case of case (c), with "$n = -1$".)

One can easily show that both cases (a) and (b) do not occur with a Picard group of rank < 4, and case (c) does not occur with a Picard group of rank < 5. We now show that both case (a) and (b) occur with a Picard group of rank 4.

We first consider case (a).

By Propositions 1.11 and 1.12 there is an algebraic $K3$ surface S with Picard group Pic $S = \mathbf{Z}D \oplus \mathbf{Z}\Gamma \oplus \mathbf{Z}\Gamma_1 \oplus \mathbf{Z}\Gamma_2$ and intersection matrix corresponding to the configuration above, and such that $L := 3D + 2\Gamma + \Gamma_1 + \Gamma_2$ is nef (whence D, Γ_1, $\Gamma_2 > 0$ by Riemann-Roch).

We calculate $D.L - D^2 - 2 = 2$. To show that L is base point free and that Cliff $L = 2$ with D as a perfect Clifford divisor, it will suffice to show that there are no divisor B on S satisfying $B^2 = 0$, $B.L = 1, 2, 3$ or $B^2 = 2$, $B.L = 6$, and that D is nef.

Write $B \sim xD + y\Gamma + z\Gamma_1 + w\Gamma_2$. Since we can assume $B \in \mathcal{A}^0(L)$, and B base point free, we easily see that

$$B.\Gamma_2 = x - 2w = 0 \quad \text{or} \quad 1,$$

and

$$B.\Gamma_1 = x + y - 2z = 0, 1 \quad \text{or} \quad 2.$$

By the Hodge index theorem one also finds

$$B.D = y + z + w = \begin{cases} 1 & \text{if } B^2 = 0, \\ 2 & \text{if } B^2 = 2. \end{cases}$$

Also, we have

$$B.L = 4x + 3z + w = \begin{cases} 1, 2, 3 & \text{if } B^2 = 0, \\ 6 & \text{if } B^2 = 2. \end{cases}$$

One checks by inspection that these four equations have no integer solutions.

By Riemann-Roch, either $\Gamma > 0$ or $-\Gamma > 0$. As in the previous example, possibly after a change of basis one can assume that $\Gamma > 0$ and that D is nef, whence perfect.

We now consider case (b).

Again by Propositions 1.11 and 1.12 there is an algebraic $K3$ surface S with Picard group Pic $S = \mathbf{Z}D \oplus \mathbf{Z}\Gamma \oplus \mathbf{Z}\Gamma_1 \oplus \mathbf{Z}\Gamma_2$ and intersection matrix corresponding to the configuration for (b) above, and such that $L := 3D + 2\Gamma + \Gamma_1 + \Gamma_2$ is nef (whence D and $\Gamma_2 > 0$ by Riemann-Roch).

We calculate $D.L - D^2 - 2 = 2$. To show that L is base point free and that Cliff $L = 2$ with D as a perfect Clifford divisor, it will again suffice to show that there are no divisor B on S satisfying $B^2 = 0$, $B.L = 1, 2, 3$ or $B^2 = 2$, $B.L = 6$, and that D is nef.

Write $B \sim xD + y\Gamma + z\Gamma_1 + w\Gamma_2$ as before. Again by the Hodge index theorem and since we can assume $B \in \mathcal{A}^0(L)$, and B base point free, we get

$$B.D = y + 2w = \begin{cases} 1 & \text{if } B^2 = 0, \\ 2 & \text{if } B^2 = 2, \end{cases}$$

$$B.\Gamma_2 = 2(x - w) = 0 \quad \text{or} \quad 2,$$

and

$$B.\Gamma_1 = y - 2z = -1, 0, \quad \text{or} \quad 1$$

(since we do not know whether it is Γ_1 or $-\Gamma_1$ which is effective). Combining these equations with

$$B.L = 2(2x + y + 2w),$$

we find no integer solutions. Again, possibly after a change of basis, we get that D is perfect and that all D, Γ, Γ_1 and $\Gamma_2 > 0$.

We can also check which curves are contracted by L.

In case (a), the only contracted curve is in general Γ, so all surfaces in that family has an A_1 singularity, and the general surface has only such a singularity. Furthermore S'' is then in general smooth.

In case (b), the only contracted curves are in general Γ and Γ_1 , so all surfaces in that family has an A_2 singularity, and the general surface has only such a singularity. Furthermore S'' is then necessarily singular.

By comparing with the table on page 112, we then find that case (a) has $b_1 = 3$ and case (b) has $b_1 = 2$.

Example 11.8. As an easy example we consider the case $g = 10$, $c = 1$, $D^2 = 0$ and the scroll type $(5, 2, 1)$. This occurs if $\mathcal{R}_{L,D} = \emptyset$ and $h^0(L - 5D) = 1$. An analysis as in the proof of Proposition 3.7 shows that $L \sim 5D + 3\Gamma_1 + 2\Gamma_2 + \Gamma_3$, with the following configuration:

$$D \text{ —— } \Gamma_1 \text{ —— } \Gamma_2 \text{ —— } \Gamma_3.$$

One can easily show that this cannot be achieved with a Picard group of rank < 4.

By Propositions 1.11 and 1.12 again there is an algebraic $K3$ surface S with Picard group Pic $S = \mathbf{Z}D \oplus \mathbf{Z}\Gamma_1 \oplus \mathbf{Z}\Gamma_2 \oplus \mathbf{Z}\Gamma_3$ and intersection marix corresponding to the configuration above, and such that $L := 5D + 3\Gamma_1 + 2\Gamma_2 + \Gamma_3$ is nef (whence D and $\Gamma_1 > 0$ by Riemann-Roch).

We calculate $D.L - D^2 - 2 = 1$. To show that L is base point free and that Cliff $L = 1$ with D as a perfect Clifford divisor, it will suffice to show that there is no divisor E on S satisfying $E^2 = 0$, $E.L = 1, 2$ and that D is nef.

By the Hodge index theorem $36E.D \leq (E + D)^2 L^2 \leq ((E + D).L)^2 \leq 25$, whence $E.D = 0$. Writing $E \sim xD + y\Gamma_1 + z\Gamma_2 + w\Gamma_3$, we get

$$E.D = y = 0,$$

whence

$$E.L = 3x + y = 3x \neq 1 \quad \text{or} \quad 2.$$

Possibly after a change of basis, we get that D is perfect and that all D, Γ_1, Γ_2 and $\Gamma_3 > 0$.

One finds that the only contracted curves with this Picard group are Γ_2 and Γ_3, so the general surface in this family has an A_2 singularity.

Example 11.9. We give a more involved example: $g = 10$, $c = 2$, $D^2 = 0$ and the scroll type $(3, 2, 1, 1)$. This occurs if $\mathcal{R}_{L,D} = \emptyset$ and $h^0(L - 3D) = 1$. By the table on page 62, we must have $b_1 = 3$ or 4, and we will now show that both these cases exist (with the number of moduli 17 and 16 respectively).

One easily sees that there is no way to achieve this situation with a Picard group of rank < 3. We will now show that it is possible with a Picard group of rank 3.

By Propositions 1.11 and 1.12 there is an algebraic $K3$ surface S with Picard group $\operatorname{Pic} S = \mathbf{Z}D \oplus \mathbf{Z}\Gamma_1 \oplus \mathbf{Z}\Gamma_2$ and intersection matrix

$$\begin{bmatrix} D^2 & D.\Gamma_1 & D.\Gamma_2 \\ \Gamma_1.D & \Gamma_1^2 & \Gamma_1.\Gamma_2 \\ \Gamma_2.D & \Gamma_2.\Gamma_1 & \Gamma_2^2 \end{bmatrix} = \begin{bmatrix} 0 & 2 & 0 \\ 2 & -2 & 1 \\ 0 & 1 & -2 \end{bmatrix},$$

and such that $L := 3D + 2\Gamma_1 + \Gamma_2$ is nef (whence by Riemann-Roch D and $\Gamma_1 > 0$).

We have $D.L - D^2 - 2 = 2$. To show that L is base point free and of Clifford index 2 with D as a perfect Clifford divisor, it suffices to show that there is no effective divisor B such that $B^2 = 0$, $B.L = 1, 2, 3$, or $B^2 = 2$, $B.L = 6$.

By the Hodge index theorem one has

$$18(B^2 + 2B.D) = L^2(B + D)^2 \leq ((B + D).L)^2 = (B.L + 4)^2,$$

which gives $B.D \leq 1$.

Writing $B \sim xD + y\Gamma_1 + z\Gamma_2$, we have $B.D = 2y$, whence $y = 0$.

Since either $\Gamma_2 > 0$ or $-\Gamma_2 > 0$ and we can assume $B \in \mathcal{A}^0(L)$, we must have $B.\Gamma_2 = y - 2z = -2z = -1, 0, 1$. We therefore get $z = 0$.

So B is a multiple of D, a contradiction.

Possibly after a change of basis, we get that D is perfect and that also $\Gamma_2 > 0$.

One finds that the only contracted curve with this Picard group is Γ_2, so that all surfaces in this family have at least an A_1 singularity, and the general such surface has such a singularity. By comparing with the table on page 112, we see that we must have $b_1 = 4$.

But there is also another family of surfaces. Again we find that there is an algebraic $K3$ surface S with Picard group $\operatorname{Pic} S = \mathbf{Z}D \oplus \mathbf{Z}\Gamma_1 \oplus \mathbf{Z}\Gamma_2 \oplus \mathbf{Z}\Gamma_3$ and intersection matrix

$$\begin{bmatrix} D^2 & D.\Gamma_1 & D.\Gamma_2 & D.\Gamma_3 \\ \Gamma_1.D & \Gamma_1^2 & \Gamma_1.\Gamma_2 & \Gamma_1.\Gamma_3 \\ \Gamma_2.D & \Gamma_2.\Gamma_1 & \Gamma_2^2 & \Gamma_2.\Gamma_3 \\ \Gamma_3.D & \Gamma_3.\Gamma_1 & \Gamma_3.\Gamma_2 & \Gamma_3^2 \end{bmatrix} = \begin{bmatrix} 0 & 2 & 1 & 1 \\ 2 & -2 & 0 & 0 \\ 1 & 0 & -2 & 0 \\ 1 & 0 & 0 & -2 \end{bmatrix},$$

and such that $L := 3D + \Gamma_1 + \Gamma_2 + \Gamma_3$ is nef.

One can show that Cliff $L = 2$ with D as a perfect Clifford divisor (again after possibly changing the basis). Furthermore, one finds that with this lattice, there are no contracted curves, whence S' is smooth. By comparing with the table on page 112, we see that we must have $b_1 = 3$.

11.5 The list of projective models of low genus

We will now summarize essential information about birational projective models S' of $K3$ surfaces of genera $5 \leq g \leq 10$. In some cases we are able to give a resolution of S' in its scroll $\mathcal{T}$. When we are not able to do this, we give the vector bundle a section of which cuts out S'' in $\mathcal{T}_0 \simeq \mathbf{P}(\mathcal{E})$ (which is the dual of the vector bundle F_1 in the resolution

$$\cdots \longrightarrow F_2 \longrightarrow F_1 \longrightarrow \mathcal{O}_{\mathcal{T}_0} \longrightarrow \mathcal{O}_{S''} \longrightarrow 0.)$$

This vector bundle is a direct sum of line bundles, which we write as a linear combination of the line bundles $\mathcal{H}$ and $\mathcal{F}$ on $\mathbf{P}(\mathcal{E})$, where $\mathcal{H} = i^*\mathcal{O}_{\mathbf{P}^g}(1)$ and $\mathcal{F} = \pi^*\mathcal{O}_{\mathbf{P}^1}(1)$, with

$$\begin{tikzcd} \mathbf{P}(\mathcal{E}) \arrow[r, "i"] \arrow[d, "\pi"] & \mathcal{T} \subseteq \mathbf{P}^g \\ \mathbf{P}^1. & \end{tikzcd}$$

Also note that we in all cases have $\mathcal{J}_{S'/\mathcal{T}} = i_*\mathcal{J}_{S''/\mathcal{T}_0}$ by Proposition 8.6, and that in most cases, by Remark 8.35, the sections of $F_1^\vee$ are constant on the fibers of i, whence they also give "equations" cutting out S' in $\mathcal{T}$ set-theoretically.

The singularity type listed in the rightmost column of the tables below indicates that for "almost all" $K3$ surfaces in question its projective model S' has singularities exactly as indicated, and that none have milder singularities. By "almost all" we here mean that the moduli of the exceptional set of $K3$ surfaces in question with different singularity type(s) is strictly smaller than the number of moduli listed in the middle column. These exceptional $K3$ surfaces will have "worse" singularities than the one(s) listed in the rightmost column.

In the tables below, c is as usual the Clifford index of L, D is a perfect Clifford divisor and A is as defined in (6.1). To find the tables we use A' and Δ' as above, but since A and A' (resp. Δ and Δ') enjoy the same intersection and cohomology properties, we can then reintroduce A (resp. Δ). In particular, the tables below are still valid if one exchanges A with A'.

$$\mathbf{g = 5}$$

The general projective model is a complete intersection of three hyperquadrics. The others are as follows:

c	D^2	scroll type	# mod.	type of L	sing.
1	0	$(1,1,1)$	18	$h^0(L-2D)=0$	sm.
1	0	$(2,1,0)$	17	$\{1,0\}^a$, $A^2=-2$	A_1

In these cases $\mathcal{O}_{S'}$ has the following $\mathcal{O}_{\mathcal{T}}$-resolution:

$$0 \longrightarrow \mathcal{O}_{\mathcal{T}}(-3\mathcal{H}+\mathcal{F}) \longrightarrow \mathcal{O}_{\mathcal{T}} \longrightarrow \mathcal{O}_{S'} \longrightarrow 0.$$

$\mathbf{g = 6}$

The general projective model is a hyperquadric section of a Fano 3-fold of index 2 and degree 5. The others are as follows:

c	D^2	scroll type	# mod.	type of L	sing.
1	0	$(2,1,1)$	18	$h^0(L-2D)=0$	sm.
1	0	$(2,2,0)$	17	$\{1,0\}^a$, $A^2=0$, $A \not> D$	A_1
1	0	$(3,1,0)$	16	$\{1,0\}^a$, $A^2=0$, $A > D$ $^{(i)}$	A_2
1	2	$(2,1,0,0)$	18	(E0)	A_1

In the three first cases $\mathcal{O}_{S'}$ has the following $\mathcal{O}_{\mathcal{T}}$-resolution:

$$0 \longrightarrow \mathcal{O}_{\mathcal{T}}(-3\mathcal{H}+2\mathcal{F}) \longrightarrow \mathcal{O}_{\mathcal{T}} \longrightarrow \mathcal{O}_{S'} \longrightarrow 0.$$

In the last case, S' has a resolution:

$$\begin{aligned} 0 &\longrightarrow \mathcal{O}_{\mathcal{T}}(-4\mathcal{H}+2\mathcal{F}) \oplus \mathcal{O}_{\mathcal{T}}(-4\mathcal{H}+\mathcal{F}) \\ &\longrightarrow \mathcal{O}_{\mathcal{T}}(-2\mathcal{H}+2\mathcal{F}) \oplus \mathcal{O}_{\mathcal{T}}(-3\mathcal{H}+\mathcal{F}) \oplus \mathcal{O}_{\mathcal{T}}(-3\mathcal{H}) \\ &\longrightarrow \mathcal{O}_{\mathcal{T}} \longrightarrow \mathcal{O}_{S'} \longrightarrow 0 \end{aligned}$$

Comments on the types of L:

(i) $L \sim 3D + 2\Gamma + \Gamma_0 + \Gamma_1$, with the following configuration:

$$\begin{tikzcd} D \arrow[r, dash] \arrow[d, dash] & \Gamma \arrow[r, dash] & \Gamma_1 \\ \Gamma_0 & & \end{tikzcd}$$

$\mathbf{g = 7}$

The general projective model is a complete intersection of 8 hyperplanes in Σ^{10}_{12}, as described in the beginning of Chapter 10.

The other projective models are as follows:

c	D^2	scroll type	# mod.	type of L	sing.
1	0	$(2,2,1)$	18	$h^0(L-2D)=2,\ h^0(L-3D)=0$	sm.
1	0	$(3,1,1)$	16	$h^0(L-2D)=2,\ h^0(L-3D)=1$	sm.
1	0	$(3,2,0)$	17	$\{1,0\}^a,\ A^2=2,\ A>D,\ A\not>2D$	A_1
1	0	$(4,1,0)$	16	$\{1,0\}^a,\ A^2=2,\ A>2D$ $^{(i)}$	A_3
2	0	$(1,1,1,1)$	18	$h^0(L-2D)=0$	sm.
2	0	$(2,1,1,0)$	17	$\{2,0\}^a,\ A^2=-2$	A_1
2	0	$(2,2,0,0)$	16	$\{2,0\}^b$ or $\{2,0\}^c,\ A\not>D$ $^{(ii)}$	$2A_1$
2	0	$(3,1,0,0)$	15	$\{2,0\}^b$ or $\{2,0\}^c,\ A>D$ $^{(iii)}$	$2A_2$
2	2	$(1,1,1,0,0)$	18	$h^0(L-2D)=0$	sm.
2	2	$(2,1,0,0,0)$	17	(E1) or (E2) $^{(iv)}$	$2A_1$

In the cases $(c,D^2)=(1,0)$ $\mathcal{O}_{S'}$ has the following $\mathcal{O}_{\mathcal{T}}$-resolution:

$$0 \longrightarrow \mathcal{O}_{\mathcal{T}}(-3\mathcal{H}+3\mathcal{F}) \longrightarrow \mathcal{O}_{\mathcal{T}} \longrightarrow \mathcal{O}_{S'} \longrightarrow 0.$$

In the cases $(c,D^2)=(2,0)$ $\mathcal{O}_{S'}$ has the following $\mathcal{O}_{\mathcal{T}}$-resolution:

$$0 \longrightarrow \mathcal{O}_{\mathcal{T}}(-4\mathcal{H}+(g-1)\mathcal{F}) \longrightarrow$$
$$\mathcal{O}_{\mathcal{T}}(-2\mathcal{H}+b_1\mathcal{F}) \oplus \mathcal{O}_{\mathcal{T}}(-2\mathcal{H}+b_2\mathcal{F}) \longrightarrow \mathcal{O}_{\mathcal{T}} \longrightarrow \mathcal{O}_{S'} \longrightarrow 0,$$

with $(b_1,b_2)=(1,1)$ or $(2,0)$ for the scroll types $(1,1,1,1)$ and $(2,1,1,0)$ and $(b_1,b_2)=(2,0)$ for the scroll types $(2,2,0,0)$ and $(3,1,0,0)$.

In the cases $(c,D^2)=(2,2)$ then S'' is cut out in $\mathcal{T}_0$ by a section (which is constant on the fibers of i) of

$$\oplus_{i=1}^4 \mathcal{O}_{\mathcal{T}_0}(2\mathcal{H}-b_i\mathcal{F}),$$

where $(b_1,b_2,b_3,b_4)=(1,1,1,0)$ or $(2,1,0,0)$ for the type $(1,1,1,0,0)$, and $(2,1,0,0)$ for the type $(2,1,0,0,0)$.

Comments on the types of L:

(i) $L \sim 4D+3\Gamma+2\Gamma_1+\Gamma_2$, with the following configuration:

$$D \;—\; \Gamma \;—\; \Gamma_1 \;—\; \Gamma_2$$

(ii) The number of moduli of the case $\{2,0\}^c$ is 15, with mildest singularity A_3.

(iii) In the case $\{2,0\}^b$ we have $L \sim 3D+2\Gamma_1+2\Gamma_2+\Gamma_1'+\Gamma_2'$, with the following configuration:

$$\begin{array}{ccccc} D & — & \Gamma_1 & — & \Gamma_1' \\ & \diagdown & & & \\ & & \Gamma_2 & — & \Gamma_2', \end{array}$$

and in the case $\{2,0\}^c$ we have $L \sim 3D + 4\Gamma_0 + 3\Gamma_1 + 2\Gamma_2 + 2\Gamma_3 + \Gamma_4$, with the following configuration:

$$\begin{tikzcd} D \arrow[r, dash] & \Gamma_0 \arrow[r, dash] \arrow[d, dash] & \Gamma_1 \arrow[r, dash] & \Gamma_3 \arrow[d, dash] \\ & \Gamma_2 & & \Gamma_4. \end{tikzcd}$$

The mildest singularity of this latter case is A_5.

(iv) The number of moduli of the case (E2) is 16, with mildest singularity A_3.

$\mathbf{g = 8}$

The general projective model is a complete intersection of 5 hyperplanes in Grass $(V^6, 2) \subseteq \mathbf{P}^{14}$.

The others are as follows:

c	D^2	scroll type	# mod.	type of L	sing.
1	0	$(2,2,2)$	18	$h^0(L-2D)=3$, $h^0(L-3D)=0$	sm.
1	0	$(3,2,1)$	17	$h^0(L-2D)=3$, $h^0(L-3D)=1$	sm.
1	0	$(4,2,0)$	17	$\{1,0\}^a$, $A^2=4$	A_1
2	0	$(2,1,1,1)$	18	$h^0(L-2D)=1$	sm.
2	0	$(2,2,1,0)$	17	$\{2,0\}^a$, $A^2=0$, $A \not> D$	A_1
2	0	$(3,1,1,0)$	15	$\{2,0\}^a$, $A^2=0$, $A > D$ $^{(i)}$	A_2
2	0	$(3,2,0,0)$	16	$\{2,0\}^b$ or $\{2,0\}^c$, $A^2=2$, $A > D$ $^{(ii)}$	$2A_1$
2	2	$(2,1,1,0,0)$	18	$h^0(L-2D)=1$	sm.
3	2	$(1,1,1,0,0,0)$	18	(CG1) or (CG2) $^{(iii)}$	sm.
3	2	$(2,1,0,0,0,0)$	16	(CG3)-(CG7) $^{(iv)}$	$3A_1$

In the cases $(c, D^2) = (1, 0)$ $\mathcal{O}_{S'}$ has the following $\mathcal{O}_{\mathcal{T}}$-resolution:

$$0 \longrightarrow \mathcal{O}_{\mathcal{T}}(-3\mathcal{H} + 4\mathcal{F}) \longrightarrow \mathcal{O}_{\mathcal{T}} \longrightarrow \mathcal{O}_{S'} \longrightarrow 0.$$

In the cases $(c, D^2) = (2, 0)$ $\mathcal{O}_{S'}$ has the following $\mathcal{O}_{\mathcal{T}}$-resolution:

$$0 \longrightarrow \mathcal{O}_{\mathcal{T}}(-4\mathcal{H} + (g-1)\mathcal{F}) \longrightarrow$$
$$\mathcal{O}_{\mathcal{T}}(-2\mathcal{H} + b_1\mathcal{F}) \oplus \mathcal{O}_{\mathcal{T}}(-2\mathcal{H} + b_2\mathcal{F}) \longrightarrow \mathcal{O}_{\mathcal{T}} \longrightarrow \mathcal{O}_{S'} \longrightarrow 0,$$

with $(b_1, b_2) = (2, 1)$, except for the type $(3, 2, 0, 0)$, where $(b_1, b_2) = (3, 0)$. In this latter case, S also contains a different perfect Clifford divisor (by the footnote on page 132), so S' can also be described as for the case $(c, D^2) = (2, 2)$.

In the cases $(c, D^2) = (2, 2)$ then S'' is cut out in $\mathcal{T}_0$ by a section (which is constant on the fibers of i) of

$$\oplus_{i=1}^{4}\mathcal{O}_{\mathcal{T}_0}(2\mathcal{H}-b_i\mathcal{F}),$$

where $(b_1,b_2,b_3,b_4)=(3,2,0,0),(3,1,1,0)$ or $(2,2,1,0)$.

In the cases (CG1) and (CG2) then S'' is cut out in $\mathcal{T}_0$ by a section of

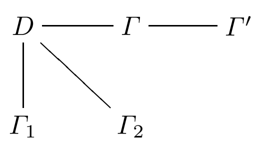
$$\begin{array}{ll}\mathcal{O}_{\mathcal{T}_0}(2\mathcal{H}-2\mathcal{F})\oplus\mathcal{O}_{\mathcal{T}_0}(2\mathcal{H}-\mathcal{F})^2\oplus\mathcal{O}_{\mathcal{T}_0}(2\mathcal{H})^5 \text{ or}\\ \mathcal{O}_{\mathcal{T}_0}(2\mathcal{H}-\mathcal{F})^4\oplus\mathcal{O}_{\mathcal{T}_0}(2\mathcal{H})^4 \qquad\text{or}\\ \mathcal{O}_{\mathcal{T}_0}(2\mathcal{H}-2\mathcal{F})\oplus\mathcal{O}_{\mathcal{T}_0}(2\mathcal{H}-\mathcal{F})^3\oplus\mathcal{O}_{\mathcal{T}_0}(2\mathcal{H})^3\oplus\mathcal{O}_{\mathcal{T}_0}(2\mathcal{H}+\mathcal{F})\end{array}$$

(which is constant on the fibers of i in the first two cases). In the cases (CG3)-(CG7) then S'' is cut out in $\mathcal{T}_0$ by a section (which is constant on the fibers of i) of

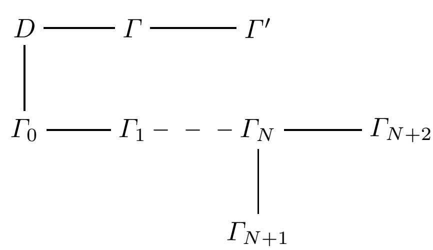
$$\mathcal{O}_{\mathcal{T}_0}(2\mathcal{H}-2\mathcal{F})\oplus\mathcal{O}_{\mathcal{T}_0}(2\mathcal{H}-\mathcal{F})^2\oplus\mathcal{O}_{\mathcal{T}_0}(2\mathcal{H})^5$$

Comments on the types of L:

(i) Here there are two subcases, one of them is: $L\sim 3D+2\Gamma+\Gamma'+\Gamma_1+\Gamma_2$, with the following configuration:



The number of moduli in this subcase is 15, with mildest singularity A_2. In the other subcase $L\sim 3D+2\Gamma+\Gamma'+2\Gamma_0+2\Gamma_1+\cdots+2\Gamma_N+\Gamma_{N+1}+\Gamma_{N+2}$, for $N\geq 0$ (in general $N=0$) with the following configuration:



The number of moduli in this subcase is 14, with mildest singularity A_2+2A_1.

(ii) The number of moduli of the case $\{2,0\}^c$ is 15, with mildest singularity A_3.

(iii) The number of moduli of the case (CG2) is 17, with mildest singularity A_1.

(iv) The number of moduli of the cases (CG4), (CG5), (CG6) and (CG7) are 15, 14, 13 and 12 respectively, with mildest singularities A_1+A_3, A_5, D_6 and E_7 respectively.

$\mathbf{g = 9}$

The general projective model is a complete intersection of 4 hyperplanes in Σ^6_{16}, as described in the beginning of Chapter 10.

The others are as follows:

c	D^2	scroll type	# mod.	type of L	sing.
1	0	$(3,2,2)$	18	$h^0(L-2D)=4,\ h^0(L-3D)=1$	sm.
1	0	$(3,3,1)$	17	$h^0(L-iD)=4,2,0$, for $i=2,3,4$	sm.
1	0	$(4,2,1)$	16	$h^0(L-iD)=4,2,1$, for $i=2,3,4^{(i)}$	A_1
1	0	$(5,2,0)$	17	$\{1,0\}^a,\ A^2=6$	A_1
2	0	$(2,2,1,1)$	18	$h^0(L-2D)=2,\ h^0(L-3D)=0$	sm.
2	0	$(3,1,1,1)$	15	$h^0(L-2D)=2,\ h^0(L-3D)=1$	sm.
2	0	$(2,2,2,0)$	17	$\{2,0\}^a,\ A^2=2,\ A \not> D$	A_1
2	0	$(3,2,1,0)$	16	$\{2,0\}^a,\ A^2=2,\ A>D^{\ (ii)}$	A_1
2	0	$(3,2,1,0)$	16	$\{2,0\}^a,\ A^2=2,\ A>D^{\ (iii)}$	A_2
2	0	$(4,2,0,0)$	17	$\{2,0\}^b\ (L\sim 4D+2\Gamma_1+2\Gamma_2)$ or $\{2,0\}^{c(iv)}$	$2A_1$
2	2	$(2,2,1,0,0)$	18	$h^0(L-2D)=2,\ h^0(L-3D)=0$	sm.
2	4	$(2,1,1,0,0,0)$	19	$L\sim 2D$	sm.
3	0	$(1,1,1,1,1)$	18	$h^0(L-2D)=0$	sm.
3	0	$(2,1,1,1,0)$	17	$\{3,0\}^a,\ A^2=-2$	A_1
3	0	$(2,2,1,0,0)$	16	$\{3,0\}^b$ or $\{3,0\}^c,\ A^2=0,\ A\not> D^{\ (v)}$	$2A_1$
3	0	$(3,1,1,0,0)$	14	$\{3,0\}^b$ or $\{3,0\}^c,\ A^2=0,\ A>D^{\ (vi)}$	$2A_2$
3	2	$(1,1,1,1,0,0)$	18	$h^0(L-2D)=0$	sm.
3	2	$(2,1,1,0,0,0)$	17	$\{3,2\}^a$	A_1

In the cases $(c,D^2)=(1,0)$ $\mathcal{O}_{S'}$ has the following $\mathcal{O}_{\mathcal{T}}$-resolution:

$$0 \longrightarrow \mathcal{O}_{\mathcal{T}}(-3\mathcal{H}+5\mathcal{F}) \longrightarrow \mathcal{O}_{\mathcal{T}} \longrightarrow \mathcal{O}_{S'} \longrightarrow 0.$$

In the cases $(c,D^2)=(2,0)$ $\mathcal{O}_{S'}$ has the following $\mathcal{O}_{\mathcal{T}}$-resolution:

$$\begin{gathered}0 \longrightarrow \mathcal{O}_{\mathcal{T}}(-4\mathcal{H}+(g-1)\mathcal{F}) \longrightarrow \\ \mathcal{O}_{\mathcal{T}}(-2\mathcal{H}+b_1\mathcal{F}) \oplus \mathcal{O}_{\mathcal{T}}(-2\mathcal{H}+b_2\mathcal{F}) \longrightarrow \mathcal{O}_{\mathcal{T}} \longrightarrow \mathcal{O}_{S'} \longrightarrow 0,\end{gathered}$$

with $(b_1,b_2)=(2,2)$ or $(3,1)$ for the scroll type $(2,2,1,1)$; $(b_1,b_2)=(2,2)$ for the scroll types $(3,1,1,1)$, $(2,2,2,0)$ and $(3,2,1,0)$ (A_2-sing.); $(b_1,b_2)=(3,1)$ for the scroll type $(3,2,1,0)$ (A_1-sing.); and $(b_1,b_2)=(4,0)$ for the scroll type $(4,2,0,0)$. In this latter case S also contains a different perfect Clifford divisor (by the footnote on page 132), so S' can also be described as in the case $(c,D^2)=(2,2)$ (with perfect Clifford divisor $2D+\Gamma_1+\Gamma_2$ or $D+\Delta_0$). (The $\{2,0\}^c$ case of the table corresponds to $(L\sim 4D+2\Delta_0)$). In the case $(c,D^2)=(2,2)$ then S'' is cut out in $\mathcal{T}_0$ by a section (which is constant on the fibers of i) of

$$\mathcal{O}_{\mathcal{T}_0}(2\mathcal{H}-3\mathcal{F}) \oplus \mathcal{O}_{\mathcal{T}_0}(2\mathcal{H}-2\mathcal{F})^2 \oplus \mathcal{O}_{\mathcal{T}_0}(2\mathcal{H}).$$

In the case $(c, D^2) = (2, 4)$ then S'' is cut out in $\mathcal{T}_0$ by a section (which is constant on the fibers of i) of:

$$\mathcal{O}_{\mathcal{T}_0}(2\mathcal{H}-2\mathcal{F})^3 \oplus \mathcal{O}_{\mathcal{T}_0}(2\mathcal{H}-\mathcal{F})^2 \oplus \mathcal{O}_{\mathcal{T}}(2\mathcal{H})^2.$$

We also have that S' is the 2-uple embedding of the quartic $\varphi_D(S)$ if and only if D is not hyperelliptic. If D is hyperelliptic, then there is an elliptic pencil $|E|$ such that $E.D = 2$. Then E is also a free Clifford divisor for L and defines a scroll $\mathcal{T}(2, E)$ containing S'. The $\mathcal{O}_{\mathcal{T}(2,E)}$-resolution of $\mathcal{O}_{S'}$ is given in Proposition 8.39.

In the cases $(c, D^2) = (3, 0)$ we have an $\mathcal{O}_{\mathcal{T}}$-resolution of $\mathcal{O}_{S'}$ of the following type:

$$\begin{array}{l} 0 \longrightarrow \quad \mathcal{O}_{\mathcal{T}}(-5\mathcal{H}+8\mathcal{F}) \quad \longrightarrow \quad \oplus_{i=1}^5 \mathcal{O}_{\mathcal{T}}(-3\mathcal{H}+a_i\mathcal{F}) \\ \quad \longrightarrow \oplus_{i=1}^5 \mathcal{O}_{\mathcal{T}}(-2\mathcal{H}+b_i\mathcal{F}) \longrightarrow \mathcal{O}_{\mathcal{T}} \quad \longrightarrow \quad \mathcal{O}_{S'} \quad \longrightarrow \quad 0, \end{array}$$

with $a_i = 3 - b_i$, for all i. For the smooth scroll type $(1, 1, 1, 1, 1)$ we have $(b_1, b_2, b_3, b_4) = (2, 1, 1, 1, 1)$ or $(2, 2, 2, 0, 0)$. For the scroll type $(2, 1, 1, 1, 0)$ we have $(b_1, b_2, b_3, b_4) = (2, 1, 1, 1, 1)$, $(2, 2, 1, 1, 0)$ or $(2, 2, 2, 0, 0)$. For the remaining two singular scroll types we have $(b_1, b_2, b_3, b_4) = (2, 2, 1, 1, 0)$ or $(2, 2, 2, 0, 0)$.

In the case $(c, D^2) = (3, 2)$ with scroll type $(1, 1, 1, 1, 0, 0)$ then S'' is cut out in $\mathcal{T}_0$ by a section (which is constant on the fibers of i) of:

$$\begin{array}{l} \mathcal{O}_{\mathcal{T}_0}(2\mathcal{H}-2\mathcal{F}) \oplus \mathcal{O}_{\mathcal{T}_0}(2\mathcal{H}-\mathcal{F})^5 \oplus \mathcal{O}_{\mathcal{T}_0}(2\mathcal{H})^2 \text{ or} \\ \mathcal{O}_{\mathcal{T}_0}(2\mathcal{H}-\mathcal{F})^7 \oplus \mathcal{O}_{\mathcal{T}_0}(2\mathcal{H}) \end{array}$$

In the case $(c, D^2) = (3, 2)$ with scroll type $(2, 1, 1, 0, 0, 0)$ then S'' is cut out in $\mathcal{T}_0$ by a section (which is constant on the fibers of i) of:

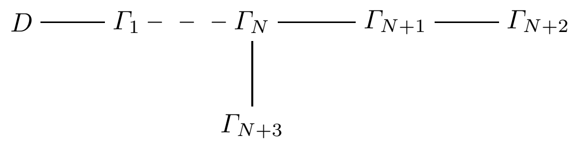
$$\begin{array}{l} \mathcal{O}_{\mathcal{T}_0}(2\mathcal{H}-2\mathcal{F})^3 \oplus \mathcal{O}_{\mathcal{T}_0}(2\mathcal{H}-\mathcal{F}) \ \oplus \mathcal{O}_{\mathcal{T}_0}(2\mathcal{H})^4 \quad \text{or} \\ \mathcal{O}_{\mathcal{T}_0}(2\mathcal{H}-2\mathcal{F})^2 \oplus \mathcal{O}_{\mathcal{T}_0}(2\mathcal{H}-\mathcal{F})^3 \oplus \mathcal{O}_{\mathcal{T}_0}(2\mathcal{H})^3. \end{array}$$

Comments on the types of L:

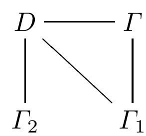
(i) $L \sim 4D + 2\Gamma_1 + \Gamma_2 + \Gamma_3$, with the following configuration:

$$\begin{tikzcd}[arrows=dash] D \arrow[r] \arrow[d] & \Gamma_1 \arrow[r] & \Gamma_2 \\ \Gamma_3, & & \end{tikzcd}$$

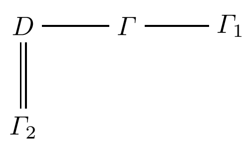
or $L \sim 4D + 3\Gamma_1 + \cdots + 3\Gamma_N + 2\Gamma_{N+1} + \Gamma_{N+2} + \Gamma_{N+3}$, for $N \geq 1$ (in general $N = 0$) with the following configuration:



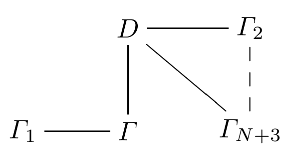
(ii) $L \sim 3D + 2\Gamma + \Gamma_1 + \Gamma_2$, with the following configuration:



(iii) $L \sim 3D + 2\Gamma + \Gamma_1 + \Gamma_2$, with the following configuration:



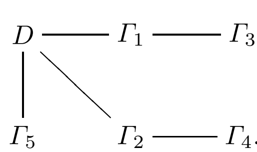
or $L \sim 3D + 2\Gamma + \Gamma_1 + \Gamma_2 + \cdots + \Gamma_{N+3}$, for $N \geq 0$ (in general $N = 0$) with the following configuration:



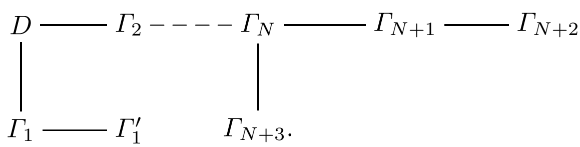
(iv) The number of moduli of the case $\{2, 0\}^c$ is 16, with mildest singularity A_3.

(v) The number of moduli of the case $\{3, 0\}^c$ is 15, with mildest singularity A_3.

(vi) In the case $\{3, 0\}^b$ we have $L \sim 3D + 2\Gamma_1 + 2\Gamma_2 + \Gamma_3 + \Gamma_4 + \Gamma_5$, with the following configuration:



or $L \sim 3D + 2\Gamma_1 + \Gamma_1' + 3\Gamma_2 + \cdots + 3\Gamma_N + 2\Gamma_{N+1} + \Gamma_{N+2} + \Gamma_{N+3}$, with the following configuration:



In the case $\{3,0\}^c$ we have $L \sim 3D + \Gamma + 4\Gamma_0 + 3\Gamma_1 + 2\Gamma_2 + 2\Gamma_3 + \Gamma_4$, with the following configuration:

$$\begin{tikzcd}[arrows=dash]
D \arrow[r] \arrow[d] & \Gamma_0 \arrow[r] \arrow[d] & \Gamma_1 \arrow[r] & \Gamma_3 \arrow[d] \\
\Gamma & \Gamma_2 & & \Gamma_4 .
\end{tikzcd}$$

The mildest singularity of this case is A_5.

$\mathbf{g = 10}$

The general projective model is a complete intersection of 2 hyperplanes in the homogeneous variety Σ^5_{18}, as described in Chapter 10.

The others are as follows:

c	D^2	scroll type	# mod.	type of L	sing.
1	0	$(3,3,2)$	18	$h^0(L-2D)=5,\ h^0(L-3D)=2,\ h^0(L-4D)=0$	sm.
1	0	$(4,2,2)$	16	$h^0(L-2D)=5,\ h^0(L-3D)=2,\ h^0(L-4D)=1$	sm.
1	0	$(4,3,1)$	17	$h^0(L-2D)=5,\ h^0(L-3D)=3,\ h^0(L-4D)=1$	sm.
1	0	$(5,2,1)$	16	$h^0(L-5D)=1$ $^{(i)}$	A_2
1	0	$(6,2,0)$	18	$\{1,0\}^a,\ L \sim 6D+3\Gamma$	A_1
2	0	$(2,2,2,1)$	18	$h^0(L-2D)=3,\ h^0(L-3D)=0$	sm.
2	0	$(3,2,1,1)$	17	$h^0(L-2D)=3,\ h^0(L-3D)=1$ $^{(ii)}$	A_1
2	0	$(3,2,1,1)$	16	$h^0(L-2D)=3,\ h^0(L-3D)=1$ $^{(iii)}$	sm.
2	0	$(3,2,2,0)$	17	$\{2,0\}^a,\ A^2=4,\ h^0(A-D)=1,\ A \not> 2D$ $^{(iv)}$	A_1
2	0	$(3,3,1,0)$	16	$\{2,0\}^a,\ A^2=4,\ h^0(A-D)=2,\ A \not> 2D$ $^{(v)}$	A_2
2	0	$(4,2,1,0)$	16	$\{2,0\}^a,\ A^2=4,\ A > 2D$ $^{(vi)}$	$2A_1$
2	2	$(3,2,1,0,0)$	19	$L \sim 3D$	sm.
3	0	$(2,1,1,1,1)$	18	$h^0(L-2D)=0$	sm.
3	0	$(2,2,1,1,0)$	17	$\{3,0\}^a,\ A^2=0$	A_1
3	0	$(2,2,2,0,0)$	16	$\{3,0\}^b$ or $\{3,0\}^c,\ A^2=2,\ A \not> D$ $^{(vii)}$	$2A_1$
3	0	$(3,2,1,0,0)$	15	$\{3,0\}^b,\ A^2=2,\ A > D$ $^{(viii)}$	$2A_1$
3	0	$(3,2,1,0,0)$	15	$\{3,0\}^b,\ A^2=2,\ A > D$ $^{(ix)}$	A_1+A_2
3	0	$(3,2,1,0,0)$	14	$\{3,0\}^c,\ A^2=2,\ A > D$ $^{(x)}$	A_3
3	0	$(3,2,1,0,0)$	14	$\{3,0\}^c,\ A^2=2,\ A > D$ $^{(xi)}$	A_4
3	2	$(2,1,1,1,0,0)$	18	$h^0(L-2D)=1$	sm.
3	4	$(2,1,1,0,0,0,0)$	18	(E0)	A_1
4	2	$(1,1,1,1,0,0,0)$	18	(CG1)' or (CG2)' $^{(xii)}$	sm.
4	2	$(2,1,1,0,0,0,0)$	16	(CG3)' or (CG4)' $^{(xiii)}$	$3A_1$

In the cases $(c, D^2) = (1,0)$ $\mathcal{O}_{S'}$ has the following $\mathcal{O}_{\mathcal{T}}$-resolution:

$$0 \longrightarrow \mathcal{O}_{\mathcal{T}}(-3\mathcal{H}+6\mathcal{F}) \longrightarrow \mathcal{O}_{\mathcal{T}} \longrightarrow \mathcal{O}_{S'} \longrightarrow 0.$$

In the cases $(c, D^2) = (2, 0)$ $\mathcal{O}_{S'}$ has the following $\mathcal{O}_{\mathcal{T}}$-resolution:

$$\begin{gathered} 0 \longrightarrow \mathcal{O}_{\mathcal{T}}(-4\mathcal{H} + (g-1)\mathcal{F}) \longrightarrow \\ \mathcal{O}_{\mathcal{T}}(-2\mathcal{H} + b_1\mathcal{F}) \oplus \mathcal{O}_{\mathcal{T}}(-2\mathcal{H} + b_2\mathcal{F}) \longrightarrow \mathcal{O}_{\mathcal{T}} \longrightarrow \mathcal{O}_{S'} \longrightarrow 0, \end{gathered}$$

with $(b_1, b_2) = (3, 2)$ or $(4, 1)$ for the scroll type $(4, 2, 1, 0)$, $(b_1, b_2) = (3, 2)$ for the scroll types $(2, 2, 2, 1)$, $(3, 2, 1, 1)$ (smooth), $(3, 2, 2, 0)$ and $(3, 3, 1, 0)$, and $(b_1, b_2) = (4, 1)$ for the scroll type $(3, 2, 1, 1)$ (A_1-sing.).

In the case $(c, D^2) = (2, 2)$ then S'' is cut out in $\mathcal{T}_0$ by a section (which is constant on the fibers of i) of

$$\mathcal{O}_{\mathcal{T}_0}(2\mathcal{H} - 4\mathcal{F}) \oplus \mathcal{O}_{\mathcal{T}_0}(2\mathcal{H} - 3\mathcal{F}) \oplus \mathcal{O}_{\mathcal{T}_0}(2\mathcal{H} - 2\mathcal{F}) \oplus \mathcal{O}_{\mathcal{T}_0}(2\mathcal{H}).$$

or of

$$\mathcal{O}_{\mathcal{T}_0}(2\mathcal{H} - 3\mathcal{F})^3 \oplus \mathcal{O}_{\mathcal{T}_0}(2\mathcal{H}).$$

In the cases $(c, D^2) = (3, 0)$ we have an $\mathcal{O}_{\mathcal{T}}$-resolution of $\mathcal{O}_{S'}$ of the following type:

$$\begin{array}{ccccc} 0 \longrightarrow & \mathcal{O}_{\mathcal{T}}(-5\mathcal{H} + 9\mathcal{F}) & \longrightarrow & \oplus_{i=1}^5 \mathcal{O}_{\mathcal{T}}(-3\mathcal{H} + a_i\mathcal{F}) & \\ \longrightarrow \oplus_{i=1}^5 \mathcal{O}_{\mathcal{T}}(-2\mathcal{H} + b_i\mathcal{F}) \longrightarrow \mathcal{O}_{\mathcal{T}} & \longrightarrow & \mathcal{O}_{S'} & \longrightarrow & 0, \end{array}$$

with $a_i = 4 - b_i$ for all i.

For the smooth scroll type $(2, 1, 1, 1, 1)$ we have $(b_1, b_2, b_3, b_4) = (2, 2, 2, 1, 1)$. For the scroll type $(2, 2, 2, 0, 0)$ we have $(b_1, b_2, b_3, b_4) = (2, 2, 2, 1, 1)$ or $(2, 2, 2, 2, 0)$. For the remaining two singular scroll types we have $(b_1, b_2, b_3, b_4) = (3, 3, 2, 0, 0)$, $(3, 3, 1, 1, 0)$, $(3, 2, 2, 1, 0)$, $(2, 2, 2, 1, 1)$ or $(2, 2, 2, 2, 0)$.

In the case $(c, D^2) = (3, 2)$ then S'' is cut out in $\mathcal{T}_0$ by a section (which is constant on the fibers of i) of

$$\mathcal{O}_{\mathcal{T}_0}(2\mathcal{H} - 2\mathcal{F})^3 \oplus \mathcal{O}_{\mathcal{T}_0}(2\mathcal{H} - \mathcal{F})^4 \oplus 2\mathcal{O}_{\mathcal{T}_0}(\mathcal{H}).$$

In the case $(c, D^2) = (3, 4)$ then S'' is cut out in $\mathcal{T}_0$ by a section (which is constant on the fibers of i) of

$$\mathcal{O}_{\mathcal{T}_0}(2\mathcal{H} - 2\mathcal{F})^3 \oplus \mathcal{O}_{\mathcal{T}_0}(2\mathcal{H} - \mathcal{F})^4 \oplus \mathcal{O}_{\mathcal{T}}(2\mathcal{H})^5.$$

In the cases (CG1)' and (CG2)' then S'' is cut out in $\mathcal{T}_0$ by a section of

$$\begin{array}{l} \mathcal{O}_{\mathcal{T}_0}(2\mathcal{H} - 2\mathcal{F}) \oplus \mathcal{O}_{\mathcal{T}_0}(2\mathcal{H} - \mathcal{F})^7 \oplus \mathcal{O}_{\mathcal{T}_0}(2\mathcal{H})^5 \text{ or} \\ \mathcal{O}_{\mathcal{T}_0}(2\mathcal{H} - \mathcal{F})^9 \oplus \mathcal{O}_{\mathcal{T}_0}(2\mathcal{H})^4 \qquad \text{or} \\ \mathcal{O}_{\mathcal{T}_0}(2\mathcal{H} - 2\mathcal{F}) \oplus \mathcal{O}_{\mathcal{T}_0}(2\mathcal{H} - \mathcal{F})^8 \oplus \mathcal{O}_{\mathcal{T}_0}(2\mathcal{H})^3 \oplus \mathcal{O}_{\mathcal{T}_0}(2\mathcal{H} + \mathcal{F}), \end{array}$$

where the last option occurs only for the case (CG2)' (the section is constant on the fibers of i in the first two cases).

In the cases (CG3)', (CG4)' then S'' is cut out in $\mathcal{T}_0$ by a section (which is constant on the fibers of i) of

$$\mathcal{O}_{\mathcal{T}_0}(2\mathcal{H}-2\mathcal{F})^2 \oplus \mathcal{O}_{\mathcal{T}_0}(2\mathcal{H}-\mathcal{F})^5 \oplus \mathcal{O}_{\mathcal{T}_0}(2\mathcal{H})^6 \text{ or}$$
$$\mathcal{O}_{\mathcal{T}_0}(2\mathcal{H}-2\mathcal{F})^3 \oplus \mathcal{O}_{\mathcal{T}_0}(2\mathcal{H}-\mathcal{F})^3 \oplus \mathcal{O}_{\mathcal{T}_0}(2\mathcal{H})^7.$$

Comments on the types of L:

(i) $L \sim 5D + 3\Gamma_1 + 2\Gamma_2 + \Gamma_3$, with the following configuration:

$$\begin{tikzcd} D \arrow[r, dash] & \Gamma_1 \arrow[r, dash] & \Gamma_2 \arrow[r, dash] & \Gamma_3 \end{tikzcd}$$

(ii) L is in general of the form $L \sim 3D + 2\Gamma_1 + \Gamma_2$, with the following configuration:

$$\begin{tikzcd} D \arrow[r, equal] & \Gamma_1 \arrow[r, dash] & \Gamma_2 \end{tikzcd}$$

(iii) L is in general of the form $L \sim 3D + \Gamma_1 + \Gamma_2 + \Gamma_3$, with the following configuration:

$$\begin{tikzcd} D \arrow[r, equal] \arrow[d, dash] \arrow[dr, dash] & \Gamma_1 \\ \Gamma_2 & \Gamma_3 \end{tikzcd}$$

(iv) L is in general of the form $L \sim 3D + 2\Gamma + \Gamma_1$, with the following configuration:

$$\begin{tikzcd} D \arrow[r, dash] \arrow[dr, equal] & \Gamma \arrow[d, dash] \\ & \Gamma_1 \end{tikzcd}$$

(v) $L \sim 3D + E + 2\Gamma + \Gamma_1$, where E is a smooth elliptic curve, with the following configuration:

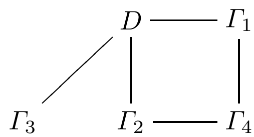
$$\begin{tikzcd} D \arrow[r, dash] \arrow[d, equal] & \Gamma \arrow[r, dash] & \Gamma_1 \\ E & & \end{tikzcd}$$

(vi) L is in general of the form $L \sim 4D + 2\Gamma + 2\Gamma_1 + \Gamma_2$, with the following configuration:

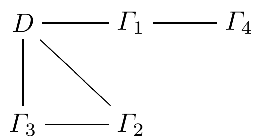
$$\begin{tikzcd} D \arrow[r, dash] \arrow[d, dash] & \Gamma_1 \arrow[r, dash] & \Gamma_2 \\ \Gamma & & \end{tikzcd}$$

(vii) The number of moduli of the case $\{3, 0\}^c$ is 15, with mildest singularity A_3.

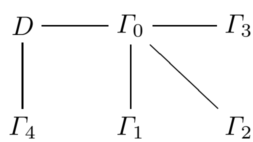
(viii) L is in general of the form $L \sim 3D + 2\Gamma_1 + 2\Gamma_2 + \Gamma_3 + \Gamma_4$, with the following configuration:



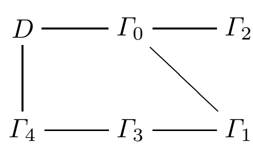
(ix) L is in general of the form $L \sim 3D + 2\Gamma_1 + 2\Gamma_2 + \Gamma_3 + \Gamma_4$, with the following configuration:



(x) L is in general of the form $L \sim 3D + 4\Gamma_0 + 2\Gamma_1 + 2\Gamma_2 + \Gamma_3 + \Gamma_4$, with the following configuration:



(xi) L is in general of the form $L \sim 3D + 4\Gamma_0 + 3\Gamma_1 + 2\Gamma_2 + 2\Gamma_3 + \Gamma_4$, with the following configuration:



(xii) The number of moduli of the case (CG2)' is 17, with mildest singularity A_1.

(xiii) The number of moduli of the case (CG4)' is 15, with mildest singularity $A_1 + A_3$.

12

Some applications and open questions

The methods developed in the previous sections for studying projective models of $K3$ surfaces can be applied and taken further in different directions.

12.1 BN generality

The case $g = 12$ can be described in basically the same way as we describe the cases $g = 5, \ldots, 10$ This is a piece of hard computational work, and one concludes with a table similar to the ones we gave in Chapter 11 for $g = 5, \ldots, 10$. The list is given by Gert M. Hana (unpublished)

As mentioned in Chapter 10 one easily sees that BN generality implies Clifford generality for all g and we proved that these concepts coincide for $g \leq 7$ and $g = 9$. Moreover, we found that the concepts do not coincide for $g = 8$ and 10. The following result is due to Hana.

Proposition 12.1. *For $g = 8$ and every $g \geq 10$ there exist polarized $K3$ surfaces that are Clifford general, but not BN general.*

Proof. By Propositions 1.11 and 1.12, we can find a $K3$ surface S with Pic $S = \mathbf{Z}L \oplus \mathbf{Z}D$, with intersection matrix

$$\begin{bmatrix} L^2 & L.D \\ D.L & E^2 \end{bmatrix} = \begin{bmatrix} 2(g-1) & \lfloor \frac{g+7}{2} \rfloor \\ \lfloor \frac{g+7}{2} \rfloor & 2 \end{bmatrix}$$

and such that L is nef. If L is not base point free, there exists by Proposition 1.10 a curve B such that $B^2 = 0$ and $B.L = 1$. An easy calculation shows that this is impossible. To prove that (S, L) is not BN general, one proves $h^0(D) \geq 3$, and $h^0(L-D) \geq (g+2) - \lfloor \frac{g+7}{2} \rfloor$, and observes that $3((g+2) - \lfloor \frac{g+7}{2} \rfloor) \geq g+1$, for the values of g in question. To prove that (S, L) is Clifford general one first proves that D is nef. Then one argues by contradiction and assumes that there exists a decomposition $L = M + (L - M)$ of L with M a free Clifford divisor inducing a non-general Clifford index. After a long and tedious computation one finds a numerical contradiction. $\square$

T. Johnsen and A.L. Knutsen: LNM 1842, pp. 155–158, 2004.

It is amusing to note that this result has the following interesting corollary for curves:

Corollary 12.2. *For $g = 8$ and every $g \geq 10$ there exists a smooth curve of genus g which is Clifford general but not Brill-Noether general.*

Moreover, for $g = 12, 13$, one can show that there are polarized $K3$ surfaces which are Clifford general but not BN general with a different Picard lattice from the ones considered in Theorem 12.1. For all $g \geq 14$ one expects such cases to occur as well.

12.2 Applications to Calabi-Yau threefolds

Recall that a Calabi-Yau threefold is a smooth 3-dimensional variety X with $\omega_X \simeq \mathcal{O}_X$ and $h^1(\mathcal{O}_X) = 0$. Such a threefold is a 3-dimensional analogue of a $K3$ surface.

In recent years Calabi-Yau threefolds have been intensively studied both by mathematicians and physicists because of their importance in string theory. In particular a lot of interest has been devoted to studying smooth rational curves in these threefolds.

Let $\mathcal{T}$ be a rational normal scroll of dimension 4 in $\mathbf{P}^N$ and of type $(e_1, \ldots, e_4)$, where the e_i are ordered in an non-increasing way and $e_1 - e_3 \leq 1$. Hence the subscroll $\mathbf{P}(\mathcal{O}_{\mathbf{P}^1}(e_1) \oplus \mathcal{O}_{\mathbf{P}^1}(e_2) \oplus \mathcal{O}_{\mathbf{P}^1}(e_3))$ is of maximally balanced type. In [J-K] we show that for positive a, and d exceeding a lower bound, depending on a, a general 3 dimensional (anti-canonical) divisor of type $4\mathcal{H} - (N-5)\mathcal{F}$ will contain an isolated rational curve of bidegree (d, a). To be more precise, we show:

Theorem 12.3. [J-K, Thm. 4.3] *Let $\mathcal{T}$ be a rational normal scroll of dimension 4 in $\mathbf{P}^N$ with a subscroll of of maximally balanced type of dimension 3 as decribed. Assume this subscroll spans a $\mathbf{P}^g$ (so $g = e_1 + e_2 + e_3 + 2$) Let $d \geq 1$ and $a \geq 1$ be integers satisfying:*

(a) If $g \equiv 1 (\mathrm{mod}\, 3)$, then either $(d, a) \in \{(\frac{g-1}{3}, 1), (2(g-1)/3, 2)\}$; or $d > \frac{(g-1)a}{3} - \frac{3}{a}$, $(d, a) \neq (2(g-1)/3 - 1, 2)$ and $3d \neq (g-1)a$.

(b) If $g \equiv 2 (\mathrm{mod}\, 3)$, then either $(d, a) \in \{(g-1, 3), (2g-2, 6)\}$; or $d > \frac{(g-1)a}{3} - \frac{3}{a}$, $(d, a) \notin \{(2(g-2)/3, 2), ((4g-5)/3, 4), ((7g-8)/3, 7)\}$ and $3d \neq (g-1)a$.

(c) If $g \equiv 0 (\mathrm{mod}\, 3)$, then either $(d, a) \in \{((g-3)/3, 1), ((2g-3)/3, 2)\}$; or $d \geq ga/3$.

Then the zero scheme of a general section of $4\mathcal{H} - (N-5)\mathcal{F}$ will be a smooth Calabi-Yau threefold and contain an isolated rational curve of bidegree (d, a).

(Recall that we say a curve C in a variety V is *isolated* in V if the space of embedded deformations of C in V is reduced and zero-dimensional. This is equivalent to $h^0(\mathcal{N}_{C/V}) = 0$.)

The main steps in the proof in [J-K] of Theorem 12.3 are as described:

(I) Set $g := e_1 + e_2 + e_3 + 2$. Using lattice-theoretical considerations we find a (smooth) $K3$ surface S in $\mathbf{P}^g$ with Pic $S \simeq \mathbf{Z}H \oplus \mathbf{Z}D \oplus \mathbf{Z}\Gamma$, where H is the hyperplane section class, D is the class of a smooth elliptic curve of degree 3 and C is a smooth rational curve of bidegree (d, a). Let $T = T_S$ be the 3-dimensional scroll in $\mathbf{P}^g$ swept out by the linear spans of the divisors in $|D|$ on S. The rational normal scroll T will be of maximally balanced type and of degree $e_1 + e_2 + e_3$.

The proof of Step I, which is the most difficult one, involves the whole formalism of Clifford divisors and associated scrolls developed in this book, and is to a great extent an application of the techniques described in this book. Indeed the elliptic curve D above will be a Clifford divisor and the surfaces constructed have Clifford index one.

The other steps are as follows.

(II) Embed $T = \mathbf{P}(\mathcal{O}_{\mathbf{P}^1}(e_1) \oplus \mathcal{O}_{\mathbf{P}^1}(e_2) \oplus \mathcal{O}_{\mathbf{P}^1}(e_3))$ (in the obvious way) in a 4 dimensional scroll $\mathcal{T} = \mathbf{P}(\mathcal{O}_{\mathbf{P}^1}(e_1) \oplus \cdots \oplus \mathcal{O}_{\mathbf{P}^1}(e_4))$ of type $(e_1, \ldots, e_4)$. Hence T corresponds to the divisor class $\mathcal{H} - e_4 F$ in $\mathcal{T}$, and S corresponds to a "complete intersection" of divisors of type $\mathcal{H} - e_4\mathcal{F}$ and $3\mathcal{H} - (g - 4)\mathcal{F}$ on $\mathcal{T}$. We deform the complete intersection in a rational family (i.e. parametrized by $\mathbf{P}^1$) in a general way. For "small values" of the parameter we obtain a $K3$ surface with Picard group of rank 2 and no rational curve on it.

(III) Take the union over $\mathbf{P}^1$ of all the K3 surfaces described in (II). This gives a threefold V, which is a section of the anti-canonical bundle $4\mathcal{H} - (g - 4 + e_4)\mathcal{F} = 4\mathcal{H} - (N - 5)\mathcal{F}$ on $\mathcal{T}$. For a general complete intersection deformation the threefold will have only finitely many singularities, none of them on C. Then C will be isolated on V.

(IV) Deform V as a section of $4\mathcal{H} - (g - 4 + e_4)\mathcal{F} = 4\mathcal{H} - (N - 5)\mathcal{F}$ on $\mathcal{T}$. Then a general deformation W will be smooth and have an isolated curve C_W of bidegree (d, a).

As mentioned in [J-K] this overall strategy is analogous to the one used in [Cl] to show the existence of isolated rational curves of infinitely many degrees in the generic quintic in $\mathbf{P}^4$, and in [E-J-S] to show the existence of isolated rational curves of bidegree $(d, 0)$ in general complete intersection Calabi-Yau threefolds in some specific biprojective spaces.

12.3 Analogies with Enriques surfaces

Let S be an Enriques surface (i.e. S is smooth with $K_S \neq 0$, $2K_S = 0$ and $h^1(\mathcal{O}_S) = 0$) and L a base point free divisor on S satisfying $L^2 > 0$. By the adjunction formula, $L^2 = 2g - 2$, where g is the arithmetic genus of all the curves in $|L|$ and $h^0(L) = g$.

A very interesting point is that any Enriques surface carries elliptic pencils, and one can always find such a pencil $|P|$ of minimal degree with respect to L, satisfying $P.L \leq 2\lfloor\sqrt{L^2}\rfloor$ [CD, Corollary 2.7.1].

If $h^0(L-P) \geq 2$ (and this is easily seen to be satisfied for $g \geq 10$), then the pencil $|P|$ defines a rational normal scroll in $\mathbf{P}^{g-1}$, into which S is mapped by the φ_L. This is the scroll which is the union of the linear spans of all the $\varphi_L(P_i)$, for the members P_i of $|P|$.

If $g \geq 6$ it is shown in [Co] that φ_L is birational, so in particular we get that any polarized Enriques surface (S, L) (with L base point free) of genus $g \geq 10$ has a projective model $\varphi_L(S) \subseteq \mathbf{P}^{g-1}$ lying in a rational normal scroll defined by an elliptic pencil of minimal degree (with respect to L). Also note that the condition $h^0(L-P) \geq 2$ is also satisfied for certain $g < 10$, depending on the value $P.L$. Scrolls containing Enriques surfaces are studied in [Han].

Note that the inequality $P.L \leq 2\lfloor\sqrt{L^2}\rfloor$ can somehow be seen as an analogue of the inequality $c \leq \lfloor\frac{g-1}{2}\rfloor$ for the Clifford index of a polarized $K3$ surface. The pencil P restricted to any member of $|L|$ induces a Clifford index $\leq 2\lfloor\sqrt{2g-2}\rfloor - 2$ on every smooth curve in $|L|$, and this is actually $\lfloor\frac{g-1}{2}\rfloor$ when $g \geq 29$. It will often, but not always, be true that the pencil P induces the Clifford index of every smooth curve in $|L|$. Moreover, contrary to the $K3$ case, it does not always hold that all the smooth curves in $|L|$ have the same Clifford index.

References

[A-C-G-H] E. Arbarello, M. Cornalba, P. A. Griffiths, J. Harris, *Geometry of Algebraic Curves, Volume I*, number 267 in *Grundlehren der Mathematischen Wissenschaften*, Springer Verlag, Berlin, Heidelberg, New York, Tokyo (1985).

[Ar] M. Artin, *Some numerical criteria for contractability of curves on algebraic surfaces*, American Journal of Math. **84**, 485–496 (1962).

[Ba] E. Ballico, *A remark on linear series on general k-gonal curves*, Boll. Un. Mat. Ital. A (7) **3**, no. 2, 195–197 (1989).

[B-P-V] W. Barth, C. Peters, A. Van de Ven *Compact Complex Surfaces*, Springer Verlag, Berlin, Heidelberg, New York, Tokyo (1984).

[Be] A. Beauville, *Determinantal hypersurfaces*, Michigan Math. Journal (Special volume dedicated to W. Fulton) **48** (1997).

[B-S] M. Beltrametti, A. J. Sommese, *Zero cycles and k-th order embeddings of smooth projective surfaces*, in *Projective surfaces and their classification*, Cortona proceedings Symp. Math. INDAM 32, pp. 33-48, New York, London: Academic press 1988.

[Bo] F. Bogomolov, *Holomorphic tensors and vector bundles on projective varieties*, Math. USSR Izv. **13**, 499-555 (1979).

[Br] J. Brawner, *Tetragonal Curves, Scrolls and $K3$-surfaces*, Trans. of the Am. Mat. Soc. **349**, No. 8, 3075–3091 (1997).

[B-E] D. Buchsbaum, D. Eisenbud *Algebra structures for finite free resolutions and some structure theorems for ideals in codimension 3*, Am. J. Math. **99**, 447–485 (1977).

[C-F] F. Catanese, M. Franciosi, *Divisors of small genus on algebraic surfaces and projective embeddings*, in *Proceedings of the Hirzebruch 65 Conference on Algebraic Geometry (Ramat Gan, 1993)*, 109–140, Israel Math. Conf. Proc. **9**, Bar-Ilan Univ., Ramat Gan (1996).

[C-P] C. Ciliberto, G. Pareschi, *Pencils of minimal degree on curves on a $K3$ surface*, J. reine angew. Math. **460**, 15–36 (1995).

[Cl] H. Clemens, *Homological equivalence, modulo algebraic equivalence, is not finitely generated*, Publ. Math. IHES **58**, 19–38 (1983).

[C-M] M. Coppens, G. Martens, *Secant Spaces and Clifford's theorem*, Comp. Math. **68**, 337–341 (1991).

T. Johnsen and A.L. Knutsen: LNM 1842, pp. 159–162, 2004.

[Co] F. R. Cossec, *Projective models of Enriques surfaces*, Math. Ann. **265**, 283–334 (1983).

[CD] F. R. Cossec, I. Dolgachev, *Enriques surfaces I*, Progress in Mathematics **76**, Birkhauser, (1989).

[D-M] R. Donagi, D. R. Morrison, *Linear systems on $K3$ sections*, J. Diff. Geom. **29**, 49–64 (1989).

[E-H] D. Eisenbud, J. Harris, *On Varieties of Minimal Degree (A Centennial Account)*, Proceedings of Symposia in Pure Mathematics **46**, 3–13 (1987).

[E-L-M-S] D. Eisenbud, H. Lange, G. Martens, F.-O. Schreyer, *The Clifford dimension of a projective curve*, Comp. Math. **72**, 173–204 (1989).

[E-J-S] T. Ekedahl, T. Johnsen, D.E. Sommervoll *Isolated rational curves on K3-fibered Calabi-Yau threefolds*, Manuscripta Math. **99**, 111–133 (1999).

[Gr] M. Green, *Koszul cohomology and the geometry of projective varieties*, J. Differ. Geom. **19**, 125–171 (1984).

[G-L1] M. Green, R. Lazarsfeld, *A simple proof of Petri's theorem on canonical curves*, In: geometry Today, progress in Math., birkhäuser (1986).

[G-L2] M. Green, R. Lazarsfeld, *On the projective normality of complete linear series on an algebraic curve*, Invent. Math. **83**, 73-90 (1986).

[G-L3] M. Green, R. Lazarsfeld, *Some results on the syzygies of finite sets and algebraic curves*, Comp. Math. **67**, 301–314 (1988).

[G-L4] M. Green, R. Lazarsfeld, *Special divisors on curves on a $K3$ surface*, Invent. Math. **89**, 357–370 (1987).

[G-L5] M. Green, R. Lazarsfeld, *The nonvanishing of certain cohomology groups*, Appendix to [Gr].

[G-H] P. Griffiths, J. Harris, *Principles of algebraic geometry*, Wiley Classics Library, John Wiley & Sons, Inc., New York, 1994.

[G-L-P] L. Gruson, R. Lazarsfeld, C. Peskine, *On a theorem of Castelnuovo and the equations defining space curves*, Invent. Math. **72**, 491–506 (1983).

[Han] G.M. Hana, *Projective Models of Enriques Surfaces in Scrolls*, Preprint, 2003.

[Har] J. Harris, *A Bound on the Geometric Genus of Projective Varieties*, Ann. Scuola Norm. Sup. Pisa Cl. Sci, **8** (4), 35–68 (1981).

[H-M] J. Harris, D. Mumford, *On the Kodaira dimension of the moduli space of curves*, Invent. Math. **67**, no. 1, 23–86 (1982).

[Hrts] R. Hartshorne, *Algebraic Geometry*, Graduate Texts in Mathematics, **52**, Springer Verlag (1977).

[I-R1] A. Iliev, K. Ranestad, *Canonical curves and varieties of sums of powers of cubic polynomials*, J. Algebra **246**, no. 1, 385–393 (2001).

[I-R2] A. Iliev, K. Ranestad, *$K3$ surfaces of genus 8 and varieties of sums of powers of cubic fourfolds*, Trans. Amer. Math. Soc. **353**, no. 4, 1455–1468 (2001).

[J-K] T. Johnsen, A. L. Knutsen, *Rational Curves in Calabi-Yau threefolds*, Comm. in Algebra **31** (8), 3917–3953 (2003).

[Kn1] A. L. Knutsen, *Exceptional curves on Del Pezzo surfaces*, Mathematische Nachrichten **256**, 1, 58–81 (2003).

[Kn2] A. L. Knutsen, *Gonality and Clifford index of curves on $K3$ surfaces*, Archiv der Mathematik **80** (3), 235–238 (2003).

[Kn3] A. L. Knutsen, *Higher order birational embeddings of Del Pezzo surfaces*, Mathematische Nachrichten **254-255**, 1, 183-196 (2003).

[Kn4] A. L. Knutsen, *On kth order embeddings of $K3$ surfaces and Enriques surfaces*, Manuscripta Math. **104**, 211–237 (2001).

[Kn5] A. L. Knutsen, *Smooth curves on projective $K3$ surfaces*, Math. Scand. **90**, No. 2, 215–231 (2002).

[La1] R. Lazarsfeld, *A sampling of vector bundle techniques in the study of linear series*, Sugaku **47**, 125-144 (1995).

[La2] R. Lazarsfeld, *Brill-Noether-Petri without degenerations*, J. Diff. Geom. **23**, 299–307 (1986).

[GMa] G. Martens, *On curves on $K3$ surfaces*, Springer LNM 1398, 174–182 (1989).

[HMa] H.H. Martens, *On the varieties of special divisors on a curve, II*, J. reine und angew. Math. **233**, 89–100 (1968).

[Mo] D. R. Morrison, *On $K3$ surfaces with large Picard number*, Invent. Math. **75**, 105–121 (1984).

[Mu1] S. Mukai, *Curves, $K3$ surfaces and Fano 3-folds of Genus* ≤ 10, Algebraic Geometry and Commutative Algebra (In Honor of M. Nagata) **1**, 357–377 (1988).

[Mu2] S. Mukai, *New development of theory of Fano 3-folds: vector bundle method and moduli problem*, Sugaku **47**, 125–144 (1995).

[Na] M. Nagata, *Local rings*, Interscience Publishers, Wiley (1989).

[Ni] V. V. Nikulin, *Integer symmetric bilinear forms and some of their geometric applications*, Izv. Akad. Nauk SSSR Ser. Mat. **43**, no. 1, 111–177, 238 (1979).

[No] M. Noether, *Über die invariante Darstellung algebraischer Funktionen einer Variablen*, Math. Ann. **88**, 163–284 (1880).

[Pa] G. Pareschi, *Exceptional linear systems on curves on Del Pezzo surfaces*, Math. Ann. **291**, 17-38 (1991).

[Pe] K. Petri, *Über die invariante Darstellung algebraischer Funktionen*, Math. Ann. **17**, 243–289 (1923).

[P-S] R. Piene, G. Sacchiero, *Duality for rational normal scrolls*, Comm. in Algebra **12** (9), 1041–1076 (1984).

[R-S] K .Ranestad, F. O. Schreyer, *Varieties of sums of powers*, J. Reine Angew. Math. **525**, 147–181 (2000).

[R-R-W] M. S. Ravi, J. Rosenthal, X. Wang, *Dynamic pole assignment and Schubert calculus*, SIAM J. of Control and Optimization, **34** (3), 813–832 (1996).

[Re1] M. Reid, *"Bogomolov's theorem* ${c_1}^2 \leq 4c_2$*"*, Internat. Sympos. on Algebraic Geometry, Kyoto, 623-642 (1977).

[Re2] M. Reid, *Chapters on Algebraic Surfaces*, In: Complex algebraic geometry (Park City 1993), IAS/Park City Math., Ser. **3**, 3–159 (1997).

[Re3] M. Reid, *Special linear systems on curves lying on a K3 surface*, J. London Math. Soc. (2) **13**, no. 3, 454-458 (1976).

[Rdr] I. Reider, *Vector bundles of rank* 2 *and linear systems on algebraic surfaces*, Annals of Math. **127**, 309–316 (1988).

[SD] B. Saint-Donat, *Projective models of $K3$ Surfaces*, Amer. J. Math. **96**, 602–639 (1974).

[Sc] F. O. Schreyer, *Syzygies of canonical curves and special linear series*, Math. Ann. **275**, 105–137 (1986).

[Se1] C. Segre, *Sulle rigate razionali in uno spazio lineare qualunque*, Atti della R. Acc. delle scienze di Torino, **XIX**, 265–282 (1883-4).

[Se2] C. Segre, *Sulle varieta a tre dimensioni composte di serie semplici razionali di piani*, Atti della R. Acc. delle scienze di Torino **XXI**, 95–115 (1885-6).

[Si] A. Simis, *Multiplicities and Betti numbers of homogeneous ideals*, In: Space Curves (Rocca di Papa 1985), Springer Lecture Notes in Mathematics, **1266**, 232–250 (1987).

[Ste] J. Stevens, *Rolling factors deformations and extensions of canonical curves*, Doc. Math. **6**, 185–226 (2001).

[Str] S. A. Strømme, *On parametrized rational curves in Grassmann varieties*, In: Space Curves (Rocca di Papa 1985), Springer LNM 1266, 251–272 (1987).

[Ty] A. N. Tyurin, *Cycles, curves and vector bundles on an algebraic surface*, Duke Math. J. **54**, no. 1, 1–26 (1987).

[Vo1] C. Voisin, *Green's canonical syzygy conjecture for generic curves of odd genus*, math.AG/0301359 (2003).

[Vo2] C. Voisin, *Green's generic syzygy conjecture for curves of even genus lying on a $K3$ surface*, J. Eur. Math. Soc. (JEMS) **4**, no. 4, 363–404 (2002).

[Wa] C. Walter, *Pfaffian Subschemes*, Jour. of Alg. Geom. **5**, 671–704 (1996).

[We] A. Weil, *Œvres Scientifiques (Collected Papers)*, II, Springer, Heidelberg (1980).

Index