Lectures on Algebraic Geometry

G. Harder

Table of contents

Table of contents **I. Categories, products, projective and inductive limits**

Introduction

Before I describing the contents of this book I have to say a few words about its history. My original plan was to write a book on the cohomology of arithmetic group and I still have this intention. Actually there exists a first version of this book in German. They resulted from a series of lectures I gave on this subject. This subject requires a lot of background from different aereas of mathematics for intance linear algebraic groups, arithmetic groups, number theory. For a while it is possible to keep this input on an elementary level because one can work with special examples like Sl_2, the action of $Sl_2(\mathbb{Z})$ on the upper half plane is not so difficult to undestand. But then the basic concepts of homological algebra and sheaf cohomology enter.

Therefore I wrote an exposition of sheaf cohomology and an introduction into homological algebra. Then it happened that I gave a course on algebraic geometry and again the same concepts were fundamental. So I used this first part as an introduction into algebraic geometry and I completed it by writing up some notes for the students which covered some algebraic geometry.

At a certain point I decided write an introduction into algebraic geometry which also included the basic facts of sheaf theory and homological algebra. Again after a while I realized that this volume would cover more than 500 pages and so I decided to split this volume again in two parts so that my plan is now to deliver three volumes.

The present volume I starts with a very informal introduction into category theory. It continues with an introduction into homological algebra. In view of the content of the third volume Chapter II is an introduction into homological algebra based on the example of cohomology of groups. Chapter III introduces into the theory of sheaves. The role of sheaves is twofold: They allow to formulate the concepts af manifolds is locally ringed spaces ($\mathcal{C}_\infty$-manifolds, complex manifolds, algebraic manifolds) this is discussed in III.2 and we have the cohomology of sheaves which is covered in Chapter IV. Up to here the book may serve as an introduction into algebraic topology but with a strong focus on applications in algebraic geometry and in the cohomology of arithmetic groups. The discussion of singular homology is rather short.

My original notes where very informal but after a while I felt the desire to give a rather self contained account and so it happened that the introduction into sheaf cohomology became rather complete up to a certain level. I included Poincare duality, spectral sequences, the cup product and Hodge theory. Only in the section on Hodge theory I refer to some analytical results which are not proved in this book.

The last chapter V we apply sheaf cohomology to the theory of compact Riemann surfaces. In the first section of Chapter V we prove the theorem of Riemann-Roch, we show that Riemann surfaces are smooth projective algebraic curves. We prove Abels theorem and we discuss the theory of Jacobians over the field $\mathbf{C}$. In the second section we discuss the theory of line bundles on these Jacobians and more general on abelian varieties.

This last chapter goes beyond homological algebra and algebraic topolgy but it shows the enormous usefulness of these concepts. The Chapter V can also be seen as a preparation for the second volume which is an introduction into algebraic geometry.

Chapter I

Categories, products, projective and inductive limits

I.1. The notion of a category and examples

I want to give a very informal introduction to the theory of categories. The main problem for a beginner is to get some acquaintance with the language and to get used to the abstractness of the subject.

A *category* $\mathcal{C}$ is

(i) a collection of objects $\mathrm{Ob}(\mathcal{C})$.

We do not insist that this collection is a set. For me this means that we do not have the notion of equality of two objects. If we write $N \in \mathrm{Ob}(\mathcal{C})$ then we mean that N is an object in the category $\mathcal{C}$.

(ii) To any two objects $N, M \in \mathrm{Ob}(\mathcal{C})$ is attached a set

$$\mathrm{Hom}_{\mathcal{C}}(N, M)$$

which is called the set of *morphisms* between these two objects. Usually we denote a morphism $\phi \in \mathrm{Hom}_{\mathcal{C}}(N, M)$ by an arrow $\phi : N \to M$.

(iii) For any three objects N, M, P we have the *composition* of morphisms

$$\mathrm{Hom}_{\mathcal{C}}(N, M) \times \mathrm{Hom}_{\mathcal{C}}(M, P) \to \mathrm{Hom}_{\mathcal{C}}(N, P)$$

$$(\phi, \psi) \mapsto \psi \circ \phi.$$

If a morphism η is a composition of ϕ and ψ then we denote this by a *commutative diagram* (or commutative triangle)

$$\begin{tikzcd} A \arrow[rr, "\phi"] \arrow[dr, "\eta"'] & & B \arrow[dl, "\psi"] \\ & P & \end{tikzcd}$$

We require that this composition is associative in the obvious sense (if we have four objects...). The reader should verify that this associativity can be formulated in terms of a tetrahedron all of whose for sides are commutative triangles. Here we use that the morphisms between objects form a set. In a set we know what equaliy between elements means.

(iv) For any object $N \in \mathrm{Ob}(\mathcal{C})$ we have a distinguished element $\mathrm{Id}_N \in \mathrm{Hom}_{\mathcal{C}}(N,N)$ which is an identity on both sides under the composition.

1.1 Examples:

Everybody has seen categories:

a) The category **Ens** of all sets where the arrows are arbitrary maps.

b) The category $\mathbf{Vect}_k$ of vector spaces over a given field k where the sets of morphisms are the k-linear maps.

c) the category $\mathbf{Mod}_A$ of modules over a ring A where the maps are A-linear maps.

We also have the category of abelian groups **Ab**, the category **Groups** of all groups where the morphisms are the homomorphisms of groups.

d) the category **Top** of topological spaces where the maps are the continuous maps.

I said in the beginning that we do not have the notion of equality in a category. But we we can say that two objects $N, M \in \mathrm{Ob}(\mathcal{C})$ are *isomorphic.* This means that we can find two arrows $\phi : N \to M$ and $\psi : M \to N$ such that $\mathrm{Id}_N = \psi \circ \phi, \mathrm{Id}_M = \phi \circ \psi$. But in general it may be possible to find many such isomorphisms between the objects and hence we have many choices to identify them. Then it is better to refrain from considering them as equal. For instance we can consider the category of finite dimensional vector spaces over a field k. Of course two such vector spaces are isomorphic if they have the same dimension. Since we may have many of these isomorphisms, we do not know how to identify them and the notion of equality does not make sense.

But if we consider the category of framed finite dimensional $k-$ vector spaces, i.e. vector spaces V equipped with a basis which is indexed by the numbers $1, 2, \ldots, n = \dim(V)$ and morphisms which are linear maps which send basis elements to basis elements and which respect the ordering, then the situation is different. In this case we can say the objects form a set: If two such objects are isomorphic then the isomorphism is unique.

It is important to accept the following fact: The axioms give us a lot of flexibility, at no point we require that the elements in $\mathrm{Hom}_{\mathcal{C}}(N,M)$ are actual maps between sets (with some additional structure). Insofar all the above examples are somewhat misleading.

The simplest example of a situation where the arrows are not maps is the following one:

e) We may start from a an ordered set $\mathcal{I} = (I, \leq)$ and we consider its elements as the objects of a category. For any pair $i, j \in I$ we say that $\mathrm{Hom}_{\mathcal{I}}(i,j)$ consists of one single element $\phi_{i,j}$ if $i \leq j$ and is empty otherwise. The composition is the obvious one obtained from the transitivity of the order relation.

The reader may say that this is not a good example, because the $\phi_{i,j}$ can be considered as maps between the two sets $\{i\}, \{j\}$ but that is the wrong point of view. To make this clear we can also construct a slightly different category $\mathcal{J}$ from our ordered set. We may define the sets of morphisms as:

$\mathrm{Hom}_{\mathcal{J}}(i,j)$ are finite sequences $\{i_0, i_1, \ldots, i_n\}$ with $i_\nu \leq i_{\nu+1}$ and $i = i_0, j = i_n$.

We leave it to the reader to verify that we have a composition and an identity. Now we may have many arrows between two objects $\{i\},\{j\}$ which are sets consisting of one element.

We may also do the following which may look strange at the first glance. If we have a category $\mathcal{C}$ we may revert the arrows and form the so called *opposite* category $\mathcal{C}^{\mathrm{opp}}$ which has the same objects but where

$$\mathrm{Hom}_{\mathcal{C}^{\mathrm{opp}}}(N,M)=\mathrm{Hom}_{\mathcal{C}}(M,N).$$

I.2. Functors:

We need the notion of a *functor* F from one category $\mathcal{C}$ to another category $\mathcal{C}'$. A functor is a rule that transforms an object $N\in\mathrm{Ob}(\mathcal{C})$ into an object $F(N)\in\mathrm{Ob}(\mathcal{C}')$ and for any two objects $N,M\in\mathrm{Ob}(\mathcal{C})$ it provides maps

$$F_{N,M}:\mathrm{Hom}_{\mathcal{C}}(N,M)\longrightarrow\mathrm{Hom}_{\mathcal{C}'}(F(N),F(M)).$$

In other words: For any $\phi:N\to M$ the functor produces an arrow

$$F_{N,M}(\phi)=F(\phi):F(N)\to F(M)$$

and this production should satisfy the obvious consistency conditions namely respect identity elements and composition. Such an F together with the collection of map between the sets of morphisms is called a *covariant* functor because direction of the arrows is preserved.

We also have the notion of a *contravariant* functor which turns the arrows backwards or what amounts to the same: it is a functor from the opposite category $\mathcal{C}^{\mathrm{opp}}$ to $\mathcal{C}'$.

Any object X of a category defines functors from this category to the category **Ens**: We attach to it the covariant functor

$$h_X(Z)=\mathrm{Hom}_{\mathcal{C}}(X,Z).$$

If we have two objects Z,Z' and $\psi:Z\to Z'$ then the composition produces $h_X(\psi)$: $\mathrm{Hom}_{\mathcal{C}}(X,Z)\to\mathrm{Hom}_{\mathcal{C}}(X,Z')$ which sends $\phi:X\to Z$ to $\psi\circ\psi$.

We may also put X into the second free place in the $\mathrm{Hom}_{\mathcal{C}}(\ ,\ )$ and consider $h^{\circ}_X(Z)=\mathrm{Hom}_{\mathcal{C}}(Z,X)$. This gives us a contravariant functor.

2.1 Examples:

a) We have a contravariant functor from the category of vector spaces into itself: We send a vector space $V\in\mathrm{Ob}(\mathbf{Vect}_k)$ to its dual space $V^{\vee}=\mathrm{Hom}_k(V,k)$.

b) A very clever example of a functor is the homology of topological space (see Chap. IV). To any topological space X (i.e an object in the category **Top**) we may attach the homology groups $H_0(X,\mathbb{Z}),H_1(X,\mathbb{Z}),\ldots,H_i(X,\mathbb{Z}),\ldots$ the index set runs over all positive integers. These homology groups are abelian groups which depend functorially on the space X: A continuous map

$$f:X\longrightarrow Y$$

between spaces induces a homomorphism between their homology groups

$$f_i : H_i(X, \mathbb{Z}) \longrightarrow H_i(Y, \mathbb{Z}) \text{ for all indices } \ i.$$

This functor transforms a very complicated object - a toplogical space- into a simpler but not too simple object namely a family of abelian group. This can be used to prove that $\mathbb{R}^n$ is not homeomorphic to $\mathbb{R}^m$ if $n \neq m$. To see this we remove the origin from $\mathbb{R}^n$ and from $\mathbb{R}^m$ and we will see that the resulting spaces will have non-isomorphic homology groups if $n \neq m$.(Chap IV, 7.3). On the other hand if we had a homeomorphism between the two spaces we could arrange that it maps the origin to the origin. Hence we would get a homeomorphism between the modified spaces which then must induce isomorphisms on the homology groups and this is impossible.

If I am right these homology groups are historically the first examples where the concept of functors has been used.

We will see many more interesting functors in the chapter on homological algebra.

I.3. Products, projective limits and direct limits in a category.

Let us assume that we have a category $\mathcal{C}$ and an ordered set $\mathcal{I} = (I, \leq)$. Furthermore we assume that to any $i \in I$ we have attached an object $X_i \in \mathrm{Ob}\,(\mathcal{C})$ and for any pair $i \leq j$ of indices we have an arrow

$$\phi_{ij} \in \mathrm{Hom}_{\mathcal{C}}(X_j, X_i)$$

We assume that always $\phi_{ii} = \mathrm{Id}_{X_i}$ and for any triple $i \leq j \leq j'$ we have

$$\phi_{ij} \circ \phi_{jj'} = \phi_{ij'}.$$

We have seen in example e) that we may consider our ordered set $(I, \leq)$ as a category $\mathcal{I}$. Then we can summarize our assumptions by saying that $i \to X_i$ is a contravariant functor from the category $\mathcal{I}$ to the category $\mathcal{C}$.

The family $(\{X_i\}_{i\in I}, \phi_{ij})$ is called a *projective* or sometimes *inverse system* of objects in $\mathcal{C}$.

For any object $Z \in \mathrm{Ob}\,(\mathcal{C})$ we define a set

$$\mathrm{Hom}_{\mathcal{C}}(Z, (\{X_i\}_{i\in I}, \phi_{ij}))$$

which consists of families $\{\phi_i\}_{\{i\in I\}}$ of morphisms

$$\phi_i : Z \to X_i$$

such that for any pair $i \leq j$ the diagram

$$\begin{array}{ccc} Z & \xrightarrow{\quad \phi_j \quad} & X_j \\ \phi_i \searrow & & \swarrow \phi_{ij} \\ & X_i & \end{array}$$

commutes. It is clear that

$$Z \to \mathrm{Hom}_{\mathcal{C}}(Z, (\{X_i\}_{i\in I}, \phi_{ij}))$$

is a contravariant functor from $\mathcal{C}$ to **Ens** because a morphism $\phi : Z' \to Z$ induces a map $\mathrm{Hom}_{\mathcal{C}}(Z, (\{X_i\}_{i\in I}, \phi_{ij})) \to \mathrm{Hom}_{\mathcal{C}}(Z', (\{X_i\}_{i\in I}, \phi_{ij}))$ which is induced by the composition. We should think of $(\{X_i\}_{i\in I}, \phi_{ij})$ as a huge diagram

$$\begin{array}{ccccccc}
\longrightarrow & X_i & \longrightarrow & X_j & & & \\
 & & & & \searrow & & \\
 & & \searrow & & & X_c & \longrightarrow \\
 & & & & \nearrow & & \\
 & & & X_a & & & \\
 & & \nearrow & & & & \\
\longrightarrow & X_b & & & \searrow & & \\
 & & \searrow & & & & \\
 & & & X_r & \longrightarrow & X_u & \longrightarrow
\end{array}$$

where we did not draw the compositions because they are redundant and make the picture complicated. Then an element $\phi \in \mathrm{Hom}_{\mathcal{C}}(Z, (\{X_i\}_{i\in I}, \phi_{ij}))$ is a system of arrows $\{\varphi_\nu : Z \to X_\nu\}_{\nu\in I}$ into this diagram:

$$\begin{array}{cccccccc}
 & \longrightarrow & X_i & \longrightarrow & X_j & & & \\
 & & & & & \searrow & & \\
 & \nearrow & & \searrow & & & X_c & \longrightarrow \\
 & & & & & \nearrow & & \\
Z & & & & X_a & & & \\
 & \searrow & & \nearrow & & & & \\
 & \longrightarrow & X_b & & & \searrow & & \\
 & & & \searrow & & & & \\
 & & & & X_r & \longrightarrow & X_u & \longrightarrow
\end{array}$$

so that every diagram induced by a $i \leq j$ commutes. Again we suppressed the compositions. (In the special diagram, are the two arrows from Z to X_i and X_b arbitrary or is there a constraint?).

An object $P \in \mathcal{C}$ together with an element

$$\Phi \in \mathrm{Hom}_{\mathcal{C}}(P, (\{X_i\}_{i\in I}, \phi_{ij}))$$

is called *projective limit* of the system $(\{X_i\}_{i\in I}, \phi_{ij}))$ if doe any $Z \in \mathrm{Ob}(\mathcal{C})$ the map

$$\begin{array}{ccc}
\mathrm{Hom}_{\mathcal{C}}(Z, P) & \longrightarrow & \mathrm{Hom}_{\mathcal{C}}(Z, (\{X_i\}_{i\in I}, \phi_{ij})) \\
\psi & \longrightarrow & \{\Phi_i \circ \psi\}_{\{i\in I\}}
\end{array}$$

is a bijection. This is the so called *universal property* of (P, Φ)

In terms of our above diagrams this means that a projective limit P is an object that is squeezed between Z and the diagram i.e. we have given our $\Phi \in \mathrm{Hom}_{\mathcal{C}}(P, (\{X_i\}_{i\in I}, \phi_{ij}))$ and any ϕ from any Z into the diagram is obtained by first giving an arrow $Z \to P$ and then composing with the universal arrow Φ.

Such a projective limit may not exist in our category. But if it exists then this gives us a first example of a *representable functor*:

Starting from the functor $Z \to \mathrm{Hom}_{\mathcal{C}}(Z, (\{X_i\}_{i\in I}, \phi_{ij}))$ we find a P such our functor is equivalent to the functor h_P° which we attached to P. More precisely we have an universal element $\Phi \in \mathrm{Hom}_{\mathcal{C}}(P, (\{X_i\}_{i\in I}, \phi_{ij}))$ such that the equivalence of the functors is given by the universal property above.

3.1 The Yoneda lemma:

We have a simple categorical argument which is called the Yoneda lemma which shows that such a (P, Φ) is unique up to a *canonical* isomorphism. If we have a second pair (P', Φ') then we get from the universal property that Φ' is obtained from a uniquely defined morphism $\psi' : P' \to P$ composed with Φ and conversely we get Φ from Φ' by composing with a unique $\psi : P \to P'$. Finally the universal property yields that the composition $\psi' \circ \psi$ and $\psi \circ \psi'$ must be the identities.

So we can conclude: If a projective limit exists it is unique up to a canonical isomorphism and is denoted by

$$P = \varprojlim_{i\in I} X_i$$

This limit is also called the inverse limit because the arrow points backwards. We also should remember the the arrows in our system $\{X_i\}$ point from objects with a larger index to objects with smaller index. The universal map u is sometimes suppressed in the notation.

I will discuss some examples of projective limits which belong to the general education of anybody working in algebra or topology.

3.2 Examples.

(1) We consider the case where $\mathcal{C} = \mathbf{Ens}$ and the order relation on I is trivial, i.e. $i \leq j$ if and only if $i = j$. Then we may take the product of these sets $P = \prod_{i\in I} X_i$ and the $\Phi_i : P \to X_i$ are the usual set theoretic projections. Then $\{P, \Phi_i\}_{i\in I}$ is also the product in the categorical sense.

(2i) We take the set of positive natural numbers $\mathbb{N}_+$ and we define as order relation the divisibility relation, i.e. $n \leq m \Leftrightarrow n \mid m$. For any m we can define the quotient rings $\mathbb{Z}/m\mathbb{Z}$ and if $m \mid m'$ then we have the projection

$$\phi_{m,m'} : \mathbb{Z}/m'\mathbb{Z} \to \mathbb{Z}/m\mathbb{Z},$$

and $\varphi_{m,m'}(x_{m'}) = x_m$ means that $x_{m'} \equiv x_m \bmod m$. We can define a ring

$$\hat{\mathbb{Z}} = \{(\ldots, x_n, \ldots)_{n\in\mathbb{N}_+} \mid x_n \in \mathbb{Z}/n\mathbb{Z}, x_{n'} = x_n \bmod n' \text{ if } n'|n\}$$

where addition and multiplication are taken componentwise, and we have the projection map

$$\hat{\mathbb{Z}} \xrightarrow{\phi_n} \mathbb{Z}/n\mathbb{Z}$$

which is the projection to the n-th component. Then $(\hat{\mathbb{Z}}, \varphi_n)_{n\in\mathbb{N}^+}$ is the projective limit in the category of rings.

2ii) We may also look at the ordered set $\{p^n\}_{\{n=1,2,\ldots\}}$ where p is a prime. Then we get the projective system

$$\mathbb{Z}/p^n\mathbb{Z} \to \mathbb{Z}/p^m\mathbb{Z} \to$$

and the projective limit

$$\mathbb{Z}_p = \{(\ldots, x_n, \ldots) \mid x_m \equiv x_n \bmod p^n \text{ if } n \leq m\}.$$

Each component x_n determines completely the x_m with $m \leq n$ but if we go backwards we get more and more refined information. We can put a topology onto $\mathbb{Z}_p$ where a basis of open sets is given by the elements of the form $y + p^k\mathbb{Z}_p$.

The ring $\mathbb{Z}_p$ contains $\mathbb{Z}$ as a dense subring, it is a local ring without zero divisors, the unique maximal ideal is $p\mathbb{Z}_p = (p)$. Its quotient field is the field $\mathbb{Q}_p$ of p-adic numbers.

It follows from elementary number theory (The Chinese remainder theorem) that

$$\hat{\mathbb{Z}} = \prod_p \mathbb{Z}_p.$$

This ring $\hat{\mathbb{Z}}$ is not integral, it has zero divisors.

(3) It is not too difficult to see that in **Ens** projective limits exist. One simply forms the product

$$\prod_{i\in I} X_i$$

and takes the subset of those elements

$$x = (\ldots, x_i, \ldots)_{i\in I}$$

which satisfy $\phi_{ij}(x_j) = x_i$.

This implies that also in such categories like the category of rings, the category of modules over a given ring products and projective limits exist.

But in the category of fields we even cannot form the product of two fields, because we cannot avoid zero divisors.

(4) A very important example of a projective limit is the Galois group of a field k. We assume that we have constructed an algebraic closure $\bar{k}$ of k, this is a field with the followind two properties

(i) Every $\alpha \in \bar{k}$ is algebraic over k, i.e. satisfies a nontrivial equation

$$\alpha^n + a_1\alpha^{n-1} + \dots a_n = 0$$

with $a_i \in k$.

(ii) The field $\bar{k}$ is algebraically closed.

(Such a field can always be constructed if we use the axiom of choice).

We have the set of finite normal extensions

$$k \subset K \subset \bar{k},$$

this is an ordered set by inclusion. For any normal extension $k \subset k_1 \subset \bar{k}$ let $\mathrm{Gal}(k_1/k)$ be the group of automorphisms of k_1 whose restriction k induces the identity. For a tower of finite normal extensions

$$k \subset K \subset L$$

we have a surjective map

$$\mathrm{Gal}(L/k) \to \mathrm{Gal}(K/k)$$

which is simply given by restriction. We can form the projective limit

$$\varprojlim_{K/k} \mathrm{Gal}(K/k)$$

of this system. It exists by the above remark. The resriction defines an isomorphism

$$\mathrm{Gal}(\bar{k}/k) \to \varprojlim_{K/k} \mathrm{Gal}(K/k).$$

This is clear if we know that every automorphism $\sigma : K \to K$ over k can be extended to an automorphism of the algebraic closure.

(5) It is of course obvious that in the category **Ensfin** of finite sets we cannot have infinite products. But if we have a family $(\{X_i\}_{i\in I}, \phi_{ij})\}$ of finite sets we can form the product in **Ens** and we define a topology on this product. This should be the coarsest topology such that the projections

$$p_i : \prod_{j\in I} X_j \to X_i$$

become continuous. (On X_i we take the discrete topology, every subset is open). Hence we get a basis for the topology if we take finite intersections

$$\bigcap_{i\in E} p_i^{-1}(\{x_i\})$$

where E is finite and $x_i \in X_i$ a point.

It is not too difficult to prove that the product endowed with this topology becomes a compact space. The same holds if we take projective limits of finite sets (groups, rings,.....), these limits are compact topological spaces (groups, rings, ...). The resulting projective limits are called profinite sets (groups, rings,.....).

For instance the ring

$$\hat{\mathbb{Z}} = \lim_{\substack{\leftarrow \\ m}} \mathbb{Z}/m\mathbb{Z}$$

is such a profinite ring. The Galois group $\mathrm{Gal}(\overline{k}/k)$ of a field k is a profinite group. The topology of this groups is called the *Krull topology.*

3.3 Representable functors

I want to say a few words about *representable* functors. We discussed the example of projective limits. But the notion of representability for a functor is much more general. It may be applied to any contravariant or covariant functor which takes values in the category of sets.

If we have a contravariant functor

$$F : \mathcal{C} \to \mathrm{Ens}$$

we may ask whether we can find an object X and an element $u \in F(X)$ such that for any $Z \in \mathrm{Ob}(\mathcal{C})$ we get a bijection

$$\mathrm{Hom}_{\mathcal{C}}(Z, X) \tilde{\to} F(Z)$$

which is given by the universal rule

$$\phi \to F(\phi)(u)?$$

If such an object X together with $u \in F(X)$ exists then the so-called Yoneda lemma asserts that it is unique up to a canonical isomorphism. This means that the data provide a distinguished isomorphism between two solutions of the problem. (The proof is basically the same as in the case of projective limits: If we have two such objects X, X' we have $\mathrm{Hom}_{\mathcal{C}}(X', X) \tilde{\to} F(X')$. Now the $u' \in F(X')$ provides a morphism in $\mathrm{Hom}_{\mathcal{C}}(X', X)$. Interchanging the two arguments gives us a morphism in the opposite direction. The compositions must be the identities.)

3.4. Direct limits:

I begin with the simplest example. If we have a family $\{X_i\}_{i \in I}$ of sets then we can form the disjoint union

$$\bigsqcup_{i \in I} X_i.$$

This construction satisfies

$$\mathrm{Hom}_{\mathbf{Ens}}(\bigsqcup_{i \in I} X_i, Z) = \prod_{i \in I} \mathrm{Hom}_{\mathcal{C}}(X_i, Z).$$

Here is becomes clear that the formation of a disjoint union and a product are dual to each other this means that the arrows are turned backwards. We formulate a principle:

3.4.1. *The product is constructed so that we know what the arrows into it are, the disjoint union so that we know what the arrows from it are.*

To describe inductive (or direct) limits we start again from an ordered set $(I, \leq)$. Now we consider a *covariant* functor which attaches to any i an $X_i \in Ob(\mathcal{C})$ and an element

$$\phi_{ij} \in \mathrm{Hom}_{\mathcal{C}}(X_i, X_j)$$

whenever $i \leq j$. So in contrast to the case of projective limits the arrows point from objects with a smaller index to objects with a larger index. Such a system (or functor) is called an *inductive* system.

This time we look at

$$\mathrm{Hom}_{\mathcal{C}}((\{X_i\}_{i\in I}, \phi_{ij}), Z),$$

these are now maps from the diagram to objects in $\mathcal{C}$. We say that an object L together with a map

$$\Psi \in \mathrm{Hom}_{\mathcal{C}}((\{X_i\}_{i\in I}, \phi_{ij}), L)$$

is a direct limit of $(\{X_i\}_{i\in I}, \phi_{ij})$ if

$$\mathrm{Hom}_{\mathcal{C}}(L, Z) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{C}}((\{X_i\}_{i\in I}, \phi_{ij}), Z)$$

and where the bijection is given by composition

$$\phi_i = \phi \circ \Psi_i$$

and $\phi_i \in \mathrm{Hom}_{\mathcal{C}}(X_i, Z), \phi_i \in \mathrm{Hom}_{\mathcal{C}}(L, Z)$. If such a limit exists we write

$$L = \varinjlim_{i\in I} X_i.$$

It is clear that in the category **Ens** direct limits exist: Starting from an inductive system of sets $(\{X_i\}_{i\in I}, \phi_{ij})$ we form the disjoint union $\bigsqcup_{i\in I} X_i$.. We introduce an equivalence relation $\sim$ on this disjoint union. This equivalence relation will satisfy $x_i \sim x_j$ whenever $\phi_{ij}(x_i) = x_j$. This is not necessarily an equivalence relation, but we simply take the equivalence relation generated by the relation. Then it is not hard to see that the quotient of the disjoint union by this relation is a direct limit.

3.5 An ordered set $(I, \leq)$ is called *directed* if for any two $i, j \in I$ we can find an element $l \in I$ such that $i < l, j < l$. If we have inductive system of sets $(\{X_i\}_{i\in I}, \phi_{ij})$ over a directed set, then the equivalence relation in our construction above can be described directly

$$x_i \sim x_j \leftrightarrow \exists\, l \in I, i \leq l, j \leq l \text{ and } \phi_{il}(x_i) = \phi_{jl}(x_j).$$

We may also look at the opposite case where the ordering relation on the set I is trivial, i.e. we have $i \leq j$ if and only if $i = j$. If we have an inductive system $(\{X_i\}_{i\in I}, \phi_{ij})$ over such a set then the inductive limit should be called *a disjoint union.*

More examples of such direct limits will be constructed in Chapter III where we shall see that stalks of sheaves are direct limits.

Generally we had projective limits as subsets of products, direct limits will be quotients of disjoint unions.

By the way in some sense this discussion of direct limits is superfluous because if we pass to the opposite category the direct limits become projective limits.

3.6 Exercise:

1) Do we have disjoint unions in the category $\mathbf{Vect}_k$? If so how does the disjoint union of two vector spaces look like.

2) We may ask the same question for the category $\mathbf{Rings}$ of rings, for the category of commutative rings and for the category of groups.

2b) In any category we can consider diagrams of the form

$$\begin{tikzcd} & B \\ A \arrow[ur] \arrow[dr] & \\ & C \end{tikzcd}$$

we can interprete this as an inductive system and we can ask whether the limit exists. If our category is the category of groups then the limit does exist and it is given by the almaganated product.

3) Let me assume we have an index set $(I, \leq)$ and a projective system $(\{X_i\}_{i\in I}, \phi_{ij})$ on it. Let us assume that the indexing set contains a maximal element m, i.e. $m \geq i$ for all elements $i \in I$. I claim that the projective limit exists. How does it look like? Can you formulate an analogous assertion for injective limits.

4) Let us assume that we have a directed set $(I, \leq)$ We assume that we have inductive system of rings $\{R_i\}_{i\in I}$. Does the direct limit exist? Hint: Forget the ring structure and consider the R_i as sets. Form the limit in the category of sets. Now you can reintroduce the ring structure on this limit by observing that any pair (or even finite set) of elements can be represented by elements in a suitable member R_i of the family.

5) We have seen that we may interpret an ordered set $(A, \leq)$ as a category. What does it mean for such a category that the product of two elements exists?

Chapter II

Basic Concepts of Homological Algebra

In this chapter I want to explain the fundamental concepts of homological algebra. I will do this for the specific case group (co-) homology.

This example will become important to us in the third volume of this book where we discuss the cohomology of arithmetic groups. But since in this particular case the basic principles of homological become very clear I have chosen this example as introduction into the subject. The cohomology of sheaves, which can serve as a second example, will be discussed in Chapter IV.

II.1. The category of Γ-modules:

In the following Γ will always be a group. A Γ-module is an abelian group M together with an action of Γ: This means we have a map $\Gamma \times M \mapsto M, (\gamma, m) \to \gamma m$, which satisfies $1_\Gamma m = m$, $(\gamma_1\gamma_2)m = \gamma_1(\gamma_2 m)$ and $\gamma(m_1 + m_2) = \gamma m_1 + \gamma m_2$. These Γ-modules are the objects of the category of Γ-modules: If we write $M \in \mathrm{Ob}(\mathbf{Mod}_\Gamma)$, then this means that M is a Γ-module.

If $M_1, M_2 \in \mathrm{Ob}(\mathbf{Mod}_\Gamma)$, then we may consider the set

$$\mathrm{Hom}_{\mathbf{Mod}_\Gamma}(M_1, M_2) = \mathrm{Hom}_\Gamma(M_1, M_2) = \left\{ \varphi : M_1 \to M_2 \;\middle|\; \begin{array}{l} \varphi \text{ homomorphism of abelian groups} \\ \varphi(\gamma m_1) = \gamma\varphi(m_1) \text{ for all } \gamma, m_1 \end{array} \right\}$$

On $\mathrm{Hom}_\Gamma(M_1, M_2)$ we have a natural structure of an abelian group: For any two elements $\varphi, \psi \in \mathrm{Hom}_\Gamma(M_1, M_2)$ we put $(\varphi + \psi)(m_1) = \varphi(m_1) + \psi(m_1)$.

Here we have another typical example of a category: We have a collection of objects — this collection is not a set in general — and for any two such objects we have a set of morphisms. (In our special case these sets of morphisms are abelian groups.) A certain bunch of axioms has to be satisfied: We have the identity $\mathrm{Id}_M \in \mathrm{Hom}_\Gamma(M, M)$, we have a composition $\mathrm{Hom}_\Gamma(M_1, M_2) \times \mathrm{Hom}_\Gamma(M_2, M_3) \to \mathrm{Hom}_\Gamma(M_1, M_3)$ and Id_M is neutral with respect to this composition. (See the introduction in Chap. I) In our special case this composition is bilinear.

The special category $\mathbf{Mod}_\Gamma$ has some extra features: Given $\varphi : M \to N$ we can form the *kernel* and the *image*

$$\ker(\varphi) = \{m \mid \varphi(m) = 0\}, \mathrm{Im}(\varphi) = \{\varphi(m) \mid m \in M\},$$

clearly these are also Γ-modules.

If $N \subset M$ is a Γ-submodule of M, then we may form the *quotient module*

$$M/N = M \mathrm{mod} N,$$

this is again a Γ-module. Finally, we have direct sums and direct products

$$\begin{aligned}
\bigoplus_{i\in I} M_i &= \{(\ldots m_i \ldots)_{i\in I} \mid m_i \in M_i,\ \text{almost all } m_i = 0\} \\
\prod_{i\in I} M_i &= \{(\ldots m_i \ldots)_{i\in I} \mid m_i \in M_i\}
\end{aligned}$$

where the addition and the action of Γ are defined componentwise.

All these properties of imply that $\mathbf{Mod}_\Gamma$ is an abelian category. The notion of abelian categories can be axiomatized (see [Go], 1.8.).

1.1 Complexes

If we have a sequence of maps between Γ-modules

$$\longrightarrow M_{\nu+1} \xrightarrow{d_{\nu+1}} M_\nu \xrightarrow{d_\nu} M_{\nu-1} \longrightarrow$$

then this is called a *(homological) complex* if $d_\nu \circ d_{\nu+1} = 0$ for all indices ν, i.e. if always

$$\mathrm{Im}(d_{\nu+1}) \subset \ker(d_\nu).$$

The maps d_ν are the *differentials* of the complex. The complex is called *exact* if we have $\mathrm{Im}(d_{\nu+1}) = \ker(d_\nu)$ for all indices ν.

We define the *homology groups* of such a complex as

$$H_\nu(M^\bullet) = \frac{\ker(d_\nu : M_\nu \to M_{\nu-1})}{\mathrm{Im}(d_{\nu+1} : M_{\nu+1} \to M_\nu)}.$$

The complex is exact if and only if its homology groups are trivial.

We can also consider complexes where the differentials raise the index by one then we write the indices ν as superscripts

$$\longrightarrow M^{\nu-1} \xrightarrow{d^{\nu-1}} M^\nu \xrightarrow{d^\nu} M^{\nu+1} \longrightarrow,$$

then this is a *cohomological* complex.

Very often we abbreviate and simply write $M^\bullet$ or $(M^\bullet, d)$ for a (co)homological complex. We define the *cohomology groups* of a cohomological complex by

$$H^\nu(M^\bullet) = \frac{\ker(M^\nu \xrightarrow{d^\nu} M^{\nu+1})}{\mathrm{im}(M^{\nu-1} \longrightarrow M^\nu)}.$$

Again we abbreviate $H^\bullet(M^\bullet)$ and this is the graded direct sum over all cohomology groups.

The elements in the kernel $Z^\nu(M) = \ker(M^\nu \xrightarrow{d^\nu} M^{\nu+1})$ are called the *cocycles* in degree ν and the elements in $B^\nu(M) = \mathrm{im}(M^{\nu-1} \longrightarrow M^\nu)$ are the *coboundaries.* Hence the cohomology is the group of cocycles modulo the coboundaries.

Actually we may also view these cohomology groups as a complex, the differentials are zero. Again is clear that a complex is exact, if and only if all its cohomology groups vanish.

A map between two complexes

$$\varphi^\bullet : M^\bullet \to N^\bullet$$

is a sequence of maps $\phi^\nu : M^\nu \to N^\nu$ which commutes with the differentials. It is clear that such a map induces a map between the cohomology groups $H^\bullet(\varphi^\bullet) : H^\bullet(M^\bullet) \to H^\bullet(N^\bullet)$.

A *short exact sequence* is an exact complex

$$0 \longrightarrow M' \xrightarrow{i} M \xrightarrow{p} M'' \longrightarrow 0,$$

i.e. i is injective, $\mathrm{Im}(i) = \ker(p)$ and p is surjective, i.e. M'' is isomorphic to the quotient of M by the submodule $i(M') \simeq M'$.

II.2. More Functors

As I explained already in the first chapter a functor is a rule that produces in a functorial way an object in a target category from an object in the source category. If for instance the source category is $\mathbf{Mod}_\Gamma$ and the target category is the category $\mathbf{Ab}$ of abelian groups, then a functor

$$F : \mathbf{Mod}_\Gamma \longrightarrow \mathbf{Ab}$$

associates to any Γ-module $M \in \mathrm{Ob}(\mathbf{Mod}_\Gamma)$ an abelian group $F(M)$. Functoriality means that for any $M_1, M_2 \in \mathrm{Ob}(\mathbf{Mod}_\Gamma)$ we have a map

$$F_{M_1,M_2} : \mathrm{Hom}_{\mathbf{Mod}_\Gamma}(M_1, M_2) \longrightarrow \mathrm{Hom}_{\mathbf{Ab}}(F(M_1), F(M_2))$$

which sends Id_M to $\mathrm{Id}_{F(M)}$ and compositions into compositions. If we require that this map is a homomorphism F_{M_1,M_2} between the abelian groups, then this functor is an *additive* functor between abelian categories.

There are two very simple functors between the category $\mathbf{Mod}_\Gamma$ and the category $\mathbf{Ab}$ of the abelian group

$$\begin{array}{lcccc} \text{Forget} & : & \mathbf{Mod}_\Gamma & \longrightarrow & \mathbf{Ab} \\ \text{Trivial} & : & \mathbf{Ab} & \longrightarrow & \mathbf{Mod}_\Gamma \end{array}$$

where the first factor “forgets” the Γ-module structure on the abelian group M and the second introduces the trivial Γ-action on an abelian group A, i.e. every element $\gamma \in \Gamma$ induces the identity on A.

These two functors are *exact* functors, this means that they transform exact sequences into exact sequences. Homological algebra owes its existence to the fact that many important additive functors are not exact.

Here comes the first example. If M is a Γ-module, we define the module $M^\Gamma = \{m \mid \gamma m = m \text{ for all } \gamma \in \Gamma\}$ of *invariants*. It is an abelian group and hence we defined a functor

$$\text{Invariants} : \mathbf{Mod}_\Gamma \longrightarrow \mathbf{Ab}$$

from the category of Γ-modules to the category of abelian groups. If A is a trivial Γ-module, then

$$\mathrm{Hom}_{\mathbf{Mod}_\Gamma}(A, M) = \mathrm{Hom}_{\mathbf{Ab}}(A, M^\Gamma),$$

and this property also characterizes the submodule M^Γ in M.

The module M_Γ of coinvariants is defined as a quotient

$$M \longrightarrow M_\Gamma$$

where M_Γ is a trivial Γ-module and for any Γ-module with trivial action by Γ we have

$$\mathrm{Hom}_{\mathbf{Mod}_\Gamma}(M, A) = \mathrm{Hom}_{\mathbf{Ab}}(M_\Gamma, A).$$

To give a different description of M_Γ we recall the notion of the group ring $R = \mathbb{Z}[\Gamma]$ of our group Γ. It consists of all finite linear combinations

$$\sum_{\gamma \in \Gamma} n_\gamma \gamma \qquad n_\gamma \in \mathbb{Z}, \quad \text{almost all } n_\gamma = 0,$$

where we add componentwise (i.e. the additive group is the free abelian group over the set), and where we multiply

$$\left(\sum_\gamma n_\gamma \gamma\right) \cdot \left(\sum_\eta m_\eta \eta\right) = \sum_{\gamma,\eta} n_\gamma m_\eta \gamma\eta = \sum_\delta \left(\sum_{\gamma\eta=\delta} n_\gamma m_\eta\right)\delta.$$

This group ring contains the so called *augmentation ideal* I_Γ which is the two sided ideal

$$I_\Gamma = \left\{\sum n_\gamma \gamma \mid \sum n_\gamma = 0\right\}.$$

It is clear that this ideal is generated by elements of the form $1-\gamma$ (as $\mathbb{Z}$-module). For any Γ-module M the module $I_\Gamma M \subset M$ is a Γ-submodule, and it is also an easy exercise that

$$M_\Gamma = M/I_\Gamma M = H_0(\Gamma, M)$$

has the desired property the module of coinvariants should have.

The following fact is the starting point of homological algebra:

In In general the functors $M \to M^\Gamma$ and $M \to M_\Gamma$ are not exact.

To be more precise: If we start from a short exact sequence

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

of Γ-modules, then the sequence

$$0 \longrightarrow (M')^\Gamma \longrightarrow M^\Gamma \longrightarrow (M'')^\Gamma$$

is exact, but the last arrow is not surjective (in general).
A similar assertion holds for M_Γ. We only get an exact sequence

$$M'_\Gamma \longrightarrow M_\Gamma \longrightarrow M''_\Gamma \longrightarrow 0.$$

We say that H^0 is a *left exact* functor and H_0 is a *right exact* functor.
The goal is to construct the so called *derived* functors which measure the deviation from exactness. We motivate this by an example:

2.1 The first cohomology group

I want to explain why the functor $M \to M^\Gamma$ is not exact and to show how this more or less automatically leads to the definition of the *derived functor*.
Let us start from an exact sequence of Γ-modules

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0.$$

We get an exact sequence of abelian groups

$$0 \longrightarrow (M')^\Gamma \longrightarrow M^\Gamma \longrightarrow (M'')^\Gamma.$$

We pick an element $m'' \in (M'')^\Gamma$, and we want to understand, why this is not necessarily in the image of M^Γ. Of course we can find an element $m \in M$ which maps to m''. But there is no reason why this element should be invariant under Γ, the only thing we know is that for all $\gamma \in \Gamma$ the difference

$$m'_\gamma = m - \gamma m \in M'.$$

We get a map

$$\begin{array}{ccc} \Gamma & \longrightarrow & M' \\ \gamma & \longmapsto & m'_\gamma \end{array},$$

and this map satisfies

$$m'_{\gamma_1} + \gamma_1 m'_{\gamma_2} = m'_{\gamma_1\gamma_2}.$$

A map $\Gamma \to M'$ satisfying this relation is called a 1-*cocycle*. On the set of all 1-cocycles we get a structure of an abelian group if we add the values and we denote by

$$Z^1(\Gamma, M'),$$

the abelian group of 1-cocycles. Our element m is in M^Γ if and only if the cocycle $m'_\gamma = m - \gamma m = 0$.

We notice that the choice of m is not unique, we may change $m \to m + m'$ with $m' \in M'$. This is the only possible modification. Then we also modify the cocycle defined by m into $\gamma \mapsto m'_\gamma + m' - \gamma m'$.

This leads to the definition of the group $B^1(\Gamma, M')$ of 1-*coboundaries*. It is the group of those cocycles $\gamma \mapsto b_\gamma$ for which we can find a $m' \in M'$ such that $b_\gamma = m' - \gamma m'$ for all γ.

Hence we see: The element $m'' \in (M'')^\Gamma$ defines an element in $Z^1(\Gamma, M')$ which is well defined up to a coboundary. We introduce the first cohomology group (provisorial definition)

$$\check{H}^1(\Gamma, M') = Z^1(\Gamma, M')/B^1(\Gamma, M'),$$

and we have seen that any $m'' \in (M'')^\Gamma$ defines a class

$$\delta(m'') \in \check{H}^1(\Gamma, M')$$

which is zero if and only if m'' is in the image of $M^\Gamma \to (M'')^\Gamma$.

It is clear that δ is a homomorphism, and that we have extended our exact sequence one step further

$$0 \longrightarrow (M')^\Gamma \longrightarrow M^\Gamma \longrightarrow (M'')^\Gamma \xrightarrow{\delta} \check{H}^1(\Gamma, M').$$

The next thing that can be checked easily is the functoriality of $M' \to \check{H}^1(\Gamma, M')$. If we have a $\varphi \in \mathrm{Hom}_{\mathbf{Mod}_\Gamma}(M', N)$ then this induces a map $\check{\varphi}^{(1)} : \check{H}^1(\Gamma, M') \to \check{H}^1(\Gamma, N)$, and our above considerations also show that we get an even longer exact sequence

$$0 \to (M')^\Gamma \to M^\Gamma \to (M'')^\Gamma \xrightarrow{\delta} \check{H}^1(\Gamma, M') \to \check{H}^1(\Gamma, M) \to \check{H}^1(\Gamma, M''),$$

the verification of exactness is left to the reader. But at the end it stops again; the last map needs not to be surjective.

We also see that this longer exact sequence depends functorially on the short exact sequence we started from. If we have a map between two exact sequences of Γ-modules

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M' & \longrightarrow & M & \longrightarrow & M'' & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & N' & \longrightarrow & N & \longrightarrow & N'' & \longrightarrow & 0 \end{array}$$

then this induces a map between the two resulting exact sequences (in the sense of maps between complexes, i.e. all diagrams commute).

In principle we can try to extend our sequence beyond $\check{H}^1(\Gamma, M'')$. We pick an element in $\check{H}^1(\Gamma, M'')$ and try to lift it to an element in $\check{H}^1(\Gamma, M)$, and then we will see what the obstruction to this lifting will be. This will suggest a definition of a cohomology group $\check{H}^2(\Gamma, M')$. But actually there is a much more elegant way to define the cohomology functor which is also universal in the sense that it applies to many other cases. This will be done in section II.3.

2.2 At this point we introduce some new notation, instead of M^Γ we also write $H^0(\Gamma, M)$ and $H_0(\Gamma, M)$ will be the same as M_Γ. This is a very suggestive notation if we use it for our exact sequence above.

Of course all this does not yet prove that $M \to M^\Gamma$ is not exact in general. For instance, it could be that $\check{H}^1(\Gamma, M) = 0$ for all Γ and all M, or it could also be that $\check{H}^1(\Gamma, M') \to \check{H}^1(\Gamma, M)$ is always injective. We will show in exercise 2.6. that for $\Gamma \neq \{1\}$ these functors are not trivial.

2.3 Exercise:

If A is a trivial Γ-module, then

$$\check{H}^1(\Gamma, A) = \mathrm{Hom}(\Gamma, A)$$

where the last Hom *is the* Hom *in the category of groups.*

This shows that for suitable A the module $\check{H}^1(\Gamma, A) \neq 0$ if Γ is not equal to its commutator group $[\Gamma, \Gamma]$.

Let us now assume that $\Gamma' \subset \Gamma$ is a subgroup. We have the important functor from the category of Γ'-modules to the category of Γ-modules which is called *induction*. For any Γ'-module Y we define an abelian group

$$\mathrm{Ind}_{\Gamma'}^{\Gamma} Y = \{f : \Gamma \to Y \mid f(\gamma'\gamma) = \gamma' f(\gamma) \text{ for all } \gamma' \in \Gamma', \gamma \in \Gamma\},$$

and we define the action of Γ on $\mathrm{Ind}_{\Gamma'}^{\Gamma} Y$ by

$$(\gamma f)(\gamma_1) = f(\gamma_1\gamma).$$

This is the *induced* Γ*-module* from the Γ'-module Y. It is very easy to check that for any Γ-module X we have an isomorphism (*Frobenius reciprocity*)

$$\mathrm{Hom}_\Gamma(X, \mathrm{Ind}_{\Gamma'}^{\Gamma} Y) \xrightarrow{\sim} \mathrm{Hom}_{\Gamma'}(X, Y)$$

which is given by

$$\varphi \longmapsto \{x \longmapsto \varphi(x)(1)\}.$$

2.4 Exercise:

We have a canonical (this means functorial in Y) isomorphism

$$\check{H}^1(\Gamma, \mathrm{Ind}_{\Gamma'}^{\Gamma} Y) \xrightarrow{\sim} \check{H}^1(\Gamma', Y).$$

This isomorphism is obtained from the following map on the level of cocycles: For any 1-cocycle

$$\{\gamma \longmapsto f_\gamma\} \in Z^1(\Gamma, \mathrm{Ind}_{\Gamma'}^{\Gamma} Y)$$

we define the 1-cocycle

$$\{\gamma' \longmapsto \overline{f}_{\gamma'}\} \in Z^1(\Gamma', Y)$$

by $\overline{f}_{\gamma'} = f_{\gamma'}(1)$. Show that this map sends coboundaries into coboundaries and induces an isomorphism on cohomology.

Hint: We have to combine several little observations:

i) We consider an 1-cocycle

$$\{\gamma \longmapsto f_\gamma\} \in Z^1(\Gamma, \mathrm{Ind}_{\Gamma'}^{\Gamma} Y),$$

and we take into account that f_γ is actually a Y-valued function on Γ. Then the cocycle relation reads

$$f_{\gamma_1\gamma_2}(x) = f_{\gamma_1}(x) + (\gamma f_{\gamma_2})(x) = f_{\gamma_1}(x) + f_{\gamma_2}(x\gamma_1).$$

If we evaluate at $x = 1$ we get

$$f_{\gamma_2}(\gamma_1) = f_{\gamma_1\gamma_2}(1) - f_{\gamma_1}(1),$$

and this relation tells us that we only need to know the values $f_\gamma(1)$. Then the cocycle relation gives us the values of the f_γ at any $x \in \Gamma$.

ii) If we have any function

$$\begin{array}{ccccc} h & : & \Gamma & \longrightarrow & Y \\ h & : & \gamma & \longrightarrow & h_\gamma, \end{array}$$

we may put (think of h_γ as being $f_\gamma(1)$)

$$H_\gamma(x) = h_{x\gamma} - h_x,$$

then H_γ is a function on Γ with values in Y. If $\gamma \to h_\gamma$ satisfies

$$h_{\gamma' x\gamma} - h_{\gamma' x} = \gamma'(h_{x\gamma} - h_x),$$

then $H_\gamma \in \mathrm{Ind}_{\Gamma'}^{\Gamma} Y$ and $\gamma \to H_\gamma$ it is a 1-cocycle.

iii) If we have a 1-cocycle $\gamma \to f_\gamma$ in $Z^1(\Gamma, \mathrm{Ind}_{\Gamma'}^{\Gamma} Y)$, then

$$\gamma' \longrightarrow f_{\gamma'}(1) \qquad\qquad \gamma' \in \Gamma'$$

is a one-cocycle in $Z^1(\Gamma', Y)$. Hence we have a map

$$Z^1(\Gamma, \mathrm{Ind}_{\Gamma'}^{\Gamma} Y) \longrightarrow Z^1(\Gamma', Y),$$

and it is clear that this map sends coboundaries into coboundaries.

iv) If we have a 1-cocycle

$$\gamma' \longrightarrow \overline{f}_{\gamma'}$$

in $Z^1(\Gamma', Y)$, then we want to construct a 1-cocycle $\gamma \to f_\gamma$ so that $\overline{f}_{\gamma'} = f_{\gamma'}(1)$. To do this we choose a system γ_i of representatives of $\Gamma' \backslash \Gamma$ where we choose the identity for the class Γ'. For $\gamma = \gamma'\gamma_i$ we put

$$f_\gamma(1) = f_{\gamma'}(1)$$

and apply (ii). The cocycle relation for $\gamma' \to f_{\gamma'}(1)$ provides the decisive relation in (ii).This proves the surjectivity of our map between 1-cocycles in (iii).
v) Finally, we have to check that $\gamma \to f_\gamma$ is a boundary if $\gamma' \to \overline{f}\gamma'$ is a boundary. We can write

$$f_{\gamma'}(1) = y - \gamma' y$$

with $y \in Y$ and for all $\gamma' \in \Gamma'$.
If we want to write $\gamma \to f_\gamma$ as a boundary, i.e.

$$f_\gamma = c - \gamma c,$$

then this reads $f_\gamma(x) = c(x) - c(x\gamma)$, and evaluation at 1 yields

$$f_\gamma(1) = c(1) - c(\gamma).$$

Hence we choose $c(1) = y$ and put

$$c(x) = y - f_x(1)$$

and verify that this c bounds f_γ.

2.5 Exercise:

Use the previous exercise and prove that for any group $\Gamma \neq \{1\}$ there is a Γ-modules M s.t.

$$\check{H}^1(\Gamma, M) \neq 0.$$

2.6 Exercise:

The group ring $\mathbb{Z}[\Gamma]$ is also a Γ-module by multiplication from the left. We get an exact sequence of Γ-modules

$$0 \longrightarrow I_\Gamma \longrightarrow \mathbb{Z}[\Gamma] \longrightarrow \mathbb{Z} \longrightarrow 0.$$

If we apply the functor H_0 to this sequence, we find

$$\begin{array}{ccccc} I_\Gamma/I_\Gamma I_\Gamma & \longrightarrow & \mathbb{Z}[\Gamma]/I_\Gamma & \xrightarrow{\sim} & \mathbb{Z} \longrightarrow 0 \\ & & \| & & \\ & & \mathbb{Z} & & , \end{array}$$

and hence the sequence is exact if and only if $I_\Gamma = I_\Gamma I_\Gamma$.
It is not difficult to prove that

$$\begin{array}{ccc} \Gamma & \longrightarrow & I_\Gamma/I_\Gamma I_\Gamma \\ \gamma & \longrightarrow & 1 - \gamma \end{array}$$

induces an isomorphism

$$\Gamma/[\Gamma,\Gamma] = \Gamma_{\mathrm{ab}} \xrightarrow{\sim} I_\Gamma/I_\Gamma I_\Gamma$$

($[\Gamma, \Gamma]$ *is the commutator subgroup*).

II.3. The derived functors

After these motivating considerations we explain the fundamental problem to be solved in homological algebra. We have the functor

$$M \longrightarrow M^{\Gamma} = H^0(\Gamma, M)$$

which is only left exact. We want to construct the *right derived functor*: This is a collection of functors

$$M \longrightarrow H^i(\Gamma, M) \qquad\qquad i = 0, 1, \ldots,$$

such that for any short exact sequence

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

we get a long exact sequence

$$\begin{aligned} 0 \longrightarrow H^0(\Gamma, M') \longrightarrow H^0(\Gamma, M) \longrightarrow H^0(\Gamma, M'') \xrightarrow{\delta} H^1(\Gamma, M') \longrightarrow \\ \longrightarrow H^1(\Gamma, M) \longrightarrow H^1(\Gamma, M'') \xrightarrow{\delta} H^2(\Gamma, M') \longrightarrow \ldots \end{aligned}$$

which depends functorically on the exact sequence in the sense explained before.

Finally we want this functor to be minimal (or universal) in the following sense.

If we have any other collection of functors

$$M \longrightarrow \tilde{H}^i(\Gamma, M) \qquad\qquad i = 0, 1, \ldots$$

with $H^0(\Gamma, M) = \tilde{H}^0(\Gamma, M)$, and the same properties as above, then we find a natural transformation

$$H^i(\Gamma, M) \longrightarrow \tilde{H}^i(\Gamma, M),$$

which is compatible with the connecting homomorphisms.

We want to indicate the main ideas how to construct these derived functors. The verification that the new construction of the H^1 gives the same result as our previous $\check{H}^1$ will be done in the exercises 4.4.

I want to explain a very simple principle that governs to the construction of these functors.

A simple observation: If we have an exact sequence of Γ-modules

$$0 \longrightarrow M' \xrightarrow{i} M \xrightarrow{p} M'' \longrightarrow 0,$$

we say that the sequence *splits* if one of the following assertions holds:

(i) We have a section to p. This is a Γ-modules homomorphism

$$s : M'' \longrightarrow M$$

for which $p \circ s = \mathrm{id}_{M''}$.
(ii) The modules M splits, i.e. we have a Γ-submodules $\tilde{M}''$ such that

$$\begin{array}{ccc} M' \oplus \tilde{M}'' & \xrightarrow{\sim} & M \\ (m', \tilde{m}'') & \longmapsto & i(m') + \tilde{m}''. \end{array}$$

(iii) We have a Γ-modules homomorphism

$$j : M \longrightarrow M'$$

s.t. $j \circ i = \mathrm{id}_{M'}$.
Now our observation is that our functors H^0,H_0 will transform split exact sequences into split exact sequences, in other words if we restrict them to split exact sequences then they are still exact.

3.1 **The simple principle:**
It is based on the assumption that we have already constructed a derived functor $\{M \to \tilde{H}^i(\Gamma, M)\}$. Let us assume we have a class of $\mathcal{C}$ of Γ-modules which are *acyclic* for this functor, this means that for any $X \in \mathcal{C}$ we have $\tilde{H}^i(\Gamma, X) = 0$ for all $i > 0$.
An *acyclic resolution* of $M \in \mathbf{Ob}(\mathbf{Mod}_\Gamma)$ by objects in $\mathcal{C}$ is an exact sequence of Γ-modules

$$0 \longrightarrow M \longrightarrow X^0 \longrightarrow X^1 \longrightarrow X^2 \longrightarrow$$

where the $X^\nu \in \mathcal{C}$.

Lemma 3.2.: *If $\mathcal{C}$ is a class of acyclic objects for the derived functor $\{M \to \tilde{H}^i(\Gamma, M)\}$, and if*

$$0 \longrightarrow M \longrightarrow X^0 \longrightarrow X^1 \longrightarrow$$

is an acyclic resolution of M by objects in $\mathcal{C}$. Then we have an isomorphism

$$\tilde{H}^i(\Gamma, M) \simeq H^i((X^\bullet)^\Gamma).$$

Proof: By induction on i. For $i = 0$ we get the exact sequence

$$0 \longrightarrow M^\Gamma \longrightarrow (X^0)^\Gamma \longrightarrow (X^1)^\Gamma \longrightarrow$$

and

$$M^\Gamma \xrightarrow{\sim} \ker((X^0)^\Gamma \longrightarrow (X^1)^\Gamma) = H^0((X^\bullet)^\Gamma).$$

Now we cut the resolution into pieces. We write

$$0 \longrightarrow M \longrightarrow X^0 \longrightarrow X^0/M \longrightarrow 0,$$

and we have a resolution by objects in $\mathcal{C}$

$$0 \longrightarrow X^0/M \longrightarrow Y^0 \longrightarrow Y^1 \longrightarrow$$

where $Y^{\nu-1} = X^\nu$.

The first sequence yields a long exact sequence which is interrupted by many zeroes which come from the $\tilde{H}^\bullet$-acyclicity of the X^ν.

$$\begin{array}{ccccc}
0 & \longrightarrow & M^\Gamma \longrightarrow (X^0)^\Gamma \longrightarrow (X^0/M)^\Gamma & \longrightarrow & \tilde{H}^1(\Gamma, M) \longrightarrow 0 \\
 & \longrightarrow & \tilde{H}^1(\Gamma, X^0/M) \qquad \xrightarrow{\quad\sim\quad} & & \tilde{H}^2(\Gamma, M) \longrightarrow 0 \\
 & & \vdots & & \vdots \\
0 & \longrightarrow & \tilde{H}^{i-1}(\Gamma, X^0/M) \qquad \xrightarrow{\quad\sim\quad} & & \tilde{H}^i(\Gamma, M) \longrightarrow 0.
\end{array}$$

We check the case $i = 1$. Here we find

$$\tilde{H}^1(\Gamma, M) \simeq (X^0/M)^\Gamma/(X^0)^\Gamma,$$

but $X^0/M \subset X^1$ is the kernel of $X^1 \to X^2$, and $(X^0/M)^\Gamma = \ker((X^1)^\Gamma \to (X^2)^\Gamma)$, and hence

$$\tilde{H}^1(\Gamma, M) \simeq \frac{\ker((X^1)^\Gamma \to (X^2)^\Gamma)}{\mathrm{im}((X^0)^\Gamma \to (X^1)^\Gamma)} = H^1((X^\bullet)^\Gamma).$$

Hence we proved our assertion for $i = 1$ and then induction is clear.

We want to apply this principle to construct the derived functors. But in some sense we are trapped: If we have not yet defined the derived functor, how can we know that certain objects are acyclic? This difficulty is resolved by the notion of *injective* modules.

A Γ-module $\mathcal{I}$ is called *injective* if it has the following property: Whenever we have a diagram of Γ-modules

$$\begin{array}{ccc}
A & \xrightarrow{\varphi} & B \\
\big\downarrow \psi & & \\
\mathcal{I} & &
\end{array}$$

where $\ker(\varphi) \subset \ker(\psi)$, then we can extend the diagram to a commutative diagram

$$\begin{array}{ccc}
A & \xrightarrow{\varphi} & B \\
\big\downarrow \psi & \swarrow \lambda & \\
\mathcal{I} & &
\end{array}$$

Our assumption on φ, ψ is valid if φ is injective. If we want to check the injectivity of a module it clearly suffices to check diagrams with $\varphi = 0$.

Injective modules have a very important property: Whenever we have a short exact sequence

$$0 \longrightarrow \mathcal{I} \longrightarrow M \longrightarrow M' \longrightarrow 0,$$

and the module $\mathcal{I}$ is injective then the sequence splits. We simply apply the defining property of injective modules to

$$\begin{array}{ccc} 0 \longrightarrow & \mathcal{I} & \longrightarrow M \\ & \big\downarrow & \\ & \mathcal{I} & \end{array}$$

where the vertical arrow is the identity. Our simple observation above implies that we get exact sequences

$$0 \longrightarrow H^0(\Gamma, \mathcal{I}) \longrightarrow H^0(\Gamma, M) \longrightarrow H^0(\Gamma, M') \longrightarrow 0.$$

$$0 \longrightarrow H_0(\Gamma, \mathcal{I}) \longrightarrow H^0(\Gamma, M) \longrightarrow H^0(\Gamma, M') \longrightarrow 0.$$

whenever the module $\mathcal{I}$ on the left is injective. Since the require that the cohomology modules should measure the deviation from exactness and that they should be minimal in this respect, we expect them to vanish for injective modules, in other words we expect that injecive modules should be acyclic. In view of our simple principle above we try to define them by *injective resolutions.*

The following lemma is the starting point:

Lemma 3.3.: *Every Γ-modules M can be embedded into an injective module $\mathcal{I}$.*

Sketch of the proof: First we consider the category **Ab** of abelian groups. This is the case $\Gamma = \{\mathrm{Id}\}$. One proves that the abelian group $\mathbb{Q}/\mathbb{Z}$ is injective (this requires Zorn's lemma), then we see that every abelian group A can be embedded into a suitable product

$$A \longrightarrow \prod_{\alpha} \mathbb{Q}/\mathbb{Z}.$$

If we have an Γ-module M we forget the Γ-module structure and embed it into an injective abelian group, i.e. $M \hookrightarrow \mathcal{J}$. Now we get $\mathrm{Ind}_{\{1\}}^{\Gamma} M \hookrightarrow \mathrm{Ind}_{\{1\}}^{\Gamma} \mathcal{J}$, and the module $\mathrm{Ind}_{\{1\}}^{\Gamma} \mathcal{J}$ is injective in the category of Γ-modules. This follows from Frobenius reciprocity. Then we have achieved our goal since we have

$$M \longrightarrow \mathrm{Ind}_{\{1\}}^{\Gamma} M \longrightarrow \mathrm{Ind}_{\{1\}}^{\Gamma} \mathcal{J} = \mathcal{I}.$$

Now the actual construction of the cohomology functor (the universal derived functor) becomes clear. We noticed that injective modules should be acyclic, i.e. $H^r(\Gamma, \mathcal{I}) = 0$ for $r > 1$. But our earlier lemmas tell us that we can find an injective resolution of M, i.e.

$$0 \longrightarrow M \longrightarrow \mathcal{I}^0 \longrightarrow \mathcal{I}^1 \longrightarrow,$$

in short $0 \to M \to I^\bullet$. Then our lemma tells us

$$H^\nu(\Gamma, M) \simeq H^\nu((\mathcal{I}^\bullet)^\Gamma) = \frac{\ker((\mathcal{I}^\nu)^\Gamma \to (\mathcal{I}^{\nu+1})^\Gamma)}{\mathrm{im}((\mathcal{I}^{\nu-1})^\Gamma \to (\mathcal{I}^\nu)^\Gamma)},$$

and in some sense that has to be explained and justified. We take this as definition of the cohomology.

Of course we have to investigate how these cohomology groups depend on the injective resolution and we have to show functoriality.

3.4 Functoriality

If we have two Γ-modules M, N and a $\varphi \in \mathrm{Hom}_{\mathbf{Mod}_\Gamma}(M, N)$, and if in additon we have chosen two injective resolutions $0 \to M \to I^\bullet$ and $0 \to N \to J^\bullet$, then I claim that we can extend the map φ to a map between the complexes

$$\begin{array}{ccccccccc} 0 & \to & M & \xrightarrow{i} & \mathcal{I}^0 & \to & \mathcal{I}^1 & \to & \dots \\ & & \downarrow\varphi & & \downarrow & & \downarrow & & \\ 0 & \to & N & \xrightarrow{j} & \mathcal{J}^0 & \to & \mathcal{J}^1 & \to & \dots \end{array}$$

The existence of this extension is proved by induction on the degree. To get the first arrow $\varphi^0 : \mathcal{I}^0 \to \mathcal{J}^0$ we apply the defining property of injective modules to get the arrow φ^0 in the diagram

$$\begin{array}{ccc} M & \xrightarrow{i} & \mathcal{I}^0 \\ j \circ \varphi \Big\downarrow & \swarrow \varphi^0 & \\ \mathcal{J}^0 & & \end{array} .$$

Then we construct φ^1 by the same principle and it is quite clear that at any step the existence of the vertical arrow follows directly from the defining property of injective modules (we only need that the $\mathcal{J}^\nu$ are injective). This extension $\varphi^\bullet : \mathcal{I}^\bullet \to \mathcal{J}^\bullet$ induces of course a map between the cohomology group

$$H^\bullet(\varphi^\bullet) : H^\bullet((\mathcal{I}^\bullet)^\Gamma) \longrightarrow H^\bullet((\mathcal{J}^\bullet)^\Gamma).$$

Now we have to worry what happens if we take two different extensions $\varphi^\bullet, \tilde{\varphi}^\bullet$ of our map φ. I want to show that these two extensions induce the same map on the cohomology. To see this we can easily reduce to the case where $\varphi = 0$, and where $\varphi^\bullet$ is an arbitrary extension of $\varphi = 0$. Then I have to show that $\varphi^\bullet$ induces the zero map on the cohomology.

I prove this by showing that under this assumption the map

$$\varphi^\bullet : \mathcal{I}^\bullet \to \mathcal{J}^\bullet$$

is actually *homotopic to zero*. This means that we can construct maps $h^\nu : \mathcal{I}^\nu \to \mathcal{J}^{\nu-1}$ ($h^0 = 0$) such that

$$\varphi^\nu = d \circ h^\nu + h^{\nu+1} \circ d$$

To construct h^1 we observe that our assumption $\varphi = 0$ implies that the kernel of $\mathcal{I}^0 \to \mathcal{I}^1$ is contained in the kernel of the vertical arrow $\mathcal{I}^0 \to \mathcal{J}^0$. Since $\mathcal{J}^0$ is injective we can construct $h^1 : \mathcal{I}^1 \to J^0$ which produces a commutative diagram

$$\begin{array}{ccc} \mathcal{I}^0 & \to & \mathcal{I}^1 \\ \downarrow & \swarrow h^1 & \\ \mathcal{J}^0 & & \end{array} .$$

Now we modify the given vertical arrow $\mathcal{I}^1 \to \mathcal{J}^1$ by subtracting the composition of h^1 and the horizontal arrow $\mathcal{I}^0 \to \mathcal{J}^1$. To this modified arrow we can apply the previous argument and it becomes clear how to construct these h^ν by induction. Again the existence of such an h^ν in any degree follows from the injectivity of the $\mathcal{J}^{\nu-1}$ and the construction of the previous ones. But if we now apply our functor (invariants under Γ) we get

$$\begin{array}{ccccccccc} 0 & \to & (\mathcal{I}^0)^\Gamma & \to & (\mathcal{I}^1)^\Gamma & \to & (\mathcal{I}^2)^\Gamma & \to & \dots \\ & h^0 \swarrow & \downarrow \varphi^0 & h^1 \swarrow & \downarrow \varphi^1 & h^2 \swarrow & \downarrow \varphi^2 & & \\ 0 & \to & (\mathcal{J}^0)^\Gamma & \to & (\mathcal{J}^1)^\Gamma & \to & (\mathcal{J}^2)^\Gamma & \to & \dots \end{array}$$

(We should have written $\varphi^{\bullet\Gamma}, h^{\bullet\Gamma}$ to be absolutely correct.)

But now it is clear that $\varphi^\bullet$ induces zero in the cohomology. If we have a cycle $c_\nu \in (\mathcal{I}^\nu)^\Gamma$ representing a given cohomology class then $\varphi^\nu(c_\nu) = d \circ h(c_\nu)$ and hence it represents the trivial class.

If we apply this to a module M, the identity and two different resolutions, then we find that we get a unique isomorphism between the resulting cohomology groups. In this sense the cohomology groups do not depend on the chosen resolution.

We also get functoriality of the cohomology groups by the same argument.

3.5 Other resolutions

If we start from an arbitrary resolution of our module M say $0 \to X^0 \to X^1 \to \dots$ and if we also choose an injective resolution as above then our considerations in 1.3.4. show that we can construct a morphism of complexes of Γ-modules

$$\begin{array}{ccccccccc} 0 & \to & M & \xrightarrow{i} & X^0 & \to & X^1 & \to & \dots \\ & & \downarrow \varphi & & \downarrow & & \downarrow & & \\ 0 & \to & N & \xrightarrow{j} & \mathcal{J}^0 & \to & \mathcal{J}^1 & \to & \dots \end{array}$$

because we only need the injectivity of the $\mathcal{J}^\bullet$. Therefore we get a canonical homomorphism

$$H^\bullet((X^\bullet)^\Gamma) \to H^\bullet((\mathcal{J}^\bullet)^\Gamma) = H^\bullet(\Gamma, M)$$

Our starting principle in 3.1. says that this homomorphism will be an isomorphism if the modules X^ν are acyclic.

Now we want to show that we get a long exact sequence in the derived functors if we start from a short exact sequence

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0.$$

3.6 Injective resolutions of short exact sequences:

We write our short exact sequence vertically and choose injective resolutions of the two modules M', M'' which we write horizontically. Imagine we have done this. Then we can write the direct sum in the middle and we get short vertical exact sequences. It will look like this:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & M' & \to & \mathcal{I}'^0 & \to & \mathcal{I}'^1 & \to & \ldots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & M & & \mathcal{I}'^0 \oplus \mathcal{I}''^0 & & \mathcal{I}' \oplus \mathcal{I}''^1 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & M'' & \to & \mathcal{I}''^0 & \to & \mathcal{I}''^1 & \to & \ldots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

The horizontal arrows in the middle are still missing. Now the injectivity of $\mathcal{I}'^0$ allows an arrow from M to $\mathcal{I}'^0$ which yields a commutative diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & \\
0 \arrow[r] & M' \arrow[r] \arrow[d] & \mathcal{I}'^0 \\
 & M \arrow[ur] &
\end{tikzcd}$$

This and the map from $M'' \to \mathcal{I}''^0$ gives us an injection

$$0 \longrightarrow M \longrightarrow \mathcal{I}'^0 \oplus \mathcal{I}''^0,$$

which we can fill into the diagram above.

This yields a diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & M' & \to & I'^0 & \to & U & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & M & \to & I'^0 \oplus I''^0 & \to & V & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & M'' & \to & I''^0 & \to & W & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

We have $U \hookrightarrow I'^1$ and $W \hookrightarrow I''^1$ and again we construct as before an arrow

$$V \hookrightarrow I'^1 \oplus I''^1.$$

This goes on forever and we get a resolution of the exact sequence 3.6.1.
Now one has to be aware that in general the homomorphisms of

$$I'^\nu \oplus I''^\nu \longrightarrow I'^{\nu+1} \oplus I''^{\nu+1}$$

are **not** the direct sum of the two homomorphisms which are already given by the resolution of the extreme modules. The construction gives that we have to add a homomorphism

$$\Psi : I''^\nu \longrightarrow I'^{\nu+1}$$

to this direct sum.
Is we apply the functor $H^0(\Gamma)$ to these complexes. We get the diagram

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & (\mathcal{I}'^{,0})^\Gamma & \longrightarrow & (\mathcal{I}'^{,1})^\Gamma & \longrightarrow & \dots \\
 & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & (\mathcal{I}'^{,0} \oplus \mathcal{I}''^{,0})^\Gamma & \longrightarrow & (\mathcal{I}'^{,1} \oplus \mathcal{I}''^{,1})^\Gamma & \longrightarrow & \dots \\
 & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & (\mathcal{I}''^{,0})^\Gamma & \longrightarrow & (\mathcal{I}''^{,1})^\Gamma & \longrightarrow & \dots \\
 & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & &
\end{array}$$

where the horizontal complexes compute the cohomology of M', M and M'' respectively and where the vertical sequences are exact. Now it is standard in homological algebra that this leads to a long exact sequence of the cohomology groups.

We constructed a derived functor using these injective resolutions. It is universal as one sees easily from the requirement that it vanished on injective modules.

Essentially the same strategy works for the left derived functor

$$M \longrightarrow H_i(\Gamma, M) \qquad\qquad i = 1, \ldots,$$

where $H_0(\Gamma, M) = M_\Gamma$. The defining property of injective modules implied that it is always a direct summand if it sits in a bigger module. The dual notion is the notion of projective modules.

A Γ-modules P is called *projective* if for any diagram

$$\begin{array}{ccccc} M & \xrightarrow{P} & N & \longrightarrow & 0 \\ & & \uparrow i & & \\ & & P & & \end{array}$$

where the top sequence is exact we can find a map $j : P \to M$ so that $p \circ j = i$, i.e. we find

$$\begin{array}{ccccc} M & \xrightarrow{P} & N & \longrightarrow & 0 \\ & j \nwarrow & \uparrow i & & . \\ & & P & & \end{array}$$

It is easily seen that free Γ-modules, i.e.

$$\bigoplus_{i \in I} \mathbb{Z}[\Gamma]$$

are projective. Hence we find

i) Every Γ-modules M has a projective resolution

$$\longrightarrow P_2 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow .$$

ii) Every projective Γ-module P which is a quotient of a Γ-module X is a direct summand, i.e. the sequence

$$0 \longrightarrow Y \longrightarrow X \longrightarrow P \longrightarrow 0$$

splits.

The assertion ii) implies that the sequence

$$0 \longrightarrow Y_\Gamma \longrightarrow X_\Gamma \longrightarrow P_\Gamma \longrightarrow 0$$

is still exact. Hence we should require

$$H_i(\Gamma, P) = 0 \qquad \text{for} \qquad i = 1, \ldots .$$

Now we may apply the same argument. For a module we choose a projective resolution

$$P_\bullet \longrightarrow M \longrightarrow 0$$

and put

$$H_\bullet(\Gamma, M) = H_\bullet((P_\bullet)_\Gamma).$$

The same arguments as before show that this gives a universal left derived functor for the functor

$$M \longrightarrow M_\Gamma.$$

All arrows are reversed.

II.4 The functors Ext and Tor:

We may look at our previous constructions from a slightly more general point of view. The category of Γ-modules is the same as the category of $R = \mathbb{Z}[\Gamma]$-modules where R is the group ring. We now consider the category $\mathbf{Mod}_R$ of modules over an arbitrary ring R.

To any pair of modules $M, N \in \mathrm{Ob}(\mathbf{Mod}_R)$ we can introduce the abelian group

$$\mathrm{Hom}_R(N, M) = \left\{ \varphi : N \to M \,\middle|\, \begin{array}{l} \varphi \text{ homomorphism of abelian groups} \\ \varphi(rm) = r\varphi(m) \text{ for all } r, m, \end{array} \right\}$$

and if we fix N then this becomes a functor from the category $\mathbf{Mod}_R$ to the category of abelian groups.

But we may also fix M and vary N, then our functor becomes a contravariant functor:

A R-module homomorphism

$$\Psi : N_1 \longrightarrow N_2$$

induces a map

$$\mathrm{Hom}_R(N_2, M) \longrightarrow \mathrm{Hom}_R(N_1, M)$$

by composition. (So far all our functors were covariant.)

It is quite clear that the covariant functor

$$M \longrightarrow \mathrm{Hom}_R(N, M)$$

is left exact and that the contravariant functor

$$N \longrightarrow \mathrm{Hom}_R(N, M)$$

is right exact.

For a given N we may try to construct the right derived functor to $M \longrightarrow \mathrm{Hom}_R(N,M)$ and for given M we may try to construct a left derived functor to the functor $N \longrightarrow \mathrm{Hom}_R(N,M)$. The same principles as before tell us that the right derived functor should be zero on injective modules (same definition), and the left derived functor should be trivial on projective modules. Hence we choose an injective resolution of

$$0 \longrightarrow M \longrightarrow I^\bullet$$

and define the right derived functor by

$$R\,\mathrm{Ext}_R^\bullet(N,M) = H^\bullet(\mathrm{Hom}_R(N,I^\bullet)).$$

I say againwhat this means: For a fixed R-module N and any exact sequence of R modules

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0,$$

we get a long exact sequence

$$0 \to \mathrm{Hom}_R(N,M') \to \mathrm{Hom}_R(N,M) \to \mathrm{Hom}_R(N,M'') \to R\,\mathrm{Ext}_R^1(N,M') \to R\,\mathrm{Ext}_R^1(N,M) \to$$

But by construction these groups $R\,\mathrm{Ext}_R^i(N,M)$ are also functorial in N if we fix M, the functors $N \to \mathrm{Ext}^i(N,M)$ are contravariant.
Analogously we choose a projective resolution

$$P_\bullet \longrightarrow N \longrightarrow 0$$

and define

$$L\,\mathrm{Ext}_R^\bullet(N,M) = H_\bullet(\mathrm{Hom}_R(P_\bullet,M)).$$

Our previous arguments show that we get a left derived functor which has all functorial properties.
It is clear that the functors

$$M \longrightarrow \mathrm{Hom}_R(P,M)\ ,\ \text{resp.}\quad N \longrightarrow \mathrm{Hom}_R(N,I)$$

are exact if P is projective (resp. I)is injective. Hence we have in this case

$$R\,\mathrm{Ext}_R^i(P,M) = 0\ ,\ L\,\mathrm{Ext}_R^i(N,I).$$

We will indicate briefly that this implies that we have a functorial isomorphism

$$L\,\mathrm{Ext}_R^i(N,M) \simeq R\,\mathrm{Ext}_R^i(N,M).$$

To see this we choose two resolutions:

$$0 \longrightarrow M \longrightarrow I^\bullet \quad , \quad P_\bullet \longrightarrow N \longrightarrow 0,$$

and we form the double complex

$$\mathrm{Hom}_R(P_\bullet \to N \to 0, 0 \to M \to I^\bullet)$$

which in full looks like

$$\begin{array}{ccccccccc}
 & & \uparrow & & \uparrow & & & & \\
 & \longrightarrow & \mathrm{Hom}_R(P_1, M) & \longrightarrow & \mathrm{Hom}_R(P_1, I^0) & \longrightarrow & \mathrm{Hom}_R(P_1, I^1) & \longrightarrow & \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \mathrm{Hom}_R(P_0, M) & \longrightarrow & \mathrm{Hom}_R(P_0, I^0) & \longrightarrow & \mathrm{Hom}_R(P_0, I^1) & \longrightarrow & \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \mathrm{Hom}_R(N, M) & \longrightarrow & \mathrm{Hom}_R(N, I^0) & \longrightarrow & \mathrm{Hom}_R(N, I^1) & \longrightarrow & \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

Now the first vertical complex computes the $L\,\mathrm{Ext}^\bullet_R(P, M)$ and the horizontal complex at the bottom computes $R\,\mathrm{Ext}^\bullet_R(P, M)$. All other vertical or horizontal complexes are exact. Then a simple diagram chase shows that the cohomology of the bottom horizontal complex and the first vertical complex are isomorphic.

We summarize

4.1 *We have defined a functor in two variables* $\mathrm{Ext}^\bullet_R(N, M)$ *which can be be computed from an an injective resolution of* M *or a projective resolution of* N.

The higher $\mathrm{Ext}^i_R(N, M)$ *for* $i > 0$ *vanish if* M *is injective or if* N *is projective.*

4.2 The derived functor for the tensor product

Another functor in two variables is given by the tensor product. Here we have to be a little bit carefull in case that our ring R is not commutative. We consider the categories $\mathbf{Mod^L}_R, \mathbf{Mod^R}_R$ of left and right R-modules. The tensor product of a right R-module N and a left R-module M is an abelian group $N \otimes_R M$ together with a map

$$\begin{aligned} \Psi : N \times M &\longrightarrow N \otimes_R M \\ \Psi : (n, m) &\longmapsto n \otimes m \end{aligned}$$

which has the following properties

(i) It is a biadditive, i.e.

$$\begin{aligned} \Psi(n_1 + n_2, m) &= (n_1 + n_2) \otimes m = n_1 \otimes m + n_2 \otimes m \\ \Psi(n + m_1 + m_2) &= m \otimes (m_1 + m_2) = n \otimes m_2 + n \otimes m_2 \end{aligned}.$$

(ii) For all $r \in R, n \in N, m \in M$ we have

$$nr \otimes m = n \otimes rm.$$

(This is the moment where we need that N is a right R-module and M is a left R-module).
(iii) This map is universal: If we have another

$$\Psi' : N \times M \to X$$

with an abelian group X which satisfies (i) and (ii) then we can find a $\varphi : N \otimes_R M \to X$ such that $\Psi' = \varphi \circ \Psi$.

It is easy to construct $N \otimes_R M$, we form the free abelian group which is generated by pairs $(n, m) \in N \times M$ and divide by the subgroup generated by elements of the form

$$(n_1 + n_2, m) - (n_1, m) - (n_2, m)$$
$$(n, m_1 + m_2) - (n, m_1) - (n, m_2)$$

and

$$(nr, m) - (n, rm).$$

If our ring R is commutative then we can give $N \otimes_R M$ the structure of an R-module: We simply define

$$r(n \otimes m) = nr \otimes m = n \otimes rm.$$

In this case of a commutative ring R we can assume that both variables N, M are left R-modules.

It is a right exact functor but in general it is not exact. This means that for a short exact sequence $0 \to M' \to M \to M'' \to 0$ the sequence

$$N \otimes_R M' \to N \otimes_R M \to N \otimes_R M'' \to 0$$

will be exact but the first arrow on the left will not be injective in general. We leave it as an exercise to the reader to verify the right exactness. In the section on flat morphisms of schemes we will discuss some examples which explain these phenomena (AS. xxx). But if the module M'' is projective then the sequence stays exact if we tensorize by any N because the sequence can be split.

This allows us to construct the derived functor. We work with a projective resolution

$$P_\bullet \longrightarrow M \to 0,$$

to define

$$R\,\mathrm{Tor}_\bullet^R(N, M) = H_\bullet(N \otimes_R P_\bullet).$$

This is a universal left derived functor of our functor above, it clear that this is a functor in the two variables N, M.

We can also choose a projective resolution $Q_\bullet \to N \to 0$ define the functor $L\,\mathrm{Tor}_\bullet^R(N, M) = H_\bullet(Q_\bullet \otimes_R M)$. Again it is not so difficult to prove that these two functors are indeed equivalent. To see this we consider the double complex defined by the two resolutions and the vertical and horizontal subcomplexes are acyclic in the "interior".

Again we summarize

4.3.1 *Hence we can define a functor in two variables* $\mathrm{Tor}_\bullet^R(N,M)$ *which can be computed by a projective resolution of* N *or a projective resolution of* M.

The higher $\mathrm{Tor}_i^R(N,M)$ *vanish for* $i>0$ *if one of the entries is a projective module.*

A left R-module M is called *flat* if the functor $N \to N \otimes_R M$ is exact. The following is obvious

4.3.2 *The left* R*-module* M *is flat if and only if* $\mathrm{Tor}_i^R(N,M)=0$ *for all* $i>0$ *and all right* R*-modules* N .

The functors cohomology and homology of groups are special cases of $\mathrm{Ext}^\bullet$ and $\mathrm{Tor}^\bullet$. We take for our ring the group ring

$$\mathbb{Z}[\Gamma],$$

and we observe: If $\mathbb{Z}$ is the abelian group $\mathbb{Z}$ with trivial Γ-action then

$$\mathrm{Hom}_{\mathbb{Z}[\Gamma]}(\mathbb{Z},M)=M^\Gamma,$$

and hence we see

$$\mathrm{Ext}^i_{\mathbb{Z}[\Gamma]}(\mathbb{Z},M)=H^i(\Gamma,M)$$

and

$$M\otimes_{\mathbb{Z}[\Gamma]}\mathbb{Z}=M_\Gamma,$$

hence

$$\mathrm{Tor}_\bullet^R(M,\mathbb{Z})=H_\bullet(\Gamma,M).$$

We conclude the section with some extra remarks and some exercises.

First we observe that we can compute the cohomology of a group also from a projective resolution

$$\begin{array}{ccccccc} \longrightarrow P_2 \longrightarrow P_1 \longrightarrow & \mathbb{Z}[\Gamma] & \longrightarrow \mathbb{Z} \longrightarrow 0 \\ & " & \\ & P_0 & . \end{array}$$

Hence we get

$$H^i(\Gamma,M)=H^i(\mathrm{Hom}_\Gamma(P_\bullet,\mathbb{Z})),$$

4.4 Exercise:

(i) Apply this to the case of a cyclic group $\Gamma=\mathbb{Z}/n\mathbb{Z}$. *Let* σ *be a generator of the group. We have the exact sequence*

$$0\longrightarrow I_\Gamma \longrightarrow \mathbb{Z}[\Gamma]\longrightarrow \mathbb{Z}\longrightarrow 0$$

and $I_\Gamma=\mathbb{Z}[\Gamma](1-\sigma)$.

ia) In the case $n=0$ *(the infinite group) we have that* I_Γ *is a free module. This gives simple formulae for the cohomology and shows* $H^\nu(\Gamma,M)=0$ *for* $\nu\ge 2$.

ib) In the case of a finite group the map

$$\begin{array}{ccc} \mathbb{Z}[\Gamma] & \longrightarrow & I_\Gamma \\ r & \longrightarrow & r(1-\sigma) \end{array}$$

has the kernel $\mathbb{Z}[\Gamma](1+\ldots+\sigma^{n-1})$. *Construct a "periodic" resolution from this and compute the cohomology.*

(ii) We still have to compare our provisorial cohomology groups $\check{H}^1(\Gamma, M)$ *and the new ones. This is not so difficult.*

a) We observe that our new cohomology groups obviously satisfy: For a subgroup $\Gamma' \subset \Gamma$ *and a* Γ'*-module*

$$H^i(\Gamma, \operatorname{Ind}_{\Gamma'}^{\Gamma} Y) = H^i(\Gamma', Y).$$

(Choose an injective resolution of the Γ'*-modules* Y *and ..... .)*

b) Hence we have $H^1(\Gamma, \operatorname{Ind}_1^\Gamma M) = 0$. *We constructed the sequence*

$$0 \longrightarrow M \longrightarrow \operatorname{Ind}_1^\Gamma M \longrightarrow (\operatorname{Ind}_1^\Gamma M)/M \longrightarrow 0,$$

and we find

$$\Big((\operatorname{Ind}_1^\Gamma M)/M\Big)^\Gamma / (\operatorname{Ind}_1^\Gamma M)^\Gamma \simeq H^1(\Gamma, M),$$

but this was also the formula for the $\check{H}^1$.

(iii) Let us consider the ring $R = k[X]/(X^2)$ *where* k *is any field. Then the category of* R*-modules is the same as the category of* k*-vector spaces* V *together with an* k*-linear endomorphism* $\alpha : V \to V$ *which satisfies* $\alpha^2 = 0$. *If* $\dim_k V = 1$, *then* α *must be zero. Compute*

$$\operatorname{Ext}_R^1(k, k).$$

Does this ring a bell?

Chapter III

Presheaves and Sheaves

III. 1. Presheaves

We start from a topological space X and we define the category $\mathbf{Off}(X)$ of open sets: The objects are the open sets $U, V \subset X$ and the morphisms

$$\mathrm{Hom}_{\mathbf{Off}(X)}(V,U) = \begin{cases} \emptyset & \text{if} \quad V \not\subset U \\ \{i\} & i \text{ is the inclusion if} \quad V \subset U. \end{cases}$$

1.1. Definition : *A presheaf on X with values in a category $\mathcal{C}$ is a contravariant functor from the category $\mathbf{Off}(X)$ with values in the category $\mathcal{C}$.*

We say again what this means: To any open set $U \subset X$ our presheaf $\mathcal{F}$ associates an object $\mathcal{F}(U) \in \mathrm{Ob}(\mathcal{C})$. Whenever we have an inclusion

$$V \xrightarrow{i} U$$

we get a so-called *restriction morphism*

$$r_{U|V} : \mathcal{F}(U) \longrightarrow \mathcal{F}(V).$$

Of course we have $r_{U|U} = \mathrm{Id}$ and for $V_1 \subset V_2 \subset U$ we get a commutative diagram

$$\begin{tikzcd} \mathcal{F}(U) \arrow[rr, "r_{U|V_2}"] \arrow[dr, "r_{U|V_1}"'] & & \mathcal{F}(V_2) \arrow[dl, "r_{V_2|V_1}"] \\ & \mathcal{F}(V_1) & \end{tikzcd}$$

which can be written awkwardly

$$r_{U|V_1} = r_{V_2|V_1} \circ r_{U|V_2}. \qquad (Sh0)$$

If this functor $\mathcal{F}$ takes values in the category $\mathbf{Ab}$ of abelian groups (, rings, modules over a ring, vector spaces, sets) we call it a presheaf of abelian groups (, rings, modules over a ring, vector spaces, sets). For us the target category will always be of one these simpler categories. This means that the objects $\mathcal{F}(U)$ will be sets equipped with some kind of additional structure and the morphisms will be maps which respect this additional structure.

Under this assumption we we know what the elements in $\mathcal{F}(U)$ are they will be called the *sections* of $\mathcal{F}$ over U.

Sometimes it is a nagging question what $\mathcal{F}(\emptyset)$ should be. Usually we can can take for $\mathcal{F}(\emptyset)$ a so called *final object* in the category, this is an object Ω such that for any other object $X \in \mathrm{Ob}(\mathcal{C})$ we have exactly one morphism from X to Ω. For the category of sets we can take any set consisting of just one element and for the category $\mathbf{Mod}_R$ we can take the zero module.

It is clear that presheaves with values in a given category $\mathcal{C}$ on X form a category $\mathcal{PS}_X$ by themselves. If we have two presheaves $\mathcal{F}, \mathcal{G}$ a morphism

$$\Psi \in \mathrm{Hom}_{\mathcal{PS}_X}(\mathcal{F}, \mathcal{G})$$

is a collection of morphisms

$$\Psi_U : \mathcal{F}(U) \longrightarrow \mathcal{G}(U)$$

which satisfies the obvious rule of consistency: whenever we have $V \subset U$ we get a commutative diagram

$$\begin{array}{ccccc} \Psi_U & : & \mathcal{F}(U) & \longrightarrow & \mathcal{G}(U) \\ & & \Big\downarrow r_{U|V} & & \Big\downarrow r_{U|V} \\ \Psi_V & : & \mathcal{F}(V) & \longrightarrow & \mathcal{G}(V). \end{array}$$

(If we were pedantic, we should also denote the $r_{U|V}$ differently ($r^{\mathcal{F}}_{U|V}$ or so).)

The category of presheaves (in a suitable target category $\mathcal{C}$) contains a (so called “full”) subcategory, this is the category of sheaves. Before I can define sheaves I need:

1.1.1 A remark about products

Let us assume we have two indexing sets I, J and two families of objects $\{X_i\}_{i\in I}$, $\{Y_j\}_{j\in J}$ in a category with products. Assume that we have a map

$$\tau : J \longrightarrow I$$

and in addition that for every $j \in J$ we have a morphism

$$f(j) : X_{\tau(j)} \longrightarrow Y_j.$$

Then we get for $j \in J$ a composition morphism

$$f(j) \circ p(\tau(j)) : \prod_{i\in I} X_i \to Y_j$$

It is the definition of the product that this gives us a unique morphism

$$f_\tau : \prod_{i\in I} X_i \longrightarrow \prod_{j\in J} Y_j$$

which produces a commutative diagram

$$\begin{array}{ccc} \prod_{i\in I} X_i & \xrightarrow{f_\tau} & \prod_{j\in J} Y_j \\ \downarrow & & \downarrow \\ X_{\tau(j)} & \xrightarrow{f(j)} & Y_j. \end{array}$$

Hence morphisms from one product into another product can be obtained from maps between the indexing sets in the opposite direction and morphisms between the objects indexed by indices related by this map. We say that this arrow is *induced by the maps between the indexing sets and the maps between the objects.*

Now we explain the extra condition a presheaf has to satisfy if it wants to be a sheaf. We need that the target category $\mathcal{C}$ has products. For our purpose it is good enough if it is a category of rings or a category of modules.

Let $\mathcal{F}$ be a $\mathcal{C}$-valued presheaf on our space X. Let $U \subset X$ be open, let $U = \cup_{\alpha\in A} U_\alpha$ be an open covering. Then we get a diagram of maps

$$\mathcal{F}(U) \xrightarrow{p_0} \prod_{\alpha\in A} \mathcal{F}(U_\alpha) \underset{p_2}{\overset{p_1}{\rightrightarrows}} \prod_{(\alpha,\beta)\in A\times A} \mathcal{F}(U_\alpha \cap U_\beta),$$

where the arrows are given as follows: The arrow p_0 is induced by the maps

$$\mathcal{F}(U) \xrightarrow{r_{U|U_\alpha}} \mathcal{F}(U_\alpha)$$

and p_1, p_2 are induced in the above sense by the two projections

$$A \times A \rightrightarrows A$$

$$(\alpha, \beta) \longmapsto \alpha$$
$$(\alpha, \beta) \longmapsto \beta$$

and the restriction maps $\mathcal{F}(U_\alpha) \to \mathcal{F}(U_\alpha \cap U_\beta)$. If we assume that our target category is the category of sets, (abelian) groups, rings ... where the product is the simple-minded product then we can see what happens to $s \in \mathcal{F}(U)$: It is mapped to

$$\left(\ldots, r_{U|U_\alpha}(s), \ldots\right)_{\alpha\in A.}$$

A section

$$(\ldots s_\alpha \ldots)_{\alpha\in A} \in \prod_{\alpha\in A} \mathcal{F}(U_\alpha)$$

is mapped to

$$\left(\ldots, r_{U_\alpha|U_\alpha\cap U_\beta}(s_\alpha) \ldots\right)_{(\alpha,\beta)\in A\times A}$$

under p_1 and to

$$(\dots, r_{U_\beta|U_\alpha\cap U_\beta}(s_\beta), \dots)_{(\alpha,\beta)\in A\times A}$$

under p_2.

In any case it is clear from $(Sh0)$ that the first arrow "equalizes" the two arrows p_1, p_2. This means that $p_1 \circ p = p_2 \circ p$.

Now we are ready to state the condition a sheaf has to satisfy. For simplicity we assume that our target category is one of the simple ones above.

1.2 Definition *A presheaf $\mathcal{F}$ is a sheaf if and only if*

(Sh1) The arrow p_0 is injective.

(Sh2) The image of p_0 is exactly the set of those elements where p_1, p_2 take the same values.

We summarize the two conditions into

$$\mathcal{F}(U) \xrightarrow{\sim} \Big(\prod_{\alpha\in A} \mathcal{F}(U_\alpha) \underset{p_2}{\overset{p_1}{\rightrightarrows}} \prod_{(\alpha,\beta)\in A\times A} \mathcal{F}(U_\alpha \cap U_\beta)\Big)[p_1 = p_2].$$

We will say that the above sequence is an *exact sequence of sets.*

Comment: In the case of an abstract target category $\mathcal{C}$ we would have to explain what injectivity of p_0 means and how we define the object $[p_1 = p_2]$ for a pair of morphisms $A \underset{p_2}{\overset{p_1}{\rightrightarrows}} B$. This is actually not so difficult.

Now we fix a target category $\mathcal{C}$. The sheaves with values in $\mathcal{C}$ form a "full" subcategory $\mathcal{S}_X$ of the category of presheaves with values in $\mathcal{C}$. This means that each sheaf is also a presheaf and for any two sheaves $\mathcal{F}, \mathcal{G}$ on X we have that the sets of morphisms in the category of sheaves and in the category of presheaves are the same, i.e.

$$\mathrm{Hom}_{\mathcal{PS}_X}(\mathcal{F}, \mathcal{G}) = \mathrm{Hom}_{\mathcal{S}_X}(\mathcal{F}, \mathcal{G})$$

1.3. Examples:

On any topological space X we have the sheaf $\mathcal{C}_0$ of continuous $\mathbb{R}$ or $\mathbb{C}$-valued functions: For any open set $U \subset X$ we put $\mathcal{C}_0(U)$ = ring of real or complex valued continuous functions on U. The restriction maps are given by the restriction of functions. The properties $(Sh1), (Sh2)$ are obvious because the continuity of a function can be checked locally.

We can also define the sheaf

$$U \to \mathbb{Z}_X(U) = \text{ locally constant } \mathbb{Z} \text{ -valued functions on } U.$$

Note that

$$U \to \text{ constant } \mathbb{Z} - \text{ valued functions on } U$$

would only define a presheaf because $(Sh2)$ will not be satisfied in general.

This makes it clear what the general rule is: whenever we have a class of functions defined by certain properties then they provide a sheaf if these properties can be checked locally.

Of course we can replace $\mathbb{Z}$ by any abelian group A and define the sheaf

$$U \longrightarrow A_X(U) = \text{locally constant } A\text{-valued functions on } U.$$

We may look at these sheaves from a different point of view. We can put the discrete topology on A, and then we see that $A_X(U)$ is simply the abelian group of continuous functions on U with values in A. Sometimes we will write $\underline{A}$ instead of A_X.

If we have a point $p \in X$ then we can define the ring of *germs* of continuous functions in this point p. A *germ* of a continous function at p is a continuous function $f : U_p \to \mathbb{C}$ defined in an open neighborhood U_p of p modulo the following equivalence relation:

$$(f : U_p \to \mathbb{C}) \sim (g : V_p \to \mathbb{C})$$

if and only if there is a neighborhood $W_p \subset U_p \cap V_p$ of p such that and $f|W_p = g|W_p$.

It is clear that the germs form a ring which is called $\mathcal{C}_{0,X,p}$. It is clear that this ring is the direct limit

$$\varinjlim_{U \ni p} \mathcal{C}_0(U) = \mathcal{C}_{0,X,p}$$

(See I.3.4,3.5).

This ring is a *local ring* which means that it has a unique maximal ideal. This maximal ideal $\mathfrak{m}_p$ is the kernel of the evaluation at p. To see this one has to observe that a germ f which does not vanish at p also does not vanish in a small neighborhood of p and on this neighborhood we can define the continuous function $1/f$. This means that f is invertible in $\mathcal{C}_{0,X,p}$ and it follows that any ideal in $\mathcal{C}_{0,X,p}$ which is not contained in $\mathfrak{m}_p$ is the entire ring. Of course such a ring of germs is pretty big.

If we do the same thing with our sheaf $\mathbb{Z}_X$ then it is clear that a germ at p is determined by its value at p. Hence in this case the ring of germs is simply $\mathbb{Z}_{X,p} = \mathbb{Z}$. This is not a local ring.

III.2. Manifolds as locally ringed spaces

At this point I want to explain that the concept of sheaves gives us a better way to think of topological ($\mathcal{C}^0$), differentiable ($\mathcal{C}^\infty$) or complex manifolds. I hope the explanation will also be helpful for the understanding of the concept of sheaves.

A *topological manifold* X is a Hausdorff space such that for each point $p \in X$ we can find an open neighborhood U_p of p which is homeomorphic to an open set in $\mathbb{R}^n$:

$$U_p \xrightarrow{\sim} U \subset \mathbb{R}^n.$$

This is also called a $\mathcal{C}^0$-manifold, on this space we can define the sheaf $\mathcal{C}_{0,X}$ of germs of continuous function with values in $\mathbb{R}$ or $\mathbb{C}$.

A non-trivial theorem in algebraic topology asserts that two non-empty open sets $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^m$ can only be homeomorphic if $n = m$. (See Chap IV.7.3) This allows us to speak of the dimension of the topological manifold provided it is connected.

A $\mathcal{C}^\infty$*-manifold of dimension* n is a topological manifold X together with a $\mathcal{C}^\infty$-atlas: This is a family

$$\{V_\alpha, u_\alpha\}_{\alpha \in A}$$

of open subsets such that

(i) $X = \bigcup_{\alpha \in A} V_\alpha$

(ii) The u_α are homeomorphisms

$$u_\alpha : V_\alpha \xrightarrow{\sim} V'_\alpha$$

where the V'_α are open subsets in $\mathbb{R}^n$.

(iii) If $V_\alpha \cap V_\beta \neq \emptyset$ then we get a diagram

$$\begin{array}{ccc} & & u_\alpha(V_\alpha \cap V_\beta) \;\; \subset V'_\alpha \\ & \overset{u_\alpha}{\nearrow} & \\ V_\alpha \cap V_\beta & & u_{\alpha\beta} \downarrow \;\; \uparrow u_{\beta\alpha} \\ & \overset{u_\beta}{\searrow} & \\ & & u_\beta(V_\alpha \cap V_\beta) \subset \;\; V'_\beta \end{array}$$

and we demand that $u_{\alpha\beta}, u_{\beta\alpha}$ are $\mathcal{C}^\infty$-maps. The maps $u_\alpha : V_\alpha \xrightarrow{\sim} V'_\alpha$ are called the local charts of the atlas. In this case it is easier to see that the dimension is well defined.

We may define a *complex manifold of dimension* n in the same way if we demand that the V'_α are open in $\mathbb{C}^n$ and the $u_{\alpha\beta}, u_{\beta\alpha}$ are holomorphic maps. Of course it is clear that a complex manifold of dimension n also carries a structure of a $\mathcal{C}^\infty$ manifold of dimsion $2n$.

Once we have the notion of $\mathcal{C}^\infty$-manifold (resp. complex manifold) we may define the sheaves of germs of $\mathcal{C}^\infty-$ (resp. holomorphic) functions:

For $U \subset X$ and $f : U \to \mathbb{C}$, we say that f is $\mathcal{C}^\infty$ (resp. holomorphic) if for any $p \in U$ and any V_α with $p \in V_\alpha$ the map

$$\tilde{f}_\alpha : f \circ u_\alpha^{-1} : u_\alpha^{-1}((V_\alpha \cap U)) \longrightarrow \mathbb{C}$$

is $\mathcal{C}^\infty$ (resp. holomorphic).

Let us denote these sheaves by $\mathcal{C}^\infty_X$ and $\mathcal{O}_X$ respectively.

After defining a $\mathcal{C}^\infty$ (resp. complex) manifold in the way we did it there is still a lot of talking about how to compare different atlases, what are equivalence classes of atlases what are maximal atlases and so on.

I want to explain that these concepts of manifolds become much clearer if we follow Grothendieck and introduce the concept of *locally ringed spaces.*

With our definition we know what it means that a map

$$h : X \to Y$$

between two $\mathcal{C}^\infty$ (resp. complex) manifolds is a $\mathcal{C}^\infty$ (resp. holomorphic) map. Such a map should be continuous and then we use the atlases to formulate what else should be true, namely that the maps induced by the local charts should be $\mathcal{C}^\infty$ (resp. holomorphic).

But we see that there is a different way of formulating that h is $\mathcal{C}^\infty$ (resp. holomorphic): Whenever we have open sets $U \subset X, V \subset Y$ such that $h(U) \subset V$, i.e.

$$h : U \to V$$

and a section $f \in \mathcal{C}_X^\infty(V)$ (resp. $f \in \mathcal{O}_X(V)$) then the composite $f \circ h$ is certainly continuous. It is not hard to check, that our map is $\mathcal{C}^\infty$ (resp. holomorphic) if and only if for any such pair U, V and any f the composite map $f \circ h$ is again $\mathcal{C}^\infty$ (resp. holomorphic), i.e. we get a map

$$\circ h : \mathcal{C}_Y^\infty(V) \longrightarrow \mathcal{C}_X^\infty(U)$$

resp.

$$\circ h : \mathcal{O}_Y(V) \longrightarrow \mathcal{O}_X(U).$$

A better formulation is obtained if we introduce the sheaf (see the following sections on f_*, f^* and the adjointness formula) $h^*(\mathcal{C}_Y^\infty)$ on X: For any open subset $U \subset X$ the space of section

$$h^*(\mathcal{C}_Y^\infty)(U)$$

consists functions $f : U \to \mathbb{C}$ which have the following property: For any point $p \in U$ we can find a neighborhood U_p of p and an open set $V_{h(p)} \subset Y$ such that

$$h(U_p) \subset V_{h(p)}$$

and we can find a section $\tilde{f} \in \mathcal{C}_X^\infty(V_{h(p)})$ so that

$$f = \tilde{f} \circ h.$$

Then we can say that a map $h : X \to Y$ is $\mathcal{C}^\infty$ (resp. holomorphic) if it is continuous and induces a map

$$\circ h : h^*(\mathcal{C}_Y^\infty) \to \mathcal{C}_X^\infty$$

resp.

$$\circ h : h^*(\mathcal{O}_Y) \to \mathcal{O}_X.$$

Of course the composition with h always induces a map

$$h^*(\mathcal{C}_Y^0) \to \mathcal{C}_X^0$$

between the sheaves of continuous functions. A $\mathcal{C}^\infty$ or holomorphic map h has to respect the distinguished subsheaves which have been defined using the atlases.

Now we come to the decisive point, we turn the whole thing around and say:

A $\mathcal{C}^\infty$- (resp. complex) manifold is a topological space X together with a subsheaf $\mathcal{C}_X^\infty$ (resp. $\mathcal{O}_X$) in the sheaf of continuous functions such that for any point $p \in X$ we have a neighborhood U_p of p and a homeomorphism h between U_p and an open subset U' of $\mathbb{R}^n$ (resp. $\mathbb{C}^n$) such that

$$(U_p, \mathcal{C}_X^\infty) \simeq (U'_p, \mathcal{C}_{U'_p}^\infty)$$

resp.

$$(U_p, \mathcal{O}_X) \simeq (U'_p, \mathcal{O}_{U'_p})$$

where U_p is open in $\mathbb{R}^n$ (resp. $\mathbb{C}^n$) and the sheaves are the $\mathcal{C}^\infty$ (resp. holomorphic) germs on U'_p and where $\simeq$ means that the composition $\circ h$ induces an isomorphism between the subsheaves.

In very simple words: A *so and so* manifold is a topological manifold on which we have a sheaf of function which locally looks like the sheaf of it so and so functions on some model. In our examples the stalks are local rings hence we get a examples of so called *locally ringed* spaces.

It is not only so that we get a much clearer concept of $\mathcal{C}^\infty$ or complex manifolds. It turns out that this concept allows generalizations to cases where we cannot work with atlases anymore. (See example 4) below and Chapter VI.) It is possible to define the category of locally ringed spaces. These are topological spaces X together with a sheaf of rings whose stalks are local rings. We will encounter these objects in Chapter VI.

I want to discuss a couple of examples and exercises.

2.1 Examples

1a): We define the structure of a complex space on the one dimensional projective space $\mathbb{P}^1(\mathbb{C})$. As a topological space this is the space of lines in $\mathbb{C}^2$ passing through the origin. This is also the space of all pairs $(z_0, z_1) \neq (0,0)$ of complex numbers divided by the equivalence relation

$$(z_0, z_1) \sim (\lambda z_0, \lambda z_1) \quad \lambda \in \mathbb{C}^*.$$

We have the two open subsets U_0 (respectively U_1) where the coordinate $z_0 \neq 0$ (respectively $z_1 \neq 0$.) On these open subsets we can normalize the non zero coordinate to one and get bijections

$$U_0 \xrightarrow{\sim} \mathbb{C}, \quad U_1 \xrightarrow{\sim} \mathbb{C},$$

$$(1, z) \mapsto z, \quad (u, 1) \mapsto u$$

Now we define the sheaf $\mathcal{O}_{\mathbb{P}^1}$: For any open subset $U \subset \mathbb{P}^1(\mathbb{C})$ the sections of $\mathcal{O}_{\mathbb{P}^1}(U)$ consist of those $\mathbb{C}$-valued functions whose restriction $U_0 \cap U$ resp. $U_1 \cap U$ are holomorphic.

1b): Of course we can define the n-dimensional projective space $\mathbb{P}^n(\mathbb{C})$. Again it is the space of lines in $\mathbb{C}^{n+1}$ passing through the origin. We can identify this to the space

$$\{(z_0, \cdots, z_n) \in \mathbb{C}^{n+1} \mid \quad \text{not all} \quad z_i = 0\}/\mathbb{C}^*$$

where $\mathbb{C}^*$ acts diagonally. We define the subset

$$U_i = \{(z_0, \cdots, z_n) \in \mathbb{C}^{n+1} \mid z_i \neq 0\}/\mathbb{C}^*$$

and identify $U_i \xrightarrow{\sim} \mathbb{C}^n$ by the map $(z_0, \cdots, z_i, \cdots, z_n) \longmapsto \left(\frac{z_0}{z_i}, \cdots, \frac{z_n}{z_i}\right)$. The sheaf of holomorphic functions on $\mathbb{P}^n(\mathbb{C})$ is the sheaf of those functions whose restriction to the U_i is holomorphic.

2): We choose a lattice

$$\Omega = \{n_1\omega_1 + n_2\omega_2 \mid n_1, n_2 \in \mathbb{Z}\}$$

in $\mathbb{C}$, where ω_1, ω_2 are linearly independent over $\mathbb{R}$. This lattice operates by translations in $\mathbb{C}$, we form the quotient $\mathbb{C}/\Omega$ as a topological space, the projection

$$\pi : \mathbb{C} \to \mathbb{C}/\Omega.$$

is locally a homeomorphism. We define

$$\mathcal{O}_{\mathbb{C}/\Omega}(U) = \{f : U \to \mathbb{C} \mid \pi^{-1}(U) \xrightarrow{f\circ\pi} \mathbb{C} \text{ is holomorphic}\}.$$

Then it is clear that this gives $\mathbb{C}/\Omega$ the structure of a complex manifold.

3): Let us assume that we have a holomorphic function $f : U \to \mathbb{C}$ where $U \subset \mathbb{C}$ is open and contains the origin. We assume $f(0) = 0$. We consider f as a germ and we assume that its n-th iteration is the identity, i.e. $f(f(\ldots f(z))\ldots) = z$. We assume the f is of exact order n, i. e. no earlier iteration gives the identity. Of course $f(z) = \zeta z + a_1 z^2 \ldots$ where $\zeta = e^{\frac{2\pi i k}{m}}$ and $(k, m) = 1$. We can find a smaller open set $D \subset U$ such that $f(D) = D$. This defines a holomorphic action of the cyclic group $G =< f^\nu >$ of order n on D and we can form the quotient under this action. This is the space $D/G = B$. Let $\pi : D \to B$ the projection map. We define a sheaf $\mathcal{O}_B$ on B : For any open set $V \subset B$ we define $\mathcal{O}_B(V)$ as the ring of holomorphic functions on the inverse image $\pi^{-1}(V) \subset D$ which are invariant under the action of the cyclic group G.

2.1.3 Exercise

Prove that this sheaf defines a structure of a one dimensional complex space on B.

Hint: Consider the special case where $U = \mathbb{C}$ and $f(z) = \zeta z$ first. Of course the problem arises only in a neighborhood of the origin 0. There the stalk of the sheaf $\mathcal{O}_B$ is ring of power series in $w = z^n$ which have a strictly positive radius of convergence. Then return to the general case and prove that you can find a germ of a function $g(z) = z + b_2 z^2 + b_3 z^3 \ldots$ such that $f(g(z)) = g(\zeta z)$ and show that this reduces the problem to the first case.

4) Let us consider $\mathbb{C}^2$ and consider the following action of our cyclic group:

$$f : (z_1, z_2) \mapsto (\zeta z_1, \zeta^{-1} z_2)$$

If we form the quotient $\pi : \mathbb{C}^2 \to \mathbb{C}^2/G = B$ we can try to play the same game. Again we get the structure of a two dimensional complex variety except at the point $\pi(0) = 0$. Here we see that the germ of our sheaf $\mathcal{O}_B$ becomes a power series ring in $u = z_1^n, v = z_2^n, w = z_1 z_2$ and we have $uv = w^n$. This means u, v, w are not independent variables anymore. At 0 our space is *singular* and not locally isomorphic to $(\mathbb{C}^2, \mathcal{O}_{\mathbb{C}^2})$. But our concepts of locally ringed spaces are strong enough to deal with this and define the more general structure of a *complex space* which may have singular points.

5a) I want to discuss an example which is much more subtle, and where I need some difficult theorems from local complex analysis. We assume that $U \subset \mathbb{C}^n$ is an open subset and $f_1(z_1, \cdots, z_n), \cdots, f_r(z_1, \cdots, z_n)$ are holomorphic functions on U. Then we can consider the ideal $I \subset \mathcal{O}_{\mathbb{C}^n}(U)$ which is generated by these functions. We can look at the subset Y of common zeroes of the f_i, i.e.

$$Y = \{\underline{z} = (z_1, \cdots, z_n) \mid f_i(\underline{z}) = 0 \text{ for all } i = 1 \cdots r\}$$

and this is of course also the set of common zeroes of all the $f \in I$.
For any open subset $V \subset Y$ we can look at the open sets $U' \subset U$ with $U' \cap Y = V$, and then we can form the quotient

$$\mathcal{O}_{\mathbb{C}^n}(U')/(f_1, \cdots, f_r)$$

where $(f_1, \cdots, f_r)$ is the $\mathcal{O}_{\mathbb{C}^n}(U')$-ideal generated by the f_i. We put

$$\mathcal{O}_Y(V) = \varinjlim_{U' : U' \cap Y = V} \mathcal{O}_{\mathbb{C}^n}(U')/(f_1, \cdots, f_r).$$

Now it is a theorem in local complex analysis that $V \to \mathcal{O}_Y(V)$ is in fact a sheaf (see ????). One checks that the stalk $\mathcal{O}_{Y,y} = \varinjlim_{V : y \in V} \mathcal{O}_Y(V)$ is a local ring and the pair

$$(Y, \mathcal{O}_Y)$$

is in fact a locally ringed space. I want to point out that we cannot interprete the rings $\mathcal{O}_Y(V)$ as rings of holomorphic functions on Y. We may for instance consider the case that $U = \mathbb{C}$, and we take the single function $f(z) = z^2$. Then $Y = \{0\}$ and the local ring is $\mathbb{C}[z]/(z^2)$. It contains nilpotent elements and cannot be interpreted as ring of holomorphic functions.
But still our space $(Y, \mathcal{O}_Y)$ is called a *complex space.*

5b) We say that our system of equations satisfies the *Jacobi criterion* in a point $y \in Y$ if the Jacobian matrix

$$\left(\frac{\partial f_i}{\partial z_j}\right)_{i,j}(y) \qquad i = 1 \cdots r, j = 1 \cdots n$$

has maximal rank r. Then this is still true in a small open neighborhood of y. The implicite function theorem says that in a small neighborhood $U_1 \subset \mathbb{C}^n$ of y we can perform a change of coordinates

$$u_i = g_i(z_1 \cdots z_n) \qquad i = 1 \cdots n$$

such that in the new coordinates our functions become $f_1(u_1 \cdots u_n) = u_1, \cdots, f_r(u_1 \cdots u_n) = u_r$. Hence we see that in this neighborhood

$$Y \cap U_1 = \{(0, \cdots, 0, u_{r+1}, \cdots, u_n) \mid u_i \quad \text{suff. small}\},$$

and then $(Y, \mathcal{O}_Y)$ is clearly an $n-r$-dimensional complex manifold in the neighborhood of $y \in Y$. In this case we do not have to invoke the above mentioned theorem.

We can turn this around and say that a subset $Y \subset U$ is a d dimensional submanifold of U if can describe it locally as the common set of zeroes of $n-d$ holomorphic functions which satisfy the Jacobi criterion.

We come back to the situation in 5a). We say that the ideal I defines a (smooth) submanifold of dimension d if the set of common zeroes Y is a submanifold of dimension d and if in addition at any point $y \in Y$ we can find $g_1, \ldots, g_{n-d} \in I$ which satisfy the Jacobi criterion at the point y. In this situation the argument in 5b) shows that these $g_1, \ldots, g_{n-d}$ generate the ideal I we we restrict it to a small neighborhood of y.

5c) A closed subset $Y \subset \mathbb{P}^n(\mathbb{C})$ is a d dimensional complex projective manifold if for any index i the intersection $Y \cap U_i$ is a d-dimensional complex submanifold of U_i.

5d) A homogenous polynomial of degree k is a polynomial

$$f(z_0, \cdots, z_n) = \sum a_{\nu_0 \cdots \nu_n} z_0^{\nu_0} \cdots z_n^{\nu_n}$$

where $a_{\nu_0 \cdots \nu_n} = 0$ unless $\sum \nu_i = k$. We cannot consider such a polynomial as a function on $\mathbb{P}^n(\mathbb{C})$. But of course it makes sense to speak of the zeroes of this polynomial on $\mathbb{P}^n(\mathbb{C})$. Therefore we may consider an ideal $I = \{f_1, \cdots, f_s\}$ which is generated by s homogenous polynomials. We can look at the common set of zeroes

$$Y = \{\underline{z} = (z_0 \cdots z_n) \mid \underline{z} \neq 0, f_i(\underline{z}) = 0\}/\mathbb{C}^*.$$

Such a set Y is called an *algebraic subset* of $\mathbb{P}^n(\mathbb{C})$.

If we restrict a homogenous polynomial f to one of the open sets U_i above, then we can interprete it as a function on U_i because we can normalize the i-th coordinate of a point to one. Hence our ideal I defines an ideal I_i of holomorphic functions on each of the U_i.

Such a subset $Y \subset \mathbb{P}^n(\mathbb{C})$ is called a *smooth projective* (algebraic) variety of dimension d if each of the ideals I_i defines a smooth submanifold of dimension d in the sense of 5b). This definition is not yet very satisfactory because it needs input from analysis (the implicit function theorem), for a definition in purely algebraic terms I refer to VII.3.

It happens the we need more than $n-d$ equations to describe a smooth projective variety of dimension d. Locally at a point y we can choose $n-d$ equations from our set of equations to describe Y but this subset may vary if the point moves around.

If we have such a complex d -dimensional submanifold $Y \subset \mathbb{C}^n$ then the coordinate functions $z_1, \ldots, z_n$ are of course holomorphic functions on $\mathbb{C}^n$ and therefore after restriction also on M. If we have a point $y \in Y$ we may consider the functions

$$\tilde{z}_i = z_i - z_i(y) \quad \text{for } i = 1, \ldots, n$$

as holomorphic functions on Y. Then it follows from 5b) that we can pick d functions from this set - let us assume that these are $\tilde{z}_1, \ldots, \tilde{z}_d$- such that locally the remaining functions can be written as convergent power series in these, i.e.

$$\tilde{z}_{d+j} = h_j(\tilde{z}_1, \ldots, z_d) \quad j = 1 \ldots n - d$$

Then the

$$\tilde{z}_i = z_i - z_i(y) \quad \text{for } i = 1, \ldots, n$$

are called a system of *local parameters at y.*

III.3. Stalks and sheafification

In our examples above we had the notion of a germ of a function at a point p. This notion can be extended to the more general classes sheaves. Let us assume that we consider the category of (pre-) sheaves on X with values in some nice category (abelian groups, rings or sets). If we have a point $p \in X$ then we consider the set $\mathfrak{U}_p$ of open sets containing our point p. We define an ordering on this set

$$V \geq U \quad \text{if and only if} \quad V \subset U. \tag{sic!}$$

Then this is an inductive system which is also directed: to any U_1, U_2 we find a V with $U_1 \leq V$, $U_2 \leq V$. If we have a (pre-)sheaf $\mathcal{F}$ on X we define the stalk in p by

$$\mathcal{F}_p = \varinjlim_{U \in \mathfrak{U}_p} \mathcal{F}(U),$$

and this limit is simply the (abelian group, ring, set) of germs of sections. It inherits the structure of an (abelian group, ring, set); this follows from this directedness and is discussed in the Exercise 4) in Chapter I.

An element $s_p \in \mathcal{F}_p$ is called a *germ of a section.* By definition it can always be represented by a section $s_U \in \mathcal{F}(U)$ where $U \in \mathfrak{U}_p$. If this is so we write $s_U|_p = s_p$ and we say that s_p is the restriction of s_U to the stalk at p.

3.1

Let s be a section over the open set U. If we have $s_p = 0$ at $p \in U$ then we find an open neighborhood V of p such that s restricted to this neighborhood is zero. Hence we can define the *support* of s: It is the closed subset of U where $s_p \neq 0$.

These stalks help to clarify the difference of the notion of sheaves and presheaves. For any presheaf we can consider the map

$$\mathcal{F}(U) \longrightarrow \prod_{p \in U} \mathcal{F}_p,$$

which is given by restricting the sections to the stalks. Then we know:

(i) This map is injective, if and only if our presheaf satisfies $(Sh1)$.

(ii) If a presheaf $\mathcal{F}$ satisfies $(Sh1)$ then it is a sheaf if and only if the following holds: A collection of germs $(\dots s_p \dots)_{p\in U}$ is the restriction of a section over U if for any p we find a $U_p \in \mathfrak{U}_p$ and a section $\tilde{s}_p \in \mathcal{F}(U_p)$ such that $\tilde{s}_p|_q = s_q$ for all $q \in U_p$.

We leave the verification of this fact to the reader.

3.2. The process of sheafification of a presheaf:

To any presheaf $\mathcal{G}$ on a space X we can construct a sheaf $\mathcal{G}^{\#}$ together with a map $j : \mathcal{G} \to \mathcal{G}^{\#}$ (in the category of presheaves) such that for any sheaf $\mathcal{F}$ we have

$$\mathrm{Hom}_{\mathcal{PS}_X}(\mathcal{G}, \mathcal{F}) = \mathrm{Hom}_{\mathcal{S}_X}(\mathcal{G}^{\#}, \mathcal{F}).$$

This can also be seen as another example of a representable functor. Our presheaf $\mathcal{G}$ defines a functor from the category $\mathcal{S}$ of sheaves (with values in the category of rings, abelian groups, sets) into the category of sets namely the functor $\mathcal{F} \to \mathrm{Hom}_{\mathcal{PS}_X}(\mathcal{G}, \mathcal{F})$. our sheaf $\mathcal{G}^{\#}$ is the representing this functor. Hence by the Yoneda-lemma it is unique up to isomorphism.

Using the stalks it is possible to define $\mathcal{G}^{\#}$ quite directly. We define

$$\mathcal{G}^{\#}(U) = \left\{ (\dots s_p \dots) \in \prod_{p\in U} \mathcal{G}_p \;\middle|\; \begin{array}{c} \text{For any point } p \in U \ \exists \text{ open } U_p \\ p \in U_p \subset U \text{ and } \tilde{s}_p \in \mathcal{G}(U_p), \text{ s. t.} \\ \tilde{s}_p \mid q = s_q \text{ for all } q \in U \end{array} \right\}.$$

The reader should verify, that this defines indeed a sheaf, this sheaf has the same stalks as our original presheaf, we have a map $\mathcal{G} \to \mathcal{G}^{\#}$ and it has the required property.

There exist some more abstract notions of sheaves on so called Grothendieck topologies, these are in some sense "spaces" which sometimes do not have points anymore. In such a case it is not possible to use the stalks, but still it is possible to construct $\mathcal{G}^{\#}$. Therefore I will give here another construction of $\mathcal{G}^{\#}$ which does not use stalks.

We consider coverings $\mathfrak{U} = \{U_i\}_{i\in I}, U = \bigcup_{i\in I} U_i$ of an open set U. We introduce the category of coverings. An arrow from a covering $\mathfrak{V} = \{V_\alpha\}_{\alpha\in A}$ to the covering $\mathfrak{U} = \{U_i\}_{i\in I}$ is a map

$$\tau : A \longrightarrow I$$

s. t. $\bigcup_{\alpha\in\tau^{-1}(i)} V_\alpha = U_i$. We write

$$\tau : \mathfrak{V} \to \mathfrak{U}$$

for such a morphism. In general the arrow τ is not determined by the two coverings, but many constructions using this arrow will give results not depending on it.

We will say that such an arrow defines a *refinement* of $\mathfrak{U}$ by $\mathfrak{V}$. Sometimes we will say that $\mathfrak{V}$ is a *refinement* of $\mathfrak{U}$ if there is an arrow from $\mathfrak{V}$ to $\mathfrak{U}$.

The arrow τ defines a map between diagrams (see the general remark about maps between products at the beginning of this section)

$$\begin{array}{ccccc} \mathcal{G}(U) & \xrightarrow{p_0} & \prod_{i \in I} \mathcal{G}(U_i) & \overset{p_1}{\underset{p_2}{\rightrightarrows}} & \prod_{(i,j)\in I\times I} \mathcal{G}(U_i \cap U_j) \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{G}(U) & \longrightarrow & \prod_{\alpha \in A} \mathcal{G}(V_\alpha) & \rightrightarrows & \prod_{(\alpha,\beta)\in A\times A} \mathcal{G}(V_\alpha \cap V_\beta) \end{array}.$$

For any covering $\mathfrak{U} = \{U_i\}_{i\in I}$ of U we define

$$\mathcal{G}^{\mathfrak{U}}(U)[p_1 = p_2] := \left\{ s \in \prod_{i\in I} \mathcal{G}(U_i) \mid p_1(s) = p_2(s) \right\}.$$

If $\mathfrak{V}$ is a refinement of $\mathfrak{U}$ then our map τ defines a map

$$\begin{array}{ccc} \mathcal{G}(U) & \longrightarrow & \mathcal{G}^{\mathfrak{U}}(U)[p_1 = p_2] \\ \| & & \downarrow \\ \mathcal{G}(U) & \longrightarrow & \mathcal{G}^{\mathfrak{V}}(U)[p_1 = p_2] \end{array}.$$

It is not difficult to see that the vertical arrow does not depend on the choice of τ. Now we need the courage to believe that we can extend the notion of direct limit to this situation where we do not have an indexing set but a category which is directed because two coverings have always a common refinement. We put

$$\mathcal{G}^+(U) = \varinjlim_{\mathfrak{U}} \mathcal{G}^{\mathfrak{U}}(U)[p_1 = p_2].$$

We check that $\mathcal{G}^+$ is again a presheaf, and it satisfies condition $(Sh1)$. Moreover if the original presheaf $\mathcal{G}$ satisfies already $(Sh1)$ then $\mathcal{G}^+$ satisfies even $(Sh2)$. Hence we see that $\mathcal{G}^{++} = \mathcal{G}^{\#}$ is always a sheaf. We have

$$i : \mathcal{G} \longrightarrow \mathcal{G}^{\#},$$

and $\mathcal{G}^{\#}$ has the required universal property.

III.4. The functors f_* and f^*:

Given two topological spaces X, Y and a continuous map $f : X \to Y$, we construct two functors f_*, f^* which transport sheaves on X to sheaves on Y and sheaves on Y to sheaves on X respectively. Let us denote by $\mathcal{S}_X$ resp. $\mathcal{S}_X$ the category of sheaves resp. presheaves on X with values in the category of abelian groups, rings or sets.

If we have a sheaf $\mathcal{F}$ on X we define the sheaf $f_*(\mathcal{F})$ on Y by

$$f_*(\mathcal{F})(V) = \mathcal{F}(f^{-1}(V))$$

for all open subsets $V \subset Y$. It is clear that $f_*(\mathcal{F})$ is a sheaf on Y, it is called the *direct image* of $\mathcal{F}$.

The functor f^* transforms sheaves on Y into sheaves on X. The idea is that the stalk of $f^*(\mathcal{G})$ in a point $x \in X$ is equal to the stalk of the original sheaf $\mathcal{G}$ in the point $y = f(x)$, i.e. $f^*(\mathcal{G})_x = \mathcal{G}_{f(x)}$. The actual construction is a little bit complicated. At first we define a presheaf $f'(\mathcal{G})$:

For $U \subset X$ we put

$$f'(\mathcal{G})(U) = \varinjlim_{V \supset f(U)} \mathcal{G}(V).$$

It is not difficult to verify that this is a presheaf and that for any covering $U = \cup_{i \in I} U_i$ we get an injective map

$$f'(\mathcal{G})(U) \longrightarrow \prod_{i \in I} f'(\mathcal{G})(U_i).$$

It satisfies $(Sh1)$ but not necessarily $(Sh2)$. We define

$$f^*(\mathcal{G}) = f'(\mathcal{G})^{\#}.$$

We recall that the stalks of the sheafification of a presheaf are equal to the stalks of the presheaf hence we get

$$f^*(\mathcal{G})_x = \varinjlim_{x \in U} \varinjlim_{V \supset f(U)} \mathcal{G}(V) = \varinjlim_{V : f(x) \in V} \mathcal{G}(V) = \mathcal{G}_{f(x)}.$$

Since the exactness of sequences of sheaves can be checked stalkwise, it is clear that f^* is an exact functor.

4.1. The adjunction formula:

The functors f_*, f^* are adjoint functors. To be more precise: The functor f^* is left adjoint to f_*. This means that we have a functorial isomorphism

$$\mathrm{Hom}_{\mathcal{S}_X}(f^*(\mathcal{G}), \mathcal{F}) = \mathrm{Hom}_{\mathcal{S}_Y}(\mathcal{G}, f_*(\mathcal{F})).$$

Here functorial means that from morphisms $u : \mathcal{G}' \to \mathcal{G}$ and $v : \mathcal{F} \to \mathcal{F}'$, we get the obvious commutative diagrams.

It is not very difficult to verify the adjointness formula. From the construction of the sheafification we have $\mathrm{Hom}_{\mathcal{PS}_X}(f'\mathcal{G}, \mathcal{F}) = \mathrm{Hom}_{\mathcal{S}_X}(f^*\mathcal{G}, \mathcal{F})$. Hence a morphism ψ in $\mathrm{Hom}_{\mathcal{PS}_X}(f'\mathcal{G}, \mathcal{F})$ is a collection of $\psi_U : f'\mathcal{G}(U) \to \mathcal{F}(U)$. It follows from the definition $f'\mathcal{G}(U)$ and the properties of the direct limit that this is nothing else than a collection of maps

$$\psi_{U,V} : \mathcal{G}(V) \to \mathcal{F}(U)$$

where U, V run over all open sets in X, Y which satisfy $f(U) \subset V$, and where the maps in this collection satisfy the obvious compatibilities. We will call $\psi_{U,V}$ the evaluation of ψ on U, V. Now we are allowed to evaluate on $U = f^{-1}(V)$ and we get a collection $\psi_{f^{-1}(V),V} = \phi_V : \mathcal{G}(V) \to f_*(\mathcal{F})(V)$, i.e. an element in $\mathrm{Hom}_{\mathcal{S}_Y}(\mathcal{G}, f_*(\mathcal{F}))$. The other direction is also clear.

Remark: It is always confusing and hard to memorize which functor is a left (right) adjoint of which. The question is whether f^* has to be placed into the source or the target of the Hom$(,)$. Here is a simple rule that helps. We have to remember that f_* gives directly a sheaf while the construction of $f^*\mathcal{G}$ involves the process of sheafification and this uses direct limits. But as I explained in the chapter on categories direct limits are made so that we know what the maps *from* them are. Hence the free place on the left in Hom(,) is the place where $f^*\mathcal{G}$ belongs.

4.2 Extensions and restrictions: We can consider the special case of an open subset $U \subset X$ and let $A = X \setminus U$ be its complement. Then we have the two inclusions $i : A \hookrightarrow X, j : U \hookrightarrow X$. For a sheaf $\mathcal{F}$ on X the sheaf $j^*(\mathcal{F})$ is very easy to understand since for an open set $V \subset U$ we have $j^*(\mathcal{F})(V) = \mathcal{F}(V)$. This is called the restriction of $\mathcal{F}$ to U. The operation $i^*(\mathcal{F})$ is much more delicate and will cause us some trouble (See IV 1.2.2.1). If we have a sheaf $\mathcal{G}$ on U then $j_*(\mathcal{G})$ is a delicate functor since it depends on the local topology in the neighborhood of boundary points (See IV ???). But for a sheaf $\mathcal{G}$ on A the $i_*(\mathcal{G})$ is easy to understand. Its stalks are zero outside of A and equal to the stalks of $\mathcal{G}$ on A. It is called the *extension by zero.*

III.5. Products

If we have a family of sheaves $\{\mathcal{F}_\alpha\}_{\alpha \in A}$ the we can define the product: For any open set $U \subset X$ we put

$$(\prod_{\alpha \in A} \mathcal{F}_\alpha)(U) := \prod_{\alpha \in A} \mathcal{F}_\alpha(U)$$

and it is easy to verify that this is again a sheaf. If our sheaves have values in the category of rings, modules, abelian groups etc. the product is again a sheaf with values in that category.

If the $\mathcal{F}_\alpha$ are abelian groups or modules we might be tempted to take the direct sum of sheaves. But this does not work in general. The naive definition gives only a presheaf because $(Sh2)$ may be violated if the indexing set A is infinite.

5.1. Perhaps here is the right place to explain that the sheaves on X with values in the category of abelian groups form an *abelian category* . First of all this says that for two such sheaves the set

$$\Psi \in \mathrm{Hom}_{\mathcal{S}_X}(\mathcal{F}, \mathcal{G})$$

are abelian groups: If we have two morphisms $\Psi = \{\Psi_U, \Phi_U\}_U, \Phi = \{\Phi_U, \Phi_U\}_U$ then $\Psi + \Phi = \{\Psi_U + \Phi_U\}_U$. This group structure is bilinear with respect to composition.

If we have a morphism $\Psi : \mathcal{F} \to \mathcal{G}$ then we can define the kernel $\ker(\Psi)$ as the subsheaf $U \to \ker(\Psi_U)$. This kernel has a categorial interpretation: For any other sheaf $\mathcal{A}$

$$\mathrm{Hom}_{\mathcal{S}_X}(\mathcal{A}, \ker(\Psi)) = \{\phi \in \mathrm{Hom}_{\mathcal{S}_X}(\mathcal{F}, \mathcal{G}) | \Psi \circ \phi = 0\}.$$

Now we may consider the preasheaf

$$\mathcal{K}(U) = \mathcal{F}(U)/\ker(\Psi)(U).$$

It is fundamental that this preasheaf is not neccesarily a sheaf and this will be explained in detail in the next Chapter. It is not hard to verify the first sheaf condition $(Sh1)$ but in general it does not satisfy the second condition $(Sh2)$. Of course we can sheafify the preasheaf $\mathcal{K}$ and we get the quotient sheaf

$$\mathcal{F}/\ker(\Psi) = \mathcal{K}^{\#}$$

This quotient has again a categorial interpretation and it is called the *coimage* of Ψ. We can also define the *image* of Ψ as a subsheaf of $\mathcal{G}$. It is simply $\mathrm{im}(\Psi)(U) = \Psi_U(\mathcal{K}^{\#}(U))$ and by construction it is isomorphic to the coimage. These to objects namely the coimage and image can be defined in a categorial context and it is one of the axioms for an abelian category that they should be canonically isomorphic (See [Mac Lane]).

In an abelian category we can define the notion of exact sequences but this will be discussed in the following chapter.

Chapter IV
Cohomology of Sheaves

We consider sheaves with values in abelian groups. We can define the notion of an exact sequence of sheaves. A sequence of sheaves on a space X

$$0 \longrightarrow \mathcal{F}' \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}'' \longrightarrow 0$$

is *exact* if for all points $x \in X$ the sequence of stalks is exact. It is easy to see that this is equivalent to

(i) For all open sets $U \subset X$ the sequence

$$0 \longrightarrow \mathcal{F}'(U) \longrightarrow \mathcal{F}(U) \longrightarrow \mathcal{F}''(U)$$

is exact and

(ii) For any $s'' \in \mathcal{F}''(U)$ we can find a covering $U = \bigcup_j U_j$ by open sets and $s_j \in \mathcal{F}(U_j)$ such that $s_j \mapsto s''|U_j$

It is the decisive point that the exactness does not imply that $\mathcal{F}(U) \longrightarrow \mathcal{F}''(U)$ is surjective. We can only find local liftings for an $s'' \in \mathcal{F}''(U)$.

Applied to $U = X$ this tells us that the functor $\mathcal{F} \to \mathcal{F}(X)$ will not be exact in general. Hence we have to construct a right derived functor to it. As in Chapter II we introduce the notation $H^0(X, \mathcal{F})$ for $\mathcal{F}(X)$ and we want construct cohomology groups $H^1(X, \mathcal{F}), H^2(X, \mathcal{F})$ which have functorial properties and such that any short exact sequence yields a long exact sequence

$$0 \longrightarrow \mathcal{F}'(X) \longrightarrow \mathcal{F}(X) \longrightarrow \mathcal{F}''(X) \to H^1(X, \mathcal{F}') \to$$

as in the previous chapter.

The following two examples are absolutely fundamental. In a nutshell we see everything that makes sheaf cohomology work. I also want to stress the almost perfect analogy between these two examples which will be explained in (1.2.1.).

IV.1 Examples

1.1 Sheaves on Riemann surfaces

In the previous section we introduced the notion of a complex manifold (See 2.1). Here I want to consider a compact Riemann surface $(X, \mathcal{O}_X)$. This means that X is a compact connected complex manifold of dimension 1. For any $P \in X$ we find an open neighborhood U_P of P such that (See III.2 and III.2.1)

$$(U_P, \mathcal{O}_X|U_P) \simeq (B, \mathcal{O}_B),$$

where $B = \{z \in \mathbb{C} \mid |z| < 1\}$ is the open unit disc and where $\mathcal{O}_B$ is the sheaf of germs of holomorphic functions on B. We assume that the homeomorphism between the spaces maps P to the origin 0 in the disc.

The element $z \in \mathcal{O}_B(B)$ yields via the isomorphism an element $z_P \in \mathcal{O}_X(U_P)$. This element z_P vanishes at P and it generates the maximal ideal $\mathfrak{m}_P$ of the stalk $\mathcal{O}_{X,P}$. Such an element is called a *uniformizer* or *uniformizing element* at P. Any power series

$$u_P = f(z_P) = az_P + bz_P^2, \ldots$$

which has a positive radius of convergence and with $a \neq 0$ can serve as an uniformizer as well.

A complex function

$$g : U_P \backslash \{P\} \to \mathbb{C}$$

is called *meromorphic* on U_P if it is holomorphic and if we can find an integer n such that $z_P^n \cdot g = h$ extends to a holomorphic function on U_P. We say that g has a *pole of order n at P* if n is the smallest value for such an integer. We write $\mathrm{ord}_P(g) = -n$ and by definition $g \in {z_P^{-n}} \mathcal{O}_{X,P}$, but $g \notin {z_P^{-n+1}} \mathcal{O}_{X,P}$.

If T is a finite subset of S and if $f : S \setminus T \to \mathbb{C}$ is a holomorphic function then we say that f is meromorphic if its singularities at the points of T are at most poles (and not essential singularities). For any point $P \in T$ we have defined $\mathrm{ord}_P(f)$ and we define the *polar divisor*

$$\mathrm{Div}_\infty(f) = \sum_{P \in T, \mathrm{ord}_P(f) > 0} \mathrm{ord}_P(f) P$$

which we consider as an an element in the *divisor group* $\mathrm{Div}(S)$, the free abelian group generated by the points of S. Since S is compact it follows from the principle of analytic continuation that f can only have a finite number of zeroes on $U = S \setminus T$ and this implies that $1/f$ is also holomorphic on some open set $U' = S \setminus T'$ where T' is finite and then $1/f$ is also meromorphic. We may also define the *zero divisor* of f as

$$\mathrm{Div}_0(f) = -\mathrm{Div}_\infty(1/f)$$

and the divisor of f as

$$\mathrm{Div}(f) = \mathrm{Div}_0(f) + \mathrm{Div}_\infty(f).$$

We have a homomorphism called the *degree* which is given by

$$\deg : \mathrm{Div}(S) \to \mathbb{Z}$$

which is given by $\deg : D = \sum n_P P \mapsto \sum n_P$.

A divisor $D = \sum_P n_P P$ which is the divisor of a meromorphic function will be called a *principal divisor.* We will see (See V.1.3.2) that for a principal divisor the degree $\deg(D) = \sum n_P = 0$.

To any divisor $D = \sum_P n_P P$ we attach the sheaf $\mathcal{O}_X(D)$ which which is defined by

$$\mathcal{O}_X(D)(U) = \{f \text{ meromorphic on } U \mid \operatorname{ord}_P(f) \geq -n_P \text{ for all } P \in U\}.$$

We could also say that $f \in z_P^{-n_P}\mathcal{O}_{X,P}$ for all P.

A divisor $D = \sum n_P P$ is called *effective* if all $n_P \geq 0$, we could also call this a *positive* divisor and write $D \geq 0$. The definition of $\mathcal{O}_X(D)(U)$ can be reformulated: It consists of those meromorphic functions f on U for which $\operatorname{Div}(f) + D \mid U \geq 0$. If D is an effective divisor we have an inclusion of sheaves $\mathcal{O}_X \subset \mathcal{O}_X(D)$. We form the quotient sheaf $\mathbb{L}_D = \mathcal{O}_X(D)/\mathcal{O}_X$. It is clear that the stalk at P is $z_P^{-n_P}\mathcal{O}_{X,P}/\mathcal{O}_{X,P}$.

For any point P

$$z_P^{-n}\mathcal{O}_{X,P}/\mathcal{O}_{X,P} = \mathbb{L}_P^{(n)}$$

is the finite dimensional vector space of Laurent expansions, an element $\ell \in \mathcal{L}_P$ can be written as

$$\ell = \frac{a_n}{z_P^n} + \frac{a_{n-1}}{z_P^{n-1}} + \dots \frac{a_1}{z_P} \quad \operatorname{mod} \quad \mathcal{O}_{X,P}.$$

If $a_n \neq 0$, we say that ℓ has a pole of order n. So the stalk of this sheaf at a point $P \in X$ is the vector space of all Laurent expansions of pole order $\leq n_P$. Especially the stalk is zero at points where $n_P = 0$ and therefore the sheaf $\mathbb{L}_D$ has only a finite number of non-zero stalks. It is called a *skyscraper sheaf.* We have the exact sequence of sheaves

$$0 \to \mathcal{O}_X \to \mathcal{O}_X(D) \to \mathbb{L}_D \to 0.$$

It is clear that the space of sections $H^0(X, \mathbb{L}_D)$ is simply the direct sum of the stalks in the points P with $n_P > 0$. There is no interaction between the different points.

The question whether the sequence

$$0 \to H^0(X, \mathcal{O}_X) \to H^0(X, \mathcal{O}_X(D)) \to H^0(X, \mathbb{L}_D) \to 0$$

is exact amounts to whether a given collection of Laurent expansions at the finitely many points P with $n_P > 0$ can be realized by a meromorphic function on X. In general the answer is no and the discrepancy is controlled by the first cohomology group $H^1(X, \mathcal{O}_X)$ which we will define later. To be more precise we will construct a map

$$\delta : H^0(X, \mathbb{L}_D) \to H^1(X, \mathcal{O}_X)$$

such that the extended sequence

$$0 \to H^0(X, \mathcal{O}_X) \to H^0(X, \mathcal{O}_X(D)) \to H^0(X, \mathbb{L}_D) \to H^1(X, \mathcal{O}_X)$$

becomes exact. The computation of $H^1(X, \mathcal{O}_X)$ is more or less equivalent to the Riemann-Roch Theorem which we will discuss in the chapters on curves and Riemann surfaces.

1.1.1. Exercise:

(1) *Prove that in the case $X = \mathbb{P}^1(\mathbb{C})$ the above sequence is always exact.*

(2) *Prove that in the case $X = \mathbb{C}/\Omega$ the above sequence is not always exact.*

1.2 Cohomology of the circle

We consider the circle S^1 and the sheaf $\underline{\mathbb{Z}}$ which is defined by

$$\underline{\mathbb{Z}}(V) = \{f : V \to \mathbb{Z} \mid f \quad \text{is locally constant}\}.$$

We pick a point $P \in S^1$ and let $U = S^1 \backslash \{P\}$. We define a sheaf $\mathbb{Z}^{(P)}$ by

$$\underline{\mathbb{Z}}^{(P)}(V) = \underline{\mathbb{Z}}(U \cap V).$$

If $i : U \to S^1$ is the inclusion then this is the sheaf $i_*(\mathbb{Z})$ (see III.4.2). Clearly we have an inclusion $\underline{\mathbb{Z}} \subset \underline{\mathbb{Z}}^{(P)}$ and for all $Q \neq P$ we have the equality of stalks

$$\underline{\mathbb{Z}}_Q = \underline{\mathbb{Z}}_Q^P = \mathbb{Z}.$$

But in the point P we have

$$\mathbb{Z} = \underline{\mathbb{Z}}_P \hookrightarrow \underline{\mathbb{Z}}_P^{(P)} = \mathbb{Z} \oplus \mathbb{Z}$$

because on a little interval I_ϵ containing P we have $\underline{\mathbb{Z}}(I_\epsilon) = \mathbb{Z}$ but $\underline{\mathbb{Z}}^{(P)}(I_\epsilon) = \underline{\mathbb{Z}}(I_\epsilon \cap U) = \mathbb{Z} \oplus \mathbb{Z}$. Hence we get an exact sequence of sheaves

$$0 \to \underline{\mathbb{Z}} \to \underline{\mathbb{Z}}^{(P)} \to \mathcal{S}_P \to 0$$

where $\mathcal{S}_P$ is the skyscraper sheaf whose stalk at P is $\mathbb{Z}$ and zero elsewhere. We get the sequence of global sections

$$\begin{array}{ccccccc}
0 \to & H^0(S^1, \underline{\mathbb{Z}}) & \tilde{\to} & H^0(S^1, \underline{\mathbb{Z}}^{(P)}) & \to & H^0(S^1, \mathcal{S}_P) \\
& \| & & \| & & \| \\
0 \to & \mathbb{Z} & \tilde{\to} & \mathbb{Z} & \to & \mathbb{Z}
\end{array}$$

and we see that the last arrow is not surjective. Again we need a non-zero $H^1(S^1, \mathbb{Z})$ to control the discrepancy.

We even can have an idea what this group $H^1(S^1, \mathbb{Z})$ should be. Intuitively we should think that the sheaf $\mathbb{Z}^{(P)}$ doubles the point P, so our circle becomes an intervall I and it is at least plausible that

$$H^1(I, \mathbb{Z}) = H^1(S^1, \mathbb{Z}^{(P)}).$$

But the intervall is contractible (see 4.4.11), and we will see then $H^1(I, \mathbb{Z}) = 0$. Hence we should expect (and we will prove this later) that

$$H^0(S^1, \mathcal{S}_P) \tilde{\to} H^1(S^1, \mathbb{Z}) \simeq \mathbb{Z}.$$

I want to stress another important point. We can ask whether the isomorphism $H^1(S^1, \mathbb{Z}) \simeq \mathbb{Z}$ is canonical. The answer is no!

This becomes clear if we recall that

$$\mathcal{S}^P = (\mathbb{Z} \oplus \mathbb{Z})/\mathbb{Z}$$

where $\mathbb{Z}$ is embedded diagonally. There is no way to distinguish between the two possibilities to identify $H^0(S^1, \mathcal{S}^P)$ to $\mathbb{Z}$.

But we can choose an orientation on S^1 (see 3.3.1), this eans tha we choose a direction (a non zero tangent vector) at each point which varies continously with the point. Then we have distinction between the two intervalls in the intersection

$$I_\varepsilon \cap U = U_\varepsilon^+ \cup U_\varepsilon^-$$

we say that U_ε^+ is the intervall which the positive tangent vector at P points to. Then

$$\underline{\mathbb{Z}}(I_\varepsilon \cap U) = \underline{\mathbb{Z}}(U_\varepsilon^+) \oplus \underline{\mathbb{Z}}(U_\varepsilon^-) = \mathbb{Z} \oplus \mathbb{Z},$$

and we now have a canonical identification

$$H^1(S, \mathbb{Z}) = \mathbb{Z}$$

where we send

$$(a, b) \bmod \mathbb{Z} \longrightarrow a.$$

1.2.1. I want to stress the analogy between the two examples: The sheaves $\mathcal{O}_X$ and $\underline{\mathbb{Z}}$ have a property in common: They are very rigid. This eans that any section over a connected open subset U is determined by its restriction to an arbitrarily small non empty open subset $V \subset U$.

The analogy goes even further. If we consider the sheaf $\underline{\mathbb{R}}$ on a manifold M, then we can characterize $\underline{\mathbb{R}}$ as a subsheaf in the sheaf $\mathcal{C}_\infty(M)$: It is the subsheaf of functions with zero derivatives. An analogous statement is true for $\mathcal{O}_X$. We can characterize $\mathcal{O}_X$ as the subsheaf in the sheaf of $\mathcal{C}_\infty$ function annihilated by the Cauchy-Riemann operators.

IV.2 The derived functor.

2.1 Introduction. We want to define a universal derived functor to the functor $\mathcal{F} \to H^0(X, \mathcal{F})$. To do this we use the same ideas as in Chapter II. We define the notion of an *injective sheaf*: A sheaf $\mathcal{I}$ is *injective* if in any diagram

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{\varphi} & \mathcal{B} \\ \Big\downarrow \psi & & \\ \mathcal{I} & & \end{array}$$

with $\ker(\varphi) \subset \ker(\psi)$ we can find a map $\eta : \mathcal{B} \to \mathcal{I}$ which make this diagram commutative.

It is rather easy to see that every sheaf $\mathcal{F}$ can be embedded into an injective sheaf. The following construction has been invented by Godement (See [Go],???). For any point $x \in X$ we embed the stalk $\mathcal{F}_x$ into an injective abelian group I_x. We define the sheaf $\mathcal{I}$ by

$$\mathcal{I}(U) = \prod_{x \in U} I_x$$

and the restriction maps $\prod_{x \in U} I_x \to \prod_{x \in V} I_x$ are induced by the inclusion $V \subset U$.

To prove the injectivity of $\mathcal{I}$ we consider our diagram above stalk by stalk and choose for each $x \in X$ an η_x such that the diagram commutes

$$\begin{array}{ccc} \mathcal{A}_x & \xrightarrow{\varphi_x} & \mathcal{B}_x \\ \Big\downarrow \psi_x & \swarrow \eta_x & \\ \mathcal{I}_x & & \end{array}$$

commutes. By construction this collection provides an embedding $\eta : \mathcal{F} \to \mathcal{I}$, for any open set $U \subset X$ the homomorphism $\eta_U : \mathcal{F}(U) \to \mathcal{I}(U)$ is induced by the maps $\mathcal{F}(U) \to \mathcal{F}_x \to I_x$. Now it is obvious that we can find an injective resolution for any sheaf $\mathcal{F}$:

$$0 \to \mathcal{F} \to \mathcal{I}^0 \to \mathcal{I}^1 \to .$$

Consequently we define

$$H^\bullet(X, \mathcal{F}) = H^\bullet(\mathcal{I}^\bullet(X)).$$

The same arguments as in the previous section show that this defines a universal right derived functor.

The reader might or should even be scared: How can we ever compute the cohomology of a sheaf if we use such "huge and bizarre" sheaves to define it.

Our strategy will be to exhibit classes of "smaller" sheaves which have the property that they are acyclic. One possibility to construct such sheaves is discussed in the following exercise.

2.1.1 Exercise.

Let us assume that we have a sheaf of commutative rings $\mathcal{R}$ on X, the rings should have an identity, especially we have $1 \in \mathcal{R}(X)$. Let us assume that we have a so called partition of 1: For any covering $X = \bigcup_{i \in I} U_i$ we can find elements $h_i \in \mathcal{R}(X)$ such that $\mathrm{Supp}(h_i) \subset U_i$, for any point $x \in X$ we have only finitely many indices such that $h_{ix} \neq 0$ and finally $1 = \sum_i h_i$.

Show that sheaves $\mathcal{F}$ of $\mathcal{R}$ modules are acyclic.

Hint: Assume we have a short exact sequence

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$$

of $\mathcal{R}$-modules. Use the partition of unity to show that $\mathcal{F}(X) \to \mathcal{F}''(X)$ is surjective.

We will see that on a $\mathcal{C}^\infty$ -manifold M the sheaves of rings of $\mathcal{C}^\infty$ functions have a partition of unity. This will imply that for any $\mathcal{C}^\infty$ vector bundle (See 3.2) $\mathcal{E}$ and the sheaf $\mathcal{C}^\infty(\mathcal{E})$ of $\mathcal{C}^\infty$ -sections in it

$$H^i(M, \mathcal{C}^\infty(\mathcal{E})) = 0 \text{ for all } i > 0$$

2.2 A direct definition of H^1:

We want to indicate briefly how we could approach the problem to define a right derived functor for $H^0(X, \mathcal{F})$ more directly. The reader should notices the analogy between this approach and the one used to define the first cohomology group in group cohomology (See II.2.1.)

Let us assume we have an exact sequence of sheaves

$$0 \longrightarrow \mathcal{F}' \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}'' \longrightarrow 0.$$

We look at

$$\mathcal{F}(X) \longrightarrow \mathcal{F}''(X)$$

and pick a section $s'' \in \mathcal{F}''(X)$. Locally we can lift this section to a section in $\mathcal{F}$. This means we can find a covering $X = \cup U_\alpha$ and sections $s_\alpha \in \mathcal{F}(U_\alpha)$ which map to $s'' \mid U_\alpha$. But the s_α do not necessarily match: The difference

$$s'_{\alpha,\beta} = s_\alpha - s_\beta \mid U_\alpha \cap U_\beta$$

is a section in $\mathcal{F}'(U_\alpha \cap U_\beta)$ because it goes to zero in $\mathcal{F}''$. The collection $\{s'_{\alpha,\beta}\}_{(\alpha,\beta) \in A \times A}$ satisfies the cocycle relation, i.e. we have

$$s'_{\alpha,\beta} - s'_{\beta,\gamma} + s'_{\gamma,\alpha} \mid U_\alpha \cap U_\beta \cap U_\gamma = 0.$$

This suggests the definition of the group of 1-cocycles with respect to a covering $\mathfrak{U} = \{U_\alpha\}_{\alpha \in A}$: These 1-cocycles are collections

$$(\ldots, t_{\alpha,\beta}, \ldots) \in \prod_{(\alpha,\beta) \in A \times A} \mathcal{F}'(U_\alpha \cap U_\beta)$$

which satisfy the relation above. They form a group which will be denoted by $Z^1(\mathfrak{U}, \mathcal{F}')$.
We may also define the group of coboundaries: An element $(\ldots t_{\alpha,\beta} \ldots)$ is a coboundary if we can find $s'_\alpha \in \mathcal{F}'(U_\alpha)$ s.t. $t'_{\alpha,\beta} = s'_\alpha - s'_\beta$.
We define $H^1(X, \mathfrak{U}, \mathcal{F}')$ to be the quotient

$$H^1(X, \mathfrak{U}, \mathcal{F}') = Z^1(\mathfrak{U}, \mathcal{F}')/B^1(\mathfrak{U}, \mathcal{F}').$$

Now it is clear that s'' defines an element

$$\delta(s'') \in H^1(S, \mathfrak{U}, \mathcal{F}'),$$

and it is clear that s'' is in the image if and only if $\delta(s'') = 0$.
If we start from a different covering $\mathfrak{U}'$ then $\mathfrak{U}$ and $\mathfrak{U}'$ have a common refinement (See III.3.1) $\tau : \mathfrak{W} \to \mathfrak{U}, \tau' : \mathfrak{W} \to \mathfrak{U}'$. We get maps

$$\begin{tikzcd}[column sep=small] H^1(X, \mathfrak{U}, \mathcal{F}') \arrow[dr] & & H^1(X, \mathfrak{U}', \mathcal{F}') \arrow[dl] \\ & H^1(X, \mathfrak{W}, \mathcal{F}'). & \end{tikzcd}$$

It is not difficult to see that these maps do not depend on the choice of of the arrows.
It is clear that these maps are compatible with δ and hence we get a boundary operator

$$\delta : \mathcal{F}''(X) \longrightarrow \varinjlim_{\mathfrak{U}} H^1(X, \mathfrak{U}, \mathcal{F}') = \check{H}^1(X, \mathcal{F}').$$

Now it is not hard to see that we have a structure of an abelian group on the limit, the boundary operator is a homomorphism and the sequence

$$0 \to H^0(X, \mathcal{F}') \to H^0(X, \mathcal{F}) \to H^0(X, \mathcal{F}'') \xrightarrow{\delta} \check{H}^1(X, \mathcal{F}') \to \check{H}^1(X, \mathcal{F}) \to \check{H}^1(X, \mathcal{F}'')$$

is exact.
Of course we need to compare this construction of cohomology groups with the other one using injective resolutions, this comes next.

2.2.1 A sheaf $\mathcal{F}$ on a space X is called *flabby* if for any open set $U \subset X$ the restriction map $\mathcal{F}(X) \to \mathcal{F}(U)$ is surjective. This is a very strange property of a sheaf. For instance the continuous functions on a space almost never have this property.

2.2.2 We want to show that injective sheaves are flabby. To do this we consider an open subset $U \subset X$ we denote its inclusion by $j : U \to X$, let $A = X \setminus U$ let us denote the inclusion of the closed set by $i : A \to X$. For any sheaf $\mathcal{F}$ we can take its restriction to A and extend this restriction again to X by using i_*. (Extension by zero: See III 4.2.)

We have a surjective homomorphism of sheaves $\mathcal{F} \to i_*i^*(\mathcal{F})$ and this gives us an exact sequence of sheaves

$$0 \to j_!(\mathcal{F}) \to \mathcal{F} \to i_*i^*(\mathcal{F}) \to 0$$

where of course $j_!(\mathcal{F})$ is just the kernel.

A short disgression: We may give a direct definition of this kernel and call it again the extension of $\mathcal{F}|U$ to X by zero. To give this direct definition we recall the notion of the support of a section (see III.3.1.) and notice that for any open set $V \subset X$ we have more or less by definition

$$j_!(\mathcal{F})(V) = \{s \in \mathcal{F}(V) \mid \text{ the support of } s \text{ does not meet } V \cap A\}.$$

This means that this sheaf is a little bit delicate. By construction we have an inclusion $j_!(\mathcal{F})(V) \hookrightarrow \mathcal{F}$. In a sense the sheaf $j_!(\mathcal{F})$ "knows" the boundary points of U.

Now we come back to our original problem, we wanted to show that injective sheaves are flabby. We have an inclusion

$$\begin{array}{ccc} j_!\mathcal{I} & \longrightarrow & j_*\mathcal{I} \\ \downarrow & & \\ \mathcal{I} & & \end{array}$$

and since $\mathcal{I}$ is injective we find a homomorphism $\phi : j_*\mathcal{I} \to \mathcal{I}$ which makes this diagram commute. If we have a section $s \in \mathcal{F}(U)$ then this is by definition the same as a section $s \in i_*(\mathcal{F})(X)$ and then $\phi(s) \in \mathcal{F}(X)$. It is clear from the diagram that $\phi(s)$ restricted to U is s. Moreover we see that our section $\phi(s)$ has support contained in the closure $\bar{U}$, the best we can expect.

2.2.3 Exercise

a) Show that for a flabby sheaf $\mathcal{F}$ we have $\check{H}^1(X, \mathcal{F}) = 0$

b) Show that $\check{H}^1(X, \mathcal{I}) = 0$ for an injective sheaf. Show that this implies that for any sheaf $\mathcal{F}$ $\check{H}^1(X, \mathcal{F}) = H^1(X, \mathcal{F})$.

c) Show that flabby sheaves are acyclic.

I discussed this construction of the first cohomology groups in detail, because here we can see how natural these constructions are. Here we meet a fundamental principle of homological algebra which is applied again and again:

We want to lift a section $s'' \in H^0(X, \mathcal{F}'')$ to a section $s \in H^0(X, \mathcal{F})$. We localize the problem by choosing a covering where we have local liftings. These are not unique and hence it can happen, that they do not match on the intersections. These differences on *the intersections yield a cocycle, and the class of this cocycle yields the obstruction to the global solution of the problem.*

*We have seen how the same principle works in group cohomology (*Chap. II.2.1.*) There we want to lift a* Γ *invariant section* $m'' \in (M'')^\Gamma$ *to a* Γ*-invariant section* $m \in M^\Gamma$. *In this context localizing means that we drop the requirement that* m *should be* $\Gamma-$*invariant. Then we find a non unique lifting. The comparison of the local sections on the intersections of the open sets in the geometric situation corresponds now to the comparison of* m *with* γm *where* γ *runs through the group. This gives the cocycles* $\gamma \to m - \gamma m \in M'$

This construction generalizes to higher cohomology groups. We can define the so called Čech cohomology by means of coverings. The cohomology defined by means of injective resolutions and the Čech cohomology coincide on reasonably spaces. We postpone this discussion.

At this point we make a short detour. Since we discussed H^1 in some detail it may be appropriate to discuss the *non abelian* H^1, this means we discuss sheaves with values in non commutative groups and their first cohomology sets.. This non abelian cohomology plays an important role in the theory of bundles and I want to loose some words on this subject.

VI.3 Fiber bundles and non abelian H^1.

3.1. Fibrations. I want to introduce the notion of fibre bundles. We consider maps between topological spaces

$$\pi : X \longrightarrow B.$$

If we have another such map $\pi' : X' \to B$ then a *map over* B is a continuous map $f : X \to X'$ which commutes with the projections.

Now we consider a space F (the fibre) and another space B (the base). A continuous map $\pi : X \to B$ is called a *(locally trivial) fibration with fibre* F, if we can find a covering $B = \bigcup_{i\in I} U_i$ such that for any i we can find a homoemorphism Ψ_i over the base U_i

$$\Psi_i \;:\; \begin{array}{ccc} \pi^{-1}(U_i) & \xrightarrow{\sim} & U_i \times F \\ & \searrow \quad \swarrow p_1 & \\ & U_i. & \end{array}$$

Locally in the base our space is a product of an open set in the base and the given fibre. We also say that $X \xrightarrow{\pi} B$ is a *fibre bundle with fibre* F.

If $V \subset B$ is open then *a section to* π *over* V is a continous map $s : V \to X$ for which $\pi \circ s = \mathrm{Id}_V$.

It is important to consider fibres F which are not only topological spaces but also carry some extra structure.

3.1.1. Vector bundles

For instance we can consider the case that F is a finite dimensional $\mathbb{R}$ or $\mathbb{C}$-vector space and where F, $\mathbb{R}$, $\mathbb{C}$ are equipped with the standard topology. For convenience we denote by $\mathbb{K}$ a field which is either $\mathbb{R}$ or $\mathbb{C}$. In this case we can make an additional assumption on our trivialization. We assume that we have a covering $B = \bigcup_{i \in I} U_i$ and

$$\Psi_i : \pi^{-1}(U_i) \xrightarrow{\sim} U_i \times F$$

as before. But in addition we assume that for any pair i, j of indices the map

$$G_{ij} = \Psi_j \mid U_i \cap U_j \circ \Psi_i^{-1} \mid U_i \cap U_j : (U_i \cap U_j) \times F \to (U_i \cap U_j) \times F$$

has the form

$$G_{ij}(u, x) = (u, g_{ij}(u)x)$$

where $g_{ij}(u)$ is a linear automorphism of our vector space F.
It is clear that $u \to g_{ij}(u)$ must be a continuous map from $U_i \cap U_j$ into the general linear group $G = Gl_n(\mathbb{K})$. Moreover, it is obvious that we have a cocycle relation for any triplet of three indices i, j, k

$$g_{ij}(u) \cdot g_{jk}(u) = g_{ik}(u) \quad \text{for all} \quad u \in U_i \cap U_j \cap U_k.$$

If this assumption is fulfilled, we say that

$$\pi : X \longrightarrow B$$

is a vector bundles. I find this definition is a little bit unsatisfactory because it needs the covering and the Ψ_i. We will give a second definition which I think is better.

Of course our data allow us to introduce the structure of a vector space on each fibre $\pi^{-1}(b)$ such that the vector space structure "varies continuously with b". What do we mean by that? Our definition also implies that we can find sections

$$e_1, \cdots, e_n : U_i \longrightarrow \pi^{-1}(U_i) = U_i \times F,$$

such that in each point $u \in U_i$ the elements $e_1(u), \cdots, e_n(u) \in \pi^{-1}(u)$ form a basis of this vector space. Now we can identify

$$\pi^{-1}(u) \xrightarrow{\sim} \mathbb{K}^n$$

by sending

$$\sum a_\nu e_\nu(u) \longmapsto (a_1, \cdots a_n),$$

and we get a map

$$\pi^{-1}(U_i) \longrightarrow U_i \times \mathbb{K}^n \ ,$$

and the phrase "the vector space structure varies continuously with b" means that this is a homeomorphism.

This allows us to give a different formulation of the concept of a vector bundle. We can say that

$$\pi : X \longrightarrow B$$

is a vector bundle if:

(a) For any $b \in B$ we have the structure of a finite dimensional $\mathbb{K}$ -vector space on the fibre $\pi^{-1}(b)$

(b) For any $b \in B$ we can find a neighborhood V of b and sections

$$e_1, \cdots, e_n : V \longrightarrow \pi^{-1}(V)$$

such that these sections evaluated at any point $v \in V$ form a basis of $\pi^{-1}(v)$.

(c) The map

$$\pi^{-1}(V) \longrightarrow V \times \mathbb{K}^n$$

sending a point $x = \sum\ a_i\ e_i(v)$ above $v \in B$ to $(v, a_1 \cdots a_n) \in V \times \mathbb{K}^n$ is a homeomorphism.

If we have such a vector bundle

$$\pi : X \longrightarrow B,$$

and if we have an open set $V \subset B$ together with the section

$$e_i : V \longrightarrow \pi^{-1}(V) \qquad i = 1 \cdots n$$

which form a basis at any point $v \in V$, then we call this a *local trivialization* of a bundle.

We can consider the sheaf of germs of sections into X, let us call it $\mathcal{C}_{0,X}$. This is a locally free module over the sheaf of germs of continous functions $\mathcal{C}_{0,B}$.

On the other hand it is rather clear that a locally free module $\mathcal{E}$ over $\mathcal{C}_{0,B}$ also gives us a vector bundle.

3.2 The non abelian H^1.

We know of course what it means that two vector bundles $X \to B$ are isomorphic. Actually it is obvious that the vector bundles over a given base space form a category: A continuous map

$$\begin{array}{ccccc} \varphi & : & X & \longrightarrow & X' \\ & & & \searrow \quad \nearrow & \\ & & & B & \end{array}$$

is a morphism of vector bundles, if φ restricted to the fibres is linear.

I want to explain the description of the set of isomorphism classes of vector bundles on B in terms of *non abelian sheaf cohomology.* Given our vector bundle we select a covering $\mathfrak{V} = \{V_i\}_{i\in I}$ of B and local trivializations

$$e_{i,\nu} : V_i \longrightarrow \pi^{-1}(V_i) \qquad \nu = 1 \cdots n.$$

If we have an ordered pair (i, j) of indices, then we get a continuous map

$$g_{ij} : V_i \cap V_j \longrightarrow GL(n, \mathbb{K})$$

such that

$$g_{ij}(v) \cdot (e_{i,\nu}(v)) = e_{j,\nu}(v)$$

i.e. $g_{ij}(v)$ sends the sections $\{e_{i,\nu}\}_v$ into the sections $\{e_{j,\nu}\}_v$.

This is clearly a one-cocycle, this means

$$g_{ij} \cdot g_{jk} = g_{ik} \quad \text{on} \quad V_i \cap V_j \cap V_k$$

and

$$g_{ii} = \mathrm{Id}\,.$$

This suggests that we introduce the set of one-cocycles with respect to our covering. We introduce the sheaf of germs of continuous maps form our space B to the group $G = GL(n, \mathbb{K})$, we denote this sheaf by $\mathcal{C}_0(G)$. Then we define as before

$$C^1(\mathfrak{V}, \mathcal{C}_0(G)) = \{\underline{c} = (\cdots g_{ij} \cdots) \in \prod_{i,j} \mathcal{C}(G)(V_i \cap V_j) \mid \underline{c} = (\cdots g_{ij} \cdots) \text{ is a 1- cocycle}\}.$$

If we modify our local trivialization, then we modify the cocycle into

$$g'_{ij} = h_i g_{ij} h_j^{-1} \qquad \text{on} \qquad V_i \cap V_j,$$

where $\underline{h} \in \prod \mathcal{C}_0(G)(V_i)$. This gives us an equivalence relation on $C^1(\mathfrak{V}, \mathcal{C}_0(G))$ and dividing by this relation we get a set

$$H^1(B, \mathfrak{V}, \mathcal{C}_0(G)).$$

Again we may change the covering, we can pass to common refinements and we end up with

$$H^1(B, \mathcal{C}_0(G)) = \varinjlim_{\mathfrak{V}} H^1(B, \mathfrak{V}, \mathcal{C}_0(G)).$$

Since our sheaf takes values in the category of non abelian groups, we cannot multiply cocycles and we only get a set. Now it follows from our considerations that:

3.2.1. *The elements in $H^1(B, \mathcal{C}_0(G))$ are in one-to-one correspondence with the set of isomorphism classes of vector bundles.*

3.3 The reduction of the structure group.

Of course we can start from any topological group G, we can consider te sheaf of G valued functions on B and we can look at the cohomology set $H^1(B, \mathcal{C}_0(G))$. This set classifies so called *principal G-bundles*, this are bundles $P \to B$ with a left action of G such that G acts simply transitively on the fibres. Then G is called the *structure group* of $P \to B$.

3.3.1 We may introduce different kinds of additinal structures on the fibres of a vector bundle $\pi : X \to B$.

For instance we may choose a euclidian metric $< \, , \, >$ on the fibres which varies differentiably with the point. Then we can choose local trivializations $e_1, \dots, e_n$ which are given by orthonormal basis vectors. If we compare two such local trivializations then our functions g_{ij} will be functions with values in the orthogonal group $O(n)$ and therefore it will correspond to an element in $H^1(B, \mathcal{C}_0(O(n)))$.

In such a situation we say that the additional structure induces a *reduction of the structure group*. In this case we hve a reduction from $Gl_n(\mathbb{R})$ to the orthogonal group $O(n)$.

Another such additional structure is an *orientation.* If we consider the highest exterior power $\Lambda^n(X/B)$, i.e we take the highest exterior power fibre by fibre, then we get a bundle of one dimensional vector spaces. On this bundle we have an action of the multiplicative group of positive real numbers $\mathbb{R}^*_{>0}$. If we divide the bundle by this action then the quotient is a bundle $\tilde{B} \to B$ with fibres consisting of two points. If we can find a global section $s : B \to \tilde{B}$, then we say that $X \to B$ is orientable. If we choose such a section then we say that $X \to B$ is *oriented.*

If we have an orientation on B then we may choose local trivializations $e_1, \dots, e_n$ for which the ordered basis is positive with respect to the orientation. If we have done this then our g_{ij} will take values in the subgroup $Gl_n(\mathbb{R})^+$ of matrices with determinant positive and thus we have another case of the reduction of the structure group.

On a $\mathcal{C}_\infty$ manifold M we have the notion of the tangent bundle T_M. (See a book ???). Locally on M we have coordinate functions $x_1, \dots, x_n$ so that any differentiable function is a differentiable function in the variables $x_1, \dots, x_n$. Then the vector fields $\partial/\partial x_1, \dots, \partial/\partial x_m$. provide a local trivialization of this tangent bundle.

If we have in addition an euclidian metric on this bundle then M is called a *Riemannian manifold.* If we have chosen an orientation (if possible) then we call M *oriented.*

3.3.2 Local systems. If B is a topological space and A an abelian group, then we attached to A the sheaf $\underline{A} = A_B$ of locally constant functions with values in A (see III, 1.3.).

We want to introduce the notion of *local A-systems* or *local systems* of A's.

If $\mathcal{A}$ is a sheaf of abelian groups on B, then we call $\mathcal{A}$ a *local A-system,* if for any point $b \in B$ we can find an open neighborhood V_b such that the restriction of $\mathcal{A}$ to V_b is isomorphic to A_{V_b}. This implies that for any point $b \in B$ the stalk $\mathcal{A}_b$ is isomorphic to A. At this point it is reasonable to assume that our space B is locally connected, i.e. for any point $b \in B$ and any open neighborhood V_b of b we can find a connected open neighborhood $U_b \subset V_b$ of b.

If we have that $\mathcal{A} \mid V_b$ is isomorphic to A_{V_b} as above, and if we replace V_b by the connected open neighborhood U_b, then

$$\mathcal{A}(U_b) \simeq A,$$

and for any point $u \in U_b$ we get an isomorphism

$$\mathcal{A}(U_b) \longrightarrow \mathcal{A}_u.$$

If we now fix a covering $B = \cup V_i$, where the V_i are connected and we have isomorphisms

$$\Psi_i : \mathcal{A} \mid V_i \tilde{\longrightarrow} A_{V_i},$$

then we may compare the Ψ_i on the intersections and we get

$$g_{ij} : V_i \cap V_j \longrightarrow \mathrm{Aut}(A)$$

which are locally constant (or continuous if $\mathrm{Aut}(A)$ is endowed with the discrete topology). Hence we see that the local A-systems are classified by the elements in

$$H^1(B, \underline{\mathrm{Aut}}(A))$$

where $\underline{\mathrm{Aut}}(A)$ is the sheaf of locally constant functions in $\mathrm{Aut}(A)$.

3.3.3 We may even consider local systems of vector spaces. In this case we endow the vector space $\mathbb{R}^d$ with the discrete topology.

Such a local system of vector spaces may also be considered as a vector bundle

$$\pi : X \longrightarrow B$$

where we have local sections $e_1, \ldots, e_n$, which are called "constant". If we pass to a different open set V' and a trivialization of X over V' by constant sections $e'_1, \cdots, e'_n$, then on the intersection

$$e'_i = \sum a_{ij} e_j$$

where now the a_{ij} are locally constant functions on $V_i \cap V_j$.

Of course we can describe the set of isomorphism classes of local systems of vector spaces in terms of non abelian cohomology. We consider the group $G_d = GL_n(\mathbb{K})_d$ which is the general linear group but endowed with the discrete topology. Then it is clear that the isomorphism classes of local systems of n-dimensional $\mathbb{K}$-vector spaces are given by

$$H^1(B, GL_n(\mathbb{K})_d).$$

These local sysytems of vector spaces are the same kind of objects as bundles with a flat connection. (See also 9.1 and 9.2)

IV.4 Fundamental properties of the cohomology of sheaves.

4.1 Introduction. I will now state some results concerning the cohomology of sheaves. They are not so easy to prove. In some cases I can give a hint to how to prove them in an exercise.

If we have any space X and an abelian group A then we may define the sheaf $\underline{A}_X$ of germs of locally constant sections. It is called the *constant sheaf attached to* A . Sometimes we simply write $\underline{A}$. We define the cohomology of X with coefficients in A as

$$H^\bullet(X, A) := H^\bullet(X, \underline{A}_X).$$

If $A = \mathbb{Z}$ then the cohomology groups

$$H^\bullet(X, \mathbb{Z}) := H^\bullet(X, \underline{\mathbb{Z}}_X).$$

are equal to the ones defined by singular cochains, if the space X is reasonable. (This is a theorem, we come back to it later).

The first important result which we will show is that the cohomology of constant sheaves vanishes on certain contractible spaces. We begin by stating a special case which is also the starting point for the more general results:

1.) If $D = \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid \Sigma x_i^2 \leq 1\}$ and $\overset{\circ}{D}$ the interior of D, then

$$H^i(D, A) = H^i(\overset{\circ}{D}, A) = 0 \text{ for } i \geq 1.$$

The following exercise treats the case $n = 1$.

4.1.1 Exercise.

Let us consider the following property ($\mathcal{E}$) *of a sheaf* $\mathcal{A}$ *on the intervall* $X = [-1, 1]$: *For any open interval* $I \subset [-1, 1]$ *the restriction map* $\mathcal{A}([-1, 1]) \to \mathcal{A}(I)$ *is surjective. (We only require that* I *is open in* $[-1, 1]$, *i.e. it may contain the boundary points. Condition* $(\mathcal{E})$ *does not mean that* $\mathcal{A}$ *is flabby!)*

1.) *Show that the sheaves* $\underline{A}_{[-1,1]}$ *and injective sheaves have property* $(\mathcal{E})$.

2.) *If* $\mathcal{A}$ *has property* $(\mathcal{E})$ *and if we have an exact sequence*

$$0 \longrightarrow \mathcal{A} \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow 0$$

then $\mathcal{F}(I) \to \mathcal{G}(I)$ *is surjective for any open interval in* $[-1, 1]$.

3.) *If we have sequence*

$$0 \longrightarrow \mathcal{A} \longrightarrow \mathcal{B} \longrightarrow \mathcal{C} \longrightarrow 0$$

where $\mathcal{A}$ *and* $\mathcal{B}$ *have property* $(\mathcal{E})$ *then* $\mathcal{C}$ *also has property* $(\mathcal{E})$.

4.) *For any sheaf* $\mathcal{A}$ *which satisfies* $(\mathcal{E})$ *we have* $H^q([-1, 1]), \mathcal{A}) = 0$ *for all* $q \geq 1$. *Especially we have* $H^q([-1, 1], \underline{A}) = 0$. *for any abelian group* A.

This is some progress, I think we justified the computation in 1.2. But we need a stronger result and this stronger result is provided by corollary 4.4.9 which says that the cohomology groups are invariant under homotopies. To get to this point we need to investigate a relative situation $f : X \to Y$.

4.2 The derived functor to f_*.

Given two spaces X, Y and a continuous map $f : X \to Y$, we constructed the two functors f_*, f^* which transport sheaves on X to sheaves on Y and sheaves on Y to sheaves on X respectively (see III.4). Let us denote by $\mathcal{S}_X$ the category of sheaves on X with values in the category of abelian groups.

If we have a sheaf $\mathcal{F}$ on X (with values in the category of abelian groups), then we defined the sheaf $f_*(\mathcal{F})$ on Y by

$$f_*(\mathcal{F})(V) = \mathcal{F}(f^{-1}(V))$$

for all open subsets $V \subset Y$. It is clear that $f_*(\mathcal{F})$ is a sheaf on Y. The functor f_* is left exact but not exact in general.

We get our previous case if we take Y to be just one point, i.e. $Y = \{pt\}$. Then the stalk of $f_*(\mathcal{F})_{pt}$ in this point is simply $\mathcal{F}(X) = H^0(X, \mathcal{F})$.

Again we define a derived functor for f_* by the same method as before. We choose an injective resolution

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{I}^0 \longrightarrow \mathcal{I}^1 \longrightarrow \dots$$

of $\mathcal{F}$, and we get a complex of sheaves on Y by taking the direct image

$$0 \longrightarrow f_*(\mathcal{I}^0) \longrightarrow f_*(\mathcal{I}^1) \longrightarrow \dots .$$

This is now a complex of sheaves on the space Y. We define the sheaves (see III.5.1)

$$R^q f_*(\mathcal{F}) = \frac{\ker(f_*(\mathcal{I}^q) \to f_*(\mathcal{I}^{q+1}))}{\operatorname{Im}(f_*(\mathcal{I}^{q-1}) \to f_*(\mathcal{I}^q))}.$$

It is clear that the stalk of $R^q f_*(\mathcal{F})$ in a point y is simply the cohomology of the complex of stalks.

As before, we show that these sheaves do not depend on the choice of the resolution and that for any morphism

$$u : \mathcal{F} \longrightarrow \mathcal{G}$$

we get the derived map

$$R^q u : R^q f_*(\mathcal{F}) \longrightarrow R^q f_*(\mathcal{G}).$$

Finally it is clear that $f_*(\mathcal{F}) = R^0 f_*(\mathcal{F})$, and that any short exact sequence

$$0 \longrightarrow \mathcal{F}' \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}'' \longrightarrow 0$$

leads to a long exact sequence

$$0 \to f_*(\mathcal{F}') \to f_*(\mathcal{F}) \to f_*(\mathcal{F}'') \to R^1 f_*(\mathcal{F}') \to R^1 f_*(\mathcal{F}) \to R^1 f_*(\mathcal{F}'') \to R^2 f_*(\mathcal{F}').$$

The intuitive idea — which in some cases is right in some cases wrong — (see 4.4.7) is that the stalk of $R^q f_*(\mathcal{F})_y$ in a point y should be the cohomology of the fibre $f^{-1}(y) \subset X$ with coefficients in the restriction of $\mathcal{F}$ to this fibre.

4.2.1. The following special case is very important (see III 4.2.). Let us assume that $A \subset X$ is a closed subspace and let $i : A \to X$ be the embedding of A into X. Then $i_*(\mathcal{F})$ is a sheaf on X. It is clear that the stalk of $i_*(\mathcal{F})$ is given by

$$i_*(\mathcal{F})_x = \begin{cases} \mathcal{F}_x & \text{if} \quad x \in A \\ 0 & \text{if} \quad x \notin A \end{cases}.$$

It is clear that $\mathcal{F} \to i_*(\mathcal{F})$ is an exact functor.

4.2.2. We also defined the functor f^*. This functor transforms sheaves on Y into sheaves on X. The idea is that the stalk of $f^*(\mathcal{G})$ in a point $x \in X$ is equal to the stalk of the original sheaf G in the point $y = f(x)$, i.e. $f^*(\mathcal{G})_x = \mathcal{G}_{f(x)}$. Since the exactness of sequences of sheaves can be checked stalkwise, it is clear that f^* is an exact functor. We know that these two functors are adjoint and I recall the adjointness formula

$$\operatorname{Hom}_{\mathcal{S}_X}(f^*(\mathcal{G}), \mathcal{F}) = \operatorname{Hom}_{\mathcal{S}_Y}(\mathcal{G}, f_*(\mathcal{F})).$$

We want to discuss the consequences of existence of f^*, f_* and the adjointness formula for the cohomology and its functorial properties.

4.2.3. Lemma. *If $f : X \to Y$ is continuous, and if $\mathcal{I}$ is an injective sheaf on X, then $f_*(\mathcal{I})$ is injective on Y.*

Proof. This follows directly from the adjointness formula and the exactness of f^*.

4.3 Functorial properties of the cohomology.

If we start from a sheaf $\mathcal{G}$ on the target space Y, and if we take an injective resolution

$$0 \longrightarrow \mathcal{G} \longrightarrow \mathcal{J}^0 \longrightarrow \mathcal{J}^1 \longrightarrow \dots,$$

then we get a resolution

$$0 \longrightarrow f^*(\mathcal{G}) \longrightarrow f^*(\mathcal{J}^0) \longrightarrow f^*(\mathcal{J}^1) \longrightarrow \dots.$$

As we have seen earlier (III.3.5), this gives us a map

$$H^q(0 \to f^*(\mathcal{J}^0)(X) \to f^*(\mathcal{J}^1) \to \dots) \longrightarrow H^q(X, f^*(\mathcal{G})).$$

On the other hand we have a map between the complexes

$$\begin{array}{ccccccc}
0 & \longrightarrow & \mathcal{J}^0(Y) & \longrightarrow & \mathcal{J}^1(Y) & \longrightarrow \\
 & & \Big\downarrow & & \Big\downarrow & \\
0 & \longrightarrow & f^*\mathcal{J}^0(X) & \longrightarrow & f^*\mathcal{J}^1(X) & \longrightarrow
\end{array}$$

(This follows from the definition of f^*.), and hence a functorial map

$$H^q(Y,\mathcal{G}) \longrightarrow H^q(X, f^*\mathcal{G}).$$

There is an especially important case of this: If $f : X \to Y$, and we consider the sheaf $\underline{\mathbb{Z}}_Y$ on Y, then we see easily

$$f^*(\underline{\mathbb{Z}}_Y) = \underline{\mathbb{Z}}_X.$$

To see this we construct a homomorphism from $f'(\underline{\mathbb{Z}}_Y)$ to $\underline{\mathbb{Z}}_X$: For $U \subset X$, U open, we have

$$f'(\mathbb{Z})(U) = \varinjlim_{V \supset f(U)} \mathbb{Z}(V).$$

For $V \subset f(U)$ we have $f^{-1}(V) \subset U$, and of course, we have maps

$$\underline{\mathbb{Z}}_Y(V) \longrightarrow \underline{\mathbb{Z}}_X(f^{-1}(V)) \longrightarrow \underline{\mathbb{Z}}_X(U),$$

and this provides a map

$$f'(\underline{\mathbb{Z}}_Y)(U) \longrightarrow \underline{\mathbb{Z}}_X(U).$$

This is a map from the praesheaf $f'(\underline{\mathbb{Z}}_Y)$ to the sheaf $\underline{\mathbb{Z}}_X$, and this provides a unique map

$$f^*(\underline{\mathbb{Z}}_Y) \longrightarrow \underline{\mathbb{Z}}_X.$$

Looking at the stalks we see that this map is an isomorphism.
This yields the functoriality of the cohomology groups $H^q(X,\mathbb{Z})$. For any map $f : X \to Y$ we get

$$f^q : H^q(Y,\mathbb{Z}) \longrightarrow H^q(X,\mathbb{Z}).$$

There is another case: We always get a map $H^q(Y, f_*\mathcal{F}) \to H^q(X, f^*f_*\mathcal{F})$ and the adjointness provides the map $f^*f_*\mathcal{F} \to \mathcal{F}$ which corresponds to the identity $f_*\mathcal{F} \to f_*\mathcal{F}$. The composition of these two maps yields a map $f^q : H^q(Y, f_*\mathcal{F}) \to H^q(X,\mathcal{F})$. For this map we have an easy theorem:

4.3.1 Theorem. *Let us assume that $f : X \to Y$ is continuous and $\mathcal{F}$ a sheaf on X. If the higher derived sheaves $R^q f_*(\mathcal{F}) = 0$ for $q \geq 1$ then we get an isomorphism*

$$f^q : H^q(X,\mathcal{F}) \xrightarrow{\sim} H^q(Y, f_*(\mathcal{F}))$$

for all $q \geq 1$.

This is clear: We start from an injective resolution

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{I}^0 \longrightarrow \mathcal{I}^1 \longrightarrow \dots.$$

Then our assumption says that

$$0 \longrightarrow f_*(\mathcal{F}) \longrightarrow f_*(\mathcal{I}^0) \longrightarrow f_*(\mathcal{I}^1) \longrightarrow$$

is a resolution, and the lemma implies that this resolution is injective. Hence

$$H^q(Y, f_*(\mathcal{F})) = H^q(0 \to f_*(\mathcal{I}^0)(Y) \longrightarrow f_*(\mathcal{I}^1)(Y) \longrightarrow \dots).$$

But the complex of sections over Y is equal to the complex

$$0 \longrightarrow \mathcal{I}^0(X) \longrightarrow \mathcal{I}^1(X) \longrightarrow \dots,$$

which gives the cohomology $H^q(X, \mathcal{F})$.

One important consequence of this theorem is the case of an embedding

$$i : A \hookrightarrow X$$

where A is a closed subspace of X. In this case we have seen that i_* is an exact functor from sheaves on A to sheaves on X, hence $R^q i_*(\mathcal{F}) = 0$ for $q \geq 1$ and

$$H^q(A, \mathcal{F}) = H^q(X, i_*(\mathcal{F})).$$

4.3.2 If we want to apply the theorem above we have to understand how to compute the sheaves $R^q f_*(\mathcal{F})$. We want to show that under certain assumptions the stalks $R^q f_*(\mathcal{F})_y = H^q(f^{-1}(y), i_y^*(\mathcal{F}))$.

To get more precise informations which will allow us to compute cohomology groups in certain cases we have to make assumptions on our spaces.

4.4 Paracompact spaces.

In general the sheaves $R^q f_*(\mathcal{F})$ may be very difficult to compute. One possibility is to relate the stalks $R^q f_*(\mathcal{F})_y$ to the cohomology groups of the fibre $f^{-1}(y)$. This is possible if our spaces have certain finiteness and seperability properties.

A covering $X = \bigcup_{\alpha \in \mathbf{A}} U_\alpha$ is called *locally finite* if for any point $x \in X$ we can find a neighborhood V_x of x such that V_x meets only finitely many of the U_α, i.e. the set of indices α for which $V_x \cap U_\alpha \neq \emptyset$ is finite.

A space X is called *paracompact* if it is Hausdorff and if for any open $U \subset X$ and any covering $U = \bigcup\limits_{i \in I} U_i$ we can find a locally finite refinement of the covering. Recall that a refinement of the covering is another covering $U = \bigcup_{j \in J} W_j$ together with a map $\tau : J \to I$ such that for all $j \in J$ we have the inclusion $W_j \subset U_{\tau(j)}$. We call such a refinement a *strong refinement* if even the closures $\bar{W}_j$ are contained in $U_{\tau(j)}$.

I claim:

If our space X is paracompact and locally compact then any covering of an open set $U = \bigcup_{i \in I} U_i$ has a strong locally finite refinement.

Since our space is Hausdorff and locally compact we know: For any point $x \in X$ and any open neighborhood V_x of x we can find an open neighborhood W_x such that its closure $\overline{W}_x$ is contained in V_x. Now it is clear how get a strong locally finite refinement of a covering $U = \bigcup_{i \in I} U_i$: We can construct a strong refinement of the covering and after that we construct a locally finite refinement of this strong refinement.

4.4.1 We have a simple criterion for paracompactness. We say that an open subset $U \subset X$ is *exhaustible by compact subsets* if we we can find a increasing sequence of compact subsets

$$\emptyset = K_0 \subset \ldots \subset K_n \subset K_{n+1} \subset \ldots$$

s. t. $U = \bigcup K_n$ and for any n the compact set K_n is contained in the interior $\overset{\circ}{K}_{n+1}$ of the next one.

4.4.2 Lemma. *A Hausdorff space X for which any open subset is exhaustible by compact subsets is paracompact.*

To see this we consider $U \subset X$ and a covering $U = \bigcup_{i \in I} U_i$ by open subsets. We choose inductively finite coverings of K_n, where each covering set is contained in $\overset{\circ}{K}_{n+1}$, in one of the U_i of the given covering and has empty intersection with K_{n-1}. It is clear that the resulting covering is locally finite: For any point $x \in U$ we have a smallest n such that $u \in K_n$ then it has an open neighborhood which is contained in some $\overset{\circ}{K}_{n+1}$ and does not meet K_{n-1} and this neighborhood clearly meets only finitely many of the sets in the covering.

We say that our space X is *exhaustible by compact sets*, if the open subset $U = X$ has this property.

4.4.3 It is not difficult to show that a Hausdorff space is paracompact if it is exhaustible by compact sets and if any open set U can be exhausted by a sequence of sets

$$W_n \subset \overset{\circ}{W}_{n+1} \subset W_{n+1}$$

where the W_n are only closed subsets of X.

To see that this is true we observe that a closed subspace $A \subset X$ is exhaustible by compact sets. This implies that any covering of A by open sets has a locally finite refinement. (Same proof as for Lemma 1.5.2.) Assume that we have a covering of U by open sets. We proceed as in the proof of Lemma 1.5.2 but now we construct locally finite coverings of the W_n (instead of finite ones) where we obey the same precautions as before.

We come to a very technical lemma which says something about extension of sections.

Assume that we have a closed embedding $i : A \hookrightarrow X$. We call this a *nice emdedding* if the following is true:

Let $\mathcal{F}$ be a sheaf on X, we consider the sheaf $i^*(\mathcal{F})$ on A. By construction it is the sheafification of the preasheaf $V \to i'(\mathcal{F})(V)$ where V is open in A and

$$i'(\mathcal{F})(V) = \varinjlim_{U \supset V} (\mathcal{F}(U)).$$

If $\tilde{s} \in \mathcal{F}(U)$ and if s is its image in $i^*(F)(U \cap A)$ then we say that s is the restriction of $\tilde{s}$ to $A \cap U = V$. Now the condition for being nice is that if any section $s \in i^*(\mathcal{F})(V)$ can be extended into some neighborhood U of V in X, this means it is in the image of $\mathcal{F}(U) \to i'(\mathcal{F})(V)$ for some U which satisfies $U \cap A \supset V$. This can be reformulated by saying that $i'(\mathcal{F})$ is already a sheaf.

4.4.4 Lemma. (Extension of sections): *If X is paracompact and locally compact then any closed embedding $i : A \hookrightarrow X$ is nice.*

Proof. We start with $V \subset A$ and our section $s \in \mathcal{F}(V)$. We know from the definition of $i^*(\mathcal{F})$ that for any point $p \in V$ the image of s in the stalk $s_p \in i^*(\mathcal{F})_p$ is the restriction of a section $\tilde{s}_p \in \mathcal{F}(U_p)$ where U_p is an open neighborhood of p in X. Hence we can find a covering $\bigcup_\alpha U_\alpha \supset V$ and sections $\tilde{s}_\alpha \in \mathcal{F}(U_\alpha)$ such that $\tilde{s}_\alpha$ maps to the restriction $s|U_\alpha \cap V$. We may assume that this covering is locally finite since our space is paracompact. Let $\{W_j\}_{j \in J}$ be a strong locally finite refinement of this covering. As usual we denote the map between the indexing sets by $\tau : J \to I$.

Let $q \in V$ we can find an open neighborhood V_q of q in X such that V_q meets only finitely many of the W_j and the U_α.. We choose an open neighborhood $D_q \subset V_q$ which is contained in W_j for all those (finitely many) j for which $q \in W_j$ and also in all those finitely many U_α with $q \in U_\alpha$. We may also choose D_q so small that $D_q \cap W_j = \emptyset$ if $q \notin \bar{W}_j$ because D_q meets only finitely many of them anyway. It follows from the defintion of i^* that we can take these D_q so small that we have $\tilde{s}_\alpha|D_q = \tilde{s}_\beta|D_q$ whenever $q \in U_\alpha \cap U_\beta$. Let $\tilde{\tilde{s}}_p \in \mathcal{F}(D_p)$ be the restriction of any of these $\tilde{s}_\alpha$. I claim that these sections $\tilde{\tilde{s}}_p, \tilde{\tilde{s}}_q$ restrict to the same section over $D_p \cap D_q$ for any pair p, q. This is clear if $D_p \cap D_q = \emptyset$ so we may assume that $D_p \cap D_q \neq \emptyset$. If $D_p \subset W_j$ then $q \in \bar{W}_j$ because otherwise we have $D_q \cap W_j = \emptyset$ by construction and this implies $D_p \cap D_q = \emptyset$, a contradiction. We have $D_p \subset W_j \subset U_{\tau(j)}$. Since the W_j formed a strong refinement of the U_α we even know that $\bar{W}_j \subset U_{\tau(j)}$. Hence $q \in U_{\tau(j)}$ and then we conclude that $D_q \subset U_{\tau(j)}$ again by construction. Consequently we have that D_p and D_q are contained in $U_{\tau(j)}$ and this implies that the sections $\tilde{\tilde{s}}_p, \tilde{\tilde{s}}_q$ are restrictions of $\tilde{s}_{\tau(j)}$. Hence the $\tilde{\tilde{s}}_p$ define a section $\tilde{s}$ over $U = \cup D_q$ and this is the element we where looking for.

4.4.5. I want to discuss a variant of this lemma. Let us assume that only the closed subset A is locally compact and paracompact. But in addition we assume that can find an subset $W \subset X$ with $W \supset A$ and such that $W \tilde{\to} W_0 \times A$ where W_0 is a topological space and we assume that the isomorphism sends $A \tilde{\to} \{w_0\} \times A$ where w_0 is a point in W_0.

I claim that under these assumptions the embedding is nice.

This can be shown by a slight modification of the proof of lemma 4.4.4. We proceed as in the proof but we choose the open sets U_α to be of the form $U_\alpha = V_\alpha \times W_\alpha$ where V_α is open in V and W_α is a neighborhood of w_0 in W_0. Then we choose a strong locally finite refinement of the covering $V = \bigcup_{\alpha \in I} V_\alpha$, let us denote this refinement by $V = \bigcup_{\beta \in J} Y_\beta$ and let $\tau : J \to I$ be the map for which $Y_\beta \subset V_{\tau(\beta)}$. This gives us a covering of V by open set in X: We have $V \subset \bigcup_{\beta \in J} Y_\beta \times W_{\tau(\beta)}$. This covering now plays the role of the covering by the W_j in the proof of the Lemma 4.4.4. We proceed essetially in the same way as before. We choose neighborhoods D_q which satisfy $D_q \subset Y_\beta \times W_{\tau(\beta)}$ if $q = (q, w_0) \in Y_\beta \times W_{\tau(\beta)}$ and $D_q \cap Y_\beta \times W_{\tau(\beta)} = \emptyset$ if - here we have a slight modification- $q \notin \bar{Y}_\beta \times W_{\tau(\beta)}$. From here on the argument is the same.

4.4.6 Lemma. *Let $i : A \hookrightarrow X$ be a nice embedding. If $\mathcal{I}$ is an injective sheaf on X then $i^*(\mathcal{I})$ is flabby and hence acyclic.*

Proof: This follows from 2.2.2

These technical considerations will be applied to prove the following difficult theorem:

4.4.7 Theorem (Base change): *Let us assume that X is paracompact, that Y is locally compact and Hausdorff and that*

$$f : X \longrightarrow Y$$

is a proper map. Then for any sheaf $\mathcal{F}$ on X and any $y \in Y$ we have

$$R^q f_*(\mathcal{F})_y = H^q(f^{-1}(y), i_y^*(\mathcal{F})).$$

Recall that f is called proper if the inverse image of a compact set in Y is again compact.

Proof. We shall need a modification of the theorem therefore we will also discuss to what extend we really need our assumptions.

Let $i_y : f^{-1}(y) \hookrightarrow X$ be the inclusion of the (closed and compact) fibre. Then we know from our assumptions that the embedding is nice (Lemma 4.4.4).

We formulate the following condition on our map f

$$\textit{For any open neighborhood of a fibre } U \supset f^{-1}(y) \textit{we find a } V_0 \textit{s.t.} f^{-1}(V_0) \subset U. \qquad \text{(Cyl)}$$

We show that (Cyl) is valid under the asssumption of the theorem. We consider the intersection

$$(X \setminus U) \cap f^{-1}(\overline{V})$$

for all closures $\overline{V}$ of compact open neighborhoods if V. Since $f^{-1}(\overline{V})$ is compact, the intersection is also compact. Since U is a neighborhood of $f^{-1}(y)$ we know that for $x \in X \setminus U$ we have $f(x) \neq y$. We may choose open neighborhoods $W_{f(x)}, V_y$ s.t. their closure is compact and $f(W_{f(x)}) \cap V_y = \emptyset$, hence $f(x) \notin \overline{V}_y$. Hence $x \notin (X \setminus U) \cap f^{-1}(\overline{V}_y)$ and therefore

$$\bigcap_{V \ni y} (X \setminus U) \cap f^{-1}(\overline{V}) = \emptyset.$$

Now it follows from a standard argument on compact spaces that there must be a V, $y \in V$ with $f^{-1}(V) \subset U$.

The following considerations prove the assertion of the theorem under the two assumptions

a) for all y the fibre $f^{-1}(y)$ is closed and the embedding is nice

b) (Cyl) holds.

By definition we have

$$f_*(\mathcal{A})_y = \varinjlim_{V: y \in V} \mathcal{A}(f^{-1}(V))$$

and (Cyl) implies that

$$\varinjlim_{V: y \in V} \mathcal{A}(f^{-1}(V)) = \varinjlim_{U: f^{-1}(y) \subset U} \mathcal{A}(U).$$

Then the fact that the embedding of the fibre is nice yields

$$\varinjlim_{U: f^{-1}(y) \subset U} \mathcal{A}(U) = i_y^*(\mathcal{A})$$

and we conclude

$$f_*(\mathcal{A})_y = i_y^*(\mathcal{A})$$

This proves our the theorem for $q = 0$. To prove it in general, we start from an injective resolution

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{I}^0 \longrightarrow \mathcal{I}^1 \longrightarrow \dots$$

on X. Then (we sometimes drop the brackets inf_*)

$$0 \longrightarrow f_*\mathcal{F} \longrightarrow f_*\mathcal{I}^0 \longrightarrow \dots$$

is a complex of injective sheaves. If we pass to the sequence of stalks at a point $y \in Y$, we get a complex of abelian groups

$$0 \longrightarrow f_*\mathcal{F}_y \longrightarrow f_*\mathcal{I}^0_y \longrightarrow \dots,$$

and the cohomology of this complex is the stalk $R^q f_*(\mathcal{F})_y$. But this complex is equal to the complex

$$0 \longrightarrow i_y^*(\mathcal{F})(f^{-1}(y)) \longrightarrow i_y^*\mathcal{I}^0(f^{-1}(y)) \longrightarrow \dots,$$

and this is the complex of global sections of the complex of sheaves on $f^{-1}(y)$:

$$0 \longrightarrow i_y^*\mathcal{F} \longrightarrow i_y^*\mathcal{I}^0 \longrightarrow \dots$$

which is a flabby and hence acyclic resolution of $i_y^*\mathcal{F}$. Hence the cohomology of the above complex of global sections is

$$H^q(f^{-1}(y), i_y^*(\mathcal{F})).$$

4.4.8 Corollary. *If X, Y and $f : X \to Y$ are as in the theorem and if*

$$H^q(f^{-1}(y), i_y^*(\mathcal{F})) = 0 \quad \text{for} \quad q \geq 1 \quad \text{and all} \quad y \in Y,$$

then

$$f^q : H^q(Y, f_*\mathcal{F}) \xrightarrow{\sim} H^q(X, \mathcal{F})$$

is an isomorphism.

This is the combination of the difficult and the easy theorem. The following corollary is not a direct consequence of the theorem.

4.4.9 Corollary. *If X is a Hausdorff space and if*

$$p : X \times [0,1] \longrightarrow X,$$

is the projection to the first factor then this projection induces isomorphisms in cohomology

$$p^\bullet : H^\bullet(X, \underline{\mathbb{Z}}) \xrightarrow{\sim} H^q(X \times [0,1], \underline{\mathbb{Z}}).$$

For any $t \in [0,1]$ the inclusion $x \mapsto x \times \{t\}$ induces an isomorphism in cohomology.

This is not a direct consequence of the the theorem as it is stated since we do not make any assumption on X except that it is Hausdorff. But first of all our modifified lemma 4.4.6 implies that for any point x_0 in X the embedding $\{x_0\} \times I \hookrightarrow X \times I$ is nice. (We need that the fibre is closed so we can get away with the weaker assumtion that points in X are closed). Secondly it is clear that the condition (Cyl) in the proof of the Base change theorem is also fulfilled. This means that the proof is valid for the projection p.

The rest is clear since $p_*\underline{\mathbb{Z}} = \underline{\mathbb{Z}}$, and since

$$H^q(\{x\} \times [0,1], \underline{\mathbb{Z}}) = 0.$$

The second assertion follows if we compose the inclusion with the projection.

4.4.10 The homotopy axiom. *Let X be a Hausdorff space. If we have two maps*

$$f, g : X \longrightarrow Y$$

which are homotopic, i.e. there is a map

$$F : X \times [0,1] \longrightarrow Y,$$

so that $F(x,0) = g(x)$, $F(x,1) = f(x)$ then

$$f^\bullet = g^\bullet : H^\bullet(Y, \underline{\mathbb{Z}}) \longrightarrow H^\bullet(X, \underline{\mathbb{Z}}).$$

Proof. Look at

$$X \underset{\text{bot}}{\overset{\text{top}}{\rightrightarrows}} X \times [0,1] \xrightarrow{F} Y$$

where the arrows are $x \to (x,0)$, $x \to (x,1)$. If we compose these arrows with F we get f, g.

4.4.11. A space X is called *contractible to a point* $p \in X$ if the two maps $f = \mathrm{Id}$ and the map g which maps all the points in X to the point p are homotopic.

If we apply the homotopy axiom to this two maps we get

For a contractible Hausdorff space X we have

$$H^i(X, \mathbb{Z}) = 0 \textit{ for all } i > 0$$

It is clear that the space $\mathbb{R}^n$ is contractible. The same thing holds for any open ball $B^n\{(x_1, \ldots, x_n) | \sum x_i^2 < 1\}$ and also its closure.

4.4.12 Application.

We have the tools to compute cohomology groups of spheres and other simple spaces.

We consider the sphere $S^n = \{(x_0, x_1, \ldots, x_n) \in \mathbb{R}^{n+1} | x_0^2 + x_1^2 + \ldots + x_n^2 = 1\}$. We cover it by the two balls $D_\pm$ which are defined by $x_n \geq 0$ or $x_n \leq 0$ respectively. We have the two inclusions $i_\pm : D_\pm \hookrightarrow S^n$. These balls are contractible, we have the sheaves $\underline{\mathbb{Z}}_{D_\pm}$ which we extend to the two sheaves $\underline{\mathbb{Z}}_\pm = i_{\pm *}(\underline{\mathbb{Z}}_{D_\pm})$. We also have maps $\underline{\mathbb{Z}}_{S^n} \to \underline{\mathbb{Z}}_\pm$ which on open sets $V \subset S^n$ are defined by the restriction $\underline{\mathbb{Z}}_{S^n}(V) \to \underline{\mathbb{Z}}_\pm(V) = \underline{\mathbb{Z}}_{D_\pm}(V \cap D_\pm)$. This gives an inclusion $\underline{\mathbb{Z}}_{S^n} \hookrightarrow \underline{\mathbb{Z}}_+ \oplus \underline{\mathbb{Z}}_-$ which is an isomorphism in all the stalks which are not over the intersection of the two balls which is the sphere S^{n-1}. In the points x in the intersection the inclusion is given by the diagonal $\underline{\mathbb{Z}}_x = \mathbb{Z} \subset (\underline{\mathbb{Z}}_+ \oplus \underline{\mathbb{Z}}_-)_x = \mathbb{Z} \oplus \mathbb{Z}$ From this we get an exact sequence

$$0 \to \underline{\mathbb{Z}}_{S^n} \to \underline{\mathbb{Z}}_+ \oplus \underline{\mathbb{Z}}_- \to \underline{\mathbb{Z}}_{S^{n-1}} \to 0,$$

where the map $s : (\underline{\mathbb{Z}}_+ \oplus \mathbb{Z}_-)_x = \mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z}_{S^{n-1},x} = \mathbb{Z}$ is the difference betweeb the $+$ and $-$ component.

The cohomology of the two balls is trivial except in degree zero, hence we get

$$H^{\nu-1}(S^{n-1}, \underline{\mathbb{Z}}) \tilde{\to} H^\nu(S^n, \underline{\mathbb{Z}})$$

if $\nu - 1 > 0$ and in degree zero we find the exact sequence

$$0 \to H^0(S^n, \underline{\mathbb{Z}}) \to H^0(D_+, \underline{\mathbb{Z}}) \oplus H^0(D_-, \underline{\mathbb{Z}}) \to H^0(S^{n-1}, \underline{\mathbb{Z}}) \to H^1(S^n, \underline{\mathbb{Z}}) \to 0.$$

Putting all this information together we can prove rather easily that

$$H^0(S^0, \underline{\mathbb{Z}}) = \mathbb{Z} \oplus \mathbb{Z}$$

and for $n > 0$ we have

$$H^\nu(S^n, \underline{\mathbb{Z}}) = \begin{cases} 0 & \text{for } \nu \neq 0, n \\ \mathbb{Z} & \text{for } \nu = 0 \text{ or } \nu = n \end{cases}$$

This is of course essentially the same calculation as the one one finds in books on topology. In these books the two essential ingredients are homotopy and the so called Mayer-Vietoris sequence. Here the Mayer-Vietoris sequence is replaced by the construction of suitable exact sequences of sheaves.

Of course we have to be aware that the isomorphism $H^n(S^n, \mathbb{Z}) \tilde{\to} \mathbb{Z}$ is not canonical (See also the example 1 (ii) at the beginning of this chapter). It depends on the choice of the homomorphism s above. To make this choice we used the explicit description of the sphere to decide what D_+ and D_- is. The point is that it suffices to choose an orientation on S^n, then we can identify $H^n(S^n, \mathbb{Z}) = \mathbb{Z}$ (See 3.3.1 and 7.3,7.4) . To see this we proceed inductively. If we have covered $S^n = D_a \cup D_b$ and $D_a \cap D_b = S^{n-1}$, and if we have choosen an orientation on S^{n-1}, then we can do the following: We choose a positively oriented basis $e_1 \cdots e_{n-1}$ of tangent vectors in a point $x \in S^{n-1}$. Now we choose a normal vector v to S^{n-1} at x such that $e_1 \cdots e_{n-1}, v$ is positively oriented. This normal vector points into D_a or D_b. Now we put $D_+ = D_a$ if it points to D_a or $D_+ = D_b$ in the other case. Then we have a unique choice for s, and we get a canonical isomorphism

$$H^{n-1}(S^{n-1}, \mathbb{Z}) \longrightarrow H^n(S^n, \mathbb{Z}).$$

Since we selected an orientation on S^{n-1} we have $H^{n-1}(S^{n-1}, \mathbb{Z}) = \mathbb{Z}$.

4.4.13. A two dimensional, compact, oriented manifold is called an oriented surface. The simplest example is the 2-sphere S^2. If we have such a surface S we can construct a new one by the following construction: We pick two different points $p, q \in S$ and we choose two small neighborhoods D_p, D_q which are homeomorphic to a two dimensional disc. The boundaries $\partial\bar{D}_p, \partial\bar{D}_q$ can be identified to the oriented circle S^1. We form a cylinder $S^1 \times [0,1]$. We remove the interior of the two disks from the surface S and map $\partial(S^1 \times [0,1]) = S^1 \times \{0\} \cup S^1 \times \{1\}$ by taking the identity on each component to the boundaries of our two discs in $S \setminus D_p \cup D_q$.

Using this map we glue the cylinder to our surface, we add a so called handle. It is a theorem in two dimensional topology that any oriented surface S can be obtained from the sphere by adding a certain number of handles.

4.4.14 Exercise.

Let S be a compact oriented surface which has been obtained from the sphere by adding g handles. Show that $H^0(S, \underline{\mathbb{Z}}) = H^2(S, \underline{\mathbb{Z}}) = \mathbb{Z}$ and $H^1(S, \underline{\mathbb{Z}}) \xrightarrow{\sim} \mathbb{Z}^{2g}$.

Hint: Construct a sequence sheaves on S which is suggested by the process of adding a handle and proceed by induction.

We can also understand the cohomology of our oriented surface without such an explicit construction.This will be discussed in the section on Poincaré duality (see 8.3.).

IV.5 Čech cohomology of sheaves

5.1. The Čech-complex:

For any space X, any sheaf $\mathcal{F}$ on X with values in the category of abelian groups and any open covering

$$\mathfrak{U} = \{U_i\}_{i \in I}, \quad X = \bigcup_{i \in I} U_i$$

of X, we will define the Čech-cohomology groups

$$\check{H}^q(X, \mathfrak{U}, \mathcal{F}), \quad q = 0, 1, \dots$$

To define these cohomology groups we introduce the so-called Čech complex. For any set of indices $(i_0, \dots, i_q) \in I^{q+1}$ we define

$$U_{i_0 \dots i_q} = U_{i_0} \cap U_{i_1} \cap \dots \cap U_{i_q}.$$

Then we put

$$C^q(X, \mathfrak{U}, \mathcal{F}) = \prod_{(i_0, \dots, i_q) \in I^{q+1}} \mathcal{F}(U_{i_0 \dots i_q})$$

for $q = 0, 1, \dots$. We define a boundary map

$$d : C^q(X, \mathfrak{U}, \mathcal{F}) \to C^{q+1}(X, \mathfrak{U}, \mathcal{F})$$

by the following formula

$$(dc)_{i_0\dots i_{q+1}} = \sum_{\nu=0}^{q+1}(-1)^\nu \operatorname{res}(c_{i_0,\dots,\hat{i}_\nu,\dots,i_{q+1}}).$$

We have to explain why this formula makes sense:
An element $c \in C^q(X,\mathfrak{U},\mathcal{F})$ is an element in a product and has components

$$c_{j_0\dots j_q} \in \mathcal{F}(U_{j_0\dots j_q}).$$

Hence dc will also have components which are indexed by elements in I^{q+2}. An element $(i_0,\dots,i_{q+1}) \in I^{q+2}$ provides $q+2$ elements in I^{q+1} which are obtained by suppressing one of the components. By $(i_0,\dots,\hat{i}_\nu,\dots,i_{q+1})$ we denote the element in in I^{q+1} where we removed i_ν.
For all these $q+2$ possibilities we have the restriction associated to $U_{i_0\dots i_{q+1}} \subset U_{i_0,\dots\hat{i}_\mu\dots,i_{q+1}}$ which we simply denote by

$$\operatorname{res} : \mathcal{F}(U_{i_0,\dots,\hat{i}_\mu,\dots,i_{q+1}}) \to \mathcal{F}(U_{i_0,\dots,i_{q+1}}).$$

Now it is clear that the formula gives the rule to compute the $(i_0\dots i_{q+1})$-component of dc. We leave it as an exercise to prove that $d\circ d = 0$. Hence $(C^\bullet(X,\mathfrak{U},\mathcal{F}),d)$ is a complex of abelian groups.
Let us look at the beginning of our complex

$$0 \to \prod_{i\in I}\mathcal{F}(U_i) \xrightarrow{d} \prod_{(i,j)\in I\times I}\mathcal{F}(U_i\cap U_j) \to \dots .$$

An element

$$c = (\dots,c_i,\dots)$$

in the first term goes to zero if and only if

$$c_i|U_i\cap U_j = c_j|U_i\cap U_j \text{ for all } i,j$$

But since $\mathcal{F}$ is a sheaf this implies that this is the case if and only if c comes from a uniquely defined global section $s \in \mathcal{F}(X)$, i.e. $s_i = s|U_i$ for all i.
We define cycles $Z^q(X,\mathfrak{U},\mathcal{F})$ to be the kernel of d and boundaries are the elements $b \in C^q(X,\mathfrak{U},\mathcal{F})$ of the form $b = dc$ with $c \in C^{q-1}(X,\mathfrak{U},\mathcal{F})$. The boundaries form a subgroup $\mathcal{B}^q(X,\mathfrak{U},\mathcal{F})$ of $Z^q(X,\mathfrak{U},\mathcal{F})$ and now we put

$$\check{H}^q(X,\mathfrak{U},\mathcal{F}) = Z^q(X,\mathfrak{U},\mathcal{F})/\mathcal{B}^q(X,\mathfrak{U},\mathcal{F}).$$

We just saw

$$\check{H}^0(X,\mathfrak{U},\mathcal{F}) = \mathcal{F}(X),.$$

5.1.1 Remark. In general these Čech cohomology groups do depend on $\mathfrak{U}$. Later on we shall see that under certain assumptions on the sheaves and on the space and the nature of the covering they will be independent of the covering.

We have the notion of a refinement of a covering (See III.3.1). If $\tau : \mathfrak{V} \to \mathfrak{U}$ is such a refinement the map τ between the indexing set yields a map between the Čech complexes $\tau^\bullet(C^\bullet(X, \mathfrak{U}, \mathcal{F}), d) \to (C^\bullet(X, \mathfrak{V}, \mathcal{F}), d)$ and we get a map

$$\check{H}^\bullet(X, \mathfrak{U}, \mathcal{F}) \to \check{H}^\bullet(X, \mathfrak{V}, \mathcal{F}).$$

It is possible to show that this map on the level of cohomology does not depend on τ, but we do not need this fact here. Since the coverings form a category we can define the Čech cohomology groups of a space as direct limit

$$\varinjlim_{\mathfrak{U}} \check{H}^\bullet(X, \mathfrak{U}, \mathcal{F}) = \check{H}^\bullet(X, \mathcal{F}).$$

5.1.2 The alternating complex.

We can also look at the so called alternating complex $C^\bullet_{\text{alt}}(X, \mathfrak{U}, \mathcal{F})$. It is defined as the subcomplex where the cochains satisfy

(i)
$$c_{i_0,\dots,x,\dots,x,\dots,i_q} = 0$$

and

(ii)
$$c_{i_0,\dots,x,\dots,y,\dots,i_q} = -c_{i_0,\dots,y,\dots,x,\dots,i_q}.$$

It is not too difficult to prove that $C^\bullet_{\text{alt}}(X, \mathfrak{U}, \mathcal{F})$ is a subcomplex, i.e. the coboundary operator maps it into itself. It is a little bit more difficult to prove that

$$C^\bullet_{\text{alt}}(X, \mathfrak{U}, \mathcal{F}) \to C^\bullet(X, \mathfrak{U}, \mathcal{F})$$

induces an isomorphism in cohomology. Sometimes it is easier to do computations using this smaller complex.

5.1.3 Exercise.

a) Prove that

$$C^\bullet_{\text{alt}}(X, \mathfrak{U}, \mathcal{F}) \to C^\bullet(X, \mathfrak{U}, \mathcal{F})$$

induces an isomorphism in cohomology.

b) Consider the simplex

$$\Delta^{n+1} = \{(x_1, \dots, x_{n+2}) \in \mathbb{R}^{n+2} | x_i \geq 0, \sum x_i = 1\}.$$

Then we get a covering $\mathfrak{U}$ of Δ^{n+1} by the open sets

$$U_i = \{(x_1, \ldots, x_{n+2}) \in \Delta n + 1 | x_i > 0\}.$$

Show that the cohomology groups

$$\check{H}^m(\Delta^{n+1}, \mathfrak{U}, \underline{\mathbb{Z}}) = \begin{cases} = 0 & \text{if } m > 0 \\ = \mathbb{Z} & \text{if } m = 0. \end{cases}$$

c) Now we remove the interior of Δ^{n+1} and we get the n-dimensional sphere

$$\partial\Delta = S^n = \{(x_1, \ldots, x_{n+2}) | \quad \text{at least one of the } x_i \text{ is zero}\}.$$

Our covering of Δ^{n+1} induces a covering $\mathfrak{U}'$ on S^n.

$$U_i = \{(x_1, \ldots, x_{n+2}) \in S^n | x_i > 0\}.$$

Show that the Čech-cohomology groups $\check{H}^\bullet(S^n, \mathfrak{U}', \underline{\mathbb{Z}})$ coincide with cohomology groups computed by injective resolutions.

A rather elegant solution of this exercise can be obtained if we use the following Lemma whose proof I give for later references.

5.1.4 Lemma. *Let $\mathfrak{U}$ be a covering of an arbitrary space X and let us assume that in our covering $\mathfrak{U} = \{U_i\}_{i \in I}$ is a member $y \in I$ for which $U_y = X$. Then we have $H^q(X, \mathfrak{U}, \mathcal{F}) = 0$ for all $q \geq 1$.*

Proof. Let us assume we have a cocycle

$$c = (\ldots c_{i_0,\ldots,i_q} \ldots)_{(i_0,\ldots,i_q) \in I^{q+1}} \in Z^q(X, \mathfrak{U}, \mathcal{F}).$$

We construct a cochain $b \in C^{q-1}(X, \mathfrak{U}, \mathcal{F})$ by

$$b_{i_0,\ldots,i_{q-1}} = c_{y,i_0,\ldots,1_{q-1}}.$$

We have to observe that

$$U_{i_0,\ldots,i_{q-1}} = U_{y,i_0,\ldots,i_{q-1}}.$$

Then

$$\begin{aligned} (db)_{i_0,\ldots,i_q} &= \Sigma(-1)^\nu b_{i_0,\ldots,\hat{i}_\nu,\ldots,i_q} \\ &= \Sigma(-1)^\nu c_{y,i_0,\ldots,\hat{i}_\nu,\ldots,i_q} = -(dc)_{y,i_0,\ldots,i_q} + c_{i_0,\ldots,i_q} = c_{i_0,\ldots,i_q}. \end{aligned}$$

To apply this to the exercise above we can consider the inclusion $U_0 \hookrightarrow \Delta^{n+1}$. The covering of Δ^{n+1} induces a covering of U_0 and these two coverings yield the same Čech complexes.

5.2 The Čech resolution of a sheaf.

5.2.1 Heuristical remark. Let $\mathcal{F}$ be a sheaf on X with coefficients in the category of abelian groups. Let us assume that we have a resolution of $\mathcal{F}$

$$0 \to \mathcal{F} \to \mathcal{G}^0 \to \mathcal{G}^1 \to \ldots \to \mathcal{G}^n \to \ldots$$

A resolution of a sheaf may be very useful for the computation of the cohomology of $\mathcal{F}$. In ChapterII we showed: If the resolution above is acyclic then we can use it to compute the cohomology groups of $\mathcal{F}$, we have :

$$H^\bullet(X, \mathcal{F}) = H^\bullet(\mathcal{G}^\bullet(X)).$$

But even if a resolution is not acyclic it may also be helpful. For instance we still have a homomorphism

$$H^\bullet(\mathcal{G}^\bullet(X)) \to H^\bullet(X, \mathcal{F})$$

which in general is neither injective nor surjective. But we have some kind of estimate for the deviation from being an isomorphism and in these estimates the cohomology groups $H^q(X, \mathcal{G}^p)$ will enter.

In the case of an acyclic resolution these cohomology groups are zero for $q > 0$ and we get another explanation of our old result. The precise formulation of the result will be given in 6.2.1.

5.2.2. I want to put to Čech complex into this context. Let $\mathfrak{U} = \{U_\alpha\}_{\alpha \in \mathbf{A}}$ be a covering of our space X let $\mathcal{F}$ be a sheaf with values in the category of abelian groups. We give the indexing set $\mathbf{A}$ a total order and we denote by $A^{q+1}_<$ the subset of those sequences $\underline{\alpha} = (\alpha_0, \alpha_1, \ldots, \alpha_q)$ where $\alpha_0 < \alpha_1 \ldots < \alpha_q$. Again we put $U_{\underline{\alpha}} = U_{\alpha_0} \cap U_{\alpha_1} \cap \ldots U_{\alpha_q}$ and let $i_{\underline{\alpha}} : U_{\underline{\alpha}} \hookrightarrow X$ be the inclusion map. We restrict $\mathcal{F}$ to $U_{\underline{\alpha}}$ and take the direct image of this restriction. This way we obtain the sheaves $\mathcal{F}^*_{\underline{\alpha}} = i_{\underline{\alpha}*} i^*_{\underline{\alpha}}(\mathcal{F})$. I recall that these sheaves are defined by the rule $i_{\underline{\alpha}*} i^*_{\underline{\alpha}}(\mathcal{F})(V) = \mathcal{F}(V \cap U_{\underline{\alpha}})$. The stalk of this sheaf is equal to $\mathcal{F}_x$ if $x \in U_{\underline{\alpha}}$ it is zero if $x \notin \bar{U}_{\underline{\alpha}}$ and it depends on the local structure of $\bar{U}_{\underline{\alpha}}$ in the boundary points $x \in \partial \bar{U}_{\underline{\alpha}}$. We have always a homomorphism $\mathcal{F}_x \to \mathcal{F}^*_{\underline{\alpha} x}$. I allow myself to write $\mathcal{F}^*_\alpha$ for $\mathcal{F}^*_{\{\alpha\}}$.

Now we construct a resolution of our sheaf $\mathcal{F}$

$$0 \to \mathcal{F} \to \prod_{\alpha \in A} \mathcal{F}^*_\alpha \to \prod_{(\alpha,\beta) \in A \times A_<} \mathcal{F}^*_{(\alpha,\beta)} \to \ldots \to \prod_{\underline{\alpha} \in A^{q+1}_<} \mathcal{F}^*_{\underline{\alpha}} \to .$$

The first map is simply

$$\mathcal{F}_x \to \prod_{\alpha \in A} \mathcal{F}^*_{\alpha x}.$$

The boundary map

$$d : \prod_{\underline{\alpha}\in A^{q+1}_<} \mathcal{F}^*_{\alpha} \to \prod_{\underline{\beta}\in A^{q+2}_<} \mathcal{F}^*_{\underline{\beta}}$$

is given by the following rule: Let $s = (\dots s_{\underline{\alpha}} \dots) \in (\prod \mathcal{F}^*_{\underline{\alpha}})_x$

$$(ds)_{\underline{\beta}} = \sum_{i=0}^{q+1} (-1)^i s_{\beta_0 \dots \hat{\beta}_i \dots \beta_{q+1}}$$

where we interpret $s_{\beta_0 \dots \hat{\beta}_i \dots \beta_{q+1}}$ as an element in $\mathcal{F}^*_{\underline{\beta},x}$ It is clear that this is a complex of sheaves.

5.2.3 Exercise.

a) Prove that this complex of sheaves is exact.

Hint: We have to check exactness in the stalks. If $x \in X$ we know that we can find an element $\gamma \in \mathbf{A}$ with $x \in U_\gamma$. Now we are in the same situation as in the above Lemma, except that we have modified the Čech complex since we have ordered the index set. But it is not difficult to adapt the Lemma to this situation here.

b) Let E be a finite totally ordered set, i.e. $E = \{0, 1, \dots, n\}$. Let A be an abelian group, for any $r \in \mathbb{Z}$ we define

$$C^r(A) = \oplus_{I \subset E, |I|=r+1} A,$$

by definition we have $C^r(A) = 0$ if $r \in \{0, \dots, n\} = E$. For a subset $I \subset E$ and $\alpha \in I$ we define $p(\alpha, I)$ as the position of α in I, i.e. $p(\alpha, I) = 0$ if α is the smallest element, $p(\alpha, I) = |I| - 1$ if α is the biggest one.

We define (co-)boundary operators

$$d : C^r(A) \to C^{r+1}(A)$$
$$\delta : C^r(A) \to C^{r-1}(A)$$

by

$$(da)_J = \sum_{\beta \in J} (-1)^{p(\beta J)} a_{J \setminus \{\beta\}}$$
$$(\delta a)_J = \sum_{\beta \notin J} (-1)^{p(\beta J \cup \{\beta\})} a_{J \cup \{\beta\}}$$

where a_J is the J-th component of

$$a = (\dots a_J \dots) \in C^\bullet(A).$$

We get two complexes

$$0 \to C^0(A) \xrightarrow{d} \dots \xrightarrow{d} C^r(A) \xrightarrow{d} C^{r+1}(A) \xrightarrow{d}$$
$$0 \to \dots C^r(A) \xrightarrow{d} C^{r-1}(A) \xrightarrow{d} \dots \xrightarrow{d} C^0(A) \to 0$$

Show that these two complexes are exact using the ideas of the part a) and the previous exercises

5.2.4. Let me come back to the heuristical remark above. I said that the complexes of sections $\mathcal{G}^\bullet(X)$ of a resolution contain some information concerning the cohomology of $\mathcal{F}$. Now we see that for the special case of the Čech resolution the resulting complex of global sections is very similar to the Čech complex attached to the covering actually it is the ordered Čech complex and it is known that the ordered Čech complex gives us the same cohomology groups.

We see that coverings allow us to construct resolutions of sheaves. We already saw some other constructions providing resolutions of sheaves. If we look back to our computation of the cohomology of the spheres (See 4.5) then we see that our first short exact sequence is a resolution. We could extend it by resolving $\underline{\mathbb{Z}}_{S^{n-1}}$ and so on. This gives us the general idea that these resolution in some sense provide a kind of cutting a space into simpler pieces. (See Exercise 4.5.2).

IV.6 The spectral sequence

6.1. Now we have seen that a covering of our space provides resolutions of sheaves. In general the sheaves in the resolution will not be acyclic but perhaps their cohomology is easier to compute because they are supported on smaller sets namely the closures of the $U_{i_0,\dots,i_q}$. The method of the spectral sequence allows us to extract informations on the cohomology of the sheaf $\mathcal{F}$ in terms of the cohomology groups of the resolving sheaves.

We consider a resolution of a sheaf $\mathcal{F}$:

$$0 \to \mathcal{F} \to \mathcal{G}^0 \to \mathcal{G}^1 \to \dots \to \mathcal{G}^n \to \dots$$

6.1.1. We break the sequence

$$0 \to \mathcal{F} \to \mathcal{G}^0 \to \mathcal{K} \to 0$$

and we have seen (II.3.6) that we can find an injective resolution of this short exact sequence

$$\begin{array}{ccccccccc}
0 \to & \mathcal{F} & \to & \mathcal{G}^0 & \to & \mathcal{K} & \to 0 \\
 & \downarrow & & \downarrow & & \downarrow & \\
0 \to & I^0 & \to & I^0 \oplus J^0 & \to & J^0 & \to 0 \\
 & \downarrow & & \downarrow & & \downarrow & \\
0 \to & I^1 & \to & I^1 \oplus J^1 & \to & J^1 & \to 0 \\
 & \downarrow & & \downarrow & & \downarrow &
\end{array}$$

We have the second half of the exact sequence

$$0 \to \mathcal{K} \to \mathcal{G}^1 \to \dots \to \mathcal{G}^n \to$$

and we can apply the same to this sequence. Proceeding in the same way forever, we get a diagram

$$\begin{array}{ccccccccccc}
 & & 0 & & 0 & & 0 & & & 0 & \\
 & & \downarrow & & \downarrow & & \downarrow & & & \downarrow & \\
0 \to & & \mathcal{F} & \to & \mathcal{G}^0 & \to & \mathcal{G}^1 & \to \cdots \to & & \mathcal{G}^n & \to \\
 & & \downarrow & & \downarrow & & \downarrow & & & \downarrow & \\
0 \to & & I^0 & \to & I^{00} & \to & I^{10} & \to \cdots \to & & I^{p0} & \to \\
 & & \downarrow & & \downarrow & & \downarrow & & & \downarrow & \\
0 \to & & I^1 & \to & I^{01} & \to & I^{11} & \to \cdots \to & & I^{p1} & \to \\
 & & \downarrow & & \downarrow & & \downarrow & & & \downarrow & \\
0 \to & & I^2 & \to & I^{02} & \to & I^{12} & \to \cdots \to & & I^{p2} & \to \\
 & & \downarrow & & \downarrow & & \downarrow & & & \downarrow &
\end{array}$$

where all the I^ν and I^{pq} are injective, all squares commute. This double complex of sheaves has two properties

a) *all horizontal sequences are exact.*

b) *The vertical complexes $I^{\nu\bullet}$ are injective resolutions of $\mathcal{G}^\nu$ and $I^\bullet$ is an injective resolution of $\mathcal{F}$.*

We apply the functor global sections to this diagram and get the *augmented* double complex $\tilde{I}^{\bullet\bullet}(X)$

$$\begin{array}{cccccccccc}
 & 0 & & 0 & & 0 & & & 0 & \\
 & \downarrow & & \downarrow & & \downarrow & & & \downarrow & \\
0 \to & \mathcal{F}(X) & \to & \mathcal{G}^0(X) & \to & \mathcal{G}^1(X) & \to \cdots \to & & \mathcal{G}^n(X) & \to \\
 & \downarrow & & \downarrow & & \downarrow & & & \downarrow & \\
0 \to & I^0(X) & \to & I^{00}(X) & \to & I^{10}(X) & \to \cdots \to & & I^{p0}(X) & \to \\
 & \downarrow & & \downarrow & & \downarrow & & & \downarrow & \\
0 \to & I^1(X) & \to & I^{01}(X) & \to & I^{11}(X) & \to \cdots \to & & I^{p1}(X) & \to \\
 & \downarrow & & \downarrow & & \downarrow & & & \downarrow & \\
0 \to & I^2(X) & \to & I^{02}(X) & \to & I^{12}(X) & \to \cdots \to & & I^{p2}(X) & \to \\
 & \downarrow & & \downarrow & & \downarrow & & & \downarrow &
\end{array}$$

We replace the vertical complex on the left and the horizontal line on the top by zero and then we get the (non augmented) double domplex $I^{\bullet\bullet}$

$$\begin{array}{cccccc}
 & 0 & & 0 & & 0 \\
 & \downarrow & & \downarrow & & \downarrow \\
0 \to & I^{00}(X) & \to & I^{10}(X) & \to \cdots \to & I^{p0}(X) \\
 & \downarrow & & \downarrow & & \downarrow \\
0 \to & I^{01}(X) & \to & I^{11}(X) & \to \cdots \to & I^{p1}(X) \\
 & \downarrow & & \downarrow & & \downarrow \\
0 \to & I^{02}(X) & \to & I^{12}(X) & \to \cdots \to & I^{p2}(X) \\
 & \downarrow & & \downarrow & & \downarrow
\end{array}$$

All squares commute and all vertical and horizontal sequences are complexes. We give a name to the differentials

$$\begin{aligned} d'_{pq} \quad &: \quad I^{pq}(X) \longrightarrow I^{p+1,q}(X) \quad \text{horizontal} \\ d''_{pq} \quad &: \quad I^{pq}(X) \to I^{p,q+1}(X) \qquad \text{vertical.} \end{aligned}$$

We get a simple complex $I^\bullet_{\text{spl}}$ from $I^{\bullet\bullet}$: We put

$$I^n_{\text{spl}} = \oplus_{p+q=n} I^{pq}$$

and we define

$$d^n : I^n_{\text{spl}} \longrightarrow I^{n+1}_{\text{spl}}$$

as

$$d^n = \sum_{p+q=n} d'_{pq} + (-1)^p d''_{pq}$$

It is clear that the commuting of the squares implies that

$$d^{n+1} \circ d^n = 0.$$

6.1.2. The following facts are more or less obvious from the construction 6.1.1.

a) The vertical complexes $(I^{p\bullet}(X), d'')$ compute the cohomology of the sheaves $\mathcal{G}^p$, i.e.

$$H^q(X, \mathcal{G}^p) = H^q(I^{p\bullet}(X), d'')$$

b) The horizontal complexes $(I^{\bullet q}, d')$ compute the cohomology of I^q and hence they are exact except in degree zero

$$H^0(X, I^q) = I^q(X) = H^0(I^{\bullet q}(X), d')$$

and

$$H^p(I^{\bullet q}(X), d') = 0 \text{ for } \; p > 0.$$

c) The inclusion

$$I^\bullet(X) \longrightarrow I^\bullet_{\text{spl}}(X)$$

given by $x_q \longrightarrow (x_q, 0, \cdots, 0)$ induces an isomorphism

$$H^\bullet(I^\bullet(X)) \simeq H^\bullet(I^\bullet_{\text{spl}}(X)$$

and hence we have

$$H^\bullet(X, \mathcal{F}) = H^\bullet(I^\bullet_{\text{spl}}(X))$$

The last assertion is not quite so obvious, it requires a little argument using b). Let us look at a class which is represented by the cocycle

$$x = (x_{0,n}, \ldots, x_{n,0}).$$

The entries of the array are placed in our complex like that:

$$\begin{array}{cccc} \dots & \dots & \dots & x_{n,0} \\ \dots & \dots & x_{n-1,1} & 0 \\ & & & \\ \dots & \ddots & \dots & \dots \\ x_{0,n} & \dots & \dots & \dots \end{array}$$

The cocycle condition implies $d'_{n0}x_{n0} = 0$. Hence we find a $b = (0, \dots, y_{n-1,0}) \in I^{n-1}_{\mathrm{spl}}(X)$ such that $d'_{n-1,0}(y_{n-1,0}) = x_{n0}$ and $x - d^{n-1}_{\mathrm{spl}}(b)$ represents the same class but has its last component in the upper right corner is equal to zero. Repeating this we get that we can represent our element by a cocycle whose components are zero except the first one in the lower left corner. This implies c).

We also have the inclusion of the complex

$$\begin{array}{ccccccc} \mathcal{G}^0(X) & \to & \mathcal{G}^1(X) & \to \dots \to & \mathcal{G}^n(X) & \to \\ \downarrow & & \downarrow & & \downarrow & \\ I^0_{\mathrm{spl}}(X) & \to & I^1_{\mathrm{spl}}(X) & \to \dots \to & I^n_{\mathrm{spl}}(X) & \to \end{array}$$

and hence we get from this construction a homomorphism

$$H^n(\mathcal{G}^\bullet(X)) \longrightarrow H^n(I^\bullet_{\mathrm{spl}}(X)) = H^n(X, \mathcal{F})$$

This is the so called *edge homomorphism.*

If the sheaves $\mathcal{G}^p$ are acyclic then II.3.2 tells us that this edge homomorphism is an isomorphism. This can also be seen by looking at the the double complex, the same argument which gave us c) in the assertion above implies that the edge homomorphism is an isomorphism.

If the $\mathcal{G}^p$ are not acyclic then the edge homomorphism is neither injective nor surjective in general. In this general case we can get information on $H^n(X, \mathcal{F})$ in terms of the cohomology groups $H^q(X, \mathcal{G}^p)$ in other words the cohomology groups of the sheaves $\mathcal{G}^p$ enter.

Before I discuss this in some detail I recommend to the reader to solve the following exercise. It shows how these mechanisms work and it deals with the computation of $H^1(I^\bullet_{\mathrm{spl}}(X))$.

The cocycles are the elements $(x_{0,1}, x_{1,0})$ which satisfy $d'x_{1,0} = 0, d''x_{0,1} = 0$ and $d''x_{1,0} + d'x_{0,1} = 0$. Now a simple calculation solves the following exercise

6.1.3 Exercise:

a) Show that the edge homomorphism

$$H^1(\mathcal{G}^\bullet(X)) \longrightarrow H^1(X, \mathcal{F})$$

is injective. It provides an isomorphism to those classes which can be represented by cocycles with $x_{0,1} = 0$.

b) Sending a class to $x_{0,1}$ induces a a homomorphism

$$H^1(X,\mathcal{F}) = H^1(I^\bullet_{\mathrm{spl}}(X)) \longrightarrow H^1(X,\mathcal{G}^0)$$

and that the kernel of this maps is the image of the map in a).

c) The image of the map $H^1(X,\mathcal{F})) \longrightarrow H^1(X,\mathcal{G}^0)$ lands in the kernel of $H^1(X,\mathcal{G}^0) \to H^1(X,\mathcal{G}^1)$ and we have a homomorphism $\ker(H^1(X,\mathcal{G}^0) \to H^1(X,\mathcal{G}^1)) \to H^2(\mathcal{G}^\bullet(X))$.

d) Show that we get even on exact sequence

$$0 \to H^1(\mathcal{G}^\bullet(X)) \to H^1(X,\mathcal{F}) \to \ker(H^1(X,\mathcal{G}^0) \to H^1(X,\mathcal{G}^1)) \to H^2(\mathcal{G}^\bullet(X))$$

Now I explain what happens in higher degrees. The formulation of the becomes more complicated. I will formulate the main result and give some indications why it is true.

6.2 The vertical filtration.

In the following discussion we will never need that the horizontal complexes of sheaves are exact. This means that in 6.1.1 we can drop the condition a) and moreover of condition b) we only need that the vertical sequences $I^{p\bullet}$ are injective resolutions of $\mathcal{G}^p$. The sheaf $\mathcal{F}$ will not play any role.

We have a filtration on the double complex, we define $F^p(I^{\bullet\bullet}(X))$ to be the subcomplex where the entries in the first $p-1$ vertical collumns are zero. By we denote $F^p(I^\bullet_{\mathrm{spl}}(X))$ we denote the resulting simple complex. The inclusion of complexes

$$F^p(I^\bullet_{\mathrm{spl}}(X)) \hookrightarrow I^\bullet_{\mathrm{spl}}(X)$$

induces a homomorphism in cohomology

$$H^n(F^p(I^\bullet_{\mathrm{spl}}(X))) \to H^n(I^\bullet_{\mathrm{spl}}(X)).$$

and we define $F^p(H^n(I^\bullet_{\mathrm{spl}}(X)))$ as the image of this homomorphism. This yields a filtration of the cohomology, we have $F^0(H^n(I^\bullet_{\mathrm{spl}}(X))) = H^n(I^\bullet_{\mathrm{spl}}(X))$ and $F^p(H^n(I^\bullet_{\mathrm{spl}}(X))) = 0$ for $p > n$. Our goal is to get some understanding of the filtration steps

$$F^p(H^n(I^\bullet_{\mathrm{spl}}(X)))/F^{p+1}(H^n(I^\bullet_{\mathrm{spl}}(X))).$$

We have an exact sequence of complexes

$$0 \to F^{p+1}(I^\bullet_{\mathrm{spl}}(X)) \to F^p(I^\bullet_{\mathrm{spl}}(X)) \to F^p(I^\bullet_{\mathrm{spl}}(X))/F^{p+1}(I^\bullet_{\mathrm{spl}}(X)) \to 0.$$

The complex on the right is simply the vertical complex given by the p-th collumn. Hence we know

$$H^n(F^p(I^\bullet_{\mathrm{spl}}(X))/F^{p+1}(I^\bullet_{\mathrm{spl}}(X))) \xrightarrow{\sim} H^{n-p}(X,\mathcal{G}^p).$$

We write the exact sequence in cohomology

$$\to H^n(F^{p+1}(I^\bullet_{\mathrm{spl}}(X))) \to H^n(F^p(I^\bullet_{\mathrm{spl}}(X))) \to H^{n-p}(X,\mathcal{G}^p) \to H^{n+1}(F^{p-1}(I^\bullet_{\mathrm{spl}}(X)))$$

which yields an inclusion

$$H^n(F^p(I^\bullet_{\mathrm{spl}}(X)))/\operatorname{Im}(H^n(F^{p+1}(I^\bullet_{\mathrm{spl}}(X))) \subset H^{n-p}(X,\mathcal{G}^p)$$

By definition we have a homomorphism

$$H^n(F^p(I^\bullet_{\mathrm{spl}}(X)))/\operatorname{Im}(H^n(F^{p+1}(I^\bullet_{\mathrm{spl}}(X))) \to F^p(H^n(I^\bullet_{\mathrm{spl}}(X)))/F^{p+1}(H^n(I^\bullet_{\mathrm{spl}}(X)))$$

which gives us
E1: *The filtration steps $F^p(H^n(I^\bullet_{\mathrm{spl}}(X)))/F^{p+1}(H^n(I^\bullet_{\mathrm{spl}}(X)))$ are isomorphic to subquotients of $H^{n-p}(X,\mathcal{G}^p)$.*
We put $n-p=q$ we write $E_1^{p,q}=H^q(X,\mathcal{G}^p)$. In addition the structure of a complex on $\mathcal{G}^\bullet$ induces a structure of a complex

$$H^q(X,\mathcal{G}^\bullet) = \quad \to H^q(X,\mathcal{G}^{p-1}) \to H^q(X,\mathcal{G}^p) \to H^q(X,\mathcal{G}^{p+1}) \to .$$

We denote these boundary operators by $d_1^{pq}: E_1^{p,q} \to E_1^{p+1,q}$ and call the complex

$$(E_1^{pq}, d_1^{pq})_{pq}$$

the E_1-term of our double complex. Since this is a complex we have the cocyles and coboundaries in it

$$B_1^{pq} \subset Z_1^{pq} \subset E_1^{pq}$$

The next question is:
How can we compute the subquotient of $E_1^{p,q}$ which is isomorphic to the subquotient $F^p(H^n(I^\bullet_{\mathrm{spl}}(X)))/F^{p+1}(H^n(I^\bullet_{\mathrm{spl}}(X)))$ of the cohomology? A subquotient of E_1^{pq} is by definition of the form $Z_\infty^{pq}/B_\infty^{pq}$ where $B_\infty^{pq} \subset Z_\infty^{pq} \subset E_1^{pq}$, we have to compute these two submodules.
First of all it is clear from the definition that Z_∞^{pq} consists of those classes $\xi_{pq} \in H^q(X,\mathcal{G}^p)$ which have a representative $x_{pq} \in I^{pq}(X)$ which is the lower left entry of a cocycle

$$x = \begin{matrix} & & & x_{p+2,q-2} \\ & & x_{p+1,q-1} & \\ & x_{pq} & & \\ 0 & & & \end{matrix}$$

i.e. given x_{pq} we can place entries $x_{p+1,q-1},\dots,x_{p+q,0}$ such that $dx=0$.

It is also clear that $\xi_{pq} \in B_\infty^{pq}$ if and only if we can find an element

$$y \in I_{\mathrm{spl}}^{n-1}(X)$$

i.e. an element

$$y = \begin{pmatrix} y_{n-1,0} \\ \\ y_{0,n-1} \end{pmatrix}$$

such that $x - dy \in F^{p+1}(I_{\mathrm{spl}}^{n}(X))$.
Our strategy is to approximate these submodules Z_∞^{pq} (resp. B_∞^{pq}) by a sequence of decreasing (resp. increasing) submodules, i.e. we will construct sequences of submodules

$$Z_1^{pq} \supset Z_2^{pq} \supset \dots Z_\infty^{pq}$$

and

$$B_1^{pq} \subset B_2^{pq} \subset \dots B_\infty^{pq}$$

such that for large indices r we have $Z_r^{pq} = Z_\infty^{pq}, B_r^{pq} = B_\infty^{pq}$.
The first step in this sequence is easy to describe.
It is of course clear, the first problem is to find an $x_{p+1,q-1} \in I^{p+1,q-1}(X)$ for which

$$d' x_{pq} = d'' x_{p+1,q-1}.$$

The element x_{pq} represents our class ξ_{pq}, and we can find such an $x_{p+1,q-1}$ if and only if the class goes to zero under

$$d_1^{pq} : H^q(X, \mathcal{G}^p) \longrightarrow H^q(X, \mathcal{G}^{p+1})$$

i.e. it lies in the kernel

$$Z_1^{pq} = \ker(d_1^{pq})$$

and this implies $Z_1^{pq} \supset Z_\infty^{pq}$.
If on the other hand our element

$$\xi_{pq} \in B_1^{pq} = \mathrm{Im}(d_1^{p-1,q} : H^q(X, \mathcal{G}^{p-1}) \to H^q(X, \mathcal{G}^p)),$$

then this means that we can find an element $y_{p-1,q}$ which represents a class in $H^q(X, \mathcal{G}^{p-1})$ and therefore satisfies $d''(y_{p-1,q}) = 0$, and which maps to x_{pq} under d'. Then we can choose our element

$$y = \begin{pmatrix} 0 \\ \\ y_{p-1,q} \\ \\ 0 \end{pmatrix}$$

and

$$dy = \begin{array}{ccc} & & 0 \\ & & x_{pq} \\ & 0 & \end{array}$$

which means that

$$\xi_{pq} \in B_\infty^{pq} \subset Z_\infty^{pq}.$$

We conclude that $B_1^{pq} \subset B_\infty^{pq} \subset Z_\infty^{pq} \subset Z_1^{pq}$, and we define

$$E_2^{pq} = Z_1^{pq}/B_1^{pq} = H^q(H^p(X, \mathcal{G}^\bullet)).$$

In other words we define E_2^{pq} as the cohomology groups of the complex (E_1^{pq}, d_1^{pq}). We get
E2: *The filtration steps $F^p(H^n(I^\bullet_{\mathrm{spl}}(X)))/F^{p+1}(H^n(I^\bullet_{\mathrm{spl}}(X)))$ are isomorphic to subquotients of E_2^{pq}.*
The decisive point is that we can define

$$d_2^{pq} : E_2^{pq} \to E_2^{p+2,q-1}.$$

such that we get a complex

$$E_2^{p-2,q+1} \xrightarrow{d_2^{p-2,q+1}} E_2^{pq} \xrightarrow{d_2^{pq}} E_2^{p+2,q-1}$$

such that the cocyles and coboundaries of this complex satisfy $B_2^{pq} \subset B_\infty^{pq} \subset Z_\infty^{pq} \subset Z_2^{pq}$.
To construct this map we consider an element $\xi_{pq} \in Z_1^{pq}$ we want to lift it to a cocycle x. Since this element maps to zero under the boundary map d_1^{pq} we can find an element $x_{p+1,q-1}$ such that $d_1^{pq}(x_{pq}) = d''(x_{p+1,q-1})$. If we apply the horizontal boundary operator we get $d'(x_{p+1,q-1}) = z_{p+2,q-1} \in I^{p+2,q-1}(X)$. This element $z_{p+2,q-1}$ represents a class in $H^{p+2,q-1}(X, \mathcal{G}^{q-1})$ which is in the kernel $Z_1^{p+2,q-1}$ of $d_1^{p+2,q-1}$ because it is a boundary under the horizontal boundary operator.
We made some choices, but it is obvious that another choice modifies $z_{p+2,q-1}$ by an element in $B_1^{p+2,q-1}$ and hence we get a homomorphism

$$d_2^{pq} : E_2^{pq} \to E_2^{p+2,q-1}.$$

It is alo easy to verify that we even get a complex

$$E_2^{p-2,q+1} \xrightarrow{d_2^{p-2,q+1}} E_2^{pq} \xrightarrow{d_2^{pq}} E_2^{p+2,q-1}$$

It is clear that we must have $d_2^{pq}(\xi_{pq}) = 0$ if $\xi_{pq} \in Z_\infty^{pq}$ i.e. if this class lifts to a class in $H^n(F^p(I^\bullet_{\mathrm{spl}}(X))$ Or in other words $Z_2^{pq} = \ker(d_2^{pq}) \supset Z_\infty^{pq}$.

I claim that a class in E_2^{pq} goes to zero in $H^n(I^\bullet_{\mathrm{spl}}(X))/F^{p+1}(H^n(I^\bullet_{\mathrm{spl}}(X)))$ if it is in the image of $d_2^{p-2,q+1}$. An element in E_2^{pq} can be represented by an element $y_{p-2,q+1} \in I^{p-2,q+1}(X)$ which goes to zero under the vertical boundary operator and whose image under the horizontal boundary opertar is of the form $d'(y_{p-2,q+1}) = d''(y_{p-1,q})$. Then $d'(y_{p-1,q}) = x_{pq}$ represents an element in the image. Then we can take

$$y = \begin{array}{ccc} & & 0 \\ & & y_{p-1,q} \\ & (-1)^{p-2} y_{p-2,q+1} & \\ 0 & & \end{array}$$

and

$$dy = \begin{array}{ccc} & & 0 \\ & x_{pq} & \\ 0 & & \end{array}$$

Hence we get $B_2^{pq} = \mathrm{Im}(d_2^{p-2,q+1}) \supset B_\infty^{pq}$.
Now we define E_3^{pq} as the cohomology of the complex, i.e.

$$E_3^{pq} = Z_2^{pq}/B_2^{pq}$$

Now it is clear - and I will not give the formal proof - that this construction can extended by induction to all r and we get

6.2.1. *Starting from*

$$E_1^{pq} = H^q(X, \mathcal{G}^q) \text{ and } d_1^{pq} : H^q(X, \mathcal{G}^p) \to \mathcal{H}^q(X, \mathcal{G}^{p+1})$$

we can define a sequence of terms

$$(E_r^{pq}, d_r) \quad d_r^{pq} : E_r^{pq} \to E_r^{p+r,q-r+1} \qquad \text{where } d_r^{p+r,q-r+1} \circ d_r^{pq} = 0$$

such that at any level

$$E_{r+1}^{pq} = \frac{\ker(d_r^{pq} : E_r^{pq} \to E_r^{p+r,q-r+1})}{\mathrm{Im}(d_r^{p-r,q+r-1} : E_r^{p-r,q+r-1} \to E_r^{p,q})}$$

and such that for all r the subquotients $F^p(H^n(I^\bullet_{\mathrm{spl}}(X)))/F^{p+1}(H^n(I^\bullet_{\mathrm{spl}}(X)))$ are isomorphic to subquotients of E_r^{pq}. Since we are in the positive quadrant, i.e. $p, q \geq 0$ the sequence of modules E_r^{pq} becomes stationary after a while.

We say that the spectral sequence converges to the target $H^n(X, \mathcal{F})$ and write

$$(E_1^{pq}, d_1^{pq}) \Rightarrow H^n(X, \mathcal{F})$$

or if one knows the E_2-term

$$(E_2^{pq}, d_2^{pq}) \Rightarrow H^n(X, \mathcal{F}).$$

6.3 The horizontal filtration.

6.3.1 The horizontal filtration in the case of a resolution

We go back to our starting point where we had a resolution of a sheaf

$$0 \to \mathcal{F} \to \mathcal{G}^0 \to \mathcal{G}^1 \to \ldots \to \mathcal{G}^n \to \ldots$$

from which we constructed the double complex. Then conditions a) and .b) in 6.1.1. are valid.

We now consider consider the filtration by subcomplexes $'F^q(I^\bullet_{\mathrm{spl}}(X)) \subset I^\bullet_{\mathrm{spl}}(X)$ where the entries in the first $q-1$ horizontal lines are zero. If we apply the same the same arguments to this horizontal filtration we get something that we have done already. Since the $I^\bullet$ are acyclic our arguments yield that

$$H^\bullet('F^q(I^\bullet_{\mathrm{spl}}(X))/'F^{q+1}(I^\bullet_{\mathrm{spl}}(X))$$

vanishes except we are in degree zero and

$$H^0('F^q(I^\bullet_{\mathrm{spl}}(X))/'F^{q+1}(I^\bullet_{\mathrm{spl}}(X)) = H^q(X, \mathcal{F}).$$

Hence we see that for this filtration

$$'E_1^{pq} = 0 \text{ unless } p = 0$$

and

$$'E_1^{0n} = H^n(X, \mathcal{F}).$$

We do not have any non trivial differentials. Hence we see again that the double complex computes the cohomology $H^\bullet(X, \mathcal{F})$. This is only true since we assumed a) in 6.1.1. If we do not have this assumption then both filtrations are non trivial (see 6.4.2). We will also encounter a case where the vertical filtration is uninteresting.

6.3.2 The general case

Assume that we have a complex $\mathcal{G}^\bullet$ which starts in degree zero and that we have an adjusted injective resolution $\mathcal{G}^\bullet \to I^{\bullet\bullet}$ (See 6.4.1). We change the notation and give the index q to the vertical complexes. (We want a certain consistency therefore we arrange things so that p is the index for the filtration F^p.)

Now the horizontal filtration will not be trivial in general. We can apply to it the same method which we applied for the vertical filtration. Let $'F^p(I^{\bullet\bullet})$ be the subcomplex where the entries in the first $p-1$ lines are zero. Now we use the specific form of the adjusted injective resolution. The horizontal complex $'F^p(I^{\bullet\bullet})/'F^{p+1}(I^{\bullet\bullet})$ is of the form

$$I_B^{q-1,p} \oplus I_H^{q-1,p} \oplus I_B^{qp} \to I_B^{qp} \oplus I_H^{qp} \oplus I_B^{q+1,p} \to I_B^{q+1,p} \oplus I_H^{q+1,p} \oplus I_B^{q+2,p}$$

where the diffrential is always zero on the first two summands and is the identity isomorphism between the third term in degree q and the first term in degree $q+1$. This makes it easy to compute the cohomology even after we applied the global section functor

$$H^q('F^p(I^{\bullet\bullet})(X)/'F^{p+1}(I^{\bullet\bullet})(X) = I_H^{qp}(X) \xrightarrow{\sim} {}'E_1^{pq}$$

and the differential ${}'d_1^{pq} : I_H^{qp}(X) \to I_H^{q,p+1}(X)$ is the differential which obtained from the differential in the resolution $\mathcal{H}^q(\mathcal{G}^\bullet) \to I_H^{q,\bullet}$ after we take the global sections. Hence we see that the E_2 term of the spectral sequence whose E_2-term is

$${}'E_2^{pq} \xrightarrow{\sim} H^p(X, \mathcal{H}^q(\mathcal{G}^\bullet))$$

and which converges to $H^\bullet(I^\bullet_{\mathrm{spl}}(X))$.

If the sheaves $\mathcal{G}^p$ are acyclic for the functor global sections then we have $\mathcal{R}^\bullet H^0_X(\mathcal{G}^\bullet) \xrightarrow{\sim} \mathcal{G}^\bullet(X)$ But still we may consider an adjusted injective resolution of $\mathcal{G}^\bullet \to I^{\bullet\bullet}$. In this case the vertical filtration is kind of uninteresting. Its E_1^{pq} term is as always $H^q(X, \mathcal{G}^p)$ but this is zero for $q > 0$. The differentials are given by $d : H^0(X, \mathcal{G}^p) \to H^0(X, \mathcal{G}^{p+1})$ and his gives us the E_2^{pq} term as

$$E_2^{pq} = \begin{cases} H^p(\mathcal{G}^\bullet(X)) & \text{for } q = 0 \\ 0 & \text{for } q > 0 \end{cases}$$

The higher differentials are zero.

6.3.3 Applications.

This method of the spectral sequence has many applications. We will discuss some of these applications lateron. Here we give some indications how such applications can look like.

If we consider again the resolution of our sheaf

$$0 \to \mathcal{F} \to \mathcal{G}^0 \to \mathcal{G}^1 \to \ldots \to \mathcal{G}^n \to \ldots$$

One typical applications provide finiteness results. For instance it we can show that E_1^{pq} or E_2^{pq} are finitely generated abelian groups or finite dimensional vector spaces, then we can conclude that the same is true for the target groups (vector spaces) $H^n(X, \mathcal{F})$.

Another typical application concerns Euler characteristics. If we know that the cohomology groups $H^n(X, \mathcal{F})$ are finite dimensional vector spaces over a field k which vanish for $n >> 0$ then we define the Euler characteristic

$$\chi(X, \mathcal{F}) = \sum (-1)^n \dim_k(H^n(X, \mathcal{F})).$$

It is of course clear that

$$\chi(X, \mathcal{F}) = \sum \sum_{p+q=n} (-1)^{p+q} \dim_k(E_\infty^{pq}).$$

But if we have for a certain level r that the total dimension of the E_r^{pq} is finite then it follows from simple principles in linear algebra that

$$\sum(-1)^{p+q}\dim_k(E_r^{pq}) = \sum(-1)^{p+q}\dim_k(E_{r+1}^{pq}).$$

Then we can conclude

$$\chi(X,\mathcal{F}) = \sum_{p,q}(-1)^{p+q}\dim_k(E_r^{pq}).$$

If already the $H^q(X,\mathcal{G}^q)$ have finite total dimension then

$$\chi(X,\mathcal{F}) = \sum_{p,q}(-1)^{p+q}\dim_k(H^q(X,\mathcal{G}^p)).$$

There are interesting cases where one knows the structure of the groups E_2^{pq} and one also knows that the d_2^{pq} are zero. Then we have $E_2^{pq} = E_3^{pq}$ and it can happen that the differentials on this level vanish again and that this goes on forever. Then we say that the spectral sequence *degenerates at level* E_2. In such a case the E_2^{pq} are equal to the subquotient in the filtration on the target. If for instance the cohomology groups are finite dimensional vector spaces then we can compute the dimensions of the cohomology $\dim_k(H^n(X,\mathcal{F})) = \sum_{p,q}\dim_k(E_r^{pq})$.
But in some cases the computation of the higher differentials becomes an extremely difficult task.

6.4. The derived category.

Instead of considering sheaves we consider complexes of sheaves on X

$$\mathcal{G}^\bullet = 0 \to \ldots \to \mathcal{G}^\nu \to \mathcal{G}^{v+1} \to \cdots \to$$

where we assume that $G^\alpha = 0$ if $\alpha << 0$, we say that the complex is bounded to the left.

We introduce the sheaves of cocycles $Z(\mathcal{G}^\nu) = \ker(\mathcal{G}^\nu \to \mathcal{G}^{\nu+1})$ and the sheaf of coboundaries $B(\mathcal{G}^\nu) = \operatorname{im}(\mathcal{G}^{\nu-1} \to \mathcal{G}^\nu)$ and the cohomology sheaves

$$\mathcal{H}^\nu(\mathcal{G}^\bullet) = \frac{\ker(\mathcal{G}^\nu \to \mathcal{G}^{\nu+1})}{\operatorname{im}(\mathcal{G}^{\nu-1} \to \mathcal{G}^\nu)}.$$

We may have cohomology in negative degrees.
If we have two such complexes we have an obvious notion of a morphism

$$\mathcal{G}_1^\bullet \xrightarrow{\psi} \mathcal{G}_2^\bullet$$

It is clear that ψ induces a morphism of between the cohomology sheaves

$$\mathcal{H}^\nu(\mathcal{G}_1^\bullet) \to \mathcal{H}^\nu(\mathcal{G}_2^\bullet)$$

Now the derived category $\mathcal{D}(\mathcal{S}_X)$ is defined as a "quotient" category of the category of complexes: A morphism

$$\psi : \mathcal{G}_1^\bullet \to \mathcal{G}_2^\bullet$$

is declared to be an isomorphism if it induces an isomorphism on the cohomology sheaves. This means that the objects are the complexes of sheaves which are bounded to the left but the sets of morphisms are the morphisms between complexes divided by an equivalence relation: Two morphisms

$$\mathcal{G}_1^\bullet \rightrightarrows \mathcal{G}_2^\bullet$$

become equal if there is third complex $\mathcal{G}_3^\bullet$ and a morphism $\mathcal{G}_2^\bullet \to G_3^\bullet$ which equalizes the two maps and which introduces an isomorphism $\mathcal{H}^\bullet(\mathcal{G}_2^\bullet) \to \mathcal{H}^\bullet(\mathcal{G}_3^\bullet)$. This categorial construction will not be discussed in further detail here. (Reference ???)

This gives us a new way to speak of resolutions. If we have a sheaf $\mathcal{F}$ we can view it as a complex

$$\mathcal{F}[0] \quad : \quad 0 \to 0 \to \cdots \to \mathcal{F} \to 0 \to 0 \to \cdots$$

where the sheaf sits in degree zero. If we have a resolution

$$0 \to \mathcal{F} \to \mathcal{G}^0 \to G^1 \to \cdots \to \mathcal{G}^n \to$$

we write this as morphism $\psi : \mathcal{F}[0] \to \mathcal{G}^\bullet$:

$$\begin{array}{ccccccc}
0 & \to \cdots \to & \mathcal{F} & \to & 0 & \to \cdots \to & \\
 & & \downarrow & & \downarrow & & \\
 & 0 \to & \mathcal{G}^0 & \to \cdots \to & \mathcal{G}^n & & \to
\end{array}$$

and the fact that $\mathcal{G}^\bullet$ is a resolution translates into the fact that ψ is an isomorphism in the derived category.

We can introduce the notion of a resolution of a complex. This is a double complex

$$\begin{array}{cccccccc}
0 \to & 0 & \to \cdots \to & 0 & \to & 0 & \to \cdots \to \\
 & \downarrow & & \downarrow & & \downarrow & \\
0 \to & \mathcal{G}^{\nu_0} & \to \cdots \to & \mathcal{G}^n & \to & \mathcal{G}^{n+1} & \to \cdots \to \\
 & \downarrow & & \downarrow & & \downarrow & \\
0 \to & \mathcal{A}^{\nu_0 0} & \to \cdots \to & \mathcal{A}^{n0} & \to & \mathcal{A}^{n+1,0} & \to \cdots \to \\
 & \downarrow & & \downarrow & & \downarrow & \\
0 \to & \mathcal{A}^{\nu_0 1} & \to \cdots \to & \mathcal{A}^{n1} & \to & \mathcal{A}^{n+1,1} & \to \cdots \to \\
 & \downarrow & & \downarrow & & \downarrow &
\end{array}$$

where the vertical complexes are exact.

We can drop the line on the top and consider the double complex of sheaves $\mathcal{A}^{\bullet\bullet}$ and we can pass to the simple complex $\mathcal{A}^{\bullet}_{\mathrm{spl}}$. Of course we have a morphism

$$r : \mathcal{G}^{\bullet} \longrightarrow \mathcal{A}^{\bullet}_{\mathrm{spl}}$$

and I leave it as an exercise to the reader to prove that this is an isomorphism in the derived category. (See 2.3.)

6.4.1 Injective resolutions of complexes The point is that we can construct injective resolutions of complexes of sheaves by double complexes. We adapt the approach we used when we constructed the double complex for the resolution $\mathcal{G}^{\bullet}$ of $\mathcal{F}$ in 6.1.1. The only difference is that we do not assume that the complex of sheaves $\mathcal{G}^{\bullet}$ is exact, we have to take the cohomology sheaves into account. (The condition 6.1.1 a) is not valid) We start at the left end of our complex and we break it

$$0 \to Z(G^{\nu_0}) \to \mathcal{G}^{\nu_0} \to \mathcal{B}(\mathcal{G}^{\nu_0+1}) \to 0$$

and resolve this by the standard construction (see Chap. Hom. Alg. ???). Our resolution looks as follows

$$\begin{array}{ccccccccc}
0 & \to & \mathcal{H}^{\nu_0}(\mathcal{G}^{\bullet}) & \to & \mathcal{G}^{\nu_0} & \to & B(\mathcal{G}^{\nu_0+1}) & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & I_{\mathcal{H}}^{\nu_0 0} & \to & I_{\mathcal{H}}^{\nu_0 0} \oplus I_B^{\nu_0+1,0} & \to & I_B^{\nu_0+1,0} & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & I_{\mathcal{H}}^{\nu_0 1} & \to & I_{\mathcal{H}}^{\nu_0 1} \oplus I_B^{\nu_0+1,1} & \to & I_B^{\nu_0+1,1} & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & &
\end{array}$$

For the following indices we always have the sequences

$$0 \to B(\mathcal{G}^q) \longrightarrow Z(\mathcal{G}^q) \to \mathcal{H}^q(\mathcal{G}^{\bullet}) \to 0$$

and

$$0 \to Z(\mathcal{G}^q) \to \mathcal{G}^q \longrightarrow B(\mathcal{G}^{q+1}) \to 0$$

We always resolve the first sequence by this method and then we use the resolution of the term in the middle for the left term in the second sequence, resolve the term on the right and then proceed by the standard construction to resolve the term in the middle.

This goes on forever and we get an *injective resolution of the complex* $\mathcal{G}^{\bullet}$.

$$\begin{array}{cccccccc}
0 \to & 0 & \to \cdots \to & 0 & \to & 0 & \to \cdots \to \\
 & \downarrow & & \downarrow & & \downarrow & \\
0 \to & \mathcal{G}^{\nu_0} & \to \cdots \to & \mathcal{G}^{n} & \to & \mathcal{G}^{n+1} & \to \cdots \to \\
 & \downarrow & & \downarrow & & \downarrow & \\
0 \to & I^{\nu_0 0} & \to \cdots \to & I^{n0} & \to & I^{n+1,0} & \to \cdots \to \\
 & \downarrow & & \downarrow & & \downarrow & \\
0 \to & I^{\nu_0 1} & \to \cdots \to & I^{n1} & \to & I^{n+1,1} & \to \cdots \to \\
 & \downarrow & & \downarrow & & \downarrow &
\end{array}$$

We call such a resolution an *adjusted* injective resolution. It is also rather clear that this construction is functorial in $\mathcal{G}^\bullet$, if we have a morphism between two complexes $\psi : \mathcal{G}_1^\bullet \to \mathcal{G}_2^\bullet$ then this extends to a morphism of the injective resolutions

$$\tilde{\Psi} : I_1^{\bullet\bullet} \longrightarrow I_2^{\bullet\bullet}$$

and if we pass to the simple complexes

$$I_{1,\mathrm{spl}}^\bullet \longrightarrow I_{2,\mathrm{spl}}^\bullet$$

this extension is unique by the definition of the derived category.

Once we have the notion of the derived category we get a new idea of derived functor should be. I explain this in the context of sheaves on spaces and the global section functor but it works in a much more general context.
If we have a complex of sheaves on our space X

$$\mathcal{G}_1^\bullet \in Ob(D(\mathcal{S}_X))$$

we can take an injective resolution

$$r : \mathcal{G}^\bullet \longrightarrow I^{\bullet\bullet}.$$

We apply the functor global sections and we get

$$r_X : \mathcal{G}^\bullet(X) \longrightarrow I^{\bullet\bullet}(X)$$

and we may pass to the simple complex on the right hand side. Then

$$I_{\mathrm{spl}}^\bullet(X)$$

is a complex of abelian groups and can be viewed as an object in the derived category of abelian groups.
Our functor $\mathcal{F} \to \mathcal{F}(X) = H^0(X, \mathcal{F}) = H_X^0(\mathcal{F})$ is a functor from the category of sheaves on X to the category of abelian groups. Now we constructed the derived functor

$$\mathcal{R}^\bullet H_X^0 : D(\mathcal{S}_X) \longrightarrow D(Ab)$$

from the derived category of sheaves on X to the derived category of abelian groups.
We apply it to our sheaf $\mathcal{F}$. We view it as a complex $\mathcal{F}[0]$ and consider

$$R^\bullet H^0(\mathcal{F}[0]).$$

We recover the cohomology groups $H^\bullet(X, \mathcal{F})$ as the cohomology groups of the complex $R^\bullet H^0(\mathcal{F}[0])$.

In general we can say that the derived functor $\mathcal{R}^\bullet H^0(\mathcal{F}[0])$ is a "higher level object" it contains more information then the cohomology groups.

6.4.2. In section II.3.1 I explained that we may - after defining the derived functor by using injective resolutions - compute it from acyclic resolutions. The same is of course true in the context of derived categories. If we have a resolution of our complex $\mathcal{G}^\bullet \to \mathcal{A}^{\bullet\bullet}$ as above and if the $\mathcal{A}^{pq}$ are acyclic for the functor $H^0(X, \quad)$ then we have

$$\mathcal{R}^\bullet H^0_X(G^\bullet) \xrightarrow{\sim} H^0(X, \mathcal{A}^{\bullet\bullet}) = \mathcal{A}^{\bullet\bullet}(X).$$

To see this we choose an injective resolution $\mathcal{G}^\bullet \to \mathcal{I}^{\bullet\bullet}$ The definition of injective sheaves allows us to construct a commutative diagram of complexes

$$\begin{array}{ccc} \mathcal{G}^\bullet & \to & \mathcal{A}^{\bullet\bullet} \\ \downarrow & & \downarrow \\ \mathcal{G}^\bullet & \to & I^{\bullet\bullet} \end{array}$$

which then induces a homomorphism of complexes

$$\mathcal{A}^\bullet(X)_{\mathrm{spl}} \to I^\bullet(X)_{\mathrm{spl}}$$

which must be an isomorphism in the derived category. To see this we look at the vertical filtration (see 6.2. ???) and find that we get the same E_1^{pq} term namely $H^q(X, \mathcal{G}^p)$ on both sides. Here we used the acyclicity of the $\mathcal{A}^{\bullet\bullet}$ resolution. Now the rest follows from a simple argument of functoriality: We get an isomorphism for the E_∞^{pq} and hence the homomorphism must be an isomorphism.

6.4.3 The composition rule.

The concept of derived categories allows a very elegant formulation of the content of the theory of spectral sequences. I want to explain this in a special case but it will be clear what happens in more general situations.

Now we assume that we have a continuous map $f : X \to Y$ between two topological spaces. Let $\mathcal{F}$ be a sheaf on X with values in the category of abelian groups. We have the functors $H^0(X, \ ) = H^0_X(\ )$ and f_*. It is clear that $H^0_X(\ )$ is the composition of f_* and $H_Y(0)$. We want to understand what this composition implies for the derived functors.

We introduced the higher direct images $R^\bullet f_*(\mathcal{F})$ as the derived functor of the direct image functor f_*. This is just a collection of sheaves on Y which are indexed by degrees.

But we defined the derived functor

$$\mathcal{R}^\bullet f_* : D(S_X) \longrightarrow D(S_Y)$$

which sends a complex of sheaves on X into an object in the derived category of sheaves on Y. The cohomology of this object gives us the derived sheaves $R^\bullet f_*(\mathcal{F})$. Now we have a little lemma which makes the whole thing click.

6.4.4 Lemma. *The direct image $f_*(I)$ of an injective sheaf I on X is a sheaf which is acyclic for the functor H^0 on Y.*

This is lemma 1.5.1. I stated the Lemma in the weaker form because the acyclicity $f_*(I)$ is what we actually need and it can be applied in more general situations.

Now come back to our complex of sheaves. We choose on injective resolution $\mathcal{G}^\bullet \to I^{\bullet\bullet}$, apply f_* and get

$$\mathcal{R}^\bullet f_*(\mathcal{G}^\bullet) \simeq f_*(I^\bullet_{\text{spl}})$$

Now we apply $\mathcal{R}^\bullet H^0_Y$ to these complexes. But since the sheaves in $f_*(I^\bullet_{\text{spl}})$ are acyclic we find

$$\mathcal{R}^\bullet H^0_Y \circ \mathcal{R}^\bullet f(\mathcal{G}^\bullet) = H^0(Y, F_*(I^\bullet_{\text{spl}})) \simeq H^0(X, I^\bullet_{\text{spl}}) \simeq \mathcal{R}^\bullet H^0_X(\mathcal{G}^\bullet)$$

Hence we get

$$H^0_X(\mathcal{G}^\bullet) = \mathcal{R}^\bullet H^0_Y \circ \mathcal{R}^\bullet f_*(\mathcal{G}^\bullet).$$

This gives us a general principle: If we pass to the derived category then the derived functor of a composition is the composition of the derived functors provided the above lemma is available.

Of course this formulation is very elegant but it may be difficult to extract explicit information from this rule. In our case here we can apply 2.4.2. and we get that we have a spectral sequence with E_2 term $H^p(Y, R^q f_*(\mathcal{F}))$ which converges to $H^n(X, \mathcal{F})$.

6.5 The spectral sequence of a fibration.

This spectral sequence is especially useful if we can apply it in combination with base change. We say that the map

$$f : X \longrightarrow Y$$

is a *cohomological fibration for the sheaf* $\mathcal{F}$ if the sheaves

$$\mathcal{R}^q f_*(\mathcal{F})$$

are local systems on Y (see 3.2.1) whose stalk in y is given by

$$H^q(f^{-1}(y), \mathcal{F}).$$

If our space Y is locally connected this means we have base change, i.e.

$$\mathcal{R}^q f_*(\mathcal{F})_y \simeq H^q(f^{-1}(y), \mathcal{F}).$$

and for any point y we can find a connected neighborhood V_y such that $\mathcal{R}^q f_*(\mathcal{F})$ restricted to V_y is isomorphic to the sheaf of locally constant sections into $\mathcal{R}^q f_*(\mathcal{F})_y$.

The intuitive meaning of this notion is that $\mathcal{R}^q f_*(\underline{A})$ is “the system of cohomology groups of the fibres”.

To produce examples of such cohomological fibrations we consider maps $f : X \to Y$ which are locally trivial fibrations with some fibre F (See 3.1). Furthermore we assume that for any local trivialization

$$\begin{tikzcd} \Psi_i \; : \; f^{-1}(U_i) \arrow[rr, "\sim"] \arrow[dr] & & U_i \times F \arrow[dl, "p_1"] \\ & U_i, & \end{tikzcd}$$

the restriction of $\mathcal{F}$ to $f^{-1}(U_i)$ is isomorphic to a pullback with respect to the projection to F of a sheaf on the fibre F. Finally we assume that X, Y are Hausdorff and Y should be locally contractible, i.e. each point $y \in Y$ has arbitrarily small contractible (see 4.4.11) neighborhoods. Then I claim

Under these assumptions $f : X \to Y$ is a cohomological fibration.

If we assume in addition that our space Y is pathwise connected and if we pick a base point $y_0 \in Y$, then we will also show (See 8.1) that our local system is basically the same object as a representation of the fundamental group

$$\pi_1(Y, y_0)$$

on $H^q(f^{-1}(y_0), \underline{A})$. Especially for a simply connected base space Y we even have

$$\mathcal{R}^q f_*(\underline{A}) = H^q\underline{(f^{-1}(y), \underline{A})}.$$

I want to discuss some special cases where this spectral sequence for a fibration becomes very useful.

Let us consider a fibre space

$$\pi : X \longrightarrow Y$$

(see ???) where the fibre F is homeomorphic to a sphere S^{n-1}. Then we have the E_2-term in the spectral sequence

$$E_2^{pq} = H^p(Y, R^q\pi_*(\mathbb{Z})),$$

and it is clear that $R^0\pi_*(\mathbb{Z}) = \underline{\mathbb{Z}}$. We have $R^q\pi_*(\mathbb{Z})$ for $q \neq 0, n-1$ and $R^{n-1}\pi_*(\mathbb{Z})$ is a local system where the stalcks are isomorphic to $\mathbb{Z}$. We say that this fibration by spheres is orientable if the local system is trivial, and we say that the fibration is oriented if we fix an isomorphism

$$R^{n-1}\pi_*(\mathbb{Z}) \simeq \underline{\mathbb{Z}}.$$

Now we consider the E_2-term of the spectral sequence. It looks like

$$\begin{array}{cccc} H^0(Y, R^{n-1}\pi_*(\mathbb{Z})) & H^1(Y, R^{n-1}\pi_*(\mathbb{Z})) & \cdots & H^p(Y, R^{n-1}\pi_*(\mathbb{Z})) \\ 0 & 0 & & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & & 0 \\ H^0(Y, \mathbb{Z}) & H^1(Y, \mathbb{Z}) & \cdots & H^p(Y, \mathbb{Z}), \end{array}$$

and the differential operator d_2 is given by an arrow that points 2 steps to the right and one step down. Thus it is zero (unless we have $n-1=1$) and stays zero for a while. So the terms $E_2^{pq} = E_3^{pq} \cdots$ for a while until we come to the differential d_n which sends

$$d_n : H^p(Y, R^{n-1}\pi_*(\mathbb{Z})) \longrightarrow H^{p+n}(Y, R^0\pi_*(\mathbb{Z})),$$

and now the $E_{n+1}^{p,q}$ may be different from E_n^{pq}. After that the spectral sequence degenerates. Therefore we get an exact sequence

$$\longrightarrow H^{p+n-1}(Y,\mathbb{Z}) \longrightarrow H^{p+n-1}(X,\mathbb{Z}) \longrightarrow H^p(Y, R^{n-1}\pi_*(\mathbb{Z})) \longrightarrow H^{p+n}(Y, R^0\pi_*(\mathbb{Z}))$$

which is the so called Gysin sequence. It contains relevant information concerning the fibration. If for instance, one of the differentials is non zero, then the map

$$H^\bullet(Y,\mathbb{Z}) \longrightarrow H^\bullet(X,\mathbb{Z})$$

is not injective. From this we can conclude that under this assumption the fibration cannot have a section

$$s : Y \longrightarrow X$$

to π.

If the bundle is oriented, then $R^{n-1}\pi_*(\mathbb{Z}) = \mathbb{Z}$, and we have a canonical generator $e \in R^{n-1}\pi_*(\mathbb{Z})$. This gives a class in $H^0(Y, R^{n-1}\pi_*(\mathbb{Z}))$ which is mapped to a class

$$e = d_n^{0,n-1}(e_0) \in H^n(Y,\mathbb{Z}),$$

and this class is the Euler class of the fibration. If it is non zero, then the bundle has no section.

6.6 Čech complexes and the spectral sequence.

I return to the Čech resolutions constructed from coverings $\mathfrak{U} = \{U_\alpha\}_{\alpha\in A}$ (See 5.2.2.):

$$\mathcal{F}_{\mathfrak{U}}^\bullet = 0 \to \mathcal{F} \to \Pi_{\alpha\in A}\mathcal{F}_\alpha \to \Pi_{(\alpha,\beta)\in A^2_<}\mathcal{F}_{\alpha_\beta} \to \cdots \to$$

In view of our previous discussion this means that we have an isomorphism in the derived category

$$\begin{array}{ccccccccc} 0 \to & \mathcal{F} & \to & 0 & \to & 0 & & \\ & \downarrow & & \downarrow & & \downarrow & & \\ 0 \to & \Pi_{\alpha\in A}\mathcal{F}_\alpha & \to & \Pi_{(\alpha,\beta)\in A^2_<}\mathcal{F}_{\alpha,\beta} & \to & \Pi_{(\alpha,\beta,\gamma)\in A^3_<}\mathcal{F}_{\alpha,\beta,\gamma} & \to \cdots \to \end{array}$$

and hence these two complexes have isomorphic derived functors.

The sheaves $\mathcal{F}_{\underline{\alpha}}$ are concentrated on the closed subsets $\overline{U}_{\underline{\alpha}} = \overline{U_{\alpha_0} \cap \ldots \cap U_{\alpha_p}}$ and our resolution is acyclic if and only if the sheaves $\mathcal{F}_{\underline{\alpha}}$ on $\bar{U}_\alpha$ are acyclic. In this case we say that the covering $\mathfrak{U}$ *provides a $\mathcal{F}$-acyclic* resolution.

We consider the vertical filtration (See 6.2 and 6.3). We get for our E_1 term

$$\prod_{\underline{\alpha}\in A_<^{p+1}} H^q(X,\mathcal{F}_{\underline{\alpha}}) = \prod_{\underline{\alpha}\in A_<^{p+1}} H^q(\overline{U}_{\underline{\alpha}},\mathcal{F}_{\underline{\alpha}}) = H^q(X,\mathcal{F}^\bullet_{\mathfrak{U}})$$

The $E_2^{p,0}$ term is the Čech cohomology and the edge homomorphism yields a homomorphism

$$\check{H}^p(X,\mathfrak{U},\mathcal{F}) \to H^p(X,\mathcal{F}).$$

It is clear

6.6.1. *If the covering provides a $\mathcal{F}$ acyclic resolution then the edge homomorphism is an isomorphism or in other words the Čech complex computes the cohomology of $\mathcal{F}$.*

In general this edge homomorphism needs not to be injective because we may have a non trivial differential

$$d_2 : E_2^{p-2,1} \to E_2^{p0} = \check{H}^p(X,\mathfrak{U},\mathcal{F}).$$

But for $p=1$ this differential is zero and it follows that edge homomorphism

$$\check{H}^1(X,\mathfrak{U},\mathcal{F}) \to H^1(X,\mathcal{F})$$

is injective. (Exercise 6.1.3)

But of course it may be non surjective, its image is the kernel of

$$H^1(X,\mathcal{F}) \to \prod_{\alpha\in A} H^1(X,\mathcal{F}_\alpha) = \prod_{\alpha\in A} H^1(\overline{U}_\alpha,\mathcal{F}_\alpha)$$

I want to consider a special case. We cover our space X by two open sets $X = U\cup V$ then our resolution becomes very short:

$$0 \to \mathcal{F} \to \mathcal{F}_U \oplus \mathcal{F}_V \to \mathcal{F}_{U\cap V} \to 0$$

where $\mathcal{F}_U = i_{U*}i_U^*(\mathcal{F})$ and so on. Then our spectral sequence has only two collums: We have as E_1 term

$$\begin{array}{ccccc}
H^q(U,\mathcal{F})\oplus H^q(V,\mathcal{F}) & \to & H^q(U\cap V,\mathcal{F}) & \to 0 \\
\vdots & \to & \vdots & \to 0 \\
\underbrace{H^0(U,\mathcal{F})\oplus H^0(V,\mathcal{F})}_{p=0} & \to & \underbrace{H^0(U\cap V,\mathcal{F})}_{p=1} & \to 0
\end{array}$$

and the horizontal boundary operator is taking the difference of the restriction maps. Then we se that the spectral sequence degenerates on E_2 level and we find that we have a long exact sequence

$$H^{q-1}(U\cap V,\mathcal{F}) \to H^q(X,\mathcal{F}) \to H^q(U,\mathcal{F}) \oplus H^q(V,\mathcal{F}) \to H^q(U\cap V,\mathcal{F}) \to$$

which is called the Mayer-Vietoris sequence. It is of course nothing else that the long exact sequence obtained from the short exact sequence which is given by the resolution.

With a slight modification we used this Mayer-Vietoris already when we computed the cohomology of spheres (See 4.5).

6.6.1.1 A *CW-complex* is a space which is obtained by succesive attachment of cells. This construction is easy to explain. We start with a point, this is the simplest CW-complex. If Y is already a CW-complex, and if

$$f : S^{n-1} \longrightarrow Y$$

is a continuous map, then we construct a new space $X = D^n \cup_f Y$ which is again a CW-complex. To construct his new space X we consider S^{n-1} as boundary of D^n and $X = D^n \cup_f Y$ is obtained from the disjoint union $D^n \sqcup Y$ by identifying $x \in S^{n-1}$ to $f(x) \in Y$. This process is called "attaching a n-cell" to Y. We can relate the cohomology of the spaces X and Y. If we consider a tubular neighborhood T of the boundary sphere (i.e. $\{(x_1, \cdots, x_n) \mid 1-\varepsilon < \sum x_i^2 \leq 1\}$) then $V = T \cup_f Y$ is open in X and clearly the inclusion $Y \hookrightarrow V$ is a homotopy equivalence. The open ball $\overset{\circ}{D^n}= U$ is also open in X and we have a covering

$$X = U \cup V.$$

The open set U is acyclic and $U \cap V$ is homotopy equivalent to S^{n-1}. Thus our spectral sequence yields for $q \geq 1$

$$\longrightarrow H^{q-1}(S^{n-1},\mathbb{Z}) \longrightarrow H^q(X,\mathbb{Z}) \longrightarrow H^q(Y,\mathbb{Z}) \longrightarrow H^q(S^{n-1},\mathbb{Z}) \longrightarrow .$$

This tells us that we have some control how the cohomology of Y changes if we attach a n-cell. More precisely we can say that we can compute the cohomology of X if we know already the cohomolgy of Y and if we understand the boundary operator on the E_1 term:

$$H^{n-1}(Y,\mathbb{Z}) \longrightarrow H^{n-1}(S^{n-1},\mathbb{Z}).$$

There is a very prominent example where this method of computing the cohomology is especially successful. We consider the n-dimensional complex projective space $\mathbb{P}^n(\mathbb{C})$ (III.2 Examples).

Exercise: *a) Show that the topological space $\mathbb{P}^n(\mathbb{C})$ is obtained from $\mathbb{P}^{n-1}(\mathbb{C})$ by attaching a $2n$-cell.*

b) Show that

$$H^\bullet(\mathbb{P}^n(\mathbb{C}),\mathbb{Z}) = \bigoplus_{i=0}^{n} \mathbb{Z}\, e_i$$

where $e_i \in H^{2i}(\mathbb{P}^n(\mathbb{C}),\mathbb{Z})$ is a free generator.

6.6.2 A criterion for degeneration.

Let us assume that our complex of sheaves has the following property: For any index q we can construct a splitting of the sequence

$$0 \longrightarrow Z(\mathcal{G}^q) \longrightarrow \mathcal{G}^q \longrightarrow B(\mathcal{G}^{q+1}) \longrightarrow 0.$$

If we now construct our adjusted resolution, then we can achieve that the vertical differentials

$$\begin{array}{ccc} I_Z^{qp} & \oplus & I_B^{q+1,p} \\ & \downarrow & \\ I_Z^{q,p+1} & \oplus & I_B^{q+1,p+1} \end{array}$$

are the direct sum of the differentials of the resolutions of $Z(\mathcal{G}^q)$ and $B(\mathcal{G}^{q+1})$ (see II, 3.6.1). This means for the adjusted resolution of the complex we get

$$\begin{array}{ccccc} I^{qp} & = & I_B^{qp} \oplus I_H^{qp} \oplus I_B^{q+1,p} & = & I_Z^{qp} \oplus I_B^{q+1,p} \\ \downarrow & & \downarrow & & \downarrow \\ I^{q,p+1} & = & I_B^{q,p+1} \oplus I_H^{q,p+1} \oplus I_B^{q+1,p+1} & = & I_Z^{q,p+1} \oplus I_B^{q+1,p} \end{array}$$

and the rightmost vertical arrow can be taken as the direct sum of the arrows in the resolution of $Z(\mathcal{G}^p)$ and $B(\mathcal{G}^{p+1})$.

I claim that this implies that the two spectral sequences obtained from the vertical and the horizontal filtration degenerate on E_2-level.

We consider the horizontal filtration. We just saw that the E_2-term is given by $H^p(X, \mathcal{H}^q(\mathcal{G}^\bullet))$. An element in this group is represented by the element

$$\xi_{qp} \in I_H^{qp}(X)$$

which is mapped to zero under the vertical boundary map

$$\begin{array}{c} I_H^{qp}(X) \\ \downarrow \\ I_H^{q,p+1}(X). \end{array}$$

But if we view it as an element in $I^{qp}(X)$, then it is mapped to an element

$$\eta_{q,p+1} \in I_B^{q,p+1}(X) \subset I_B^{q,p+1}(X) \oplus I_H^{q,p+1}(X) \oplus I_B^{q+1,p+1}(X).$$

We look at the boundary map

$$\begin{array}{ccc} I^{q-1,p+1}(X) & \to & I^{q,p+1}(X) \\ \| & & \| \\ I_B^{q-1,p+1}(X) \oplus I_H^{q-1,p+1}(X) \oplus I_B^{q,p+1}(X) & \to & I_B^{q,p+1}(X) \oplus I_H^{q,p+1}(X) \oplus I_B^{q+1,p+1}(X), \end{array}$$

and we see that our element $\eta_{q,p+1}$ is the image of the element

$$\tilde{\eta}_{q-1,p+1} = (0, 0, \eta_{q,p+1}) \in I^{q-1,p+1}(X)$$

under this boundary map. Now our assumption implies that this element goes to zero under the vertical differential, because this vertical differential respects the decomposition

$$I^{q-1,p+1}(X) = I_Z^{q-1,p+1}(X) \oplus I_B^{q,p+1}(X).$$

But then the element

$$\tilde{\xi}_{qp} = \xi_{qp} + (-1)^{p-1}\tilde{\eta}_{q-1,p+1}$$

is a cocycle. This implies that

$$E_\infty^{p,q} = E_2^{p,q},$$

and this is the degeneration of the spectral sequence.

6.6.3. We even get more. We know that the E_2 term is a step in the filtration and hence

$$H^p(X, \mathcal{H}^q(\mathcal{G}^\bullet)) \xrightarrow{\sim} {}'F^p H^n(X, \mathcal{G}^\bullet)/{}'F^{p+1} H^n(X, \mathcal{G}^\bullet)$$

but we just constructed a homomorphism

$$i_{pq} : H^p(X, \mathcal{H}^q(\mathcal{G}^\bullet)) \longrightarrow H^n(X, \mathcal{G}^\bullet)$$

because to any class ξ_{qp} we constructed a cocycle $\tilde{\xi}_{qp}$ in $I_{spl}^n(X)$. Hence we even get a splitting

$$H^n(X, \mathcal{G}^\bullet) = \bigoplus_{p+q=n} H^p(X, \mathcal{H}^q(\mathcal{G}^\bullet)).$$

This splitting is not canonical because it may depend on the choice of the splitting $\mathcal{G}^p = Z(\mathcal{G}^p) \oplus B(\mathcal{G}^{p+1})$ since this choice influences the correction term η. But the images of the $H^p(X, \mathcal{H}^q(\mathcal{G}^\bullet))$ are well defined modulo the horizontal filtration.

6.6.4 An application.

We consider a space $Z = X \times Y$, and we assume that Y has a finite covering $Y = \bigcup_{\alpha \in I} U_\alpha$ by open sets which is $\underline{\mathbb{Z}}$ acyclic (See 6.4.5). Now we consider the Čech resolution of the sheaf $\underline{\mathbb{Z}}$ on Z:

$$0 \longrightarrow \underline{\mathbb{Z}} \longrightarrow \prod_\alpha \underline{\mathbb{Z}}_\alpha \longrightarrow \prod_{(\alpha,\beta)} \underline{\mathbb{Z}}_{\{\alpha,\beta\}} \longrightarrow$$

as in 5.2.2. We abbreviate the notation and denote the Čech complex simply by $\underline{A}^\bullet$. Then

$$\begin{array}{ccccccc} & & 0 & & & & \\ & & \downarrow & & & & \\ & & \underline{\mathbb{Z}} & & & & \\ & & \downarrow & & & & \\ 0 & \longrightarrow & \underline{A}^0 & \longrightarrow & \underline{A}^1 & \longrightarrow & \cdots \end{array}$$

is an isomorphism in the derived category and get

$$H^\bullet(Z,\underline{\mathbb{Z}}) \simeq H^\bullet(Z,A^\bullet).$$

Now it follows from our assumptions on the covering that the sheaves $\underline{A}^p$ are acyclic for the projection map p_1 to the factor X. We have that

$$p_{1,*}(\underline{A}^p) = \underline{\prod_{\alpha} \underline{\mathbb{Z}}_{\alpha}(X)} = \underline{\prod_{\alpha} \mathbb{Z}(U_{\alpha_0}\cap\ldots\cap U_{\alpha_r})}$$

i.e. it is the complex of locally constant sheaves on X associated to the abelian groups $\prod_\alpha \mathbb{Z}(U_{\alpha_0}\cap\cdots U_{\alpha_r})$. This is a complex of finite generated free $\underline{\mathbb{Z}}$ modules, we denote it by $B^\bullet$. Then we know

$$H^\bullet(Z,\underline{\mathbb{Z}}) = H^\bullet(X,\underline{B}^\bullet).$$

We apply the previous observation. Since the complex $B^\bullet$ is a complex of finitely generated free $\mathbb{Z}$ modules, we can conclude that the quotient $A^p/Z(A^p)$ is also free and hence we know that we can split off the boundaries. Hence we know that the spectral sequence degenerates, and we get an isomorphism

$$K : \bigoplus_{p+q} H^p(X,H^q(Y,\underline{\mathbb{Z}}))\tilde{\to}H^n(Z,\underline{\mathbb{Z}}).$$

This isomorphism may depend on the splitting because this spl;itting influences the choice of the correction term η above.

Under our assumptions the modules $H^q(Y,\mathbb{Z})$ are finitely generated abelian groups. This allows us to write these groups as quotient of two finitely generated free abelian groups, i.e. we have an exact sequence

$$0 \longrightarrow M_1 \longrightarrow M_2 \longrightarrow H^q(Y,\mathbb{Z}) \longrightarrow 0$$

where M_1, M_2 are finitely generated and free. Now we have obviously

$$H^p(X,M_i) = H^p(X,\mathbb{Z})\otimes M_i,$$

and hence we get an exact sequence

$$\begin{aligned} H^p(X,\mathbb{Z})\otimes M_1 \longrightarrow & H^p(X,\mathbb{Z})\otimes M_2 \longrightarrow H^p(X,H^q(X,\mathbb{Z})) \longrightarrow \\ & \longrightarrow H^{p+1}(X,\mathbb{Z})\otimes M_1 \longrightarrow H^{p+1}(X,\mathbb{Z})\otimes M_2. \end{aligned}$$

This yields a short exact sequence. We observe that the first arrow on the left yields a kokernel

$$H^q(X,\mathbb{Z})\otimes M_2/M_1 = H^q(X,\mathbb{Z})\otimes H^p(Y,\mathbb{Z}),$$

and the last arrow on the right has the kernel $\mathrm{Tor}^1_{\mathbb{Z}}(H^{q+1}(X,\mathbb{Z}),H^p(Y,\mathbb{Z}))$ (see Chap II.4.3), and hence our short exact sequence will be

$$0 \to H^q(X,\mathbb{Z})\otimes H^p(Y,\mathbb{Z}) \to H^q(X,H^p(Y,\mathbb{Z})) \to \mathrm{Tor}^1_{\mathbb{Z}}(H^{q+1}(X,\mathbb{Z}),H^p(Y,\mathbb{Z})) \to 0.$$

If we make the further assumption that $H^\bullet(X,\mathbb{Z})$ is finitely generated the module on the right is finite. Then the restriction of K to the tensor products gives us a homomorphism

$$\bigoplus_{p+q=n} H^p(X,\mathbb{Z})\otimes H^q(Y,\mathbb{Z}) \longrightarrow H^n(X\times Y,\mathbb{Z})$$

which is injective and has a finite cokernel. This is the so called Künneth homomorphism. This homomorphism does not depend on the choice of the splitting. To see that this is the case we assume that our space X has a finite $\mathbb{Z}$-acyclic covering $\{V_\beta\}_{\beta\in B}$ by open sets. In this case we can consider our locally constant sheaves $\underline{A}^p$ on X and take their Čech resolution provided by $\{V_\beta\}_{\beta\in B}$. Taking sections we get a double complex in which the (p,q) component is

$$\prod_{\underline{\beta}\in B_<^{p+1}} \prod_{\alpha\in A_<^{q+1}} \mathbb{Z}((X_{\underline{\beta}}\times Y)\cap(X\times U_{\underline{\alpha}}),$$

and where the vertical and horizontal boundary operators are induced from the boundary operators in the Čech complezes. But then it is clear: If we have cocycles

$$\xi^p\in\prod_{\underline{\beta}\in B_<^{p+1}}\mathbb{Z}(V_{\underline{\beta}}),\eta^q\in\prod_{\underline{\alpha}\in A<^{q+1}}\mathbb{Z}(U_{\underline{\alpha}}),$$

then we can define

$$\xi^p\eta^q=(\cdots\xi^p_{\underline{\beta}}\eta^q_{\underline{\alpha}}\cdots),$$

and this is a cocycle for the resulting simple complex which computes $H^\bullet(Z,\mathbb{Z})$. Hence we see that we do not need the correction in 6.6.4 which shows that the class does not depend on the splitting.

In the next section where I discuss products in a more general context and then we will see that K does not depend on the choice of the covering.

We apply the same reasoning to the vertical filtration. A slightly different argument gives us another construction of the canonical homomorphism

$$K:\bigoplus_{p+q=n} H^p(X,\mathbb{Z})\otimes H^q(Y,\mathbb{Z}) \longrightarrow H^n(X\times Y,\mathbb{Z}).$$

.

We may interchange the role of X,Y this means we study the spectral sequence attached to the map $p_2: X\times Y\to Y$. Now we assume that X also has a $\mathbb{Z}$-acyclic covering by open sets. Then the E_2 term is $H^q(X,H^p(Y,\mathbb{Z}))$ and we get homomorphisms

If we compute the cohomology starting from Čech coverings, and if we interchange the two spaces, then the two simple complexes resulting from the double complexes are actually isomorphic. We simply have to reflect along the diagonal. But we have to observe the sign convention in the definition of the differentials. This forces us to put signs. This eventually results in the formula: If we look at the two product maps

$$\begin{array}{ccc} H^p(X,\mathbb{Z}) \otimes H^q(Y,\mathbb{Z}) & & \\ & \searrow^{i_1} & \\ & & H^n(X \times Y, \mathbb{Z}), \\ & \nearrow_{i_2} & \\ H^q(Y,\mathbb{Z}) \otimes H^p(X,\mathbb{Z}) & & \end{array}$$

then we have

$$i_1(\alpha \otimes \beta) = (-1)^{pq} i_2(\beta \otimes \alpha).$$

6.7 Products

We want to discuss products in a more general context. We start with a commutative ring R with identity and we consider sheaves of R-modules on topological spaces. If we have two such sheaves $\mathcal{F}, \mathcal{G}$ on a space X, then we can consider the tensor product sheaf $\mathcal{F} \otimes_R \mathcal{G}$ on X. It is plausible that this should be defined as the sheaf attached to the presheaf

$$U \longrightarrow \mathcal{F}(U) \otimes_R \mathcal{G}(U)$$

(see III.3.1.), and it is really not too hard to show that the stalk of this sheaf is given by

$$(\mathcal{F} \otimes_R \mathcal{G})_x = \mathcal{F}_x \otimes_R \mathcal{G}_x$$

for all points $x \in X$.

Now we consider two spaces X, Y amd the two projections p_1, p_2 from $X \times Y$ to X and Y respectively. If now $\mathcal{F}$ and $\mathcal{G}$ are sheaves of R-modules on X and Y respectively, then we can define the exterior tensor product

$$\mathcal{F} \tilde{\otimes}_R \mathcal{G} = p_1^*(\mathcal{F}) \otimes_R p_2^*(\mathcal{G})$$

as a sheaf on $X \times Y$.

We want to construct a R-module homomorphism

$$m : H^i(X, \mathcal{F}) \otimes H^j(Y, \mathcal{G}) \longrightarrow H^{i+j}(X \times Y, \mathcal{F} \tilde{\otimes}_R \mathcal{G}).$$

It is not so entirely obvious how this can be done because if we take injective resolutions

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{F} & \longrightarrow & 0 & & \\ & & \downarrow & & & & \\ 0 & \longrightarrow & I^0 & \longrightarrow & I^1 & \longrightarrow & \end{array}$$

and

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{G} & \longrightarrow & 0 & & \\ & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & J^0 & \longrightarrow & J^1 & \longrightarrow & \end{array}$$

then the resulting morphism of complexes

$$\mathcal{F}\tilde{\otimes}_R\mathcal{G} \longrightarrow (I^\bullet\tilde{\otimes}_R J^\bullet)_{\text{ spl}}$$

needs not to be an isomorphism in the derived category. In other words, the simple complex of sheaves on the right hand side is not necessarily exact because the tensor product is not exact.

Therefore it seems to be reasonable to assume that one of the sheaves is flat and admits a flat acyclic resolution, say

$$\begin{array}{ccccccc} & & 0 & & & & \\ & & \downarrow & & & & \\ & & \mathcal{G} & & & & \\ & & \downarrow & & & & \\ 0 & \longrightarrow & \mathcal{A}^0 & \longrightarrow & \mathcal{A}^1 & \longrightarrow & \dots \end{array}$$

where flat means of course that the stalks $\mathcal{A}^i_x$ are flat R-modules. Then we find that the double complex

$$\begin{array}{ccccccc} \mathcal{F}\tilde{\otimes}_R\mathcal{G} & \longrightarrow & \mathcal{F}\tilde{\otimes}_R\mathcal{A}^0 & \longrightarrow & \mathcal{F}\tilde{\otimes}_R\mathcal{A}^1 & \longrightarrow \\ \downarrow & & \downarrow & & \downarrow & \\ I^0\tilde{\otimes}_R\mathcal{G} & \longrightarrow & I^0\tilde{\otimes}_R\mathcal{A}^0 & \longrightarrow & I^0\tilde{\otimes}_R\mathcal{A}^1 & \longrightarrow \\ \downarrow & & \downarrow & & \downarrow & \\ I^1\tilde{\otimes}_R\mathcal{G} & \longrightarrow & I^0\tilde{\otimes}_R\mathcal{A}^0 & \longrightarrow & I^1\tilde{\otimes}_R\mathcal{A}^1 & \longrightarrow \\ \downarrow & & \downarrow & & \downarrow & \end{array}$$

has exact rows and exact columns and hence we get a resolution of $\mathcal{F}\tilde{\otimes}_R\mathcal{G}$ by the simple complex $(I^\bullet\tilde{\otimes}_R\mathcal{A}^\bullet_{\text{spl}})$ which we write down

$$0 \longrightarrow \mathcal{F}\tilde{\otimes}_R\mathcal{G} \longrightarrow I^0\tilde{\otimes}_R\mathcal{A}^0 \longrightarrow I^1\tilde{\otimes}_R\mathcal{A}^0 \oplus I^0\tilde{\otimes}_R\mathcal{A}^1 \longrightarrow \dots .$$

Hence we get a map

$$H^n((I^\bullet\tilde{\otimes}_R\mathcal{A}^\bullet)_{\text{spl}}(X\times Y) \longrightarrow H^n(X\times Y, \mathcal{F}\tilde{\otimes}_R\mathcal{G}).$$

We get a map

$$H^n((I^\bullet\tilde{\otimes}_R\mathcal{A}^\bullet)_{\text{spl}}(X\times Y)) \longrightarrow H^n(X\times Y, \mathcal{F}\tilde{\otimes}_R\mathcal{G}).$$

By construction we have a map

$$m' : H^i(I^\bullet(X)) \otimes H^j(\mathcal{A}^\bullet(Y)) \to H^n((I^\bullet\tilde{\otimes}_R\mathcal{A}^\bullet)(X\times Y))$$

if $i+j=n$ the composition of these two is the map which we want to construct.

It is not clear that this product is independent of the resolution, so it may not be canonical. But if we assume in addition that also $\mathcal{F}$ is flat and has a acyclic resolution by flat sheaves, then it is clear that this product does not depend on the resolution. To see that this is so we first consider exact sequences

$$0 \longrightarrow \mathcal{F}_1 \longrightarrow \mathcal{F}_2 \longrightarrow \mathcal{F}_3 \longrightarrow 0$$

of sheaves on X. This provides without any assumption on the $\mathcal{F}_i$ an exact sequence

$$0 \longrightarrow \mathcal{F}_1 \tilde{\otimes}_R \mathcal{G} \longrightarrow \mathcal{F}_2 \tilde{\otimes}_R \mathcal{G} \longrightarrow \mathcal{F}_3 \tilde{\otimes}_R \mathcal{G} \longrightarrow 0,$$

and we get two exact sequences

$$\longrightarrow H^{i-1}(X, \mathcal{F}_2) \longrightarrow H^{i-1}(\mathcal{F}_3) \xrightarrow{\delta} H^i(X, \mathcal{F}_1) \longrightarrow,$$

and

$$H^{i-1+j}(X \times Y, \mathcal{F}_2 \tilde{\otimes}_R \mathcal{G}) \xrightarrow{\delta} H^{i-1+j}(X \times Y, \mathcal{F}_3 \tilde{\otimes}_R \mathcal{G}) \longrightarrow H^{i+j}(X \times Y, \mathcal{F}_1 \tilde{\otimes}_R \mathcal{G}).$$

Now the formula

$$m(\delta(\xi) \otimes \eta) = \delta(m(\xi \otimes \eta))$$

for $\xi \in H^{i-1}(X, \mathcal{F}_3)$ and $\eta \in H^j(Y, \mathcal{G})$ is obvious by construction.
This reduces the problem of the uniqueness of the map m to the assertion that

$$m : H^0(X, \mathcal{F}) \otimes H^j(Y, \mathcal{G}) \longrightarrow H^j(X \times Y, \mathcal{F} \tilde{\otimes}_R \mathcal{G})$$

is independent of the resolution. But this is obvious because in this case m is the following map: Any element $s \in H^0(X, \mathcal{F})$ induces a morphism

$$m(s) : p_2^*(\mathcal{G}) \to p_1^*(\mathcal{F}) \otimes_R p_2^*(\mathcal{G})$$

which is given by multiplication and clearly

$$m(s \otimes \xi) = m(s)^j(\xi)$$

for all $\xi \in \mathcal{H}^j(X, \mathcal{G})$.
Now it is clear that the general considerations fit into the context of our earlier discussion of the Künneth-formula and the cup product.
If we consider spaces X, Y which have a nice acyclic covering, then the acyclic resolutions

$$0 \longrightarrow \mathbb{Z} \longrightarrow \prod_{\alpha} \underline{\mathbb{Z}}_{\{\alpha\}} \longrightarrow \prod_{(\alpha,\beta)} \underline{\mathbb{Z}}_{\{\alpha,\beta\}} \longrightarrow \dots$$

are resolutions by free $\mathbb{Z}$-modules and therefore they are also flat. Since we have

$$\underline{\mathbb{Z}} \tilde{\otimes}_{\mathbb{Z}} \underline{\mathbb{Z}} = \underline{\mathbb{Z}},$$

we see that the above considerations generalize the previous ones.

6.7.1 The Cupproduct.

We may take $X = Y$, and we consider the product

$$H^p(X, \mathbb{Z}) \otimes H^q(X, \mathbb{Z}) \longrightarrow H^{p+q}(X \times X, \mathbb{Z}).$$

Now we consider the diagonal $X \to X \times X$, and we can consider the restriction

$$\Delta^* i(\alpha \otimes \beta) = \alpha \cup \beta,$$

and this is the cupproduct of the two classes.

Now we have seen – at least for reasonable spaces – that the cohomology groups

$$H^\bullet(X, \underline{\mathbb{Z}}) = \bigoplus_p H^p(X, \underline{\mathbb{Z}})$$

carry the additional structure of a graded anticommutative algebra. We want to determine the structure of this algebra in some special cases.

6.7.2 An example.

Let us consider a n-dimensional vector space V over $\mathbb{R}$ and let $\Gamma \subset V$ be a lattice, i.e. a free submodule of rank n such that V/Γ becomes a compact space. We can choose a basis $e_1, \cdots, e_n$ of Γ, this is also a basis for V and we get an isomorphism

$$V/\Gamma \simeq (\mathbb{R}/\mathbb{Z})^n = (S^1)^n.$$

The Künneth formula yields

$$H^\bullet((S^1)^n, \mathbb{Z}) = H^\bullet(S^1, \mathbb{Z}) \otimes \cdots \otimes H^\bullet(S^1, \mathbb{Z}),$$

and we see that the cohomology

$$H^\bullet((S^1)^n, \mathbb{Z}) = \bigoplus_\nu H^\nu((S^1)^n, \mathbb{Z})$$

is a free module of rank 2^n over $\mathbb{Z}$. It remains to determine the structure as a graded algebra.

First of all we notice that $H^1(\mathbb{R}/\mathbb{Z}, \underline{\mathbb{Z}}) = \mathbb{Z}$ (see ....). Now we consider the cohomology in degree p. If we have a class $\xi \in H^p(V/\Gamma, \underline{\mathbb{Z}})$, then we can attach to it an alternating p-linear map

$$\varphi_\xi \in \mathrm{Hom}^p_{\mathrm{alt}}(\Gamma, \mathbb{Z}).$$

To define this element we have to give the value $\varphi_\xi(\gamma_1, \cdots, \gamma_p)$ for any p-tuple $\underline{\gamma} = (\gamma_1, \cdots, \gamma_p)$ of elements in Γ. We take these elements and construct a homomorphism

$$\alpha_{\underline{\gamma}} : \mathbb{R}^p/\mathbb{Z}^p \longrightarrow V/\Gamma$$

which is given by

$$\alpha_{\underline{\gamma}}(x_1, \cdots, x_p) = x_1\gamma_1 + \cdots + x_p\gamma_p.$$

Then we can restrict our class ξ to $\mathbb{R}^p/\mathbb{Z}^p$. We consider $\alpha^*_{\underline{\gamma}}(\xi) \in H^p(\mathbb{R}^p/\mathbb{Z}^p, \underline{\mathbb{Z}}) = \mathbb{Z}$, and this is our definition

$$\varphi_\xi(\gamma_1, \cdots, \gamma_p) = \alpha^*_{\underline{\gamma}}(\underline{\xi}).$$

We see rightaway that this value is zero if $\gamma_1 \cdots \gamma_p$ are linearly dependent because then the image of $\alpha_{\underline{\gamma}}$ is a $(S^1)^{p'}$ with $p' \subset p$.

We have to show that the map φ_ξ is p-linear. This is easily reduced to the following special case: We consider $\mathbb{R}^{p+1}/\mathbb{Z}^{p+1}$, and we consider the three inclusions $i_1, i_2, \Delta : \mathbb{R}^p/\mathbb{Z}^p \to \mathbb{R}^{p+1}/\mathbb{Z}^{p+1}$ given by

$$\begin{array}{ccccc} I_1 & : & (x_1 . \cdots, x_p) & \mapsto & (x_1, 0, x_2, \cdots x_p) \\ i_2 & : & (x_1, \cdots x_p) & \mapsto & (0, x_1, x_2, \cdots x_p) \\ \Delta & : & (x_1, \cdots, x_p) & \to & (x_1, x_1, x_2 \cdots x_p) \end{array}$$

and for a class $\xi \in H^{p+1}(\mathbb{R}^{p+1}/\mathbb{Z}^{p+1}, \underline{\mathbb{Z}})$ we have to show that

$$i_1^*(\xi) + i_2^*(\xi) = \Delta^*(\xi).$$

Both sides are linear in ξ and hence we have to check this equality for classes

$$\xi_1 \in H^1(\mathbb{R}/\mathbb{Z}, \underline{\mathbb{Z}}) \otimes H^0(\mathbb{R}/\mathbb{Z}, \mathbb{Z}) \otimes H^{p-1}(\mathbb{R}^{p-1}/\mathbb{Z}^{p-1}, \underline{\mathbb{Z}})$$

and

$$\xi_2 \in H^0(\mathbb{R}/\mathbb{Z}, \underline{\mathbb{Z}}) \otimes H^1(\mathbb{R}/\mathbb{Z}, \underline{\mathbb{Z}}) \otimes H^{p-1}(\mathbb{R}^{p-1}/\mathbb{Z}^{p-1}, \underline{\mathbb{Z}}),$$

and then it is obviously true. This gives us a homomorphism of graded modules

$$\alpha : H^\bullet(V/\Gamma, \underline{\mathbb{Z}}) \longrightarrow \mathrm{Hom}^\bullet_{\mathrm{alt}}(\Gamma, \mathbb{Z}).$$

It is a well known elementary fact that the right hand side has the structure of an anti-commutative graded algebra where the product is given by

$$(\varphi \wedge \psi)(\gamma_1 \cdots \gamma_m) = \sum_t (-1)^{\varepsilon(t)} \varphi(\gamma_{i_1} \cdots \gamma_{i_p}) \cdot \psi(\gamma_{j_1} \cdots \gamma_{j_q})$$

where φ is a p-form, ψ is a q-form $m = p + q$ and so on.

Perhaps it is not so much of a surprise that:

The homomorphism

$$\alpha : H^\bullet(V/\Gamma, \underline{\mathbb{Z}}) \longrightarrow \mathrm{Hom}^\bullet_{\mathrm{alt}}(\Gamma, \mathbb{Z})$$

is an isomorphism of graded algebras.

To verify this we wrtie $V/\Gamma = (\mathbb{R}/\mathbb{Z})^n$, and we have the following basis for the cohomology: We look at ordered subsets $i_1 < i_2 \cdots < i_p$ of $\{1, \cdots n\}$ and form

$$1 \otimes \cdots \otimes 1 \otimes e_{i_1} \otimes \cdots e_{i_2} \cdots e_{i_p} \otimes 1 \cdots \otimes 1 = \xi_{\underline{i}}$$

where $e_{i_\nu} \in H^1(\mathbb{R}/\mathbb{Z}, \underline{\mathbb{Z}})$ is the canonical generator. The e_i can be viewed as basis elements for Γ at the same time, then

$$\varphi_{\xi_{\underline{i}}}(e_{j_1}, \cdots, e_{j_p}) = \begin{cases} 1 & \text{if } \underline{i} \text{ and } i_1 = j_1 \cdots i_p = j_p \\ 0 & \text{else} \end{cases}$$

and clearly

$$\xi_{\underline{i}} \cup \xi_{\underline{j}'} = \begin{cases} 0 & \text{if } \underline{i} \text{ and } \underline{i}' \text{ are not disjoint} \\ (-1)^{\varepsilon(\underline{i}, \underline{i}')} \xi_{\underline{i} \cup \underline{i}'} & \text{else} \end{cases}$$

This proves the assertion.

6.8 An excursion into homotopy theory.

We want to discuss briefly an application of the spectral sequence which is not directly related to the goals of this book but which is certainly important and beautiful.

If we consider a pathwise connected space X together with a base point x_0 we can define the homotopy groups

$$\pi_n(X, x_0).$$

On the other hand we have the singular homology groups $H_i(X, \mathbb{Z})$ which are not discussed here (exept in the chapter on cohomology of manifolds), and we always have the so called Hurewič homomorphism

$$\pi_n(X, x_0) \longrightarrow H_n(X, \mathbb{Z}).$$

A famous theorem of W. Hurewič asserts:

Let X be pathwise connected with base point x_0. Let $n > 0$ be an integer and let us assume that

$$H_i(X, \mathbb{Z}) = 0 \qquad \text{for} \qquad 0 < i < n.$$

Then the Hurewič homomorphism

$$\pi_n(X, x_0) \longrightarrow H_n(X, \mathbb{Z})$$

is an isomorphism if $n > 1$ and for $n = 1$ we get an isomorphism

$$\pi_1(X, x_0)_{ab} = \pi_1(X, x_0)/[\pi_1(X, x_0), \pi_1(X, x_0)] \xrightarrow{\sim} H_1(X, \mathbb{Z}).$$

Here $\pi_1(X,x_0)_{ab}$ is the abelianized fundamental group, i.e. the maximal abelian quotient.

We cannot prove this theorem here, since we did not define the homology groups. But we can also define the singular cohomology group $H^n_{\mathrm{sing}}(X,A)$ and for reasonable spaces we have

$$H^n_{\mathrm{sing}}(X,A) \simeq H^n(X,A)$$

i.e. the singular cohomology with coefficients in A is isomorphic to sheaf cohomology. Under these circumstances we can show

$$\mathrm{Hom}(H_i(X,\mathbb{Z}),\mathbb{Q}/\mathbb{Z}) \simeq H^i(X,\underline{\mathbb{Q}/\mathbb{Z}}),$$

and hence we can reformulate the Hurewič theorem:

An element $[\varphi] \in \pi_n(X,x_0)$ is represented by a homotopy class of maps of pointed spaces (the basepoints are pt and x_0)

$$\phi : (S^n, pt) \to (X,x_0)$$

and hence it defines a map

$$\varphi^* : H^n(X,\underline{\mathbb{Q}/\mathbb{Z}}) \longrightarrow H^n(S^n,\underline{\mathbb{Q}/\mathbb{Z}}) = \mathbb{Q}/\mathbb{Z}.$$

Now the map $\xi \mapsto \varphi^*(\xi)$ defines a homomorphism

$$H^n(X,\underline{\mathbb{Q}/\mathbb{Z}}) \longrightarrow \mathrm{Hom}(\pi_n(X,x_0),\mathbb{Q}/\mathbb{Z})$$

(the dual of the Hurewič map), and this map is an isomorphism is $H^i(X,\underline{\mathbb{Q}/\mathbb{Z}}) = 0$ for $0 < i < n$.

Indication of proof: We introduce the space $(\Sigma X, x_0)$ of continuous path starting at x_0, i.e. the space of all σ

$$\begin{array}{rcl} \sigma : [0,1] & \longrightarrow & X \\ \sigma(0) & = & x_0. \end{array}$$

Then this space is contractible and we have a map

$$\begin{array}{rcccl} e & : & (\Sigma X, x_0) & \longrightarrow & X \\ e & : & \sigma & \longrightarrow & \sigma(1). \end{array}$$

This map is a fibration, the fibre over x_0 is the loop space $\Omega(X,x_0)$. Hence we can say

$$H^\bullet(\Sigma X,\underline{\mathbb{Q}/\mathbb{Z}}) = H^\bullet(X,\mathcal{H}^\bullet(\Omega(X,x_0))$$

where $\mathcal{H}^\bullet(\Omega(X,x_0))$ is the local system of cohomology groups of the loop space.

We prove the Hurewič theorem by induction on n. The key is the observation that $H^i(\Sigma X, \underline{\mathbb{Q}/\mathbb{Z}}) = 0$ for $i > 0$ since ΣX is contractible.

If $n = 1$, then we consider the E_2-term in our spectral sequence in degree one

$$H^0(X, \mathcal{H}^1(\Omega(X, x_0))$$

$$H^1(X, \mathcal{H}^0(\Omega(X, x_0))$$

It follows that $H^1(X, \mathcal{H}^0(\Omega(X, x_0)) = 0$ and

$$H^0(X, \mathcal{H}^1(\Omega(X, x_0)) \longrightarrow H^2(X, \mathcal{H}^0(\Omega(X, x_0))$$

is an isomorphism.

The local system $\mathcal{H}^0(\Omega(X, x_0))$ is easy to compute, it is a module under the fundamental group $\Gamma = \pi_1(X, x_0)$. We have a continuous map

$$\Omega(X, x_0) \longrightarrow X$$

where $\tilde{X}$ is the universal convering and the fibres of this map are exactly the connected components of $\Omega(X, x_0)$. Then it is easy to see that $\mathcal{H}^0(\Omega(X, x_0))$ is the local system given by the Γ module

$$\mathcal{C}(\Gamma, \mathbb{Q}/\mathbb{Z})$$

of all $\mathbb{Q}/\mathbb{Z}$-valued functions on Γ where Γ acts by translations. This module contains the constant functions and hence we get an exact sequence

$$0 \to \underline{\mathbb{Q}/\mathbb{Z}} \to \mathcal{H}^0(\Omega(X, x_0)) \to M \to 0,$$

where M is the quotient sheaf. We get a long exact sequence in cohomology

$$0 \to H^0(X, \underline{\mathbb{Q}/\mathbb{Z}}) \to H^0\left(X, \mathcal{H}^0(\Omega(X, x_0))\right) \to H^0(X, M) \to H^1(X, \mathbb{Q}/\mathbb{Z}) \to 0.$$

For the local systems the sections $H^0(\ )$ are simply the invariants under Γ, it is easy to see that we get an exact sequence

$$(\mathbb{Q}/\mathbb{Z})^\Gamma \xrightarrow{\sim} (\mathcal{C}(\Gamma, \mathbb{Q}/\mathbb{Z}))^\Gamma \to M^\Gamma \xrightarrow{\sim} H^1(X, \mathbb{Q}/\mathbb{Z}) \to 0$$

Now it follows from our results on the cohomology of groups that

$$M^\Gamma \simeq H^1(\Gamma, \mathbb{Q}/\mathbb{Z}) = \mathrm{Hom}(\Gamma, \mathbb{Q}/\mathbb{Z}),$$

and hence we proved the result for $n = 1$.

For $n > 1$ we apply the same method. Now we know that $\Omega(X, x_0)$ is connected and hence we see that

$$\mathcal{H}^0(\Omega(X, x_0)) \simeq \underline{\mathbb{Q}/\mathbb{Z}}.$$

Then we find many zeroes in the bottom row of the spectral sequence and the system of cohomology groups $\mathcal{H}^i(\Omega(X, x_0))$ will be trivial. This shows that the E_2-term in the spectral sequence looks as follows

$$\begin{array}{llll}
H^0(X, \mathcal{H}^{n-1}(\Omega(X, x_0))) & & & \\
\quad\vdots & & & \\
H^0\left(X, \mathcal{H}^1(\Omega(X, x_0))\right) & & & \\
H^0\left(X, \mathcal{H}^0(\Omega(X, x_0))\right) & 0 & \cdots 0 & H^n\left(X, \mathcal{H}^0(\Omega(X, x_0))\right).
\end{array}$$

We conclude that $\mathcal{H}^i(\Omega(X, x_0)) = 0$ and that the differential

$$d : H^0\left(X, \mathcal{H}^{n-1}(\Omega(X, x_0))\right) \longrightarrow H^n\left(X, \mathcal{H}^0(\Omega(X, x_0))\right)$$

must be an isomorphism.

This implies that

$$\mathcal{H}^{n-1}(\Omega(X, x_0))_{x_0} \simeq H^n(X, \underline{\mathbb{Q}/\mathbb{Z}})$$

i.e.

$$H^{n-1}(\Omega(X, x_0), \underline{\mathbb{Q}/\mathbb{Z}}) \xrightarrow{\sim} H^n(X, \underline{\mathbb{Q}/\mathbb{Z}}).$$

Now we have the exakt sequence for homotopy groups which say that

$$\pi_{i-1}(\Omega(X, x_0)) \simeq \pi_i(X, x_0),$$

and the Hurewič theorem follows.

It is quite amusing to consider the special case of $X = S^n$. In this case we find

$$H^i(\Omega S^n, \mathbb{Q}/\mathbb{Z}) = \begin{cases} \mathbb{Q}/\mathbb{Z} & \text{for } i = k(n-1) \\ 0 & \text{else} \end{cases}$$

IV.7 Cohomology with compact supports

7.1. The definition Let X be a locally compact space and $\mathcal{F}$ a sheaf of abelian groups. If we have a section $s \in H^0(X, \mathcal{F})$ then its support is the set of $x \in X$ with $s_x \neq 0$. It is always closed.

We can define the submodule $H^0_c(X, \mathcal{F})$ of sections with compact support. This yields a left exact functor and we define the cohomology with compact supports as the right derived functor of $H^0_c(X, \mathcal{F})$. In accordance with our general principles we choose an injective resolution

$$0 \to \mathcal{F} \to I^0 \to I^1 \to$$

of $\mathcal{F}$ and define

$$H^i_c(X, \mathcal{F}) = H^i(H^0_c(X, I^\bullet)).$$

The cohomology with compact supports has different properties from the ordinary cohomology, it does not satisfy the homotopy axiom. We will see in the section on the cohomology of manifolds that it is dual to the ordinary cohomology. Of course on a compact space X we have $H^\bullet_c(X, \mathcal{F}) = H^\bullet(X, \mathcal{F})$. If we have open sets $U \subset V \subset X$ then we have natural map

$$H^i_c(U, \mathcal{F}) \to H^i_c(V, \mathcal{F}) \to H^i_c(X, \mathcal{F})$$

here we see that the restriction maps which we had in the theory of sheaves are turned backwards. On the other hand if we have a map $f : X \to Y$ then we will not be able to define a map from $H^i_c(Y, \underline{\mathbb{Z}})$ to $H^i_c(X, \underline{\mathbb{Z}})$ unless the map is proper.

Let us assume that $U \overset{i}{\hookrightarrow} X$ is an open subset of our space let us assume that its closure is compact, then its boundary $\partial \bar{U} \setminus U$ is also compact. Let $\mathcal{F}$ be a sheaf on U. We defined two new sheaves on X: The direct image $i_*(\mathcal{F})$ where

$$i_*(\mathcal{F})(V) = \mathcal{F}(U \cap V)$$

and the the extension by zero

$$i_!(\mathcal{F})(V) = \{s \in \mathcal{F}(V) \mid |s| \text{ does not meet } V \cap \partial \bar{U}\}$$

One checks easily that $i_!(\mathcal{F})$ has the stalks

$$\begin{array}{lll} i_!(\mathcal{F})_x = \mathcal{F}_x & \text{if} & x \in U \\ i_!(\mathcal{F})_y = 0 & \text{if} & y \notin U \end{array}.$$

We have a morphism of sheaves $i_!(\mathcal{F}) \to i_*(\mathcal{F})$ which is an isomorphism in all stalks except the ones on the boundary $\partial \bar{U}$.

7.1.1 Proposition. *If X is a space and $i : U \hookrightarrow X$ an open subset with compact closure and if $\mathcal{F}$ is a sheaf of abelian groups on U then*

$$H^\bullet_c(U, \mathcal{F}) = H^\bullet(X, i_!(\mathcal{F})).$$

Proof. This is almost clear from the definition. We choose an injective resolution of the sheaf $\mathcal{F}$ on U

$$0 \to \mathcal{F} \to I^0 \to I^1 \to \dots$$

and we notice that

$$i_!(I^\bullet)(X) = H_c^0(U, I^\bullet).$$

7.2. An example Now we consider the sheaf $\underline{\mathbb{Z}}$ on the open ball $D^n \subset \mathbb{R}^n$. We want to compute $H_c^\bullet(D^n, \underline{\mathbb{Z}})$. To do this we embed $D^n \overset{i}{\hookrightarrow} \bar{D}^n$. On $\bar{D}^n$ we have an exact sequence

$$0 \to i_!(\underline{\mathbb{Z}}) \to i_*(\underline{\mathbb{Z}}) \to i_*(\underline{\mathbb{Z}})/i_!(\underline{\mathbb{Z}}) \to 0.$$

The sheaf $i_*(\underline{\mathbb{Z}})$ is $\underline{\mathbb{Z}}_{\bar{D}^n}$ and the sheaf $i_*(\underline{\mathbb{Z}})/i_!(\underline{\mathbb{Z}})$ is concentrated on $\bar{D} \setminus D = S^{n-1}$ and on this space it is simply $\underline{\mathbb{Z}}_{S^{n-1}}$. We write the long exact sequence in cohomology, exploit our proposition and get

$$\to H_c^\nu(D, \underline{\mathbb{Z}}) \to H^\nu(\bar{D}, \underline{\mathbb{Z}}) \to H^\nu(S^{n-1}, \underline{\mathbb{Z}}) \to\to H_c^{\nu+1}(D, \underline{\mathbb{Z}}).$$

We have $H^\nu(\bar{D}, \underline{\mathbb{Z}}) = 0$ for $\nu > 0$ and hence we get for $\nu = 0$

$$\begin{array}{ccccc} 0 & \to H^0(\bar{D}^n, \underline{\mathbb{Z}}) & \to H^0(S^{n-1}, \underline{\mathbb{Z}}) & \to H_c^1(D^n, \underline{\mathbb{Z}}) & \to 0 \\ & \| & & & \\ & \mathbb{Z} & & & \end{array}$$

and for $\nu \geq 0$

$$H^\nu(S^{n-1}, \underline{\mathbb{Z}}) \xrightarrow{\sim} H_c^{\nu+1}(D^n, \underline{\mathbb{Z}}).$$

Our computation of the cohomology groups of spheres yields

$$H_c^\nu(D^n, \underline{\mathbb{Z}}) \xrightarrow{\sim} \begin{cases} \mathbb{Z} & \text{for} \quad \nu = n \\ 0 & \text{for} \quad \nu \neq n \end{cases}$$

If we have a homeomorphism $f : D_n \xrightarrow{\sim} D_n$ then it induces necessarily an isomorphism $f^* : H_c^d(D^n, \underline{\mathbb{Z}}) \xrightarrow{\sim} H_c^d(D^n, \underline{\mathbb{Z}})$ which can only be multiplication by ± 1. If we take for instance the homoeomorphism that sends $(x_1, x_2, \dots, x_n) \mapsto (x_1, x_2, \dots, -x_n)$, i.e. we change the sign of the last coordinate then we get multiplication by -1 on $H_c^d(D^n, \underline{\mathbb{Z}})$. We say that f preserves the orientation, if it induces the identity on $H_c^d(D^n, \underline{\mathbb{Z}})$. (???? Noch was murmeln)

7.2.1. We want to consider a relative situation. Let us assume that we have a diagram

$$\begin{tikzcd} X \arrow[r, hook, "i"] \arrow[d, "\pi"'] & \overline{X} \arrow[dl, "\bar{\pi}"] \\ Y, & \end{tikzcd}$$

and we want to assume this is some kind of fibration by n-dimensional balls. By this I means that locally in Y we can trivialize our diagram

$$\begin{array}{ccc} V \times D^n & \stackrel{i}{\hookrightarrow} & V \times \bar{D}^n \\ \downarrow & \swarrow & \\ V. & & \end{array}$$

If we choose a covering $Y = \cup V_\alpha$ such that we have trivializations over the V_α, then we get identifications (see 3.1.1.). For $v \in V_\alpha \cap V_\beta$ we have homeomorphisms

$$g_{\alpha,\beta}(v) : (D^n, \bar{D}^n) \longrightarrow (D^n, \bar{D}^n)$$

which means $g_{\alpha,\beta}(v)$ is a homeomorphism of $\bar{D}^n$ which maps the interior to the interior and the boundary to the boundary.

We call this fibration orientable if the $g_{\alpha,\beta}(v)$ preserve the orientation, and if we selected a consistent orientation on the fibres . We consider the sheaf $\mathbb{Z}$ on X and its extension $i_!(\mathbb{Z})$ to $\bar{X}$. We want to compute the cohomology $H^\bullet(\bar{X}, i_!(\mathbb{Z}))$. We apply the spectral sequence for a fibration (see 6.6.2), and we have the E_2-term $H^p(Y, \mathcal{R}^q\pi_*(i_!(\mathbb{Z}))$. Our computation in the previous section yields

$$\mathcal{R}^q\pi_*(i_!(\mathbb{Z})) = \begin{cases} 0 & q \neq n \\ \underline{\mathbb{Z}} & q = n \end{cases}$$

(remember that we have the orientation) and consequently the spectral sequence degenerates and

$$H^{p+n}(\bar{X}, i_!(\mathbb{Z})) = H^p(Y, \mathcal{R}^n\pi_*(i_!\mathbb{Z})) = H^p(Y, \mathbb{Z}).$$

7.2.2 Formulae for cup products : We want to explain some formulae for prducts and for this purpose it is convenient to replace the open (resp. closed) ball D^n (resp. $\bar{D}^n$) by the open (resp. closed) box

$$B^n = \{(x_1, \cdots, x_n) \mid |x_i| < 1\} \subset \bar{B}^n = \{(x_1, \cdots, x_n) \mid |x_i| \leq 1\}$$

because the pairs $(D^n, \bar{D}^n)$ and $(B^n, \bar{B}^n)$ are homeomorphic.

Let us assume that we have two numbers d_1, d_2 with $d_1 + d_2 \geq n$. We consider products

$$\bar{B}^{d_1} \times B^{n-d_1} \qquad \text{and} \qquad \bar{B}^{d_2} \times B^{n-d_2},$$

and we consider embeddings

$$\begin{array}{lcl} i_1 & : & \bar{B}^{d_1} \times B^{n-d_1} \longrightarrow \bar{B}^n \\ i_1 & : & ((x_1, \cdots, x_{d_1}), (y_1, \cdots, y_{n-d_1})) \longmapsto (x_1, \cdots, x_{d_1}, y_1, \cdots, y_{n-d_1}) \end{array}$$

and

$$\begin{array}{rcl} i_2 & : & B^{n-d_2} \times \bar{B}^{d_2} \longrightarrow \bar{B}^n \\ i_2 & : & ((y_1, \cdots, y_{n-d_2}), (x_1, \cdots, x_{d_2})) \mapsto (y_1, \cdots, y_{n-d_2}, x_1, \cdots, x_{d_2}). \end{array}$$

We get the sheaves $i_{1,!}(\mathbb{Z})$ and $i_{2,!}(\mathbb{Z})$ on $\bar{B}^n$, and clearly

$$\begin{array}{rcl} H^{n-d_1}(\bar{B}^n, i_{1,!}(\mathbb{Z})) & \xrightarrow{\sim} & \mathbb{Z} \\ H^{n-d_2}(\bar{B}^n, i_{2,!}(\mathbb{Z})) & \xrightarrow{\sim} & \mathbb{Z} \end{array}$$

where we chose the orientations as given by the ordering of the y-coordinates. We want to consider the cup product

$$H^{n-d_1}(\bar{B}^n, i_{1,!}(\mathbb{Z})) \times H^{n-d_2}(\bar{B}^n, i_{2,!}(\mathbb{Z})) \longrightarrow H^{2n-d_1-d_2}(\bar{B}^n, i_{1,!}(\mathbb{Z}) \otimes i_{2,!}(\mathbb{Z})).$$

We must understand the tensor product of the two sheaves. We observe that we have an embedding

$$\begin{array}{rcl} i_{1,2} & : & \bar{B}^{d_1+d_2-n} \times B^{2n-d_1-d_2} \longrightarrow \bar{B}^n \\ i_{12} & : & ((x_1, \cdots, x_{d_1+d_2-n}), (y_1, \cdots, y_{2n-d_1-d_2})) \longrightarrow \\ & & (y_1, \cdots, y_{n-d_2}, x_1, \cdots, x_{d_1+d_2-n}, y_{n-d_2+1}, \cdots, y_n) \end{array}$$

and an isomorphism provided by the multiplication on the stalks

$$i_{1,!}(\mathbb{Z}) \otimes i_{2,!}(\mathbb{Z}) \xrightarrow{\sim} i_{12,!}(\mathbb{Z})$$

We choose the orientation on $B^{2n-d_1-d_2}$ which is given by the ordering of the coordinates. Then all the cohomology groups in

$$H^{n-d_1}(\bar{B}^n, i_{1,!}(\mathbb{Z}))) \times H^{n-d_2}(\bar{B}^n, i_{2,!}(\mathbb{Z})) \longrightarrow H^{2n-d_1-d_2}(\bar{B}^n, i_{12,!}(\mathbb{Z}))$$

are identified to $\mathbb{Z}$. Now I claim:

Under these identifications the cup product is given by the multiplication.

I have the feeling that the proof can be left as an exercise to the reader. First of all we can restrict to the case $d_1 + d_2 = n$, and now we have enough flexibility to reduce to the case $d_1 = n-1, d_2 = 1$. Then we recall that

$$H^1_c(B^1, \mathbb{Z}) = H^1(\bar{B}^1, i_!(\mathbb{Z}))$$

can be computed from the exact sequence

$$0 \to i_!(\mathbb{Z}) \longrightarrow \underline{\mathbb{Z}} \longrightarrow \underline{\mathbb{Z}}/i_!(\mathbb{Z}) \longrightarrow 0$$

which reduces it to a cup product with an H^0 in it.

7.3 The fundamental class.

Let M be a connected $\mathcal{C}_0$-manifold of dimension n. If we have a point $p \in M$ then we can find a neighborhood V_p of p which is homeomorphic to an open ball $D \subset \mathbb{R}^n$. Then we have $H^d_c(V_p, \underline{\mathbb{Z}}) \xrightarrow{\sim} \mathbb{Z}$ but this isomorphism is not canonical. If we have two points p, q

and two small open neighborhoods V_p, V_q of these points, then we have no consistent way to identify $H_c^d(V_p, \underline{\mathbb{Z}})$ and $H_c^d(V_q, \underline{\mathbb{Z}})$. But if we choose a path $\gamma : [0,1] \to M$ which starts at p and ends at q, then we get an identification along the path (see the discussion of this argument in the section on local systems). We say that M is orientable if for any two points this identification does not depend on the path. If M is orientable then we can choose a generator in $H_c^d(V_p, \underline{\mathbb{Z}})$ for all p which is consistent with the above identifiction along paths. Once we have choosen such generators we call the manifold oriented.

If our manifold has a differentiable structure, then we have another notion of orientation on M. (see ....). In this case it is easy to see that the two concepts of being oriented coincide.

In the next chapter we will prove that for a connected and oriented $\mathcal{C}^\infty$- manifold M of dimension n a point $p \in M$ and any open ball $D_p \subset M$ containing p the map $H_c^d(D_p, \underline{\mathbb{Z}}) \to H_c^d(M, \underline{\mathbb{Z}})$ is an isomorphism. This is the starting point to get Poincaré duality. The image of the generator in $H_c^d(D_p, \underline{\mathbb{Z}})$ is the so called *fundamental class.*

IV.8 Cohomology of manifolds

8.1 Local systems.

I want to study the cohomology of $\mathcal{C}^\infty$-manifolds with coefficients in local systems See 3.3.2).

We will know that the stalks at two different points x, y are always isomorphic but in general we do not have the possibility to identify them in a consistent way. The following argument is the same that everybody has seen during the discussion of the principle of analytic continuation in theory of complex functions. Since M is connected, we can choose a path $\gamma : [0,1] \to M$ with $\gamma(0) = x$ and $\gamma(1) = y$. We cover the path by finitely many sufficiently small open sets U_i, on which $\mathcal{V}$ is trivial. This gives us a subdivision $0 = t_0 < t_1 < \ldots < t_n = 1$ such that the $\gamma[t_i, t_{i+1}]$ are entirely in one of the covering sets U_ν and hence we can identify $\mathcal{V}_{\gamma(t_i)} = \mathcal{V}_{\gamma(t_{i+1})} = \mathcal{V}(U_\nu)$. This sequence of identifications yields an identification

$$\Psi_\gamma : \mathcal{V}_x \xrightarrow{\sim} \mathcal{V}_y.$$

This identification depends on the path, but it is not difficult to see that it only depends on the homotopy class $[\gamma]$ of the path.

If we choose a base point $x_0 \in M$ and consider closed paths which start and end at x_0, then we get a representation of the fundamental group

$$\begin{aligned} \rho : \pi_1(M, x_0) &\to \mathrm{Aut}(\mathcal{V}_{x_0}) \\ \rho : [\gamma] &\mapsto (\Psi_{[\gamma]} : \mathcal{V}_{x_0} \to \mathcal{V}_{x_0}) \end{aligned}.$$

It is not hard to see that the local system can be reconstructed from this representation: We consider the set of pairs $([\gamma], v)$ where $[\gamma]$ is a homotopy class of path from x_0 to x and $v \in \mathcal{V}_{x_0}$. The stalk at a point $x \in M$ will be this set divided by the equivalence relation

$$([\gamma], v) \sim ([\gamma_1], v_1)$$

if and only if

$$\rho([\gamma_1]^{-1} \circ [\gamma])(v_1) = v.$$

We can express this by saying that we have an equivalence of categories:

Abelian groups V together with an action of $\pi_1(M, x_0)$ and local systems $\mathcal{V}$ whose stalk at x_0 is isomorphic to V.

If we have a local system $\mathcal{V}$ on our manifold M then under certain assumptions we can construct a dual local system $\mathcal{V}^\vee$. We want to study

$$H^i(M, \mathcal{V}) \quad \text{and} \quad H^i_c(M, \mathcal{V}^\vee)$$

and we will again under certain assumptions construct a duality between $H^i(M, \mathcal{V})$ and $H^{d-i}_c(M, \mathcal{V}^\vee)$ where d is the dimension of M. This will be the Poincaré duality.

8.2 Čech resolutions of local systems.

If M is a $\mathcal{C}^\infty$-manifold then we can find a covering $M = \bigcup_{\alpha \in A} U_\alpha$ by open sets which has the following two properties

(1) The covering is locally finite, i.e. to any point $p \in M$ we can find an open neighborhood V_p containing p such that we have only finitely many $\alpha \in A$ such that $U_\alpha \cap V_p \neq \emptyset$.

(2) For any finite set $\alpha_0, \ldots, \alpha_q \in A$ the intersection $U_{\alpha_0} \cap \ldots \cap U_{\alpha_q}$ is diffeomorphic to an open ball of dimension n and the intersection $\bar{U}_{\alpha_0} \cap \ldots \cap \bar{U}_{\alpha_q}$ is contractible.

I want to explain briefly why we can find such a covering It is part of the definition of a manifold, that it should be paracompact (See 4.4) . This means that any covering admits a locally finite refinement. This follows if we assume that M is *countable at infinity*, i.e. we can find an increasing sequence of relatively compact open sets

$$W_n \subset W_{n+1}$$

where $\bar{W}_n \subset W_{n+1}$ for all n and which exhausts the manifold M, i.e. $\bigcup W_n = M$.

We can use the paracompactness to introduce a Riemannian metric on M. To do this we construct a *partition of unity* on M. This is a family $\{h_i\}_{i \in I}$ of positive $\mathcal{C}^\infty$ functions which has the following properties.

(i) The support of such an h_i is small so that we can find an open set $U_i \subset M$ which is $\mathcal{C}^\infty$-isomorphic to an open ball $D \subset \mathbb{R}^d$ and $\mathrm{Supp}(h_i) \subset U_i$.

(ii) For any point $x \in M$ there are only finitely many indices $j \in I$ with $h_j(x) \neq 0$.

(iii) We have

$$\sum_{i \in I} h_i = 1.$$

The construction of such a partition of unity is standard and quite easy.

We can construct a Riemannian g_i metric on each of the U_i simply by transporting the standard metric on the ball by means of the diffeomorphism. We multiply this metric by h_i then it extends to a positive quadratic form on the tangent bundle of M which is positive definite on the support of h_i. Adding up gives the desired metric.

Now we invest some differential geometry. Any point $x \in M$ has an open neighborhood V_x which is diffeomorph to a ball and which has the property that it is convex: Any two points $y, z \in V_x$ can be joined by a unique geodesic. We may simply take a small ball $B(x,\epsilon) = V_x$, these are all those points which have distance $< \epsilon$ from x. The closure of such a ball is diffeomorphic to a closed ball in $\mathbb{R}^d$, the boundary $\overline{\partial B(x,\epsilon)}$ is a sphere. It is a smooth hypersurface in M.

Now I come back to the construction of a covering with the required properties. I assume that M is countable at infinity. We can exhaust it by a sequence of relatively compact open sets W_n which in addition have the property that $\bar{W}_n \subset W_{n+1}$.

We start at an index n and cover $\bar{W}_n$ by a finite family of such small balls as above. We require that these balls are contained in W_{n+1}. Now we proceed with $\bar{W}_{n+1}$ but we require in addition that these balls have empty intersection with $\bar{W}_{n-1}$. Then it is clear that the union of all these families provides a covering $\{U_\alpha\}_{\alpha \in A}$ of M. The intersections $U_{\alpha_0} \cap \ldots \cap U_{\alpha_q}$ are diffeomorphic to open balls in $\mathbb{R}^n$. If we have a point in the boundary

$$x \in \overline{U_{\alpha_0} \cap \ldots \cap U_{\alpha_q}} \setminus U_{\alpha_0} \cap \ldots \cap U_{\alpha_q} = \partial(U_{\alpha_0} \cap \ldots \cap U_{\alpha_q})$$

then the intersection $B(x,\epsilon) \cap \overline{U_{\alpha_0} \cap \ldots \cap U_{\alpha_q}}$ is contractible: each point in the intersection is joined to x by a unique geodesic in this intersection.

We give the indexing set A a total order, in other word we identify it to $\{0, 1, \ldots, n\}$ or $\mathbb{N}$. We consider $q+1$-tuples $\underline{\alpha} = (\alpha_0, \ldots, \alpha_q) \in A^{q+1}$ where the indices are increasing. Let us denote this subset of indices by $A^{q+1}_<$. Then we have for such an $\underline{\alpha} = (\alpha_0, \ldots, \alpha_q)$ the inclusion

$$i_{\underline{\alpha}} : U_{\alpha_0} \cap \ldots \cap U_q \hookrightarrow M$$

and as in 2.2.we form the sheaf

$$\mathcal{V}^*_{\underline{\alpha}} = i_{\underline{\alpha}*} i^*_{\underline{\alpha}}(\mathcal{V}).$$

This sheaf has non-zero stalks only in the points $x \in \overline{U_{\alpha_0} \cap \ldots \cap U_q}$ and in such a point the stalk is equal to $\mathcal{V}_x$. Here we need that for any point $x \in \overline{U_{\alpha_0} \cap \ldots \cap U_q}$ and a small ball $B(x,\epsilon)$ that

$$\mathcal{V}(B(x,\epsilon)) = i^*_\alpha(\mathcal{V})(B(x,\epsilon) \cap \overline{U_{\alpha_0} \cap \ldots \cap U_{\alpha_q}}).$$

Outside of $\overline{U_{\alpha_0} \cap \ldots \cap U_q}$ the sheaf has been mowed.

Now we consider the Čech a resolution of our sheaf $\mathcal{V}$ (see 5.2 and 6.5):

$$0 \to \mathcal{V} \to \prod_{\alpha \in A} \mathcal{V}^*_\alpha \to \prod_{(\alpha,\beta) \in A \times A_<} \mathcal{V}^*_{(\alpha,\beta)} \to \ldots \to \prod_{\underline{\alpha} \in A^{q+1}_<} \mathcal{V}^*_{\underline{\alpha}} \to .$$

This is now an acyclic resolution since all the sheaves $\mathcal{V}^*_\alpha$ are acyclic by the homotopy axiom.

Hence we see that the cohomology groups $H^\nu(M,\mathcal{V})$ can be computed from the complex of global sections (See 6.5.1)

$$\begin{array}{ccccc}
\prod_{\alpha\in A}\mathcal{V}^*_\alpha(M)\to & \prod_{(\alpha\beta)\in A\times A_<}\mathcal{V}^*_{(\alpha,\beta)}(M)\to\dots\to & \prod_{\alpha\in A^{q+1}_<}\mathcal{V}^*_{\underline{\alpha}}(M)\to \\
\| & \| & \| \\
\prod_{\alpha\in A}\mathcal{V}(U_\alpha)\to & \prod_{(\alpha,\beta)\in A\times A_<}\mathcal{V}(U_\alpha\cap U_\beta)\to\dots\to & \prod_{\alpha\in A^{q+1}_<}\mathcal{V}(U_{\alpha_0}\cap\dots\cap U_{\alpha_q})\to
\end{array}$$

which is the Čech-complex attached to the resolution.
We introduce the Čech coresolution. To do this we consider the sheaves

$$i_{\alpha!}i^*_{\underline{\alpha}}(\mathcal{V})=V^!_{\underline{\alpha}}.$$

These sheaves are zero outside of the open sets $U_{\alpha_0}\cap\dots\cap U_{\alpha_q}$ and on these sets they coincide with $\mathcal{V}$.
On these open sets our sheaves $\mathcal{V}$ are isomorphic to a constant sheaf. We computed the cohomology with compact supports (Prop. 7.2)

$$H^\nu(M,\mathcal{V}^!_\alpha)=\begin{cases}0 & \text{for } \nu\neq d\\ \mathcal{V}(U_\alpha) & \text{for } \nu=d\end{cases}.$$

Now we can define a complex of sheaves (???)

$$\to\prod_{\underline{\alpha}\in A^{q+1}_<}\mathcal{V}^!_{\underline{\alpha}}\to\dots\to\prod_{(\alpha,\beta)\in A\times A_<}\mathcal{V}^!_{\alpha,\beta}\to\prod_{\alpha\in A}\mathcal{V}^!_\alpha\to\mathcal{V}\to 0$$

where the boundary operator is given by

$$(ds_x)_{\alpha_0,\dots,\alpha_q}=\sum_\beta(-1)^{\epsilon(\beta,\underline{\alpha})}s_{x,\alpha_0,\dots,\beta,\dots,\alpha_q}$$

where β runs over those indices which do not occur in $\underline{\alpha}$ where $\epsilon(\beta,\underline{\alpha})$ gives us the position of β with respect to the ordering, where $s_{x,\alpha_0,\dots,\beta,\dots,\alpha_q}$ is an element in the stalk $\mathcal{V}^!_{\alpha_0,\dots,\beta,\dots,\alpha_q,x}$. The last homomorphism on the right is simply summation $\sum s_{x,\alpha}$. Again it is clear that this is an exact complex of sheaves.(See ???).

8.2.1 Comment.

In a sense we will see that this is an acyclic resolution for the right exact functor

$$\mathcal{V}\to H^d_c(M,\mathcal{V})$$

but one has to define a suitable category of sheaves for which this makes sense. For a local system on a manifold the $H^d_c(M,\mathcal{V})$ plays the same role as the global sections for the ordinary cohomology. Except that in addition all the arrows point in opposite directions and this is duality.

Now I claim that we have $H_c^\nu(M, \prod_{\underline{\alpha}\in A_<^{q+1}} \mathcal{V}^!_\alpha) = 0$ if $\nu \neq d$ and

$$H_c^d(M, \prod_{\underline{\alpha}\in A_<^{q+1}} \mathcal{V}^!_\alpha) = \bigoplus_{\underline{\alpha}\in A_<^{q+1}} H^d(M, \mathcal{V}^!_\alpha).$$

To see this we take the injective resolution constructed by Godement for the $\mathcal{V}^!_{\underline{\alpha}}$ (See 2.1) Then the product of the sheaves in the resolution gives a resolution of the product:

$$0 \to \prod_{\underline{\alpha}\in A_<^{q+1}} \mathcal{V}^!_{\underline{\alpha}} \to \prod_{\underline{\alpha}\in A_<^{q+1}} I^0_{\underline{\alpha}} \to \prod_{\underline{\alpha}\in A_<^{q+1}} I^1_{\underline{\alpha}} \to$$

and to compute the cohomology with compact support we look at the resulting complex of global sections with compact support. But since any compact set meets only finitely many of the open sets U_α we see that

$$H_c^0(M, \prod_{\underline{\alpha}\in A_<^{q+1}} I^q_{\underline{\alpha}}) = \bigoplus_{\underline{\alpha}\in A_<^{q+1}} H_c^0(M, I^q_{\underline{\alpha}}).$$

To see this we have to take into account that the stalks of the sheaves $I^q_{\underline{\alpha}}$ are zero ouside $U|_{\underline{\alpha}}$ which is clear from the construction. Then

$$H_c^\nu(M, \prod_{\underline{\alpha}\in A_<^{q+1}} \mathcal{V}^!_{\underline{\alpha}}) = \bigoplus_{\underline{\alpha}\in A_<^{q+1}} H^\nu(M, \mathcal{V}^!_{\underline{\alpha}})$$

We apply the functor H_c^d applied to the complex yields a complex

$$\to \bigoplus_{\alpha\in A_<^{q+1}} H_c^d(M, \mathcal{V}^!_\alpha) \to \ldots \to \bigoplus_{\alpha\in A} H_c^d(M, \mathcal{V}^!_\alpha) \to 0.$$

I want to make the additional assumption that the number of indices α for which U_α contains a given x is not only finite but even bounded independently of x. This has the consequence that our complex of sheaves is even finite.

We numerate the complex so that the boundary operator has degree -1 and the complex starts at zero at the right hand side, i.e. it looks like

$$\to Y_m \to \ldots Y_1 \to Y_0 \to 0.$$

8.2.2 Theorem. *The cohomology of this complex is the cohomology with compact supports*

$$H^{d-i}(M, \mathcal{V}) = H^i(\to \bigoplus_{\alpha\in A_<^{q+1}} H^d(M, \mathcal{V}^!_\alpha) \to \ldots \to \bigoplus_{\alpha\in A} H^d(M, \mathcal{V}^!_\alpha) \to 0.)$$

Proof. We used the same arguments which we used when we proved that we can compute cohomology groups by acyclic resolutions. We break the complex of sheaves into pieces

$$\to \bigoplus_{\underline{\alpha}\in A^{q+1}_<} V^!_{\underline{\alpha}} \to \ldots \to \bigoplus_{\alpha,\beta\in A^2_<} V^!_{(\alpha,\beta)} \to \mathcal{G} \to 0$$

and

$$0 \to \mathcal{G} \to \bigoplus_{\alpha\in A} V^!_\alpha \to \mathcal{V} \to 0.$$

The second complex gives us a long exact sequence if we apply the cohomology with compact supports. Since the sheaf in the middle has only cohomology with compact supports in degree d we get

$$H^{i-1}_c(M,\mathcal{V}) \simeq H^i_c(M,\mathcal{G}) \quad \text{for} \quad i \neq d-1, d$$

and

$$0 \to H^{d-1}_c(M,\mathcal{V}) \to H^d_c(M,\mathcal{G}) \to \bigoplus_{\alpha\in A} H^d(M,\mathcal{V}^!_\alpha) \to H^d_c(M,\mathcal{V}) \to H^{d+1}_c(M,\mathcal{G}) \to 0.$$

At first we want to conclude that $H^m_c(M,\mathcal{V}) = 0$ for $m > n$. If not we would have $H^m_c(M,\mathcal{G}) \neq 0$ for some $m > n$. But $\mathcal{G}$ sits in a short exact sequence

$$0 \to \mathcal{G}_1 \to \bigoplus_{\alpha,\beta\in A^2_<} \mathcal{V}^1_{\alpha,\beta} \to \mathcal{G} \to 0$$

and $\mathcal{G}_1$ is the end of the complex

$$\to \bigoplus_{(\alpha,\beta,\gamma)\in A^3_<} \mathcal{V}^1_{\alpha,\beta,\gamma} \to \mathcal{G}_1 \to 0.$$

We would get $H^m_c(M,\mathcal{G}_1) \neq 0$ for some $m > n$ and applying the same procedure again and again we get a contradiction because the complex is finite to the left. Hence we get in degree n

$$0 \to H^{d-1}_c(M,\mathcal{V}) \to H^d_c(M,\mathcal{G}) \to \bigoplus_{\alpha\in A} H^d(M,\mathcal{V}^!_\alpha) \to H^d_c(M,\mathcal{V}) \to 0.$$

Induction on the length of the complex gives us that the complex

$$0 \to \ldots \to \bigoplus_{\underline{\alpha}\in A^{q+1}_<} H^d(M,\mathcal{V}^!_\alpha) \to \ldots H^d(M,\mathcal{V}^!_{(\alpha,\beta)}) \to 0$$

computes the cohomology $H^{d-\bullet}_c(M,\mathcal{G})$ and the theorem follows.

8.3 Poincaré-Duality.

We assume that we have a local system $\mathcal{V}$ on M which has values in the category of R-modules where R is a commutative ring with identity. We can also consider the dual local system $\mathcal{V}^\vee = \mathrm{Hom}_R(\mathcal{V}, R)$. We compute the cohomology $H^\bullet(M,\mathcal{V})$ and the cohomology with compact support $H_c^\bullet(M,\mathcal{V}^\vee)$ by means of the two complexes which we obtain from a suitable covering. We write the complexes

$$\begin{array}{ccccc}
0 \to \prod_{\alpha\in A} \mathcal{V}(U_\alpha) & \to \prod_{(\alpha,\beta)\in A^2_<} \mathcal{V}(U_\alpha\cap U_\beta) & \to \dots \to \prod_{\alpha\in A^{q+1}_<} & \to \\
\| & \| & \| & \\
X^0 \xrightarrow{d} & X^1 \to \dots & \xrightarrow{d} X^q & \to
\end{array}$$

and

$$\begin{array}{ccccc}
\bigoplus_{\underline{\alpha}\in A^{q+1}_<} \mathcal{V}^\vee(U_{\underline{\alpha}}) \dots & \to \bigoplus_{(\alpha.\beta)\in A^2_<} \mathcal{V}^\vee(U_\alpha\cap U_\beta) & \to \bigoplus_{\alpha\in A} \mathcal{V}^\vee(U_\alpha) & \to 0 \\
\| & \| & \| & \\
Y^q \xrightarrow{\delta} & \xrightarrow{\delta} Y^1 & \xrightarrow{\delta} Y^0 & \to
\end{array}$$

where we made the identification $\mathcal{V}^\vee(U_{\underline{\alpha}}) = H_c^d(M,\mathcal{V}^{\vee,!})$. We have a pairing

$$<,>: Y^q \times X^q \to R$$

which is given by the formula

$$\begin{aligned}
s &= (\dots s_{\underline{\alpha}} \dots) \in \prod \mathcal{V}(U_{\underline{\alpha}}) \\
t &= (\dots t_{\underline{\alpha}} \dots) \in \bigoplus \mathcal{V}^\vee(U_{\underline{\alpha}}) \\
< s,t > &= \sum_{\underline{\alpha}} s_{\underline{\alpha}} \cdot t_{\underline{\alpha}}
\end{aligned}$$

where $s_{\underline{\alpha}} \cdot t_{\underline{\alpha}}$ is the pairing induced by the pairing on the coefficient systems. The expression makes sense because t has only finitely many non-zero entries.

We have

$$< ds, t > = < s, \delta t >$$

for $s \in X^q, t \in Y^{q+1}$. From this we get a pairing

$$H^q(M,\mathcal{V}) \times H_c^{d-q}(M,\mathcal{V}^\vee) \to R.$$

We want to discuss the properties of this pairing.

8.3.1 Theorem: Poincaré-duality. *We assume that $\mathcal{V}$ is a local system of finite dimensional k-vector spaces. We also assume that we can find a convex covering such that the function $x \to \#\{\alpha | x \in U_\alpha\}$ is bounded. Furthermore we assume that*

$$\dim_k H^i(M,\mathcal{V}) \quad \text{and} \quad \dim_k H_c^i(M,\mathcal{V})$$

are finite for all i. Then the pairing

$$H^q(M,\mathcal{V}) \times H_c^{d-q}(M,\mathcal{V}^\vee) \to k$$

is non-degenerate for all q.

If R is a discrete valuation ring and if $\mathcal{V}$ is a local system of free R-modules of finite rank and if M admits a finite acyclic covering then the pairing

$$H^q(M,\mathcal{V})/\mathrm{Tors} \times H_c^{d-q}(M,\mathcal{V}^\vee)/\mathrm{Tors} \to R$$

is non-degenerate.

Proof. We start from the computation of the cohomology by means of the two complexes

$$\begin{array}{llll} \to \prod_{\underline{\alpha}\in A^q_<} H^0(M,\mathcal{V}^*_{\underline{\alpha}}) & \to \dots & \to \prod_{\alpha\in A} H^0(M,\mathcal{V}^*_\alpha) & \to \\ \to \bigoplus_{\underline{\alpha}\in A^q_<} H^d(M,\mathcal{V}^{\vee !}_\alpha) & \to \dots & \to \bigoplus_{\alpha\in A} H^d(M,\mathcal{V}^{\vee !}_\alpha) & \to \end{array}.$$

We abbreviate these complexes by $X^\bullet$ and $Y^\bullet$ respectively. We have the pairing

$$<,>: Y^q \times X^q \to k.$$

These spaces may be of infinite dimension. We say that a linear form $\lambda : X^q = \prod_{\alpha\in A^q_<} H^0(M,\mathcal{V}^*_\alpha) \to R$ is continuous if it factors over a quotient $\prod_{\alpha\in E_\lambda} H^0(M,\mathcal{V}^*_\alpha)$ where E_λ is a finite subset of $A^q_<$. Then it is clear that Y^q is the space of continuous linear forms on X^q. It is also clear that X^q is the space of all linear forms on Y^q.

In X^q (resp. Y^q) we have the subspaces of cocycles of coboundaries

$$\begin{array}{l} B^q(X^\bullet) \subset Z^q(X^\bullet) \subset X^q \\ B^q(Y^\bullet) \subset Z^q(Y^\bullet) \subset Y^q \end{array}.$$

Since $B^q(X^\bullet) = d(X^{q-1})$ by definition we find that

$$Z^q(Y^\bullet) = \{y \in Y^q | \delta y = 0\} = B^q(X^\bullet)^\perp = \{y \in Y^q | < B^q(X^\bullet), y >= 0\}$$

and by the same argument we find that

$$Z^q(X^\bullet) = B^q(Y^\bullet)^\perp.$$

The spaces X^q, Y^q are in perfect duality. If they were finite dimensional we could conclude, that for any subspace W of one of them we have $(W^\perp)^\perp = W$. This is also true in our situation here for subspaces $W \subset Y^q$ because X^q is the space of all linear forms on Y^q and for an element $y \notin W$ we can find an $x \in X^q$ with $< W, x >= 0$ and $< y, x >\neq 0$. (Zorn's Lemma).

But for subspaces $W \subset X^q$ the same argument is only true for closed subspaces, which means

$$W = \{x \in X^q \mid \lambda(x) = 0 \text{ for continuous linear forms } \lambda \text{ which vanish on } W\}.$$

Now we use a little

Lemma. *The space of cocycles $Z^q(X^\bullet)$ is closed. If*

$$\dim_k Z^q(X^\bullet)/B^q(X^\bullet) < \infty$$

then $B^q(X^\bullet)$ is also closed.

Proof. Exercise

Since we assumed that the various cohomology groups should be finite dimensional, we conclude that all the $B^q(X^\bullet)$ are closed and hence

$$\begin{array}{c} B^q(X^\bullet)^{\perp\perp} = B^q(X^\bullet) \\ B^q(Y^\bullet)^{\perp\perp} = B^q(Y^\bullet) \end{array}.$$

But then it is obvious that the pairing

$$H^q(M, \mathcal{V}) \times H_c^{d-q}(M, \mathcal{V}^\vee) \to k$$

is non-degenerate.

Now we come to the second half of the theorem, we assume that $\mathcal{V}$ is a local system of free R-modules of finite rank where R is a discrete valuation ring. We also assume that we have an acyclic covering which is finite. In this case we have that the modules X^q, Y^q are free of finite rank and the conclusion

$$\begin{array}{c} Z^q(X^\bullet) = B^q(Y^\bullet)^\perp \\ Z^q(Y^\bullet) = B^q(X^\bullet)^\perp \end{array}$$

is still valid. But we cannot conclude that $B^q(Y^\bullet)^{\perp\perp} = B^q(Y^\bullet)$ and $B^q(X^\bullet)^{\perp\perp} = B^q(X)$. But $Z^q(X^\bullet), Z^q(Y^\bullet)$ are direct summands in X^q resp. Y^q and therefore

$$X^q/Z^q(X^\bullet) \quad \text{and} \quad Y^q/Z^q(Y^\bullet)$$

are torsion free.

I claim that $B^q(X^\bullet)^{\perp\perp}/B^q(X^\bullet)$ and $B^q(Y^\bullet)^{\perp\perp}/B^q(Y^\bullet)$ must be torsion. Let K is field of fractions of R. The claim follows because we know that $B^q(X^\bullet)^{\perp\perp} \otimes_R K = B^q(X^\bullet \otimes_R K)^{\perp\perp}$ It follows that

$$\begin{array}{l} = B^q(X^\bullet)^{\perp\perp} \subset Z^q(X^\bullet) \\ = B^q(Y^\bullet)^{\perp\perp} \subset Z^q(Y^\bullet) \end{array}$$

and if we look at

$$Z^q(X^\bullet)/B^q(X^\bullet) = H^q(X^\bullet) \to Z^q(X^\bullet)/B^q(X^\bullet)^{\perp\perp}$$

then the kernel of this map is the torsion subgroup. The same holds for $Y^\bullet$.
Now it is an easy exercise to prove that the pairing

$$Z^q(X^\bullet)/B^q(X^\bullet)^{\perp\perp} \times Z^q(Y^\bullet)/B^q(Y^\bullet)^{\perp\perp} \to R$$

is non-degenerate.

8.4 The cohomology in top degree and the homology.

We start from a local system $\mathcal{V}$ and we assume that we obtained it form an action of the fundamental group $\pi = \pi_1(M, x_0)$ on an abelian group (or R-module) V. We have $\mathcal{V}_{x_0} = V$. We want to compute the cohomology with compact support in top degree. We will see that this can be expressed completely in terms of the action of π on V. Let I_π be the augmentation ideal, we introduced the module of coinvariants $V/I_\pi = V_\pi$. We will see that $H^d_c(M, \mathcal{V}) \tilde{\to} V_\pi$. This makes it clear that the cohomology with compact supports behaves like homology.

We start from our complex

$$\bigoplus_{(\alpha,\beta)\in A^2_<} H^d(M, \mathcal{V}^!_{\alpha\beta}) \to \bigoplus_{\alpha\in A} H^d(M, \mathcal{V}_\alpha) \to H^d_c(M, \mathcal{V}) \to 0.$$

Let α_0 be an index such that $x_0 \in U_{\alpha_0}$. I claim that the map

$$H^d(M, \mathcal{V}^!_{\alpha_0}) \simeq V \to H^d_c(M, \mathcal{V})$$

is surjective and induces an isomorphism

$$V/I_\pi V \xrightarrow{\sim} H^d_c(M, \mathcal{V}).$$

Let α be any other index. We can choose a sequence $\alpha_0, \alpha_1, \ldots, \alpha_r = \alpha$ of indices such that $U_{\alpha_i \cap U_{\alpha_{i+1}}} \neq \emptyset$ for all i. For any pair of consecutive indices α_i, α_{i+1} we restrict the boundary operator

$$\delta : \bigoplus_{(\alpha,\beta)\in A^2_<} H^d(M, \mathcal{V}^!_{\alpha\beta}) \to \bigoplus_{\gamma\in A} H^d(M, \mathcal{V}^!_\gamma)$$

to the direct summand

$$H^d(M, \mathcal{V}^!_{\alpha_i\alpha_{i+1}})$$

(we assume $\alpha_i < \alpha_{i+1}$ otherwise we interchange the indices). It is clear that the image of this restriction in the target module lies in the submodule

$$H^d(M, \mathcal{V}^!_{\alpha_i}) \bigoplus H^d(M, \mathcal{V}^!_{\alpha_{i+1}}).$$

We have a natural isomorphism

$$\Psi_{\alpha_i,\alpha_{i+1}} : H^d(M, \mathcal{V}^!_{\alpha_{i+1}}).$$

which is the composition of the isomorphisms

$$\begin{aligned} H^d(M, \mathcal{V}^!_{\alpha_i\alpha_{i+1}}) &\xrightarrow{\sim} H^d(M, \mathcal{V}^!_{\alpha_i}) \\ H^d(M, \mathcal{V}^!_{\alpha_i\alpha_{i+1}}) &\xrightarrow{\sim} H^d(M, \mathcal{V}^!_{\alpha_{i+1}}) \end{aligned}$$

which are induced by the inclusions $U_{\alpha_i} \cap U_{\alpha_{i+1}} \hookrightarrow U_{\alpha_i}$ and $U_{\alpha_i} \cap U_{\alpha_{i+1}} \hookrightarrow U_{\alpha_{i+1}}$. It is clear from the definition that the image of δ restricted to $H^d(M, \mathcal{V}^!_{\alpha_i\alpha_{i+1}})$ is the submodule

$$\mathcal{V}_i, -\Psi_{\alpha_i\alpha_{i+1}}(\mathcal{V}_i)),$$

and hence we see that this submodule is in the kernel of

$$H^d(M, \mathcal{V}^!_{\alpha_i}) \bigoplus H^d(M, \mathcal{V}^!_{\alpha_{i+1}}) \to H^d_c(M, \mathcal{V}).$$

Now our claim of indices gives us by composition an isomorphism

$$\Psi_{\alpha_0,\alpha_1} \ldots, \alpha_r = \alpha : H^d(M, \mathcal{V}^!_{\alpha_0}) \to H^d(M, \mathcal{V}^!_{\alpha})$$

and it is clear that the elements

$$(\mathcal{V}, -\Psi_{\alpha_0,\alpha_1\ldots,\alpha_v=\alpha}(\mathcal{V}))$$

lie in the kernel of

$$H^d(M, \mathcal{V}^!_{\alpha_0}) \bigoplus H^d(M, \mathcal{V}^!_{\alpha}) \to H^d_c(M, \mathcal{V}).$$

From this it follows that the summand $H^d(M, \mathcal{V}^!_{\alpha_0}) \simeq V$ is mapped surjectively to $H^d_c(M, \mathcal{V}^)$.

We can take $\alpha = \alpha_0$ then we can construct a path

$$\gamma : [0, 1] \to M$$

with $\gamma(0) = \gamma(1) = x_0$ which is obtained by joining x_0 inside of U_{α_0} to a point in $U_{\alpha_0} \cap U_{\alpha_1}$, this point to a point in $U_{\alpha_1} \cap U_{\alpha_2}$ and so on and finally joining the point in $U_{\alpha_{r-1}} \cap U_{\alpha_{\alpha_r}}$ to x_0. The homotopy class of this path is uniquely determined by the chain of indices.

Then it is clear from the construction of the local system from the action $\rho : \pi \to \mathrm{Aut}(V)$ that $\Psi_{\alpha_0,\alpha_1} \ldots, \alpha_r = \alpha = \rho([\gamma])$. Hence we see that all elements of the form $(\mathrm{Id} - \rho([\gamma])\mathcal{V}$ lie in the kernel of

$$H^d(M, \mathcal{V}^!_{\alpha_0}) \to H^d_c(M, \mathcal{V})$$

and the surjective map factors

$$\begin{array}{ccc} V & \longrightarrow & H^d_c(M,\mathcal{V}) \\ & \searrow \qquad \nearrow & \\ & V/I_\pi V & \end{array} .$$

But now it follows that

$$V/I_\pi V \xrightarrow{\sim} H^d_c(M,\mathcal{V})$$

must be an isomorphism, because the group

$$\bigoplus_{(\alpha,\beta)\in A^2_<} H^d_c(M,\mathcal{V}^!_{\alpha\beta})$$

is generated by its direct summands.

We can also define the singular homology groups $H_i(M,\mathcal{V})$. To do this we consider map

$$\sigma : \Delta_1 \to M$$

where $\Delta_q = \{(t_0,\dots,t_q) \in \mathbb{R}^{q+1}_+ \,|\, \sum t_i = 1\}$ is the q-dimensional standard simplex. We can consider the pull back $\sigma^*(\mathcal{V})$ of our local system and since Δ_q is contractible, we have

$$\sigma^*(\mathcal{V})(\Delta_q) = \sigma^*(\mathcal{V})_p$$

where p is any point in our simplex. We form linear combinations

$$\sum m_\sigma \cdot \sigma$$

where $m_\sigma \in \sigma^*(\mathcal{V})(\Delta_q)$. These linear combinations form an abelian group

$$C_q(M,\mathcal{V}).$$

We define a boundary operator

$$\partial_q : C_q(M,\mathcal{V}) \to C_{q-1}(M,\mathcal{V}).$$

To do this we observe that we have face maps

$$\begin{aligned} \tau_i :& \Delta_{q-1} \to \Delta_q \\ \tau_i :& (t_0 \dots t_q) \to (t_0,\dots,t_{i-1},t_i,\dots t_q) \end{aligned}$$

and we put

$$\partial_1(m_\sigma \sigma) = \sum (-1)^i m_\sigma \cdot \sigma \circ \tau_i$$

where we use the fact that

$$(\sigma \circ \tau_i)^*(\mathcal{V})(\Delta_{q-1}) = \sigma^*(\mathcal{V})(\Delta_q).$$

An easy computation yields $\partial_{q-1} \circ \partial_q = 0$ hence we get the chain complex with coefficients in $\mathcal{V}$

$$\to C_q(M, \mathcal{V}) \to C_{q-1}(M, \mathcal{V}) \to \cdots \to C_0(M, \mathcal{V}) \to 0$$

and the homology groups of this complex are the homology groups of M with coefficients in $\mathcal{V}$:

$$H_q(M, \mathcal{V}) = H_q(C_\bullet(M, \mathcal{V})).$$

It is clear what $H_0(M, \mathcal{V})$ is: we see that $C_0(M, \mathcal{V})$ is the group given by linear combinations

$$\sum_{x \in M} m_x \cdot x$$

where $m_x \in \mathcal{V}_x$. Of course we see that

$$m_x x - m_y y \quad \text{is a boundary}$$

if we can find a path $\gamma : [0, 1] \to M$ with $\gamma(0) = x, \gamma(1) = y$ and $[\gamma] m_x = m_y$. Hence it is clear that

$$H_0(M, \mathcal{V}) \simeq H_c^d(M, \mathcal{V}) = V/I_\pi V.$$

This suggests that we have

$$H_i(M, \mathcal{V}) \simeq H_c^{d-i}(M, \mathcal{V})$$

and this is actually very easy to prove.

8.4.1. If we want to understand the cohomology $H_c^\bullet(M, \mathcal{V})$ it is sometimes very useful to embed M into a compact space. Let us consider an open embedding

$$i : M \hookrightarrow \overline{M},$$

where $\overline{M}$ is compact. Then we can consider the sheaf $i_!(\mathcal{V})$, and we know

$$H_c^\bullet(M, \mathcal{V}) = H^\bullet(\overline{M}, i_!(\mathcal{V})).$$

We may also consider the direct image $i_*(\mathcal{V})$. Here we have to be careful because the functor i_* is not exact in general. But if we assume that our local system is acyclic with respect to i_*, then we know that

$$H^\bullet(M, \mathcal{V}) = H^\bullet(\overline{M}, i_* \mathcal{V}).$$

Especially we may have the situation that $\overline{M}$ is an oriented manifold with boundary and M is the interior of $\overline{M}$. Then it is clear that $M \hookrightarrow \overline{M}$ is a homotopy equivalence and a local system $\mathcal{V}$ on M extends to a local system on $\overline{M}$, which we denote by $\overline{\mathcal{V}}$. Under these circumstances we have

$$H^\bullet(M,\mathcal{V}) = H^\bullet(\overline{M}, i_*\mathcal{V}) = H^\bullet(\bar{M},\overline{\mathcal{V}}).$$

If now $\mathcal{V}$ is a local system of free R-modules of finite rank and $\mathcal{V}^\vee = \mathrm{Hom}(\mathcal{V},R)$ the dual system, then we have the Poincaré pairing

$$H^q_c(M,\mathcal{V}) \times H^{m-q}(M,\mathcal{V}^\vee) \longrightarrow H^m_c(M,R) \simeq R$$

which we may also write as

$$H^q(\bar{M}, i_!(\mathcal{V})) \times H^{m-q}(\bar{M}, i_*(\mathcal{V}^\vee)) \longrightarrow R.$$

It should not be too much of a surprise that this pairing can also be expressed in terms of the cup product.

We start with the observation that both sheaves $i_!(\mathcal{V})$ and $i_*(\mathcal{V}^\vee)$ have flat acyclic resolutions. In this situation we defined the product (see 6.6.1)

$$H^q(\overline{M}, i_!(\mathcal{V})) \times H^{m-q}(\overline{M}, i_*(\mathcal{V}^\vee)) \longrightarrow H^m(\overline{M}, i_!(\mathcal{V})\tilde{\otimes} i_*(\mathcal{V}^\vee)),$$

and we have the pairing

$$i_!(\mathcal{V})\tilde{\otimes} i_*(\mathcal{V}^\vee) \longrightarrow i_!(R).$$

Now the cup product provides a pairing

$$H^q(\overline{M}, i_!(\mathcal{V})) \times H^{m-q}(\overline{M}, i_*(\mathcal{V}^\vee)) \longrightarrow H^m(\overline{M}, i_!(R)) = H^m_c(M,R) = R.$$

We want to convince ourselves that this pairing must be the Poincaré pairing.

To see that this is the case we compute the cohomology groups from Čech resolutions. Our situation is a little bit different from the previous one since now our manifold has a boundary. But we may put a Riemannian metric as before and at first we cover a neighborhood of ∂M by small open "half-balls" with center on the boundary. Then the complement of the union of these balls is compact, and we cover it by small balls whose closure does not hit the boundary. Let us denote this covering by $\{U_\alpha\}_{\alpha\in A}$.

For any $\alpha_0\cdots\alpha_q$ we consider $U_{\alpha_0}\cap\cdots\cap U_{\alpha_q} = U_{\underline{\alpha}}$, and we remove the boundary points from it (if there are any) and call the result $\overset{\circ}{U}_{\underline{\alpha}}$. Then

$$i_{\underline{\alpha}} : \overset{\circ}{U} \longrightarrow \overline{M}$$

is the inclusion, and we define

$$i_{\underline{\alpha},!} i^*_{\underline{\alpha}}(\mathcal{V}) = \mathcal{V}^!_\alpha.$$

Now we compute our cohomology groups from the Čech resolution and the coresolution as before. We have

$$\longrightarrow \bigoplus_{\alpha,\beta} V^{v,!}_{\{\alpha,\beta\}} \longrightarrow \bigoplus \mathcal{V}^{\vee,!}_{\alpha} \longrightarrow i_!(\mathcal{V}^{\vee}) \longrightarrow 0,$$

and

$$0 \longrightarrow \mathcal{V} \longrightarrow \prod_{\alpha} \mathcal{V}^*_{\alpha} \longrightarrow \prod_{(\alpha,\beta)} \mathcal{V}^*_{(\alpha,\beta)} \longrightarrow \cdots.$$

Of course it is clear that the Poincaré pairing

$$H^0(\overline{M}, i_*(\mathcal{V})) \times H^m(\overline{M}, i_!(\mathcal{V}^{\vee})) \longrightarrow R$$

is given by the cup product. Then we proceed by induction on the degree. We breack the two resolutions

$$0 \to \mathcal{G} \to \bigoplus \mathcal{V}^{\vee,!}_{\alpha} \to i_!(\mathcal{V}^{\vee}) \to 0$$

$$0 \to \mathcal{V} \to \prod_{\alpha} \mathcal{V}^*_{\alpha} \to \mathcal{H} \to 0$$

and we get the folowing pieces of long exact sequences

$$0 \longrightarrow H^{n-1}(\overline{M}, i_!(\mathcal{V}^{\vee})) \longrightarrow H^n(\overline{M}, \mathcal{G}) \longrightarrow \cdots$$

and

$$H^0(\overline{M}, \mathcal{H}) \longrightarrow H^1(\overline{M}, i_*(\mathcal{V})) \longrightarrow 0.$$

The pairing $i_!(\mathcal{V}^{\vee}) \times i_*(\mathcal{V}) \to i_!(R)$ induces a pairing

$$\left(\bigoplus \mathcal{V}^{v,!}_{\alpha}\right) \times \prod_{\alpha} \mathcal{V}^*_{\alpha} \longrightarrow i_!(R)$$

(see 8.3), and this induces a pairing

$$\mathcal{G} \times \mathcal{H} \longrightarrow i_!(R).$$

If we look at the definition of the Poincaré pairing of two classes $\xi \in H^{n-1}(\overline{M}, i_!(\mathcal{V}^{\vee}))$ and $\eta = \delta_{\psi} \in H^1(\overline{M}, i_*(\mathcal{V}))$, then we have

$$\langle \xi, \eta \rangle = \langle \delta\xi, \psi \rangle.$$

But the right hand side is also the cup product of the classes $\delta\xi \in H^0(\overline{M}, \mathcal{G})$ and $\psi \in H^n(\overline{M}, \mathcal{H})$. But we have to see that also the cup product satisfies the rule

$$\xi \cup \eta = \xi \cup d\psi = \delta\xi \cup \psi,$$

and we have seen that the cup product and the Poincaré pairing coincide on $H^{n-1}_c \times H^1$. The general case follows by the same argument inductively.

8.5 The fundamental class of a submanifold : The homology groups can be defined for any space X and they provide a covariant functor from spaces to abelian groups: if we have a continuous map $f : X \to Y$, then we get a homomorphism

$$f_{*,i} : H_i(X,\mathbb{Z}) \longrightarrow H_i(Y,\mathbb{Z})$$

for all degrees i.

This suggests that we should also have this kind of functoriality for the cohomology with compact supports on an oriented manifold M.

I want to discuss a special case where we see this functoriality. We consider a manifold M and a oriented submanifold $N \subset M$, let m, n be the dimensions of M and N respectively. I allow that N has several connected components. We assume that we have a relative orientation, this means that we have an orientation of the normal bundle $T_{M/N}$.

Let $\mathcal{V}$ be a local system on M, let $\mathcal{V}'$ be its restriction to N. If we consider the homology groups then we get get - directly from the definition - a homomorphism

$$H_i(N,\mathcal{V}') \longrightarrow H_i(M,\mathcal{V}).$$

On our manifolds the homology groups are isomorphic to cohomology groups with compact support and hence we get a homomorphism

$$H_c^j(N,\mathcal{V}') \longrightarrow H_c^{m-n+j}(M,\mathcal{V}),$$

and I want to construct this homomorphism directly.

We choose an auxiliary Riemannian metric. This Riemannian metric splits the tangent bundle of M along N into $T_M = T_N \oplus T_{M/N}$. Using the exponential map we can construct a tubular neighborhood $i_N : TN \subset M$ (See [ ]). We have the projection $\pi : TN \to N$ where the fibres $\pi^{-1}(b)$ can be identified to small open balls in $T_{M/N,b}$. By $\overline{TN}$ we denote the closure of TN. This gives us a fibration by open and closed balls as in 7.3.1.

Let $\mathcal{V}_N$ be the restriction our local system $\mathcal{V}$ to the open subset TN, then we put $\mathcal{V}_N^! = i_{N,!}(\mathcal{V}_N)$ we have an inclusion $\mathcal{V}_N^! \hookrightarrow \mathcal{V}^!$ and therefore a homomorphism

$$H_c^\bullet(TN,\mathcal{V}_N) = H^\bullet(M,\mathcal{V}_N^!) \to H^\bullet(M,\mathcal{V}^!)$$

I claim that we have a canonical iomorphism

$$H^j(N,\mathcal{V}') \xrightarrow{\sim} H_c^{j+m-n}(TN,\mathcal{V}_N) = H^{j+m-n}(\overline{TN},\mathcal{V}_N^!)$$

We apply the same argument as in 7.3.1 and get

$$H^{j+m-n}(\overline{TN},\mathcal{V}_N^!) \xrightarrow{\sim} H^j(N, R^{m-n}\pi_*(\mathcal{V}))$$

and then it is obvious that $\mathcal{V}' = R^{m-n}\pi_*(\mathcal{V})$. We assume in addition that M and N are oriented and the relative orientation is compatible with these orientations. Finally we assume that $\mathcal{V} = \mathbb{Z}$ is the trivial local system.

Especially we have the map

$$\mathbb{Z} = H^0(N, \mathbb{Z}) \longrightarrow H_c^{m-n}(M, \mathbb{Z}),$$

and the image of 1 under this map is a class $[N] \in H_c^{m-n}(M, \mathbb{Z})$. It is called the fundamental class of N in M.

Let ω be cohomology class in the complementary dimension $n = \dim N$, then we can consider the cup product

$$[N] \cup \omega \in H^m(M, \mathbb{Z}) = \mathbb{Z}$$

or we can restrict ω to N and get a class $i^*(\omega) \in H^n(N, \mathbb{Z}) = \mathbb{Z}$.

It follows directly from the definition and the compatibility of the Poincaré-pairing and the cup product that

$$i^*(\omega) = [N] \cup \omega. \qquad 8.5.1$$

8.6 Cup product and intersections Let us assume we have two oriented compact submanifolds N_1, N_2 of codimension d_1, d_2 in our oriented manifold M. We get two classes $[N_1], [N_2]$ in the cohomology with compact support, they sit in degrees d_1, d_2. We want to understand the cup product of these two classes.

We assume that our two submanifolds *intersect transversally.* This means that in any point p of $N_1 \cap N_2$ the intersection of the two tangent spaces $T_{N_1,p} \cap T_{N_2,p}$ has dimension $\dim M - d_1 - d_2$. This implies that the intersection $N_1 \cap N_2$ is again a compact submanifold of codimension $d_1 + d_2$. It may have several connected components. We write

$$N_1 \cap N_2 = \bigcup C_j$$

where the C_j are the connected components. For any point $p \in C_j$ we get an exact sequence of tangent spaces

$$0 \to T_{C_j,p} \to T_{N_1,p} \oplus T_{N_2,p} \to T_{M,p} \to 0$$

where the arrow from the direct sum to the tangent space of M is given by : first component minus second component. This gives us an isomorphism

$$\Lambda^c(T_{C_j,p}) \otimes \Lambda^m(T_{M,p}) \xrightarrow{\sim} \Lambda^{n_1}(T_{N_1,p}) \otimes \Lambda^{n_1}(T_{N_2,p})$$

and this puts an orientation O_j on C_j for all j. Now I claim

$$\sum [C_j] = [N_1] \cup [N_2] \qquad 8.6.1$$

We can look at the special case where $d_1 + d_2 = n$. In this case the cup product lands in $H_c^m(M, \mathbb{Z})$ and hence it is a number. If we keep the assumption of transversality then the intersection is a finite number of points. Now the tangent space of a point has always a canonical orientation. If now $c \in N_1 \cap N_2$ then we define

$$m(c) = \begin{cases} 1 & \text{if the orientatation } O_c \text{ is canonical} \\ -1 & \text{if not} \end{cases}$$

Our formula becomes

$$[N_1] \cup [N_2] = \sum_{c \in N_1 \cap N_2} m(c)$$

It is purely local problem to verify these formalae. According to (8.5.1) we have to restrict the class $[N_2]$ to N_1. We recall the construction of $[N_2]$, this class was the image of a class in $H_c^{m-d_2}(TN_2, \mathbb{Z}) = H^{m-d_2}(M, \mathbb{Z}^!_{N_2})$. If we restrict this class we get a class in $H^{m-d_2}(N_1, \mathbb{Z} \otimes \mathbb{Z}^!_{N_2})$ the rest is clear but has to be written.

8.7 Compact oriented surfaces:

Let S be a compact oriented 2 dimensional manifold.We have seen that for any ring R

$$H^0(S, R) = R \text{ and } H^2(S, R) = R$$

and the only unknown cohomology sits in degree one. For any prime p we have the exact sequence of sheaves

$$0 \to \underline{\mathbb{Z}} \to \underline{\mathbb{Z}} \to \mathbb{F}_p \to 0$$

and in the resulting long exact sequence we find the piece

$$0 \to H^1(S, \mathbb{Z}) \to H^1(S, \mathbb{Z}) \to H^1(S, \mathbb{F}_p) \to 0$$

we have zeroes at both ends because $H^0(S, \mathbb{Z}) \to H^0(S, \mathbb{F}_p)($ resp. $H^2(S, \mathbb{Z}) \to H^2(S, \mathbb{Z}))$ is surjective (resp. injective). This implies that $H^1(S, \mathbb{Z})$ is torsion free. Since we also know that these cohomology groups are finitely generated we conclude that $H^1(S, \mathbb{Z})$ is free of some rank.

Now we have the Poincare duality pairing

$$H^1(S, \mathbb{Z}) \times H^1(S, \mathbb{Z}) \to \mathbb{Z}$$

which is non degenerate and alternating. A well known result from elementary algebra tells us that we can find a basis $e_1, \ldots, e_g, f_1, \ldots, f_g$ of $H^1(S, \mathbb{Z})$ such that the duality pairing is given by

$$e_i \cup f_j = \delta_{ij},\ e_i \cup e_j = 0\ , f_i \cup f_j = 0$$

The number g is called the genus of the surface. For a curve of genus 3 we can draw the following picture

We three pairs of 1-cycles which form a basis in homology. But we also may view these cycles as submanifolds isomorphic to S^1 which are oriented by the arrows. These submanifolds have fundamental classes $e_1, f_1, e_2, f_2, e_3, f_3$ in $H^1(S, \mathbb{Z})$ and if we numerate them in the right way we have the above values of the intersection pairing.

8.8 The cohomology ring of $\mathbb{P}^n(\mathbb{C})$ We are now able to determine the structure of the cohomology ring $H^\bullet(\mathbb{P}^n(\mathbb{C}), \mathbb{Z})$ (See 6.5.1.1 Exercise). The fundamental class of the hyperplane $\mathbb{P}^{n-1}(\mathbb{C}) \subset \mathbb{P}^n(\mathbb{C})$ gives us a multiple of the generator $ae_1 \in H^2(\mathbb{P}^n(\mathbb{C}), \mathbb{Z})$ (Since we are dealing with complex manifolds, all manifolds have a canonical orientation.) Now we can put n such hyperlanes in general position such that they intersect transversally and $L_1 \cap \cdots \cap L_n$ is a point. The fundamental class of a point is the generator in $H^{2n}(\mathbb{P}^n(\mathbb{C}), \mathbb{Z})$. We conclude that $a^n e_1 \cup e_1 \ldots e_1$ is this generator, it follows that $a = 1$, e_1 is the fundamental class of the hyperplane and

$$H^\bullet(\mathbb{P}^n(\mathbb{C}), \mathbb{Z}) = \mathbb{Z}[e_1]/(e_1^{n+1}).$$

IV.9. The de-Rham and the Dolbeault Isomorphism

9.1 Real manifolds.

Let M be a $\mathcal{C}^\infty$-manifold and let $\mathcal{V}$ be a local system consisting of finite dimensional $\mathbb{R}$ or $\mathbb{C}$-vector spaces. (See 3.3.2). Locally on small connected open subsets $U \subset M$ we have a trivialization of $\mathcal{V}$ by constant sections $e_1, \cdots, e_n$ and

$$\mathcal{V}(U) = \left\{ \sum_{i=1}^{n} a_i e_i \mid a_i \in \mathbb{K} \right\}.$$

We define

$$\mathcal{V}_\infty(U) = \left\{ \sum_{i=1}^{n} f_i e_i \mid f_i \in \mathcal{C}^\infty(U) \right\},$$

and this gives us the sheaf of $\mathcal{C}^\infty$-sections in $\mathcal{V}$, we forget the discrete topology on the fibres.

Let $\Omega^p_{M,\infty}$ be the sheaf of $\mathcal{C}^\infty - p$-forms on M (3.3.2). We can define a differential

$$d : \mathcal{V}_\infty(U) \longrightarrow \mathcal{V}_\infty(U) \otimes \Omega^1_{M,\infty}(U)$$

by

$$d : \sum_{i=1}^{n} f_i e_i \longmapsto \sum_{i=1}^{n} e_i \otimes df_i.$$

If we pass to another open set U' and if we choose a trivialization $e'_1, \cdots, e'_n$ over U' then we get expressions

$$e_i = \sum a_{ij} e'_j$$

over $U \cap U'$ where the a_{ij} are locally constant. Thus we see that we can define a global differential

$$d : \mathcal{V}_\infty \longrightarrow \mathcal{V}_\infty \otimes \Omega^1_{M,\infty}.$$

It is clear from the definition that for any open set $U_1 \subset M$

$$\mathcal{V}(U_1) = \{ s \in \mathcal{V}_\infty(U_1) \mid ds = 0 \}.$$

We can extend our differential to forms of higher degree

$$s : \mathcal{V}_\infty \otimes \Omega^p_{M,\infty} \longrightarrow \mathcal{V}_\infty \otimes \Omega^{p+1}_{M,\infty}$$

by

$$d(\sum s_i \otimes \omega_i) = \sum s_i \otimes d\omega_i + \sum ds_i \wedge d\omega_i$$

where $ds_i = \sum g_{ij} \otimes \omega'_j$ and $ds_i \wedge d\omega_i = \sum_i g_{ij} \otimes \omega'_j \wedge \omega_i$. It is well known that $dd = 0$, and we get the so called de-Rham complex of sheaves

$$0 \to \mathcal{V} \to \mathcal{C}^\infty(\mathcal{V}) \to \mathcal{C}^\infty(\mathcal{V}) \otimes \Omega^1_{M,\infty} \to \ldots \to \mathcal{C}^\infty(\mathcal{V}) \otimes \Omega^d_{M,\infty} \to 0.$$

A form $\omega \in \mathcal{C}^\infty(\mathcal{V})(U) \otimes \Omega^d_{M,\infty}(V)$ is called *closed* if $d\omega = 0$.

9.1.1 Remark. If we have a $\mathcal{C}^\infty$-vector bundle $\mathcal{E}$ over M (see 3.1.1, here the g_{ij} have to be $\mathcal{C}^\infty$-functions), then we may consider differentials

$$d : \mathcal{C}^\infty(\mathcal{E}) \longrightarrow \mathcal{C}^\infty(\mathcal{E}) \otimes \Omega^1_{M,\infty}$$

which satisfy

$$\begin{aligned} d(s_1 + s_2) &= ds_1 + ds_2 \\ d(fs_1) &= fds_1 + s_1 \otimes df \end{aligned}$$

for local sections s_1, s_2 and local $\mathcal{C}^\infty$-functions f. Such differentials are called *connections* on $\mathcal{E}$. Such a connection is called a *flat connection* if

$$d(ds_1) = d(\sum s_i \otimes \omega_j) = \sum s_j \otimes d\omega_j + ds_i \wedge \omega_j = 0.$$

We saw that starting from a local coefficient system $\mathcal{V}$ we have a canonical flat connection on $\mathcal{V}_\infty$. But in turn, if we have a flat connection, then we can attach a local system $\mathcal{E}_0$ to $\mathcal{E}$ by defining

$$\mathcal{E}_0(U) = \{s \in \mathcal{C}^\infty(\mathcal{E})(U) \mid ds = 0\}.$$

It is of course clear that the flatness of the connection is necessary for the construction of the de-Rham complex.

It is not very hard to see that the de-Rham complex is exact. This follows from the well-known Lemma of Poincaré which says that a closed p-form ω on a convex open set U in $\mathbb{R}^d$ can be written as $d\psi = \omega$ with $\psi \in \Omega^{p-1}_\infty(\mathcal{V})(U)$. Hence the de-Rham complex gives us a resolution of the sheaf $\mathcal{V}$. We want to introduce the notation $\Omega^p_\infty(\mathcal{V})$ for $\mathcal{V}_\infty \otimes \Omega^p_{M,\infty}$. I claim that this resolution is also acyclic, we have

$$H^i(M, \Omega^p_\infty(\mathcal{V})) = 0 \text{ for } i \geq 1 \text{ and all } p \geq 0.$$

To see that this is the case we apply the Exercise 2.1.1. We have seen (Seee 8.2) that we have a partition of unity for the sheaf $\mathcal{C}^\infty(M)$. If we have any $\mathcal{C}^\infty(M)-$ vector bundle $\mathcal{E}$ on M then the sheaf $\mathcal{C}^\infty(\mathcal{E})$ is a sheaf of $\mathcal{C}^\infty(M)$ modules. Then our exercise 2.1.1 yields that the higher cohomology groups $\mathcal{C}^\infty(\mathcal{E})$ vanish.

We apply the functor global sections and then the resulting complex of global differential forms computes the cohomology (See II.3.1). We get the de-Rham-Isomorphism:

$$H^i(M, \mathcal{V}) = H^i(\Omega^\bullet_\infty(\mathcal{V})(M)) = \frac{\{\omega \in \Omega^p_\infty(\mathcal{V})(M) | d\omega = 0\}}{\{d\psi | \psi \in \Omega^{p-1}_\infty(\mathcal{V})(M)\}}.$$

The argument in Exercise 2.1.1 also applies to the cohomolgy with compact supports and get by the same token

$$H^i_c(M,\mathcal{V}) = H^i(H^0_c(M,\Omega^\bullet_\infty(\mathcal{V})).$$

If for instance we take the trivial system $\mathbb{R}$ on $M=\mathbb{R}^d$ then a closed form ω of degree $p>0$ on $\mathbb{R}^d$ can be written as $d\psi$ with $\psi\in\Omega^{p-1}_\infty(\mathbb{R}^d)$. If $p=0$ then a closed form is a constant function $f=c$ thus we get $H^0(\mathbb{R}^n,\mathbb{R})=\mathbb{R}$ and $H^i(\mathbb{R}^n,\mathbb{R})=0$ for $i>0$. (See 4.4.11)

If we consider the cohomology with compact supports then a closed form in degree zero which has compact support must vanish. Hence we get $H^0_c(\mathbb{R}^d,\mathbb{R})=0$. But if we have a form ω with compact support on $\mathbb{R}^d$ which is of degree d then we may not be able to write it as $\omega=d\psi$ where also ψ has compact support. If we could do so we would have

$$\int_{\mathbb{R}^d}\omega=\int_D\omega=\int_{\partial D}\psi=0$$

where D is a big closed ball which contains the support of ω and ψ. Hence we get a surjective linear form

$$\begin{aligned} H^d_c(\mathbb{R}^n,\mathbb{R}) &\to \mathbb{R} \\ [\omega] &\mapsto \int_{\mathbb{R}^d}\omega \end{aligned}.$$

Since it is easy to see that a form ω with compact support for which in addition $\int_{\mathbb{R}^d}\omega=0$ can be written as $\omega=d\psi$ with $\psi\in\Omega^{d-1}_c(\mathbb{R}^d)$ we get that the above map is an isomorphism.

9.1.2: We want to discuss the product structure of the cohomology in the context of the de-Rham isomorphism. If we have two manifolds M and N, then the resolution of the sheaf $\underline{\mathbb{R}}$ by the two de-Rham complexes are flat (Comp. the discussion in 6.7.). If we consider the product $M\times N$ and the two projections p_1,p_2, then we have a homomorphism of complexes

$$p_1^*(\Omega^\bullet_{M,\infty})\otimes_{\mathbb{R}} p_2^*(\Omega^\bullet_{N,\infty}) = \Omega^\bullet_{M,\infty}\tilde{\otimes}_R\Omega^\bullet_{N,\infty}\longrightarrow\Omega^\bullet_{M\times N,\infty}$$

which is given by the exterior multiplication of the differential forms. Hence it is clear that the product

$$\begin{aligned} H^p(M,\mathbb{R})\times H^q(N,\mathbb{R}) &\longrightarrow H^{p+q}(M\times N,\mathbb{R}) \\ (\alpha,\beta) &\longrightarrow \alpha\tilde{\otimes}_R\beta \end{aligned}$$

is induced by the exterior multiplication of the differential forms which represent the classes α,β.

Especially it becomes clear that the cup product on $H^\bullet(M,\mathbb{R})$ is induced by the structure of exterior algebra on the differential forms.

Finally it is clear that the Poincaé duality pairing on an open oriented manifold M

$$H^i_c(M,\mathbb{R})\times H^{m-i}(M,\mathbb{R})\longrightarrow H^m_c(M,\mathbb{R})\longrightarrow\mathbb{R}$$

is given by

$$[\omega] \times [\eta] \longrightarrow \int_m \omega \wedge \eta$$

if the classes $[\omega] \in H_c^i(M, \mathbb{R})$ and $[\eta] \in H^{m-i}(M, \mathbb{R})$ are represented by ω and η.

9.1.3: The de-Rham isomorphism also provides a different way looking at the notion of the fundamental class and the formulae for the cup product (see 8.5, 8.6). Let us consider an open ball $D^n \subset M$ in our manifold of dimension n. We assume it to be oriented. If we remove the origin from D, then we have a diffeomorphism

$$D^n \setminus \{0\} \simeq (0,1) \times S^{n-1}$$

which is given by

$$(x_1, \cdots, x_n) \longrightarrow \left(\sqrt{x_1^2 + \cdots + x_n^2}, \left(\frac{x_1}{\sqrt{x_1^2 + \cdots + x_n^2}}, \cdots, \frac{x_n}{\sqrt{x_1^2 + \cdots + x_n^2}} \right) \right) = (r, \varphi).$$

On S^{n-1} we have a unique differential form ω in degree $n-1$ which is invariant under the orthogonal group $SO(n)$, and which satisfies

$$\int_{S^{n-1}} \omega = 1.$$

Now we choose a $\mathcal{C}^\infty$-function $h(r)$ which is identically equal to 1 if r is close to 0 and identically equal to zero if r is close to one and this provides the differential form

$$h(r)\omega = \psi$$

on $D \setminus \{0\}$. If we take its exterior derivative

$$d\psi = \frac{\partial h(r)}{\partial r} \cdot dr \wedge \omega = \omega,$$

then ω is a form on $D^n \setminus \{0\}$ which vanishes identically in a small open ball around zero and near the boundary of D^n. Therefore we can extend it to a differential form on M and clearly we have

$$\int_M \omega = 1.$$

Thus we constructed a form which represents the canonical generator in $H^n(M, \mathbb{Z})$.

Let us assume that M is an oriented manifold and $N \subset M$ is an oriented submanifold. We construct a tubular neighborhood TN of N such that we have the projection

$$\pi : TN \longrightarrow N$$

and such that locally in N we have

$$\begin{array}{ccc}\pi^{-1}(V) & \xrightarrow{\sim} & V \times D^{m-n} \\ \downarrow & & \\ V. & & \end{array}$$

On $V \times D^{m-n}$ we construct a $m-n$-form ω_V which is the pullback of a form ω on D^{m-n} which is constructed as above.

Now we choose a covering $N = \bigcup_{i \in I} V_i$ which is locally finite and which trivializes $\pi : TN \to N$, and we choose a portion of unity $1 = \sum h_i$ with $\mathrm{supp}(h_0) \subset V_i$. On each $\pi^{-1}(V_i)$ we construct ω_i and we put

$$\omega = \sum h_i \omega_i.$$

For any point in $x \in TN$ we have

$$(d\omega)_x = \sum (dh_i)_x \wedge \omega_i - d(\Sigma h_i)_x \wedge \omega_i = 0,$$

and we see that $\omega_{N,\infty} \in \omega^{m-n}(M)$ is a closed form. It is clear that this form represents the fundamental class

$$[N] \in H^{m-n}(M, \mathbb{R}).$$

If now N_1, N_2 are two oriented submanifolds in M, and if we assume that one of them is compact, then we have the two classes

$$[N_1] = [\omega_{N_1}], [N_2] = [\omega_{N_2}]$$

where one of the forms has compact support. We just saw that

$$[N_1] \cup [N_2] = \omega_{N_1} \wedge \omega_{N_2}.$$

If now these two submanifolds are if complementary dimension, and if they intersect transversally, then it is easy to see that

$$\int_{M,\infty} \omega_{N_1} \wedge \omega_{N_2} = \sum_{c \in N_1 \cap N_2} \int_{D(c)} \omega_{N_1} \wedge \omega_{N_2}$$

where $D(c)$ is a small ball containing the local support of $\omega_{N_1} \wedge \omega_{N_2}$. Now it is easy to verify that these contributions from the points are equal to $m(c)$ (see 8.6.1)

9.2. Cohomology of holomorphic bundles on complex manifolds

Let M be a complex manifold. From our discussion in IV.3.2 it is rather clear what a homolorphic vector bundle $\mathcal{E}$ on M is. This is of course a bundle $\pi : \mathcal{E} \to M$ of $\mathbb{C}-$ vector spaces for which the transition functions $g_{ij} : V_i \cap V_j \to GL(n,\mathbb{C})$ are holomorphic. This allows us to define the sheaf of holomorphic sections which also denote by $\mathcal{E}$. It follows from our general principles in IV.3.3 that the holomorphic vector bundles are classified by $H^1(M, GL_n(\mathcal{O}_{M,\infty}))$ where $GL_n(\mathcal{O}_{M,\infty})$ is the sheaf of holomorphic functions from M to $GL_n(\mathbb{C})$.

To such a holomorphic vector bundle $\mathcal{E}$ we have the sheaf of germs of holomorphic sections which will be denoted by the same letter. This sheaf will be a locally free $\mathcal{O}_M$-module and a locally free $\mathcal{O}_M$ gives in turn a holomorphic vector bundle.

We also can define the sheaf $\mathcal{E}_\infty$ of $\mathcal{C}^\infty$- sections in the bundle, we have the inclusions of sheaves $\mathcal{E} \hookrightarrow \mathcal{E}_\infty$.

The following discussion will show that this concept is completely analogous to the concept of local systems. (See 9.2.4)

9.2.1 The tangent bundle: We pick a point $p \in M$ and an open neighborhood U_p of p such that

$$(U_p, \mathcal{O}_{M|U_p}) \simeq (D_p, \mathcal{O}_{D_p})$$

where D_p is an open ball in $\mathbb{C}^d$ whose center is $p = (0,\ldots,0)$. The tangent T_M is of course a holomorphic bundle which over U_p can be trivialized by the derivations $\frac{\partial}{\partial z_1}, \ldots, \frac{\partial}{\partial z_d}$. We write the complex coordinates by their real and imaginary parts

$$(z_1, \ldots, z_d) = (x_1 + iy - 1, \ldots, x_d + iy_d).$$

Then the tangent bundle of the $\mathcal{C}^\infty$-manifold M (we forget the complex structure) has a basis locally at p which is given by

$$\frac{\partial}{\partial x_1}, \frac{\partial}{\partial y_1}, \ldots, \frac{\partial}{\partial x_d}, \frac{\partial}{\partial y_d}.$$

These section are only sections in $T_{M,\infty}$. This bundle of $2d$-dimensional real vector spaces has the structure of a bundle of d dimensional complex vector spaces where locally the multiplication by i is given by

$$\begin{array}{llll} I : \frac{\partial}{\partial x_1} & \mapsto \frac{\partial}{\partial y_1}; \frac{\partial}{\partial y_1} & \mapsto -\frac{\partial}{\partial x_1} \\ \vdots & & & . \\ I : \frac{\partial}{\partial x_d} & \mapsto \frac{\partial}{\partial y_d}; \frac{\partial}{\partial y_d} & \mapsto -\frac{\partial}{\partial x_d} \end{array}$$

We have a priviledged orientation on the underlying $\mathcal{C}^\infty$-manifold which determined by requiring that $dx_1 \wedge dy_1 \wedge \ldots dx_d \wedge dy_d$ is positive.

We can take the tensor product

$$T_M \otimes_{\mathbb{R}} \mathbb{C} = T_{M,\mathbb{C}}$$

and get a bundle of $2d$-dimensional complex vector spaces. On this bundle of complex $2d$-dimensional vector spaces we still have the linear transformation I above and $T_{M,\mathbb{C}}$ decompose into two eigenspaces which are the eigenspaces with eigenvalues i and $-i$ for I:

$$T_{M,\mathbb{C}} = T^{10}_{M,\mathbb{C}} \oplus T^{01}_{M,\mathbb{C}}$$

where $T^{10}_{M,\mathbb{C}}$ is the eigenspace for the eigenvalue i for I and $T^{01}_{M,\mathbb{C}}$ is the eigenspace for the eigenvalue $-i$ for I. It is easy to see that locally on M we $T^{10}_{M,\mathbb{C}}$ has the basis (fibre by fibre)

$$\begin{array}{c} \dfrac{\partial}{\partial z_1} = \dfrac{\partial}{\partial x_1} - i \otimes \dfrac{\partial}{\partial y_1} \\ \vdots \\ \dfrac{\partial}{\partial z_d} = \dfrac{\partial}{\partial x_d} - i \otimes \dfrac{\partial}{\partial y_d} \end{array}.$$

This provides a structure of a holomorphic vector bundle on $T^{10}_{M,\mathbb{C}}$ the local trivialization is given by the above basis. The composition map $T_M \to T_M \otimes \mathbb{C} \to T^{10}_{M,\mathbb{C}}$ induces an isomorphism of complex vector bundles. Hence we can also view $T_{M,\infty}$ as a holomorphic bundle. The composition $T_M \to T_M \otimes \mathbb{C} \to T^{01}_{M,\mathbb{C}}$ is antilinear.

9.2.2 Here we apply some very simple principles of linear algebra which on the other hand can be confusing and their application requires some care.

If we have a $\mathbb{C}$-vector space V, we may define the complex conjugate space $\bar{V}$. Its underlying abelian group is V but the scalar multiplication

$$\mathbb{C} \times \bar{V} \to \bar{V}$$

is given by

$$(z, v) \longmapsto \bar{z} \cdot v,$$

where the dot on the right hand side denotes the scalar multiplication of $v \in V$ by $\bar{z} \in \mathbb{C}$. Hence we see that the identity map Id $: V \to \bar{V}$ is antilinear.

If we consider our complex vector space V over $\mathbb{C}$ as a real vector space together with a linear transformation I with $I^2 = -\mathrm{Id}$, then we can extend I to a linear transformation on $V \otimes \mathbb{C}$ and decompose into the eigenspace $V^{10}_{\mathbb{C}}$ (resp. V^{01}) with eigenvalues i (resp. $-i$).

The vector spaces $V, \bar{V}$ considered as real vector spaces are isomorphic by the identity map. In the following diagram the composition of the horizontal maps

$$\begin{array}{lcccc} & V & \longrightarrow & V \otimes_{\mathbb{R}} \mathbb{C} & \stackrel{pr_{10}}{\longrightarrow} & V^{10}_{\mathbb{C}} \\ \downarrow \mathrm{Id} & & & & & \\ & \bar{V} & \longrightarrow & V \otimes_{\mathbb{R}} \mathbb{C} & \stackrel{pr_{01}}{\longrightarrow} & V^{01}_{\mathbb{C}} \end{array}$$

are isomorphisms of $\mathbb{C}$-vector spaces.

The thing that may cause confusion is the following fact: On $V \otimes_{\mathbb{R}} \mathbb{C}$ we have the complex conjugation on the coefficients which may also be denoted by $v \to \bar{v}$. Then we get obviously

$$\overline{V_{\mathbb{C}}^{10}} = V_{\mathbb{C}}^{01},$$

but now putting a bar on $V_{\mathbb{C}}^{10}$ has a different meaning, we get a different underlying set in contrast to our convention above.

On the other hand we can say that we constructed canonical isomorphisms

$$\begin{array}{ccc} V & \xrightarrow{\sim} & V_{\mathbb{C}}^{10} \\ \bar{V} & \xrightarrow{\sim} & V_{\mathbb{C}}^{01} \end{array}$$

which allow us to identify V to $V_{\mathbb{C}}^{10}$ and $\bar{V}$ to $V_{\mathbb{C}}^{01}$. If we insert the map given by complex conjugation on the right end of our diagram above, then we ge a commutative diagram and the inconsistency in notation dissolves.

9.2.3 We can form the dual bundle Ω_M^1. Attached to this bundle we have the sheaf of $\mathcal{C}^\infty$ sections in this bundle which is also denoted by $\Omega_{M,\infty}^1$. We have a decomposition

$$\Omega_{M,\mathbb{C}}^1 = \Omega_M^1 \otimes \mathbb{C} \xrightarrow{\sim} \Omega_M^{10} \oplus \Omega_M^{01}.$$

The sheaf Ω_M^1 is locally generated by $dz_1 \dots dz_d$, we have

$$\begin{aligned} &< dz_\nu, \frac{\partial}{\partial x_\mu} - i \otimes \frac{\partial}{\partial y_\mu} > = 2\delta_{\nu\mu} \\ &< dz_\nu, \frac{\partial}{\partial x_\mu} + i \otimes \frac{\partial}{\partial y_\mu} > = 0 \end{aligned} .$$

We can define the fibres of $\Omega_{M,\mathbb{C}}^1$ at a point p simply as

$$\Omega_{M,\mathbb{C},p}^1 = \mathrm{Hom}_{\mathbb{R}}(T_{M,p}, \mathbb{C})$$

and then $\Omega_{M,p}^{10} = \{\omega | \omega(It_p) = i\omega(t_p)\}$ for all tangent vectors $t_p \in T_{M,p}$, in other words

$$\Omega_{M,p}^{10} = \mathrm{Hom}_{\mathbb{C}}(T_{M,p}, \mathbb{C}).$$

Analogously we have that $\Omega_{M,p}^{01}$ are the antilinear 1-forms. If we have a local section $\omega \in \Omega_M^{10}(U)$ then the complex conjugate $\bar{\omega}$ is given by

$$\bar{\omega}(t_p) = \overline{\omega(t_p)},$$

where $t_p \in T_{M,p}$ is a tangent vector at the point $p \in U$.
Again we can form the complex of differential forms

$$\Omega_{M,\mathbb{C},\infty}^{n-1} \to \Omega_{M,\mathbb{C},\infty}^{n} \to \Omega_{M,\mathbb{C},\infty}^{n+1}$$

but this sheaf (vector bundle) of n-forms decomposes

$$\Omega^n_{M,\mathbb{C},\infty} = \bigoplus_{p+q=n} \Omega^{p,q}_{M,\infty}$$

where

$$\Omega^{p,q}_{M,\infty} = \Lambda^p \Omega^{10}_{\infty} \otimes \Lambda^q \Omega^{01}_{\infty}.$$

(Once we have a superscript p, q we know that we are dealing with complex vector spaces, we omoit the subscript $\mathbb{C}$) Locally we get that a (p, q) form can be written as

$$\omega = \sum_{\alpha,\beta} f_{\alpha,\beta} dz_{\alpha_1} \wedge \ldots \wedge dz_{\alpha_p} \wedge d\bar{z}_{\beta_1} \wedge \ldots \wedge d\bar{z}_{\beta_q}$$

where $f_{\alpha,\beta}$ is a complex valued $\mathcal{C}^\infty$ function on U (the open set where we have these local coordinates). We get a decomposition of the operator

$$d : \Omega^n_{M,\infty} \to \Omega^{n+1}_{M,\infty}$$

as $d = \frac{1}{2}(d' + d'')$ where

$$d'\omega = \sum_\gamma \frac{\partial f_{\alpha,\beta}}{\partial z_\gamma} dz_\gamma \wedge \ldots \wedge dz_{\alpha_p} \wedge d\bar{z}_{\beta_1} \wedge \ldots \wedge d\bar{z}_{\beta_0}$$

and

$$d''\omega = (-1)^p \sum_\delta \frac{\partial f_{\alpha,\beta}}{\partial \bar{z}_f} dz_{\alpha_1} \wedge \ldots \wedge dz_{\alpha_p} \wedge d\bar{z}_\delta \wedge \ldots d\bar{z}_{\beta_q}$$

where the factor $\frac{1}{2}$ is explained by the fact that $\frac{\partial}{\partial z_v}, \frac{\partial}{\partial, \bar{z}_\mu}$ and $dz_v, d\bar{z}_\mu$ are not exactly dual bases of each other. (Nochmal nachrechnen).

We have

$$\begin{aligned} d' :& \Omega^{p,q}_{M,\infty} \to \Omega^{p+1,q}_{M,\infty} \\ d'' :& \Omega^{p,q}_{M,\infty} \to \Omega^{p,q+1}_{M,\infty} \end{aligned}.$$

9.2.4 Now we come back to our holomorphic vector bundle $\mathcal{E}$. We can embed the sheaf $\mathcal{E}$ of holomorphic sections into the sheaf of $\mathcal{C}^\infty$-sections, we write

$$0 \to \mathcal{E} \to \mathcal{E}_\infty = \Omega^0_\infty(\mathcal{E}).$$

As in the case of local systems we can characterize the subsheaf of holomorphic sections by a differential equation. We observe that we can define the operator

$$d'' : \mathcal{C}^\infty(\mathcal{E}) \to \Omega^{01}_{M,\infty}(\mathcal{E}_\infty) = \mathcal{E}_\infty \otimes \Omega^{01}_{M,\infty}.$$

To do this we write a local section on U in $\mathcal{C}^\infty(\mathcal{E})$ in the form

$$s = \sum f_i s_i$$

where the f_i are $\mathcal{C}^\infty$-functions and the s_i form a basis of holomorphic sections. Then we put

$$d''s = \sum \frac{\partial f_i}{\partial \bar{z}_v} s_i \otimes d\bar{z}_v.$$

This is well-defined because just as in the case of local systems where the corresponding s_i were constant we put $d''s_i = 0$ and this is consistent with the change of trivializations because holomorphic functions f are characterized by $\frac{\partial f}{\partial \bar{z}_v} = 0$. As in the case of local systems we get a complex of sheaves

$$0 \to \mathcal{E} \to \Omega^0_\infty(\mathcal{E}) \to \Omega^0_\infty(\mathcal{E}) \to \Omega^{01}_\infty(\mathcal{E}) \to \Omega^{0d}_\infty(\mathcal{E}) \to 0.$$

We need an analogon of the lemma of Poincaré, this is the lemma of Dolbeault:

9.2.5 *This complex of sheaves is exact*

Hier muss ich noch was zu sagen.

Combined with our previous observation, namely that the sheaves $\Omega^{pq}_\infty(\mathcal{E})$ are acyclic, gives us an acyclic resolution of the sheaf $\mathcal{E}$. From our general principles we get the Dolbeault Isomorphism:

$$H^i(M, \mathcal{E}) = H^i(\Omega^{0\bullet}_\infty(\mathcal{E})(M)).$$

9.3 Chern classes : We consider holomorphic line bundles $\mathcal{L}$ on our compact complex manifold M. The isomorphism classes of these line bundles form a group under the tensor product and this group is the first cohomology $H^1(M, \mathcal{O}^*_{M,\infty})$ (IV.3.3). We have a homomorphism from the sheaf of holomorphic functions $\mathcal{O}_M$ to $\mathcal{O}^*_M$ which is given by the exponential function

$$\begin{array}{ccc} \mathcal{O}_M(U) & \longrightarrow & \mathcal{O}^*_M(U) \\ & & \\ f & \mapsto & e^{2\pi i f}, \end{array}$$

and this is a surjective homomorphism of sheaves. The kernel is the sheaf of locally constant functions in $\mathbb{Z}$, thus we get an exact sequence of sheaves

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathcal{O}_M \longrightarrow \mathcal{O}^*_M \longrightarrow 1$$

which leads to the exact sequence in cohomology

$$\longrightarrow H^1(M, \mathcal{O}_M) \longrightarrow H^1(M, \mathcal{O}^*_M) \xrightarrow{\delta} H^2(M, \mathbb{Z}) \longrightarrow .$$

If we have a line bundle $\mathcal{L}$, and its isomorphism class corresponds to $[\mathcal{L}] \in H^1(M, \mathcal{O}^*_M)$, then the image under δ is called the (first)-Chern class of $\mathcal{L}$, i.e.

$$\delta([\mathcal{L}]) = c_1(\mathcal{L}) \in H^2(M, \mathbb{Z}).$$

We want to give a geometric interpretation of this class. We assume that our holomorphic bundle has a non zero section

$$s \in H^1(M, \mathcal{L})$$

which has an additional property, namely it defines a "smooth divisor". By this I mean the following: for any open set $U \subset M$ over which our bundle becomes trivial and where we selected a nowhere vanishing section $1_U \in H^0(U, \mathcal{L})$. Our section s can be written as

$$s = f_U \cdot 1_U$$

where f_U is a holomorphic function. Now we require that the differential df_U is non zero in all the points where f_U – and therefore s – is zero. The implicit function theorem implies that the set of zeroes of s is a submanifold $Y \subset M$ which is of complex codimension one. This is our smooth divisor.

Since we are in the complex case, we know that M and Y have natural orientations, and this also defines a relative orientation (see 8.5). In this situation we attached a fundamental class

$$[Y] \in H^2(M, \mathbb{Z})$$

to Y.

Proposition 9.3.1 *Under these conditions we have the equality*

$$[Y] = c_1(\mathcal{L}).$$

Proof. We choose a Riemannian metric in M and a tubular neighborhood TY of Y as in 8.5. We shrink this tubular neighborhood slightly by making the discs a little bit smaller. We cover the shrinked neighborhood

$$T_\varepsilon Y = \cup\, Y_\alpha \times D_\alpha = U_\alpha$$

where Y_α is open in Y and D_α is a disc. The indices run over an indexing set A. Now we cover $M \setminus T_\varepsilon Y$ by open sets U_γ such that $\overline{U}_\gamma \cap Y = \emptyset$ and γ runs through C. We assume that we can trivialize the bundle on each of these open sets and on the $U_\gamma, \gamma \in C$ we may trivialize by using the section s. Then we get our 1-cocycle $g_{\alpha\beta} \in \mathcal{O}^*_M(U_\alpha \cap U_\beta)$, and $g_{\beta\gamma} = 1$ if $\beta, \gamma \in C$.

We assume that the covering sets are convex (See 8.2) that we can form $h_{\alpha\beta} = \frac{1}{2\pi i} \log g_{\alpha\beta}$ on $U_\alpha \cap U_\beta$, and we choose $h_{\alpha\beta} = 0$ if $\alpha, \beta \in C$. Then we get the $\underline{\mathbb{Z}}$ valued two cocycle

$$c_{\alpha\beta\delta} = h_{\alpha\beta} - h_{\beta\delta} + h_{\alpha\delta} \quad \text{on} \quad U_\alpha \cap U_\beta \cap U_\delta$$

and this 2-cocycle represents our class $c_1(\mathcal{L})$. But we notice that $c_{\alpha\beta\delta} = 0$ if all three open sets lie in the complement of Y. This means that $c_{\alpha\beta\delta} \neq 0$ implies that at least one of the indices lies in A. Consequently

$$U_\alpha \cap U_\beta \cap U_\delta \subset T_\varepsilon Y.$$

Now we consider the sheaf $i_!(\mathbb{Z})$ on M where $i_! : TY \to M$ (this is the larger tubular neighborhood), and we see that our cocycle lies in

$$C^2(\mathfrak{A}, i_!\mathbb{Z}).$$

We have the inclusion $i_!(\mathbb{Z}) \to \underline{\mathbb{Z}}$ and we conclude that $c_1(\mathcal{L})$ is the image of the class

$$c_1^Y(\mathcal{L}) \in H^2(M, i_!\mathbb{Z})$$

which is the class represented by this cocycle. But we know that

$$H^2(M, i_!\mathbb{Z}) = H^0(Y, \mathcal{R}^2\pi_*(i_!\mathbb{Z}))$$

(see 8.5.). Since we have a relative orientation we have $\mathcal{R}^2\pi_*(i_!(\mathbb{Z})) = \underline{\mathbb{Z}}$ on Y and by definition

$$[Y] = 1_Y = \text{constant } \mathbb{Z}\text{-valued function } 1.$$

We want to show that $c_1^Y(\mathcal{L}) = [Y]$. This can be checked locally on Y.
If we want to do this, we consider a point $p \in Y$ and a neighborhood U_p of p in M such that we have an isomorphism

$$(U_p, \mathcal{O}_{U_p}) \simeq (B, \mathcal{O}_B)$$

where $B \subset \mathbb{C}^n$ is an open polydisc, say

$$B = \{(z_1, \cdots, z_m) \mid |z_i| < 1\}.$$

We take U_p so small that we can trivialize our line bundle $\mathcal{L} \mid U_p$ by a section $s_1 \in H^0(U_p, \mathcal{L})$ which is non zero everywhere. Our global section $s \in H^0(M, \mathcal{L})$ is of the form

$$s = fs_1,$$

if we restrict it to U_p. We assumed that $df \neq 0$ at p and therefore we may even assume that $f(z_1, \cdots, z_m) = z_1$ and then $Y \cap U_p = \{(0, z_2, \cdots, z_m) \mid z_i \in \mathcal{C}\}$. Of course it may be necessary to pass to a smaller neighvorhood and after doing this we can still think that

$$U_p \simeq \{(z_1, z_2, \cdots, z_m) \mid |z_i| < 1\}$$

and $s = z_1 s_1$. We cover the disc $D = \{z_1 \mid |z_1| < 1\}$ by open sets. The first one is $V_0 = \{z \mid |z| < r\}$ where $r < 1$ but close to one. The set

$$V_1 = \{z_1 \mid \mathrm{Re}(z_1) > \varepsilon, z_1 \in D\}$$

where $\varepsilon > 0$ on small. Then

$$V_2 = e^{\frac{2\pi i}{3}}\ V_1 \quad , \quad V_3 = e^{\frac{4\pi i}{3}}\ V_1.$$

This yields a covering of U_p if we multiply by the disc

$$D' = \{(z_2, \ldots, z_m) \mid |z_i| < 1\}.$$

We compute the 1-cocycle by the recipe given in our discussion above. We get

$$g_{ij} = 1 \quad \text{if} \quad 1 \leq i, j \leq 3$$

and

$$g_{0i} = z \quad \text{for} \quad i = 1, 2, 3.$$

We have to write these $g_{\alpha\beta}$ as $e^{2\pi i h_{\alpha\beta}}$ with some function $h_{\alpha\beta}$ on $V_{\alpha\beta}$. Of course we take $h_{ij} = 0$ for $1 \leq i, j \leq B$. To define the h_{0j} we take a path γ from 1 to a point $z \in V_0 \cap V_j$ which goes counterclockwise around zero, and

$$h_{0j} = \frac{1}{2\pi i} \int_\gamma \frac{d\zeta}{\zeta}.$$

We have to compute the differences

$$c_{0ij} = h_{0i} - h_{0j} + h_{ij}$$

and get

$$c_{012} = c_{123} = 0$$

but

$$c_{013} = 1.$$

Now it is clear that this 2-cocycle with values in $i_!\mathbb{Z}$ yields the positive generator in

$$H^2(U_p, i_!\mathbb{Z}) = H^0(Y \cap U_p, \mathcal{R}^2\pi_* i_!\mathbb{Z})$$

and this proves the proposition.

9.3.2 This formula gives us a very nice interpretation for Chern classes. Let us assume that $\dim M = d$ and let us assume that $\mathcal{L}_1, \ldots, \mathcal{L}_d$ are line bundles. We assume that each of these line bundle has a section $s_i \in H^0(M, \mathcal{L})$ and let us assume furthermore that the zero sets are smooth divisors and $Y_i = [s_i = 0]$ intersect transversally (See 8.6). This has the consequence that the intersections $Y_1 \cap Y_2 \ldots \cap Y_k = Z_k$ are smooth complex submanifolds. Let us consider a point p in the intersection of all the Y_i and local trivializations $t_i \in H^0(U_p, \mathcal{L}_i)$ of the line bundles at p. Then locally at p we have $s_i = f_i t_i$ where f_i is holomorphic at p and $f_i(p) = 0$. Then our transversality assumption implies that $f_1, f_2, \ldots, f_d$ is a sytem of local coordinates at p. The point is isolated in the intersection. We can invoke our formula 8.6.1. It tells us that the cup product of the Chern classes is a class in $H^{2d}(M, \mathbb{Z})$ and hence a number and

$$c_1(\mathcal{L}_1) \cup c_1(\mathcal{L}_2) \ldots \cup c_1(\mathcal{L}_d) = |Y_1 \cap Y_2 \cap \ldots \cap Y_d|.$$

Of course we may always form the above cup product of d Chern classes of line bundles and we call the result the *intersection number* of the line bundles. We may even take one line bundle $\mathcal{L}$ and call $c_1(\mathcal{L})^d$ the *d-fold or total selfintersection number* of the line bundle. We will discuss later (See V.3.1.2) on projective smooth varieties this cup product can always be interpreted as an intersection number of smooth divisors.

9.3.3 The line bundles $\mathcal{O}_{\mathbb{P}^n(\mathbb{C})}(k)$:

I want to outline the construction of a familiy of line bundles $\mathcal{O}_{\mathbb{P}^n(\mathbb{C})}(k)$ on $\mathbb{P}^n(\mathbb{C})$. I begin with construction of $\mathcal{O}_{\mathbb{P}^n(\mathbb{C})}(1)$. We consider the coordinate functions $z_i : \mathbb{C}^{n+1} \to \mathbb{C}$ as linear forms on $\mathbb{C}^{n+1}$. Starting from these linear forms we construct the bundle $\mathcal{O}_{\mathbb{P}^n(\mathbb{C})}(1)$. This bundle becomes trivial when we restrict it to one of the open subsets U_i and over this subset z_i is a trivializing section, i.e. it it is nowhere zero. For any pair i, j of indices we have the two trivializing sections z_i, z_j on $U_i \cap U_j$. They are related by the equation $z_i = (z_i/z_j)z_j$ and $z_i/z_j = g_{ij}$ is a holomorphic nowhere vanishing function on $U_i \cap U_j$. These quotients define the transition functions (See III. 3.2 and 9.2) defining the bundle $\mathcal{O}_{\mathbb{P}^n(\mathbb{C})}(1)$. It is clear that z_i defines in fact a global section in $H^0(\mathbb{P}^n(\mathbb{C}), \mathcal{O}_{\mathbb{P}^n(\mathbb{C})}(1))$ and this section which defines a smooth divsisor $[z_i = 0]$ and this is the hyperplane at infinity for those people who live in U_i. Hence we see that the Chern class of the bundle $\mathcal{O}_{\mathbb{P}^n(\mathbb{C})}(1)$ is the fundamental class of an arbitrary hyperplane in $\mathbb{P}^n(\mathbb{C})$.

In view of our considerations in IV.8.8 this means that the Chern class $c_1(\mathcal{O}_{\mathbb{P}^n(\mathbb{C})}(1))$ is a generator in $H^2(\mathbb{P}^n(\mathbb{C}), \mathbb{Z})$.

Now the other bundles are simply the tensor products

$$\mathcal{O}_{\mathbb{P}^n(\mathbb{C})}(1)(n) = \mathcal{O}_{\mathbb{P}^n(\mathbb{C})}(1)^{\otimes n}$$

and their Chern classes are given by n-times the generator.

IV.10. Hodge Theory

10.1. Real manifolds.

In this section I describe some very powerful analytical tools which provide insight into the structure of cohomology groups. They are based on the construction of certain linear elliptic differential operators (Laplace operators) which arise if we try to write down an inverse for the operators d, d', d'' in the de-Rham or Dolbeault complexes. We need some results on elliptic linear differential operators which we not prove here.

We go back to the situation where we have an oriented manifold M, and a local system of finite dimensional $\mathbb{R}$ or $\mathbb{C}$-vector spaces $\mathcal{V}$ on M. Let d be the dimension of M.

We have the de-Rham complex

$$0 \to \mathcal{V} \to \mathcal{V}_\infty \to \mathcal{V}_\infty \otimes \Omega^1_M \to \ldots \to \mathcal{V}_\infty \otimes \Omega^d_M \to 0.$$

If we take global sections and if we drop the first term the resulting complex computes the cohomology groups $H^\nu(M, \mathcal{V})$.

We have seen that we can construct a Riemannian $<,>$ metric on M and using the same method we construct an euclidean (or hermitian) metric $< , >_h$ on $\mathcal{V}$. The metric on the tangent bundle provides a metric on the bundle of differential forms Ω^p_M. This allows us to define a scalar product on the sections $f \in \mathcal{V}_\infty \otimes \Omega^p_M(M)$: It is clear that the metric on $\mathcal{V}_\infty$ and the Riemannian metric together give us a metric on the tensor product of fibres $\mathcal{V}_x \otimes \Omega^p_{Mx}$ at any point x. Hence we get a function $x \to< v_{1x} \otimes \omega_{1x}, v_{1x} \otimes \omega_{2x} >$ on M for any two $v_1 \otimes \omega_1, v_2 \otimes \omega_2 \in \mathcal{V}_\infty \otimes \Omega^p_M(M)$. Since our manifold M is oriented and Riemannian we have a unique section $\omega_{\rm top} \in \Omega^d_M(M)$ which has length one and is positive with respect to the orientation. Hence we can integrate

$$< v \otimes \omega_1, v_2 \otimes \omega_2 >= \int_{M,\infty} < v_{1x}, v_{2x} >< \omega_{1x}, \omega_{2x} > \omega_{\rm top}.$$

Here we have to assume that the integral converges. This is centainly so if M is compact. Otherwise we have to introduce the notion of introduce integrable sections.

There is another way to describe this scalar product. We have the $*$-operator on the bundle of forms

$$* : \Omega^p_M \to \Omega^{n-p}_M$$

which is defined pointwise by the requirement

$$\omega_1 \wedge *\omega_2 =< \omega_1, \omega_2 > \cdot \omega_{\rm top}.$$

Hence we get as our scalar product for two sections $v_1 \otimes \omega_1, v_2 \otimes \omega_2 \in \mathcal{V}_\infty \otimes \Omega^p_M(M)$ the formula

$$< v_1 \otimes \omega_1, v_2 \otimes \omega_2 >= \int_{M,\infty} < v_1, v_2 >_h \omega_1 \wedge *\omega_2.$$

Now it becomes clear that we can define an adjoint operator

$$\delta : \mathcal{V}_\infty \otimes \Omega^p_M(M) \to \mathcal{V}_\infty \otimes \Omega^{p-1}_M(M)$$

which we simply put

$$\delta = (-1)^{d(p-1)+1} * d * .$$

We have to verify that for $v_1 \otimes \omega_1 \in \mathcal{V}_\infty \otimes \Omega^{p-1}_M(M), v_2 \otimes \omega_2 \in \mathcal{V}_\infty \otimes \Omega^p_M(M)$ we have

$$< dv_1 \otimes \omega_1, v_2 \otimes \omega_2 >=< v_1 \otimes \omega_1, \delta v_2 \otimes \omega_2 > .$$

To see this we perform a simple calculation

$$< dv_1 \otimes \omega_1, v_2 \otimes \omega_2 >= \int_{M,\infty} dv_1 \otimes \omega_1 \wedge *v_2 \otimes \omega_2 =$$

$$\int_{M,\infty} d \cdot (v_1 \otimes \omega_1 \wedge *v_2 \otimes \omega_2) - (-1)^{p-1} v_1 \otimes \omega_1 \wedge d * v_2 \otimes \omega_2.$$

From this moment on we assume that M is compact, then $\int_{M,\infty} d(\omega_1 \wedge d * v_2 \otimes \omega_2) = 0$ and hence

$$< d\omega_1, v_2 \otimes \omega_2 >= (-1)^{p+(d-p+1)(p-1)} \int_{M,\infty} d\omega_1 \wedge *v_2 \otimes \omega_2 =$$

$$(-1)^{p+1+(d+p)p} \int_{M,\infty} < \omega_1 \wedge * * d * v_2 \otimes \omega_2 =< \omega_1, *d * v_2 \otimes \omega_2 > .$$

We define the Laplace operator

$$\Delta = d\delta + \delta d$$

which sends p-forms to p-forms. It is clear that this is a linear operator of second order and it is elliptic.

[Erkläre was ein ellipt. Operator für beliebige Vektorbündel ist

$$D : \mathcal{E}_\infty(M) \to \mathcal{E}_\infty(M).]$$

The theory of elliptic operators tells us that:

10.1.1 *We have a decomposition into eigenspace*

$$"\mathcal{V}_\infty \otimes \Omega^p_M(M) = \sum_\lambda \mathcal{V}_\infty \otimes \Omega^p_M(M)(\lambda)"$$

where

$$\mathcal{V}_\infty \otimes \Omega^p_M(M)(\lambda) = \{\omega \in \mathcal{V}_\infty \otimes \Omega^p_M(M) \mid \Delta\omega = \lambda\omega\}.$$

The eigenspaces have a finite dimension and the eigenvalues tend to infinity, i.e. on any finite interval $[0, T]$ we have only finitely many eigenvalues λ. The sign $\sum_\lambda$ means that any ω can be written as

$$\omega = \sum_\lambda \omega_\lambda$$

where the convergence is uniform on M and stays uniform if we apply a finite number of derivatives.

Once we believe this we can compute the cohomology very easily: The operators d and δ respect the decompositions, they send eigenspaces into eigenspaces to the same eigenvalue.

Now we can compute the cohomology. Let $\omega \in \mathcal{V}_\infty \otimes \Omega^p_M(M)(M)$ be a closed form. Write

$$\omega = \omega_0 + \omega'$$

where $\Delta\omega_0 = 0$ is the harmonic component, the one in the eigenspace to $\lambda = 0$. Then

$$d\omega = d\omega_0 + d\omega' = 0$$

and hence $d\omega_0 = 0$ and $d\omega' = 0$. But

$$\omega' = \sum_{\lambda \neq 0} \omega'_\lambda$$

where $d\omega'_\lambda = 0$. Hence

$$\omega'_\lambda = \frac{1}{\lambda}\Delta\omega'_\lambda = \frac{1}{\lambda}(d\delta + \delta d)\omega'_\lambda = \frac{1}{\lambda}\delta\omega'_\lambda$$

and therefore

$$\omega' = d(\sum_{\lambda \neq 0} \frac{1}{\lambda}\delta\omega'_\lambda).$$

This means that ω_0 represents the same cohomology class as ω and we have the

10.1.2 Theorem: *The harmonic forms satisfy $d\omega = \delta\omega = 0$. Sending a harmonic form to its cohomology class provides an isomorphism*

$$\mathbb{H}^p(\mathcal{V}_\infty \otimes \Omega^p_M)(M) = \{\omega \in \mathcal{V}_\infty \otimes \Omega^p_M(M) | \Delta\omega = 0\} \xrightarrow{\sim} H^p(M, \mathcal{V}).$$

We almost proved this theorem. We observe that Δ is a positive operator. We have $< \Delta\omega, \omega >=< d\omega, d\omega > + < \delta\omega, \delta\omega >\geq 0$. If $\Delta\omega = 0$ then we conclude

$$0 =< \delta\omega, \delta\omega >=< d\omega, d\omega >$$

this implies the first assertion. Since harmonic forms are closed they define cohomology classes. If ω is harmonic and $\omega = d\Psi$ then $< \omega, \omega >=< \omega, d\Psi >=< \delta\omega, \Psi >= 0$ and hence $\omega = 0$. The map from harmonic forms to cohomology is injective. The surjectivity has been shown above.

10.1.3 : We can give some indications how these results can be proved. Since we introduced the scalar product on $\mathcal{V}_\infty \otimes \Omega^p_M(M)$ we may take the completion with respect to this scalar product, and we get the Hilbert space

$$L^2(\mathcal{V}) \otimes \Omega^p_{M,(2)}(M)$$

of quadratically integrable differential forms with values on $\mathcal{V}$.

If we have a closed form $\omega \in \mathcal{V}_\infty \otimes \Omega^p_M(M)$, then we can modify it by a form $d\psi$ and we can try to minimize the square of the norm

$$\|\omega + d\psi\|^2 = \int_M \langle \omega + d\psi, \omega + d\psi \rangle.$$

We look at the lim inf of all the real numbers $\|\omega + d\psi\|_2^2$ where ψ varies. We can find a sequence

$$\omega + d\psi_n = \omega_n$$

such that $\|\omega_n\|_2^2$ converges to this infimum. Since the unit ball in our Hilbert space is weakly compact, we can find a weakly convergent subsequence, i.e. we may assume that ω_n converges weakly to a form

$$\omega_0 \in L^2(\mathcal{V}) \otimes \Omega^p_{M,(2)}(M).$$

We would like to prove that ω_0 is a $\mathcal{C}^\infty$-form, that it is harmonic and that this form represents the given class, i.e. $\omega_0 = \omega + d\psi_0$.

Assume that we know that ω_0 is a harmonic form. This means that it is $\mathcal{C}^\infty$ and satisfies $d\omega_0 = \delta\omega_0 = 0$. Then we have

$$\begin{array}{rcccl} \langle d\omega_0, \eta \rangle & = & \langle \omega_0, \delta\eta \rangle & = & 0 \\ \langle \delta\omega_0, \psi \rangle & = & \langle \omega_0, d\psi \rangle & = & 0 \end{array}$$

for all $\psi \in \mathcal{V}_\infty \otimes \Omega_M^{p-1}(M), \eta \in \mathcal{C}_\infty(\mathcal{V}) \otimes \Omega_M^{p+1}(M)$. Of course the equalities

$$\langle \omega_0, \delta\eta \rangle = 0 \qquad \langle \omega_0, d\psi \rangle = 0$$

make sense for any square integrable form. I claim that they are true. The first one follows because

$$\langle \omega_0, \delta\eta \rangle = \lim_{n \to \infty} \langle \omega + d\psi_n, \delta\eta \rangle = \langle d\omega + dd\psi, \eta \rangle = 0$$

(this is the definition of weak convergence) and the second one follows from the minimality of the norm $\|\omega_0\|_2^2$.

The really deep input from analysis is that the validity of the two equations

$$\langle \omega_0, \delta\eta \rangle = \langle \omega_0, d\psi \rangle$$

for all η, ψ implies that ω_0 must be indeed $\mathcal{C}^\infty$ and then it follows that ω_0 must be harmonic. The rest is easy again. We need to know that ω_0 still represents the given cohomology class. This follows from Poincaré-duality. We consider the dual local system $\mathcal{V}^\vee$. We have the non degenerate pairing

$$H^p(M, \mathcal{V}) \times H^{n-p}(M, \mathcal{V}^\vee) \longrightarrow \mathbb{R}$$

which in terms of differential forms is given by integration over M. Hence we see that for any cohomology class $[\omega'] \in H^{n-p}(M, \mathcal{V}^\vee)$ which is represented by a $\mathcal{C}^\infty - (n-p)$-form ω' that

$$[\omega] \cup [\omega'] = \int_{M,\infty} \mathrm{tr}(\omega \wedge \omega') = \int \mathrm{tr}((\omega + d\psi_n) \wedge \omega'$$

and weak convergence gives that this integral is equal to

$$\int \mathrm{tr}(\omega_0 \wedge \omega') = [\omega_0] \cup [\omega'].$$

This theorem has consequences.

(i) We conclude that for a compact oriented $\mathcal{C}^\infty$-manifold the cohomology $H^p(M, \mathcal{V}_\infty)$ has finite dimension for any local system of finite dimensional $\mathbb{R}$ or $\mathbb{C}$ vector spaces.

(ii) If we have such a local system $\mathcal{V}_\infty$ and if $\mathcal{V}^\vee$ is the dual system then ????.

Both these consequences are known to us, they even hold for more general local systems. But in the next section where we discuss the analogous situation of holomorphic bundles on complex manifolds and in this case these consequences are only obtained by the analytic methods.

10.2. Complex manifolds:

Now we consider a compact complex manifold M. We introduce a hermitian metric h on the tangent bundle $T_{M,\infty}$.

10.2.1 Some linear algebra.

I have to recall some simple facts from linear algebra which concern these metrics. Therefore I start from a complex vector space V of finite dimension d. In the following I view V as a real vector space of dimension $2d$ which is endowed with a linear transformation $I : V \to V$ which satisfies $I^2 = -Id$. The structure as a $\mathbb{C}$-vector space is regained if we define scalar multiplication of $v \in V$ by i by $v \to I(v)$.

If we have a hermitian form h on V then we can write

$$h(v_1, v_2) = \operatorname{Re} h(v_1, v_2) + i \cdot \operatorname{Im} h(v_1, v_2)$$

and it is clear that

$$\begin{array}{ll} \operatorname{Re} h : V \times V \to \mathbb{R} & \text{is symmetric} \\ \operatorname{Im} h : V \times V \to \mathbb{R} & \text{is alternating} \end{array}.$$

Since $h(Iv_1, Iv_2) = h(iv_1, iv_2) = h(v_1, v_2)$ we see that both components satisfy

$$\begin{array}{l} \operatorname{Re} h : (Iv_1, Iv_2) = \operatorname{Re} h(v_1, v_2) \\ \operatorname{Im} h : (Iv_1, Iv_2) = \operatorname{Im} h(v_1, v_2) \end{array},$$

in other words I is an isometry for the real part and the imaginary part.

But we may also recover h from either part. We simply write

$$h(v_1, Iv_2) = \operatorname{Re} h(v_1, Iv_2) + i \operatorname{Im} h(v_1, Iv_2)$$

and since $h(v_1, Iv_2) = -ih(v_1, Iv_2)$ this yields

$$h(v_1, v_2) = -\operatorname{Im} h(v_1, Iv_2) + i \operatorname{Re} h(v_1, Iv_2)$$

and from this we get

$$\begin{array}{l} \operatorname{Re} h(v_1, v_2) = -\operatorname{Im} h(v_1, Iv_2) \\ \operatorname{Im} h(v_1, v_2) = \operatorname{Re} h(v_1, Iv_2) \end{array}.$$

Hence we see that a sesquilinear form h on V (this is a hermitian form without the requirement that it should be positive definite) is the same thing as a symmetric form or an alternating form

$$\operatorname{Re} h = V \times V \to \mathbb{R}, \operatorname{Im} h = V \times V \to \mathbb{R}$$

for which I is an isometry. The form h is hermitian (positive definite) if and only if $\operatorname{Re} h$ is euclidean.

We complexify V and extend $\operatorname{Re} h$ to a bilinear form

$$\operatorname{Re} h_{\mathbb{C}} : V_{\mathbb{C}} \times V_{\mathbb{C}} \to \mathbb{C}.$$

We have the decomposition

$$V_{\mathbb{C}} = V_{\mathbb{C}}^{10} \oplus V_{\mathbb{C}}^{01}$$

and it is clear that $V_{\mathbb{C}}^{10}, V_{\mathbb{C}}^{01}$ are isotropic with respect to $\operatorname{Re} h_{\mathbb{C}}$, i.e.

$$\operatorname{Re} h_{\mathbb{C}}(V^{10}, V^{10}) = \operatorname{Re} h_{\mathbb{C}}(V^{01}, V^{01}) = 0.$$

This follows from the definition of the V^{10}, V^{01} as eigenspaces for I with eigenvalue $i, -i$. But the pairing

$$\operatorname{Re} h_{\mathbb{C}} : V^{10} \times V^{01} \to \mathbb{C}$$

will be not trivial in general. If for instance the form h is positive definite then this pairing is a perfect duality.

We have an isomorphism of complex vector spaces

$$j : V \to V^{10}$$

which is obtained by the embedding of V into $V_{\mathbb{C}}$ followed by the projection. Under this isomorphism we send

$$j : v \to \frac{1}{2}(v - Iv \otimes i)$$

and we can recover the hermitian form h from $\operatorname{Re} h_{\mathbb{C}}$ by the formula

$$h(v_1, v_2) = \frac{1}{2} \operatorname{Re} h_{\mathbb{C}}(j(v_1), \overline{j(v_2)})$$

where $\overline{}$ is of course the antilinear isomorphism from V^{10} to V^{01} introduced by complex conjugation on the factor $\otimes \mathbb{C}$.

We introduce a so-called Hodge structure on the pair vector space V together with its complex structure I. This is a homomorphism

$$h_{\mathcal{D}} : \mathbb{C}^* \to GL(V)$$

and it is defined as

$$h_{\mathcal{D}}(z) = h_{\mathcal{D}}(a + bi) = a \cdot Id + b \cdot I.$$

It is clear that this map is a homomorphism.

With respect to the euclidean metric on V it has the property that

$$< h_{\mathcal{D}}(z)v_1, h_{\mathcal{D}}(z)v_2 >= z\bar{z} \cdot < v_1, v_2 >,$$

it is not an isometry but a similitude.

If we complexify the space to $V_{\mathbb{C}}$ then it is obvious that

$$\begin{aligned} V^{10} &= \{v | h_{\mathcal{D}}(z)v = zv\} \\ V^{01} &= \{v, h_{\mathcal{D}}(z)v = \bar{z}v\} \end{aligned}.$$

The action of $\mathbb{C}^*$ commutes with complex conjugation we have $\overline{h_{\mathcal{D}}(z)v} = h_{\mathcal{D}}(z)\bar{v}$ on $V_{\mathbb{C}}$. We can extend this action of $\mathbb{C}*$ to the exterior powers $\Lambda^n V$ and $\Lambda^n V_{\mathbb{C}}$ simply by

$$h_{\mathcal{D}}(z)(v_1 \wedge \ldots \wedge v_n) = h_{\mathcal{D}}(z)v_1 \wedge \ldots \wedge h_{\mathcal{D}}(z)v_n$$

and it is clear that we can characterize the subspace $\Lambda^p V^{10} \otimes \Lambda^q V^{01} \subset \Lambda^{p+q} V_{\mathbb{C}}$ as

$$\Lambda^p V^{10} \otimes \Lambda^q V^{01} \{\omega \in \Lambda^{p+q} V_{\mathbb{C}} | h_{\mathcal{D}}(z)\varphi = z^p \bar{z}^q \varphi\}.$$

Of course if we extend $\operatorname{Re} h_{\mathbb{C}}$ to a bilinear form on $\Lambda^n V_{\mathbb{C}}$ by

$$\operatorname{Re} h_{\mathbb{C}}(\varphi, \psi) = \operatorname{Re} h_{\mathbb{C}}(v_1 \wedge \ldots \wedge v_n, w_1, \wedge \ldots \wedge w_n) = \det(\operatorname{Re} h_{\mathbb{C}}(v_1, w_j))$$

then we have

$$\operatorname{Re} h_{\mathbb{C}}(h_{\mathcal{D}}(z)\varphi, h_{\mathcal{D}}(z)\psi) = (z\bar{z})^n \operatorname{Re} h_{\mathbb{C}}(\varphi, \psi).$$

This implies that the $*$-operator sends $* : \Lambda^p V^{10} \otimes \Lambda^q V^{01} \to \Lambda^{d-p} V^{10} \otimes \Lambda^{d-q} V^{01}$. This must be so because the product $v_{pq} \wedge * w_{pq}$ is in top degree and $h_{\mathcal{D}}(z) v_{pq} \wedge * w_{pq} = (z\bar{z})^d h_{\mathcal{D}}(z) v_{pq} \wedge * w_{pq}$

If can extend our hermitian form h to a hermitian form on $\Lambda^p V^{10} \otimes \Lambda^q V^{01}$ by

$$h(\varphi, \psi) = \operatorname{Re} h_{\mathbb{C}}(\varphi, \bar{\psi}).$$

10.2.2. Now we come back to our compact complex manifold M of dimension d, we assume that we have introduced a hermitian metric $<,>_h$ on $T_{M,\infty}$. This introduces a hermitian metric on $\Omega^1_{M,\infty}$. We have the decomposition

$$\Omega^1_{M,\mathbb{C}} = \Omega^{10}_{M,\infty} \bigoplus \Omega^{01}_{M,\infty}$$

and

$$\Lambda^n \Omega_{M,\mathbb{C}} = \bigoplus \Lambda^p \Omega^{10}_{M,\infty} \otimes \Lambda^q \Omega^{01}_{M,\infty} = \oplus_{p+q=n} \Omega^{pq}_{M,\infty}.$$

We have the euclidean metric $\operatorname{Re} h$ on $\Omega^1_{M,\infty}$ and it introduces a star operator

$$* : \Lambda^n \Omega^1_{M,\infty} \to \Lambda^{2d-n} \Omega^1_{M,\infty}.$$

We have seen in the above section on linear algebra that we should extend this antilinearily to

$$* : \Lambda^n \Omega^1_{M,\mathbb{C}} \to \Lambda^{2d-n} \Omega^1_{M,\mathbb{C}}$$

and that this operator sends

$$* : \Omega^{pq}_{M,\infty} \to \Omega^{d-p,d-q}_{M,\infty}.$$

We define the scalar product on the sections $\Omega^{pq}_{M,\infty}(M)$ by

$$< \omega_1, \omega_2 >= \int_{M,\infty} \omega_1 \wedge * \omega_2.$$

Now we are able to define the adjoint operators to d' and d'', we put

$$\begin{aligned} \delta' &= - * d' * \\ \delta'' &= - * d'' * \end{aligned}.$$

The sign factor simplifies because our manifold has an even dimension when we consider it as a real manifold. Of course we have to verify the adjointness formular

$$\begin{aligned} < d'\omega_1, \omega_2 > &=< \omega_1, \delta'\omega_2 > \\ < d''\omega_1, \omega_2 > &=< \omega_1, \delta''\omega_2 > \end{aligned}.$$

To see this we observe that it is enough to check the case where $\omega \in \Omega^{p-1,q}(M), \omega_2 \in \Omega^{r,s}(M)$. Let us consider the first case. We see that both sides are zero unless $p = d-r, q = d-s$. So we assume that this is the case. Now we perform the same calculation as in the real case where we at certain places we have to replace d' by d and then again d by d'. We observe that

$$< d'\omega_1, \omega_2 >=< d\omega_1, \omega_2 >=< \omega_1, \delta\omega_2 >=< \omega_1, \delta'\omega_2 > .$$

This allows us to define the Laplace operator

$$\Delta' = d'\delta' + \delta'd' \quad \text{and} \quad \Delta'' = d''\delta'' + \delta''d''.$$

We want to compare these operators to the real Laplacian. Here we find

$$\begin{aligned} \Delta &= (d' + d'')(\delta' + \delta'') + (\delta' + \delta'')(d' + d'') = \\ &= \Delta' + \Delta'' + \delta''d' + d'\delta'' + \delta'd'' + d''\delta' \end{aligned}.$$

This is not so good unless we know that the mixed contributions disappear. This is indeed the case if our metric h satisfies a certain condition, which I want to explain.

Our metric h on the tangent bundle has its imaginary part

$$\operatorname{Im} h : T_{M,\infty} \times T_{M,\infty} \to \mathbb{R}.$$

Hence the imaginary part defines a 2-form ω_h on the manifold. If we complexify the tangent bundle and if we observe that I is an isometry for ω_h then we see that ω_h is a form of type (1,1) because it must be zero on $T^{10} \otimes T^{10}$ and $T^{01} \otimes T^{01}$. This is the so-called Kähler form of the metric. Kähler discovered the following

10.2.3 Theorem: *If the form ω_h is closed, i.e. $d\omega_h = 0$ then the two mixed terms are zero and we have*

$$\Delta = \Delta' + \Delta''.$$

I will not prove this theorem here. But in our later discussion of the special case of Riemann surfaces - in this case we have automatically $d\omega_h = 0$ - I will carry out the necessary calculations.

This theorem has the following important consequence:

The Laplace operator Δ respects the decomposition

$$\Omega^n_{M,\mathbb{C}}(M) = \oplus \Omega^{p,q}_{M,\infty}(M)$$

and the harmonic forms

$$\omega \in \Omega^n_{M,\mathbb{C}}(M)$$

are sums of harmonic forms

$$\omega = \sum_{p+q=n} \omega_{p,q}$$

and

$$\Delta\omega = 0 \iff \Delta'\omega = \Delta''\omega = 0.$$

A form ω is harmonic if and only if it satisfies all the equations

$$d\prime\omega = d''\omega = \delta\prime\omega = \delta''\omega$$

This follows by the same positivity argument which we used in the real case.
This provides us the famous *Hodge decomposition* of the cohomology of a compact complex manifold which admits a Kähler metric:

$$H^n(M, \mathbb{C}) = \oplus_{p+q=n} H^{p,q}(M, \mathbb{C}).$$

Now we apply the same methods to compute the cohomology of a holomorphic bundle $\mathcal{E}$ on M. We choose a hermitian metric on the tangent bundle and on the bundle $\mathcal{E}$ itself. We consider the Dolbeault complex

$$0 \to \mathcal{C}^\infty(\mathcal{E})(M) \xrightarrow{d''} \mathcal{C}^\infty(\mathcal{E}) \otimes \Omega^{01}(M) \xrightarrow{d''} \mathcal{C}^\infty(\mathcal{E}) \otimes \Omega^{02}(M) \to .$$

Again we introduce a scalar product on the sections by

$$< s_1 \otimes \omega_1, s_2 \otimes \omega_2 >= \int_{M,\infty} < s_1, s_2 > \omega_1 \wedge *\omega_2$$

where $< s_1, s_2 >$ is the hermitian form on the bundle $\mathcal{E}$. We can construct the adjoint operator

$$\delta'' : \mathcal{C}^\infty(\mathcal{E}) \otimes \Omega^{0q}_{M,\infty}(M) \to \mathcal{C}^\infty(\mathcal{E}) \otimes \Omega^{0,q-1}_{M,\infty}(M)$$

it is given by

$$\delta'' = (-1)^? * d'' *$$

and we have the Laplacian

$$\Delta'' = \delta'' d'' + d'' \delta''.$$

Again we conclude

The cohomology groups

$$H^p(M,\mathcal{E}) = \mathbb{H}^p(\mathcal{C}^\infty(\mathcal{E})\otimes\Omega^{0p}_{M,\infty}(M)) = \{\omega\in\mathcal{C}^\infty(\mathcal{E})\otimes\Omega^{0p}_{M,\infty}(M) | \Delta''\omega = 0\}.$$

Especially we can conclude that these groups are finite dimensional.

The finite dimensionality is fundamental and there is no easy way to get it. (In contrast to the case of $\mathcal{C}^\infty$-manifolds and cohomology with coefficients in local systems).

Let us consider the dual bundle $\mathcal{E}^\vee = \mathrm{Hom}(\mathcal{E},\mathcal{O}_{M,\infty})$. We introduce the dual metric on $\mathcal{E}^\vee$ (the dual basis to an orthonormal basis of a fibre $\mathcal{E}_x$ should be an orthonormal basis of $\mathcal{E}_x^v$).

10.3. Hodge theory on tori: We have a special case where the two main theorems of Hodge theory are easy to prove. We consider the lattice $\Gamma\subset\mathbb{R}^n$ (see 6.5.7), and we consider the $\mathcal{C}^\infty$-manifold

$$M = \mathbb{R}^n/\Gamma.$$

For any point $p\in M$ we have a canonical identification $T_pM = \mathbb{R}^n$. If we take the standard euclidian metric $<\,,\,>$ on $\mathbb{R}^n$ then we get a Riemannian metric on M. If $x_1\cdots x_n$ are the coordinates on $\mathbb{R}^n$, then the differential forms can be written as

$$\omega = \sum f_{i_1\cdots i_p}\, dx_{i_1}\wedge\cdots\wedge dx_{i_p}.$$

I want to consider complex valued differential forms, the $f_{i_1\cdots i_p}$ are complex valued $\mathcal{C}^\infty$-functions.

A basically simple and straightforward computation yields a formula for the Laplace operator:

$$\Delta\omega = \sum\left(-\sum\frac{\partial^2 f_{i_1\cdots i_p}}{\partial x_i^2}\right) dx_{i_1}\wedge\cdots\wedge dx_{i_p}.$$

Now we consider the dual lattice

$$\Gamma^\vee = \{\varphi\in\mathbb{R}^n \mid \langle\varphi,\Gamma\rangle\subset\mathbb{Z}\}$$

and for $\varphi\in\Gamma^\vee$ the function

$$e_\varphi(x) = e^{2\pi i\langle\varphi,x\rangle}$$

on M is an eigenfunction for the Laplacian

$$\Delta e_\varphi(x) = 4\pi^2\langle\varphi,\varphi\rangle e_\varphi(x).$$

Now any $\mathcal{C}^\infty$-function on M has a Fourier expansion

$$f = \sum_{\varphi\in\Gamma^\vee} a_\varphi e_\varphi(x)$$

where the absolute values $|a_\varphi|$ tend to zero very rapidly. Consequently any differential form can be written as convergent infinite sum

$$\omega = \sum_\varphi \omega_\varphi$$

where $\Delta\omega_\varphi = +4\pi^2\langle\varphi,\varphi\rangle\omega_\varphi$. This is the "decomposition" in 10.1.1.. It has the required property: There are only finitely many φ which satisfy $4\pi^2\langle\varphi,\varphi\rangle \le T$. We apply our arguments in the "proof" of theorem 10.1.2. . The harmonic forms are the constant forms

$$\omega_0 = \sum a_{i_1\cdots i_p} dx_{i_1} \wedge \cdots \wedge dx_{i_p}$$

where $a_{i_1\cdots i_p} \in \mathbb{C}$. We conclude that the cohomology ring $H^\bullet(M,\mathbb{C})$ is the exterior algebra of the complexified dual tangent bundle $\mathrm{Hom}(T_pM,\mathbb{C})$. This agrees with 6.5.7 but the result overthere is slightly sharper because it gives the structure over $\mathbb{Z}$.

If we consider a complex torus

$$M = \mathbb{C}^n/\Gamma$$

where Γ is a lattice of rank $2n$, then M is a complex manifold and the tangent space is the complex vector space $\mathbb{C}^n$ in any point of M. On this tangent space we introduce the standard hermitian metric

$$\sum_{v=1}^n z_v\overline{z}_v = h(z,z).$$

Again we perform a simple computation and find

$$\Delta' = \Delta'' : \omega = \sum_{\underline{\alpha},\underline{\beta}} f_{\underline{\alpha}\underline{\beta}} dz_{\underline{\alpha}} \wedge d\overline{z}_{\underline{\beta}} \to \sum_{\underline{\alpha},\underline{\beta}} \left(- \sum_\nu \frac{\partial^2 f_{\underline{\alpha}\underline{\beta}}}{\partial z_\nu \partial \overline{z}_v} \right) dz_{\underline{\alpha}} \wedge d\overline{z}_{\underline{\beta}}.$$

Again we can introduce the dual lattice

$$\Gamma^\vee = \{\varphi \in \mathbb{C}^n \mid \mathrm{Re}\, h(\varphi,\gamma) \in \mathbb{Z} \text{ for all } \gamma \in \Gamma\},$$

and we can expand $\mathcal{C}^\infty$-functions

$$f(z) = \sum a_\varphi e^{2\pi i\, \mathrm{Re}\, h\langle\varphi,z\rangle}.$$

Now we argue as before. We have the Dolbeault complex

$$0 \longrightarrow \mathcal{O}_{M,\infty} \longrightarrow \mathcal{C}^\infty(M) \xrightarrow{d''} \Omega^{01}_\infty(M) \longrightarrow \cdots,$$

and we have the adjoint δ''. The operator $\Delta'' = \delta'' d'' + d''\delta''$ has the form above, we can "decompose" into eigenspaces. If we take global sections, we find that

$$H^\bullet(M,\mathcal{O}_{M,\infty}) = \mathbb{H}^\bullet(M,\Omega^{0,\bullet}_\infty(M)) = \{\omega \in H^0(M,\Omega^{0,\bullet}_\infty(M)) \mid \Delta''\omega = 0\}$$

and again the harmonic forms are the constants.

We conclude that

$$H^\bullet(M,\mathcal{O}_{M,\infty}) = \mathrm{Hom}^{\mathrm{alt}}_{\mathbb{C}}(\overline{T_p(M)},\mathbb{C}),$$

and this will be used in the next chapter.

Chapter V

Compact Riemann Surfaces and Abelian Varieties

V.1 Compact Riemann Surfaces

1.0 Introduction: A *compact Riemann surface* is a compact complex manifold of dimension 1. Let S be such a surface. It has a canonical orientation (See IV.9.2) In IV 4.4.13 and IV.8.7 we have seen that the cohomology groups of such a surface are given by

$$\begin{aligned} H^0(S,\mathbb{Z}) &= \mathbb{Z} \\ H^1(S,\mathbb{Z}) &= \mathbb{Z}^{2g} \\ H^2(S,\mathbb{Z}) &= \mathbb{Z} \end{aligned}$$

and the Poincaré-Duality gives us in addition an alternating perfect pairing

$$<,>_{\text{cup}}: H^1(S,\mathbb{Z}) \times H^1(S,\mathbb{Z}) \to \mathbb{Z}.$$

The number g is called *genus* of the surface. The genus g is also a measure fo the complexity of the Riemann surface. We will show that a Riemann surface S of genus $g = 0$ is isomorphic to the so called Riemann sphere $\mathbb{P}^1(\mathbb{C})$ (see III.2.1 1a) and 1.3.6 in this chapter). In the exercise in IV. 6.5.1 we showed that $H^1(\mathbb{P}^1(\mathbb{C}),\mathbb{Z}) = 0$ and therefore $\mathbb{P}^1(\mathbb{C})$ has genus zero.

Of course it is clear that a holomorphic function on a compact Riemann surface S must be constant. We will work very hard to show that on any compact Riemann surface we can find a non constant meromorphic function (see 1.3).

Once we have a non constant meromorphic function f we can cover S by the two open sets U_0, U_1 where f or f^{-1} are holomorphic. We get a holomorphic map

$$\begin{array}{cccccc} f & : & U_0 & \longrightarrow & \mathbb{C} \\ & & u & \longmapsto & f(w) \end{array}$$

and

$$\begin{array}{cccccc} f^{-1} & : & U_1 & \longrightarrow & \mathbb{C} \\ & & u & \longrightarrow & f^{-1}(u) \end{array}$$

and it is clear that these two maps provide a surjective map which is also denoted by

$$f : S \longrightarrow \mathbb{P}^1(\mathbb{C})$$

(see III, 1.2.1a). It will turn out that this map has finite fibres and the number of points in the fibres (counted with the right multiplicities) is equal to the degree of the polar divisor (See IV.1.1.) This will become a decisive tool for the understanding of Riemann surfaces (see 1.4.).

1.1 The Hodge structure on $H^1(S,\mathbb{C})$

1.1.1 : We study the cohomology with coefficients in $\mathbb{C}$. I want to change the notation slightly. On our Riemann surface Ω^1_S will be the sheaf of holomorphic one forms. The sheaves of $\mathcal{C}_\infty$ differential forms will be denoted by $\Omega^\bullet_\infty$. We consider the de-Rham complex

$$0 \to \mathbb{C} \to \Omega^0_\infty \to \Omega^1_\infty \to \Omega^2_\infty \to 0.$$

Then

$$H^i(S,\mathbb{C}) = H^i(\Omega^\bullet_\infty(S)).$$

We recall our results from IV.9.2. We have a complex structure on the tangent bundle this is a linear transformation $I : T_S \to T_S$ which satisfies $I^2 = -Id$. We get a decomposition

$$T_{S,\mathbb{C}} = T_S^{10} \oplus T_S^{01}.$$

This provides a decomposition of the complex of differential forms, which only effects 1-forms:

$$0 \to \mathbb{C} \to \Omega^0_\infty \xrightarrow{d'+d''} \Omega^{10}_\infty \oplus \Omega^{01}_\infty \xrightarrow{d'+d''} \Omega^{11}_\infty \to 0.$$

The sheaf Ω^{10}_∞ contains the sheaf Ω^1_S of holomorphic 1-forms. (See IV,??).
In local coordinates at a point p we have

$$\omega = \omega' + \omega'' = f dz + g d\bar{z}$$

(see IV.9.2.1). For a function f we have

$$df = d'f + d''f = \frac{\partial f}{\partial z} dz + \frac{\partial f}{\partial \bar{z}} d\bar{z},$$

and for a 1-form $\omega = f dz + g d\bar{z} = \omega' + \omega''$ we get

$$d\omega = d''\omega' + d'\omega'' = \big(\frac{\partial f}{\partial \bar{z}} - \frac{\partial g}{\partial z}\big) dz \wedge d\bar{z}.$$

Especially we see that a 1-form $\omega = f dz = \omega'$ is holomorphic if and only if $d''\omega' = 0$

We introduce a hermitian metric h on the tangent bundle T_S (see IV.10.2.1). As I explained in general discussion such a hermitian metric is the same as an euclidean metric $\mathrm{Re}\, h = h_0$ on the tangent bundle T_S which satisfies $h_0(x,y) = h_0(Ix, Iy)$ for any two tangent vectors $x, y \in T_{S,p}$ and any point p. This induces a metric on the dual bundle T_S^* which we will denote by $<,>$.

If we pick a point $p \in S$ and a local coordinate z at p then it identifies a neighborhood U_p to a disc around zero in $\mathbb{C}$. The differential dz is a generator of the $\mathcal{O}_S(U_p)$ module of holomorphic differentials $\Omega^1(U_p)$. In the neighborhood U_p of p our hermitian metric is given by a strictly positive function (see section on linear algebra in section on Hodge theory)

$$u \to \operatorname{Re} h_{\mathbb{C}} < dz(u), d\bar{z}(u) >$$

which we simply denote by $< dz, d\bar{z} >$. Since we can view U_p as an open disc in $\mathbb{C}$ we have $dz = dx + idy$ and $< dz, d\bar{z} >=< dx, dx > + < dy, dy >= 2 < dx, dx >$ because the complex structure I which sends dx to dy is an isometry. The metric and the orientation give us a distinguished form ω_{top} in degree 2 which is positive with respect to he orientation and has length 1 with respect to the metric. It is given by

$$\omega_{\text{top}} = i\frac{dz \wedge d\bar{z}}{< dz, d\bar{z} >} = i\frac{(dx + idy) \wedge (dx - idy)}{< dx, dx > + < dy, dy >} = i(-i)\frac{dx \wedge dy}{< dx, dx >} = \frac{(dx \wedge dy)}{< dx, dx >}.$$

Of course $d\omega_{\text{top}} = 0$ and hence we see that our Riemann surface is a Kähler manifold (See IV.10.2.3).

Now it is rather easy to check that the $*$ operator does the following

$$\begin{aligned}
&* : f \to i\bar{f}\frac{dz \wedge d\bar{z}}{< dz, d\bar{z} >} \\
&* : fdz \to i\bar{f}d\bar{z} \\
&* : gd\bar{z} \to -i\bar{g}dz \\
&* : dz \wedge d\bar{z} \to -i < dz, d\bar{z} > .
\end{aligned} \tag{1.1.2}$$

We can introduce the adjoint operators δ', δ'' (See IV.10.2.3) and define the Laplacian

$$\begin{aligned}
\Delta &= (d' + d'')(\delta' + \delta'') = \\
&= d'\delta' + \delta'd' + d''\delta'' + \delta''d'' + d'\delta'' + \delta''d' + d''\delta' + \delta'd'' \\
&= \Delta' + \Delta'' + \quad \text{extra terms.}
\end{aligned}$$

The extra terms add up to zero because the metric is a Kähler metric. (See IV.10.2.3). Since I stated this result without proof in the general case I will carry out the calculation for special situation.

On the forms of degree 0 or 2 this is rather clear. If we consider for instance an $f \in \Omega^0_\infty(S)$ then

$$\delta''d'f = \delta''\frac{\partial f}{\partial z}dz = *d'' * \frac{\partial f}{\partial z}dz = -i * d''\overline{\frac{\partial f}{\partial z}}d\bar{z} = 0$$

and the same principle works for the other combinations. But for forms of degree one we have to work a little bit. Let us consider $\omega = fdz$ then we see easily that for two of the four terms that they must vanish simply by looking at the degree:

$$\delta''d'\omega = d'\delta''\omega = 0$$

For the other two terms we have to compute.

$$\delta' d'' f dz = \delta' \frac{\partial f}{\partial \bar z} d\bar z \wedge dz = - * d' * \frac{\partial f}{\partial \bar z} dz \wedge d\bar z$$

$$*id' \frac{\partial \bar f}{\partial z} < dz, d\bar z > = *i(\frac{\partial^2 \bar f}{\partial z^2} < dz, d\bar z > + \frac{\partial \bar f}{\partial z}\frac{\partial}{\partial z} < dz, d\bar z >) \wedge dz$$
$$= (\frac{\partial^2 f}{\partial \bar z^2} < dz, d\bar z > + \frac{\partial f}{\partial \bar z}\overline{\frac{\partial}{\partial z} < dz, d\bar z >}) d\bar z$$

Since $< dz, d\bar z >$ is positive and therefore real we have

$$\overline{\frac{\partial}{\partial z} < dz, d\bar z >} = \frac{\partial}{\partial \bar z} < dz, d\bar z >$$

Now we treat the second term

$$d''\delta' f dz = -d'' * d' * f dz = -d'' * (-i) d' \bar f d\bar z$$
$$= -id'' * \frac{\partial \bar f}{\partial z} dz \wedge d\bar z = -d'' \frac{\partial f}{\partial \bar z} < dz, d\bar z >=$$
$$= -(\frac{\partial^2 f}{\partial \bar z^2} < dz, d\bar z > + \frac{\partial f}{\partial \bar z}\frac{\partial}{\partial \bar z} < dz, d\bar z >) d\bar z.$$

Hence we see that the two terms add up to zero and $(\delta' d'' + d''\delta') f dz = 0$.

We apply our general theorem in the section on Hodge theory to this case. We are mainly interested the first cohomology group. We get that it is given by the harmonic forms in degree one and these harmonic forms are sums of harmonic forms in the degrees $(10), (01)$. A form $\omega = f dz \in \Omega^{10}_\infty(S)$ is harmonic if and only if $d'\omega = d''\omega = \delta'\omega = \delta''\omega = 0$. Two of these equations are automatically fulfilled the other two are equivalent to ω being holomorphic. We get the Hodge decomposition

$$H^1(S, \mathbb{C}) = H^0(S, \Omega^1_S) \oplus \overline{H^0(S, \Omega^1_S)} \tag{1.1.3}$$

The $\mathbb{C}$ vector space $\underline{H^1(S, \mathbb{C})} = H^1(S, \mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C}$ has the complex conjugation on it as an antilinear map and $\overline{H^0(S, \Omega^1_S)}$ is the complex conjugate of $H^0(S, \Omega^1_S)$ under this complex conjugation.(See IV. 10.2.1 especially the discussion concerning the formation of the complex conjugate space of a $\mathbb{C}$-vector space)

1.1.4 : I want to give an indication how this consequence of the general Hodge theory can be proved in this special situation. Of course there is no problem in degree zero and degree 2. So we look at the case $p = 1$ and start from a one form

$$\omega = \omega^{10} + \omega^{01}$$

which is closed and represents a cohomology class $[\omega] \in H^1(S, \mathbb{C})$. We have seen that we can construct a weakly convergent sequence $\omega_n = \omega + d\psi_n$ (see IV.10.1.3) such that the weak limit

$$\omega_0 \in \Omega^1_{(2)}(S)$$

satisfies

$$\langle \omega_0, d\psi \rangle = 0$$

for all $\psi \in \mathcal{C}_\infty(S)$. We also have

$$\langle \omega_0, \delta\eta \rangle = 0$$

for all $\eta \in \Omega''_\infty(S)$ because this is true for alle $\omega + d\psi_n$.
Now we decompose

$$\omega_0 = \omega_0^{10} + \omega_0^{01}.$$

I claim that even

$$\langle \omega_0^{10}, d'\psi \rangle = \langle \omega_0^{10}, \delta'\eta \rangle = 0$$

for all $\psi \in \mathcal{C}_\infty(S), \eta \in \Omega''_\infty(S)$. We have

$$\langle \omega_0^{10}, d'\psi \rangle + \langle \omega_0^{01}, d''\psi \rangle = 0$$

and

$$\langle \omega_0^{10}, \delta''\eta \rangle + \langle \omega_0^{01}, \delta'\eta \rangle = 0$$

for all ψ, η. We take $\eta = *\overline{\psi}$ and then we get from our local formulae ((See 1.1.2))

$$\begin{array}{rcl} \delta''\eta & = & -id'\psi \\ \delta'\eta & = & id''\psi \end{array}$$

and the second line becomes

$$i\langle \omega_0^{10}, d'\psi \rangle - i\langle \omega_0^{01}, d''\psi \rangle = 0.$$

We find that in the first line both terms are zero, i.e.

$$\langle \omega_0^{10}, d'\psi \rangle = \langle \omega_0^{01}, d''\psi \rangle = 0$$

for all $\psi \in \Omega^0_\infty(S)$.
We want to conclude that ω_0^{10} is itself a holomorphic 1-form. The holomorphicity is a local property of ω_0^{10}. We choose a point p and a neighborhood U_p such that we can identify $(U_p, \mathcal{O}_{U_p})$ with the disc $(D, \mathcal{O}_D)$. Let z be the coordinate function on D. Our differential form can be written

$$\omega_0^{10} = f(z)dz$$

and since the restriction of f to D must be square integrable, we have

$$\int_D |f(z)|^2 i \,\frac{dz \wedge d\bar{z}}{\langle dz, d\bar{z} \rangle} < \infty.$$

Since the fuction $< dz, d\bar{z} >$ is bounded and bonded away from zero the square integrability condition is equivalent to

$$\int_D |f(z)|^2 i\ dz \wedge d\bar{z} < \infty.$$

Now we exploit the orthogonality relation $\langle \omega_0^{10}, d'\psi \rangle = 0$ for $\mathcal{C}_\infty$-functions ψ with compact support in D. Our local formulae for the $*$-operator implies that the space of forms $*d'\psi$ is the same as the space of forms $d''\psi$ amd therefore the orthogonality condition is equivalent to

$$\int_D \omega_0^{10} \wedge d''\psi = 0 \text{ for all compactly supported } \psi \in \Omega_\infty^0(D)$$

We introduce polar coordinates and write

$$f(z) = f(r, \varphi) = \sum_{m \in \mathbb{Z}} a_m(r) e^{im\varphi},$$

and we have that $a_m(r)$ is square integrable on $[0,1]$ with respect to rdr and we have

$$\sum_m \int_0^1 |a_m(r)|^2 r dr < \infty.$$

We can choose our function ψ, and we take

$$\psi(z) = b(r) e^{-in\varphi}$$

where $b(r)$ is $\mathcal{C}_\infty$ on $[0,1)$ and has compact support in $[0,1)$. Then an easy computation shows

$$\frac{\partial}{\partial \bar{z}} \psi(z) = \frac{1}{2}\ e^{i\varphi} \left(\frac{\partial}{\partial r} b(r) + \frac{n}{r} b(r) \right) e^{-in\varphi} = \frac{1}{2} \left(\frac{\partial}{\partial r} b(r) + \frac{n}{r} b(r) \right) e^{-i(n-1)\varphi}.$$

Consequently our assumption implies

$$\int_0^1 a_{n-1}(r) \left(\frac{\partial}{\partial r} b(r) + \frac{n}{r} b(r) \right) r dr = 0$$

for all such choices of $b(r)$ and n. This means that

$$\int_0^1 \left(-r \frac{\partial}{\partial r} a_{n-1}(r) + n a_{n-1}(r) \right) b(r) dr = 0$$

and therefore we can conclude that

$$a_{n-1}(r) = c_{n-1} r^{n-1}$$

with some constant c_{n-1}. It follows that $a_n(r) = 0$ for $n < 0$ because in this case

$$\int_0^1 r^{2n} r dr = \infty,$$

and we get

$$f(z) = \sum_{n=0}^{\infty} c_n r^n e^{in\varphi}$$

and

$$\sum |c_n|^2 \cdot \frac{1}{(2n+2)^2} < \infty.$$

This is good enough to show that f is holomorphic on the disc.

The cup product $<,>_{\mathrm{cup}}$: $H^1(S,\mathbb{Z}) \times H^1(S,\mathbb{Z}) \to H^2(S,\mathbb{Z}) = \mathbb{Z}$ extends to a bilinear pairing $<,>_{\mathrm{cup}}$: $H^1(S,\mathbb{C}) \times H^1(S,\mathbb{C}) \to H^2(S,\mathbb{C}) = \mathbb{C}$ and we know (See IV. 9.1) that this pairing is given by

$$< [\omega_1],[\omega_2] >_{\mathrm{cup}} = \int_S \omega_1 \wedge \omega_2$$

where ω_1, ω_2 are closed forms which represent the classes $[\omega_1],[\omega_2]$ in the cohomology.

With respect to this pairing the two subspaces $H^0(S,\Omega^1_S), \overline{H^0(S,\Omega^1_S)}$ are maximal isotropic spaces and hence the cup product induces a perfect bilinear pairing

$$<,>_{\mathrm{cup}}: H^0(S,\Omega^1_S) \times \overline{H^0(S,\Omega^1_S)} \to \mathbb{C}.$$

1.2. Cohomology of holomorphic bundles

1.2.1 For any holomorphic vector bundle on $\mathcal{E}$ on S we consider the Dolbeault complex

$$0 \to \mathcal{E} \xrightarrow{d''} \Omega^0_\infty(\mathcal{E}) \xrightarrow{d''} \Omega^{01}_\infty(\mathcal{E}) \to 0.$$

The cohomology group of $\mathcal{E}$ are computed from the complex

$$0 \to \Omega^0_\infty(\mathcal{E})(S) \xrightarrow{d''} \Omega^{01}_\infty(\mathcal{E})(S) \to 0.$$

Now choose in addition a hermitian metric $<,>_h$ the bundle $\mathcal{E}$ and on T_S.

The metrics on $\mathcal{E}$ and on T_S provide an adjoint operator $\delta'' : \Omega^{01}_\infty(\mathcal{E})(S) \to \Omega^0(\mathcal{E})(S)$ and now Hodge theory implies (See IV.10.2.4 noch einfuegen)

$$\begin{aligned} H^0(S,\mathcal{E}) &= \{s \in \Omega^0_\infty(\mathcal{E}) | d''s = 0\} \\ H^1(S,\mathcal{E}) &= \{s \in \Omega^{01}_\infty(\mathcal{E}) | \delta''s = 0\}. \end{aligned} \tag{1.2.2}$$

and the cohomology groups are finite dimensional.

I emphazise that the proof of this finite dimensionality needs the full strength theory of elliptic operators. Actually for the cohomology in degree zero $H^0(S,\mathcal{E})$ the finite dimensionality is not difficult but the H^1 is by no means easy.

For compact Riemann surfaces these results are much easier to prove and therefore I will outline the proofs here.

At first I consider the assertion $\dim H^0(S,\mathcal{E}) < \infty$. We proceed by induction on the rank of the bundle. Let us assume that we have a non zero section $s \in H^0(S,\mathcal{E})$. Then we show

1.2.3 Lemma: *To this non zero section s we can attach a line subbundle $\mathcal{L} \subset \mathcal{E}$ such that $s \in H^0(S,\mathcal{L})$.*

Proof: The section s provides a map

$$\begin{array}{ccc} \mathcal{O}_S & \longrightarrow & \mathcal{E} \\ f & \longmapsto & fs \end{array}$$

for any holomorphic function f on some open subset $U \subset S$. This yields indeed a line subbundle $\mathcal{L}'$ but it is not yet the one we want. If we are at a point $x \in S$ where $s(x) = 0$, then we can choose a neighborhood U_x and a local trivialization of $\mathcal{E}$ by local sections $e_1 \cdots e_n$ which are nowhere zero on U_x. Our section s can be written as

$$s = \sum_{i=1}^{n} f_i e_i$$

with f_i holomorphic at x and $f_i(x) = 0$ for all $i = 1 \cdots n$. This implies that the set of zeroes of s is a finite subset of S. But since $\dim S = 1$ we have a local uniformizer $\pi_x \in \mathfrak{m}_x \subset \mathcal{O}_{S,x}$ and $f_i = \pi_x^{n_i} h_i$ where $h_i \in \mathcal{O}_{S,x}^*$. Let m be the minimum of the n_i. Then

$$\pi_x^{-m} s = \sum \pi_x^{-m} f_i e_i$$

extends to a section in $\mathcal{E}$ which is defined iover U_x. This section defines subbundle $\mathcal{L}^{(x)} \subset \mathcal{E}|_{U_x}$. But this line subbundle coincides with the above bundle $\mathcal{L}'$ if we restrict to the complement of the point x. Hence we see that we can glue the $\mathcal{L}'$ and the $\mathcal{L}^{(x)}$ to a line bundle $\mathcal{L}$ on S. We have $\mathcal{L} \subset \mathcal{E}$, the quotient $\mathcal{E}/\mathcal{L}$ is a vector bundle of smaller rank and $s \in H^0(S,\mathcal{L})$. This reduces the proof of the finite dimensionality to the case of line bundles. But if we have a line bundle $\mathcal{L}$ and a section $s \neq 0$, then we look again at the inclusion $\mathcal{O}_S \to \mathcal{L}$ induced by the sectiuon, and we get an exact sequence

$$0 \longrightarrow \mathcal{O}_S \longrightarrow \mathcal{L} \longrightarrow \mathcal{L}/\mathcal{O}_S \longrightarrow 0$$

and now $\mathcal{L}/\mathcal{O}_S$ is a sky scraper sheaf (see IV.1.1). Since $H^0(S,\mathcal{O}_C) = \mathbb{C}$ and obviously $\dim H^0(S,\mathcal{L}/\mathcal{O}_S)$ is finite dimensional, we are through.

The proof of the second assertion is much more difficult.

In a first step we construct an antilinear map $H^1(S,\mathcal{E}) \longrightarrow H^0(S,\mathcal{E}^\vee \otimes \Omega^1)$, where the target space has finite dimension. In a second step we show that this map is injective.

The argument in the first step ia similar to the one in 1.1.3. Of course an section $\omega \in \Omega^{01}_\infty(\mathcal{E})(S)$ defines a class in $H^1(S,\mathcal{E})$. On these sections we have the scalar product

$$\langle \omega, \omega_1 \rangle = \int_S \omega \wedge *\omega_1$$

and again we introduce the Hilbert space of square integrable sections

$$\Omega^{01}_{(2)}(\mathcal{E})(S).$$

We can modify a given section ω by an element $d''\psi$, and we can find a sequence $\omega = \omega + d''\psi_n$ such that

$$\|\omega_n\|_2^2 \longrightarrow \inf_\psi \|\omega + d''\psi\|_2^2.$$

Again we choose a subsequence which converges weakly to an element $\tilde{\omega} \in \Omega^{01}_{(2)}(\mathcal{E})(S)$, and then we have again

$$\langle \tilde{\omega}, d''\psi \rangle = 0$$

for all $\psi \in \Omega^0(\mathcal{E})(S)$. The hermitian metric yields an antilinear map $j_h = \mathcal{E} \to \mathcal{E}^\vee$ and together with the hermitian metric on the tangent bundle we get

$$* = \tilde{j}_h : \Omega^{01}_{(2)}(\mathcal{E})(S) \longrightarrow \Omega^{10}_{(2)}(\mathcal{E}^\vee)(S).$$

We have of course $\langle *\tilde{\omega}, \delta''\eta \rangle = 0$ for all $\eta \in \Omega^{11}_\infty(\mathcal{E}^\vee)(S)$ and basically the same argument which we used in 1.1.3 implies that $*\tilde{\omega} = \tilde{j}_h(\tilde{\omega})$ is holomorphic. We see that we get a map

$$H^1(S,\mathcal{E}) = \Omega^{01}_\infty(\mathcal{E})(S)/d''\Omega^0_\infty(\mathcal{E})(S) \longrightarrow H^0(S, \mathcal{E}^\vee \otimes \Omega^1)$$

which is surjective because the star operator above yields a bijection. We know already that $H^0(S,\mathcal{E}^\vee \otimes \Omega^1)$ is finite dimensional so it remains to show that this map is injective (our second step). We have the two operators

$$\begin{aligned} d'' &: \Omega^0(\mathcal{E})(S) \longrightarrow \Omega^{01}(\mathcal{E})(S) \\ \delta'' &: \Omega^{01}(\mathcal{E})(S) \longrightarrow \Omega^0(\mathcal{E})(S) \end{aligned}$$

and we take the two orthogonal decomposions of the two Hilbert spaces

$$\begin{aligned} \Omega^0_{(2)}(\mathcal{E})(S) &= \mathcal{E}(S) \oplus \tilde{\Omega}^0_{(2)}(\mathcal{E})(S) \\ \Omega^{01}_{(2)}(\mathcal{E})(S) &= \ker(\delta'') \oplus \tilde{\Omega}^{01}_{(2)}(\mathcal{E})(S). \end{aligned}$$

Since $\mathcal{E}(S)$ and $\ker(\delta'')$ consist of holomorphic sections, we get the same decomposition for $\mathcal{C}_\infty$-sections

$$\begin{aligned} \Omega^0_\infty(\mathcal{E})(S) &= \mathcal{E}(S) \oplus \tilde{\Omega}^0_\infty(\mathcal{E})(S) \\ \Omega^{01}_\infty(\mathcal{E})(S) &= \ker(\delta'') \oplus \tilde{\Omega}^{01}_\infty(\mathcal{E})(S), \end{aligned}$$

and we get

$$d'' \quad : \quad \tilde{\Omega}^0_\infty(\mathcal{E})(S) \longrightarrow \tilde{\Omega}^{01}_\infty(\mathcal{E})(S)$$

$$\delta'' \quad : \quad \tilde{\Omega}^{01}_\infty(\mathcal{E})(S) \longrightarrow \tilde{\Omega}^0(\mathcal{E})(S).$$

These restrictions of $d''.\delta''$ to $\tilde{\Omega}^\bullet_\infty$ are injective and they have seen that there images are dense. But we need that these restricted operators provide isomorphisms and this follows from the decomposition therem in Hodge theory (See IV.10.1.1)

In our special situation here there is an argument which proves

$$d'' : \tilde{\Omega}^0_\infty(\mathcal{E})(S) \xrightarrow{\sim} \tilde{\Omega}^{01}_\infty(\mathcal{E})(S)$$

more directly.

Our argument is based on some simple principles of functional analysis. We consider the local problem, this means we choose a small disc D around a point p, we assume that we identified it to the unit disc and we assume that z is a local parameter. We consider the Hilbert spaces $\Omega^\bullet_{(2)}(\mathcal{E})(D)$.

Our first principle says the following:

1.2.4 Let us assume we have a form $\omega \in \Omega^{01}_\infty(\mathcal{E})(D)$ and let us assume that this form is square integrable. Then we can find an $\eta \in \Omega^0_\infty(\mathcal{E})(D)$ such that $d''\eta = \omega$ and we can bound the L^2 norm of η

$$\|\eta(z)\|_{2,D} \leq C\|\omega\|_{2,D}$$

Since our bundle is locally trivial we may asum that its restriction to D is trivial. It is also clear that the validity of the L^2 estimates does not depend on the hermtian metric. Thesetwo facts together allow us to restrict to the case where D is the unit disc and where $\mathcal{E}|D \xrightarrow{\sim} \mathcal{O}_D$ and the metric is the trivial metric. Then we have to show: If $f : D \to \mathbb{C}$ is a $\mathcal{C}_\infty$ -function on D which is square integrable, i.e.

$$\|f\|_2^2 = \int_D |f(z)|^2 i dz \wedge d\bar{z} < \infty$$

then we can find $\mathcal{C}_\infty$-function u on D which satisfies

$$\frac{\partial u}{\partial \bar{z}} = f$$

and

$$\|u\|_2 \leq C\|f\|_2.$$

The point is that we can write down an explicit solution for this diferential equation:

$$u(z) = \frac{1}{2\pi i}\int_D \frac{f(\zeta)}{\zeta - z} d\zeta \wedge d\bar{\zeta}.$$

We leave it as anexercise to the reader to show that this is a solution for our equation. It is quite amusin to show that for$f = 1$ we get $u(z) = \bar{z}$ and then the general formula is easy.

It is easy to see that this function is $\mathcal{C}_\infty$. If $z_0 \in D$, then we can find a $\mathcal{C}_\infty$-function χ on D which is one on a small neighborhood of z_0 and zero on a small neighborhood of the boundary of D. Then

$$u(z) = \frac{1}{2\pi i}\int_D \frac{\chi(\zeta)f(\zeta)}{\zeta - z}d\zeta \wedge d\bar{\zeta} + \frac{1}{2\pi i}\int_D \frac{(1-\chi(\zeta))f(\zeta)}{\zeta - z}d\zeta \wedge d\bar{\zeta}.$$

The second summand is holomorphic at z_0 and the first summand is

$$\frac{1}{1\pi i}\int_D \frac{\chi(\zeta + z)f(\zeta + z)}{\zeta}d\zeta \wedge d\bar{\zeta},$$

and now we can differentiate under the integral sign because $\chi \cdot f$ has compact support and the singularity disappears if we change to polar coordinates.
It is an amusing exercise to show that for $f = 1$ we have $u(z) = \bar{z}$ and from this it follows easily that u satisfies the differential equation.
I think it is also very easy to see that the integral

$$\frac{1}{2\pi i}\int \frac{1}{|\zeta - z|}d\zeta \wedge d\bar{\zeta}$$

is bounded by a constant not depending on z. We may work with polar coordinates around z.
To get the L^2-estimate we start from

$$\begin{aligned} |u(z)|^2 &= \frac{1}{4\pi^2}\left| \int_D \frac{f(\zeta)}{\zeta - z}d\zeta \wedge d\bar{\zeta}\right|^2 \leq \\ &= \frac{1}{4\pi^2}\left| \int_D \frac{|f(\zeta)|}{|\zeta - z|^{1/2}} \cdot \frac{1}{|\zeta - z|^{1/2}}d\zeta \wedge d\bar{\zeta}\right|^2 \end{aligned}$$

This is the square of the scalar product of two L^2 functions, and we get by Cauchy's integral formula that the right hand side

$$\leq \frac{1}{4\pi^2}\left(\int_D \frac{|f(\zeta)|^2}{|\zeta - z|}d\zeta \wedge d\bar{\zeta}\right)\left(\int_D \frac{1}{|\zeta - z|}d\zeta \wedge d\bar{\zeta}\right).$$

We mentioned already that second factor is bounded by a constant not depending on z and hence we get

$$|u(z)|^2 \leq C\frac{1}{2\pi i}\int_D \frac{|f(\zeta)|^2}{|\zeta - z|}d\zeta \wedge d\bar{\zeta}.$$

Consequently we get

$$\frac{1}{2\pi i}\int_D |u(z)|^2 dz\wedge d\bar z \le C\int_D\int_D \frac{|f(\zeta)|^2}{|z-\zeta|}d\zeta\wedge d\bar\zeta\cdot dz\wedge d\bar z,$$

and if we change integration and use our above estimate a second time, then we get

$$\|u\|_2^2 \le C'\cdot \|f\|_2^2.$$

Now we come to a second principle. Inside of our Hilbert space $\Omega^0_{2,D}(\mathcal{E})(D)$ we can consider the holomorphic square integrable functions. The second principle says that this subspace is closed and even better:

1.2.5 *Any weakly convergent sequence of holomorhic functions η_n with bounded $\Omega^0_{2,D}$-norm in $\Omega^1_{2,D}(\mathcal{E})(D)$ converges locally uniformly to a holomorphic function on D.*
This is an immedeate consequence of Cauchys integral formula. If we pick a point $Q\in D$ and we put three concentric discs around Q:

$$Q\in D_1\subset D_2\subset D_3\subset D$$

each of them is slightly bigger than the previous one. If we have a circle $\Gamma\subset D_3\setminus D_2$ then we get from Cauchys formula for $z\in D_2$

$$\eta_n(z)=\frac{1}{2\pi i}\int_\Gamma \eta_n(\zeta)\frac{1}{\zeta-z}d\zeta.$$

Now we integrate over all Γ_τ between D_2 and D_3 and consider $z\in D_1$. We get

$$\eta_n(z)=\frac{c}{2\pi i}\int_{D_3\setminus D_2}\eta_n(\zeta)\frac{1}{\zeta-z}d\zeta\wedge d\bar\zeta,$$

where c is a consant depending on the width of the annulus. We can read this as a skalar product, since the sequence η_n is weakly convergent to η we see that η_n converges pointwise to the function

$$\tilde\eta : z\mapsto \frac{c}{2\pi i}\int_{D_3\setminus D_2}\eta(\zeta)\frac{1}{\zeta-z}d\zeta\wedge d\bar\zeta$$

which is holomorphic on D_1. But now it the Cauchy-formula also gives us that the η_n are equicontinous and then it follows that the convergence $\eta_n\to\tilde\eta$ is locally uniform and that $\eta=\tilde\eta$

We choose an $\omega\in\tilde\Omega^{01}_\infty(\mathcal{E})(S)$, and we know that we can find a sequence of functions $\psi_n\in\tilde\Omega^0_\infty(\mathcal{E})(S)$ such that

$$\|d\psi_n-\omega\|_2\longrightarrow 0.$$

I claim that the sequence of L^2-norms $\{\|\psi_n\|\}_{n\in\mathbb{N}}$ is bounded. We cover S by a finite family of discs

$$S=\bigcup_{\alpha\in A} D_\alpha.$$

We restrict the members of the family $\{\psi_n\}_{n\in\mathbb{N}}$ to these discs and call the restrictions $\psi_n^{(\alpha)}$. Now we can decompose the restriction

$$\Omega^0_{(2)}(\mathcal{E})(D_\alpha) = \Omega^{0,\prime}_{(2)}(\mathcal{E})(D_\alpha) \oplus \Omega^{(0,\mathrm{hol})}_{(2)}(\mathcal{E})(D_\alpha)$$

where the second component is the Hilbert space of holomorphic square integrable sections on D_α and accordingly we have

$$\psi_n^{(\alpha)} = \psi_n^{(\alpha,\prime)} + \psi_n^{(\alpha,\mathrm{hol})}.$$

We get

$$d''\psi_n^{(\alpha)} = d''\psi_n^{(\alpha,\prime)}.$$

Since we have an η_n^α for which $d''\eta_n^\alpha = d''\psi_n^\alpha$ and $\|\eta_n^\alpha\|_{2,D_\alpha} \leq C\|d''\psi_n^\alpha\|_{2,D_\alpha}$ we can conclude that $\|\psi_n^{(\alpha,\prime)}\|_{2,D_\alpha}$ stays bounded because $\|\psi_n^{(\alpha,\prime)}\|_{2,D_\alpha} \leq \|\eta_n^\alpha\|_{2,D_\alpha}$.

Hence we see: If $\|\psi_n\|_2$ is unbounded, then there must be an α such that $\|\psi_n^{(\alpha,\mathrm{hol})}\|_{2,D_\alpha}$ is unbounded and even

$$\|\psi_n^{(\alpha,\mathrm{hol})}\|_{2,D_\alpha} \geq C'\|\psi_n\|_2$$

with some constant C'. Now we consider the sequence of functions $\psi_n/\|\psi_n\|_2$. We can extract a subsequence which is weakly convergent, on any D_α this sequence has the same limit as $\psi_n^{(\alpha,\mathrm{hol})}/\|\psi_n\|_2$ hence it converges to a holomorphic function. This function must be zero because our ψ_n where chosen from the orthogonal complement of the holomorphic sections. It follows from our second principle that the sequence $\psi_n/\|\psi_n\|$ converges uniformly to zero. This cannot be the case because the L^2-norm of the members of the sequence is one. We get a contradiction.

So we see that the sequence of norms $\|\psi_n\|$ is bounded. Of course we extract a weakly convergent subsequence. If ψ is the limit of this subsequence we found the element which satisfies $d\psi = \omega$. This finishes the proof of 1.2.2. (Genau genug ???)

If we write locally $s = f \otimes d\bar{z}$, then

$$\delta'' s = - * d'' * (f \otimes d\bar{z}) = - * d''(*f) \otimes dz = - * (\frac{\partial}{\partial \bar{z}} * f) d\bar{z} \wedge dz.$$

As I explained in the general section on Hodge theory the $*$-operator provides an antilinear map from the bundle $\mathcal{E}$ into its dual bundle.(Nachsehen, ob das da schon steht) Hence we see that $*f \otimes dz \in \mathcal{C}_\infty(\mathcal{E}^\vee) \otimes \Omega^{10}(S)$ and the condition $\frac{\partial}{\partial \bar{z}} * f \otimes dz = 0$ means that $*f \otimes dz$ is holomorphic and hence we get an antilinear isomorphism

$$H^1(S, \mathcal{E}) \to H^0(S, \mathcal{E}^\vee \otimes \Omega^1_S).$$

This antilinear isomorphism will depend on the metric. But if our line bundle is the structure sheaf $\mathcal{O}_S$ then we may of course choose the standard metric which gives the value 1 to the constant section 1 at every point. This gives us a canonical identification

$$\overline{H^0(S, \Omega^1_S)} \xrightarrow{\sim} H^1(S, \mathcal{O}_S).$$

Remark: In the context of algebraic geometry we had a duality pairing for line bundles $\mathcal{L}$ on curves:

$$H^1(C,\mathcal{L}) \times H^0(C,\mathcal{L}^\vee \otimes \Omega^1_S) \to \mathbb{C}.$$

This is related to the assertion above. But in the analytic context we get on $\mathbb{C}$-antilinear isomorphism instead of a pairing. This is so because we have chosen an hermitian metric on $\mathcal{L}$. We will compare these pairings later after the discussion of the theorem of Riemann-Roch.

1.3. The theorem of Riemann-Roch

We consider the group $\mathrm{Pic}(S)$ isomorphism classes of line bundles on our Riemann surface. This group is isomorphic to $H^1(S,\mathcal{O}_S^*)$. The exact sequence of sheaves (see IX. 9.3 ????)

$$0 \to \mathbb{Z} \to \mathcal{O}_S \to \mathcal{O}_S^* \to 1$$

on our Riemann surface provides a long exact cohomology sequence

$$\begin{aligned} &0 \to \mathbb{Z} \to \mathbb{C} \to \mathbb{C}^* \to 0 \\ &\quad \to H^1(S,\mathbb{Z}) \to H^1(S,\mathcal{O}_S) \to H^1(S,\mathcal{O}_S^*) \to H^2(S,\mathbb{Z}) \to 0. \end{aligned}$$

(We have seen that $H^2(S,\mathcal{O}_S) = 0$ because the Dolbeault complex stops in degree 1).

In IV.1.2 we attached a sheaf $\mathcal{O}_S(D)$ to any divisor D, it is obvious that $\mathcal{O}_S(D)$ is a line bundle. It is also clear that $\mathcal{O}_S(D) \otimes \mathcal{O}_S(D_1) \xrightarrow{\sim} \mathcal{O}_S(D+D_1)$ and that $\mathcal{O}_S(D) \xrightarrow{\sim} \mathcal{O}_S$ if and only if D is principal and hence we get

$$\mathrm{Div}(S)/\text{ principal divisors } \xrightarrow{\sim} \mathrm{Pic}(S)$$

.

For a line bundle $\mathcal{L}$, which we can view as an element $[\mathcal{L}]$ in $H^1(S,\mathcal{O}_S^*)$ we define the degree $\deg(\mathcal{L}) = \delta([\mathcal{L}]) \in H^2(S,\mathbb{Z}) = \mathbb{Z}$. (We have a canonical orientation on S). This degree has various properties.

1.3.1 Lemma : *If we have a line bundle $\mathcal{L}$ on S and a point P then*

$$\deg(\mathcal{L}(P)) = \deg(\mathcal{L}) + 1$$

Proof: This is a special case of proposition IV.9.3.1

Exercise:

(i) If we choose a hermitian metric h on our line bundle $\mathcal{L}$, if we pick a point P and a neighborhood U_P and a local section $s \in \mathcal{L}(U_p)$ which is a generator for all points in U_P then we can form the expression

$$d''d' \log h(s,s) = \omega_h.$$

This is a (1,1)-form on S which is closed and it does not depend on the choice of the generator s. I claim that the class of this form in $H^2(S,\mathbb{C})$ is the Chern class $c_1(\mathcal{L}) = \deg(\mathcal{L})$.

(ii) I refer to the proof of our Lemma: We consider the holomorphic 1-form function $z_P^{-1}dz_P$ on the annulus $U_1 \cap U_2$. It is clear that we can extend this form to a $\mathcal{C}_\infty$-form η on the disc $U_2 = D_P$, (we simply multiply it by a $\mathcal{C}_\infty$-function which is one on the annulus and zero in a neighborhood of P). If we consider $d''\eta$ we get a $(1,1)$ form on the disc D_P which has compact support because it vanishes on the annulus. Hence it defines a class in $H^2_c(D_P,\mathbb{C})$, this maps to $H^2(S,\mathbb{C})$

Show that this is the class $\delta([\mathcal{O}_S(P)])$. This way we can construct a form of type $(1,1)$ which represents the degree. This form can be written as a boundary on any open set in S which misses a small disc around P.

(iii) Let us assume that we have an arbitrary compact complex manifold X and a divisor $D \subset X$ which is locally given on the open set of a covering $\mathfrak{U} = \{U_\alpha\}$ by one equation $f_\alpha = 0$. We choose a hermitian metric on X. We choose our covering in such a way that we place small balls around the points on D and choose a finite subcovering $\mathfrak{U} = \{U_\alpha\}$ of D. Then we supplement it by an open set U_0 which is the set of points having distance $> \epsilon$ from D.

Construct a $(1,1)$ form ω_D which has support in the complement of U_0 and which represents the class $c_1(\mathcal{O}_X(D))$. Show that this form is a boundary outside of the support of D.

(iv) If we have divisors $D_1, D_2, \dots, D_d$ $(d = \dim X)$ which intersect nicely then we can consider the intersection number $D_1 \cdot D_2 \cdot \ldots \cdot D_d$.

Show that this intersection number can also be computed by the integral

$$\int_X \omega_{D_1} \wedge \omega_{D_2} \wedge \ldots \wedge \omega_{D_d}$$

(v) Of course we can attach to any line bundle $\mathcal{L}$ its Chern class $c_1(\mathcal{L}) \in H^2(X,\mathbb{Z})$. If we have d such bundles $\mathcal{L}_1, \dots, \mathcal{L}_d$ we can compute their intersection number and we can take the cup product of their Chern classes which gives an element in $H^{2d}(X,\mathbb{Z}) = \mathbb{Z}$. Our exercise iv) gives us the equality of these numbers

$$\mathcal{L}_1 \cdot \ldots \cdot \mathcal{L}_d = c_1(\mathcal{L}_1) \cup \ldots \cup c_1(\mathcal{L}_d)$$

Show the equality of these numbers without using the de-Rham isomorphism ???.

1.3.2 The above Lemma implies: If we have a line bundle $\mathcal{L}$ and a non-zero section $s \in H^0(S,\mathcal{L})$ then on a suitably small open set U we can write $s = ft$ where t is a local generator and f is holomorphic. This function f defines a divisor on U, it is the divisor of its zeroes (See IV.1.1). Since we can do this everywhere we get a divisor $D = \mathrm{div}(s)$ and it is clear that $\mathcal{L} \sim \mathcal{O}_S(D)$. Then it follows from Lemma 1.3.1 that

$$\deg(\mathcal{L}) = \deg(D) \qquad (1.3.3.a)$$

. Hence we can conclude that the degree of a line bundle which has non zero sections must be positive.

If D is the divisor of a meromorphic function f then this function defines a section in $\mathcal{O}(D)$ and f^{-1} defines a section in $\mathcal{O}(-D)$ and consequently we must have

$$\deg(\mathrm{Div}(f)) = 0 \tag{1.3.3.b}$$

1.3.4. We also can conclude that a line bundle of degree zero has a non zero section if and only if it is trivial.

We can formulate the

Theorem of Riemann-Roch: *If $\mathcal{L}$ is a line bundle on a compact Riemann surface S then*

$$\dim_{\mathbb{C}} H^0(S,\mathcal{L}) - \dim_{\mathbb{C}} H^1(S,\mathcal{L}) = \deg(\mathcal{L}) + 1 - g.$$

We have

$$\dim_{\mathbb{C}} H^1(S,\mathcal{L}) = \dim_{\mathbb{C}} H^0(S,\mathcal{L}^{-1}\otimes\Omega^1_S).$$

Furthermore we have $\deg(\Omega^1_S) = 2g-2$ *and consequently* $\dim_{\mathbb{C}} H^1(S,\mathcal{L}) = 0$ *if* $\deg(\mathcal{L}) \geq 2g-1$.

This is now more or less obvious. We proved the finite dimensionality and the equality of the dimensions of the H^0 and the H^1 in the previous section. We write $\chi(\mathcal{L})$ for the left hand side.

The assertion is true for $\mathcal{L} = \mathcal{O}_S$ by definition. If we want to prove it for our given sheaf $\mathcal{L}$ we pick a point P and consider the exact sequence

$$0 \to \mathcal{L} \to \mathcal{L}(rP) \to \mathcal{L}(rP)/\mathcal{L} \to 0$$

for a large value of r. Then the dimension of the space of sections of the skyscraper sheaf becomes large and this space of sections is mapped to the finite dimensional $H^1(S,\mathcal{L})$. This implies that eventually $H^0(S,\mathcal{L}(rP))$ will be non zero. But then a non zero section gives us an inclusion $\mathcal{O}_S \hookrightarrow \mathcal{L}(rP)$ with a scyscraper quotient $\mathcal{S}$.(See proof of lemma 1.2.3). We have the exact sequence

$$0 \to \mathcal{O}_S \to \mathcal{L}(rP) \to \mathcal{S} \to 0$$

and a glance at the resulting exact sequence yields that $\chi(\mathcal{L}) - \chi(\mathcal{O}_S) = \dim H^0(S,\mathcal{S})$. This is also the degree of $\mathcal{L}$. (Lemma 1.3.1 iterated). Hence we have proved the first formula for $\mathcal{L}(rP)$. Then the same argument applied backwards proves it for $\mathcal{L}$.

It remains to prove the formula for the degree of Ω^1_S. To get this we apply the first formula in the theorem to the sheaf Ω^1_S We get

$$\dim_{\mathbb{C}} H^0(S,\Omega^1_S) - \dim_{\mathbb{C}} H^1(S,\Omega^1_S) = \deg(\Omega^1_S) + 1 - g.$$

The left hand side is equal $g-1$ and the theorem is proved.

I would like to stress again that the real difficulty is to show that $H^1(S,\mathcal{O}_S)$ is finite dimensional.

1.3.5. At this point we have proved a very strong finiteness result. We know that any line bundle $\mathcal{L}$ on a compact Riemann surface S has a very simple acyclic resolution: We simply take an effective divisor $D = \Sigma n_p P$ with sufficiently large degree and then

$$0 \longrightarrow \mathcal{L} \longrightarrow \mathcal{L}(D) \longrightarrow \mathbb{L}_D \longrightarrow 0$$

is an acyclic resolution of $\mathcal{L}$ (see IV.1.1.). We get the exact sequence

$$0 \longrightarrow H^0(S,\mathcal{L}) \longrightarrow H^0(S,\mathcal{L}(D)) \longrightarrow H^0(S,\mathbb{L}_D) \longrightarrow H^1(S,\mathcal{L}) \longrightarrow 0.$$

We have seen that $\dim H^1(S,\mathcal{L}) = \dim H^0(S,\mathcal{L}^{-1}\otimes\Omega^1_S)$ but we can prove a stronger result. We construct a non degenerate bilinear pairing

$$H^1(S,\mathcal{L}) \times H^0(S,\mathcal{L}^{-1}\otimes\Omega^1_S) \longrightarrow \mathbb{C}.$$

To get this pairing we respresent an element $\xi \in H^1(S,\mathcal{L})$ as the image unter the boundary map. We lift it to an element

$$\tilde{\xi} = (\dots\tilde{\xi}_p\dots)_{P\in|D|} \in H^0(S,\mathbb{L}_D)$$

where $|D|$ is the support of D. We choose small discs D_P around these P such that we can trivialize the bundle $\mathcal{L}$ over these discs by non vanishing sections $t_P \in H^0(D_P,\mathcal{L})$. Then the components $\tilde{\xi}_P$ can be written as

$$\tilde{\xi}_P = \left(\frac{b_{-n}}{z_P^n} + \dots + \frac{b_{-1}}{z_P}\right) t_P$$

where z_p is a local parameter at P. If now $\eta \in H^0(S,\mathcal{L}^{-1}\otimes\Omega^1_S)$, we can write the restriction of η to D_p in the form

$$\eta\mid_{D_P} = t_P^{-1}\cdot f(z_P)dz_P,$$

and we can consider the product

$$\tilde{\xi}_P\eta = \left(\frac{b_{-n}}{z_P^n} + \dots + \frac{b_{-1}}{z_P}\right)\cdot f(z_P)dz_P = \left(\frac{a_{-n}}{z_P^n} + \dots \frac{a_{-1}}{z_P} + \dots\right) dz_P = \omega_P,$$

this is a holomorphic 1-form on $D_p\setminus\{P\}$ which may have a pole (a meromorphic 1-form). To such a meromorphic 1-form we attach its residue at P, it is given by

$$\mathrm{Res}_P\left(\frac{a_{-n}}{z_P^n} + \dots \frac{a_{-1}}{z_P}\dots\right) dz_P = a_{-1}.$$

It is not clear a priori that this residue is well defined but everybody should know the formula

$$a_{-1} = \frac{1}{2\pi i}\int_\Gamma \omega_p$$

where Γ is a path in $D_P \setminus \{P\}$ which winds counterclockwise aroung P just once.
Then we define

$$\langle \xi, \eta \rangle = \sum_P \mathrm{Res}_P(\tilde{\xi}_P \eta) = \sum_P \mathrm{Res}_P(\omega_P).$$

We have to show that the value of this pairing does not depend on the choice of the lifting. If we replace $\tilde{\xi}$ by $\tilde{\xi} + f$ where $f \in H^0(S, \mathcal{L}(D))$, then $f\eta = \omega$ is a meromorphic 1-form on S, it is an element in $H^0(S, \Omega^1_S(D))$. For such a form it is clear that the sum of the residues vanishes. We simply observe that we can take the D_P so small that they do not intersect and for the path Γ_P we take their boundaries with counterclockwise orientation. Then

$$\sum_P \mathrm{Res}_P(\omega) = \frac{1}{2\pi i} \sum \int_{\Gamma_P} \omega = \frac{1}{2\pi i} \int_{S \setminus \cup D_P} d\omega = 0.$$

This proves that we get a well defined pairing

$$H^1(S, \mathcal{L}) \times H^0(S, \mathcal{L}^{-1} \otimes \Omega^1_S) \longrightarrow \mathbb{C}.$$

But it is also clear that any non zero element $\eta \in H^0(S, \mathcal{L}^{-1} \otimes \Omega^1_S)$ induces a non zero linear form on $H^1(S, \mathcal{L})$. To see that this is so we simply compute this linear form on $\mathbb{L}_D$, and then it is obviously non zero.

This non degenerate pairing is called Serre *duality pairing* but in this special case it was certainly known to Riemann. It expresses the fact that the existence of holomorphic differentials on a Riemann surface of higher genus provides an obstruction for a collection of Laurent expansions

$$\tilde{\xi} \in H^0(S, \mathcal{O}_S(D)/\mathcal{O}_S)$$

to come from a meromorphic function (see IV, Exercise 1.1.1(2)). It is stated in Riemann's works (see [ ], ...) that $\tilde{\xi}$ comes from a meromorphic function if and only if for all holomorphic differentials ω we have

$$\sum_P \mathrm{Res}_P(\tilde{\xi}\omega) = 0.$$

1.3.6. If the genus of the Riemann surface S is equal to zero and if P is any point, then it follows from Riemann Roch and Serre duality that

$$\dim H^0(S, \mathcal{O}_S(P)) = 2,$$

and we conclude that we can find a meromorphic function f which is holomorphic everywhere except at the point P and at P it has a simple pole.

We saw already that this function gives us a map

$$f : S \longrightarrow \mathbb{P}^1(\mathbb{C}),$$

I claim that this map is an isomorphism between Riemann surfaces. To see this we observe that there is exactly one point – namely the point P – which goes to infinity. If $U = S\backslash\{P\}$, then we get for the restriction

$$f : U \longrightarrow \mathbb{C} = \{(z,1) \mid z \in \mathbb{C}\} \subset \mathbb{P}^1(\mathbb{C})$$

(see ......). For any $c \in \mathbb{C}$ we know that the polar divisor of $f - c$ is $-P$. Hence the zero divisor is of degree one and is equal to Q where $f(Q) = c$. Since S is compact it follows that this map is a homeomorphism.

For any point Q we can find a neighborhood $D_Q \subset S$ such that

$$(D_Q, \mathcal{O}_{D_Q}) \simeq (D, \mathcal{O}_D)$$

where D is the unit disc in $\mathbb{C}$. Let z be the resulting uniformizing element. Under the map f this neighborhood is mapped to an open set $f(D_Q)$ which contains $f(Q) \in \mathbb{P}^1(\mathbb{C})$. We choose a uniformizing element z_Q, this is a holomorphic function defined in a neighborhood of $f(Q)$ and which has a first order zero at z_Q. Then $z_Q \circ f$ is a holomorphic function on a smaller disc $D'_Q \subset D_Q$ and hence a power series in z. Since the function $z_Q \circ f$ is injective we can conclude that

$$z_Q \circ f = az + \cdots$$

with $a \neq 0$.

Now $\mathcal{O}_{\mathbb{P}^1, f(Q)}$ is the ring of convergent power series in z_Q and $\mathcal{O}_{S,Q}$ is the ring of convergent power series in z. We see that the map

$$\mathcal{O}_{\mathbb{P}^1, f(Q)} \longrightarrow \mathcal{O}_{S,Q}$$

(see ....) is an isomorphism and this proves our assertion.

We can also give examples of Riemann surfaces of genus one. If $\Omega \subset \mathbb{C}$ is a lattice, then the quotient $S = \mathbb{C}/\Omega$ is a compact Riemann surface (see III. 1.1.1). It is homeomorphic to $\mathbb{R}^2/\mathbb{Z}^2$, and hence we have $H^1(S,\mathbb{Z}) = \mathbb{Z}^2$ (see IV, ...) and hence we see that S has genus 1. We know that the space of holomorphic differentials is of dimension one and clearly the form $\omega = dz$ is a generator.

If in turn S is a compact Riemann surface of genus one, then we may do the following: We pick a point $s_0 \in S$ and we consider the following space

$$\tilde{S} = \{(s,\gamma) \mid s \in S, \gamma \text{ homotopy class of a point starting in } s_0 \text{ and ending in } s\}.$$

We have the projection

$$\pi : \tilde{S} \longrightarrow S$$

and locally this projection is a homeomorphism. (This construction can be done for any connected Riemann surface, then $\tilde{S}$ is the so called universal cover of S). It is also clear that we have a structure of a Riemann surface on $\tilde{S}$.

Now we can construct a holomorphic map h from $\tilde{S}$ to $\mathbb{C}$. We simply send

$$h : \tilde{s} = (s, \gamma) \longmapsto \int_\gamma \omega$$

where we choose a differentiable path in the homotopy class. I leave it as an exercise to the reader to show that this map is an isomorphism between $\tilde{S}$ and $\mathbb{C}$. It is also not difficult to show that $h^{-1}(s_0) = \Omega$ is a lattice in $\mathbb{C}$ and the map factorizes over an isomorphism

$$\begin{array}{ccc} \tilde{S} & \xrightarrow{h} & \mathbb{C} \\ \pi \downarrow & & \downarrow \\ S & \xrightarrow{\tilde{h}} & \mathbb{C}/\Omega \end{array}$$

This makes it clear that all compact Riemann surfaces are of the form $\mathbb{C}/\Omega$.

1.4. The algebraicity of Riemann surfaces:

1.4.0: We are now able to show that compact Riemann surfaces may be considered as purely algebraic objects. More precisely we can say that compact Riemann surfaces are the same objects as smooth, connected, projective curves over $\mathbb{C}$. It will be discussed in the second volume of this book what this exactly means.

The starting point is that the meromorphic functions on S form a field $K = \mathbb{C}(S)$. We will show that this field is finitely generated over $\mathbb{C}$ and it is of trancendence degree 1. We will reconstruct the Riemann surface S from this field.

If for instance we consider the Riemann sphere $S = S^2 = \mathbb{P}^1(\mathbb{C})$ (III.2.1. 1a) or 1.3.6)) then $\mathbb{C}(S^2) = \mathbb{C}(z)$ is the rational function field in one variable. It is the quotient field of the polynomial ring $\mathbb{C}[z]$ which is the ring of meromorphic functions which are holomorphic on $U_0 = \mathbb{P}^1(\mathbb{C}) \setminus \{\infty\}$.

1.4.1 If we pick a point $P \in S$ and if we consider the line bundle $\mathcal{O}_S(nP)$ for $n >> 0$ then we see that we can find a non-constant function $f \in H^0(S, \mathcal{O}_S(nP))$. As I explained in the introduction to this chapter a non-constant meromorphic function f on S provides a surjective map $f : S \to \mathbb{P}^1(\mathbb{C})$. We put $U_0 = S \setminus \{P\}$ and U_1 will be the complement of the set of zeroes of f.

1.4.1.1 : If we have a point $s \in S$ where f is holomophic, then the differential df is holomorphic at this point. If it is non-zero at s then we know from the theorem of implicit functions that f yields a biholomorphic map from a neighborhood of s to a neighborhood of $f(s)$. We say that f is *unramified* or *not ramified* if df is nt zero at this point s. If f has a pole at s then we take $g = \frac{1}{f}$. If $dg \neq 0$ then we have the analogous assertion for the functon g. In terms of f this can be reformulated: If f has a pole at s and if the pole of df is of second order, then f yields a biholomorphic map from a neighborhood of s to a neighborhood of $\infty \in \mathbb{P}^1(\mathbb{C})$. Again we say that f is unramified at s.

The map f is called *unramified* at a point $x \in \mathbb{P}^1(\mathbb{C})$ if it is unramified in all points of the fibre $f^{-1}(x)$. It is clear that the set of points where f is ramified is finite.

Changing the coordinates allows us to assume that our map is unramified at 0 and ∞. We give it a new name and write

$$\pi : S \to \mathbb{P}^1(\mathbb{C}).$$

Let V_0 (resp. V_1) be the complement of 0 (resp. ∞), let $U_0 = \pi^{-1}(V_0), U_1 = \pi^{-1}(V_1)$.

For any set $V \subset S$ (or in $\mathbb{P}^1(\mathbb{C})$) which is the complement of a finite number of points we define $\mathcal{O}_S^{\mathrm{mer}}(V)$ to be the ring of those holomorphic functions on V, which have at most poles in the points $S \setminus V$. For $V_0, V_1 \subset \mathbb{P}^1(\mathbb{C})$ this yields

$$\begin{aligned}\mathcal{O}_{\mathbb{P}^1(\mathbb{C})}^{\mathrm{mer}}(V_0) &= \mathbb{C}[f]\\ \mathcal{O}_{\mathbb{P}^1(\mathbb{C})}^{\mathrm{mer}}(V_1) &= \mathbb{C}[f^{-1}].\end{aligned}$$

We may also consider the rings $\mathcal{O}_S^{\mathrm{mer}}(U_0), \mathcal{O}_S^{\mathrm{mer}}(U_1)$ and these two rings are modules for $\mathcal{O}_{\mathbb{P}^1(\mathbb{C})}^{\mathrm{mer}}(V_0)$ and $\mathcal{O}_{\mathbb{P}^1(\mathbb{C})}^{\mathrm{mer}}(V_0)$ respecively.

1.4.2 Proposition: *a) The modules $\mathcal{O}_S^{\mathrm{mer}}(U_0)$ (resp. $\mathcal{O}_S^{\mathrm{mer}}(U_1)$) over $\mathcal{O}_{\mathbb{P}^1(\mathbb{C})}^{\mathrm{mer}}(V_0)$ (resp. $\mathcal{O}_{\mathbb{P}^1(\mathbb{C})}^{\mathrm{mer}}(V_1)$) are finitely generated.*

b) If $\{a_1, \ldots, a_t\} \subset V_0$ is a finite subset and $V_0' = V_0 \setminus \{a_1, \ldots, a_t\}$ and $U_0' = \pi^{-1}(V_0')$ then

$$\mathcal{O}_S^{\mathrm{mer}}(U_0') = \mathcal{O}_S^{\mathrm{mer}}(U_0) \cdot \mathcal{O}_{\mathbb{P}^1(\mathbb{C})}^{\mathrm{mer}}(V_0').$$

Proof: We show that $\mathcal{O}_S^{\mathrm{mer}}(U_0)$ is a finitely generated $\mathcal{O}_{\mathbb{P}^1(\mathbb{C})}^{\mathrm{mer}}(V_0)$ module. We consider the divisor $D_\infty = \sum_{P \in \pi^{-1}(\infty)} P$, it is the divisor of poles of the function f pulled back to S.(Here we use that π is unramified at ∞, actually this is only technical). For $n > 0$ we consider the vector spaces $H^0(S, \mathcal{O}_S(nD_\infty))$. They form an increasing sequence of vector spaces exhausting $\mathcal{O}_S^{\mathrm{mer}}(U_0)$ if $n \to \infty$. The dimension of these spaces is given by the theorem of Riemann-Roch: If $n >> 0$ then

$$\dim_{\mathbb{C}} H^0(S, \mathcal{O}_S(nD_\infty) = n\deg(D_\infty) + 1 - g.$$

We observe that the multiplication by z yields a linear map (multiplication by f??????)

$$\times z : H^0(S, \mathcal{O}_S(nD_\infty) \to H^0(S, \mathcal{O}_S((n+1)D_\infty)$$

and I claim that this map becomes surjective if we divide the space on the right hand side by the subspace $H^0(S, \mathcal{O}_S(nD_\infty)$. We pick a funcion $h \in H^0(S, \mathcal{O}_S((n+1)D_\infty)$ its polar divisor is of the form $D = \sum_{P \in \pi^{-1}(\infty)} m_P P$ with $m_P \leq n+1$. If even $m_P \leq n$ for all n then this function is in the subspace which we divide out. Now we observe that it follows from our assumption $n >> 0$ that

$$H^0(S, \mathcal{O}_S(nD_\infty) \to H^0(S, \mathcal{O}_S(nD_\infty)/\mathcal{O}_S((n-1)D_\infty))$$

is surjective. Therefore we can find a function $f \in H^0(S, \mathcal{O}_S(nD_\infty)$ which has an n-th order pole at a given point P where $m_P = n+1$ and has at most a $n-1$-th order pole at all the other points in $\pi^{-1}(\infty)$ For a suitable combination $h - azf$ the number of m_P which are equal to $n+1$ drops by one and our assertion follws by induction. Our claim implies that the $\mathcal{O}^{\mathrm{mer}}_{\mathbb{P}^1(\mathbb{C})}(V_0)$ module $\mathcal{O}^{\mathrm{mer}}_S(U_0)$ is generated by $H^0(S, \mathcal{O}_S(n_0 D_\infty)$ for some sufficiently large n_0 and a) follows.

Now b) is is not difficult anymore. Let f be meromorphic function in $\mathcal{O}^{\mathrm{mer}}_S(U_0')$. We can find a function $h \in \mathcal{O}_{\mathbb{P}^1(\mathbb{C})}(V_0')$ which has a zero in the points $a_1, \dots, a_t$ and nowhere else (take the inverse of a function which has poles in exacly these points). If we pull it back to U_0 it has zeroes in all points in the fibres $\pi^{-1}(a_i)$ i.e. in all points in $U_0 \setminus U_0'$ and nowhere else. Hence $f \cdot h^N$ will be holomorphic in all points of $U_0 \setminus U_0'$ and this means $fh^N \in \mathcal{O}^{\mathrm{mer}}_S(U_0')$.

Now we consider the function field $K = \mathbb{C}(S)$ of meromorphic functions. It is clear that the function field of the Riemann sphere $\mathbb{C}(\mathbb{P}^1((\mathbb{C})) = \mathbb{C}(z) = \mathbb{C}(f)$ is the rational function field in one variable. The assertion b) in our proposition above implies that any meromorphic function h on S can be written as a quotient $h = g/f$ where $g \in \mathcal{O}^{\mathrm{mer}}_S(U_0)$ and f is a meromorphic function in $\mathbb{C}(\mathbb{P}^1(\mathbb{C}))$. Therefore we can conclude: If $y_1, \dots y_d$ is a set of generators of the $\mathcal{O}^{\mathrm{mer}}_{\mathbb{P}^1(\mathbb{C})}(V_0)$ module $\mathcal{O}^{\mathrm{mer}}_S(U_0)$ then $\mathbb{C}(S)$ is generated by these elements as a $\mathbb{C}(\mathbb{P}^1(\mathbb{C}))$-vector space. We conclude

1.4.3 Theorem *The field of meromorphic functions on a compact Riemann surface is a finite extension of a rational funtion field $\mathbb{C}(h)$ where h is any non constant meromorphic function on S.*

The only thing we have to observe that a field K which is a finite extension of a rational function field $\mathbb{C}(x)$ is also finite over $\mathbb{C}(h)$ for any $h \notin \mathbb{C}$.

We explain how we can reconstruct S from K. To do this we will use in an ad hoc manner some arguments from commutative algebra which will be explained in a more systematic way in chapter VII.

1.4.3.1: Integral elements and integral closures: The finiteness of $\mathcal{O}^{\mathrm{mer}}(U_0)$ as a $\mathcal{O}^{\mathrm{mer}}_{\mathbb{P}^1}(V_0)$ module implies by a standard argument of commutative algebra that any element $h \in \mathcal{O}^{\mathrm{mer}}(U_0)$ is *integral over* $\mathcal{O}^{\mathrm{mer}}_{\mathbb{P}^1}(V_0)$ and this means that it satisfies an equation

$$h^n + a_1 h^{n-1} \dots + a_n = 0$$

where the $a_i \in \mathcal{O}^{\mathrm{mer}}_{\mathbb{P}^1(\mathbb{C})}(V_0)$ and $n > 0$ (See Chap. VII ???? or [Ei], ???). But if in turn $h \in \mathbb{C}(S)$ is integral over $\mathcal{O}^{\mathrm{mer}}_{\mathbb{P}^1(\mathbb{C})}(V_0)$ then it must be holomorphic on U_0, we simply look at the possible order of a pole. We conclude

1.4.3.2: The ring $\mathcal{O}^{\mathrm{mer}}_S(U_0)$ is the *integral closure* of $\mathcal{O}^{\mathrm{mer}}_{\mathbb{P}^1}(V_0)$ in K and this means that it consists of all the elements in K which are integral over $\mathcal{O}^{\mathrm{mer}}_{\mathbb{P}^1}(V_0)$.

The principal observation is that a point $P \in S$ defines a subring $\mathcal{O}^{\mathrm{mer}}_P \subset K$, it is the ring of meromorphic functions which are regular at P. This ring is a *valuation ring* with quotient field K and this means:

a) for any $f \in K$ we have f or f^{-1} is in $\mathcal{O}_P^{\text{mer}}$.

b) The ring is not equal to K and it contains the constant functions $\mathbb{C}$.

Such a ring $\mathcal{O}_P^{\text{mer}}$ has a unique maximal ideal which consists of the elements

$$\mathfrak{m}_P = \{f \in \mathcal{O}_P^{\text{mer}} | f^{-1} \notin \mathcal{O}_P^{\text{mer}}\} = \{f \in K | f \text{ vanishes at } P\}$$

This maximal ideal is non zero and it is generated by any function which has a first order zero at P. This means that the ring is even a *discrete valuation* ring. The elements which are not in the maximal ideal are invertible. The property a) implies that the quotient field of such a ring is K.

Our aim is to show that we can identify S with the set $\text{Val}(K)$ of all valuation rings in K, i.e. those subrings which satisfy a) and b).

1.4.4 Starting from the function field: We forget our Riemann surface completely and start from a field K over $\mathbb{C}$ which is a finite extension of a rational function field $\mathbb{C}(x)$. We can write

$$K = \mathbb{C}(x)[y]$$

where y satisfies an irreducible polynomial equation

$$y^n + a_1(x)y^{n-1} \cdots a_n(x) = 0$$

with $a_i(x) \in \mathbb{C}(x)$. Of course we know what $\text{Val}(K)$ is.

Propositon 1.4.4.1 *All $A \in \text{Val}(K)$ are discrete valuation rings, i.e. the maximal ideal $\mathfrak{m}_A$ is always a principal ideal. The composition $\mathbb{C} \to A \to A/\mathfrak{m}_A$ is an isomorphism and this means that the residue field is canonically isomorphic to $\mathbb{C}$. This also means that we can evalute an $f \in A$ at A. The value $f(A) = f \bmod \mathfrak{m}_A$. Furthermore we will see that for any $f \in K$ the set of A such that $f \notin A$ is finite.*

In accordance with our previous definitions we say that f is *regular* at A if $f \in A$.

Again we have to invest a little bit of commutative algebra. If $K_0 = \mathbb{C}(x)$ then an $A \in \text{Val}(K_0)$ contains $\mathbb{C}[x]$ or $\mathbb{C}[x^{-1}]$. Let us assume that $A \supset \mathbb{C}[x]$. The maximal ideal $\mathfrak{m}_A$ intersected with $\mathbb{C}[x]$ gives us a non zero prime ideal in $\mathbb{C}[x]$. It is an elementary fact that the non zero prime ideals in $\mathbb{C}[x]$ are of the form $(x - \alpha)$. This implies that the elements of $\text{Val}(K_0)$ are in one to one correspondence with the points in $\mathbb{P}^1(\mathbb{C}) = \mathbb{C} \cup \{\infty\}$: For any $\alpha \in \mathbb{C}$ we have the ring

$$A_\alpha = \left\{ f = \frac{P(x)}{Q(x)} \mid Q(\alpha) \neq 0 \right\},$$

and for ∞ we have

$$A_\infty = \left\{ f = \frac{P(x)}{Q(x)} \mid \deg(P) \leq \deg(Q) \right\}$$

where P, Q are polynomials and $f(A_a) = f(a)$. Clearly the valuation rings are discrete valuation rings. This must be well known.

We consider the integral closures B_0 (resp. B_∞) of $\mathbb{C}[x]$ (resp. $\mathbb{C}[x^{-1}]$) in the field K. Then the theory of Dedeking rings (see [Ei], [Neu], or Chap. VII, 2.4.4) implies that these integral closures are finitely generated modules over $\mathbb{C}[x]$ (resp. $\mathbb{C}[x^{-1}]$). Since the polynomial rings have unique factorization, it follows that these modules are even free of rank $[K : \mathbb{C}(x)]$.

This fact has the following consequence:

1.4.4.2 If $\mathfrak{p}$ is a prime ideal in B_0 then $\mathfrak{p}$ is maximal and $B_0/\mathfrak{p} = \mathbb{C}$. The ring

$$A = \{\frac{f}{g} \mid f, g \in B_0, g \notin \mathfrak{p}\}$$

is a discrete valuation ring.

To see that this is so we consider $\mathfrak{p}_0 = \mathfrak{m}_A \cap \mathbb{C}[x]$. It is clear that $\mathfrak{p}_0$ is non zero. Then $B_0/\mathfrak{p}$ is an integral domain and a finite dimensional vector space over $\mathbb{C} = \mathbb{C}[x]/\mathfrak{p}_0$. This implies that $B_0/\mathfrak{p} = \mathbb{C}[x]/\mathfrak{p}_0 = \mathbb{C}$. For the last assertion we refer to [Ei],??? or [Neu], or Chap. VII)

Now we pick an $A \in \mathrm{Val}(K)$ and let us assume $A \supset \mathbb{C}[x]$, otherwise it contains the other ring. I claim that this implies $A \supset B_0$. If $f \in B_0$ we write down the polynomial equation

$$f^n + a_1(x) f^{n-1} \cdots a_n(x) = 0$$

with $a_i(x) \in \mathbb{C}[x]$.

If now $f \notin A$ then f^{-1} lies in the maximal ideal $\mathfrak{m}_A$ of A and our polynomial equation yields

$$1 = -a_1(x) f^{-1} \cdots - a_n(x) f^{-n}$$

which gives a contradiction. Now $A \supset B_0$, we consider the prime ideal $\mathfrak{p} = \mathfrak{m}_A \cap B_0$ and and $\mathfrak{p}$ must be maximal. Then

$$A = \{\frac{f}{g} \mid f, g \in B_0, g \notin \mathfrak{p}\}.$$

This together with our considerations above make it clear that the elements of U_0 are in one-to-one correspondence with the maximal ideals in B_0 and these maximal ideals are also just the homomorphisms $B_0 \to \mathbb{C}$ which are $\mathbb{C}$ linear, i.e. the identity in $\mathbb{C}$. This proves the second assertion in 1.4.4.1 above. It is also clear that the complements of U_0 and U_∞ in $\mathrm{Val}(K)$ are finite because it is rather obvious that there are only finitely many prime ideals in B_0 (resp. B_∞) which lie above (x) (resp.(x^{-1}). This implies the finiteness assertion if we apply our consideration to $x = f$. Hence the proposition is proved.

We define a topology on $\mathrm{Val}(K)$. The open sets $U \subset \mathrm{Val}(K)$ are defined as the complement of finite sets, and of course we have to add the empty set. This topology is called the *Zariski topology.*

We can define the sheaf of meromorphic functions. For any open set $U \subset \mathrm{Val}(K)$ we put

$$\mathcal{O}(U) = \bigcap_{A \in U} A,$$

this is the ring of functions which are regular on U and meromorphic on S. This gives $(\mathrm{Val}(K), \mathrm{Zar}, \mathcal{O})$ the structure of a locally ringed space.

If we take any $f \in K$ which is not constant, i.e. $f \notin \mathbb{C}$ then D_f is the set of points where f is regular. Then

$$\mathcal{O}(D_f) = \text{the integral closure of } \mathbb{C}[f] \text{ in } K$$

and this follows from the fact that a Dedekind ring is the intersection of the discrete valuation in the quotient field which contain it (???)

This object $(\mathrm{Val}(K), \mathrm{Zar}, \mathcal{O})$ is almost what is called a smooth, projective, connected curve over $\mathbb{C}$. The only thing missing is the so called *generic* point. This generic point is simply the field K. We can just drop the assumption $A \neq K$ for our valuation rings and put $\tilde{\mathrm{Val}}(K) = \mathrm{Val}(K) \cup \{K\}$. We define the Zariski topology on $\tilde{\mathrm{Val}}(K)$, the open sets are the complements of finite subsets in $\mathrm{Val}(K)$ and the empty set. We define the sheaf as before and now $(\tilde{\mathrm{Val}}(K), \mathrm{Zar}, \mathcal{O})$ is a locally ringed space and this is now really a smooth, connected, projective curve. The stalks of the structure sheaf are discrete valuation rings in the closed points and the stalk in $\{K\}$ is K.

1.4.5 Back to the Riemann surface: Now we assume again that K is the field of meromorphic functions on our compact Riemann surface S. We observed earlier that we have a map

$$S \to Val(K)$$

and we want to show that this is a bijection. Here it is clear that we have to use the compactness of the Riemann surface. This compactness will enter in the form that any holomorphic function on S must be constant.

We pick a valuation ring $A \subset K$, let

$$\mathfrak{m}_A = \{f \in A \mid f^{-1} \notin A\}$$

be its maximal ideal. Our goal is to show that there is a unique point $P \in S$ such that $A = \mathcal{O}_P^{\mathrm{mer}}$. We will show that this point P is the common zero of the $f \in \mathfrak{m}_A$ and it also be characterized as the unique point where all the elements of A are regular.

We pick a non zero $f \in \mathfrak{m}_A$. We consider the intersection $A \cap \mathbb{C}[f]$ then $\mathfrak{m}_A \cap \mathbb{C}[f] = (f)$ because the principal ideal (f) is maximal. The intersection

$$B = A \cap \mathbb{C}(f) = \{g/h \mid g, h \in \mathbb{C}[f], h \notin (f)\}$$

We consider the diagram

$$\begin{array}{ccc} \pi : S & \to & \mathbb{P}^1(\mathbb{C}) \\ \downarrow & & \downarrow \\ \tilde{\pi} : \mathrm{Val}(K) & \to & \mathrm{Val}(\mathbb{C}(f)) \end{array}$$

induced by f. As before $V_0 = \mathbb{P}^1(\mathbb{C}) \setminus \{\infty\}$ and $U_0 = \pi^{-1}(V_0)$. Then our ring B consists of those meromorphic functions on $\mathbb{P}^1(\mathbb{C})$ which are regular in 0. We have to show that the map $\pi^{-1}(0) \to \tilde{\pi}(0)$ is surjective. The integral closure A_1 of B in K is a free module of rank $[K : \mathbb{C}(f)]$ and

$$A_1/A_1 f = \sum_{\mathfrak{p}} A_1/\mathfrak{p}^{e_{\mathfrak{p}}}$$

where the $\mathfrak{p}$ are the maximal ideals in A_1, they are in one to one correspondence to the elements in $\tilde{\pi}(0)$, we have $\sum_{\mathfrak{p}} e_{\mathfrak{p}} = [K : \mathbb{C}(x)]$. The divisor of f is of the form $\sum_{\mathfrak{p} \in \pi^{-1}(0)} e_{\mathfrak{p}}$ and it has degree $[K : \mathbb{C}(x)]$ and from this the equality of the two fibres follows.

1.4.6 The recovery of the analytic topology: The set S has some further structure, it has a topology and a sheaf of complex valued functions on it. We want to reconstruct this structure starting from K.

Our Riemann surface is also a locally ringed space, and it is clear that the map

$$(S, \mathcal{O}_S) \longrightarrow (\mathrm{Val}(K), \mathrm{Zar}, \mathcal{O}^{\mathrm{mer}})$$

is a morphism between locally ringed spaces. This is of course not an isomorphism because on the left hand side we have many more functions, the ring $\mathcal{O}_{S,P}$ is much larger than $\mathcal{O}^{\mathrm{mer}}_{S,P} = A$ if P maps to A.

We still go one step further. Again we forget the compact Riemann surface S, and we start from a function field

$$K = \mathbb{C}(x)[y]$$

where

$$y^n + a_1(x)y^{n-1} = + \cdots + a_n(x) = 0.$$

We put $S = \mathrm{Val}(K)$ on this set, we have the Zariski topology and our sheaf $\mathcal{O}_S$ with respect to the Zariski topology. We want to construct a finer topology on S and S together with this finer topology will be called S_{an}. Of course the identity $S_{an} \to S$ will now be continuous. Furthermore we want to construct a sheaf $\mathcal{O}_S^{an}$ of $\mathbb{C}$ valued functions on S_{an} such that we get a locally ringed space and such that

$$(S_{an}, \mathcal{O}_S^{an})$$

will be a compact Riemann surface.

Finally we can restrict meromorphic functions $f \in \mathcal{O}_S(U)$ to the open sets in S_{an}, and this will induce a morphism of locally ringed spaces

$$(S_{an}, \mathcal{O}_S^{an}) \to (S, \mathrm{Zar}, \mathcal{O}_S)$$

(see III,......).

We come to the construction of the analytic topology. For any open subset $U \subset S$ we have the ring $\mathcal{O}_S(U)$, and we can interpret $\mathcal{O}_S(U)$ as ring of $\mathbb{C}$-valued functions on U. We introduce the coarsest topology on U for which all these functions are continuous.

If we have two different points $A, B \in S$, then it is clear that we cannot have $A \supset B$ or $B \subset A$. Hence we can find an $f \in A$ for which $f \notin B$. Since we can add a constant, we can assume $f \notin \mathfrak{m}_A$. Then $g = 1/f \in A$ but $g \notin \mathfrak{m}_A$ and $g \in \mathfrak{m}_B$. In other words, the element g is regular at A and at B and $g(A) \neq 0$ and $g(B) = 0$. Hence we have $A, B \in D_f$ and from the definition of the analytic topology follows that we can find neighborhoods of A and B whose intersection is empty and we have proved that our analytic topology is Hausdorff.

We want to describe a neighborhood of a point $A \in S$, and we want to show that A has neighborhoods isomorphic to a disc in $\mathbb{C}$.

This is of course clear if $K = \mathbb{C}(x)$, in this case we could identify

$$\mathrm{Val}(K) = \mathbb{P}^1(\mathbb{C})$$

and the analytic topology is of course the usual topology on $\mathbb{P}^1(\mathbb{C})$.

We reduce the general case to this one. We have our point $A \in S$. We choose an element $f \in \mathfrak{m}_A$ which generates the ideal. Again we consider the integral closure $\mathcal{O}(D_f)$ of $\mathbb{C}[f]$ in K. We have

$$\begin{array}{ccc} \mathbb{C}[f] \cdot f & \subset & \mathbb{C}[f] \\ \cap & & \cap \\ (f) & \subset & \mathcal{O}(D_f). \end{array}$$

In our special situation the $\mathbb{C}[f]$-module $\mathcal{O}(D_f)$ is free of rank n where $n = [K : \mathbb{C}(f)]$ (see ......).

$$\mathcal{O}(D_f) = \oplus \mathbb{C}[f] y_i = \mathbb{C}[f, y_1,_2 \dots y_n]$$

where the elements $f, y_1, \dots, y_n$ satisfy some polynomial identities

$$P(f, y_1, \dots, y_n) = 0$$

with some polynomials $P(F, Y_1, \dots, Y_n)$ from the polynomial ring $\mathbb{C}[F, Y_1, \dots Y_n]$. If is I the ideal generated by all these polynomals then we get an isomorphism

$$\mathbb{C}[F, Y_1, \dots Y_n]/I \tilde{\to} \mathcal{O}(D_f)$$

We introduce the evaluation map

$$\begin{gathered} E : D_f \to \mathbb{C}^{a+1} \\ E : u \mapsto (f(u), y_1(u), \dots, y_n(u)) \end{gathered}$$

The the elements of $\mathcal{O}(D_f)$ separate the points in D_f because the ponts corresond to the mximal ideals of $\mathcal{O}(D_f)$. Therefore the evaluation map is injective. The image consists of those points in $(z_0, z_1, \dots, z_n) \in \mathbb{C}^{n+1}$ which satisfy $P(z_0, z_1, \dots, z_n) = 0$ for all elements $P \in I$.

Our point A is mapped to an element $(0, a_1, \dots, a_n) = (f(A), y_1(A), \dots, y_n(A))$. We have a finite set of distinct points $A = A_0, A_1, \dots, A_m$ in S for which $f(A_1) = \dots = f(A_m) = 0$. We can find an $r > 0$ such that for all i, α, β we have $|y_i(A_\beta) - y_i(A_\alpha)| > 2r$ whenever these two numbers are not equal. We consider the open set $U \subset S$ which is defined by the requirement

$$U = \{B \mid |y_i(B) - y_i(A)| < r \text{ for all } 1 = 1, 2, \dots n\}.$$

We consider the projection to the first coordinate

$$p : U \to \mathbb{C}$$

$$B \mapsto f(B)$$

and this projection is by construction a homeomorphism to the image. Now we observe that we can write any of our y_i in the form

$$y_i = y_i(A) + \gamma_i f + R_i$$

where $\gamma_i \in \mathbb{C}$ and $R_i = f^2 g_i / h_i$ where $g_i, h_i \in \mathcal{O}(D_f)$ and $h_i(A) \neq 0$. We represent these elements bv polynomials $G_i, H_i \in \mathbb{C}[F, Y_1, \dots Y_n]$ and then we know that the ideal I above contains elements of the form

$$L_i = H_i(F, Y_1, \dots, Y_n)(Y_i - y_i(A)) - H_i(F, Y_1, \dot{;} Y_n)\gamma_i F - F^2 G(F, Y_1, \dots, Y_n) \text{ for } i = 1, \dots, n.$$

The independent variables are F and the Y_i for $i = 1 \dots, n$ and the partial Jacobi matrix

$$(\frac{\partial L_i}{\partial Y_i})(A)_{i,j}$$

is a diagonal matrix with non zero entries on the diagonal and therefore it has maximal rank. Hence we can conclude from the theorem of implicit functions that for a suitably small $\varepsilon > 0$ we can construct an inverse to the projection p above

$$q : D(\varepsilon) \to U$$

$$z \mapsto (z, y_1(z), \dots, y_n(z))$$

where now $y_1(z), \dots, y_n(z)$ are convergent power series and q identifies $D(\varepsilon)$ to an open neigborhood $U(\varepsilon)$ of A in S. On this open neighborhood we can define the sheaf $\mathcal{O}_S^{\mathrm{an}}(U(\varepsilon))$ of holomorphic functions, this is simply the sheaf of holomorphic functions on our small disc. Hence we constructed the structure of a compact Riemann surface $(S_{\mathrm{an}}, \mathcal{O}_S^{\mathrm{an}})$ and clearly the identity map

$$(S_{\mathrm{an}}, \mathcal{O}_S^{\mathrm{an}}) \to (S, \mathrm{Zar}, \mathcal{O}_S)$$

is a morphism of locally ringed spaces.

One word concerning the notation. Here we think that the algebraic object $(S, \mathrm{Zar}, \mathcal{O}_S)$ is given first and to denote the analytic object we put the sub and superscripts and write $S_{\mathrm{an}}, \mathcal{O}_S^{\mathrm{an}}$. In the beginning of this section we did the oppostite. There the Riemann surface was given and we had to introduce the sub- and superscripts $\mathrm{Zar}, \mathrm{mer}$.

Finally I want to say a few words about the connection to algebraic geometry. I come back to the description of

$$\mathcal{O}(D_f) = \mathbb{C}[f, y_1, \cdots, y_n] = \mathbb{C}[F, Y_1, \cdots, Y_n]/I.$$

We described the image of D_f under the evaluation map as a set of solutions of polynomial equations

$$Y = E(D_f) = \{(a_0, a_1, \cdots, a_n) \mid P(a_0, a_1, \cdots, a_n) = 0 \quad \text{for all} \quad P \in I\},$$

and this means (by defintion) that this image is an affine algebraic variety over $\mathbb{C}$. I claim that for any point $B = (a_0, \cdots, a_n) \in Y$ we can pick an index i such that $y_i - a_i = \tilde{y}_i$ is a local parameter: In a small neighborhood the other coordinates of a point $b \in Y$ can be expressed as holomorphic functions in $\tilde{y}_i$. We simply apply our arguments above to B. Therefore our variety is in fact one dimensional and smooth (See III.2 Example 5).

Actually we can say even more. Since $\mathcal{O}(D_f)$ is the integral closure of $\mathbb{C}[f]$ in the function field we know that the elements $\tilde{y}_i$ satisfy an equation

$$\tilde{y}_i^{n_i} + a_1 \tilde{y}_i^{n_i - 1} \ldots + a_{n_i} = 0$$

where the coefficients $a_i \in \mathbb{C}[f|$. We may assume that this polynomial is irreducible. We must have $a_{n_i}(A) = 0$. It is not entirely obvious but true that the previous coefficient $a_{n_i - 1}$ does not vanish at A. We can conclude that for the points B in our small neighborhood of A the polynomial

$$Y^{n_i} + a_1(B) Y^{n_i - 1} \ldots + a_{n_i}(B)$$

has exactly one root which is close to one. This means in classical terms that $\tilde{y}_i^{n_i}$ is an algebraic function in the variable $z = f(B)$, it is a root of the polynomial which is distinguished and depends analytically on z.

Of course a few points are missing, namely, the points in $S \setminus D_f$. But we can find an element $g \in K$ which is regular at these missing points. We have a second evaluation map which identifies

$$D_g \xrightarrow{\sim} Y_1 \subset \mathbb{C}^{m+1}$$

and $\mathcal{O}(D_g) = \mathbb{C}[g, u_1, \cdots, u_m]$. In Y we have the open subset Y_g where g is regular and in Y_1 the open subset $Y_{1,f}$ where f is regular and these two open sets are identifed to $D_f \cap D_g$ under the evaluation maps.

We have to say in terms of the two data what the regular functions on $D_f \cap D_g$ are. I claim that

$$\mathcal{O}_S(D_g \cap D_f) = \mathbb{C}[g, u_1, \cdots, u_m, f, y_1, \cdots, y_n],$$

and this means that the regular functions on $D_g \cap D_f$ can be written as sums of products of elements in $\mathcal{O}(D_f)$ and $\mathcal{O}(D_g)$. If $h \in \mathcal{O}_S(D_g \cap D_f)$, then this function may have poles in $T_f \cup T_g$ where $T_f = S \setminus D_f$, $T_g = S \setminus D_g$. We want to modify h by sums of products of functions $u_1 u_2$ where u_1 has poles only in T_f and u_2 has poles only in T_g. Let us pick a point $t \in T_f$ with $t \notin T_g$ and $s \in T_g$, $s \notin T_f$ such that h has a pole at t. If such a pair does not exist there is nothing to prove.

We produce a function u_1 which has a pole at t and nowhere else. This is possible by Riemann-Roch. We produce a function u_2 which has a pole at s and nowhere else but which in addition has a simple zero at t. Then $u_1 u_2^m$ has a simple pole at t for a suitable choice of m. Now we can modify h by subtracting a suitable power of $u_1 u^m$

$$h - \gamma \cdot (u_1 u^m)^k$$

such that the pole order of h at t drops. This means that the total pole order at points in $T_f \setminus (T_f \cap T_g)$ drops. We repeat this process until h does not have any pole in the set $T_f \setminus (T_f \cap T_g)$, and then the modified function has only poles in T_g. Then we achieved our goal.

I summarize: Our space S together with the sheaf $\mathcal{O}_S$ is covered by two affine varieties (or affine schemes) and the ring of regular functions is generated by the regualar functions on the two pieces. With a corn of salt this means that we constructed a separated scheme. (see Chap. VI ??) . Actually it is even projective this will be discussed later.

We want to have a brief look at the case of Riemann surfaces of genus one. We have seen that they are of the form $S = \mathbb{C}/\Omega$ where Ω is a lattice in $\mathbb{C}$.

In this case it is not so difficult to produce meromorphic functions on S, in a first semester course on function theory it is taught that we have the two meromorphic functions

$$\wp(z) = \frac{1}{z^2} + \sum_{\omega \in \Omega, \omega \neq 0} \left(\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2}\right)$$

$$\wp'(z) = -2 \sum_{\omega \in \Omega} \frac{1}{(z-\omega)^3}$$

on S and they are related by an equation

$$\wp'(z)^2 = 4\wp(z)^3 + g_2(\Omega)\wp(z) + g_3(\Omega)$$

where the coefficients $g_2(\Omega), g_3(\Omega)$ can be expressed in terms of the lattice. (They are modular forms see for instance [ ]). Furthermore we know that these two functions generate the field of meromorphic functions on S.

We also get an embedding into the projective space, we map

$$z \mapsto (\wp'(z), \wp(z), 1) = (1, \frac{\wp(z)}{\wp'(z)}, \frac{1}{\wp'(z)}) \in \mathbb{P}^2(\mathbb{C})$$

this map provides an analytic isomorphism

$$S \xrightarrow{\sim} \{(x,y,z) \in (\mathbb{C}^3 \setminus 0)/\mathbb{C}^* \, | y^2z - x^3 - g_2(\Omega)xz^2 - g_3(\Omega)z^3 = 0\} \subset \mathbb{P}^2(\mathbb{C})$$

1.4.7 Geometrie Analytique et Geometrie Algebrique

An analytic sheaf $\mathcal{E}^{an}$ on S_{an} is called a *coherent sheaf* if it is a sheaf of $\mathcal{O}_S^{an}$–modules, and if for any point $P \in S$ the $\mathcal{O}_{S,P}^{an}$–module $\mathcal{E}_P^{an}$ is finitely generated. We have the same notion for Zariski sheaves on S and clearly any coherent Zariski sheaf $\mathcal{E}$ provides an analytic sheaf $\mathcal{E}^{an} = \mathcal{E} \otimes_{\mathcal{O}_S^{mer}} \mathcal{O}_S^{an}$.

Here we encounter the simplest case of the so called GAGA–principle. In our situation this principle says that this construction provides an equivalence of categories.

1.4.7.1 Proposition *For any coherent sheaf $\mathcal{E}^{an}$ on S we can find a unique coherent Zariski sheaf $\mathcal{E}$ such that*

$$\mathcal{E}^{an} = \mathcal{E} \otimes_{\mathcal{O}_S^{mer}} \mathcal{O}_S^{an}.$$

For any pair $\mathcal{F}, \mathcal{G}$ of coherent Zariski sheaves the map

$$\mathrm{Hom}_S(\mathcal{F}, \mathcal{G}) \longrightarrow \mathrm{Hom}_{S^{an}}(\mathcal{F}^{an}, \mathcal{G}^{an})$$

is a bijection.

Proof: In either situation a coherent torsion sheaf is a sheaf where each stalk consists of torsion elements. Let us assume that our sheaf is a torsion sheaf. We pick a point $P \in S$ and the generators $t_1, t_2, \ldots, t_s$ as above. They are annihilated by a non zero element $f \in \mathcal{O}_P$ regardless in which case we are. But then this element f can be extendet into a small neighborhood D'_P such that it is non zero at any point $Q \neq P, Q \in D'_P$. Hence this restriction has finite support in P and we conclude that torsion sheaves are the scysraper sheaves. Now we observe that for any point P and any positive integer $a > 0$ we have the equality

$$\mathcal{O}_{S,P}^{an}/(\mathfrak{m}_P^{an})^a = \mathcal{O}_{S,P}/(\mathfrak{m}_P)^a,$$

and therefore analytic and Zariski torsion sheaves are the same objects.

Since $\mathcal{O}_{S,P}^{an}$ is a discrete valuation ring we can find generators $u_1, \ldots, u_m$ such tht the stalk $\mathcal{E}_{S,P}^{an}$ is the direct sum

$$\mathcal{E}_{S,P}^{an} = \oplus \mathcal{O}_{S,P}^{an} u_i.$$

If we restrict to a still smaller neighborhood then we can express the u_i in terms of the t_i and vice versa. We may assume that the coefficients in these epressions are defined over this neighborhood and this means that can view the u_i as restrictions of some $\tilde{u}_i$ which are defined in this neighborhood. We can replace the t_i by te $\tilde{u}_i$. Now some of the u_i are torsion elements and these elements define a torsion subsheaf if we restrict to this neighborhood. If we still shrink this neighborhood further then this torsion subsheaf has support in P and the quotient is free. This happens in a small neighborhood of an arbitrary point P shows us that we can define a finite skyscraper sheaf $\mathcal{E}^{an}_{tors} \subset \mathcal{E}^{an}$ and the quotient $\mathcal{E}^{an}/\mathcal{E}_{tors} = \mathcal{E}'^{an}$ is locally free.

But if we have a locally free sheaf $\mathcal{E}^{an}$ and a section s which is defined in a punctured disc $\dot{D}_P = D_P \setminus \{P\}$, then we know what it means that s has at most a pole at P. Hence for any Zariski open subset $U \subset S$ we can define the $\mathcal{O}_S(U)$–module of meromorphic sections $\mathcal{E}^{mer}(U) = \mathcal{E}(U)$. We have defined $\mathcal{E}$ and the assertion $\mathcal{E}^{an} = \mathcal{E} \otimes_{\mathcal{O}_S} \mathcal{O}_S^{an}$ follows from our strong finiteness results.

Now it is clear what we do in the general case. We have

$$\mathcal{E}'^{an} = \mathcal{E}^{an}/\mathcal{E}_{tors},$$

and we define $\mathcal{E}(U)$ as the space of sections which go to meromorphic sections on $\mathcal{E}'(U)$.

The rest is clear.

Here we encountered a special case of a general principle which is called the GAGA-principle (see the headline of this section). In a very rough form it says that compact complex manifolds are in fact algebraic, provided they have enough meromorphic functions. In such a case the coherent algebraic and the coherent analytic sheaves form equivalent categories (See [Se]). We will come back to this principle in the second half of this chapter.

1.4.8 Comparison of two pairings

We have by the Hodge-Dolbeault theorem that

$$H^1(S, \mathcal{O}_S) \simeq \overline{H^0(S, \Omega^1_S)}.$$

We have the Hodge decomposition of $H^1(S, \mathbb{C}) \tilde{\to} H^0(S, \Omega^1_S) \oplus \overline{H^0(S, \Omega^1_S)}$ (1.1.3). If we compute the cohomology $H^1(S, \mathbb{C})$ using the de-Rham complex then the cup product

$$H^1(S, \mathbb{C}) \times H^1(S, \mathbb{C}) \to \mathbb{C}$$

on the cohomology is giving by integrating cup products of representing forms.(See IV.9.1.3.) If we consider the above decompostion the two summands are isotropic and we get the $\mathbb{C}$-linear pairing

$$H^0(S, \Omega^1_S) \times \overline{H^0(S, \Omega^1_S)} \to \mathbb{C}$$

which is given by

$$(\omega_1, \bar{\omega}_2) \mapsto < \omega_1, \bar{\omega}_2 > = \int \omega_1 \wedge \bar{\omega}_2.$$

The combination of the isomorphism above and the pairing yields a $\mathbb{C}$-bilinear pairing

$$H^0(S, \Omega^1_S) \times H^1(S, \mathcal{O}_S) \to \mathbb{C}.$$

We will call this pairing the *analytic* pairing. We constructed the Serre duality pairing in section 1.3.5.

1.4.9 Theorem: *The analytic pairing is $-2\pi i$ times the Serre duality pairing.*

To see this we need some simple considerations which in principle concern the comparison between Čzech-cohomology of sheaves and the cohomology groups obtained by injective (or acyclic) resolutions for instance the de-Rham resolution.
We pick a point $P \in S$ and an $n >> 0$ such that the map

$$H^0(S, \mathcal{O}_S(nP)/\mathcal{O}_S) \to H^1(S, \mathcal{O}_S)$$

becomes surjective. We choose an element $\eta \in H^1(S, \mathcal{O}_S)$ and we lift it to an element $\xi \in H^0(S, \mathcal{O}_S(nP)/\mathcal{O}_S)$. We choose a disc D_P around P and a local coordinate z_P which is zero at P. Now we represent an element ξ by a Laurent series

$$f(z) = \frac{a_n}{z^n} + \ldots + a_0 + a_1 z + \ldots.$$

We cover S by two open sets, one of them $U_1 = D_p$ and U_2 is the complement of a smaller closed disc $\overline{D_p(\epsilon)}$ around P, hence $U_1 \cap U_2$ is an annulus. We have that

$$f \in \mathcal{O}_S(U_1 \cap U_2)$$

and it defines a 1- cocycle for he covering $S = U_1 \cap U_2$. This cocycle maps to η under the edge homomorphism (See IV, 6.6.1) Now we proceed and use the de Rham resolution, we get a diagram

$$\begin{array}{ccc}
\mathcal{O}_S(U_1) \oplus \mathcal{O}_S(U_2) & \longrightarrow & \mathcal{O}_S(U_1 \cap U_2) \\
\downarrow & & \downarrow \\
\Omega^0_\infty(U_1) \oplus \Omega^0_\infty(U_2) & \longrightarrow & \Omega^0_\infty(U_1 \cap U_2) \\
\downarrow d'' & & \downarrow d'' \\
\Omega^{01}_\infty(U_1) \oplus \Omega^{01}_\infty(U_2) & \longrightarrow & \Omega^{01}_\infty(U_1 \cap U_2).
\end{array}$$

We send f to $\Omega^0_\infty(U_1 \cap U_2)$ and I claim that we may write $f|U_1 \cap U_2$ as the restriction of a $\mathcal{C}_\infty$-function h_1 on $U_1 = D_p$. To see this we simply multiply the function f which is actually defined on he punctured disk by a $\mathcal{C}_\infty$-function which is identically equal to 1 on the annulus and which isidentically zero in a neighborhood of zero. This C_∞-function on the disc is holomorphic on the annulus but if we go into the interior it certainly loses this property. This means that $d''h_1 = \psi$ is an element in $\Omega^{01}_\infty(U_1)$ which has compact support and therefore can be extended it by zero to S. Then $\psi \in \Omega^{01}_\infty(S)$, it is closed and it represents our given class in $H^1(S, \mathcal{O}_S)$ via the Dolbeault isomorphism.

The integral

$$\int_S \omega \wedge \psi$$

for a holomorphic 1-form ω on S gives the value of the analytic pairing between ξ and ω. We compute this integral. We observe that

$$\int_S \omega \wedge \psi = \int_{D_p} \omega \wedge \psi$$

and the integrand has compact support in D_p. We choose a circle $\partial D_p(r)$ which lies in the annulus, we still have

$$\int_{D_p} \omega \wedge \psi = \int_{D_p(r)} \omega \wedge \psi.$$

But now we write again $\psi = d''h_1$ and we have $\omega \wedge dh_1 = \omega \wedge (d'h_1 + d''h_1) = \omega \wedge d''h_1$. Therefore

$$\begin{aligned} \int_{D_p(r)} \omega \wedge d''h_1 &= \int_{D_p(r)} \omega \wedge dh_1 \\ -\int_{D_p(r)} d(\omega \wedge h_1) &= -\int_{\partial D_p(r)} h_1 \omega \\ &= -\int_{\partial D_p(r)} f\omega = -2\pi i \mathrm{Res}_P(f\omega). \end{aligned}$$

and now the right hand side is by definition the value of the Serre duality pairing multiplied by $2\pi i$.

Since our pairings are non degenerate we conclude that we have two different ways of producing an identification $H^1(S, \mathcal{O}_S) \xrightarrow{\sim} H^0(S, \Omega^1_S)^\vee$ which differ by a factor $-2\pi i$. We could call the one produced by the cupproduct the *analytic* identification and the other one the *algebraic* identification. We will mostly use the analytic identification.

1.5. The Jacobian of a compact Riemann surface

Let S be a compact Riemann surface. We defined $\mathrm{Pic}(S) = H^1(S, \mathcal{O}_S^*)$ to be the group of isomorphism classes of holomorphic line bundles on S. Our exact sequence in 1.3. provides the homomorphism

$$H^1(S, \mathcal{O}_S^*) = \mathrm{Pic}(S) \to H^2(S, \mathbb{Z}) = \mathbb{Z}$$

The kernel is denoted by $\mathrm{Pic}^0(S)$ and it is called the *Jacobian* of the curve and sometimes we write $J = \mathrm{Pic}^0(S)$. The exact sequence yields

$$\mathrm{Pic}^0(S) = H^1(S, \mathcal{O}_S)/H^1(S, \mathbb{Z}).$$

Here we divide a g dimensional $\mathbb{C}$-vector space by a free $\mathbb{Z}$-module of rank $2g$, I claim that we are in fact dividing by a lattice, i.e. the submodule is in fact discretely embedded. To see this we recall the Hodge decomposition (1.1.3) and get inclusions

$$H^1(S, \mathbb{Z}) \hookrightarrow H^1(S, \mathbb{R}) \hookrightarrow H^1(S, \mathbb{C}) = H^0(S, \Omega^1_S) \oplus \overline{H^0(S, \Omega^1_S)}.$$

Since $H^1(S,\mathbb{R}) = H^1(S,\mathbb{Z}) \otimes \mathbb{R}$, we see that $H^1(S,\mathbb{Z})$ is a lattice in $H^1(S,\mathbb{R})$. On the other hand it is clear that the projection of $H^1(S,\mathbb{R})$ to any of the two summands in the decomposition of $H^1(S,\mathbb{C})$ is an isomorphism since the summands are complex conjugate. This implies that the inclusions followed by the projection

$$H^1(S,\mathbb{Z}) \hookrightarrow \overline{H^0(S,\Omega^1_S)} = H^1(S,\mathcal{O}_S)$$

maps $H^1(S,\mathbb{Z})$ isomorphically to a lattice Γ in $H^1(S,\mathcal{O}_S)$. We want to denote this isomorphism by

$$j : H^1(S,\mathbb{Z}) \tilde{\to} \Gamma.$$

Of course it is clear the the multiplication of line bundles in $\mathrm{Pic}^0(S)$ induces the addition on $H^1(S,\mathcal{O}_S)/\Gamma$ and hence we see that the quotient

$$J = \mathrm{Pic}^0(S) = H^1(S,\mathcal{O}_S)/\Gamma$$

has a natural structure of a connected, compact complex-analytic group of dimenson g. Such a group is called a *complex torus.*

1.6. The classical version of Abel's Theorem

In the previous section we described the group of line bundles Pic^0 in terms of the cohomology group $H^1(S,\mathcal{O}_S^*)$. Our main tool was the exact sequence

$$0 \to H^1(S,\mathbb{Z}) \to H^1(S,\mathcal{O}_S) \to H^1(S,\mathcal{O}_S^*) \to H^2(S,\mathbb{Z})$$

which allowed us to define the degree of the line bundle and gave us the description

$$\mathrm{Pic}^0(S) = H^1(S,\mathcal{O}_S)/H^1(S,\mathbb{Z}).$$

Now we recall (see 1.3.)that the group of line bundles may also be described as the group of divisor classes

$$\mathrm{Div}(S)/\text{ principal divisors } \tilde{\to} \mathrm{Pic}(S).$$

We have seen that the degree of the line bundle $\delta(\mathcal{O}_S(D)) = \deg(\mathcal{O}_S(D)) = \deg(D) = \sum n_P$ and by composition we get the isomorphism

$$\mathrm{Div}^0(S)/\text{ principal divisors } \tilde{\to} H^1(S,\mathcal{O}_S)/H^1(S,\mathbb{Z}).$$

We want to compute this isomorphism, i.e. if $\mathcal{O}_S(D) \in \mathrm{Pic}^0(S)$, how can we compute the corresponding element in $H^1(S,\mathcal{O}_S)/H^1(S,\mathbb{Z})$?

We reformulate our problem slightly. The analytic pairing gives us an identification $H^1(S,\mathcal{O}_S) \tilde{\to} H^0(S,\Omega^1_S)^\vee$, the Poincare duality gives an identification $H_1(S,\mathbb{Z}) \tilde{\to} H^1(S,\mathbb{Z})$. The resulting embedding $i_1 : H_1(S,\mathbb{Z}) \hookrightarrow H^0(S,\Omega^1_S)^\vee$ is obtained by the following rule: We represent a homology class $[c]$ by a cycle c and to this class we attach the linear form

$$\varphi_c : \omega \mapsto \int_c \omega.$$

Then the homomorphism $[c] \mapsto \varphi_c$ is our embedding i_1.

Hence our problem is to compute the isomorphism

$$\mathrm{Div}^0(S)/ \text{ principal divisors } \tilde{\to} H^0(S, \Omega^1_S)^\vee / H_1(S, \mathbb{Z}).$$

Let D be a divisor of degree zero. Then we can find a 1-chain $\mathfrak{z}_D$ whose boundary $\partial \mathfrak{z}_D = D$. This 1-chain provides a map

$$\varphi_{D,\mathfrak{z}_D} : H^0(S, \Omega^1_S) \to \mathbb{C}$$
$$\varphi_{D,\mathfrak{z}_D} : \omega \mapsto \int_{\mathfrak{z}_D} \omega.$$

If we have a second 1-cycle $\mathfrak{z}'_D$ which also satisfies $\partial \mathfrak{z}'_D = D$ then $\mathfrak{z}'_D = \mathfrak{z}_D + c_D$ where c_D is a closed one- cycle, i.e. $\partial c_D = 0$. Hence we see that

$$\varphi_{D,\mathfrak{z}_D} - \varphi_{D,\mathfrak{z}'_D} \in H_1(S, \mathbb{Z}) \subset H^0(S, \Omega^1_S)^\vee$$

Hence we see that D defines a well-defined element

$$\tilde{\varphi}_D \in H^0(S, \Omega^1_S)^\vee / H_1(S, \mathbb{Z}).$$

Theorem of Abel: *The isomorphism*

$$\mathrm{Div}^0(S)/ \text{ principal divisors } \tilde{\to} H^0(S, \Omega^1_S)^\vee / H_1(S, \mathbb{Z}).$$

is given by $[D] \mapsto \tilde{\varphi}_D$.

To prove this it suffices to consider the case of two points P, Q on our Riemann surface S which lie in a small disc D_P. This is clear because our map $D \to \varphi_D$ is a homomorphism from the group of divisors of degree zero to $H^0(S, \Omega^1_S)^\vee / H_1(S, \mathbb{Z})$ and these divisors generate the group of divisors of degree zero.

We assume that our local coordinate z is zero at P and 1 at Q. We want to compute the class of the line bundle $O_S(Q - P)$ in $H^1(S, \mathcal{O}_S)/H^1(S, \mathbb{Z})$. To be more precise we want to find a representative of this class in $\Omega^{01}_\infty(S)$ and identify it as a linear form on the space of holomorphic differentials.

We draw the straight path γ from P to Q in our disc and we cover S by $U_2 = S \backslash \gamma$ and $U_1 = D_P$. The meromorphic function $\frac{z}{z-1}$ trivializes our bundle on U_1 and the costant function 1 trivializes it on U_2. Hence the holomorphic function $\frac{z}{z-1}$ on $U_1 \cap U_2 = D \backslash \gamma$ defines a Czech cocycle with values in $\mathcal{O}^*_S$ and its image in $H^1(S, \mathcal{O}^*_S)$ is the class of

$O_S(Q-P)$. I claim that we can define the function $\log\frac{z}{z-1}$ on $D\backslash\{\gamma\}$. This is so because we can write down integrals

$$\int_a^z \frac{dz}{z} \text{ and } \int_a^z \frac{dz}{z-1}$$

along a path from a point a to z which avoids γ. The values of these integrals depend not only on z but also on the homotopy class of the path. But the multivaluedness drops out if we take the difference of the integrals, which then gives us the function $\log\frac{z}{z-1}$. The element $\frac{1}{2\pi i}\log\frac{z}{z-1} \in \mathcal{O}_S(D\backslash\{\gamma\})$ is a 1-cocycle with values in $\mathcal{O}_S$. It defines a class in $\xi_{P,Q} \in H^1(S,\mathcal{O}_S)$ which maps to the class of $\mathcal{O}_S(Q-P)$ in $H^1(S,\mathcal{O}_S^*)$. This class $\xi_{P,Q}$ can be represented by a closed form of type $(0,1)$. To find such a form we shrink the set U_2 a little bit to a set U_2' so that it is the complement of a little neighborhood N of γ. (In the picture below this neigborhood is the "cigar" containing the path from P to Q and U_2' is the complement of the "cigar". This "cigar" is obtained by drawing have circles of radius $\epsilon > 0$ around P, Q and then joining the endpoints by straight lines parallel to γ. The boundary is a $\mathcal{C}_1$ - manifold.)

By the same argument as in 1.4.8 we extend the restriction of $\frac{1}{2\pi i}\log\frac{z}{z-1}$ to $U_2' \cap D_P$ to a $\mathcal{C}_\infty$-function h on D_P and put $\mu = d''h$. This form μ has compact support in U, hence it can be extended to a $(0,1)$–form on S which then represents $\xi_{P,Q} \in H^1(S,\mathcal{O}_S)$ (A special case of the argument in VI.6.1.1). Again we have that the pairing of this class with a holomorphic 1-form ω is given by

$$\int_S \omega \wedge \mu.$$

To compute this integral we observe that $\omega \wedge \mu$ has support in the neighborhood N of γ hence it suffices to integrate over this neighborhood. But now we can write $\omega \wedge \mu = \omega \wedge dh$ and our integral becomes

$$-\int_{\partial N} \omega \wedge h$$

where $h = \frac{1}{2\pi i} \log \frac{z}{z-1}$ on the boundary of N.

Letting this neighborhood shrink to γ the values of $\log \frac{z}{z-1}$ differ by $2\pi i$ on the two sides of our path γ. Hence we get that

$$< \xi_{P,Q}, \omega >= \int_\gamma \omega$$

and this is Abel's theorem in the case that our divisor is $Q - P$, and P, Q close to each other.

This theorem of Abel is the source for the so called self- duality of the Jacobian, which will be discussed in detail in a later section (V.2.3). We pick a point P_0 and consider the morphism

$$i_{P_0} : S \to J$$

which is given by

$$i_{P_0} : P \mapsto (P) - (P_0).$$

This is clearly a holomorphic map and we want to explain how Abels theorem gives us its differential. The tangent space of J at any point is $H^1(S, \mathcal{O}_S)$ and hence we see that the space of holomorphic one forms on J is $H^0(S, \Omega_S)$. Hence i_{P_0} yields a $\mathbb{C}$-linear map between the spaces of holomorphic 1-forms

$$i_{P_0}^* : H^0(J, \Omega_J^1) = H^0(S, \Omega_S^1) \to H^0(S, \Omega_S^1)$$

and I claim that this map must be the identity. If $\omega \in H^0(J, \Omega_J)$ is a holomorphic 1-form and if $X \in T_P$ is a tangent vector at P_0 we compute $i_{P_0}^*(\omega)_P(X)$. We choose a local coordinate z at P, then we may assume that $X = \frac{\partial}{\partial z}$.

The $i_{P_0}^*(P + h \cdot \frac{\partial}{\partial z})$ is the linear form

$$\omega \to \int_{P_0}^{P+h\frac{\partial}{\partial z}} \omega = \int_{P_0}^{P} \omega + \int_{P}^{P+h\frac{\partial}{\partial z}} \omega$$

and this yields

$$i_{P_0}^*(\omega)_P(X) = i_{P_0}^*(\omega)_P(\frac{\partial}{\partial z}) = \omega_P(\frac{\partial}{\partial z})$$

and hence $i_{P_0}^*(\omega) = \omega$.
Hence we see that i_{P_0} gives us an holomorphic embedding of the curve into its Jacobian. This map also induces a homomorphism between the Picard groups

$$i_{P_0}^* : \mathrm{Pic}(J) \to Pic(S).$$

We will define the subgroup $\mathrm{Pic}^0(J)$ (V.2.1.1) and we will prove that the restriction

$${}^t i_{P_0} : \mathrm{Pic}^0(J) \to \mathrm{Pic}^0(S)$$

is an isomorphism. This is the so called self-duality of J.

1.7. Riemann period relations

The cup product $<,>_\cup$ defines a non degenerate alternating pairing on our lattice Γ . On the other hand we have $\Gamma \otimes \mathbb{R} \tilde{\to} H^1(S, \mathcal{O}_S)$ and this identification provides a complex structure I on $\Gamma \otimes \mathbb{R}$, namely the one which is induced by the multiplication by i on $H^1(S, \mathcal{O}_S)$.
We will show that the complex structure I is an isometry for the extension of $<,>_\cup$ to $\Gamma_\mathbb{R}$. Therefore this pairing is the imaginary part of a hermitian form h on $(\Gamma \otimes \mathbb{R}, I)$ (See IV.10.2.1). We compute this form h and in addition it will turn out to be positive definite.
We can define a hermitian scalar product on $\overline{H^0(S, \Omega_S^1)}$. If we have two antiholomorphic forms $\bar{\omega}_1, \bar{\omega}_2 \in \overline{H^0(S, \Omega_S^1)}$ we put

$$h < \bar{\omega}_1, \bar{\omega}_2 >= -2i \int_S \bar{\omega}_1 \wedge \omega_2.$$

If we write locally

$$\bar{\omega}_1 = f_1 d\bar{z}, \quad \bar{\omega}_2 = f_2 d\bar{z}$$

then the integrand becomes

$$f_1(dx - idy) \wedge \bar{f}_2(dx + idy) = 2if_1\bar{f}_2 dx \wedge dy$$

hence we see that h is a positive definite hermitian form.
Now we take to cohomology classes $\xi, \eta \in H^1(S, \mathbb{Z})$. Using the de-Rham isomorphism, we can represent them by differential forms which we can decompose

$$\omega_\xi = \overline{\omega_\xi^{01}} + \omega_\xi^{01}$$
$$\omega_\eta = \overline{\omega_\eta^{01}} + \omega_\eta^{01}.$$

The cup product pairing is given by integrating the representing differential forms

$$< \xi, \eta >_{\mathrm{cup}} = \int_S \omega_\xi \wedge \omega_\eta = -\int_S \omega_\eta^{01} \wedge \overline{\omega_\xi^{01}} + \int_S \omega_\xi^{01} \wedge \overline{\omega_\eta^{01}} = -\operatorname{Im} h(\omega_\xi^{01}, \omega_\eta^{01}).$$

We have the isomorphism

$$j : H^1(S,\mathbb{Z}) \xrightarrow{\sim} \Gamma \subset H^1(S,\mathcal{O}_S) \xrightarrow{\sim} \overline{H^0(S,\Omega^1_S)}$$

and it is clear the the classes $j(\xi)$ (resp.$j(\eta)$) are represented by ω_ξ^{01} (resp. ω_η^{01}). We can transport the cup product pairing via j to Γ then we get the famous

Riemann period relations: *The restriction of the imaginary part of the hermitian form h to Γ is the cup product times -1. Especially we can conclude that the values of* $\operatorname{Im} h$ *on* $\Gamma \times \Gamma$ *are integers and this form is a perfect pairing.*

V.2. Line bundles on complex tori

2.1 Construction of line bundles:

These period relations are of great importance, because they allow the construction of line bundles on J. The positivity of the form h will ensure that these bundles will be *ample* and this means roughly that high positive powers of this bundle have many sections (See below ???). To explain this construction we consider a more general situation.

Let V be a complex vector space of dimension g and let $\Gamma \subset V$ be a lattice in V, this means that Γ is a free $\mathbb{Z}$-module of rank $2g$ which sits in V as a discrete submodule. The quotient $A = V/\Gamma$ is a compact complex analytic variety which also carries the structure of a complex analytic abelian group, it is a complex torus (V.1.2). We have $\Gamma_{\mathbb{R}} \tilde{\to} V$ as real vector space and as usual we denote by I the complex structure on $\Gamma_{\mathbb{R}}$ induced by this isomorphism.

We change our point of view slightly. Our starting point is a free abelian group Γ of rank $2g$ on which we have an alternating 2-form

$$\langle\ ,\ \rangle : \Gamma \times \Gamma \longrightarrow \mathbb{Z}.$$

A second datum is a complex structure $I : \Gamma_{\mathbb{R}} \to \Gamma_{\mathbb{R}}$ which is an isometry for the pairing, i.e. $\langle Ix, Iy\rangle = \langle x, y\rangle$ for all $x, y \in \Gamma_{\mathbb{R}}$. Then we can put $V = (\Gamma_{\mathbb{R}}, I)$ and consider it as a complex vector space and V/Γ is our complex torus. Let H on $V = (\Gamma_{\mathbb{R}}, I)$ be the hermitian form obtained from $(\langle\ ,\ \rangle, I)$ (See IV.10.2.1).

The pairing $\langle\ ,\ \rangle$ allows us to construct certain vector bundles

$$\mathcal{L}(\langle\ ,\ \rangle, \eta, \varphi)$$

which depend on additional data φ and η where

$$\varphi \in \mathrm{Hom}(\Gamma, \mathbb{C})$$

and where η is a map

$$\eta : \Gamma/2\Gamma \to \frac{1}{2}\mathbb{Z}/\mathbb{Z}$$

which satisfies the compatibility relation

$$\frac{1}{2}\langle\gamma_1, \gamma_2\rangle \mod \mathbb{Z} + \eta(\gamma_1 + \gamma_2) - \eta(\gamma_1) - \eta(\gamma_2) = 0 \qquad (***)$$

for all $\gamma_1, \gamma_2 \in \Gamma$. We say that η is *adapted* to the alternating form $<\ ,\ >$.

To define such a bundle we consider an open connected neighborhood U of a point $x \in A$ which is so small that the connected components U_α in the inverse image of U map isomorphically to U under the projection

$$p : V \to A.$$

For any two such components $U_\alpha, U_\beta \subset p^{-1}(U)$ there is exactly one element $\gamma \in \Gamma$ such that $\gamma + U_\alpha = U_\beta$. We define a sheaf $\mathcal{L}_H(\langle\ ,\ \rangle, \eta, \varphi) = \mathcal{L}(\langle\ ,\ \rangle, \eta, \varphi)$ whose sections over U are the holomorphic functions

$$f : p^{-1}(U) \to \mathbb{C}$$

which satisfy the transformation rule

$$f(z+\gamma) = f(z) e^{\pi(H(z,\gamma)+\frac{1}{2}H(\gamma,\gamma))+2\pi i(\varphi(\gamma)+\eta(\gamma))}.$$

The reader should notice that $e^{2\pi i \eta(\gamma)}$ is well defined and is equal to ± 1.

I claim that giving such a function is the same as giving a holomorphic function on any of the connected components U_α in $p^{-1}(U)$ and then extending it to the other components by the transformation rule. To see this we have to check consistency which means we have to verify that

$$f(z+\gamma_1+\gamma_2) = f((z+\gamma_1)+\gamma_2).$$

To do this we compute both sides:

$$\begin{aligned} f(z+\gamma_1+\gamma_2) &= f(z) e^{\pi(H(z,\gamma_1+\gamma_2)+\frac{1}{2}H(\gamma_1+\gamma_2,\gamma_1+\gamma_2))+2\pi i(\varphi(\gamma_1+\gamma_2)+\eta(\gamma_1+\gamma_2))} \\ &= f(z) e^{\pi(H(z,\gamma_1)+H(z,\gamma_2)+\frac{1}{2}(H(\gamma_1,\gamma_1)+H(\gamma_2,\gamma_2)+H(\gamma_1,\gamma_2)+H(\gamma_2,\gamma_1)))+2\pi i(\varphi(\gamma_1)+\varphi(\gamma_2)+\eta(\gamma_1+\gamma_2))}. \end{aligned}$$

For the other side we get

$$\begin{aligned} f((z+\gamma_1)+\gamma_2) &= f(z+\gamma_1) e^{\pi(H(z+\gamma_1,\gamma_2)+\frac{1}{2}H(\gamma_2,\gamma_2))+2\pi i(\varphi(\gamma_2)+\eta(\gamma_2))} \\ &= f(z) \cdot e^{\pi(H(z,\gamma_1)+\frac{1}{2}H(\gamma_1,\gamma_1)+H(z,\gamma_2)+H(\gamma_1,\gamma_2)+\frac{1}{2}H(\gamma_2,\gamma_2))+2\pi i(\varphi(\gamma_1)+\varphi(\gamma_2)+\eta(\gamma_1)+\eta(\gamma_2))}. \end{aligned}$$

The exponential factors are equal because their quotient is

$$\begin{aligned} & e^{\pi(\frac{1}{2}(H(\gamma_2,\gamma_1)-H(\gamma_1,\gamma_2)))+2\pi i(\eta(\gamma_1+\gamma_2)-\eta(\gamma_1)-\eta(\gamma_2))} \\ &= e^{2\pi i(\frac{1}{2}\operatorname{Im} H(\gamma_2,\gamma_1)+\eta(\gamma_1+\gamma_2)-\eta(\gamma_1)-\eta(\gamma_2))}. \end{aligned}$$

and this is equal to 1 since we required $(***)$ for η.

If we consider the map

$$\begin{gathered} C_H(\varphi,\eta) : \gamma \to \mathcal{O}_V(V) \\ \gamma \mapsto \pi(H(z,\gamma)+\tfrac{1}{2}H(\gamma,\gamma)) + 2\pi i(\varphi(\gamma)+\eta(\gamma)) \end{gathered}$$

as a map from the group Γ to the holomorphic functions on V then our computation says that this is a 1-cocycle modulo the constant functions with values in $2\pi i\mathbb{Z}$.

Hence we see that

$$\mathcal{L}_H(\langle\ ,\ \rangle, \eta, \varphi)(U) \sim \mathcal{O}(U_\alpha)$$

for any component in $U_\alpha \subset p^{-1}(U)$. and this means that $\mathcal{L}_H(\langle\ ,\ \rangle, \eta, \varphi)$ is a line bundle. Now we see why the integrality of $\operatorname{Im} H$ on $\Gamma \times \Gamma$ is so important.

The data H and $(< \, , \, >, I)$ determine eachother, therefore we may suppress the subscript H in the notation. On the other hand it follows from IV.10.2.1 that the pair of $\mathbb{R}$-bilinear forms $< \, , \, >, H$ determines the complex structure I which is not directly visible in the definition of the line bundle. Hence it may be sometimes useful to keep the H.

In IV.6.6.2 we have shown that the second cohomology group (See ???)

$$H^2(A, \mathbb{Z}) = H^2(V/\Gamma) = \mathrm{Hom}(\Lambda^2\Gamma, \mathbb{Z}).$$

We have the exact sequence

$$0 \to H^1(A, \mathbb{Z}) \to H^1(A, \mathcal{O}_A) \to H^1(A, \mathcal{O}_A^*) \to H^2(A, \mathbb{Z})$$

and we leave it as an exercise to the reader to verify that

$$c_1(\mathcal{L}_H(\langle \, , \, \rangle, \eta, \varphi)) = \mathrm{Im}\, H|\Gamma \times \Gamma.$$

It is not too difficult to show that we can find an η for a given $\langle \, , \, \rangle$. It is not unique, but it is not hard to see that for two choices η_H, η'_H we can find a homomorphism $\delta : \Gamma \to \mathbb{C}$ so that $\delta(\Gamma) \subset \frac{1}{2}\mathbb{Z}$ and $\delta(\gamma) = \eta_H(\gamma) - \eta'_H(\gamma) \in \mathbb{Z}$. Then it is clear that $\mathcal{L}_H(\langle \, , \, \rangle, \eta, \varphi) \tilde{\to} \mathcal{L}_H(\langle \, , \, \rangle, \eta', \varphi + \delta)$

We want to show that this construction gives us all line bundles on A. In any case it is clear that the group of Chern classes of line bundles is the kernel of the homomorphism

$$H^2(A, \mathbb{Z}) \longrightarrow H^2(A, \mathcal{O}_A).$$

We have seen that $H^2(A, \mathbb{Z}) = \mathrm{Hom}(\Lambda^2, \mathbb{Z})$ and it is an easy exercise in linear algebra that an element

$$c \in H^2(A, \mathbb{Z}) = \mathrm{Hom}(\Lambda^2\Gamma, \mathbb{Z})$$

goes to zero in $H^2(A, \mathcal{O}_A)$ if and only if the extension

$$c_{\mathbb{R}} : (\Gamma \otimes \mathbb{R}) \wedge (\Gamma \otimes \mathbb{R}) \to \mathbb{R}$$

satisfies $c_{\mathbb{R}}(Ix, Iy) = c_{\mathbb{R}}(x, y)$, i.e. the complex structure is an isometry. But for those classes c we gave an explicit construction of line bundles with Chern classe c. We can take any $\mathcal{L}(c, \eta, \varphi)$.

2.1.1 *This means that our construction at least provides enough line bundles to fill up the module of possible Chern classes of line bundles.*

It is of course clear that these alternating forms c which satisfy $c_{\mathbb{R}}(Ix, Iy) = c_{\mathbb{R}}(x, y)$ form a finitely generated subgroup $NS(A)$ of $\mathrm{Hom}(\Lambda^2\Gamma, \mathbb{Z})$. This group is called the *Neron-Severi group.*

We should be aware that this group $NS(A)$ can be trivial, actually this is the case for a generic choice of the complex structure on $\Gamma_{\mathbb{R}}$.

To get the group of all line bundles we return to its description as $H^1(A, \mathcal{O}_A^*)$ and we define

$$\mathrm{Pic}^0(A) = \ker(\delta : \mathrm{Pic}(A) \to H^2(A, \mathbb{Z}))$$

From our familiar exact sequence we get

$$\mathrm{Pic}^0(A) = H^1(A, \mathcal{O}_A)/H^1(A, \mathbb{Z}).$$

Again we get from IV.6.6.2 that $H^1(A, \mathbb{Z}) = H^1(\mathbb{C}^g/\Gamma; \mathbb{Z}) = \mathrm{Hom}(\Gamma, \mathbb{Z})$. To compute $H^1(A, \mathcal{O}_A)$we consider the Dolbeault complex

$$0 \to \Omega^0_\infty(\mathcal{O}_A)(A) \to \Omega^{01}_\infty(\mathcal{O}_A)(A) \to \Omega^{02}_\infty(\mathcal{O}_A)(A) \to$$

The tangent bundle of A is trivial. Using the translations we can identify the tangent space at any point to $T_{A,0} \simeq V$ the tangent space at zero.
Hence the bundle of differentials is also trivial and at any point

$$\Omega^1_{A,x} = \mathrm{Hom}_{\mathbb{C}}(V, \mathbb{C}).$$

It is clear that the bundle $\Omega^{01}_{A,}$ is also trivial and if we give a basis to V and write $z = (\ldots, z_\alpha \ldots) \in \mathbb{C}^g = V$ then the global sections $\Omega^{01}_\infty(A)$ are given by

$$\omega = \sum f_\alpha d\bar{z}_\alpha$$

where f_α is a $\mathcal{C}_\alpha$-function on A. We apply the principles of Hodge theory: we choose a positive definite hermitian form on the tangent bundle, which we get from a hermitian form on $V = T_{A,0}$. We choose it in such a way that the basis vectors above form an orthonormal basis. Then it is an easy computation to show that

$$\Delta''\omega = (d''\delta'' + \delta''d'')\omega = \left(\sum_\beta \frac{\partial^2 f_\alpha}{\partial z_\beta \partial \bar{z}_\beta}\right) d\bar{z}_\alpha$$

and in IV.10.3 we proved that

$$\mathbb{H}^{01}(\Omega^{01}_\infty(A)) = \{\sum_\alpha c_\alpha d\bar{z}_\alpha | c_\alpha \in \mathbb{C}\}.$$

and that we get an isomorphism

$$\{\sum c_\alpha d\bar{z}_\alpha | c_\alpha \in \mathbb{C}\} \xrightarrow{\sim} H^1(A, \mathcal{O}_A).$$

It does not depend on the metric, it is induced from the embedding of the translation invariant differential form into the space of all differential forms.

The space on the left is canonically isomorphic to $\mathrm{Hom}_{\mathbb{C}-\mathrm{antilin}}(V,\mathbb{C})$ (See IV9.2.2) and we obtain for our complex torus $A = V/\Gamma$

$$\mathrm{Hom}_{\mathbb{C}-\mathrm{antilin}}(V,\mathbb{C}) = \mathrm{Hom}_{\mathbb{C}}(\bar{V},\mathbb{C}) = \bar{V}^{\vee} = H^1(A,\mathcal{O}_A)$$

To get the group $\mathrm{Pic}^0(A)$ we have to divide by the subgroup $H^1(A,\mathbb{Z})$ and we have to remind ourselves how this group sits inside $\bar{V}^{\vee} = H^1(A,\mathcal{O}_A)$. This is clear: We know that $H^1(A,\mathbb{Z}) = \mathrm{Hom}(\Gamma,\mathbb{Z})$ and

$$\mathrm{Hom}(\Gamma,\mathbb{Z}) \subset \mathrm{Hom}(\Gamma,\mathbb{C}) = \mathrm{Hom}(\Gamma\otimes_{\mathbb{Z}}\mathbb{C},\mathbb{C})$$

and by construction

$$\Gamma\otimes_{\mathbb{Z}}\mathbb{C} = V\oplus\bar{V} = V^{10}\oplus V^{01}$$

Now the embedding $H^1(A,\mathbb{Z}) \hookrightarrow H^1(A,\mathcal{O}_A)$ is given by extending $\phi \in \mathrm{Hom}(\Gamma,\mathbb{Z})$ to an element in $\mathrm{Hom}(\Gamma\otimes_{\mathbb{Z}}\mathbb{C},\mathbb{C})$ and then restricting it to the summand $\bar{V} = V^{01}$. Hence we get an isomorphism

$$c: \mathrm{Pic}^0(A) \xrightarrow{\sim} \mathrm{Hom}(V^{01},\mathbb{C})/\mathrm{Hom}(\Gamma,\mathbb{Z}) \xrightarrow{\sim} \mathrm{Hom}(\Gamma,\mathbb{C})/(\mathrm{Hom}_{\mathbb{C}}(V,\mathbb{C}) + \mathrm{Hom}(\Gamma,\mathbb{Z})).$$

We want to invert this isomorphism. We constructed the line bundles $\mathcal{L}_H(\langle\ ,\ \rangle,\eta,\varphi)$ where $\varphi\in\mathrm{Hom}(\Gamma,\mathbb{C})$. We denote the restrictions of φ to V and $\bar{V}$ respectively by $\varphi_{10},\varphi_{01}$ and hence $\varphi = (\varphi_{10},\varphi_{01})$. It is clear from the construction that

2.1.2 Lemma : *a) The two line bundles $\mathcal{L}_H(\langle\ ,\ \rangle,\eta,\varphi)$ and $\mathcal{L}_H(\langle\ ,\ \rangle,\eta,\varphi)$ are isomorphic if $\varphi-\varphi' = (\psi,0)$.*
b) The bundles $\mathcal{L}_H(\langle\ ,\ \rangle,\eta,\varphi)$ and $\mathcal{L}_H(\langle\ ,\ \rangle,\eta,\varphi')$ are actually the same line bundles if

$$\varphi-\varphi' \in \mathrm{Hom}(\Gamma,\mathbb{Z})$$

Proof: To see a) we observe that $e^{2\pi i\psi(z)}$ is holomorphic on V and multiplication by this function provides an isomorphism between $\mathcal{L}_H(\langle\ ,\ \rangle,\eta,\varphi)$ and $\mathcal{L}_H(\langle\ ,\ \rangle,\eta,\varphi)$.
The assertion b) is obvious because $e^{2\pi i\varphi(\gamma)} = e^{2\pi i\varphi'(\gamma)}$ for all $\gamma\in\Gamma$.

If now the alternating form $\langle\ ,\ \rangle = O$ is the trivial nullform then we choose $\eta_O = 0$. We find

$$\mathcal{L}(O,0,\varphi)\otimes\mathcal{L}(O,0,\varphi') = \mathcal{L}(O,0,\varphi+\varphi'),$$

our construction of line bundles yields a homomorphism

$$\mathrm{Hom}(\Gamma,\mathbb{C}) \to \mathrm{Pic}^0(A).$$

which by the previous Lemma factors through $\mathrm{Hom}_{\mathbb{C}}(V,\mathbb{C}) + \mathrm{Hom}(\Gamma,\mathbb{Z})$. Hence our construcion yields a homomorphism

$$d : \mathrm{Hom}(\Gamma,\mathbb{C})/(\mathrm{Hom}_{\mathbb{C}}(V,\mathbb{C}) + \mathrm{Hom}(\Gamma,\mathbb{Z})) \to \mathrm{Pic}^0(A).$$

I leave it as an exercise to show

Proposition 2.1.3 : *These two homomorphisms c, d are inverse to each other.*

Corollary 2.1.4: *If $A = V/\Gamma$ is a complex torus then the group $\mathrm{Pic}^0(A)$ has again the structure of a complex torus and is canonically isomorphic to $\bar{V}^\vee/\Gamma^\vee$.*

This torus is called the *dual torus* and denoted by $A^\vee$.

Our considerations also imply that the bundles with a given Chern class form a principal homogeneous space under

$$\mathrm{Hom}_{\mathbb{C}}(\Gamma,\mathbb{C})/(\mathrm{Hom}_{\mathbb{C}}(V,\mathbb{C}) \oplus \mathrm{Hom}(\Gamma,\mathbb{Z})).$$

But this description requires a choice of an η adapted to $<\ ,\ >$. We have seen that changing η can be corrected by the modification of the linear form φ.

Now it is clear that all line bundles $\mathcal{L}$ on A are of the form $\mathcal{L}_H(\langle\ ,\ \rangle, \eta, \varphi)$.

2.1.5 We want to construct *families of line bundles.* We have the isomorphism

$$\mathrm{Pic}^0(A) \xrightarrow{\sim} \mathrm{Hom}(\Gamma^{01}_{\mathbb{C}},\mathbb{C})/\Gamma^\vee.$$

We consider the line bundles

$$\mathcal{L}(\langle\ ,\ \rangle, \eta, (0,\psi))$$

where $\psi \in \mathrm{Hom}(\Gamma^{01}_{\mathbb{C}},\mathbb{C})$. It is clear that these line bundles can be seen as members of a family of line bundles on $A \times \mathrm{Hom}(\Gamma^{01}_{\mathbb{C}},\mathbb{C})$: We consider the line bundle $\tilde{\mathcal{L}}(\langle\ ,\ \rangle, \eta)$ on $A \times \mathrm{Hom}(\Gamma^{01}_{\mathbb{C}},\mathbb{C})$ whose sections over an open set

$$U \times \mathrm{Hom}(\Gamma^{01}_{\mathbb{C}},\mathbb{C})$$

are given by holomorphic functions

$$h : \pi^{-1}(U) \times \mathrm{Hom}(\Gamma^{01}_{\mathbb{C}},\mathbb{C}) \longrightarrow \mathbb{C}$$

which satisfy

$$h(u+\gamma,\psi) = e^{\pi(H(u,\gamma)+\frac{1}{2}H(\gamma,\gamma))+2\pi i(\psi(\gamma)+\eta(\gamma))}h(u,\psi).$$

Let

$$p_2 : A \times \mathrm{Hom}(\Gamma^{01}_{\mathbb{C}},\mathbb{C})/\Gamma^\vee = A \times A^\vee$$

be the projection on the second factor, we want to "push" this bundle down to a bundle $\mathcal{P}(\langle\ ,\ \rangle,\eta)$ on the quotient, i.e. we want that

$$p_2^*(\mathcal{P}(\langle\ ,\ \rangle,\eta))\tilde{\to}\tilde{\mathcal{L}}(\langle\ ,\ \rangle,\eta)$$

To see how this can be done we have to find out what happens if we modify ψ by an element λ_{01} in $\Gamma^\vee$. Such an element is the image of an element $\lambda \in \mathrm{Hom}(\Gamma,\mathbb{Z})$ under the restriction map $\mathrm{Hom}(\Gamma_{\mathbb{C}},\mathbb{C}) \to \mathrm{Hom}(\Gamma_{\mathbb{C}}^{01},\mathbb{C})$. We can write

$$\lambda = \lambda_{10} + \lambda_{01},$$

and we saw in Lemma 2.1.2 that

$$\mathcal{L}(\langle\ ,\ \rangle,\eta,(0,\psi)) = \mathcal{L}(\langle\ ,\ \rangle,\eta,(0,\psi)+\lambda)$$

In the same lemma above we showed that we can write an explicit isomorphism

$$\psi_\lambda : \mathcal{L}(\langle\ ,\ \rangle,\eta,(\lambda_{10},\psi+\lambda_{01})) \xrightarrow{\sim} \mathcal{L}(\langle\ ,\ \rangle,\eta,(0,\psi+\lambda_{01}))$$

which on the sections was given by multiplication by $e^{-2\pi i\lambda_{10}(z)}$. Now it is clear that these isomorphisms are consistent. If we consider a sum $\lambda'+\lambda''=\mu$, then $\psi_\mu = \psi_{\lambda'}\circ\psi_{\lambda''}$. This allows to "push" down the bundle. We can define the bundles

$$\mathcal{P}(\langle\ ,\ \rangle,\eta,y)$$

where now $y \in A^\vee = \mathrm{Hom}(\Gamma_{\mathbb{C}}^{01},\mathbb{C})/\Gamma^\vee$ and they form a family $\mathcal{P}(<\ ,\ >,\eta)$ on $A\times A^\vee$.

2.2 Homomorphisms between complex tori.

If we have two such tori

$$V_1/\Gamma_1 = A_1, V_2/\Gamma_2 = A_2$$

then an analytic homomorphism $\phi : A_1 \to A_2$ is of course the same thing as a $\mathbb{C}$-linear map $\phi : V_1 \to V_2$ which maps the lattice Γ_1 into Γ_2. We may also view ϕ as an element $\phi : \Gamma_1 \to \Gamma_2$ which after extension to a linear map $\Gamma_1\otimes\mathbb{R} \to \Gamma_2\otimes\mathbb{R}$ respects the complex structures on $\Gamma_1\otimes\mathbb{R} = V_1, \Gamma_2\otimes\mathbb{R} = V_2$.

We summarize:

2.2.1 *The module* $\mathrm{Hom}(A_1,A_2)$ *is a submodule of* $\mathrm{Hom}(\Gamma_1,\Gamma_2)$. *It consists of those elements which after extension to* $\mathbb{R}$ *commute with the complex structures.*

A homomorphism $\varphi : A_1 \to A_2$ also induces a homomorphism between the Picard groups

$$\varphi^* : \mathrm{Pic}(A_2) \longrightarrow \mathrm{Pic}(A_1)$$

which is induced by the pull back of line bundles.

We can restrict this homomorphism to the groups $\mathrm{Pic}^0(A_2) = A_2^\vee$ and $\mathrm{Pic}^0(A_1) = A_1^\vee$ and denote this restriction by

$$\varphi^\vee : A_2^\vee \longrightarrow A_1^\vee.$$

A priori this is an homomorphism between abstract groups but from the explicit description of the isomorphism $\mathrm{Pic}^0(A_i) \xrightarrow{\sim} A_i^\vee$ it becomes clear:

2.2.1.1. Proposition: The element $\varphi^\vee$ is a homomorphism of complex tori. This homomorphism– viewed as an element in $\mathrm{Hom}(\Gamma_2^\vee, \Gamma_1^\vee)$- is simply the adjoint of the element $\varphi \in \mathrm{Hom}(\Gamma_1, \Gamma_2)$. Especially we see that the function $\phi \mapsto \phi^\vee$ is additve.

To see that this is true, we consider an element $x \in \mathrm{Pic}^0(A_2)$. We gave an explicit construction of a line bundle $\mathcal{L}_x$ corresponding to x starting from a linear map

$$\lambda_x : \Gamma_2 \longrightarrow \mathbb{C},$$

which after extension to $\Gamma_2 \otimes \mathbb{C}$ and restriction to $\bar{V}_2$ maps to x:
For an open set $V \subset A_2$ the space of sections

$$\mathcal{L}_x(V) = \left\{ f : \pi^{-1}(V) \to \mathbb{C} \mid \ f \text{ is holomorphic and } f(z+\gamma) = f(z)e^{2\pi i \lambda_x(\gamma)} \right\}$$

and the fibre of $\mathcal{L}_x$ in a point $y \in H^1(S, \mathcal{O}_S)/\Gamma$ is given by

$$(\mathcal{L}_x)_y = \left\{ f : \pi^{-1}(y) \to \mathbb{C} \mid f(y+\gamma) = f(y)e^{2\pi i \lambda_x(\gamma)} \right\}.$$

If now $\varphi : A_1 \to A_2$ and if $y_1 \in A_1$, then

$$\varphi^*(\mathcal{L}_x)_{y_1} = (\mathcal{L}_x)_{\varphi(y_1)}.$$

If we consider the diagram

$$\begin{array}{ccccc} \Gamma_1 & \xrightarrow{\varphi} & \Gamma_2 & \xrightarrow{\lambda_x} & \mathbb{C} \\ \downarrow & & \downarrow & & \\ V_1 & \xrightarrow{\varphi} & V_2 & & \end{array}$$

then we see that $\varphi^*(\mathcal{L})$ is the line bundle defined by the composition

$$\lambda_x \circ \varphi : \Gamma_1 \longrightarrow \mathbb{C}.$$

and this proves the desired formula.

We may also consider the induced map

$$\overline{\varphi}^* : NS(A_2) \longrightarrow NS(A_1).$$

An element $e \in NS(A_2)$ is an alternating form $e : \Gamma_2 \times \Gamma_2 \to \mathbb{Z}$ and $\varphi^*(e)$ is simply the form on $\Gamma_1 \times \Gamma_1$ induced by φ, i.e.

$$\varphi^*(e)\langle \gamma_1, \gamma_1' \rangle = e\langle \varphi(\gamma_1), \varphi(\gamma_2) \rangle.$$

This implies of course that the function $\phi \mapsto \bar{\phi}^*$ is not linear, it is in fact quadratic which means that we have

$$\overline{\phi + \psi}^* = \bar{\phi}^* + \bar{\psi}^* + < \phi, \psi > \qquad 2.2.1.2$$

where $(\phi, \psi) \to < \phi, \psi >$ *is bilinear.*

2.2.2 We come to another interpretation of the Neron-Severi group. If we pick an element $e \in NS(A)$, then this element defines a homomorphism

$$\begin{array}{rcccl} \phi_e & : & \Gamma & \longrightarrow & \Gamma^\vee \\ \phi_e & : & \gamma & \longrightarrow & \{\gamma' \to e\langle \gamma, \gamma' \rangle\}. \end{array}$$

It is clear that the condition that I is an isometry for the extension $e_\mathbb{R}$ implies that it extends to a $\mathbb{C}$ linear homomorphism

$$\psi_e : V \to \bar{V}^\vee$$

We have the inclusions $\Gamma \subset V$ and $\Gamma^\vee \subset \bar{V}^\vee$ and it is clear that ψ_e maps Γ into $\Gamma^\vee$ and induces ϕ_e on the lattices.
Therefore we see that we have a canonical homomorphism

$$\Phi : NS(A) \longrightarrow \mathrm{Hom}(A, A^\vee).$$

Any element $\phi : \Gamma \to \Gamma^\vee$ has a transpose

$$\phi^\vee : \Gamma^{\vee\vee} = \Gamma \to \Gamma^\vee.$$

We can define the alternating elements $\mathrm{Hom}_{alt}(A, A^\vee)$ to be the elements which satisfy $\phi^\vee = -\phi$ and it is an easy exercise in linear algebra to show that our above map Φ provides an isomorphism

$$\Phi : NS(A) \longrightarrow \mathrm{Hom}_{alt}(A, A^\vee). \qquad (2.2.2.1)$$

The inverse of this homomorphism is given by the map that sends an alternating element ϕ to the form

$$e_\phi < \gamma, \gamma' >= \phi(\gamma)(\gamma').$$

(I hope that I got the sign right).

2.2.3 We also have a homomorphism $\Psi : \mathrm{Hom}(A, A^\vee) \to NS(A)$.
To construct this homomorphism we start from the alternating form

$$\tau : (\Gamma \oplus \Gamma^\vee) \times (\Gamma \oplus \Gamma^\vee) \to \mathbb{Z},$$

which is defined by

$$\tau < (\gamma_1, \psi_1), (\gamma_2, \psi_2) >= \psi_2(\gamma_1) - \psi_1(\gamma_2).$$

If now $\phi : \Gamma \to \Gamma^\vee$ then this yields the homomorphism

$$\mathrm{Id} \times \phi : \Gamma \to \Gamma \oplus \Gamma^\vee$$

and we can restrict τ via this inclusion to Γ. The resulting form

$$< \gamma_1, \gamma_2 >_\phi = \phi(\gamma_1)(\gamma_2) - \phi(\gamma_2)(\gamma_1) = e_\phi < \gamma_1, \gamma_2 > - e_{\phi^\vee} < \gamma_1, \gamma_2 >$$

on Γ is alternating. It depends only on the alternating component of ϕ and for alternating ϕ the map $\phi \mapsto < \ , \ >_\phi$ is twice the inverse of Φ.

We give a different construction of the homomorphism Φ which works with the bundles themselves rather than with their Chern classes.

To our element $e \in NS(A)$ we choose a line bundle $\mathcal{L}$ with $c_1(\mathcal{L}) = e$, in other words we choose an adapted η and a $\varphi : \Gamma \to \mathbb{C}$ and consider the line bundle

$$\mathcal{L} = \mathcal{L}(e, \eta, \varphi).$$

Any element $x \in A$ induces a translation $T_x : y \mapsto x + y$ on A and we can consider the line bundle $T_x^*(\mathcal{L}) \otimes \mathcal{L}^{-1}$. To compute this line bundle we choose an element $\tilde{x}$ in the fibre $p^{-1}(x)$. Let H_e the attached hermitian form then the fibre of $T_x^*(\mathcal{L})$ at a point z is equal to the fibre of $\mathcal{L}$ at $x + z$ and therefore it is given by the functions which satisfy

$$f(\tilde{z} + \tilde{x} + \gamma) = f(\tilde{z} + \tilde{x}) e^{\pi(H_e(\tilde{z}+\tilde{x},\gamma) + \frac{1}{2} H_e(\gamma,\gamma)) + 2\pi i(\varphi(\gamma) + \eta_{H_e}(\gamma))}$$

for all $\tilde{z} \in p^{-1}(z)$.
Comparing this to the fibre of $\mathcal{L}$ at z yields that the fibre of $T_x^*(\mathcal{L}) \otimes \mathcal{L}^{-1}$ is given by the functions

$$f(\tilde{z} + \gamma) = f(\tilde{z}) e^{\pi H_e(\tilde{x},\gamma)}$$

This line bundle is obtained from the linear form $\varphi_{\tilde{x}} : \gamma \mapsto H_e(\tilde{x}, \gamma)$, in other words it is isomorphic to $\mathcal{L}_O(\varphi_{\tilde{x}})$. An easy calculation shows that this linear form is of the type $(0, \varphi_{10})$ in other words it is trivial on the first component in the decomposition

$$\mathrm{Hom}_{\mathbb{C}}(\Gamma \otimes \mathbb{C}, \mathbb{C}) = \mathrm{Hom}_{\mathbb{C}}(V \oplus \bar{V}, \mathbb{C})$$

The same calculation shows that the linear form $\psi_{\tilde{x}} : \gamma \mapsto H(\gamma, \tilde{x})$ is of type $(\psi_{01}, 0)$. Therefore we do not change the isomorphism class of the line bundle if we replace $\varphi_{\tilde{x}}$ by $\phi_{\tilde{x}}(\gamma) = H(\tilde{x}, \gamma) - H(\gamma, \tilde{x}) = 2i \operatorname{Im} H_e(\tilde{x}, \gamma) = 2ie < \tilde{x}, \gamma >$. Hence we see that $T_x^*(\mathcal{L}) \otimes \mathcal{L}^{-1} \xrightarrow{\sim} \mathcal{L}(O, 0, e < \tilde{x}, \quad >)$ where $e < \tilde{x}, \quad >$ is a linear map from Γ to $\mathbb{C}$.

Therefore it is clear that

2.2.4 *The map*

$$x \mapsto T_x^*(\mathcal{L}) \otimes \mathcal{L}^{-1}$$

from A to $A^\vee$ is a homomorphism and this homomorphism is equal to $\Phi(e)$.

This new description of Φ has the advantage that it is constructed in terms of the bundles rather than in terms of the Chern classes. It is of course important that this homomorphism depends only on the Chern class of the line bundle $\mathcal{L}$.

We have also a description of $2\times$ the inverse of this homomorphism in terms of line bundles. We observe that the alternating form τ on $\Gamma \oplus \Gamma^\vee$ defines an element in $NS(A \times A^\vee)$ in other words the complex structure on $(\Gamma \oplus \Gamma^\vee)_\mathbb{R}$ is an isometry. Hence we can construct a line bundle $\mathcal{N}$ on $A \times A^\vee$ whose Chern class is τ. If now $\phi : A \to A^\vee$ we get a homomorphism $\mathrm{Id} \times \phi : A \to A \times A^\vee$. The pullback $(\mathrm{Id} \times \phi)^*(\mathcal{N})$ gives us a line bundle $\mathcal{N}_\phi$ and $\Phi(\mathcal{N}_\phi) = 2\phi$.

2.2.5 If an element $c = c_1(\mathcal{L})$ of the Neron-Severi group is called *rationally non degenerate* if the alternating pairing $c_\mathbb{Q} : \Gamma_\mathbb{Q} \times \Gamma_\mathbb{Q} \to \mathbb{Q}$ is non degenerate then it is clear that the induced homomorphism $\phi_c : \Gamma \to \Gamma^\vee$ is injective and that the image $\phi_c(\Gamma) \subset \Gamma^\vee$ has finite index. From our description of the complex tori it is immedeately clear that the kernel of

$$x \mapsto T_x^*(\mathcal{L}) \otimes \mathcal{L}^{-1}$$

is canonically isomorphic to $\Gamma^\vee/\phi_c(\Gamma)$ we have an isomorphism

$$\ker(\phi_c) \xrightarrow{\sim} \Gamma^\vee/\phi_c(\Gamma).$$

2.3. The self duality of the Jacobian.

We specialize these considerations to the Jacobian J of our Riemann surface S. I recall

$$J = H^1(S, \mathcal{O}_S)/\Gamma = \overline{H^0(S, \Omega_S^1)}/\Gamma$$

where we identify $H^1(S, \mathcal{O}_S) = \overline{H^0(S, \Omega_S^1)}$ by means of the Dolbeault isomorphism and Hodge theory. The submodule Γ is the image of $H^1(S, \mathbb{Z})$ under the homomorphism $j : H^1(S, \mathbb{Z}) \to \Gamma$ (see 1.5).

On this module Γ we have the priviledged alternating form given by the cup product $e_0 : \Gamma \times \Gamma \to \mathbb{Z}$. It provides an isomorphism

$$\varphi_{e_0} : \Gamma \longrightarrow \Gamma^\vee.$$

We need to know that this isomorphism is compatible with the complex structures. This follows from the Riemann period relations (see 1.7) they imply that e_0 is the imaginary part of a hermitian form

$$h_{e_0} : \overline{H^0(S, \Omega_S^1)} \times \overline{H^0(S, \Omega_S^1)} \longrightarrow \mathbb{C}.$$

which then gives an isomorphism of the complex vector spaces - also denoted by φ_{e_0} - namely

$$\varphi_{e_0} : \overline{H^0(S.\Omega^1_S)} \longrightarrow H^0(S,\Omega^1_S)^\vee$$

and this alltogether gives us an isomorphism

$$\begin{array}{cccccc} \Phi(e_0) & : & J & \longrightarrow & J^\vee \\ & & \| & & \| \\ & & H^1(S,\mathcal{O}_S)/\Gamma & \longrightarrow & H^0(S,\Omega^1_S)^\vee/\Gamma^\vee \end{array}$$

Our isomorphism $\varphi_{e_)}$ is of course the composition of the Dolbeault isomorphism and the analytic duality (See 1.4.8):

$$H^1(S,\mathcal{O}_S) \xrightarrow{\sim} \overline{H^0(S,\Omega^1_S)} \xrightarrow{\sim} H^0(S,\Omega^1_S)^\vee$$

The isomorphism - or what is the same - the class $e_0 \in NS(J)$ is called the *canonical polarization* of J. It is an additional datum attached to the complex torus.

At the end of the discussion of Abel's theorem we discussed the embedding

$$i_{P_0} : S \longrightarrow J$$

which provided a homomorphism

$$\begin{array}{ccccc} {}^t i_{P_0} & : & \mathrm{Pic}^0(J) & \longrightarrow & \mathrm{Pic}^0(S) \\ & & \| & & \| \\ & & J^\vee & \longrightarrow & J \end{array}$$

and now it is clear from these computations that ${}^t i_{P_0}$ is the inverse of the canonical polarization.

2.4 Ample line bundles and the algebraicity of the Jacobian. Let us assume that we have an alternating form $e =<\ ,\ >$ for which I is an isometry. So far it did not play any role that the hermitian form H attached to this form e was positive definite. We want to discuss the implication of te positivity and we will see that it implies that sufficiently high powers of this bundle will have many sections.

I refer to the section on Kähler manifolds (See IV.10.2.2) There we attached a 2-form ω_h to any (positive definite) hermitian form h on the tangent bundle. In our case here the tangent bundle of $A = V/\Gamma$ is trivial and isomorphic to $\Gamma \otimes \mathbb{R}$ at the origin. Then our 2-form ω_h on A is invariant by translation and is our form e at the origin. It is clear that ω_h is closed, it defines a class $[\omega_h] \in H^2(M,\mathbb{Z})$ and of course

$$[\omega_h] = e.$$

If now in addition the hermitian form $H_{<,>}$ is positive definite, then $<,>$ gives us a Kähler metric on $M = V/\Gamma$ whose class is integral.

Now I want to formulate the famous embedding theorem of Kodaira.

2.4.1 Theorem: *Let X be a compact complex manifold. Let us assume that we have a hermitian metric h on T_X whose corresponding class ω_h is closed and defines an integral class in $H^2(X,\mathbb{Z})$. Then we can find a line bundle $\mathcal{L}$ on X whose Chern class $c_1(\mathcal{L}) = [\omega_h]$. For $n >> 0$ we have that $H^q(X,\mathcal{L}^{\otimes n}) = 0$ for all $q > 0$ and we get a projective embedding*

$$\begin{aligned}\Theta_n : X &\longrightarrow \mathbb{P}(H^0(X,\mathcal{L}^{\otimes n}))\\ x &\longrightarrow H_x = \{s \in H^0(X,\mathcal{L}^{\otimes n}) | s(x) = 0\}.\end{aligned}$$

We have a tautological example for this theorem. If our manifold X is the projective space $\mathbb{P}^n(\mathbb{C})$ and the bundle is $\mathcal{L} = \mathcal{O}_{\mathbb{P}^n(\mathbb{C})}(1)$ then we can take $n = 1$ and the bundle provides an embedding and a closer look shows that this embeddeding is simply the identity.

In the general case it is clear from the construction that the restriction by Θ_n of the bundle $\mathcal{O}_{\mathbb{P}(H^0(X,\mathcal{L}^{\otimes n}))}(1)$ to X is our bundle $\mathcal{L}^{\otimes n}$

This theorem also applies to our complex tori, we need that our class $e =< \, , > H = H_{<,>}$ is positive definite. Then any bundle $\mathcal{L} = \mathcal{L}_H(\langle \, , \, \rangle, \eta, \varphi)$ is of the type as in the theorem. If we can find such an alternating form e for which the attached hermitian form is positive definite, then we say that our complex torus is an *abelian variety.*

We will almost prove the above theorem of Kodaira in the special case of abelian varieties. This will be done by showing that the bundles have a lot of sections. After that we will make it more precise what a projective embedding is.

2.4.2 The spaces of sections and Theta functions: We want to compute the space of global sections in our line bundles $\mathcal{L}(\langle \, , \, \rangle, \eta, \varphi)$. To do this we give a different description of these bundles: we modify the cocycle $C_H(z,\omega)$ by a boundary.

To get this modifiction we choose a sublattice $G \subset \Omega$ of rank g such that Ω/G is also free and the alternating form $<,>$ is trivial on G. This is possible because our form is alternating. Then our hermitian form $H_{<,>}$ restricted to G takes real values and is symmetric. Since we have $G \oplus IG = V$ we can extend this restriction to a symmetric $\mathbb{C}$-bilinear form h on V, so we have $H(z,\omega) = h(z,\omega)$ for all $\omega \in G$.

For simplicity I want to assume that the restrictions of η and φ to G are trivial. Actually we can assume this without loss of generality. The function $\eta|G$ satisfies $\eta(e_1 + e_2) = \eta(e_1) + \eta(e_2)$ for $e_1, e_2 \in G$. We can construct a linear form $\varphi' : \Omega \to \frac{1}{2}\mathbb{Z}$ such that $\eta|G = \varphi'|G \bmod 2$. Now we modify η by φ' such that $\eta(\omega) = 0$ for all $\omega \in G$. Once we have done this we also modified φ. Now we can restrict the form $\varphi : \Omega \to \mathbb{C}$ to G and extend this φ to a linear $\mathbb{C}$-form ψ on V. We have seen that $\mathcal{L}(< \, , \, >, \eta, \varphi) \simeq \mathcal{L}(< \, , \, >, 0, \varphi - \psi)$ and hence we may also assume that φ restricted to G is trivial.

We look at our 1- cocycle $\bmod 2\pi i\mathbb{Z}$

$$\omega \mapsto \pi(H(z,\omega) + \frac{1}{2}H(\omega,\omega)) + 2\pi i(\varphi(\omega) + \eta(\omega)) = C_H(z,\omega).$$

This cocycle is uniquely determined by $\Omega, < \, , \, >$ and the complex structure I and η. We change our notation slightly and we denote the resulting bundle by $\mathcal{L}(C_H, \varphi)$.

Now we consider global sections in this bundle and this means that we consider holomorphic functions which which satisfy

$$f(z+\omega) = f(z) e^{C_H(z,\omega)+2\pi i \varphi(\omega)}$$

We modify these functions and consider

$$\tilde{f}(z) = f(z) \cdot e^{-\frac{\pi}{2} h(z,z)}.$$

These functions can be considered as sections in a new bundle which is isomorphic to the given one but which is described by a different 1-cocycle. If we put

$$C_{\mathrm{hol}}(z,\omega) = \pi(H(z,\omega) - h(z,\omega)) + \frac{\pi}{2}(H(\omega,\omega) - h(\omega,\omega))$$

then the section of the bundle $\mathcal{L}(C_{\mathrm{hol}}, \varphi)$ are functions which satisfy

$$\tilde{f}(z+\omega) = \tilde{f}(z) \cdot e^{C_{\mathrm{hol}}(z,\omega)+2\pi i(\varphi(\omega)+\eta(\omega))}.$$

This new 1-cocycle has the disadvantage that it depends on the choice of G but it has several advantages.

1) We have $H(z,\omega) = h(z,\omega)$ for all $z \in V, \omega \in G$ and $\eta(\omega) = 0, \varphi(\omega) = 0$ for all $\omega \in G$. Hence we see that $\tilde{f}(z+\omega) = \tilde{f}(z)$ for all $\omega \in G$, the function $\tilde{f}$ is periodic with respect to the sublattice G.

2) We will show that the cocycle depends "holomorphically" on I and this means that we can view the abelian varieties together with the bundles as a holomorphic family.

Now I give an indication how we can describe the space of sections in these bundles. We choose a basis $e_1 \dots e_g$ of G, this is also a $\mathbb{C}$-basis of V. We write the elements of V as linear combinations $z = \sum e_\nu z_\nu = (z_1, \dots, z_g)$.

The periodicity of $\tilde{f}$ with respect to G means that $\tilde{f}(z_1+n_1, \dots, z_g+n_g) = \tilde{f}(z_1, \dots, z_g)$ for all $n_1, \dots, n_g \in \mathbb{Z}^g$. Hence we may introduce the variables $u_\nu = e^{2\pi i z_\nu}$ and write

$$\tilde{f}(z_1, \dots, z_g) = h(u_1, \dots, u_g).$$

We expand

$$h(u_1, \dots, u_g) = \sum a_{m_1 \dots m_g} u_1^{m_1} \dots u_g^{m_g}.$$

We apply the transformation formula defining the line bundle. The elements in Ω/G will provide recursion formulae for the coefficients. We will see that some of the coefficients can be chosen arbitrarily and then the other ones can be computed from these via recursion and depend holomorphically on I.

Before I carry out this computation in detail I want to make a remark on the moduli space of principally polarized abelian varieties.

2.4.3 The symmetric domain

To explain this I start from a principally polarized abelian variety. I recall that this is a triplet

$$A = (\Omega, \langle\ ,\ \rangle, I)$$

where
(i) Ω is a free $\mathbb{Z}$-module of rank $2g$ and $\langle\ ,\ \rangle$ is a skew symmetric form

$$\langle\ ,\ \rangle : \Omega \times \Omega \longrightarrow \mathbb{Z}$$

which is non degenerate over $\mathbb{Z}$. This means that we can write our lattice

$$\Omega = \bigoplus_{\nu=1}^{g} \mathbb{Z}e_i \oplus \mathbb{Z}f_\nu$$

where $\langle e_\nu, f_\nu\rangle = -1 = -\langle f_\nu, e_\nu\rangle$ and where all other $\langle\ ,\ \rangle$ between basis elements are zero.

(ii) The element I is a complex structure on $\Omega_\mathbb{R}$, we have $I^2 = -\mathrm{Id}$ and it respects the alternating form $\langle\ ,\ \rangle_\mathbb{R} : \Omega_\mathbb{R} \times \Omega_\mathbb{R} \to \mathbb{R}$.

(iii) On the complex vector space $V = (\Omega_\mathbb{R}, I)$ we can define a hermitian form H_I on V by

$$\mathrm{Im}\, H_I(x, y) = \langle x, y\rangle$$

for all $x, y \in \Omega_\mathbb{R}$. It is part of our assumption that this form is positive definite.

We want to explain that these data can be viewed as points in a complex manifold.

If we consider such an element I, and if we extend the scalars to $\mathbb{C}$, then the $\mathbb{C}$-vector space $\Omega_\mathbb{C} = \Omega \otimes \mathbb{C}$ decomposes

$$\Omega_\mathbb{C} = \Omega_\mathbb{C}^{10} \oplus \Omega_\mathbb{C}^{01},$$

where $\Omega_\mathbb{C}^{10}$ is the eigenspace for I with eigenvalue i and $\Omega_\mathbb{C}^{01}$ is the eigenspace with eigenvalue $-i$.

There exists an object which is called the Grassmann variety of maximal isotropic subspaces. I do not want to discuss this in detail. Its complex points $\mathcal{X}(\mathbb{C})$ are the maximal isotropic subspaces of $\Omega \otimes \mathbb{C}$ and in fact we have the structure of a complex manifold on this set. (We will give some local coordinates later.)

To any element I we can attach a point in $\mathcal{X}(\mathbb{C})$. Actually we have two choices – namely we can attach $\Omega_\mathbb{C}^{10}$ or $\Omega_\mathbb{C}^{01}$ to I – but in our situation we choose

$$I \longrightarrow \Omega_\mathbb{C}^{01} = \{u \in \Omega_\mathbb{C} \mid Iu = -i \otimes u\}.$$

On $\mathcal{X}(\mathbb{C})$ we have complex conjugation, it interchanges the two space in the decomposition and sends the element I into $-I$. This means that the two parabolic subgroups (the stabilizer of $\Omega_{\mathbb{C}}^{10}$ and $\Omega_{\mathbb{C}}^{01}$) are in opposition.

If in turn we have a point $z \in \mathcal{X}(\mathbb{C})$, and the corresponding parabolic subgroup P_z, and if P_z and $P_{\overline{z}}$ are in opposition, then we get a decomposition

$$\Omega_{\mathbb{C}} = W \oplus \overline{W}$$

where $W = z$. Then we can consider the automorphism J which acts by multiplication by i on W and $-i$ on $\overline{W}$. Clearly this defines a complex structur on $\Omega_{\mathbb{R}}$: the elements of $\Omega_{\mathbb{R}}$ are the elements of the form

$$\omega = w + \overline{w}$$

and $I\omega = w \otimes i + \overline{w} \otimes (-i) = w \otimes i + \overline{w \otimes i}$. We conclude that we have a bijection

$$\{I \mid I^2 = -\operatorname{Id}, \langle Ix, Iy\rangle = \langle x, y\rangle\} \xrightarrow{\sim} \mathcal{X}_0(\mathbb{C})$$

where $\mathcal{X}_0(\mathbb{C})$ is the set of points z for which z and $\overline{z}$ are in opposition. This induces a complex structure on the set of all I.

On $\mathcal{X}_0(\mathbb{C})$ we have an action of $G(\mathbb{R})$ by conjugation, we want to determine the orbits.

Recall that an element I defines a hermitian form H_I on the complex vextor space $(\Omega_{\mathbb{R}}, I)$ and the stabilizer of the element I is the unitary group $U_I \subset G(\mathbb{R})$ of the hermitian form This hermitian form H_I has a signature (p, q) with $p + q = g$ and $H_I \simeq U(p, q)$. Now it is an easy — or perhaps better — a well known theorem that:

The orbits under $G(\mathbb{R})$ on $\mathcal{X}(\mathbb{C})$ are given by the signatures (p, q) of the hermitian forms H_I.

Especially we have the open orbit $X \subset \mathcal{X}_0(\mathbb{C})$ where the form H_I is positive definite. This is the orbit which is hit by the principally polarized abelian varieties.

We see that two such principally polarized abelian varieties $(\Omega, <\ ,\ > I), (\Omega, <\ ,\ > I')$ are isomorphic if we can find an automorphism of $(\Omega, <\ ,\ >)$ which sends I into I'. The group of these automorphisms is the symplectic group $G(\mathbb{Z}) = Sp_g(\mathbb{Z})$ and this gives us a hint that the abelian varieties with a given type of polarization are parametrized by

$$G(\mathbb{Z})\backslash X = \text{universal family of principally polarized abelian varieties}$$

We had chosen a basis $e_1 \cdots e_g, f_g \cdots f_1$ for Ω. Now we write

$$f_\nu = \sum (x_{\nu\mu} + y_{\nu\mu} I) e_\mu, \tag{$\dagger$}$$

we put $\tau_{\nu\mu} = x_{\nu\mu} + y_{\nu\mu} \cdot i$.

The element I gives the decomposition $\Omega_{\mathbb{C}} = \Omega_{\mathbb{C}}^{10} \oplus \Omega_{\mathbb{C}}^{01}$. We attached the space $\Omega_{\mathbb{C}}^{01} \in \mathcal{X}(\mathbb{C})$ to I, and we want to write "coordinates" this point (or space). To compute this space we have to observe that we have

$$\Omega_{\mathbb{C}} = G_{\mathbb{C}} \oplus \Omega_{\mathbb{C}}^{01}.$$

The map $V \to \Omega_{\mathbb{C}}/\Omega^{01}_{\mathbb{C}}$ is by construction an isomorphism and hence

$$f_\nu - \sum \tau_{\nu\mu} \otimes e_\mu \in \Omega^{01}_{\mathbb{C}}. \tag{††}$$

The assumption that $\Omega^{01}_{\mathbb{C}}$ is maximal isotropic is equivalent to the symmetry of the matrix

$$(\tau_{\nu\mu})$$

and our hermitian form is positive definite if and only if the real part of this matrix is positive definite. Therefore the $\tau_{\nu\mu}$ are the holomorphic coordinates for the possible choices of I..

We see that the space of possible complex structures I which provide a positive definte form H_I can be identified to the points in the Siegel half space

$$\mathcal{H} = \{Z | Z = X + iY\}$$

where Z is symmeric and Y is positive definite.

2.4.4 We consider the cocycle $\omega \to C_{\text{hol}}(z,\omega)$ and I want to explain that this cocycle depends holomorphically on I. To be more precise we can fix an element $\omega \in \Omega$ and consider this cocycle as a function in the variables z and I. Then we want to show that this cocycle is holomorphic in both variables.

We have the two forms

$$\begin{array}{rcccc} H & : & \Omega_{\mathbb{R}} \times \Omega_{\mathbb{R}} & \longrightarrow & \mathbb{C} \\ h & : & \Omega_{\mathbb{R}} \times \Omega_{\mathbb{R}} & \longrightarrow & \mathbb{C}, \end{array}$$

which H is hermitian with respect to the element I and where h is linear in both variables with respect to I. Now we extend these forms to $\Omega_{\mathbb{C}} = \Omega_{\mathbb{R}} \otimes \mathbb{C}$ bilinearly, i.e. we have

$$\begin{array}{rcl} H_{\mathbb{C}}(\omega \otimes z, \eta \otimes w) & = & zw\ H(\omega,\eta) \\ h_{\mathbb{C}}(\omega \otimes z, \eta \otimes w) & = & zw\ h(\omega,\eta). \end{array}$$

for $\omega, \eta \in \Omega$ and $z, w \in \mathbb{C}$

We observe that the inclusion $\Omega_{\mathbb{R}} \to \Omega_{\mathbb{C}}$ induces an isomorphism

$$\Omega_{\mathbb{R}} \longrightarrow \Omega_{\mathbb{C}}/\Omega^{01}_{\mathbb{C}},$$

and this map is $\mathbb{C}$-linear if we give $\Omega_{\mathbb{R}}$ the complex structure where multiplication by i is given by I.

We can decompose

$$\Omega_{\mathbb{C}} = G_{\mathbb{C}} \oplus \Omega^{01}_{\mathbb{C}},$$

and hence we can write any element $\omega \in \Omega_{\mathbb{C}}$ as a sum

$$\omega = \omega_G + \omega_{01}.$$

Now we consider the expression

$$H_{\mathbb{C}}(z,\omega) - h_{\mathbb{C}}(z,\omega),$$

and we observe that this depends only on $\omega \bmod G_{\mathbb{C}}$ in the second variable. On the other hand we see: If $z \in \Omega_{\mathbb{C}}$ is in $\Omega_{\mathbb{C}}^{01}$, then

$$H_{\mathbb{C}}(z,\omega) = h_{\mathbb{C}}(z,\omega) = 0.$$

this is clear because $z = \eta + I\eta \otimes i$ with some $\eta \in \Omega_{\mathbb{R}}$ and

$$H_{\mathbb{C}}(\eta + I\eta \otimes i, \omega) = H(\eta,\omega) + i\, H(I\eta,\omega) = H(\eta,\omega) - H(\eta,\omega) = 0,$$

and the same holds for $h_{\mathbb{C}}$. Hence we conclude that

$$H_{\mathbb{C}}(z,\omega) - h_{\mathbb{C}}(z,\omega)$$

defines a bilinear form

$$\Omega_{\mathbb{C}}/\Omega_{\mathbb{C}}^{01} \times \Omega_{\mathbb{C}}/G_{\mathbb{C}} \longrightarrow \mathbb{C}.$$

We can express this form in terms of the original alternating form: We write as above

$$\omega = \omega_G + \omega_{01},$$

then I claim that

$$H_{\mathbb{C}}(z,\omega) - h_{\mathbb{C}}(z,\omega) = 2i\langle z, \omega_{01}\rangle_{\mathbb{C}}$$

where $\langle\ ,\ \rangle_{\mathbb{C}}$ is of course the bilinear extension of $\langle\ ,\ \rangle$ to $\Omega_{\mathbb{C}}$.
To see that this is the case it suffices to show that

$$H_{\mathbb{C}}(z,\omega_{01}) - h_{\mathbb{C}}(z,\omega_{01}) = 2i\langle z, \omega_{01}\rangle_{\mathbb{C}}.$$

In this case

$$\omega_{01} = \eta + I\eta \otimes i$$

and because $h_{\mathbb{C}}$ is bilinear with respect to I, we get

$$h_{\mathbb{C}}(z, \eta + I\eta \otimes i) = 0$$

as above. Hence we have to show that

$$H_{\mathbb{C}}(z,\omega_{01}) = 2i\langle z, \omega_{01}\rangle_{\mathbb{C}}.$$

We may assume that $z \in \Omega_{\mathbb{R}}$ and again we write $\omega_{01} = \eta + I\eta \otimes i$. Then

$$H_{\mathbb{C}}(z, \eta + I\eta \otimes i) = H(z,\eta) + i \cdot H(z, I\eta) = i\, H(z, I\eta) + H(z,\eta).$$

We invoke our formulae for H and get

$$\begin{aligned} i(\langle z,\eta\rangle + & i\langle z, I\eta\rangle) - \langle z, I\eta\rangle + i\langle z,\eta\rangle = \\ & 2i\langle z,\eta\rangle - 2\langle z, I\eta\rangle = 2\left(\langle z,\eta\otimes i\rangle_{\mathbb{C}} - \langle z, I\eta\rangle\right) = \\ & 2i\left(\langle z,\eta\rangle + \langle z, I\eta\otimes i\rangle_{\mathbb{C}}\right) = 2i\langle z,\omega_{01}\rangle_{\mathbb{C}}. \end{aligned}$$

This function $(z,I)\mapsto 2i\langle z,\omega_{01}\rangle_{\mathbb{C}}$ is now clearly holomorphic in the variables z, I. Now we pick an ω and we get for our 1-cocycle

$$C_{\mathrm{hol}}(z,\omega) + 2\pi i\varphi(\omega) = 2\pi i\langle z,\omega_{01}\rangle_{\mathbb{C}} + \pi i\langle\omega,\omega_{01}\rangle_{\mathbb{C}} + 2\pi i(\eta(\omega)+\varphi(\omega))$$

We write $z=\sum z_\nu e_\nu \in \Omega_{\mathbb{R}}$ and we want exploit the formula

$$f(z+\omega) = f(z)e^{C_{\mathrm{hol}}(z,\omega)+2\pi i(\varphi(\omega)+\eta(\omega))}$$

for the section of the bundle. If $\omega\in G$ then the factor on the right hand side is equal to 1 and this means that f is periodic in G which in our coordinates means that $f(z_1+n_1,\dots,z_g+n_g) = f(z_1,\dots .z_g)$ where the n_i are integers. Hence we have to consider elements $\omega=\sum_i n_i f_i$. Then the relation (††) says that

$$p_{01}(\omega) = \sum n_\mu (f_\mu - \sum \tau_{\mu\nu} e_\nu).$$

If $z=\sum z_\mu e_\mu$, then it follows that

$$H(z,\omega) - h(z,\omega) = 2i\sum z_\mu n_\mu,$$

and

$$\begin{aligned} H(\omega,\omega) - & h(\omega,\omega) = 2i\langle\omega, p_{01}(\omega)\rangle = \\ & 2i\langle\sum_\mu n_\mu f_\mu, \sum_\mu n_\mu(f_\mu - \sum_\nu \tau_{\mu\nu}e_\nu) = -2i\sum_{\nu,\mu} n_\nu n_\mu \tau_{\nu\mu}. \end{aligned}$$

and hence our recursion formula will be

$$f(z+\omega) = f(z)e^{2\pi i(\sum_\mu z_\mu n_\mu) - \pi i(\sum_{\nu,\mu} n_\nu n_\mu \tau_{\nu\mu}) + 2\pi i(\varphi(\omega)+\eta(\omega))}$$

We introduce the new variables

$$u_\nu = e^{2\pi i z_\nu} \text{ and } q_{\nu\mu} = e^{\pi i \tau_{\nu\mu}}.$$

The q- variables are not independent, we have the relation

$$q_{\nu\mu} = q_{\mu\nu}.$$

We return to the problem of the computation of the space of sections. We start from our data $A = (\Omega, \langle\ ,\ \rangle, I)$, and we consider the d-th power of our line bundle $\mathcal{L}(C_{\mathrm{hol}}, \varphi)^{\otimes d} = \mathcal{L}(dC_{\mathrm{hol}}, d\varphi)$.

The periodicity with respect to G allows us to write our sections f as functions in the variables u_ν

$$f(z_1, \ldots, z_g) = h(u_1, \ldots, u_g)$$

and the transformation rules becomes for an element $\omega = n_1 f_1 + \ldots + n_g f_g$

$$\begin{aligned} &h(u_1 \prod_\nu q_{\nu 1}^{2n_\nu}, \ldots, u_g \prod_\nu q_{\nu g}^{2n_\nu}) \\ &= h(u_1, \ldots, u_g) \prod_{\nu=1}^{g} u_\nu^{dn_\nu} \prod_{\lambda\kappa} q_{\lambda\kappa}^{-n_\lambda dn_\kappa} e^{2\pi d(\varphi(\omega)+\eta(\omega))} \end{aligned}$$

If we now expand the function h into a Laurent series

$$h(u_1, \ldots, u_g) = \sum a_{m_1,\ldots,m_g} u_1^{m_1} \ldots u_g^{m_g}$$

then transformation rule for our element $\omega = \sum_\nu n_\nu f_\nu$ yields the following recursion

$$a_{m_1,\ldots,m_g} \prod_{\nu\mu} q_{\nu\mu}^{2n_\nu m_\mu} = a_{m_1-dn_1,\ldots,m_g-dn_g} \prod_{\lambda\kappa} q_{\lambda\kappa}^{-n_\lambda dn_\kappa} e^{2\pi i(\varphi(\omega)+\eta(\omega))}$$

From this we conclude that the coefficients $a_{\nu_1,\ldots,\nu_g}$ for $0 \le \nu_i \le d-1$ determine the rest of the coefficients. On the other hand we can choose values for the coefficients $a_{\alpha_1}, \ldots, a_{\alpha_g}$ for the indices $0 \le \alpha_i \le d-1$ and then we vary the n_ν we get

$$a_{\alpha_1-dn_1,\ldots,\alpha_g-2dn_g} = a_{\alpha_1,\ldots,\alpha_g} q_{\nu\mu}^{n_\nu \alpha_\mu + dn_\nu n_\mu} e^{2\pi id(\varphi(\omega)+\eta(\omega))}$$

Now we make the fundamental observation that the positive definiteness of our matrix Y above implies that we can prove an estimate

$$|a_{\alpha_1-dn_1,\ldots,\alpha_g-dn_g}| < e^{-c(n_1^2+\ldots n_g^2)}$$

with some constant $c > 0$ depending on Y. This implies that the Laurent series will be convergent for all $u_1, \ldots, u_g \in \mathbb{C}^*$ and we conclude

2.4.5 *We can write down explicitly all sections in a line bundle $\mathcal{L}(C_{\mathrm{hol}}, \varphi)^{\otimes d}$ as infinite Laurent series, if the corresponding hermitian form is positive definite. The dimension of the space of section is d^g.*

With a little bit more efforts in linear algebra linear algebra it is not difficult to show

Theorem: *If $e \in NS(J)$ is an alternating form on Γ which for which the corresponding hermitian form H_e is positive definite, then*

$$\dim H^0(V/\Gamma, \mathcal{L}_{H_e}(\varphi, \eta_H)) = \det <,> .$$

If we return to our Jacobian J then we have the cup product pairing on $H^1(S,\mathbb{Z}) \simeq \Gamma$. If H_0 is the corresponding hermitian form then we can form the line bundle $\mathcal{P} = \mathcal{L}_{H_0}(\varphi, \eta_H)$ with an arbitrary φ and suitable η_H. Our theorem yields

$$\dim_{\mathbb{C}} H^0(J,\mathcal{P}) = 1.$$

If we take powers of this line bundle then $\det(r <,>) = r^{2g}$ and it follows that

$$\dim_{\mathbb{C}} H^0(J,\mathcal{P}^{\otimes r}) = r^{2g}.$$

This can be used to construct an embedding into the projective space

Theorem (Lefschetz): *If we take $r = 3$ then the morphism*

$$\begin{aligned} J &\longrightarrow \mathbb{P}(H^0(J,\mathcal{P}^{\otimes 3})) \\ x &\longrightarrow H_x = \{s \in H^0(J,\mathcal{P}^{\otimes 3}) | s(x) = 0\} \end{aligned}$$

is everywhere defined and yields an embedding of J into the projective space.

I want to comment on this theorem without proving it, its proof will be discussed in the second volume in the section on Jacobians. At first we need to know is that for any $x \in J$ we can find a section $s \in H^0(J,\mathcal{P}^{\otimes 3})$ which does not vanish at this point. Secondly we have prove that for any pair of points $x \neq y$ we can find a section which vanishes at x but not at y. And finally we need to know the following: If we pick a point x and a section s_0 which does not vanish at x then the ratios s/s_0 are function on J which are defined in a suitable neighborhood of x. Then we have to show that we can find sections $s_1, \ldots, s_d$ which vanish at x such that the differentials $d(s_1/s_0), \ldots, d(s_d/s_0)$ generate the dual tangent space. This implies that the local ring $\mathcal{O}_(J,x)$ is the ring of convergent power series in $s_1/s_0, \ldots, s_d/s_0$.

If all this is shown then it is clear that

the image of J under Θ is a complex analytic submanifold in $Y \subset \mathbb{P}(H^0(J,\mathcal{P}^{\otimes 3}))$ and

$$\Theta : J \xrightarrow{\sim} Y$$

is in fact an analytic isomorphism.

Now we use the theorem of Chow which says that a smooth and closed submanifold of $\mathbb{P}^n(\mathbb{C})$ is in fact a smooth projective algebraic variety. (See III.1.2.1,5d). Hence we can define the image $\Theta(J) = Y$ as the set of common zeroes of a finite number of homogeneous polynomials. Then the pair $(J,\mathcal{P})$ becomes an object in algebraic geometry. To make this precise we have to say a few words about the comparison between algebraic and analytic geometry.

As in the case of Riemann surfaces we define a new topology on Y, namely the Zariski toplogy. If we have a homogeneous polynomials $f(z_0, \ldots, z_n)$ then we can look at the set

$V(f) \subset \mathbb{P}^n(\mathbb{C})$ where it vanishes and the set $D(f) \subset \mathbb{P}^n(\mathbb{C})$ where it does not vanish. These sets $D(f)$ form a basis for the Zariski topology on $\mathbb{P}^n(\mathbb{C})$, i.e. the Zariski open subsets in $\mathbb{P}^n(\mathbb{C})$ are unions of sets of the form $D(f)$.

The Zariski open subsets in Y are the intersections of Zariski open subsets in $\mathbb{P}^n(\mathbb{C})$ with Y. As in the case of Riemann surfaces we know that the identity map $Y_{an} \to Y_{Zar}$ is continous.

If now $U \subset \mathbb{P}^n(\mathbb{C})$ is a Zariski open subset we say that a holomorphic function $f : U \to \mathbb{C}$ is meromorphic if for any point $y \in U$ we can find homogenous polynomals g, h of the same degree, such that $h(y) \neq 0$ and such that $f = g/h$ on the open set $U \cap D(h)$. We put as before

$$\mathcal{O}^{\text{mer}}_{\mathbb{P}^n(\mathbb{C})}(U) = \{f : U \to \mathbb{C} | f \text{ is meromorphic}\}$$

We can do the same thing with Y and define the sheaf $\mathcal{O}_Y^{\text{mer}}$.

As in the case of Riemann surfaces the identity map

$$(Y_{an}, \mathcal{O}_Y) \to (Y_{Zar}, \mathcal{O}_Y^{\text{mer}})$$

is a morphism between locally ringed spaces.

A meromorphic function on Y is an element in some $\mathcal{O}_Y^{\text{mer}}(U)$ where $U \neq \emptyset$ is Zariski open in Y. Now Y was special namely it was the image of J under Θ. Hence it is connected as a toological space and from this it follows easily that the intersection of two non empty Zariski open sets is again non empty. This allows us to define the field $\mathbb{C}(Y) = \mathbb{C}(J)$ of meromorphic functions on J.

We state without proof:

The field of meromorphic functions on J is a finitely generated extension of trancendance degree d

As in the case of Riemann surfaces we can define coherent sheaves of $\mathcal{O}_Y$-modules (resp. $\mathcal{O}_Y^{\text{mer}}$) modules on Y_{an} (resp.Y_{Zar}). In both cases this are sheaves of modules under the structure sheaf which locally are finitely generated.

It is the content of Serre's paper "Geometry algebrique ét geometry analytique" (in short GAGA) that these two categories are equivalent. In short words: To any coherent $\mathcal{O}_{\mathbb{P}^n,an}$-sheaf $\mathcal{F}_{an}$ on $\mathbb{P}^n(\mathbb{C})$ we can find a unique subsheaf $\mathcal{F}$ of $\mathcal{O}_{\mathbb{P}^n}$-modules (i.e. $\mathcal{F}(U)$ is an $\mathcal{O}_{\mathbb{P}^n}$-modules for any $U \subset \mathbb{P}^n$, Zariski open) such that $\mathcal{F}_{an} = \mathcal{F} \otimes_{\mathcal{O}_{\mathbb{P}^n}} \mathcal{O}_{\mathbb{P}^n,an}$.

A first consequence of the GAGA-principle get that the sheaf I_{an} which defines the analytic subspace Y is the extension of a sheaf of ideals $I \subset \mathcal{O}_{\mathbb{P}^n}$, and this is of course the statement of Chows theorem.

If we now consider line bundles on J we have the freedom to look at them as complex analytic bundles or as line bundles on the scheme $(J, \mathcal{O}_J)$. Hence we will not make any distinction between these two kinds of line bundles, we identify

$$\begin{aligned}
\mathrm{Pic}_{\mathrm{Zar}}(J) &= \mathrm{Pic}(J) = \mathrm{Pic}(J_{an}) \\
H^1_{\mathrm{Zar}}(J, \mathcal{O}_J^*) &= H^1(J, \mathcal{O}_{J,an}^*)
\end{aligned}$$

where actually $H^1(J, \mathcal{O}_{J,an}^*)$ was exactly what we called $H^1(J, \mathcal{O}_J^*)$ before.

2.4.6 Degenerations of abelian varieties

At this point we achieved also something else. We can consider the $\tau_{\nu\mu}$ as complex analytic variables and then then we can consider our abelian varieties as varying in a holomorphic family of bundles. And we have in addition a holomorphic family of lines bundles and a holomorphic family of sections. If the matrix Y tends to infinity then the $q_{\nu\mu}$ tend to zero and our abelian variety degenerates into a product of $\mathbb{C}^*$ in a certain sense we have written down a universal degenerating family of abelian varieties.

The point is that this degeneration can be given an arithmetic meaning. This will be explained in a special case.

2.4.7 The case of genus 1

I want to discuss these constructions in the special case of curves of genus one. We can assume that the Jacobian is of the form

$$J = \mathbb{C}/\{1, \tau\}$$

where $\tau \in \mathbb{C}$, $\mathrm{Im}(\tau) > 0$ and where $\{1, \tau\} = \Omega$ is the $\mathbb{Z}$-lattice generated by the elements $1, \tau$.

We have an alternating pairing $\langle\ ,\ \rangle$ on Ω, it is determined by its value on the basis elements, and we put

$$\langle 1, \tau \rangle = -1.$$

All other alternating pairings are of the form $d\langle\ ,\ \rangle$ with some integer d. In this case it is clear that $\langle\ ,\ \rangle$ is the imaginary part of a hermitian form H on $\mathbb{C}$. If $y = \mathrm{Im}(\tau)$, then this form is given by

$$H(z_1, z_2) = \frac{1}{y} z_1 \overline{z}_2.$$

Now we consider line bundles

$$\mathcal{L}(d\langle\ ,\ \rangle, \varphi, \eta)$$

where we have changed the notation in the sense of 6.1.1 (???).

Let us look at the case $d = 1$ first. In this case we must have a non trivial η. One possibility is to take

$$\eta(1) = \eta(\tau) = \eta(1+\tau) = \frac{1}{2},$$

and there are 3 other choices, namely, taking the value $\frac{1}{2}$ on exactly one of the elements $\{1, \tau, 1+\tau\}$ and zero on the two others.

I want to stick to the first choice, it is in a sense the most canonical. We investigate the line bundle

$$\mathcal{L}(C_H, 0),$$

i.e. we take a linear form to be trivial. We have to look at functions which satisfy

$$f(z+\omega) = e^{\frac{\pi}{y} z\overline{\omega} + \frac{\pi}{2y}\omega\overline{\omega} + 2\pi i \eta(\omega)} f(z).$$

Now we apply the strategy explained in 6.4. We choose the submodule G to be the module generated by 1. Now η is not trivial on the vector 1 this forces us to make some minor modifications. Again we put

$$\tilde{f}(z) = e^{-\frac{\pi}{2y} z^2} f(z)$$

and then we find

$$\tilde{f}(z+1) = -\tilde{f}(z).$$

(Here we have to take into account that $\eta(1) = \frac{1}{2}$.)

A simple computation shows

$$\begin{aligned}\tilde{f}(z+n\tau) &= e^{-\frac{\pi}{2y}(z+n\tau)^2} f(z+n\tau) = e^{-\frac{\pi}{2y}(z+n\tau)^2} e^{\frac{\pi}{y} zn\overline{\tau} + \frac{\pi}{2y} n^2 \tau\overline{\tau} + \pi i n} f(z) \\ &= e^{-2\pi i n z} e^{-\pi i n^2 \tau + \pi i n} f(z)\end{aligned}$$

In this situation we modify the definitions of q and u and we put $u = e^{\pi i z}$ and $q = e^{\pi i \tau}$, then our first relation above says that our function f has a Laurent expansion

$$\sum_{m = 1 \bmod 2} a_m u^m,$$

and as in 6.4. the second relation gives a recursion for the coefficients a_m.

It says

$$\tilde{f}(uq^n) = (-1)^n u^{-2n} \cdot q^{-n^2} \tilde{f}(u)$$

and looking at the expansion we get

$$a_m u^n q^{mn} = (-1)^n u^{-2n} q^{-n^2} (\ldots + a_{m+2n} u^{m+2n} \ldots),$$

and hence for any choice n, m

$$a_{m+2n} = (-1)^n q^{mn+n^2} a_m.$$

Since the coefficients with an even index are vanishing, we see that the coefficient a_1 determines all the others. We put it equal to one and then we get

$$\tilde{f}(u) = \sum_{m\in\mathbb{Z}} (-1)^m \cdot q^{m+m^2} u^{1+2m}.$$

Since we have $\mathrm{Im}(\tau) > 0$ we have $|q| < 1$ and hence our function converges for all $u \in \mathbb{C}^*$.

This function is one of the Jacobi Theta functions. We change the notation and write

$$\vartheta(u, q) = \sum (-1)^m q^{m+m^2} u^{1+2m}.$$

We have seen that in modern language this ϑ-function is a section in a line bundle on the Jacobian $\mathbb{C}/\{1, \tau\}$.

We can ask ourselves whether we have a different description of this line bundle. Clearly it is of degree one so on our Riemann surface J it should be of the form $\mathcal{O}_J(P)$ with some point $P \in \mathbb{C}/\{1, \tau\}$. The bundle $\mathcal{O}_J(P)$ has a non trivial section which vanishes at P. Hence we see that our $\vartheta(u, q)$ must vanish for some value of u. A simple computation yields

$$\vartheta(1, q) = 0,$$

and hence we conclude

$$\mathcal{L}(\langle\ ,\ \rangle, 0, \eta) \simeq \mathcal{O}_J(O)$$

where $O \in \mathbb{C}/\{1, \tau\}$ is the zero element.

Now it becomes clear what the other choices of η will give. In section 6.1. I explained that different choices of η can be compensated by changing φ. In this case we can consider $\varphi : \Omega \to \mathbb{C}$ such that $\varphi(\Omega) \subset \frac{1}{2}\mathbb{Z}$, of course what matters is the resulting homomorphism

$$\overline{\varphi} : \Omega/2\Omega \longrightarrow \frac{1}{2}\mathbb{Z}/\mathbb{Z}.$$

We have three non zero such homomorphisms and

$$\mathcal{L}(\langle\ ,\ \rangle, 0, \eta') = \mathcal{L}(\langle\ ,\ \rangle, \varphi, \eta)$$

if $\eta = \overline{\varphi} + \eta$. We could carry out the same calculations and get three more ϑ functions

$$\begin{array}{rcl} \vartheta_{01} & = & \sum (-1)^m q^{m^2} u^{2m} \\ \vartheta_{10} & = & \sum q^{m^2+m} u^{2m+1} \\ \vartheta_{11} & = & \sum q^{m^2} u^{2m} \end{array}$$

and they correspond to the linear forms with $\overline{\varphi}_{ij}(1) = \frac{i}{2} \bmod \mathbb{Z}$, $\overline{\varphi}_{ij}(\tau) = \frac{j}{2} \bmod \mathbb{Z}$. These give the four Jacobi θ-functions. The kernel of φ_{ij} defines a two torsion point $P_{ij} \in J$, and we must have that ϑ_{ij} is a non zero section in

$$H^0(J, \mathcal{L}(\langle\ ,\ \rangle, \varphi_{ij}, \eta)$$

and
$$\mathcal{L}(\langle\ ,\ \rangle, \varphi_{ij}, \eta) = \mathcal{O}_J(P_{ij}).$$
We make another change of notation and put
$$\Theta_1 = \vartheta.$$
we know how we can write down sections in
$$H^0(J, \mathcal{L}(dC_{\mathrm{hol}}, 0,)).$$
We know from the Riemann-Roch theorem that this space of sections has dimension d which we also get from our considerations above.

We always have for sections in $H^0(J, \mathcal{L}(d\langle\ ,\ \rangle, 0, d\eta))$
$$\begin{array}{rcl} \tilde{f}(-u) & = & (-1)^d \tilde{f}(u) \\ \tilde{f}(uq^m) & = & (-1)^{dm} u^{-2md} q^{-dm^2} \tilde{f}(u) \end{array}$$
and if we expand into a Laurent series in u the non zero coefficients with even (resp. odd) indices vanish if d is odd (resp. even).
Later we will consider the case where d is even more closely. Then it is usefull to replace d by $2d$ and
$$u \quad \text{by } u = e^{2\pi i z} \text{ and } q \text{by} q = e^{2\pi i \tau}$$
then the recursion formula becomes
$$f(uq^m) = u^{-2md} q^{-dm^2} \tilde{f}(u)$$

If $d = 2$ then
$$\Theta_1^2 = \ldots - \big(\sum_{m\in\mathbb{Z}} q^{2m^2+2m}\big)u^0 + \big(\sum_{m\in\mathbb{Z}} q^{2m^2}\big)u^2 + \ldots,$$
and we find a second section where the coefficient $a_0 = 1$ and $a_2 = 0$.
$$\Theta_2 = \sum_{m\in\mathbb{Z}} q^{2m^2} u^{4m}.$$

We consider $d = 3$, we have already two sections, namely, Θ_1^3 and $\Theta_1\Theta_2$, and we can write a third section
$$\Theta_3 = \sum_{m\in\mathbb{Z}} (-1)^m q^{3m^2+m} u^{1+6m}.$$

We are now in exactly the same situation as in ???. We have the sections
$$\begin{array}{rcccl} \theta_1 & \in & H^0(J, \mathcal{O}_J(O)) & \subset & H^0(J, \mathcal{O}_J(3O)) \\ \theta_2 & \in & H^0(J, \mathcal{O}_J(2O)) & \subset & H^0(J, \mathcal{O}_J(3O) \end{array}$$

and

$$\theta_3 \in H^0(J, \mathcal{O}_J(3O)).$$

We must have linear relations among the monomials $\theta_3^2, \theta_3\theta_2\theta_1, \theta_3\theta_1^3, \theta_2^3, \theta_2^2\theta_1^2, \theta_2\theta_1^4, \theta_1^6$. Now we take into account that our curve depends on a parameter τ and hence on $e^{\pi i\tau} = q$, and we want to find out what these relations are. I refer to the discussion of the Weierstraß equation in section ??? .

We have the two division points $\frac{1}{2}$, $\frac{\tau}{2}$ and $\frac{1+\tau}{2}$ in $\mathbb{C}/\{1,\tau\}$, we call them P_{10}, P_{01} and P_{11} respectively. Then $P_{00} = O$. The ratios

$$\xi_{\nu\mu} = \frac{\vartheta_{\nu\mu}^2}{\vartheta^2}$$

are meromorphic functions on $\mathbb{C}/\{1,\tau\}$ and $\mathrm{div}(\xi_{\nu\mu}) = 2P_{\nu\mu} - 2O$.

We choose one of these points, say P_{10}, and we put $x = \xi_{10}$ and we consider the function

$$x(x - x(P_{01})) \cdot (x - x(P_{11})).$$

The function $y = \frac{\vartheta_{10}\vartheta_{01}\vartheta_{11}}{\vartheta^3}$ has the divisor $P_{01} + P_{10} + P_{11} - 3O$ and hence

$$\mathrm{div}(y^2) = 2P_{01} + 2P_{10} + 2P_{11} - 6O.$$

We have $\mathrm{div}(x - x(P_{01})) = P_{01} + Q - 2O$, but then we conclude $Q = P_{01}$ because if not the divisor could not be a principal divisor.(??) The same argument holds for the third factor. We get $\mathrm{div}(y^2) = \mathrm{div}(x(x - x(P_{01}))(x - x(P_{11}))$.

We conclude that

$$y^2 = a \cdot x(x - x(P_{01}))(x - x(P_{11})).$$

Our division points are $P_{10} = \frac{1}{2}$, $P_{11} = \frac{1+\tau}{2}$ and $P_{01} + \frac{\tau}{2}$. A simple calculation shows that $\vartheta_{10}(P_{10}) = 0$. Hence $x = \frac{\vartheta_{10}^2}{\vartheta^2}$ (this shows that the choice of indices is correct) and we have

$$\begin{aligned}\vartheta_{10}\left(\tfrac{\tau}{2}\right) &= \textstyle\sum_{m\in\mathbb{Z}} q^{m^2+m} e^{\frac{\pi i\tau}{2}(2m+1)} = \sum_{m\in\mathbb{Z}} q^{m^2+m} e^{\pi i\tau m} e^{\frac{\pi i\tau}{2}} = \\ &\quad \left(\textstyle\sum_{m\in\mathbb{Z}} q^{m^2+2m}\right) e^{\frac{\pi i\tau}{2}} = (\sum q^{m^2}) e^{-\frac{\pi i\tau}{2}}\end{aligned}$$

$$\begin{aligned}\vartheta_{10}(\tfrac{1+\tau}{2}) &= \textstyle\sum q^{m^2+m} e^{\pi i\frac{\tau+1}{2}\cdot(2m+1)} = (\sum q^{m^2+m} e^{\pi i\tau m} \cdot e^{\pi i m}) e^{\frac{\pi i}{2}} e^{\frac{\pi i}{2}} = \\ &\quad (\textstyle\sum (-1)^m q^{m^2+2m}) \cdot e^{\frac{\pi i\tau}{2}} \cdot e^{\frac{\pi i}{2}}) = (\sum (-1)^{m+1} q^{m^2}) e^{-\frac{\pi i\tau}{2}} e^{\frac{\pi i}{2}}.\end{aligned}$$

The same calculation for ϑ yields

$$\begin{aligned}\vartheta(P_{10}) &= \left(\textstyle\sum q^{m^2+m}\right) e^{\frac{\pi i}{2}} \\ \vartheta(P_{01}) &= \left(\textstyle\sum (-1)^m q^{m^2+2m}\right) e^{\frac{\pi i\tau}{2}} \\ \vartheta(P_{11}) &= \left(\textstyle\sum q^{m^2+2m}\right) e^{\frac{\pi i\tau}{2}} e^{\frac{\pi i}{2}}.\end{aligned}$$

We get

$$y^2 = a \cdot x \left(x - \frac{(\sum q^{m^2})^2}{(\sum (-1)^{m+1} q^{m^2})^2} \right) \left(x - \frac{(\sum (-1)^{m+1} q^{m^2})^2}{(\sum q^{m^2})^2} \right).$$

We can compute the factor a. As in (???) we look at the leading term in the expansion for $x + \left(\frac{\vartheta_{10}}{\vartheta}\right)^2$ and $y + \left(\frac{\vartheta_{10}\vartheta_{11}\vartheta_{01}}{\vartheta}\right)^3$ at $z = 0$. If

$$\vartheta(z) = \beta z + \text{higher order terms}$$

then

$$\begin{aligned} x &= \tfrac{\vartheta_{10}(0)}{\beta^2 z^2} + \ldots \text{higher order terms} \\ y &= \tfrac{\vartheta_{10}\vartheta_{11}\vartheta_{01}(0)^3}{\beta^3 z^3} + \text{higher order terms} \end{aligned}$$

and we get

$$a = \frac{\vartheta_{11}(0)^2 \cdot \vartheta_{01}(0)^2}{\vartheta_{10}(0)^4}.$$

If we now replace

$$\begin{aligned} y \quad &\text{by} \quad y \quad \cdot \quad \vartheta_{10}(0)^2 \cdot \vartheta_{01}(0)\vartheta_{11}(0) \cdot \tfrac{\vartheta(P_{11})^3}{\vartheta_{10}(P_{11})^3} \\ x \quad &\text{by} \quad x \quad \cdot \quad \tfrac{\vartheta(P_{11})^2}{\vartheta_{10}(P_{11})^2} \end{aligned}$$

then our equation becomes

$$y^2 = x(x-1) \left(x - \frac{(\Sigma q^{m^2}(-1)^n)^4}{(\Sigma q^{m^2})^4} \right).$$

This is now an equation in Legendre normal form. We put

$$\lambda(q) = \frac{(\Sigma q^{m^2}(-1)^m)^4}{(\Sigma q^{m^2})^4}.$$

We can change our point of view. So far we consider q as a complex number or perhaps better as a function depending on a complex variable τ. But the power series $\lambda(q)$ is in fact a power series in $\mathbb{Z}[[q]]$ and we have

$$\mathbb{Z}[\lambda(q)] \subset \mathbb{Z}[[q]].$$

We can comfortably interprete our θ-series as elements in the ring of Laurent series

$$\mathbb{Z}[[q]][[u, u^{-1}]].$$

We have constructed a curve – namely

$$\mathcal{E} : y^2 = x(x-1) \left(x - \frac{(\sum_m q^{m^2}(-1)^m)^4}{(\sum_m q^{m^2})^4} \right)$$

over $\mathbb{Z}[[q]]$ and its discriminant is

Platz fuer Formel

We conclude that this curve is smooth over the ring $\mathbb{Z}\left[\frac{1}{2}\right][[q]]\left[\frac{1}{q}\right]$. In principle we can say that we are in the same situation as in (???) where we wrote the Legendre normal form for an elliptic curve with a 2-torsion point and a 1-form ω.

But here we can write down a formula for the points. So we may for instance consider the morphism

$$\begin{array}{ccc} \mathcal{E} & & \\ \downarrow & \nwarrow^{\theta} & \\ \operatorname{Spec}\left(\mathbb{Z}\left[\frac{1}{2}\right][[q]]\right) & \longleftarrow & \operatorname{Spec}\left(\mathbb{Z}\left[\frac{1}{2}\right][[q]][[u,u^{-1}]]\right) \end{array}$$

and the θ-series; can be seen as a tautological section θ. We can write down points with values in extensions of $\mathbb{Z}\left[\frac{1}{2}\right][[q]]$, so for instance we get n-torsion points on $\mathcal{E}$ if we evaluate at roots of unity ζ or elements of the form $\zeta q^{1/2n}$.

If we consider a homomorphism

$$\mathbb{Z}\left[\frac{1}{2}\right] \longrightarrow k$$

where k is any field, then we can consider the base change of our curve to $k[[q]]$ and to the quotient field $K = k[[q]]\left[\frac{1}{q}\right]$, then it can be shown that

$$\mathcal{E}(K) \simeq K^*/\langle q^2 \rangle$$

(Tate's theorem), and we have an explicit description of the group of rational points.

V.3. Towards the algebraic theory

3.1 Introduction During our discussion of the Jacobian J of a Riemann surface S and the description of the Picard group of J we made heavily use of transcendental methods. This means we worked in the category of (compact) complex manifolds. We used the cohohomology groups $H^i(J, \mathbb{Z})$ and the construction of line bundles was done in the analytic context.

On the other hand we have seen that the Riemann surface can be viewed as the set of $\mathbb{C}$ valued points of a non singular projective curve $C/\operatorname{Spec}(\mathbb{C})$ and we also have stated the result that J is projective algebraic variety which is equipped with a line bundle $\mathcal{P}$ which has very specific properties.

In this section we will aim at an algebraic formulation of our central results, we still use the trancendental arguments in the proofs.

It is the content of chapter X in the second volume that the main results of the present chapter here can be formulated and proved in algebraic terms and hence they belong to algebraic geometry. Especially they make sense in characteristic $p > 0$.

The key to a deeper understanding of J and also of the Riemann surface itself is the investigation of the Picard group of varieties of the form

$$S \times S, S \times J, J \times J \text{ and } J \times J^{\vee}.$$

Let X be any smooth, projective, connected variety over $\mathbb{C}$ (see III,???), we denote its set of complex valued points also by X. We use the above mentioned principles from GAGA. Then we have

$$0 \to H^1(X, \mathbb{Z}) \to H^1(X, \mathcal{O}_X) \to H^1(X, \mathcal{O}_X^*) \to H^2(X, \mathbb{Z}) \to$$

and from here

$$0 \to \operatorname{Pic}^0(X) \to \operatorname{Pic}(X) \xrightarrow{\delta} H^2(X, \mathbb{Z}).$$

The class $\delta(\mathcal{L})$ is the Chern class of $\mathcal{L}$ and the subgroup generated by the Chern classes is called the Neron-Severi group $NS(X)$.

3.1.1 The algebraic definition of the Neron-Severi group: If X is any smooth prjective algebraic variety then he group $H^2(X, \mathbb{Z})$ is of course a trancendental object, it needs the concept of continuity in its definition. But if we believe in GAGA then the group $\operatorname{Pic}(X)$ is defined in the context of algebraic geometry. We also can give an algebraic definition of subgroups which are close to $\operatorname{Pic}^0(X)$. For instance we can define the subgroup $\operatorname{Pic}^{00}(X)$ of those line bundles which are algebraically equivalent to zero: We say that a line bundle $\mathcal{L}$ on X *is algebraically equivalent to zero* if we can find a connected projective algebraic variety T over $\mathbb{C}$ and a line bundle $\tilde{\mathcal{L}}$ on $X \times T$ such that there are two poits t_1, t_0 on T for which $\mathcal{L}_{t_1} = \tilde{\mathcal{L}}|X \times t_1 \xrightarrow{\sim} \mathcal{L}$ and $\mathcal{L}_{t_0} = \tilde{\mathcal{L}}|X \times t_0 \xrightarrow{\sim} \mathcal{O}_X$. Naively speaking this means that we can deform our bundle into the trivial bundle. It is of course clear that during such a deformation process the Chern classes do not change. This means that the group $\operatorname{Pic}^{00}(X)$ of lines bunles algebraically equivalent to zero is always contained in $\operatorname{Pic}^0(X)$.

If we divide Pic(X) by this subgroup we get a modified Neron-Severi group which is defined in the context of algebraic geometry.

Our results (for instance 2.1.5) imply that for abelian varieties A over $\mathbb{C}$ we have in fact $\mathrm{Pic}^0(A) = \mathrm{Pic}^{00}(A)$ and thus we have an algebraic definition of $NS(A)$ for abelian varieties.

3.1.2 : The algebraic definition of the intersection numbers. At this point I want to outline how we can define in purely algebraic terms the intersection numbers of line bundles on a smooth connected projective variety $X \subset \mathbb{P}^d(\mathbb{C})$.

If we have d line bundles $\mathcal{L}_1 \cdots \mathcal{L}_d$ on X, then we have their Chern classes $c_1(\mathcal{L}_1), \cdots, c_1(\mathcal{L}_d)$. We can form the cup product

$$c_1(\mathcal{L}_1) \cup \cdots \cup c_d(\mathcal{L}_d) \in H^{2d}(X, \mathbb{Z}) = \mathbb{Z},$$

and the result is a number. We have already seen that under certain favorable circumstances we can interprete this number as the number of points in the intersection of d smooth divisors (see IV, 9.3.2)

$$c_1(\mathcal{L}_1) \cup \cdots \cup c_1(\mathcal{L}_d) = Y_1 \cap \cdots \cap Y_d.$$

I want to explain that it is always possible to interprete this cup product of Chern classes as intersection numbers. I have to appeal to some theorems in projective algebraic geometry which will be discussed in more detail later.

Our projective space $\mathbb{P}^d(\mathbb{C})$ has the line bundle $\mathcal{O}_{\mathbb{P}^d(\mathbb{C})}(1) = \mathcal{H}$ on it. We will show that for any bundle $\mathcal{L}$ on X we can find an integer $n > 0$ and a non zero section $s \in H^0(X, \mathcal{L} \otimes \mathcal{H}^{\otimes n})$ such that $[s = 0]$ is a smooth divisor (see IV ...). We take our bundle $\mathcal{L}_1$ and construct such a section

$$s_1 \in H^0(X, \mathcal{L}_1 \otimes \mathcal{H}^{\otimes n})$$

and another section

$$t_1 \in H^0(X, \mathcal{H}^{\otimes n})$$

which also provides a smooth divisor on X. If we look at the cup product of the Chern classes, we find

$$c_1(\mathcal{L}_1) \cup \cdots \cup c_d(\mathcal{L}_d) = c_1(\mathcal{L}_1 \otimes \mathcal{H}^{\otimes n}) \cup \cdots \cup c_d(\mathcal{L}_d) - c_1(\mathcal{H}^{\otimes n}) \cup \cdots \cup c_d(\mathcal{L}_d).$$

Now the two divisors $[s_1 = 0]$ and $[t_1 = 0]$ are again smooth projective varieties. They are perhaps not connected but their connected components $Z_1 \cdots Z_k \cdots$ are also smooth projective varieties by the theorem of Chow. We can restrict the remaining line bundles $\mathcal{L}_1 \cdots L_d$ to these components.

Now we assume by induction that we have an algebraic definition of the intersection number $\mathcal{L}_2' \cdots \mathcal{L}_d$ of $d-1$ line bundles on smooth projective varieties of dimension $d-1$. Then the above argument gives us a definition for d line bundles on X.

Here we have to observe that in view of our result in IV. 9.3.2 we know that this definition does not depend on the choices of n and of the sections s_i and t_1 because the intersection numbers are also given by the cup product. But later in the context of algebraic geometry, when now cohomology groups are available, then we have to work a little bit more.

If X is equal to one of our four varieties above and if we write $X = Y \times Z$ then we get for the cohomology in degree one
We get from 2.3 and the Künnet formula

$$\begin{array}{rcl} H^1(J, \mathcal{O}_J) & = & H^1(S, \mathcal{O}_S) \\ H^1(Y \times Z, \mathcal{O}_{Y \times Z}) & = & H^1(Y, \mathcal{O}_Y) \oplus H^1(Z, \mathcal{O}_Z) \end{array}$$

The second isomorphism yields

$$\mathrm{Pic}^0(X) = \mathrm{Pic}^0(Y \times Z) \tilde{\to} \mathrm{Pic}^0(Y) \oplus \mathrm{Pic}^0(Z)$$

We say that Pic^0 is linear.
We recall the notation $\Gamma \simeq H^1(S, \mathbb{Z})$ and then

$$\begin{aligned} H^2(J, \mathbb{Z}) &= \mathrm{Hom}(\Lambda^2\Gamma, \mathbb{Z}) \\ H^2(S \times S, \mathbb{Z}) &= H^2(S, \mathbb{Z}) \oplus H^1(S, \mathbb{Z}) \otimes H^1(S, \mathbb{Z}) \oplus H^2(S, \mathbb{Z}) = \\ & \quad \mathbb{Z} \oplus \Gamma \otimes \Gamma \oplus \mathbb{Z}. \\ H^2(S \times J, \mathbb{Z}) &= H^2(S, \mathbb{Z}) \oplus H^1(S, \mathbb{Z}) \otimes H^1(J, \mathbb{Z}) \oplus H^2(J, \mathbb{Z}) = \\ & \quad \mathbb{Z} \oplus \Gamma \otimes \Gamma^\vee \oplus \mathrm{Hom}(\Lambda^2\Gamma, \mathbb{Z}). \end{aligned}$$

We have to find out what the Neron-Severi group will be in our four cases. It decomposes into

$$NS(Y) \oplus A(H^1(Y, \mathbb{Z}) \otimes H^1(Z, \mathbb{Z})) \oplus NS(Z).$$

To see this we observe that we have the Chern classses of the pull backs $p_1^*(\mathcal{L}_1), p_2^*(\mathcal{L}_2)$ of line bundles on the two factors, which have Chern classes $(c_1, 0, 0), (0, 0, c_2)$ with respect to the above decomposition. On the other hand we can choose point $y_0 \in Y, z_0 \in Z$ and restrict a bundle $\mathcal{L}$ on $Y \times Z$ to $y_0 \times Z, Y \times z_0$. The Chern classes of these restrictions do not depend on the selected points (because Y, Z are connected) and if we modify $\mathcal{L}$ by the product of the inverses of the pull backs we get a bundle whose Chern class is $(0, c_2, 0)$.
We put

$$NS'(Y \times Z) = \{\underline{c} \in NS(Y \times Z) | \underline{c} = (0, c_2, 0)\}$$

The first and the third summand are considered as less interesting at this point since they are filled up by the Chern classes of line bundles which are pull backs from the two factors. We are interested in the summand in the middle.

We have the morphisms

$$S \times S \xrightarrow{\mathrm{Id} \times i_{P_0}} S \times J \xrightarrow{i_{P_0} \times \mathrm{Id}} J \times J \xrightarrow{\mathrm{Id} \times PD} J \times J^\vee.$$

which induces a sequence of isomorphisms

$$\Gamma \otimes \Gamma \longleftarrow \Gamma \otimes \Gamma^\vee \longleftarrow \Gamma^\vee \otimes \Gamma^\vee \longleftarrow \Gamma^\vee \otimes \Gamma^\vee$$

where the isomorphism is always the tensor product of the identity and the Poincaré duality *pd*. It is clear that this sequence of isomorphisms also induces homomorphisms between the corresponding subgroups $NS'(Y \times Z)$ and

3.1.3 Proposition: *With the obvious notation we get a sequence of isomorphisms*

$$A(\Gamma \otimes \Gamma) \longleftarrow A(\Gamma \otimes \Gamma^\vee) \longleftarrow A(\Gamma^\vee \otimes \Gamma^\vee) \longleftarrow A(\Gamma^\vee \otimes \Gamma^\vee)$$

To see that this is indeed the case we recall that the Neron-Severi group is always the kernel of

$$H^2(X, \mathbb{Z}) \to H^2(X, \mathcal{O}_X).$$

In our situation we have to apply the Künneth-formula and look at the kernel of

$$H^1(Y, \mathbb{Z}) \otimes H^1(Z, \mathbb{Z}) \to H^1(Y, \mathcal{O}_Y) \otimes H^1(Z, \mathcal{O}_Z)$$

and then the claim follows because the maps

$$H^1(J^\vee, \mathcal{O}_{J^\vee}) \tilde{\to} H^1(J, \mathcal{O}_J) \to H^!(S, \mathcal{O}_S)$$

are isomorphisms.

There is a slightly different way of looking at this. We have seen that have to study the alternating 2-forms on $\Gamma \oplus \Gamma, \Gamma \oplus \Gamma^\vee, \Gamma^\vee \oplus \Gamma^\vee$. If our two summands are Γ_1, Γ_2 then we denote the space of those alternating 2-forms which are trivial on the two summands by $\mathrm{Alt}'_2(\Gamma_1 \oplus \Gamma_2, \mathbb{Z})$. Then we have an ispmorphism

$$\Gamma_1^\vee \otimes \Gamma_2^\vee \tilde{\to} \mathrm{Alt}'_2(\Gamma_1 \oplus \Gamma_2, \mathbb{Z})$$

An element $\psi_1 \otimes \psi_2 = \Psi$ is sent to the form

$$e_\Psi : ((\gamma_1, \gamma_2), (\gamma_1', \gamma_2')) \mapsto \psi_1(\gamma_1)\psi_2(\gamma_2') - \psi_1(\gamma_1')\psi_2(\gamma_2).$$

To get the Neron-Severi group we have to look at those alternating forms which are after tensorzation by $\mathbb{R}$ compatible with the complex structure and then we have to translate what this means for the corresponding elements in $\Gamma_1^\vee \otimes \Gamma_2^\vee$

If for instance $\Gamma_1 = \Gamma^\vee$ and $\Gamma_2 = \Gamma$ then the we get $\Gamma \otimes \Gamma^\vee = \mathrm{End}(\Gamma)$.

Now it is obvious that the subspace of alternating forms which satisfy in addition condition i) are exactly those which under the above map Ψ correspond to elements in $\mathrm{End}(\Gamma)$ which are compatible with the complex structure. Hence we have shown

$$NS'(J \times J^\vee) \xrightarrow{\sim} \mathrm{End}(J)$$

(To get this identification we did not use the polarization).

Now we consider the case $J \times J$, in this case we have to look at $\Gamma \otimes \Gamma$ and this is the module of bilinear forms on $\Gamma^\vee$ and via Poincaré duality this is the same as the module of bilinear forms on Γ. Following the identifications we see that an element $\gamma_1 \otimes \gamma_2 \in \Gamma \otimes \Gamma$ gives us the bilinear form $(\eta_1, \eta_2) \mapsto e_0 < \gamma_1, \eta_1 > e_0 < \gamma_2, \eta_2 >$.

Now it is an easy exercise that the identification

$$\Gamma \otimes \Gamma \xrightarrow{\sim} \Gamma \otimes \Gamma^\vee \xrightarrow{\sim} \mathrm{End}(\Gamma)$$

the element $\mathrm{Id} \in \mathrm{End}(\Gamma)$ corresponds to the polarization form $e_0 \in \Gamma \otimes \Gamma \xrightarrow{\sim} \Gamma^\vee \otimes \Gamma^\vee$

This element e_0 therefore defines an element E_0 in $\mathrm{Alt}'_2(\Gamma \oplus \Gamma, \mathbb{Z})$ which is given by

$$E_0 \langle \langle (\gamma_1, \gamma_2), (\gamma'_1, \gamma'_2) \rangle \rangle = e_0 \langle \gamma_1, \gamma'_2 \rangle - e_0 \langle \gamma_2, \gamma'_1 \rangle.$$

The alternating form provides the line bundle

$$\mathcal{L}(E_0, 0, 0) = \mathcal{N} \qquad \text{on} \qquad J \times J$$

whose Chern class is E_0.

More generally it is now obvious that under the identification $\Gamma \otimes \Gamma \xrightarrow{\sim} \Gamma \otimes \Gamma^\vee \xrightarrow{\sim} \mathrm{End}(\Gamma)$ an element $\alpha \in \mathrm{End}(\Gamma)$ corresponds to the bilinear form $E_\alpha < \gamma_1, \gamma_2 >= E_0 < \gamma_1, \alpha(\gamma_2) >$ We can summarize this discussion and say

3.1.4 Theorem: *We have a canonical identification*

$$NS'(J \times J) \xrightarrow{\sim} \mathrm{End}(J)$$

which is given by the map which sends an elemet $\psi \in \mathrm{End}(J)$ *to the line bundle* $(\mathrm{Id} \times \psi)^*(\mathcal{N})$.∎

This should be seen in conjunction with our earlier result

$$NS(J) \xrightarrow{\sim} \mathrm{End}_{\mathrm{sym}}(J)$$

I want to point out that the construction of our line bundle $\mathcal{N}$ on $J \times J$ an algebraic terms only using the bundle $\mathcal{P}$.: We consider the product and three maps

$$J \times J \qquad \begin{matrix} \xrightarrow{p_1} \\ \xrightarrow{m} \\ \xrightarrow[p_2]{} \end{matrix} \qquad J$$

and we put

$$\mathcal{N} = m^*(\mathcal{P}) \otimes p_1^*(\mathcal{P})^{-1} \otimes p_2^*(\mathcal{P})^{-1}.$$

It is quite clear that this bundle has a Chern class, the class E_0. But it is also clear that this bundle does not depend on the choice of $\mathcal{P}$: If we modify $\mathcal{P}$ by a line bundle $\mathcal{L}$ which has Chern class zero then this amounts to changing the linear form in the construction . But this change cancels in the construction of $\mathcal{N}$, this means that $\mathcal{N}$ is a canonical bundle on $J \times J$.

The bundles $\mathcal{P}$ provides the isomorphism

$$\varphi_{\mathcal{P}} : J \longrightarrow J^\vee = \mathrm{Pic}^0(J)$$

which was given by

$$x \longrightarrow T_x(\mathcal{P}) \otimes \mathcal{P}^{-1}.$$

This isomorphism can also be interpreted in terms of $\mathcal{N}$: If we restrict $\mathcal{N}$ to $J \times \{x\}$, then this restriction is a line bundle $\mathcal{N}_x$ on J and clearly by definition we have

$$\mathcal{N}_x \simeq T_x(\mathcal{P}) \otimes \mathcal{P}^{-1}.$$

The bundle $\mathcal{N}$ gives us a still better understanding of the self duality of the Jacobian. The map

$$\begin{array}{ccc} y & \longmapsto & \mathcal{N}' \mid S \times \{y\} \\ J & \longrightarrow & \mathrm{Pic}^0(S) = J \end{array}$$

is the identity.

But the bundle $\mathcal{N}$ has a still stronger property which is called universality. Let us assume that we have a line bundle $\mathcal{L}$ on $J \times T$ where T is a complex analytic variety. We assume that T is connected, and we also assume that $\mathcal{L} \mid J \times t_o$ is in $\mathrm{Pic}^0(J)$ for some point t_0. Now we can define a map

$$\psi : T \longrightarrow J^\vee = \mathrm{Pic}^0(J)$$

which is defined by

$$\mathcal{L} \mid J \times \{t\} \simeq \mathcal{N}_{\psi(t)}.$$

I claim that ψ is indeed an analytic map, and that in addition for any point $t_1 \in T$ we can find a neighborhood V of t_1 such that

$$\mathcal{L} \mid J \times V \simeq (\mathrm{Id} \times \psi^*)(\mathcal{N}) \mid J \times V.$$

We introduce the following notation: If we have two line bundles $\mathcal{L}_1, \mathcal{L}_2$ on $X \times T$ then we write

$$\mathcal{L}_1 \sim_T \mathcal{L}_2$$

is these bundles are isomorphic locally in T which means that for any point $t \in T$ we can find a neighborhood V such that

$$\mathcal{L}_1 \mid X \times V \simeq \mathcal{L}_2 \mid X \times V.$$

This amounts to that we can find a lim bundle $\mathcal{M}$ on T such that

$$\mathcal{L}_1 \simeq \mathcal{L}_2 \otimes p_2^*(\mathcal{M}).$$

This looks very plausible but it is not so easy to prove, and I will not give the proof here. The proof requires some deeper results in complex analysis, for instance some finiteness results.

In the following chapter (???) we will prove an analogous statement in the context of algebraic geometry. In that case the truth of the assertion will be a consequence of the construction, and we will need the full strength of the finiteness results in algebraic geometry.

3.2 The structure of $\mathrm{End}(J)$:

Since we have an inclusion $\mathrm{End}(J) \subset \mathrm{Hom}(\Gamma, \Gamma)$, we know that $\mathrm{End}(J)$ is a finitely generated algebra over $\mathbb{Z}$.

For any $\varphi \in \mathrm{End}(J)$ defined an endomorphism

$$\varphi^* : \mathrm{Pic}(J) \longrightarrow \mathrm{Pic}(J)$$

of the Picard group which is given by the pull back of line bundles. We denote by ${}^t\varphi$ the restriction of φ^* to the subgroup $\mathrm{Pic}^0(J)$, and use the selfduality of the Jacobian we get the transposed

$${}^t\varphi : J \longrightarrow J.$$

I want to point out that this result makes sense in the context of algebraic geometry. We have seen that the group $NS'(J \times J)$ has an algebraic definition and this is also the case for $\mathrm{End}(J)$.

We have seen that ${}^t\varphi$ corresponds to the transpose of $\varphi : \Gamma \to \Gamma$ with respect to e_0.

The map $\varphi \to^t \varphi$ is called the Rosati involution (with respect to the standard polarization). It has the property

$$\begin{aligned} {}^t(\varphi + \psi) &= {}^t\varphi +^t \psi \\ {}^t(\varphi\psi) &= {}^t\psi^t\varphi \end{aligned}$$

It is of course clear that φ^* also induces an endomorphism

$$\overline{\varphi}^* : NS(J) \longrightarrow NS(J).$$

We had the identification

$$NS(J) \simeq \mathrm{End}_{\mathrm{sym}}(J)$$

(see ???) and we saw that

$$\overline{\varphi}^* : \psi \longrightarrow \varphi\psi^t\varphi.$$

Especially it follows that $\varphi \to \overline{\varphi}^*$ is quadratic. This means that we can consider $\varphi_1 + \varphi_2$ and then

$$(\overline{\varphi_1 + \varphi_2})^* = \overline{\varphi}_1^* + \overline{\varphi}_2^* + \langle \varphi_1, \varphi_2 \rangle$$

where $\langle \varphi_1, \varphi_2 \rangle : NS(J) \to NS(J)$ depends bilinearily on the two variables.

To any $\varphi \in \mathrm{End}(J)$ we can define $tr(\varphi)$ and $\deg(\varphi)$ simply as the trace and the determinant of φ considered as endomorphism of Γ, i. e.

$$\begin{aligned} tr(\varphi) &= tr(\varphi : \Gamma \to \Gamma) \\ \deg(\varphi) &= \det(\varphi : \Gamma \to \Gamma). \end{aligned}$$

These functions have the obvious properties

$$\begin{aligned} \mathrm{tr}({}^t\varphi) &= \mathrm{tr}(\varphi) \\ \det({}^t\varphi) &= \det(\varphi) \\ \det(\varphi_1\varphi_2) &= \det(\varphi_1)\det(\varphi_2) \end{aligned}$$

We have the following

3.2.1 Theorem (Positivity of the Rosati-Involution): *For any $\varphi \in \mathrm{End}(J)$, $\varphi \neq 0$ we have*

$$tr(\varphi^t\varphi) > 0.$$

Proof: We give a proof which uses the lattice Γ. We have seen in IV.8.7 that we can find a basis

$$\{e_1, f_1, e_2, f_2, \cdots e_g, f_g\}$$

such that $Ie_{\dot{v}} = f_{\dot{v}}$ and $\langle e_{\dot{v}}, f_{\dot{v}} = -1$ where the other scalar products vanish. Then it is clear that for any endomorphism ψ we have

$$tr(\psi) = -\sum \langle \psi(e_i), f_i \rangle + \sum \langle \psi(f_i), e_i \rangle.$$

If $\psi =^t \varphi\varphi$ then we get

$$\begin{aligned} tr({}^t\varphi\varphi) &= -\textstyle\sum_{i=1}^g \langle \varphi(e_i), \varphi(f_i) \rangle + \sum_{i=1}^g \langle \varphi, f_i \rangle, \varphi(e_i) \rangle \\ &= -2\textstyle\sum_{i=1}^g \langle \varphi, e_i \rangle, \varphi(Ie_i) \rangle. \end{aligned}$$

Since φ commutes with I, the last sum is equal to

$$-2\sum\langle\varphi(e_i), I\varphi(e_i)\rangle = +\sum 2h_{\langle\ ,\ \rangle}(\varphi(e_i),\varphi(e_i)).$$

The terms are ≥ 0 and since at least one of the $\varphi(e_i) \neq 0$ $(\varphi(f_i) = \varphi(Ie_i) = \varphi(e_i)!)$ we conclude that the sum must be strictly positive.

The existence of the trace and the positivity of the Rosati involution have a dramatic influence on the structure of the algebra $\mathrm{End}(J)$ this will be discussed in the second volume Chap. ???

We defined the degree of an endomorphism as $\deg(\varphi) = \det(\varphi|\Gamma)$, this definition uses the trancendental object Γ. But we can easily define this degree in purely algebraic terms. We consider

$$\begin{array}{ccc} J & \xrightarrow{\varphi} & J \\ \uparrow & & \uparrow \\ H^1(S,\mathcal{O}_S)/\Gamma & \xrightarrow{\varphi} & H^1(S,\mathcal{O}_S)/\Gamma. \end{array}$$

and then we see easily:

3.2.2 Proposition: *The degree of φ is non zero if and only if the morphism φ is finite. If the degree of φ is non zero, then we have*

$$\deg(\varphi) = [\mathbb{C}(J) :_\varphi \mathbb{C}(J)] = \text{number of points in } \varphi^{-1}(0).$$

Proof: This is rather clear: The determinant of φ is equal to the index of $\varphi(\Gamma)$ in Γ. If $\Gamma' \subset H^1(\mathcal{O}_S,\mathbb{Z})$ is the inverse image of Γ, then $\Gamma'/\Gamma \simeq \Gamma/\varphi(\Gamma)$ and this proves that the order of the kernel $\varphi^{-1}(0)$ is also equal to $\det(\varphi)$.

It is also clear that φ induces an isomorphism of the tangent spaces at zero and hence we can conclude that the fibre $\varphi^{-1}(0)$ is reduced. The morphism φ induces an inclusion of the field of franctions $\mathbb{C}(J) \hookrightarrow \mathbb{C}(J)$ and the subfield is the field of invariants under Γ'/Γ, the last assertion follows from Galois theory.

We defined the degree $\deg(\varphi)$ and the trace $tr(\varphi)$ on J using the lattice Γ but now we found an algebraic definition of the degree. From the definition of the degree as a determinant it follows that

$$\deg(\varphi + n\,\mathrm{Id}) = a_0(\varphi) + \cdots a_{2n-1}(\varphi)n^{2g-1} + n^{2g},$$

then

$$tr(\varphi) = a_{2n-1}(\varphi).$$

But it not clear how we can derive this formula from the algebraic definition as the degree of a field extension.

The point in the following considerations will be that for an element $\psi \in \mathrm{End}(A)$ the element

$$\psi^*(c) \in NS(A)$$

contains decisive information on the endomorphism ψ. Especially we will see that we can express the degree of the endomorphism in terms of this class. The fundamental point is that its g-th power with respect to the cup product is an element in $H^{2g}(J,\mathbb{Z}) = \mathbb{Z}$ by Poincare duality. But if we think of $\psi^*(c) \in NS(A)$ then we can define the g fold selfintersection of this line bundle in the context of algebraic geometry (See ???) this is also a number. We explained in (IV.8.6) show us that these numbers are the same.

The element c is an alternating form on $\Gamma^\vee$. If $\dim A = g$, then we can raise this element into the g'th power in the cohomology ring and we have seen (See IV.6.6.2.1) that this means that we have to take its g'th exterior power

$$c^g = c \wedge c \ldots \wedge c \in \mathrm{Hom}^{2g}_{\mathrm{alt}}(\Gamma^\vee, \mathbb{Z}) \simeq \mathbb{Z},$$

(this is the selfintersection number of the class c.) Now the class

$$\psi^*(c)(\gamma_1, \gamma_2) = c(\psi(\gamma_1), \psi(\gamma_2)),$$

and it is an elementary excercise in linear algebra that

$$\psi^*(c)^g = \deg(\psi) \cdot c^g,$$

where $\det(\psi)$ is of course the determinant of the endomorphism ψ on the free module Γ which is of rank $2g$.

Now the formula above gives us the degree of ψ in terms of the elements c and $\psi^*(c)$ which are in $NS(A)$ and this makes sense- as explained above- in the context of algebraic geometry.

Of course this formula will give us an expression for the trace of $\psi \in \mathrm{End}(A)$. We assume that A has a polarization $c \in NS(A)$ such that a corresponding bundle is ample. Now we consider the map

$$(\psi + n\,\mathrm{Id})^* : NS(A) \longrightarrow NS(A).$$

We have seen that this map is

$$\psi^* + n\langle \psi, \mathrm{Id}\rangle + n^2 \cdot \mathrm{Id}\,.$$

We have the formula for the degree

$$\begin{aligned}\deg(\psi + n\,\mathrm{Id}) \cdot c^g &= ((\psi + n\,\mathrm{Id})(c))^g = \\ &= (\psi^*(c) + n\langle \psi, \mathrm{Id}\rangle(c) + n^2 \cdot c)^g = \\ &\cdots + g n^{2g-1} c^{g-1} \wedge \langle \psi, \mathrm{Id}\rangle(c) + n^{2g} \cdot c^g\end{aligned}$$

and hence we get the formula

$$tr(\psi)\cdot c^g = gc^{g-1}\wedge\langle\psi,\mathrm{Id}\rangle(c),$$

and this gives us the trace as a cup product of classes in the cohomology evaluated on the fundamental cocycle. Let us assume that $A = J$ and c is the canonical polarization.
Let us assume that the endomorphism ψ is a product of the form $\psi = {}^t\varphi\varphi$. Then

$$\begin{aligned} c\langle(\psi+\mathrm{Id})\gamma_1,(\psi+Id)\gamma_2\rangle &= (\psi+\mathrm{Id})^*c\langle\gamma_1,\gamma_2\rangle = \\ &= \psi^*c\langle\gamma_1,\gamma_2\rangle + c\langle\gamma_1,\gamma_2\rangle + c\langle\gamma_1,\psi\gamma_2\rangle + c\langle\psi\gamma_1,\gamma_2\rangle \end{aligned}$$

and the sum of the last two terms is $\langle\psi,\mathrm{Id}\rangle(c)\langle\gamma_1,\gamma_2\rangle$. Hence we get

$$\langle\psi,\mathrm{Id}\rangle(c)\langle\gamma_1,\gamma_2\rangle = 2c\langle\varphi\gamma_1,\varphi\gamma_2\rangle,$$

and this means that

$$\langle\psi,\mathrm{Id}\rangle(c) = \varphi^*(c).$$

This gives us the formula

$$tr({}^t\varphi\varphi)\cdot c^g = 2gc^{g-1}\wedge\varphi^*(c).$$

We return to the interpretation in terms of algebraic geometry. We assumed that c should be the class of an ample line bundle and we have seen that the highest intersection numbers of line bundles are equal to the highest cup product of their Chern classes. Hence we can say that

$$tr({}^t\varphi\varphi)\cdot\mathcal{P}^g = 2g\mathcal{P}^{g-1}\wedge\varphi^*(\mathcal{P}).$$

Since the right hand side must be positive for $\varphi\neq 0$ we get a formula in the context of algebraic geometry which implies the positivity of the Rosati involution.

Finally we want to give a formula for the trace of an endomorphism in terms of intersection number of two divisors on these surface $S\times S$. We return to our bundle

$$\mathcal{N} = m^*(\mathcal{P})\otimes p_1^*(\mathcal{P})^{-1}\otimes p_1^*(\mathcal{P})^{-1}$$

on $J\times J$ has Chern class zero when restricted to $e\times J$ and $J\times e$ and its Chern class is E_0.

If we pick an element $\psi\in\mathrm{End}(J)$, then we can consider the bundle

$$(\mathrm{Id}\times\psi)^*(\mathcal{N})$$

on $J\times J$. We have the inclusion

$$i_{P_0}\times i_{P_0} : S\times S\longrightarrow J\times J$$

and get the line bundle

$$(i_{P_0}\times i_{P_0})^*\circ(\mathrm{Id}\times\psi)^*(\mathcal{N}) = \mathcal{L}_\psi$$

on $S \times S$. The Chern class of this line bundle sits in $NS'(S \times S) \subset H^1(S,\mathbb{Z}) \otimes H^1(S,\mathbb{Z}) = \Gamma \otimes \Gamma$.

Now we have the famous trace formula

$$\Delta \cdot \mathcal{L}_\psi = -\operatorname{tr}(\psi).$$

This is a rather formal consequence of the definitions. We have seen that the intersection product of two divisors is equal to the cup product of the Chern classes and this evaluated on the fundamental class. The cup product of the classes

$$\xi_1 \otimes \eta_1, \xi_2 \otimes \eta_2 \in \Gamma \otimes \Gamma \subset H^2(S_1 \times S_1, \mathbb{Z})$$

is given by

$$-\xi_1, \xi_2 \cup \eta_1\eta_2$$

where now $\xi_1\xi_2 \in H^2(S,\mathbb{Z}) \otimes H^0(S,\mathbb{Z})$ and $\eta_1\eta_2 \in H^0(S,\mathbb{Z}) \otimes H^2(S,\mathbb{Z})$.

Since we have the alternating 2-form, we can choose as a standard basis on Γ a $\mathbb{Z}$-basis

$$u_1 \cdots u_g \qquad v_1 \cdots v_g$$

such that $e_0\langle u_i, v_i\rangle = -e_0\langle v_i, u_i\rangle = 1$ and all other products are zero.

Under this identification the element

$$E = \sum u_i \otimes v_i - \sum v_i \otimes u_i \in \Gamma \otimes \Gamma$$

becomes the identity element in $\Gamma^\vee \otimes \Gamma = \operatorname{Hom}(\Gamma, \Gamma)$: The element E applied to an element $\gamma \in \Gamma$ yields

$$E(\gamma) := \sum \langle u_i, \gamma\rangle v_i - \sum \langle v_i, \gamma\rangle u_i.$$

Then it is clear that $E(u_i) = u_i$ and $E(v_i) = v_i$.

Then the Chern class of $\mathcal{L}_\psi$ is given by

$$c_1(\mathcal{L}_\psi) = \sum u_i \otimes \psi(v_i) - \sum v_i \otimes \psi(u_i)$$

and

$$\begin{aligned} E \cup c_1(\mathcal{L}_\psi) &= \Big(\sum u_i \otimes v_i - \sum v_i \otimes u_i\Big) \cup \Big(\sum u_i \otimes \psi(v_i) - \sum v_i \otimes \psi(u_i)\Big) \\ &= +\sum \langle u_i, v_i\rangle \cdot \langle v_i, \psi(u_i)\rangle = \sum \langle u_i, v_i\rangle \cdot \langle u_i, \psi(v_i)\rangle \\ &= \sum \langle v_i, \psi(u_i)\rangle - \sum \langle u_i, \psi(v_i)\rangle = -\operatorname{tr}(\psi). \end{aligned}$$

Let us consider the Abel map

$$i_{p_0} : S \longrightarrow J$$

which induces a map on the first cohomology

$$\begin{array}{ccccc} i_{P_0}^* & : & H^1(J,\mathbb{Z}) & \longrightarrow & H^1(S,\mathbb{Z}) \\ & & \| & & \| \\ & & \Gamma^\vee & \longrightarrow & \Gamma \end{array}$$

which we identified as the inverse of the polarization map. It induces a map

$$\begin{array}{ccc} H^2(J,\mathbb{Z}) & \longrightarrow & H^2(S,\mathbb{Z}) \\ \| & & \| \\ \Lambda^2\Gamma^\vee & \longrightarrow & \mathbb{Z} \end{array}$$

and this map is just the evaluation by the dual form

$$\varphi \wedge \psi \longmapsto e^\vee(\varphi,\psi).$$

This linear form on $H^2(J,\mathbb{Z})$ can be viewed as an element $[S]$ in $H_2(J,\mathbb{Z})$ or as an element in $H^{2g-2}(J,\mathbb{Z})$. This class $[C]$ is of course the Chern class of our Riemann surface S, it is the image of the fundamental class in $H_2(S,\mathbb{Z})$ in $H_2(J,\mathbb{Z})$ under the map induced by the Abel map.

If we choose a basis $e_1,\cdots,e_g,f_1\cdots f_g$ on Γ such that $\langle e_i,f_i\rangle = 1 = -\langle f_i,e_i\rangle$ and all other pairings give zero, then the $-f_1,\cdots,-f_g,e_1,\cdots,e_g$ are the elements of the dual basis if we identify Γ and $\Gamma^\vee$ by the polarization map. Then the form e_v is given by

$$\sum \quad e_i \wedge f_i$$

again and our form in $H^{2g-2}(J,\mathbb{Z})$ which is the fundamental class of S is given by

$$\sum_i \quad e_1 \wedge f_i \cdots \widehat{e_i \wedge f_i} \wedge \cdots e_g \wedge f_g,$$

i.e. the factor $e_i \wedge f_i$ is left out.

The polarization class $e^\vee \in \Lambda^2\Gamma^\vee$ itself maps to g in $H^2(S,\mathbb{Z})$, and it is clear that

$$(e^\vee)^{g-1} = (\sum \ e_i \wedge f_i) \wedge \cdots \wedge (\sum \ e_i \wedge f_i) = (g-1)![S]$$

and

$$(e^\vee)^g = g!.$$

Now recall the formula for the trace

$$\mathrm{tr}(\psi) = g\frac{\mathcal{P}^{g-1}\cdot\langle \mathrm{Id},\psi\rangle(\mathcal{P})}{\mathcal{P}^g}.$$

We have seen that $\mathcal{P}^{g-1} = (g-1)![S]$ and $\mathcal{P}^g = g!$, hence we get

$$\mathrm{tr}(\psi) = [S] \cdot \langle \mathrm{Id}, \psi \rangle (\mathcal{P}) =$$

I think that at this point the role of the polarization becomes clear. First of all we have seen that it allows the constructionof a line bundle $\mathcal{P}$ on J which is ample and hence it provides a projective embedding of J and hence $(J, \mathcal{P})$ becomes an object in algebraic geometry. The reader should be aware that the bundle $\mathcal{P}$ is not unique since it depends on the choice of a linear form.

Now we consider the diagram

$$\begin{array}{lcl} S & \xrightarrow{\Delta} & S \times S \\ \downarrow i_{P_0} & & \downarrow i_{P_0} \times i_{P_0} \\ J & \xrightarrow{\Delta} & J \times J \end{array}$$

The ring of correspondences

We have the isomorphism $\mathrm{Pic}(S \times S)/p_1^*(\mathrm{Pic}(S)) + p_2^*(\mathrm{Pic}(S)) \simeq \mathrm{End}(J)$. We want to explain how we can define a ring structure on the left hand side directly.

If we have an irreducible divisor $D \subset S \times S$ we can look at it as a so called correspondence: To any point $z \in S$ we can consider the points$(z, z_i) \in D$ and call the points z_i counted with multiplicity as the points corresponding to z. We can form the free group of these divisors and mod out by the divisors of the form $S \times D'$ or $D' \times S$ where D' (resp. D'') is a divisor in the first or second factor, let us call this $\mathcal{R}$. After we have done this, we can introduce a product on this group: We choose suitable representatives D_1, D_2 of two elements and consider the divisors in

$$D_1 \times S, S \times D_2 \quad \text{on} \quad S \times S \times S.$$

Now we take the intersection – this makes sense if we made a careful choice – and project this intersection to the two outer factors.

This induces a ring structure on $\mathcal{R}$ with identity which is given by the class of the diagonal.

It is clear that this ring has an involution which is obtained by interchanging the two factors.

We can also define a trace: For any $[D] \in \mathcal{R}$ we choose a representative D for which $D \mid z_0 \times S$ and $D \mid z_0 \times S$ are both in $\mathrm{Pic}^0(S)$. Then we put

$$-\mathrm{tr}([D]) = \Delta \cdot D.$$

Now it is at least plausible that

$$-\operatorname{tr}({}^t[D] * [D]) = ({}^t[D] \times [D]) \cdot \Delta = D \cdot D,$$

and this last number is strictly < 0 if $D \neq 0$ by the Hodge index theorem. (See ???)

We know of course that $\mathcal{R} \simeq \operatorname{End}(J)$. These considerations show that we can define this ring without reference to the Jacobian.

(Vielleicht später an eine andere Stelle!)

3.3 An algebraic sustitute for the cohomology I think that I convinced the reader that the cohomology groups $H^1(S,\mathbb{Z}) = \Gamma$, $H^1(J,\mathbb{Z}) = \Gamma^\vee$ play a fundamental role in understanding the structure of S and J. Therefore we should have a substitute for these cohomology groups in the algebraic context. This will be explained later (see ???) but we give an indication how we can get cohomology in the algebraic context.

We have

$$J = H^1(S,\mathcal{O}_S)/H^1(S,\mathbb{Z}) = H^1(S,\mathcal{O}_S)/\Gamma.$$

The module Γ does not make sense in the context of algebraic geometry. Now we consider the endomorphism

$$n \operatorname{Id} : J \longrightarrow J,$$

and we consider the kernel

$$J[n] = \ker(n \operatorname{Id} : J \longrightarrow J).$$

This kernel is obviously isomorphic to

$$\frac{1}{n}\,\Gamma/\Gamma \simeq (\mathbb{Z}/n\mathbb{Z})^{2g}.$$

But this kernel can be defined in the context of algebraic geometry. We consider J as a projective variety over $\mathbb{C}$ which has the structure of an abelian algebraic group and then the kernel of $n \operatorname{Id}$ is a finite group scheme over $\mathbb{C}$.

Once we have done this, we observe that we have for $n \mid n_1$

$$J[n] \longrightarrow J[n_1],$$

and we can define

$$\operatorname{Tors}(J) = \varinjlim_n \ J[n],$$

where the ordering on $\mathbb{N}$ is given by divisibility. Of course it is clear that

$$\operatorname{Tors}(J) = \Gamma \otimes \mathbb{Q}/\mathbb{Z}$$

and we conclude:

The module Γ cannot be defined in terms of algebraic geometry, but the module

$$\Gamma \otimes \mathbb{Q}/\mathbb{Z}$$

is an algebraic geometric object.

We can pass to the dual module, we consider

$$\mathrm{Hom}(\mathrm{Tors}(J), \mathbb{Q}/\mathbb{Z}).$$

It is an elementary fact that

$$\mathrm{Hom}(\mathbb{Q}/\mathbb{Z}, \mathbb{Q}/\mathbb{Z}) \simeq \lim_{\leftarrow} \ \mathbb{Z}/n\,\mathbb{Z} = \hat{\mathbb{Z}},$$

and therefore we get

$$\mathrm{Hom}(\mathrm{Tors}(J), \mathbb{Q}/\mathbb{Z}) = \Gamma \otimes \hat{\mathbb{Z}},$$

and this is the so called Tate-module

$$T(J) = \lim_{\substack{\leftarrow \\ n}} \ J[n]$$

where now for n/n_1 the map

$$J[n_1] \longrightarrow J[n]$$

is given by multiplication by n_1/n.

(Verweise auf Kap. I ???)

The Chinese remainder theorem yields

$$\mathbb{Q}/\mathbb{Z} = \bigoplus_{\ell:\ell \mathrm{pine}} \mathbb{Q}_\ell/\mathbb{Z}_\ell$$

where ℓ runs over the primes and $\mathbb{Q}_\ell$ is the ℓ-adic completion and $\mathbb{Z}_\ell$ is the ring of ℓ-adic integers. This yields a decomposition

$$\mathrm{Tors}(J) = \bigoplus_\ell \ \mathrm{Tors}(J)_\ell$$

where $\mathrm{Tors}(J)_\ell = \lim \ J[\ell^\alpha]$ and dually

$$\hat{\mathbb{Z}} = \prod_\ell \ \mathbb{Z}_\ell$$

and

$$T(J) = \prod_\ell \ T_\ell(J).$$

Chapter VI
Basic notions in algebraic geometry

VI.1. Affine schemes

We consider commutative rings $A, B \ldots$ with identity $(1A, 1_B, \ldots)$, morphisms $\phi : A \to B$ are homomorphisms sending the identity of A into the identity of B. We always assume that the identity in a ring is different from zero. For any such ring A we have the group of *invertible elements (units)*: $A^* = \{a \in A \mid \exists\, a' \in A$ such that $aa' = 1_A\}$. A *proper* ideal $\mathfrak{a} \subset A$ is an ideal with $1_A \notin \mathfrak{a}$, prime ideals are always proper. For any ring and any $f \in A$ we use the standard notation (f) for the principal ideal Af.

VI.1.1 Localization

If we have a subset $S \subset A$ which is closed under multiplication and contains the identity $1_A \in S$ we can define a quotient ring A_S and a map $\phi_S : A \to A_S$ such that the elements of S become invertible.

To do this we consider pairs $(a, s) \in A \times S$ and introduce an equivalence relation

$$(a, s) \sim (a', s') \Longleftrightarrow \exists\, s'' \in S \text{ such that } (as' - a's) \cdot s'' = 0.$$

We consider the quotient A_S of $A \times S$ by this relation, let $\pi_S : A \times S \to A_S$ be the projection to this quotient. We define a ring structure on A_S by

$$\pi_S((a, s)) + \pi_S((a'.s')) = \pi_S((as' + a's, ss'))$$

$$\pi_S((a, s)) \cdot \pi_S((a'.s')) = \pi_S((aa', ss')).$$

We have a homomorphism of rings

$$\phi_S : \begin{array}{ccc} A & \longrightarrow & A_S \\ a & \longmapsto & \pi_S((a, 1)) \end{array}.$$

We will write the elements of A_S simply as $\pi_S((a, s)) = \dfrac{a}{s} = as^{-1}$. Of course

$$\frac{a}{s} = \frac{as'}{ss'} = \frac{as's''}{ss's''} \ldots.$$

The quotient ring has a universal property: For any ring B we can consider

$$\mathrm{Hom}_{\text{Rings, } S \text{ to units}}(A, B) = \{\phi : A \to B | \phi(S) \in B^* \text{ for all}\, s \in S\}$$

and this set of homomorphisms is equal to

$$\mathrm{Hom}_{\mathrm{Rings}}(A_S, B)$$

where the identification is given by the diagram

$$\begin{array}{ccc} A & \xrightarrow{\phi_S} & A_S \\ \phi \searrow & & \swarrow \phi' \\ & B & \end{array}.$$

If $0 \in S$ then $A_S = \{0\}$. If $f \in A$ then we write

$$A_f = A_{\{f^n\}_{n=0,1\ldots}}.$$

VI.1.2 The spectrum of a ring A.

If A is a commutative ring with identity then we define the spectrum of A as

$$\mathrm{Spec}(A) = \{\mathfrak{p} | \ \mathfrak{p} \quad \text{prime ideal in} \quad A\}.$$

The spectrum of A is ordered the ordering is given by the inclusion among prime ideals. It is functorial in A, if we have a homomorphism

$$\phi : A \to B$$

then it induces

$$^t\phi : \mathrm{Spec}(B) \to \mathrm{Spec}(A)$$
$$^t\phi(\mathfrak{p}) = \phi^{-1}(\mathfrak{p}) = \{f \mid \phi(f) \in \mathfrak{p}\}$$

and $^t\phi$ respects the order relation.

The spectrum $\mathrm{Spec}(A)$ always contains as a subset the set of *maximal* ideals. A maximal ideal $\mathfrak{m} \subset A$ is an ideal with $1_A \notin \mathfrak{m}$ and for any ideal $\mathfrak{m}'$ with

$$\mathfrak{m} \subseteq \mathfrak{m}' \subseteq \ A$$

we have $\mathfrak{m} = \mathfrak{m}'$ or $\mathfrak{m}' = A$. We have a different characterization

Lemma VI.1.2.1 : *An ideal* $\mathfrak{m} \subset A$ *with* $1_A \notin \mathfrak{m}$ *is maximal if and only if* $A/\mathfrak{m}$ *is a field. Maximal ideals are prime ideals.*

I think this should be clear.

The set of maximal ideals is denoted by $\mathrm{Specmax}(A) \subset \mathrm{Spec}(A)$.

Zorn's lemma implies:

1.2.1.1.*If A is a commutative ring with $1_A \neq 0$, then* $\mathrm{Specmax}(A) \neq \emptyset$.

I want to give brief indications of the proof: A *chain* of proper ideals is a totally ordered subset $\mathfrak{K}$ of the set of proper ideals, this means that for any pair $\mathfrak{a}, \mathfrak{b} \in \mathfrak{K}$ we have $\mathfrak{a} \subseteq \mathfrak{b}$ or $\mathfrak{b} \subseteq \mathfrak{a}$. For any chain we can form

$$\bigcup_{\mathfrak{a} \in K} \mathfrak{a} = \mathfrak{a}^*,$$

this is an ideal with $1_A \notin \mathfrak{a}^*$ and $\mathfrak{a}^* \supset \mathfrak{a}$ for all $\mathfrak{a} \in \mathfrak{K}$. Hence we see that for any chain of proper ideals we can find a proper ideal which contains all elements of the chain. Now it is simply the assertion of Zorn's lemma that this implies the existence of a (proper) maximal ideal. This has as a consequence:

Lemma VI.1.2.2 : *If $f \in A$ is not nilpotent, then* $\mathrm{Spec}(A_f) \neq \emptyset$. *Hence we get*

$$\mathrm{Rad}\,(A) = \textit{Ideal of nilpotent elements} = \bigcap_{\mathfrak{p} \in \mathrm{Spec}(A)} \mathfrak{p}.$$

A commutative ring is called a *local ring* if it has a unique maximal prime ideal.

If $\mathfrak{p} \in \mathrm{Spec}(A)$ then the complement $S = A \setminus \mathfrak{p}$ is closed under multiplication. Then we write (abuse of notation)

$$A_{(A \setminus \mathfrak{p})} =: A_{\mathfrak{p}}.$$

The ring $A_{\mathfrak{p}}$ is called the *local ring at* $\mathfrak{p}$. The ideal

$$\mathfrak{m}_{\mathfrak{p}} = \{\frac{f}{g} | f \in \mathfrak{p}, g \notin \mathfrak{p}\}.$$

is the unique maximal. The field $k(\mathfrak{p}) = A_{\mathfrak{p}}/\mathfrak{m}_{\mathfrak{p}}$ is called the residue field at $\mathfrak{p}$.

Lemma VI.1.2.3 : *If we consider any multiplicatively closed $S \subset A$ and the localization $\phi : A \to A_S$ then ${}^t\phi$ is an inclusion. We get*

$${}^t\phi : \mathrm{Spec}(A_S) \xrightarrow{\sim} \{\mathfrak{p} \in \mathrm{Spec}(A) | \mathfrak{p} \cap S = \emptyset\}.$$

If especially $S = \{f^n\}_{n=0,1\ldots}$ then

$$\mathrm{Spec}(A_f) = \{\mathfrak{p} \in \mathrm{Spec}(A) | f \notin \mathfrak{p}\}.$$

If $\mathfrak{p} \in \mathrm{Spec}(A)$ *then*

$$\mathrm{Spec}(A_{\mathfrak{p}}) = \{\mathfrak{q} \in \mathrm{Spec}(A) \mid \mathfrak{q} \subset \mathfrak{p}\}.$$

The proof is left to the reader.

VI.1.2.4 *Heuristical remarks*:

(i) The spectrum of a ring is a geometric object attached to the ring. At this point it is simply an ordered set, but soon we will put a topology onto this space (*The Zariski Topology*). We already spoke of maximal ideals. If our ring A is integral then the zero ideal (0) is also a prime ideal which then is the unique minimal element. This ideal (0) is called the *generic point* of $\mathrm{Spec}(A)$.

(ii) Intuitively we want to consider A as a ring of functions on $\mathrm{Spec}(A)$. This is not quite the case because these functions do not have a common domain of values. But it makes sense to say that $f \in A$ "vanishes" at $\mathfrak{p} \in \mathrm{Spec}(A)$: By this we mean that $f \in \mathfrak{p}$. Sometimes we will write $f(\mathfrak{p}) = 0$, (resp. $f(\mathfrak{p}) \neq 0$) for $f \in \mathfrak{p}$ (resp. $f \notin \mathfrak{p}$).

Examples.

(1) The ring $\mathbb{Z}$ and the polynomial ring $k[X]$ are principal ideal domains. This implies immediately that the maximal ideals are of the form

$$\mathfrak{p} = (p) \,\text{resp.}\, \mathfrak{p} = (p(x))$$

where p is a prime number (resp. $p(X) \in k[X]$ is a non-constant irreducible polynomial). Both rings contain one more prime ideal namely $\mathfrak{p} = (0)$ because they are integral. Hence

$$\begin{array}{r} \mathrm{Spec}(\mathbb{Z}) = (0) \cup \{(2), (3), (5), \ldots\} \\ \mathrm{Spec}(k[X]) = (0) \cup \{(X), (X-1), \ldots\} \end{array}.$$

(Of course not all irreducible polynomials are linear, but I cannot write down any other polynomial which is irreducible regardless what the field k is).

(2) Let us assume that k is a field. We consider the polynomial ring $A = k[X_1, X_2, \ldots, X_n]$ in n variables. For any point $P = (a_1, a_2, \ldots, a_n) \in k^n$ we get an evaluation homomorphism

$$\begin{array}{c} \phi_P : A \to k \\ \phi_P : f \mapsto f(P), \end{array}$$

whose kernel is the maximal ideal $\mathfrak{m}_P = (X_1 - a_1, X_2 - a_2, \ldots, X_n - a_n)$.

If our field k is algebraically closed then the Nullstellensatz of Hilbert (It will be discussed in the next section on commutative algebra) says that we get an identification

$$\mathrm{Specmax}(k[X_1, X_2, \ldots, X_n]) = k^n.$$

In other words the maximal ideals are exactly the ideals of the form $\mathfrak{m} = \mathfrak{m}_P$.

Exercise (1):

(i) Prove the Nullstellensatz in the case of a polynomial ring in one variable.

(ii) Try to prove it in the case of a polynomial ring $A = k[X, Y]$ in two variables.

Hint: Let $\mathfrak{m}$ be a maximal ideal. It cannot be the zero ideal.

Step 1:

Assume it contains an element of the form $F(X, Y) = Y^m + g_1(X)Y^{m-1} + \ldots g_m(X)$ where the g_i are polynomials in X. Now we get a diagram

$$\begin{tikzcd} A \arrow[rr] \arrow[dr] & & A/(F) = B \arrow[dl] \\ & A/\mathfrak{m} & \end{tikzcd}$$

The ring B contains the polynomial ring $k[X] = B_0$ and over this ring it is generated by an element y which satisfies the relation $y^m + g_1(X)y^{m-1} + \ldots g_m(X) = 0$. The maximal ideal $\mathfrak{m} \subset A$ has as its image a maximal ideal $\bar{\mathfrak{m}}$ in B.

Prove that $\bar{\mathfrak{m}} \cap k[X]$ is a maximal prime ideal! (In this case it suffices to show it is not zero.) Hence $B_0/\bar{\mathfrak{m}} = k$ and since $B/\mathfrak{m}$ is a finite extension of k it follows that $B/\mathfrak{m} = k$.

Step 2: We know that $\mathfrak{m}$ contains a non zero polynomial

$$F(X,Y) = \sum_{\nu\mu} a_{\nu,\mu} X^\nu Y^\mu.$$

Write this as a polynomial in Y with coefficients polynomials in X. Now it will not be the case in general that the highest power of Y occuring in the polynomial has a constant coefficient not involving X. But if we make a substitution

$$X \to X + Y^m = X'$$

$$Y \to Y$$

then $k[X', Y = k[X,Y]$ for $m >> 0$ the new polynomial will satisfy the assumption in step 1 .

It is known that a polynomial ring $k[X_1, \ldots, X_n]$ has unique factorization (See [Ei],???). This implies that any non constant irreducible polynomial $f \in k[X_1, \ldots, X_n]$ defines a prime ideal $\mathfrak{p} = (f)$. This means of course that for instance $\mathrm{Spec}(k[X,Y])$ contains many more elements than just (0) and the maximal ones: Any irreducible polynomial

$$p(X,Y) = X + Y \text{ or } X^2 + Y^3 \text{ or} \ldots$$

defines a prime ideal $\mathfrak{p} = (p(X,Y))$.

If k is algebraically closed then the Nullstellensatz allows us to identify $\mathfrak{p} = (p(X,Y))$ with

$$V(\mathfrak{p}) = \{(a,b) \in k^2 | f(a,b) = 0 \quad \text{for all} \quad f \in \mathfrak{p}\}$$

this is the set of common zeroes of the elements in $\mathfrak{p}$ or the set of zeroes of $p(X,Y)$ and also the set of maximal ideals containing $\mathfrak{p}$. Hence we get an injection into the power set

$$\mathrm{Spec}(k[X,Y]) \longrightarrow \mathfrak{P}(\mathrm{Specmax}(k[X,Y]))\mathfrak{p} \longmapsto V(\mathfrak{p}).$$

Then the maximal ideals correspond to the sets consisting of one element, the prime ideals $\mathfrak{p} = (p(X,Y))$ give hypersurfaces and $\mathfrak{p} = (0)$ gives us the entire plane.

(3) Let k be arbitrary and

$$A = k[X,Y](XY) = k[x,y].$$

The elements x, y satisfy $xy = 0$. Hence this ring has zero divisors.

A prime ideal $\mathfrak{p}$ in A has to contain either x or y. On the other hand the principal ideals $\mathfrak{p} = (x)$ and $\mathfrak{q} = (y)$ are prime because after dividing by them we get polynomial rings in the other variable.

We see that

$$\mathrm{Spec}(k[x,y]) = \mathrm{Spec}(k[x]) \cup_{(0,0)} \mathrm{Spec}(k[y])$$

where the two spectra are identified at $(x, y) = (0, 0)$.

This is an example of a *reducible* spectrum.

VI.1.3. The Zariski Topology on $\mathrm{Spec}(A) = X$.

We define a topology on the space X. To do so we have to define what open sets are. At first we declare the sets of the form

$$X_f = \mathrm{Spec}(A_f) \subset X$$

open. We saw that X_f was the set of prime ideals $\mathfrak{p}$ which do not contain f. In our remark (VI.1.2.4..) we said that this means "f does not vanish at $\mathfrak{p}$." Hence our topology has the property that the sets where a given $f \in A$ is not zero are open set.

This system of sets is closed under finite intersection because

$$X_{f_1} \cap \ldots \cap X_{f_s} = X_{f_1 \ldots f_s}.$$

These open sets are called *affine open sets* the reason is that they are again equal to a spectrum of a ring. This system of affine open sets forms a basis for the Zariski topology and this means that a set $U \subset X$ is open if and only if it is the union of the affine open sets which are contained in U.

A subset $Y \subset X$ is closed if the complement $X \backslash Y$ is open. Of course this means that Y is the set of common zeroes of a collection of elements in A.

The Zariski topology is not Hausdorff in general. It has other strange properties one has to get used to:

Exercise (2):

If $\mathfrak{p} \in \mathrm{Spec}(A)$ *then the closure of the set* $\{\mathfrak{p}\}$ *is given by*

$$\overline{\{\mathfrak{p}\}} = \{\mathfrak{q} \in \mathrm{Spec}(A) | \mathfrak{q} \supset \mathfrak{p}\}.$$

A point $\mathfrak{p} \in \mathrm{Spec}(A)$ *is closed if and only if* $\mathfrak{p}$ *is maximal.*

If $\mathfrak{q}$ is in the closure of $\{\mathfrak{p}\}$ then we say that $\mathfrak{q}$ is a *specialization* of $\mathfrak{p}$.

Examples: In our rings $\mathbb{Z}$ and $k[X]$ the closed points are the principal ideals (p) resp. $(p(x))$ where p is a prime (resp. $p(x)$ a non-constant irreducible polynomial). The generic point (0) is dense in $\mathrm{Spec}(A)$ in both cases. The open sets are the complements of finite sets of closed points and the empty set.

Here it becomes quite clear that $\mathrm{Spec}(\mathbb{Z})$ and $\mathrm{Spec}(k[X])$ are not Hausdorff.

For an integral ring A the generic point (0) always dense in the space $\mathrm{Spec}(A)$. Every prime ideal $\mathfrak{p} \in \mathrm{Spec}(A)$ is a specialization of the generic point.

VI.1.3.1.General remark:

We have put some further structure onto the set $\mathrm{Spec}(A)$: Now it is a topological space. But still this space does not yet contain a lot of information on the original ring A. If for instance A is a field then $\mathrm{Spec}(A)$ is a single point which will never be able to recover the field A.

This is different for finitely generated algebras $A = k[x_1, x_2, \ldots, x_n]$ over an algebraically closed field k. In the next section we will see that the Nullstellensatz implies (See CA.1.3.)

$$\bigcap_{\mathfrak{m} \in \mathrm{Specmax}(A)} \mathfrak{m} = \mathrm{Rad}(A).$$

If our k-algebra is reduced, i.e. if $\mathrm{Rad}(A) = 0$ then this implies that we can view A as an an algebra of k-valued functions on $\mathrm{Specmax}(A)$ This discussion is resumed in VI. 2.5. and 2.6.

Exercise (3):

Let k be an algebraically closed field and let

$$A = k[x_1, x_2, \ldots, x_n], B = k[y_1, y_2, \ldots, y_m]$$

be two finitely generated reduced k-algebras. Let $\phi : A \to B$ be a homomorphism which is the identity on k (a k- algebra homomorphism). Show that this induces a map $\phi^* : \mathrm{Specmax}(B) \to \mathrm{Specmax}(A)$. We assume in addition that A and B do not contain nilpotent elements.

Prove: (a) ϕ^* is injective $\Longleftrightarrow$ ϕ is surjective.

(b) If ϕ^* is surjective then ϕ is injective.

(c) The map ϕ^* determines ϕ.

The following property of the topological space $\mathrm{Spec}(A)$ is very important and perhaps a little bit surprising at the first glance

Proposition VI.1.3.2: *The space $X = \mathrm{Spec}(A)$ is quasicompact, this means that for any covering $\bigcup_{i \in I} U_i$ by open sets U_i we can find a finite subcovering, i. e. we can find a finite subset $E \subset I$ such that $X = \bigcup_{i \in E} U_i$*

Proof: The U_i are open, hence we can cover each of them by open sets of the form $X_{f_{i,\nu}}$. Therefore it is clear that we may assume that the U_i themselves are of this form $U_i = X_{f_i}$. Now we consider the ideal generated by the f_i it consists of the finite linear combinations

$$\mathfrak{a} = \{\sum_{i \in I} g_i f_i \mid \text{ almost all } g_i = 0\}$$

This cannot be a proper ideal because otherwise we could find a maximal ideal $\mathfrak{m}$ containing $\mathfrak{a}$ (See 1.2.1.1.). Then we have $f_i(\mathfrak{m}) = 0$ for all $i \in I$ and hence $\mathfrak{m} \notin \bigcup_{i \in I} X_{f_i}$. This implies that $\mathfrak{a} = A$ and hence the identity 1_A is in our ideal. We can find a finite linear combination $1_A = \sum_{i \in E} g_i f_i$ with $E \subset I$ finite. But then it is clear that $X = \bigcup_{i \in E} X_{f_i}$ because if there would be a $\mathfrak{p}$ not contained in this union then we would have $f_i(\mathfrak{p}) = 0$ for all $i \in E$ and hence $1_a(\mathfrak{p}) = 0$ which cannot be the case if our ring is not the zero ring. But for this last ring the spectrum is empty so the claim is also clear.

Our next goal will be to put more structure on $X = \mathrm{Spec}(A)$. Since it is already a topological space we have the notion of a sheaf on this space. We will construct the sheaf of regular functions on $\mathcal{O}_X = \mathrm{Spec}(A)$ and then $(X, \mathcal{O}_X)$ will be a locally ringed space.

VI.1.4 The structure sheaf on $\mathrm{Spec}(A)$:

We want to introduce the structure of a locally ringed space on $X = \mathrm{Spec}(A)$. This means that we want to construct a sheaf of rings $\tilde{A}$ on X which plays the role of the sheaf of regular functions on X. It will turn out- but this will be a theorem- that the ring of regular functions on the total space is again A.

We make the following Ansatz: If we have an open set $X_f \subset X$ then the element $1/f \in A_f$ should be a regular function on the affine open set X_f. Hence we define $\tilde{A}'(X_f) = A_f$. If we have $h = gf$ then $A_h = (A_f)_g$ then the map $\phi_{\{g^n\}_{\{n=0,1,\dots\}}} : A_f \to A_h$ gives us a restriction map $\tilde{A}'(X_f) \to \tilde{A}'(X_{gf}) = X_h$. This obviously satisfies the transitivity relation for presheaves this means that the restriction from $\tilde{A}'(X_f)$ to X_{fg} and then composed with the restriction to X_{fgl} is equal to the restriction from X_f to X_{fgl}. We will denote the restriction map $\phi_{\{g^n\}_{\{n=0,1,\dots\}}} : A_f \to A_h$ also by $\mid X_h$. Hence our $\tilde{A}'$ is something like a presheaf except that it is not yet defined it on all open set but only on the affine open sets of the form X_f.

We now have a proposition which says that $\tilde{A}'$ satisfies the axioms (Sh1),(Sh2) when we restrict them to these special open sets.

Proposition VI.1.4.1..: *If we have an arbitrary covering $X = \bigcup_{i\in I} X_{f_i}$ then the sequence*

$$\tilde{A}'(X) \xrightarrow{p_0} \prod_{i\in I} \tilde{A}'(X_{f_i}) \underset{p_2}{\overset{p_1}{\rightrightarrows}} \prod_{(i,j)\in I\times I} \tilde{A}'(X_{f_i f_j}),$$

is exact, this means that the first arrow is injective and its image is exactly the set of elements which become equal under p_1 and p_2.

This proposition is really central.

Proof: Since our space is quasicompact, we can find a finite subset $E \subset I$ such that already $X = \bigcup_{i\in E} = X$. I assume that I proved exactness for this finite covering. I want to show that then we have exactness for the original covering. We get a map from the diagram above to the corresponding diagram for our finite covering.

$$\begin{array}{ccccc}
\tilde{A}'(X) & \xrightarrow{p_0} & \prod_{i\in I} \tilde{A}'(X_{f_i}) & \underset{p_2}{\overset{p_1}{\rightrightarrows}} & \prod_{(i,j)\in I\times I} \tilde{A}'(X_{f_i f_j}) \\
\downarrow & & \downarrow & & \downarrow \\
\tilde{A}'(X) & \xrightarrow{p_0} & \prod_{i\in E} \tilde{A}'(X_{f_i}) & \underset{p_2}{\overset{p_1}{\rightrightarrows}} & \prod_{(i,j)\in E\times E} \tilde{A}'(X_{f_i f_j}),
\end{array}$$

The injectivity of the first arrow this is quite obvious, because the arrow p_0 for the second diagram is the composite of the p_0 for the first diagram and the projection from the product over I to the product over E. Now let us take an element $(\dots, \frac{g_i}{f_i^{n_i}}, \dots)_{i\in I}$ in the first diagram with $p_1((\dots, \frac{g_i}{f_i^{n_i}}, \dots)) = p_2((\dots, \frac{g_i}{f_i^{n_i}}, \dots))$. If we project it to the second diagram then the image is also equalized by the corresponding two arrows. Hence by our assumption it comes from an element $g \in A$, this means that the image of g in A_{f_i} is equal to $\frac{g_i}{f_i^{n_i}}$ for all $i \in E$. We have to show that this implies that g actually maps to

$(\ldots,\frac{g_i}{f_i^{n_i}},\ldots)_{i\in I}$. This means that we have show that g maps to $\frac{g_i}{f_i^{n_i}}$ for all $i\in I$. Let us pick an $i\in I$. We know that $X_{f_i}=\bigcup_{e\in E}(X_{f_i}\cap X_{f_e})=\bigcup_{e\in E}X_{f_if_e}$. But then we have $g\mid X_{f_if_e}=(g\mid X_{f_e})\mid X_{f_ef_i}=\frac{g_e}{f_e^{n_e}}\mid X_{f_ef_i}=\frac{g_i}{f_i^{n_i}}\mid X_{f_ef_i}$ for all $e\in E$. Now we need a little remark. The open set X_{f_i} is again the spectrum of a ring. Hence everything we proved for X is also valid for X_{f_i}. Especially we can assume that $A_{f_i}\to\prod_{e\in E}A_{f_if_e}$ is injective. We have seen that $g\mid X_{f_if_e}=\frac{g_i}{f_i^{n_i}}\mid X_{f_ef_i}$ for all $e\in E$ hence we conclude $g\mid X_{f_i}=\frac{g_i}{f_i^{n_i}}\mid X_{f_i}$ for all $i\in I$. Hence the reduction to the case of a finite covering is complete and therefore we assume now that our covering is finite.

If the homomorphism p_0 is not injective then we have have an element $f\in A$ and $f|X_{f_i}=0$ for all $i\in I$. This means that we can find exponents n_i so that $ff_i^{n_i}=0$ in the ring A. Since I is finite we can assume that all these exponents are equal to a fixed integer n.

We have seen in the proof of the compactness of X that we can find $g_i\in A$ such that

$$\sum g_if_i=1.$$

Raising this to a suitable high power N we get a relation

$$\sum G_if_i^n=1$$

and hence $f=f1=\sum_{i\in I}G_if_i^nf=0$ and proves injectivity.

Now let us assume we have an element

$$(\ldots,\frac{g_i}{f_i^{n_i}},\ldots)_{i\in I}$$

for which

$$\frac{g_i}{f_i^{n_i}}|X_{f_i,f_j}=\frac{g_i}{f_j^{n_j}}\mid X_{f_i,f_j},$$

for all pairs $(i,j)\in I\times I$. Again we may assume that all n_i are equal.

Then the equality means that we can find an integer N so that

$$(g_if_j^n-g_jf_i^n)(f_if_j)^N=0.$$

We are searching an element $g\in A$ which satisfies

$$g|X_{f_i}=\frac{g_i}{f_i^n}$$

for all $i\in I$. This is certainly true if

$$f_i^ng=g_i\quad\text{for all}\quad i\in I.$$

But again we can find $H_\nu \in A$ for $\nu \in I$ such that

$$\sum_{\nu\in I} H_\nu f_\nu^n = 1$$

and we see that $g = \sum_{\nu\in I} H_\nu g_\nu$ solves our problem.

Still we have not yet defined our sheaf $\tilde{A}$. For an arbitrary open set $U \subset X$ we choose a covering $U = \bigcup_{i\in I} X_{f_i}$ and define (See SH xxx)

$$\tilde{A}(U) = (\prod_{i\in I} A_{f_i} \rightrightarrows \prod_{(i,j)\in I\times I} A_{f_i f_j})[p_1 = p_2].$$

We have to verify that this does not depend on the covering and really defines a sheaf. I will not do this in detail, the proof is a little bit tedious. To prove the independence of the covering we first pass to a refinement of the covering: We have $\tau : J \to I$ and

$$X_{f_i} = \bigcup_{\nu\in\tau^{-1}(i)} X_{f_i h_\nu}.$$

We put $\tilde{h}_\nu = f_i h_\nu$ (the index ν determines the index i) and $X = \bigcup_{\nu\in J} X_{\tilde{h}_\nu}$.

We get a diagram

$$\begin{array}{ccc}
\prod_{i\in I} A_{f_i} & \rightrightarrows & \prod_{(i,j)\in I\times I} A_{f_i f_j} \\
\downarrow & & \downarrow \\
\prod_{\nu\in J} A_{\tilde{h}_\nu} & \rightrightarrows & \prod_{(\nu,\mu)\in J\times J} A_{\tilde{h}_\nu,\tilde{h}_\mu}
\end{array}$$

and is easy to check that if we apply the proposition VI.1.4.1. to the vertical arrows that we get an isomorphism between the $[p_1 = p_2]$ subrings.

Then we may compare two coverings by passing to a common refinement.

The fact that $U \to \tilde{A}(U)$ is actually a sheaf can be derived by similar arguments as those used in the sheafification process. The intuitive meaning of $\tilde{A}(U)$ is clear: These are the "regular functions" on U and these are "functions" which can locally be written in the form $\frac{g_i}{f_i^{n_i}}$ in such a way that $\frac{g_i}{f_i^{n_i}}, \frac{g_j}{f_j^{n_j}}$ match on the intersection of their "domains of definition" which of course is $X_{f_i f_j}$.

Our proposition says that $\tilde{A}(X_{f_i}) = A_{f_i}$ and this means that a "regular function" on affine open sets X_{f_i} has always a kind of "global" description which uses only denominators of the form $f_i^{n_i}$. Especially we have $\tilde{A}(X) = A$.

VI.1.5. **Quasicoherent sheaves**

Our considerations can be generalized. If we have an A-module M a set $S \subset A$ which is closed under multiplication and contains 1_A, then we define

$$M_S = \{(m,s) | m \in M, s \in S\} / \sim$$

where the equivalence relation is

$$(m,s) \sim (m',s') \iff \exists s" \in S \quad \text{such that} \quad (ms' - m's)s" = 0.$$

It is quite clear that this defines an A_S-module M_S.

Now we can construct a sheaf $\tilde{M}$ of $\tilde{A}$-modules just by defining

$$\tilde{M}(X_f) = M_f$$

then verifying the proposition (just replace A by M everywhere) and then we put

$$\tilde{M}(U) = (\prod_{i \in I} M_{f_i} \overset{\longrightarrow}{\longrightarrow} \prod_{(i,j) \in I \times I} M_{f_i f_j})[p_1 = p_2].$$

The stalk of the sheaf $\tilde{A}$ at a point $\mathfrak{p}$ is the local ring $A_\mathfrak{p}$, the stalk of $\tilde{M}$ in $\mathfrak{p}$ is the $A_\mathfrak{p}$-module $M_\mathfrak{p} = M_{(A \setminus \mathfrak{p})}$.

It can happen that the stalk $M_\mathfrak{p}$ vanishes in some points. This is so if for any $m \in M$ we can find an $s \in A \setminus \mathfrak{p}$ such that $sm = 0$. The module M defines an ideal

$$\mathrm{Ann}(M) = \{f \in A | fM = 0\}.$$

It is clear that $M_\mathfrak{p} \neq 0$ is equivalent to $\mathfrak{p} \supset \mathrm{Ann}(M)$. The set of these $\mathfrak{p}$ is called the *support* of $\tilde{M}$ and is a closed subset in $\mathrm{Spec}(A)$.

It is not so that any sheaf $\mathcal{M}$ of $\tilde{A}$-modules is automatically of the form $\tilde{M}$ with some A-module M. On $\mathrm{Spec}(\mathbb{Z}_{(p)})$ we have the sheaf

$$\mathcal{M}(\{(0)\}) = \mathbb{Q}$$
$$\mathcal{M}(\mathrm{Spec}(\mathbb{Z}_{(p)})) = 0$$

which is not of this form.

The sheaves $\tilde{M}$ which are obtained from an A-module M are called the *quasicoherent* sheaves on $\mathrm{Spec}(A)$. We can recover the A-module from the sheaf since $M = \tilde{M}(X)$.

We will say that $\tilde{M}$ is the sheaf *obtained from* M

We get quasicoherent sheaves of ideals on $X = \mathrm{Spec}(A)$ by starting from an ideal $I \subset A$, this is an A-module and the sheaf

$$\tilde{I} \subset \tilde{A}$$

is a quasicoherent sheaf of ideals.

VI.1. 6 **Schemes as locally ringed spaces:**

An *affine scheme* is a locally ringed space of the form $(X, \mathcal{O}_X) = (\mathrm{Spec}(A), \tilde{A})$. (See SH x.x.x)

I recall the definition of a morphism between two locally ringed spaces. A morphism is a pair

$$(f,h):(X,\mathcal{O}_X)\to(Y,\mathcal{O}_Y)$$

where f is a continuous map from X to Y and h is a map of sheaves of rings

$$h:f^*(\mathcal{O}_Y)\to\mathcal{O}_X$$

which induces local homomorphisms

$$h_x:\mathcal{O}_{Y,f(x)}=f^*(\mathcal{O}_Y)_x\to\mathcal{O}_{X,x}$$

on the stalks. This means that the maximal ideal of $\mathcal{O}_{Y,f(x)}$ is mapped into the maximal ideal of $\mathcal{O}_{X,x}$. The locally ringed spaces form a category we have an obvious way of composing two morphisms.

VI.1.6.1.*Heuristical remark*:

The difficulty is as always that the sections of the sheaves are not actual functions, they are elements in very abstract rings. In our previous examples ($\mathcal{C}_\infty$-manifolds, complex manifolds (See SH xxx)) a continous map $f:X\to Y$ between the spaces gave us a map h_0 from the sheaves of continuous functions on Y to the continuous functions on X. Then we made requirements that this map should respect certain distinguished subsheaves of functions which define a "so and so" structure on X and Y. If this was the case we called f a "so and so" map. The map h_0 was determined by f in such a case. Especially it is clear in these examples that a germ f at a point $y\in Y$ with $f(y)=0$ is mapped by h_0 to a germ at $x\in f^{-1}(y)$ which vanishes at x. This means the h_0 is automatically local.

Here the situation is different, the map is $h:f^*(\mathcal{O}_Y)\to\mathcal{O}_X$ is an extra datum.

What is left from the notion of functions is the notion of vanishing of a section $f\in\mathcal{O}_X(U)$ in a point $x\in U$ (VI.1.2.4.). Now the requirement that h induces local homomorphism in the stalks becomes clear: A germ in $\mathcal{O}_{Y,f(x)}$ which vanishes in $f(x)$ must be sent by h to a germ in $\mathcal{O}_{X,x}$ which vanishes at x. The reader should observe that a germ in $\mathcal{O}_{Y,f(x)}$ which does not vanish at $f(x)$ is a unit and hence it goes automatically to a germ in $\mathcal{O}_{X,x}$ which does not vanish in x.

End of the remark

The following theorem is fundamental.

Theorem. VI.1.6.2 *Let* $(X,\mathcal{O}_X)=(\mathrm{Spec}(A),\tilde{A})$ *and* $(Y,\mathcal{O}_Y)=(\mathrm{Spec}(B),\tilde{B})$ *be affine schemes. A morphism*

$$(f,h):(X,\mathcal{O}_X)\to(Y,\mathcal{O}_Y)$$

defines a map $h_X:\mathcal{O}_Y(Y)\to\mathcal{O}_X(X)$ *i.e. a homomorphism*

$$h_X:B\to A.$$

The map

$$\mathrm{Hom}_{\text{affine schemes}}((X,\mathcal{O}_X),(Y,\mathcal{O}_Y))\to\mathrm{Hom}_{\mathrm{Rings}}(B,A)$$

given by

$$(f,h) \to h_X$$

is a bijection.

Proof: We start by constructing a map in the other direction and then we show that the maps are inverse to each other.

Given $\phi : B \to A$ we have defined a map

$$\begin{aligned} &{}^t\phi : \mathrm{Spec}(A) \to \mathrm{Spec}(B) \\ &{}^t\phi(\mathfrak{p}) = \phi^{-1}(\mathfrak{p}) \end{aligned}.$$

If we have an element $b \in B$ we get an open set $Y_b = \{\mathfrak{q} | b \notin \mathfrak{q}\}$ in Y and it is clear that

$${}^t\phi^{-1}(Y_b) = X_{\phi(b)}.$$

Hence our map is continuous and we get maps

$$\phi_b : \mathcal{O}_Y(Y_b) = B_b \to f_*(\mathcal{O}_X)(Y_b) = \mathcal{O}_X(X_{\phi(b)}) = A_{\phi(b)}$$

which by the adjointness formula is nothing else than a map

$$\tilde{\phi} : f^*(\mathcal{O}_Y) \to \mathcal{O}_X$$

and hence we constructed a morphism $({}^t\phi, \tilde{\phi})$ between locally ringed spaces.

We have to show that these maps are inverse to each other. At first we start from $\phi : B \to A$ we get $({}^t\phi, \tilde{\phi})$. From this we construct again a homomorphism from $B \to A$. According to our rules we have to evaluate $\tilde{\phi}$ on the pair X, Y and get

$$\tilde{\phi}_X : B \to A$$

which is our original map.

Now we start from (f,h) The map h can be evaluated on X, Y this gives us

$$h_X : B \to A.$$

We have to prove at first that the map

$${}^t h_X : X \to Y$$

is equal to f. We have ${}^t h_X(\mathfrak{p}) = h_X^{-1}(\mathfrak{p}) = \mathfrak{q}$.

Since we know that h induces a morphism between the sheaves we get a diagram

$$\begin{array}{ccc} h_X : B & \to & A \\ \downarrow & & \downarrow \\ h_{\mathfrak{p}} : B_{f(\mathfrak{p})} & \to & A_{\mathfrak{p}} \end{array}.$$

This implies that h_X has to map elements $b \in B \backslash f(\mathfrak{p})$ to elements $h_X(b) \in A \backslash \mathfrak{p}$ because b becomes invertible in $B_{f(\mathfrak{p})}$ and hence $h_{\mathfrak{p}}$ has to map it to a unit in $A_{\mathfrak{p}}$. This implies

$$B \backslash f(\mathfrak{p}) \subset B \backslash h_X^{-1}(\mathfrak{p})$$

and hence $h_X^{-1}(\mathfrak{p}) \subset f(\mathfrak{p})$. We also know that $h_{\mathfrak{p}}$ maps the maximal ideal $\mathfrak{m}_{f(\mathfrak{p})}$ into the maximal ideal $\mathfrak{m}_{\mathfrak{p}}$. Hence it maps the elements of $f(\mathfrak{p})$ into $\mathfrak{p}$ and this implies $f(\mathfrak{p}) \subset h_X^{-1}(\mathfrak{p})$ and we have the desired equality for $f(\mathfrak{p}) = h_X^{-1}(\mathfrak{p}) =^t h_X(\mathfrak{p})$.

The rest is clear, the map $\tilde{h}_X$ which we construct from h_X is obviously equal to h since these two coincide on the global sections.

VI 1.6.1 **Closed subschemes**:

We start from an ideal $I \subset A$. We have the projection map $\pi : A \to A/I$ and we have $\operatorname{Spec}(A/I) = \{\mathfrak{p} \mid \mathfrak{p} \supset I\} = V(I)$. If $i : V(I) \to \operatorname{Spec}(A)$ is the inclusion then we consider the map

$$(i, \tilde{\pi}) : (V(I), \widetilde{A/I}) \to (Spec(A), \tilde{A})$$

as a closed subscheme of $(\operatorname{Spec}(A), \tilde{A})$.

At this point the reader might wonder: We made a lot of effort to show the following: something seemingly simple namely the category of commutative rings with identity is antiequivalent category of locally ringed spaces and this category consists of rather complicated objects and the morphisms are also not so easy to define.

The reason why we do this will become clear: These concepts allow us to glue affine schemes together so that we can build larger objects namely schemes. Locally these schemes look like affine schemes but globally they look different.

VI. 2. Schemes:

VI 2.1 The definition of a scheme:

An arbitrary *scheme* is a locally ringed space $(X, \mathcal{O}_X)$ which is locally isomorphic to an affine scheme. In other words we can find a covering $X = \bigcup_\nu U_\nu$ by open sets such that $(U_\nu, \mathcal{O}_X \mid U_\nu)$ is affine.

This implies of course that $(U_\nu, \mathcal{O}_X \mid U_\nu) = (\operatorname{Spec}(A_\nu), \tilde{A}_\nu)$ where $A_\nu = \mathcal{O}_X(U_\nu)$ is the ring of regular functions on U_ν. But in contrast to the case of affine scheme the ring of regular functions on X may be to small to contain enough information to recover the scheme $(X, \mathcal{O}_X)$. This will be demonstrated in the next section on projective schemes.

Example VI.2.1.1:

Let us consider the polynomial ring $A = k[f, g]$, let $X = \operatorname{Spec}(A)$. We remove the point $(0, 0)$ from X, the resulting space U inherits a topology and the structure of a locally ringed space. It is clearly a scheme since we may cover it by

$$U = X_f \cup X_g$$

.

But U is not an affine scheme since we have $\mathcal{O}_U(U) = A$ and $U \neq \mathrm{Spec}(A)$. We are not able to produce a regular function on U which is not regular on X.

This example shows that any open subset of an affine scheme is a scheme but not an affine scheme in general.

2.1.3 :Again we have the notion of a *quasicoherent* $\mathcal{O}_X$*-module* $\mathcal{M}$. This means that for all open subsets U the sections $\mathcal{M}(U)$ form an $\mathcal{O}_X(U)-$module and for affine open subsets U the restriction $\mathcal{M} \mid U$ is obtained from the $\mathcal{O}_X(U)$ module $\mathcal{M}(U)$ (See 1.5.1.).

2 .1.4. At his point it is rather clear what a *closed subscheme* of a general scheme is. We know how to define a quasicoherent sheaf of ideals $\mathcal{I} \subset \mathcal{O}_X$: It is a sheaf of ideals, i.e. for any open subset $U \subset X$ the sections $\mathcal{I}(U) \subset \mathcal{O}_X(U)$ form an ideal in $\mathcal{O}_X(U)$ and for an affine U the restriction $\mathcal{I} \mid U$ is the sheaf associated to $\mathcal{I}(U)$. On this open affine subset U we have the closed subscheme $((V(\mathcal{I}(U)), \tilde{(}A(U)/\mathcal{I}(U)))), \tilde{(}A(U)/\mathcal{I}(U))) \hookrightarrow (U, \tilde{A}(U))$. We define $V(\mathcal{I})$ to be the union of all these subsets $(V(\mathcal{I}(U)), \tilde{(}A(U)/\mathcal{I}(U)))$ and a quotient sheaf $\mathcal{O}_X/\mathcal{I}$ by its restriction to the affine pieces. This yields the closed subscheme $(V(\mathcal{I}), \mathcal{O}_X/\mathcal{I}) \hookrightarrow (X, \mathcal{O}_X)$.

2.1.5. If we have a morphism $f : X \to Y$ between schemes ????

VI.2.2:**Construction of quasicoherent sheaves**

We have an important way of constructing quasicoherent $\mathcal{O}_X-$modules on X. Let us assume we have a covering $\mathfrak{U} = \{U_\nu\}_{\nu \in N}$ of a scheme $(X, \mathcal{O}_X)$ by affine subschemes. Let us also assume that we have given an $\mathcal{O}_X(U_\nu)-$module M_ν for all $\nu \in N$. Each of them defines a quasicoherent sheaf $\tilde{M}_\nu$ on the corresponding subscheme U_ν. Now let us assume that for any pair (ν, μ) of indices we have an isomorphism

$$g_{\nu,\mu} : \tilde{M}_\nu \mid U_\nu \cap U_\mu \tilde{\to} \tilde{M}_\mu \mid U_\nu \cap U_\mu$$

such that this system of isomorphism satisfies

a) $g_{\nu,\nu} = \mathrm{Id}$ for all $= \nu$

b)$g_{\nu,\mu} \circ g_{\nu,\mu} = \mathrm{Id}$ for all pairs ν, μ

c) and for any three indices ν, μ, λ we have the relation $g_{\nu,\mu} \circ g_{\mu,\lambda} = g_{\nu,\lambda}$ on $U_\nu \cap U_\mu \cap U_\lambda$.

Then we can construct a sheaf $\tilde{M} = (M_\nu, \mathfrak{U}, g_{\nu,\mu})$ on X by the glueing process: For an open set $V \subset X$ which is contained in at least one of the U_ν we define $\tilde{M}(V)$ to be the set of vectors $m = (\ldots, m_\nu, \ldots, m_\mu, \ldots)$ where the indices run over the subset of indices λ for which $U_\lambda \supset V$, where the $m_\nu \in \tilde{M}_\nu(V)$ and where

$$g_{\nu,\mu}(m_\nu) = m_\mu \text{ for all pairs } \nu, \mu$$

Of course any of the components determines all the others). Then for an arbitrary V we may cover it by the $V \cap U_\nu$ and define $\tilde{M}(V)$ by the conditions (SH1), (SH2) for sheaves (See 1.1.2.).

I will not discuss an example for this kind of construction, for this I refer to the chapter on projective spaces.

VI 2.3 Fibered products

The schemes form a category. It is very important to study relative situations. We consider a fixed scheme S. Then a *scheme over* S is a scheme X together with a morphism

$$\begin{tikzcd} X \arrow[d, "\pi"] \\ S \end{tikzcd} \quad .$$

Sometimes we write X/S, the morphism π is called the *structure morphism* and S is called the base scheme. (For more motivation why we need this general concept I refer to the end of this section)

The schemes over S form a category again. If we have two schemes $X/S, Y/S$ then the S-morphisms

$$\mathrm{Hom}_S(X, Y)$$

are those morphisms φ from X to Y which render the following diagram

$$\begin{tikzcd} X \arrow[rr, "\varphi"] \arrow[dr] & & Y \arrow[dl] \\ & S & \end{tikzcd}$$

commutative.

Given two schemes $X/S, Y/S$ we have the notion of the *fibered product* of these schemes over S. This fibered product is nothing else than the product in the category of schemes over S.

Hence the fibered product is an object Z/S together with (See Cat xx)

$$\begin{tikzcd} Z \arrow[rr, "p_1"] \arrow[dr] & & X \arrow[dl] \\ & S & \end{tikzcd} \qquad \begin{tikzcd} Z \arrow[rr, "p_2"] \arrow[dr] & & Y \arrow[dl] \\ & S & \end{tikzcd}$$

such that for any scheme T/S we have

$$\mathrm{Hom}_S(T, Z) \xrightarrow{\sim} \mathrm{Hom}_S(T, X) \times \mathrm{Hom}_S(T, Y)$$

where the identification is given by

$$\Psi \longmapsto (p_1 \circ \Psi, p_2 \circ \Psi).$$

We can do this for any category. The reader is advised to consider the construction of fibered product in the category of sets as an example.

Theorem VI.2.3 : *In the category of schemes fibered products exist.*

This theorem will not be proved here, I will prove it for affine schemes in this section and at the end of the proof I will give some indication how to do it in general. (See also Harthshornes book Chap. II, 3.3 or to EGA I, 3.2.6.) In the next section on projective schemes I will prove that the product of projective schemes is again projective and the existence of products in that case will be a byproduct.

We consider the category of affine schemes. If $X = \mathrm{Spec}(A)$ and $S = \mathrm{Spec}(R)$ then $\pi : X \to S$ is the same thing as a homomorphism of rings

$$\varphi : R \to A.$$

At this point two remarks are in order
(i) The datum $\varphi : R \to A$ is the same as giving the additive group A the structure of an R-module, i.e. giving a composition

$$\cdot : R \times A \to A,$$

which satisfies the usual rules, especially we want $1_R \cdot a = a$ and we have to require in addition $r \cdot (a_1 a_2) = (r \cdot a_1)a_2$. This is clear because starting from φ we put

$$r \cdot a = \varphi(r) \cdot a.$$

On the other hand if we have given the R-module structure on A then

$$\Psi(r) = r \cdot 1_A$$

gives us the ring homomorphism.

(ii) To simplify the notation we will drop the name of the morphism, this means we will only write $R \to A$ instead of $\varphi : R \to A$. In view of the first remark this means: If we see $R \to A$ then this allows us to write $r \cdot a$ for $r \in R$ on $a \in A$ and this satisfies the obvious rules.

If we have $R \to A$ we say that A is a ring over R or a R-ring sometimes we also say that A is an R-algebra.

If we have given R-rings A, B then

$$\mathrm{Hom}_R(A, B)$$

are exactly those homomorphisms which are linear with respect to R, i.e.

$$\varphi \in \mathrm{Hom}_R(A, B)$$

means that φ satisfies in addition $\varphi(r \cdot a) = r \cdot \varphi(b)$. (Now the two $\cdot$ have different meaning).

Of course we don't make any assumption that $R \to A$ should be injective. For instance $\mathbb{Z} \to \mathbb{Z}/p$ make $\mathbb{Z}/p\mathbb{Z}$ to a ring over $\mathbb{Z}$. Actually it is clear that any ring A is in a unique way a ring over $\mathbb{Z}$, we simply send $1 \to 1_A$.

I come back to the construction of fibered products in the category of affine schemes. We describe the problem in the category of rings and therefore we turn the arrows around. We have

$$\begin{tikzcd} A & & B \\ & R \arrow[ul] \arrow[ur] & \end{tikzcd}.$$

We are looking for a ring C over R together with two R-homomorphism $\alpha : A \to C$, $\beta : B \to C$ such that for any other ring T over R we get: In the following diagram

$$\begin{tikzcd} & C & \\ A \arrow[ur, "\alpha"] & & B \arrow[ul, "\beta"'] \\ & R \arrow[ul] \arrow[ur] \arrow[uu] \arrow[r] & T \end{tikzcd}$$

a R homomorphism from C to T is the same thing as a pair of R-homomorphisms $f : A \to T, g : B \to T$.

How do we get such a C? Starting from f, g we get a map

$$\begin{aligned} A \times B &\longrightarrow T \\ (a, b) &\longmapsto f(a) \cdot g(b) \end{aligned}.$$

This is an R-bilinear map from $A \times B$ to T. We have to verify

$$\begin{tikzcd} (r \cdot a, b) \arrow[dr] & \\ & r \cdot (f(a)g(b)) \\ (a, r \cdot b) \arrow[ur] & \end{tikzcd}$$

but

$$\begin{aligned} (r \cdot a, b) \to f(r \cdot a)g(b) &= f((r \cdot 1_A)g(b) \\ &= f(r \cdot 1_A)f(a) \cdot g(b) = (r \cdot f(1_A))f(a)g(b) \\ &= (r \cdot 1_C)f(a)g(b) = r \cdot (f(a)g(b)) \end{aligned}$$

and the distributivity is clear.

But this tells us that the pair f, g provides us an R-linear map

$$f \otimes g : A \otimes_R B \longrightarrow T$$

where $A \otimes_R B$ is of course the tensor product of the two R-modules A, B.

We define a ring structure on $A \otimes_R B$: The elements of the tensor product are finite sums

$$a_1 \otimes b_1 + a_2 \otimes b_2 \ldots + a_s \otimes b_s$$

where we have the following rules

$$\begin{gathered}(r \cdot a) \otimes b - a \otimes r \cdot b = 0 \\ (a_1 + a_2) \otimes b - a_1 \otimes b - a_2 \otimes b = 0\,. \\ a \otimes (b_1 + b_2) - a \otimes b_1 - a \otimes b_2 = 0\end{gathered}$$

We introduce as multiplication

$$(a \otimes b)(a' \otimes b') = aa' \otimes bb',$$

we extend this by distributivity. Then we have to check that this is compatible with the rules above.

We put $C = A \otimes_R B$ with this ring structure, we have the homomorphism

$$r \longrightarrow r \cdot 1_A \otimes 1_b = 1_A \otimes r \cdot 1_b,$$

we have

$$\begin{aligned}\alpha : A &\to A \otimes_R B \\ a &\to a \otimes 1_B \\ \beta : B &\to A \otimes_R B \\ b &\to 1_A \otimes b.\end{aligned}$$

Starting from f, g we already had the R-linear map from the R-module $A \otimes_R B$ to T. But the ring structure on $A \otimes_R B$ is made in such a way that $f \otimes g$ is a ring homomorphism. On the other hand, if $h : A \otimes_R B \to T$ is a R-homomorphism then we may put $f = h \circ \alpha, g = h \circ \beta$ and it is clear that

$$\begin{aligned}h(a \otimes b) &= h((a \otimes 1_B) \cdot (1_A \otimes b)) \\ &= h((a \otimes 1_B)) \cdot h((1_A \otimes b)) = f(a) \cdot g(b) \\ &= (f \otimes g)(a \otimes b).\end{aligned}$$

After all this it should be clear that the diagram

$$\begin{tikzcd} & \mathrm{Spec}(A \otimes_R B) \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ \mathrm{Spec}(A) \arrow[dr] & & \mathrm{Spec}(B) \arrow[dl] \\ & \mathrm{Spec}(R) & \end{tikzcd}$$

is a fibered product of $\mathrm{Spec}(A)$ and $\mathrm{Spec}(B)$ over the base $\mathrm{Spec}(R)$.

2.3.1. Examples:

1)If A and B are polynomial rings over R, i.e.

$$A = R[X_1 \dots X_n]$$
$$B = R[Y_1 \dots Y_m]$$

then A, B are free R modules with a basis consisting of monomials

$$X_1^{\nu_1} \dots X_n^{\nu_n}, Y_1^{\mu_1} \dots Y_m^{\mu_m}.$$

Then $A \otimes_R B$ is free again and has as basis

$$X_1^{\nu_1} \dots X_n^{\nu_n} \otimes Y_1^{\mu_1} \dots Y_m^{\mu_m}.$$

But then it is obvious that $A \otimes_R B$ is actually isomorphic to the polynomial ring in $X_1 \dots X_n, Y_1 \dots Y_m$, i.e.

$$R[X_1 \dots X_n] \otimes R[Y_1 \dots Y_m] = R[X_1 \dots X_n, Y_1, \dots Y_m].$$

The scheme $\mathrm{Spec}(R[X_1 \dots X_n])$ is called *the n-dimensional affine space over* $\mathrm{Spec}(R)$ and if $S = \mathrm{Spec}(R)$ we write $\mathbb{A}_S^n$ for this scheme. Hence we get the truly exciting formula

$$\mathbb{A}_S^n \times_S \mathbb{A}_S^m = \mathbb{A}_S^{n+m}.$$

2) If we have two R-algebras $R \to A, R \to B$ and we have given two ideals $I \subset A, J \subset B$ then these ideal define closed subschemes

$$\begin{tikzcd} \mathrm{Spec}(A/I) \arrow[r, hook] \arrow[dr] & \mathrm{Spec}(A) \arrow[d] \\ & \mathrm{Spec}(R) \end{tikzcd} \quad \begin{tikzcd} \mathrm{Spec}(B/J) \arrow[r, hook] \arrow[dr] & \mathrm{Spec}(B) \arrow[d] \\ & \mathrm{Spec}(R) \end{tikzcd} .$$

Hence we get a morphism from the fibered products

$$\begin{tikzcd} \mathrm{Spec}(A/I) \times_{\mathrm{Spec}(R)} \mathrm{Spec}(B/J) \arrow[rr] \arrow[dr] & & \mathrm{Spec}(A) \times_{\mathrm{Spec}(R)} \mathrm{Spec}(B) \arrow[dl] \\ & \mathrm{Spec}(R) & \end{tikzcd}$$

and I claim that this is again a closed embedding. I leave it to the reader as an exercise to show, that the arrow gives us an isomorphism of the fibered product of the subschemes to the subscheme defined by the ideal $(A \otimes_R B)(1_A \otimes_R J) + (A \otimes_R)(I \otimes_R B) \subset A \otimes_R B$.

3) If k is a field and K/k is a finite extension, then we have a map

$$\begin{tikzcd} \mathrm{Spec}(K) \arrow[d] \\ \mathrm{Spec}(k) \end{tikzcd}$$

which on the underlying sets is just a map from a point to a point. But we have different rings of regular functions on these points hence this morphism is not an isomorphism.

If we take the fibered product we get

$$\mathrm{Spec}(K)\times_{\mathrm{Spec}(k)}\mathrm{Spec}(K)=\mathrm{Spec}(K\otimes_k K)$$

and $K\otimes_k K$ will not be a field in general. If for instance K/k is a separable normal extension then

$$K\otimes_k K=\bigoplus_{\sigma\in\mathrm{Hom}_k(K,K)} K.$$

(Main theorem of Galois theory) and hence

$$\mathrm{Spec}(\bigoplus_{\sigma\in\mathrm{Hom}_k(K,K)} K)=\mathrm{Hom}_k(K,K)$$

as a set. Hence we have an example where the underlying set of $X\times_S Y$ may differs from the set theoretic fibered product which in our case is still a point.

As I explained earlier, we always have a canonical morphism $\mathrm{Spec}(A)\to\mathrm{Spec}(\mathbb{Z})$ and we may define the absolute product of two affine schemes as

$$\mathrm{Spec}(A)\times\mathrm{Spec}(B)=\mathrm{Spec}(A\otimes_{\mathbb{Z}} B).$$

We can consider the situation that we have an R-algebra R_1 and two R_1 algebras, i.e. we have a digram

$$\begin{array}{ccc} & & A \\ & \nearrow & \\ R\to R_1 & & \\ & \searrow & \\ & & B. \end{array}$$

Then we have a morphism

$$\rho: A\otimes_R B\to A\otimes_{R_1} B$$

(in the ring on the right hand side we have the rule $r_1 a\otimes b=a\otimes r_1 b$ for $r_1\in R$ which is not valid in the ring on the right hand side).

The map ρ is clearly surjective, hence we have

$$\mathrm{Spec}(A)\times_{\mathrm{Spec}(R_1)}\mathrm{Spec}(B)\hookrightarrow\mathrm{Spec}(A)\times_{\mathrm{Spec}(R)}\mathrm{Spec}(B).$$

2.3.2. I want to say a few words concerning the construction of fibered products for general schemes. First of all we have to cover the base scheme S by affines and to construct the fibered product over these affine subschemes and to glue the fibered products over the intersections. Over an affine base S we cover the schemes $X/S, Y/S$ by affines and constructs the fibered products for the pairs of affine covering sets. These will be glued together. For the details I refer to the references given above.

In the following section on projective schemes I will discuss this construction in a special case.

VI.2.4 Points, T-valued points and geometric points

In the theory of schemes we have to be careful with the notion of a point. If we have a scheme $(X, \mathcal{O}_X)$ then X is a set and the points of the scheme are the elements of this set. But we have seen that these points do not behave well under fibered products.(2.3.1. Example 3)

If we have a scheme X/S and another scheme T/S, i.e. we have a diagram

$$\begin{array}{ccc} X & & \\ \downarrow & & \\ S & \leftarrow & T \end{array}$$

then *the T-valued points* of X are simply the S-arrows from T to X i.e.

$$X_S(T) = \mathrm{Hom}_S(T, X).$$

If $S = \mathrm{Spec}(R)$ and $T = \mathrm{Spec}(B)$ then we denote the set of $T = \mathrm{Spec}(B)$-valued points also by $X(B)$ and speak of B-valued points.

Therefore we see that a scheme X/S defines a contravariant functor

$$F_X : \mathit{Schemes}/S \to \mathit{Ens}.,$$

where $F_X(T) = X_S(T)$.

It is the definition of the fibered product that the T valued points behave well under fibered products. We have

$$(X \times_S Y)_S(T) = X_S(T) \times Y_S(T).$$

I want to discuss this concept in a couple of examples. Let k be a field and we consider

$$A = k[X_1 \dots X_n]/I = k[x_1, \dots x_n]$$

where I is an ideal in the polynomial ring which is generated by polynomials

$$F_1(X_1 \dots X_n), \dots F_r(X_1 \dots X_n].$$

We have

$$\begin{array}{ccc} \mathrm{Spec}(A) & & \\ \downarrow & & \\ \mathrm{Spec}(k) & \xleftarrow{id} & \mathrm{Spec}(k)(= T) \end{array}.$$

In this case the T valued points are the k-homomorphisms

$$\varphi : k[x_1 \dots x_n] \to k.$$

Such a φ is determined by its values $(\varphi(x_1), \ldots, \varphi(x_n)) = (a_1, \ldots, a_n) \in k^n$. But we have constraints on the n-tuples of values because

$$F_1(x_1, \ldots, x_n) = \ldots = F_r(x_1, \ldots, x_n) = 0$$

and hence also $(a_1, \ldots a_n)$ has to satisfy

$$F_1(a_1, \ldots a_n) = \ldots = F_r(a_1, \ldots a_n) = 0.$$

Hence the point $a = (a_1, \ldots a_n)$ has to be a solution of the polynomial equations $F_1 = \ldots = F_r = 0$. If in turn $a = (a_1, \ldots a_n)$ solves the system of equations then we can look at the diagram

$$\begin{array}{ccc} k[X_1 \ldots X_n] & \longrightarrow & k[X_1 \ldots X_n]/I = A \\ \Psi a \searrow\nwarrow & & \nearrow \\ & k & \end{array}$$

where $\Psi_a(X_i) = a_i$. This homomorphism vanishes on the generators $(F_1, \ldots, F_r)$ of the ideal I and hence its factors over A. All this tells us that

$$\operatorname{Spec}(A)(k) = \{a \in k^n | F_1(a) = \ldots = F_r(a) = 0\},$$

the k-valued points are the solutions of the system of polynomial equations given by the $F_1, \ldots, F_r$.

2.4.1. *Geometric points*: We can embed our field k into an algebraic closure $\bar{k}$. Then this defines a morphism $\operatorname{Spec}(\bar{k}) \to \operatorname{Spec}(k)$ hence we have the diagram

$$\begin{array}{ccc} X & & \\ \downarrow & & \\ \operatorname{Spec}(k) & \leftarrow & \operatorname{Spec}(\bar{k}) \end{array}.$$

Then $X(\bar{k})$ is the set of solutions of the system of equations over $\bar{k}$. These are the so-called *geometric points* of the scheme $X \to \operatorname{Spec}(k)$.

Of course it is much easier to find geometric points than k-valued points. If we have a polynomial $y^2 = x^3 - x - 1$ in $\mathbb{Q}[x, y]$ then we may start from a value $a \in \mathbb{Q}$ for x but then it requires a good portion of luck if we want to find a point $(a, b) \in \mathbb{Q}$ satisfying the equation because we have to find a square root of $a^3 - a - 1$ in $\mathbb{Q}$. But if we look for geometric points we do not have this problem.

VI.2.4.2. To conclude this section I want to say a few words why we want to consider such relative situations

$$\begin{array}{c} X \\ \Big\downarrow \pi \\ S \end{array}.$$

For any point $s \in S$ we can consider the stalk $\mathcal{O}_{S,s}$ this is a local ring and let $\mathfrak{m}_s$ be its maximal ideal. Then $k(s) := \mathcal{O}_{S,s}/\mathfrak{m}_s$ is a field and we have a morphism $S \leftarrow \operatorname{Spec}(k(s))$

and hence we get a base change $X \times_s \operatorname{Spec}(k(s))$ and this is a scheme over $\operatorname{Spec}(k(s))$. It is called the *fiber* of X/S over s. Hence we get a family of schemes which is parametrized by the points of S. We may even go one step further and embed $k(s)$ into an algebraically closed field $\bar{k}$. This gives us a morphism $\operatorname{Spec}(k(s)) \leftarrow \operatorname{Spec}(\bar{k})$ and the composition $S \leftarrow \operatorname{Spec}(k)$ is a geometric point of S. The base change $S \times_S \operatorname{Spec}(\bar{k})$ is the geometric fiber over the geometric point.

VI2.5.**Closed points and geometric points on schemes of finite type**:

A scheme $X = \operatorname{Spec}(k[X_1, X_2, \ldots, X_n]/(F_1, \ldots, F_r)) = \operatorname{Spec}(A)$ is called an *affine scheme of finite type.* If we have a geometric point $P : A \to \bar{k}$ then the kernel of P is a maximal ideal $\mathfrak{m}_P$ of A hence it is also a closed point. Hence we get a map $X(\bar{k}) \to \operatorname{Specmax}(A)$. If we have in turn a maximal ideal $\mathfrak{m}$ in A then it follows from The Nullstellensatz that $k(\mathfrak{m}) = A/\mathfrak{m}$ is a finite extension of k. We get a diagram

$$\begin{array}{ccccc} A & & \to & & A/\mathfrak{m} = k(\mathfrak{m}) \\ & \nwarrow & & \nearrow & \\ & & k \quad \to \bar{k} & & \end{array}$$

and it is clear that the k-homomorphisms from $k(\mathfrak{m}) \to \bar{k}$ correspond one to one to the geometric points of X which lie above $\mathfrak{m}$. Hence we get from the results of Galois theory that

2.5.1. *(1) The map* $X(\bar{k}) \to \operatorname{Specmax}(A)$ *is surjective with finite fibers. The points in the fibre are in one to one correspondence with the prime ideals in the local* $\bar{k}$ *-algebra* $k(\mathfrak{m}) \otimes \bar{k}$.

(2) The cardinality of the fibre of $\mathfrak{m}$ *of this map divides the degree* $[k(\mathfrak{m}) : k]$ *and it is equal to this degree if and only if* $k(\mathfrak{m})/k$ *is a separable extension.*

(3) The Galois group of $\bar{k}/k$ *acts transitively on the geometric points in the fibre over* $\mathfrak{m}$.

We may also speak of integral solutions. If we have a scheme

$$A = \mathbb{Z}[X_1, \ldots, X_n]/I$$

then we may consider

$$\begin{array}{ccc} X = \operatorname{Spec}(A) & & \\ \downarrow & & \\ \operatorname{Spec}(\mathbb{Z}) & \leftarrow & \operatorname{Spec}(\mathbb{Z}) \end{array}.$$

and $X(\mathbb{Z}) = X(\operatorname{Spec}(\mathbb{Z}))$ are the integral solutions of the system of equations. For instance we can try to find $\mathbb{Z}$-valued points on $\operatorname{Spec}(\mathbb{Z}[x,y]/(y^2 - x^3 + x + 1))$ which is even harder than finding $\mathbb{Q}$-valued points.

VI.2.6 **Morphisms, T-valued points and geometric points** We start from the following diagram of schemes

$$\begin{array}{ccc} X & & Y \\ \searrow & & \swarrow \\ & S \leftarrow T. & \end{array}$$

It is clear that morphisms $\phi \in \mathrm{Hom}_S(X, Y)$ induce maps between the sets of T-valued points : $\phi(T) : X(T) \to Y(T)$. On the other hand if we have a consistent system of maps $\phi(T) : X_S(T) \to Y_S(T)$ (consistent means that the obvious diagrams commute) then we get from this a morphism $\phi \in \mathrm{Hom}_S(X.Y)$. We simply plough in X itself for T and get the morphism as the image of the identity in $\mathrm{Hom}_S(X, X)$. (This is a Yoneda type argument).

At this point one can always ask the question whether a given functor

$$\mathcal{F} : Schemes/S \to Ens$$

is representable. Developing tools to answer this question in concrete situations turns out to be a fundamental problem in algebraic geometry.

Hence if we have two schemes $X/S, Y/S$ and a system of maps between their T valued points **for all possible** T, then this is induced by a unique morphism between the schemes. In some cases it is not necessary to have a map between the T valued points for all T.

If for instance $S = \mathrm{Spec}(k)$ where k is an algebraically closed field. Let us assume in addition that $X = \mathrm{Spec}(k[x_1, x_2, \ldots, x_n]), Y = \mathrm{Spec}(k[y_1, y_2, \ldots, y_m])$ are affine schemes of finite type over $\mathrm{Spec}(k)$ and we assume moreover that the two k-algebras $A = k[x_1, x_2, \ldots, x_n]$, $B = k[y_1, y_2, \ldots, y_m]$ do not have nilpotent elements (they are reduced).

Then the Nullstellensatz guarantees that the sets of geometric points $X(\bar{k}) = \mathrm{Specmax}(A)$, $Y(\bar{k}) = \mathrm{Specmax}(B)$. In this case we can interprete the elements of the rings A, resp. B as $\bar{k}$ valued functions on $X(\bar{k}) = \mathrm{Specmax}(A)$, resp. $Y(\bar{k}) = \mathrm{Specmax}(B)$. (To see this we have to notice that an element f in one of these algebras which vanishes in all geometric points lies in the intersection of the maximal ideals. It follows from the Nullstellensatz that the intersection of the maximal ideals is the radical and hence zero. (How ?).)

Now it is clear that a map ψ between the sets of geometric points comes from morphism $X \to Y$ if and only if the induced map between the k-valued functions maps the elements of B into elements of A. Hence we get a homomorphism of rings which then in turn induces the map between the geometric points from which we started.

Exercises : *(1) Let us assume that k is a field of characteristic $p > 0$. We take $A = B = k[X]$. Then the set of geometric points is $\bar{k}$ and we have the bijective map $x \to x^p$ on the set of geometric points. Show that this map comes from a morphism but its inverse does not.*

This teaches us that a morphism between affine schemes of finite type over k which induces a bijection between the sets of geometric points is not necessarily an isomorphism.

(2) We go back to the general situation that we have two reduced affine schemes of finite type over $\mathrm{Spec}(k)$. We assume now in addition that k is perfect (this means that the map $x \to x^p$ is surjective and hence bijective if our field has positive characteristic $p > 0$.) Let us assume that we have a morphism $\phi_{\bar{k}} : X \times_k \bar{k} \to Y \times_k \bar{k}$ which induces a map ${}^t\phi : X(\bar{k}) \to Y(\bar{k})$. Now we can define an action of the Galois group $\mathrm{Gal}(\bar{k}/k)$ on the two set of geometric points. Show that $\phi_{\bar{k}}$ is defined over k, i.e. comes from a morphism $\phi : B \to A$ if and only if it commutes with the action of the Galois group.

To summarize this section we can say that for reduced finitely generated k-algebras over an algebraically closed field the set of geometric points together with the algebra A of regular k - valued functions contains all the information. This is of course tautological, because the algebra A determines everything but its realization as algebra of k-valued functions gives us some geometric picture which is less abstract than the concept of a locally ringed space.(Exercise (3), VI.1.2.)

If our ground field is not algebraically closed but perfect we get this information if we consider the geometric points together with the action of the Galois group. But if our ground field is not perfect or if the schemes are not reduced this breaks down.

VI.2.7.**Flat morphisms**

Let us consider a commutative ring A with identity and its category of A-modules. In this category we have the tensor product of two modules, which is again an A-module.

Small remark: Here we need the commutativity of A since we want

$$m \otimes (fg)n = (fg)m \otimes n = f(gm) \otimes n = gm \otimes fn = m \otimes g(fn) = m \otimes (gf)n.$$

If we fix a module M we can consider the functor

$$N \to N \otimes_A M$$

of the category into itself. This is a so-called right exact functor: If we have an exact sequence

$$0 \to N' \to N \to N'' \to 0$$

then the sequence

$$N' \otimes_A M \to N \otimes_A M \to N'' \otimes_A M \to 0$$

is still exact. But the functor is not exact in general, it can happen that the first arrow is not injective anymore.

Exercise.

1) Prove the right exactness (I think one has to use the definition by generators and relations).

(2) To construct an example which shows that the functor is not exact, let us consider an element $m \in M, m \neq 0$ *which has a non-trivial annulator in* A*, i.e. there is an* $f \in A, f \neq 0,$ and $fm = 0.$ *Consider the sequence of* A *modules*

$$0 \to Af \to A \to A/Af \to 0.$$

Show: The element $f \otimes_A m \in Af \otimes M$ *goes to zero in* $A \otimes_A M = M$*. Construct an example where you know that* $f \otimes_A m \neq 0$*.*

2.7.1. *The concept of flatness*

An A-module M is called *flat* if the functor $N \to N \otimes_A M$ is exact. A simple example of a flat module is the free A-module over an arbitrary index set I

$$A^I = \{(\ldots, a_i, \ldots)_{i \in I} | \quad \text{almost all} \quad a_i = 0\}.$$

This is the direct sum

$$A^I = \oplus_{i\in I} A$$

but the direct product $\prod_{i\in I} A$ is also flat.

An A-module M is called *faithfully flat* if it is flat and if in addition $N \otimes_A M = 0$ implies $N = 0$.

The above examples of the direct sum and direct product are indeed faithfully flat. We will see flat modules which are not faithfully flat in a minute.

Since an A-algebra $A \to B$ is also an A-module, we can speak of *flat* A-algebras. Since we may view $A \to B$ also as a morphims of affine schemes we can speak of *flat morphisms* $(\mathrm{Spec}(B), \tilde{B}) \to (\mathrm{Spec}(A), \tilde{A})$ of affine schemes. We also observe that for an A-module N the module $B \otimes_A N$ has an obvious natural structure as a B-module, we simply put $b(b_1 \otimes_A n) = bb_1 \otimes_A n$

A morphism of affine schemes is called *faithfully flat* if the A-module B is faithfully flat.

2.7.1.1. *Examples and exercises :*

(1) Of course the polynomial ring $A[X_1, \ldots, X_n]$ is faithfully flat over A because it is free as an A-module.

(2) An important case of a flat algebra is given by the localization. It is not so difficult to check that for any subset $S \subset A$, which is closed under multiplication the A algebra

$$A \to A_S$$

is flat. To see this one should also observe that $N \otimes_A A_S \simeq N_S$ where the isomorphism is simply given by

$$n \otimes \frac{f}{s} \mapsto \frac{fn}{s}.$$

Now it is clear that if $N' \subset N$ then

$$N'_S \subset N_S.$$

(3) Let us assume that $S = \{f^n\}$ where $f \in A$ and not nilpotent. We have just seen that

$$A \to A_f$$

is flat. But it will not be faithfully flat in general. To see this one should prove

$$A/fA \otimes_A A_f = 0.$$

But on the other hand:

(4) If we have a covering $X = \mathrm{Spec}(A) = \bigcup_{i\in I} X_{f_i}$ where we may assume that I is finite (prop,VI.1.3.2.) then

$$A \to \prod_{i\in I} A_i$$

is faithfully flat. This corresponds to a map

$$X \leftarrow \bigsqcup X_{f_i}.$$

and this is a faithfully flat morphism of schemes.

(5) If B is the quotient of A by an ideal

$$A \to B = A/I,$$

then $\mathrm{Spec}(B) = \mathrm{Spec}(A/I) \to \mathrm{Spec}(A)$ *is a closed subscheme. In this case one can show that this is almost never flat. (I think that in the case that A is noetherian this morphism is flat if and only if* $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ *is also open.)*

(6) If A is a Dedekind ring (See VII. 2.3.1 *), then an A-module N is flat if and only if it is torsion free, i.e. if $fn = 0$ with $f \in A, n \in N$ then f or n is zero.*

If we have $A \to B \to C$ and an A-module N then we have a canonical isomorphism

$$C \otimes_B (B \otimes_A N) \tilde{\to} C \otimes_A N$$

We leave it to the reader to verify

7). If we have $A \to B \to C$ and B is a flat A-algebra and C a flat B-algebra then C is also a flat A-algebra.

8). If we have $A \to B \to C$ and if C is a flat A-algebra and if C is a faithfully flat B-algebra then B is a flat A-algebra.

The second assertion is important because it allows us to check flatness locally.

Proposition VI.2.7.2.

$$A \to B \to \prod_{i \in I} B_{f_i}$$

with some f_i in B such that the $X_{fi} \subset X = \mathrm{Spec}(B)$ form a covering of X and if $A \to B_{fi}$ is flat for all i then B is a flat A-algebra.

This is obvious from the exercise.

The same principle can be applied if we want to check whether an A-module is flat: If $A \to B$ is faithfully flat, then an A-module M is flat if and only if the B-module $M \otimes_A B$ is flat.

Now we can define when a morphism between to schemes

$$\begin{array}{c} X \\ \downarrow \pi \\ S \end{array}$$

is (faithfully) flat: It is *flat* if for any open affine subscheme $V \subset S$ and any open affine subscheme $U \subset \pi^{-1}(V)$ the $O_S(V)$-algebra $\mathcal{O}_X(U)$ is flat. It is *faithfully flat* if it is also surjective.

To justify the last definition we need to solve the following

2.7.2.1 Exercise: *Show that for a flat A-algebra B the two conditions are equivalent*

(1) B is faithfully flat

(2) $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is surjective.

In section VI.2.5. we explained that a scheme

$$\begin{array}{c} X \\ \downarrow \pi \\ S \end{array}$$

provides a contravariant functor from the category of schemes over S to the category of sets

$$T \to \mathrm{Hom}_S(T, X) = X(T).$$

This functor determines X/S up to a canonical isomorphism. We also mentioned the problem to decide whether such a functor is representable.

Now we have the following fundamental fact (See III.1.2.)

Theorem VI.2.7.3:

Let X/S be a scheme. Then for any faithfully flat morphism

$$S \leftarrow S'$$

we get an exact sequence of sets

$$X(S) \xrightarrow{p_0} X(S') \underset{p_2}{\overset{p_1}{\rightrightarrows}} X(S' \times_S S')$$

Proof: It is clear that it suffices to prove this for affine schemes. Then we have $S = \mathrm{Spec}(A)$, $X = \mathrm{Spec}(C)$ and $A \to C$. If now $S' = \mathrm{Spec}(B) \to S$ is faithfully flat this means that $A \to B$ is faithfully flat. We have to show that the sequence

$$\mathrm{Hom}_A(C, A) \xrightarrow{p_0} \mathrm{Hom}_A(C, B) \underset{p_2}{\overset{p_1}{\rightrightarrows}} \mathrm{Hom}_A(C, B \otimes_A B)$$

is exact. Of course $A \to B$ is injective, so the first arrow is indeed an injection. Now we have to show that an element $\phi : C \to B$ in $\mathrm{Hom}_A(C, B)$ which is sent to the same homomorphisms under the two arrows

$$b \begin{array}{l} \mapsto b \otimes 1 \\ \\ \mapsto 1 \otimes b \end{array}$$

is actually an element in $\mathrm{Hom}_A(C,A)$. But this is clear once we know that

$$A=\{b\in B\mid b\otimes 1-1\otimes b=0\}$$

We put $\delta(b): b\mapsto b\otimes 1-1\otimes b$ then last assertion says that the sequence

$$0\to A\to B\xrightarrow{\delta} B\otimes_A B$$

is exact. Since $A\to B$ is faithfully flat this is equivalent to the exactness of the same sequence tensorized by B:

$$0\to B\to B\otimes_A B\to B\otimes_A B\otimes_A B.$$

where the homomorphisms are $b\mapsto 1\otimes b$ and $b_1\otimes b_2\mapsto (b_1\otimes 1-1\otimes b_1)\otimes b_2$. This is now a sequence of B modules where B acts always on the last factor. We have an isomorphism

$$B\otimes_A B\otimes_A B\xrightarrow{\sim} B\otimes_A B\otimes_B B\otimes_A B$$

which is given by $b_1\otimes b_2\otimes b_3\mapsto b_1\otimes 1\otimes b_2\otimes b_3$. (In the fourfold tensor product we are allowed to move the entry at place 2 to place 4 and backwards since we take the tensor product over B). Hence we have to prove the exactness of

$$0\to B\to B\otimes_A B\to B\otimes_A B\otimes_B B\otimes_A B.$$

But now the inclusion $B\to B\otimes_A B$ admits a splitting given by $b_1\otimes b_2\mapsto 1\otimes b_1b_2$ and hence we have $B\otimes_A B=B\oplus Y$ where Y is the kernel of that splitting. The elements of $y\in Y$ are mapped to

$$\delta(y)=1\otimes y+y\otimes 1+\ldots\in B\otimes Y\oplus Y\otimes B\oplus Y\otimes Y$$

and now it is clear that δ is injective on Y.

This theorem plays an enormous role since it gives us a necessary condition for the representability of a functor (See I.3.3)

$$\mathcal{F}:\mathrm{Schemes}/S\to\mathrm{Ens}.$$

Recall that this means that can find a scheme X/S and an element (the *"identity section"*)

$$e_X\in\mathcal{F}(X))$$

such that map

$$\begin{array}{ccc}\mathrm{Hom}_S(T,X) & \longrightarrow & \mathcal{F}(T)\\ \varphi & \longmapsto & \varphi^*(e_X)\end{array}$$

is a bijection for all $T\to S$. Here φ^* is an abbreviation for $\mathcal{F}(\varphi)$ and e_X is called the "identity section", because it is equal to $\varphi^*(\mathrm{Id}_X)$.

The reader should have noticed that the sequence above is rather similar to the condition which have to be satisfied by sheaves on topological spaces. To make this analogy clear we start from a covering of a topological space $X = \bigcup_{i\in I} U_i$ by open sets. We consider the disjoint union of the open set we get a continous map $X \xleftarrow{p} X' = \bigsqcup U_i$. We can form the fibered product

$$X' \underset{p_2}{\overset{p_1}{\leftleftarrows}} X' \times_X X'$$

Then the conditions SH1,SH2 for a presheaf $\mathcal{F}$ is exactly the exactness of the sequence

$$\mathcal{F}(X) \xrightarrow{p_0^*} \mathcal{F}(X') \underset{p_2^*}{\overset{p_1^*}{\rightrightarrows}} \mathcal{F}(X' \times_X X'),$$

where the p^* are the restriction maps.

A. Grothendieck introduced a much more general concept of topologies. For instance we have the flat topology on a scheme S. This means roughly that we replace coverings by Zariski open sets just by faithfully flat morphisms $S \leftarrow S'$. In this more general context we can still define sheaves: These are functors

$$\mathcal{F} : (\text{Faithfullyflatschemes } S' \to S) \to \text{Ens}$$

which satisfy the extra condition that for all $S' \xrightarrow{p_0} S$ we get an exact sequence

$$\mathcal{F}(S) \xrightarrow{p_0^*} \mathcal{F}(S') \underset{p_2^*}{\overset{p_1^*}{\rightrightarrows}} \mathcal{F}(S' \times_S S'),$$

where again the p^* are called the restriction maps, these are the maps induced by the functor.

We can summarize:

If we have a scheme S and a contravariant functor

$$\begin{array}{ccccc} \mathcal{F} & : & \text{Schemes}/S & \longrightarrow & \text{Ens} \\ \mathcal{F} & : & T & \longrightarrow & \mathcal{F}(T) \end{array}$$

then a necessary condition for this functor to be representable is that its restriction to faithfully flat topology is a sheaf.

We can say even more. For any such functor $\mathcal{F}$ and any scheme $T \xrightarrow{f} S$ we can define the restriction

$$\mathcal{F}_T : \text{Schemes}/T \longrightarrow \text{Ens},$$

which is defined by the obvious definition

$$\mathcal{F}_T(T') = \mathcal{F}(T')$$

for any scheme $T' \to T$, which then by the composition with f becomes a scheme over S. We hve tautologically

If our functor is representable by a scheme X/S, and if $T \to S$ is a scheme over S, then the restriction $\mathcal{F}_T$ is represented by $X \times_S T$.

In the last chapter of this book we will discuss the representability of the Picard functor or more precisely a modified Picard functor. There we will encounter a problem of the following type.

Let S be a scheme and let $\mathcal{F}$ be a contravariant functor from schemes over S to Ens

$$\mathcal{F} : T \longrightarrow \mathcal{F}(T)$$

which satisfies our sheaf condition for the faithfully flat topology.

Let us assume that we can find a faithfully flat scheme $S' \to S$ such that the restriction $\mathcal{F}' = \mathcal{F}_{S'}$ of our functor becomes representable over S', i.e. we have a scheme X'/S' and an element the *"identity section"* $e_{X'} \in \mathcal{F}'(X')$ which represent the functor in the above sense. Under what conditions can we conclude the already $\mathcal{F}$ itself is representable?

This question has been analyzed by Grothendieck and we will describe the techniques which allow to construct- under certain conditions - an X/S which represents $\mathcal{F}$.

We introduce some notation, we put

$$S' \times_S S' = S'', S' \times_S S' \times_S S' = S'''$$

We have the two projections

$$S' \quad \overset{p_1}{\underset{p_2}{\leftleftarrows}} \quad S'',$$

and we can take the pullback of X'/S' by these two arrows, i.e. we consider

$$X_1'' = X' \times_{S',p_1} S'' \qquad \text{and} \qquad X_2'' = X' \times_{S',p_2} S''.$$

These two schemes together with the restrictions $e_{X_1''}, e_{X_2''}$ of $e_{X'}$ represent the two restrictions of $\mathcal{F}'$ to S'' . But these restrictions are also the restrictions of $\mathcal{F}$ to S'' via the composition of p_1 and p_2 with $p : S' \to S$. Since these two compositions are equal we see that $\mathcal{F}_1''(T) = \mathcal{F}_2''(T)$ for any object $T \to S''$. This means that the two restrictions of functors are the same, hence have uniquely determined isomorphisms of schemes (inverse to each other)

$$X_1'' \overset{\varphi_{12}}{\underset{\varphi_{21}}{\leftrightarrows}} X_2''$$

which send the restrictions $e_{X_1''}, e_{X_2''}$ of the section $e_{X'}$ into eachother.

Now we go one step further and consider

$$S' \overset{p_1}{\longleftarrow} \underset{\longleftarrow}{\overset{p_2}{}} S'' \quad \begin{matrix} \overset{p_{12}}{\longleftarrow} \\ \overset{p_{13}}{\longleftarrow} \\ \overset{p_{23}}{\longleftarrow} \end{matrix} \quad S'''$$

and can consider the pullback

$$p_{ij}^*(\varphi_{12}) : (X' \times_{S',p_1} S'') \times_{S'',p_{ij}} S''' \to (X' \times_{S',p_2} S'') \times_{S'',p_{ij}} S'''$$

and the same for $p_{ij}^*(\varphi_{21})$.

The composition $p_\nu \circ p_{ij}$ is always a projection to a factor (the α-th factor)

$$\pi_\alpha : S''' \to S'.$$

and we have

$$\begin{array}{rclrclrcl} p_1 \circ p_{12} & = & \pi_1 & p_1 \circ p_{13} & = & \pi_1 & p_1 \circ p_{23} & = & \pi_2 \\ p_2 \circ p_{12} & = & \pi_2 & p_2 \circ p_{13} & = & \pi_3 & p_2 \circ p_{23} & = & \pi_3. \end{array}$$

This allows us to identify always two of the two step pullbacks to a one step pullback, we get

$$(X' \times_{S',p_1} S'') \times_{S'',p_{12}} S''' = X' \times_{S',\pi_1} S''' = (X' \times_{S',p_1} S'') \times_{S'',p_{13}} S''',$$
$$(X' \times_{S',p_2} S'') \times_{S'',p_{12}} S''' = X' \times_{S',\pi_2} S''' = (X' \times_{S',p_1} S'') \times_{S'',p_{23}} S''',$$
$$(X' \times_{S',p_2} S'') \times_{S'',p_{13}} S''' = X' \times_{S',\pi_3} S''' = (X' \times_{S',p_2} S'') \times_{S'',p_{23}} S''',$$

Our φ_{ij} induce isomorphisms amoung these S''' schemes, for example

$$\begin{array}{ccc} X' \times_{S',\pi_1} S''' & \overset{p_{12}^*(\varphi_{12})}{\longrightarrow} & X' \times_{S',\pi_2} S''' \\ \searrow & & \swarrow \\ & S''' & \end{array}$$

and the assumption that X' represents the restriction $\mathcal{F}'$ of our functor $\mathcal{F}$ implies that the resrictions of the "identity sections" are mapped into eachother. Hence we get the following 1-cocycle relation:

$$p_{13}^*(\varphi_{12})^{-1} \circ p_{23}^*(\varphi_{12}) \circ p_{12}^*(\varphi_{12}) = \mathrm{Id}\,.$$

Let us forget about the functor $\mathcal{F}$ for a moment, let us assume that we have a faithfully flat scheme

$$S' \longrightarrow S$$

and a scheme $f' : X' \to S'$.

If we have an isomorphism

$$\varphi_{12} : X' \times_{S',p_1} S'' \longrightarrow X' \times_{S',p_2} S''$$

which satisfies the cocycle rule

$$p_{13}^*(\varphi_{12})^{-1} \circ p_{23}^*(\varphi_{12}) \circ p_{12}^*(\varphi_{12}) = \mathrm{Id}$$

where we made the 3 identifications between the corresponding two step pullbacks, then we call this after A. Grothendieck a *descend datum.*

Such a descend datum is called *effective* if we can find a scheme $X \xrightarrow{f} S$ and an isomorphism

$$\begin{tikzcd} X \times_S S' \arrow[rr, "h"] \arrow[dr] & & X' \arrow[dl, "f'"] \\ & S' & \end{tikzcd}$$

such that: Firstly

$$X \times_S S'' = (X \times_S S') \times_{S',p_1} S'' = (X \times_S S') \times_{S',p_2} S'',$$

and secondly the diagram

$$\begin{tikzcd} & X' \times_{S',p_1} S'' \arrow[dd, "\varphi_{12}"] \\ X \times_S S'' \arrow[ur] \arrow[dr] & \\ & X' \times_{S',p_2} S'' \end{tikzcd}$$

is commutative.

Then we say that $(X'/S', \varphi_{12})$ descends to S and (X, h) is the *realization* of the descend datum. To be able to say this, we should convince ourselves that such a realization is unique up to a canonical isomorphism.

Let us assume we have two such realizations $(X/S, h)$ and $(X_1/S, h_1)$. Then we get from the definition an isomorphism

$$h_1^{-1} \circ h : X \times_S S' \longrightarrow X_1 \times_S S'$$

and from the compatibility with the descend datum we conclude that the two pullbacks

$$p_1^*(h_1^{-1} \circ h_2) : (X \times_S S') \times_{S',p_1} S'' \to (X_1 \times_S S') \times_{S',p_1} S''$$

and

$$p_2^*(h_1^{-1} \circ h_2) : (X \times_S S') \times_{S',p_2} S'' \to (X_1 \times_S S') \times_{S',p_2} S''$$

must be equal. Hence we get from 2.7.3 that $h_1^{-1} \circ h$ is the pullback of a uniquely determined isomorphism.

Now we come back to our functor $\mathcal{F}$ whose restriction to $X \times_S S'$ was supposed to be representable by a scheme X'/S'. Then we constructed a descent datum for X'/S' from this information and we can summarize:

Proposition 2.7.3.1: *If our functor*

$$\mathcal{F}: \text{ Schemes}/S \longrightarrow Ens$$

is a sheaf for the flat topology and if it is representable by a scheme X'/S' for some faithfully flat $S' \to S$, then it is representable by a scheme X/S if and only if the descent datum is effective.

Here we can also discuss the descend for sheaves. If we have a flat morphism $S' \to S$ and if we have a sheaf for the flat topology $\mathcal{F}'/S'$. We want to know under which conditions we can find a sheaf$\mathcal{F}/S$ which restricts to $\mathcal{F}'$. We may also define descend data: We take the two restriction $\mathcal{F}'_1, \mathcal{F}'_2$ of $\mathcal{F}'$ by the two morphisms p_1, p_2 from $S' \times_S S'$ to S' and we assume that we have and isomorphism of sheaves

$$\varphi_{12} : \mathcal{F}'_1 \xrightarrow{\sim} \mathcal{F}'_2$$

We say that this isomorphism is a descent datum for the sheaf $\mathcal{F}'$ if its pulbacks via the different projections from $S' \times_S S' \times_S S'$ to $S' \times_S S'$ satisfy the 1-cocycle relation

$$p_{13}^*(\varphi_{12})^{-1} \circ p_{23}^*(\varphi_{12}) \circ p_{12}^*(\varphi_{12}) = \mathrm{Id}\,.$$

It is now easy to see that now the sheaf $\mathcal{F}'$ descends to a sheaf $\mathcal{F}$, if we have such a descend datum on it. The sheaf $\mathcal{F}$ is unique up to a canonical isomorphism.

We can now look at our previous discussion of descend data on schems from a different point of view. If we have a scheme X'/S' then it defines a scheme for the flat topology. A descend datum defines also a descend datum for the sheaf. Then the sheaf descends to a sheaf over S. Now the question whether the descend datum is effective is whether this resulting sheaf is representable.

Remarks:

1) Of course it is clear that for a scheme X/S we have a descend datum on $X \times_S S'$, we can simply take the pullbacks of the identity.

2) Just to avoid possible misunderstandings: If we have a scheme X'/S', and if we can find a scheme X/S such that $X \times_S S' \xrightarrow{\sim} X'$, then this scheme X/S is not necessarily unique. Only if the isomorphism $X \times_S S' \xrightarrow{\sim} X'$ is compatible with the given descend datum, then we have uniqueness.

We have a simple case where we have effectiveness of descend data. If all our schemes $S = \operatorname{Spec}(A)$, $S' = \operatorname{Spec}(A')$ and $X' = \operatorname{Spec}(B')$ are affine, then we have the diagram

$$\begin{array}{ccc} & & B' \\ & & \uparrow \\ A & \longrightarrow & A'. \end{array}$$

We have the two homomorphisms and we assume that we have an isomorphism of $A' \otimes_A A'$-algebras

$$B' \otimes_{A',i_1} A' \otimes_A A' \xrightarrow{\varphi} B' \otimes_{A',i_2} A' \otimes_A A'.$$

We consider the pullbacks

$$\varphi \otimes i_{v\mu} : (B' \otimes_{A',i_1} A' \otimes_A A') \otimes_{i_{v\mu}} A' \otimes A' \otimes A' \to B' \otimes_{A',i_2} A' \otimes_A A'.$$

The $i_{v\mu}$ send an $a' \otimes a''$ to a three fold tensor with $a1_{A'}$ at the right place. Now we say that φ is a descend datum (we simply have to translate) if

$$(\varphi \circ i_{12}) \circ (\varphi \otimes i_{23}) \circ (\varphi \otimes i_{13})^{-1} = \operatorname{Id}.$$

Now we want to show that such a descend datum is always effective. We consider the algebra $B \subset B'$ consisting of elements (???)

$$B = \{b \in B' \mid \varphi(b \otimes i_1) = (b \otimes i_2)\}.$$

This is an A-algebra and checks easily that

$$B \otimes_A A' \simeq B'$$

In this book we need only a very special case for applying this method of descend. This case is also the historic origin of the method.

Let us assume that we have a field k and a finite separable normal extension L/k. These are our two schemes $S = \operatorname{Spec}(k)$, $S' = \operatorname{Spec}(L)$.

Let us assume that we have an affine scheme $X'/\operatorname{Spec}(L)$ it is given as an affine L-algebra A', i.e. $X' = \operatorname{Spec}(A')$. Let us even go one step further: We consider X' as a closed subscheme of A^n_L/L, this means that

$$X' = \operatorname{Spec}(L[X_1 \cdots X_n]/I')$$

where $I' \subset L(X_1 \cdots X_n]$ is the defining ideal. Now we look at the action of the Galois group $\operatorname{Gal}(L/k)$. It is clear that it acts on $L[X_1 \cdots X_n]$ via the action on the coefficients and we can define the conjugate of X' as

$$(X')^\sigma = \operatorname{Spec}(L[X_1 \cdots X_n]/I^\sigma)$$

where of course

$$I^\sigma = \left\{ \Sigma a^\sigma_{v_1 \cdots v_n} X_1^{v_1} \cdots X_n^{v_n} \mid \Sigma a_{v_1 \cdots v_n} X_1^{v_1} \cdots X_n^{v_n} \in I \right\}.$$

If we consider the set of geometric point $X'(\overline{k})$, then we see

$$(X')^\sigma(\overline{k}) = (X'(\overline{k}))^{\tilde{\sigma}}$$

where $\tilde{\sigma} \in \mathrm{Gal}(\overline{k}/k)$ maps to σ. If we want that $X'/\operatorname{Spec}(L)$ is obtained by a base extension of an affine scheme over k, then the two affine L-schemes X' and $(X')^\sigma$ must be isomorphic for any $\sigma \in \mathrm{Gal}(L/k)$.

What does it mean to have an isomorphism? This means that we should have an L-algebra isomorphism

$$\begin{array}{ccccc} f_\sigma & : & L[X_1 \cdots X_n]/I & \longrightarrow & L[X_1 \cdots X_n]/I^\sigma \\ & & " & & " \\ f_\sigma & : & A & \longrightarrow & A^\sigma. \end{array}$$

Now we see that the element in the Galois group defined an isomorphism

$$\sigma : h \longmapsto h^\sigma$$

between the two rings A and A^σ, the point is that this isomorphism is not L-linear, but only $L-\sigma$-linear, this means that

$$\sigma(\lambda h) = \lambda^\sigma h^\sigma$$

for all $\lambda \in L$. But this remark allows us to write the L-algebra in a different way.

We say A^σ is the L-algebra which as a ring is equal to A but where the scalar $\lambda \in L$ acts by

$$\lambda *_\sigma h = \lambda^\sigma h$$

where on the right hand side λ^σ act by the original L-algebra structure. The point is that we gave a definition of the conjugate scheme X'^σ without reference to the embedding.

Now we reformulate the concept of descend datum for this special case. We say that a Galois-descend datum is a family of isomorphisms

$$f_\sigma : A \longrightarrow A^\sigma$$

which satisfies a compatibility condition.

We observe that for any τ and any σ we get an isomorphism

$$f_\sigma^\tau : A^\tau \longrightarrow (A^\tau)^\sigma = A^{\sigma\tau}$$

which is given by

$$h \longmapsto f_\sigma(h)$$

(we have $f_\sigma^\tau(\lambda^\tau h) = f_\sigma(\lambda^\tau h) = (\lambda^\tau)^\sigma f_\sigma(h)) = \lambda^{\sigma\tau} f_\sigma(h))$. Then we require

(1) $f_1 = \mathrm{Id}$.

(2) For any pair σ, τ of elements in the Galois group the diagram

$$\begin{array}{ccc} A & \xrightarrow{f_\tau} & A^\tau \\ & \searrow^{f_{\sigma\tau}} \quad \swarrow^{f_\sigma^\tau} & \\ & (A^\tau)^\sigma & \end{array}$$

commutes.

Now I want to explain two things:

(i) If we have such a Galois-descend datum, then the k-algebra

$$A_0 = \{a \in A \mid f_\sigma(a) = a\}$$

defines an affine scheme over k (of course) and

$$\begin{array}{rcl} A_0 \otimes_k L & \longrightarrow & A \\ h \otimes \lambda & \longmapsto & \lambda h \end{array}$$

is an L-algebra isomorphism.

(ii) Such a Galois-descend datum is nothing else than a descent datum in the previous sense.

The first assertion is a consequence of the main theorem in Galois theory. If $h \in A$ and if $\lambda \in L$, then we can form

$$\sum_{\sigma \in \mathrm{Gal}(L/k)} \lambda^\sigma f_\sigma(h).$$

If $\tau \in \mathrm{Gal}(L/k)$, then

$$f_\tau(\sum_\sigma \lambda^\sigma f_\sigma(h)) = \sum_\sigma (\lambda^\sigma (\lambda^\sigma)^\tau f_\tau f_\sigma(h) =$$
$$\sum \lambda^{\tau\sigma} f_\tau^\sigma f_\sigma(h) = \sum \lambda^{\tau\sigma} f_{\tau\sigma}(h)$$

and hence we see that this element lies in A_0. But we know that we can find $d = [L:k]$ elements $\lambda_1 \cdots \lambda_d \in L$ such that the determinant of the matrix

$$(\lambda_i^\sigma)_{i=1\cdots d, \sigma \in \mathrm{Gal}(L/k)}$$

is non-zero. (Linear independence of the elements in the Galois group.) Hence we can write h as a unique linear combination of the elements

$$\sum \lambda_i^\sigma f_\sigma(h),$$

and hence the assertion is clear.

We can show: *A Galois-descend datum for affine schemes is always effective.*

We come to assertion (ii): Again we apply the main theorem of Galois theory which can be summarized to

$$L \otimes_k L \xrightarrow{\sim} \bigoplus_{\sigma \in \mathrm{Gal}(L/k)} L$$

where

$$a \otimes_k b \longrightarrow (\cdots, a\sigma(b), \cdots)_{\sigma \in \mathrm{Gal}(L/k)}.$$

If we look at the two arrows

$$i_1 : L \longrightarrow L \otimes L, i_2 : L \longrightarrow L \otimes L$$

then

$$\begin{aligned} i_1(\lambda) &= (\cdots \lambda \cdots)_{\sigma \in \mathrm{Gal}(L/k)} \\ i_2(\lambda) &= (\cdots \lambda^\sigma \cdots)_{\sigma \in \mathrm{Gal}(L/k)} \end{aligned}$$

and $k = \{\lambda \mid i_1(\lambda) = i_2(\lambda)\}$ (see ???). Now I skip some details. I say it is quite clear for an L-algebra A the datum of a descend datum

$$\varphi : A \otimes_{L,i_1} L \otimes L \longrightarrow A \otimes_{L,i_2} (L \otimes_k L)$$

is exactly the same as the datum of a Galois-descent datum, we simply have to use the above description of $L \otimes_k L$.

Giving a descent datum on a scheme X'/L of finite (???) can also be interpreted in a different way.

Let us assume that X'/L is reduced. It is of course clear that we have an action of the Galois group $\mathrm{Gal}(\overline{k}/L)$ on the set $X'(\overline{k})$ of geometric points. We can speak of L-rational points. Obviously a point $x \in X'(\overline{k})$ which has already in $X'(k_s)$ is defined over L if and only if it is a fixed point under the action of $\mathrm{Gal}(\overline{k}/L)$.

If we want that X'/L descends to a scheme over k or in other words, if we want to construct a scheme X/k such that $X \times_k L \simeq X'/L$, then we have to be able to extend the Galois action of $\mathrm{Gal}(\overline{j}/L)$ to an action of $\mathrm{Gal}(\overline{k}/k)$.

First of all we can define a "conjugation" map

$$\sigma : X'(\overline{k}) \longrightarrow (X')^\sigma(\overline{k})$$

which on affine pieces is defined as follows: If $U' \subset X'$ is affine on $U' = \mathrm{Spec}(A')$, then a geometric point is an L-homomorphism

$$\begin{array}{ccccc} x & : & A' & \longrightarrow & \overline{k} \\ & & \nwarrow & & \swarrow \\ & & & L & \end{array}$$

Then the composition $\sigma \circ x : A' \to \overline{k}$, $(\sigma \circ x)f = x(\varphi(f))$ is clearly a homomorphism of the L-algebra $(A')^\sigma$ to $\overline{k}$. This gives us our conjugation for affine pieces and hence also for X'/L itself. But of course our descend datum $\{f_\sigma\}_{\sigma \in \mathrm{Gal}(L/k)}$ yields a bijection $X'(\overline{k}) \overset{f_\sigma}{\to} (X')^\sigma(\overline{k})$. For any $\tilde{\sigma} \in \mathrm{Gal}(\overline{k}/k)$ with restriction σ to $\mathrm{Gal}(L/k)$ we define

$$\tilde{\sigma}(x) = f_\sigma^{-1} \circ \sigma(x)$$

for all $x \in X(\overline{k})$. Then it is clear that the cocycle condition for the descend datum implies that this defines an action of $\mathrm{Gal}(\overline{k}/k)$ on $X(\overline{k})$.

It is also clear that in case of effectiveness of our descend datum this action of $\mathrm{Gal}(\overline{k}/k)$ on $X'(\overline{k})$ is exactly the action which is provided by $h : X \times_k L \to X'$ and the action of $\mathrm{Gal}(\overline{k}/k)$ on $X(\overline{k})$.

Moreover it is also clear that the descend datum is uniquely determined by the extension of the action to $\mathrm{Gal}(\overline{k}/k)$ if our scheme is reduced. A morphism between reduced schemes is determined by what it does on geometric points. (Do we need absolutely reduced ???)

We want to say a word on Galois descend on absolutely reduced schemes X/k which are of finite type. It is quite clear that we can prove effectiveness for a descend datum

$$(X'/L, f_\sigma)_{\sigma \in \mathrm{Gal}(L/k)}$$

if we know in addition that we can find a finite covering $X' = \{U_i'\}_{i \in E}$ by affine open subschemes which is compatible with the descend datum, i.e.

$$f_\sigma : U_i' \longrightarrow (U_i')^\sigma$$

for all σ and all indices $i \in E$. This is clear because then we apply that the datum on the affine pieces is effective, and we glue the descended affine pieces together. The detailed proof is pure routine.

There is a simple criterion that tells us that we can verify this assumption. If our scheme X'/k is absolutely reduced and separated, if we know that any finite set $\{x_1, \cdots, x_m\} \in$

$X'(\overline{k})$ is contained in the set of geometric points of an affine open subset $U'/L \subset X'/L$, then our assumption is true. We simply start with any point $x \in X'(\overline{k})$ and look at its finitely many conjugates $\{\tilde{\sigma}(x)\}_{\tilde{\sigma}\in \mathrm{Gal}(\overline{k}/k)}$. We pick an open affine set $U'' \subset X'/L$ which contains all these points. It is clear that the conjugates $\tilde{\sigma}(U''(\overline{k}))$ are also the sets of geometric points of an affine open subset, namely $f_\sigma^{-1}((U'')^\sigma)$ and we look at the intersection

$$U' = \bigcap_{\sigma\in \mathrm{Gal}(L/k)} \tilde{\sigma}(U'').$$

It is affine and it contains our given point $x \in X'(\overline{k})$. Moreover it is clear that $f_\sigma : U' \to (U')^\sigma$, and we are finished.

2.7.4. **The Artin-Rees theorem:** Before I formulate the theorem I discuss its consequences and I also put it into its general framework.

Now we assume that A is a noetherian local ring (See CA.1.4.) with maximal ideal $\mathfrak{m}$. We can provide it with the so called $\mathfrak{m}$-adic topology: The open neighborhoods of $m \in M$ are the sets $m + \mathfrak{m}^N$.

We have the *completion*

$$i : A \to \hat{A} = \lim_{\leftarrow} A/\mathfrak{m}^N.$$

We have also the completion of an A-module M we define $\hat{M} = \lim_{\leftarrow} M/\mathfrak{m}^N M = M \otimes_A \hat{A}$. The theorem of Artin-Rees implies

If M is a finitely generated module over the noetherian local ring A then $M \to \hat{M}$ is injective

This is of course equivalent to

$$\bigcap_{N\geq 1} \mathfrak{m}^N M = (0).$$

Then the assertion above says that the $\mathfrak{m}$-adic topology is Hausdorff. The reader should verify that the module $\hat{M}$ is indeed complete with respect to the $\mathfrak{m}$-adic topology, i.e. any Cauchy sequence is convergent.

We consider finitely generated A-modules $M, M' \ldots$.

The main implication of the Artin-Rees theorem is:

On the category of finitely generated A-modules the functor

$$M \to \hat{M} = M \otimes_A \hat{A}$$

is faithfully flat which means that it is exact and $\hat{M} = 0 \Rightarrow M = 0$.

The faithfulness is Nakayama's lemma (See CA.1.2.) To prove flatness we have to show: If M', M are finitely generated A-modules and if

$$M' \to M$$

is injective then

$$\hat{M}' \to \hat{M}$$

is again injective.

If one tries to prove this one sees that the following has to be shown: If we consider $M' \bigcap \mathfrak{m}^N M$ for a very large N then this must be contained in $\mathfrak{m}^n M'$ where n is also large. We cannot prove our theorem if we are not able to show that n goes to infinity if N goes to infinity.

The answer to our problem is the actual Artin-Rees theorem

There exists an $r_0 > 0$ such that for all $r > r_0$ we have

$$\mathfrak{m}^r M \bigcap M' = \mathfrak{m}^{r-r_0}(\mathfrak{m}^{r_0} M \bigcap M') \subset \mathfrak{m}^{r-r_0} M'$$

For the proof I refer to the literature.(Ref ?????)

It is clear that it is the technical answer to our problem and it implies that $A \to \hat{A}$ is faithfully flat on finitely generated modules.

We leave it as an exercise to verify that it also gives $\bigcap_{N \geq 1} \mathfrak{m}^N M = (0)$.

VI 2.8.**Formal schemes and infinitesimal schemes**

I want to discuss a situation which is in some sense opposite to the one discussed above. Let k be a field, consider a polynomial ring $A = k[X_1, \ldots, X_n]$ and we localize it at the maximal ideal $\mathfrak{m} = (X_1, X_2, \ldots, X_n)$. This local ring $A_\mathfrak{m}$ is the ring of germs of regular functions at zero and hence we say that $\mathrm{Spec}(A_\mathfrak{m})$ is the germ of our affine space at zero. This ring is not of finite type over k anymore but it is still a direct limit of finitely generated k-algebras. This local ring sits in the ring $\hat{A} = k[[X_1, \ldots, X_n]]$ of power series in the variables $X_1, \ldots, X_n$ as a subring: If $Q(X_1, \ldots, X_n) \in A$ and $a_0 = Q(0, \ldots, 0) \neq 0$ then we can write $Q = a_0 + R$ where R is a polynomial without constant term and

$$\frac{1}{Q(X_1, \ldots, X_n)} = \frac{1}{a_0 + R} = \frac{1}{a_0}(1 - R/a_0 + (R/a_0)^2 \ldots).$$

It is easy to see that this power series ring can also be obtained as the projective limit $\lim_{\leftarrow N} A_{\mathfrak{m}_P}/\mathfrak{m}_P^N$. It is also the completion of $A_\mathfrak{m}$ with respect to the $\mathfrak{m}$-adic topology. (See subsection on the Artin Rees theorem)

The verification of the following assertions are left to the reader.

(1)

$$\mathrm{Hom}_k(A_\mathfrak{m}, k) = \mathrm{Hom}_k(\hat{A}, k) = (0) \in k^n$$

(we lost all our geometric points except the origin. The geometric points are not dense and this does not contradict the Nullstellensatz because the local ring is not of finite type.)

Let us assume that B is a k-algebra which is local with maximal ideal $\mathfrak{n}$ and $B/\mathfrak{n} = k$. Then

(2)

$$\mathrm{Hom}_k(A_{\mathfrak{m}}, B) = \mathfrak{n}^n$$

where the identification is provided by $\phi \mapsto (\phi(X_1), \ldots, \phi(X_n))$.

(3) We can construct rings B as above such that we still have

$$\mathrm{Hom}_k(\hat{A}, B) = (0)$$

(4)

If in addition the ideal $\mathfrak{n}$ in our k algebra B consists of nilpotent elements then we have

$$\mathrm{Hom}_k(A_{\mathfrak{m}}, B) = \mathrm{Hom}_k(\hat{A}, B) = \mathfrak{n}^n$$

.

We have the inclusion of rings:

$$A \hookrightarrow A_{\mathfrak{m}} \hookrightarrow k[[X_1, \ldots, X_n]]$$

which in turn induces morphism between schemes

$$\mathrm{Spec}(k[[X_1, \ldots, X_n]]) \hookrightarrow \mathrm{Spec}(A_{\mathfrak{m}}) \hookrightarrow \mathrm{Spec}(A)$$

We call an algebra B as above for which the ideal $\mathfrak{n}$ consists of nilpotent elements an *infinitesimal* algebra. Its spectrum is a single point. But if we consider $\mathrm{Spec}(B)$-valued points of $\mathrm{Spec}(k[[X_1, \ldots, X_n]])$ then we get n-tuples $(\epsilon_1, \epsilon_2, \ldots, \epsilon_n) \in \mathfrak{n}^n$. These are considered to be points which are infinitesimally small in the sense that the coordinates are not zero but if we raise them into a suitably high power they become zero. This leads us to the idea that $\mathrm{Spec}(k[[X_1, \ldots, X_n]])$ should be considered as an infinitesimal neighborhood of the origin in $\mathrm{Spec}(k[X_1, \ldots, X_n]) = \mathbb{A}^n$.

The scheme $\mathrm{Spec}(k[[X_1, \ldots, X_n]])$ is also called the formal completion of $\mathrm{Spec}(k[X_1, \ldots, X_n])$ $\blacksquare$ at the origin.

Chapter VII
Some commutative algebra

I want to collect some standard facts from commutative algebra. Here I will be rather scetchy because there are many good references available. On the other hand I think the reader might be able to fill the gaps itself.

VII. 1. **Finite A-algebras**

A morphism $\phi : A \to B$ between two affine schemes is called *finite* if one of the following two conditions is satisfied

(i) The $A-$module B is finitely generated.

This means that we can find $b_1, b_2, \dots, b_r$ such that any $b \in B$ can be written as linear combination $b = a_1 b_1 + \dots a_r b_r$ with $a_i \in A$.

(ii) The A-ring B is finitely generated by elements $x_1, \dots, x_s$ where each of the generators is a zero of a unitary polynomial in $A[X]$.

This means that the A-module B is generated by the monomials $x_1^{\nu_1} \dots x_s^{\nu_s}$ and that for any i we have a polynomial $P_i|X] = a_0 + a_1 X \dots + X^{n_i} \in A[X]$ such that $P_i(x_i) = 0$. (We say that the x_i are integral over A.)

I think that is immediately clear that (ii) implies (i) since we can use the polynomials to reduce the degree of the generating monomials.

The proof that (i) implies (ii) is amusing, we leave it as an exercise. Item a) in the following exercise gives a hint.

Exercise 1:

a) We have to show that any $b \in B$ is a zero of a unitary polynomial in $A[X]$, i. e. it is integral over A. To see this we multiply the generators b_i by b and express the result again as A-linear combination of the b_i. This gives us a $r \times r$-matrix M with coefficients in A. If $\underline{b}$ is the column vector formed by the b_i we get a relation $b\underline{b} = M\underline{b}$ or $(M - b\,\mathrm{Id})\underline{b} = 0$. From this we have to conclude that $\det(M - b\,\mathrm{Id}) = 0$. This would be clear if A would be integral. But it is also true in the general case (I refrain from giving a hint). Hence we see that b is a zero of the characteristic polynomial of the matrix M but this polynomial equation has highest coefficient 1 and the other coefficients are in A.

b) This argument generalizes: Let us consider any A-algebra $A \to B$. Show that an element $b \in B$ is integral over A if we can find a finitely generated A-submodule $Y \subset B$ which satisfies some extra condition (?) with $bY \subset Y$

c) Let us assume that $A \to B$ are both integral and that $K \to L$ is the corresponding extension of their quotient fields. Let us assume that L/K is a finite extension. Let us also assume that A is integrally closed in K (any element in K that is integral over A is already in A, such rings are called $normal\ rings$). Then an element $x \in L$ induces by multiplication a linear transformation L_x of the K-vector space L and it is well known that x is a zero of the characteristic polynomial of L_x. Show that x is integral if and only if this polynomial has coefficients in A.

d) Under the assumptions of c) we have $\mathrm{trace}_{L/K}(x) \in A$ *for any element* $x \in L$ *which is integral over* A.

e) Again under the assumptions of c) we can say: For any $x \in L$ *we can find a non zero element* $a \in A$ *such that* ax *becomes integral over* A

We have the following

VII.1.2 **Theorem**:

Assume that the ring homomorphism $\phi : A \to B$ *is finite and injective. Then the induced map* ${}^t\phi : \mathrm{Spec}(B) \to \mathrm{Spec}(A)$ *is surjective, has finite fibers and the elements in the fibers are incomparable with respect to the order on* $\mathrm{Spec}(B)$

This means in other words: For any $\mathfrak{p} \in \mathrm{Spec}(A)$ we can find a $\mathfrak{q} \in Spec(B)$ such that $A \cap \mathfrak{q} = \mathfrak{p}$. The number of such $\mathfrak{q}$ is finite, whenever we have two of them $\mathfrak{q}_1, \mathfrak{q}_2$ we have $\mathfrak{q}_1 \not\subset \mathfrak{q}_2$. This theorem summarizes the so called going up and going down theorems in the books on commutative algebra.

To prove the theorem we need another famous result from commutative algebra

Lemma of Nakayama: *Let* A *be a local ring with maximal ideal* $\mathfrak{m}$ *and let* M *be a finitely generated* A*-module. If*

$$M \otimes (A/\mathfrak{m}) = M/\mathfrak{m}M = 0$$

then $M = 0$.

To see this one uses the same trick as above: Express a system of generators of M as a linear combination of these generators but now with coefficients in $\mathfrak{m}$. We find that 1_A is a zero of a characteristic polynomial of a matrix with coefficients in $\mathfrak{m}$ which is only possible for the 0×0 -matrix.

Now I sketch the proof of the theorem. We pick a prime $\mathfrak{p} \in \mathrm{Spec}(A)$. The residue class ring $A/\mathfrak{p}$ is integral, we have $\mathrm{Spec}(A/\mathfrak{p}) \hookrightarrow \mathrm{Spec}(A)$ and the zero ideal (0) is mapped to $\mathfrak{p}$. We localize at (0) and we get the quotient field $(A/\mathfrak{p})_{(0)}$. Taking fibered products we get a diagram of affine schemes

$$\begin{array}{ccc}
\mathrm{Spec}(A) & \leftarrow & \mathrm{Spec}(B) \\
\uparrow & & \uparrow \\
\mathrm{Spec}(A/\mathfrak{p}) & \leftarrow & \mathrm{Spec}(B) \otimes_A (A/\mathfrak{p}) \\
\uparrow & & \uparrow \\
\mathrm{Spec}(A/\mathfrak{p})_{(0)} & \leftarrow & \mathrm{Spec}(B) \otimes_A (A/\mathfrak{p})_{(0)}
\end{array}$$

The vertical arrows are inclusions and it is clear that the prime ideals $\mathfrak{q} \in \mathrm{Spec}(A)$ for which $\mathfrak{q} \cap A$ are exactly the elements in $\mathrm{Spec}(B) \otimes_A (A/\mathfrak{p})_{(0)}$ To prove the surjectivity we have to show that this scheme is not empty. This follows from the lemma of Nakayama because we can obtain $\mathrm{Spec}(A/\mathfrak{p})_{(0)}$ also as the residue field of the local ring $A_\mathfrak{p}/\mathfrak{m}_\mathfrak{p}$. We have $B \otimes_A (A_\mathfrak{p}) \neq 0$ (only the zero divisors of $S = A \setminus \mathfrak{p}$ go to zero in this tensor product) and hence we get by Nakayama that $B \otimes_A A_\mathfrak{p}/\mathfrak{m}_\mathfrak{p} \neq 0$ and his implies $B \otimes_A A_\mathfrak{p}/\mathfrak{m}_p \neq \emptyset$. Now we have that $B \otimes_A (A/\mathfrak{p})_{(0)}$ is a finite dimensional vector space over the field $(A/\mathfrak{p})_{(0)}$ it is a finite $(A/\mathfrak{p})_{(0)}-$ algebra. This implies that any prime ideal $\mathfrak{q} \in \mathrm{Spec}(B \otimes_A (A/\mathfrak{p})_{(0)}$

is maximal because the residue ring is automatically a field. Then it is also clear that $\operatorname{Spec}(B) \otimes_A (A/\mathfrak{p})_{(0)}$ must be finite. (The map $B \otimes_A (A/\mathfrak{p})_{(0)} \to \prod_{\mathfrak{q}} B \otimes_A (A/\mathfrak{p})_{(0)}/\mathfrak{q}$ is easily seen to surjective.) Hence we have proved that the fibers are finite and non empty and we have seen that the prime ideals in the fibers are incomparable.

The following lemma is extremely important, we leave the proof to the reader a special case of it was discussed in exercise (1) V.1.2

Noethersches Normalisierungslemma:

Let $A = k[x_1, x_2, \ldots, x_n]$ be a finitely generated k-algebra (k a field. Then we can replace the system of generators by another system $y_1, \ldots y_r, y_{r+1}, \ldots, y_m$ such that $B = k[y_1, y_2, \ldots, y_r]$ is a polynomial ring in the variables $y_1, \ldots, y_r$ and the other variables (or the algebra A) are integral over B .

For the proof I refer to the standard books. The basic idea is explained in the exercise mentioned above.

The theorem above has several important applications. One of them is the

Nullstellensatz von Hilbert:

Let k be a field and let $A = k[x_1, x_2, \ldots, x_r]$ be a finitely generated k-algebra. For any maximal ideal $\mathfrak{m} \in \operatorname{Spec}(A)$ the residue field $A/\mathfrak{m}$ is a finite field extension of k. Especially if k is algebraically closed then $A/\mathfrak{m} = k$.

We divide by $\mathfrak{m}$ in other words we may assume that A itself is already a field. We proceed by induction on r. If $r = 0$ then the theorem is obvious. Let $r \geq 1$. We apply the lemma above and find that that A is finite over a polynomial ring $B = k[y_1, \cdots, y_s] \subset A$ If $s \geq 1$ then B has non trivial prime ideals which by our theorem VII.1.2. extend to non zero prime ideals in A which is not possible. Hence we see that $B = k$ and therefore A is a finite field extension of k.

VII. 1.3 The Nullstellensatz implies :

For a finitely generated k-algebra A/k the intersection of the maximal prime ideals is equal to the radical

Proof: This has been mentioned earlier. Note that A_f is still a finitely generated k-algebra.

VII 1.2. **Dimension theory for finitely generated k algebras**

If such a k-algebra A is integral we define the *dimension* $\dim(A)$. The number $\dim(A)+1$ is equal the maximal length a chain of prime ideals $(0) \subset \mathfrak{p}_1 \subset \ldots \subset \mathfrak{p}_r$ where all the inclusions are proper. We say say that the dimension is ∞ if chains of arbitrary length exist.

A maximal chain is a chain that cannot be refined. We also define the *height* $h(\mathfrak{p})$ of a prime ideal $\mathfrak{p}$. The number $h(\mathfrak{p}) + 1$ is the maximal length of a chain $(0) \subset \mathfrak{p}_1 \subset \ldots \subset \mathfrak{p}$ ending with $\mathfrak{p}$. There exists a theory of dimensions for arbitrary noetherian rings (See for instance At-McD) which is more general than what we are doing here.

Before we prove the main theorem in this section I want to mention

Proposition 1.2.1. *If we have an inclusion $i : A \to B$ of two commutative rings with identify and if B is integral over A then for any chain $(0) \subset \mathfrak{p}_1 \subset \ldots \subset \mathfrak{p}_r$ of prime ideals in A we can find a chain $(0) \subset \mathfrak{p}'_1 \subset \ldots \subset \mathfrak{p}'_r$ of prime ideals such that $\mathfrak{p}_i = \mathfrak{p}'_i \cap A$. Especially we can say that the two rings have the same dimension.*

Proof: Obvious from VII. 1.2.

VII.1.3 **Theorem**

The dimension of an integral finitely generated k-algebra A is finite and any maximal chain has the same length. The dimension of the polynomial ring $k[X_1, X_2, \ldots, X_n]$ is n.

To see this we proceed again by induction on the number of generators. If we have two integral k-algebras $A \subset B$ and if B is finite over A then $\dim(A) = \dim(B)$. Using the Noether Normalisierung we can write our algebra A as a finite extension of an algebra with less generators as long as we have non trivial relations among the generators. Hence we are reduced to the case of a polynomial ring $A = k[X_1, \ldots, X_n]$. At this point the reader should be aware that it is not clear at this point that we can find minimal non zero prime ideals, it could be possible that we can always to smaller and smaller non zero prime ideals. This would of course imply that the dimension is ∞.

To see that this is not the case we pick a an arbitrary non zero element $F(X_1, X_2, \ldots, X_n) \in A$. We consider prime ideals $\mathfrak{p}$ containing the principal ideal $(F(X_1, X_2, \ldots, X_n))$.

We will show:

We can find minimal prime ideals $\mathfrak{p} \supset (F(X_1, X_2, \ldots, X_n))$ and each such ideal is also a minimal non zero prime ideal (Hauptidealsatz of Krull).

Proof: Again we apply the Noether Normalisierung which tells us that after some change of variables we can assume that this polynomial is of the form

$$F(X_1, X_2, \ldots, X_n) = a_0(X_1, \ldots, X_{n-1}) + a_1(X_1, \ldots, X_{n-1})X_n + \ldots + X_n^m,$$

i.e. it is unitary in X_n over the ring $k[X_1, \ldots, X_{n-1}]$.

We rewrite our polynomial ring by increasing the number of generators

$$A = k[X_1, \ldots, X_n, Y]/(F(X_1, X_2, \ldots, X_n) - Y).$$

This gives us an embedding

$$k[X_1, \ldots, X_{n-1}, Y] \subset k[X_1, \ldots, X_n, Y]/(F(X_1, X_2, \ldots, X_n) - Y)$$

where now $k[X_1, \ldots, X_{n-1}, Y]$ is again a polynomial ring. (This requires a little argument.) We exchanged the variable X_n against the polynomial F. Now it is very easy to see that the principal ideal $(Y) = Yk[X_1, \ldots, X_{n-1}, Y]$ is a non zero prime ideal and it is minimal with this property. Our original algebra A is integral over $k[X_1, \ldots, X_{n-1}, Y]$. Our theorem tells us that we can find a prime ideal $\mathfrak{p}$ for which we have $\mathfrak{p} \cap k[X_1, \ldots, X_{n-1}, Y] = (Y)$ and hence $\mathfrak{p} \supset F(X_1, \ldots, X_n)$ and $\mathfrak{p}$ is minimal with this property. It is even clear that $\mathfrak{p}$ is a minimal non zero prime ideal because a smaller one would induce a non zero prime ideal contained in (Y). Now we see that chains of prime ideals in which the non zero members contain $F(X_1, \ldots, X_n)$ have the same length as the induced chains in $k[X_1, \ldots, X_{n-1}, Y]$ where the non zero members contain (Y). But these last chains are chains in $k[X_1, \ldots, X_{n-1}]$ and it follows from the induction hypothesis that the maximal chains in this ring have length n. Hence we see that maximal chains in A have length $n+1$ and the theorem is proved.

Finally I want to comment briefly on the *Hauptidealsatz von Krull.* We proved in the course of the proof of the previous theorem for a polynomial ring, but the general case follows easily again by Noether normalization. In our situation it says that

Hauptidealsatz von Krull

For a finitely generated k-algebra A which is integral and a non zero element $f \in A$ and a minimal prime ideal $\mathfrak{p} \supset (f)$ we have $\dim A - 1 = \dim(A/\mathfrak{p})$.

The general version of this theorem holds for noetherian rings.(See Atiyah- Macdonald p. 122)

1.3.1. We need the integrality of our algebra because otherwise $\operatorname{Spec}(A)$ can if several irreducible components (See VII. 2.) and these components may have different dimensions. This would have the effect that we can find non refinable chains of prime ideals which have different length.

On the other hand we know that $\operatorname{Spec}(A) = \operatorname{Spec}(A/\operatorname{Rad}(A))$ and this implies that we need is the integrality of $A/\operatorname{Rad}(A)$ to define the dimension.

VII.1.4.**Rings with finiteness conditions**

In this section I collect some facts about rings. I will not give any proof because these facts are easily available in the literature. On the other hand it may be a good exercise if the reader tries to find the proofs itself.

A commutative ring A with identity is called *noetherian* if it satisfies one of the following four conditions
(1) Any ideal $\mathfrak{a} \subset A$ is finitely generated.

(2) Any submodule N of a finitely generated A-module M is finitely generated.

(3) Any ascending chain

$$\mathfrak{a}_\nu \subseteq \mathfrak{a}_{\nu+1} \subseteq \ldots \mathfrak{a}_n \subseteq \ldots$$

becomes stationary, i.e. there exists an n_0 such that

$$\mathfrak{a}_{n_0} = \mathfrak{a}_{n_0+1} \cdots = \mathfrak{a}_{n_0+m}$$

for all $m \geq 0$.

(4) Any ascending chain of A-submodules

$$N_\nu \subseteq N_{\nu+1} \subseteq \ldots$$

of a finitely generated A-module M becomes stationary.

The ring $\mathbb{Z}$ is noetherian and of course we know that fields are so too.

Hilbertscher Basissatz:

If A is a noetherian ring then the polynomial ring $A[X]$ is also noetherian.

This implies that polynomial ring $A[X_1, \ldots, X_n]$ are noetherian.

VII.2. Minimal prime ideals and decomposition into irreducibles

For any noetherian ring A we can consider the set of minimal prime ideals. Of course for an integral ring A the set of minimal prime ideals consists of just one element $(0) \in \operatorname{Spec}(A)$, this is the generic point. In the general case it is not a priory clear that minimal prime ideals exist

A ring is called *reduced* if $\operatorname{Rad}(A) = 0$, we know $\operatorname{Spec}(A) = \operatorname{Spec}(A/\operatorname{Rad}(A))$ hence if we want to say something about the ordered topological space $\operatorname{Spec}(A)$ we may very well restrict our attention to the case that A is reduced. I state a theorem which is a weak form of a theorem proved by E. Lasker.

Theorem VII.2.1. *Let A be a reduced noetherian ring. Then the set of minimal prime ideals is finite. To any minimal prime ideal $\mathfrak{p}$ we can find an $f \in A$ such that*

$$(\mathfrak{p}) = Ann_A(f) = \{x \in A | xf = 0\}.$$

I want indicate the steps of the proof and leave it to the reader to fill the gaps.

1) We prove that there exist minimal prime ideals. This is clear if A is integral. If not we find $f, g \in A \setminus \{0\}$ such that $fg = 0$.

1a) Consider $\operatorname{Ann}_A(f) = \mathfrak{a}$ and prove: If $\mathfrak{a}$ is not prime then we can find an $x \in A$ such that $f_1 = xf \neq 0$ and such that $\mathfrak{a}_1 = \operatorname{Ann}_A(f_1)$ is strictly larger than $\mathfrak{a} = \operatorname{Ann}_A(f)$.

1b) Show that this implies that we can find an $y \in A$ such that $\operatorname{Ann}_A(fy) = \mathfrak{p}$ is a prime ideal and that this prime ideal is minimal. Hence we see that minimal primes exist.

Let us write $yf = f_\mathfrak{p}$. It is clear that $f_\mathfrak{p} \notin \mathfrak{p}$.

2) Prove that any prime ideal $\mathfrak{q}$ contains a minimal prime ideal of the form $\operatorname{Ann}_A(f_\mathfrak{p})$ and hence all minimal prime ideals are of this form.

Let us assume we picked an $f_\mathfrak{p}$ for any minimal prime ideal.

3) Prove that for two minimal prime ideals $\mathfrak{p} \neq \mathfrak{p}_1$ the product $f_\mathfrak{p} f_{\mathfrak{p}_1} = 0$.

4) Consider the ideal generated by these $f_\mathfrak{p}$ and combine the fact that this ideal is finitely generated and 3) above to show that these $f_\mathfrak{p}$ form a finite set.

5) Let A be an arbitrary noetherian ring, let $\mathfrak{p}_1, \ldots, p_r$ be the set of minimal prime ideals. Let us also assume that the spaces $\operatorname{Spec}(A/\mathfrak{p}_i)$ are disjoint. Then there is a unique collection of elements $e_1, \ldots, e_r$ such that

$$\begin{gathered} e_i \notin \mathfrak{p}_i \text{and } e_i \in \mathfrak{p}_j \text{for all } j \neq i \\ e_i^2 = e_i \text{for all } i \\ e_i e_j = 0 \text{ for all } i \neq j \\ \sum e_i = 1_A \end{gathered}$$

I want to give a hint: Our assumption that the spaces $\mathrm{Spec}(A/\mathfrak{p}_i)$ *are disjoint implies that we can find* e_i' *such that* $e_i' \equiv 1 \bmod \mathfrak{p}_i$ *and* $e_i \in \mathfrak{p}_j$ *for all* $j \neq i$. *These* e_i' *satisfy all the relations if we compute modulo the radical* $\mathrm{Rad}(A)$. *Now we can modify* $e_i' \to e_i' + r_i = e_i$ *such that we have the idempotency* $e_i^2 = e_i$. *(See the following exercise 6)). Then all the other requirements are also fulfilled.*

6) If we have any noetherian ring R *and if we consider the homomorphism* $R \to R/\mathrm{Rad}(R)$ *then the group* R^* *of units of* R *is the inverse image of the units in* $R/\mathrm{Rad}(R)$*)*

This decomposition of $1_A = e_1 + \ldots + e_r$ is called the decomposition into orthogonal idempotents. It gives a decomposition of the ring

$$A = \bigoplus_i Ae_i$$

If our ring has no radical the the $f_{\mathfrak{p}_i}$ are equal to the e_i.

An affine scheme $X = \mathrm{Spec}(B)$ is called *irreducible* if cannot be written in a non trivial way as a union of two closed subschemes $X = X_1 \cup X_2$. (Non trivial means both subschemes X_1 and X_2 are not equal to X.)

If $A = B/\mathrm{Rad}(B)$ is not integral then we get a non trivial decomposition $\mathrm{Spec}(B) = \cup \mathrm{Spec}(A/\mathfrak{p}_i)$ and hence we see that $\mathrm{Spec}(B)$ is not irreducible in this case. If on the other hand we have a non trivial decomposition $\mathrm{Spec}(B) = X_1 \cup X_2$ then we can find elements f_1, f_2 which are non zero such that f_i vanishes on X_i. Then $f_1 f_2$ vanishes on $\mathrm{Spec}(B)$, hence it must be in the radical. (Ref ???). This implies that A has zero divisors. We proved

2.1.2. *The spectrum* $\mathrm{Spec}(B)$ *of a noetherian ring* B *is irreducible if and only if* $B/\mathrm{Rad}(B)$ *is integral.*

It follows from Laskers theorem that for a noetherian ring A we have a unique finite decomposition of $\mathrm{Spec}(A)$ into irreducible subschemes $\mathrm{Spec}(A/\mathfrak{p}_i)$. These irreducible subschemes are called the irreducible components. If our ring A has no radical then the elements $f_\mathfrak{p}$ have the property that the vanish on all components except the one which they define. This makes it clear why we have 3) in the exercise above.

2.1.3. For our given noetherian ring A we consider the natural homomorphism

$$\pi : A \to \prod A/\mathfrak{p}_i$$

where the $\mathfrak{p}_i$ run over the minimal prime ideals. The kernel is the radical of A but it is not always surjective because the irreducible components may have non empty intersections.

A non zero element $f \in A$ is called a *non zero divisor* if $fg = 0$ implies $g = 0$. The non zero divisors form a multiplicatively closed subset $S \subset A$. It is clear that $f \in A$ is a non zero divisor if and only if all its components under the projection map $\pi$are non zero or what amounts to the same if f is not contained in any of the minimal prime ideals.

The *total quotient ring* of A is the localization $A_S = \mathrm{Quot}(A)$. The spectrum of A_S is simply the set of generic points of the irreducible components. If we pick a minimal prime ideal $\mathfrak{p}_i$ then the localization $A_{\mathfrak{p}_i}$ is a local *artinian* ring this means that its unique maximal ideal is also minimal (and the ring itself is noetherian). We get

The ring A_S is the direct product of the finitely many local artinian rings $A_{\mathfrak{p}_i}$ and our map π_S is defined componentwise as

$$\prod A_{\mathfrak{p}_i} \to \mathrm{Quot}(A/\mathfrak{p}_i)$$

It is surjective the kernel is $\mathrm{Rad}(A)_S$

In principle this is the situation of 5) in our exercise, since we localize at the generic points of the components, we make the irreducible components disjoint. Instead of taking this radical step we could have chosen non zero divisors $f_{ij} \subset A$ which are zero on $\mathrm{Spec}(A/\mathfrak{p}_i) \cap \mathrm{Spec}(A/\mathfrak{p}_j)$. If we take the product of all these f_{ij} we get a non zero divisor $F \in A$ and then $\mathrm{Spec}(A_F)$ has now the virtue that its irreducible components are disjoint. Hence we may apply exercise 5) and write

$$A_F \xrightarrow{\sim} \bigoplus A_F e_i.$$

Here we replaced the generic points by actual open sets which looks a little bit more geometric.

An element $f \in \mathrm{Quot}(A)$ can be written as $f = g/h$ where h is a no zero divisor. Then we have of course $f \in A_h$ and f is a regular function on $\mathrm{Spec}(A_h)$. We we look at all different ways to write f as a quotient, the we see that f is defined on the union of all the $\mathrm{Spec}(A_h)$. his open set is called the *domain of definition* for f. It is a dense subset of $\mathrm{Spec}(A)$.

2.1.4. **Decomposition into irreducibles for noetherian schemes**: If we have an arbitrary scheme X we say that this scheme is a *noetherian scheme* if we have a finite open covering $X = \bigcup U_i$ by affine schemes $U_I = \mathrm{Spec}(A_i)$ where the rings A_i are noetherian rings. Again we can speak of irreducible schemes. If one of the covering set U_i is not irreducible then we write it non trivially as $U_i = V_i \cup W_i$. If we take the closures of these subschemes in X we clearly get two closed subschemes which are not equal to X. Then we get a non trivial decomposition of $X = \bar{V}_i \cup \bar{W}_i \cup Z_i$ where Z_i is the complement of U_i.

Hence we see: Our noetherian scheme $X = \cup U_i$ is irreducible if and only if all the U_i are irreducible.

From this we can deduces very easily that X itself has always a unique finite decomposition into irreducible subschemes.

This is usually proved directly without reference to affine schemes.

If we have an affine scheme $X = \mathrm{Spec}(k[X_1, \dots, X_n]/I) = \mathrm{Spec}(A)$ then it is automatically noetherian. Hence our previous considerations apply.

VII.2.2.**Affine schemes over k and change of scalars** In this section I want to consider k-algebras A/k and I call such a k-algebra an *affine k-algebra* if it is finitely generated over k as an algebra.

For these schemes over a field we always have the option of base change to $\bar{k}$. This means that we can consider the algebra $A \otimes_k \bar{k}$. We have to discuss what happens to irreducibility, reducedness and dimensions under such a base change. We also want to discuss their behavior under the formation of products.

First of all it can happen that $A = K$ is finite field extension. In this case $\mathrm{Spec}(K)$ is irreducible but $K \otimes_k \bar{k}$ is not if the extension is non trivial and separable. Hence irreducibility is not preserved under base change to $\bar{k}$.

We call A/k *absolutely irreducible* if $A \otimes_k \bar{k}$ is irreducible.

We call a k-algebra A/k *reduced* if it does not contain non zero nilpotent elements. Again it can happen that a reduced k-algebra A/k becomes non reduced if we extend the ground field. To see this we start from a non separable finite extension K/k and then we know that $K \otimes_k \bar{k}$ contains nilpotent elements. We call call A/k *absolutely reduced* if $A \otimes_k \bar{k}$ does not contain non trivial nilpotent elements.

2.2.1..A k algebra is called an *affine k- variety* if it is absolutely irreducible and absolutely reduced.

In the following exercise we will show that these examples are typical: If an integral k-algebra becomes reducible or reduced after extension of scalars, then we can find a finite extension of k which is contains in the field of fractions of A and which is responsible for that.

Exercise 2

Let A/k be an integral finitely generated algebra. If it is not absolutely irreducible or not absolutely reduced then we can find a non zero $f \in A$ such that the localization A_f contains a non trivial finite extension of k.

Hint: Reduce the problem to a base change $A \otimes_k L$ where $L = k[X]/(p(X))$ with an irreducible polynomial $p(X) \in k[X]$.

a) Investigate under which conditions the algebra $A \otimes_k L$ can become reducible or non reduced. This means that we can find two polynomials $g_1(X), g_2(X) \in A[X]$ whose degrees are less than the degree of $p(X)$ such that $p(X)$ divides the product $g_1(X)g_2(X)$. Look at all such pairs and pick one where the degree of say $g_1(X)$ is minimal. Using the arguments you learned in your first algebra course to show that there must be a non zero element $f \in A$ such that $g_1(X)$ divides $fp(X)$. This means that $g_1(X)/f \in A_f[X]$ divides $p(X)$. Now conclude that the coefficients of $g_1(X)/f$ must be algebraic over k. But not all of them are in k and we found the non trivial extension.

b) We call an element $y \in A_f$ a *pseudoconstant* if it generates a finite extension of k, i.e. if it is algebraic over k. We call it a *constant* if it is actually in A .(It looks a little bit bizzar that we can have pseudoconstants which are not constants, but this may be the case). I claim:

b) The pseudoconstants generate a finite algebraic extension of K/k.

To see this we apply the Normalisierungslemma and write A as a finite extension of a polynomial ring B. Hence A is a finite B module let m be the number of elements in a system of generators of this B-module. It is easy to see that for any non zero $f \in A$ we can find an $F \in B$ such that $A_f \subset A_F$ and A_F is still integral over B_F. If our field k is infinite then we can find a k rational point $a \in \mathrm{Spexmax}(B_F)$ let $\mathfrak{m}_a$ be the corresponding maximal ideal. We get an injection of the field of constants of A_F to $A_F/\mathfrak{m}_a A_F$ and the degree of this extension is bounded by m. Hence we see that the degree of the field of pseudoconstants is bounded. This proves the assertion if k is infinite. If k is *finite then we can replace k by an extension k_1/k of prime degree where this prime is larger than m and such that we find a k_1 valued point a of B_F. Then we see that the tensor product $A_F \otimes_k k_1$ is still a field and injects into $A_F \otimes B_F/\mathfrak{m}_a$ and we have the same argument as before.*

To give an example we consider the $\mathcal{R}$-algebra $A = \mathbb{R}[x, y|/(x^2 + y^2)$. Then our scheme $\mathrm{Spec}(A)$ is irreducible but $A \otimes \mathbb{C}$ is not. The elements $x/y, y/x$ are pseudoconstants but they are not constants. The geometric points are given by two lines intersecting in the origin. If we remove the origin, the value of say x/y is $\pm i$ on the two lines. In the origin the function x/y is just a little bit confused and does not know which value to take.

If k is a non perfect field of characteristic 2 and if $a \in k$ is not a square then $A = k[x, y|/(x^2 + ay^2)$ is integral. Again we see that $x/y, y/x$ are pseudoconstants but not constants.

2.2.1.1 It is clear that a pseudoconstant has a dense domain of definition. Hence we see that we have an open dense subset $U \subset \mathrm{Spec}(A)$which is the common domain of all pseudoconstants.

2.2.2.. If we consider an arbitrary finitely generated k-algebra we can consider its total quotient $\mathrm{Quot}(A)/k$ and define the subalgebra of pseudoconstants $L/k \subset \mathrm{Quot}(A)/k$ as being the ring of elements which are finite over k. It is obvious that this k-algebra L/k is exactly the inverse image of the pseudoconstants under the map

$$\pi : \mathrm{Quot}(A) \to \bigoplus_{\mathfrak{p}_i \text{ minimal}} \mathrm{Quot}(A/\mathfrak{p}_i),$$

where now on the right hand side we have a sum of fields.

We conclude using part b) of the exercise that L/k is finite dimensional, because the radical is of finite dimension over k. It is also clear that the map

$$\mathrm{Spec}(\mathrm{Quot}(A)) \to \mathrm{Spec}(L)$$

is bijective, the left hand side is just the set of generic points of the irreducible components of $\mathrm{Spec}(A)$.

We also see that all the pseudoconstants lie already in a suitable subalgebra A_f where f is a non zero divisor.

Now we have the following

Lemma 2.2.2.1. *The k-algebra L/k of pseudoconstants is preserved under change of scalars, i.e. $L \otimes \bar{k}$ is the $\bar{k}$-algebra of pseudoconstants of $A \otimes \bar{k}$. We can find a finite separable extension K/k such that the irreducible components of $A \otimes K$ will be absolutely irreducible.*

We can give this a slightly different formulation. We can consider the k-algebra homomorphism

$$L \otimes \bar{k} \to \mathrm{Quot}(A) \otimes \bar{k}.$$

The lemma says that this induces a bijections

$$\mathrm{Spec}(L \otimes \bar{k}) \leftarrow \mathrm{Spec}(\mathrm{Quot}(A) \otimes \bar{k})$$
$$\mathrm{Rad}(L \otimes \bar{k}) \rightarrow \mathrm{Rad}(\mathrm{Quot}(A) \otimes \bar{k})$$

The proof is based on an observation which will also play a role in the next section.

2.2.2.2. Let us consider an element $F \in A \otimes \bar{k}$. We can write this element in the form

$$F = \sum f_i \otimes \alpha_i$$

where the α_i are taken from a finite normal extension K/k. This extension has a maximal separable subextension K_{sep} in it. We know that any element $\alpha \in K$ raised to a sufficiently high power of the characteristic p of our field k will fall into K_{sep}. Hence we get that

$$F^{p^r} = \sum f_i^{p^r} \otimes \alpha_i^{p^r} \in A \otimes K_{\mathrm{sep}}$$

If we know form the norm of this power, this means we form the product over all conjugates by elements $\sigma \in \mathrm{Gal}(K_{\mathrm{sep}}/k)$ then its norm

$$\prod_\sigma \sigma(F^{p^r}) = \prod_\sigma (\sum f_i^{p^r} \otimes \sigma(\alpha_i^{p^r})) \in A.$$

This has consequences: If for instance F is a non zero divisor then we see that we can find a multiple $FF_1 = G \in A \subset A \otimes \bar{k}$ which is still a non zero divisor. Hence we see that $(A \otimes \bar{})_F \subset A_G \otimes \bar{k}$ and this implies that

$$\mathrm{Quot}(A) \otimes \bar{k} = \mathrm{Quot}(A \otimes \bar{k}).$$

Now let us assume that $F \in \mathrm{Quot}(A) \otimes K$ is a pseudoconstant. This is so if and only if $G = F^{p^r}$ is a pseudoconstant. We rewrite G in the form

$$G = \sum g_i \otimes \beta_i$$

where the $\beta_i \in K_{\mathrm{sep}}$ form a basis over k. We have the trace map $\mathrm{tr} : K_{\mathrm{sep}} \rightarrow k$ and consider the elements

$$\mathrm{tr}(G\beta_j) = \sum_i g_i \, \mathrm{tr}(\beta_i \beta_j).$$

These elements are pseudoconstants and since the determinant of the matrix $\mathrm{tr}(\beta_i\beta_j)$ is non zero we get that the coefficients $g_i \in \mathrm{Quot}(A)$ must be pseudoconstants. We have the decomposition of $\mathrm{Quot}(A) \otimes \bar{k}$ into a direct sum of artinian algebras and we have the decomposition of the identity into orthogonal idempotents $1_A = \sum e_i$. The e_i are clearly pseudoconstants in $\mathrm{Quot}(A) \otimes \bar{k}$. We apply our previous argument and find that suitably high powers $e_i^{p^r}$ lie in $L \otimes \bar{k}$. But then we have $e_i \in L \otimes \bar{k}$ and more precisely we see that the e_i are already in $L \otimes K$ where K/k is separable.

In our discussion above we gave an argument which also can be used to show exercise

c) If K/k is separable then $\mathrm{Rad}(A \otimes K) = \mathrm{Rad}(A) \otimes K$

The only thing that has to be proved is the equality concerning the radicals in the reformulation of the Lemma. For this we may assume that our field k is already separably closed and that A/k is integral. Now we repeat the argument in the exercise above. We reduce the problem to the case of an extension $K = k(\alpha)$ where $\alpha^p = a \in k$. Then this argument yields: If $A \otimes K$ has nilpotent elements then we can find a $\beta \in \mathrm{Quot}(A)$ such that $\beta^p = \alpha^p = a$. This shows the desired equality of radicals for this small extension. Then we proceed by looking at $(A \otimes K)/\mathrm{Rad}(A \otimes K)$ and apply the same argument.

2.2.3. Let us consider two affine $\bar{k}$-algebras $A_1/\bar{k}, A_2/\bar{k}$ then we have the following

Lemma2.2.3.1. *We have*

$$\mathrm{Rad}(A_1 \otimes A_2) = \mathrm{Rad}(A_1) \otimes A_2 + A_1 \otimes \mathrm{Rad}(A_2)$$

and especially the tensor product $A_1 \otimes A_2$ is reduced if the factors are so.

If the two algebras are irreducible then then tensor product $A_1 \otimes A_2$ is also irreducible.

Proof: Choose a basis $\{g_1, g_2, \ldots\}$ of A_2/k where the first say t basis vectors form a basis of $\mathrm{Rad}(A_2)$. Let $h = \sum f_i \otimes g_i$ be an element in the radical. We evaluate this element at geometric points $x \in \mathrm{Specmax}(A_1)$ and then the element $\sum f_i(x) g_i \in A_2$ must be in the radical of A_2. This implies that for all such points x we have $f_i(x) = 0$ for all indices $i > t$. The Nullstellensatz gives us $f_i \in \mathrm{Rad}(A_1)$ for all these $i > t$ and this proves the first assertion.

The proof of the second assertion is similar. We can assume that both algebras are actually integral. We take the same basis and consider a pair of zero divisors

$$(\sum f_i \otimes g_i)(\sum f_i' \otimes g_i) = 0$$

If we evaluate at the geometric points x as above then for each such x we must have that one of the factors vanishes. (Irreducibility of $\mathrm{Spec}(A_2)$). Hence we may consider the two closed subschemes of $\mathrm{Spec}(A_1)$ which are defined by the vanishing of of the f_i or the f_i' respectively. There union contains all geometric points hence they cover $\mathrm{Spec}(A_1)$. Since $\mathrm{Spec}(A_1)$ is irreducible we get that one of them must be the whole $\mathrm{Spec}(A_1)$ and this proves that the corresponding factor in the product must be zero because the product is absolutely reduced.

Proposition 2.3.2. *If A_1, A_2 are two irreducible $\bar{k}$-algebras then we have*

$$\dim(A_1 \otimes A_2) = \dim(A_1) + \dim(A_2).$$

Proof: This is rather clear. We apply Noether Normalisierungssatz two our algebras and write them as finite extensions of polynomial rings. Then the number of variables is equal to the dimension in either case (VII. prop. 1.2.1. and Thm. VII. 1.3.). The tensor product is finite over the polynomial ring in the disjoint union of the variables.

VII. 2.3. Local irreducibility

Now we assume that we have a geometric point $P \in \mathrm{Hom}_k(A, \bar{k})$. It induces a maximal ideal $\mathfrak{m}_P$ in $A \otimes_k \bar{k}$. We say that $A \otimes \bar{k}$ is *locally integral* at P if the local ring $(A \otimes_k \bar{k})_{\mathfrak{m}_P}$ is integral. The point P also induces a maximal ideal $\mathfrak{m}_P^0$ in A. Under these assumptions we have the following technical Lemma

Lemma 2.3.1. *If $A \otimes \bar{k}$ is locally integral at P then the local ring $A_{\mathfrak{m}_P^0}$ is also integral, we have an injection $i : A_{\mathfrak{m}_P^0} \to (A \otimes_k \bar{k})_{\mathfrak{m}_P}$ and equality of dimensions $\dim(A_{\mathfrak{m}_P^0}) = \dim((A \otimes_k \bar{k})_{\mathfrak{m}_P}))$. If $\mathfrak{p}^0$ is the unique minimal prime ideal in $A_{\mathfrak{m}_P^0}$ which is contained in $\mathfrak{m}_P^0$ then these dimensions are also equal to dim$(A/\mathfrak{p}^0)$.*

Proof: We prove the integrality of $A_{\mathfrak{m}_P^0}$, Let $x \in A_{\mathfrak{m}_P^0}$ be an element that maps to zero in $(A \otimes_k \bar{k})_{\mathfrak{m}_P}$. We can assume that $x \in A$ and via the inclusion $A \to A \otimes_k \bar{k}$ we can view it as element in the ring $A \otimes_k \bar{k}$. We now that we can find an element $y_P \notin \mathfrak{m}_P$ such that $xy_P = 0$. We have a finite number of geometric points $P = P_1, P_2, \ldots, P_s$ which lie above $\mathfrak{m}_P^0$ and the Galois group of $\bar{k}/k$ permutes these points transitively (See 2.5.). We can find an element $z_P \notin \mathfrak{m}_P$ such that $z_P \in \mathfrak{m}_{P_i}$ for all $i = 2, \ldots, s$. If we take an element σ_i in the Galois group which maps P to P_i then $\sigma(y_P z_P) = y_{P_i} z_{P_i} \notin \mathfrak{m}_{P_i}$ but it lies in all the other $\mathfrak{m}_{P_j}$ for $j \neq i$. We always have $xy_{P_i} z_{P_i} = 0$ and if we put $s = \sum y_{P_i} z_{P_i}$ then we have $s \notin \mathfrak{m}_{P_i}$ for all i and of course we still have $xs = 0$. We apply the argument from 2.2.2.1. to s and see that we can produce an element $t^{p^N} = \mathrm{Norm}(s^{p^N}) \in A$.Then we still have $xt^{p^N} = 0$ and $t^{p^N} \notin \mathfrak{m}_{P_i}$ for all i and hence $t^{p^N} \notin \mathfrak{m}_P^0$. But this tells us that x maps to zero in $A_{\mathfrak{m}_P^0}$ and since we assumed A integral we must have $x = 0$.

Now we prove the assertions concerning the equality of dimensions. It is clear that we can find an $f \notin \mathfrak{m}_P^0$ such that A_f is integral and then we have $\dim(A_f) = \dim(A_{\mathfrak{m}_P^0})$ this follows from Theorem VII.1.3. and the fact that $\mathrm{Spec}(A_{\mathfrak{m}_P^0}) = \{\mathfrak{p} \in \mathrm{Spec}(A_f) \mid \mathfrak{p} \in \mathfrak{m}_P^0\}$. (ref/????). Now we have seen that we can find a finite extension K/k such that $A_f \otimes K = \sum_i A_f \otimes K e_i$ such that the summands are absolutely irreducible. For this we may have to change the f. But now we have that $A_f \to A \otimes K$ is finite, then it is clear that (??? e_i conjugate under galois) $A \to Ae_i$ is injective and finite and hence we have $\dim(A) = \dim(Ae_i)$.Now we have exactly one index i such that $e_i(P) = 1$ and then we have $(A \otimes_k \bar{k})_{\mathfrak{m}_P} = (Ae_i)_{\mathfrak{m}_{\mathfrak{P}}}$ and the equality of dimensions follows.

But if we extend the ground field to $\bar{k}$ then this is a union of finite extensions and the equality follows from our first theorem in this chapter.

The property that A is locally integral at the geometric point P implies that P lies on exactly one irreducible component of $\mathrm{Spec}(A \otimes_k \bar{k})$ and this implies that it also lies on exactly one irreducible component of $\mathrm{Spec}(A)$. The converse is not true since the local ring at P may still have nilpotent elements. But for the definition of the dimension at P we only need that P lies on exactly one component. In this case we will say that A is *locally irreducible at P.*

2.3.1.2. If we have an integral affine k-algebra A/k and a maximal ideal $\mathfrak{m}$ then we know that the dimension of A is equal to the dimension of the local ring $A_{\mathfrak{m}}$(Thm VII.1.3.) If our k-algebra is not integral and if $\mathfrak{m}$ is a maximal ideal then we may still speak of $\dim(A_{\mathfrak{m}})$ provided this local ring is irreducible which means that $\mathfrak{m}$ lies on exactly one irreducible component, i.e. if there is exactly one minimal prime ideal $\mathfrak{p}_i \subset \mathfrak{m}$. If we pick an element

$f \notin \mathfrak{m}$ which lies in all the other minimal prime ideals then $\mathrm{Spec}(A_f)$ is open and irreducible in $\mathrm{Spec}(A)$. We again have the equality of dimensions $\dim(A_f) = \dim(A_\mathfrak{m})$. If we have another maximal ideal $\mathfrak{m}_1 \supset \mathfrak{p}_i$ and which does not lie in any other irreducible component then we can choose f so that $f \notin \mathfrak{m}_1$. This shows that the local dimension $\dim(A_\mathfrak{m})$ stays constant as long as moves within one component and avoids the set of points which lie in several components.

If A/k is irreducible then we can speak of $\dim(A)$. But what happens if A/k is not absolutely irreducible. If claim that nothing happens. We have seen that we can find a finite separable extension K/k such that the irreducible components of $A \otimes K$ will be absolutely irreducible (Lemma 2.2.2.). Then we see as in the proof Lemma 2.3.1. that $\dim(A) = \dim(A \otimes K)/\mathfrak{p}$ for any minimal prime ideal. We can summarize by saying that the dimension is stable under the extension of scalars.

VII.2.4. **Low dimensional rings**

A noetherian ring is of dimension zero if every prime ideal is maximal (and minimal). In this case it is clear from the assertion above that $\mathrm{Spec}(A) = \{\mathfrak{p}_1, \dots, \mathfrak{p}_t\}$ is a finite set. Then the local rings $A_{\mathfrak{p}_i}$ are also of dimension zero and $A_{\mathfrak{p}_i}$ has only one prime ideal namely $m_{\mathfrak{p}_i}$ and hence $m_{\mathfrak{p}_i}$ is also the radical of this local ring. We have a map

$$A \longrightarrow \prod_{i=1}^{t} A_{\mathfrak{p}_i}.$$

This map is an isomorphism. We have done this in even greater generality in 2.1.3. but here the irreducible components are automatically disjoint.

The rings $A_{\mathfrak{p}_i}$ and hence also A are *artinian* which means that every descending chain of ideals becomes stationary.

Now we consider integral rings A with $\dim(A) = 1$. This means that every non-zero prime ideal $\mathfrak{p}$ is already maximal. If we have any ideal $(0) \neq \mathfrak{a} \neq A$, then $\dim(A/\mathfrak{a}) = 0$ and

$$\mathrm{Spec}(A/\mathfrak{a}) \subset \mathrm{Spec}(A)$$

is a finite subset by the previous results.

Hence we see that for a one dimensional ring A the open sets $U \subset \mathrm{Spec}(A)$ are the complements of a finite set of closed points (maximal prime ideals) and of course the empty set.

If A is integral and of dimension one and local, then $\mathrm{Spec}(A)$ consists of two points $\{\mathfrak{p}, (0)\}$. Such a ring is A is called a *discrete valuation ring* if $\mathfrak{p}$ is a principal ideal, i.e. we can find an element $\pi_\mathfrak{p}$ such that

$$\mathfrak{p} = A \cdot \pi_\mathfrak{p} = (\pi_\mathfrak{p}).$$

The element $\pi_\mathfrak{p}$ is called an *uniformizing element.* It is of course not unique in general, it can be multiplied by a unit and is still an uniformizing element. It is quite clear that any element $a \in A$ can be written as

$$a = \epsilon \cdot \pi_\mathfrak{p}^{\nu_\mathfrak{p}(a)}$$

where ϵ is a unit and where $\nu_{\mathfrak{p}}(a)$ is an integer. This exponent is called the order of a and can be considered as the order of vanishing a at $\mathfrak{p}$.

The elements of the quotient field K are of the form

$$x = \frac{b}{c} = \frac{\epsilon \cdot \pi_{\mathfrak{p}}^{\nu_{\mathfrak{p}}(b)}}{\epsilon' \cdot \pi_{\mathfrak{p}}^{\nu_{\mathfrak{p}}(c)}} = \epsilon" \cdot \pi_{\mathfrak{p}}^{\nu_{\mathfrak{p}}(a)-v_{\mathfrak{p}}(b)} = \epsilon'' \cdot \pi_{\mathfrak{p}}^{\mathrm{ord}_{\mathfrak{p}}(x)}.$$

We clearly have $\nu_{\mathfrak{p}}(x) \geq 0$ if and only if $x \in A$. We may say that x has a pole of order $-\nu_{\mathfrak{p}}(x)$ if $\nu_{\mathfrak{p}}(x) < 0$.

A very important class of one dimensional rings is provided by the *Dedekind rings*. We have the following

2.3.1.**Proposition/Definition**:

A one-dimensional integral ring A is called a Dedekind ring if one of the following conditions is satisfied.

(1) The ring A is integrally closed in its quotient field K, i.e. if $x = \frac{b}{c} \in K$ and we can find a polynomial

$$x^n + a_1 x^{n-1} + \ldots a_n = 0$$

with $a_i \in A$ then $x \in A$.

(2) For every prime ideal $\mathfrak{p} \neq (0)$ the local ring $A_{\mathfrak{p}}$ is a discrete valuation ring.

The inclusion (2) $\Rightarrow$ (1) is quite clear. We consider our element $x \in K$. I claim that for any $\mathfrak{p} \neq (0)$ we must have $x \in A_{\mathfrak{p}}$. Otherwise we could write $x = \epsilon \pi_{\mathfrak{p}}^{-n}$ with $n > 0$ and ϵ a unit in $A_{\mathfrak{p}}$. But then it is not possible that x satisfies the polynomial equation. But if $x \in A_{\mathfrak{p}}$ for all $\mathfrak{p}$ then this means that is regular at all points in $\mathrm{Spec}(A)$ and the assertion follows from proposition V. 1.4.1.

The direction (1) $\Rightarrow$ (2) is a little bit tricky. Of course we may assume that A is already local. If $\mathfrak{p}$ is the maximal ideal then we consider the A-module $\mathfrak{p}^{-1}$ of all elements $x \in K$ which satisfy $x\mathfrak{p} \subset A$. We clearly have $\mathfrak{p}^{-1} \supset A$. The decisive point is to show that we can find an element $y \in \mathfrak{p}^{-1}$ which is not in A. To see this we pick a non zero element $b \in \mathfrak{p}$. The ring $A/(b)$ has dimension zero and therefore the image of $\mathfrak{p}$ in this ring is equal to the radical. This implies that a suitable power $\mathfrak{p}^n \subset (b)$ we choose n minimal with this property. Then we know that we can find elements $p_1 \ldots, p_{n-1} \in Liep$ such that the element $y = p_1 p_2 \ldots p_{n-1}/b \notin A$. But if we multiply by any further element element in $\mathfrak{p}$ then the result lies in A. Now we conclude $y\mathfrak{p} = A$ or $y\mathfrak{p} = \mathfrak{p}$. But the second case is impossible, because by exercise 1) b) in VII. 1. this would imply that y is integral over A and since A is integrally closed we get $y \in A$ in contrast to what we know. The rest is clear: We can find a $\pi \in \mathfrak{p}$ such that $y\pi = 1$. Now it is clear that $\mathfrak{p} = (\pi)$ because if $p \in \mathfrak{p}$ then $yp = a \in A$ and this gives $p = \pi a$.

This proposition is fundamental for the foundation of the theory of algebraic numbers.

If we have a Dedekind ring A and a non-zero ideal $(0) \neq \mathfrak{a} \subset A$, then the quotient $A/\mathfrak{a}$ has dimension zero and we just saw that

$$A/\mathfrak{a} = \prod_{\mathfrak{p} \supset \mathfrak{a}} (A/\mathfrak{a})_{\mathfrak{p}}.$$

If $\mathfrak{a}_\mathfrak{p}$ is the image of $\mathfrak{a}$ in the localization $A_\mathfrak{p}$ then $(A/\mathfrak{a})_\mathfrak{p} = A_\mathfrak{p}/\mathfrak{a}_\mathfrak{p}$. Now we know that $A_\mathfrak{p}$ is a discrete valuation ring hence we have $\mathfrak{a}_p = (\pi_p^{\nu_\mathfrak{p}(\mathfrak{a})})$ and $\nu_\mathfrak{p}(\mathfrak{a})$ is called the order of $\mathfrak{a}$ at $\mathfrak{p}$. It is not difficult to show that $A/\mathfrak{p}^{\nu_\mathfrak{p}(\mathfrak{a})} = A_\mathfrak{p}/(\pi_p^{\nu_\mathfrak{p}(\mathfrak{a})})$ and hence we get

$$A/\mathfrak{a} = \prod_{\mathfrak{p}\supset\mathfrak{a}} A/\mathfrak{p}^{\nu_\mathfrak{p}(\mathfrak{a})}.$$

Exercise 3:

a) this assertion implies $\mathfrak{a} = \prod_{\mathfrak{p}\supset\mathfrak{a}} \mathfrak{p}^{\nu_\mathfrak{p}(\mathfrak{a})}$. *(Hint: What is in general the relation between the product* $\mathfrak{a}\mathfrak{b}$ *and the intersection of two ideals* $\mathfrak{a}, \mathfrak{b}$ *in an arbitrary ring* A*? Show that there is always an inclusion in one direction. Then verify that this inclusion becomes an equality if the two ideals generate the ring, or in other words if* $\mathrm{Spec}(A/\mathfrak{a}) \cap \mathrm{Spec}(A/\mathfrak{b}) = \emptyset$.*)*

b) To any maximal prime ideal $\mathfrak{p}$ *we can find an* $x \in K$ *(the field of fractions) such that* $\mathrm{ord}_\mathfrak{p}(x) = -1$ *and* $\mathrm{ord}_\mathfrak{q}(x) \geq 0$ *for all the other maximal ideals. Then* $xp \in A$ *for all* $p \in \mathfrak{p}$.

A *fractional* ideal $\mathfrak{b}$ of a Dedekind ring A is a finitely generated A non zero submodule in the field of fractions K. For any such fractional ideal $\mathfrak{b}$ we can find an $x \in K^*$ such that $x\mathfrak{b} \subset A$ becomes an integral (ordinary) ideal. We can multiply such fractional ideals and our previous results imply that

2.3.1.1 *The fractional ideals in a Dedekindring form a group under multiplication.*

The neutral element is obviously given by the ring A itself and (exercise 2 b) above gives the inverse $\mathfrak{p}^{-1} = (1, x)$. This group is the free abelian group generated by the prime ideals. It is also called the group of *divisors* $\mathrm{Div}(A)$. This group of divisors contains the subgroup of principal divisors $P(A)$, these are the ideals of the form (x) with $x \neq 0$. The quotient group

$$\mathrm{Pic}(A) = \mathrm{Div}(A)/P(A)$$

is the so called ideal class group of A. It is an important invariant of the ring. By definition it is trivial if and only if A is a principal ideal domain.

If we have an Dedekindring A with quotient field K and if L/K is an extension of finite degree, then we may consider the integral closure of A in L. This is the ring B consisting of those elements b which satisfy an equation $b^n + a_1 b^{n-1} + \dots a_0 = 0$ with $a_i \in A$. (These elements can also be characterized as those elements which leave a finitely generated non zero A-submodule $M \subset L$ invariant under multiplication, i.e. $aM \subset M$. Then it is easy to see that the form indeed a ring).

We have the

2.4.2.**Theorem**(Krull - Akizuki) *The integral closure of a Dedekindring in a finite extension of its quotient field is again a Dedekindring*

This is not an easy theorem for the proof I refer to the book of J. Neukirch (Algebraische Zahlentheorie).The main problem is to show that B is again noetherian.

The following fundamental theorem is easier

2.4.3.*a) If* A *is a Dedekindring with quotient field* K *and if* L/K *is a finite separable extension, the the integral closure* B *of* A *in* L *is a finitely generated* A*- module. Hence* B *is clearly noetherian and therefore again a Dedekindring.*

b) If A is a Dedekindring which is a finitely generated algebra over a field k, i. e. $A = k[x_1, \ldots, x_n]$ and if L is any finite extension of the quotient field K of A then the integral closure B of A in L is again a finitely generated algebra over k and hence noetherian and Dedekind.

Exercise 4

To see a) we start from a basis $a_1, \cdots, a_n$ of the field L over K which consists of integral elements over A. Write an element $b \in B$ as linear combination $b = a_1x_1 + a_2x_2 \ldots a_nx_n$ with $x_i \in K$. We use the seperapabilty to invert this system of equations for the x_i. The traces $\mathrm{trace}_{L/K}(ba_\nu)$ are integral (Use Exercise 1 c)) and we find the relations

$$\mathrm{trace}_{L/K}(ba_\nu) = \sum \mathrm{trace}_{L/K}(a_ia_\nu)x_i.$$

Conclude that we can find an element $a \in A$ which does not depend on b such that $a_ia \in A$, hence $B \subset A\frac{a_1}{a} + \ldots + A\frac{a_n}{a}$ and therefore is finitely generated.

To prove b) we check that we may assume that L/K is normal. Then we have a maximal inseparable subextension L_i/K which is purely inseparable. This is obtained by successive extraction of p-th roots. Hence we prove the assertion for extensions of the form $L = K[r^{1/p}]$ (not so easy) and proceed by induction and apply a) at the end.

The theorem above has the following implication: Let us assume that we have a Dedekindring A with quotient field K and a finite extension L/K and we assume that the assumptions of a) or b) are valid. Then we know that the integral closure B of A in L is a finitely generated A module. Let us pick a maximal prime ideal $\mathfrak{p} \subset A$. We consider the $A/\mathfrak{p}$ algebra $B/\mathfrak{p}B$. First of all I claim that the dimension of $B/\mathfrak{p}B$ as a $A/\mathfrak{p}$-vector space is equal to the degree $[L : K] = \dim_K L$. To see this we may assume that A is local and then B must be a free A-module of rank $[L : K]$. Now we have seen that

$$B/\mathfrak{p}B = \prod_{\mathfrak{P} \supset \mathfrak{p}B} B/\mathfrak{P}^{\nu_{\mathfrak{P}}(\mathfrak{p}B)}.$$

For a $\mathfrak{P} \supset \mathfrak{p}$ we get a finite extension of residue fields $(B/\mathfrak{P})/(A/\mathfrak{p})$ and we denote its degree by $f_{\mathfrak{P}} = [B/\mathfrak{P} : A/\mathfrak{p}]$. Moreover we know that for any integer m that the quotient $\mathfrak{P}^m/\mathfrak{P}^{m+1}$ is a $B/\mathfrak{P}$ vector space of dimension one and hence a $A/\mathfrak{p}$ vector space of dimension $f_{\mathfrak{P}}$. Hence we get that $B/\mathfrak{P}^{\nu_{\mathfrak{P}}}(\mathfrak{p}B)$ is a $A/\mathfrak{p}$ vector space of dimension $f_{\mathfrak{P}}\nu_{\mathfrak{P}}(\mathfrak{p}B)$ We call the numbers $\nu_{\mathfrak{P}}(\mathfrak{p}B) = e_{\mathfrak{P}}$ they are the so called ramification indices. Counting the dimensions yields the formula

$$[L : K] = \sum_{\mathfrak{P} \supset \mathfrak{p}B} f_{\mathfrak{P}} e_{\mathfrak{P}}$$

A finite extension L of $\mathbb{Q}$ is called an algebraic number field. The ring $\mathbb{Z}$ is a Dedekindring and hence its integral closure $\mathcal{O}_L$ in L is so too. This is the called the ring of integers in L. The study of these rings of integers is the subject of algebraic number theory.

Let me just state the theorem that $\mathrm{Pic}(\mathcal{O}_L)$ is a finite abelian group.

VII.2.4.**Finiteness properties of** Tor

The functor Tor^R. has a certain finiteness property which will become important later.

If we have an R-module M which is not flat, then this means that we can find another R-module N such that $Tor_1^R(N,M)\neq 0$.

Proposition 1: *If M,N arc R-modules such that $Tor_1^R(N,M)\neq 0$ then we can find a finitely generated submodule $N'\subset N$ such that $Tor_1^R(N',M)\neq 0$.*

Proof: We start from a projection resolution of N

$$0\to X\to P_0\to N\to 0$$

and we get from our assumption that

$$X\otimes_R M\to P_0\otimes_R M$$

is not injective.

Hence we find an element $y=\sum \times_\nu\otimes m_\nu\in X\otimes_R M$ which is not zero but which goes to zero in $P_0\otimes_R M$. Using the construction of the tensor product by generators and relations we see that we can find a finitely generated submodule $Y\subset P_0$ which contains the X_ν and such that y goes already to zero in $Y\otimes_R M$. But then we get a sequence

$$0\to Y\cap X\to Y\to N'\to 0$$

where $N'\subset N$ is finitely generated. Our element $y\in(Y\cap X)\otimes M$ is still non zero and maps to zero in $Y\otimes_R M$. Hence we see that

$$(Y\cap X)\otimes_R M\to Y\otimes_R M$$

is not injective and this implies $Tor_1^R(N',M)\neq 0$.

This has the consequence that for a non flat A-module M we can find an ideal $\mathfrak{a}\subset A$ such that $Tor_1^R(A/\mathfrak{a},M)\neq 0$.

If we have a flat A-module M which is not faithfully flat, then we get by similar arguments that we can find an ideal $\mathfrak{a}\subset A,\neq A$ such that $(A/\mathfrak{a}\otimes M)=0$.

Hence we can say that the modules of the form $A/\mathfrak{a}$ recognize flat resp. faithfully flat A-modules.

Proposition 2.4.1.:*Let A be a noetherian ring and assume that M is a finitely generated A-module. For $\mathfrak{p}\in\mathrm{Spec}(A)$ the $A_\mathfrak{p}$ module $M_\mathfrak{p}=M\otimes A_\mathfrak{p}$ is free if and only if it is flat. Then the set of points $\mathfrak{p}\in\mathrm{Spec}(A)$ where $M_\mathfrak{p}$ is a free $A_\mathfrak{p}$-module is an open subset $U\subset\mathrm{Spec}(A)$.*

Proof: We pick a point $\mathfrak{p}$ then $\mathcal{M}\otimes k(\mathfrak{p})$ is a vector space of finite dimension over $k(\mathfrak{p})$. If we lift the elements of a basis of this vector space to elements $m_1,\dots,m_s$ then these elements generate the $A_\mathfrak{p}$ module $M_\mathfrak{p}$ (Lemma of Nakayama). Hence we get an exact sequence

$$0 \to R \to A_{\mathfrak{p}}^s \to M \to 0.$$

It is clear that $M_{\mathfrak{p}}$ is free if and only if $R = 0$ and then this module is also flat.

Now we assume that $M_{\mathfrak{p}}$ is flat. Taking the tensor product with $k(\mathfrak{p})$ gives us an exact sequence

$$0 \to R \otimes k(\mathfrak{p}) \to A_{\mathfrak{p}}^s \otimes k(\mathfrak{p}) \to M \otimes k(\mathfrak{p}) \to 0$$

where now the second arrow in this sequence is an isomorphism by construction. Hence we that the first arrow is the zero map. Our assumption implies that $R \otimes k(\mathfrak{p}) = 0$. But since R is finitely generated (remember we assumed that A is noetherian) we get $R = 0$.

Now the rest follows because it is clear that if $M_{\mathfrak{p}}$ is free then we can find an open neighborhood $U = \mathrm{Spec}(A_f)$ of $\mathfrak{p}$ such that $M_f = M \otimes_A A_f$ is free.

VII.2.5. **Dominant morphisms:**

If we have an arbitrary morphism $\pi : X \to Y$ between two schemes $X/k, Y/k$ of finite type (???) then we would like to have some answers for the following questions:

(a) What is the structure of the image?

(b) What can we say about the dimension of the fibers?

2.5.1 To attack these questions we have the following reduction process.

(i)e cover X, Y by affine schemes U_i, V_i such that $\pi : U_i \to V_i$, then we see that we can reduce these questions to the case of affine morphisms.

(ii) If now $\phi : A \to B, \mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is a morphism of affine schemes then we see that we may assume that ϕ is injective, because it factors over $\mathrm{Spec}(A/\mathfrak{a})$, where $\mathfrak{a}$ is the kernel of ϕ.

(iii) Now we decompose both afine schemes into irreducible components and then we see that we may assume that our the algebras are integral.

Let us assume that we have two integral affine k-algebras and a morphism

$$\begin{array}{cccc} \varphi\colon & A & \longrightarrow & B \\ & \nwarrow & & \nearrow \\ & & k & \end{array}$$

Let us assume in addition that the morphism φ is injective, under these assumptions the morphism φ is called *dominant*. We get a morphism $X = \mathrm{Spec}(B) \to Y = \mathrm{Spec}(A)$. We can pass to the quotient fields K of A and L of B and we get an injection

$$\begin{array}{cccc} \varphi_1\colon & K & \longrightarrow & L \\ & \nwarrow & & \nearrow \\ & & k & \end{array}$$

and now we get a surjective $Spec(L) \to Spec(K)$ map and the extension L/K is faithfully flat.

The following proposition asserts that these two facts remain true after the restriction to suitable nonempty subsets $U \subset X, V \subset Y$, i.e. we get a faithfully flat morphism $U \to V$.

Proposition 2.5.1: *Let $\varphi: A \to B$ be a dominant morphism between two affine k-algebras. Then we can find an $f \neq 0$ in A and a $g \neq 0$ in B such that*

$$A_f \longrightarrow B_{fg}$$

is faithfully flat and then $Spec(B_{fg}) \to Spec(A_f)$ is surjective.

Proof: We consider the field of fractions K of A. Then $B \otimes_A K$ becomes a finitely generated K-algebra, it is still integral. We have $B \otimes_k K = K[x_1, \ldots, x_m]$ let $x_1, \ldots, x_r$ a maximal subset of elements such that $K[x_1, \ldots, x_r]$ is the polynomial ring in r variables. The remaining generators $x_{r+1}, \ldots, x_m$ satisfy algebraic equations

$$P_{\nu 0}(x_1, \ldots, x_r)x_\nu^{m_\nu} + P_{\nu 1}(x_1, \ldots, x_r)x_\nu^{m_\nu - 1} + \ldots P_{\nu m_\nu}(x_1, \ldots, x_r) = 0.$$

where $\nu = r+1, \ldots, m$ and the $P_{\nu 0}(x_1, \ldots, x_r) \in K[x_1, \ldots, x_r]$ are not zero. If $H(x_1 \ldots x_r)$ is the product of the $P_{\nu 0}$ then we can find $f \in A, f \neq 0$ such that $fH(x_1, \ldots, x_r) \in A[x_1 \ldots x_r]$ and hence we see that $B_{fH} = B \otimes A_f[x_1, \ldots, x_r, \frac{1}{fH}]$ is integral over $A_f[x_1, \ldots, x_r, \frac{1}{fH}]$. We look at the coefficients of fH, which are elements in A_f. At least one of these coefficients c is not zero. We replace f by fc then one of the coefficients of $fH = g$ is a unit.

Now we have

$$A_f \subset A_f[x_1, \ldots, x_r, \frac{1}{g}] \subset B_{fg}$$

and B_{fg} is finite over $A_f[x_1 \ldots x_r, \frac{1}{g}]$.

I claim that under these conditions the A_f-algebra $A_f[x_1, \ldots, x_r, \frac{1}{g}]$ is faithfully flat.

First of all we observe that

$$A_f[x_1, \ldots, x_r, \frac{1}{g}] = A_f[x_1, \ldots, x_r, y]/(yg - 1)$$

The ideal $(yg - 1)$ is a free $A_f[x_1, \ldots, x_r, y]-$ module and hence we see that the A_f-modules $A_f[x_1, \ldots, r, y]$ and $(yg - 1)$ are free. Hence we have the sequence of A_f-modules

$$0 \to (yg - 1) \to A_f[x_1, \ldots, x_r, y] \to A_f[x_1, \ldots, x_r, \frac{1}{g}] \to 0$$

To check flatness it suffices to show that this sequence stays exact if we tensorize by quotients $A/\mathfrak{a}$. We get

$$(yg - 1) \otimes A/\mathfrak{a} \to A/\mathfrak{a}[x_1, \ldots, x_r, y] \to A/\mathfrak{a}[x_1 \ldots x_r, \frac{1}{g}] \to 0$$

but now it is not difficult to see that the first arrow is still injective because $yg-1$ is not a zero divisor in $A/\mathfrak{a}[x_1,\ldots,x_r,y]$

To check that $A_f[x_1,\ldots,x_r,\frac{1}{g}]$ is faithfully flat, we need that $A/\mathfrak{a}[x_1,\ldots,x_r\frac{1}{g}]\neq 0$ for all ideals $\mathfrak{a}\in A,\mathfrak{a}\neq A$. This is obviously the case since we assumed that $g\not\equiv 0 \bmod \mathfrak{a}$ for all ideals of this kind.

Now we observe that the quotient field of B_{fg} is a finite extension of the quotient field of $A_f[x_1,\ldots,x_r,\frac{1}{g}]$. Hence we see that B_{fg} is a free in the generic point of $\mathrm{Spec}(A_f[x_1,\ldots,x_r,\frac{1}{g}])$ and hence it is is flat over a non empty open subset of $\mathrm{Spec}(A_f[x_1,\ldots,x_r,\frac{1}{g}])$ and then it is automatically locally free hence faithfully flat. This open set contains some $\mathrm{Spec}(A_f[x_1,\ldots,x_r,\frac{1}{g}])_{g_1}$ but then we see that this again contains a subset of the form $\mathrm{Spec}(A_{f'}[x_1,\ldots,x_r,\frac{1}{g},])$. Hence we proved the assertion concerning faithful flatness.

The surjectivity follows from the faithfulness but can also easily derive from the fact that B_{fg} is integral over $A_f[x_1,\ldots,x_r,\frac{1}{g}]$.

Of course our consideration imply:

Proposition 2.5.2: *If $\varphi\colon A\to B$ be a dominant morphism between two affine k-algebras, if r is the difference of the trancendence degrees of their fraction fields. Then we can find a non empty open subset $V=\mathrm{Spec}(A_f)\subset\mathrm{Spec}(A)$ such that for any point $y\in V$ the dimensions of the irreducible components of the fiber $f^{-1}(y)$ are greateror equal to r.*

Proof: This follows from the proof of the previous proposition. We can even say that the components of $f^{-1}(y)$ which itersect non trivial with $\mathrm{Spec}(B_{fg})$ have dimension $=r$.

2.5.3 : Now we can derive a general statement concerning the image of a morphism between arbitrary k schemes of finite type. Actually what we would like to have

Under some reasonable conditions the image of $f:X\to Y$ is open in the Zariski closure of the image.

Let us start from a dominant morphism $f:X\to Y$ between affine schemes. Then we have seen that the image contains a non empty open subset $U\subset Y$. Let us consider the complement Y' of U, this is a closed subscheme, we get a morphism $X'=f^{-1}(Y')\to Y'$. We consider two irreducible componets Y_1',X_1' of Y',X' respectively, such that $X_1'\to Y_1'$ and we take open affine subsets V_1',U_1' such that $U_1'\to V_1'$ and we get a homomorphism between the corresponding integral affine algebras $A'\to B'$. But now there is no reason why this morphism is still dominant, we get a diagram

$$\begin{array}{ccc} A'/\mathfrak{p} & \hookrightarrow & B' \\ & \nwarrow \quad \nearrow & \\ & A' & \end{array}$$

and this tells us that our morphism may factor through a proper closed irreducible subscheme V_1'' of V_1'. This morphism $U_1'\to V_1''$ is again dominant and therefore its image contains a non empty open subset $\overset{\circ}{V_1''}\subset V_1''$. Again we can take its complement and proceed.

This consideration tells us that the image under a morphism between schemes of finite type over a field k is always a so called *constructible* subset of Y. The family of constructible subsets is the smallest family of subsets which contains the open subsets and is closed under finite intersections and the formation of complements. So open and closed subsets are constructible but also open subsets of closed subsets and so on.

Exercise *Construct an example of the form*

$$k[U,V] \hookrightarrow k[U,V,Z]/(P(U,V,Z))$$

where the image is the affine plane minus (one of the axes minus the point $(0,0)$*)*

2.5.4 If we analyse our reasoning in 2.5.3. and the example in the exercise. Then we see that taking the restriction from $Y = \mathrm{Spec}(A)$ to the complement of the open set U we may loose the dominance. Keeping the notations from 2.5.3 this means that the tensor product with the quotient field of A'

$$A'_{(0)} \otimes_{A'} B'$$

may become zero. But now we look at the sequence

$$0 \to A' \to A'_{(0)} \to A'_{(0)}/A' \to 0$$

and take the tensor product with B'. We get

$$0 \to B' \to A'_{(0)} \otimes_{A'} B' \to A'_{(0)}/A' \to 0.$$

Now if B' is not zero (i.e. $X' \neq \emptyset$) then we see that B' is **not a flat** B' algebra if $A'_{(0)} \otimes_{A'} B' = 0$. This gives us the decisive hint.

Proposition 2.5.5 *If $f : X \to Y$ is a flat morphism between to k-Schemes of finite type, then the image $f(X)$ is open in Y.*

Proof: Again we easily reduce this to the case $X = \mathrm{Spec}(A), Y = \mathrm{Spec}(B)$ and $\phi : A \to B$ is flat. It is a formal consequence of the concept of flatness that for any ideal $\mathfrak{a} \subset A$ the algebra homomorphism $A/\mathfrak{a} \to B \otimes_A A/\mathfrak{a}$ is flat. It is also clear that we can restrict the morphism to the inverse image of a connected component of Y. A component is given by a minimal prime ideal $\mathfrak{p}$. We remarked that $A/\mathfrak{p} \to B \otimes_A A/\mathfrak{p}$ is flat. Then we may have that $B \otimes_A A/\mathfrak{p} = 0$, then the inverse image of $\mathrm{Spec}(A/\mathfrak{p})$ is empty and nothing has to be proved. Otherwise we observed that $(A/\mathfrak{p})_{(0)} \otimes_{(A/\mathfrak{p})} B \neq 0$ and hence we see that B contains a minimal prime ideal $\mathfrak{q}$ such that ${}^t\phi(\mathfrak{q}) = \mathfrak{p}$. I other words $A/\mathfrak{p} \to B/\mathfrak{q}$ is injective. We are in the situation of proposition 2.5.1. and can conclude that $\mathrm{Spec}(B) \to \mathrm{Spec}(A/\mathfrak{p})$ has an open image. Now we proceed as indicated, we pass to the complement of the open image and use the fact that the restriction of the morphism to this complement is still flat.

VII.3. Smooth points

Let k be a field and X/k be a scheme of finite type. This means that we can find a finite covering $X = \bigcup U_i$ by affine schemes whose rings of regular functions $\mathcal{O}_X(U_i)$ are finitely generated k-algebras. We consider the set of geometric points $X(\bar{k}) = \mathrm{Hom}_{\mathrm{Spec}(k)}(X, \mathrm{Spec}(\bar{k}))$ of X.

I want to introduce the notion of a *smooth* point $P \in X(\bar{k})$. To give a first idea of what it should be a give examples of *non smooth* or *singular* points.

Examples:

Consider the two algebras $A = k[x,y]/(x^2 - y^3)$ and $B = k[x,y]/(xy)$. The point $P = (0,0)$ is a geometric point on both of them and it is singular. If you now take $k = \mathbb{R}$ and draw a picture of the $\mathbb{R}$-valued points then you get an idea what a singular point is.

The property of a point to be smooth will be local, this means that by definition a point will be smooth if and only if for all affine open neighborhoods $U \subset X, P \in U(\bar{k})$ the point is smooth on U. Hence we may assume that $X = \mathrm{Spec}(A)$ we could also pass to the germ of X at P. The notion of a smooth point is also "geometric": If we embed $k \hookrightarrow \bar{k}$ and form the base extension $\bar{X} = X \times_{\mathrm{Spec}(k)} \bar{k} = \mathrm{Spec}(A \otimes_k \bar{k})$ then P is also a geometric point of $\bar{X}$. The definition of a smooth point will be such that P is smooth as a geometric point on X if and only if it is smooth as a geometric point of $\bar{X}$. It will even turn out that the smoothness of the point is a property of the formal completion of our scheme at P (See section on infinitesimal schemes in the section on affine schemes).

Our point P is now a homomorphism

$$\begin{array}{ccccc} P: & A \otimes_k \bar{k} & \rightarrow & \bar{k} & \\ & \nwarrow & & \nearrow & . \\ & & k & & \end{array}$$

It defines a maximal ideal $\mathfrak{m}_P \subset A \otimes_k \bar{k}$. We have a decomposition of the $\bar{k}$-vector space

$$A \otimes_k \bar{k} = \bar{k} \oplus \mathfrak{m}_P,$$

which is given by $f = f(P) + (f - f(P))$ for $f \in A \otimes_k \bar{k}$. Now we give the

VII.3.1. Definition: *The point $P \in \mathrm{Hom}_k(A, \bar{k})$ is smooth if and only if the local ring $(A \otimes \bar{k})_{\mathfrak{m}_P}$ is integral and if the dimension of the $\bar{k}$-vector space $\mathfrak{m}_P/\mathfrak{m}_P^2$ is equal to the dimension of the local ring $(A \otimes_k \bar{k})_{\mathfrak{m}_P}$.*

3.1.1. The integrality of the local ring implies that our point lies on exactly one irreducible component, i.e.the local ring is irreducible. Actually we will see that it would be sufficient to assume that the local ring is irreducible. Then we can speak of its dimension (See VII.???) and we will see that the required equality of dimensions will imply that the local ring is reduced. hence integral.

We also know that $\dim((A \otimes \bar{k})_{\mathfrak{m}_P}) = \dim((A \otimes \bar{k})/\mathfrak{p})$ (See 2.3.1.2.). The maximal ideal $\mathfrak{m}_P$ induces also a maximal ideal $\mathfrak{m}_P^0$ in A itself. I recall that we proved that the local ring $A_{\mathfrak{m}_P^0}$ is also integral, we have an injection $i : A_{\mathfrak{m}_P^0} \to (A \otimes_k \bar{k})_{\mathfrak{m}_P}$ and equality of dimensions $\dim(A_{\mathfrak{m}_P^0}) = \dim((A \otimes_k \bar{k})_{\mathfrak{m}_P}))$. If $\mathfrak{p}^0$ is the unique prime ideal in $A_{\mathfrak{m}_P^0}$ which is contained in $\mathfrak{m}_P^0$ then these dimensions are also equal to $dim(A/\mathfrak{p}^0)$. (See VII.2.3.1. and 2.3.1.2.) It is quite clear that we can say that the point $\mathfrak{m}_P^0 \in \mathrm{Specmax}(A)$ is smooth if one or all the geometric points lying above this point are smooth. (See V 2.5.1)

Now a want to discuss the implications of the smoothness of a point. For the following discussion we assume that $k = \bar{k}$ and hence $A \otimes_k \bar{k} = A$.

I want to explain the meaning of the equality

$$\dim_k \mathfrak{m}_P/\mathfrak{m}_P^2 = \dim(A_{\mathfrak{m}_P}) = d.$$

We pick elements $t_1 \dots t_d \in \mathfrak{m}_P$ which provide a basis $\bar{t}_1 \dots \bar{t}_d$ in the quotient $\mathfrak{m}_P/\mathfrak{m}_P^2$. These elements generate the A-module $\mathfrak{m}_P$ by Nakayamas lemma. We consider the polynomial ring with the same number of generators $B = k[X_1, \dots, X_d]$ and we consider its maximal ideal $\mathfrak{m}_0 = (X_1, \dots, X_d)$. We get a k-algebra homomorphism $B_{\mathfrak{m}_0} \to A_{\mathfrak{m}_P}$ sending X_i to t_i. We compute modulo powers of the maximal ideals, i.e. we look at the rings

$$B_{\mathfrak{m}_0}/\mathfrak{m}_0^N, A_{\mathfrak{m}_P}/\mathfrak{m}_P^N.$$

Clearly our homomorphism induces maps $B_{\mathfrak{m}_0}/\mathfrak{m}_0^N \to A_{\mathfrak{m}_P}/\mathfrak{m}_P^N$. It is easy to see that these maps are surjective: The t_i generate the $A_{\mathfrak{m}_P}$−module $\mathfrak{m}_P$ hence any $f \in A_{\mathfrak{m}_P}$ can be written as $f = a + \sum g_i t_i$ with $a \in k$. Apply the same to the g_i we get $f = a + \sum b_i t_i + \textit{quadratic terms}$ with $b_i \in k$. Since this goes on forever and the assertion becomes clear.

We have projective systems

$$\begin{array}{ccccc} B_0/\mathfrak{m}_0^{N+1} & \longrightarrow & B_0/\mathfrak{m}_0^N & \longrightarrow & \\ \downarrow & & \downarrow & & \downarrow \\ A_{\mathfrak{m}_P}/\mathfrak{m}_P^{N+1} & \longrightarrow & A_{\mathfrak{m}_P}/\mathfrak{m}_P^N & \longrightarrow & \end{array}$$

and if we consider the completions $\hat{B} = \underleftarrow{\lim} B_0/\mathfrak{m}_0^N$ and $\hat{A}_{\mathfrak{m}_P} = \underleftarrow{\lim} A_{\mathfrak{m}_P}/\mathfrak{m}_P^N$. The ring $\hat{B}_{\mathfrak{m}_0}$ is isomorphic to the ring of formal power series $k[[X_1, \dots, X_d]]$. (Ref. to Affine schemes)

We get a diagram

$$\begin{array}{ccc} B_{\mathfrak{m}_0} & \to & A_{\mathfrak{m}_P} \\ \downarrow & & \downarrow \\ i_P : k[[X_1, \dots, X_d]] & \to & \hat{A}_{\mathfrak{m}_P} \end{array}$$

Theorem: VII.3.2 *Let A/k be an affine algebra over the algebraically closed field k, let P be a geometric point such that $A_{\mathfrak{m}_P}$ is irreducible, i.e. P lies on exactly one irreducible component. We always have*

$$\dim_k \mathfrak{m}_P/\mathfrak{m}_P^2 \geq \dim(A_{\mathfrak{m}_P}) = d.$$

The vertical maps in the diagram above are always injective. We have equality in our inequality if and only if the horizontal arrow in the bottom line i_P is an isomorphism. This is the case if and only if P is smooth.

This would be obvious if we had proved that the ring of formal power series is noetherian and if we had the general theory of dimensions of noetherian rings. But since we do not have this we give a different argument.

Proof: The injectivity of the vertical maps follows from the Artin-Rees theorem (without any assumption on P). It is also clear that the bottom line arrow is surjective since we have the surjective maps $B_0/\mathfrak{m}^N \to A_{\mathfrak{m}_P}/\mathfrak{m}_P^N$.

Now we prove the inequality. We proceed by induction on $d' = \dim(A_{\mathfrak{m}_P})$. Here we use our assumption that P lies on exactly one irreducible component. Let $\mathfrak{a}$ be the kernel of i_P. For $d' = 0$ the maximal ideal $\mathfrak{m}_P$ is nilpotent and we have $d \geq d' = 0$. If $d' \geq 1$ then $\mathfrak{m}_P \neq \mathrm{Rad}(A_{\mathfrak{m}_P})$ we pick one of the generators say t_1 which is not in the radical. We choose a minimal prime ideal $\mathfrak{m}_P \supset \mathfrak{p} \supset (t_1)$. The Hauptidealsatz tells us that $\dim((A/\mathfrak{p})_{\mathfrak{m}_P}) = d' - 1$. Now the number of generators of the image of $\mathfrak{m}_P$ in the local ring $(A/\mathfrak{p})_{\mathfrak{m}_P}$ has dropped by a number $r \geq 1$. Hence we get $d - r \geq d' - 1$ and this proves that always $d \geq d'$.

If our point P is smooth then it is also a smooth point on the subscheme $A/\mathfrak{p}$ because in this case we have $d - r \geq d' - 1$ and since $d = d'$ we must have $r = 1$. We apply our construction above to $(A/\mathfrak{p})_{\mathfrak{m}_P}$ and get a homomorphism

$$k[[X_2, \ldots, X_d]] \to (A/\hat{\mathfrak{p}})_{\mathfrak{m}_P}.$$

This homomorphism is a bijection by the induction hypothesis. The kernel of

$$k[[X_1, X_2, \ldots, X_d]] \to k[[X_2, \ldots, X_d]]$$

is the principal ideal (X_1) and hence $\mathfrak{a} \subset (X_1)$. Assume $\mathfrak{a} \neq (0)$. If $d = 1$ we get that $\mathfrak{a} = (X_1^m)$ for some m, but this is impossible because this implies $t_1^m = 0$ and the dimension of $A_{\mathfrak{m}_P}$ would be zero. If $d > 1$ then we could replace t_1 by any linear combination of the generators t_i and conclude that $\mathfrak{a}$ is contained in any principal ideal (f) where $f = a_1X_1 + a_2X_2 + \ldots a_dX_d +$ *higher order terms* with non zero linear term. But this tells us that any element $h \in \mathfrak{a}$ must be divisible by any such power series f which is clearly impossible unless $h = 0$. Once we proved that i_P is an isomorphism we conclude that the local ring $(A/\mathfrak{p})_{\mathfrak{m}_P}$ must be integral and we proved that P is smooth.

If in turn i_P is an isomorphism then we observe that X_1 is sent to $t_1 \in \mathfrak{m}_P$. Hence we see that (t_1) is a prime ideal and

$$k[[X_2, \ldots, X_d]] \to (A/(\hat{t}_1)_{\mathfrak{m}_P}$$

is an isomorphism. By induction this implies $d' - 1 = d - 1$ hence $d' = d$ and our point must be smooth.

The theorem means that in an "infinitesimal neighborhood" of a smooth point a scheme X/k over an algebraically closed field k looks like the "infinitesimal neighborhood" of the origin of the the an affine space of dimension $\dim(X)$. (V.2.8.)

Now we assume that our k-algebra is given as a quotient of a polynomial ring, in other words we consider $X = \mathrm{Spec}(A)$ as embedded into an affine space. We write

$$0 \to I \to k[X_1, \dots, X_n] \to A \to 0$$

where I is the defining ideal. Again we look at our smooth point $P : A \to k$ and for simplicity we assume that $P(X_i) = 0$ for all i, i.e. our point P is the origin. We want to show that the smoothness of P has the consequence that the system of equations which defines X locally at P as a subscheme has certain nice properties (Jacobi criterion).

Let $\mathfrak{m}_0$ be the maximal ideal in A defined by P and $\mathfrak{M}_0 = (X_1, \dots, X_n)$ the maximal ideal in the polynomial ring. We get an exact sequence (localization is flat)

$$0 \to I_{\mathfrak{M}_0} \to k[X_1, \dots, X_n]_{\mathfrak{M}_0} \to A_{\mathfrak{m}_0} \to 0.$$

Let $d = \dim(A_{\mathfrak{m}_0})$ then we may assume that $X_1, \dots X_d$ map to a basis in $\mathfrak{m}_0/\mathfrak{m}_0^2$. We write x_i for the image of X_i in $A_{\mathfrak{m}_0}$. We get a diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & I_{\mathfrak{M}_0} & \longrightarrow & k[X_1, \dots, X_n]_{\mathfrak{M}_0} & \longrightarrow & A_{\mathfrak{m}_0} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \hat{I}_{\mathfrak{M}_0} & \longrightarrow & k[[X_1, \dots, X_n]] & \longrightarrow & k[[x_1, \dots, x_d]] & \longrightarrow & 0 \end{array}$$

where $\hat{I}_{\mathfrak{M}_0}$ is the completion of $I_{\mathfrak{M}_0}$ and the exactness on the left in the second sequence is again the Artin-Rees theorem. We also use our previous theorem which gives us that $\hat{A}_{\mathfrak{m}_P}$ is indeed the ring of power series in the given d variables.

Now we can find certain specific elements in $I_{\mathfrak{M}_0}$ which turn out to be generators of the ideal. We observe that for any $i = d+1, \dots, n$ the element $x_i \in \mathfrak{m}_0$ must be up terms of higher order (i.e. elements in $\mathfrak{m}_0^2$) a linear combination of the x_ν with $\nu = 1, \dots, d$. In other words for any such i we get a relation in $A_{\mathfrak{m}_0}$:

$$x_i = L_i(x_1, \dots x_d) + G_i(x_1, \dots, x_n)$$

where L_i is linear in the $x_1, \dots, x_d$ and where G_i is an expression which contains only terms of degree ≥ 2. If we lift this to the variable $X_1, \dots, X_n$ this provides elements in $I_{\mathfrak{M}_0}$

$$F_i = X_i - L_1(X_1, \dots, X_d) - G_i(X_1, \dots, X_n) \in I_{\mathfrak{M}_0} \quad i = d+1, \dots, n.$$

I claim:

The elements $F_{d+1}, \dots, F_n$ generate the ideal $I_{\mathfrak{M}_0}$.

This is rather clear: If we look at the completion then we see that for any $i = d+1, \dots, n$ we can express the image of x_i as power series in the variables $x_1, \dots, x_d$. (Just substitute the expression for x_i into the G_i and you get an expression for the x_i in terms of a quadratic term in the $x_1, x_2, \dots, x_d$ and an expression in all x_i where the terms are of degree ≥ 3. and so on) Hence we can write the x_i for $i = d+1, \dots, n$ as power series in the $x_1, \dots, x_d$. Hence we get that the F_i viewed as elements in the ideal $\hat{I}_{\mathfrak{M}_0}$ can be written as

$$F_i(X_1, \ldots, X_n) = X_i - P_i(X_1, X_2, \ldots, X_d) \quad i = d+1, \ldots, n,$$

where the $P_i(X_1, X_2, \ldots, X_d)$ are power series. If we divide the power series ring $k[[X_1, \ldots, X_n]]$ by these F_i then the resulting ring is the power series ring in the variables $X_1, \ldots, X_d$ and this is also the result if we divide by $\hat{I}_{\mathfrak{M}_0}$. This makes it clear that these F_i generate the ideal $\hat{I}_{\mathfrak{M}_0}$. Now $I_{\mathfrak{M}_0}/\mathfrak{M}_0 I_{\mathfrak{M}_0} = \hat{I}_{\mathfrak{M}_0}/\mathfrak{M}_0 \hat{I}_{\mathfrak{M}_0}$ and the Nakayama lemma implies that the ideal $I_{\mathfrak{M}_0}$ is also generated by these F_i

This has several consequences. It is clear that we can find an affine open neighborhood $U \subset \mathbb{A}^n_k$ of our point P such that the restriction $\tilde{I}(U)$ of I to U contains the F_i and is generated by them. Hence

(1) In a suitable Zariski neighborhood of a smooth point P can define a scheme $X/k \hookrightarrow \mathbb{A}^n_k$ by $n - \dim(X)$ equations, namely the above F_i.

(2) If we form the Jacobi matrix of the relations F_i at our smooth point P then the matrix

$$(\frac{\partial F_i}{\partial X_j}(P))_{i=d+1,\ldots,n; j=1,\ldots n}$$

has rank $n - d$.

This leads us to the famous Jacobi criterion which gives a characterization of the smooth points of an embedded affine scheme $X \subset \mathbb{A}^n$.

VII.3.3. The Jacobi criterion :From now on we drop the assumption that k should be algebraically closed. We consider an affine scheme $A = k[X_1, \ldots, X_n]/I$. A geometric point $P \in \mathrm{Hom}_k(A, \bar{k})$ defines a maximal ideal $\mathfrak{m}_P$ in A but it can happen that different P give the same maximal ideal. We have also the maximal ideal $\mathfrak{M}_P$ in the polynomial ring which lies above $\mathfrak{m}_P$. We want to formulate a criterion for the smoothness of P in terms of the data over k , i.e. we do not extend the base to $\bar{k}$. This will imply for instance that all geometric point lying over a given maximal ideal $\mathfrak{m}$ will be smooth if one of them is so.

Jacobi criterion:

Let I be an ideal in $k[X_1, X_2, \ldots, X_n]$ and $A = k[X_1, X_2, \ldots, X_n]/I$. Let P be a geometric point of $\mathrm{Spec}(A)$, *let $\mathfrak{m}_P$ (resp $\mathfrak{M}_P$) be the corresponding maximal ideal in A (resp. $k[X_1, X_2, \ldots, X_n]$). Let $F_1, \ldots, F_r$ be a system of generators of $I_{\mathfrak{M}_P}$. We assume that the Jacobi matrix*

$$(\frac{\partial F_i}{\partial X_j}(P))_{i=1,\ldots,r; j=1,\ldots n}$$

has rank r. Then P is smooth and $\dim(A_{\mathfrak{m}_P}) = n - r$.

If conversely P is smooth and $\dim(A_{\mathfrak{m}_P}) = n - r$ *then we can find generators $G_1, \ldots, G_r$ of the ideal $I_{\mathfrak{M}_P}$ such that the Jacobi matrix built in these generators has rank r.*

Before I give an outline of the proof, I want to make a few comments. First of all we remark that the evaluation of an element $F \in k[X_1, \ldots, X_n]$ at P has the result is in $\bar{k}$.

Hence we did not keep our promise not to extend the ground field. But our statement concerning the rank is equivalent to the assertion that a determinant of a suitable $r \times r$-matrix is non zero. This matrix has entries $\frac{\partial F_i}{\partial X_j} \in k[X_1, \ldots, X_n]$ and hence the determinant is an element $F \in k[X_1, \ldots, X_n]$. Now $F(P) \neq 0$ is equivalent to $F \notin \mathfrak{M}_P$ we restated our condition on the Jacobi matrix without extending the ground field. Then it is also clear that all the geometric points lying over a given maximal ideal are all smooth or none of them is.

Finally I want to say that this criterion tells us that in a Zariski neighborhood of a smooth maximal ideal we can describe a subscheme $X/k \subset \mathbb{A}^n_k$ always by $n - \dim(A_{\mathfrak{m}_P})$ equations which satisfy the Jacobi criterion and vice versa a set of equations satisfying the Jacobi criterion in an open set $U \subset X$ tells us that all the geometric points in U are smooth.

Proof: Since we have a geometric definition of the smoothness of a point we have to extend the ground field to $\bar{k}$. We consider the ideal $I \otimes_k \bar{k} \subset \bar{k}[X_1, X_2, \ldots, X_n]$. After making a substitution we may assume that $P = (0, \ldots, 0)$. Hence it is reasonable to denote the maximal ideals in $A \otimes_k \bar{k}$ resp. the polynomial ring by $\mathfrak{m}_0$ resp. $\mathfrak{M}_0$. We put $d = n - r$. If we renumber our variables we can assume that the partial Jacobi matrix

$$(\frac{\partial F_i}{\partial X_j}(P))_{i=1,\ldots,r;j=d+1,\ldots n}$$

has rank r. If we expand the polynomials at $P = (0, \ldots, 0)$ we get expressions

$$F_i(X_1, \ldots, X_n) = \sum_{j=1}^{n} \frac{\partial F_i}{\partial X_j}(P) X_j + \textit{higher order terms}$$

for $i = 1, \ldots, r$. Let us denote the homomorphism of the polynomial ring to our algebra A by Φ. Then we get in $(A \otimes_k \bar{k})_{\mathfrak{m}_P}$ the relations

$$0 = \sum_{j=1}^{n} \Phi(\frac{\partial F_i}{\partial X_j}) x_j + \textit{higher order terms in the } x_i$$

If we ignore higher terms and evaluate at P then we get a system of linear for the $x_i \in \mathfrak{m}_0/\mathfrak{m}_0^2$ and our assumption on the partial Jacobi matrix implies that we can express the images x_i for $i = d+1, \ldots, n$ as linear combinations of the x_i with $i = 1, \ldots, d$. We get that the x_i with $i = 1, \ldots, d$ generate $\mathfrak{m}_0/\mathfrak{m}_0^2$.

To prove the smoothness of P it suffices to show that $(A \otimes_k \bar{k})_{\mathfrak{m}_0}$ is integral, it has dimension d and that x_i with $i = 1, \ldots, d$ form a basis $\mathfrak{m}_0/\mathfrak{m}_0^2$.

To prove these facts we embed $(A \otimes_k \bar{k})_{\mathfrak{m}_0}$ into its $\mathfrak{m}_0$- adic completion $(A \hat{\otimes}_k \bar{k})_{\mathfrak{m}_0}$. In this completion our defining equations just say that the x_i with $i = d+1, \ldots, n$ can be expressed as power series in the x_i with $i = 1, \ldots, d$ and hence it is clear that $(A \hat{\otimes}_k \bar{k})_{\mathfrak{m}_0}$ is the power series ring in the variables x_i with $i = 1, \ldots, d$. This implies integrality of $(A \otimes_k \bar{k})_{\mathfrak{m}_0}$ and it shows that $\dim(\mathfrak{m}_0/\mathfrak{m}_0^2) = d$.

As in our previous considerations we can can construct an injective homomorphism from the localized polynomial ring

$$\bar{k}[X_1, \ldots, X_d]_{\mathfrak{M}_0^0} \to (A \otimes_k \bar{k})_{\mathfrak{m}_0}$$

Then it is clear (see 2.5.3.) that $\dim((A \otimes_k \bar{k})_{\mathfrak{m}_0}) \geq d$.Now it follows from our theorem above that $\dim(A_{\mathfrak{m}_0}) = d$. This proves the first half of the theorem.

To prove the converse we observe that we have seen that the ideal $I_{\mathfrak{M}_P} \otimes \bar{k}$ can be generated by $G_1, \ldots, G_r \in \bar{k}[X_1, \ldots, X_n]$ such that these have non vanishing Jacobi matrix at P. Let $F_1, \ldots, F_t \in I$ be a system of generators. Then we can find an $H \in \bar{k}[X_1, \ldots, X_n]$ with $H(P) \neq 0$ such that we can write

$$G_i H = \sum_j L_{ij} F_j$$

where $L_{ij} \in \bar{k}[X_1, \ldots, X_n]$. Taking partial derivatives and evaluating at P yields a system of equations

$$H(P) \frac{\partial G_i}{\partial X_\nu}(P) = \sum_j L_{ij}(P) \frac{\partial F_j}{\partial X_\nu}(P)$$

This makes it clear that the Jacobi matrix built out of the F_j evaluated at P has rank r. But this means that the Jacobi matrix $(\frac{\partial F_j}{\partial X_\nu})$ has rank r if we send it to the residue field $A/\mathfrak{m}_P$. Taking a submatrix having the right number of columns gives us a subset of the set of generators say $F_1, \ldots, F_r$ which satisfies the Jacobi criterion. This subset will necessarily generate $I_{\mathfrak{M}_P}$.

3.3.1. The Jacobi criterion implies that the set of smooth geometric points of a scheme X/k of finite type is always open. To see this we consider the case of an affine k-algebra $A = k[X_1, \ldots, X_n]/I$. We may assume that it is irreducible because smooth point lie on exactly one irreducible component. We pick generators $F_1, F_2, \ldots, F_t$ of our ideal and consider the open set where the Jacobi matrix has rank equal to $n - \dim(A)$. This set can be empty. If for instance $A = k[X]/(X^2)$ then the only geometric point is $x = 0$ the dimension is zero but the number $d = 1$.

VII.3.4. Theorem: Generic smoothness

Let A/k be a finitely generated k-algebra, assume that A/k is absolutely reduced. Then we can find a non empty open subset $U \subset \operatorname{Spec}(A)$ such that the morphism $\pi : U \to \operatorname{Spec}(k)$ is smooth.

Proof: Since localization does not destroy reducedness we may assume that A is integral. We also may assume that $\bar{k} = k$. We write $A = k[x_1, x_2, \ldots, x_n] = k[X_1, X_2, \ldots X_n]/I$ and let $F_1, F_2, \ldots F_t$ be a set of generators of the ideal I. Let $\frac{\partial F_i}{\partial x_j}$ be the image of $\frac{\partial F_i}{\partial X_j} \in k[X_1, X_2, \ldots, X_n]$ modulo the ideal I. Let r be the rank of the Jacobi matrix $(\frac{\partial F_i}{\partial x_j})$ considered as a matrix in the field of fractions of A and assume that

$$(\frac{\partial F_i}{\partial x_j})_{i=1,\ldots,r;j=n-r+1,\ldots n}$$

has non zero determinant. Then this determinant is a unit in a suitable localization $\mathcal{O}_X(U) = \tilde{A}(U)$. Let $d = n - r$. I claim that the subalgebra $k[x_1, \ldots, x_d]$ is a polynomial ring in d variables. If not the the intersection $k[X_1, \ldots, X_d] \cap I \neq 0$. For all F in $k[X_1, \ldots, X_d] \cap I$ we have $\frac{\partial F}{\partial x_j} = 0$ for all $j = 1, \ldots, d$ because otherwise the rank of the Jacobian would be greater than r. This is equivalent to the assertion that for all $F \in k[X_1, \ldots, X_d] \cap I$ we have $\frac{\partial F}{\partial X_j} \in I$ for all $j = 1, \ldots, d$. Let us consider a non zero element $F \in k[X_1, \ldots, X_d] \cap I$ with lowest total degree, i.e. the sum of the exponents in the highest monomial occuring in F is minimal. Then $\frac{\partial F}{\partial X_j} \in I$ for all $j = 1, \ldots d$ and these polynomials have a strictly smaller total degree. Hence they are zero and F must be constant or of the form

$$F(X_1, \ldots, X_d) = \sum a_\nu X^{p\nu}$$

where p is the characteristic of k and $\nu = (\nu_1, \ldots, \nu_d)$ is a multiindex. Of course F cannot be constant. But then we find nilpotent elements in $\bar{k}[x_1, \ldots, x_d]$ because we can write

$$F(X_1, \ldots, X_d) = \sum a_\nu X^{p\nu} = (\sum a_\nu^{1/p} X^\nu)^p.$$

The element $(\sum a_\nu^{1/p} X^\nu)^p \notin k[X_1, \ldots, X_d] \cap I$ because it has smaller total degree than F. Its image in A is a non zero nilpotent element. This gives a contradiction to our assumption that A is absolutely reduced.

But once we know that $k[x_1, \ldots, x_d]$ is a polynomial ring we see that the dimension of A is greater or equal to d (2.5.3.) and the Jacobi criterion yields that the open subscheme where the above determinant is not zero must be smooth.

I want to discuss a consequence of the smoothness of a geometric point for the original algebra A and the induced maximal ideal $\mathfrak{m}_P$. Let us assume we have an ideal $I \in k[X_1, \ldots, X_n]$ which is generated by $F_1, \ldots, F_r$. Let X/k be the subscheme defined by the ideal, i.e. $X = \mathrm{Spec}(k[X_1, \ldots, X_n]/I) = \mathrm{Spec}(A)$. We assume that these generators satisfy the Jacobi criterion at a point $P \in X(\bar{k})$, hence we know that this point is smooth. Let $\mathfrak{m}_P$ be the maximal ideal induced by P in our algebra A, we know that $A_{\mathfrak{m}_P}$ is integral. Again we put $d = n - r$. Now we assume for the coordinates $X_{d+1}, \ldots, X_n$ that

$$\det((\frac{\partial F_j}{\partial X_\mu}(P))_{j=1\ldots,r,\mu=d+1,\ldots,n}) \neq 0$$

Let us denote by x_ν the image of the X_ν in A. The we consider the subalgebra $B \subset A$ generated by the elements $x_1, \ldots, x_d$. It will become clear in a minute that this algebra is indeed the polynomial algebra in these variables. We have the diagram

$$\begin{array}{ccc} k[x_1,\dots,x_d] & \to & A \\ \downarrow & & \downarrow \\ \bar{k}[x_1,\dots,x_d] & \to & A\otimes_k \bar{k} \end{array}$$

The maximal ideal $\mathfrak{m}_P$ induces a maximal ideal (geometric point) $\mathfrak{n}_P$ in $k[x_1,\dots,x_d]$. Let $\bar{\mathfrak{n}_P}$ (resp. $\bar{\mathfrak{m}_P}$ be the maximal ideals induced by P in $\bar{k}[x_1,\dots,x_d]$ (resp. $A\otimes_k \bar{k}$) We have seen in the beginning of this section that after passing to the completions we get an isomorphism

$$\bar{k}[x_1, \hat{\dots}, x_d]_{\bar{\mathfrak{n}_P}} \tilde{\to} (A\otimes_k \bar{k})\hat{}_{\bar{\mathfrak{m}_P}}.$$

and it follows from this that the subalgebra $B\subset A$ is the polynomial algebra.

We have the inclusion of local rings

$$B_{\mathfrak{n}_P} \hookrightarrow A_{\mathfrak{m}_P}$$

We consider the residue field $B/\mathfrak{n}_P = k(\mathfrak{n}_P)$.

I claim:

(i) The tensor product

$$A_{\mathfrak{m}_P}\otimes_B k(\mathfrak{n}_P) = A_{\mathfrak{m}_P}/A_{\mathfrak{m}_P}\cdot \mathfrak{n}_P$$

is a field, it is a finite separable extension of $k(\mathfrak{n}_P)$

(ii) Moreover we have $A_{\mathfrak{m}_P}\cdot\mathfrak{n}_P = \mathfrak{m}_P$ *and the residue field* $k(\mathfrak{m}_P) = A_{\mathfrak{m}_P}\otimes_B k(\mathfrak{n}_P)$

Proof: The second assertion follows from the first. To prove the first assertion we put $K = k(\mathfrak{n}_P)$ and $\bar{A} = A_{\mathfrak{m}_P}\otimes_B k(\mathfrak{n}_P)$. We have that $\bar{A} = K[\bar{X}_{d+1},\dots,\bar{X}_n]/(\bar{F}_1,\dots,\bar{F}_r)$ where the bar means taking the residue classes. Our point P induces a maximal ideal $\bar{\mathfrak{m}_P}\subset\bar{A}$. Of course we have that the generators $\bar{F}_i, i=1,\dots,r$ satisfy the Jacobi criterion at P and this means

$$\det\big(\big(\frac{\partial\bar{F}_j}{\partial\bar{X}_\mu}(P)_{j=1\dots,r,\mu=d+1,\dots,n}\big)\neq 0.$$

Then it follows that all points P_i which lie above P are smooth and the scheme $\bar{A}\otimes_K\bar{K}$ is of dimension zero and reduced. But this means that $\bar{A}$ is a separable extension of K.

In our situation above we will call the elements $x_1,\dots,x_d$ a system of *local parameters* at the point P. We can interprete the inclusion $B\subset A$ as the morphism from X/k to $\mathbb{A}^r/k$ which is induced by the projection of the ambient affine space $\mathbb{A}^n/k$ to the first r coordinates. If we extend the ground field to $\bar{k}$ then this projection becomes an isomorphism between the infinitesimal neighborhoods of P in $X\times\bar{k}$ and its image in $\mathbb{A}^r/\bar{k}$. This is the algebraic version of the theorem of implicit functions. If we want to formulate what happens over k itself then this becomes a little bit less intuitive, we have to be aware that we may have non trivial residue fields which even may be non separable.

Our notion of a system of local parameters is given in terms of an embedding of our scheme X/k into an affine space and with the help of the Jacobi criterion. In the next section we will give a much more elegant formulation using differentials.

VII.3.4. Relative differentials: For any morphism

$$\pi : X \to Y$$

we will define a quasicoherent sheaf $\Omega^1_{X/Y}$ on X which called the sheaf of relative differentials. Since it is a quasicoherent sheaf it is enough to define it in the case where $X = \mathrm{Spec}(B), Y = \mathrm{Spec}(A)$. Let $\phi : A \to B$ be the homomorphism of rings corresponding to π.

We consider the fibered product of X/Y by itself

$$\begin{array}{ccc} B \otimes_A B & & X \times_Y X \\ \uparrow & & \downarrow \\ A & & Y \end{array}$$

and we have the diagonal $X \to X \times_Y X$ defined by $(\mathrm{Id}, \mathrm{Id})$. The diagonal corresponds to the multiplication

$$B \otimes_A B \xrightarrow{m} B$$

and is defined by the ideal I which is the kernel von m, i.e. we have an exact sequence

$$0 \to I \to B \otimes_A B \to B \to 0.$$

Lemma 3.4.1. : *The ideal I is generated as a $B \otimes_A B$-module by the elements $f \otimes 1 - 1 \otimes f$. The ideal I/I^2 is a B-module which is generated by the elements $dg = f \otimes 1 - 1 \otimes f \bmod I^2$*

Proof: If $\Sigma f_\nu \otimes g_\nu \in I$, i.e. $\Sigma f_\nu g_\nu = 0$ then

$$\Sigma f_\nu \otimes g_\nu = \Sigma (f_\nu \otimes 1)(1 \otimes g_\nu) = \Sigma (f_\nu \otimes 1 - 1 \otimes f_\nu)(1 \otimes g_\nu).$$

The B-module structure on I/I^2 is induced by the $B \otimes_A B$ module structure of I.

The B-module I/I^2 is denoted by $\Omega^1_{B/A}$ and is called the module of relative differentials. We have the product rule

$$dfg = gdf + fdg$$

and

$$da = 0 \quad \text{for} \quad a \in A$$

which follows from

$$fg \otimes 1 - 1 \otimes gf = fg \otimes 1 - f \otimes g + f \otimes g - 1 \otimes fg = (f \otimes 1)(g \otimes 1 - 1 \otimes g) + (g \otimes g)(f \otimes 1 - 1 \otimes f)$$

and

$$a \otimes 1 = 1 \otimes a.$$

We collect some facts where proofs can be found in the book by Matsumura but since they are all not difficult to prove, the reader could try to find them himself.

a) The B-module $\Omega^1_{B/A}$ is a universal module of A-differentials: Whenever we have a B-module M and an A-linear map $d_1 : B \to M$ such that $d_1(fg) = fd_1g + gd_1f$ then we get a commutative diagram

$$\begin{tikzcd} \Omega^1_{B/A} \arrow[rr, "\varphi"] & & M \\ & B \arrow[ul, "d"] \arrow[ur, "d_1"'] & \end{tikzcd}$$

where φ is unique and B-linear. It is clear that $\phi(f \otimes 1 - 1 \otimes f) = d_1(f)$ but it is not so clear why it is well defined. This can be checked by using d) below and reducing to the case of a polynomial ring.

b) We start from our algebra homomorphism $A \to B$ and we pick an element $f \in B$. Then we have $A \to B \to B_f$ and hence we know what $\Omega^1_{B_f/A}$ is. It is easy to see that we have a canonical isomorphism $\Omega^1_{B_f/A} \xrightarrow{\sim} (\Omega^1_{B/A})_f$ which in principle comes from the rule

$$d\frac{1}{f} = -\frac{df}{f^2}$$

Therefore we can view $\Omega^1_{B/A}$ as a quasicoherent sheaf on $\mathrm{Spec}(B)$ and then we can define $\Omega^1_{X/Y}$ for an arbitrary morphism of schemes $\pi : X \to Y$.

c) The module of relative differentials is compatible with base change. If we have a diagram

$$\begin{tikzcd} B \arrow[r] & B' = B \otimes_A A \\ A \arrow[u] \arrow[r] & A' \arrow[u] \end{tikzcd}$$

then we have a canonical isomorphism

$$\Omega^1_{B/A} \otimes_B B' \xrightarrow{\sim} \Omega^1_{B'/A}.$$

d) Finally we consider the situation that our algebra B is a quotient of an A-algebra C by an ideal J, i.e. we have

$$\begin{tikzcd} 0 \arrow[r] & J \arrow[r] & C \arrow[r, "\pi"] & B \arrow[r] & 0 \\ & & A \arrow[u] \arrow[ur] & & \end{tikzcd}.$$

It is clear that we have a surjective C-module homomorphism

$$\Omega^1_{C/A} \to \Omega^1_{B/A}$$

which sends $gdh \in \Omega^1_{C/A}$ to $\pi(g)d\pi(h)$. On the other hand we can map J to $\Omega^1_{C/A}$ by sending an $f \in Y$ to df. This yields a sequence of B-modules

$$J/J^2 \to \Omega^1_{C/A} \otimes_C B \to \Omega_{B/A} \to 0$$

and I claim that this sequence is exact. We have to show that the elements in $\Omega^1_{C/A} \otimes_C B$ which go to zero come from J/J^2. To see this we start with the observation that

$$\Omega^1_{C/A} \otimes_C B = \Omega^1_{C/A}/J\Omega^1_{C/A}.$$

If we look at

$$C \otimes_A C \to B \otimes_A B$$

then we have seen that the kernel is the ideal $J \otimes_A C + C \otimes_A J$ (See V.2.3.1.??). On the other hand we have the kernels of the multiplication maps

$$\begin{aligned} I_C &= \ker(C \otimes_A C \to C) \\ I_B &= \ker(B \otimes_A B \to B) \end{aligned}.$$

We consider elements in I_C which go to zero in $I_B/I_B^2 = \Omega^1 f_{B/A}$. But it is so clear that $I_C^2 \to I_B^2$ is surjective hence we can represent an element in $\Omega^1_{C/A}$ which goes to zero in $\Omega^1_{B/A}$ by an element

$$h \in I_C \cap (J \otimes_A C + C \otimes_A J).$$

Let us write this element

$$\sum j_\nu \otimes c_\nu + \sum c'_\mu \otimes j'_\mu$$

where the $j_\nu, j'\mu \in J$. We have $\sum j_\nu c_\nu + \sum j'_\mu c'_\mu = 0$. The sum is equal to

$$\begin{aligned} \sum j_\nu \otimes c_\nu + \sum j'_\mu \otimes c'_\mu + \sum (c'_\mu \otimes j'_\mu - j'_\mu \otimes c'_\mu) = \\ \sum_\nu (1 \otimes c_\nu)(j_\nu \otimes 1 - 1 \otimes j_\nu) + \sum_\mu (1 \otimes c'_\mu)(j'_\mu \otimes 1 - 1 \otimes j'_\mu) \\ + \sum_\mu (1 \otimes j'_\mu)(c'_\mu \otimes 1 - 1 \otimes c'_\mu) - \sum_\mu (1 \otimes c'_\mu)(j'_\mu \otimes 1 - 1 \otimes j'_\mu) \end{aligned}.$$

This yields an element in $\Omega^1_{C/A}$ namely

$$\sum_\nu c_\nu dj_v + \sum_\mu c'_\mu dj'_\mu + \sum j'_\mu dc'_\mu - \sum c'_\mu dj'_\mu$$

which is clearly in the image of $J \bmod J\Omega^1_{C/A}$.

Example:

1) If $B = A[X_1 \ldots X_n]$ then

$$B \otimes_A B = A[X_1 \ldots X_n, Y_1, \ldots Y_n]$$

where $X_i = X_i \otimes 1$ and $Y_i = 1 \otimes X_i$. The ideal I is generated by the elements $X_i - Y_i = X_i \otimes 1 - 1 \otimes X_i$ as a $B \otimes_A B$-module and I/I^2 is the free B-module generated by $dX_i = X_i - Y_i \bmod I^2$.

(2) Assume that

$$B = A[X_1, \ldots, X_n]/ < F_1, \ldots, F_r >$$

where $F_i = \sum_\nu a_{i\nu} X^\nu \in A[X_1, \ldots, X_n]$. Assume we have an $f \in B$ such that for all maximal ideals $\mathfrak{m} \in \operatorname{Spec}(B_f)$ the rank of

$$(\frac{\partial F_i}{\partial X_j})_{i=1,\ldots,r, j=n-r+1,\ldots,n} \bmod \mathfrak{m}$$

is maximal, i.e. equal to r. Then $\Omega^1_{B_f/A}$ is locally free of rank $d = n - r$. If at our maximal ideal $\mathfrak{m}$ the subdeterminant

$$\det(\frac{\partial F_i}{\partial X_j})_{i=1,\ldots,r, j=n+1-r,\ldots,r-n} \not\equiv 0 \bmod \mathfrak{m}$$

then the differentials $dx_1, \ldots, dx_d$ (where the x_i are the images of the X_i in B) are free generators of $\Omega^1_{B/A,\mathfrak{m}}$.

This is clear from our consideration above.

3) We look at our examples of non-smooth points and we will see how the fact that they are not smooth is reflected by the sheaf of relative differentials.

a) If

$$A = k[X, Y]/(XY) = k[x, y]$$

then the set of geometric points is the union of the X and the Y-axis in $\mathbb{A}^2$. The origin is not smooth. The A-module of differentials is generated by dx and dy and we have the relation

$$xdy + ydx = 0.$$

In this example the origin is not smooth because already the condition that the local ring should be integral is violated. But we could easily modify the example by looking at

$$A_1 = k[X, Y]/(XY + X^5 + Y^7) = k[x, y].$$

Now the local ring at (O, O) is integral and we have

$$(y + 5x^4)dx + (x + 7y^6)dy = 0.$$

The dimension of A_1 is one but the module of differentials is not free of rank one. We will see later that this is another criterion for that the origin is not smooth.

This example is interesting for a different reason. If we look at the completions of our two rings then they become isomorphic

$$\hat{A} \simeq \hat{A}_1.$$

We leave this as an exercise to the reader, one has to show that one can construct power series

$$\tilde{X} = X + P(X,Y)$$
$$\tilde{Y} = Y + Q(X,Y)$$

such that the relation $XY + X^5 + Y^7 = 0$ becomes $\tilde{X}\tilde{Y} = 0$.

The reader should also prove that for $k = \mathbb{C}$ and a small number $\epsilon > 0$ the complex space

$$\{(x,y) \in \mathbb{C}^2 \,|xy + x^5 + y^7 = 0\} \bigcap B(0,\epsilon)$$

where $B(0,\epsilon) = \{(x,y) \in \mathbb{C}^2 \,||x|^2 + |y|^2 < \epsilon\}$ is indeed the union of two discs intersecting transversally at the origin.

Hence we see that depending on the microscope which we use to look at our singular point, the local ring may be integral (Zariski topology) or non-integral (analytic topology). The two branches of our complex space above in $B(0,\epsilon)$ will come together very far out again.

Such a singular point is called an ordinary double point.

b)

$$A = k[X,Y]/(X^2 - Y^3).$$

Again we see that $\Omega^1_{A/k}$ at the origin is generated by dx, dy with the relation $2xdx - 3y^2dy = 0$ the module of relative differentials is not free of rank 1.

In our three examples the scheme is smooth outside the origin and this is also the open set where the sheaf of differentials is free of rank one. We will see that this indeed a very general fact which will allow us define smooth morphisms in a very general context.

If we are in the situation of example (2) and if we consider a point

$$\mathrm{Spec}(k(y)) \hookrightarrow J, \quad \text{i.e.} \quad \Psi : A \to k(y),$$

then we can consider the base change

$$\begin{array}{ccc} B & \longrightarrow & B \otimes_A k(y) \\ \uparrow & & \uparrow \\ A & \longrightarrow & k(y) \end{array}$$

.

This scheme is called the fibre over the point y this is now a scheme over a field. It is clear that this scheme

$$\operatorname{Spec}(B \otimes_A k(y)) \to \operatorname{Spec}(k(y))$$

is smooth of dimension d. Moreover it also follows that if we have given a system of local parameters at a closed point as above - say $x_1, \ldots, x_d$ - then the differentials $dx_1, \ldots, dx_d$ form a basis of $\Omega^1_{B\otimes_A k(y))/\operatorname{Spec}(k(y))}$ at this point.

Now we have enough insight to formulate our general definition.

Let us assume we have an arbitrary ring A and a finitely generated A- algebra

$$B = A[x_1, \ldots, x_n] = A[X_1, \ldots, X_n]/I$$

Definition and Theorem VII.3.4.1f :

The morphism

$$\begin{array}{c} \operatorname{Spec}(B) = X \\ \pi \downarrow \\ \operatorname{Spec}(A) = S \end{array}$$

is called smooth if

a) it is flat

b) For all point $s \in S$ the fibre $X_s \to \operatorname{Spec}(k(s))$ is smooth.

The condition b) is equivalent to

b1) The fibres X_s are locally integral at each point and at any point $x \in X_s$ the sheaf $\Omega^1_{X/S}$ is locally free of rank $\dim(\mathcal{O}_{X_s,x})$.

Before I come to the proof I want to make two comments:

1) We do not require that the fibres are integral since we never did that before. Hence it can happen that the fibres are disjoint unions of irreducible ones which then may even have different dimensions

2) Again it is clear that the notion of smoothness can be checked locally at the points. Hence it is clear that our definition extends to an arbitrary morphism $\pi : X \to S$ where X is of finite type over S.

To prove the theorem we start with the case that $S = \operatorname{Spec}(k)$ is a field. In this case the assumption a) is automatically fulfilled. We have seen in the discussion of example 2) that b) $\Rightarrow$ b1).

If (b1) is satisfied we write our algebra $B = k[x_1, \ldots, x_n]$ as a quotient

$$k[x_1, \ldots, x_n]/I \xrightarrow{\sim} B.$$

Let us assume that we have a geometric point P which induces the maximal ideals m_p (resp. $\mathfrak{M}_p$) in B (resp. $C = k[X_1, \ldots, X_n]$). The module $\Omega^1_{B/k}$ is locally free at P and it is of course generated by $dx_1, \ldots, dX_n$. We can choose a basis locally at P let us assume it is of the form

$$dx_1, \ldots, dx_d \quad \text{where} \quad d = \dim B_{m_p}.$$

As usual we put $r = n - d$. It is clear (see (d) above) that we can find $F_1, \ldots, F_r \in I$ such that $dF_1, \ldots, dF_r$ together with $dX_1, \ldots, dX_r$ provide a basis of $\Omega^1_{C/k}$ at P. But this implies that

$$\operatorname{rank}\left(\frac{\partial F_i}{\partial X_\mu}\right) \bmod \mathfrak{M}_p = r.$$

These $F_1, \ldots, F_r$ generate an ideal J and we get

$$\begin{array}{ccccc} C_{\mathfrak{M}_p} & = k[X_1, \ldots, X_n]_{\mathfrak{M}_p} & & \longrightarrow & B_{m_p} \\ & & \searrow & & \nearrow \Psi \\ & & & \tilde{B}_{\tilde{m}_p} & \end{array}$$

where $\tilde{B}_{\tilde{m}_p}$ is the quotient by the ideal J. We know that $\tilde{B}_{\tilde{m}_p}$ is smooth at P by the Jacobi criterion and $\dim \tilde{B}_{\tilde{m}_p} = r$. But then it is clear that Ψ must be an isomorphism and it follows that B is smooth at P.

Now we pass to the general case. From what we know it is clear that b1) $\Rightarrow$ b) because b) can be checked fibre by fibre.

Now we assume b), we pick a point $s \in S$ and a closed point $P \in X_s$. We want to show that $\Omega^1_{B/A}$ is locally free and its rank is $\dim(\mathcal{O}_{X_s,P})$.

We may assume that $S = \operatorname{Spec}(A)$ is local and s is closed. Then we can write

$$A[X_1, \ldots, X_n]/I = B = A[x_1, \ldots, x_n].$$

If we pass to the special fibre we get an exact sequence

$$0 \to I_0 \to k(s)[X_1, \ldots, X_n] \to k(s)[x_1, \ldots, x_n] = B \otimes_A k(y).$$

Now B is a flat A-algebra (we localized in between but this preserves flatness) and hence we get

$$I \otimes k(s) \xrightarrow{\sim} I_0.$$

Now we know that we can find

$$F_1^{(0)}, \ldots, F_r^{(0)} \in I_0$$

which generate I_0 and which satisfy the Jacobi criterion at P. These elements can be lifted to elements $F_1, \ldots, F_r$ in I and by Nakayama's lemma they generate I locally at P. Hence we see that locally at P we have

$$B \simeq A[X_1, \ldots, X_n]/(F_1, \ldots, F_r)$$

and we have seen in example 2) that this implies that

$$\Omega^1_{B/A}$$

is locally free of rank d at P.

We are now able to give a very intrinsic definition of a system of local parameters.

If we have a smooth morphism
$$A \to B$$
and a point P then we say that
$$x_1, \ldots, x_d \in B_f \quad \text{with} \quad f(P) \neq 0$$
is a system of local parameters at P, if the $dx_1, \ldots, dx_d$ form a basis of $\Omega^1_{B_f/A}$. We get a diagram
$$\begin{tikzcd} A \arrow[rr] \arrow[dr] & & A[X_1, \ldots, X_d] \arrow[dl] \\ & B_f & \end{tikzcd}$$
where we send the X_i t to the x_i. We have seen in our previous proof that we can write
$$B_f = A[X_1, \ldots, X_d, X_{d+1}, \ldots, X_n]/I$$
where $I = (F_1, \ldots, F_r)$ and where we have
$$(\frac{\partial F_i}{\partial X_\mu})(P)_{i=1,r,\mu=d+1,\ldots,n} \neq 0.$$
(Actually if we modify f we can assume that this is true at all points in $\mathrm{Spec}(B_f)$).

Now we can show that
$$A[X_1, \ldots, X_r] \hookrightarrow B_f$$
is flat, smooth and $\Omega^1_{B_f/A[X_1,\ldots,X_r]}$ is zero. This implies that the fibres are reduced zero dimensional.

This is an example of an etale morphism. A morphism $A \to B$ is called *etale* if it is smooth, if finite type and if the sheaf of relative differentials is zero.

If we have a closed point $\mathfrak{n} \in \mathrm{Spec}(B)$ then $k(\mathfrak{n}) \to B \otimes_B k(\mathfrak{n})$ is still smooth and the connected components are zero dimensional. Hence it follows that $k(\mathfrak{n}) \to B \otimes_B k(\mathfrak{n})$ is a finite separable algebra, i.e a finite sum of separable field extensions of $k(\mathfrak{n})$.

Chapter VIII
Projective schemes

VIII.1 Geometric constructions

1.1 The projective space $\mathbb{P}^n_A$

Let A be a commutative ring with 1_A. We consider the following $n+1$ affine schemes

$$U_i = \mathrm{Spec}(A[T_{i0}\dots T_{in}]/(T_{ii}-1)).$$

The ring of regular functions of U_i is the polynomial ring in $n+1$ independent variables variables $T_{i0}, T_{i1}\dots T_{i,i-1}, T_{i,i}, T_{i,i+1}\dots T_{in}$ and we divide by the relation $T_{ii}=1$. This means that all the U_i are copies of $\mathbb{A}^n_{\mathrm{Spec}(A)}$.

We denote by t_{ij} the images of the T_{ij} in the quotient ring

$$A[t_{i0}\dots t_{in}],$$

this means that we consider for a given i the $t_{i0}, t_{i1}\dots t_{i,i-1}, t_{i,i+1}\dots t_{in}$ as independent and $t_{ii}=1$.

For any index j we define the open subscheme

$$\begin{aligned} U_{ij} &= \mathrm{Spec}(A[T_{i0}\dots T_{in}]/(T_{ii}-1))_{T_{ij}} \\ &= \mathrm{Spec}(A[T_{i0}\dots T_{in}, T_{ij}^{-1}]/(T_{ii}-1)) \end{aligned}.$$

We have an isomorphism

$$f_{ij}: U_{ij} \to U_{ji}$$

given on the level of rings by

$$\begin{aligned} \phi_{ij}: A[T_{j0}\dots T_{jn}, t_{ji}^{-1}]/(T_{jj}-1) &\to A[T_{i0}\dots T_{in}, t_{ij}^{-1}]/(T_{ii}-1) \\ \phi_{ij}(T_{j\nu}) &= T_{i\nu}\cdot t_{ij}^{-1} \end{aligned}.$$

We see that $\phi_{ij}(t_{jj}) = \phi_{ij}(1) = t_{ij}\cdot t_{ij}^{-1} = 1$ and $\phi_{ij}(t_{ji}^{-1}) = t_{ii}\cdot t_{ij} = t_{ij}$ as it must be. We allow $i=j$ and then ϕ_{ii} is the identity on U_i.

Given three indices i,j,k we get a commutative diagram

$$\begin{array}{ccc} U_{ij} \supset U_{ij}\cap U_{ik} & \xrightarrow{\phi_{ij}} & U_{ji}\cap U_{jk} \subset U_{ji} \\ \searrow \phi_{ik} & & \swarrow \phi_{jk} \\ & U_{ki}\cap U_{kj} & \end{array}$$

of isomorphisms. This allows us to define an equivalence relation on the space

$$\bigsqcup_{i=0,\ldots,n} U_i$$

namely $u_i \sim u_j$ if and only if $u_i \in U_j, u_j \in U_{ji}$ and $\phi_{ij}(u_i) = u_j$. We define the space

$$\bigsqcup U_i / \sim = \mathbb{P}^n_A.$$

The projection map

$$\pi : \bigsqcup U_i \to \mathbb{P}^n_A$$

provides a homeomorphism from U_i to an open subset in $\mathbb{P}^n_A$, we identify U_i with this open subset, i.e. we consider U_i as an open subset in $\mathbb{P}^n_A$.

Now we define a sheaf $\mathcal{O}_{\mathbb{P}^n_A}$ on the space $\mathbb{P}^n_A$ simply by putting

$$\mathcal{O}_{\mathbb{P}^n_A}|U_i = A[T_{i0}, \widetilde{\ldots}, T_{in}]/(T_{ii} = 1)$$

and we use the ϕ_{ij} to glue $\mathcal{O}(U_i)|U_i \cap U_j$ with $\mathcal{O}(U_j)|U_i \cap U_j$.

After constructing $\mathbb{P}^n_A$ we can think of it in the following way: The scheme $\mathbb{P}^n_A$ admits an open covering

$$\mathbb{P}^n_A = \cup_{i=0\ldots n} U_i$$

such the U_i are affine spaces. The ring of regular functions on U_i is

$$\mathcal{O}_{\mathbb{P}^n_A}(U_i) = A[t_{i0} \ldots t_{in}].$$

where $t_{ii} = 1$ and $t_{i0}, \ldots, t_{i,i-1}, t_{i,i+1} \ldots, t_{in}$ are independent.

The intersection $U_i \cap U_j$ is affine and

$$\mathcal{O}_{\mathbb{P}^n_A}(U_i \cap U_j) = A[t_{i0} \ldots t_{in}, t_{ij}^{-1}] = A[t_{j0} \ldots t_{jn}, t_{ji}^{-1}]$$

and

$$t_{i\nu} = t_{j\nu} \cdot t_{ji}^{-1}, t_{j\mu} = t_{i\nu} \cdot t_{ij}^{-1}.$$

Theorem 1.1.1: *The regular functions on $\mathbb{P}^n_A$ are the constants, this can be stated briefly as*

$$\mathcal{O}_{\mathbb{P}^n_A}(\mathbb{P}^n_A) = A.$$

Proof: Let $f \in \mathcal{O}_{\mathbb{P}^n_A}(\mathbb{P}^n_A)$. We restrict f to the open sets U_i and get a polynomial

$$f|U_i = P_i(t_{i0} \ldots t_{in}) \in A[t_{i0} \ldots t_{in}]$$

For any pair of indices we have $P_i|U_i \cap U_j = P_j|U_i \cap U_j$ and this means that $\phi_{ij}(P_j) = P_i$ and hence

$$P_i(t_{i0} \ldots t_{in}) = P_j(t_{i0} \cdot t_{ij}^{-1}, \ldots, t_{in} \cdot t_{ij}^{-1}).$$

Now we write P_j as polynomial

$$\sum a_{\nu_0\ldots\nu_n} t_{j0}^{\nu_0} \ldots t_{j,j-1}^{\nu_{j-1}} t_{j,j+1}^{\nu_{j+1}} \ldots t_{jn}^{\nu_n}.$$

We get

$$P_i(t_{i0} \ldots t_{in}) = \sum a_{\nu_0\ldots\nu_n} \left(\frac{t_{i0}}{t_{ij}}\right)^{\nu_0} \ldots \left(\frac{t_{in}}{t_{ij}}\right)^{\nu_n}$$

this is a polynomial in $t_{i0} \ldots t_{i,i-1}, t_{i,i+1} \ldots t_{in}$.

But if $i \neq j$ then this is only possible if all exponents $\nu_0 = \ldots \nu_{i-1}, \nu_{i+1}, \ldots \nu_n = 0$ in other words, P_i has to be constant.

This proves the theorem.

1.1.2. We have a different way of looking at this argument.

We consider $n+1$ new variables $X_0, X_1, \ldots, X_{n+1}$ and we endow the polynomial ring $A[X_0, X_1, \ldots, X_n]$ with its standard graduation: To any monomial $X_0^{r_0} X_1^{r_1} \ldots X_n^{r_n}$ we attach the degree $d = \sum r_i$, then we define $A[X_0, X_1, \ldots, X_n](d)$ to be the A-module generated by the monomials of degree d. This is the module of homogeneous polynomials (or forms) of degree d. We have the direct sum decomposition

$$\bigoplus_{d=0}^{\infty} A[X_0, X_1, \ldots, X_n](d) = A[X_0, X_1, \ldots, X_n].$$

We can localize this ring by inverting any of the variables and in any of these localizations we can consider the subring of elements of degree zero, i.e. we consider

$$A[X_0, X_1, \ldots, X_n]_{X_i}^{(0)} = \{\frac{F}{X_i^d} \mid F \in A[X_0, X_1, \ldots, X_n](d)\}.$$

It is clear that the homomorphism sending $t_{ij} \mapsto X_j/X_i$ induces an isomorphism

$$A[t_{i0} \ldots t_{in}] \tilde{\to} A[X_0, X_1, \ldots, X_n]_{X_i}^{(0)}$$

and the ϕ_{ij} correspond to the obvious inclusion maps. We can always think that we have the relation $t_{ij} = X_j/X_i$.

Now our theorem above says that

$$\bigcap_{i=0\ldots n} A[X_0, X_1, \ldots, X_n]_{X_i}^{(0)} = A$$

1.1.3. Points: It is of course clear what an A-valued point on $\mathbb{P}_A^n$ is. Any such A-valued point $x \in \mathbb{P}_A^n(A)$ is represented by an element

$$x_i = (a_{i0}, \ldots, 1, \ldots, a_{in})$$

where $a_{iv} \in A$. Such a point lies also in U_j if a_{ij} is a unit in A and then

$$x_j = \left(\frac{a_{i0}}{a_{ij}}, \dots, 1, \dots, \frac{1}{a_{ij}}, \dots \frac{a_{in}}{a_{ij}}\right)$$

represents the same point.

Of course we can also say that we can represent a point by an element in

$$\mathbb{A}^{n+1}_* = \{(a_0, \dots, a_n) \mid \text{ at least one entry is a unit in } \quad A\},$$

and we divide by the equivalence relation

$$(a_0, \dots, a_n) \sim (a_0 b, \dots, a_n b) \quad \text{with} \quad b \in A^*.$$

1.2. Closed subschemes: Now we know of course what a closed subscheme of $\mathbb{P}^n_A$ is (See V 2.1.2.). We simply pick a quasicoherent sheaf of ideals $\mathcal{I}$ in $\mathcal{O}_{\mathbb{P}^n_A}$, then $((V(\mathfrak{I}), \mathcal{O}_{\mathbb{P}^n_A})$ is a closed subscheme of $\mathbb{P}^n_A$. We get a commutative diagram

$$\begin{array}{ccc} (V(\mathfrak{I}), \mathcal{O}_{\mathbb{P}^n_A}/\mathfrak{I}) & \hookrightarrow & \mathbb{P}^n_A \\ & \searrow & \downarrow \\ & & \mathrm{Spec}(A) \end{array}$$

hence this subscheme is automatically a scheme over $\mathrm{Spec}(A)$.

Such a quasicoherent sheaf of ideals $\mathcal{I}$ is simply a collection of ideals $I_i \subset \mathcal{O}_{\mathbb{P}^n}(U_i)$ such that I_i and I_j generate the same ideal in $\mathcal{O}_{\mathbb{P}^n}(U_i \cap U_j)$.

We call an ideal $\tilde{\mathcal{I}} \subset A[X_0, X_1, \dots, X_n]$ homogeneous if for any $F \in \tilde{\mathcal{I}}$ its homogeneous components F_d are also in the ideal.

We can get quasicoherent sheaves of ideals from homogeneous ideals $\tilde{I}$ in $A[X_0, X_1, \dots, X_n]$. This is not difficult to see. To any homogeneous polynomial $F \in A[X_0, X_1, \dots, X_n](d) \in \tilde{I}$ of degree d we attach a collection of elements $\{f_i \in \mathcal{O}_{\mathbb{P}^n}(U_i)\}_{i=0,1\dots n}$ simply by substituting $t_{i,\nu}$ for X_ν into the polynomial. These f_i generate an ideal I_i in $\mathcal{O}_{\mathbb{P}^n}(U_i)$. It is clear that the restrictions of f_i, f_j to $U_i \cap U_j$ satisfy $t_{ij}^{-d} f_i | U_i \cap U_j = f_j | U_i \cap U_j$ and since t_{ij} is a unit in $\mathcal{O}_{\mathbb{P}^n}(U_i \cap U_j)$ it follows that I_i and I_j generate the same ideal in $\mathcal{O}_{\mathbb{P}^n}(U_i \cap U_j)$. If F runs over the homogeneous elements of $\tilde{\mathcal{I}}$ these F give us a quasicoherent sheaf of ideals $\mathcal{I}$.

As in the affine case we can attach the closed subset $V(\tilde{I})$(resp. $V(\mathcal{I})$) which is the set of common zeroes of $\tilde{I}$ (resp. $\mathcal{I}$).

If in turn $Z \subset \mathbb{P}^n_A$ is a closed subset, we may consider the ideal $\tilde{I}$ generated by homogeneous polynomials which vanish on Z. It is clear that $V(\tilde{I}) = Z$ (???).

Lemma 1.2.1. *For a homogeneous ideal $\tilde{I} \subset A[X_0, X_1, \dots, X_n]$ we have $V(\tilde{I}) = \emptyset$ if and only if $\tilde{I}(d) = A[X_0, X_1, \dots, X_n](d)$ for d sufficiently large.*

Proof: Of course we know that $V(\tilde{I}) = \emptyset$ if and only if $V(\tilde{I}) \cap U_I = \emptyset$ for $i = 0 \dots n$. This is equivalent to the assertion: For all i we can find $g_\nu \in \mathcal{O}_{\mathbb{P}^n}(U_i)$ and $f_\nu \in I_i$ such that

$$\sum g_\nu f_\nu = 1$$

(See ???). Now we remember that we should think of $t_{i,j}$ as being X_j/X_i then we see that we can multiply this relation by a power of X_i and it yields a relation

$$\sum G_\nu F_\nu = X_i^{m_i}$$

where the G_ν, F_ν are homogeneous and $F_\nu \in I_i$. Hence we conclude that $V(\tilde{I}) = \emptyset$ implies that for all i a suitable power of X_i is in the ideal which proves one direction of the Lemma. The other direction is obvious.

For our considerations above it was not necessary to assume that the base scheme $\mathrm{Spec}(A)$ is affine. All the constructions work over an arbitrary base scheme S.

We could have started with the ring $A = \mathbb{Z}$ and construct the scheme $\mathbb{P}^n_{\mathbb{Z}}$. If S is any scheme then we have the absolute morphism $S \to \mathrm{Spec}(\mathbb{Z})$ (???) and we could define $\mathbb{P}^n_S = \mathbb{P}^n_{\mathbb{Z}} \times S$.

1.2.2. Points again: Of course it is again very easy to describe A-valued points of closed subscheme $X \subset \mathbb{P}^n_A$ if this subscheme is given by a homogeneous ideal $I \subset A[X_0, \dots, X_n]$. Then the B-valued points of X are given by

$$X(B) = \{(b_0, \dots, b_n) \in B_*^{n+1} \mid f(b_0, \dots, b_n) = 0 \quad \text{for all homogeneous} \quad f \in I\}/B^*.$$

1.3. Projective morphisms and projective schemes: We call a morphism $X \to S$ *projective* if we can find a commutative diagram

$$\begin{array}{ccc} i : X & \to & \mathbb{P}^n_S \\ & \searrow & \downarrow \\ & & S \end{array}$$

where i is a closed embedding, i.e. an isomorphism to a closed subscheme in $\mathbb{P}^n$. We also say that X/S is a *projective scheme* over S.

Now I want to show that the fibered product of projective schemes exists and is again a projective scheme.

I want to change the notation slightly. We write $S = \mathrm{Spec}(A)$. We have the two schemes

$$\begin{array}{ccc} \mathbb{P}^n_S & & \mathbb{P}^m_S \\ \searrow & & \swarrow \\ & S & \end{array}$$

It will turn out that this fibered product can be written as a closed subscheme of $\mathbb{P}_S^{nm+n+m}$.

To construct this closed subscheme I start from the usual covering by affine spaces. But here I will change the numeration: We had $n+1$ open affine spaces to cover $\mathbb{P}_S^n$. Here we have $nm+n+m = (n+1)(m+1)-1$ hence we need $(n+1)(m+1)$ open affine spaces for the covering.

We write for $0 \le i \le n, 0 \le j \le m$

$$U_{ij} = \mathrm{Spec}(A[\ldots x_{ij\nu\mu} \ldots])$$

where of course $0 \le \nu \le n, 0 \le \mu \le m$, the $x_{ij\nu\mu}$ except that we have the relation $x_{ijij} = 1$. We cover $\mathbb{P}_S^n$ by

$$\begin{aligned} U_i &= \mathrm{Spec}(A[t_{i0} \ldots t_{in}]) \\ V_j &= \mathrm{Spec}(A[t'_{j0} \ldots t'_{jn}]) \end{aligned}.$$

We have already constructed the fibered product

$$U_i \times_S V_j = \mathrm{Spec}(A[t_{i0} \ldots t_{in}, t'_{j0} \ldots t'_{jn}]).$$

We construct an isomorphism from $U_i \times V_j$ with a closed subscheme of U_{ij}. To do this we construct an A-homomorphism of rings

$$A[\ldots x_{ij\nu\mu} \ldots] \to A[t_{i0} \ldots t_{in}, t'_{j0} \ldots t'_{jn}])$$

and this homomorphism is given by

$$x_{ij\nu\mu} \to t_{i\nu} \cdot t'_{j\mu}.$$

We observe that x_{ijij} maps to 1. We also observe that

$$\begin{aligned} x_{iji\mu} &\to t'_{j\mu} \\ x_{ij\nu j} &\to t_{i\nu} \end{aligned}$$

and hence it is clear that the kernel of this homomorphism has as its kernel the ideal I_{ij} generated by

$$x_{iji\mu} \cdot x_{ij\nu j} - x_{ij\nu\mu}.$$

If we divide by the ideal generated by these polynomials, then we get a polynomial ring generated by $x_{ij\nu\mu}$ with $(\nu, \mu) \neq (i, j)$).

Hence this ideal defines a closed subscheme $V(I_{ij}) \subset U_{ij}$ which is isomorphic to $U_i \times_S V_j$. This ideal defines a quasicoherent sheaf $\tilde{I}_{ij}$ on U_{ij} and we will show that

$$\tilde{I}_{ij} | U_{ij} \cap U_{i'j'} = \tilde{I}_{i'j'} | U_{ij} \cap U_{i'j'}.$$

To see this we consider the following diagram

$$\begin{array}{ccccc}
U_i \times_S V_j & \xrightarrow{\sim} & V(I_{ij}) \subset & & U_{ij} \\
 & & \cup & & \cup \\
 & & V(I_{ij}) \cap V(I_{i'j'}) & \subset & U_{ij} \cap U_{i'j'} . \\
 & & \cap & & \cap \\
U_i' \times_S V_j & \xrightarrow{\sim} & V(I_{i'j'}) \subset & & U_{i'j'}
\end{array}$$

We can construct the fibered product $(U_i \cap U_{i'}) \times_S (V_j \cap V_{j'})$ since both factors are affine and of course we can place it into the middle of the left column of the diagram above and we get open subschemes $(U_i \cap U_{i'}) \times_S (V_j \cap V_{j'}) \subset U_{i'} \times_S V_j$ and $(U_i \cap U_{i'}) \times_S (V_j \cap V_{j'})$. Of course we want a horizontal arrow from $(U_i \cap U_{i'}) \times_S (V_j \cap V_{j'})$ to $V(I_{ij}) \cap V(I_{i'j'})$ which should be an isomorphism. To see that this works we rewrite the diagram in terms of rings, we drop the column in the middle

$$\begin{array}{ccc}
A[\dots t_{i\nu}, \dots, t'_{j\mu} \dots] & \longleftarrow & A[\dots x_{ij\nu\mu} \dots] \\
\cap & & \cap \\
A[\dots t_{i\nu}, \dots, t'_{j'\mu}, t_{ii'}^{-1}, t_{jj'}^{-1}] & \longleftarrow & A[\dots x_{ij\nu\mu} \dots, x_{iji'j'}^{-1}] \\
\| & & \| \\
A[\dots t_{i'\nu}, \dots, t'_{j'\mu}, t_{i'i}^{-1}, t_{j'j}^{-1}] & \longleftarrow & A[\dots x_{i'j'\nu\mu} \dots, x_{i'j'ij}^{-1}] \\
\cup & & \cup \\
A[\dots t_{i'\nu}, \dots, t'_{j'\mu} \dots] & \longleftarrow & A[\dots x_{i'j'\nu\mu} \dots]
\end{array} .$$

The two arrows in the middle are obtained from the arrow on the top and the bottom by extending. We send $x_{iji'j'}^{-1}$ (resp. $x_{i'j'ij}^{-1}$) to $t_{ii'}^{-1} \cdot t_{jj'}^{-1}$ (resp. $t_{i'i}^{-1} \cdot t_{j'j}^{-1}$). But using the rules for changing the coordinates yield that the two arrows in the middle are equal. The kernel of this arrow in the middle is clearly $(I_{ij})_{x_{iji'j'}} = (I_{i'j'})_{x_{i'j'ij}}$ and this proves $\tilde{I}_{ij} | U_{ij} \cap U_{i'j'} = \tilde{I}_{i'j'} | U_{ij} \cap U_{i'j'}$. For later references we write our diagram again, but now we write the *completed diagram*

$$\begin{array}{ccccc}
U_i \times_S V_j & \xrightarrow{\sim} & V(I_{ij}) \subset & & U_{ij} \\
\cup & & \cup & & \cup \\
(U_i \cap U_{i'} \times_S (V_j \cap V_{j'}) & \xrightarrow{\sim} & V(I_{ij}) \cap V_{I_{i'j'}}) & \subset & U_{ij} \cap U_{i'j'} . \\
\cap & & \cap & & \cap \\
U_i' \times_S V_j & \xrightarrow{\sim} & V(I_{i'j'}) \subset & & U_{i'j'}
\end{array}$$

Hence the I_{ij} define a coherent sheaf of ideals on $\mathbb{P}_S^{nm+n+m}$ and this defines a closed subscheme y of $\mathbb{P}_S^{nm+n+m}$.

We can define projection maps

$$\begin{array}{ccc}
p_1 : y & \longrightarrow & \mathbb{P}_S^n \\
 & \searrow \quad \swarrow & \\
 & S &
\end{array}
\qquad
\begin{array}{ccc}
p_2 : y & \hookrightarrow & \mathbb{P}_S^m \\
 & \searrow \quad \swarrow & \\
 & S &
\end{array}$$

which on $U_{ij} \cap y = U_i \times_S V_j$ are the projections to the first and second factor. (It needs a little computation that these projections match on $U_{ij} \cap U_{i'j'}$).

Now it is clear that

$$\begin{tikzcd}[column sep=small] & y \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ \mathbb{P}^n_S \arrow[dr] & & \mathbb{P}^m_S \arrow[dl] \\ & S & \end{tikzcd}$$

is indeed a fibered product. If we have a scheme $T \to S$ and a pair of arrows

$$\begin{tikzcd} T \arrow[rr, "f"] \arrow[dr] & & \mathbb{P}^n_S \arrow[dl] \\ & S & \end{tikzcd} \qquad \begin{tikzcd} T \arrow[rr, "g"] \arrow[dr] & & \mathbb{P}^m_S \arrow[dl] \\ & S & \end{tikzcd}$$

then we find for any $t \in T$ an open neighborhood V such that

$$f(V) \subset U_i, g(V) \subset V_j$$

for some i, j. Hence on V we get a map

$$\begin{tikzcd} V \arrow[rr] \arrow[dr] & & U_i \times_S V_j \simeq U_{ij} \cap y \arrow[dl] \\ & S & \end{tikzcd}$$

and if we cover T by such $V's$ then the maps must match on the intersections. This follows from the completed diagram in our proof.

We could also use the description of subschemes by homogeneous ideals. We introduce the ring $A[Z_{ij}]$ where $i = 0, 1, \ldots, n$, $j = 0, 1, \ldots, n$ and in this ring we have the homogeneous ideal $\tilde{\mathcal{P}}$ generated by the polynomials

$$Z_{ij} Z_{ab} - Z_{aj} Z_{ib}$$

for all quadruples of indices. Then the process of passing from homogeneous ideals to quasicoherent sheaves of ideals gives us the ideal describing $\mathbb{P}^n_S \times_S \mathbb{P}^m_S$ as a subscheme of $\mathbb{P}_S^{(n+1)(m+1)-1}$. This is the Segre embedding.

1.3.1. Locally free sheaves on $\mathbb{P}^n$

At this point I return to the construction of sheaves by the gluing process. We want to construct locally free sheaves and line bundles on $\mathbb{P}^n_A$. To do this we start from the collection of free modules of a fixed rank m on the open schemes U_i:

$$\mathcal{O}_{\mathbb{P}^n}(U_i)^m = A[t_{i0}, \ldots, t_{in}]^m = M_i.$$

They define sheaves $\tilde{M}_i$ on the affine schemes U_i. Now we choose $\mathcal{O}_{\mathbb{P}^n}(U_i \cap U_j)$ - linear isomorphisms

$$g_{ij} : \tilde{M}_i(U_i \cap U_j) \xrightarrow{\sim} \tilde{M}_j(U_i \cap U_j),$$

this is nothing else than a collection of matrices

$$g_{ij} \in Gl_m(\mathcal{O}_{\mathbb{P}^n}(U_i \cap U_j))$$

which should satisfy the cocycle relation

$$\begin{aligned} g_{ii} &= \mathrm{Id} \\ g_{ij} g_{ji} &= Id \\ g_{ij} \cdot g_{je} &= g_{ie} \quad \text{on} \quad U_i \cap U_j \cap U_e \end{aligned} .$$

Then we get a locally free sheaf (vector bundle, see below) on $\mathbb{P}^n_A$ by gluing:

$$\tilde{M} = (M_i, g_{ij})_{i=0,\dots,u;j=0,\dots,n}.$$

I recall that for an open set a section $s \in \tilde{M}(V)$ is a collection $s = (s_0, \dots, s_i, \dots, s_n)$ where $s_i \in \tilde{M}_i(U_i \cap V)$ and where $g_{ij} s_i = s_j$ on $V \cap U_i \cap U_j$. Especially we have $\tilde{M}(U_i) = M_i$ because in this case the i-th component determines the others.

In the simple case $m = 1$ our matrix becomes a unit in $\mathcal{O}_{\mathbb{P}^n}(U_i \cap U_j)$ and we may choose for instance an integer r and define

$$g_{ij} = t_{ij}^{-r}.$$

This yields a locally free sheaf of rank one (a line bundle or invertible sheaf) which is called

$$\mathcal{O}_{\mathbb{P}^n}(r)$$

on $\mathbb{P}^n_A$. Again I point out that $\mathcal{O}_{\mathbb{P}^n}(r)(U_i) = \mathcal{O}_{\mathbb{P}^n}(U_i) = A[t_{i0}, \dots, t_{in}]$.

1.3.2. Exercise

(1) Compute $H^0(\mathbb{P}^n_A, \mathcal{O}_{\mathbb{P}^n}(r))$ and show that this is isomorphic to the A-module of homogeneous polynomials in $n+1$-variables of degree r. To be more precise: A section in $H^0(\mathbb{P}^n_A, \mathcal{O}_{\mathbb{P}^n}(r))$ is by definition a collection

$$s = (f_0, \dots, f_n)$$

where $f_i \in A[t_{i0}, \dots, t_{in}]$ which satisfies

$$t_{ij}^{-r} \cdot f_i = f_j.$$

on $U_i \cap U_j$. A homogeneous polynomial

$$F(X_0, \dots, X_n) = \Sigma a_{\nu_0,\dots,\nu_n} X_0^{\nu_0}, \dots, X_n^{\nu_n}$$

with $\Sigma \nu_i = r$ provides such a section if we define

$$f_i(t_{i0}, \dots, t_{in}) = F(t_{i0}, \dots, 1, \dots, t_{in}).$$

Show that this gives us the isomorphism.

(2) Any homogeneous polynomial $F(X_0, \dots, X_n)$ defines a sheaf of ideals

$$\mathcal{O}_{\mathbb{P}^n}(F) \hookrightarrow \mathcal{O}_{\mathbb{P}^n}$$

which is defined by $\mathcal{O}_{\mathbb{P}^n}(F)(U_i) = \mathcal{O}_{\mathbb{P}^n}(U_i)f_i$, in other words: the restriction of the ideal to the open sets U_i is the principal ideal (f_i).

Show: If for all $\mathfrak{p} \in \operatorname{Spec}(A)$ this polynomial is non-zero in $A/\mathfrak{p}[X_0, \dots, X_n]$ then this is a locally free sheaf on $\mathbb{P}^n_A$ which is isomorphic to $\mathcal{O}_{\mathbb{P}^n}(-d)$ where $d =$ degree of F.

Hint: Write a section $\mathcal{O}_{\mathbb{P}^n}(F)(V)$ as acollection of elements $(\dots, h_i f_i, \dots)$ and show that we must have $t_{ij}^d h_i = h_j$. Where do we use our assumption. Can it be replaced by a weaker assumption.

I refer to the exercise (2) The sheaf of ideals defines a closed subscheme $V(F)$ which is called a hypersurface of degree d. If we take for instance simply $F = X_0$ then this closed subscheme is the reduced scheme $H_0 = \mathbb{P}^n \setminus U_0$, it is isomorphic to $\mathbb{P}^{n-1}/\operatorname{Spec}(A)$. It is called the hyperplane at infinity (from the point of view of somebody who lives in U_0). We as well consider the other hyperplanes $H_i = V(X_i)$.

The tensor product of locally free sheaves is again a locally free sheaf and especially we have $\mathcal{O}_{\mathbb{P}^n}(r) \otimes \mathcal{O}_{\mathbb{P}^n}(s) \xrightarrow{\sim} \mathcal{O}_{\mathbb{P}^n}(r+s)$. The inclusion $\mathcal{O}_{\mathbb{P}^n}(X_0) \subset \mathcal{O}_{\mathbb{P}^n}$ combined with the isomorphism $\mathcal{O}_{\mathbb{P}^n}(X_0) \xrightarrow{\sim} \mathcal{O}_{\mathbb{P}^n}(-1)$ gives us an inclusion $\mathcal{O}_{\mathbb{P}^n}(-1) \hookrightarrow \mathcal{O}_{\mathbb{P}^n}$ and this induces after taking tensor products $\mathcal{O}_{\mathbb{P}^n}(r-1) \hookrightarrow \mathcal{O}_{\mathbb{P}^n}(r)$ and we get a chain of inclusions

$$\mathcal{O}_{\mathbb{P}^n} \hookrightarrow \mathcal{O}_{\mathbb{P}^n}(1) \hookrightarrow \mathcal{O}_{\mathbb{P}^n}(2) \hookrightarrow \dots \mathcal{O}_{\mathbb{P}^n}(d) \dots$$

where all the embeddings are obtained from $\mathcal{O}_{\mathbb{P}^n}(X_0) \hookrightarrow \mathcal{O}_{\mathbb{P}^n}$.

Of course it should be clear that we have many ways of mapping the sheaf $\mathcal{O}_{\mathbb{P}^n}$ into $\mathcal{O}_{\mathbb{P}^n}(d)$. To be more precise we can look at

$$\operatorname{Hom}_{\mathcal{O}_{\mathbb{P}^n}}(\mathcal{O}_{\mathbb{P}^n}, \mathcal{O}_{\mathbb{P}^n}(d))$$

and to give a homomorphism among sheaves we only need to know what happens to $1 \in H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n})$ because this section generates the stalk on each point. Hence we get

$$\operatorname{Hom}_{\mathcal{O}_{\mathbb{P}^n}}(\mathcal{O}_{\mathbb{P}^n}, \mathcal{O}_{\mathbb{P}^n}(d)) = H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d))$$

and this is the A-module of homogeneous forms of degree d in $A[X_0, \dots, X_n]$. Hence giving such an embedding $\mathcal{O}_{\mathbb{P}^n} \hookrightarrow \mathcal{O}_{\mathbb{P}^n}(d)$ amounts to pick a form of degree d. We have chosen at this point the form X_0^d.

1.3.3. $\mathcal{O}_{\mathbb{P}^n}(d)$ as sheaf of "meromorphic functions"

If we look at

$$\mathcal{O}_{\mathbb{P}^n} \hookrightarrow \mathcal{O}_{\mathbb{P}^n}(d)$$

and restrict this map to U_i then we get

$$\mathcal{O}_{\mathbb{P}^n}(U_i) \longrightarrow \mathcal{O}_{\mathbb{P}^n}(d)(U_i) = \mathcal{O}_{\mathbb{P}^n}(U_i)$$

and this map is given by

$$f_i \longrightarrow t_{0i}^d f_i.$$

We can embed $\mathcal{O}_{\mathbb{P}^n}(U_i)$ into the module $\frac{1}{t_{0i}^d}\mathcal{O}_{\mathbb{P}^n}(U_i)$ which is the $\mathcal{O}_{\mathbb{P}^n}(U_i)$-module of "meromorphic functions on U_i" which are regular on $U_i \cap U_0$ and have at most "a pole of order d along the hyperplane $U_i \backslash U_i \cap U_0 = U_i \cap V(X_0)$".

Then the map above gives an isomorphism

$$\frac{1}{t_{i_0}^d}\mathcal{O}_{\mathbb{P}^n}(U_i) \xrightarrow{\sim} \mathcal{O}_{\mathbb{P}^n}(d)(U_i).$$

Especially for $i = 0$ we have $t_{00} = 1$ which means that $V(X_0)$ does not meet U_0. We glue these modules $\frac{1}{t_{0i}^d}\mathcal{O}_{\mathbb{P}^n}(U_i)$ over the intersections of two affine sets and hence we get the sheaf $\mathcal{O}_{\mathbb{P}^n}(-dH_0)$ and call this the sheaf of "meromorphic functions on $\mathbb{P}^n$" which are regular on U_0 and have at most a pole of order d along the hyperplane at infinity. We have a diagram

$$\mathcal{O}_{\mathbb{P}^n} \quad \hookrightarrow \quad \mathcal{O}_{\mathbb{P}^n}(-dH_0) \xrightarrow{\sim} \quad \mathcal{O}_{\mathbb{P}^n}(d)$$

This also gives a slightly different view of the above chain of inclusions: The chains arise simply because we allow higher and higher orders of poles.

Of course we have also an interpretation for $r < 0$: In this case the sections are the germs of regular functions which have a zero of order $\geq -r$ along the hyperplane.

1.3.4. Vector bundles attached to locally free modules:

If we have any scheme X and a locally free $\mathcal{O}_X-$module $\tilde{M}$ of finite constant rank n then we may cover X by suitable affine open subschemes V_j such that $\tilde{M}|V_j$ becomes trivial in other words we may choose a basis $e_{1,j}, \dots, e_{n,j}$ of the $\mathcal{O}_X(V_j)-$module $\tilde{M}(V_j)$. If we use this basis to identify $\psi_j : \mathcal{O}(V_j)^n \xrightarrow{\sim} M(V_j)$ by sending the canonical basis to the $e_{\nu,j}$, then $g_{j,l} = \psi_l \circ \psi_j^{-1} : \mathcal{O}_X(V_j \cap V_l)^n \xrightarrow{\sim} \mathcal{O}_X(V_j \cap V_l)^n$ gives gluing data which in turn give a locally free module which is isomorphic to $\tilde{M}$. Hence any locally free module is obtained by gluing.

We stick to the situation above. We consider the dual module $\tilde{M}^\vee$. Locally on the V_j we may consider the symmetric tensor algebra $\mathrm{Sym}(M^\vee(V_j))$ which in this situation is nothing else than the polynomial algebra $\mathcal{O}(V_j)[X_{1j}, \dots, X_{nj}]$ where the $X_{\nu j}$ form a dual basis of the basis $e_{\mu j}$. Hence we get a scheme

$$\begin{array}{ccc} \mathrm{Spec}(\mathrm{Sym}(M^\vee(V_j))) & \xrightarrow{\sim} & \mathrm{Spec}(\mathcal{O}(V_j)[X_{1j}, \dots, X_{nj}]) \\ & \searrow \quad \swarrow & \\ & \mathrm{Spec}\,\mathcal{O}(V_j) & \end{array}$$

We have $\mathrm{Spec}(\mathrm{Sym}(M^\vee(V_j \cap V_l)) \subset \mathrm{Spec}(\mathrm{Sym}(M^\vee(V_j), \mathrm{Spec}(\mathrm{Sym}(M^\vee(V_l)$ as an open subscheme and this gives us again by a gluing process a scheme

$$\begin{array}{c} V(\tilde{M}) \\ \downarrow \\ X \end{array}$$

which is locally over the V_j equal to $\mathrm{Spec}(\mathrm{Sym}(M^\vee(V_j)$. This scheme is called the *vector bundle* attached to $\tilde{M}$.

Let us consider the projective space $\mathbb{P}^n/\,\mathrm{Spec}(k)$. We have seen that the set of k-valued points of the projective space is the set of lines in k^{n+1} passing through zero. We want to construct the tautological line bundle over $\mathbb{P}^n/\,\mathrm{Spec}(k)$. A k-valued point should be a pair (x, ζ) where $x \in \mathbb{P}^n(k)$ and $\zeta \in x$. Of course it is very easy to describe this line bundle as a closed subscheme of $\mathbb{P}^n \times_{\mathrm{Spec}(k)} \mathbb{A}^{n+1}_k$.

It is left to the reader to verify that this bundle is $\mathcal{O}_{\mathbb{P}^n}(-1)$

1.3.5 The tangent bundle of $\mathbb{P}^n$

It is of course obvious that the scheme

$$\begin{array}{c} \mathbb{P}^n_A \\ \downarrow \\ \mathrm{Spec}(A) \end{array}$$

is always smooth. Locally on one of the U_i the module of differentials is the free module generated by the $dt_{i0}, \dots, dt_{i,i-1}, dt_{i,i+1}, \dots, dt_{i,n}$, of course $dt_{ii} = 0$.

We consider the n-th exterior power of this module

$$\Lambda^n \Omega_{\mathbb{P}^n/A} = \Omega^n_{\mathbb{P}^n/A}.$$

The restriction $\Omega^n_{\mathbb{P}^n/A} \mid U_i$ is the trivial line bundle generated by

$$dt_{i0} \wedge \dots dt_{i,i-1} \wedge dt_{i,i+1} \dots dt_{in},$$

and on $U_i \cap U_j$ we have

$$t_{iv} = \frac{1}{t_{ij}} t_{jv},$$

and hence

$$dt_{iv} = \frac{1}{t_{ij}} dt_{jv} - \frac{t_{jv}}{t_{ji}^2} \cdot dt_{ji}.$$

Taking the highest exterior power we get

$$dt_{i0} \wedge \dots \wedge dt_{i,i-1} \wedge dt_{i,i+1} \wedge \dots \wedge dt_{in} = \left(\frac{dt_{j0}}{t_{ji}} - \frac{t_{j0}}{t_{ji}^2} dt_{ji} \right) \wedge \dots \wedge \left(\frac{dt_{jn}}{t_{ji}} - \frac{t_{jn}}{t_{ji}^2} dt_{ji} \right),$$

where on the right hand side we h ave to leave out the factor with index $\nu = i$, and where we observe that the factor $\nu = j$ simplifies to

$$-\frac{1}{t_{ji}^2}\ dt_{ji}.$$

Hence we see that the right hand side is

$$(-1)^{i-j}\frac{1}{t_{ji}^{n+1}}\ dt_{j0} \wedge \ldots \wedge dt_{j,j-1} \wedge dt_{j,j+1} \ldots \wedge dt_{jn}.$$

Therefore we see that the line bundle $\Omega_{\mathbb{P}^n_A/A}$ is obtained from the cocycle

$$(-1)^{i-j} t_{ij}^{n+1} = g_{ij}$$

in the sense of 1.3.1. But we may of course change our generator of $\Omega^n_{\mathbb{P}^n_A/A}$ over U_i by the sign $(-1)^i$ and the cocycle modifies into $g_{ij} = t_{ij}^{n+1}$ which then implies

$$\Omega^n_{\mathbb{P}^n_A/A} \simeq \mathcal{O}_{\mathbb{P}^n_A/A}(-n-1).$$

If we consider the dual sheaf of $\Omega_{\mathbb{P}^n/A}$ then we get the sheav of tangent vectors. We can pass to the associated vector bundle, and we get the tangent bundle

$$\begin{array}{ccc} T\mathbb{P}^n & \longrightarrow & \mathbb{P}^n \\ & \searrow \quad \swarrow & \\ & \mathrm{Spec}(A). & \end{array}$$

This tangent bundle has now a very simple geometric description. In 1.3.4 we consider the trivial bundle

$$\mathbb{P}^n \times_{\mathrm{Spec}(A)} \mathbb{A}^{n+1}_{\mathrm{Spec}(A)},$$

and we described the tautological line bundle in it. This means that we have

$$\begin{array}{ccc} V(\mathcal{O}_{\mathbb{P}^n}(-1)) & \longrightarrow & \mathbb{P}^n \times \mathbb{A}^{n+1} \\ & \searrow \quad \swarrow & \\ & \mathbb{P}^n, & \end{array}$$

and we can devide the trivial bundle by the tautological bundle. It is rather clear that the result is the tangent bundle $T\mathbb{P}^n$ (This has to be verified but from the geometric intuition it is clear.) Hence we get an exact sequence of vectorbundles

$$0 \to (\mathcal{O}_{\mathbb{P}^n}(-1)) \to \mathbb{P}^n \times \mathbb{A}^{n+1} \to T\mathbb{P}^n \to 0$$

over $\mathbb{P}^n$. If we now take the $(n+1)$-th exterior power of the trivial bundle

1.4 Seperated and proper morphisms:

The property of a morphism to be projective is a global property in contrast to the properties *smooth* and *flat* which can be checked locally.

There are two other global properties of morphism which are global in nature namely a morphism $X \to S$ can have the property to be *separated* and it can be *proper* and I think here is the right place to discuss them.

If we have a scheme $\pi : X \to S$ then we can form the fibered product $X \times_S X/S$ and the identity $\mathrm{id} : X \to X$ provides an element in $(\mathrm{id}, \mathrm{id}) \in \mathrm{Hom}_S(X, X) \times \mathrm{Hom}_S(X, X)$. By the universal property this is nothing else than an element $\Delta_X \to X \times_S X$.

The morphism π is called *separated* if Δ_X is a closed embedding.

It is not too hard to see that this property is local in the base, hence if we discuss this notion we may assume that $S = \mathrm{Spec}(A)$. Then

Lemma :*A morphism $\pi : X \to \mathrm{Spec}(A) = S$ is separated if for any two affine open subsets $U, V \subset X$ the intersection $U \cap V$ is affine again and $\mathcal{O}_X(U \cap V)$ is generated by the restriction of $\mathcal{O}_X(U)|U \cap V$ and $\mathcal{O}_X(V)|U \cap V$.*

This is rather clear because as I mentioned at the end of the section on fibered products we can cover $X \times_S X$ by open affine subsets $U \times_S V$. The morphism Δ is a closed embedding if for any such pair $U \cap V \to U \times_S V$ is a closed embedding, But then $U \cap V$ is a closed subscheme of the affine scheme $U \times_S V$ and its ring of regular functions $\mathcal{O}_X(U \cap V)$ is a quotient of $\mathcal{O}_X(U) \otimes_A \mathcal{O}_X(V)$.

We should notice that a morphism $\pi : X \to \mathrm{Spec}(A) = S$ is separated if we can find some covering $\mathfrak{U} = \{U_i\}_{i \in I}$ by affine subschemes such that for any pair of indices i, j the affine schemes U_i, U_j satisfy the condition in the lemma above.

We have given an explicit construction of $\mathbb{P}^n_S \times_S \mathbb{P}^m_S$ as a projective subscheme of some $\mathbb{P}^N_S$. We apply this to the case $n = m$. In this case we can either verify that the system of affine sets $\{U_i\}_{i=0,\ldots,n}$ satisfies the condition in the Lemma because the the elements in $\mathcal{O}_{\mathbb{P}^n}(U_i)$ which we have to invert to get $\mathcal{O}_{\mathbb{P}^n}(U_i \cap U_j)$ lie in $\mathcal{O}_{\mathbb{P}^n}(U_j)$. Hence we see that that $\mathbb{P}^n \to S$ is separated. We could also argue that we can describe the diagonal as a closed subscheme of the product just by adding the polynomials $Z_{ij} - Z_{ji}$ to the ideal which describes the product as a subscheme of $\mathbb{P}_S^{(n+1)(m+1)-1}$.

An open subscheme $y \subset X/S$ of a projective scheme X/S is called *quasiprojective* over S. It is now clear that quasiprojective schemes y/S are also separated.

Finally I want to introduce the notion of a *proper* morphism. A morphism $\pi : X \to S$ is called proper if it is separated and if it is universally closed. This means that for any base change $S' \to S$ and any closed subscheme $Z \subset X \times_S S'$ the image of Z under the projection $\pi \times_S S'$ is closed.

1.4.1 Theorem: *A projective morphism*

$$\pi : X \longrightarrow S$$

is always proper.

Proof: We have just seen that it is separated. We recall the definition of a projective morphism. It means that we have a diagram

$$\begin{array}{ccc} X & \overset{i}{\hookrightarrow} & \mathcal{P}^n_S \\ & \pi \searrow & \downarrow \\ & & S \end{array}$$

where i is a closed embedding. We have to show that for any base change $S' \to S$ and any closed subset $y \subset X \times_S S'$ the image of π is closed. Since this closed subset is also a closed subset in $\mathbb{P}^n_{S'}$ it suffices to show that for any closed subset $Z \subset \mathbb{P}^n_S$ its image in S is closed.

The question is local in the base, we assume that $S = \mathrm{Spec}(A)$. Now we know that we can describe Z as the set of zeroes of a homogeneous ideal $\tilde{I} \subset A[X_0, X_1, \dots, X_n]$ (see 1.1.2). Now we pick a point $s \in S$ which is not in the image of Z. We localize A at s, let $\mathfrak{m}_s$ be the maximal ideal of this local ring $A_{\mathfrak{m}_s}$. Now we have $Z_s = \emptyset$ and our Lemma 1.1.2.1 tells us that for a sufficiently large $d >> 0$ we have

$$\tilde{I}(d) \otimes A_{\mathfrak{m}_s}/\mathfrak{m}_s \simeq A_{\mathfrak{m}_s}/\mathfrak{m}_s[X_0, X_1, \dots, X_n](d)$$

or in other words

$$A_{\mathfrak{m}_s}[X_0, X_1, \dots, X_n](d)/\tilde{I}(d) \otimes A_{\mathfrak{m}_s}/\mathfrak{m}_s = (0).$$

By the lemma of Nakayama it follows that

$$A_{\mathfrak{m}_s}[X_0, \dots, X_n](d)/\tilde{I}(d) = 0.$$

We are basically through but since we passed to the localization at s, we need still a little finiteness argument. We know that any monomial $X_0^{r_0} \dots X_n^{r_n}$ of degree d can be written in the form

$$X^{r_0} \dots X_n^{r_n} = \sum G_{\mathfrak{u}} \cdot F_{\mathfrak{u}}$$

where $F_{\mathfrak{u}} \in \tilde{I}$ and $G_{\mathfrak{u}} \in A[X_0, \dots, X_n]$. The local ring $A_{\mathfrak{m}_s}$ is obtained from our original A by localization. But to write down the $G_{\mathfrak{u}}$ we need only finitely many denominators which means that we can replace A by a localization A_f with some f with $f(s) \neq 0$. Hence we see that already

$$\tilde{I}(d) \otimes A_f \simeq A_f[X_0, X_1 \dots X_n](d)$$

which proves that the image of Z does not intersect with the open neighborhood $\mathrm{Spec}(A_f)$ of s. This proves the theorem.

1.5 The valuative criteria

I want to state criteria for separatedness and properness which are extremely important, and which give a very intuitive idea of this notion. I will not give the proofs here, I refer to Grothendieck's book EGA, II, §7. This section of the book is relatively selfcontained, so it can be read directly.

Since we know that the question whether a morphism $\pi : X \to y$ is separated (resp. proper) is local in the base, we consider the following situation. Let $y = \mathrm{Spec}(A)$ where A is noetherian, let $\pi : X \to y$ be of finite type, i.e. we can cover X by affines $U_\alpha = \mathrm{Spec}(B_\alpha)$ where B_α is a finitely generated A-algebra.

Now we have:

Theorem 1.5.1: *Under our conditions above the morphism* $\pi : X \to \mathrm{Spec}(A)$ *is*

a) *separated if for any discrete valuation ring* C *with quotient field* K *and any morphism*

$$f : \mathrm{Spec}(C) \longrightarrow \mathrm{Spec}(A)$$

two $\mathrm{Spec}(C)$ *valued points*

$$\begin{array}{ccccc} \mathrm{Spec}(C) & & \rightrightarrows & & X \\ & f \searrow & & & \downarrow \\ & & \mathrm{Spec}(A) & & \end{array}$$

which become equal if we restrict them to $\mathrm{Spec}(K)$ *are already equal,*

b) *proper if for any such* C *and any* f *a* $\mathrm{Spec}(K)$ *valued point extends (uniquely) to a* $\mathrm{Spec}(C)$ *valued point.*

For the proof see EGA, II, §7, 7.2.3 and Remarks 7.2.4 and 7.3.8.

1.5.2. The valuative criterion for the projective space:

The following property of a projective space is an algebraic substitute for the fact that the complex projective space $\mathbb{P}^{(}\mathbb{C})$ is compact.

Let us start from a discrete valuation ring R. We want to study the $S = \mathrm{Spec}(R)$ valued points of $\mathbb{P}^n_S \to S$. Let π be a uniformizing element of R and let K be its quotient field. We have a diagram

$$\begin{array}{ccc} \mathbb{P}^n & & \\ \downarrow & & \\ \mathrm{Spec}(R) & \leftarrow & \mathrm{Spec}(K) \end{array}$$

The valuative criterion asserts that the K valued points and the R valued points are the same. But this is almost obvious. We have seen that (1.1.3.)

$$\mathbb{P}^n_S(\mathrm{Spec}(K)) = \mathbb{P}^n_S(K) = (K^{n+1} \setminus \{0\})/K^*.$$

But for any $x = (x_0, x_1 \ldots, x_n) \in K^{n+1} \setminus \{0\}$ we write $x_i = u_i \pi^{n_i}$ and pick the index i_0 for which n_{i_0} is minimal. Then $(\pi^{-n_{i_0}} x_0, \pi^{-n_{i_0}} x_1 \ldots, \pi^{-n_{i_0}} x_n)$ represents the same point, but now the coordinates are in R and one of the coordinates is a unit. Therefore this is a R-valued point and have have shown

If R is a discrete valuation ring with quotient field K then $\mathbb{P}^n_{\mathrm{Spec}(R)}(R) = \mathbb{P}^n_{\mathrm{Spec}(R)}(K)$

This is the valuative criterion for the projective space. This of course extends immediately to projective schemes over R.

This expresses in algebraic terms the "compactness" of the projective space. We should look at $\mathrm{Spec}(R)$ as a small disc, the generic point $(0) \in \mathrm{Spec}(R)$ corresponds to the disc minus the origin and the closed point (π) corresponds to the origin. The analogous object in function theory is a disc $D = \{z||z| < 1\}$ in the complex plane. Then we know from function theory that a meromorphic map $f : D\setminus\{0\} \to U_i(\mathbb{C}) \subset \mathbb{P}^n(\mathbb{C})$ extends in a unique way to a holomorphic map $f : D \to \mathbb{P}^n(\mathbb{C})$.

1.6. The construction Proj(R)

We have a construction of projective schemes starting from a graded algebra. Let A be an arbitrary ring and let R be a graded A-algebra. This means that R is an A-algebra and we have a direct sum decomposition

$$R = R_0 \oplus R_1 + \ldots \oplus R_n \oplus \ldots$$

such that the R_i are A-modules and $R_i R_j \subset R_{i+j}$. The identity element is in R_0 and the algebra homomorphism from A to R factor through R_0. We assume that $A \to R_0$ is surjective and we assume that R_1 is a finitely generated A-module which generates the A-algebra.

We define the set $\mathrm{Proj}(R)$ (the projective spectrum) to be the set of homogeneous prime ideals of R which do not contain R_1. If we have such a prime ideal $\mathfrak{p}$ we pick on $f \in R_1$ such that $f \notin \mathfrak{p}$ and we define

$$\mathrm{Proj}(R)_f = \{\mathfrak{q} \in \mathrm{Proj}(R) | f \notin \mathfrak{q}\}.$$

We form the ring

$$R_f^{(0)} = \{\frac{g}{f^n} | g \in R_n\}$$

i.e. the ring of elements of degree zero in the quotient ring R_f. It is very easy to see that

$$\mathrm{Proj}(R)_f = \mathrm{Spec}(R_f^{(0)}).$$

We use this to define a topology on $\mathrm{Proj}(R)$ and a structure of a ringed space. The open sets $V \subset \mathrm{Proj}(R)$ are those for which $V \cap \mathrm{Proj}(R)_f$ is open for all f.

We define the sheaf of regular function so that its restriction to the $\mathrm{Proj}(R)_f$ is simply the sheaf $\widetilde{R_g^{(0)}}$ on $\mathrm{Spec}(R_g^{(0)})$. Hence we have defined a scheme $(\mathrm{Proj}(R), \tilde{\mathcal{O}})$ for any such graded A-algebra.

If we take for instance the polynomial graded algebra

$$A[X_0, \ldots, X_n]$$

then $\mathrm{Proj}(A[X_0, \ldots, X_n], \mathcal{O}) = (\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n})$. If our A-algebra R is generated by elements $x_0, \ldots, x_n$ we have a homomorphism of graded A-algebras

$$\begin{tikzcd} A[X_0, \ldots, X_n] \arrow[rr] & & R_0[x_0, \ldots, x_n] \\ & A \arrow[ul] \arrow[ur] & \end{tikzcd}$$

and we see that $\mathrm{Proj}(R)$ is a closed subscheme

$$\begin{tikzcd} \mathrm{Proj}(R) \arrow[rr] \arrow[dr] & & \mathbb{P}^n_A \arrow[dl] \\ & \mathrm{Spec}(A) & \end{tikzcd} .$$

This generalizes easily to the case where we replace $\mathrm{Spec}(A)$ by an arbitrary scheme S and where R is a graded sheaf of O_S-algebras.

1.6.1 The assumption that R is generated by homogeneous elements in degree one is not essential. Let us assume that R is finitely generated by elements $x_0, x_1, \ldots, x_n$ which are of degree d_0, $d \geq 1$ respectibely. Then we can define

$$R_{x_i}^{(0)} = \left\{ \frac{f}{x_i} \mid \deg(f) = d_i m \right\} \subset R_{x_i},$$

and we can consider $\mathrm{Spec}(R_{x_i}^{(0)}$. Now we define $\mathrm{Proj}(R)$ as the space of homogeneous prime ideals $\mathfrak{p}$ which do not contain all the x_i. Then we have

$$\mathrm{Proj}(R) = \bigcup_{i=1}^{n} \mathrm{Spec}(R_{x_i}^{(0)}$$

and we can proceed as before.

1.7. Ample and very ample sheaves

Let $S = \mathrm{Spec}(A)$ be a noetherian scheme. We want to discuss certain constructions which allow us to show that a scheme $X \to S$ is projective or which provide projective embeddings of this scheme.

Let us go back briefly to the case of affine schemes. If we have an arbitrary scheme X and if we want to show that this scheme is affine we have only one chance: We consider the ring of global sections $B = \Gamma(X, \mathcal{O}_X)$ and we try to prove that $X \xrightarrow{\sim} \mathrm{Spec}(B)$.

This may not be so easy. Of course we always have a morphism

$$\pi : X \to \mathrm{Spec}(B)$$

which sends a point $x \in X$ to the prime ideal $\mathfrak{p}_x$ of functions which vanish at x and then we get an inclusion $h_x : B_{\mathfrak{p}_x} \hookrightarrow \mathcal{O}_{X,x}$. (See V.1.6.) Both maps have to be bijective. It is clear what kind of information we need if we want to be successful. We need that we have enough regular functions on X. So for instance we certainly need the information that the regular functions *separate points* , this means: If we have two point $x, y \in X$ and if y is not in the closure of $\{x\}$ (???) then we can find an $f \in B$ such that $f(x) = 0$ and $f(y) \neq 0$. This would tell us that π is injective but this is by far not good enough. For instance the example V.2.1.1 tells us that π does not need to be surjective. Hence we have to assume the surjectivity of π or to make some assumptions which allow to conclude that π is surjective.

I want to discuss some instances where our strategy will be successful.

First I want to discuss an affine situation. Let $X \to \mathrm{Spec}(A)$ be a separated scheme. I assume it can be covered by finitely many affine subschemes.

Let us assume that we have a A-subalgebra $B \subset \Gamma(X, \mathcal{O}_X)$ so that we get a diagram

$$\begin{tikzcd} X \arrow[rr, "\pi"] \arrow[dr] & & \mathrm{Spec}(B) = y \arrow[dl] \\ & \mathrm{Spec}(A) = S & \end{tikzcd}$$

Finally I assume that π is closed and any fibre $\pi^{-1}(y)$ is contained in an open affine subset of X.

Under these assumptions I claim: *the scheme X is affine.*

Proof: We pick a point $y \in Y$ and we choose an affine subset U which conyains the fibre $\pi^{-1}(y)$. We consider the complement of U in X and by our assumption we know that the image of this complement is a closed subset $Z \subset y$ which of course does not contain y. Hence we find a regular function $g \in B$ such that $g(y) \neq 0$ and $g|Z = 0$. Then we see that $\pi^{-1}(y_g) \subset U$ and this is an affine scheme because $\pi^{-1}(y_g) = U_g$. We conclude that any point $y \in y$ has an open neighborhood such that the inverse image of this neighborhood is affine. We can cover Y by affine subsets Y_{g_i} such that the $\pi^{-1}(Y_{g_i}) = X_i$ are affine. We get homomorphisms

$$\mathcal{O}_y(y_i) \longrightarrow \mathcal{O}_X(X_i).$$

The algebras $\mathcal{O}_y(y_i)$ are localizations B_{g_i} and since $B \to \Gamma(X, \mathcal{O}_X)$ is injective, we get inclusions

$$B_{g_i} \to \Gamma(X, \mathcal{O}_X)_{g_i} \longrightarrow \mathcal{O}_X(X_i).$$

We need that the last arrow is an isomorphism. This follows from a little

Lemma: *Let X be a separated scheme which can be covered by a finite set of affine schemes. Let $g \in \Gamma(X, \mathcal{O}_X)$ and let $h \in \mathcal{O}_X(X_g)$. Then we can find an integer $n > 0$ such that $g^n h$ is the restriction of an element in $\Gamma(X, \mathcal{O}_X)$ to X_g.*

Proof: Let us write $X = \bigcup_{i \in E} U_i$ with U_i affine and E finite. We consider the restriction h_i of h to $X_g \cap U_i = U_{i,g}$. Now it is clear by construction that $h_i = \frac{f_i}{g^{n_i}}$ for some $f_i \in \Gamma(U_i, \mathcal{O}_X)$ and hence we can find an index n, and functions $F_i \in \mathcal{O}_X(U_i)$ such that

$$g^n h = F_i \mid U_i \cap X_g.$$

Now we compare F_i and F_j on $U_i \cap U_j$. We have

$$F_i \mid U_i \cap U_j \cap X_g = F_j \mid U_i \cap U_j \cap X_g.$$

Since we assumed that X is separated, we can conclude that we can find an m such that

$$g^m F_i \mid U_i \cap U_j = g^m F_j \mid U_i \cap U_j$$

and then the $g^m F_i$ are restrictions of a function $F \in \Gamma(X, \mathcal{O}_X)$.

(If we want to avoid to assume that X is separated, we can assume instead that any open set has a finite covering by affines.)

Now it is rather clear that

$$X \longrightarrow \mathrm{Spec}(\Gamma(X, \mathcal{O}_X))$$

must be an isomorphism. It is obvious that $\Gamma(X, \mathcal{O}_X)$ separated points because B separates the points in y and the $\Gamma(X_i, \mathcal{O}_X)$ separate the points on X_i. It is also clear that for any $x \in X$ the map

$$h : \Gamma(X, \mathcal{O}_X)_{\mathfrak{p}_x} \longrightarrow \mathcal{O}_{X,x}$$

is an isomorphism.

It rests to prove the surjectivity. Again we exploit the assumption that π is closed. From this we get easily that π is surjective. If $x \in \mathrm{Spec}(\Gamma(X, \mathcal{O}_X))$ we consider its image in y and observe that $x \in X_i = \mathrm{Spec}(\Gamma(X_i, \mathcal{O}_X))$.

We have seen that the assumption that $\pi : X \to \mathrm{Spec}(B) = y$ is surjective is vital and it is not so clear how we can verify in a given case that it is fulfilled.

If we have a projective scheme over $X \to \mathrm{Spec}(A)$ this strategy cannot work since we do not have enough regular functions. The fundamental principle is to replace the regular functions by the sections in the positive powers of a suitable line bundle $\mathcal{L}$. If we have a section $s \in H^0(X, \mathcal{L})$ then we can trivialize $\mathcal{L}$ on the open set X_s where s does not vanish and hence we can view elements in $H^0(X, \mathcal{L}^{\otimes n})$ as regular functions on X_s and therefore a line bundle provides a tool to construct regular functions on certain open sets. For a certain class of line bundles - the so called *very ample* bundles - this method can be used to construct projective embeddings and I will explain this construction.

If we start from $X \to S$ and if we have an embedding

$$\begin{tikzcd} X \arrow[rr, hook, "i"] \arrow[dr, "\pi"'] & & \mathbb{P}^n_A \arrow[dl] \\ & \mathrm{Spec}(A) & \end{tikzcd}$$

then we have the line bundle $\mathcal{O}_{\mathbb{P}^n_A}(1)$ on $\mathbb{P}^n_A$ and we may consider its pullback $\mathcal{L} = i^*(\mathcal{O}_{\mathbb{P}^n}(1))$. Starting from this line we can reconstruct an embedding from X into a projective space.

Now our starting point will be a scheme $X \to S$ of finite type and let $\mathcal{L}$ be a line bundle on X. We consider the direct image sheaf $R_1 = \pi_*(\mathcal{L})$. Since we assume that S is affine $R_1 = \pi_*(\mathcal{L})$ is an A-module.

We put $R_0 = A$ and form the graded A-algebra

$$R' = R_0 \oplus R_1 \oplus \oplus_{n \geq 2} \pi_*(\mathcal{L}^{\otimes n})$$

and in this algebra we consider the subalgebra

$$R = R_0 \oplus R_1 \oplus R_2 \oplus \ldots$$

which is generated by R_1.

Now we make the assumption that our line bundle has *no base point.* This means that for any point $s \in S$ we can find an affine neighborhood V such that the sections

$$\pi_*(\mathcal{L})(V) = \mathcal{L}(\pi^{-1}(V))$$

generate the stalk of the line bundle at any point $x \in \pi^{-1}(s)$. This is the same as saying that for any x we can find a section $t \in H^0(\pi^{-1}(V), \mathcal{L})$ which does not vanish at x, i.e. it does not go to zero in $\mathcal{L} \otimes_{\mathcal{O}_x} k(x)$.

The assumption that the line bundle $\mathcal{L}$ over X has no base point gives us a morphism

$$\begin{tikzcd} j : X \arrow[rr] \arrow[dr] & & \mathrm{Proj}(R) \arrow[dl] \\ & \mathrm{Spec}(A). & \end{tikzcd}$$

To see this morphism, we pick a point x and a section $f \in H^0(\pi^{-1}(V), \mathcal{L})$ with $f(x) \neq 0$. Then the line bundle $\mathcal{L}$ and and its powers $\mathcal{L}^{\otimes n}$ become trivial over X_f, hence any section $t \in H^0(\pi^{-1}(V), \mathcal{L}^{\otimes n})$ can be written as

$$t = h \cdot f^n$$

with a unique $h \in \mathcal{O}_S(X_f)$.

This provides an injective homomorphism

$$R_f^{(0)} \to \mathcal{O}_X(X_f)$$

of A-algebras and hence a morphism

$$X_f \to \mathrm{Proj}(R_f^{(0)}).$$

These morphism match on the intersections $X_f \cap X_g = X_{fg}$ and hence we get the morphism

$$\begin{tikzcd} X \arrow[rr] \arrow[dr] & & \mathrm{Proj}(R) \arrow[dl] \\ & \mathrm{Spec}(A) & \end{tikzcd} .$$

It seems to me clear that this morphism must be surjective because we assumed that $X \to S$ should be projective.

Locally on the base S the sheaves $\pi_*(\mathcal{L})$ are finitely generated modules hence we may write them as quotients of free modules. We have seen that this gives us a morphism of graded algebras

$$A[X_0, \dots, X_n] \to R$$

and a diagram

$$\begin{tikzcd} X \arrow[rr] \arrow[dr] & & \mathrm{Proj}(R) \arrow[dl] \arrow[r, hook] & \mathbb{P}^n_S \\ & \mathrm{Spec}\, S & & \end{tikzcd}$$

hence we see that our base point free sheaf provides a morphism

$$\begin{tikzcd} X \arrow[rr] \arrow[dr] & & \mathbb{P}^n_S \arrow[dl] \\ & S & \end{tikzcd} .$$

We may ask under which circumstances this morphism has nice properties, for instance when is it an embedding?

If for instance we go back to the case where we start from an embedding

$$\begin{array}{ccc} i : X & \hookrightarrow & \mathbb{P}^n_S \\ & \searrow \quad \swarrow & \\ & S & \end{array}$$

and take for $\mathcal{L}$ the sheaf $\mathcal{L} = i_*(\mathcal{O}_{\mathbb{P}^n_S}(1))$ then our construction gives us back the given embedding.

If we start from a sheaf $\mathcal{L}$ on X/S and if our construction provides a projective embedding

$$X \hookrightarrow \mathrm{Proj}(R) = \mathrm{Proj}(A \oplus \pi_*(\mathcal{L}) \oplus \dots)$$

then the sheaf $\mathcal{L}$ is called *relatively very ample.* If we have a line bundle $\mathcal{L}$ for which a high power $\mathcal{L}^{\otimes n}$ provides such an embedding then it is called relatively ample.

We consider a proper scheme $\pi : X \to$ where S is locally noetherian, actually we aasume without loss of generality that $S = \mathrm{Spec}(A)$ where A is a noetherian ring.

We make the assumption that for any line bundle $\mathcal{L}$ on X the direct image is a coherent $\mathcal{O}_S$-module. This is not really an assumption, it follows from a general finiteness theorem by Grothendieck (see VIII, ???). This theorem will not be proved in full generality in this book, but in some applications where we need some implications of this result we can circumvent the use of this general theorem.

We start from a line bundle $\mathcal{L}$ on X. then $\pi_*(\mathcal{L})$ is coherent and hence by our assumption a finitely generated A-module $R_1 = H^0(X, \mathcal{L})$. We consider the graduated ring

$$R = A \oplus R_1 \oplus R_2 \oplus \dots$$

where $R_i \subset H^0(X, \mathcal{L}^{\otimes i})$ is the submodule generated by the products of elements in R_1.

We assume that R_1 has no base point, and we consider the morphism

$$\begin{array}{ccc} r_{\mathcal{L}} : X & \longrightarrow & \mathrm{Proj}(R) \\ & \searrow \quad \swarrow & \\ & S & \end{array}$$

If we pick a point $x \in X$ and an $s \in R_1$ such that $s(x) \neq 0$, then we can consider the localization

$$r_{\mathcal{L},s} : X_s \longrightarrow \mathrm{Spec}(R_s^{(0)}).$$

Now we can make several assumptions:

b1) For any choice of (x, s) this morphism has finite fibres.

b2) The algebra $R_s^{(0)}$ seperates points on X_s.

(This assumption is equivalent to the assumption that $H^0(X,\mathcal{L})$ separates points.)

The strongest assumption is

b3) The algebra $R_s^{(0)}$ separetes points and for any $x \in X_s$ the homomorphism

$$R_{s,y_x}^{(0)} \longrightarrow \mathcal{O}_{X,x}/\mathfrak{m}_{X,x}^2$$

is surjective.

Finally to make a technical assumption

c) Any finite subset in X lies in an affine open subscheme.

Theorem: *(i) Under the assumptions b1) and c) or under the assumption b2) the X_s are affine and the scheme $\pi : X \to S$ is projective and the morphism $r_{\mathcal{L}} : X \to \mathrm{Proj}(R)$ is finite.*

(ii) If we assume b3), then the morphism $r_{\mathcal{L}}$ is an isomorphism.

Proof: Since $X \to S$ is proper, this morphism is universally closed. If we consider the fibered product $X \times_S \mathrm{Proj}(R)$, then any closed subscheme $Z \subset X \times_S \mathrm{Proj}(R)$ has a closed image under the projection to $\mathrm{Proj}(R)$. We apply this to the graph $\Gamma_{r_{\mathcal{L}}}$ and conclude that the image of X in $\mathrm{Proj}(R)$ is closed. Since by construction the map between the rings

$$R_s^{(0)} \longrightarrow \Gamma(X_s, \mathcal{O}_{X_s})$$

is injective. We conclude that $r_{\mathcal{L}}$ must be surjective.

Then it is clear that our previous considerations imply that the X_s must be affine, this holds for the two cases. Hence we also know that $X_s = \mathrm{Spec}(\Gamma(X_s, \mathcal{O}_X))$.

The direct image $r_{\mathcal{L},*}(\mathcal{O}_X)$ (???, sehr dubioses Argument) is a coherent $\mathcal{O}_{\mathrm{Proj}(R)}$ module because $\pi_*(\mathcal{O}_X)$ is a coherent $\mathcal{O}_S$-module. Hence we see that the algebras $\Gamma(X_s, \mathcal{O}_{X_s})$ are finite over $R_{s_i}^{(0)}$.

Now we assume b3). Since the algebra $R_s^{(0)}$ separates points, we get that

$$X_s \longrightarrow \mathrm{Spec}(R_s^{(0)}$$

is always bijective. We have to show that (see ???)

$$R_{s,\mathfrak{p}_x}^{(0)} \longrightarrow \mathcal{O}_{X_s,x}$$

is always an isomorphism. We localize at $\mathfrak{p}_x$ and get the algebra

$$\Gamma(X_s, \mathcal{O}_X) \otimes R_{s,\mathfrak{p}_x}^{(0)}$$

which is finite over $R_{s,\mathfrak{p}_x}^{(0)}$ and hence its maximal ideals lie over $\mathfrak{p}_x$ (see Chap. CA, ???) and our assumption b3) implies that there is only one such maximal ideal. We get

$$\Gamma(X_s, \mathcal{O}_X) \otimes R_{s,\mathfrak{p}_x}^{(0)} \subset \mathcal{O}_{X,x}.$$

We reduce modulo the maximal ideal $\mathfrak{m}_{\mathfrak{p}_x} \subset R^{(0)}_{s.\mathfrak{p}_x}$ and consider the algebra

$$\Gamma(X_s, \mathcal{O}_X) \otimes R^{(0)}_{s,\mathfrak{p}_x}/\mathfrak{m}_{\mathfrak{p}_x},$$

this is a local finite algebra over the residue field. Its maximal $\overline{\mathfrak{m}}_x$ ideal is the image of $\mathfrak{m}_x$ and if the rank of this algebra over the residue field is > 1, then we must have $\overline{\mathfrak{m}}_x \neq \overline{\mathfrak{m}}_x^2$ and this contradicts the surjectivity of $\mathfrak{m}_{\mathfrak{p}_x} \to \mathfrak{m}_x/\mathfrak{m}_x^2$. We conclude that the rank must be one. It follows that

$$\Gamma(X_s, \mathcal{O}_X) \otimes R^{(0)}_{s,\mathfrak{p}_x}/\mathfrak{m}_{\mathfrak{p}_x} = \mathcal{O}_{X,x}/\mathfrak{m}_x$$

and Nakayama's lemma gives the desired result.

Now we assume b1) and c) or b2). We begin with the observation that we have a finite number of sections s_i such that

$$\bigcup X_{s_i} = X.$$

The algebras $\Gamma(X_{s_i}, \mathcal{O}_X)$ are finite over $R^{(0)}_{s_i}$ and ?????

Our problem is that $\mathcal{L}$ does not have enough sections. The sections of $\mathcal{L}$ produce the affine algebras $R^{(0)}_s$ which are too small, but the algebras $\Gamma(X_s, \mathcal{O}_X)$ are finite over $R^{(0)}_s$. Hence we can chose a set of generators $s_0, \dots, s_n$ of $H^0(X, \mathcal{L})$, and we can choose finite sets $f_{i,0} \dots f_{i,m_i}$ of generators of the $R^{(0)}_{s_i}$-module $\Gamma(X_{s_i}, \mathcal{O}_X)$.

Our strategy will be to construct a line bunde $\mathcal{H}$ on $Y = \mathrm{Proj}(R)$ which contains the trivial bundle $\mathcal{O}_Y$. We take the pullback of this bundle $r^*_{\mathcal{L}}(\mathcal{H})$ on X. Then we get $\mathcal{L} \hookrightarrow \mathcal{L} \otimes r^*_{\mathcal{L}}(\mathcal{H})$ on X and $H^0(X, \mathcal{L}) \hookrightarrow H^0(X, \mathcal{L} \otimes r^*_{\mathcal{L}}(\mathcal{H}))$. Now we can consider functions t/s_i with $t \in H^0(X, \mathcal{L} \otimes r^*_{\mathcal{L}}(\mathcal{H}))$, and we want that $\mathcal{H}$ is so big that this module of functions on X_{s_i} contains all the generators. If we succeed in doing this we know that the elements t/s_i with $t \in H^0(X, \mathcal{L} \otimes r^*_{\mathcal{L}}(\mathcal{H}))$ generate $\Gamma(X_{s_i}, \mathcal{O}_X)$.

To construct this bunde $\mathcal{X}$, we write the ring R as quotient of the graded polznomial ring $A[X_0, \dots, X_n]$ we send $x_i \to s_i$. This gives us an embedding

$$\begin{tikzcd} Y \arrow[rr, "i"] \arrow[dr] & & \mathbb{P}^n_A \arrow[dl] \\ & \mathrm{Spec}(A) & \end{tikzcd}$$

(see ???), and we recall that $i^*(\mathcal{O}_{\mathbb{P}^n_A}(1)) \simeq r_{\mathcal{L}_x}(\mathcal{L})$.

On $\mathbb{P}^n_A$ we have the sheaf

$$\mathcal{O}_{\mathbb{P}^n_A}(-X_0) \otimes \dots \otimes \mathcal{O}_{\mathbb{P}^n_A}(-X_n) = \mathcal{H}_0$$

(see ???), and the restriction of a high positive power of this sheaf to Y will be our $\mathcal{H}$.

We consider an element $f \in \Gamma(X_{s_i}, \mathcal{O}_X)$. If we restrict this element to an open subset $X_{s_i} \cap S_{s_j}$, then we can find an element $g_j \in \Gamma(X_{s_j}, \mathcal{O}_X)$ such that

$$f \mid X_i \cap X_j = \frac{g_j}{s_i^{d_j}}.$$

We can assume that all these d_j are equal to a fixed number d and now we can niew $\frac{1}{s_j^d}$ as a section in

$$r_{\mathcal{L}}^*(i^*(\mathcal{O}_{\mathbb{P}_A^n}(-dX_j))$$

(see ???). Hence we conclude that the section f can be extended to a section in

$$H^0(X_{s_i} \cup X_{s_j}, \mathcal{L} \otimes \mathcal{H}_0^{\otimes d}).$$

But if we compare two such extensions on the intersections, they become equal, maz be we have to pass to a still higher power of the bundle. But finally we conclude that f extends to a section in $H^0(X, \mathcal{L} \otimes \mathcal{H}^{\otimes d})$.

Čech cohomology of quasicoherent sheaves

We want to apply Čzech cohomology to the following situation. Let A be a ring and let us put $S = \mathrm{Spec}(A)$. We consider a separated scheme

$$\begin{array}{c} X \\ \downarrow \pi \\ S \end{array},$$

recall that this means that for any two affine open subschemes $U, V \subset X$ we have that $U \cap V$ is again affine and $\mathcal{O}_X(U \cap V)$ is generated by the images of $\mathcal{O}_X(U)$ and $\mathcal{O}_X(V)$ in $\mathcal{O}_X(U \cap V)$. We have seen in the section on projective spaces that $\mathbb{P}^n_S \to S$ is separated and from this we concluded that any quasiprojective scheme

$$\begin{array}{ccc} X & \hookrightarrow & \mathbb{P}^n_s \\ & & \\ \searrow & & \swarrow \\ & & \\ & S & \end{array}$$

is separated too.

We consider quasicoherent sheaves $\mathcal{F}', \mathcal{F}, \ldots$ on X. We choose a covering of our scheme X by affine subschemes: $\mathfrak{U} = \{U_i\}_{i \in I}$. We have defined the cohomology groups

$$H^q(X, \mathfrak{U}, \mathcal{F}) \qquad q = 0, 1, \ldots$$

using the Čzech-complex. We have seen that

$$H^0(X, \mathfrak{U}, \mathcal{F}) = \mathcal{F}(X).$$

These cohomology groups are A- modules. It is also clear that these cohomology groups depend functorially on $\mathcal{F}$, if we have an $\mathcal{O}_X$-module homomorphism of quasicoherent sheaves $\varphi : \mathcal{F} \to \mathcal{F}'$ then it induces a morphism on the cohomology groups.

We want to show

Theorem: *(1) The cohomology groups do not depend on the choice of the covering: Given another covering $\mathfrak{V}$ we can construct a functorial isomorphism*

$$H^\bullet(X, \mathfrak{U}, \mathcal{F}) \xrightarrow{\sim} H^\bullet(X, \mathfrak{V}, \mathcal{F}).$$

(2) Any short exact sequence of quasicoherent sheaves

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$$

gives rise to a long exact sequence of cohomology groups

$$\begin{array}{lllll} 0 & \to H^0(X, \mathfrak{U}, \mathcal{F}) & \to H^0(X, \mathfrak{U}, \mathcal{F}) & \to H^0(X, \mathfrak{U}, \mathcal{F}'') & \to \\ & \to H^1(X, \mathfrak{U}, \mathcal{F}') & \to H^1(X, \mathfrak{U}, \mathcal{F}) & \to H^1(X, \mathfrak{U}, \mathcal{F}'') & \to \\ & \to H^2(X, \mathfrak{U}, \mathcal{F}') & \to H^2(X, \mathfrak{U}, \mathcal{F}) & \to H^2(X, \mathfrak{U}, \mathcal{F}'') & \to \end{array}$$

which depends functorially on the sequence.

(3) If X is an affine scheme then

$$H^q(X, \mathfrak{U}, \mathcal{F}) = \begin{cases} \mathcal{F}(X) \ \textit{for } q = 0 \\ \qquad 0 \ \textit{for } q > 1. \end{cases}$$

Before I come to the proof I want to make comments:

It is clear that (3) is actually a consequence of (1) since we can use the covering $X = \cup_{\{i\}} U_i$ where $U_i = X$. For this covering (3) is clear if we use the alternating Čzech complex. But it is important for the understanding that one sees that (3) should be true for philosophical reasons. If we have $X = \mathrm{Spec}(B)$ then the quasicoherent sheaves on X are the sheaves $\tilde{M}$ obtained from the B-modules M. We have seen that we may recover M from $\tilde{M}$ just as the modules of global sections: $M = \tilde{M}(X)$ (see section on affine schemes). Now it is clear that an exact sequence of B-modules gives rise to an exact sequence of sheaves. This is the flatness of the localization. But if we have an exact sequence of quasicoherent sheaves

$$0 \to \tilde{M}' \to \tilde{M} \to \tilde{M}'' \to 0$$

then the sequence

$$0 \to M' \to M \to M'' \to 0$$

must also be exact. If it were not exact we would get an exact sequence

$$0 \to M' \to M \to M'' \to N \to 0$$

with $N = \tilde{N}(X)$ and hence $\tilde{M} \to \tilde{M}''$ would not be surjective.

The reason for introducing the cohomology groups is that they should control the deviation from exactness of the functor $\mathcal{F} \to H^0(X, \mathcal{F})$. (That is also the reason why these functors are called the right **derived** functors of H^0). But for X affine the functor $\mathcal{F} \to H^0(X, \mathcal{F})$ is exact on quasicoherent sheaves, we do not need the higher cohomology groups.

Now I come to the proof of the theorem. We start discussing (2). An exact sequence of quasicoherent sheaves $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ gives rise to a sequence of complexes

$$\begin{array}{ccccc}
 & 0 & 0 & 0 & \\
 & \downarrow & \downarrow & \downarrow & \\
0 & \to C^0(X, \mathfrak{U}, \mathcal{F}') & \to C^1(X, \mathfrak{U}, \mathcal{F}') & \to C^2(X, \mathfrak{U}, \mathcal{F}') & \to \\
 & \downarrow & \downarrow & \downarrow & \\
0 & \to C^0(X, \mathfrak{U}, \mathcal{F}) & \to C^1(X, \mathfrak{U}, \mathcal{F}) & \to C^2(X, \mathfrak{U}, \mathcal{F}) & \to \\
 & \downarrow & \downarrow & \downarrow & \\
0 & \to C^0(X, \mathfrak{U}, \mathcal{F}'') & \to C^1(X, \mathfrak{U}, \mathcal{F}'') & \to C^2(X, \mathfrak{U}, \mathcal{F}'') & \to \\
 & \downarrow & \downarrow & \downarrow & \\
 & 0 & 0 & 0 &
\end{array}$$

The first thing we check is that the vertical complexes are exact. To see this we recall that for any of the three sheaves

$$C^q(X,\mathfrak{U},\mathcal{G}) = \prod_{(i_0,\ldots,i_q)\in I^{q+1}} \mathcal{G}(U_{i_0,\ldots,i_q})$$

and hence we have to check the exactness of (we turn by 90°)

$$0 \to \mathcal{F}'(U_{i_0,\ldots,i_q}) \to \mathcal{F}(U_{i_0,\ldots,i_q}) \to \mathcal{F}''(U_{i_0,\ldots,i_q}) \to 0$$

for any $(i_0,\ldots,i_q) \in I^{q+1}$. But since our scheme is separated we know that $U_{i_0,\ldots,i_q} = U_{i_0} \cap \ldots \cap U_{i_q}$ is affine and now it follows from the comment above that the sequence is exact.

The horizontal lines compute the cohomology and the long exact cohomology sequence is now obtained by the standard argument in homological algebra.

Now it is clear that we can prove (3) if we are able to prove the *effaceability* of our functor. his means that for any quasicoherent sheaf $\mathcal{F}$ on an affine scheme X we can find an embedding

$$\mathcal{F} \hookrightarrow \mathcal{G}$$

into a quasicoherent sheaf $\mathcal{G}$ such that

$$H^q(X,\mathfrak{U},\mathcal{F}) \to H^q(X,\mathfrak{U},\mathcal{G})$$

is the zero map for all $q \geq 1$.

If we succeed in proving the existence of such a $\mathcal{G}$ for any $\mathcal{F}$ then we get an exact sequence

$$0 \to H^0(X,\mathfrak{U},\mathcal{F}) \to H^0(X,\mathfrak{U},\mathcal{G}) \to H^0(X,\mathfrak{U},\mathcal{G}/\mathcal{F}) \to H^1(X,\mathfrak{U},\mathcal{F}) \to 0.$$

The first three terms are modules of global sections of a quasicoherent sheaf on an affine scheme and this gives us that $H^0(X,\mathfrak{U},\mathcal{G}) \to H^0(X,\mathfrak{U},\mathcal{G}/\mathcal{F})$ is surjective and hence $H^1(X,\mathfrak{U},\mathcal{F}) = 0$ for any $\mathcal{F}$. Of course we can apply this to $\mathcal{G}/\mathcal{F}$ and we get also $H^1(X,\mathfrak{U},\mathcal{G}/\mathcal{F}) = 0$. This implies $H^2(X,\mathfrak{U},\mathcal{F}) = 0$ for any $\mathcal{F}$ and so on.

To construct such a $\mathcal{G}$ we start with a little

Lemma: *The scheme X is still affine and let us assume that in our covering $\mathfrak{U} = \{U_i\}_{i\in I}$ there is a member $y \in I$ with $U_y = X$. Then we have $H^q(X,\mathfrak{U},\mathcal{F}) = 0$ for all $q \geq 1$.*

Proof: Let us assume we have a cocycle

$$c = (c_{i_0,\ldots,i_q})_{(i_0,\ldots,i_q)} \in Z^q(X,\mathfrak{U},\mathcal{F}).$$

We construct a $b \in C^{q-1}(X,\mathfrak{U},\mathcal{F})$ by

$$b_{i_0,\ldots,i_{q-1}} = c_{y,i_0,\ldots,1_{q-1}}.$$

We have to observe that

$$U_{i_0,\ldots,i_{q-1}} = U_{y,i_0,\ldots,i_{q-1}}.$$

Then

$$\begin{aligned}(db)_{i_0,\ldots,i_q} &= \Sigma(-1)^\nu b_{i_0,\ldots,\hat{i}_\nu,\ldots,i_q}\\ &= \Sigma(-1)^\nu c_{y,i_0,\ldots,\hat{i}_\nu,\ldots,i_q} = -(dc)_{y,i_0,\ldots,i_q} + c_{i_0,\ldots,i_q} = c_{i_0,\ldots,i_q}.\end{aligned}$$

We come back to our problem to construct $\mathcal{G}$. For any of the open sets $U_i \subset X$ we construct a sheaf

$$\mathcal{F}_{U_i} : V \to \mathcal{F}(U_i \cap V)$$

on X. This sheaf is quasicoherent because if V is affine then it is obtained from the $\mathcal{O}_X(V)$-module $\mathcal{F}(U_i \cap V)$. We consider the cohomology groups

$$H^q(X, \mathfrak{U}, \mathcal{F}_{U_i})$$

and to do this we look at the Čzech-complex.

$$C^q(X, \mathfrak{U}, \mathcal{F}_{U_i}) = \prod_{(i_0,\ldots,i_q)} \mathcal{F}_{U_i}(U_{i_0} \cap \ldots \cap U_{i_q}).$$

But then it is clear that this is isomorphic to the Čzech complex

$$C^q(U_i, \mathfrak{U} \cap U_i, \mathcal{F})$$

which computes the cohomology

$$H^q(U_i, \mathfrak{U} \cap U_i, \mathcal{F}).$$

But now U_i itself is a member of the covering hence our lemma implies

$$H^q(U_i, \mathfrak{U} \cap U_i, \mathcal{F}) = 0$$

for $q \geq 1$.

We have a map

$$\mathcal{F} \to \mathcal{F}_{U_i}$$

induced by the restriction

$$\mathcal{F}(V) \to \mathcal{F}_{U_i}(V) = \mathcal{F}(U_i \cap V)$$

and hence we get an injection

$$\mathcal{F} \to \mathcal{F}_{\mathfrak{U}} = \prod_{i \in I} \mathcal{F}_{U_i}$$

and it is clear that $\mathcal{G} = \mathcal{F}_{\mathfrak{U}}$ does the job.

Hence we have proved (3). But then it is also clear how we can prove (1).

I want to investigate the cohomology of the sheaves $\mathcal{O}_{\mathbb{P}^n}(r)$ on $\mathbb{P}^n/S$ where $S = \operatorname{Spec}(A)$. Our main tool will be the exact sequences obtained from

$$\begin{array}{ccccccc} 0 & \to \mathcal{O}_{\mathbb{P}^n}(X_0) & \to \mathcal{O}_{\mathbb{P}^n} & \to \mathcal{O}_{\mathbb{P}^{n-1}} & \to 0 \\ & \| & & & \\ & \mathcal{O}_{\mathbb{P}^n}(-1) & & & \end{array}.$$

tensoring this with $\mathcal{O}_{\mathbb{P}^n}(r)$ we get

$$0 \to \mathcal{O}_{\mathbb{P}^n}(r-1) \to \mathcal{O}_{\mathbb{P}^n}(r) \to \mathcal{O}_{\mathbb{P}^{(n-1)}}(r) \to 0$$

for all integers r.

In the section on locally free sheaves we had the chain of inclusions

$$\mathcal{O}_{\mathbb{P}^n} \hookrightarrow \mathcal{O}_{\mathbb{P}^n}(1) \hookrightarrow \mathcal{O}_{\mathbb{P}^n}(2) \hookrightarrow \ldots \mathcal{O}_{\mathbb{P}^n}(d) \ldots$$

We consider the limit

$$\mathcal{O}_{\mathbb{P}^n}(\infty) = \lim_{d \to \infty} \mathcal{O}_{\mathbb{P}^n}(d).$$

We get

$$\mathcal{O}_{\mathbb{P}^n}(\infty)(\mathbb{P}^n) = \mathcal{O}_{\mathbb{P}^n}(U_0)$$

and hence we have for any open set $V \subset \mathbb{P}^n$

$$\mathcal{O}_{\mathbb{P}^n}(\infty)(V) = \mathcal{O}_{\mathbb{P}^n}(U_0 \cap V).$$

This implies that

$$H^i(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(\infty)) = 0 \quad \text{for all} \quad i > 0$$

simply because this cohomology is the same as $H^i(U_0, \mathcal{O}_{\mathbb{P}^n})$. (This argument is the same as the one to prove the effaceability of the Čzech cohomology).

We proceed by analyzing the information provided by the long exact cohomology sequence

$$\begin{array}{ccccc} 0 & \to H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(r-1)) & \to H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(r)) & \to H^0(\mathbb{P}^{n-1}, \mathcal{O}_{\mathbb{P}^{n-1}}(r)) & \xrightarrow{\delta} \\ & H^1(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(r-1)) & \to H^1(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(r)) & \to H^1(\mathbb{P}^{n-1}, \mathcal{O}_{\mathbb{P}^{n-1}}(r)) & \xrightarrow{\delta} \end{array}$$

We know that

$$H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(r)) \to H^0(\mathbb{P}^{n-1}, \mathcal{O}_{\mathbb{P}^{n-1}}(r))$$

is surjective unless we have $n = 1$ and $r < 0$. This follows from the Exercise (1) above. The map between the modules of homogeneous polynomials is given by putting $X_0 = 0$. Hence we get for $n = 1$

$$\begin{array}{ccccc} 0 & \to H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(r-1)) & \to H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(r)) & \to H^0(\mathbb{P}^0, \mathcal{O}_{\mathbb{P}^0}(r)) & \to \\ & H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(r-1)) & \to H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(r)) & \to 0 & \to \\ & H^2(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(r-1)) & \to H^2(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(r)) & \to 0 & \to . \end{array}$$

We turn our attention to H^i with $i \geq 2$. We see that for any r the map

$$H^i(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(r-1)) \to H^i(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(r))$$

is injective for $i \geq 2$. Since $H^i(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(\infty)) = 0$ for $i \geq 1$ a simple argument shows that this implies

$$H^2(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(r)) = H^3(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(r)) = \ldots = 0.$$

(Look at čzech cohomology and show that a class in $H^i(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(r))$ which vanishes in $H^i(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^n}(\infty))$ must already vanish in some $H^i(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(r+m))$ with $m >> 0$).

The same argument applies for $H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(r))$ as long as we have surjectivity in the first line, i.e. $r \geq 0$. Hence we conclude

$$H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(r)) = 0$$

for all $r \geq 0$. For $r = -1$ our sequence becomes

$$\begin{array}{ccccc} 0 \to 0 \to 0 \to & H^0(\mathbb{P}^0, \mathcal{O}_{\mathbb{P}^0}(-1)) & \to H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-2)) \to 0 \\ & \| & \\ & A & \end{array}$$

and hence

$$H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-2)) \simeq A.$$

For $r \leq -2$ we get a short exact sequence

$$\begin{array}{ccccc} & A & & & \\ & \| & & & \\ 0 & \to H^0(\mathbb{P}^0, \mathcal{O}_{\mathbb{P}^0}(r)) \xrightarrow{\delta} & H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(r-1)) \to H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(r)) \to & 0 \end{array}$$

and hence: For $r \leq -2$

$$H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(r)) = A^{(-r-1)}.$$

Now we state the theorem.

Theorem: *For $n \geq 1$ we have*

$$H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(r)) = \quad \text{Module of homogeneous polynomials of degree } r \text{ in } A[X_0, \ldots, X_n].$$

Especially if it is zero for $r < 0$

$$H^i(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(r)) = 0 \quad \text{for} \quad 0 < i < n \quad \text{or} \quad i > n.$$

$$H^n(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(-r)) = \begin{cases} 0 & \text{for} \quad r < n+1 \\ A & \text{for} \quad r = n+1 \\ A^m & \text{for} \quad r \geq n+1 \end{cases}$$

where $m =$ *rank of* $H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(-n-1+r))$.

We have proved this for $n = 1$. We get easily by induction that

$$H^i(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(r-1)) \to H^i(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(r))$$

is injective for all $0 < i \leq n-1$ and $i > n$. Hence by the same argument as above we find $H^i(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(r)) = 0$ for all $i \neq 0, n$. For $i = n$ we get

$$0 \to H^{n-1}(\mathbb{P}^{n-1}, \mathcal{O}_{\mathbb{P}^{n-1}}(r)) \to H^n(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(r-1)) \to H^n(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(r)) \to 0.$$

For $r > -n$ the first term is zero hence we get an isomorphism between the second and third term. This implies again by the limit argument

$$H^n(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(r)) = 0$$

for $r > -n$. For $r = -n$ we get an isomorphism

$$H^{n-1}(\mathbb{P}^{n-1}, \mathcal{O}_{\mathbb{P}^{n-1}}(-n)) \xrightarrow{\sim} H^n(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(-n-1-))$$

and for still smaller r we get

$$0 \to H^{n-1}(\mathbb{P}^{n-1}, \mathcal{O}_{\mathbb{P}^{n-1}}(-n-s)) \to H^n(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(-n-1-s)) \to H^n(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(-n-s)) \to 0$$

and one checks easily that the recursion for the rank as a function of s is the same as for $H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(s))$.

Cohomology of coherent sheaves.

From now on we assume that our groundring A is noetherian. A quasicoherent sheaf $\mathcal{F}$ on $\mathbb{P}^n$ is *coherent* if one of the following conditions is fulfilled:

(1) For all U_i the $\mathcal{O}_{\mathbb{P}^n}(U_i)$-module $\mathcal{F}(U_i)$ is finitely generated.

(2) For any affine open set $V \subset \mathbb{P}^n$ the $\mathcal{O}_{\mathbb{P}^n}(V)$- module $\mathcal{F}(V)$ is finitely generated.

(3) The stalks $\mathcal{F}_\mathfrak{p}$ are finitely generated $\mathcal{O}_{\mathbb{P}^n,\mathfrak{p}}$-modules for all points $\mathfrak{p} \in \mathbb{P}^n$.

We leave it as an exercise to prove that these conditions are equivalent.

In this section I want to prove the celebrated coherence theorem.

Theorem: *Let A be a noetherian ring and let $\mathcal{F}$ be a coherent sheaf on $\mathbb{P}^n/\operatorname{Spec}(A) = \mathbb{P}^n/S$. Then the A-modules $H^i(\mathbb{P}^n, \mathcal{F})$ are finitely generated. They are zero for $i > n$.*

The proof requires a series of steps. Again we start from the exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^n}(r-1) \to \mathcal{O}_{\mathbb{P}^n}(r) \to \mathcal{O}_{\mathbb{P}^{n-1}}(r) \to 0.$$

We will consider the tensor products $\mathcal{F}(r) = \mathcal{F} \otimes \mathcal{O}_{\mathbb{P}^n}(r)$, but we have to be aware of a minor technical complication: Since the tensor product is not an exact functor the sequence

$$0 \to \mathcal{O}_{\mathbb{P}^n}(X_0) \to \mathcal{O}_{\mathbb{P}^n} \to \mathcal{O}_{\mathbb{P}^{n-1}} \to 0$$

yields only an exact sequence

$$0 \to \mathcal{G} \to \mathcal{O}_{\mathbb{P}^n}(X_0) \otimes \mathcal{F} \to \mathcal{O}_{\mathbb{P}^n} \otimes \mathcal{F} \to \mathcal{O}_{\mathbb{P}^{n-1}} \otimes \mathcal{F} \to 0$$

where $\mathcal{G}$ is the kernel of the next arrow so the exactness is true by definition. Since $\mathcal{O}_{\mathbb{P}^n}(r)$ is locally free, we get for all $r \in \mathbb{Z}$an exact sequence

$$0 \to \mathcal{G}(r) \to \mathcal{O}_{\mathbb{P}^n}(r-1) \otimes \mathcal{F} \to \mathcal{O}_{\mathbb{P}^n}(r) \otimes \mathcal{F} \to \mathcal{O}_{\mathbb{P}^{n-1}} \otimes \mathcal{F} \to 0.$$

The first step in the proof of the coherence theorem will be the

Theorem: *Under the assumptions of the theorem above we can find an $r_0 > 0$ (depending on $\mathcal{F}$) such that for all $r \geq r_0$ $H^0(\mathbb{P}^n, \mathcal{F}(r))$ is a finitely generated A-module and $H^i(\mathbb{P}^n, \mathcal{F}(r)) = 0$ for all $i > 0$.*

The proof of this second theorem will basically be obtained by induction on n. We examine our sequence above. The first thing we have to do is to find out how the sheaf $\mathcal{G}$ looks like. Here we observe that locally we are in the following situation: We have an A-algebra B an element $f \in B$ and we consider the sequence of B modules

$$0 \to fB \to B \to B/fB \to 0.$$

This will be tensorized by a B-module N (the local $\mathcal{F}$) and we get

$$fB \otimes N \to N \to N/fN \to 0$$

and the first arrow is $f \otimes n \mapsto fn$ and hence the kernel is exactly the annihilator of f in N. This means that we get

$$0 \to \mathrm{Ann}_N(f) \to Bf \otimes N \to N \to N/fN \to 0.$$

Now it is clear that $\mathrm{Ann}_N(f)$ is an A/fA-module (which is of course finitely generated).

This implies for our sequence above that the kernel $\mathcal{G}$ is in fact a $\mathcal{O}_{\mathbb{P}^{n-1}}$-module sheaf, hence it is a coherent sheaf on the hyperplane at infinity.

We break the four term exact sequences into two pieces

$$\begin{array}{l} 0 \to \mathcal{G}(r) \to \mathcal{O}_{\mathbb{P}^n}(r-1) \otimes \mathcal{F} \to \mathcal{F}'(r-1) \to 0 \\ 0 \to \mathcal{F}'(r-1) \to \mathcal{O}_{\mathbb{P}^n}(r) \otimes \mathcal{F} \to \mathcal{O}_{\mathbb{P}^{n-1}}(r) \otimes \mathcal{F} \to 0. \end{array}$$

Assuming that we have proved our theorem of $\mathbb{P}^{n-1}$ we get

$$H^i(\mathbb{P}^n, \mathcal{G}(r)) = H^i(\mathbb{P}^{n-1}, \mathcal{G}(r)) = 0 \text{ for all } r >> 0$$

and

$$H^i(\mathbb{P}^{n-1}, \mathcal{O}_{\mathbb{P}^{n-1}}(r) \otimes \mathcal{F}) = 0 \text{ for all } r >> 0, i > 0$$

Hence we get
$$H^i(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(r-1)\otimes\mathcal{F}) = H^i(\mathbb{P}^n, \mathcal{F}'(r-1))$$
from the first exact sequence. We substitute this into the long exact sequence for the second short exact sequence of sheaves and get

$$\begin{array}{lll} H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(r)\otimes\mathcal{F}) & \to H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^{n-1}}(r)\otimes\mathcal{F}) & \to \\ H^1(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(r-1)\otimes\mathcal{F}) & \to H^1(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(r)\otimes\mathcal{F}) & \to 0 \\ \quad\vdots & & . \\ \to H^i(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(r-1)\otimes\mathcal{F}) & \xrightarrow{\sim} H^i(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(r)\otimes\mathcal{F}) & \to \end{array}$$

Now the limit argument saying that $H^i(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(\infty)\otimes\mathcal{F}) = 0$ implies that $H^i(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(r)\otimes\mathcal{F}) = 0$ for $i \geq 2$. But for $i = 1$ we still are not finished because we do not know whether the arrow in degree 0 is surjective.

To get the case $i = 1$ we use a different method.

We need the following

Lemma: *For any coherent sheaf $\mathcal{F}$ on $\mathbb{P}^n/S$ we can find an integer $r_0 > 0$ such that for all $r \geq r_0$ the sheaf $\mathcal{F}(r) = \mathcal{F}\otimes\mathcal{O}_{\mathbb{P}^n}(r)$ is generated by a finite number of global sections. In other words we can find $s_1, \ldots, s_N \in H^0(\mathbb{P}^n, \mathcal{F}(r))$ such that for any point $\mathfrak{p} \in \mathbb{P}^n$ the stalk $\mathcal{F}(r)_{\mathfrak{p}}$ is generated by the restrictions of these sections to this stalk as an $\mathcal{O}_{\mathbb{P}^n,\mathfrak{p}}$-module.*

Proof : We consider the restriction of $\mathcal{F}$ to the open sets U_i, we know that $\mathcal{F}|U_i = \tilde{\mathcal{F}(U_i)}$. The $\mathcal{O}_{\mathbb{P}^n}(U_i)$-modules $\mathcal{F}(U_i)$ are finitely generated. We write by $s_{i1}, \ldots, S_{i\nu_i}$ for these generators. They also generate the stalks in the points inside U_i. We have the embedding $\mathcal{O}_{\mathbb{P}^n} \subset \mathcal{O}_{\mathbb{P}^n}(-dH_i)$ and this induces a morphism among sheaves $\mathcal{F}\otimes\mathcal{O}_{\mathbb{P}^n} \to \mathcal{F}\otimes\mathcal{O}_{\mathbb{P}^n}(-dH_i)$. This morphism is an isomorphism on the stalks inside U_i. If we pass to the limit we see that $\mathcal{F}\otimes\mathcal{O}_{\mathbb{P}^n}(-\infty H_i)(\mathbb{P}^n) = \mathcal{F}\otimes\mathcal{O}_{\mathbb{P}^n}(U_i)$ and hence we see that all the sections $s_{i\nu}$ extend to sections in the limit.

But then these extensions must already lie in some $\mathcal{F}\otimes\mathcal{O}_{\mathbb{P}^n}(-rX_i)$ and since we have $\mathcal{O}_{\mathbb{P}^n}(-rX_i)\tilde{\to}\mathcal{F}(r) = \mathcal{F}\otimes\mathcal{O}_{\mathbb{P}^n}(r)$ we see that at least the stalks inside U_i can be generated by global sections in $\mathcal{F}(r)$. Since this is so for any i the lemma follows.

It follows from the Lemma that we get a morphism of sheaves
$$\mathcal{O}_{\mathbb{P}^n}^N \to \mathcal{F}(r)$$
simply by sending $(0, \ldots, 1, \ldots, 0) \to s_i, \ldots$. This morphism is surjective and has a kernel $\mathcal{G}$ which again is coherent. We get an exact sequence which we still can twist by $\mathcal{O}_{\mathbb{P}^n}(s)$ for an arbitrary integer s. Hence we obtain
$$0 \to \mathcal{G}(s) \to (\mathcal{O}_{\mathbb{P}^n}(s))^N \to \mathcal{F}(r+s) \to 0$$

which gives us for $s >> 0$

$$
\begin{array}{lllll}
0 & \to H^0(\mathbb{P}^n, \mathcal{G}(s)) & \to H^0(\mathbb{P}^n, \mathcal{O}^N_{\mathbb{P}^n}(s)) & \to H^0(\mathbb{P}^n, \mathcal{F}(r+s)) & \xrightarrow{\delta} \\
 & \to H^1(\mathbb{P}^n, \mathcal{G}(s)) & \to 0 & \to H^1(\mathbb{P}^n, \mathcal{F}(r+s)) & \xrightarrow{\delta} \\
 & H^2(\mathbb{P}^n, \mathcal{G}(s)) & \to 0 & & \\
 & \quad \vdots & & & \\
 & & 0 & H^i(\mathbb{P}^n, \mathcal{F}(r+r)) & \xrightarrow{\delta} \\
 & \to H^{i+1}(\mathbb{P}^n, \mathcal{G}(s)) & & & ,
\end{array}
$$

where all the boundary maps δ except the one in degree zero are isomorphisms.

We know already that $H^q(\mathbb{P}^n, \mathcal{G}(s)) = 0$ for $q \geq 2$ and $s >> 0$. Hence we get $H^i(\mathbb{P}^n, \mathcal{F}(r+s)) = 0$ for $i \geq 1$ and $r >> 0, s >> 0$ which of course implies the second assertion in the theorem.

To get the first assertion we simply observe that in our exact sequence above also $H^1(\mathbb{P}^n, \mathcal{G}(s)) = 0$ for $s >> 0$ and hence we get

$$H^0(\mathbb{P}^n, \mathcal{O}^N_{\mathbb{P}^n}(s)) \to H^0(\mathbb{P}^n, \mathcal{F}(r+s))$$

will be surjective for $r+s >> 0$. Then the assertion concerning the finite generation follows from our computation of the cohomology of the sheaves $\mathcal{O}_{\mathbb{P}^n}(s)$.

Now the coherence theorem becomes easy. Given our coherent sheaf $\mathcal{F}$ we produce an acyclic resolution. We embed $\mathcal{F} \to \mathcal{F}(r)$ such that $H^i(\mathbb{P}^n, \mathcal{F}(r)) = 0$ for $i \geq 0$ and $H^0(\mathbb{P}^n, \mathcal{F}(r))$ is a finitely generated A-module. We get an exact sequence

$$0 \to \mathcal{F} \to \mathcal{F}(r) \to \mathcal{F}_1 \to 0$$

and do the same thing to $\mathcal{F}_1$. This yields an exact sequence of coherent sheaves

$$0 \to \mathcal{F} \to \mathcal{F}(r) \to \mathcal{F}_1(r_1) \to \mathcal{F}_2(r_2) \to$$

which we write as

$$0 \to \mathcal{F} \to \mathcal{F}^1 \to \mathcal{F}^2 \to \ldots \mathcal{F}^m \to \ldots$$

where the $\mathcal{F}^i$ are acyclic and $H^0(\mathbb{P}^n, \mathcal{F}^i)$ is a finitely generated A-module. Hence we have constructed an acyclic resolution of $\mathcal{F}$ by coherent sheaves. Taking global sections over $\mathbb{P}^n$ we get a complex $H^0(\mathbb{P}^n, \mathcal{F}^\bullet)$ of A-modules. I refer to the *simple principle* in the section on homological algebra. This principle tells us that this complex computes the cohomology groups. Hence

$$H^\bullet(\mathbb{P}^n, \mathcal{F}) = H^\bullet(H^0(\mathbb{P}^n, \mathcal{F}^\bullet))$$

and since the individual members of the complex are finitely generated A-modules the theorem follows.

The reader should observe that the two parts of the proof have an overlap. In the first part we fail to say something in degree 1, in the second we cannot say what happens in large degrees ($\geq n+1$). If we would allow to work with the alternating čzech complex, we would see $H^i(\mathbb{P}^n, \mathcal{F}) = 0$ for $i > n$ directly.

Nevertheless, the above discussion shows that after we have defined the cohomology by using Čzech complexes we can very successfully treat them if we only use their properties.

Base change:

If we have a projective scheme

$$\begin{array}{ccc} X & \hookrightarrow & \mathbb{P}^n_S \\ & \searrow & \downarrow \\ & & S \end{array}$$

or more generally a coherent sheaf $\mathcal{F}$ on $\mathbb{P}^n_S$ then the intuitive idea is that this should be viewed as a family of projective schemes parametrized by the points in S or a family of sheaves parametrized by the points in $s \in S$.

To be a little bit more precise, for any morphism of the spectrum of a field $s : \operatorname{Spec}(K) \to S$ we have the base change

$$\begin{array}{ccc} X & \longleftarrow & X \times_S \operatorname{Spec}(K) = X_s \\ \downarrow & & \downarrow \\ S & \longleftarrow & \operatorname{Spec}(K) : s \end{array} .$$

The natural question is how certain data attached to the fibres X_s vary if we vary s. We may for instance ask how the dimensions of the irreducible components of the fibres X_s behave.

I give an example which shows that this behavior may be rather irregular. We consider the affine space over a field k, i.e. $S = \operatorname{Spec}(k[Y_1, \dots, Y_n])$. In the projective space $\mathbb{P}^{n-1}/S$ we define a subscheme Z by the system of homogeneous (even linear equations)

$$Y_i X_j - Y_j X_i = 0.$$

(The $X_1, \dots, X_n$ are the homogeneous variables for $\mathbb{P}^{n-1}$). Now it is clear that for any $s : \operatorname{Spec}(K) \to S$, i.e. a homomorphism $\varphi : k[Y_1, \dots, Y_n] \to K$, the fibre consists of a single point if $(\varphi(Y_1), \dots, \varphi(Y_n)) \neq (0)$, because in this case the solutions are given by the line determined by $(\varphi(Y_1), \dots, \varphi(Y_n))$. But if $(\varphi(Y_1), \dots, \varphi(Y_n)) = 0$ then suddenly the system of equations degenerates to $0X_i - 0X_j = 0$ and the fibre is the full $\mathbb{P}^{n-1}/\operatorname{Spec}(K)$.

General educational remark: This process is called "blowing up a point" and is of considerable importance in algebraic geometry. The intuitive meaning is that the point is replaced by the projective space attached to its tangent space. To see how useful this may be we give two examples.

(1) Let us assume we have the meromorphic function

$$f(X, Y) = \frac{X + Y}{X - Y}$$

in the function field of the affine plane $S = \operatorname{Spec}(k[X, Y])$ where k is a field and $\operatorname{char}(k) \neq 2$. It is regular outside the diagonal $X = Y$ and provides a morphism

$$f : \operatorname{Spec}(k[X, Y]) \backslash \Delta \to \operatorname{Spec}(k[T]) = \mathbb{A}^1_k .$$

We can extend this to a morphism into $\mathbb{P}^1_k$ if we send the points on the diagonal Δ to infinity as long as we avoid $(X,Y)=(0,0)$. But in the origin we do not know what to do.

But if we pull back the map to Z then a point in the fibre over $(0,0)$ tells us in "which direction" we approach $(0,0)$ and we can take a "limit". This allows us to extend f to a regular morphism $\tilde{f}: Z \to \mathbb{P}^1$.

(2) If we have a plane curve $C \subset \mathrm{Spec}(k[X,Y])$, for example something like

$$f(X,Y) = XY + X^7 + Y^{28} + \ldots = 0$$

then it has a so-called ordinary double point at $O=(0,0)$. The curve has two tangents at $O=(0,0)$. If we blow up the origin then the inverse image of our curve has two irreducible components, one is the "proper transform" $\tilde{C}$ and the other is the fibre over O. The curve $\tilde{C}$ meets the fibre in two points (the two tangents) and is now smooth in these two points. We have resolved the singularity.

End of the remark.

We also look at more general base changes which in our situation amounts to choosing an A-algebra A', i.e. $A \to A'$. Then we can consider

$$\mathbb{P}^n \times_{\mathrm{Spec}(A)} \mathrm{Spec}(A') = \mathbb{P}^n_{S'}$$

and we get a sheaf $\mathcal{F}'$ on $\mathbb{P}^n_{S'}$ which is the pull-back, in other words $\mathcal{F}'(U_i') = \mathcal{F}(U_i) \otimes_A A'$. This tells us that the Čzech-complex which computes $H^\bullet(\mathbb{P}^n_{S'}, \mathcal{F}')$ is the tensor product of $C^\bullet(\mathbb{P}^n, \mathfrak{U}, \mathcal{F}) \otimes_A A'$ and hence we get a map between the complexes

$$C^\bullet(\mathbb{P}^n_S, \mathfrak{U}, \mathcal{F}) \to C^\bullet(\mathbb{P}^n_{S'}, \mathfrak{U}, \mathcal{F}')$$

and therefore a map between the cohomology groups

$$H^\bullet(\mathbb{P}^n_S, \mathcal{F}) \to H^\bullet(\mathbb{P}^n_{S'}, \mathcal{F}').$$

Here we consider both sides as abelian groups, the left hand side carries the structure of A-modules, the right hand side cohomology groups are A'-modules. It is obvious that we get from the homomorphism above an A'-module homomorphism

$$H^\bullet(\mathbb{P}^n_S, \mathcal{F}) \otimes_A A' \to H^\bullet(\mathbb{P}^n_{S'}, \mathcal{F}).$$

This map between the cohomology groups is the *base change homomorphism.* In general we cannot say very much about this base change homomorphism.

We have to make some assumptions, and here we have two options

(a) The base change $A \to A'$ is flat.

(b) The sheaf $\mathcal{F}$ is flat over A (we say it is A-flat or $S = \mathrm{Spec}(A)$-flat).

We consider the case (a) first. It is much easier. Here we can start from a more general situation, we can drop the assumption that A noetherian and our sheaf $\mathcal{F}$ needs only to be quasi-coherent.

Actually we can even assume that we have a quasiprojective scheme

$$\begin{array}{ccc} X & \hookrightarrow & \mathbb{P}^n_S \\ \searrow \pi & & \swarrow \\ & S = \mathrm{Spec}(A) & \end{array}$$

and a quasicoherent sheaf $\mathcal{F}$ on X.

In this case we have

Theorem: *If $S \leftarrow S'$ is flat, i.e. if $A \to A'$ is flat then the base change homomorphism is an isomorphism.*

To see this we have to show if

$$K^\bullet : 0 \to K^0 \to K^1 \to \dots$$

is a complex of A-modules and if $A \to A'$ is flat then

$$H^\bullet(K^\bullet \otimes_A A') = H^\bullet(K^\bullet) \otimes_A A'.$$

This is clear since in a given degree i we have

$$\begin{array}{ccc} 0 \to Z^{i-1} \to K^{i-1} \to & B^i & \to 0 \\ & \downarrow & \\ & Z^i & \to 0 \\ & \downarrow & \\ & H^i(K^\bullet) & \\ & \downarrow & \\ & 0 & \end{array}.$$

Now since Z^i is the kernel of $d : K^i \to K^{i+1}$ we see that

$$Z^i \otimes_A A' = \ker(K^i \otimes_A A' \to K^{i+1} \otimes_A A')$$

and

$$B^i \otimes_A A' = \mathrm{im}(K^{i+1} \otimes_A A' \to K^i \otimes_A A')$$

because $A \to A'$ is flat. Hence

$$H^i(K^\bullet \otimes A') = Z^i(K^\bullet \otimes_A A')/B^i(K^\bullet \otimes A') = Z^i \otimes_A A'/B^i \otimes_A A'$$

and the last quotient is equal to $H^i(K^\bullet \otimes_A A')$ which is clear if we tensorize the above vertical sequence by A' over A.

This rather simple fact has an important consequence. We may consider the special case that $A' = A_f$ is a localization, in that case

$$S' = \mathrm{Spec}(A_f) \to S = \mathrm{Spec}(A)$$

is an affine open subscheme of S. We just have seen that

$$H^i(X \times_S \mathrm{Spec}(A_f), \mathcal{F}) \otimes A_f = H^i(\mathbb{P}^n_{\mathrm{Spec}(A))}\mathcal{F}) \otimes A_f$$

and this means that we can view these cohomology groups as sections of a sheaf

$$\widetilde{H^i(\mathbb{P}^n_{\mathrm{Spec}(A)}, \mathcal{F})}$$

on $\mathrm{Spec}(A)$. This allows us to drop the assumption that the base S is an affine scheme. For any scheme S we may consider sheaves $\mathcal{F}$ on quasicoherent

$$\begin{array}{c} X \\ \downarrow \pi \\ S \end{array}$$

where X over S is quasiprojective. We can cover S by open affine subschemes $S = \cup U_\nu$ and consider the cohomology groups

$$H^i(X \times_S U_\nu, \mathcal{F} \times U_\nu)$$

and from the sheaves $\widetilde{H^i(X \times_S U_\nu}, \mathcal{F} \times U_\nu)$ over U_ν. The remark above yields that the sheaves restricted to affine open subsets in $U_\nu \cap U_\mu$ are obtained by localization and hence they form a sheaf on S.

These sheaves get a new name. They are called the higher direct images of $\mathcal{F}$ under π and are denoted by

$$R^i\pi_*(\mathcal{F}).$$

Of course $R^0\pi_*(\mathcal{F}) = \pi_*(\mathcal{F})$.

Now we come to the case b). Here we are interested in situations where the fibres or the sheaves vary in a rather gentle manner. The technical concept is that of *flatness.* We will consider coherent sheaves $\mathcal{F}$ on $\mathbb{P}^n/S$ which are flat over S. I recall that this means that for all U_i the A-module $\mathcal{F}(U_i)$ is flat. We call it S-flat or A-flat. I want to prove

Theorem: *If $S = \mathrm{Spec}(A)$ and A noetherian and if $\mathcal{F}$ is an A-flat coherent sheaf on $\mathbb{P}^n$, then we can find an $r >> 0$ such that*

(1) $H^i(\mathbb{P}^n \times_A A', \mathcal{F} \times S'(r)) = 0$ for all $i > 0$ and all schemes $S' = \mathrm{Spec}(A') \to S = \mathrm{Spec}(A)$.

(2) The A module $H^0(\mathbb{P}^n, \mathcal{F}(r))$ is locally free.

(3) For any $A \to A'$ the base change homomorphism

$$H^0(\mathbb{P}^n \times_A A', \mathcal{F}'(r)) \xrightarrow{\sim} H^0(\mathbb{P}^n, \mathcal{F}(r)) \otimes_A A'$$

is an isomorphism.

To prove the theorem we use the same ideas as in the section before. We write $\mathcal{F}(r)$ for $r >> 0$ as a quotient of $\mathcal{O}_{\mathbb{P}^n}^N$ and we get an exact sequence

$$0 \to \mathcal{G} \to \mathcal{O}_{\mathbb{P}^n}^N \to \mathcal{F}(r) \to 0.$$

Since $\mathcal{F}$ is an A-flat sheaf and since $\mathcal{O}_{\mathbb{P}^n}$ is even free we see that also $\mathcal{G}$ must be an A-flat sheaf. Hence we see that this sequence stays exact after any base change

$$\mathrm{Spec}(A) = S \leftarrow \mathrm{Spec}(A') = S'$$

so we can write

$$0 \to \mathcal{G}' \to \mathcal{O}_{\mathbb{P}/S'}^N \to \mathcal{F}'(r) \to 0$$

for any $S \leftarrow S'$. This yields a long exact sequence in cohomology and since we have

$$H^i(\mathbb{P}^n_{S'}, \mathcal{O}^N_{\mathbb{P}^n_S}) = 0 \quad \text{for all} \quad i > 0$$

we get isomorphisms

$$H^i(\mathbb{P}^n_{S'}, \mathcal{F}'(r)) \simeq H^{i+1}(\mathbb{P}^n_{S'}\mathcal{G}').$$

Since we may also tensorize the sequence still by $\mathcal{O}_{\mathbb{P}^n}(r_1)$ we get

$$\begin{array}{llll} 0 & \to H^0(\mathbb{P}^n_{S'}, \mathcal{G}'(r_1)) & \to H^0(\mathbb{P}^n_{S'}, \mathcal{O}^N_{\mathbb{P}^n/S'}(r_1)) & \to H^0(\mathbb{P}^n_{S'}\mathcal{F}'(r+r_1)) \\ & \to H^1\mathbb{P}^n_{S'}\mathcal{G}'(r_1)) & \to 0 & \to H^1(\mathbb{P}^n_{S'}, \mathcal{F}'(r+r_1))\,. \\ & \to H^2\mathbb{P}^n_{S'}, \mathcal{G}'(r_1)) & \to 0 & \to \end{array}$$

Since $H^i(\mathbb{P}^n_{S'}, \mathcal{G}'(r_1)) = 0$ for $i > n$ we conclude that

$$H^i(\mathbb{P}^n_{S'}, \mathcal{F}'(r+r_1)) = 0 \quad \text{for all} \quad i \geq n$$

provided $r, r_1 >> 0$ and any S'. But since $\mathcal{G}$ is also flat we may apply this to $\mathcal{G}$ and get that for all $S \leftarrow S'$

$$H^i(\mathbb{P}^n_{S'}, \mathcal{G}(r_1+r_2)) = 0 \quad \text{for all} \quad i \geq n$$

provided $r_1, r_2 >> 0$ and this in turn gives

$$H^i(\mathbb{P}^n_{S'}, \mathcal{F}'(r+r_1+r_2)) = 0 \quad \text{for all} \quad i \geq n-1$$

if $r, r_1, r_2 >> 0$. Eventually we conclude by downwards induction

$$H^i(\mathbb{P}^n_{S'}, \mathcal{F}'(r)) = 0 \quad \text{for all} \quad i > 0 \quad \text{and all} \quad S \leftarrow S'$$

provided r is sufficiently large. Again we apply this to $\mathcal{G}$ and we get for all $S \leftarrow S'$ an exact sequence

$$0 \to H^0(\mathbb{P}^n_{S'}, \mathcal{G}'(r_1)) \to H^0(\mathbb{P}^n_{S'}, \mathcal{O}_{\mathbb{P}^n_{S'}}(r_1)) \to H^0(\mathbb{P}^n_{S'}, \mathcal{F}(r+r_1)) \to 0$$

provided r_1, r are large enough.

Now we want to prove the assertion (2). To do this we may assume that A is a local ring with maximal ideal $\mathfrak{m}$ and we consider the base change $A \to A/\mathfrak{m}$. $S' = \mathrm{Spec}(A/\mathfrak{m})$.

We consider the following diagram

$$\begin{array}{cccccccc}
 & & & A^M & & & & \\
 & & & \| & & & & \\
0 & \to H^0(\mathbb{P}^n_{S'}, \mathcal{G}(r_1)) & & \to H^0(\mathbb{P}_S, \mathcal{O}^N_{\mathbb{P}^n_S}(r_1)) & & \to H^0(\mathbb{P}^n_S, \mathcal{F}(r+r_1)) & & \to 0 \\
 & \downarrow & & \downarrow & & \downarrow & & \\
 & H^0(\mathbb{P}^n_S, \mathcal{G}(r_1)) \otimes A/\mathfrak{m} & & \to (A/\mathfrak{m})^M & & \to H^0(\mathbb{P}^n_S, \mathcal{F}(r+r_1)) \otimes A/\mathfrak{m} & & \to 0 \\
 & \downarrow \rho_{\mathcal{G}} & & \| & & \downarrow \rho_{\mathcal{F}} & & \\
0 & \to H^0(\mathbb{P}^n_{S'}, \mathcal{G}'(r_1)) & & \to H^0(\mathbb{P}^n_{S'}, \mathcal{O}_{\mathbb{P}^n_{S'}}(r_1))^N & & \to H^0(\mathbb{P}^n_{S'}, \mathcal{F}(r+r_1)) & & \to 0
\end{array} .$$

The map $\rho_{\mathcal{F}}$ in the lower right corner is surjective. This is true for any A-flat coherent sheaf provided $r + r_1$ is large enough. But then we conclude that also $\rho_{\mathcal{G}}$ in the lower left corner must be surjective at least if we enlarge r_1 still further. But then it follows that $\rho_{\mathcal{F}}$ is even bijective and hence also $\rho_{\mathcal{G}}$. (This proves assertion (3) for this special base change).

We summarize that for r_1, r large enough the two vertical arrows $\rho_{\mathcal{G}}$ and $\rho_{\mathcal{F}}$ are bijections. But if $\rho_{\mathcal{G}}$ is bijective we can conclude the supplemented sequence in the middle is exact

$$0 \to H^0(\mathbb{P}^n_{S'} \mathcal{G}'(r_1)) \otimes A/\mathfrak{m} \to (A/\mathfrak{m})^M \to H^0(\mathbb{P}^n_S, \mathcal{F}(r+r_1)) \otimes A/\mathfrak{m} \to 0$$

is exact. We pick a basis of $(A/\mathfrak{m})^M$

$$< y_1, \ldots, y_a, x_1, \ldots, x_b >$$

such that the y's form a basis for

$$H^0(\mathbb{P}^n_{S'} \mathcal{G}'(r_1)) \otimes A/\mathfrak{m}$$

and the x's map to a basis of $H^0(\mathbb{P}^n_{S'} \mathcal{F}'(r+r_1)) \otimes A/\mathfrak{m}$. Lifting this to a $H^0(\mathbb{P}^n_S, \mathcal{O}^N_{\mathbb{P}^n}(r_1))$ yields by Nakayama's lemma a system of $a+b = M$ generators and hence we get a basis of $H^0(\mathbb{P}^n_S, \mathcal{O}^N_{\mathbb{P}^n}(r_1))$. This proves that the modules of sections $H^0(\mathbb{P}^n_S, \mathcal{F}(r))$ are locally free A-modules and therefore we proved (2). To prove the assertion (3) we work with the same diagram as above but we replace $A' = A/\mathfrak{m}$ by an arbitrary A-algebra $A \to A'$. But now we know that the horizontal sequence in the middle is exact even if we supplement it by $0 \to$ on the left. This is so because $H^0(\mathbb{P}^n_S, \mathcal{F}(r))$ is locally free. Picking the same kind of basis as above for $(A')^M$, we see that the images of the x_i in $H^0(\mathbb{P}^n_S, \mathcal{F}(r_1+r))$ must form a basis.

The theorem has important consequences. We stick to our assumption that $S = \mathrm{Spec}(A)$ is local.

We look at the hyperplane H_0 at infinity and have the morphism of $\mathcal{O}_{\mathbb{P}^n}$-sheaves

$$\mathcal{F} \to \mathcal{F} \otimes \mathcal{O}_{\mathbb{P}^n}(-rH_0).$$

This needs not to be injective because the tensor product is taken over $\mathcal{O}_{\mathbb{P}^n}$ and our sheaf $\mathcal{F}$ is only A-flat. But if we restrict this morphism to U_0 it becomes an isomorphism hence it is injective on U_0. Therefore we get an embedding

$$\mathcal{F} \to \oplus_{i=0}^n \mathcal{F} \times \mathcal{O}_{\mathbb{P}^n}(-rH_i) = \mathcal{F}^0$$

and we know for $r >> 0$ that

a) $H^i(\mathbb{P}^n \times_S S', \mathcal{F}^0_x) = 0$ for all $i > 0$ and all $S \leftarrow S'$.

b) $H^0(\mathbb{P}^n \times S, \mathcal{F}^0)$ is a free A-module of finite rank.

c) $H^0(\mathbb{P}^n, \mathcal{F}^0) \otimes A' = H^0(\mathbb{P}^n \times_{\mathrm{Spec}(A)} \mathrm{Spec}(A'), \mathcal{F}^0 \otimes A')$.

Then we get an exact sequence of sheaves

$$0 \to \mathcal{F} \to \mathcal{F}^0 \to \mathcal{H} \to 0$$

where $\mathcal{H}$ is again flat. We apply the same process to $\mathcal{H}$ and get a resolution

$$0 \to \mathcal{F} \to \mathcal{F}^0 \to \mathcal{F}^1 \to \dots$$

where all the sheaves $\mathcal{F}^\nu$ satisfy a) b) c) above.

Remark: I want to point out that we used the same principle twice. In the section on homological algebra we explained that for the computation of the derived functors of a function we should start from an acyclic resolution. In general such resolutions are very big and for instance injective modules or sheaves are very big and their modules of sections don't have a chance to be finite in some sense. But in the proof of the coherence theorem we showed that a coherent sheaf $\mathcal{F}$ has an acyclic resolution

$$0 \to \mathcal{F} \to \mathcal{F}^0 \to \mathcal{F}^1 \to$$

by *coherent* sheaves and we showed that the modules of global sections are finitely generated A-modules.

Here we showed that for a coherent flat sheaf $\mathcal{F}$ we have an acyclic resolution where the $\mathcal{F}^i$ satisfy even more requirements.

The appropriate way of formulating this is provided by the language of *derived categories*.

The derived category here will be the category of complexes of quasicoherent sheaves. We rewrite our resolutions

$$\begin{array}{ccccc} & 0 & & & \\ & \downarrow & & & \\ 0 & \to \mathcal{F} & \to 0 & \to 0 & \to \\ & \downarrow & \downarrow & \downarrow & \\ 0 & \to \mathcal{F}^0 & \to \mathcal{F}^1 & \to \mathcal{F}^2 & \to \end{array}$$

we have here a morphism between the two horizontal complexes. These complexes of sheaves have cohomology groups which are again sheaves we compute the cohomology stalkwise.

The fact that we have a resolution is equivalent to saying that this map between complexes induces an isomorphism in cohomology. But then this map is by the construction of the derived category on isomorphisms, this is built into the construction. Hence we can say:

In the derived category of quasicoherent sheaves a coherent (resp. coherent and A-flat) sheaf is isomorphic to

$$0 \to \mathcal{F}^0 \to \mathcal{F}^1 \to \dots$$

where the $\mathcal{F}^i$ are acyclic (resp. acyclic after any base change) and have some further properties which I stated above.

Let us call this a flat, universally acyclic resolution. We know that for any $S \leftarrow S'$ the complex

$$0 \to H^0(\mathbb{P}^n_{S'}, \mathcal{F}'^0) \to H^0(\mathbb{P}_{S'}, \mathcal{F}'^1) \to \dots$$

computes the cohomology groups $H^i(\mathbb{P}^n_{S'}\mathcal{F}')$, and we also know that

$$H^0(\mathbb{P}^n_{S'}, \mathcal{F}') = H^0(\mathbb{P}^n_S, \mathcal{F}) \otimes_A A'.$$

Hence we see that the above flat universally acyclic resolution gives us a complex of finitely generated free A-modules

$$K^\bullet = 0 \to H^0(\mathbb{P}^n_S, \mathcal{F}^0) \to H^0(\mathbb{P}^n_S, \mathcal{F}^1) \to \dots$$

such that for any $S \leftarrow S'$, i.e. $A \to A'$ the complex

$$K^\bullet \otimes_A A'$$

computes the cohomology groups

$$H^i(\mathbb{P}^n_{S'}, \mathcal{F}').$$

From this we can obtain all the information about the behavior of the cohomology if we vary the fibre by simple linear algebra.

Let us assume that A is local with maximal ideal $\mathfrak{m}$. Let $\mathfrak{p}$ be another prime ideal in A let $A_\mathfrak{p}$ be localization at $\mathfrak{p}$ and $\mathfrak{m}_\mathfrak{p}$ the maximal ideal. We have

$$\begin{array}{ccc} A & \longrightarrow & A_\mathfrak{p} \\ \downarrow & & \\ A/\mathfrak{m} & & . \end{array}$$

We assume that we have a flat sheaf $\mathcal{F}$ on $\mathbb{P}^n_{\mathrm{Spec}(A)}$ and we have the complex

$$0 \to H^0(\mathbb{P}^n_{\mathrm{Spec}(A)}, \mathcal{F}^0) \to H^0(\mathbb{P}^n_{\mathrm{Spec}(A)}, \mathcal{F}^1) \dots .$$

If we tensorise this complex by $A/\mathfrak{m}$ we get the cohomology of

$$H^i(\mathbb{P}^n_{\mathrm{Spec}(A/\mathfrak{m})}, \mathcal{F} \otimes A/\mathfrak{m})$$

and if we tensorise by $A_\mathfrak{p}/\mathfrak{m}_\mathfrak{p}$ we get the cohomology $H^i(\mathbb{P}^n_{\mathrm{Spec}(A_\mathfrak{p})/\mathfrak{m}_\mathfrak{p}}\mathcal{F} \otimes A_\mathfrak{p}/\mathfrak{m}_\mathfrak{p})$.

Now we have to remember some linear algebra. If we have a map

$$\varphi : A^{M_1} \to A^{M_2}$$

and reduce it $\bmod \mathfrak{m}$ then we can find a basis

$$x_1, \dots, x_a, y_1, \dots, y_b$$

of $(A/\mathfrak{m})^{M_1}$ such that the $\varphi(x_i)$ go to zero and the $\varphi(y_i)$ are linearly independent in $(A/\mathfrak{m})^{M_2}$.

We lift this to a basis

$$\tilde{x}_1, \dots, \tilde{x}_a, \tilde{y}_1, \dots, \tilde{y}_b$$

in A^{M_1} and the $\varphi(\tilde{y}_1), \dots, \varphi(\tilde{y}_b)$ will form a part of a basis in A^{M_2}. With respect to these two bases we write the matrix for φ and get

$$C = \mathrm{Mat}(\varphi) = \begin{pmatrix} 1 & & & \\ & \ddots & & \\ & & 1 & \\ * & * & * & B \end{pmatrix}$$

where B is a matrix with coefficients in $\mathfrak{m}$, the $*$ are in $\mathfrak{m}$, and all other entries are zero.

Now this matrix has rank equal to b if we map it to $A/\mathfrak{m}$ but its rank may become bigger if we map it to $A_{\mathfrak{p}/\mathfrak{m}}$.

Actually we have for any homomorphism of A into a field K

$$\Psi : A \to K$$

$$\mathrm{Rank}(\Psi(C)) = b + \mathrm{Rank}(\Psi(B)).$$

This implies

$$\begin{aligned} \dim_{A_\mathfrak{p}/\mathfrak{m}_\mathfrak{p}} \ker((A_\mathfrak{p}/\mathfrak{m}_\mathfrak{p})^{M_1} \to (A_\mathfrak{p}/\mathfrak{m}_\mathfrak{p})^{M_2})) \leq \dim_{A/\mathfrak{m}} \ker((A/\mathfrak{m})^{M_1} \to (A/\mathfrak{m})^{M_2}) \\ \dim_{A_\mathfrak{p}/\mathfrak{m}_\mathfrak{p}} \mathrm{Im}((A_\mathfrak{p}/\mathfrak{m}_\mathfrak{p})^{M_1} \to (A_\mathfrak{p}/\mathfrak{m}_\mathfrak{p})^{M_2}) \geq \dim_{A/\mathfrak{m}} \mathrm{Im}((A/\mathfrak{m})^{M_1} \to (A/\mathfrak{m})^{M_2}) \end{aligned} .$$

This applied to our complex yields that the dimensions of the boundaries drop and the dimensions of the cycles go up if we specialize $\mathfrak{p} \to \mathfrak{m}$. (See Chap. on aff. schemes, sect. Zariski topology.)

Hence we get

Semicontinuity:

If $\mathcal{F}$ is a coherent A-flat sheaf on $\mathbb{P}^n(\mathrm{Spec}(A))$ and if $\mathfrak{m} \supset \mathfrak{p}$ are two prime ideals then

$$\dim_{A/\mathfrak{m}} H^i(\mathbb{P}^n_{\mathrm{Spec}(A/\mathfrak{m})}, \mathcal{F} \otimes A/\mathfrak{m}) \geq \dim_{A_\mathfrak{p}/\mathfrak{m}_\mathfrak{p}} H^i(\mathbb{P}^n_{\mathrm{Spec}(A_\mathfrak{p}/\mathfrak{m}_\mathfrak{p})}, \mathcal{F} \otimes A_\mathfrak{p}/\mathfrak{m}_\mathfrak{p}).$$

Usually one finds this in a slightly different form in the literature. We consider an arbitrary noetherian base scheme S and a S-flat coherent sheaf $\mathcal{F}$ on a projective scheme $\pi : X \to S$. Then we consider the sheaves $R^i\pi_*(\mathcal{F})$ on S. We consider the functions $s \to \dim_{k(s)} H^i(X_s, \mathcal{F}_s)$. Then this function is upper semicontinous as a function in s with respect to the Zariski topology on S. I find it easier to remember that the function may jump up if we specialize from a point $s \to s'$. But this formulation is obviously the same as the one above because the assertion is local on S.

Invariance of the Euler characteristic

Let $\pi : X \to S$ be a projective scheme over a noetherian base scheme S and let $\mathcal{F}$ be a S flat coherent sheaf. Then the function

$$s \mapsto \chi(X_s, \mathcal{F}) = \sum_{i=0}^{d} (-1)^i \dim_{k(s)} H^i(X_s, \mathcal{F}_s)$$

is constant. Here d is large enough such that the cohomology groups vanish in degrees beyond d.

Local constancy:

We say that the direct image $R^i\pi_*(\mathcal{F})$ is *locally constant* at $s \in S$ if this sheaf is locally free at s and if furthermore the base change homomorphism $R^i\pi_*(\mathcal{F})_s \otimes_{\mathcal{O}_{X,s}} k(s) \xrightarrow{\sim} H^i(X_s, \mathcal{F}_s)$ is an isomorphism. To see under which conditions this will be the case we switch to our previous notations, we assume that $S = \mathrm{Spec}(A)$ where A is noetherian local and $s = \mathfrak{m}$. Of course we should assume that $\mathcal{F}$ is flat over A then we are in our previous situation since also assume that π is projective.

It is quite clear what we have to do if we want to prove local constancy. We have to look at the complex

$$H^{i-1}(X, \mathcal{F}^{i-1}) \xrightarrow{d^{i-1}} H^i(X, \mathcal{F}^i) \xrightarrow{d^i} H^{i+1}(X, \mathcal{F}^{i+1})$$

which is obtained from an universally acyclic resolution. We apply our linear algebra arguments and we see that we get local constancy if for the boundary maps d^{i-1}, d^i the $\mathfrak{m}$ valued matrices B_{i-1}, B_i are actually equal to zero. If we assume that the ring A is reduced we can conclude that these matrices must be zero if we have

$$\dim_{A/\mathfrak{m}} H^i(\mathbb{P}^n_{\mathrm{Spec}(A/\mathfrak{m})}, \mathcal{F} \otimes A/\mathfrak{m}) = \dim_{A_\mathfrak{p}/\mathfrak{m}_\mathfrak{p}} H^i(\mathbb{P}^n_{\mathrm{Spec}(A_\mathfrak{p}/\mathfrak{m}_\mathfrak{p})}, \mathcal{F} \otimes A_\mathfrak{p}/\mathfrak{m}_\mathfrak{p})$$

for any minimal prime ideal $\mathfrak{p}$. This is clear because if one of these B_j is not zero then it is not zero in some residue field $A_\mathfrak{p}/\mathfrak{m}_\mathfrak{p}$ for a minimal prime ideal $\mathfrak{p}$ and the the dimension of the cohomology group in the fibre over this $\mathfrak{p}$ must necessarily drop because either the rank of the modules of boundaries will go up or the rank of the group of cycles drops.

Hence we see that we have local constancy for $R^i\pi_*(\mathcal{F})$ in a point $s = \mathfrak{m}$ if the local ring at s is reduced and if the rank of the cohomology stays constant in an open neighborhood of s.

Chapter IX

Curves and the theorem of Riemann Roch

A curve over field k is a scheme C/k of finite type over k all whose irreducible components are of dimension 1. In other words our scheme has a finite covering by affine schemes $U_i = \mathrm{Spec}(\mathcal{O}_C(U_i))$ where the $\mathcal{O}_C(U_i)$ are finitely generated k-algebras and $\dim \mathcal{O}_C(U_i) = 1$.

To give simple examples we can take a non zero homogeneous polynomial of degree d say

$$f(x,y,z) = \sum a_{\nu_1\nu_2\nu_2} x^{\nu_1} y^{\nu_2} z^{\nu_3} \quad a_{\nu_1\nu_2\nu_3} \in k, \nu_1 + \nu_2 + \nu_3 = d.$$

Then the ideal (f) defines a curve

$$\begin{array}{ccc} C = V(f) & \hookrightarrow & \mathbb{P}^2 \\ & \searrow \quad \swarrow & \\ & \mathrm{Spec}(k) & \end{array}$$

To see that this defines a curve we restrict to one of the affine planes which cover $\mathbb{P}^2$. This means we put one of the variables equal to 1 and divide the polynomial ring in the remaining 2 variables by the resulting polynomial. We have to show that the irreducible components of C intersected with this plane are of dimension one. This intersection is non empty if and only if this resulting polynomial is not constant. But then the irreducible components correspond to the minimal prime ideals containing the polynomial and these are of height one (Krull Hauptidealsatz) and hence the quotient ring by this ideal is of dimension one

Polynomial rings over fields are factorial (See ???). Therefore a principal ideal defined by an irreducible polynomial $F(X_1, \dots, X_n) \in k[X_1, \dots, X_n]$ is a prime ideal. In our case here we see that an irreducible polynomial f defines an irreducible curve C. It may happen that f is irreducible but can be factorized over a bigger field. If for instance we take $k = \mathbb{Q}$ then $(x + y - \sqrt{2}z)(x + y + \sqrt{2}z) = (x + y)^2 - 2z^2$ is irreducible over $\mathbb{Q}$ but factors over $\mathbb{Q}[\sqrt{2}]$. If this occurs the curve is irreducible but not absolutely irreducible. In our example the curve $C \otimes_{\mathbb{Q}} \bar{\mathbb{Q}}$ is the union of two lines which are interchanged by the Galois group. They intersect in one point, which is not smooth.

It is also possible that a curve C/k is smooth, irreducible but not absolutely irreducible. This happens if we start from an absolutely irreducible curve C/k. Let k_0 be a subfield of k such that k/k_0 is a finite separable non trivial extension. Then we have $C \to \mathrm{Spec}(k) \to \mathrm{Spec}(k_0)$ and we can view C/k_0 as a curve over k_0. This curve will still be smooth, but it is not absolutely irreducible. To see this we simply look at $C \times_{k_0} \bar{k}$ and verify that this is a disjoint union of $[k : k_0]$ smooth curves.

There is a slightly different way of looking at this phenomenon. If we have a smooth irreducible curve C/k and an element $f \in \mathcal{O}_C(U)$ which is not in k it can happen that this element is still algebraic over k. It generates a finite extension L/k contained in $\mathcal{O}_C(U)$ and now $\mathcal{O}_C(U) \otimes_k \bar{k}$ must have zero divisors and hence our curve is not absolutely irreducible.

We call such elements f which are algebraic over k the *constant elements* or simply the *constants* (???). They form a finite extension of the ground field. A smooth irreducible curve is absolutely irreducible if and only if the field of constants is equal to k.

Exercise:

1) We return to the case of a curve defined by a homogeneous polynomial f as above. Use the results in the previous section (???) to compute the cohomology groups $H^\bullet(C,\mathcal{O}_C)$ and show especially that $H^0(C,\mathcal{O}_C)$ is a k-vector space of dimension one. Conclude that C cannot decompose into disjoint closed subschemes and show that C must be absolutely irreducible if it is smooth.

Remark: We can construct a universal curve of degree d. If we consider forms of a fixed degree and we remove the trivial form, then we see that we can view the coefficients $(a_{\nu_1\nu_2\nu_3})$ as the k-valued point of a projective scheme $\mathcal{S}_d = \mathrm{Proj}(\mathbb{Z}[\dots,a_{\nu_1\nu_2\nu_3},\dots])$ where we consider the $a_{\nu_1\nu_2\nu_3}$ as independent variables in degree one. We can define the universal curve of degree d which is a subscheme of

$$\begin{array}{ccc}\mathcal{C} & \hookrightarrow & \mathcal{S}_d \times_{\mathrm{Spec}(\mathbb{Z})} \mathbb{P}^2 \\ & \searrow \quad \swarrow & \\ & \mathcal{S}_d & \end{array}$$

Exercise 2:

a) Prove that there is a closed subscheme $\mathcal{S}_d^{\mathrm{sing}} \subset \mathcal{S}_d$ such that the universal curve restricted to the complement $\mathcal{S}_d^{\mathrm{smooth}} := \mathcal{S}_d \setminus \mathcal{S}_d^{\mathrm{sing}}$ is smooth and that all the fibres over $\mathcal{S}^{\mathrm{sing}}$ are singular.

b) Compute $\mathcal{S}_d^{\mathrm{smooth}}(\mathbb{Z})$ for $d=1, d=2$. (It is a deep theorem that $\mathcal{S}_d^{\mathrm{smooth}}(\mathbb{Z}) = \emptyset$ for $d \geq 3$)

Proposition: *For a smooth curve C/k the local rings $\mathcal{O}_{C,\mathfrak{p}}$ at a closed point $\mathfrak{p}$ is a discrete valuation ring. For any non empty affine open set $U \subset C$ the ring $\mathcal{O}_C(U)$ is a Dedekind ring.*

Proof: Let P be a geometric point, let $\mathfrak{p}$ be the corresponding point on C. We can choose a local parameter f at P. Recall that this is an element in $f \in \mathcal{O}_{C,\mathfrak{p}}$ such that the differential df generates $\Omega^1_{C/k}$ at $\mathfrak{p}$. We have seen in the section on smooth points (???) that we get an embedding of the polynomial ring

$$\begin{gathered} k[X] \hookrightarrow \mathcal{O}_{C,\mathfrak{p}} \\ X \mapsto f. \end{gathered}$$

If $\mathfrak{p}_0 = k[X] \cap \mathfrak{p}$ then the embedding of local rings

$$k[X]_{\mathfrak{p}_0} \hookrightarrow \mathcal{O}_{C,\mathfrak{p}}$$

is étale (???):

(i) The maximal ideal $\mathfrak{p}_0 \subset k[X]_{\mathfrak{p}_0}$ generates the maximal ideal $\mathfrak{p} \subset \mathcal{O}_{C,\mathfrak{p}}$.

(ii) The extension

$$k[X]_{\mathfrak{p}_0}/\mathfrak{p}_0 \hookrightarrow \mathcal{O}_{C,\mathfrak{p}}/\mathfrak{p}$$

is a finite separable extension. (This is explained at the end of the section on smooth points, just before the section on flat morphisms).

Since the ring $k[X]$ is principal, we have $\mathfrak{p}_0 = (p(X))$ with an irreducible polynomial $p(X)$. But the maximal ideal of the local ring $\mathcal{O}_{C,\mathfrak{p}}$ is also generated by $p(X)$ and the proposition follows. Since we had the habit to denote a uniformizing element of the maximal ideal $\mathfrak{p}$ of a discrete valuation ring by $\pi_\mathfrak{p}$ we can choose $\pi_\mathfrak{p} = \pi_{\mathfrak{p}_0} = p(X)$.

The above étale morphism provides in a certain sense a good approximation of $\mathcal{O}_{C,\mathfrak{p}}$ by $k[X]_{\mathfrak{p}_0}$. If we assume for instance that $k(\mathfrak{p}_0) = k(\mathfrak{p})$ then it is clear that for any n we have $k[X]/\mathfrak{p}_0^n = \mathcal{O}_{C,\mathfrak{p}}/\mathfrak{p}^n$ and hence we get an isomorphism between the completions

$$\hat{k[X]}_{\mathfrak{p}_0} = \lim_{\leftarrow} k[X]/\mathfrak{p}_0^n \xrightarrow{\sim} \lim_{\leftarrow} \mathcal{O}_{C,\mathfrak{p}}/\mathfrak{p}^n\mathcal{O}_{C,\mathfrak{p}} = \hat{\mathcal{O}}_{C,\mathfrak{p}}.$$

Under our assumptions the extension $k(\mathfrak{p})/k(\mathfrak{p}_0)$ is separable, but the first step $k(\mathfrak{p}_0)/k$ can be inseparable. But

If $\mathfrak{p}$ is a closed point on the smooth curve C/k and if the extension of residue fields $k(\mathfrak{p})/k$ is separable, and if $\pi_\mathfrak{p}$ is a uniformizing element at $\mathfrak{p}$, then $d\pi_\mathfrak{p}$ generates the module of differentials at $\mathfrak{p}$.

We choose our X as above. Then our assumption implies that $p'(X)$ is a unit in $k[X]_{\mathfrak{p}_0}$ and then $d\pi_{\mathfrak{p}_0} = d\pi_\mathfrak{p} = p'(X)dX$ is a generator for the module $\Omega^1_{C,\mathfrak{p}}$ of differentials.

The structure of $\hat{\mathcal{O}}_{C,\mathfrak{p}}$:

We need a little bit of information concerning the structure of $\hat{\mathcal{O}}_{C,\mathfrak{p}}$. We have the diagram

$$\begin{tikzcd} \hat{\mathcal{O}}_{C,\mathfrak{p}} \arrow[r, "\Psi"] & k(\mathfrak{p}) \\ k. \arrow[u] \arrow[ur] & \end{tikzcd}$$

If the extension of residue fields $k(\mathfrak{p})/k$ is separable then we have a unique section $s : k(\mathfrak{p}) \to \hat{\mathcal{O}}_{C,\mathfrak{p}}$ with $s|k = \mathrm{Id}_k$ and $\Psi \circ s = \mathrm{Id}_{k(\mathfrak{p})}$. To get this section we write $k(\mathfrak{p}) = k(\theta)$ where θ is the zero of an irreducible and separable polynomial $f(X) \in k[X]$. We lift θ to an element $\tilde{\theta} \in \hat{\mathcal{O}}_{C,\mathfrak{p}}$. Then $F(\theta) \equiv 0 \bmod \mathfrak{p}$. Let $\pi_\mathfrak{p}$ be a uniformizing element. We modify $\tilde{\theta}$ into $\tilde{\theta} + \alpha\pi_\mathfrak{p}$ where $\alpha \in \hat{\mathcal{O}}_{C,\mathfrak{p}}$. We evaluate F at this new argument and get $F(\tilde{\theta} + \alpha\pi_\mathfrak{p}) = F'(\tilde{\theta}) + F'(\tilde{\theta})\alpha\pi_\mathfrak{p}$. We know that $F'(\tilde{\theta})$ is a unit in $\hat{\mathcal{O}}_{C,\mathfrak{p}}$ and hence we see that we can choose α in such a way that

$$F(\tilde{\theta} + \alpha\pi_\mathfrak{p}) \equiv 0 \bmod \mathfrak{p}^2$$

. We modify again by adding a $\beta\pi_\mathfrak{p}^2$ and improve our solution to a solution $\bmod\, \mathfrak{p}^3$. This yields a sequence which converges to an exact solution. This argument is called the Lemma of Hensel and it is the p-adic version of Newtons method.

We identify $s(k(\mathfrak{p}))$ to $k(\mathfrak{p})$ and our diagram above becomes

$$\begin{array}{ccc} k(\mathfrak{p})[[\pi_\mathfrak{p}]] & \longrightarrow & k(\mathfrak{p}) \\ \uparrow & & \\ & \nearrow & \\ k & & \end{array}$$

in other words our ring $\hat{\mathcal{O}}_{C,\mathfrak{p}}$ is the power series ring in one variable $\pi_\mathfrak{p}$ over the residue field. The quotient field is $k(\mathfrak{p})[[\pi_\mathfrak{p}]][1/\pi_\mathfrak{p}]$, it is sometimes called the field of Laurent expansions at $\mathfrak{p}$.

If the residue field extension is not separable then the structure of $\hat{\mathcal{O}}_{C,\mathfrak{p}}$ is not so nice and will cause us some trouble.

If the extension $k(\mathfrak{p})/k$ purely inseparable then it is clear that the L-algebra $k(\mathfrak{p}) \otimes_k L$ is local and hence we have only one prime ideal $\mathfrak{p}'$ in the fibre over $\mathfrak{p}$.

Base change:

We have to investigate systematically what happens if we extend the field of constants k, i.e. we choose an extension L/k and we consider the curve $C \times_{\mathrm{Spec}(k)} \mathrm{Spec}(L)$ or simply $C \times_k L$. We have a morphism (the base change morphism)

$$C \leftarrow C \times_k L.$$

We considered this at the end of the section on affine schemes for the case $L = \bar{k}$ but it is clear that the considerations carry over to this case. The base change morphism induces a map on the closed points and the fibre over a closed point $\mathfrak{p}$ is the set of prime ideals of the finite L-algebra $k(\mathfrak{p}) \otimes_k L$. If we consider a closed point $\mathfrak{p}$ and an affine neighborhood U of $\mathfrak{p}$ then our base change morphism corresponds to the inclusion

$$A(U) \to A(U) \otimes_k L.$$

Let us assume that L/k is finite. Then the prime ideal $\mathfrak{p} \subset A(U)$ decomposes in $A(U) \otimes_k L$, i.e.

$$\mathfrak{p}(A(U) \otimes_k L) = \mathfrak{p}_1^{e_1} \cdot \ldots \cdot \mathfrak{p}_s^{e_s}$$

or

$$(A(U) \otimes L)/\mathfrak{p}A(U) \otimes L \simeq A(U) \otimes_k L/\mathfrak{p}_1^{e_1} \oplus \ldots \oplus (A \otimes_k L)/\mathfrak{p}_s^{e_s}.$$

The e_ν are called the ramification indices. There are some special cases

(1) If the residue field $k(\mathfrak{p}) = k$, then it is clear that $\mathfrak{p}$ stays prime. (This tells us also that our assumption that L/k is finite is inessential. After passing to a suitable finite extension nothing essential happens after that (if we stick to one $\mathfrak{p}$)).

(2) If the extension $k(\mathfrak{p})/k$ is separable then the ramification exponents e_ν are one, the algebra $(A(U) \otimes L)/\mathfrak{p}A(U) \otimes L = k(\mathfrak{p}) \otimes_k L$ has no nilpotent elements.

(3) But if for instance $k(\mathfrak{p}) = k(\sqrt[p]{a})$ where $p = \mathrm{char}(k)$ and $a \notin k^p$ and $A(U) = k[X]/(X^p - a)$ and if $L = k(\sqrt[p]{a})$ then

$$k[X] \otimes_k k(\sqrt[p]{a})/_{(X^p-a)} = k(\sqrt[p]{a})[X]/(X - \sqrt[p]{a})$$

and hence we get nilpotent elements.

(4) If more generally $k(\mathfrak{p})/k$ is purely inseparable we choose L big enough so that it can contain the k-algebra $k(\mathfrak{p})$. Then the ramification index $e = [k(\mathfrak{p}) : k]$ in other words in the group of fractional ideals of $A(U) \otimes_k L$ we have $\mathfrak{p} = \mathfrak{p}'^{[k(\mathfrak{p}):k]}$. The completion $\mathcal{O}_{C \times_k L, \mathfrak{p}'}$ is again a power series ring.

If we pass to the completions, i.e. the infinitesimal neighborhoods of $\mathfrak{p}$ then we get

$$\hat{\mathcal{O}}_{C,\mathfrak{p}} \otimes_k L = (\varprojlim A(U)/\mathfrak{p}^n \otimes L) \simeq \hat{=}\mathcal{O}_{C \times_k L, \mathfrak{p}_1} \oplus \ldots \oplus \hat{\mathcal{O}}_{C \times_k L, \mathfrak{p}_s}.$$

If the extension $k(\mathfrak{p})/k$ is separable then we get

$$\hat{\mathcal{O}}_{C,\mathfrak{p}} \otimes_k L = k(\mathfrak{p})[[\pi_\mathfrak{p}]] \otimes_k L \simeq (k(\mathfrak{p}) \otimes_k L)[[\pi_\mathfrak{p}]]$$

where now the points above $\mathfrak{p}$ correspond to the points of $\mathrm{Spec}(k(\mathfrak{p}) \otimes_k L)$.

Curves and their function fields

For an irreducible curve C/k we consider the function field $k(C)$ of meromorphic functions. This is simply the stalk $\mathcal{O}_{C,\eta}$ of the sheaf $\mathcal{O}_C$ in the generic point η. For any affine non-empty open subset $U \subset C$ the ring $\mathcal{O}_C(U)$ is a one-dimensional, integral ring and $k(C)$ is its quotient field. We pick an $f \in \mathcal{O}_C(U)$, which is not constant. By definition this element cannot be algebraic over k. It yields an embedding of the polynomial ring

$$k[f] \hookrightarrow \mathcal{O}_C(U)$$

and an embedding of fields

$$k(f) \hookrightarrow k(C).$$

I claim

The field $k(C)$ is a finite extension of $k(f)$ and hence of transcendence degree one.

This is rather clear, if we had an $g \in k(C)$ which is not algebraic over $k(f)$ then $k[f, g]$ would be a polynomial ring in two variables sitting in some Dedekind ring $\mathcal{O}_C(V)$ which is absurd.

Let us denote by $D_f \supset U$ the set of points $\mathfrak{p}$ where f is regular, i.e. we have $f \in \mathcal{O}_{C,\mathfrak{p}}$ Since $\mathcal{O}_C$ is a sheaf then we have $f \in \mathcal{O}_C(D_f)$. Now we consider the integral closure A of $k[f]$ in $k(C)$. In the section on Dedekind rings we indicated that A is again a finitely generated k-algebra and a Dedekind ring. (The extension $k(C)/k(f)$ needs not to be separable anymore). It is clear that the elements of A are integral at all points of D_f in other words we have $A \hookrightarrow \mathcal{O}_C(D_f)$. We get a diagram

$$\begin{array}{ccc} D_f & \to & \mathrm{Spec}(A) \\ & \searrow & \\ & & C \end{array}$$

Now we assume in addition that our curve C/k is projective. Then it is clear that the morphism $D_f \hookrightarrow C$ extends uniquely to $\mathrm{Spec}(A) \to C$. Our diagram can be completed by a vertical arrow:

$$\begin{array}{ccc} D_f & \to & \mathrm{Spec}(A) \\ & \searrow & \downarrow \\ & & C. \end{array}$$

If $\mathfrak{p}$ is a closed point in the image of $\mathrm{Spec}(A)$ then we have $A \subset \mathcal{O}_{C,\mathfrak{p}}$ hence we have $f \in \mathcal{O}_{C,\mathfrak{p}}$ and this implies $\mathfrak{p} \in D_f$. Hence our diagram becomes

$$\begin{array}{cccc} D_f & \to & \mathrm{Spec}(A) & \\ & \searrow \mathrm{Id} & \downarrow & \\ & & D_f & \subset C. \end{array}$$

Looking at the resulting diagrams for rings we see that the inclusion $A \hookrightarrow \mathcal{O}_C(D_f)$ must be an isomorphism.

We summarize

Let C/k be a projective smooth curve. If we pick a non constant $f \in k(C)$ and consider the set $D_f \subset C$ where f is regular. Then $\mathcal{O}_C(D_f)$ is the integral closure of $k[f]$ in $k(C)$.

We may also consider $k[f^{-1}]$ and by the same procedure we get $k[f^{-1}] \subset \mathcal{O}_C(D_{f^{-1}})$. We have seen how to glue $\mathrm{Spec}(k[f])$and $\mathrm{Spec}(k[f^{-1}])$ over $\mathrm{Spec}(k[f, f^{-1}])$ and construct $\mathbb{P}^1/k$. Hence we see that our element f defines a morphism

$$\Phi_f : C \to \mathbb{P}^1.$$

This morphism is finite, i.e. on any open affine $Y \subset \mathbb{P}^1$ the morphism $\mathcal{O}_{\mathbb{P}^1}(Y) \hookrightarrow \mathcal{O}_C(\Phi_f^{-1}(Y))$ is finite.

This tells us that for a smooth, projective and absolutely irreducible curve we can reconstruct the curve from its field of meromorphic functions, i.e. from $\mathcal{O}_{C,\eta} = k(C)$. The set of closed points of C can be identified to the set $\mathcal{V}$ of all discrete valuation rings V in $k(C)$ which contain the field of constants k and have quotient field $k(C)$. To see this we have to show that any such discrete valuation ring $V \subset k(C)$ is the stalk of $\mathcal{O}_C$ at a closed point. We pick an $f \in V$ which is not in k then $V \supset k[f]$. Then we must have $V \supset \mathcal{O}_C(D_f)$. Hence V is the discrete valuation ring at a point of D_f. We put a topology onto the set $\mathcal{V}$: The non empty open sets are the sets $\mathcal{V}_f = \{V | f \in V\}$. Then we see that the bijection $\mathcal{V} \xrightarrow{\sim}$ closed points of C becomes a homeomorphism. Eventually we define a sheaf $\mathcal{O}(\mathcal{V}_f) = \bigcap_{V \in \mathcal{V}_f} V$, we get a ringed space and if we add a generic point we get an isomorphism of ringed spaces.

If we have two smooth, absolutely irreducible projective curves C_1, C_2 then we can consider morphisms $\phi : C_1 \to C_2$. Such a morphism is not constant (i.e. it does not map C_1 to a point) then it maps the generic point to the generic point. Hence it induces a map between the function fields

$$
\begin{array}{ccc}
k(C_2) & \stackrel{t\phi}{\to} & k(C_1) \\
\searrow & & \swarrow \\
& k &
\end{array}
$$

and it is not difficult to see, that we can recover ϕ from ${}^t\phi$. Hence we see that non constant morphisms

$$\varphi : C_1 \longrightarrow C_2$$

between two smooth, absolutely irreducible projective curves are in one to one correspondence to morphisms

$${}^t f : k(C_2) \longrightarrow k(C_1)$$

between the function fields which are the identity on the constants.

This is an exceptional phenomenon in dimension one.

We may even start from a field K/k of trancendence degree one. If we assume in addition

a) we can find an $x \in K$ which is transcendental over k and which is such that $K/k(x)$ is a finite separable extension.

b) The field k is algebraically closed in K which means that for any $\alpha \in K$ which is algebraic over k we have $\alpha \in k$,

then we can construct a smooth, absolutely irreducible, projective curve C/k with $k(C) = K$. The set of closed points will be the set of discrete valuation rings in K which contain k. The non empty open sets will be the sets of closed points containing a given $f \in K$ plus the generic point and the ring of holomorphic functions on such an open set will be the integral closure of $k[f]$ in K.

Let $\varphi : C_1 \to C_2$ be a separable, finite morphism between two smooth and absolutely irreducible projective curves, this means that

$${}^t\varphi : k(C_2) \hookrightarrow k(C_1)$$

is a finite separable extension. For any affine open subset $U \subset C_2$ we know that

$$B = \mathcal{O}_{C_1}(\varphi^{-1}(U)) \supset \mathcal{O}_{C_2}(U) = A$$

and B is the integral closure of A in $k(C_1)$. We introduced the concept of ramification in (???), we said that B is ramified at a point $\mathfrak{p} \in \mathrm{Spec}(A)$ if and only if the $A/\mathfrak{p}$-algebra $B/\mathfrak{p}B = B \otimes_A A/\mathfrak{p}$ has nilpotent elements. This is obviously equivalent to the assertion that the trace pairing

$$\mathrm{tr}_{(B/\mathfrak{p}B)/(A/\mathfrak{p})} \quad (xy)$$

is degenerate.

This leads to the definition of the fractional ideal

$$\mathfrak{D}_{B/A}^{-1} = \left\{x \in K_1 \mid \mathrm{tr}_{K_1/K_2}(xA) \subset A\right\}.$$

Since K_1/K_2 is separable we see that

$$\begin{array}{cccc} \mathrm{tr} & : & K_1 \times K_1 & \longrightarrow & K_2 \\ \mathrm{tr} & : & (x,y) & \longrightarrow & \mathrm{tr}_{K_1/K_2}(xy) \end{array}$$

is non degenerate and we conclude that we have $B \subset \mathfrak{D}_{B/A}^{-1}$ and that $\mathfrak{D}_{B/A}^{-1}$ is indeed the inverse of a non zero ideal $\mathfrak{D}_{B/A} \subset B$.

This ideal can be written as

$$\mathfrak{D}_{B/A} = \prod_{i=1}^{t} \mathfrak{p}_i^{m_i},$$

and it gives us the ramified primes and it also measures the ramification.

By definition we have a homomorphism

$$\psi : \mathfrak{D}_{B/A}^{-1} \to \mathrm{Hom}_A(B, A)$$

$$x \mapsto \{b \mapsto \underset{B/A}{\mathrm{tr}}(xb)\}$$

and I claim that this map is an isomorphism.

To see this we pick any non zero prime ideal $\mathfrak{p}$ in A and localize at this prime ideal. If now $x \in \mathfrak{D}_{B/A}^{-1} \setminus \mathfrak{D}_{B/A}^{-1}\mathfrak{p}$ then we can find a $b \in B$ s.t. $\mathrm{tr}_{B/A}(xb) \not\equiv 0 \bmod \mathfrak{p}$. Otherwise we would have $x/\pi_{\mathfrak{p}} \in \mathfrak{D}_{B/A}^{-1}$ in contradiction to our assumption on x.Hence we see that

$$\mathfrak{D}_{B/A}^{-1} \otimes A/\mathfrak{p} \hookrightarrow \mathrm{Hom}_A(B, A/\mathfrak{p})$$

and since these two vector spaces have the same dimension the claim follows.

If we vary the open set U we can put these different ideals together and get a sheaf

$$\mathfrak{D}_{C_1/C_2}$$

which measures the ramification of $\psi : C_1 \to C_2$.

Now we consider the sheaf of differentials $\Omega^1_{C_1/k}$ and $\Omega^1_{C_2/k}$. Both of them are line bundles because we made the assumption that C_1, C_2 are smooth.

If we pull back the sheaf $\Omega^1_{C_2/k}$ via φ to the sheaf $\varphi^*(\Omega_{C_2/k})$, then we have an obvious inclusion between the two line bundles

$$\varphi^*(\Omega^1_{C_2/k}) \hookrightarrow \Omega^1_{C_1/k},$$

and we have the

Riemann-Hurwitz formula.

This inclusion extends to an isomorphism

$$\varphi^*(\Omega^1_{C_2/k}) \otimes \mathfrak{D}^{-1}_{C_1/C_2} \xrightarrow{\sim} \Omega^1_{C_1/k}.$$

This is a local formula: If we choose an affine open set $U \subset C_2$ such that $\Omega^1_{A/k}$, $\Omega^1_{B/k}$ and $\mathfrak{D}_{B/A}$ become free modules, and if ω_A is generator of $\Omega^1_{A/k}$ and F a generator of $\mathfrak{D}_{B/A}$, then $F^{-1} \cdot \omega = \omega'$ is a generator of $\Omega^1_{B/k}$.

We have seen for the sheaves of differentials that they behave well under extensions of the ground field. Hence we perform the base change $\mathrm{Spec}(k) \leftarrow \mathrm{Spec}(\overline{k})$, then

$$\begin{array}{rcl} \Omega^1_{A\otimes_k \overline{k}/\overline{k}} & = & \Omega^1_{A/k} \otimes_k \overline{k} \\ \Omega^1_{B\otimes_k \overline{k}/\overline{k}} & = & \Omega^1_{B/k} \otimes_k \overline{k} \end{array}$$

Now we look at the discriminant $\mathfrak{D}_{A/B}$. Our assumption is that $A \otimes_k \overline{k}$, $B \otimes_k \overline{k}$ are still fields.

We showed that $\psi : \mathfrak{D}^{-1}_{B/A} \to \mathrm{Hom}_A(B, A)$ is an isomorphism, hence

$$\mathfrak{D}^{-1}_{B/A} \otimes_k k' \longrightarrow \mathrm{Hom}_{A\otimes_k k'}(B \otimes_k k', A \otimes_k k')$$

is an isomorphism. This shows

$$\mathfrak{D}^{-1}_{B/A} \otimes_k k' = \mathfrak{D}^{-1}_{B\otimes_k k'/A\otimes_k k'}.$$

Hence it suffices to prove the Riemann-Hurwitz formula. We can assume that our ground field is algebraically closed.

Since we may also pass to the completion, we are reduced to the case

$$\hat{\mathcal{O}}_{C_2,\mathfrak{p}} = k[[x]] \longrightarrow \hat{\mathcal{O}}_{C_1,\mathfrak{p}} = k[[y]]$$

where the extension is finite.

We can write x as a power series in y

$$x = Q(y) = y^m + a_{m+1}y^{m+1} + \cdots = y^m(1 + a_{m+1}y + \cdots)$$

which implies that $\mathrm{ord}_{(y)}(x) = m$. Then it is obvious that the elements $1, y, \cdots y^{m-1}$ form a basis of the $k[[x]]$-module $k[[y]]$, and we have an equation

$$y^m + a_1(x)y^{m-1} + \cdots a_m(x) = 0$$

where the $a_i(x) \in k[[x]]$ and $a_i(x) \equiv \mathrm{mod}(x)$. Looking at the order of vanishing yields $a_m(x) = \alpha_0 x + \alpha_1 x^2 \cdots$ with $\alpha_0 \neq 0$. If $P(Y) = Y^{,} + a_1(x)Y^{m-1} \cdots a_m(x) \in k[[x]][Y]$, we find a relation for the differentials

$$\frac{\partial P}{\partial Y}(y)dy + \left(\sum \frac{\partial a_v}{\partial x} y^{m-v}\right) dx = 0.$$

Since $\frac{\partial a_m}{\partial x} = \alpha_0 \neq 0$ we see that $\left(\sum \frac{\partial a_v}{\partial x} y^{m-v}\right) dx$ generates $\Omega^1_{A/k} \otimes B$, we see that $\Omega^1_{B/k}$ is generated by

$$\frac{dx}{\frac{\partial P}{\partial Y}(y)}.$$

Now everything boils down to show that

$$\mathfrak{D}_{B/A} = \left(\frac{\partial P}{\partial Y}(y)\right)$$

which is very easy.

Kommt noch (???)

One of the important consequences of our considerations is that we can define a trace map

$$\mathrm{tr} : f_*(\Omega^1_{C_1/k}) \longrightarrow \Omega^1_{C_2/k}.$$

This is clear. Locally we can write a differential $\omega = f dx$ where dx is a generator of $\Omega^1_{C_2/k}$ and $f \in \mathfrak{D}^{-1}_{C_1/C_2}$. Then

$$\mathrm{tr}_{C_1/C_2}(\omega) = \mathrm{tr}_{C_1/C_2}(f)dx$$

and $\mathrm{tr}_{C_1/C_2}(f)$ is regular by definition of the different

Line bundles and Divisors

The following considerations are valid in a more general framework. Let X be any scheme. A *line bundle* on X is simply a locally free $\mathcal{O}_X$-module of rank one. The structure sheaf itself is a line bundle. It is called the trivial bundle. If we have two line bundles $\mathcal{L}_1, \mathcal{L}_2$, we can form the tensor product $\mathcal{L}_1 \otimes \mathcal{L}_2$. We can form the line bundle $\mathcal{L}^{-1} = \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{L}, \mathcal{O}_X)$

and we have $\mathcal{L} \otimes \mathcal{L}^{-1} \cong \mathcal{O}_X$. Hence it is rather clear that the isomorphism classes of line bundles form a group under the multiplication. It is the so-called Picard group $\mathrm{Pic}(X)$. To define this group we do not need any assumption on X.

Under certain assumptions we can identify this group to the so called divisor class group.

At first we define the *group of divisors* as the free group generated by the irreducible subschemes $\mathfrak{p}$ of codimension one. If we want to do this we need the concepts from dimension theory and hence we assume that our scheme is covered by affine noetherian schemes (See???). I even want to assume that it is noetherian itself, i.e. it can be covered by a finite number of affine schemes. We are mainly interested in the case where X is covered by a finite number of affine schemes over a field k. For this special case we discussed the relevant results in dimension theory in section (???).

We decompose X into irreducibles $X = \bigcup X_i$ let η_i be the generic point of X_i. Then a meromorphic function f on X is simply an element in the stalk $\mathcal{O}_{X_i,\eta_i}$. We want to attach a divisor $\mathrm{Div}(f)$ to the function f. Since everything is happening on X_i I assume that X is already irreducible.

We can find a covering $X = \cup U_i$ by affine integral schemes such that on U_i we can write

$$f = \frac{g_i}{h_i}$$

with $f_i, h_i \in \mathcal{O}_X(U_i)$. We consider the prime ideals which contain (f_i) resp (g_i) and amoung those we we consider those which are minimal. Then the Hauptidealsatz of Krull asserts that there is a finite number of such minimal prime ideals $\mathfrak{p}_1, \dots, \mathfrak{p}_s \supset (f_i)$ and $\mathfrak{q}_1, \dots, \mathfrak{q}_r \supset (g_i)$ and these have height one ( ) [Ei], , ???). But this is not enough to attach a divisor to f, we need the multiplicities of the zeroes and poles.

To define these multiplicities we make the additional assumption that the scheme X should be normal: this means that the affine rings $\mathcal{O}_X(U_i)$ are integrally closed in there field of meromorphic functions. This implies that for all prime ideals $\mathfrak{p}$ of height one the local rings $\mathcal{O}_{X,\mathfrak{p}}$ are discrete valuation rings. (I gave a reference and an indication of the proof of this fact in the section on low dimensional rings.)

If now $\mathfrak{p}_\nu \supset (f_i)$ (resp. $\mathfrak{q}_\mu \supset (g_i)$ then we can write the principal ideals local rings $(f_i) \subset \mathcal{O}_{X,\mathfrak{p}_\nu}$ (resp. $(g_i) \subset \mathcal{O}_{X,\mathfrak{q}_\nu}$) as

$$(f_i) = \mathfrak{p}_\nu^{\mathrm{ord}_{\mathfrak{p}_\nu}(f_i)}, (g_i) = \mathfrak{q}_\mu^{\mathrm{ord}_{\mathfrak{q}_\mu}(g_i)}.$$

Now we attach a divisor to the restriction of f to U_i namely

$$\sum \mathrm{ord}_{\mathfrak{p}_\nu}(f_i)\mathfrak{p}_\nu - \sum \mathrm{ord}_{\mathfrak{q}_\mu}(g_i)\mathfrak{q}_\mu.$$

Comparing these divisors on the intersections $U_i \cap U_j$ gives us a divisor on X.

Of course it can happen that we have some cancellations, i.e. that the same $\mathfrak{p}$ occurs amoung the $\mathfrak{p}_\nu$ and the $\mathfrak{q}_\mu$. We say that the function f has a zero (resp. pole) of order $n \geq 0$ at an irreducible subscheme $\mathfrak{p}$ of codimension 1 if it occurs amoung the $\mathfrak{p}_\nu, \mathfrak{q}_\mu$ and if $n = \mathrm{ord}_{\mathfrak{p}}(f_i) - \mathrm{ord}_{\mathfrak{p}}(g_i)$ resp. $n = \mathrm{ord}_{\mathfrak{p}}(g_i) - \mathrm{ord}_{\mathfrak{p}}(f_i)$ (for some i).

Hence we see that we can define the group of *principal divisors* on a noetherian, integral normal scheme as the group of divisors of meromorphic functions.

We define the *divisor class group* as the group of divisors modulo principal divisors (See ???)

Under certain conditions we have an isomorphism between $\mathrm{Pic}(X)$ and the divisor class group. To get this isomorphism we have to use some results from the commutative algebra of noetherian rings, which are not in our script. In the special case of rings of dimension one they are proved in the section on "Low dimensional rings".

Assume again that X is noetherian, integral and normal. Let $\mathcal{L}$ be a line bundle on X. Then the stalk $\mathcal{L}$ at the generic point η is a one dimensional $\mathcal{O}_{X,\eta}$- vector space. Let $s \in \mathcal{L}_\eta$ be a generator. It is a meromorphic section in the line bundle. We want to attach a divisor to this meromorphic section.

We can find a covering $X = \cup U_i$ by affine integral schemes such that $\mathcal{L}|U_i$ becomes trivial, i.e.

$$\mathcal{L}(U_i) = (\mathcal{O}_X|U_i) \cdot t_i$$

where $t_i \in H^0(U_i, \mathcal{L})$. Then we have

$$s = g_i t_i$$

where g_i is an element in the field of meromorphic functions $\mathcal{O}_{X,\eta}$. We can write

$$g_i = \frac{f_i}{h_i}$$

with $f_i, h_i \in \mathcal{O}_X(U_i)$. As before we see that there is a finite number of minimal prime ideals $\mathfrak{p}_1, \ldots, \mathfrak{p}_s \supset (f_i)$ and $\mathfrak{q}_1, \ldots, \mathfrak{q}_r \supset (g_i)$ and these have height one.

Again we use the fact that the local rings $\mathcal{O}_{X,\mathfrak{p}_\nu}, \mathcal{O}_{X,\mathfrak{q}_\nu}$ are discrete valuation rings. Hence we can write in these local rings

$$(f_i) = \mathfrak{p}_\nu^{\mathrm{ord}_{\mathfrak{p}_\nu}(f_i)}, (g_i) = \mathfrak{q}_\nu^{\mathrm{ord}_{\mathfrak{q}_\nu}(g_i)}.$$

The prime ideals $\mathfrak{p}_\nu, \mathfrak{q}_\nu$ define irreducible subschemes of codimension one and we attach to $(\mathcal{L}, s, t_i)$ a divisor on U_i namely

$$\sum \mathrm{ord}_{\mathfrak{p}_\nu}(f_i)\mathfrak{p}_\nu - \sum \mathrm{ord}_{\mathfrak{q}_\mu}(g_i)\mathfrak{q}_\mu.$$

If we compare D_i and D_j on the intersection $U_i \cap U_j$ we see that they must coincide and hence we see that we can attach a divisor D on X to our data $(\mathcal{L}, s, t_i)$. If we change then t_i then we modify g_i by a unit and D_i stays the same. If we modify the meromorphic section s, i.e. we multiply s by a meromorphic function g then the divisor is changed by the divisor of a meromorphic function.

Hence we get a homomorphism

$$\text{Pic (X)} \longrightarrow \text{Divisors modulo principal divisors.}$$

It can be shown that under our assumptions this homomorphism is injective: If we can choose the sections s and t_i in such a way that the divisor becomes zero, then $s = g_i t_i$ where the divisor of g_i is zero. We conclude that g_i and g_i^{-1} lie in the intersection of all discrete valuation rings $\mathcal{O}_X(U_i)_\mathfrak{p}$ where $\mathfrak{p}$ runs over the prime ideals of height one. Now we know that this intersection is equal to $\mathcal{O}_X(U_i)$ (See [Ei], Cor. 4.11, in dimension one also in "Low dimensional rings") Hence g_i is a unit. But then $t_i = g_i^{-1} s$ and the meromorphic section s is an element in $H^0(X, \mathcal{L})$ which generates the free $\mathcal{O}_X$-module $\mathcal{L}$.

This homomorphism is not surjective in general. If we want this we need to assume that X is *locally factorial*. This means that for any point $x \in X$ and any prime ideal $\mathfrak{p} \subset \mathcal{O}_{X,x}$ of height one we find an $f_\mathfrak{p} \in \mathcal{O}_{X,x}$ such that $\mathfrak{p} = (f_\mathfrak{p})$.

Under this assumption the homomorphism becomes bijective. To see this we start from a divisor $D = \sum_\mathfrak{p} n_\mathfrak{p} \mathfrak{p}$. For any point $x \in X$ we can find an open neighborhood U_x and an element $f_x \in \mathcal{O}_X(U_x)$ such that $\mathrm{Div}(f_x)|U_x = D|X$. Now we define the line bundle $\mathcal{O}_X(-D)$: Its sections over U_x are

$$\mathcal{O}_X(U_x) = \{h \in \mathcal{O}_{X,\eta} | f_x h \in \mathcal{O}_X(U_x)\}.$$

Then it is clear that this defines a line bundle because locally at x the function $1/f_x$ trivializes the bundle. The corresponding divisor class is the class of D (Our global meromorphic section s is simply the constant function 1 and for the t_i we choose f_x^{-1}. The the g_i become the f_x.)

If we have a line bundle $\mathcal{L}$ on X which has a non zero global section $s \in H^0(X, \mathcal{L})$ then we can define *the scheme* $V(s)$ *of zeroes of* s. We simply observe that locally $\mathcal{L}$ is trivial, hence locally s is nothing else than a non zero regular function.

Our divisor is called *effective* if all the multiplicities are ≥ 0. If D is effective then $\mathcal{O}_X(-D)$ is the sheaf of germs of meromorphic functions which may have poles of order $\leq n_\mathfrak{p}$ along the $\mathfrak{p}$. In this case we have $1 \in H^0(X, \mathcal{O}_X(-D))$.

It is a theorem in commutative algebra that X is X is locally factorial if X/k is smooth (See ???). If X is of dimension one this follows from (???).

Divisors on curves :

Now we come back to the case where X is a smooth, projective and absolutely irreducible curve. The irreducible subschemes of codimension 1 are the closed points.

The group of divisors $\mathrm{Div}(C)$ is the free abelian group generated by the closed points. We write

$$D = \sum n_\mathfrak{p} \mathfrak{p}.$$

We define the *degree of a divisor* as

$$\deg(D) = \sum n_\mathfrak{p} [k(\mathfrak{p}) : k]$$

where $[k(\mathfrak{p}) : k]$ is the degree of the extension of the residue field.

If we pick a non zero element $f \in k(C)$ then we know that for any closed point $\mathfrak{p}$ we may write $f = \epsilon \pi_\mathfrak{p}^{\nu_\mathfrak{p}(f)}$ in the local ring $\mathcal{O}_{C,\mathfrak{p}}$. Hence we can attach a divisor to our element f

namely $\mathrm{Div}(f) = \sum_{\mathfrak{p}} \mathrm{ord}_{\mathfrak{p}}(f)\mathfrak{p}$. A divisor which has such a presentation is called *principal.* The principal divisors form a subgroup of the group of all divisors.

Theorem: *If C/k is a smooth, absolutely irreducible projective curve, then a principal divisor has degree zero.*

Proof: This follows from the results which we explained in the section on Dedekind rings. We assume that f is not constant. We constructed the morphism (Dies muß noch erklärt werden!)

$$\Phi_f : C \to \mathbb{P}^1$$

and it is clear from the construction that $f \subset k(\mathbb{P}^1)$. As an element in the function field of the projective line its divisor is $(0) - (\infty)$ where (0) is the closed point defined by $(f) \subset k[f]$ and (∞) is the closed point defined by $(f^{-1}) \subset k[f^{-1}]$. This divisor has degree zero. Now we consider the divisor of f on C/k. We decompose the divisor into the divisor of zeroes and the divisor of poles:

$$\mathrm{Div}(f) = \mathrm{Div}_0(f) + \mathrm{Div}_\infty(f) = \sum_{\mathfrak{p},\mathrm{ord}_{\mathfrak{p}}(f)>0} \mathrm{ord}_{\mathfrak{p}}(f)\mathfrak{p} + \sum_{\mathfrak{p},\mathrm{ord}_{\mathfrak{p}}(f)<0} \mathrm{ord}_{\mathfrak{p}}(f)\mathfrak{p}.$$

In the section on Dedekind rings we studied the behavior of prime ideals under extension of Dedekind rings. Our formula there says

$$\deg(\mathrm{Div}_0(f)) = \sum_{\mathfrak{p},\mathrm{ord}_{\mathfrak{p}}(f)>0} \mathrm{ord}_{\mathfrak{p}}(f)[k(\mathfrak{p}) : k] = [k(C) : k(f)]$$

(Our (f) is the prime ideal $\mathfrak{p}$ there and the $\mathfrak{P}$ their correspond to the $\mathfrak{p}$ here.)

The same holds for the pole divisor and the theorem is clear.

If we have a base change $C \leftarrow C \otimes_k L$ then this induces a homomorphism on the group of divisors. To see this we have to check what happens for prime divisors. We discussed what happens in the above section on base change and using the notation there we send $\mathfrak{p}$ to $\sum e_i\mathfrak{p}_i$. This extends to the group of divisors and we see that this homomorphism preserves the degree.

All our assumptions are valid in the case of smooth, absolutely irreducible curves hence we define as $\mathrm{Pic}(C/k)$ the group of line bundles of our curve C/k and we identify it to the group of divisors modulo linear equivalence. Especially we can now define the degree of a line bundle, it is simply the degree of the corresponding divisor class. The degree defines a homomorphism

$$\deg : \mathrm{Pic}(C/k) \to \mathbb{Z}.$$

The kernel of this homomorphism is denoted by $\mathrm{Pic}^0(C/k)$. It is a aim of these notes to give a proof, that this group is actually the group of rational points of a so called abelian variety of dimension g over k. An abelian variety is a connected projective variety which in addition has a structure of a group.

In the case of curves over $\mathbb{C}$ this goes back to Abel, Riemann and Jacobi.

properties of the degree

Let C/k be a smooth curve, let us consider an effective divisor $D = \sum n_{\mathfrak{p}}\mathfrak{p}$. In this casethe sections of $\mathcal{O}_C(-D)$ over an affine open set $V \subset C$ are the meromorphic functions on U whose order of poles at $\mathfrak{p}$ is estimated by $\leq n_{\mathfrak{p}}$.

We notice that we have $\mathcal{O}_C \subset \mathcal{O}_C(-D)$ and the quotient sheaf $\mathcal{O}_C(-D)/\mathcal{O}_C$ has non zero stalk only at the points $\mathfrak{p}$ with $n_{\mathfrak{p}} > 0$ this set is called the support of the divisor D and sometimes denoted by $|D|$. If V is an affine open set containing the support of D then $\mathcal{O}_C(V)$ is a Dedekind ring and we can interprete D as a fractional ideal for this Dedekind ring, it is clear from the definition (see section on Dedekind rings) that $\mathcal{O}_C(-D)(V) = \prod_{\mathfrak{p}\in V} \mathfrak{p}^{-n_{\mathfrak{p}}}$. Now it is clear that for the global sections

$$\mathcal{O}_C(-D)/\mathcal{O}_C)(C) = (\mathcal{O}_C(-D))/\mathcal{O}_C)(V) = \sum \mathfrak{p}^{-n_{\mathfrak{p}}}/\mathcal{O}_{C,\mathfrak{p}}.$$

This direct sum $\sum \mathfrak{p}^{-n_{\mathfrak{p}}}\mathcal{O}_{C,\mathfrak{p}}/\mathcal{O}_{C,\mathfrak{p}}$ is a finite dimensional vector space over k and the dimension of this vector space is $\sum n_{\mathfrak{p}}[k(\mathfrak{p}) : k] = \deg(D)$, we summarize

$$\dim_k(H^0(C, \mathcal{O}_C(-D)/\mathcal{O}_C) = \deg(D)$$

This can be generalized to arbitrary line bundles. If we have a line bundle $\mathcal{L}$ and an effective divisor D, then we can define $\mathcal{L}(-D) = \mathcal{L} \otimes \mathcal{O}_C(-D)$ and we have $\mathcal{L} \subset \mathcal{L}(-D)$. On the other hand if we have two line bundles one of them contained in the other $\mathcal{L} \subset \mathcal{L}_1$ then there is a unique divisor D such that $\mathcal{L}_1 = \mathcal{L}(-D)$. In these cases we have the formula

$$\deg(\mathcal{L}_1) = \deg \mathcal{L}(-D) = \deg(\mathcal{L}) + \deg(D) = \deg(\mathcal{L}) + \dim_k((\mathcal{L}_1/\mathcal{L})(C)).$$

Vector bundles over a curve

A locally free coherent $\mathcal{O}_C-$module $\mathcal{E}$ is called a vector bundle. Let n be its rank. It is an easy exercise in algebra to prove the following: If we have a subspace $V \subset \mathcal{E}_\eta$ in the $\mathcal{O}_{C,\eta}$ vector space $\mathcal{E}_\eta$ then we can find a submodule $\mathcal{F} \subset \mathcal{E}$ such that $\mathcal{F}_\eta = V$ and the quotient $\mathcal{E}/\mathcal{F}$ is again locally free. This subbundle is unique one could denote it by $\mathcal{E} \cap V$. This implies that our vector bundle $\mathcal{E}$ admits a complete flag $(0) = \mathcal{L}_0 \subset \mathcal{L}_1 \subset \mathcal{L}_2 \ldots \mathcal{L}_{n-1} \subset \mathcal{E}$ of subbundles such that the quotient of two successive bundles is a line bundle. We can define the n-th exterior power, this is a line bundle $\det(\mathcal{E}) = \Lambda^n(\mathcal{E})$. We define the degree by $\deg(\mathcal{E}) = \deg(\det(\mathcal{E}))$ and it is clear that we can express the degree in terms of a flag as $\deg(\mathcal{E}) = \sum \deg(\mathcal{L}_i/\mathcal{L}_{i-1})$.

For any vector bundle we can define the dual bundle as $\mathrm{Hom}_{\mathcal{O}_C}(\mathcal{E}, \mathcal{O}_C) = \mathcal{E}^\vee$. Again it is easy to see that we have $\deg(\mathcal{E}) + \deg(\mathcal{E}^\vee) = 0$. Again can derive it easily from the case of line bundles.

Our formula above generalizes to vector bundles. If we have two vector bundles $\mathcal{E} \subset \mathcal{E}_1$ of the same rank, then $\mathcal{E}_1/\mathcal{E}$ is a finitely generated torsion sheaf and

$$\deg(\mathcal{E}_1) = \deg(\mathcal{E}) + \dim_k((\mathcal{E}_1/\mathcal{E})(C)$$

To see this we trivialize both bundles on a suitably small non empty affine open set V. Then the matrices which transform the two bases into each other have non zero determinants in $\mathcal{O}_{C,\eta}$ which are units over a still smaller but still non empty affine open set $V_1 \subset V$. This means that $\mathcal{E}|V_1 = \mathcal{E}_1|V_1$. This shows that the stalks of $\mathcal{E}_1/\mathcal{E}$ are non zero only in the finite set $\mathfrak{p} \in C \setminus V_1$. But for these points it is clear that we can find $m_\mathfrak{p}$ such that $\mathfrak{p}^{m_\mathfrak{p}}\mathcal{E}_{1,\mathfrak{p}} \subset \mathcal{E}_\mathfrak{p}$ and hence it is clear that the quotient is a finitely generated torsion sheaf. To see the assertion concerning the degree we may proceed as follows. We choose an affine set V containing the support of $\mathcal{E}_1/\mathcal{E}$. If we take this set to be sufficiently small then we may assume that both bundles are trivial and the theorem on elementary divisors tells us that we can find a basis $e_1, e_2 \dots e_n$ for $\mathcal{E}_1(V)$ and non zero elements $a_1, a_2, \dots a_n$ in $\mathcal{O}_C(V)$ such that $a_1e_1, a_2e_2 \dots a_ne_n$ form a basis for $\mathcal{E}(V)$. Now these bases define complete flags in both vector bundles. These are the flags which are induced by the subspaces $V_i = \mathcal{O}_{C,\eta}e_1 \oplus \dots \oplus \mathcal{O}_{C,\eta}e_i$ of the generic fibre. We used these flags to define the degree: The degree was the sum of the degrees of the successive quotient line bundles induced by the flag. But if we compare these quotient line bundles for both vector bundles then we see the following: If $\mathcal{M}_i$ resp. $\mathcal{M}_{1,i}$ is such a quotient line bundle obtained by the flag in $\mathcal{E}$ resp. $\mathcal{E}_1$ then the stalks $(\mathcal{M}_i)_\mathfrak{q} = (\mathcal{M}_{1,i})_\mathfrak{q}$ for all $\mathfrak{q}$ not in the support of $\mathcal{E}_1/\mathcal{E}$. For the points $\mathfrak{p}$ in the support of $\mathcal{E}_1/\mathcal{E}$ and even for the $\mathfrak{p} \in V$ we have $a_i^{-1}\mathcal{M}_{i,\mathfrak{p}} = \mathcal{M}_{1,i,\mathfrak{p}}$. Hence we see $\deg(\mathcal{M}_{1,i}) = \deg(\mathcal{M}_i) + \dim_k(a_i^{-1}\mathcal{O}_C(V)/\mathcal{O}_C(V))$. Hence we get $\deg(\mathcal{E}_1) = \deg(\mathcal{E}) + \sum_i \dim_k(a_i^{-1}\mathcal{O}_C(V)/\mathcal{O}_C(V))$. On the other hand we see that

$$\mathcal{E}_1/\mathcal{E}(C) = \mathcal{E}_1/\mathcal{E}(V) = \oplus_i a_i^{-1}\mathcal{O}_C(V)/\mathcal{O}_C(V,$$

this implies our formula.

We consider vector bundles over the projective line $\mathbb{P}^1_k$. In this case we have

Theorem: Any vector bundle $\mathcal{E}/\mathbb{P}^1_k$ is a direct sum of line bundles, i.e.

$$\mathcal{E} \simeq \mathcal{O}_{\mathbb{P}^1}(d_1) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^1}(d_n)$$

with some integers $d_1 \cdots d_n$. The integers are well defined up to order.

Proof: We consider the case rank $\mathcal{E} = 2$. In this case we may tensorize by a line bundle $\mathcal{L} \simeq \mathcal{O}_{\mathbb{P}^1}(r)$ such that $\deg(\mathcal{E}) = 0$ or 1. Now it follows from the theorem of Riemann-Roch that

$$\dim H^0(\mathbb{P}^1, \mathcal{E}) - \dim H^1(\mathbb{P}^1, \mathcal{E}) = \deg(\mathcal{E}) + 2 > 0,$$

and we find a non zero section $s \in H^0(\mathbb{P}^1, \mathcal{E})$. The restriction of s to the generic point defines a subspace $V \subset \mathcal{E}_\eta$, and this subspace extends to a unique line bundle $\mathcal{L} \simeq \mathcal{O}_{\mathbb{P}^1}(m)$ such that $\mathcal{E}/\mathcal{L}$ is torsion free. We must have $\deg(\mathcal{L}) \geq 0$ and hence if

$$0 \longrightarrow \mathcal{L} \longrightarrow \mathcal{E} \longrightarrow \mathcal{E}/\mathcal{L} \longrightarrow 0$$

we see that $\deg((\mathcal{E}/\mathcal{L})^\vee \otimes \mathcal{L}) \geq -1$. If we tensorize by $(\mathcal{E}/\mathcal{L})^\vee$, then we get

$$0 \longrightarrow \mathcal{L} \otimes (\mathcal{E}/\mathcal{L})^\vee \longrightarrow \mathcal{E} \otimes (\mathcal{E}/\mathcal{L})^\vee \longrightarrow \mathcal{O}_{\mathbb{P}^1} \longrightarrow 0,$$

and we have the section $1 \in H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1})$ which is every where $\neq 0$. Since $\deg((\mathcal{E}/\mathcal{L})^\vee \otimes \mathcal{L}) \geq -1$ we see that

$$H^1(\mathbb{P}^1, (\mathcal{E}/\mathcal{L})^\vee \otimes \mathcal{L}) = 0$$

and the section 1 lifts to a section $s_0 \in H^0(\mathbb{P}^1, (\mathcal{E}/\mathcal{L})^\vee \otimes \mathcal{L})$ which does not have any zero. We conclude that we have an inclusion $\mathcal{O}_{\mathbb{P}^1} \hookrightarrow \mathcal{E} \otimes (\mathcal{E}/\mathcal{L})^\vee$ such that its composition with the map to $\mathcal{O}_{\mathbb{P}^1}$ is the identity and therefore

$$\mathcal{E} \otimes (\mathcal{E}/\mathcal{L})^\vee = \mathcal{L} \otimes (\mathcal{E}/\mathcal{L})^\vee \oplus \mathcal{O}_{\mathbb{P}^1}.$$

Since our assertion is insensitive to tensorization by line bundles, the claim follows.

The general case follows by induction. Let rank $(\mathcal{E}) = r$. We consider a line subbundle $\mathcal{L} \subset \mathcal{E}$, s.t. $\mathcal{E}/\mathcal{L}$ is a vector bundle and such that $\deg(\mathcal{L}) = d_1$ is maximal.

Then we have

$$0 \longrightarrow \mathcal{L} \longrightarrow \mathcal{E} \longrightarrow \mathcal{E}/\mathcal{L} \longrightarrow 0$$

and

$$\begin{array}{ccl} \mathcal{E}/\mathcal{L} & \simeq & \bigoplus_{\nu=2}^r \mathcal{O}_{\mathbb{P}^1}(d_\nu) \\ \mathcal{L} & \simeq & \mathcal{O}_{\mathbb{P}^1}(d_1). \end{array}$$

I claim that $d_1 \geq d_\nu$ for all $\nu \geq 2$. If we had an index ν_0 s.t. $d_{\nu_0} > d_1$, then we could consider the rank 2 bundle $\mathcal{E}' \subset \mathcal{E}$ which is the inverse image of the line subbundle $\mathcal{O}_{\mathbb{P}^1}(d_{\nu_0})$ in $\mathcal{E}/\mathcal{L}$. This bundle would decompose

$$\mathcal{E}' = \mathcal{O}_{\mathbb{P}^1}(a) \oplus \mathcal{O}_{\mathbb{P}^1}(b)$$

where $a + b = d_1 + d_{\nu_0} = \deg(\mathcal{E}')$. One of the summands has to map non trivially to $\mathcal{E}'/\mathcal{L} = \mathcal{O}_{\mathbb{P}^1}(d_{\nu_0})$. If this is $\mathcal{O}_{\mathbb{P}^1}(a)$, then we conclude $a \leq d_{\nu_0}$. We cannot have equality because then $\deg \mathcal{O}_{\mathbb{P}^1}(a) = d_{\nu_0}$, and this contradicts the choice of $\mathcal{L}$. But then $b > d_1$ and this is again a contradiction.

Now we show that the sequence

$$0 \longrightarrow \mathcal{L} \longrightarrow \mathcal{E} \longrightarrow \mathcal{E}/\mathcal{L} \longrightarrow 0$$

must split. Basically we argue as in the rank 2 case but we modify our argument slightly. We can cover $\mathbb{P}^1$ by affines U_1, U_2 such that on these affines we have sections $s_i : \mathcal{E}/\mathcal{L} \to \mathcal{E}$. These local sections may differ by a homomorphism

$$\varphi_{12} : \mathcal{E}/\mathcal{L} \mid U_1 \cap U_2 \longrightarrow \mathcal{L}$$

and φ_{12} gives us a class in
$$H^1(\mathbb{P}^1, \mathrm{Hom}(\mathcal{E}/\mathcal{L}, \mathcal{L})).$$
But $\mathrm{Hom}(\mathcal{E}/\mathcal{L}) = \bigoplus_{v=2}^{r} \mathcal{O}_{\mathbb{P}^1}(d_1 - d_\nu)$ and hence this class vanishes and we can bound the cocycle.

We add the following observation. Once we have

$$\mathcal{E} = \bigoplus_{\nu=1}^{r} \mathcal{O}_{\mathbb{P}^1}(d_\nu)$$

we can tensorize by $\mathcal{O}_{\mathbb{P}^1}(-d_{\nu_0})$ where now d_{v_0} is maximal. Then we get

$$\mathcal{E} \otimes \mathcal{O}_{\mathbb{P}^1}(-d_{\nu_0}) = \mathcal{O}_{\mathbb{P}^1}^{r_0} \oplus \bigoplus_{\nu=\nu_0+1}^{r} \mathcal{O}_{\mathbb{P}^1}(d_\nu - d_{\nu_0})$$

where we assumed that $d_1 = \cdots = d_{\nu_0} = d_{\nu_0}$, and the other d_v are smaller. Then we conclude that
$$H^0(\mathbb{P}^1, \mathcal{E} \otimes \mathcal{O}_{\mathbb{P}^1}(-d_{\nu_0})) = k^{r_0}$$
and that these sections generate the subbundle $\mathcal{O}_{\mathbb{P}^1}^{r_0}$. This implies that d_{ν_0} is determined by $\mathcal{E}$ and that the subbundle

$$\bigoplus_{\nu=1}^{r_0} \mathcal{O}_{\mathbb{P}^1}(d_\nu) \qquad d_\nu = d_{\nu_0}$$

is unique.

This shows: If we order these numbers

$$d_1 = \cdots = d_{r_0} > d_{r_0+1} = \cdots = d_{r_1} > d_{r_1+1} \cdots,$$

then the resulting sequence of numbers is determined by $\mathcal{E}$ and that the flag

$$\bigoplus_{\nu=1}^{r_0} \mathcal{O}_{\mathbb{P}^1}(d_\nu) \subset \bigoplus_{\nu=1}^{r_1} \mathcal{O}_{\mathbb{P}^1}(d_\nu) \subset \cdots$$

is also determined by $\mathcal{E}$.

The theorem above and the consequences are called Grothendieck's theorem but it has been known to Dedekind (see ???).

The Theorem of Riemann-Roch

We consider line bundles on a smooth, projective, absolutely irreducible curve C/k. Such a line bundle $\mathcal{L}$ has cohomology group $H^0(C,\mathcal{L}), H^1(C,\mathcal{L}),\dots$ which are finite dimensional k-vector spaces. We shall see that $H^i(C,L)=0$ for $i\geq 2$ and we will give a formula for

$$\chi(C,\mathcal{L}) = \dim_k H^0(C,\mathcal{L}) - \dim_k H^1(C,\mathcal{L}).$$

The first observation is

$$\dim_k H^0(C,\mathcal{O}_C) = 1.$$

If $f\in H^0(C,\mathcal{O}_C)$ then we have $f^2,f^3,\dots\in H^0(C,\mathcal{O}_C)$ and since this is a finite dimension vector space we see that f must be algebraic over k. If $f\notin k$ then we find that

$$k(C)\otimes_k \bar{k}$$

has zero divisors and this contradicts the assumption that C/k should be absolutely irreducible.

We define the *genus* of the curve as

$$g = \dim_k H^1(C,\mathcal{O}_C).$$

Now we can state

Theorem of Riemann-Roch (first version)

For any line bundle $\mathcal{L}$ on C we have

$$\chi(C,\mathcal{L}) = \dim_k H^0(C,\mathcal{L}) - \dim_k H^1(C,\mathcal{L}) = \deg(\mathcal{L}) + 1 - g.$$

The proof is not too difficult. (The main difficulty is to prove that $\dim_k H^1(C,\mathcal{L})$ is finite dimensional but this follows from our general results in the previous chapter). The theorem is true for $\mathcal{L}=\mathcal{O}_C$ by definition.

If we have a closed point $\mathfrak{p}$ then we have an exact sequence of sheaves

$$0\to\mathcal{O}_C\to\mathcal{O}_C(-\mathfrak{p})\to\mathcal{O}_C(-\mathfrak{p})/\mathcal{O}_C\to 0.$$

The quotient is a skyscraper sheaf, its only non-zero stalk is the stalk at $\mathfrak{p}$ and there it is a one dimensional $k(\mathfrak{p})$-vector space.

Since $\mathcal{L}$ is locally free we get an exact sequence by tensoring by $\mathcal{L}$

$$0\to\mathcal{L}\to\mathcal{L}\otimes\mathcal{O}_C(-\mathfrak{p})\to\mathcal{O}_C(-\mathfrak{p})/\mathcal{O}_C\to 0.$$

We observe that

$$\dim_k(H^0(C,\mathcal{O}_C(-\mathfrak{p})/\mathcal{O}_C)) = [k(\mathfrak{p}):k]$$

and

$$H^i(C,\mathcal{O}_C(-\mathfrak{p})/\mathcal{O}_C)) = 0 \quad\text{for}\quad i\geq 1.$$

We write the long exact sequence in cohomology

$$0 \to H^0(C,\mathcal{L}) \to H^0(C,\mathcal{L}\otimes\mathcal{O}_C(-\mathfrak{p})) \to H^0(C,\mathcal{O}_C(-\mathfrak{p})/\mathcal{O}_C) \to$$
$$\to H^1(C,\mathcal{L}) \to H^1(C,\mathcal{L}\otimes\mathcal{O}_C(-\mathfrak{p})) \to 0$$

and and higher degrees we get

$$H^i(C,\mathcal{L}) \xrightarrow{\sim} H^i(C,\mathcal{L}\otimes\mathcal{O}_C(-\mathfrak{p})) \qquad \text{for all} \quad i \geq 2.$$

From this we get easily that our assertions are true for $\mathcal{L}\otimes\mathcal{O}_C(-\mathfrak{p})$ if and only if they are true for $\mathcal{L}$ itself. The rest is more or less clear. We have seen that $H^i(C,\mathcal{O}_C(-\infty\mathfrak{p}))=0$ for $i\geq 2$ in the section on cohomology of coherent sheaves (see exercise below). Since we know that our line bundle is isomorphic to some $\mathcal{O}_C(-D)$ the theorem follows.

Now it is an easy argument in homological algebra that we have a Riemann-Roch formula for vector bundles:

$$\chi(C,\mathcal{E}) = \dim_k H^0(C,\mathcal{E}) - \dim_k H^1(C,\mathcal{E}) = \deg(\mathcal{E}) + n(1-g).$$

(Use the flag to prove it by induction)

Exercise:

1) *I cheated a a little bit when I said that $H^i(C,\mathcal{O}_C(-\infty\mathfrak{p}))=0$ for $i\geq 2$. To see this we need to know that $C\setminus\mathfrak{p}$ is affine. Derive this from the theorem of Riemann-Roch. Why is this not a circulus vitiosus?*

Differentials and Residues

On our curve C/k we have two privileged line bundles (???). The first one is the structure sheaf $\mathcal{O}_C$ and the other one is the sheaf $\Omega^1_{C/k}$ of differentials. It is called the *canonical bundle*. We know the degree of $\mathcal{O}_C$ and we computed the cohomology groups $H^0(C,\mathcal{O}_C)=k$ and $\dim_k H^1(C,\mathcal{O}_C)=g$ where the second assertion is tautological.

Our next aim is the study of $\Omega^1_{C/k}$. We will show that

$$\deg(\Omega^1_{C/k}) = 2g-2$$

and that we have a canonical isomorphism

$$\mathrm{Res}: H^1(C,\Omega^1_{C/k}) \to k.$$

Here we mean by canonical that this map is consistent with the trace map which we defined for separable morphisms. If we have C_1/k, C_2/k and if $f: C_1\to C_2$ is a finite separable morphism, then we defined the $\mathrm{tr}_{C_1/C_2}: f_*(\Omega^1_{C_1/k}) \longrightarrow \Omega^1_{C_2/k}$. and we require that the resulting k-linear map yields a commutative diagram

$$\begin{array}{ccc} H^1(C_1,\Omega^1_{C_1/k}) & \xrightarrow{\mathrm{tr}} & H^1(C_2,\Omega^1_{C_2/k}) \\ & \mathrm{Res}\searrow \quad \swarrow\mathrm{Res} & \\ & k & . \end{array}$$

Of course we also want compatibility with base change in the obvious sense.

These assertions are much more difficult to prove.

If we accept that they are true then we can take any line bundle $\mathcal{L}$ and define a pairing

$$H^0(C,\mathcal{L})\otimes H^1(C,\mathcal{L}^{-1}\otimes\Omega^1_{C/k})\to H^1(C,\Omega^1_{C/k})=k$$

which turns out to be non degenate. As a special case we get that

$$H^0(C,\Omega^1_{C/k})=H^1(C,\mathcal{O}_{C/k})^\vee$$

and

$$\dim_k(H^0(C,\Omega^1_{C/k})=g.$$

To understand the pairing we recall how we computed $H^1(C,\mathcal{O}_C)$. We looked at effective divisors

$$D=\sum n_\mathfrak{p}\mathfrak{p}$$

with $n_\mathfrak{p}\geq 0$, and at the resulting exact sequence

$$0\to\mathcal{O}_C\to\mathcal{O}_C(-D)\to\mathcal{O}_C(-D)/\mathcal{O}_C\to 0.$$

The quotient sheaf $\mathcal{O}_C(-D)/\mathcal{O}_C$ is a collection of Laurent expansions at the points in the support of D. We introduce the notation $\mathbb{L}(-D)$ for it. For any closed point $\mathfrak{p}$ in $|D|$ the stalk of $\mathbb{L}(-D)$ at $\mathfrak{p}$ is

$$\{f\in k(C)|\operatorname{ord}_\mathfrak{p}(f)\geq -n_\mathfrak{p}\}=\pi^{-n_\mathfrak{p}}\mathcal{O}_{C,\mathfrak{p}}/\mathcal{O}_{C,\mathfrak{p}}$$

where $\pi_\mathfrak{p}$ is a generator of the maximal ideal $\mathfrak{p}\subset\mathcal{O}_{C,\mathfrak{p}}$. We may (or may not) pass to the direct limit

$$\varinjlim_D\mathcal{O}_C(-D)/\mathcal{O}_C=\mathbb{L}$$

where $\mathbb{L}$ is the sheaf of all Laurent expansions: For an open set U its sections are given by

$$\mathbb{L}(U)=\bigoplus_{\mathfrak{p}\in U,\ \mathfrak{p}\ \text{closed}}K/\mathcal{O}_{C,\mathfrak{p}}.$$

We introduce the completions of the local rings $\varprojlim(\mathcal{O}_{C,\mathfrak{p}}/\mathfrak{p}^n)=\hat{\mathcal{O}}_{C,\mathfrak{p}}$ and the quotient fields of these completions $\hat{K}_\mathfrak{p}$. We define the module of differentials $\hat{\Omega}^1_{C_1,\mathfrak{p}}=\Omega^1_{C,\mathfrak{p}}\otimes\hat{\mathcal{O}}_{C,\mathfrak{p}}$. Of course this is again a free module of rank one over $\hat{\mathcal{O}}_{C,\mathfrak{p}}$ and it is not difficult to see that it is a universal separated module for continuous differentials. We may also introduce the infinitesimal meromorphic differentials $\hat{\Omega}^1_{C,\mathfrak{p}}=\Omega^1_{C,\mathfrak{p}}\otimes\hat{K}_\mathfrak{p}$

Now we choose an effective divisor with $\deg(D) >> 0$ and we consider the long exact sequence in cohomology. We have seen that $H^1(C, \mathcal{O}_C(-D)) = 0$ provided $\deg(D) >> 0$ and hence we get

$$0 \to H^0(C, \mathcal{O}_C) \to H^0(C, \mathcal{O}_C(-D)) \to H^0(C, \mathcal{O}_C(-D)/\mathcal{O}_C) \to H^1(C, \mathcal{O}_C) \to 0$$

if $\deg(D) >> 0$, we could as well look at the same sequence where we passed to the limit over all D.

Hence we can interprete $H^1(C, \mathcal{O}_C)$ as a quotient of a space of Laurent expansions modulo the space of those Laurent expansions which come from a meromorphic function. This means that have to understand the obstruction for a collection of Laurent expansions

$$\xi \in \oplus_{\mathfrak{p}} \pi_{\mathfrak{p}}^{-n_{\mathfrak{p}}} \mathcal{O}_{C,\mathfrak{p}}/\mathcal{O}_{C,\mathfrak{p}}$$

to come from a meromorphic function. The point is that the holomorphic differentials produce such obstructions.

We introduce the sheaf $\mathbb{L}(-D) = \mathcal{O}_C(-D)/\mathcal{O}_C$, and we identify it with its space of global sections, i.e. we write

$$H^0(C, \mathbb{L}(-D)) \simeq \mathbb{L}(-D).$$

For any $\mathfrak{p} \in |D|$ we have

$$\mathbb{L}(-D)_{\mathfrak{p}} = \pi_{\mathfrak{p}}^{-n_{\mathfrak{p}}} \mathcal{O}_{C,\mathfrak{p}}/\mathcal{O}_{C,\mathfrak{p}},$$

and we will define a residue map

$$\mathrm{res}_{\mathfrak{p}} : \varinjlim \pi_{\mathfrak{p}}^{-n} \mathcal{O}_{C,\mathfrak{p}}/\mathcal{O}_{C,\mathfrak{p}} \otimes \Omega^1_{C,\mathfrak{p}} = \varinjlim \pi^{-n} \Omega^1_{C,\mathfrak{p}}/\Omega^1_{C,\mathfrak{p}} \xrightarrow{res_{\mathfrak{p}}} k.$$

The definition of this residue map requires some work especially if our ground field is of positive characteristic.

Once we have this map, then we can define

$$\begin{aligned} \mathbb{L}(-D) \otimes H^0(C, \Omega^1_C) &\longrightarrow k \\ \underline{\xi} \otimes \omega &\longrightarrow \sum_{\mathfrak{p} \in |D|} \mathrm{res}_{\mathfrak{p}}(\xi_{\mathfrak{p}} \otimes \omega) \end{aligned}$$

and we will show:

An element $\underline{\xi} \in \mathbb{L}(-D)$ is the Laurent expansion of an element $f \in H^0(C, \mathcal{O}_C(-D))$ if and only if

$$\sum_{\mathfrak{p}} \mathrm{res}_{\mathfrak{p}}(f\omega) = 0.$$

It is clear that this says that we get a nondegenerate pairing

$$H^1(C, \mathcal{O}_C) \times H^0(C, \Omega^1_C) \longrightarrow k.$$

The elements

$$\omega' = f\omega \in H^0(C, \Omega^1_C(-D))$$

are called meromorphic differentials and part of the assertion above is that

$$\sum_{\mathfrak{p}} \mathrm{res}_{\mathfrak{p}}(\omega') = 0$$

for any meromorphic differential.

We come to the definition of the residue map

$$\mathrm{res}_{\mathfrak{p}} : \lim_{\rightarrow} \ \pi_{\mathfrak{p}}^{-n}\mathcal{O}_{C,\mathfrak{p}}/\mathcal{O}_{C,\mathfrak{p}} \otimes \Omega^1_{C,\mathfrak{p}} \longrightarrow k.$$

We notice that we can replace $\mathcal{O}_{C,\mathfrak{p}}$ by the completion because

$$\pi_{\mathfrak{p}}^{-n}\mathcal{O}_{C,\mathfrak{p}}/\mathcal{O}_{C,\mathfrak{p}} = \pi_{\mathfrak{p}}^{-n}\hat{\mathcal{O}}_{C,\mathfrak{p}}/\hat{\mathcal{O}}_{C,\mathfrak{p}}.$$

At first we define this residue map only for rational points, i.e. those for which $k(\mathfrak{p}) = k$. We change the notation slightly and denote the local parameter by X, then we have(???)

$$\hat{\mathcal{O}}_{C,\mathfrak{p}} = k[[X]],$$

the quotient field is

$$k((X)) = k[[X]]\left[\frac{1}{X}\right],$$

and our differentials are of the form

$$\omega = f(X)dX = \left(\frac{a_{-n}}{X^n} + \frac{a_{n-1}}{X^{n-1}} + \cdots + \frac{a_{-1}}{X} + \cdots\right) dX,$$

and we define

$$\mathrm{res}_{\mathfrak{p}}(\omega) = \mathrm{res}_{\mathfrak{p}}(f \otimes dX) = a_{-1}.$$

We have to show that this definition does not depend on the choice of the local parameter X. If we replace X by another local parameter

$$Y = tX + u_2X^2 + \cdots + u_nX^n + \cdots$$

where $t \neq 0$ and otherwise $P(X)$ is an arbitrary power series, then

$$\omega = \left(\frac{a'_{-n}}{Y^n} + \cdots + \frac{a'_{-1}}{Y'} \cdots\right) dY,$$

and we want (invariance of the residue)

$$a_{-1} = a'_{-1}.$$

This is indeed the case, but it is not so easy to prove.

If we want to prove the invariance of the residue, we have to write X as a power series in Y

$$X = \tau Y + v_2 Y^2 + \cdots + v_m Y^m + \cdots = Q(Y)$$

where $\tau = t^{-1}$ and the v_μ/τ are polynomial in the u_ν/t.

Then we have to expand

$$\frac{dX}{X^m} = \frac{\tau + 2v_2 Y + \cdots m v_m Y^{m-1} + \cdots}{\tau^m Y^m (1 + \frac{v_2}{\tau} Y + \frac{v_m}{\tau} \cdots Y^{m-1})^m}\, dY = \left(\cdots \frac{a'_{-1}}{Y} \cdots \right)\, dY.$$

We have to show that in this expansion the coefficient a'_{-1} of $\frac{dY}{Y}$ is

(i) equal to one if $m = 1$

and

(ii) it is zero if $m \geq 2$.

The first assertion is clear because the factor τ cancels. But the second assertion is not so clear. (I recommend to do the calculation for some small values of m.)

It is clear that our coefficient is a polynomial

$$P_m \left(\frac{v_1}{\tau}, \cdots, \frac{v_m}{\tau} \right)$$

times τ^{1-m}, hence we have to show that this polynomial is identically zero if $m \geq 2$. (We can view the $\frac{v_\nu}{\tau}$ as indeterminates.)

Now we can prove the invariance of the residue by an argument which goes back to H. Hasse.

If the characteristic of k is equal to zero, then we can write for $m \geq 2$

$$\frac{dX}{X^m} = -\frac{1}{m-1} dX^{1-m}$$

but then

$$\left(\frac{a'_{-m}}{Y^m} + \cdots \frac{a'_{-1}}{Y} \cdots \right) dY = -\frac{1}{m-1} dQ(Y)^{1-m},$$

and this is only possible if $a'_{-1} = 0$.

This argument fails if $\mathrm{char}(k) = p > 0$. For instance

$$\frac{dX}{X^{p+1}}$$

is not of the form $dG(X)$. But we have to show that the polynomial $P_m\left(\frac{v_1}{\tau}, \cdots, \frac{v_m}{\tau}\right)$ vanishes identically. If we start from the case $k = \mathbb{Q}$, then we see easily that $P_m\left(\frac{v_1}{\tau}, \cdots, \frac{v_m}{\tau}\right)$ has integer coefficients, hence it lies in

$$\mathbb{Z}\left[\frac{v_1}{\tau}, \cdots, \frac{v_m}{\tau}\right].$$

Moreover it is obvious that the corresponding polynomial in arbitrary characteristic is obtained by reducing the coefficients $\bmod p$, it lies in

$$\mathbb{F}_p\left[\frac{v_1}{\tau}, \cdots, \frac{v_m}{\tau}\right] \subset k\left[\frac{v_1}{\tau}, \cdots, \frac{v_m}{\tau}\right].$$

Since the polynomial in $\mathbb{Z}\left[\frac{v_1}{\tau}, \cdots, \frac{v_m}{\tau}\right]$ must become the zero polynomial in $\mathbb{Q}\left[\frac{v_1}{\tau}, \cdots, \frac{v_m}{\tau}\right]$, it must be identically zero itself. This proves (ii) in general.

We observe that our problem to construct $\mathrm{res}_{\mathfrak{p}}$ is purely local, it concerns the structure of the space of meromorphic differentials in an infinitesimally small neighborhood of $\mathfrak{p}$.

We keep in mind that so far we constructed the residue map only for rational points.

In the next section we will give a second proof of the invariance of the residue which uses global arguments and which I like better. In this second proof we will use the fact that for differentials which have only a first order pole the invariance of the residue is obvious. (Assertion (i) above)

The other assertions which we made in the beginning of this section are global in nature. First we will prove for the special case $\mathbb{P}^1/k$. In the course of this proof we will see that the well definedness of $\mathrm{res}_{\mathfrak{p}}$ follows from global information.

Hence we concentrate on the special case $C = \mathbb{P}^1$. We have seen $H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}) = 0$ hence $g = 0$, we know that

$$\Omega^1_{\mathbb{P}^1} \simeq \mathcal{O}_{\mathbb{P}^1}(-2)$$

(see ???) and hence we have

$$\deg(\Omega^1_{\mathbb{P}^1}) = 0 - 2 = -2.$$

We want to construct the canonical isomorphism

$$\mathrm{Res} : H^1(\mathbb{P}^1, \Omega^1_{\mathbb{P}^1}) \longrightarrow k.$$

To do this we apply a principle which also works in the general situation. So let us return to the case of an arbitrary smooth, projective and absolutely irreducible curve $C = C/k$. We will show later:

Proposition xxx:*For any non zero effective divisor D we have $H^1(C, \Omega^1_C) = 0$ and especially for any k rational point $a \in C(k)$ we have an isomorphism*

$$\delta : H^0(C, \mathbb{L}(-a) \tilde{\to} H^1(C, \Omega^1_C).$$

The map δ is of course the boundary morphism.

This proposition is of course clear for the case $C = \mathbb{P}^1$ because $\Omega^1_{\mathbb{P}^1}, (-a) \simeq \mathcal{O}_{\mathbb{P}^1}(-1)$ and

$$H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-1)) = H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-1)) = 0.$$

So we pick a point $a \in \mathbb{P}^1(k)$ and consider the sequence

$$0 \to \Omega^1_{\mathbb{P}^1} \to \Omega^1_{\mathbb{P}^1}(-a) \to \mathbb{L}(-a) \otimes \Omega^1_{\mathbb{P}^1} \to 0$$

which gives us an isomorphism

$$\delta_a : \mathbb{L}(-a) \otimes \Omega^1_{\mathbb{P}^1} \tilde{\longrightarrow} H^1(\mathbb{P}^1, \Omega^1_{\mathbb{P}^1})$$

Now we noticed that we can define the map

$$\mathrm{res}_a : \mathbb{L}(-a) \otimes \Omega^1_{\mathbb{P}^1} \longrightarrow k,$$

and we put

$$\mathrm{Res} = \mathrm{res}_a \circ \delta_a^{-1}.$$

Of course something would be wrong if we did not have the problem of well definedness again. Let us pick a point $b \in \mathbb{P}^1(k)$ which is different from a. Then we get a diagram

$$\begin{array}{ccc}
\mathbb{L}(-a) \otimes \Omega^1_{\mathbb{P}^1} & & \\
& \searrow^{i_a} & \\
& & \mathbb{L}(-a-b) \otimes \Omega^1_{\mathbb{P}^1} \xrightarrow{\delta} H^1(\mathbb{P}^1, \Omega^1_{\mathbb{P}}) \\
& {}_{i_b} \nearrow & \\
\mathbb{L}(-b) \otimes \Omega^1_{\mathbb{P}^1} & &
\end{array}$$

Now the map δ has a nontrivial kernel and it is clear that this kernel is spanned by the form

$$\omega_{a,b} = \frac{dx}{x-a} - \frac{dx}{x-b}$$

which is a generator of $H^0(\mathbb{P}^1, \Omega^1_{\mathbb{P}^1}(-a-b))$. Since $\mathrm{res}_a(\omega_{a,b}) + \mathrm{res}_b(\omega_{a,b}) = 0$. But $\delta_a = \delta \circ i_a$ and $\delta_b = \delta \circ i_b$ and it is clear that

$$\mathrm{res}_a \circ \delta_a^{-1} = \mathrm{res}_b \circ \delta_b^{-1}.$$

Now we consider the special point $\infty \in \mathbb{P}^1(k)$. We have

$$\operatorname{Spec}(k[X]) \cup \operatorname{Spec}(k[Y]) = \mathbb{P}^1$$

where $XY = 1$ and ∞ is given by $0 \in \operatorname{Spec}(k[Y])$. We consider the exact sequence

$$0 \to \Omega^1_{\mathbb{P}^1} \to \Omega^1_{\mathbb{P}^1}(-n\infty) \to \mathbb{L}(-n\infty) \otimes \Omega^1_{\mathbb{P}^1} \to 0$$

and get

$$\delta : \mathbb{L}(-n\infty) \otimes \Omega^1_{\mathbb{P}^1} \to H^1(\mathbb{P}^1, \Omega^1_{\mathbb{P}^1}) \to 0.$$

Hence we try a new definition

$$\operatorname{res}'_\infty = \operatorname{Res} \circ \delta$$

for the local residue map.

We want to give a local formula for $\operatorname{res}'_\infty$, this is easy. We observe that the elements of $\mathbb{L}(-n\infty) \otimes \Omega^1_{\mathbb{P}^1}$ can be written as

$$\omega = + \left(\frac{a_1}{Y} + \frac{a_2}{Y^2} \cdots \frac{a_n}{Y^n} \right) \otimes dY = -(a_1 X + \cdots a_n X^n) \otimes \frac{dX}{X^2}.$$

By definition we have $\operatorname{res}'_\infty(\omega) = 0$ if and only if the element ω is the image of

$$H^0(\mathbb{P}^1, \Omega^1_{\mathbb{P}^1}(-n\infty)) \longrightarrow \mathbb{L}(-na) \otimes \Omega^1_{\mathbb{P}^1},$$

and this is the case if and only if $a_1 = 0$. We can also say that the form

$$-(a_1 X + \cdots + a_n X^n) \frac{dX}{X^2}$$

is holomorphic on $\operatorname{Spec}(k[X]) \setminus \{0\}$ and has a pole of order ≤ 1 at zero with residue $-a_1$ at 0. Hence it is clear that

$$\operatorname{res}'_\infty(\omega) = a_1,$$

and this is our old definition of the residue with respect to the parameter Y at infinity.

Now we give a second proof for the independence of the local parameter of the residue map.

We have seen that it suffices to show that the differentials

$$\frac{dY}{Y^m} \in \mathbb{L}(-n\infty) \otimes \Omega^1_{\mathbb{P}^1}$$

which have residue zero for $m > 1$ still have residue zero if we make a change of local parameters

$$Y = tU + v_2 U^2 \cdots + v_r U^r + \cdots$$

where $t \neq 0$.

But we know how to characterize the elements in $\mathbb{L}(-n\infty) \otimes \Omega^1_{\mathbb{P}^1}$ which have residue zero. These are the elements which lie on the image of

$$H^0(\mathbb{P}^1, \Omega^1_{\mathbb{P}^1}(-n\infty)) \longrightarrow \mathbb{L}(-n\infty) \otimes \Omega^1_{\mathbb{P}^1}.$$

We approximate the power series which defines the change of parameters by a polynomial of degree $r \geq n$, only the first in coefficients are relevant. Since the degree can be chosen to be larger, we may assume that the polynomial

$$F(U) = tU + v_2U^2 + \cdots + v_rU^r$$

is separable.

It provides an inclusion $k[X] \subset k[U]$ and a morphism

$$\Phi_F : \mathbb{P}^1 \longrightarrow \mathbb{P}^1$$

where the first $\mathbb{P}^1$ is $\mathrm{Spec}(k[U]) \subset \mathrm{Spec}\, k[U^{-1}]$ and the second one is $\mathrm{Spec}(k[X]) \subset \mathrm{Spec}\, k[X^{-1}]$. We can consider the pull back $\Phi_F^*(\frac{dX}{X^m}) = \omega$, this form has poles of order m in the point $U = 0$ and in the other $r-1$ points $u_2 \cdots u_r$ where F vanishes. (We assume that F is separable.)

In the point $u = 0$ its Laurent expansion is

$$\frac{F'(U)dU}{F(U)^m},$$

and we want to show that the residue at $u = 0$ with respect to the local parameter U is zero.

But it has some more poles at $u_2 \cdots u_r$. The total order of the pole is rm. Hence ω must have a zero divisor Z of degree $rm - 2$ because the degree of Ω^1 is -2. We consider the line bundle

$$\Omega^1_{\mathbb{P}^1}(-Z + ma_2 + \cdots + ma_r).$$

If we multiply ω by a section $f \in H^0(\mathbb{P}^1, \Omega^1_{\mathbb{P}^1}(-Z + ma_2 \cdots + ma_r))$, then we get a meromorphic form $f\omega$ which now has only one pole at $U = 0$, the other poles are cancelled.

Of course we may have changed the expansion at $U = 0$. But if we map f to $\mathcal{O}_{\mathbb{P}^1,0}/(U)^{m-1}\mathcal{O}_{\mathbb{P}_1,0}$, then we get an exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^1}(-Z+ma_2+\cdots+ma_r+(m-1)(0)) \to \mathcal{O}_{\mathbb{P}^1}(-Z+ma_2+\cdots+ma_r) \to \mathcal{O}_{\mathbb{P}^1}/(U)^{m-1}\mathcal{O}_{\mathbb{P}^1} \to 0,$$

and the line bundle on the left has degree -1, hence it has no cohomology. This yields that

$$H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-Z + ma_2 + \cdots + ma_r) \xrightarrow{\sim} \mathcal{O}_{\mathbb{P}^1}/(U)^{m-1}\mathcal{O}_{\mathbb{P}^1}$$

is an isomorphism. This means that we find an $f \in H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-Z + ma_2 + \cdots + ma_r))$ such that its expansion at $U = 0$ with respect to U is

$$f(U) = 1 + a_m U^m + \cdots .$$

Now $f\omega$ has a pole only at $U = 0$ and its polar expansion is the same as the polar expansion of ω. Since this polar expansion is the expansion of $f\omega$ the residue must be zero.

Historical and heuristical remark:

If our ground field is the field of complex numbers then the set of complex points $S = C(\mathbb{C})$ carries the structure of a compact Riemann surface. (Complex manifold in dimension one it has a natural orientation!) . In this case everything is much easier. A meromorphic differential ω which has a pole at a point P can be written locally as

$$\Omega^1_{C/k} = f(z)dz$$

where z is a uniformizing element at P and f is meromorphic. In this case we look at a small disc around P and we know that

$$\frac{1}{2\pi i}\oint \omega = \mathrm{res}_p(\omega)$$

where the left hand side is defined intrinsically as the integral over the boundary of the disc taken counterclockwise and where the right hand side is computed in terms of the Laurent expansion of $f(z)$. Hence it is clear that the right hand side does not depend on the choice of the local parameter z.

A meromorphic differential has a finite number of poles and we can compute the sum of the residues at these poles as the sum of the integrals over the boundary of these discs. Then we consider the complement U of the open discs. This is a compact manifold with boundary ∂U= union of the boundaries of the discs. We can apply Stokes theorem which says the the integral over this boundary is equal to the integral of $d\omega$ over U. But ω is holomorphic on U hence we have $d\omega = 0$, and we see that the sum over the residues is zero.

It is interesting to look into the original paper by Riemann and to find out what Riemann and Roch actually proved.

For them cohomology did not exist, hence they could not define $H^1(S, \mathcal{O}_S)$ and $g = \dim H^1(S, \mathcal{O}_S)$.

On the other hand they could of course study the cokernel of

$$H^0(S, \mathcal{O}_S(-D)) \longrightarrow \mathbb{L}(-D)$$

which is this cohomology if $\deg(D) >> 0$.

Of course it was clear that the holomorphic 1-forms produced linear forms on this cokernel. They could define $g = \dim H^0(S, \Omega^1_S)$ and it was clear to them that $2g$ is the the first Betti- number. It was also proved that the common kernel of linear forms produced by the differentials described the image of the map above. This is the content of Serre duality, which is much deeper. Therefore it i clear that they proved more that just the first version of Riemann-Roch.

We have to understand the sheaf $\Omega^1_{C/k}$. To do this we consider a finite morphism

$$\pi : C \longrightarrow \mathbb{P}^1$$

which is induced by a choice of a function $f \in k(C)$ for which $k(C)/k(f)$ is separable.

In this case the functor π_* which sends coherent sheaves on C to coherent sheaves on $\mathbb{P}^1$ is acyclic because the fibres are finite. Hence we know that for any coherent sheaf $\mathcal{F}$ on C

$$H^\bullet(C, \mathcal{F}) \simeq H^\bullet(\mathbb{P}^1, \pi_*(\mathcal{F})).$$

For any line bundle $\mathcal{L}$ on C the bundle $\pi_*(\mathcal{L})$ is a vector bundle of rank $d =$ degree (π) over $\mathbb{P}^1$.

We apply Grothendieck's theorem to this vector bundle and write

$$\pi_*(\mathcal{L}) = \mathcal{O}_{\mathbb{P}^1}(a_1) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^1}(a_d)$$

with $a_1 \geq a_2 \cdots \geq a_d$.

If $\mathcal{L} = \mathcal{O}_C$ we have $H^0(C, \mathcal{O}_C) = k$ and this implies

$$a_1 = 0 \qquad \text{and} \qquad a_\nu < 0 \qquad \text{for all} \qquad \nu \geq 2.$$

Since by definition

$$\dim H^1(C, \mathcal{O}_C) = \dim H^1(\mathbb{P}^1, \pi_*(\mathcal{O}_C)) = g$$

we must have (see ???)

$$g = \sum_{\nu=2}^{d} (-a_\nu - 1) = -\sum_{\nu=2}^{d} a_\nu - (d-1).$$

We invoke the Riemann-Hurwitz formula

$$\Omega^1_{\mathbb{P}^1} \otimes \mathfrak{D}^{-1}_{C/\mathbb{P}^1} \xrightarrow{\sim} \Omega^1_C$$

By construction the vector bundle $\pi_*(\mathfrak{D}^{-1}_{C/\mathbb{P}^1})$ is dual to the bundle $\pi_*(\mathcal{O}_C)$. We wrote

$$\pi_*(\mathcal{O}_C) = \mathcal{O}_{\mathbb{P}^1} \oplus \bigoplus_{\nu=2}^{d} \mathcal{O}_{\mathbb{P}^1}(a_\nu),$$

hence we see that

$$\pi_*(\mathfrak{D}^{-1}_{C/\mathbb{P}^1}) = \mathcal{O}_{\mathbb{P}^1} \oplus \bigoplus_{\nu=2}^{d} \mathcal{O}_{\mathbb{P}^1}(-a_\nu)$$

and

$$\pi_*(\Omega^1_C) = \mathcal{O}_{\mathbb{P}^1}(-2) \oplus \bigoplus_{\nu=2}^{d} \mathcal{O}_{\mathbb{P}^1}(-a_\nu - 2).$$

Since the $a_\nu < 0$ we see $-a_\nu - 2 \geq -1$. We get

$$H^1(C, \Omega^1_C) \simeq H^1(C, \pi_*(\Omega^1_C)) = H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-2)) \simeq k$$

and

$$\dim_k H^0(C, \Omega^1_C) = \sum_{v=2}^{d} (-a_\nu - 1) = g.$$

For any line bundle $\mathcal{L}$ on C and any effective divisor D we can consider the exact sequence

$$0 \longrightarrow \mathcal{L} \longrightarrow \mathcal{L}(-D) \longrightarrow \mathbb{L}(-D) \longrightarrow 0$$

which yields the exact sequence

$$0 \longrightarrow \pi_*(\mathcal{L}) \longrightarrow \pi_*(\mathcal{L}(-D)) \longrightarrow \pi_*(\mathbb{L}(-D)) \longrightarrow 0$$

and $\pi_*(\mathbb{L}(-D))$ is a torsion sheaf on $\mathbb{P}^1$, its space of section has

$$\dim_k \pi_*(\mathbb{L}(-D)) = \deg D.$$

Hence we see that (see our fourmula ???)

$$\deg(\pi_*(\mathcal{L}(-D)) = \deg(\pi_*(\mathcal{L})) + \deg D.$$

If we apply this to the exact sequence

$$0 \longrightarrow \mathcal{O}_C \longrightarrow \mathfrak{D}^{-1} \longrightarrow \mathfrak{D}^{-1}/\mathcal{O}_{\mathcal{C}} \to 00$$

we get

$$\deg(\pi_*(\mathfrak{D}^{-1})) = \deg(\pi_*(\mathcal{O}_C)) + \dim_k(\mathfrak{D}^{-1}/\mathcal{O}).$$

On the other hand

$$\deg(\pi_*(\mathcal{O}_C)) + \deg(\pi_*(\mathfrak{D}^{-1})) = 0$$

because these bundlex are dual to eachother and hence

$$\dim_k(\mathfrak{D}^{-1}/\mathcal{O}) = 2\deg(\pi_*(\mathfrak{D}^{-1})) = 2\left(-\sum_{\nu=2}^{d} a_\nu\right) = 2(g+d-1).$$

This can be read on C. We get

$$\deg(\mathfrak{D}^{-1}) = 2(g-d+1).$$

On the other hand we know that $\deg(\pi^*(\Omega^1_{\mathbb{P}^1})) = -2d$ and therefore

$$\deg(\Omega^1_C) = -2d + 2(g+d-1) = 2g-2.$$

We want to construct an explicit linear map

$$\mathrm{Res}: H^1(C, \Omega^1_C) \longrightarrow k.$$

To get this map we proceed in the same way as in the case $C = \mathbb{P}^1$. We pick a point $x \in C(k)$ and consider the line bundle $\Omega^1_C(-x)$. I claim that

$$H^1(C, \Omega^1_C(-x)) = 0.$$

To see that this is so we choose a meromorphic f such that df generates the differential at x and we consider the map induced by f

$$\pi_f = \pi : C \longrightarrow \mathbb{P}^1,$$

it is étale at the point a. It is clear that the bundles

$$\pi_*(\mathcal{O}_C(x)) \qquad \text{and} \qquad \pi_*(\mathfrak{D}^{-1}(-x))$$

are dual to each other. We have

$$\pi_*(\mathcal{O}_C(x)) \subset \pi_*(\mathcal{O}_C) = \mathcal{O}_{\mathbb{P}^1} \oplus \bigoplus_{v=2}^{d} \mathcal{O}_{\mathbb{P}^1}(-a_\nu),$$

and the degree of this subbundle drops by one. Since $\pi_*(\mathcal{O}_C(x))$ has no non trivial section, we conclude that

$$\pi_*(\mathcal{O}_C(x)) \simeq \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \bigoplus_{v=2}^{d} \mathcal{O}_{\mathbb{P}^1}(-a_\nu)$$

and hence because of duality

$$\pi_*(\mathfrak{D}^{-1}(-x)) = \mathcal{O}_{\mathbb{P}^1}(1) \oplus \bigoplus_{v=2}^{d} \mathcal{O}_{\mathbb{P}^1}(a_\nu).$$

But now

$$H^1(\mathbb{P}^1.\pi_*(\pi^*(\Omega^1_{\mathbb{P}^1}) \otimes \mathfrak{D}^{-1}(-x))) = H^1(C, \Omega^1_C(-x)) = 0.$$

We have the exact sequence

$$0 \longrightarrow \Omega^1_C \longrightarrow \Omega^1_C(-x) \longrightarrow \mathbb{L}(-x) \otimes \Omega^1_C \longrightarrow 0$$

and obtain the exact sequence in cohomology

$$H^0(C, \Omega^1_C) \to H^0(C, \Omega^1_C(-x)) \to H^0(C, \mathbb{L}(-x) \otimes \Omega^1_C) \xrightarrow{\delta} H^1(C, \Omega^1_C) \to 0.$$

The map δ must be an isomorphism, we have

$$\mathrm{res}_x : H^0(C, \mathbb{L}(-x) \otimes \Omega^1_C) \xrightarrow{\sim} k$$

and define

$$\mathrm{Res} = \mathrm{res}_x \circ \delta^{-1}.$$

To see that this morphism does not depend on the choice of a, we choose a second point b. Now we consider the exact sequence

$$0 \to \Omega^1_C \to \Omega^1_C(-a-b) \to \mathbb{L}(-a-b) \otimes \Omega^1_C \to 0,$$

and find that there exists a 1-form $\omega'_{a,b}$ on C which has a simple pole at a and b and is holomorphic elswhere. We choose an $f \in k(C)$ such that df generates Ω^1_C in the two points a, b and consider the resulting morphism

$$\pi = \pi_F : C \longrightarrow \mathbb{P}^1.$$

This morphism π induces isomorphisms between the completions $\hat{\mathcal{O}}_{C,a} \simeq \hat{\mathcal{O}}_{\mathbb{P}^1,\pi(a)}, \hat{\mathcal{O}}_{C,b} \simeq \hat{\mathcal{O}}_{\mathbb{P}^1,\pi(a)}$, and it is clear that

$$\begin{aligned}
\mathrm{res}_a(\omega'_{a,b}) &= \mathrm{res}_{\pi(a)}\left(\mathrm{tr}_{C/\mathbb{P}^1}(\omega'_{a,b})\right) \\
\mathrm{res}_b(\omega'_{a,b}) &= \mathrm{res}_{\pi(b)}\left(\mathrm{tr}_{C/\mathbb{P}^1}(\omega'_{a,b})\right).
\end{aligned}$$

Of course it is clear that $\mathrm{tr}_{C/\mathbb{P}^1}(\omega'_{a,b})$ is a non zero multiple of the 1-form $\omega_{\pi(a),\pi(b)}$ which we constructed on $\mathbb{P}^1$, hence we get

$$\mathrm{res}_a(\omega'_{a,b}) + \mathrm{res}_b(\omega'_{a,b}) = 0.$$

This argument shows that $\mathrm{Res} : H^1(C, \Omega^1_C) \to k$ is well defined but it also shows that it is compatible with a map

$$\pi : C \longrightarrow \mathbb{P}^1$$

as above: By construction we habe the commutative diagram

$$\begin{array}{ccc} H^0(C, \mathbb{L}(-a) \otimes \Omega^1_C) & \longrightarrow & H^1(C, \Omega^1_C) \\ \downarrow \mathrm{tr}_{C/\mathbb{P}^1} & & \downarrow \mathrm{tr}_{C/\mathbb{P}^1} \\ H^0(\mathbb{P}^1, \mathbb{L}(-\pi(a)) \otimes \Omega^1_{\mathbb{P}^1}) & \longrightarrow & H^1(\mathbb{P}^1, \Omega^1_{\mathbb{P}^1}) \end{array}$$

Which then implies the commutativity

$$\begin{array}{ccc} H^1(C, \Omega^1_C) & & \\ & \searrow^{\mathrm{Res}} & \\ \downarrow \mathrm{tr}_{C/\mathbb{P}^1} & & k \\ & \nearrow^{\mathrm{Res}} & \\ H^1(\mathbb{P}^1, \Omega^1_{\mathbb{P}^1}) & & \end{array}$$

and then it is easy to derive the general compatibility for arbitrary separable $d : C_1 \to C_2$.

It is also clear that in the diagram above we can replace $\mathbb{L}(-a)$ by $\mathbb{L}(-na)$ with any $n > 0$, then we get a commutative diagram

$$\begin{array}{ccc} H^0(C, \mathbb{L}(-na)) & \xrightarrow{\delta} & H^1(C, \Omega^1_C) \\ & \searrow^{\mathrm{res}_a} \quad \swarrow^{\mathrm{Res}} & \\ & k & . \end{array}$$

Hence we see that under the assumption that k is algebraically closed, we have for any effective divisor $D \neq 0$ that

$$\begin{array}{ccccc} H^0(C, \omega^1_C(-D)) & \to & H^0(C, \mathbb{L}(-D) \otimes \Omega^1_C) & \to & H^1(C, \Omega^1_C) \to \\ & & \sum_{a \in |D|} \mathrm{res}_a \searrow & & \swarrow \mathrm{Res} \\ & & & k & \end{array}$$

is an exact sequence and the triangle in the bottom is commutative.

Finally we have to remove our assumption that k is algebraically closed.

We needed a rational point $a \in C(k)$ to construct $\mathrm{Res} : H^1(C, \Omega^1_C) \to k$ and so far we defined the residue map

$$\mathrm{res}_{\mathfrak{p}} : \mathbb{L}(-\infty\mathfrak{p}) \otimes \Omega^1_{C,\mathfrak{p}} \longrightarrow k$$

only fo rational points $\mathfrak{p} \in C(k)$. The decisive point is that we can extend the definition easily to "separable" points, i.e. points for which $k(\mathfrak{p})/k$ is a separable extension and in addition we always have many "separable" points on C.

How do we get separable points? We choose an $f \in k(C)$ such that $k(C)/k(f)$ becomes separable. Then we have the resulting $\pi_f : C \to \mathbb{P}^1$. We have a non empty affine open set $V \subset \mathbb{P}^1$ such that $\pi^{-1}(V) = U \to V$ is unramified.

Now if k is finite, then any point $\mathfrak{p} \in C$ is separable. Otherwise it is clear that $V(k)$ is infinite and for any $\mathfrak{p}_0 = X_0 \in V(k)$ the points $\mathfrak{y} \in C$ lying over $\mathfrak{p}_0$ are separable.

(Vielleicht hier Ref. auf früher ???)

Now we want to define $\mathrm{res}_{\mathfrak{p}}$ for a separable point $\mathfrak{p} \in C$.

Text schon vorhanden!

Now we define $\mathrm{Res} : H^1(C, \Omega^1_C) \to k$ as before: We pick a separable point $\mathfrak{p} \in C$ and consider the exact sequence in cohomology

$$H^0(C, \Omega^1_C(-\mathfrak{p})) \to H^0(C, \mathbb{L}(-\mathfrak{p}) \otimes \Omega^1_C) \xrightarrow{\delta} H^1(C, \Omega^1_C) \to 0.$$

Here the map δ is not necessarily an isomorphism. But if we extend our base field to the algebraic closure, then see that the kernel of our map

$$\mathrm{res} : H^0(C, \mathbb{L}(-\mathfrak{p}) \otimes \Omega^1_C) \longrightarrow k$$

becomes the kernel of the sum of the residues, hence $\ker(\delta) = \ker(\mathrm{res}_{\mathfrak{y}})$. Hence we can define Res by the diagram

$$\begin{tikzcd} H^0(C, \mathbb{L}(-\mathfrak{p}) \otimes \Omega^1_C) \arrow[rr] \arrow[dr, "\mathrm{res}_{\mathfrak{p}}"'] & & H^1(C, \omega^1_C) \arrow[dl, "\mathrm{Res}"] \\ & k & \end{tikzcd}$$

But once we defined Res we can define $\mathrm{res}_{\mathfrak{p}}$ for arbitrary points using the same diagram.

It is clear that our definition of Res has the right functoriality principles with respect to separable morphisms and extension of the ground field.

Especially we know now that for any effective divisor D the image of

$$H^0(C, \Omega^1_C(-D)) \longrightarrow H^0(C, \mathbb{L}(-D) \otimes \Omega^1_C)$$

consists of the elements $\underline{\xi} \otimes \omega$ for which

$$\sum_{\mathfrak{p}} \mathrm{res}_{\mathfrak{p}}(\xi_{\mathfrak{p}} \otimes \omega) = 0.$$

Finally we can state the Serre duality theorem

Theorem: *For any line bundle $\mathcal{L}$ on C we have a perfect pairing*

$$H^0(C,\mathcal{L}) \times H^1(C,\mathcal{L}^{-1}\otimes\Omega^1_C) \longrightarrow k$$

which is given by the cup product. Of course that means that a global section $s\in H^0(C,\mathcal{L})$ defines by multiplication a map

$$\langle s,\ \rangle : H^1(C,\mathcal{L}^{-1}\otimes\Omega^1_C) \longrightarrow H^1(C,\Omega^1_C) \xrightarrow{\mathrm{Res}} k.$$

Proof:

Now we can formulate the final version of the Riemann-Roch theorem. We start from a line bundle $\mathcal{L}$ on C and we pick a point $\mathfrak{p}$ (or an effective divisor). We compute the cohomology $H^1(C,\mathcal{L})$ and to do this we start from the sequence

$$0\to\mathcal{L}\to\mathcal{L}(-\infty\mathfrak{p})\to\mathcal{L}(-\infty\mathfrak{p})/\mathcal{L}\to 0.$$

We have the cohomology sequence

$$H^0(C,\mathcal{L}(-\infty\mathfrak{p}))\to H^0(C,\mathcal{L}(-\infty\mathfrak{p})/\mathcal{L})\to H^1(C,\mathcal{L})\to 0.$$

Now we consider the sheaf of differentials with coefficients in $\mathcal{L}^{-1}$ namely $\mathcal{L}^{-1}\otimes\Omega$. If $\mathfrak{p}\in H^0(C,\mathcal{L}(-\infty\mathfrak{p}))$ and $\omega\in H^0(\mathcal{L}^{-1}\otimes\Omega^1_{C/k})$ then $\xi_\mathfrak{p}\omega$ is a Laurentexpansion of a meromorphic differential at $\mathfrak{p}$ and $\mathrm{res}_\mathfrak{p}(\xi_\mathfrak{p}\omega)$ is defined. If $\xi\mathfrak{p}$ comes from a meromorphic section $s\in H^0(C,\mathcal{L}(-\infty\mathfrak{p}))$ then $s\omega$ is a meromorphic differential which is holomorphic outside $\mathfrak{p}$. Hence $\mathrm{res}_\mathfrak{p}(s\omega)=0$ and we get again a pairing

$$\mathrm{Res}: H^1(C,\mathcal{L})\times H^0(C,\mathcal{L}^{-1}\otimes\Omega^1_{C/k})\to k.$$

This generalizes the pairing we had for $\mathcal{L}=\mathcal{O}_C$.

Theorem (Serre Duality or final version of Riemann-Roch):

The pairing

$$\mathrm{Res}: H^1(C,\mathcal{L})\times H^0(C,\mathcal{L}^{-1}\otimes\Omega^1_{C/k})\to k$$

is non-degenerated.

Proof: It is quite clear that a non zero element $\alpha\in H^0(C,\mathcal{L}^{-1}\otimes\Omega^1_{C/k})$ will induce a non trivial linear form on $H^1(C,\mathcal{L})$. Hence we conclude $\dim_k H^1(C,\mathcal{L})\geq\dim_k H^0(C,\mathcal{L}^{-1}\otimes\Omega^1_{C/k})$. This last dimension is equal to (first version of Riemann-Roch)

$$\dim_k H^1(C,\mathcal{L}^{-1}\otimes\Omega^1_{C/k})-\deg(\mathcal{L})+2g-2+1-g$$

We apply our argument a second time and get $\dim_k H^1(C,\mathcal{L}^{-1}\otimes\Omega^1_{C/k})\geq\dim_k H^0(C,\mathcal{L})$ Hence we see that

$$\dim_k H^1(C,\mathcal{L}) \geq \dim_k H^0(C,\mathcal{L}) - \deg(\mathcal{L}) + g - 1$$

.

But the first version of Riemann-Roch tells us that we must have equality in this last inequality. Hence we see that all the inequalities in between where actually equalities. But if the dimensions are equal it follows that the pairing is non degenerate.

Finally we conclude: If we have a line bundle $\mathcal{L}$ with $\deg(\mathcal{L}) < 0$ then we have $H^0(C,\mathcal{L}) = 0$, because if we had a section $s \neq 0$ then $\deg(\mathrm{div}(s)) = \deg(\mathcal{L})$ and $\mathrm{div}(s)$ is effective.

Now we can say: If $\mathcal{L}$ is a line bundle with $\deg(\mathcal{L}) > 2g-2$ then $H^1(C,\mathcal{L}) = H^0(C,\mathcal{L}^{-1}\otimes \Omega^1_{C/k}) = 0$ and

$$\dim H^0(C,\mathcal{L}) = \deg(\mathcal{L}) + 1 - g.$$

Comments on the first version of the theorem of Riemann-Roch.

I begin with a very general remark. If we have a projective scheme $X \to \mathrm{Spec}(k)$ and a coherent sheaf $\mathcal{F}$ on X, then we have seen that the cohomology groups $H^i(X,\mathcal{F})$ are finite dimensional k-vector spaces and the cohomology vanishes, if $i >> 0$. This allows us to define the Euler-characteristic

$$\chi(X,\mathcal{F}) = \sum_i (-1)^i \dim_k H^i(X,\mathcal{F}).$$

A short exact sequence of sheaves

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}" \to 0$$

yields a long exact sequence in cohomology. Now it is an easy exercise in linear algebra to show that the long exact sequence provides the addivity of the Euler characteristic

$$\chi(X,\mathcal{F}) = \chi(X,\mathcal{F}') + \chi(X,\mathcal{F}").$$

Now we consider the special case where $X/k = C/k$ is a smooth curve. In this case, we have seen that for any locally free sheaf $\mathcal{E}$ on C we can define $\mathrm{rank}(\mathcal{E})$ and $\deg(\mathcal{E})$. The rank is simply the dimension of the generic fibre as a vector space over $\mathcal{O}_{C,\eta}$ and the degree can be defined inductively: If $0 \to \mathcal{E}' \to \mathcal{E} \to \mathcal{E}" \to 0$ is an exact sequence of locally free sheaves then

$$\deg(\mathcal{E}) = \deg(\mathcal{E}') + \deg(\mathcal{E}").$$

Then we show that any locally free sheaf admits a filtration (complete flag):

$$(0) = \mathcal{F}_0 \subset \mathcal{F}_1 \subset \mathcal{F}_2 \subset \mathcal{F}_{n-1} \subset \mathcal{F}_n = \mathcal{E}$$

such that $\mathcal{F}_{i+1}/\mathcal{F}_i = \mathcal{M}_i$ is a line bundle. We reduced the definition of the degree to the case of line bundles. One has to check that if $n = \mathrm{rank}(\mathcal{E})$

$$\Lambda^n \mathcal{E} \simeq \otimes \mathcal{F}_i/\mathcal{F}_{i-1} = \otimes \mathcal{M}_i.$$

The theorem of Riemann-Roch gives as a formula for $\chi(C,\mathcal{E})$ in terms of $\operatorname{rank}(\mathcal{E}), \deg(\mathcal{E})$ namely

$$\chi(C,\mathcal{E}) = \deg(\mathcal{E}) + \operatorname{rank}(\mathcal{E})(1-g).$$

To prove this we observe that both sides behave additively under exact sequences and then the existence of flags reduces the problem to the case of line bundles.

But there is a different way to look at the theorem of Riemann - Roch. We could consider all coherent sheaves on C and ask for a formula for $\chi(C,\mathcal{F})$ in terms of $\deg(\mathcal{F}), \operatorname{rank}(\mathcal{F})$. The first problem is that we do not have a notion of $\deg(\mathcal{F})$ and $\operatorname{rank}(\mathcal{F})$ for arbitrary coherent sheaves.

But let us have a closer look at the coherent sheaves on C. Locally they are finitely generated modules over Dedekind rings. If A is a Dedekind ring and M a finitely generated A-module then we have an exact sequence

$$0 \to M_{\text{tors}} \to M \to M/M_{\text{tors}} \to 0$$

where M_{tors} is the module of torsion elements and M/M_{tors} is locally free. Hence we see that any coherent sheaf on C sits in such a sequence

$$0 \to \mathcal{E}_{\text{tors}} \to \mathcal{E} \to \mathcal{E}/\mathcal{E}_{\text{tors}} \to 0$$

where $\mathcal{E}/\mathcal{E}_{\text{tors}}$ is locally free. Now it is clear that a torsion module over a discrete valuation ring $\mathcal{O}_{C,\mathfrak{p}}M = \oplus \mathcal{O}_{C,\mathfrak{p}}/\mathfrak{p}^{m_i}$ and this implies that any torsion module $\mathcal{M}$ on C can be written as quotient of a vector bundle by a subbundle of the same rank. Hence it sits in an exact sequence

$$0 \to \mathcal{E}' \to \mathcal{E} \to \mathcal{M} \to 0.$$

This suggests the definition

$$\begin{aligned} \deg(\mathcal{M}) &= \deg(\mathcal{E}) - \deg(\mathcal{E}') = \dim_k H^0(C,\mathcal{M}) \\ \operatorname{rank}(\mathcal{M}) &= \operatorname{rank}(\mathcal{E}) - \operatorname{rank}(\mathcal{E}') = 0. \end{aligned}$$

The first formula has been verified earlier. We define the degree and the rank for arbitrary coherent sheaves by

$$\begin{aligned} \deg(\mathcal{E}) &= \deg(\mathcal{E}/\mathcal{E}_{\text{tors}}) + \deg(\mathcal{E}_{\text{tors}}) \\ \operatorname{rank}(\mathcal{E}) &= \operatorname{rank}(\mathcal{E}/\mathcal{E}_{\text{tors}}) \end{aligned}$$

and this gives a more general Riemann-Roch formula.

For any coherent sheaf $\mathcal{F}$ on C we have

$$\chi(C,\mathcal{F}) = \deg(\mathcal{F}) + (1-g)\operatorname{rank}(\mathcal{F}).$$

The proof is almost obvious but I write it down in a slight sophisticated form. I introduce the group $K'(C)$. This group is generated by the isomorphism classes of coherent sheaves. For such a sheaf $\mathcal{E}$ let $[\mathcal{E}]$ be its class in $K'(C)$. We introduce the relations: For any exact sequence

$$0 \to \mathcal{E}' \to \mathcal{E} \to \mathcal{E}\text{"} \to 0$$

we have

$$[\mathcal{E}] = [\mathcal{E}'] + [\mathcal{E}"].$$

The Euler characteristic, the degree and the rank provide homomorphisms

$$\begin{gathered}\chi : K'(C) \to \mathbb{Z} \\ \deg : K'(C) \to \mathbb{Z} \\ \operatorname{rank} : K'(C) \to \mathbb{Z}\end{gathered}$$

and the theorem of Riemann-Roch says

$$\chi = \deg + (1-g) \cdot \operatorname{rank}.$$

ur previous considerations make it clear that $K'(C)$ is generated by line bundles and torsion sheaves. But since any line bundle is of the form $\mathcal{L} = \mathcal{O}_C(-D)$ we see that the group $K'(C)$ is actually generated the structure sheaf and the torsion sheaves. But now it is clear that

$$\chi(\mathcal{O}_C) = \deg(\mathcal{O}_C) + (1-g)\operatorname{rank}(\mathcal{O}_C) = 1-g$$

and for torsion sheaves

$$\chi(\mathcal{M}) = \deg(\mathcal{M})$$

and this proves the formula.

The structure of $K'(C)$

The structure of this group is rather complicated. We have the surjective homomorphism $(\deg, \operatorname{rank}) : K'(C) \to \mathbb{Z} \oplus \mathbb{Z}$ which means that the group has a very simple quotient. But the kernel of this group is very complicated. We leave it as an exercise to prove that the kernel is the group $\operatorname{Pic}^0(C)$ which is the group of line bundles of degree zero.

Applications of the Riemann-Roch Theorem

I want to begin by discussing the cases of curves with low genus. Let C/k be an absolutely irreducible, smooth and projective curve. Let us assume the genus of this curve is 0. If this curve has a rational point $P \in C(k)$, then we can consider the line bundle $\mathcal{L} = \mathcal{O}_C(-P)$ and as in VII 3.1, we can consider the morphism

$$r_{\mathcal{L}} : C \longrightarrow \operatorname{Proj}\left(\bigoplus_{n=0}^{\infty} H^0(C, \mathcal{L}^{\otimes n})\right),$$

The vector space $H^0(C,\mathcal{L})$ has rank 2, it is generated by the constant function $1 \in H^0(C,\mathcal{L})$ which we call X_0 and another function X_1 which has a first order pole at P. Then it is clear that $H^0(C, \mathcal{L}^{\otimes n})$ is spanned by the homogeneous polynomials of degree n in X_0, X_1, and these form a basis. Hence we see that

$$\bigoplus_{n=0}^{\infty} H^0(C, \mathcal{L}^{\otimes n}) = k[X_0, X_1].$$

I leave it as an exercise to the reader to show that $r_{\mathcal{L}}$ provides an isomorphism

$$r_{\mathcal{L}} : C \xrightarrow{\sim} \mathbb{P}^1$$

(see VII ...).

If the curve does not have a rational point, then we can say at least the following. We have the sheaf $\Omega^1_{C/k}$ and this is a line bundle of degree -2. Hence the dual of this line bundle has degree 2 and therefore we can find a non zero section $t \in H^0(C, (\Omega^1_{\mathbb{P}^1})^\vee)$. This section must have zeroes. The divisor of zeroes has degree 2, hence it must be a point P of degree 2 or of the form $P_1 + P_2$ with two rational points P_1 and P_2. In the second case we see that we have a rational point. In the first case the point P has a residue field $k(P)$ which is of degree 2 over k. Hence we conclude that we can always find a quadratic extension over which C will split.

Of course the first case can happen if k is not algebraically closed. If we consider a quadratic form over a field of characteristic $\neq 2$

$$ax^2 + by^2 + cz^2,$$

then it defines a curve of genus zero if it is not reducible (Exercise 1 ???). Now it is clear that this curve does not have a rational point if this form does not represent zero (i.e. we can not find $x_0, y_0, z_0) \in k^3 \backslash \{0\}$ such that $ax_0^2 + by_0^2 + cz_0^2 = 0$).

It is not difficult to see that the line bundle $\mathcal{L} = (\Omega^1_{\mathbb{P}^1})^\vee$ provides an embedding

$$r_{\mathcal{L}} : C \longrightarrow \mathbb{P}^2$$

where the image is described by a quadratic form.

Now we consider curves of genus one. Again we assume that we have a point $P \in C(k)$, and we consider the line bundle $\mathcal{L} = \mathcal{O}_C(-P)$. We have the inclusions

$$\mathcal{O}_C \subset \mathcal{O}_C(-P) \subset \mathcal{O}_C(-2P) \subset \mathcal{O}(-3P).$$

Now we consider the graded ring

$$\bigoplus_n H^0(C, \mathcal{L}^{\otimes n}).$$

In contrast to our precious situations this ring is not generated by elements in degree one, but we will see that this does not really matter.

The constant function 1 yields a section $x_0 \in H^0(C, \mathcal{L})$ and this section spans this space. Then we get from the Riemann-Roch theorem that we have a section $x_1 \in H^0(C, \mathcal{L}^{\otimes 2})$

which is independent of x_0^2, and we have a section $x_2 \in H^0(C, \mathcal{L}^{\otimes 3})$ which is independent of $x_0^3, x_0^2 x_1$. Now it is not difficult to see that the element x_0 (in degree one), x_1 (in degree 2) and x_2 (in degree 3) generate the graded ring. It follows from the Riemann-Roch theorem that the elements

$$\{x_0^6, x_0^4 x_1, x_0^2 x_1^2, x_1^3, x_0^3 x_2, x_0 x_1 x_2, x_2^2\}$$

must be linearly dependent because the space $H^0(C, \mathcal{L}^{\otimes 6})$ has dimension 6. Therefore we find a linear relation amoung them. Before we write it down we want to derive some information about it. Let π_P be a uniformizing element at $\mathbb{P}$, then

$$x_2 = \frac{a}{\pi_P^3} + \ldots \qquad a \in k^*$$

and

$$x_1 = \frac{b}{\pi_P^2} + \ldots \qquad b \in k^*.$$

It is clear that x_2^2, x_1^3 are the only terms which have a 6-th order pole, hence our searched for relation has to cancel that pole.

On the other hand we can modify x_1, x_2 by a scalar factor such that the above numbers $a = b = 1$. Then we can conclude that our relation must be of the form

$$x_2^2 + a_1 x_0 x_1 x_2 + a_3 x_0^3 x_2 = x_1^3 + a_2 x_0^2 x_1^2 + a_4 x_0^4 x_1 + a_6 x_0^6.$$

this is a relation among elements in $H^0(C, \mathcal{L}^{\otimes 6}) = H^0(C, \mathcal{O}_C(-6P))$. Since x_0 is the constant function 1 viewed as element in $H^0(C, \mathcal{O}_C(-6P))$ we see that the exponent in $x_0^\vee$ is irrelevant and we can modify our relation to a relation in variables of degree one

$$x_0 x_2^2 + a_1 x_0 x_1 x_2 + a_3 x_0^2 x_2 + x_1^3 + a_2 x_0 x_1^2 + a_4 x_0^2 x_1 + a_6 x_0^3,$$

and this is a homogeneous relation of degree three among elements in $H^0(C, \mathcal{L}^{\otimes 3})$. (The point of this trick is that we see that we do not need all monomials of degree 3 in $x_0, x_1, x_2 \in H^0(C, \mathcal{L}^{\otimes 3})$). Hence we get the morphism

$$\begin{array}{ccccc} r_{\mathcal{L}^{\otimes 3}} & : & C & \longrightarrow & \mathbb{P}^2 \\ & & & \searrow \quad \swarrow & \\ & & & \mathrm{Spec}(k). & \end{array}$$

Again I leave it to the reader to show that this is a closed embedding.

If we do not have a point on C then we have difficulties. In this case we are not able to produce a point $P \in C$ whose residue field has a degree which can be estimated independently of C.

The moduli space

At this point I want to give an outlook to more advanced topics.

We consider an elliptic curve C over a base scheme S which for simplicity I assume is affine, i.e. $S = \mathrm{Spec}(A)$. By this I mean that C is projective and smooth over S, I assume that all the fibres are absolutelz irreducible and of dimension one and also of genus one. Finally I assume that we have a section s

$$C$$

$$\pi \downarrow\uparrow s$$

$$S.$$

As before, we view the image of S under the section s as a divisor P on C. It defines a line bundle $\mathcal{L} = \mathcal{O}_C(-P)$ on C. If we evaluate at a point $x \in S$, then we get the curve

$$\begin{array}{l} C \times \mathrm{Spec}(k(x)) \\ \quad\downarrow \\ \quad \mathrm{Spec}(k(x)), \end{array}$$

the point P gives us a $k(x)$-rational point, and we are in the previous situation.

Now we consider the direct images of our line bundle $\pi_*(\mathcal{L}^{\otimes n})$. The Riemann-Roch theorem tells us that the $k(x)$-vector spaces

$$H^0(C \times_S k(x), \mathcal{L}^{\otimes n} \mid C \times_S k(x))$$

have dimension n if x varies. Then our semicontinouity implies that $\pi_*(\mathcal{L}^{\otimes n})$ are locally free of rank n over A.

Let us make the further assumption that any locally free module over A is actually free. (There are interesting rings having this property.) Then we can proceed as before. We find sections

$$x_0 \in H^0(C, \mathcal{L}), x_1 \in H^0(C, \mathcal{L}^{\otimes 2}), x_3 \in H^0(C, \mathcal{L}^{\otimes 3})$$

such that they provide respectively a basis in

$$H^0(C, \mathcal{L}) \subset H^0(C, \mathcal{L}^{\otimes 2}) \subset H^0(C, \mathcal{L}^{\otimes 3}).$$

If $J \subset \mathcal{O}_C$ is the ideal defining P the module J/J^2 is free or rank one over A. We may choose an uniformizing element π_P which generates this A-module. The elements $\pi_P^2 x_1$ and $\pi_P^3 x_2$ are elements which are regular along P, we can evaluate at P, the result is a unit in A. Now we can require

$$\pi_P^2 x - 2(P) = \pi_P^3 x_3(P) = 1.$$

Then we conclude that we have a relation

$$x_2^2 + a_1 x_0 x_1 x_2 + a_3 x_0^3 x_2 = x_1^3 + a_2 x_0^2 x_1^2 + a_4 x_0^4 x_1 = a_6 x_0^6$$

with some coefficients in A. Again we perform the change of the bundle, we replace $\mathcal{L}$ bz $\mathcal{L}^{\otimes 3} = \mathcal{O}_C(-3P)$ and we consider x_0, x_1, x_2 as sections of this bundle, then get degree one and consider the relation

$$x_0 x_2^2 + a_1 x_0 x_1 x_2 + a_3 x_0^2 x_2 = x_1^3 + a_2 x_0 x_1^2 + a_4 x_0^2 x_1 + a_6 x_0^3$$

and the homogeneous relation of degree 3 describes C as a closed subscheme of $\mathbb{P}^2 = \operatorname{Proj}\ A[X_0, X_1, X_2]$. Now we have to take into account that the choices of x_0, x_1, x_2 as sections in $H^0(C, \mathcal{O}_C(-3P))$ are not unique. Of course the choice of x_0 as constant function 1 is canonical. If we stick to our choice of π_P, then we can modify

$$\begin{array}{lcll} x_1 & \longrightarrow & x_1 + \alpha x_0 & = \ x_1' \\ x_2 & \longrightarrow & x_2 + \beta x_1 + \gamma x_0 & = \ x_2'. \end{array} \qquad (S)$$

Then we will get a new relation with new coefficients, in other words the coefficeints of the relationare by no means determined by the curve and the choice of π_p.

The following statements have to be varified by computations and a little bit of thinking.

Assume that 2 and 3 are invertible in A. Then we can find a unique substitution as above such that the new relation will be of the form

$$x_0 x_2^2 + x_1^3 + a_4' x_0^2 x_1 + a_6' x_0^3 \qquad (***)$$

where a_4' and a_6' are the following expressions in the a_1, a_3, a_2, a_4, a_6:

$$a_4' = \frac{a_1}{48} + \frac{a_1^2 a_2}{6} + \frac{a_2^2}{3} - \frac{a_1 a_3}{2} - a_4$$
$$a_6' = \frac{a_1^6}{864} + \frac{a_1^4 a_2^2}{72} + \frac{a_1^2 a_2^2}{18} + \frac{2a_2^2}{27} - \frac{a_1^3 a_3}{24} - \frac{a_1 a_2 a_3}{6} + \frac{a_3^2}{4} - \frac{a_1^2 a_4}{12} - \frac{a_2 a_4}{3} + a_6$$

Now it is quite clear that the curve defined by the equation

$$x_0 x_2^2 = x_1^3 + a_4' x_1 x_0^2 + a_6' x_0^3$$

is smooth if and only if the discriminant

$$-4(a_4')^3 - 27(a_6')^2$$

is a unit in A. If we write down this expression, then we get an expression which is a sum of monomials in the a_1, a_3, a_2, a_4, a_6 which is homogeneous of degree 12 and where the coefficients are rational numbers which have only powers of 2 in their denominator and where the largest denominator is 16. We define

$$\Delta(a_1, a_3, a_2, a_4, a_6) = 16 \cdot (-4(a_4')^3 - 27(a_6')^2).$$

This is again a sum of such monomials where now the coefficients are integral and where we have some monomials with coefficient ± 1.

Finally we can apply the same process to a_4' and a_6' and put

$$C_4 = 48 \cdot a_4' \quad , \quad C_6 = 862 \cdot a_6',$$

here the monomials have integral coefficients.

We have $48 = 4 \cdot 12$ and $862 = 12^3/2$. Then it becomes clear that

$$\triangle(a_1, a_3, a_2, a_4, a_6) = \frac{1}{12^3} \cdot (-C_4^3 - C_6^2),$$

and with the help of a computer I found

$$\begin{aligned}\Delta(a_1, a_3, a_2, a_4, a_6) = &-a_1^4 a_2 a_3^2 - 8a_1^2 a_2^2 a_3^2 - 16a_2^3 a_3^2 + a_1^3 a_3^3 + 36a_1 a_2 a_3^3 - \\ &27a_3^4 + a_1^5 a_3 a_4 + 8a_1^3 a_2 a_3 a_4 + 16a_1 a_2^2 a_3 a_4 - 30a_1^2 a_3^2 a_4 + 72a_2 a_3^2 a_4 + a_1^4 a_4^2 + 8a_1^2 a_2 a_4^2 + \\ &16a_2^2 a_4^2 - 96a_1 a_3 a_4^2 - 64a_4^3 - a_1^6 a_6 - 12a_1^4 a_2 a_6 - 48a_1^2 a_2^2 a_6 - 64a_2^3 a_6 + 36a_1^3 a_3 a_6 + \\ &144a_1 a_2 a_3 a_6 - 216a_3^2 a_6 + 72a_1^2 a_4 a_6 + 288a_2 a_4 a_6 - 432a_6^2.\end{aligned}$$

Since we have aunique substitution which transforms a general relation into the specail form (***), we conclude that a_4', a_6' are invariant expressions under our substitutions.

Now we observe that the expressions C_4, C_6 and Δ can be written down without the assumption that $\frac{1}{2}, 1|over3 \in A$. And it is clear that:

The expressions for C_4, C_6 and Δ are invariant under the substitutions induced by substitutions of the form (S).

The following theorem is almost clear from our considerations above¿

Theorem: *The equation*

$$x_0 x_2^2 + a_1 x_0 x_1 x_2 + a_3 x_2 x_0^2 = x_1^3 x_0 + a_4 x_1 x_0^2 + a_6 x_0^3$$

defines a smooth projection curve over $\mathrm{Spec}(A)$ *if and only if* $\Delta(a_1, a_3, a_2, a_4, a_6) \in A^*$. *It contains the point*

$$P = (0, 0, 1).$$

The following considerations are more geometric in nature, hence we assume at this point that C/k is a smooth, projective and irreducible curve over an algebraically closed field k. Let g be the genus of our curve.

We pick g points $P_1, \ldots, P_g$ on our curve. We form the divisor $D = \sum P_i$ and ask

Is there a non-constant meromorphic function $f \in H^0(C, \mathcal{O}_C(-D))$?

a) What does the Riemann-Roch theorem tell us?

We see that the simple version of the Riemann-Roch is not good enough.

b) What do we need to know about the behavior of the holomorphic differentials at D if we want to answer our question?

c) Prove that we can find g points $P_1, \ldots, P_g$ such that there is no a non-zero differential $\omega \in H^0(C, \Omega_C)$ which vanishes in all these points.

d) If we have such a g-tupel, as in c) what do we know about our original question?

We want to discuss this kind of questions in a more sytematic way. We consider the product of our curve by itself $C \times_k C$, then we may consider the diagonal $\Delta \subset C \times_k C$. This diagonal can locally be described by one equation and hence we can look at the line bundle $O_{C \times C}(-\Delta)$. (See section on divisors).

We now form the product

$$C \times C^g = C \times \underbrace{C \times C \times \ldots \times C}_{g-\text{times}} = X.$$

On this variety we have the line bundles $\mathcal{O}_X(-\Delta_i)$ which is obtained by the diagonal in the factor C in front and the i-th factor in the product.

This gives us a line bundle $\mathcal{L} = \otimes_{i=1}^g \mathcal{O}_X(-\Delta_i)$ on X.

Now we look at the projection to the second factor

$$\begin{array}{ccc} C & \times & C^g \\ & & \big\downarrow \pi\,. \\ & & C^g \end{array}$$

The sheaf $\mathcal{L}$ is flat over C^g. If we pick a point $P = (P_1, \ldots, P_g) \in C^g$ and restrict $\mathcal{L}$ to the fibre $\pi^{-1}(P) \simeq C$ then this restriction is exactly the line bundle $\mathcal{O}_C(-P_1 \ldots - P_g)$ on C.

e) Prove that we have a non-empty Zariski open subset $U \subset C^g$ such that

$$\dim_k H^0(C, \mathcal{O}_C(-P_1 \ldots - P_g)) = 1 \quad \textit{if only if} \quad P = (P_1, \ldots, P_g) \in U.$$

f) Prove that $U \neq C^g$ and show that the complement is of codimension one.

The next exercise is difficult, thats the reason why it has two stars.

*g**) Show that if $g \geq 2$ we can find a point P for which we can find a holomorphic differential form $\omega \neq 0$ which has a zero of order g in P.*

(This is at least plausible, because the assertion means that the intersection of the total diagonal in C^g and the complement of U have an non empty intersection. But there is no theorem that guarantees non emptyness of intersections just from dimensional reasons.)

Hint: Consider $C \times C$. Let $\Omega^{(1)}$ be the pullback of the differentials under a projection to the first factor. Consider the sheaf

$$\mathcal{L} = \Omega^{(1)} \otimes \mathcal{O}_{C\times C}((g-1)\Delta).$$

What happens if we restrict this sheaf to a fibre $C \times \{P\}$. How can we describe the resulting sheaf?

Prove that the Riemann-Roch theorem shows that

$$\dim_k H^0(C, \Omega_C((g-1)P)) \geq 1,$$

this means we can always find a one form $\neq 0$ which has a $(g-1)$-th order zero at a given point. Show that our claim follows if we can find a point P_0 where

$$\dim_k H^0(C, \Omega_C((g-1)P_0)) \geq 2.$$

Hence to prove our theorem, we may assume that

$$\dim_k H^0(C, \Omega_C((g-1)P)) = 1$$

for all $P \in C(k)$.

Now we exploit the semi-continuity. Let p_2 be the projection to the second factor. The sheaf $p_{2*}(\mathcal{O}_{C\times C}((g-1)\Delta))$ is a line bundle on the curve. Hence wecan find for any $P_0 \in C(k)$ an open neighborhood V and a section $\omega \in p_{2*}(\mathcal{O}_{C\times C}((g-1)\Delta))(V)$ such that ω_P generates $H^0(C, \Omega^1 C((g-1)P)$ at all points P in V.

Then ω_P has $2g-2$ zeroes, hence it has $g-1$ zeroes apart from the $(g-1)$-th order zero at P. We consider the subset $Z \subset C(k) \times C(k)$ which consists of pairs

$$(Q, P) \in C \times C$$

where Q is a zero of the section $\omega_P \in H^0(C, \Omega^1_C((g-1)P))$. This is by construction a divisor on $C \times_k C$. Now we must prove that Z has a non empty intersection with the diagonal! Again we are in the same situation as above, we cannot conclude this from dimensional reasons.

To proceed we need the concept of the *intersection number* $\mathcal{L}_1 \cdot \mathcal{L}_2$ of two line bundles on $C \times_k C$ which I will not discuss in detail. The definition of the intersection number is easy in special cases. If for instance the line bundle $\mathcal{L}_1$ has a non zero global section $s_1 \in H^0(C \times_k C, \mathcal{L}_1)$ such that the scheme $V(s)$ of zeroes (see section on divisors) is a smooth curve then we define $\mathcal{L}_1 \cdot \mathcal{L}_2$ as the degree of the line bundle $\mathcal{L}_2$ restricted to $V(s)$.

Of course we have to verify that this number does not depend on the choice of s. Here we aply this to the line bundle $\mathcal{O}_{C\times C}(-Z)$ on $C \times C$. We consider the intersection number $\mathcal{O}_{C\times C}(-Z)\cdot\mathcal{O}_{C\times C}(-\Delta) = Z\cdot\Delta$. It is clear that this restriction is trivial if $Z\cap\Delta=\emptyset$. Hence we can conclude that the intersection is not empty if we show that the intersection number is non zero. (This argument is valid independently of whether we have shown that the intersection number is well defined)

Actually much more is true:

If Z,Δ have no component in common then the intersection number is equal to the number of points in $Z\cap\Delta$ counted with the right multiplicities.

(For details on intersection numbers I refer to Hartshornes book. From now I will use on the concept of intersection numbers.)

But how can we compute the intersection number $\mathcal{O}_{C\times C}(-Z)\cdot\mathcal{O}_{C\times C}(-\Delta)$??

To do this we compute the line bundle $p_{2*}(\Omega^{(1)}\otimes\mathcal{O}_{C\times C}((g-1)\Delta)$ (of course still under the above assumption). This can be done inductively by applying the direct image functor to the exact sequences $(0\le a<g)$

$$0\to\Omega^{(1)}\otimes\mathcal{O}_{C\times C}(a\Delta)\to\Omega^{(1)}\otimes\mathcal{O}_{C\times C}((a-1)\Delta)\to\frac{\Omega^{(1)}\otimes\mathcal{O}_{C\times C}((a-1)\Delta)}{\Omega^{(1)}\otimes\mathcal{O}_{C\times C}(a\Delta)}\to 0.$$

The term on the right is a sheaf concentrated on the diagonal and can be computed (see also the discussion of the Grothendieck-Riemann-Roch theorem following later). From this one can compute $\mathcal{O}_{C\times C}(-Z)$ by exploiting the definition of Z. Then it is easy to compute the intersection number. (It is a cubic polynomial in g which has three integral zeroes, one of them does not occur, the other two are necessary because our therem is false for these values.)

End of the exercise g)**

If we pick a point $P\in C(k)$ and consider the spaces $H^0(C,\mathcal{O}_C(-aP))$ for $a=1,\dots,\dots$ then we get from the Riemann-Roch formula

$$\dim_k H^0(C,\mathcal{O}_C(-aP))-\dim_k H^1(C,\mathcal{O}_C(-aP))=a+1-g.$$

We know that $1\in H^0(C,\mathcal{O}_C(-aP))$ hence $\dim_k H^0(C,\mathcal{O}_C(-aP))\ge 1$.

On the other hand it is clear that

$$\dim_k H^0(C,\mathcal{O}_C(-(a+1)P))\le\dim_k H^0(C,\mathcal{O}_C(-aP))+1$$

the dimension may jump by one if we go from a to $a+1$ or it may stay constant. Finally we have

$$\dim H^0(C,\mathcal{O}_C(-(2g-1)P))=g.$$

This means that if we go up from $a = 0$ to $a = 2g - 1$ the dimension will jump exactly $g - 1$ times. It is at least plausible that for P in some non-empty open set $U \subset C$ we have the first jump if we go from $a = g$ to $g + 1$.

h) Show that for a point P for which $H^0(C, \Omega_C(gP)) = 0$ we have

$$\dim_k H^0(C, \mathcal{O}_C(-aP)) = \begin{cases} 1 & \text{for } 0 \leq a \leq g \\ a + 1 - g & \text{for } g + 1 \leq a \end{cases}.$$

Find out - perhaps by research in the literature - whether there is always a point P for which

$$H^0(C, \Omega_C(gP)) = 0.$$

Show that we have always points P for which

$$\dim_k H^0(C, \mathcal{O}_C(gP)) \geq 2.$$

i) Of course we know that a non-zero differential cannot have a zero of order $\geq 2g - 1$. But I do not know what the record is. We could ask several questions

1) For a given g can we find a curve such that $H^0(C, \Omega_C((g+1)P)) = 0$ for all points $P \in C(k)$?

2) Which is the maximal number $a_g (g \leq a_g < 2g - 1)$ for which we can find a curve of genus g such that

$$\dim_k H^0(C, \Omega_C(a_g g)) \geq 1.$$

3) How early can it happen that

$$\dim_k H^0(C, \Omega_C(aP)) \geq g - a + 1,$$

which is the minimal a for which this happens for a special curve or the generic curve.

We consider line bundles of degree $g + 1, g + 2, \ldots$ and ask whether they are base point free or provide an embedding of the curve into projective space.

j) Any line bundle $\mathcal{L}$ of degree $d \geq g + 1$ is of the form

$$\mathcal{L} = \mathcal{O}(-P_1 \ldots - P_d).$$

What does it mean that the bundle $\mathcal{L}$ is base point free?

Prove that there is a non-empty subset $U \subset C^{g+1}$ such that $\mathcal{O}(-P_1 \ldots - P_{g+1})$ is a base point free for $(P_1 \ldots, P_{g+1}) \in U$.

k) Try to prove: For a non-open set $U \subset C^{g+2}$ the line bundle $\mathcal{O}(-P_1 \ldots - P_{g+2})$ provides a closed immersion

$$i : C \to \mathbb{P}^2,$$

this means that on a non-empty open set the morphism i is injective and its differential is always non-zero. (Show that not all curves can be realized as plane projective curves).

l) Prove that for a non-empty open set $U \subset C^{g+3}$ the line bundle

$$\mathcal{L} = \mathcal{O}(-P_1 \ldots - P_{g+3})$$

provides a closed embedding of C into $\mathbb{P}^3$.

A special case of the Grothendieck-Riemann-Roch Thereom

We form the product of our curve by itself, we put $X = C \times_k C$ we have the two projections $p_1, p_2 : X \rightrightarrows C$. Of course we may again ask for a formula for the Euler characteristic

$$\chi(X, \mathcal{L}) = \sum \dim_k (-1)^\nu H^\nu(X, \mathcal{L})$$

in terms of certain data which are attached to the line bundle $\mathcal{L}$. This is the classical Riemann-Roch problem for surfaces. It is treated in Hartshornes book in Chapter IV for arbitrary smooth projective surfaces X. Here I want to discuss the special case where X is the product of a curve by itself, but I want to discuss a stronger theorem which is a special case of a Riemann- Roch theorem in the sense of Grothendieck.

In the supplementary section on the properties of the degree function I introduced the group $K'(C)$ and I showed that the formula of Riemann-Roch gave an expression for the Euler characteristic of an element in $K^{(}C)$ in terms of its degree and its rank. Here we recall that we interpreted the Euler chacteristic, the degree and the rank as homomorphisms

$$\begin{gathered} \chi_C : K'(C) \to \mathbb{Z} \\ \mathrm{rank}_C : K'(C) \to \mathbb{Z} \\ \deg_C : K'(C) \to \mathbb{Z}. \end{gathered}$$

We may ask the following question: Let $\mathcal{L}$ be a line bundle on X then we can consider the coherent sheaves $R^\nu p_{2*}(\mathcal{L})$ on the curve and we may consider the Euler characteristic

$$\chi(X, R^\bullet p_{2*}\mathcal{L}) = \sum (-1)^\nu [R^\nu p_{2*}(\mathcal{L})]$$

as an element in $K'(C)$. If we are able to "compute" this element then we can also compute $\chi(X, \mathcal{L})$.

This follows from the spectral sequence

$$\chi_C(\chi(X, R^\bullet p_{2*}\mathcal{L})) = \chi(X, \mathcal{L}) = \sum \dim_k (-1)^\nu H^\nu(X, \mathcal{L})$$

Hence it is clear that we have a better theorem if we can prove a formula for $\chi(X, R^\bullet p_{2*}(\mathcal{L}))$ such a formula will imply the classical Riemann-Roch formula for $C \times_k C$.

Again we should be much more consequent and ask for a formula for $\chi(X, R^\bullet p_{2*}(\mathcal{F}))$ for any coherent sheaf on X. We can define the group $K'(X)$ and $\mathcal{F} \to \chi(X, R^\bullet p_{2*}(\mathcal{F}))$ is a homomorphism $\chi : K'(X) \to K'(C)$.

I will explain how we can get such a formula a special situation. Let us assume that our line bundle is of the form $\mathcal{L} = \mathcal{O}_X(-Z)$ where Z is a smooth irreducible curve in X which is not one of the fibres. Then we proceed in exactly the same way as in the case of curves. We write the exact sequence

$$0 \to \mathcal{O}_X \to \mathcal{O}_X(-Z) \to \mathcal{O}_X(-Z)/\mathcal{O}_X \to 0.$$

This gives us the formula

$$\chi(X, R^\bullet p_{2*}(\mathcal{L})) = \chi(X, R^\bullet p_{2*}(\mathcal{O}_X)) + \chi(R^\bullet p_{2*}(\mathcal{O}_X(-Z)/\mathcal{O}_X))$$

But now the situation is similar (but much more complicated) as in the case of curves. The sheaf $\mathcal{O}_X(-Z)/\mathcal{O}_X)$ is supported on Z and the morphism $p_{2*} : Z \to C$ is finite. Actually the sheaf $\mathcal{O}_X(-Z)/\mathcal{O}_X)$ is a line bundle on Z and since $p_{2*} : Z \to C$ is finite the higher direct images vanish. Therefore we have $\chi(R^\bullet p_{2*}(\mathcal{O}_X(-Z)/\mathcal{O}_X) = [p_{2*}(\mathcal{O}_X(-Z)/\mathcal{O}_X)]$ and since it is clear that $\chi(X, R^\bullet p_{2*}(\mathcal{O}_X)) = -[\mathcal{O}_C^{g-1}]$ our formula simplifies

$$\chi(X, R^\bullet p_{2*}(\mathcal{L})) = -[\mathcal{O}_C^{g-1}] + [p_{2*}(\mathcal{O}_X(-Z)/\mathcal{O}_X)].$$

Hence we are left with the "computation" of $[p_{2*}(\mathcal{O}_X(-Z)/\mathcal{O}_X)]$.

Here we have to stop for a second. What does it mean to compute this element in $K'(C)$? This group is much to complicated to identify this individual object. But remember what we actually want. Eventually we want to compute the Euler characteristic $\chi_C(\chi(X, R^\bullet p_{2*}\mathcal{L}))$ and to do this we only need to know the degree and the rank of $\chi(X, R^\bullet p_{2*}\mathcal{L})$. Therefore we will be content if we can compute the two numbers

$$\deg p_{2*}(\mathcal{O}_X(-Z)/\mathcal{O}_X) \text{ and } \operatorname{rank} p_{2*}(\mathcal{O}_X(-Z)/\mathcal{O}_X)$$

.

In principle we learned how to do that. We did this in the special case when we discussed the degree of the module of differentials. It is quite clear that the rank $\operatorname{rank} p_{2*}(\mathcal{O}_X(-Z)/\mathcal{O}_X)$ is degree $d_2(Z)$ of the morphism $p_{2*} : Z \to C$. The computation of the degree is more subtle.

Some geometric considerations

These considerations are valid for arbitrary smooth surfaces X and a smooth curve $Z \subset X$. The $\mathcal{O}_X$ - module sheaf $\mathcal{N} = \mathcal{O}_X(-Z)/\mathcal{O}_X$ is a line bundle when we restrict it to Z. To see this we recall that we can write Z locally by one equation and hence the ideal $\mathcal{I}(Z)$ which defines Z is a line bundle, it is the line bundle $\mathcal{O}_X(Z)$. Then $\mathcal{I}(Z) \otimes \mathcal{O}_Z = \mathcal{I}(Z)/\mathcal{I}(Z)^2$ is a line bundle on Z, it is called the conormal bundle. The bundle $\mathcal{N} = \mathcal{O}_X(-Z)/\mathcal{O}_X$ is obviously dual to $\mathcal{I}(Z)/\mathcal{I}(Z)^2$.

If we restrict the tangent bundle T_X to Z then we get an exact sequence

$$0 \to T_Z \to T_X|Z \to T_X|_Z/T_Z \to 0$$

But know one sees that $I(Z)/\mathcal{I}(Z)^2$ is the dual bundle to the quotient $T_X|Z/T_Z$. This is so because the differentials df for $f \in \mathcal{I}(Z)$ can be evaluated on the tangent vectors $t_z \in T_{X,z}$ they vanish on the subspace $T_{Z,z}$ and this gives the pairing. Hence it is clear that $\mathcal{N} = T_X|_Z/T_Z$, the bundle $\mathcal{N}$ is the normal bundle.

We get a formula

$$\Lambda^2(T_X|_Z) = T_Z \otimes \mathcal{N}$$

which is called the adjunction formula. Very often is is written in dual form. The second exterior power $\Lambda(T_X)$ is the dual bundle to the so called canonical bundle $K_X = \Lambda(\Omega^1_X)$ and then we get

$$K_X|_Z = \Omega^1_Z \otimes \mathcal{I}(Z)/\mathcal{I}(Z)^2$$

. This yields in terms of degrees

$$\deg(K_X|_Z) = \deg(\Omega^1_Z) + \deg(\mathcal{I}(Z)/\mathcal{I}(Z)^2)$$

and this can be interpreted in terms of intersection numbers as (see the considerations following below)

$$K_X \cdot Z = 2g_Z - 2 - Z \cdot Z$$

.

We return to our discussion of the Grothendieck-Riemann-Roch theorem, we assume again that $X = C \times_k C$. The degree of the line bundle $\mathcal{N}$ on Z is called the selfintersection $Z \cdot Z = \mathcal{O}_X(-Z) \cdot \mathcal{O}_X(-Z)$. Hence we have in $K'(Z)$ that $[\mathcal{N}] = [\mathcal{O}_Z] + [\mathcal{M}]$, where $\mathcal{M}$ is a virtual torsion sheaf of degree $Z \cdot Z$. Hence we get

$$p_{2*}(\mathcal{O}_X(-Z)/\mathcal{O}_X) = p_{2*}(\mathcal{N}) = [p_{2*}(\mathcal{O}_Z)] + [p_{2*}\mathcal{M}]$$

The second term is torsion and therefore we have that its rank is zero and the degree is $Z \cdot Z$. We are not yet at the end we need to compute $p_{2*}(\mathcal{O}_Z)$. This can be done by the same method which we applied when we computed the degree of the sheaf of differentials. Again we introduce the "different"-module $\mathcal{D}_{Z/C}$ using the same definition as before, we replace $\mathbb{P}^1$ by C and C by Z. We have the perfect duality of $\mathcal{O}_C$-modules

$$p_{2*}(\mathcal{O}_Z) \times p_{2*}(\mathcal{D}^{-1}_{Z/C}) \to \mathcal{O}_C$$

which implies that the degrees of these two modules add up to zero. Again we have the Hurwitz formula

$$\Omega^1_Z = p_2^*(\Omega^1_C) \otimes \mathcal{D}^{-1}_{Z/C}$$

This gives a formula for the degree of $\mathcal{D}^{-1}_{Z/C}$ and this formula yields

$$\deg(p_{2*}(\mathcal{O}_Z)) = -\frac{1}{2}\deg(\mathcal{D}^{-1}_{Z/C}) = d_2(Z)(g-1) - (g_Z - 1)$$

This formula is still not completely satisfactory, we want to compute the genus g_Z of the curve Z. To get this we recall that we have seen that $T_Z \otimes \mathcal{N} = \Lambda^2(T_X|_Z)$ which implies $2g_Z - 2 = Z \cdot Z - \deg(\Lambda^2(T_X|_Z)$

But still it is not clear what this means, because the result will depend on Z and we need a way to produce numbers from the divisor Z. Collecting all the terms gives us

$$(\mathrm{rank}, \deg)(\mathcal{O}_X(-Z)/\mathcal{O}_X) = (d_2(Z), d_2(Z)(g-1) + \frac{1}{2}(Z \cdot Z + \deg(\Lambda^2(T_X|_Z)))$$

The tangent bundle T_X is the direct sum of the two pullbacks of the tangent bundle on C hence it is clear that $\deg(\Lambda^2(T_X|_Z)) = (2-2g)(d_1(Z)+d_2(Z))$ and our formula simplfies to

$$(\mathrm{rank}, \deg)(\mathcal{O}_X(-Z)/\mathcal{O}_X) = (d_2(Z), -d_1(Z)(g-1) + \frac{1}{2}Z \cdot Z)$$

To get the final formula we remember that $\mathcal{O}_X(-Z)/\mathcal{O}_X$ was only one term in our exact sequence. For our original line bundle $\mathcal{L} = \mathcal{O}_X(-Z)$ we get

$$(\mathrm{rank}, \deg)(R^\bullet p_{2*}(\mathcal{O}_X(-Z))) = (-(g-1) + d_2(Z), -d_1(Z)(g-1) + \frac{1}{2}Z \cdot Z)$$

At this point it is clear how to get a formula for arbitrary line bundles on X. On the product X we have the two special divisors $H_2 = x_0 \times C, H_1 = C \times x_0$ where x_0 is just an arbitrary point. Then it is clear that the degrees $d_1(Z) = Z \cdot H_1, d_2(Z) = Z \cdot H_2$. Hence the general formula should be

$$(\mathrm{rank}, \deg)(\mathcal{L}) = (-(g-1) + \mathcal{L} \cdot H_2, -(g-1)\mathcal{L} \cdot H_1 + \frac{1}{2}\mathcal{L} \cdot \mathcal{L})$$

To prove it we consider exact sequences $(a > 0)$

$$0 \to \mathcal{L} \to \mathcal{L}(-aH_1) \to \mathcal{L}(-aH_1)/\mathcal{L} \to 0$$

$$0 \to \mathcal{L} \to \mathcal{L}(-aH_2) \to \mathcal{L}(-aH_2)/\mathcal{L} \to 0$$

and we apply $R^\bullet p_{2*}$ to both sequences. It is not difficult to show that in both cases the resulting long exact sequence shows that the formula above is true for $\mathcal{L}$ if and only if it is true for $\mathcal{L}(-aH_1)$ resp. $\mathcal{L}(-aH_2)$.

Then it is clear that it suffices to prove the Riemann-Roch formula for $\mathcal{L}(-a_1H_1 - a_2H_2)$ where $a_1, a_2 >> 0$. But then the bundle $\mathcal{L}(-a_1H_1 - a_2H_2)$ will be very ample and provide a projective embedding of $C \times_k C$. Then we get $\mathcal{L}(-a_1H_1 - a_2H_2) = \mathcal{O}_X(-Z)$ where Z is a section with a hyperplane. Now we invest the theorem of Bertini which says that we can choose the hyperlane so that Z is smooth and now we are in the case which we treated above.

Of course we can now easily derive the formula for the Euler characteristic of $H^\bullet(X, \mathcal{L})$:

$$\sum(-1)^\nu \dim_k H^\nu(X, \mathcal{L}) = (g-1)^2 - (g-1)\mathcal{L}\cdot(H_1 + H_2) + \frac{1}{2}\mathcal{L}\cdot\mathcal{L}$$

It is quite clear that our approach has a serious defect. In our argument we used that we have a Riemann -Roch formula for arbitrary coherent sheaves, but on our surface we proved it only for line bundles. This is not a consistent way to proceed. We should do something for arbitrary coherent sheaves on X. I will give a scetchy description how we can proceed. Again we introduce the group $K'(X)$. But we need a way to produce numbers. To get these we introduce the Chow ring

$$A^\bullet(X) = A^0(X) \oplus A^1(X) \oplus A^2(X)$$

of our surface. The group $A^\nu(X)$ is generated by irreducible subvarieties of codimension ν. The elements $Z = \sum n_i Y_i$ are called codimension ν cycles. But we have to introduce an equivalence relation

$$Z = 0 \text{ if and only if } Z \cdot Z' = 0 \text{ for all codimension } 2-\nu \text{ cycles}$$

.

This Chow ring can be introduced for arbitrary smooth projective schemes. It has a very intuitive geometric meaning, but it is very difficult to understand. It is technically a little bit difficult to define the intersection of cycles. Remember that we have an equivalence on the linear combinations of cycles and the intersection has to be compatible with that.

Here the situation is a little bit simpler than in general: We have $A^0(X) = \mathbb{Z}$, $A^2(X) = \mathbb{Z}$ and the only complicated group is $A^1(X)$.

Now we define a homomorphism (the Chern character)

$$\mathrm{ch} : K'(X) \to A^\bullet(X)$$

The construction of this homomorphism requires the theory of Chern classes.

To any vector bundle $\mathcal{E}$ on X we can attach Chern classes

$$(c_1(\mathcal{E}), c_2(\mathcal{E}), \ldots) \in A^1(X) \oplus A^2(X) + \ldots$$

(Here X could also be of greater dimension.) We encode these Chern classes by writing the Chern polynomial

$$P(\mathcal{E}, t) = 1 + c_1(\mathcal{E})t + c_2(\mathcal{E})t^2 + \ldots$$

The fundamental properties of these Chern classes are:

(1) If we have an exact sequence of vector bundles

$$0 \to \mathcal{E}' \to \mathcal{E} \to \mathcal{E}" \to 0$$

then

$$P(\mathcal{E},t) = P(\mathcal{E}',t)P(\mathcal{E}",t),$$

here we use the ring structure of $A(X)$.

(2) If $\mathcal{L} = \mathcal{O}_X(-D)$ is a line bundle given by a divisor $D = \sum n_i \mathbb{Z}_i$ where Z_i are irreducible of codimension one then

$$P(\mathcal{L},t) = 1 + (\sum n_i Z_i)t$$

where we now consider D as an element in $A^1(X)$.

Now it is possible to show that for a smooth variety the group $K'(X)$ is isomorphic to $K(X)$ which is the group generated by vector bundles and relations coming from exact sequences. (We have proved this in the case of curves).

Now we define $\mathrm{ch}(\mathcal{E}) \in A(X)$ for a vector bundle $\mathcal{E}$ by the following rule.

If rank $(\mathcal{E}) = d$ then

$$1 + c_1(\mathcal{E})t + \ldots c_d(\mathcal{E})t^d = (1 - \lambda_1 t)\ldots(1 - \lambda_d t)$$

and

$$\mathrm{ch}(\mathcal{E}) = (\sum_{i=1}^{d} \lambda_i^0, \sum_{i=1}^{d} \lambda_i, \frac{1}{2!}\sum_{i=1}^{d} \lambda_i^2, \ldots, \frac{1}{n!}\sum_{i=1}^{d} \lambda_i^n) \in A^\bullet(X)$$

in other words

$$\mathrm{ch}(\mathcal{E}) = (\mathrm{rank}(\mathcal{E}), c_1(\mathcal{E}), \frac{1}{2}c_1(\mathcal{E})^2 - c_2(\mathcal{E}), \ldots).$$

It is clear that we get for an exact sequence of vector bundles

$$0 \to \mathcal{E}' \to \mathcal{E} \to \mathcal{E}" \to 0$$

that the property 1) translates into

$$\mathrm{ch}(\mathcal{E}) = \mathrm{ch}(\mathcal{E}') + \mathrm{ch}(\mathcal{E}").$$

If we have a line bundle $\mathcal{L} = \mathcal{O}_X(-D)$ then

$$\mathrm{ch}(\mathcal{L}) = (1, D, \frac{1}{2}D \cdot D, \ldots, \frac{1}{n!}D^n, \ldots).$$

If D is effective then we had the sequence

$$0 \to \mathcal{O}_X \to \mathcal{O}_X(-D) \to \mathcal{O}_X(-D)/\mathcal{O}_X \to 0$$

and we find

$$\mathrm{ch}(\mathcal{O}_X(-D)/\mathcal{O}) = (0, D, \frac{1}{2}D \cdot D, \ldots).$$

We return to the case that X is a surface. Let us assume that we have two irreducible subvarieties $Z_1, Z_2 \subset X$ which intersect in smooth points. If J_1, J_2 are the ideals defining these subvarieties then we get an exact sequence of sheaves

$$0 \to \mathcal{O}_X/\mathcal{O}_X(Z_1+Z_2) \to \mathcal{O}_X/\mathcal{O}_X(Z_1) \oplus \mathcal{O}/\mathcal{O}_X(Z_2) \to J \to 0$$

where J is a torsion sheaf whose stalk is non zero only at the intersection points of Z_1, Z_2. At these points we have

$$J_X = \mathcal{O}_{X,x}/(I_1, I_2).$$

Hence we see that

$$\dim_k H^0(X, J) = Z_1 \cdot Z_2$$

and

$$\mathrm{ch}(J) = (0, 0, Z_1 \cdot Z_2).$$

Therefore we must have

$$\mathrm{ch}(\mathcal{O}_X/\mathcal{O}_X(Z_1+Z_2)) + \mathrm{ch}(J) = \mathrm{ch}(\mathcal{O}_X/I_1) + \mathrm{ch}(\mathcal{O}_X/I_2)$$

and this just turns out to be right

$$(0, -Z_1 - Z_2, \frac{(Z_2+Z_2)^2}{2}) + (0, 0, Z_1 \cdot Z_2) = (0, -Z_1, \frac{{Z_1}^2}{2}) + (0, -Z_2, \frac{Z_2^2}{2}).$$

We have a map (remember $X = C \times C$)

$$p_2^\bullet : A^\bullet(X) \to A^\bullet(C)$$

which simply is given by the intersection

$$Z \to (x_0 \times C) \cdot Z$$

and hence we get a diagram

$$\begin{array}{ccc} K'(X) & \xrightarrow{\mathrm{ch}} & A^\bullet(X) \\ R^\bullet p_* \downarrow & & p_2^\bullet \downarrow \\ K'(C) & \xrightarrow{\mathrm{ch}} & A^\bullet(C) \end{array} .$$

This diagram does not commute, but we have the Grothendieck-Riemann-Roch formula

$$p_2^\bullet(\mathrm{ch}(\mathcal{F}) \cdot (\mathcal{T}(T_X)) = \mathrm{ch} \circ Rp_{2*}(\mathcal{F}) \cdot \mathcal{T}(T_C)).$$

Here $\mathcal{T}(T_X)$ resp $\mathcal{T}(T_C)$ is the so called Todd-genus of the tangent bundle $\mathcal{T}_X$ resp $\mathcal{T}_C$ It is defined as follows: We write as above

$$1 + c_1(T_X)t + \dots c_d(T_X)t^d = (1-\lambda_1 t)\dots(1-\lambda_d t)$$

and then

$$\mathcal{T}(T_X) = \prod \frac{\lambda_i}{(1-e^{-\lambda_i})}$$

This is now a general formulation of the Riemann-Roch for a morphism

$$p_2 : X \to C$$

between any two projective smooth varieties over k.

Actually we have seen a special case of this formula before. Let us assume that we have a separable finite morphism $f : C_1 \to C_2$ between two smooth curves. Then we have

$$\begin{array}{ccc} K'(C_1) & \xrightarrow{\text{ch}} & A^\bullet(C_1) \\ f_* \downarrow & & f^\bullet \downarrow \\ K'(C_2) & \xrightarrow{\text{ch}} & A^\bullet(C_2) \end{array}$$

and

$$(\text{ch} \circ f_*(\mathcal{F})) \cdot (\mathcal{T}(T_{C_2})) = f^\bullet(\text{ch}(\mathcal{F}) \cdot (\mathcal{T}(T_{C_1}))$$

and this is the Hurwitz formula.

I want to discuss an application of the Riemann-Roch theorem. By definition we have a non degenerate pairing

$$A(X) \otimes \mathbb{Q} \times A(X) \otimes \mathbb{Q} \to \mathbb{Q}$$

which is given by the intersection pairing. In $A(X)$ we have the two classes given by H_1, H_2, it is clear that $H_1 \cdot H_1 = H_2 \cdot H_2 = 0$ and $H_1 \cdot H_2 = 1$. Hence $\mathbb{Q}H_1 \oplus \mathbb{Q}H_2$ is a hyperbolic plane in $A(X) \otimes \mathbb{Q}$. Hence it has an orthogonal complement $A^1_0(X)$ and

$$A(X) \otimes \mathbb{Q} = \mathbb{Q}H_1 \oplus \mathbb{Q}H_2 \oplus A^1_0(X)$$

Now he have

Hodge index theorem:

The intersection form is negative definite on $A^1_0(X)$

Proof: Let $\mathcal{L}$ be a line bundle which corresponds to a class $\xi \in A^1_0(X)$. Then we get from the Riemann-Roch theorem that

$$(\text{rank}, \deg)(R^\bullet p_{2*}(\mathcal{L})) = (-(g-1), \frac{1}{2}\mathcal{L} \cdot \mathcal{L})$$

It is clear that $p_{2*}(\mathcal{L})$ has no torsion and since the restriction of $\mathcal{L}$ to the fibre of p_2 has degree zero we get that $p_{2*}(\mathcal{L})$ has at most rank one. We may apply the same argument to $R^1p_{2*}(\mathcal{L})$. In a fibre we have $\dim_k(R^1p_{2*}(\mathcal{L})_y) \leq g$ and henc we see that the rank of $R^1p_{2*}(\mathcal{L})$ is bounded. What can happen is that $R^1p_{2*}(\mathcal{L})$ has a large torsion subgroup. Now we replace $\mathcal{L}$ by $\mathcal{L}^n$. Our previous considerations still apply. I claim that we cannot have $\mathcal{L}\cdot\mathcal{L} > 0$. Otherwise we would get that $\mathcal{L}^{\otimes n}\cdot\mathcal{L}^{\otimes n} = n^2\mathcal{L}\cdot\mathcal{L}$ would become a large positive number. But the only thing on the left hand side that can become large is the torsion subgroup in $R^1p_{2*}(\mathcal{L}^{\otimes n})$ but it contributes by a negative term since it sits in degree one. Hence $\mathcal{L}\cdot\mathcal{L}\leq 0$ and the form is negative. Then it must be negative definite because the form is non degenerate.

Curves over finite fields.

We consider the special situation of a projective, smooth and absolutely irreducible curve $C/\mathbb{F}_q$, where $\mathbb{F}_q$ is the field with q elements. This special case is historically the origin for the theory of curves over arbitrary fields. The classical geometers always studied the case of curves over $\mathbb{C}$ and the theory of Riemann surfaces.

As I explained, we can consider the function field $K = \mathbb{F}_1(C)$, this is a finite separable expension of a rational function field $\mathbb{F}_q[f]$ (see ???). The condition that $C/\mathbb{F}_q$ says that the field of constants is equal to $\mathbb{F}_q$. We can recover the curve $C/\mathbb{F}_q$ from the field K as we explained in ???.

These function fields attracted the attention of number theorists since they are analogous to number fields. The closed points $\mathfrak{p}$ of the curve C are in one to one correspondence to the discrete valuation ring $\mathcal{O}_\mathfrak{p}\subset K$. If we remove one point ∞ from C, then $C\setminus\{\infty\}$ is the spectrum of the Dedekind ring $\mathcal{O}_C(C\setminus\{\infty\})$, and this has to be seen in analogy to the number field case where we have $\mathrm{Spec}(\mathcal{O})$ where $\mathcal{O}$ is the ring of integers.

We can attach a ζ-function to our curve which we define as

$$\zeta_K(s) = \zeta_C(s) = \prod_{\mathfrak{p}:\text{closed point}} \frac{1}{1-\frac{1}{N\mathfrak{p}^s}}$$

where $N\mathfrak{p} = \#(\mathcal{O}_\mathfrak{p}/\mathfrak{p})$ is the number of elements in the residue field. This ζ-function is analogous to the Dedekind ζ-function of a number field, and it is easy to see that the product converges for $\mathrm{Re}(s) > 1$.

We will show that the Riemann-Roch theorem implies that this ζ-function is a rational function in the variable $t = q^{-s}$ and more precisely we have

$$\zeta_C(s) = \frac{P_1(q^{-s})}{(1-q\cdot q^{-s})(1-q^{-s})}$$

where $P_1(q^{-s}) = 1 + a_1q^{-s}\cdots + q^g\cdot q^{-2gs}$ is a polynomial of degree $2g$ in q^{-s} with integer coefficients. We will see that the theorem of Riemann-Roch implies that we have a functional equation

$$\zeta(s) = q^{(2-g2)s}q^{1-g}\zeta(1-s).$$

This is in a perfect analogy to the situation of a number field.

Now it has been observed by Artin and Hasse that we can formulate the analogon of the Riemann hypothesis which would say that the zeroes of $\zeta(s)$ have imaginary part $\mathrm{Re}(s) = \frac{1}{2}$. This assertion can be formulated in terms of the polynomial $P_1(q^{-s}) = P_1(t)$. We may factor it over $\overline{\mathbb{Q}} \subset \mathbb{C}$ and get

$$P_1(t) = \prod_{i=1}^{2g}(1 - \omega_i t)$$

where the ω_i are algebraic integers. It is easy to see that the functional equation allows a grouping of these numbers ω_i so that we have

$$\omega_1 \cdots \omega_g \ , \ \omega_{g+1} \cdots \omega_{2g}$$

and $\omega_\nu \cdot \omega_{v+g} = q$.

Then the Riemann hypothesis is equivalent to the assertion that all the ω_i have absolute value

$$|\omega_i| = q^{\frac{1}{2}}.$$

This has been conjectured by Artin in his thesis (???), and he verified it in several cases. But actually Gauss knew it as special cases in a somewhat disguised form and also Artin's thesis advisor Herglotz had proved in it in a special case.

This special case occurs in Gauss' Tagebuch, where he states it as a fact which was found experimentally but he does not claim that he proved it.

The Riemann hypothesis was then proved by Hasse (1934) for curves of genus one and A. Weil announced the proof in the general case in 1941. The final proof appeared in 19.., and it is based on the theory of the Jacobian of curves.

Elementary properties of the ζ-function.

In this section I call elementary properties onf the ζ-function those which follow from the Riemann-Roch theorem for the curve.

As in the case of the Riemann ζ-function we can expand the product

$$\prod_{\mathfrak{p}} \frac{1}{1 - \frac{1}{N\mathfrak{p}^s}} = \sum_{\mathfrak{a}} \frac{1}{N(\mathfrak{a})^s}$$

where $\mathfrak{a}$ runs over the effective divisors

$$\mathfrak{a} = \sum n_i \mathfrak{p}_i \qquad n_i \leq 0,$$

where $N(\mathfrak{a}) = \prod (N\mathfrak{p}_i)^{n_i} = q^{\deg(\mathfrak{a})}$. For the following discussion we have to assume that our curve has a divisor of degree one. This assumption is equivalent to the assumption that the degrees $f_{\mathfrak{p}}$ of the prime divisors are coprime. It is a theorem of F. K. Schmidt that this is always the case.

If we make this assumtion (or if we believe F. K. Schmidt's theorem), then it is clear that the divisor classes of degree n for any n form a principal homogeneous space under $\mathrm{Pic}^0(C)(\mathbb{F}_q)$, this is the group of divisor classes of degree 0. The number h of these divisor classes is finite and it is called the class number. We get

$$\zeta_C(s) = \sum_{n=0}^{\infty} \frac{c(n)}{q^{ns}}$$

where $c(n) = \#$ of effective divisors of degree n. (This number is clearly finite.) For $n > 2g-2$ we have a formula for $c(n)$: If $\mathfrak{a}$ is an effective divisor of degree n, then we can consider the line bundle $\mathcal{L} = \mathcal{O}_C(-\mathfrak{a})$, and we have

$$\dim H^0(C, \mathcal{O}_C(\mathfrak{a})) = n + 1 - g$$

(see ???). A non zero section $s \in H^0(C, \mathcal{L})$ defines the divisor $\mathfrak{b} = \mathrm{div}(s)$. and we see that the divisors $\mathfrak{b}$ which we get, if s varies are exactly those divisors which are linearly equivalent to $\mathfrak{a}$. Then

$$c(n) = h \ \frac{q^{n+1-g} - 1}{q-1}.$$

(As a byproduct we proved the finiteness of the class number.)

We make the substitution $q^{-s} = t$, and write $\zeta_C(s) = Z_C(t)$. Then we define a new function $Z_C^*(t)$ by

$$Z_C(t) = Z_C^*(t) + \frac{h}{q-1} \cdot \left(q^g \frac{t^{2g-1}}{1-qt} - \frac{1}{1-t} \right),$$

and

$$Z_C^*(t) = \sum_{n=0}^{2g-1} c^*(n) t^n.$$

The coefficiants $c^*(n)$ are equal to zero for $n \geq 2g-1$, and we have

$$c(n) = c^*(n) - \frac{h}{q-1} \quad \text{for} \quad 0 \leq n \leq 2g-2.$$

The correcting term

$$R(t) = \frac{h}{q-1} \left(q^g \frac{t^{2g-1}}{1-qt} - \frac{1}{1-t} \right)$$

satisfies the functional equation

$$R(t) = t^{2g-2} q^{1-g} R(\frac{1}{qt}),$$

and hence we have to show that $Z_C^*(t)$ satisfies this functional equation.

Now we observe that we have a precise formula for $c(n)$. We sum over the classes of bundles of degree n and count the effective divisors in a class. If $\mathcal{L}$ is a line bundle of degree n, then the number of effective divisors in this class is

$$\frac{1}{q-1}\left(q^{\dim H^0(C,\mathcal{L})} - 1\right),$$

and hence

$$c(n) = \frac{1}{q-1} \sum_{\mathcal{L}:\deg(\mathcal{L})=n} \left(q^{\dim H^0(C,\mathcal{L})} - 1\right).$$

Now we assume that for all n this sum is not empty. This follows from the theorem of F. K. Schmidt that we can always find a divisor of degree one. Then we can conclude that the number of terms in the sum is equal to h and hence

$$c^*(n) = \sum_{\mathcal{L}:\deg(\mathcal{L})=n} q^{\dim H^0(C,\mathcal{L})}.$$

Our problem is that we do not know the dimension of the space of sections. Now we have in involution n, the set of divisor classes of degree between 0 and $2g-2$.

$$\mathcal{L} \longrightarrow \mathcal{L}^{-1} \otimes \Omega = \mathcal{L}',$$

and then $\deg(\mathcal{L}) \to 2g-2-\deg(\mathcal{L})$. We see that

$$c^*(2g-2-n) = \sum_{\mathcal{L}':\deg(\mathcal{L}')=2g-2-n} q^{\dim H^0(C,\mathcal{L}')}.$$

If now $\mathcal{L}' = \mathcal{L}^{-1} \otimes \Omega$, then

$$\dim H^0(C,\mathcal{L}) - \dim H^0(C,\mathcal{L}') = n+1-g,$$

hence this difference depends only on the degree and not on the class. This implies

$$c^*(n) = q^{n+1-g} c^*(2g-2-n),$$

and this implies the functional equation for $Z_C^*(t)$.

Now we can conclude that

$$Z_C(t) = \frac{1 + a_1 t \cdots + g^g t^{2g}}{(1-t)(1-qt)}.$$

The coefficients a_i must be integers, since the $c(n)$ are integers.

We can write

$$P_1(t) = \prod_{i=1}^{2g} (1 - \omega_i t),$$

the $\omega_i \in \mathbb{C}$ are algebraic and the functional equation implies that the collection

$$\{\omega_1 \cdots \omega_{2g}\}$$

is invariant under the substitution

$$\omega_i \longrightarrow \omega_i^{-1} q.$$

At this moment it is not yet clear that they are algebraic integers and that we have the Riemann hypothesis

$$|\omega_i| = q^{\frac{1}{2}} \qquad \text{for} \qquad i = 1 \cdots 2g.$$

These so called elementary results are all due to F. K. Schmidt. By the way F. K. Schmidt was clever enough to remove the assumption that we have a divisor of degree one, he just proved that this is always the case.

Can you also prove it?

The Riemann hypothesis.

We recall the multiplicative definition of the ζ-function, and we remember that

$$N\mathfrak{p}^{-s} = q^{-f_\mathfrak{p} s} = t^{f_\mathfrak{p}}$$

where $[\mathbb{F}_q(\mathfrak{p}) : \mathbb{F}_q] = q^{f_\mathfrak{p}}$, i.e. $f_\mathfrak{p}$ is the degree of the residue field at $\mathfrak{p}$.

Now we consider the expression

$$t \cdot \frac{Z_C'(t)}{Z_C(t)} = \sum_{\mathfrak{p}} \frac{f_\mathfrak{p} t^{f_\mathfrak{p}}}{1 - t^{f_\mathfrak{p}}}.$$

If we expand the expression on the right, we get an infinite sum

$$\sum \; a_n t^n$$

and clearly

$$a_n = \sum_{f_{\mathfrak{p}} | n} f_{\mathfrak{p}}.$$

But $f_{\mathfrak{p}}$ is exactly the number of geometric points over $\mathfrak{p}$, and these geometric points lie in

$$C(\mathbb{F}_{q^{f_{\mathfrak{p}}}}) \subset C(\mathbb{F}_{q^n}),$$

and hence we see that the right hand side must be

$$t \cdot \frac{Z_C'(t)}{Z_C(t)} = \sum \#C(\mathbb{F}_{q^n}) t^n.$$

Going back to our expression for $Z_c(t)$ as a rational function we find

$$t \cdot \frac{Z_C'(t)}{Z_C(t)} = \frac{qt}{1-qt} + \frac{t}{1-t} - \sum_{i=1}^{2g} \frac{\omega_i t}{1-\omega_i t},$$

and from here we get

$$\#C(\mathbb{F}_{q^n}) = q^n + 1 - \sum_{i=1}^{2g} \omega_i^n$$

for all n.

(This implies that $\sum_{i=1}^{2g} \omega_i^n$ is an integer for all numbers n, and this implies that the ω_i must be integers themselves.)

The following proof of the Riemann hypothesis is due to A. Grothendieck ([ ]) and is based on intersection theory and the Hodge index theorem.

Our curve $C/\mathbb{F}_q$ is defined over $\mathbb{F}_q$, and this allows us to define the Frobenius morphism

$$\Phi_q : C \longrightarrow C.$$

To define this morphism we consider an affine open subset $U \subset C$, then the ring of regular function $\mathcal{O}_c(U)$ is an $\mathbb{F}_q$-algebra and the map

$$\Psi_q : \mathcal{O}_c(U) \longrightarrow \mathcal{O}_c(U)$$

which sends $f \in \mathcal{O}_c(U)$ to f^q is an $\mathbb{F}_q$-algebra homomorphism. This defines the restriction of Φ_q to U, and since we can cover C by such open affine sets, we have defined it every where.

If we have an embedding of our curve

$$\begin{array}{ccc} i : C & \longrightarrow & \mathbb{P}^n_{\mathrm{Spec}(\mathbb{F}_q)} \\ & \searrow \ \swarrow & \\ & \mathrm{Spec}(\mathbb{F}_q). & \end{array}$$

Then the action of the Frobenius Φ_q on the geometric points is given by

$$\underline{x} = (x_0, \cdots, x_n) \longrightarrow x^q = x^q = (x_0^q, \cdots, x_n^q)$$

for any point $\underline{x} \in BP^n(\mathbb{F}_p)$.

Now we pass to a geometric situation, we replace C by $\overline{C} = C \times_{\mathbb{F}_q} \overline{\mathbb{F}}_q$ and our information that $\overline{C}$ comes from the curve $C/\mathbb{F}_q$ is encoded in the datum of the Fronenius morphism

$$\Phi_q : \overline{C} \longrightarrow \overline{C}$$

which also can be viewed as given by its graph

$$\Gamma_\Phi \subset \overline{C} \times \overline{C} = \overline{X}.$$

Then it is clear by definition that

$$C(\mathbb{F}_q) = \Gamma_\Phi \cap \Delta \subset C(\overline{\mathbb{F}}_q) \times C(\overline{\mathbb{F}}_q).$$

The tangent space of $\overline{X}$ in any geometric point $p = (x, y)$ is given by

$$T_{\overline{X},p} = T_{\overline{C},x} \oplus T_{\overline{C},y},$$

and the tangent space of Γ_Φ at $p = (x, \Phi(x))$ consist of vectors in

$$T_{\Gamma_\Phi,p} = T_{\overline{C},x}.$$

The tangent space of Δ is an point (x, x) is the diagonal, hence we see that Γ_Φ and Δ intersect transversally, and this gives us

$$\# C(\mathbb{F}_q) = \Gamma_\Phi \cdot \Delta.$$

If we pick any two points $x_0 \in C(\overline{\mathbb{F}}_q)$, $y_0 \in C(\overline{\mathbb{F}}_q)$, then we may consider the divisors

$$H_1 = \{x_0\} \times \overline{C} \ , \ \overline{C} \times \{y_0\} = H_2$$

on $\overline{C} \times \overline{C}$. We get an orthogonal decomposition of the Chow group

$$A^1(\overline{X}) = \tilde{A}^1(\overline{X}) \oplus \mathbb{Z}[H_1] \oplus \mathbb{Z}[H_2].$$

Since $H_1 + H_2$ is ample on $\overline{X}$, we can conclude that the intersection form restricted to $\tilde{A}^1(\overline{X})$ is negative definite.

We write in $A^1(\overline{X})$

$$[\Delta] = [A'] + [H_1] + [H_2],$$

then $[\Delta'] \in \tilde{y}^1(\overline{X})$.

More generally we can say: If $\Gamma \subset \overline{C} \times \overline{C}$ is the graph of a morphism $f : \overline{C} \to \overline{C}$, then

$$\begin{aligned} \Gamma \cdot (\{x_0\} \times \overline{C}) &= \Gamma \cdot H_2 &= 1 \\ \Gamma \cdot (\overline{C} \times \{y_0\}) &= \Gamma \cdot H_1 &= \deg(f) \end{aligned}$$

and

$$[\Gamma'] = [\Gamma] - (\deg f)[H_2] - [H_1] \in \tilde{A}(\overline{X}).$$

We need the selfintersection $\Gamma \cdot \Gamma$. To get this number we recall the adjunction formula. The tangent bundle $T_{\overline{X}} \simeq p_1^*(T_{\overline{C}}) + p_2^*(T_{\overline{C}})$ and hence

$$\Lambda^2 T_{\overline{X}} = p_1^*(T_{\overline{C}}) \otimes p_2^*(T_{\overline{C}}).$$

Now we have

$$\Gamma \cdot \Lambda^2 T_{\overline{X}} = \Gamma \cdot p_1^*(T_{\overline{C}}) + \Gamma \cdot p_2^*(T_{\overline{C}}) = (2g-2) \cdot (\deg(f) + 1).$$

Then the adjunction formula

$$\Gamma \cdot \Lambda^2 T_{\overline{X}} = 2g_\Gamma - 2 - \Gamma \cdot \Gamma,$$

and $g_\Gamma = g$ because the projection p_1 provides an isomorphism $p_1 : \Gamma \to \overline{C}$. Hence we get

$$\Gamma \cdot \Gamma = (2 - 2g) \cdot \deg(f)$$

and this yields

$$\Gamma' \cdot \Gamma' = -2g \deg(f).$$

We look at the case $\Gamma = \Gamma_\Phi$. We have

$$\Gamma_\Phi \cdot \Delta = \Gamma_\Phi \cdot ([\Delta'] + [H_1] + [H_2]) = \Gamma_\Phi \Delta' + q + 1,$$

and hence we get

$$-\sum_{i=1}^{2g} \omega_i = \Gamma_\Phi \cdot \Delta' = \Gamma'_\Phi \cdot \Delta'.$$

We know that

$$(n\Gamma'_\Phi + m\Delta') \cdot (n\Gamma'_\Phi + m\Delta') \geq 0$$

for all integers $n.m$, and this says that

$$-n^2 \cdot 2gq + 2nm\Gamma'_\Phi \cdot \Delta' - m^2 \cdot 2g \geq 0$$

for all $n.m$. This is equivalent to the inequality

$$(\Gamma'_\Phi \cdot \Delta')^2 = \left(\sum_{i=1}^{2g} \omega_i \right)^2 \leq 4g^2 q,$$

and hence we get the estimate

$$|\sum_{i=1}^{2g} \omega_i| \leq 2g\sqrt{q}.$$

This is in principle the Riemann hypothesis. The only thing we have to say is that we can replace $\mathbb{F}_q$ by any field $\mathbb{F}_{q^r}$, i.e. we get the inequality

$$|\sum_{i=1}^{2g} \omega_i^r| \leq 2g\sqrt{q^r}$$

for all $r = 1, 2, \cdots$. Since we have

$$\prod_{i=1}^{2g} \omega_i = q^g,$$

this can only be the case, if we have

$$|\omega_i| = \sqrt{q}$$

for all $i = 1 \cdots 2g$.

Chapter X

The Picard functor for curves

Introduction: In the previous chapter we constructed the Jacobian for compact Riemann surfaces and we heavily relied on transcendental methods. We got an complex torus and gave indications why it is in fact algebraic.

The goal of this chapter is the construction of a Jacobian $J_C/k = J/k$ for any smooth, projective and absolutely irreducible curve C/k where k is an arbitrary field. Of course we want that J is the "parameter space" for the line bundles of degree zero on C/k but we have to make it precise what we mean by that.

Once we constructed J/k we also want to investigate the group of line bundles on J/k, we want to construct the dual $J^\vee$ of J/k and we want to show that we have a canonical identification $J \to J^\vee$.

At the end of the last chapter we formulated a universality property for the line bundle $\mathcal{N}$ on $J \times J$ and $S \times J$, which we did not prove. This universality property will become the basic principle for the construction of J/k.

1. The construction of the Jacobian

1.1 **Generalities and heuristics :** If we consider any scheme X over a base S, we can define the Picard functor

$$\begin{array}{rcl} \mathrm{Pic}_{X/S} : \text{Schemes } S & \longrightarrow & \text{Abelian groups} \\ T & \longrightarrow & \mathrm{Pic}(X \times_S T) \end{array}$$

where $\mathrm{Pic}(X \times_S T) = \mathrm{Pic}_{X/S}(T)$ is the group of isomorphism classes of line bundles over $X \times_S T$. We may ask whether this functor is representable? (See ???). I recall what this means: We can find a group scheme $\mathrm{Pic}(X)/S$ and a "universal" line bundle

$$\mathcal{L} \quad \text{on} \quad X \times_S \mathrm{Pic}(X)$$

such that for any scheme $T \to S$ and any line bundle $\mathcal{L}_1$ over $X \times_S T$ we have a unique

$$\psi : T \to \mathrm{Pic}(X)$$

such that we can find an isomorphism

$$\eta : \mathcal{L}_1 \xrightarrow{\sim} (\mathrm{Id} \times \psi)^*(\mathcal{L}).$$

The answer to this question is simply "No".

The reason for this negative answer is that line bundles have non trivial automorphisms. This has the effect that we do not have a canonical way to identify two isomorphic line bundles $\mathcal{L}_1$, $\mathcal{L}_2$ on a scheme Y. Hence we cannot say that they are equal, in other words: The line bundles on $X \times_S T$ form a category but not a set.

We explain this difficulty in a different way. Let us assume that our base scheme is $S = \mathrm{Spec}(k)$ and k is an algebraically closed field. Then the intuitive meaning of the "universal" bundle $\mathcal{L}$ on $X \times_S \mathrm{Pic}(X)$ is the following: For any $p \in \mathrm{Pic}(X)(k)$ the restriction of $\mathcal{L}$ to $X \times \{p\}$ gives us a line bundle $\mathcal{L}_p$ on X. We want that $\mathrm{Pic}(X)$ parameterizes the line bundles on $X/\mathrm{Spec}(k)$ and by this we mean that $p \to [\mathcal{L}_p]$ is a "natural" bijection between $\mathrm{Pic}_{X/S}(k)$ and the isomorphism classes of line bundles on X. By "natural" we mean that this parametrization should at least have the following property: If we have a scheme T/k of finite type over k, and if we have a line bundle $\mathcal{L}_1$ on $X \times_k T$ then a point $t \in T(k)$ provides a line bundle $\mathcal{L}_{1,t}$ on X/k. To this line bundle corresponds a point $p = \psi_0(t)$ on $\mathrm{Pic}(X)(k)$ such that $\mathcal{L}_{1,t} \simeq \mathcal{L}_p$. At this point ψ_0 is just a map $\psi_0 : T(k) \to \mathrm{Pic}(X)(k)$ between the two sets of geometric points. We want that this map between the sets of geometric points is induced by a morphism $\psi : T \to \mathrm{Pic}(X)$. But still this cannot be enough since in general this morphism will not be determined by its effect on the geometric points if T is not reduced. (See ???).

1.1.1. We still want more. Let us now assume in addition that X/k is projective, reduced and connected. If we have such a ψ as above, then the line bundles $\mathcal{L}_1$ and $(\mathrm{Id} \times \psi)^*(\mathcal{L})$ are "isomorphic point by point" on $X \times T$. We want an isomorphism $\eta : \mathcal{L}_1 \xrightarrow{\sim} (\mathrm{Id} \times \psi)^*(\mathcal{L})$ between the bundles on $X \times T$. But now it is clear that we cannot have such a property: If we tensorize $\mathcal{L}_1$ by any line bundle $\mathcal{M}$ on $X \times T$ which is the pullback of a line bundle on T, we get isomorphic fibres $(\mathcal{L}_1 \otimes \mathcal{M})_t \simeq \mathcal{L}_{1,t}$ if t varies in $T(k)$. This means that the morphism ψ does not change. But we cannot expect that $\mathcal{L}_1$ is isomorphic to $\mathcal{L}_1 \otimes \mathcal{M}$.

But we can hope that a slightly weaker property is satisfied. We can form the tensor product $\mathcal{L}_1 \otimes ((\mathrm{Id} \times \psi)^*(\mathcal{L}))^{-1}$ which is a line bundle on $X \times_k T$. It is trivial at any geometric point $t \in T(k)$. Since we assumed that X/k is projective reduced and connected we have

$$\dim_k H^0(X \times \{t\}, \mathcal{L}_1 \otimes ((\mathrm{Id} \times \psi)^*(\mathcal{L}))_t^{-1}) = 1$$

for all the closed points $t \in T(k)$. (See ???) The semi-continuity theorem (See ???) implies that the direct image $p_{2,*}(\mathcal{L}_1 \otimes ((\mathrm{Id} \times \psi)^*(\mathcal{L}))^{-1}$ for the projection p_2 is a line bundle of rank one on T (we should assume that T is reduced). Hence we can find a covering $T = \cup T_\alpha$ by open subschemes such that $\mathcal{L}_1 \otimes ((\mathrm{Id} \times \psi)^*(\mathcal{L}))^{-1}$ is trivial on $X \times T_\alpha$. So we extended our point by point isomorphisms locally into some neighborhood of any point t_0.

1.1.1.1. In general we will call a line bundle $\mathcal{L}$ on $X \times T$ *locally trivial in* T if we can cover T by open sets T_α such that $\mathcal{L}|X \times T_\alpha$ becomes trivial. We call two line bundles $\mathcal{L}_1, \mathcal{L}_2$ on $X \times T$ *locally isomorphic* in T if we can cover T by open sets T_α such that $\mathcal{L}_1|X \times T_\alpha \xrightarrow{\sim} \mathcal{L}_2|X \times T_\alpha$

This means that the best we can hope for is to show that our functor is *locally representable.*

This means that we want to construct a scheme $\mathrm{Pic}(X)$ over k and a line bundle $\mathcal{P}$ on $X \times \mathrm{Pic}(X)$ such that for any bundle $\mathcal{L}$ on $X \times T$ with T of finite type over k we find a unique $\psi : T \to \mathrm{Pic}(X)$ such that

$$(\mathrm{Id} \times \psi)^*(\mathcal{P}) \sim_T \mathcal{L},$$

and this means that the bundles are locally isomorphic in T.

If X is reduced, projective and connected and let p_2 be the projection to the second factor T. Then it is clear that

$$p_{2,*}((\mathrm{Id} \times \psi)^*(\mathcal{P}) \otimes \mathcal{L}^{-1}) = \mathcal{M}$$

is locally free of rank one and we have

$$(\mathrm{Id} \times \psi)^*(\mathcal{P}) \xrightarrow{\sim} \mathcal{L} \otimes p_2^*(\mathcal{M})$$

1.1.1.2 Once we have achieved this goal we will say that $\mathrm{Pic}_{X/k}$ *is locally represented by the pair* $(\mathrm{Pic}(X), \mathcal{P})$.

Of course it seems to be reasonable to put some finiteness conditions on T we will need them. But the condition for T/k to be of finite type is perhaps to strong, we should also allow localizations of such schemes.

1.1.1.3. We can state the results of our considerations by saying the Picard functor is not a sheaf for the Zariski topology (See ???). If we have a covering of T by open subsets then we cannot expect that the sequence

$$\mathrm{Pic}_{X/S}(T) \to \prod \mathrm{Pic}_{X/S}(T_\alpha) \rightrightarrows \prod \mathrm{Pic}_{X/S}(T_\alpha \cap T_\beta)$$

is exact.

On the other hand we have seen in (???) that a necessary condition for representability of a functor is, that it is a sheaf for the faithfully flat topology.

1.1.2. The situation becomes better, if we assume that X is projective, absolutely reduced and connected over a field k and it comes with a distinguished point $P \in X(k)$. Now we define a modified functor

$$\widetilde{\mathrm{Pic}}_{X/k}(T) = \left\{ (\mathcal{L}, s) | \quad \begin{array}{l} \mathcal{L} \text{ line bundle on } X \times T, \\ s \in H^0(\{P\} \times T, \mathcal{L}) \text{ which} \\ \text{trivializes } \mathcal{L}|\{P\} \times T \end{array} \right\}.$$

This means we look at the group of isomorphism classes of line bundles on $X \times T$ which are trivial on $\{P\} \times T$ and which are equipped with

$$\eta_P : \mathcal{L}|\{P\} \times T \xrightarrow{\sim} \mathcal{O}_{\{P\} \times T}$$

where we require that $\eta_P(s) = 1$.

If we have two such pairs $(\mathcal{L}_1, s_1)$ and $(\mathcal{L}_2, s_2)$ with isomorphic line bundles then we have exactly one isomorphism which sends s_1 into s_2. Hence we can say that the two pairs $(\mathcal{L}_1, s_1)$, $(\mathcal{L}_2, s_2)$ are not only isomorphic, they are even equal. Here we used of course that $X/\operatorname{Spec}(k)$ is projective, reduced and connected.

1.1.2.1. Now it is also not difficult to show that the modified functor $\widetilde{\operatorname{Pic}}_{X/k}$ is in fact a sheaf for the faithfully flat topology.

We see that our chances to represent the functor $\widetilde{\operatorname{Pic}}_{X/k}$ are much better. We will prove representability of $\widetilde{\operatorname{Pic}}_{C/k}$ for curves C/k. The case of general smooth projective schemes X/k can be reduced to the case of curves.

But if we compare the two functors $\widetilde{\operatorname{Pic}}_{X/k}$ and $\operatorname{Pic}_{X/k}$ then we see that the difference is not so dramatic. We have a homomorphism between abelian groups

$$\widetilde{\operatorname{Pic}}_{X/k}(T) \longrightarrow \operatorname{Pic}_{X/k}(T).$$

The image consists of those isomorphism classes of line bundles which are trivial over $\{P\} \times T$. The homomorphism is injective since if we have two pairs $(\mathcal{L}_1, s_1)$, $(\mathcal{L}_2, s_2)$ such that the bundles $\mathcal{L}_1$ and $\mathcal{L}_2$ are isomorphic, then we can find an unique isomorphism which sends s_1 to s_2.

I recall that representability of $\widetilde{\operatorname{Pic}}_{X/k}$ gives us a universal triplet

$$(X \times \widetilde{\operatorname{Pic}}_{X/k}, \mathcal{P}, s)$$

where is $\widetilde{\operatorname{Pic}}_{X/k}$ a scheme, $\mathcal{P}$ a line bundle on

$$X \times \widetilde{\operatorname{Pic}}_{X/k},$$

and $s_0 \in H^0(\{P\} \times \widetilde{\operatorname{Pic}}_{X/k}, \mathcal{P})$ a section which is everywhere non zero. I do not formulate the universality property, it is now obvious.

But if we consider $\mathcal{P}$ on $X \times \widetilde{\operatorname{Pic}}_{X/k}$, and if we forget the section s, then we see easily, that this pair gives us local representability of the Picard functor. If we have a line bundle $\mathcal{L}$ on $X \times_k T$ (T over $\operatorname{Spec}(k)$ of finite type), then we can find a covering $T = \cup T_\alpha$ such that $\mathcal{L} \mid \{P\} \times T_\alpha$ is trivial. We choose sections $s_\alpha \in H^0(\{P\} \times T_\alpha, \mathcal{L})$ which trivializes this restriction, and we find unique

$$\psi_\alpha : T_\alpha \longrightarrow \widetilde{\operatorname{Pic}}_{X/k}$$

such that

$$(\operatorname{Id} \times \psi_\alpha)^*(\mathcal{P}) \mid X \times T_\alpha \simeq \mathcal{L} \mid X \times T_\alpha.$$

Then these ψ_α coincide on the intersections $T_\alpha \cap T_\beta$ and give us a $\psi : T \to \widetilde{\mathrm{Pic}}_{X/k}$.

Hence we see that representability of $\widetilde{\mathrm{Pic}}_{X/k}$ implies that the Picard functor is locally representable by a pair $\mathrm{Pic}(X)$ and line bundle $\mathcal{P}$ on $X \times \mathrm{Pic}(X)$. The scheme $\mathrm{Pic}(X)$ is unique up to a canonical isomorphism: If we have another pair $(\mathrm{Pic}^*_{X/k}, \mathcal{P}^*)$, then we have a unique

$$\psi : \mathrm{Pic}^*_{X/k} \longrightarrow \mathrm{Pic}_{X/k}$$

such that $(\mathrm{Id} \times \psi)^*(\mathcal{P}) \sim_{\mathrm{Pic}_{X^*/k}} \mathcal{P}$ and vice versa. Hence we see that $\mathrm{Pic}_{X/k}$ is unique, but $\mathcal{P}$ is not.

I remind you that the definition of $\widetilde{\mathrm{Pic}}_{X/k}$ depends on the choice of a point $P \in X(k)$ where the definition of the Picard functor does not.

Once we have local representability for the Picard functor, i.e. we have $\mathrm{Pic}(X)$ and the bundle $\mathcal{P}$ on $X \times \mathrm{Pic}(X)$, and if we also selected a $P \in X(k)$ (if this is possible), then we see that we have representability for $\widetilde{\mathrm{Pic}}_{X/k}$. We simply modify

$$\mathcal{P} \longrightarrow \mathcal{P} \otimes p_2^*(\mathcal{P} \mid \{P\} \times \mathrm{Pic}(X))^{-1} = \mathcal{P}'$$

where $p_2 : X \times \mathrm{Pic}(X) \to \mathrm{Pic}(X)$ is the projection. Then $\mathcal{P}'$ is trivial on $\{P\} \times \mathrm{Pic}(X)$ by construction, and we choose a trivializing section.

1.2. General properties of the functor Pic

1.2.1. : We consider a reduced, projective and connected scheme X/k and line bundles on $X \times T$ where T should be of finite type. In the following considerations it is always possible to replace these schemes T by the affine schemes $\mathrm{Spec}(A)$ since our questions will be local in T. We will also allow ourselves to pass to local rings at points in T.

We need some finiteness condition for T because we have to apply the finiteness theorems from (???). Since the Picard functor will be of finite type our condition above seems to be natural. But the passage to local rings does not hurt.

Lemma 1.2.2: *Let X/k be an irreducible, reduced projective scheme $T \to \mathrm{Spec}(k)$ of finite type. Let $\mathcal{L}$ be a line bundle on $X \times_{\mathrm{Spec}(k)} T$. Then there exists a "largest" closed subscheme $T_1 \subset T$ such that $\mathcal{L}_1 = \mathcal{L}|S \times_{\mathrm{Spec}(k)} T_1$ is locally trivial in T_1 and "largest" means for any closed subscheme $T' \subset T$ for which $\mathcal{L}|X \times_k T'$ is locally trivial in T', the inclusion $T' \hookrightarrow T$ factors over T_1.*

This lemma would of course be clear, if we had representability. Let $T' \subset T$ be a subscheme such that $\mathcal{L}_{T'} = \mathcal{L}|X \times_k T'$ is locally trivial. Our assumptions on X imply that the projection $p : X \times_k T' \to T'$ yields sheaves

$$p_*(\mathcal{L}_{T'}) \qquad \text{and} \qquad p_*(\mathcal{L}_{T'}^{-1})$$

which are locally constant of rank one.

But if in turn these two sheaves are locally constant of rank one, then $\mathcal{L}_{T'}, \mathcal{L}_{T'}^{-1}$ are locally trivial in T'. To see this, we can consider two such local generators

$$s \in H^0(V_t, p_*(\mathcal{L}_{T'}))\ ,\ s_- \in H^0(V_t, p_*(\mathcal{L}_{T'}^{-1})).$$

(Here V_t is an open neighborhood of a point $t \in T'$) and their product gives an element

$$ss_- \in H^0(X \times V_t, \mathcal{O}_{X \times V_t}),$$

By definition of local constancy (see ???) we know that the restrictions of these sections to the special fibre over t generate the spaces of sections $H^0(X \times \{t\}, \mathcal{L}_t)$ and $H^0(X \times \{t\}, \mathcal{L}_t^{-1})$ which are both of dimension one. Then the product ss_- restricted to the fibre is not zero, but since $X \times \{t\}$ is projective and connected, it follows that they never vanish. The closed subset in $X \times V_t$ where ss_- vanishes does not meet the fibre $X \times \{t\}$. Its projection to V_t is closed in V_t (???) and does not contain t. Hence we find an open neighborhood W_t of t such that s, s_- are nowhere zero on $X \times W_t$ and hence $\mathcal{L}_{T'}$ and $\mathcal{L}_{T'}^{-1}$ are trivial on $X \times W_t$.

We assume that we have a point $t \in T$ such that $\mathcal{L}|X \times \{t\}$ is trivial. Let A be the local ring $\mathcal{O}_{T,t}$. We restrict our sheaf to $X \times \mathrm{Spec}(A)$, let $\mathfrak{m}$ be the maximal ideal. We want to show that there is a smallest ideal I_0 such that our sheaf restricted to $X \times \mathrm{Spec}(A/I_0)$ becomes trivial.

In the section (semicontinuity) we have seen that we can construct a resolution of $\mathcal{L}$ on $X \times_k T$

$$0 \to \mathcal{L} \to \mathcal{E}_0 \to \mathcal{E}_1 \to \cdots,$$

such that the $\mathcal{E}_i$ are flat, acyclic for p_* and such that the $p_*(\mathcal{E}_i) = \mathcal{M}_i$ are locally free on T. We may assume that $T = \mathrm{Spec}(A)$, such that the $\mathcal{M}_i$ are actually free, and we get

$$0 \to p_*(\mathcal{L}) \to \mathcal{M}_0 \to \mathcal{M}_1 \to \cdots.$$

We have $\mathcal{M}_0 = A^{N_0}$, $\mathcal{M}_1 = A^{N_1}$ and the map $\alpha_0 : \mathcal{M}_0 \to \mathcal{M}_1$ is given by a $N_0 \times N_1$-matrix.

Now we choose an element $m_1 \in A^{N_0}$ which reduces to a basis element of the kernel

$$(A/\mathfrak{m})^{N_0} \longrightarrow (A/\mathfrak{m})^{N_1}.$$

We can assume that m_1 is the first element of a basis of A^{N_0}. We consider m_1 as a column vector

$$m_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

The linear map $A^{N_0} \to A^{N_1}$ is given by a matrix $M(\alpha_0)$. Our matrix operates by multiplication from the left, then it will be of the form

$$M(\alpha_0) = \begin{pmatrix} m_{11} & * & \cdots & * \\ m_{12} & & & \\ \vdots & & B & \\ m_{1,N} & & & \end{pmatrix}$$

where the $m_{1,\nu}$ are in the maximal ideal $\mathfrak{m}$. We reduce $\bmod\, \mathfrak{m}$. The matrix $B \bmod \mathfrak{m}$ maps the space of vectors

$$\begin{pmatrix} 0 \\ x_2 \\ \vdots \\ x_{N_0} \end{pmatrix} \qquad \text{with} \qquad x_i \in A/\mathfrak{m}$$

injectively into $(A/\mathfrak{m})^{N_1}$. This implies that B contains an invertible $(N_0 - 1) \times (N_0 - 1)$ submatrix. This allows us to modify the basis in the target such that the vectors

$$\begin{pmatrix} 0 \\ 1 \\ \vdots \\ \vdots \\ 0 \end{pmatrix} \cdots \begin{pmatrix} 0 \\ \vdots \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

are the images of the $(N_0 - 1)$-vectors

$$\begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}, \dots, \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}$$

and then our matrix $B$is of the form

$$\begin{pmatrix} m_{11} & 0 & \cdots & 0 \\ m_{12} & 1 & 0 & \\ \vdots & 0 & \ddots & 1 \\ m_{1,N_1} & & & \end{pmatrix}.$$

Now we take an ideal $I \subset A$ and ask: when is the kernel

$$(A/I)^{N_0} \longrightarrow (A/I)^{N_1}$$

free of rank one and surjects to the kernel of

$$(A/\mathfrak{m})^{N_0} \longrightarrow (A/\mathfrak{m})^{N_1}?$$

Clearly this is the case if and only if this kernel is spanned by the element

$$\begin{pmatrix} 1 \\ -m_{12} \\ \vdots \\ -m_{1,N_0} \end{pmatrix}.$$

The first condition to be satisfied is that $m_{11} \in I$. Then the second, third up to N_0-th component of the image is zero. Now the vanishing of the N_0+1-th up to the N_1-th entry means that the $m_{1,\mu}$ with $\mu = N_0 + 1 \ldots$ to N_1 satisfy

$$m_{1,\mu} - \sum_{j=2}^{j=N_0} b_{\mu j} m_{1,j} = 0.$$

This gives us a collection of elements which must be in I and in turn if I contains these elements, then the kernel of $(A/I)^{N_0} \to (A/I)^{N_1}$ is free of rank one.

Hence we see that we can take for I_0 the ideal generated by these elements

$$\left\{ m_{11}, m_{1,\mu} - \sum b_{\mu j} m_{1,j} \right\} = I_0$$

and $\mathrm{Spec}(A/I_0)$ is the largest subscheme of $\mathrm{Spec}(A)$ on which $\mathcal{L}|X \times \mathrm{Spec}(A/I_0)$ becomes trivial.

1.2.3. Infinitesimal properties: In our chapter on Riemann surfaces we exploited the exact sequence of sheaves

$$0 \to \mathbb{Z} \to \mathcal{O}_S \to \mathcal{O}_S^* \to 1,$$

which give rise to the sequence in cohomology

$$0 \to H^1(S,\mathbb{Z}) \to H^1(S,\mathcal{O}_S) \to H^1(S,\mathcal{O}_S^*) \to H^2(S,\mathbb{Z}) \to$$

(actually we can do this over any compact complex algebraic variety). We got from this sequence that $H^1(S,\mathcal{O}_S^*)$ is a complex analytic group with tangent space $H^1(S,\mathcal{O}_S)$. We showed that $H^1(S,\mathbb{Z})$ is a lattice in $H^1(S,\mathbb{Z})$ and $H^1(S,\mathcal{O}_S)/H^1(S,\mathbb{Z}) = \mathrm{Pic}^0(S)$ is a compact complex analytic group. The homomorphism

$$H^1(S,\mathcal{O}_S) \to H^1(S,\mathcal{O}_S)/H^1(S,\mathbb{Z}) = \mathrm{Pic}^0(S)$$

should be viewed as the exponential map from the tangent space to this complex analytic group.

We want to save some parts of this argument to our abstract situation. For instance we want to make it clear that the "additive" coherent cohomology group $H^1(X,\mathcal{O}_X)$ is the tangent space of our functor $\mathrm{Pic}_{X/k}$.

To understand the infinitesimal properties we assume that $T = \mathrm{Spec}(A)$ where A is a local artinian k-algebra, let $\mathfrak{m}$ be its maximal ideal and hence $A = k \oplus \mathfrak{m}$. We want to consider the line bundles on $X \times_k \mathrm{Spec}(A)$ which are trivial on the special fibre $X = X \times_k A/\mathfrak{m}_0$. This means that we want to understand the kernel $\mathrm{Pic}(X \times_k \mathrm{Spec}(A))_e = H^1(X \times_k A, \mathcal{O}^*_{X\times A})_e$ of the homomorphism

$$H^1(X \times_k A, \mathcal{O}^*_{X\times A}) \longrightarrow H^1(X, \mathcal{O}^*_X).$$

For any integer n we put $A_n = A/\mathfrak{m}^n$ and hence $k = A_1$. We put $X_n = X \times_k \mathrm{Spec}(A_n)$, then our assumption implies that $H^0(X_n, \mathcal{O}^*_{X_m}) = (A/\mathfrak{m}^n)^*$.

For any pair of integers $n > n_1$ we can consider the embedding

$$X_{n_1} \hookrightarrow X_n$$

as a closed subscheme, it is an isomorphism on the topological spaces and we have the restriction

$$(\mathcal{O}_{X_n})^* \longrightarrow (\mathcal{O}_{X_{n_1}})^*$$

which is surjective. For $n = n_1 + 1$ we can see easily that the kernel is isomorphic to $\mathcal{O}_X \otimes \mathfrak{m}^n/\mathfrak{m}^{n+1}$ and since $H^0(X_n, \mathcal{O}^*_{X_n}) = (A/\mathfrak{m}^n)^* \to H^0(X_{n-1}, \mathcal{O}^*_{X_{n-1}}) = (A/\mathfrak{m}^{n-1})^*$ is surjective we get an exact sequence

$$0 \to H^1(X, \mathcal{O}_X \otimes \mathfrak{m}^{n-1}/\mathfrak{m}^n) \to H^1(X_n, \mathcal{O}^*_{X_n}) \to H^1(X_{n-1}, \mathcal{O}^*_{X_{n-1}})$$

We have

$$H^1(X, \mathcal{O}_X \otimes \mathfrak{m}^{n-1}/\mathfrak{m}^n) \xrightarrow{\sim} H^1(X, \mathcal{O}_X) \otimes \mathfrak{m}^{n-1}/\mathfrak{m}^n$$

and since we know that $H^1(X, \mathcal{O}_X)$ is finite dimensional, the we can conclude that

1.2.3.1. The abelian group $\mathrm{Pic}(X \times_k \mathrm{Spec}(A))_e$ *is an extension of finite dimensional k-vector spaces.*

The sum of the dimensions which I want to call the *size* of $\mathrm{Pic}(X \times_k \mathrm{Spec}(A))_e$ can be estimated by $H^1(X, \mathcal{O}_X)$ and the structure of A. Especially if $A = k[\epsilon]$ is the k algebra of dual numbers then

$$\mathrm{Pic}(X(k[\epsilon])_e \xrightarrow{\sim} H^1(X, \mathcal{O}_X).$$

Of course in general we do not know whether

$$H^1(X_n, \mathcal{O}^*_{X_n}) \longrightarrow H^1(X_{n-1}, \mathcal{O}^*_{X_{n-1}})$$

is surjective. We could say that the functor is *smooth* if this is the case.

1.3. The theorem of the cube.

We combine the above results and Lemma 1.2.2. Again we consider a line bundle $\mathcal{L}$ on $X \times \mathrm{Spec}(A)$ where A is still local. Let $I_1 \subset A$ be the ideal which defines the maximal subscheme $\mathrm{Spec}(A/I_1)$ such that $\mathcal{L}|X \times \mathrm{Spec}(A/I_1)$ is trivial. Let $\mathfrak{m} \subset A$ be the maximal ideal, then $I_1/\mathfrak{m}I_1$ is a finite dimensional $A/\mathfrak{m}$-vector space. Any non-zero linear form $\Psi : I_1/\mathfrak{m}I_1 \to A/\mathfrak{m}$ defines an ideal $I_\Psi \subsetneq I_1$ and $\mathcal{L}|X \times \mathrm{Spec}(A/I_\Psi)$ is not trivial anymore.

We want to compute the obstruction to trivialize $\mathcal{L}|X \times \mathrm{Spec}(A/I_\Psi)$, i.e. we try to extend a trivialization to this larger scheme, and we will get a non zero element $\xi_\Psi \in H^1(X \times \mathrm{Spec}(A/\mathfrak{m}), \mathcal{O}_X)$ which tells us that this is impossible.

Let I be any ideal such that $\mathcal{L}|X \times \mathrm{Spec}(A/I)$ is trivial. Let $I' = I\mathfrak{m}$. We have the inclusion

$$X \times \mathrm{Spec}(A/I) \hookrightarrow X \times \mathrm{Spec}(A/I'),$$

and our line bundle $\mathcal{L}|X \times \mathrm{Spec}(A/I)$ has a section $s \in H^0(X \times \mathrm{Spec}(A/I), \mathcal{L})$ which is a generator at all points. We can cover $X \times \mathrm{Spec}(A/I')$ by affine open sets U_i such that the section s extends to a section $s_i \in H^0(U_i, \mathcal{L})$ which then will be a generator at all points. We have

$$s_i = g_{ij} s_j$$

with $g_{ij} \in H^0(U_i \cap U_j, \mathcal{O}^*_{U_i \cap U_j})$. These g_{ij} are of the form

$$g_{ij} = 1 + h_{ij}$$

where $h_{ij} \in I$. Since we compute modulo I', we have $I/I' \simeq k(t_0)$ and $\{h_{ij}\}$ defines a class in $H^1(X \times k(t_0), \mathcal{O}_{X \times k(t_0)})$. This makes it clear that for any point $t_0 \in \mathrm{Spec}(A/I)$ we get a linear map

$$\delta_{\mathcal{L}} : I/\mathfrak{m}_{t_0} I \longrightarrow H^1(X \times k(t_0), \mathcal{O}).$$

We may for instance take as our ideal I simply the maximal ideal attached to a point $t_0 \in \mathrm{Spec}(A)$ and get a linear map

$$\delta_{\mathcal{L}} : \mathfrak{m}_{t_0}/\mathfrak{m}_{t_0}^2 \longrightarrow H^1(X \times k(t_0), \mathcal{O})$$

which can be interpreted as the map from the tangent space of $\mathrm{Spec}(A)$ in the point t_0 to the "tangent space" of $\mathrm{Pic}(X)(k(t_0))$ in the "point" $\mathcal{L}$.

It is clear: If $I \subset A$ is the maximal ideal such that $\mathcal{L}|X \times \mathrm{Spec}(A/I)$ is trivial, then the map

$$\delta_{\mathcal{L}} : I/\mathfrak{m}_{t_0} I \longrightarrow H^1(X \times k(t_0), \mathcal{O}_X)$$

is injective for all $t_0 \in \mathrm{Spec}(A/I)$. This has the following consequence

1.3.1. Theorem of the cube: *Let X, Y be projective schemes over k. Let T be a connected scheme of finite type over k, and let $\mathcal{L}$ be a line bundle on*

$$X \times Y \times T.$$

Let us assume that we have points $x_0 \in X(k)$, $y_0 \in Y(k)$, $t_0 \in T(k)$ such that $\mathcal{L}$ restricted to

$$x_0 \times Y \times T \;\; X \times y_0 \times T \;,\;\; X \times Y \times t_0$$

becomes trivial. Then the bundle itself is trivial.

Proof: We assume $T = \mathrm{Spec}(A)$, where A is local with maximal ideal t_0. Let us consider the maximal subscheme $\mathrm{Spec}(A/I)$ such that

$$\mathcal{L}|X \times Y \times \mathrm{Spec}(A/I)$$

is trivial. We want to show $I = 0$. If not, then $I/\mathfrak{m}_{t_0} I \neq 0$, and we get an injective map

$$\delta_{\mathcal{L}} : I/\mathfrak{m}_{t_0} I \longrightarrow H^1(X \times Y \times k(t_0), \mathcal{O}_{X \times Y}).$$

But now we have the Künneth formula which tells us

$$\begin{aligned} H^1(X \times Y \times k(t_0), \mathcal{O}_{X \times Y}) &= H^1(X \times k(t_0), \mathcal{O}_X) \oplus H^1(Y \times k(t_0), \mathcal{O}_Y) \\ &= H^1(X \times y_0 \times k(t_0), \mathcal{O}_X) \oplus H^1(X \times y_0 \times k(t_0), \mathcal{O}_Y). \end{aligned}$$

But $\mathcal{L}|X \times y_0 \times T$ and $\mathcal{L}|x_0 \times Y \times T$ are assumed to be trivial. This means $\delta_{\mathcal{L}} = 0$. This is a contradiction.

Now we have seen that we can find an open neighborhood V of t_0 such that $\mathcal{L} \mid X \times Y \times V$ is trivial. On the other hand we know (Lemma 1.2.1) that we have a maximal closed subscheme $T_1 \subset T$, $t_0 \in T_1$ such that $\mathcal{L} \mid X \times Y \times T_1$ is locally closed in T_1. We must have $T_1 = T$.

The Picard functor has two natural subfunctors. We say that a line bundle $\mathcal{L}$ on our projective scheme X/k is *numerically equivalent* to zero, if its restriction to any curve $C \hookrightarrow X/k$ has degree zero.

We say that $\mathcal{L}$ is *algebraically equivalent* to zero, if we can find a connected scheme of finite type T/k with two points $t_0, t_1 \in T(k)$ on it and a line bundle $\tilde{\mathcal{L}}$ over $X \times_k T$ such that

$$\begin{aligned} &\tilde{\mathcal{L}}|X \times t_0 \;\; \text{is trivial} \\ &\tilde{\mathcal{L}}|X \times t_1 \;\; \text{is isomorphic to } \mathcal{L}. \end{aligned}$$

The intuitive meaning of this concept is: A line bundle is algebraically equivalent to zero, if it can be "deformed" into the trivial bundle.

1.3.2 Proposition: *If a line bundle $\mathcal{L}$ on X is algebraically equivalent to zero, then it is numerically equivalent to zero.*

Proof: Let us consider the case of a non singular projective curve C/k first. If we have a connected family $\tilde{\mathcal{L}}$ on $C \times T$ with $\tilde{\mathcal{L}}_{t_0} \simeq \mathcal{O}_c$, $\tilde{\mathcal{L}}_{t_1} \simeq \mathcal{L}$ then the function

$$t \longrightarrow \dim_{k(t)} H^0(C \times k(t), \tilde{\mathcal{L}}) - \dim_{k(t)} H^1(C \times k(t), \tilde{\mathcal{L}})$$

is constant on T (???). But the right hand side is equal to $\deg(\tilde{\mathcal{L}}_t) + 1 - g$, therefore the degree is constant and equal to zero because $\deg(\tilde{\mathcal{L}}_{t_0}) = 0$.

We define the subfunctors. We put

$$\begin{aligned} \mathrm{Pic}^0(X \times_k T) =& \text{Isomorphism classes of line bundles on } X \times_k T \\ & \text{such that } \mathcal{L}_t \text{ is numerically equivalent to zero for all } t \in T. \end{aligned}$$

The second functor is defined on the category of connected schemes T/k of finite type, equipped with a base point $t_0 \in T(k)$ and the morphisms respecting the base point. Then

$$\begin{aligned} \mathrm{Pic}^{00}(X \times_k T) =& \text{Isomorphism classes of line bundles } \mathcal{L} \text{ on } X \times_k T \\ & \text{for which } \mathcal{L}|X \times \{t_0) \simeq \mathcal{O}_X \end{aligned}$$

It is a central point is that this last functor is “linear”:

1.3.3. For a product $X \times Y$ of two projective schemes which are also equipped with base points $x_0 \in X(k)$ and $y_0 \in Y(k)$ we have

$$\mathrm{Pic}^{00}(X \times Y \times T) \simeq \mathrm{Pic}^{00}(X \times T) \oplus \mathrm{Pic}^{00}(Y \times T).$$

This is a direct consequence of the theorem of the cube. If we have a line bundle $\mathcal{L}$ over $X \times Y \times T$, then the restrictions

$$\begin{array}{rcl} \mathcal{L}_X & = & \mathcal{L}|X \times y_0 \times T \\ \mathcal{L}_Y & = & \mathcal{L}|x_0 \times Y \times T \end{array}$$

provide line bundles on $X \times T$ and $Y \times T$. Now we consider the two projections

$$\begin{array}{rcccl} p_X & : & X \times Y \times T & \to & X \times T \\ p_Y & : & X \times Y \times T & \to & Y \times T, \end{array}$$

and we consider

$$\mathcal{L} \otimes p_X^*(\mathcal{L}_X)^{-1} \otimes p_Y^*(\mathcal{L}_Y)^{-1}.$$

This line bundle on $X \times Y \times T$ is trivial by the theorem of the cube.

We could think of $\mathrm{Pic}^{00}(X)$ as being the "connected component " of the identity of our functor, the quotient $\mathrm{Pic}(X)/\mathrm{Pic}^{00}(X)$ gives us discrete invariants. They can viewed as elements in the second cohomology (Chern classes See ???)) and the second cohomology of a product is bigger than the sum of the second cohomology groups and hence $\mathrm{Pic}(X)/\mathrm{Pic}^{00}(X)$ should not be linear but quadratic.

1.3.4. The basic principle of the construction

The principal aim of this chapter is the construction of $\mathrm{Pic}(C)$ for a smooth, projective and absolutely irreducible curve C/k.

It is of course clear that this functor is a "disjoint union"

$$\mathrm{Pic}(C) = \bigsqcup \mathrm{Pic}^r(C),$$

where $\mathrm{Pic}^r(C)$ are the line bundles of degree r on C.

It will turn out that $\mathrm{Pic}^0(C)$ is an abelian variety, it is called $J = J_C$, the Jacobian of the curve. We will show that J and the $\mathrm{Pic}^r(C)$ are smooth projective and irreducible. In the second section of this chapter we will also discuss the Picard functor of J.

The basic idea of the construction is simple and goes back to Jacobi. The first object which we will construct is $\mathrm{Pic}^g(C)$.

Let us assume that $k = \overline{k}$ is algebraically closed. Let us consider line bundles $\mathcal{L}$ of degree g on C. The Riemann-Roch theorem says

$$\dim H^0(C, \mathcal{L}) = g + 1 - g + \dim H^1(C, \mathcal{L}) = 1 + \dim H^1(C, \mathcal{L}).$$

We introduce the notion of a "generic" line bundle of degree g. A bundle of degree g is generic if $\dim H^1(C, \mathcal{L}) = 0$ which implies that $\dim H^0(C, \mathcal{L}) = 1$.

The leading principle will be that for a "generic" line bundle $\mathcal{L}$ have a non zero global section $s \in \dim H^0(C, \mathcal{L})$ which is unique up to a non zero scalar. The set of zeroes of s is a divisor (ref ???)

$$\mathrm{Div}(s) = P_1 + \cdots + P_g,$$

where $P_i \in C(k)$. This collection of g points is unique up to permutations. Therefore we can view it as a point in $C^g/\Sigma_g(k)$ where C^g/Σ_g is the quotient of C^g divided by the symmetric group Σ_g.

We will discuss the construction of the quotient C^g/Σ_g in section 1.4. and then we will show:

1.3.5. A non empty open set in C^g/Σ_g is the "parameter space" for the "generic" line bundles of degree g on C.

Hence we have to do the following:

a) We have to define and construct C^g/Σ_g and investigate its properties. (section 1.4.)

b) We have to make precise in what sense a suitable open subset $U \subset C^g/\Sigma_g$ parameterizes bundles of degree g. (section ???)

c) We have to define some "translates" of these open sets U and glue them together to the parameter scheme $\mathrm{Pic}^g(C)$.

This is a long way to go.

1.4 Symmetric powers

Let k be a field and let X/k be a quasi-projective scheme. For any integer r we can form the r-fold product $X^r = X \times_k X \cdots \times X$ (r factors) and on this product we have an action of the symmetric group Σ_r. We will construct a quotient X^r/Σ_r together with a projection

$$\pi : X^r \longrightarrow X^r/\Sigma_r$$

such that we have the obvious universal property: For any scheme T and any morphism $h : X^r \to T$ which commutes with the action of Σ_r (i.e. $h \circ \sigma = h$ for all $\sigma \in \Sigma_r$) we have a unique morphism $\bar{h} : X^r/\Sigma_r \to T$ such that $h = \bar{h} \circ \pi$.

The construction of this quotient is easy. If $X = \mathrm{Spec}(A)$, where A is an affine k-algebra, we have $X^r = \mathrm{Spec}(A^{\otimes r})$ and

$$X^r/\Sigma_r = \mathrm{Spec}\left((A^{\otimes r})^{\Sigma_r}\right).$$

If X/k is quasi-projective, then we write

$$\begin{array}{ccc} X & \hookrightarrow & \mathbb{P}^n \\ \searrow & & \swarrow \\ & \mathrm{Spec}(k) & \end{array}$$

and we observe that a point $P \in X^r$ gives us an r-tuple of points $(P_1, \cdots, P_r)$ with $P_i \in X$. If k is infinite we can find a hyperplane $H \subset \mathbb{P}^n$ such that $P_i \notin H$ for all i. This implies that we can find an open affine subset $U \subset X$ such that all our points lie in U. If k is finite we can find a non zero section $s \in H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(m))$ for some $m >> 0$ such that non of our points lies on the zero set of s and the same argument works. Hence it is clear that for any point $P \in X^r$ we can find an affine open subset $U \subset X$ such that $P_1, \cdots, P_r \in U$. and we can cover X^r by a finite number of open affine subsets $U_\nu \times \ldots U_\nu$. We can form the quotients U_ν^r/Σ_r and now it is obvious how to glue (see ...??) these pieces together to get X^r/Σ_r.

I claim that this construction has the following properties:

(i) *It commutes with extensions of the field of scalars, i.e. for any $k \to L$ we have*

$$(X \times_k L)^r/\Sigma_r = X^r/\Sigma_r \times_k L.$$

(ii) The scheme X^r/Σ_r is again quasi-projective, and it is affine if X/k is affine and it is projective if X/k is projective.

To see the first assertion we assume that $X = \mathrm{Spec}(A)$ where A is an affine k-algebra. This algebra is free as a k-vector space, let $e_o, e_1, \cdots, e_m, \cdots$ be a basis. The elements

$$e_{\underline{i}} = e_{i_1} \otimes \cdots e_{i_r} \qquad \text{for} \qquad (i_1, \cdots, i_r) \in \mathbb{N}^r$$

form a basis of the k-vector space $A^{\otimes r}$.

The group Σ_r acts upon $\mathbb{N}^r$, let $\Sigma_{r,i}$ be the stabilizer of $\underline{i}$ in Σ_r. We put

$$E_{\underline{i}} = \sum_{\sigma \in \Sigma_r / \Sigma_{r,\underline{i}}} e_{\sigma(\underline{i})},$$

and these elements form a basis of the k-vector space $(A^{\otimes r})^{\Sigma_r}$. Since the $e_0, \cdots, e_m, \cdots$ also form a basis of $A \otimes_k L$ over L, our assertion (i) is clear for affine k-algebras. But then it is obviously also true for any quasi-projective X/k. The assertion (ii) is obvious for affine schemes. It suffices to prove it for projective schemes. But now we may write for instance

$$X = \mathrm{Proj}(k[x_0, \cdots, x_n])$$

where $k[x_0, \cdots, x_n]$ is a graded k-algebra and where the x_i have degree one. We have seen that we can write

$$X^r = \mathrm{Proj}(k[\cdots u_{\underline{\alpha}} \cdots])$$

where $\underline{\alpha} = (\alpha_1, \cdots, \alpha_r) \in [0, n]^r$ and

$$u_{\underline{\alpha}} = x_{\alpha_1} \otimes x_{\alpha_2} \cdots \otimes x_{\alpha_r}.$$

Then we get

$$X^r / \Sigma_r = \mathrm{Proj}\left((k[\cdots x_\alpha \cdots])^{\Sigma_r} \right).$$

Proposition 1.4.1. : *If A/k is an affine k-algebra, then the algebra $(A^{\otimes r})^{\Sigma_r}/k$ is generated by the elementary symmetric functions*

$$\sigma_\mu(f) = \sum 1 \otimes \cdots f \otimes 1 \cdots \otimes f \cdots \otimes 1,$$

where we sum over all the possible placements of $\mu (\leq r)$ factors f in the above tensor product.

To see this we start from our basis $e_0, e_1, \cdots e_m \cdots$ above where now in addition $e_0 = 1$. We consider a basis element $E_{\underline{i}}$ as above, and we want to show that it is in the algebra generated by the elementary symmetric functions. We proceed by downward induction on the number of times 0 occurs in $\underline{i} = (i_1, \cdots, i_r)$. If 0 occurs r-times the assertion is clear. Now we write

$$\underline{i} = (\underbrace{0, \cdots, 0}_{\nu_0 - \text{indices}}, \underbrace{i, \cdots, i}_{\nu_1 - \text{indices}}, \cdots, \underbrace{j, \cdots, j}_{\nu_m - \text{indices}})$$

where $0, i, \cdots, j$ are pairwise different.

Then

$$e_{\underline{i}} = (1 \otimes \cdots e_i \otimes \cdots \otimes e_i \otimes 1 \cdots 1) \cdot (1 \otimes \cdot \otimes e_i \otimes e_i \cdots e_i \otimes 1 \cdots \otimes 1)(1 \otimes 1 \cdots 1 \otimes e_j \otimes \cdots e_j).$$

Consider

$$E_{\underline{i}} - \sigma_{\nu_1}(e_i) \cdots \sigma_{\nu_m}(e_j)$$

which is in $(A^{\otimes r})^{\Sigma r}$. If we look carefully at

$$(+ \cdots 1 \otimes \cdots \otimes e_i \otimes \cdots 1 \otimes e_i \otimes 1 \cdots) \cdots (\cdots 1 \otimes e_j \otimes \cdots 1 \otimes e_j \cdots \otimes 1 \cdots),$$

and expand, then we see that $E_{\underline{i}}$ is exactly that part of the expansion where we multiply tensors which have the $e_i \cdots e_j$ at different places. So this part cancels. But for the remaining terms we see that we get some

$$\cdots \otimes e_\nu e_\mu \otimes \cdots$$

in the product. This $e_\nu e_\mu$ can be expressed by linear combinations of the $e_i \cdots e_j$ and hence the number of 1's in the tensor product goes up in these terms. This proves the proposition.

It is clear that the morphism

$$\pi : X^r \longrightarrow X^r/\Sigma_r$$

is finite. This implies that we have a surjective map on the set of geometric points

$$X^r(\overline{k}) \longrightarrow X^r/\Sigma_r(\overline{k}).$$

It is obvious that points which are equivalent under the action of the symmetric group map to the same point in $X^r/\Sigma_r(\bar{k})$ hence the map factorizes

$$X^r(\overline{k})/\Sigma_r \to X^r/\Sigma_r(\overline{k})$$

Exercise 1.4.2. *Show that the above map is a bijection.*

But over an arbitrary field the map $X^r(k) \to X^r/\Sigma_r(k)$ may not be surjective. A point $Q \in X^r/\Sigma_r(k)$ can be lifted to a point $P = (P_1, \cdots, P_r) \in X^r(\overline{k})$ but then we have to answer the question

When does a geometric point $P = (P_1, \cdots, P_r) \in X^r(\overline{k})$ give a point $\pi(P) \in X^r/\Sigma_r(k)$?

To answer this question we may assume that $X = \mathrm{Spec}(A)$. Then our point $P = (P_1 \cdots P_r) = (\psi_1, \cdots, \psi_r)$ where

$$\begin{array}{ccc} B & & \\ \uparrow & \searrow^{\psi_i} & \\ k & \longrightarrow & \overline{k} \end{array}$$

this is the same thing as $\otimes\psi_i : B^{\otimes r} \to \overline{k}$. The image of $(\psi_1, \cdots, \psi_r)$ lies in $X^r/\Sigma_r(k)$ if and only if the restriction $\otimes\psi_i : (B^{\otimes r})^{\Sigma_r} \to k$. Now it follows from proposition 1.4.1

Proposition 1.4.3. *We have* $\otimes\psi_i : (B^{\otimes r})^{\Sigma_r} \to k$ *if and only if for all* $f \in A$ *the polynomial*

$$(Y - \psi_1(f)) \cdots (Y - \psi_r(f)) \in k[Y].$$

Now we want to assume that our scheme is a smooth projective curve C/k. A point $P = (P_1, \cdots, P_r) \in C^r(\overline{k})$ gives us a divisor

$$\overline{D} = \sum_{i=1}^{r} P_i = \sum_{P \in C(\overline{k})} n_P P$$

on the curve $C \times_k \overline{k}$. We want to discuss the question whether this divisor is the base extension of a divisor on C/k

$$D = \sum_{\mathfrak{p}} n_{\mathfrak{p}} \mathfrak{p}.$$

In this case we say that our divisor is rational over k. Of course we must have

$$\deg(\overline{D}) = \sum n_P = \deg(D) = \sum n_{\mathfrak{p}} [k(\mathfrak{p}) : k].$$

We have the action of the Galois group $\mathrm{Gal}(\bar{k}/k)$ on the points in the support of our divisor. An orbit of this action consist of the points P which lie above one point $\mathfrak{p}$, i.e. of the points $P \to \mathfrak{p}$. It is clear that we can assume that we have just one orbit which means that all the points P_i lie over one closed point $\mathfrak{p}$ on C/k.

Hence we get: The base extension of the divisor $\mathfrak{p}$ on C/k to the geometric curve is

$$n_{\mathfrak{p}}^{(i)} (\sum_{P \to \mathfrak{p}} P),$$

where the Galois acts transitively on the $P \to \mathfrak{p}$ and where $n_{\mathfrak{p}}^{(i)}$ is the degree of inseparability of the extension $k(\mathfrak{p})/k$. (Haben wir das nicht schon frueher gesehen???)

We conclude: If we look at our divisor $\overline{D}=\sum_{P\in C(\bar{k})} n_P P$, and if we divide the points in its support under the action of the Galois group, then the orbits correspond to points $\mathfrak{p}$ on C/k. Our divisor

$$\overline{D}=\sum_{\mathfrak{p}} n_{\mathfrak{p}}(\sum_{P\to\mathfrak{p}} P),$$

is rational over k if and only if

$$n_{\mathfrak{p}}^{(i)}|n_{\mathfrak{p}}.$$

But now it is clear from our considerations in (p. 221 Chapter on curves Base change ??? Verweis auf obige proposition))) that the condition $n_{\mathfrak{p}}^{(i)}|n_{\mathfrak{p}}$ is equivalent to the condition that

$$(\cdots\underbrace{P,\cdots,P}_{n_{\mathfrak{p}}\ \text{times}}\cdots)\in C^r/\Sigma_r(k).$$

To see that this is the case we can assume that our points lie in an affine open subset $U\subset C$ and $U=\mathrm{Spec}(A)$. Let $k\hookrightarrow\bar{k}$ be an algebraic closure, let k_s,k_i be the maximal separable (inseparable) subextension respectively. Our points are k-homomorphism $\psi_\nu: A\to\bar{k}$, they form an orbit under the action of $\mathrm{Gal}(\bar{k}/k)$. For any $f\in A$ the polynomial

$$\prod_\nu(Y-\psi_\nu(f))\in k_i[Y]$$

because it is invariant under the action of the Galois group.

Lemma 1.4.4. :*The degree of inseparability* $n_{\mathfrak{p}}^{(i)}=p^e$ *is the smallest power of* p *for which all the* $\psi_\nu(f)^{p^e}\in k_s$.

Let us assume this Lemma. Then we know that p^e is the smallest power of p such that for all $f\in A$ the polynomial

$$\prod_\nu(Y-\psi_\nu(f))^{p^e}\in k[Y],$$

and now it follows from proposition ??? that p^e is the smallest power of p such that

$$(\cdots\underbrace{P,\cdots,P}_{p^e\ \text{times}}\cdots)_{P\to\mathfrak{p}}\in C^r/\Sigma_r(k).$$

It remains to prove the Lemma. We choose an $F\in A$ such that the differential dF generates the module of differentials at $\mathfrak{p}$. (See ???). Then we know that the extension $k[F]\hookrightarrow A$ is etale at $\mathfrak{p}$. This implies that $\psi_\nu(f)^{p^e}\in k_s$ for all $f\in A$ if we have $\psi_\nu(F)^{p^e}\in k_s$ (Verweis???). But now $\mathfrak{p}\cap k[F]=(p(F))$ where $p(F)\in k[F]$ is an irreducible polynomial. Now it is known from elementary algebra that

$$p(F)=F^{p^e n}+a_1F^{p^e(n-1)}+\ldots a_{n-1}F^{p^e}+a_n$$

with some coefficients in k. This polynomial is equal to

$$\prod_\nu (F - \psi_\nu(F))^{p^e} \in k[F].$$

If we know remember that this polynomial is irreducible we get the assertion of the Lemma.

Hence we come to the conclusion that the effective divisors on C/k of degree r are the points on $C^r/\Sigma_r(k)$.

In our special case we have

Theorem 1.4.5. : *For a smooth curve C/k the quotient C^r/Σ_r is again smooth.*

We may assume that k is algebraically closed. We can pick a geometric point $\mathcal{P} \in C^r/\Sigma_r$, we find a finite number of points $Q_1, \dots, Q_s$ in $C^r(k)$ lying above this point. We can write $Q_1 = (P_1, \dots, P_r)$ and the other Q_i are obtained by permuting the coordinates of Q_1. Now we have two possibilities to proceed. The first one is to pick a meromorphic function f on C which is regular at these points $P_1, \dots, P_r$ and which has the property that df is a generator of the differentials in these points. We can find an affine scheme $\mathrm{Spec}(A) \subset C$ such that our points lie in $\mathrm{Spec}(A)$. Now we know that $k[f] \hookrightarrow A$ is etale (See ???) and from this we conclude that $(k[f]^{\otimes r})^{\Sigma_r} \hookrightarrow (A^{\otimes r})^{\Sigma_r}$ (boring argument omitted). Hence we see that we have to prove our theorem for the special case $C = \mathrm{Spec}(k[f])$. But now the theorem of the elementary symmetric functions says

$$(k[f]^{\otimes r})^{\Sigma_r} \xrightarrow{\sim} k[X_1, \dots, X_r]^{\Sigma_r} = k[\sigma_1, \dots, \sigma_r]$$

where the σ_i are the elementary symmetric functions. This is a polynomial ring and hence smooth.

The second possibility is to investigate the completion of the local ring $\mathcal{O}_{C^r/\Sigma_r,P}$ at the point P. We have two extreme possibilities. If all the coordinates of$P = (P_1, \dots, P_r)$ are pairwise different then it clear that for the completions

$$\hat{\mathcal{O}}_{C^r/\Sigma_r,P} = \hat{\mathcal{O}}_{C^r,(P_1,\dots,P_r)}$$

and smoothness becomes clear. The opposite case is that all the coordinates are equal say to P'. Then this point $(P', \dots, P')$ is the only point in $C^r(k)$ which lies over P and it is clear

$$(\hat{\mathcal{O}}_{C^r,(P',\dots,P')})^{\Sigma_r} = \hat{\mathcal{O}}_{C^r/\Sigma_r,P}.$$

But now the ring on the left is the power series ring in r variables and then the ring of invariants is the power series ring in the elementary symmetric functions. To treat the general case one has to mix the arguments.

1.5. The actual construction of the Jacobian of a curve.

Now want to prove the existence of the Jacobian. We assume that we have a smooth, projective and connected curve C/k which comes with a point $P_0 \in C(k)$

Theorem I: *The functor $T \to \mathrm{Pic}^0_{P_0}(C \times_k T)$ is representable: We can construct a projective scheme J/k and an isomorphism of functors*

$$J(T) \to \mathrm{Pic}^0_{P_0}(C \times_k T)$$

for all $T \to \mathrm{Spec}(k)$ of finite type.

In our heuristical discussion in the beginning of this chapter we compared this representability to the local representability. To prove the above theorem we will prove

Theorem II:*Let C/k be smooth, projective and connected and let g be the genus of C/k. The functor $T \to \mathrm{Pic}^g(C \times_k T)$ is is locally representable.*

Of course this implies that we construct a line bundle $\mathcal{P}$ on $C \times \mathrm{Pic}^g(C)$ but this bundle cannot be unique, it can be multiplied by a bundle $\mathcal{M}$ which is the pullback of a bundle on $\mathrm{Pic}^g(C)$. (See 1.5.7.(

In exercise ... in section ... we constructed a line bundle on

$$C \times C^g$$

which was the product

$$\mathcal{L}' = \otimes \mathcal{O}_{C \times C^g}(-\Delta_i),$$

where Δ_i was the inverse image of the diagonal under the projection

$$pr_{0i} : C \times C^g \longrightarrow C \times C$$

to the zeroe'th and the i'th factor. The symmetric group Σ_g acts on the second factor and we have the quotient map

$$\Pi : C \times C^g \to C \times C^g/\Sigma_g.$$

I want to construct a line bundle $\mathcal{L}$ on $C \times C^g/\Sigma_g$ whose inverse image $\Pi^*(\mathcal{L}) \tilde{\to} \mathcal{L}'$. If $P \in C \times C^g/\Sigma_g$ is a point and if we lift this point to a point $\tilde{P} \in C \times C^g$ then we can project it to the components and we get a $g+1$ tuple $(P_0, P_1, \ldots, P_g)$. We can find a function in $k(C)$ which is holomorphic at all the points $P_0, \ldots, P_g$ and which has the additional property that df generates the differentials at all these points. In a suitable neighborhood of $\tilde{P}$ the inverse image of the diagonal $\Delta \subset C \times C$ under the projection pr_{0i} is defined by the ideal generated by

$$F_i = f \otimes 1 \ldots \otimes 1 - 1 \otimes 1 \ldots \otimes f \ldots \otimes 1$$

where we put the f into the i-th place (See ???). The inverse of this function trivializes the bundle $\mathcal{O}_{C \times C^g}(-\Delta_i)$ in our given neighborhood. This means that $\mathcal{L}'$ is trivialized locally at $\tilde{P}$ by the product $(\prod_{i=1}^{i=g} F_i)^{-1}$ and this is a meromorphic function on $C \times C^g/\Sigma_g$. This system of trivializing sections defines $\mathcal{L}$. (See ???)

We left it as an exercise (See ???) to the reader to show that the set

$$U' \subset C^g = \dim_{k(u)} H^0(C \times_k k(u), \mathcal{L}_u) = 1.$$

is a non empty open subset in C^g. It is clearly invariant under the action of the symmetric group on $C \times U'$ we can form the quotient by this action, we get an open subset $U = U'/\Sigma_g \subset C^g/\Sigma_g$.

Our line bundle $\mathcal{L}$ over $C \times C^g/\Sigma_g$, has a specific properties: If we project $\pi_0 : C \times C^g/\Sigma_g \to C^g/\Sigma_g$, and if we restrict this projection to $C \times U \to U$ then the sheaf $\pi_0^*(\mathcal{L})$ will be locally free of rank one over U. (See ???). We want to denote the restriction of $\mathcal{L}$ to $C \times U$ by $\mathcal{P}'$. Since our sheaf $\mathcal{P}'$ contains the structure sheaf $\mathcal{O}_{C\times C^g/\Sigma_g}$ we can even say that the $\mathcal{O}_U$ module $\pi_0^*(\mathcal{P}')$ is generated by the element $1 \in H^0(C \times U, \mathcal{O}_{C\times U})$.

The following proposition says that our line bundle $\mathcal{P}'$ has a universality property.

Proposition 1.5.1. *Let T be a scheme of finite type over k, let us assume that we have a line bundle $\mathcal{L}_1$ over $C \times_k T$, such that for any point $t \in T$ we have* $\deg \mathcal{L}_{1,t} = \deg \mathcal{L}_1 | C \times_k k(t) = g$ *and*

$$\dim_{k(t)} H^0(C \times_k k(t), \mathcal{L}_{1,t}) = 1.$$

Then there is a unique morphism

$$\psi : T \longrightarrow U,$$

such that for any point $t \in T$ we can find an open subset V_t of t in T such that we have an isomorphism

$$\eta : (\mathrm{Id} \times \psi)^*(\mathcal{P}') | C \times V_t \xrightarrow{\sim} \mathcal{L}_1 | C \times_k V_t$$

In other word we have

$$(\mathrm{Id} \times \psi)^*(\mathcal{P}') | C \times T \sim_T \mathcal{L}_1 | C \times T.$$

I want to comment on this proposition. It gives us already a large part of Theorem II, it says that the subfunctor of $\mathrm{Pic}^g(C)$ given by the generic bundles is locally represented by $(C \times U, \mathcal{P}')$.

The second comment is that we proved the proposition in a special case. I recommend strongly to do the following

Exercise 1.5.2. *Assume that our scheme T is integral then it has a field $L = k(T)$ of meromorphic function which is the residue field of the generic point. We can restrict our line bundle to $C \times \mathrm{Spec}(L)$.*

a) Show that our considerations in 1.4. imply that this restriction gives us an L valued point in $C^g/\Sigma_g(L)$.

b) Show that this L valued point gives us a morphism $\psi' : T' \to C^g/\Sigma_g$ from some non empty open $T' \subset T$ such that $(\mathrm{Id} \times \psi')^*(\mathcal{P}')|C \times T' \tilde{\to} \mathcal{L}_1|C \times_k T'$.

c) Let us assume we have a line bundle $\mathcal{L}_0$ on C/k which is generic, it yields a point $P \in C^g/\Sigma_g(k)$. Assume that this point lifts to a point $P = (P_1, P_2, \cdots, P_g) \in C^g(k)$ we all the components are pairwise different. We consider line bundles $\mathcal{L}$ on $T = \mathrm{Spec}(k[\epsilon])$ whose restriction to $C \times \mathrm{Spec}(k)$ give $\mathcal{L}_0$. We have seen in ??? that these $\mathcal{L}$ form a torseur under the group of line bundles on $C \times T$ which are trivial on $C \times \mathrm{Spec}(k)$ and this group is isomorphic to to $H^1(C, \mathcal{O})$. The tangent space to C^g/Σ_g at P is (see ???) the direct sum of tangent spaces T_{P_i} at the points $P_i \in C(k)$. Hence our proposition tells us that we should have an isomorphism

$$H^1(C, \mathcal{O}) \tilde{\to} \sum T_{P_i}.$$

Write down this isomorphism!

(Es muss noch irgendwo im Buch der Zusammenhang zwischen dualen Zahlen und Tangentialräumen hergestellt werden!!!!!!)

We come to the proof of the proposition, it will occupy us for a while. Our assertion is local in the base scheme T, therefore we may assume that $T = \mathrm{Spec}(A)$ where A is an affine $k-$algebra. We pick a point $t_0 \in \mathrm{Spec}(A)$. In the following considerations we have to replace A several times by a localization $\mathrm{Spec}(A_f)$ with $f \notin \mathfrak{m}_0$ and this new ring will again be called A. If we make such a replacement we will say " by passing to a neighborhood of t_0". (We could of course pass to the local ring at t_0 directly, but then our algebra is not of finite type and formulating an extension of the theorem to this larger class of rings may also cause problems).

We consider the projection

$$p_2 : C \times_k \mathrm{Spec}(A) \to \mathrm{Spec}(A).$$

Our first aim is to show

1.5.3. Under our assumptions above the sheaf $p_{2*}(\mathcal{L}_1)$ is a locally free module of rank 1 over A.

Our assumption implies that the fibres $\pi_*(\mathcal{L}_{1,t})$ are one-dimensional vector spaces over $k(t)$ for all $t \in \mathrm{Spec}(A)$. Now we assume that the k-algebra A is reduced. Then the semicontinuity theorem (See ???) implies that $\pi_*(\mathcal{L}_1)$ is a locally free module of rank 1 over A. After passing to a neighborhood of t_0 we may assume that it is free. Now I claim that we can drop the assumption that A is reduced. Let $\mathfrak{a}$ be the radical of A, let us put $A_0 = A/\mathfrak{a}$. We have the inclusion

$$X \times \mathrm{Spec}(A_0) \hookrightarrow X \times \mathrm{Spec}(A).$$

We can restrict the sheaf $\mathcal{L}_1$ to $X \times \mathrm{Spec}(A_0)$ let us call the result $\mathcal{L}_{1,0}$. We get an exact sequence of sheaves

$$0 \to \mathcal{L}_1 \otimes \mathfrak{a} \to \mathcal{L}_1 \to \mathcal{L}_{1,0} \to 0.$$

We know already that $H^0(X \times \mathrm{Spec}(A_0), \mathcal{L}_{1,0})$ is free of rank one over A_0. But now our assumption on $\mathcal{L}_1$ imply that $H^1(X \times k(t), \mathcal{L}_{1,t}) = 0$ for all points $t \in T$. Again we invoke the semicontinuity theorems (See ???, dies muss nochmal genau geprueft werden)) and we conclude $R^1\pi_*(\mathcal{L}_1) = 0$. We see that the generator $s_0 \in p_*(\mathcal{L}_1)(\mathrm{Spec}(A_0))$ lifts to an element $s \in p_*(\mathcal{L}_1)(\mathrm{Spec}(A)$ and this clearly is a free generator of this module.

Let $V(\mathcal{L}_1)$ be the bundle of one-dimensional vector space over $C \times \mathrm{Spec}(A)$ obtained from $\mathcal{L}_1$ (see PJ....) let p be the projection morphism. Then we have the zero section in this bundle

$$\begin{array}{ccc} C \times_k \mathrm{Spec}(A) \times \{0\} & \overset{i_0}{\hookrightarrow} & V(\mathcal{L}_1) \\ p \circ i_0 \searrow & & p \downarrow \\ & & C \times_k \mathrm{Spec}(A) \end{array}$$

and we use the isomorphism $p_0 \circ i_0$ to identify the zero section and $C \times_k \mathrm{Spec}(A)$. Our section s defines a subscheme in $V(\mathcal{L}_1)$ which is defined locally by one equation. This subscheme intersected with the zero section defines via our identification a subscheme

$$[s = 0] \subset C \times_k \mathrm{Spec}(A).$$

We know that this subscheme is of degree g in any fibre, this means that for any $t \in \mathrm{Spec}(A)$ the subscheme

$$[s_t = 0] := [s = 0] \times_k \mathrm{Spec}(k(t)) \subset C \times_k \mathrm{Spec}(k(t))$$

is finite and of degree g (see ???). But we have something much more precise

Lemma 1.5.4. : *After passing to a neighborhood of t_0 the scheme $[s = 0]$ is flat over* $\mathrm{Spec}(A)$ *and therefore the A-algebra $B_0 = B \otimes A/I$ is free of rank g (as an A-module).*

Proof: For this proof we may and do replace A by the local ring at t_0. In a first step we show that $[s = 0]$ is a finite affine scheme over $\mathrm{Spec}(A)$. We consider a finite subscheme $F \subset C$, $F \neq \emptyset$ such that $F \cap [s_{t_0} = 0] = \emptyset$. We put $V = C \setminus F$, this is affine and we consider

$$\big(F \times_k \mathrm{Spec}(A)\big) \cap [s = 0].$$

This must be empty, because if we project it to $\mathrm{Spec}(A)$ we get a closed subscheme which does not contain t_0 (see ....). Since A is local, it follows that this closed subscheme is empty.

We see that the scheme $[s = 0]$ is an affine scheme. But of course the restriction $\pi_0 :$ $[s = 0] \to \mathrm{Spec}(A)$ is also projective we know that $\pi_{0,*}(\mathcal{O}_{[s=0]})$ is coherent and therefore $[s = 0]$ is finite over $\mathrm{Spec}(A)$. We put $B = \mathcal{O}_C(V)$, then $V = \mathrm{Spec}(B) \subset C$, and

$$[s = 0] = \mathrm{Spec}(B \otimes A/I),$$

where I is an ideal which is locally principal.

In the second step we prove the flatness. Assume we know that $\mathrm{Tor}_1^A(B\otimes A/I, k(t_0)) = 0$. Then we choose a basis for the $k(t_0)$ vector space $(B\otimes A/I)\otimes k(t_0)$ and lift these basis elements to elements $h_1\cdots, h_g$ in $B\otimes A/I$, which then will be generators by Nakayama's lemma. Hence we get a surjective homomorphism and an exact sequence

$$0\to R\to A^g\to A\otimes B/I\to 0,$$

where R is the A-module of relations. If we tensorize by $A/\mathfrak{m}_0 = k(t_0)$ the sequence is still exact, because of the vanishing of the Tor^1. The arrow $(A/\mathfrak{m}_0)^g\to(B\otimes A/I)\otimes k(t_0)$ becomes an isomorphism and hence we conclude that $R\otimes A/\mathfrak{m}_0 = 0$. Again we apply Nakayama and get the result.

Now we have to show the vanishing of the Tor_1. We consider the sequence

$$0\to I\to B\otimes A\to B\otimes A/I\to 0,$$

and since $B\otimes A$ is free over A we get an exact sequence

$$0\to \mathrm{Tor}_1^A(B\otimes A/I, A/\mathfrak{m}_0)\to I\otimes A/\mathfrak{m}_0\to B\otimes A/\mathfrak{m}_0\to .$$

We are through, if we show that

$$I\otimes A/\mathfrak{m}_0\to B\otimes A/\mathfrak{m}_0$$

is injective. We may assume that I is generated by an element f. (We may take our V above so small that $\mathcal{L}_1$ becomes trivial on an open neighborhood of $V\times\mathrm{Spec}(A/\mathfrak{m})$). Then we have to consider elements

$$yf\otimes 1\in I\otimes A/\mathfrak{m}_0,$$

whose image yf in $B\otimes A/\mathfrak{m}_0$ is zero. But $B\otimes A/\mathfrak{m}_0$ is integral and since f is non-zero in $B\otimes A/\mathfrak{m}_0$, we can conclude y goes to zero in $B\otimes A/\mathfrak{m}_0$. But then it follows that $yf\otimes 1$ is zero and this is the injectivity.

1.5.4.1 At this point we introduce the notion of a *relative divisor* on $C\times\mathrm{Spec}(A)$. By this we mean a subscheme $Y\subset C\times\mathrm{Spec}(A)$ which is finite and flat over $\mathrm{Spec}(A)$ and where the sheaf of ideals defining Y is locally principal. The argument in the lemma above shows that such a subscheme is always contained in an open subscheme $\mathrm{Spec}(B)\times\mathrm{Spec}(A)$ where B is the algebra of regular functions on a suitable affine subset $V\subset C$, and hence

$$Y = \mathrm{Spec}(B\otimes A/I)$$

where the ideal I is locally principal.

The $A/\mathfrak{m}_0$ algebra $(B\otimes A/I)\otimes(A/\mathfrak{m}_0)$ has a rank r which is called the *degree* of the relative divisor. Our A-algebra $(B\otimes A)/I$ is free of rank r (Lemma 1.5.5.)

1.5.4.2. Our aim is to show that a relative divisor of degree r is nothing else than a Spec(A) valued point on C^r/Σ_r. I would like to point out that this is clear for $r = 1$ because in this special case our assumptions imply that $B \otimes A/I = A$. Especially the following proposition is obvious for the case $g = 1$ and all the considerations in the next 12 pages are superflous if $g = 1$.

In the folling considerations our relative divisors have degree g this number could be replaced by an arbitrary r.

Our $[s = 0] \subset C \times \mathrm{Spec}(A)$ is such a relative divisor, but we may also apply our construction to the line bundle $\mathcal{L}$ on $C \times C^g/\Sigma_g$ which provides the diagram

$$\begin{array}{ccc} C \times C^g/\Sigma_g & \hookleftarrow & [\Delta = 0] \\ \downarrow & \swarrow & \\ & C^g/\Sigma_g. & \end{array}$$

Here $[\Delta = 0]$ is of course the scheme (divisor) of zeroes of the section $1 \in H^0(C \times C^g/\Sigma_g, \mathcal{L})$.

Now we can our goal in 1.5.4.2. in a more precise form: We want to show that this diagram provides a universal relative divisor of degree g and this means that we can interprete $[s = 0]$ as an A-valued point of C^g/Σ_g, i.e. a morphism $\psi : \mathrm{Spec}(A) \to C^g/\Sigma_g$ over Spec(k). The following assertion (B) makes this precise and we will see in a minute that (B) implies proposition 1.5.1.

(B) *There exists a unique morphism*

$$C^g/\Sigma_g \longleftarrow \mathrm{Spec}(A) : \psi,$$

such that in the diagram

$$\begin{array}{ccc} C \times C^g/\Sigma_g & \xleftarrow{\mathrm{Id} \times \psi} & C \times \mathrm{Spec}(A) \\ \cup & & \cup \\ [\Delta = 0] & & [s = 0] \\ \downarrow & & \downarrow \\ C^g/\Sigma_g & \longleftarrow & \mathrm{Spec}(A) \end{array}$$

the scheme $[s = 0]$ *is the pull back of* $[\Delta = 0]$ *under* $\mathrm{Id} \times \psi$, *i.e.*

$$(\mathrm{Id} \times \psi)^{-1}[\Delta = 0] = [s = 0].$$

I want to introduce a terminology. If $[s = 0]$ is given and if we found a morphism ψ which fulfills $(\mathrm{Id} \times \psi)^{-1}[\Delta = 0] = [s = 0]$ then we say that ψ *realizes* $[s = 0]$. This formulation only concerns the existence. If we have also uniqueness then we could simply write $[s = 0] = \psi$

Before we show this assertion I want to explain that implies proposition 1.5.1. and conversely that a ψ as in proposition 1.5.1. must be a realization of $[s = 0]$.

Assume we have a ψ as in proposition 1.5.1.. Then we have an isomorphism

$$\mathcal{L}_1 \xrightarrow{\sim} (\mathrm{Id} \times \psi)^*(\mathcal{P}').$$

Let $\psi(t_0) = x_0$, then we may localize at x_0. The direct image $\pi_*(\mathcal{P}')$ is locally a free module generated by 1. The isomoprhism between our sheaves sends the section 1 to s up to a unit. Hence $[s = 0]$ is the pullback of $[\Delta = 0]$ which is the subscheme defined by the section 1.

Conversely, if we have a realization ψ, we can easily construct an isomorphism between

$$\mathcal{L}_1 \quad \text{and} \quad (\mathrm{Id} \times \psi)^*(\mathcal{P}').$$

Outside of $[s = 0]$ we identify by mapping s to 1. In a neighborhood of $[s = 0]$ we know that the sections differ by a unit and this gives the identification over $[s = 0]$.

Now we come to the proof of (B).

We have reduced our problem to a problem in commutative algebra: We have the affine algebra B of an open subset V of the curve C/k and the locally principal ideal $I \subset B \otimes_k A$ such that

$$A \longrightarrow B \otimes_k A/I$$

is free of rank g. In terms of the algebras we have to show the following:

First of all it is clear from the definition of the subscheme $[\Delta = 0]$ that our morphism ψ must factor through

$$\mathrm{Spec}\left((B^{\otimes g})^{\Sigma_g}\right) \subset C^g/\Sigma_g.$$

We translate our assertion (B) where we now introduce the convention that the homomorphism between affine algebras induced by ψ is denoted by $\tilde{\psi}$. Then we have to show:

We can find a unique homomorphism of k-algebras

$$\tilde{\psi} : \left(B^{\otimes g}\right)^{\Sigma_g} \longrightarrow A,$$

such that for

$$(\mathrm{Id} \times \tilde{\psi}) : B \otimes \left(B^{\otimes g}\right)^{\Sigma_g} \longrightarrow B \otimes A$$

image of I_Δ generates I, i.e.

$$(\mathrm{Id} \times \tilde{\psi})(I_\Delta) \cdot (B \otimes A) = I.$$

We say that our relative divisor is *very simple* if the quotient algebra $(B \otimes A)/I$ is of the following simple form

$$(B \otimes A)/I \xrightarrow{\sim} A[X]/(F(X))$$

where $F(X) = X^g + a_1 X^{g-1} + \dots a_g \in A[X]$ and where in addition the polynomial $F(X)$ decomposes completely $F(X) = (X - \alpha_1) \dots (X - \alpha_g)$ with $\alpha_i \in A$. The reader should be aware that this decompositions is not necessarily unique because we allow that A has zero divisors.

1.5.5.1 The strategy: We will apply a localization technique.

1) For any relative divisor $[s = 0] \subset C \times T$ we will construct a faithfully flat $T' \to T$ which is of finite type such that the pullback of our relative divisor becomes a very simple divisor. (This is the localization of our problem.)

2) We will prove our assertion for the special case of very simple divisors.

3) Then we will show that the localization does not destroy information. Let us asume have a relative divisor $[s = 0] \subset C \times T$, a morphism $\psi : T \to C$ and a faithfully flat $\phi : T' \to T$. Then we can take the pullback $[s' = 0] = (\mathrm{Id} \times \phi)^{-1}[s = 0] \subset C \times T'$ which is again a relative divisor. If then $\psi' = \psi \circ \phi$ realizes $[s' = 0]$ then already ψ realizes $[s = 0]$.

This is actually almost obvious and follows from exercise ??? in ???

4) Finally we notice that the notion of a very simple divisor is obviously stable under pullback by morphisms $p : T'' \to T'$. Then we start from $[s = 0]$ and construct a faithfully flat base change $T' \to T$ such that the pullback $[s' = 0]$ is realized by $\psi' : T' \to C$. Since at this point we know uniqueness we could also say $[s' = 0] = \psi'$. Now we have the two projections $p_1, p_2 : T' \times_T T' \to T'$ and the two morphisms $\psi' \circ p_1, \psi' \circ p_2$ which both realize

$$[s'' = 0] = (p \circ p_1)^{-1}[s = 0] = (p \circ p_2)^{-1}[s = 0]$$

and since the realization is unique it follows

$$\psi' \circ p_1 = \psi' \circ p_2$$

which then implies that we can find a ψ such that $\psi' = \psi \circ p$. Then it follows from 3) that $[s = 0] = \psi$.

I want to comment on this strategy. Here we are dealing with very general principles.

First of all we learn that we can learn that we can solve problems by passing to localizations.

Secondly we learn that localization does not necessarily mean that we pass to small open subset in a toplogical space, there are more general concepts. (See also ???)

And finally we learn that what a fundamental role the uniqueness of the solution after localization plays. Our case here is in contrast to the situation in Chapter ??? where we applied the same strategy and where we saw that the non uniqueness created the cohomology groups.

Now we are ready for the hard work and tackle 2)

We still have the projection $q : C^g \to C^g/\Sigma_g$ and if we pull back our relative divisor $[\Delta = 0]$ to $C \times C^g$, then we know that it is the product of g divisors

$$(\mathrm{Id} \times q)^*[\Delta = 0] = \prod [\Delta_\nu = 0],$$

and where these $[\Delta_\nu = 0]$ are the inverse images of the diagonal in $C \times C$ under the projection

$$C \times C^g \xrightarrow{p_{0\nu}} C \times C.$$

These diagonals are relative divisors over C of degree one. We have the morphism

$$\Psi = (\ldots, \psi_\nu, \ldots) : \mathrm{Spec}(A) \to C^g$$

such that $(\mathrm{Id} \times \Psi)^*([\Delta_\nu = 0]) = (\tilde{b} - \alpha_\nu)$. We simply define $\tilde{\psi}_\nu : B \to A$ as the composition

$$B \otimes A \to B \otimes A/I \tilde{\to} A[X]/(F(X)) \overset{X \mapsto \alpha_\nu}{\to} A$$

Then $\psi = q \circ \Psi$ has the required property.

Of course it looks as if we have a problem with the uniqueness since our construction depends on the choice of a decomposition of F. This means that our morphism Ψ might be not unique. But the morphism Ψ is the same thing as $\tilde{\Psi} = (\ldots, \tilde{\psi}_\nu, \ldots) : B^{\otimes g} \to A$. We show that the restriction ψ of $\tilde{\Psi}$ to the subring of symmetric elements does not depend on this choice. An element $b \in B$ is mapped to a polynomial $g(X) = b_m X^m + \ldots + b_0 \in A[X]/(F(X))$ under the composition $B \to B \otimes A \to B \otimes A/I$ and $\psi_\nu(b) = g(\alpha_\nu)$. Now we refer to the theorem of elementary symmetric functions. The algebra $(B^{\otimes g})^{\Sigma_g}$ is generated by elements of the form $\sigma_\mu(b)$ and such an element is mapped to the corresponding coefficient in the polynomial $(X - g(\alpha_1)) \ldots (X - g(\alpha_g))$. But this polynomial is independent of the choice of the α_i, its coefficients can be expressed in terms of the coefficients of the polynomial $F(X) = (X - \alpha_1) \ldots (X - \alpha_g)$.

Still this argument does not prove uniqueness.

To get uniqueness we first assume that our ψ lifts to a $\Psi : \mathrm{Spec}(A) \to C^g$ as above in terms of algebras and we assume that we have a diagram

$$\begin{array}{ccc}
B \otimes B^{\otimes g} & & \\
\uparrow & \searrow^{\mathrm{Id} \times \tilde{\Psi}} & \\
B \otimes (B^{\otimes g})^{\Sigma_g} & \xrightarrow{\mathrm{Id} \times \tilde{\psi}} & B \otimes A \\
\uparrow & & \uparrow \\
(B^{\otimes g})^{\Sigma_g} & \xrightarrow{\tilde{\psi}} & B
\end{array}$$

The ideal I'_Δ in $B \otimes B^{\otimes g}$ decomposes into a product of locally principal ideals $I'_\Delta = \prod I_{\Delta_\nu}$, the image $(\mathrm{Id} \times \tilde{\Psi})(I'_\Delta)$ generate I which then decomposes it to a locally principal ideal

$$I = \prod I'_\nu$$

with $I'_\nu = (\mathrm{Id} \times \tilde{\Psi})(I_{\Delta_\nu}) \cdot B \otimes A$.

Now we have (We still assume that our divisor is very simple.)

$$\begin{array}{ccccccc} B & \to & B \otimes A & \to & B \otimes A/I & = & A[X]/(F(X)) \\ & & & & \downarrow & \swarrow h_\nu & \\ & & A & = & B \otimes A/I_\nu & & \end{array}$$

and $A \to B \otimes A/I_\nu$ must be an isomorphism since I_ν is of degree one. This implies that the morphism $A[X]/(F(X)) \xrightarrow{h_\nu} A$ must send X to a zero of F, and hence we see that $I_\nu = (X - \beta_\nu)$ such that

$$F(X) = \prod_{\nu=1}^{g} (X - \beta_\nu)^{\cdot}$$

But now our previous remark tells us that this implies that ψ is unique.

We have to remove the hypothesis that ψ lifts to a morphism Ψ. To do this we observe:

If we constructed a $\psi : T \to C^g/\Sigma_g$, then we may consider $\Pi_2 : C^g \to C^g/\Sigma_g$ and T via ψ as schemes over C^g/Σ_g. The morphism Π_2 is flat (See ???). We can form the fibered product

$$T' = C^g \times_{C^g/\Sigma_g} T$$

and get a diagram

$$\begin{array}{ccc} C \times C^g & \xleftarrow{\mathrm{Id} \times \psi'} & C \times T' \\ \downarrow \mathrm{Id} \times q_0 & & \downarrow \mathrm{Id} \times q \\ C \times C^g/\Sigma_g \supset [\Delta = 0] & \xleftarrow{\mathrm{Id} \times \psi} & C \times T \supset [s = 0] \\ \downarrow & & \downarrow \\ C^g/\Sigma^g & \xleftarrow{\psi} & T \end{array}$$

where ψ' is the projection.

Now the uniqueness of ψ follows from the argument 4) in 1.5.5.

Now we know that we can prove (B) for very simple relative divisors and it is clear that he notion of a very simple relative divisor is stable under base change.

We will prove

Lemma 1.5.5.1.: For any relative divisor we can find a finitely presented faithfully flat A-algebra

$$A \longrightarrow A'$$

such that

$$\begin{array}{c} \mathrm{Spec}(B \otimes A'/I') \quad \supset [s=0] \\ \downarrow \\ \mathrm{Spec}(A') \end{array}$$

is very simple.

Once we proved this we get by our strategy above the assertion (B) in general.

Now we prove that any relative divisor

$$\begin{array}{c} B \otimes A \longrightarrow B \otimes A/I \\ \uparrow \qquad\qquad \\ A \qquad\qquad \end{array}$$

becomes very simple after a suitable faithfully flat base change of finite type.

Our problem is local in A. Let us pick a closed point $t_0 \in \mathrm{Spec}(A)$, then the residue field $k(t_0)$ is finite over k. We look at the fibre $[s_{t_0} = 0] \subset \mathrm{Spec}(B \otimes_k k(t_0))$. This scheme is finite, and we can find a finite normal extension $L \supset k(t_0) \supset k$ such that all the residue fields $k(\mathfrak{p})$ of the points $\mathfrak{p} \in [s_{t_0} = 0]$ can be embedded into L.

We now consider as a first step the base change $A \to A \otimes_k L = A'$. We have finitely many closed points $t_0' \in \mathrm{Spec}(A')$ which lie above t_0. This means that after a suitable base change we can assume that $k(t_0) = k$ and that all points in $[s_{t_0} = 0]$ have a residue field k, i.e. they are rational over k. Now we can find an element $\bar{y} \in B \otimes_k k(t_0)$ such that $d\bar{y} \neq 0$ at all points $\mathfrak{p} \in [s_{t_0} = 0]$ and such that the values $\bar{y}(\mathfrak{p})$ are different at the different points $\mathfrak{p} \in [s_{t_0} = 0]$. Now we know that $\bar{y} - \bar{y}(\mathfrak{p})$ is a uniformizing element at $\mathfrak{p}$, and it is clear that

$$B \otimes_k k(t_0)/(f_{t_0}) \simeq k(t_0)[X]/(\prod_{\mathfrak{p}} (X - \bar{y}(\mathfrak{p}))^{m_{\mathfrak{p}}}$$

(see CA ....). We lift $\bar{y}$ to an element $y \in B$. Then we know that after a suitable localization in the base $\mathrm{Spec}(A)$ around t_0 the images of the element $1, y, \cdots, y^{g-1}$ yield $B \otimes A/I$ of this A-module and we have

$$y^g = -a_0 - a_1 y - \cdots - a_{g-1} y^{g-1}.$$

This means that after a localization in $\mathrm{Spec}(A)$ we have

$$B \otimes A/I \simeq A[X]/(F(X))$$

where $F(X) = X^g + a_{g-1}X^{g-1} + -a_0 \in A[X]$. The rest is clear: We can find a faithfully flat extension $A \to A'$ of finite type such that

$$F(X) = (X - \alpha_1) \cdots (X - \alpha_g) \in A'[X]$$

with $\alpha_i \in A'$.

1.5.6 **The gluing**

Our goal was to construct a scheme $\mathrm{Pic}^g(C)/k$ and a line bundle $\mathcal{P}$ over $C \times \mathrm{Pic}^g(C)/k$ such that for any line bundle $\mathcal{L}_1$ over a scheme $C \times_k T$ for which $\deg \mathcal{L}_t = g$ for all points $t \in T$ we can find a

$$\psi : T \longrightarrow \mathrm{Pic}^g(C),$$

such that $(\mathrm{Id} \times \psi)^*(\mathcal{P})$ is locally on T isomorphic to $\mathcal{L}$.

We reached this goal for a certain subfunctor: We considered only those families of bundles of degree g for which $\dim H^0(C \times k(t), \mathcal{L}_t) = 1$.

So far we could work over our given ground field but now we have to extend the ground field k to an algebraically closed field. Then we have

1.5.6.1. Proposition: *If we have a smooth projective curve C/k of genus g over an algebraically closed field k, then we can find a finite family of degree zero line bundles*

$$\mathcal{F}_1, \cdots, \mathcal{F}_r$$

on C/k such that for any line bundle $\mathcal{L}'$ on $C \times_k L$ of degree g (here L is an arbitrary field extension of k), we can find an index i such that

$$\dim H^0(C \times_k L, \mathcal{L}' \otimes \mathcal{F}_i) = 1$$

i.e. $\mathcal{L}' \otimes \mathcal{F}_i$ is generic.

Proof: Our proof is based on a general principle namely that the line bundles of degree g on C/k form a *bounded family* [See Lit]. This means in our case that we have the bundle $\mathcal{L}$ on $C \times C^g/\Sigma_g$ and for any line bundle $\mathcal{L}_1$ on C/k we can find a point $u \in C^g/\Sigma_g(k)$ such that $\mathcal{L}_u \xrightarrow{\sim} \mathcal{L}_1$. (Every line bundle $\mathcal{L}_1$ "occurs" (perhaps several times) in the family $\mathcal{L}$ on $C \times C^g/\Sigma_g$). If such an $\mathcal{L}_1$ is given then we can find a line bundle $\mathcal{F}$ on C such that $\mathcal{L}_1 \otimes \mathcal{F}$ is generic. Now we pick an u such that $\mathcal{L}_1 \xrightarrow{\sim} \mathcal{L}_u$. Then we can find an open set $V \subset C^g/\Sigma_g$ containing the point u such that for all $v \in V$ the line bundle $\mathcal{L}_v \otimes \mathcal{F}$ is generic. This yields a covering of C^g by open sets V_i and bundles $\mathcal{F}_i$ such that $\mathcal{L} \otimes \mathcal{F}_i$ is generic on $C \times V_i$ and this covering has a finite subcovering and the finite list of $\mathcal{F}_i$. For $i = 1$ we choose $V_1 = U$ and $\mathcal{F}_1 = \mathcal{O}_C$.

Now we come back to our line bundle $\mathcal{L}'$ on $C \times L$. Of course we can assume that L is the quotient field of a finitely generated k-algebra B, and that $\mathcal{L}'$ is obtained by base change from a line bundle $\mathcal{L}^*$ on $C \times \mathrm{Spec}(B)$. The pullbacks of $\mathcal{F}_i$ by p_1 are still called $\mathcal{F}_i$. Now we consider a closed point $t_0 \in \mathrm{Hom}_k(B, k)$ and consider the line bundle $\mathcal{L}^*_{t_0}$ on C/k. We can find an index i such that $\mathcal{L}^*_{t_0} \otimes \mathcal{F}_i$ is generic. Then it follows from semicontinuity that $\mathcal{L} \otimes \mathcal{F}_i$ is generic. This finishes the proof.

This makes it clear how to proceed with the construction of $\mathrm{Pic}^g(C)$. If we have line bundles $\mathcal{L}_1$ on $C \times_k T$, where T is of finite type, then we have a finite covering $T = \cup\, T_i$ by open sets such that $\mathcal{L}_1 \otimes \mathcal{F}_i$ restricted to $C \times T_i$ is generic in all points $t \in T_i$. Then we find a

$$\psi_i : T_i \longrightarrow U$$

such that $(\mathrm{Id} \times \psi_i)^*(\mathcal{P}')$ is locally isomorphic to $\mathcal{L}_1 \otimes \mathcal{F}_i$ or in other words

$$(\mathrm{Id} \times \psi_i)^*(\mathcal{P}' \otimes \mathcal{F}_i^{-1}) \sim_{T_i} \mathcal{L}_1 \mid C \times T_i.$$

Hence we do the following: We consider the open set $U \subset C^g/\Sigma_g$ and the line bundle $\mathcal{P}'$ on $C \times U$. We consider r copies of $C \times U$ and on these copies we put the line bundles $\mathcal{P}' \otimes \mathcal{F}_i^{-1}$. Our previous consideration shows that $C \times U$ together with $\mathcal{P}' \otimes \mathcal{F}_i^{-1}$ is a universal bundle for families of line bundles of degree g for which $\mathcal{L}_1 \otimes \mathcal{F}_i$ is generic.

We form the disjoint union

$$\bigcup_{i=1}^{r} C \times U = C \times U \times [1, \dots, r]$$

on which we consider the line bundle $\mathcal{L}_i = \mathcal{P}' \otimes \mathcal{F}_i^{-1}$ on the i'-th component. Remember that for $i = 1$ we put $\mathcal{F}_1 = \mathcal{O}_C$. Let us put $U_i = U \times \{i\}$.

For any of the components $(C \times U_i, \mathcal{P}' \otimes \mathcal{F}_i^{-1})$ and for any other index j we may consider the open subset $C \times U_{ij} \subset C \times U_i$ where $u \in U_{ij}$ if and only if $\mathcal{P}'_u \otimes \mathcal{F}_i^{-1} \otimes \mathcal{F}_j$ is generic. Then it follows from our previous arguments that we have unique isomorphisms

$$\psi_{ij} : U_{ij} \xrightarrow{\sim} U_{ji} \subset U_j$$

such that we have

$$(\mathrm{Id} \times \psi_{ij})^*(\mathcal{P}' \otimes \mathcal{F}_j^{-1}) \sim_{U_{ij}} \mathcal{P}' \otimes \mathcal{F}_i^{-1}.$$

It is clear that this family of morphisms satisfies $\Psi_{ik} \circ \psi_{ij} = \psi_{ik}$, and

$$\psi_{ji} \circ \psi_{ij} = \mathrm{Id}\,.$$

We get an equivalence relation on our disjoint union and this allows us to glue these copies of U via the identifications ψ_{ij} to a scheme $\mathrm{Pic}^g(C)/k$ (See ???).

By construction we have a covering of $\mathrm{Pic}^g(C)$ by open sets U_i and we have line bundles $\mathcal{L}_i$ on the products $C \times U_i$ We would also like to glue the bundles to a bundle $\mathcal{P}$ on $C \times C^g/\Sigma_g$ but before we can do this we have to work a little bit.

We have the following provisorial statement:

1.5.6.2. For any scheme T/k of finite type and for any line bundle $\mathcal{L}'$ on $C \times T/k$ such that $\mathcal{L}'_t$ is of degree g for all $t \in T$ we have a unique morphism

$$\psi : T \longrightarrow \mathrm{Pic}^g(C)$$

such that for any $t \in T$ we can find a neighborhood V_t of T such that $\psi : V_t \to \mathrm{Pic}^g(C)$ factor through one of the U_i, and

$$\eta : (\mathrm{Id} \times \psi)^*(\mathcal{L}_i) \tilde{\to} \mathcal{L}' \mid C \times V_t.$$

We say that ψ *realizes* $\mathcal{L}'$

We can apply this to our bundle $\mathcal{L}$ on $C \times C^g/\Sigma_g$. Hence we get a morphism

$$\psi : C^g/\Sigma_g \longrightarrow \mathrm{Pic}^g(C)$$

which has the property above. Of course we can restrict Ψ to our open set $U \subset C^g/\Sigma_g$ and on this set our map Ψ is the identity.

Now it is clear that we have an action of the group $\mathrm{Pic}^0(C)(k)$ on $\mathrm{Pic}^g(C)$: For any line bundle of degree zero $\mathcal{F}$ on C and any of our open sets $U_i \subset \mathrm{Pic}^g(C)$ we can define a new line bundle $\mathcal{L}_i \otimes \mathcal{F}$ on $C \times U_i$ which is of degree g in any point. Hence we get a unique morphism

$$T_{\mathcal{F}} : U_i \longrightarrow \mathrm{Pic}^g(C)$$

which realizes $\mathcal{L}_i \otimes \mathcal{F}$. It is clear that these morphisms are compatible on the intersections $U_i \cap U_j$ and therefore these morphisms fit together to an automorphism

$$T_{\mathcal{F}} : \mathrm{Pic}^g(C) \longrightarrow \mathrm{Pic}^g(C),$$

and it is clear that $T_{\mathcal{F}} \circ T_{\mathcal{G}} = T_{\mathcal{F} \otimes \mathcal{G}}$.

Under this action a geometric point $u \in \mathrm{Pic}^g(C)$ which gives us an isomorphism class of line bundles $\mathcal{L}_u$ is mapped to $T_{\mathcal{F}}(u)$ and $\mathcal{L}_u \otimes \mathcal{F} = \mathcal{L}_{T_{\mathcal{F}}(u)}$. Since this action is transitive we can conclude

1.5.6.3 The scheme $\mathrm{Pic}^g(C)/k$ is smooth and separated

Proof: We write $\mathrm{Pic}^g(C) = X$. Of course the non empty open subscheme $U \subset X$ is smooth and separated. The non smooth points form a closed subset in $Z \subset X$. If $Z \neq \emptyset$ then $Z(k) not = \emptyset$. Since $\mathrm{Pic}^0(C)(k)$ acts transitively on $X(k)$ and since $Z(k)$ is invariant under this action it follows that $Z(k) = \emptyset$ because $U(k)) \neq \emptyset$. A similar argument works for separatedness. We have to show that the diagonal Δ_X is closed in $X \times X$. Assume it is not, then we can find a geometric point $(P, Q) \in \bar{\Delta}_X(k) \subset X(k) \times X(k)$ with $P \neq Q$. We have the diagonal action of $\mathrm{Pic}^g(C)(k)$ on $X(k) \times X(k)$ and clearly $\Delta_X(k)$ and $\bar{\Delta}_X(k)$ are invariant under this action. We can find an element $u \in \mathrm{Pic}^0(C)(k)$ such that $T_u(P, Q) \in U \times U$. But since U/k is separated we see that the diagonal is closed in $U \times U$. This is a contradiction.

Now I claim

1.5.6.4. Our open set $U \subset \mathrm{Pic}^g(C)$ has a complement of codimension ≥ 2 and the line bundle $\mathcal{P}'$ on $C \times U$ extends uniquely to a line bundle $\mathcal{P}$ on $C \times \mathrm{Pic}^g(C)$. This line bundle $\mathcal{P}$ has the property that $\mathcal{P}|C \times U_i \sim_{U_i} \mathcal{L}_i$

Proof : We consider the morphism $\psi : C^g/\Sigma_g \to \mathrm{Pic}^g(C)$. The fibre of a point $u \in \mathrm{Pic}^g(C)$ consists of those points $(P_1 \cdots P_g) \in C^g/\Sigma_g(k)$ for which

$$\mathcal{L}_u \simeq \mathcal{O}(-P_1 - \cdots - P_g),$$

and hence we see that the fibre has dimension

$$\dim H^0(C \times k(u), \mathcal{L}_u) - 1.$$

The map ψ identifies the open set $U \subset C^g/\Sigma_g$ with its image in $\mathrm{Pic}^g(C)$ by construction, hence we see that for points $u \in \mathrm{Pic}^g(C) \setminus U$ the dimension of the fibre is ≥ 1. (Ref ???) This implies the first assertion.

To prove the extendability we use a general principle which says that on smooth, irreducible schemes isomorphism between line bundles or sections in line bundles extend over closed subset of codimension ≥ 2.

To be more precise: Let V/k be a smooth. irreducible scheme of finite type, let $\mathcal{L}$ be a line bundle over V and let $U \subset V$ be an open subset such that the complement has codimension ≥ 2. Let $s \in H^0(U, \mathcal{L})$ be a section. Then this section extends to a unique section on V. To see that this is so we pick a point $p \in V \setminus U$ and we choose a local section $s_p \in H^0(V_p, \mathcal{L})$. Then we have $s = hs_p$ over $V_p \cap U$. Now it follows from theorem ??? in [Ei] that h extends uniquely to a regular function in $\mathcal{O}_X(V_p)$.

If we have two line bundles $\mathcal{L}_1, \mathcal{L}_2$ over V which are isomorphic over U then this isomorphism is given by non zero sections in $s_1 \in H^0(U, \mathcal{L}_1 \otimes \mathcal{L}_2^{-1}), s_2 \in H^0(U, \mathcal{L}_2 \otimes \mathcal{L}_1^{-1})$ hose product is one. Our argument above shows that these sections extend to sections on V and their product is still one.

This implies that an extension from $\mathcal{P}'$ on $C \times U$ to a line bundle $\mathcal{P}$ on $C \times \mathrm{Pic}^0 g(C)$ is unique.

Finally we extend $\mathcal{P}'$ from $C \times U$ to a line bundle $\mathcal{P}$ on $C \times \mathrm{Pic}^g(C)$. Let us assume that we extended $\mathcal{P}'$ to a line bundle-still called $\mathcal{P}'$- on $C \times U'$ where $U' \supset U$ is open. Let $a \in \mathrm{Pic}^g(C) \setminus U'$. We find an index i such that $a \in U_i$. We consider the open subset $U_i' \subset U'$ where $\mathcal{P}' \otimes \mathcal{F}_i$ is generic. By construction we have $U' \subset U_i$ and $\mathcal{P}'|C \times U_i' \sim_{U_i'} \mathcal{L}_i|C \times U_i'$. (The morphism ψ which realizes $\mathcal{P}'|C \times U_i'$ is now the inclusion). But now we can say that we can find a line bundle $\mathcal{M}$ on $\mathcal{U}_i'$ such that

$$\mathcal{P}'|C \times U_i' \xrightarrow{\sim} \mathcal{L}_i|C \times U_i' \otimes p_2^*(\mathcal{M})$$

(see ???) and hence we can glue the bundles $\mathcal{P}'$ on $C \times U'$ and $\mathcal{L}_i \otimes p_2^*(\mathcal{M})$ on $C \times U_i$ over $C \times U_i'$ to a larger extension of $\mathcal{P}'$. This process stops after a finite number of steps.

1.5.7. Hence have constructed the the scheme $C \times \mathrm{Pic}^g(C)$ and a line bundle $\mathcal{P}$ which is the unique extension of $\mathcal{P}'$ on $C \times U$ such that we have the following universality property:

For any line bundle $\mathcal{L}'$ on $C \times_k T$ where T/k is of finite type and where for all $\deg(\mathcal{L}_t) = g$ *$t \in T$, we have a unique morphism*

$$\Psi : T \longrightarrow \mathrm{Pic}^g(C)$$

such that

$$\mathcal{L}' \simeq (\mathrm{Id} \times \psi)^*(\mathcal{P}) \otimes \mathcal{M},$$

where $\mathcal{M}$ is a line bundle on T.

This means of course that we have local representability for $\mathrm{Pic}^g(C)$. We are aware that we needed to extend the ground field to $\bar{k}$. Actually we can find a finite separable extension k_1 of k such that our auxiliary bundles $\mathcal{F}_i$ are defined over k_1 and then we can define $\mathrm{Pic}(C)$ over k_1. But it is clear that $\mathrm{Pic}^g(C)/k_1$ comes with a distinguished descend datum (see ???) and hence we can expect that we have a canonical descend to k. Before we can carry out this argument, we need more information on $\mathrm{Pic}^g(C)$ (for instance projectivity See ???) and these will be proved later.

If we keep our assumption that $k = \overline{k}$, or if we assume that we have a rational point $P_0 \in C(k)$. We want to show that we can replace g by any number r. We construct the line bundle

$$\mathcal{P}_r = \mathcal{P} \otimes p_1^*(\mathcal{O}_C(-\nu P_0)) \qquad \text{on} \qquad C \times X,$$

and it is clear that this bundle has the universal property for bundles of degree $rg - \nu$. If r is given we rebaptize X and call it $\mathrm{Pic}^r(C)$ and

$$\mathcal{P}_r \qquad \text{on} \qquad C \times \mathrm{Pic}^r(C),$$

is the universal bundle for bundles of degree r.. Especially we can construct a line bundle $\mathcal{P}_0$ on $C \times \mathrm{Pic}^0(C)$. Of course all these $\mathrm{Pic}^r(C)$ are isomorphic, but the identification depends on the choice of P_0.

If we consider the scheme

$$C \times \mathrm{Pic}^r(C) \times \mathrm{Pic}^s(C),$$

then we can consider the line bundle

$$p_{12}^*(\mathcal{P}_r) \otimes p_{13}^*(\mathcal{P}_s)$$

on it and this line bundle has degree $r + s$ in any point $y \in \mathrm{Pic}^r(C) \times \mathrm{Pic}^s(C)$. Therefore we get a unique morphism

$$m : \mathrm{Pic}^r(C) \times \mathrm{Pic}^s(C) \longrightarrow \mathrm{Pic}^{r+s}(C)$$

such that

$$p_{12}^*(\mathcal{P}_r) \otimes p_{13}^*(\mathcal{P}_s) \sim (\mathrm{Id} \times m)^*(\mathcal{P}_{r+s}).$$

It is clear that this defines a structure of a group scheme (See ???) on

$$\mathrm{Pic}(C) = \bigsqcup \ \mathrm{Pic}^r(C)$$

which is a scheme over k which is not of finite type.

As a special case this gives us a group scheme structure on $\mathrm{Pic}^0(C)$, and all the $\mathrm{Pic}^r(C)$ are principal homogeneous spaces under the action of $\mathrm{Pic}^0(C)$.

2. The Picard functor on X and on J

2.1. Some heuristical remarks

We want to get a better understanding of the scheme $\mathrm{Pic}(C)$. Since all its connected components are isomorphic, we can concentrate on the components. For us it is sometimes convenient to consider $\mathrm{Pic}^g(C)$ and $\mathrm{Pic}^0(C)$. Therefore I introduce the notations

$$\begin{array}{rcl} X & = & \mathrm{Pic}^g(C) \\ J & = & \mathrm{Pic}^0(C) \end{array}$$

We have seen that X is a smooth projective scheme. Then J is also smooth projective, and it is a group scheme. Smooth projective group schemes are called *abelian varieties.*

It is already clear that $\mathrm{Pic}^0(C)/k$ is a proper scheme (see ???). To see this we recall that we have the surjective morphism

$$\pi : C^g/\Sigma_g \longrightarrow \mathrm{Pic}^g(C),$$

and that we have identifications $\mathrm{Pic}^0(C) \simeq \mathrm{Pic}^g(C)$. The morphism π is birational, because it induces an isomorphism on the open subset U.

As in the transcendental case our main object are the Picard schemes $\mathrm{Pic}(X)$ and $\mathrm{Pic}(J)$. The understanding of the structure of these Picard schemes is the key to many beautiful results.

2.1.1. I remind you that the fundamental tool for this study was the existence of the polarization and the resulting line bundles. (See ???)

We assume that k has rational points, we pick a point $P_0 \in C(k)$. Then we can restrict the bundle $\mathcal{P}$ to $\{P_0\} \times X \simeq X$ and this gives us a line bundle $\mathcal{P}_{P_0}$ on X. This bundle will play the role of - or is - the principal polarization. We have the action of J on X and this allows us to translate line bundle and divisors. For any $x \in J(k)$ we have the translation $T_x : X \to X$ and we can consider the translated bundle $T_x^*(\mathcal{P}_{P_0})$ and we compare it to $\mathcal{P}_{P_0}$ by forming the quotient

$$T_x^*(\mathcal{P}_{P_0}) \otimes \mathcal{P}_{P_0}^{-1}.$$

Now we can view x as a variable or better we will construct a bundle

$$m^*(\mathcal{P}_{P_0}) \otimes p_2^*(\mathcal{P}_{P_0}^{-1}) \text{ on } X \times J$$

(see ???) which evaluated at x gives the above bundle. Our main result will be that this bundle gives as a local representation of $\mathrm{Pic}^{00}(X)$ (see ???).

2.2. The homomorphisms $\varphi_{\mathcal{M}}$:

Let us consider any line bundle $\mathcal{M}$ on X. We can consider the map

$$\varphi_{\mathcal{M}} : J(k) = \mathrm{Pic}^0(C)(k) \to P^{00}(k)$$

defined by

$$\varphi_{\mathcal{M}}(x) = T_x(\mathcal{M}) \otimes \mathcal{M}^{-1},$$

and we want to show:

Proposition 2.2.1.: *This map $\varphi_{\mathcal{M}}$ is a homomorphism, and if $\mathcal{M}$ is algebraically equivalent to zero, then this homomorphism is trivial.*

Proof: We reduce the proof to the second assertion. Let us pick a point x, then

$$\mathcal{L} = T_x^*(\mathcal{M}) \otimes \mathcal{M}^{-1}$$

is algebraically equivalent to zero. Then the second assertion says

$$T_y^*(\mathcal{L}) \simeq \mathcal{L}$$

therefore

$$T_y^*(T_x^*(\mathcal{M}) \otimes \mathcal{M}^{-1}) \simeq T_x^*\mathcal{M} \otimes \mathcal{M}^{-1}$$

and rearranging yields

$$T_{x+y}^*(\mathcal{M}) \otimes \mathcal{M}^{-1} \simeq T_x^*(\mathcal{M}) \otimes \mathcal{M}^{-1} \otimes T_y^*(\mathcal{M}) \otimes \mathcal{M}^{-1}.$$

Now we consider the case of an $\mathcal{M}$ which is algebraically equivalent to zero. This means that we have a line bundle

$$\tilde{\mathcal{M}}|X \times Z$$

where Z is of finite type and connected over k and such that

$$\begin{array}{ccccc} \tilde{\mathcal{M}}|X & \times & z_1 & \simeq & \mathcal{M} \\ \tilde{\mathcal{M}}|X & \times & z_0 & \simeq & \mathcal{O}_X. \end{array}$$

for some points $z_1, z_0 \in Z$. We can pick a point $x_0 \in X(k)$ and tensorize $\tilde{\mathcal{M}}$ by $p_2^*(\tilde{\mathcal{M}}|x_0 \times Z)^{-1})$; then $\tilde{\mathcal{M}}|x_0 \times Z \simeq \mathcal{O}_Z$. We have two morphisms

$$\begin{array}{rcccc} p_{13} & : & X \times J \times Z & \longrightarrow & X \times Z \\ m_{12} \times \mathrm{Id}_Z & : & X \times J \times Z & \longrightarrow & X \times Z, \end{array}$$

the first one is the projection, the second one multiplication in the first to factors (times identity). We consider the bundle

$$(m_{12} \times \mathrm{Id}_Z)^*(\tilde{\mathcal{M}}) \otimes p_{13}^*(\tilde{\mathcal{M}})^{-1} = \mathcal{N}$$

which evaluated at a point $x \in J(k)$ and $z \in Z(k)$ gives us

$$T_x^*(\tilde{\mathcal{M}}_z) \otimes \tilde{\mathcal{M}}_z^{-1}$$

where $\mathcal{M}_z$ is of course $\tilde{\mathcal{M}}|X \times \{z\}$. Now we know that

$$\mathcal{N}|x_0 \times J \times Z \qquad \text{and} \qquad \mathcal{N}|X \times e \times Z \qquad \text{and} \qquad \mathcal{N}|X \times J \times z_0$$

are trivial. Now the theorem of the cube would imply that $\mathcal{N}$ is locally trivial in Z. But to apply it we need to know that X, J are projective. We don't know this yet, even worse we need our proposition to prove the projectivity. Therefore we have to make a detour at this point.

We remember that we have a morphism

$$\pi \times \mathrm{Id}_Z : C^g/\Sigma_g \times C^g/\Sigma_g \times Z \longrightarrow X \times J \times Z$$

which is birational and where the fibres are product of projective spaces. We pull our line bundle back and apply the theorem of the cube upstairs. To do this we need a little

Lemma 2.2.2. *For any line bundle $\mathcal{L}$ on $X \times J \times Z$ we have $(\pi \times \mathrm{Id}_Z)_*(\pi \times \mathrm{Id}_Z)^*(\mathcal{L}) \xrightarrow{\sim} \mathcal{L}$*

Proof: We consider the pullback of the bundle

$$\tilde{\mathcal{L}} = (\pi \times \mathrm{Id}_Z)^*(\mathcal{L}) \mid C^g/\Sigma_g \times C^g/\Sigma_g \times Z.$$

It follows from the construction (See V. 2.1.5.) that we have a morphism

$$j : \mathcal{L} \to (\pi \times \mathrm{Id}_Z)_*((\pi \times \mathrm{Id}_Z)^*(\mathcal{L}))$$

We want to show that this is an isomorphism. This is a local question on $X \times J \times Z$ and therefore we can assume that $\mathcal{L}$ is the trivial bundle $\mathcal{O}_{X \times J \times Z}$. Then the pullback is by definition the structure sheaf $\mathcal{O}_{C^g/\Sigma_g \times C^g/\Sigma_g \times Z}$. We have to show that

$$(\pi \times \mathrm{Id}_Z)_*(\mathcal{O}_{C^g/\Sigma_g \times C^g/\Sigma_g \times Z}) = \mathcal{O}_{X \times J \times Z}$$

Of course this follows if we can show the corresponding assertion for the morphism

$$\pi : C^g/\Sigma_g \times C^g/\Sigma_g \longrightarrow X \times J$$

because the sheaves in question are simply the pullbacks via the projection p_{12} to the first and second factor. Now we have seen that the morphism π is projective and hence we can conclude that $\pi_*(\mathcal{O}_{C^g/\Sigma_g \times C^g/\Sigma_g})$ is a coherent sheaf on $X \times J$ (See ???). Hence we see that for any open subset $V \subset X \times J$ the algebra $\pi_*(\mathcal{O}_{C^g/\Sigma_g \times C^g/\Sigma_g})(V)$ is finite over $\mathcal{O}_{X\times J}(V)$. If we pass to the stalks at a point x then we have $\mathcal{O}_{X\times J,x} \subset (\mathcal{O}_{C^g/\Sigma_g \times C^g/\Sigma_g,x}$ the extension is finite and the two local rings have the same field of fractions, namely the field of meromorphic functions on $X \times J$ (??? vorher was dazu sagen). But since $X \times J$ is smooth we know that the local rings $\mathcal{O}_{X\times J,x}$ are closed in their field of fraction (See [Ei], ???). Hence we conclude that the local rings are equal and this proves the Lemma.

Now we know that the first to factors are projective. and if we take inverse images of our points $\tilde{x}_0$ and $\tilde{e}$, then we still have the triviality conditions. Hence we can conclude $\tilde{\mathcal{N}}$ is locally trivial in Z, i.e. we can cover Z by open schemes Z_α s.t.

$$\tilde{\mathcal{N}} \mid C^g/\Sigma_g \times C^g/\Sigma_g \times Z_\alpha$$

is trivial. But then the lemma above yields that already $\mathcal{N}$ is locally trivial in Z.

2.3. The projectivity of X and J

Now we are prepared to prove that the scheme X/k is projective. To do this we construct an ample line bundle on X.

We start from our line bundle $\mathcal{P}_{P_0}$. We have seen that it can be written as

$$\mathcal{P}_{P_0} = \mathcal{O}_X(-D)$$

where D is an effective divisor (??? Ref. dafür, auch für effektiv?). Then we have

$$\begin{array}{rcl} T_x^*\mathcal{P}_{P_0} & = & \mathcal{O}_x(-T_x^*D) \\ T_{-x}^*\mathcal{P}_{P_0} & = & \mathcal{O}_X(-T_{-x}^*D) \end{array}$$

and

$$T_x^*\mathcal{P}_{P_0} \otimes T_{-x}^*\mathcal{P}_{P_0} \simeq \mathcal{P}_{P_0} \otimes \mathcal{L}_x \otimes \mathcal{P}_{P_0} \otimes \mathcal{L}_{-x} \simeq \mathcal{P}_{P_0}^{\otimes 2}.$$

For any point $u \in X$ we find an $x \in \mathrm{Pic}^0(C)$ such that

$$u \notin T_x^*D \cup T_{-x}^*D,$$

and hence we see that u is not a base point.

Then we have seen that we get a morphism

$$r_{\mathcal{P}_{P_0}^{\otimes 2}} : X \longrightarrow \mathbb{P}^N = \mathbb{P}(H^0(X, \mathcal{P}_{P_0}^{\otimes 2})),$$

and I claim that this morphism is finite. First we assume that we have a fibre $r^{-1}_{\mathcal{P}_{P_0}^{\otimes 2}}(y)$ which has positive dimension. This fibre contains an irreducible curve Z. We can find a section $s \in H^0(X, \mathcal{P}_{P_0}^{\otimes 2})$ which does not vanish at a particular point $x_1 \in Z(k)$ since we do not have base points, but then this section will not vanish at any point of Z which implies that $\mathcal{P}_{P_0}^{\otimes 2} \mid Z$ is the trivial bundle.

Therefore we know that $\mathcal{P}_{P_0}$ and all its translates $T_x^*(\mathcal{P}_{P_0})$ have degree zero on Z. (Ref. ???) Since $\mathcal{P}_{P_0} = \mathcal{O}_X(-D)$ we see that for any translate of D we have

$$T_x^*(D) \cap Z = \begin{cases} \emptyset \\ Z \end{cases}$$

or in other words we have $T_x^*(Z) \subset |D|$ or $T_x^*(Z) \cap |D| = \emptyset$.

Now we show that for any two points $z_1, z_2 \in Z(k)$ we have

$$T_{z_2 - z_1}^*(|D|) = |D|.$$

Let $d_0 \in |D|(k)$ then $d_0 = T_{d_0 - z_1}^*(z_1)$ and it follows

$$T_{d_0 - z_1}^*(Z) \subset |D|.$$

Now $T_{d_o - z_1}^*(z_2) = T_{d_0 - z_1}^*(T_{z_2 - z_1}^*(z_1)) = T_{z_2 - z_1}^*(T_{d_0 - z_1}^*(z_1)) = T_{z_2 - z_1}^*(d_0) \in |D|$.

But this is impossible since it would imply that $T_{z_2 - z_1}^*(\mathcal{P}_{P_0}) \simeq \mathcal{P}_{P_0}$ for all $z_2, z_1 \in Z$ and this contradicts our theorem ??? and we proved that all the schemes $\mathrm{Pic}^r(C)$ are projective.

2.3.1 Now we want to show that the homomorphisms $\phi_{\mathcal{M}}$ are not just homomorphisms on the groups of geometric points but also between the functors.

In 1.3.2. we defined the subfunktor $\mathrm{Pic}^{00}(X)(T)$. It is defined on connected schemes T/k of finite type which are equipped with a base point t_0. In the following considerations we evaluate this functor on local rings $A = \mathcal{O}_{Y,t_0}$ where Y is of finite type over k and the base point is t_0.

The action $m : X \times J \to X$ gives us for any T an action of the group $J(T)$ on the scheme $X \times T$ and hence on $\mathrm{Pic}(X \times T)$. If we denote again by T_x the automorphism induced by $x \in J(A)$ on the scheme then we denote by T_x^* the action on the line bundles. Again we pick a bundle $\mathcal{M}$ on X. For $T = \mathrm{Spec}(A)$ where A is local as above we consider

$$x \to \mathrm{Pic}(X)(A)$$

$$x \mapsto T_x^*(\mathcal{M}) \otimes \mathcal{M}^{-1}$$

Again this is a homomorphism. To see this we apply the theorem of the cube. We consider

$$X \times J \times J,$$

we have three morphisms to $X \times J$, namely, the projections p_{12}, p_{13} and $\mathrm{Id}_X \times m_{23}$ where m_{23} is the multiplication in the second and third factor.

Then we have the two morphisms

$$X \times J \underset{p_1}{\overset{m}{\rightrightarrows}} X,$$

where m is the action of J on X. For any line bundle $\mathcal{M}$ on X we can consider the line bundle

$$\begin{aligned}\mathcal{N} = (m \circ m_{23})^*(\mathcal{M}) \otimes (m \circ p_{12})^*(\mathcal{M})^{-1} \otimes (m \circ p_{13})^*(\mathcal{M})^{-1} \otimes \\ \otimes (p_1 \otimes m_{23})^*(\mathcal{M})^{-1} \otimes (p_1 \circ p_{12})^*(\mathcal{M}) \otimes (p_1 \circ p_{13})^*(\mathcal{M})\end{aligned}$$

on $X \times J \times J$.

We pick a point $x_0 \in X(k)$ and evaluate this bundle on

$$x_0 \times J \times J \qquad X \times e \times J \qquad e \times X \times J,$$

and find that its restriction to the second and the this subscheme is trivial. Let $\mathcal{N}_{x_0}$ be its restriction to the first subscheme which we identify to $J \times J$. Let

$$\tilde{\mathcal{N}}_{x_0} = p_{23}^*(\mathcal{N}_{x_0})$$

then we see that

$$\mathcal{N} \otimes p_{23}^*(\mathcal{N}_{x_0})^{-1}$$

is trivial on all three subschemes. Hence we see by the theorem of the cube that

$$\mathcal{N} \simeq p_{23}^*(\mathcal{N}_{x_0}).$$

This can be formulated differently by saying $\mathcal{N} \sim_{J \times J} \mathcal{O}_{X \times J \times J}$.

If now $x, y \in J(A)$ and if x, y then $(x, y) \in J \times J(A)$

$$T_{x+y}^*(\mathcal{M}) \otimes T_x^*(\mathcal{M})^{-1} \otimes T_y^*(\mathcal{M})^{-1} \otimes \mathcal{M}$$

on $X \times_k T$ is trivial, because the contribution of $\mathcal{N}_{x_0}$ is trivial since A is local. This means that we get a homomorphism

$$\psi_{\mathcal{M}} : J(A) \longrightarrow \mathrm{Pic}(X)(A).$$

We will apply this to the special bundle $\mathcal{P}_{P_0}$. We can now interprete this as a homomorphism of functors

$$\phi_{\mathcal{P}_{P_0}} : J \to \mathrm{Pic}^{00}(X)$$

2.4. Maps from the curve C to X

Now we construct certain morphisms from the curve into X and study the restriction (via these maps) of $\mathcal{P}_{P_0}$ and of other line bundles on X to C. We have seen in Ch. RS 6.3 (???) that such morphisms explain the selfduality of the Jacobian.

In this section we assume that k is algebraically closed.

2.4.1. I explain the morphisms naively: We choose points

$$(Q_1, \cdots, Q_g) \in C^g(k)$$

We still choose another point P_0, and we assume that the images of the points

$$(Q_1, \cdots, Q_g) \qquad \text{and} \qquad (P_0, Q_1, \cdots, \hat{Q}_i, \cdots, Q_g)$$

in the quotient C^g/Σ_g lie always in U. From these data we construct a morphism

$$j_{P_0,\underline{Q}} : C \longrightarrow X$$

which sends a point $P \in C(k)$ to the line bundle $\mathcal{O}_C(P - P_0 - Q_1 - \cdots - Q_g)$.

To give the correct definition of this morphism we consider the curve $C \times C$ with its two projections p_1, p_2 to C. On this surface we have the line bundle

$$p_1^*(O_C(-P_0 - Q_1 - \cdots - Q_g)) \otimes O_{C\times C}(\Delta) = \mathcal{L},$$

where $\Delta \subset C \times C$ is the diagonal. This gives us a family of line bundles on the first factor which is parameterized by the second factor. For a point $P \in C(k)$ in the second factor the restriction of $\mathcal{L}$ to $C \times \{P\}$ is the bundle $\mathcal{O}_C(P - P_0 - Q_1 - \cdots - Q_g)$. Hence we have a unique morphism - and this is our $j_{P_0,\underline{Q}}$ - such that locally in the second component

$$(\mathrm{Id} \times j_{P_0,\underline{Q}}(\mathcal{P})) \sim \mathcal{L}.$$

I claim that under our above assumptions the morphism $j_{P_0,\underline{Q}} : C \to \mathrm{Pic}^g(C)$ factors through the open subset U. To see this we have to show that

$$\dim H^0(C, \mathcal{O}_C(P - P_0 - Q_1 \cdots - Q_g)) = 1$$

for all points $P \in C(k)$. This is of course clear if P is one of our points P_0 or $Q_1 \cdots Q_g$ because this is our assumption. If P is not equal to any of these points, and if we had

$$\dim H^0(C, \mathcal{O}_C(P - P_0 - Q_1 \cdots - Q_g)) \geq 2$$

we could conclude that

$$\dim H^0(C, \mathcal{O}_C(-P_0 - Q_1 \cdots - Q_g)) \geq 3,$$

because this space contains 1 and 1 is not zero at P. But then we could conclude that

$$\dim H^0(C, \mathcal{O}_C(-Q_1 \cdots - Q_g)) \geq 2$$

and this is again a contradiction to our assumption.

We want to compute the line bundle

$$j^*_{P_0,\underline{Q}}(\mathcal{P}_{P_0}),$$

this is a line bundle on C. The morphism $j_{P_0,\underline{Q}}$ sends the point P to a point $j_{P_0,\underline{Q}}(P) = (Q_1(P), \cdots, Q_g(P))$ where $(Q_1(P) \cdots Q_g(P)) \in C^g(\overline{k})$ is well defined up to an element in the symmetric group. It is determined by the relation

$$\mathcal{O}_C(P - P_0 - Q_1 \cdots - Q_g) \simeq \mathcal{O}_C(-Q_1(P) \cdots - Q_g(P)).$$

Now we recall the definition of $\mathcal{P}$ and $\mathcal{P}_{P_0}$. The restriction of $\mathcal{P}$ to $\{P_0\} \times U$ is the line bundle which is induced by the divisor

$$D' = \Sigma \;\; C \times \cdots C \times P_0 \times C \cdots C$$

on C^g. This divisor descends to a divisor D on C^g/Σ_g and the line bundle $\mathcal{P}_{P_0}|U = \mathcal{O}_U(-D)$.

Then it is clear that the curve $j_{P_0,\underline{Q}}(C)$ intersects D in the points $j_{P_0,\underline{Q}}(Q_i)$ with multiplicity one and this yields the innocent looking but fundamental

$$j_{P_0,\underline{Q}}(\mathcal{P}_{P_0}) \simeq \mathcal{O}_C(-Q_1 \cdots - Q_g). \tag{2.4.2.}$$

2.4.3. Now we do what we did already very often, we consider the point $Q \in U$ as a variable, more precisely we consider it as a point in X.

We consider the diagram

$$\begin{array}{ccccc} & \xrightarrow{p_{12}} & C \times X & & \\ C \times X \times C & & & & \\ & \xrightarrow{p_{13}} & C \times C & \xrightarrow{p_1} & C \end{array}$$

and the line bundle

$$p^*_{12}(\mathcal{P}) \otimes p^*_{13}(p^*_1(\mathcal{O}_C(-P_0)) \otimes \mathcal{O}_{C \times C}(\Delta)) = \tilde{\mathcal{L}}$$

on the on $C \times X \times C$. If we evaluate at the point $\underline{Q} \in U(k) \subset X(k)$, then we get our line bundle $\mathcal{L}$ above. We view this as a family of degree g bundles on the first factor which is parameterized be the product of the second and third factor. The universal property gives us a unique morphism

$$j_{P_0} : X \times C \longrightarrow X$$

such that $(\mathrm{Id}_C \times j_{P_0})^*(\mathcal{P})$ isomorphic locally in $X \times C$ to the line bundle $\tilde{\mathcal{L}}$.

Of course it is quite clear to describe this map on geometric points. If (u, P) is a geometric point on $X \times C$ then u is the isomorphism class of a line bundle $\mathcal{L}_u$ on $C \times \bar{k}$. Then $\mathcal{L}_u \otimes \mathcal{O}_C(P - P_0)$ is a line bundle of degree g on C, hence a geometric point on X and this point is the image.

This description yields a little piece of extra information

2.4.3.1. The morphism j_{P_0} commutes with the action of J on both sides

This map j_{P_0} gives us a line bundle

$$j_{P_0}^*(\mathcal{P}_{P_0})$$

on $X \times C$. We consider this as a family of line bundles on C which now is parameterized by the first factor and find a unique morphism

$$\psi_{P_0} : X \longrightarrow X$$

such that $(\psi_{P_0} \times \mathrm{Id}_C)^*(\mathcal{P}) \sim_X j_{P_0}(\mathcal{P}_{P_0})$ where in this case $\sim$ means that the bundles are locally isomorphic in the first variable X.

But for a dense set of geometric points $\underline{Q} \in U(k)$ we have shown that

$$\psi_{P_0}(\underline{Q}) = \underline{Q}$$

and since X is reduced we can conclude that $\psi_{P_0} = \mathrm{Id}_X$ (Ref ???). But we have of course $(\mathrm{Id}_X \times \mathrm{Id}_C)^*(\mathcal{P}) \simeq \mathcal{P}$, and conclude

$$\mathcal{P} \sim j_{P_0}^*(\mathcal{P}_{P_0}) \tag{2.4.4.}$$

locally in X on $X \times C$.

2.4.5.: Now we have constructed the homomorphism

$$\phi_{\mathcal{P}_{P_0}} : \mathrm{Pic}^0(C)(k) \to \mathrm{Pic}^{00}(X)(k)$$

and we have the resriction

$$j_{P_0,\underline{Q}}^* : \mathrm{Pic}^0(C)(k) \to \mathrm{Pic}^{00}(X)(k) \to \mathrm{Pic}^0(C)(k)$$

and I claim that the composition of these to homomorphisms is the identity. To see that this is the case we have to show

$$j_{P_0,\underline{Q}}^* \left(T_x^*(\mathcal{P}_{P_0}) \otimes \mathcal{P}_{P_0}^{-1}\right) \simeq \mathcal{L}_x$$

where $\mathcal{L}_x$ is a line bundle corresponding to x. For the computation of left hand side we use the invariance under translations (2.4.3.1.) and (2.4.2 ???): We get

$$T_x^*(j^*_{P_0,\underline{Q}}\left(\mathcal{P}_{P_0} \otimes \mathcal{P}_{P_0}^{-1}\right) \xrightarrow{\sim} T_x^*(\mathcal{P}) \otimes \mathcal{P}^{-1} \xrightarrow{\sim} \mathcal{L}_x$$

and this is the claim.

We have the two morphisms

$$X \times \operatorname{Pic}^0(C) \underset{p_1}{\overset{m}{\rightrightarrows}} X,$$

and we can consider the bundle

$$m^*(\mathcal{P}_{P_0}) \otimes p_1^*(\mathcal{P}_{P_0})^{-1}$$

on $X \times J$. Then for any $\underline{Q}$ the bundle

$$(j_{P_0,\underline{Q}})^*(m^*(\mathcal{P}_{P_0}) \otimes p_1^*(\mathcal{P}_{P_0})^{-1})$$

is a line bundle on $C \times J$ which has degree zero at any closed point $t \in J$. We remember that we denoted a universal bundle on $C \times J$ by $\mathcal{P}_0$. We see that we have a unique morphism

$$\psi_{P_0,\underline{Q}} : J \longrightarrow J$$

such that

$$(\operatorname{Id} \times \psi_{P_0,\underline{Q}})^*(\mathcal{P}_0) \sim_J (j_{P_0,\underline{Q}})^*(m^*(\mathcal{P}_{P_0}) \otimes p_1^*(\mathcal{P}_{P_0})^{-1}).$$

But if we evaluate at points $x \in J(k)$ then the bundle on the left hand side and the bundle on the right are both isomorphic to $\mathcal{L}_x$. For the bundle on the left this is the definition of a universal bundle, for the bundle on the right it is our formula above.

This says that the restriction of $(m^*(\mathcal{P}_{P_0}) \otimes p_1^*(\mathcal{P}_{P_0})^{-1}$ to $C \times J$ by any $j_{P_0.\underline{Q}}$ is a universal bundle on $C \times J = C \times \operatorname{Pic}^0(C)$.

We will show that the composition with $j_{P_0,\underline{Q}} : C \to X$ does not depend on $\underline{Q}$ and yields the identity on $J(A)$.

On $X \times J$ we constructed the line bundle

$$m^*(\mathcal{P}_{P_0}) \otimes p_1^*(\mathcal{P}_{P_0})^{-1} = \tilde{\mathcal{P}},$$

and I claim:

Theorem: *The scheme $X \times J$ together with the line bundle $\tilde{\mathcal{P}}$ represent the functor* $\operatorname{Pic}^{00}(X)$*, i.e. for any line bundle on*

$$\mathcal{L} \mid X \times T,$$

where T is connected and of finite type such that $\mathcal{L} \mid X \times t_0 \simeq \mathcal{O}_X$ for some point $t_0 \in T$. We have a unique morphism

$$\psi : T \longrightarrow J$$

such that $\mathcal{L} \sim (\mathrm{Id} \times \psi)^*(\tilde{\mathcal{P}})$.

Proof: Our previous assertion says that for any $j_{\underline{Q},P_0} : C \to X$ the pullback of the line bundle $\tilde{\mathcal{P}}$ to $C \times J$ is the universal bundle on $C \times J$. Hence we can take the pullback of $\mathcal{L}$ on $X \times T$ via $j_{P_0,\underline{Q}}$ to $C \times T$, and we see that we have a unique morphism

$$\psi : T \longrightarrow J$$

such that

$$(\mathrm{Id}_C \times \psi)^*(\tilde{\mathcal{P}}) \simeq (j_{P_0,\underline{Q}} \times \mathrm{Id}_T)^*(\mathcal{L})$$

here we assume that T should be local. We have to show that already

$$(\mathrm{Id}_X \times \psi)^*(\tilde{\mathcal{P}}) \simeq \mathcal{L}.$$

Let $T = \mathrm{Spec}(A)$ be the local ring at t_0. We have the two elements

$$(\mathrm{Id}_X \times \psi)^*(\tilde{\mathcal{P}}), \mathcal{L} \in \mathrm{Pic}^{00}(X)(A)$$

which are trivial in the special fibre and which become isomorphic if we restrict via $j_{P_0,\underline{Q}}$. We have to show that they are actually equal. Let $\mathfrak{m}$ be the maximal ideal of A. It is clear that it suffices to prove that they are equal if we send them to $\mathrm{Pic}^{00}(X)(A/\mathfrak{m}^N)$ for any N. For $N = 1$ both line bundles are trivial by assumption. We apply our infinitesimal considerations from ???. We put $X_N = A \otimes A/\mathfrak{m}^N$

We apply this sequence to the diagram

$$X \longleftarrow C^g/\Sigma_g \longleftarrow C^g$$

and put $C^g/\Sigma_g = Y$, $C^g = Z$. Then we get

$$\begin{array}{ccccccc}
0 & \to & H^1(X, \mathcal{O}_X) \otimes \mathfrak{m}^N/\mathfrak{m}^{N-1} & \to & H^1(X_N, \mathcal{O}^*_{X_N})(0) & \to & H^1(X \times A_N, \mathcal{O}^*_{X_{N-1}})(0) \to \\
 & & \downarrow & & \downarrow & & \downarrow \\
0 & \to & H^1(Y, \mathcal{O}_Y) \otimes \mathfrak{m}^N/\mathfrak{m}^{N-1} & \to & H^1(Y_N, \mathcal{O}^*_{Y_N})(0) & \to & H^1(Y \times A_N, \mathcal{O}^*_{Y_{N-1}})(0) \\
 & & \downarrow & & \downarrow & & \downarrow \\
0 & \to & H^1(Z, \mathcal{O}_Z)^{\Sigma_g}(0) \otimes \mathfrak{m}^N/\mathfrak{m}^{N-1} & \to & H^1(Z_N, \mathcal{O}^*_{Z_N})^{\Sigma_g} & \to & H^1(Z \times A_N, \mathcal{O}^*_{Z_{N-1}})(0)
\end{array}$$

Now we observe that the morphisms

$$H^1(X, \mathcal{O}_X) \longrightarrow H^1(Y, \mathcal{O}_Y) \longrightarrow H^1(Z, \mathcal{O}_Z)^{\Sigma_g}$$

are isomorphisms. To see this for the first morphism we recall that

$$\Psi : Y \longrightarrow X$$

is birational and the fibres are projective spaces. This implies that

$$\Psi_*(\mathcal{O}_Y) = \mathcal{O}_X,$$

and for any point $x \in X$ we have

$$H^1(\Psi^{-1}(x), \mathcal{O}_{\Psi^{-1}(x)}) = 0.$$

From this it follows that $R^1\Psi_*(\mathcal{O}_Y) = 0$ (???) and then the assertion follows from the spectral sequence.

The fact that the second map is an isomorphism follows from the Künneth formula. (???)

This implies that

$$H^1(X, \mathcal{O}_X) \xrightarrow{\sim} H^1(C, \mathcal{O}_C)$$

(see Chap. ???).

Now we have our diagram

$$\begin{tikzcd} J(A) \arrow[r] \arrow[dr, "\mathrm{Id}"'] & \mathrm{Pic}^{00}(X)(A) \arrow[d] \\ & J(A) \end{tikzcd}$$

and our infinitesimal considerations show that $Pic^{00}(X)(A)$ and $J(A)$ are of the same size. This implies that the vertical arrow must be an isomorphism.

2.4.4. Descend to the ground field.

We go back to 1.5.7. We have shown that for our given curve C/k the functor Pic^r is locally representable if we extend our ground field to the algebraic closure $\bar{k}$. We want to discuss the question to what extend we can carry out such a construction over k.

The situation is easier for the functor $\widetilde{\mathrm{Pic}}_{C/k,P_0}$. This functor requires a k-rational point. We know that this functor is representable if we extend our groundfield to $\bar{k}$ (See 1.5.7.)

But now we apply our considerations from (V, ????), we have our functor defined over $S = \mathrm{Spec}(k)$ and it becones representable over $S' = \mathrm{Spec}(\bar{k})$. We have a descend datum, which is effective since we know that the components of our representing scheme are projective (See V, ??? and ???).

Actually we could also apply some less abstract arguments if we remember that we can also represent the functor already over a finite separable extension (See 1.5.7.) Then we can apply the arguments of Galois descend in Chap. ???.

Now we come to the functor $\mathrm{Pic}^r(C)$. We cannot apply the previous arguments since the functor is not representable. The situation is more delicate.

But I claim that we still have a canonical descend datum for the scheme $\mathrm{Pic}^r(C)$. In the following discussion I assume that we found a separable extension such that we have local representability by a scheme X' and a line bundle $\mathcal{P}_r$ on $(C \times L) \times X'$.

To get the descend datum, we consider elements $\sigma \in \mathrm{Gal}(L/k)$ and form the scheme $C \times L \times (X')^\sigma$ and the line bundle $\mathcal{P}_r^\sigma$ on it. The universality property tells us that we can find a unique

$$f_\sigma : X' \longrightarrow (X')^\sigma$$

such that $(\mathrm{Id} \times f_\sigma)^*(\mathcal{P}_r^\sigma)$ is locally in X' isomorphic to $\mathcal{P}_r$. Because these morphisms are always unique we can conclude that they have to satisfy the cocycle condition

$$f_{\sigma\tau} = f_\tau^\sigma \circ f_\sigma,$$

and therefore find that we can construct a projective scheme X/k together with an isomorphism

$$h : X \times_k L \longrightarrow X'$$

such that $f_\sigma = h^\sigma \circ h^{-1}$ (see ...). This scheme together with h is unique up to a canonical isomorphism.

But we want to do better, we want a line bundle $\mathcal{P}_r$ on $C \times_k X$ such that $(C \times X, \mathcal{P}_r)$ is representing our funcor locally. Here we have a problem because there is no way to get a descend datum which includes also the bundles. This is really not so good because now we cannot characterize X/k in terms of the functor.

We have such a bundle if $r = g$. In this case we have a bundle $\mathcal{L}$ on $C \times C^g/\Sigma_g$, and we have shown that over the extension L we have the birational morphism

$$\pi' : C^g/\Sigma_g \times_k L \longrightarrow X'$$

such that $\mathcal{L}_L$ is isomorphic to the bundle $(\mathrm{Id} \times \pi')^*(\mathcal{P}'_g)$. We have the canonical descend of X'/L to a scheme X/k together with $h : X \times_k L \to X'$. It is clear that we have $\pi : C^g/\Sigma_g \to X$ sucht that

$$\pi' = h \circ \pi.$$

Now we construct the bundle $\mathcal{P}_g$ on $C \times X$ in the same way as before (see ...). It is the unique bundle on $C \times X$ such that $(\mathrm{Id} \times \pi)^*(\mathcal{P}_g) \simeq \mathcal{L}$. I claim that this bundle has the universal property. If we extend k to L, then N will be isomorphic to $\mathcal{P}'_g$ on $C_L \times X'$.

But now assume we have a scheme T/k of finite type and a bundle $\mathcal{L}$ on $C \times_k T$ such that for any point $t \in T$ the bundle $\mathcal{L}|C \times \{t\}$ has degree g. We pass to the extension L/k, and we know that we have a unique

$$\psi' : X' \longrightarrow T \times_k L$$

such that $(\mathrm{Id}\times\psi)^*(\mathcal{P}'_g)$ is locally in T isomorphic to the extension of $\mathcal{L}$ to $C \times X \times L \simeq C_L \times X'$ (via the maps h). It is clear that ψ' is obtained from a $\psi : X \to T$, i. e. $\psi' = h \circ \psi \times_k L$. Then we are in the situation that the bundles $(\mathrm{Id}\times\psi)^*(\mathcal{P}_g)$ and $\mathcal{L}$ became locally isomorphic in T after we extend the ground field from k to L. But then they must be already locally isomorphic in T on $C \times X$.

This argument finishes the proof of the theorem ... which says that Pic^g is locally representable by a projective scheme.

Of course we would like to have the same assertion for all r. This is of course easy if our curve has a rational point. If not then this is not possible. To see this we simply consider a curve C/k of genus one, which does not have a rational point. Then we have the line bundle $\mathcal{P}_1$ which is given by the diagonal $\mathcal{O}_{C\times C}(-\Delta)$ on the product. But it is not difficult to see, that we can not have a line bundle $\tilde{\Theta} = \mathcal{P}_0$ on $C \times C$.

2.5. The ℓ-adic modules $T_\ell(J)$ and the ring of endomorphisms $\mathrm{End}(J)$.

2.5.1 We start from a smooth projective curve C/k which has a point $P \in C(k)$ and we consider its jacobian J. On the product $C \times J$ we have the universal bundle $\mathcal{P}_0$. We restrict $\mathcal{P}_0$ to $\{P\} \times J$ and the resulting line bundle will now be called Θ it is the socalled Theta divisor.

For our Jacobian J we consider the endomorphism ring $\mathrm{End}(J)$. The understanding of the structure of this endomorphism ring is fundamental for the entire field of algebraic geometry.

I recomend to compare the following consideration to the analogous reasoning in the transcendental case. (Chap. RS, 6.2.)

Any endomorphism $\varphi : J \to J$ induces an endomorphism

$$\tilde{\varphi}^* : \mathrm{Pic}(J)(k) \longrightarrow \mathrm{Pic}(J)(k),$$

and we have to understand the function $\varphi \to \tilde{\varphi}^*$.

First of all it is clear that we can restrict $\tilde{\varphi}^*$ to the geometric points of the abelian variety $\mathrm{Pic}^{00}(J) = J^\vee$, this restriction is denoted by

$$\varphi^\vee : J^\vee(k) \longrightarrow J^\vee(k).$$

We will show that - as in the transcendental case - this restriction is given by a endomorphism of $J^\vee = \mathrm{Pic}^{00}(C)$ and it is linear, i.e. we will see

$$(\varphi + \psi)^\vee = \varphi^\vee + \psi^\vee.$$

In analogy to our discussion in the complex analytic case we define

$$NS(J) = \mathrm{Pic}(J)/\mathrm{Pic}^{00}(J),$$

and $\tilde{\varphi}^*$ induces an endomorpism

$$\varphi^* : NS(J) \longrightarrow NS(J).$$

In the case of Riemann surfaces the group $\mathrm{End}(J)$ could be considered as a subgroup of the endomorphisms of the first homology group (Chap. RS 6.2.1.) but these homology groups are (not yet) available, hence we have to look at this object as it is and get our insights from elsewhere.

We also should have the idea that $NS(J)$ is related to the second cohomology, which in turn is related to the tensor product of the first cohomology groups (Chap RS , 6.3 ??). Having the idea that φ induces endomorphism in the cohomology in degree 1 and these should depend linearly on φ, we expect that $\varphi \to \tilde{\varphi}^*$ is a quadratic function. This means that

$$\widetilde{(\varphi + \psi)}^* - \tilde{\varphi}^* - \tilde{\psi}^* = \langle \varphi, \psi \rangle$$

where $\langle \varphi, \psi \rangle \in \mathrm{End}(J)$ and depends bilinearily on φ, ψ.

2.5.2. We want to understand why $\varphi^\vee$ is a morphism. The key is the representability of $\mathrm{Pic}^{00}(J)$. We have the divisor Θ on J and the three morphisms

$$J \times J \begin{array}{c} \xrightarrow{p_1} \\ \xrightarrow{m} \\ \xrightarrow{p_2} \end{array} J.$$

We have shown in the previous section that the bundle

$$\tilde{\Theta} = m^*(\Theta) \otimes p_1^*(\Theta)^{-1} \otimes p_2^*(\Theta)^{-1}$$

is universal:

The functor $\mathrm{Pic}^{00}(J)$ *is locally represented by the line bundle* $\tilde{\Theta}$ *on* $J \times J$

Especially this implies that on the set of geometric points the homomorphism

$$\varphi_\Theta : x \mapsto T_x^*(\Theta) \otimes \Theta^{-1}$$

yields an isomorphism $J(k) \to \mathrm{Pic}^{00}(J)(k)$.

If now $\varphi : J \to J$ is an endomorphism of J then $\varphi^\vee$ should be an endomorphism of $\mathrm{Pic}^{00}(J)$. Since this is given by J our $\varphi^\vee$ must be an endomorphism of J itself and now we explain how we get this endomorphism:

The endomorphism φ induces an endomorphism $(\varphi \times \mathrm{Id}) : J \times J \to J \times J$ and we can consider the pull back $(\varphi \times \mathrm{Id})^*(\tilde{\Theta})$ on $J \times J$. This means that we can find an endomorphism ${}^t\varphi : J \to J$ such that

$$(\varphi \times \mathrm{Id})^*(\tilde{\Theta}) \xrightarrow{\sim} (\mathrm{Id} \times {}^t\varphi)(\tilde{\Theta}) \otimes \mathcal{M}$$

where $\mathcal{M}$ is the pullback of a line bundle on the second factor. But then we can evalute at $e \times J$ and find that $\mathcal{M}$ must be trivial. This gives us the defining formula for ${}^t\varphi$

$$(\varphi \times \mathrm{Id})^*(\tilde{\Theta}) \xrightarrow{\sim} (\mathrm{Id} \times {}^t\varphi)(\tilde{\Theta}) \tag{2.5.2.1}$$

On the geometric points this specializes to

$$\varphi^\vee(T_x^*(\Theta)\otimes\Theta^{-1})\tilde{\to}T^*_{t_{\varphi(x)}}(\Theta)\otimes\Theta^{-1}$$

Theorem. 2.5.2.2 *Let $\varphi,\psi,\eta\in\mathrm{End}(J)$ and let $\mathcal{L}$ be a line bundle on J. Then the bundle*

$$\begin{aligned}(\varphi+\psi+\eta)^*(\mathcal{L})\otimes(\varphi+\psi)^*(\mathcal{L})^{-1}\otimes(\varphi+\eta)^*(\mathcal{L})^{-1}\otimes(\psi+\eta)(\mathcal{L})^{-1}\otimes\\ \otimes\varphi^*(\mathcal{L})\otimes\psi^*(\mathcal{L})\otimes\eta^*(\mathcal{L})\otimes(O^*\mathcal{L})^{-1}\end{aligned}$$

is trivial (here $0:J\to J$ is the zero homomorphism).

Proof. This is again a consequence of the theorem of the cube. We consider the threefold product $J\times J\times J$, and we consider the following 8 homomorphisms from $J\times J\times J$ to J

$m_{123}:J\times J\times J\to J$ sum of all components

$m_{23}\circ p_1, m_{13}\circ p_2, m_{12}\circ p_3$

p_{12},p_{13},p_{23}

and

0

We consider the bundle

$$\begin{aligned}\mathcal{N}_0 =&m_{123}^*(\mathcal{L})\otimes(m_{23}\circ p_1)^*(\mathcal{L})^{-1}\otimes(m_{12}\otimes p_3)^*(\mathcal{L})^{-1}\otimes(m_{13}\circ p_2)^*(\mathcal{L})^{-1}\otimes\\ &\otimes p_{12}^*(\mathcal{L})\otimes p_{13}^*(\mathcal{L})\otimes p_{23}^*(\mathcal{L})\otimes(0^*\mathcal{L})^{-1}\end{aligned}$$

If we restrict this bundle to one of the subvarieties

$$e\times J\times J \quad , \quad J\times e\times J \quad , \quad J\times J\times e \quad ,$$

then in any case two of the 8 maps become equal and occur with opposite signs in the product. Hence the restriction becomes trivial.

Then the theorem of the cube tells us that the bundle is trivial.

To get our theorem we use φ,ψ,η to map

$$(\varphi,\psi,\eta):J\longrightarrow J\times J\times J,$$

and the bundle in question is the pullback of $\mathcal{N}_0$, hence trivial.

If our above bundle $\mathcal{L}$ is in $\mathrm{Pic}^{00}(J)=J^\vee$, then we need only two factors. In this case we know by definition that we can find a connected scheme of finite type and a line bundle $\tilde{\mathcal{L}}$ on $J\times T$ such that

$$\begin{aligned}\tilde{\mathcal{L}}\mid J\times\{t_0\} &\simeq \mathcal{O}_J\\ \tilde{\mathcal{L}}\mid J\times\{t_1\} &\simeq \mathcal{L}\end{aligned}$$

for suitable points $t_0, t_1 \in T(k)$. Now we look at the four maps

$$J \times J \times T \longrightarrow J \times T$$

given by summation in the first two summands, the two projections to the first two summands and finally the zero map. We consider

$$m_{12}^*(\tilde{\mathcal{L}}) \otimes p_1^*(\tilde{\mathcal{L}})^{-1} \otimes p_2^*(\tilde{\mathcal{L}})^{-1} \otimes 0^*(\mathcal{L}).$$

Again we find that this bundle is trivial on

$$\{e\} \times J \times T\ ,\ J \times \{e\} \times T \quad \text{and} \quad J \times J \times \{t_0\},$$

and again by the theorem of the cube we get that the bundle is trivial on $J \times J \times \{t_1\}$.

If we have a pair of endomorphisms φ, ψ

$$(\varphi, \psi) = (\varphi \times \psi) \circ \Delta : J \longrightarrow J \times J$$

then then composition with the multiplication gives $\varphi + \psi$ and the composition with the projections gives φ and ψ. Hence

$$(\varphi + \psi)^*(\mathcal{L}) = \varphi^*(\mathcal{L}) \otimes \psi^*(\mathcal{L})$$

and this is the linearity

$${}^t(\varphi + \psi) = {}^t\varphi + {}^t\psi.$$

For any pair of endomorphisms $\varphi, \psi \in \mathrm{End}(J)$ we can look at the difference

$$(\varphi + \psi)^* - \varphi^* - \psi^* = \langle \varphi, \psi \rangle$$

where of course $\langle \varphi, \psi \rangle$ is again an endomorphism of $\mathrm{Pic}(J)$.

Perhaps this is a good place to summarize the properties of $\varphi \to \varphi^*$.

2.5.2.3. i) If we have two endomorphism φ, ψ then

$$(\varphi \circ \psi)^* = \psi^* \circ \varphi^*.$$

ii) The pairing

$$\begin{array}{ccc} \mathrm{End}(J) \times \mathrm{End}(J) & \longrightarrow & \mathrm{End}(\mathrm{Pic}(J)) \\ (\varphi, \psi) & \longrightarrow & \langle \varphi, \psi \rangle \end{array}$$

is bilinear in both variables.

iii) The endomorphism $< \varphi, \psi >$ is trivial on $\mathrm{Pic}^{00}(J)$

Proof: The assertion i) is obvious, iii) is the linearity of $\varphi \rightarrow^t \varphi$ it remains to prove ii). Let us replace φ by $\varphi_1 + \varphi_2$ and apply our formula above to the sum $\varphi_1 + \varphi_2 + \psi$. Then we see

$$(\varphi_1 + \varphi_2 + \psi)^*(\mathcal{L}) \otimes (\varphi_1 + \varphi_2)^*(\mathcal{L})^{-1} \otimes \psi^*(\mathcal{L})^{-1} =$$
$$(\varphi_1 + \psi)^*(\mathcal{L}) \otimes \varphi_1^*(\mathcal{L})^{-1} \otimes \psi(\mathcal{L})^{-1} \otimes (\varphi_2 + \psi)^*(\mathcal{L}) \otimes \varphi_2^*(\mathcal{L})^{-1} \otimes \psi(\mathcal{L})^{-1}$$

the term on the left hand side is

$$\langle \varphi_1 + \varphi_2, \psi \rangle(\mathcal{L}),$$

and on the right hand side we get

$$\langle \varphi_1, \psi \rangle(\mathcal{L}) \otimes \langle \varphi_2, \psi \rangle(\mathcal{L}).$$

Since the expression $\langle \varphi, \psi \rangle$ is symmetric, the rest is clear.

The map $\varphi \rightarrow^t \varphi$ from $\mathrm{End}(J)$ into itself is called the *Rosati involution* . This involution depends on the choice of the line bundle Θ. In principle we could choose other ample bundles on J which then would define other involutions (See [Mu], ???)

2.5.3. For an endomorphism $\varphi : J \to J$ we can define the kernel $\ker(\varphi) = \varphi^{-1}(0)$. By definition this is a subgroup scheme (See ???). If this kernel is a fintite group scheme then it is a finte group scheme over k and in this case we define the degree of φ as

$$\deg(\varphi) = \mathrm{rank}(\varphi^{-1}(0))$$

If the kernel is not finite then we put $\deg(\varphi) = 0$.

An endomorphism $\varphi : J \to J$ with finite kernel is called an *isogeny* of J. The following is quite clear:

If $\varphi : J \to J$ is an isogeny then the morphism φ is finite and locally free of rank $\deg(\varphi)$. *For any point $y \in J$ the fibre $\varphi^{-1}(y)$ is finte of rank* $\deg(\varphi)$ *over $k(y)$.*

We apply our formula for the degree: Let $\mathcal{L}$ be an ample bundle on J then

$$\deg(\varphi)\mathcal{L}^g = (\varphi^*(\mathcal{L}))^g$$

We want to compute $(n\,\mathrm{Id})^*$. It is clear how to do this, we have

$$(n\,\mathrm{Id})^* = ((n-1)\,\mathrm{Id} + \mathrm{Id})^* = ((n-1)\,\mathrm{Id})^* + \mathrm{Id}^* + (n-1)\langle \mathrm{Id}, \mathrm{Id} \rangle$$

or since $\mathrm{Id}^* = \mathrm{Id}_{\mathrm{Pic}(J)} = \mathrm{Id}$

$$(n\,\mathrm{Id})^* - ((n-1)\,\mathrm{Id})^* = \mathrm{Id} + (n-1)\langle \mathrm{Id}, \mathrm{Id} \rangle.$$

Since $(\mathcal{O}\cdot \mathrm{Id})^* = 0$ we get

$$(n\,\mathrm{Id})^* = n\,\mathrm{Id} + \frac{n(n-1)}{2}\langle \mathrm{Id}, \mathrm{Id}\rangle.$$

We can also say something about $\langle \mathrm{Id}, \mathrm{Id}\rangle$. We have

$$\mathrm{Id} + (-\mathrm{Id}) = 0$$

as endomorphism on J. Then

$$0 = (\mathrm{Id} + (-\mathrm{Id}))^* = \mathrm{Id} + (-\mathrm{Id})^* - \langle \mathrm{Id}, \mathrm{Id}\rangle,$$

and hence we see

$$\langle \mathrm{Id}, \mathrm{Id}\rangle = \mathrm{Id}^* + (-\mathrm{Id})^*.$$

We want to evaluate this formula on a line bundle $\mathcal{L}$. We put $\mathcal{L}_- = (-\mathrm{Id})^*(\mathcal{L})$and our formula yields

$$(n\,\mathrm{Id})^*(\mathcal{L}) \tilde{\to} \mathcal{L}^{\otimes n} \otimes \mathcal{L}^{\otimes \frac{n(n-1}{2}} \otimes \mathcal{L}_-^{\otimes \frac{n(n-1}{2}}$$

This allows us to compute the degree of $n\,\mathrm{Id}$. As we explained. we may take any ample line bundle $\mathcal{L}$ on J, we may even assume it to be very ample. We may replace $\mathcal{L}$ by $\mathcal{L} \otimes (-\mathrm{Id})^*(\mathcal{L})$, since $-\mathrm{Id}$ is an automorphism, it is clear that $(-\mathrm{Id})^*(\mathcal{L})$ is very ample and hence the tensor product is so too.

Then we have the formula

$$\mathcal{L}^g \deg(n\,\mathrm{Id}) = (n\,\mathrm{Id})^*(\mathcal{L})^g$$

and $(n\,\mathrm{Id})^*(\mathcal{L}) = \mathcal{L}^n \otimes \mathcal{L}^{\frac{n^2-n}{2}} \otimes \mathcal{L}^{\frac{n^2-n}{2}}$ (because $(-\mathrm{Id})^*(\mathcal{L}) = \mathcal{L}$) and hence

$$(n\,\mathrm{Id})^*(\mathcal{L}) = \mathcal{L}^{\otimes n^2}.$$

And now $(\mathcal{L}^{\otimes n^2})^g = \mathcal{L}^g \cdot n^{2g}$ and it follows

$$\deg(n\,\mathrm{Id}) = n^{2g}.$$

Since we have seen that the group low $J \times J \to J$ induces the addition on the tangent space $T_{J,e}$ (see ???) we conclude that the multiplication by n on the tangent space. We conclude:

The kernel of the multiplication by n is a finite group scheme

$$J[n] \longrightarrow \mathrm{Spec}(k)$$

of rank n^{2h}. If the characteristic p of k does not divide n, then this group scheme is etale. In this case

$$J[n](\overline{k}) \simeq (\mathbb{Z}/n\mathbb{Z})^{2g}.$$

We consider the function

$$n \longrightarrow \deg(\psi + n\,\mathrm{Id}),$$

we know how to express it in terms of intersection number. We choose an ample line bundle $\mathcal{L}$ on J and then we have

$$\deg(\psi + n\,\mathrm{Id}) \cdot \mathcal{L}^g = ((\psi + n\,\mathrm{Id})^*(\mathcal{L}))^g.$$

We can expand the right hand side and find

$$\deg(\psi + n\,\mathrm{Id}) \cdot \mathcal{L}^g = \cdots n^{2g-1}\mathcal{L}^{g-1} \cdot \langle \psi, \mathrm{Id}\rangle(\mathcal{L}) + n^{2g} \cdot \mathcal{L}^g.$$

This expression looks pretty much like a characreristic polynomial of an endomorphism and in the next section we will see that this is indeed the case. In view of this expectation we define

$$\mathrm{tr}(\psi) = \frac{\mathcal{L}^{g-1}\langle \psi, \mathrm{Id}\rangle(\mathcal{L})}{\mathcal{L}^g}.$$

In any case it is clear that $\deg(\psi + n\,\mathrm{Id})$ is a polynomial in n of degree $2g$ with rational coefficients and this polynomial takes integer values.

If we have two endomorphisms φ, ψ then we can look at the diagram

$$J \xrightarrow{\varphi\times\psi\circ\Delta} J \times J \begin{array}{c} \xrightarrow{p_1} \\ \xrightarrow{m} \\ \xrightarrow{p_2} \end{array} J$$

and for any line bundle $\mathcal{L}$ on J we have the formula

$$< \psi, \varphi > (\mathcal{L}) \tilde{\to} m^*(\mathcal{L}) \otimes p_1^*(\mathcal{L})^{-1} \otimes p_2^*(\mathcal{L})^{-1}$$

by definition. If we apply this to Θ we get

$$< \psi, \varphi > (\Theta) \tilde{\to} ((\varphi \times \psi) \circ \Delta))^*(\tilde{\Theta}) = \Delta^* \circ (\varphi \times \psi)^*(\tilde{\Theta})$$

We have the defining relation for the transpose (???) and get

$$< \psi, \varphi > (\Theta) \tilde{\to} \Delta^* \circ ({}^t\psi\varphi \times \mathrm{Id})^*(\tilde{\Theta}) \tilde{\to} \Delta^* \circ (\mathrm{Id} \times {}^t\varphi\psi)^*(\tilde{\Theta}) \tilde{\to} <^t \psi\varphi, \mathrm{Id} > (\Theta) \tilde{\to} <^t \varphi\psi, \mathrm{Id} > (\Theta)$$

If we take $\psi = \mathrm{Id}$ then we get

$$\mathrm{tr}({}^t\varphi) = \mathrm{tr}(\varphi).$$

Now we are ready for the famous

Theorem : Positivity of the Rosati involution

The bilinear form

$$(\varphi, \psi) \mapsto \operatorname{tr}({}^t\varphi\psi)$$

is a symmetric positive definite form

Proof: We have to prove the positivity. From the formula just above the statement of the theorem we get

$$< \psi, \psi > (\Theta) \tilde{\to} <^t \varphi\varphi, \mathrm{Id} > (\Theta).$$

We multiply by Θ^{g-1} and observing the definition of the trace we get

$$\Theta^{g-1} < \psi, \psi > (\Theta) = \Theta^{g-1} <^t \varphi\varphi, \mathrm{Id} > (\Theta) = \operatorname{tr}({}^t\varphi\varphi)\Theta^g$$

But obviously $< \psi, \psi > (\Theta) \tilde{\to} \psi^*(\Theta)^{\otimes 2}$ and hence we find

$$\Theta^g \operatorname{tr}({}^t\psi\psi) = 2\Theta^{g-1}\psi^*(\Theta)$$

Since Θ is ample this expression is strictly positive if $\psi \neq 0$.(na ja).

The ℓ-adic modules

Now we can pick a prime ℓ which is different from the characteristic of k, and we define

$$T_\ell(J) = \lim_{\substack{\rightarrow \\ \alpha}} J[\ell^\alpha]$$

as before.

The group of geometric points is

$$T_\ell(J)(\overline{k}) = \mathbb{Z}_\ell^{2g},$$

but we have to observe that the Galois group acts upon this module.

This is now the replacement for the cohomology groups which were available in the transcendental case. But these ℓ-adic cohomology groups have the defect that they are $\mathbb{Z}_\ell$-modules and not $\mathbb{Z}$-modules.

(It has been pointed out by J.-P. Serre that cohomology groups which are free $\mathbb{Z}$-modules of rank $2g$ cannot exist.)

Of course it is clear that we get a homomorphism

$$\mathrm{End}(J) \longrightarrow \mathrm{End}(T_\ell(J)),$$

and hence we can define a trace

$$\operatorname{tr}(\varphi) = \operatorname{tr}(\varphi_\ell \mid T_\ell(J))$$

and a determinant

$$\det(\varphi) = \det(\varphi_\ell \mid T_\ell).$$

The problem is that these numbers are ℓ-adic numbers and what we want are integers. But

Theorem: *The number* $\operatorname{tr}(\varphi_\ell \mid T_\ell)$ *and* $\det(\varphi_\ell \mid T_\ell)$ *are integers which do not depend on* ℓ. *We have*

$$\det(\varphi_\ell \mid T_\ell) = \deg(\varphi).$$

Before we can prove this theorem we have to prove the following fundamental theorem:

Theorem: *The* $\mathbb{Z}$*-module* $\operatorname{End}(J)$ *is finitely generated and for any prime* ℓ *the natural map*

$$\operatorname{End}(J) \otimes \mathbb{Z}_\ell \longrightarrow \operatorname{End}(T_\ell(J))$$

is an inclusion.

Proof: Let $M \subset \operatorname{End}(J)$ be any finitely generated submodule which is stable under the involution. The trace defines an integer valued pairing

$$\begin{array}{rcl} \operatorname{tr} & : & M \times M \longrightarrow \mathbb{Z} \\ \operatorname{tr} & : & \langle \varphi_1, \varphi_2 \rangle \longrightarrow \operatorname{tr}(\varphi_1 \, {}^t\varphi_2) \end{array}$$

which is positive definite. We see that this bilinear pairing is non degenerate over $\mathbb{Q}$, i.e. if we take a basis $\varphi_1 \cdots \varphi_k$ of the $\mathbb{Z}$-module M we can conclude that

$$\det(\operatorname{tr}(\varphi_i \, {}^t\varphi_j))$$

is a non zero integer.

Let us assume that $M \otimes \mathbb{Z}_\ell$ does not embed into $\operatorname{End}(T_\ell(J))$, this means that we can find $\alpha_1, \cdots, \alpha_k \in \mathbb{Z}_\ell$ which are not all congruent zero mod ℓ such that the linear combination $\sum_{i=1}^k \alpha_i \varphi_i$ is zero. If we approximate the α_i by integers n_i such that $\alpha_i \equiv n_i \bmod \ell^\alpha$, then the element

$$\psi = \sum n_i \varphi_i \in \operatorname{End}(J)$$

is zero on the group of ℓ^α division points. this implies that $\psi = \ell^\alpha \psi'$ where $\psi' \in \operatorname{End}(J)$.

(We have the diagram

$$\begin{array}{ccc} J & \xrightarrow{\ell^\alpha} & J \\ & \psi \searrow & \\ & & J \end{array}$$

and since ψ is zero on the kernel of ℓ^α, we can see easily that we can find a ψ' completing the diagram.)

We get a system of linear equations for the n_i:

$$\sum n_i \ \operatorname{tr}(\varphi_i \, {}^t\varphi_j) = \ell^\alpha \cdot \operatorname{tr}(\psi' \, {}^t\varphi_j)$$

where the $\operatorname{tr}(\psi' \, {}^t\varphi_j)$ are integers. We solve this system for the n_i using Cramers rule and find

$$n_i = \ell^\alpha \ \frac{A_i}{\det(\operatorname{tr}(\varphi_i \, {}^t\varphi_j))}$$

where the A_i are integers. The maximal power of ℓ dividing the denominator does not depend on α. Hence we get a contradiction to the assumption that not all of the n_i are divisible by ℓ. For this we conclude that $\operatorname{End}(J)$ is a finitely generated $\mathbb{Z}$-module.

The homomorphism $\operatorname{End}(J) \otimes \mathbb{Z}_\ell \to \operatorname{End}(T_\ell(J))$ is called the ℓ-adic representation.

Of course it follows that $\operatorname{End}(J)_\mathbb{Q}$ is a finite dimensional $\mathbb{Q}$-algebra. The positivity of the trace implies

The algebra $\operatorname{End}(J)_\mathbb{Q}$ *is semisimple.*

To see this, we consider the radical $\mathfrak{a}$ which is the maximal two-sided ideal consisting of nilpotent elements. Clearly, $\mathfrak{a}$ is stable under the Rosati involution, and hence we see: If $\mathfrak{a} \neq 0$, then we find non zero $\varphi \in \mathfrak{a}$ with ${}^t\varphi = \pm\varphi$. But then $\varphi^2 \neq 0$ since $\operatorname{tr}(\varphi^2) = \pm \operatorname{tr}(\varphi \, {}^t\varphi)$, and this is non zero by the positivity of the Rosati involution. But then φ cannot be nilpotent, we have a contradiction to $\mathfrak{a} \neq 0$.

The structure theory of algebras implies that we can find a set of central elements $e_1 \cdots e_r \in \operatorname{End}(J)_\mathbb{Q}$, such that

$$\begin{aligned} 1 &= \sum e_i \\ e_i e_j &= \begin{cases} e_i & \text{if } i = j \\ 0 & \text{else} \end{cases} \end{aligned}$$

and such that

$$\operatorname{End}(J)_\mathbb{Q} = \bigoplus \operatorname{End}(J)_\mathbb{Q} e_i = \bigoplus_{i=1}^r A_i$$

where the A_i are fields.

The e_i are not endomorphisms of J but, if we multiply them by an integer $d_i \neq 0$, then

$$E_i = d_i \ e_i$$

will be an endomorphism of J. Now we can consider the endomorphisms

$$\Phi_i = \sum_{j \neq i} \ E_j$$

and let $\tilde{J}_i$ be the connected component of the kernel of Φ_i. Then we have of course that $\tilde{J}_i$ is an abelian subvariety of J and it is clear that the map

$$\prod \tilde{J}_i \longrightarrow J$$

is surjective with finite kernel, in other words, this map is an isogeny. The algebra $A_{i,\mathbb{Z}} = A_i \cap \mathrm{End}(J)$ acts trivially on the $\tilde{J}_j$ with $j \neq i$ and it injects into $\mathrm{End}(\tilde{J}_i)$ and is clearly of finite index in this ring of endomorphisms. It is clear that the Rosati involution is the identity on the e_i and hence it induces involutions on the A_i and on $\mathrm{End}(\tilde{J}_i)$.

Theorem: *For any $\varphi \in \mathrm{End}(J)$ the numbers $\mathrm{tr}(\varphi_\ell(T_\ell(J))$ and $\det(\varphi_\ell)$ are independent of ℓ and we have*

$$\begin{aligned} \det(\varphi_\ell) &= \deg(\varphi) \\ \mathrm{tr}(\varphi_\ell) &= \mathrm{tr}(\varphi). \end{aligned}$$

Proof: We have the decomposition

$$\mathrm{End}(J)_{\mathbb{Q}} = \bigoplus A_i = \bigoplus \mathrm{End}(J)_{\mathbb{Q}} e_i,$$

and it is clear that $\mathrm{End}(\tilde{J}_i)_{\mathbb{Q}} = A_i$. In other words, for any $\varphi \in \mathrm{End}(J)_{\mathbb{Q}}$ we find a non zero integer m such that

$$m\varphi = \varphi_1 + \cdots + \varphi_r$$

where $\varphi_i \in \mathrm{End}(\tilde{J}_i)$. This means that we have a diagram

$$\begin{array}{ccc} \prod \tilde{J}_i & \longrightarrow & J \\ \downarrow \underline{\Psi} & & \downarrow m\varphi \\ \prod \tilde{J}_i & \longrightarrow & J \end{array}$$

where $\underline{\Psi} = (\cdots \Psi_i \cdots)$ and $\Psi_i \in \mathrm{End}(\tilde{J}_i)$. This implies that $\deg(m\varphi) = \prod \deg(\Psi_i)$. If we choose $\mathbb{Q}$-independent generators φ_{ij} of the $\mathrm{End}(\tilde{J}_i)$ and consider

$$\varphi = \sum_{i,j} m_{ij}\ \varphi_{ij}$$

where $\sum_j m_{ij}\varphi_{ij} = \Psi_i \in \mathrm{End}(\tilde{J}_i)$, then we know that the function

$$\varphi \longrightarrow \deg(\varphi) = \deg(\sum m_{ij}\varphi_{ij})$$

is a homogeneous polynomial of degree $2g$ in the m_{ij}.

It is also clear (???) that the individual factor $\deg(\sum_j m_{ij}\varphi_{ij})$ are homogeneous polynomials in degree $2\dim \tilde{J}_i$. This reduces our problem to the consideration of one of these factors.

We want to show that we have the formula

$$\deg(\varphi) = \det(\varphi_\ell \mid T_\ell(J)).$$

To prove this formula we look at integral linear combinations of endomorphisms and use the observation that

$$\deg(n_1\varphi_1 + \cdots + n_r\varphi_r)$$

and

$$\det(n_1\varphi_{1,\ell} + \cdots n_r\varphi_{r,\ell} \mid T_\ell(J))$$

are both homogeneous polynomials of degree $2g$ in the variable $n_1 \cdots n_r$.

On the other hand we have just seen that our algebra

$$\mathrm{End}(J) \otimes \mathbb{Q} = \bigoplus_{i=1}^{t} M_n(D_i)$$

where the D_i are division algebras over their centre $F_i/\mathbb{Q}$.

It is well known that we have homomorphisms

$$N_i : M_n(D_i)^* \longrightarrow \mathbb{Q}^*$$

which are compositions of the reduced norm

$$\text{Norm red} : M_n(D_i)^* \longrightarrow F_i^*$$

and the norm

$$N_{F_i/\mathbb{Q}} : F_i^* \longrightarrow \mathbb{Q}^*$$

which have the property that

$$N_i(n_1\alpha_1 + \cdots n_r\alpha_r)$$

is homogeneous of degree $nd_i \cdot [F_i : \mathbb{Q}] = f_i$ where $d_i = \sqrt{[D_i : F_i]}$. Moreover we know that any homomorphism

$$N_i' : M_n(D_i)^* \longrightarrow \mathbb{Q}^*$$

which has the homogeneity property with some degree m_i is of the form

$$N_i' = N_i^{m_i/f_i}$$

i.e. we must have $f_i \mid m_i$.

Hence we can conclude that

$$\deg : (\mathrm{End}(J) \otimes \mathbb{Q})^* \longrightarrow \mathbb{Q}^*$$

is of the form

$$\det = \prod_{i=1}^{t} N_i^{g_i}$$

where $\sum f_i g_i = 2g$.

We apply the same consideration to

$$\det : (\mathrm{End}(J) \otimes \mathbb{Q}_\ell)^* \longrightarrow \mathbb{Q}_\ell^*.$$

We have to observe that now it can happen that some of our field F_i may decompose

$$F_i \otimes \mathbb{Q}_\ell = \bigoplus_{\mu=1}^{a_i} D_{i\mu}$$

and

$$M_n(D_i) \otimes \mathbb{Q}_\ell = \bigoplus_{\mu=1}^{a_i} M_{n_{i\mu}}(D_{i\mu})$$

and we conclude

$$\det = \prod_{i=1}^{t} N_{i\mu}^{g_{i\mu}}$$

where now

$$N_{i\mu} : M_{n_{i\mu}}(D_{i\mu})^* \longrightarrow \mathbb{Q}_\ell^*.$$

We need to show that the $g_{i\mu}$ do not depend on μ, i.e. $g_{i\mu} = g_i'$ for all $\mu = 1 \cdots a_i$ and that $g_i' = g_i$ for all i.

This is now easy (na ja!) if we use an approximation argument and the fact that

$$|\deg \varphi|_\ell = |\det(\varphi_\ell)|_\ell$$

which is obvious from the definition of the degree.

The Weil Pairing

We consider a Jacobian J/k where we assume that k is algebraically closed. For any integer $n > 0$ which is not divisible by the characteristic p of our ground field we consider the kernel

$$J[n] = \ker(J \xrightarrow{n} J).$$

We have seen that $n = n\,\mathrm{Id}$ is an étale morphism, the kernel is a finite étale group scheme and

$$J[n] \simeq (\mathbb{Z}/n\mathbb{Z})^{2g}.$$

We can also say that $J[n]$ is the Galois group of the covering $J \xrightarrow{n} J$, this means that any $a \in J[n] = J[n](k)$ induces a translation

$$\begin{array}{ccc} J & & \\ & \searrow n & \\ \downarrow T_a & & J \\ & \nearrow n & \\ J & & \end{array}$$

and for any affine open subset $V \subset J$, the open set $(n\,\mathrm{Id})^{-1}(V) = V' \subset J$ is affine, and

$$\mathcal{O}_J(V) = \mathcal{O}_J(V')^{J[n]},$$

i.e. the algebra downstairs is the algebra of invariants under $J[n]$.

Our aim is the construction of a non degenerate alternating pairing

$$w_0 : J[n] \times J[n] \to \mu_n = \text{ group of n'th roots of unity.}$$

(Compare the discussion in the transcendental context ???)

If we pick a point $\xi \in J[n]$, then this gives us a line bundle

$$\mathcal{L}_\xi = \mathcal{N} \mid J \times \xi.$$

This line bundle is algebraically equivalent to zero and satisfies $\mathcal{L}_\xi^{\otimes n} = \mathcal{O}_J$.

We take the pull back of this line bundle under

$$n : J \longrightarrow J$$

and clearly we have $n^*(\mathcal{L}_\xi) = \mathcal{L}_\xi^{\otimes n} = \mathcal{O}$.

Hence we see that $n^*(\mathcal{L}_\xi)$ is trivial and we conclude that $H^0(J, n^*(\mathcal{L}_\xi))$ is a one dimensional k-vector space.

Since $n^*(\mathcal{L}_\xi)$ is a pull back of a bundle on J under n, we see that we have an action of $J[n]$ (the Galois group) on $n^*(\mathcal{L}_\xi)$ and hence an action of $J[n]$ on $H^0(J, n^*(\mathcal{L}_\xi))$ which then defines a character

$$\alpha_\xi : J[n] \longrightarrow \mu_n \subset k^*.$$

One thing is clear: This character is trivial if and only if $\mathcal{L}_\xi$ is trivial in other words if $\xi = 0$.

We put

$$w_0(\xi, \eta) = \alpha_\xi(\eta)$$

for $(\xi, \eta) \in J[n] \times J[n]$. It is linear in η by definition. But if $\xi = \xi_1 + \xi_2$, then we have

$$\mathcal{L}_\xi = \mathcal{L}_{\xi_1} \otimes \mathcal{L}_{\xi_2}$$

and this provides a non zero bilinear map

$$H^0(J, n^*(\mathcal{L}_{\xi_1})) \times H^0(J, n^*(\mathcal{L}_{\xi_2})) \to H^0(J, n^*(\mathcal{L}_\xi))$$

which commutes with the action of $J[n]$. Hence we see that

$$\alpha_{\xi_1+\xi_2} = \alpha_{\xi_1} \cdot \alpha_{\xi_2}$$

and this shows that w_0 is bilinear.

The most difficult step is to show that $w_0(\xi, \xi) = \alpha_\xi(\xi) = 1$. Once we have shown this, then it is clear that the map is alternating and non degenerate.

This last property follows from the symmetry of $\mathcal{N}$. Since we know that the adjoint of $n = n\,\mathrm{Id}$ is again n we have an isomorphism

$$\Phi : (n \times \mathrm{Id})^*(\mathcal{N}) \longrightarrow (\mathrm{Id} \times n)^*(\mathcal{N}).$$

We can restrict Φ to the diagonal, then it yields

$$\begin{array}{cccccc} \Phi_{x,x} & : & \mathcal{N}_{(nx,x)} & \longrightarrow & \mathcal{N}_{(x,nx)} \\ & & \| & & \| \\ & & (n \times \mathrm{Id})^*(\mathcal{N})_{(x,x)} & \longrightarrow & (\mathrm{Id}\, xn)^*(\mathcal{N})_{(x,x)} \end{array}$$

and we can compose this with

$$\Delta_{(x,nx)} : \mathcal{N}_{(x,nx)} \longrightarrow \mathcal{N}_{(nx,x)}$$

to get

$$(\Delta \circ \Phi)_{(x,x)} : \mathcal{N}_{(nx,x)} \longrightarrow \mathcal{N}_{(nx,x)},$$

this morphism is a number and because of our normalization this number is equal to one.

We normalize Φ such that

$$\Phi : (n \times \mathrm{Id})^*(\mathcal{N})_{(e,e)} = \mathcal{N}_{e,e} \longrightarrow (\mathrm{Id} \times n)^*(\mathcal{N})_{e,e} = \mathcal{N}_{e,e}$$

is the identity.

Now we consider two n-division points (ξ, η) and consider the isomorphism

$$\begin{array}{cccccc} \Phi_{\xi,\eta} & : & (n \times \mathrm{Id})^*(\mathcal{N})_{(\xi,\eta)} & \longrightarrow & (\mathrm{Id} \times n)^*(\mathcal{N}_{(\xi,\eta)} \\ & & \| & & \| \\ & & \mathcal{N}_{(e,\eta)} & \longrightarrow & \mathcal{N}_{(\xi,e)}. \end{array}$$

Since $\mathcal{N} \mid e \times J$ and $\mathcal{N} \mid J \times e$ are both trivial, we have

$$\begin{array}{rcl} H^0(J, \mathcal{N} \mid e \times J) & \simeq & k \\ H^0(J, \mathcal{N} \mid J \times e) & \simeq & k \end{array}$$

and this gives us an identification

$$\begin{array}{rcl} \mathcal{N}_{(e,y)} & = & \mathcal{N}_{(e,e)} \\ \mathcal{N}_{(x,e)} & = & \mathcal{N}_{(e,e)} \end{array}$$

and hence we get

$$\begin{array}{ccc} \mathcal{N}_{(e,\eta)} & \longrightarrow & \mathcal{N}_{(\xi,e)} \\ \| & & \| \\ \mathcal{N}_{(e,e)} & = & \mathcal{N}_{(e,e)}. \end{array}$$

The isomorphism in the top row gives us a number (the diagram is not necessarily commutative) and a little bit of thinking yields that this number is $w_0(\xi, \eta)$ or it's inverse (???)

The restriction to the diagonal gives the desired result.

I explained already that we always can define the subgroup $\mathrm{Pic}^{00}(X)$, and we can define the Neron-Severi groups as

$$NS(X) = \mathrm{Pic}(X)/\mathrm{Pic}^{00}(X).$$

If we consider $X = J$ where J is a Jacobian together with the bundle $\mathcal{P}$, then we constructed the homomorphism

$$\varphi_{\mathcal{P}} : J \longrightarrow J^{\vee} = \mathrm{Pic}^{00}(J)$$

which on the points is given by

$$x \longrightarrow T_x(\mathcal{P}) \otimes \mathcal{P}^{-1},$$

and we have shown that this is an isomorphism.

In fact we proved that for any line bundle $\mathcal{L}$ we have

$$\varphi_{\mathcal{L}} : J \longrightarrow J^{\vee} \xrightarrow{\varphi_{\mathcal{P}}^{-1}} J,$$

and hence we get a homomorphism

$$\Phi : NS(J) \longrightarrow \mathrm{End}(J).$$

In the transcendental context we proved that Φ provides an isomorphism between $NS(J)$ and the submodule of symmetric elements.

We want to investigate this map in the context of algebraic geometry, and we will show that:

The map is injective (modulo 2-torsion which actually does not exist), and the image lies in $\mathrm{End}_{\mathrm{sym}}(J)$ *and between* $2\,\mathrm{End}_{\mathrm{sym}}(J)$ *and* $\mathrm{End}_{\mathrm{sym}}(J)$*.*

To get this we also consider the Neron-Severi group $NS(J\times J)$. As in characteristic zero we have the pull backs

$$NS(J) \rightrightarrows NS(J\times J)$$

given by the two projections and we have the two homomorphisms

$$NS(J\times J) \rightrightarrows NS(J)$$

given by restriction to $\{e\}\times J$ and $J\times\{e\}$.

This allows us to write

$$NS(J\times J) = N(J) \oplus NS'(J\times J) \oplus NS(J).$$

Let $\mathcal{L}$ be a line bundle on $J\times J$ whose Chern class (the image of $\mathcal{L}$ in the Neron-Severi group) lies in the summand $NS'(J\times J)$, then we know that

$$\mathcal{L} \mid J\times e \quad \text{and} \quad \mathcal{L} \mid e\times J$$

lie in $\mathrm{Pic}^{00}(J)$. After tensorization by bundles of the form $p_i^*(\mathcal{M}_i)$ we may assume that these restrictions are even trivial.

Now we have seen that we can find a homomorphism

$$\psi : J \longrightarrow J$$

(Hier muß vielleicht früher was gesagt werden.) such that

$$\mathcal{L} \simeq (\mathrm{Id}\times\psi)^*(\mathcal{N}).$$

The map $\psi \mapsto (\mathrm{Id}\times\psi)^*(\mathcal{N})$ gives us a homomorphism (???)

$$\mathrm{End}(J) \longrightarrow NS'(J\times J)$$

which is surjective (as we just saw).

It is also injective: If $(\mathrm{Id}\times\psi)^*(\mathcal{N})$ is zero in $NS'(J\times J)$, then this means that

$$(\mathrm{Id}\times\psi)^*(\mathcal{N}) \simeq p_1^*(\mathcal{M}_1)\otimes p_2^*(\mathcal{M}_2).$$

Evaluation at $e \times J$ and $J \times e$ shows that $\mathcal{M}_1, \mathcal{M}_2$ must be trivial, hence

$$(\mathrm{Id} \times \psi)^*(\mathcal{N}) \simeq \mathcal{O}_{J \times J}.$$

But is $\psi \neq 0$, then

$$(\mathrm{Id} \times \psi)^*(\mathcal{N}) \mid J \times y = \mathcal{N} \mid J \times \psi(y)$$

which is not trivial.

Hence we proved

$$NS'(J \times J) \xrightarrow{\sim} \mathrm{End}(J)$$

a result which in the transcendental context was reduced to linear algebra in $\Lambda^\bullet \Gamma$.

Now we consider the diagonal embedding

$$\Delta : J \longrightarrow J \times J,$$

and we put

$$\mathcal{L}_\psi = \Delta^* \circ (\mathrm{Id} \times \psi)^*(\mathcal{N}),$$

this yields a homomorphism

$$\mathrm{End}(J) \longrightarrow NS'(J \times J) \longrightarrow NS(J).$$

We have the following formula

$$\Phi(\mathcal{L}_\psi) = \psi + {}^t\psi$$

(??? wird später eingefügt). If on the other hand $\mathcal{L}$ is a bundle on J, then we may consider

$$\begin{array}{ccc} & \xrightarrow{p_1} & \\ J \times J & \xrightarrow{m} & J \\ & \underset{p_2}{\longrightarrow} & \end{array}$$

and look at

$$\tilde{\mathcal{L}} = m^*(\mathcal{L}) \otimes p_1^*(\mathcal{L})^{-1} \otimes p_2^*(\mathcal{L})^{-1}.$$

Again it is clear that $\tilde{\mathcal{L}} \mid J \times e \simeq \mathcal{O}_J$ and $\hat{\mathcal{L}} \mid e \times J \simeq \mathcal{O}_J$, and hence we find a $\psi : J \to J$ such that

$$\tilde{\mathcal{L}} = (\mathrm{Id} \times \psi)^*(\mathcal{N}).$$

Restriction to the diagonal yields

$$\Delta^*(\tilde{\mathcal{L}}) = \mathcal{L}^{\otimes 2} = \mathcal{L}_\psi.$$

The shows that the homomorphism

$$NS'(J \times J) \simeq \mathrm{End}(J) \longrightarrow NS(J)$$

is surjective up to 2-torsion which means that the image contains $2NS(J)$.

Then it becomes clear that

$$NS(J) \longrightarrow \mathrm{End}_{\mathrm{sym}}(J) \subset \mathrm{End}(J)$$

because $\mathrm{End}(J)$ is torsion free and $2NS(J)$ is mapped into the symmetric tensors.

It is also clear that the kernel of this map is 2-torsion and hence it is injective because:

The group $NS(J)$ is torsion free.

We only have to exclude 2-torsion.

Let us consider a line bundle $\mathcal{L}$ on J for which the $2c(\mathcal{L}) = 0$, i.e.

$$\mathcal{L}^{\otimes 2} = \mathcal{M} \in \mathrm{Pic}^{00}(J).$$

We can write $\mathcal{M} = \mathcal{M}_1^{\otimes 2}$ with $\mathcal{M}_1 \in \mathrm{Pic}^{00}(J)$ (???) and hence we get

$$\left(\mathcal{L} \otimes \mathcal{M}_1^{-1}\right)^{\otimes 2} = \mathcal{O}_J$$

in other words we may assume that $c(\mathcal{L}) \neq 0$ but $\mathcal{L}^{\otimes 2} = \mathcal{O}_J$.

Etale Cohomology

In the previous sections I gave some indications that the modules $T_\ell(J)$ should be considered a substitute for the first homology. If our ground field is $\mathbb{C}$, and if

$$J(\mathbb{C}) = \mathbb{C}^g/\Gamma,$$

then $\Gamma = H_1(J(\mathbb{C}), \mathbb{Z})$ and

$$\Gamma \otimes \mathbb{Z}_\ell \simeq T_\ell(J).$$

If we are over an arbitrary ground field k, and if we pick a prime ℓ which is different from $\mathrm{char}(k) = p$, then we have seen that

$$J[\ell^\alpha](\overline{k}) \simeq (\mathbb{Z}/\ell^\alpha\mathbb{Z})^{2g},$$

and since the map $\ell^\alpha : J \to J$ is separable, we even see that all the ℓ^α-division points are actually defined over the separable closure k_s of k.

Hence we get an action of the Galois group $\mathrm{Gal}(k_s/k) = \mathrm{Gal}(\overline{k}/k)$ on $J[\ell^\alpha](\overline{k})$ and passing to the limit we get a continous homomorphism

$$\rho : \mathrm{Gal}(\overline{k}/k) \to GL(T_\ell(J)).$$

This means that $T_\ell(J)$ provides a much richer structure than just a free $\mathbb{Z}_\ell$-module.

The cyclotomic character.

There is a much simpler construction of such Galois modules. We consider the multiplicative group scheme G_m. This is by definition

$$G_m = \mathrm{Spec}(k[T, T^{-1}]),$$

and we have a multiplication

$$m : G_m \times G_m \longrightarrow G_m$$

which on the level of k-algebras is induced by the map

$$\begin{array}{rcl} k[T, T^{-1}] & \longrightarrow & k[T, T^{-1}] \otimes k[T, T^{-1}] \\ T & \longrightarrow & T \otimes T. \end{array}$$

This induces a group structure on the rational points in any k-algebra B: We have $G_m(B) = B^*$, and the group structure is multiplication

$$\begin{array}{rcl} B^* \times B^* & \longrightarrow & B^* \\ (b_1, b_2) & \longrightarrow & b_1 b_2. \end{array}$$

Again we can pick a prime $\ell \neq p = \mathrm{char}(k)$, and we know that

$$G_m[\ell^\alpha](\overline{k}) = \{\zeta \in \overline{k}^* \mid \zeta^{\ell^\alpha} = 1\} \simeq \mathbb{Z}/\ell^\alpha\mathbb{Z}.$$

This gives us a representation of the Galoisgroup

$$\alpha : \mathrm{Gal}(\overline{k}/k) \longrightarrow GL(T_\ell(G_m)),$$

and since $T_\ell(G_m) = \lim\limits_{\leftarrow} G_m[\ell^\alpha] \simeq \mathbb{Z}_\ell$, we see that α yields a character

$$\alpha : \mathrm{Gal}(\overline{k}/k) \longrightarrow \mathbb{Z}_\ell^*.$$

This character is called the Tate character or the cyclotinic character. We denote by $\mathbb{Z}_\ell(1)$ the module $\mathbb{Z}_\ell$ on which $\mathrm{Gal}(\overline{k}/k)$ acts by the character α.

A. Grothendieck introduced the étale cohomology group for schemes. We want to give some very general ideas how this can be done.

The decisive step is to extend the notion of a topological space or the notion of a topology on a space.

Let us fix a field k and let us consider schemes $X/\operatorname{Spec}(k)$ of finite type. We define some new "topology" on X by saying that the open sets are morphisms

$$\tilde{U} \longrightarrow U \subset X$$

where $U \subset X$ is Zariski open and $\tilde{U} \to U$ is an étale covering. These new open sets form a category in an obvious sense. A morphism is a diagram

$$\begin{array}{ccc} \tilde{U}_1 & \longrightarrow & \tilde{U}_2 \\ \downarrow & & \downarrow \\ U_1 & \longrightarrow & U_2 \end{array}$$

where $U_1 \to U_2$ is an inclusion and the diagram is commutative (the vertical arrows being étale coverings).

This category has fibered products. For an ordinary topology this fibered product of two objects (open subsets in the space) is given by the intersection of the two open suebsets. In the general context this becomes more complicated. Another difference is this: For an ordinary topological space the set of morphisms between two subjects is either empty or consists of jest one element. This is no longer true.

The étale topology is already extremely interesting for $X = \operatorname{Spec}(k)$. There are not so many Zariski open subsets, but an étale open set is simply a finite separable extension $\operatorname{Spec}(k') = X' \to X$. The fibered product is

$$X' \times_X X'' = \operatorname{Spec}(k' \otimes_k k'')$$

which then suddenly may consist of several points. If for instance k'/k is a finite normal extension, then

$$k' \otimes_k k' \simeq \bigoplus_{\substack{k' \to k' \\ \sigma: \nwarrow\ \nearrow \\ k}} k'.$$

We define Et_X to be the category of these morphisms

$$\tilde{U} \longrightarrow U \subset X.$$

If we have a morphism $X' \xrightarrow{f} X$, and if U is open in X, then $f^{-1}(U)$ is open in X' and if $\tilde{U} \to U$ is an étale covering, then $\tilde{U}' = U' \times_U \tilde{U}$ is an $r\acute{m}e$tale covering of U'.

For an object $\tilde{U} \to U$ is Et_X we can introduce the notion of a covering. This is a family of

$$\tilde{U}_i \longrightarrow U_i \subset \tilde{U} \qquad i \in I$$

such that $\cup\, U_i = \tilde{U}$.

Now we can define sheaves. This are contravariant functors

$$\mathcal{F} : Et_X \longrightarrow \mathcal{C}$$

where $\mathcal{C}$ is any reasonable category (for instance sets, rings, abelian groups and so on) which satisfy the two sheaf conditions. For any covering $\{\tilde{U}_i\}$ of $U \in Et_X$ we have

$$\mathcal{F}(U) \longrightarrow \prod_i \mathcal{F}(\tilde{U}_i) \longrightarrow \prod_{(i,j)} \mathcal{F}(U_i \times_U U_j)$$

is an exact sequence.

This notion is already non trivial for the case $X = \mathrm{Spec}(k)$. What does it mean that $\mathcal{M}$ is a sheaf on X, say with values in the category of abelian groups.

This means that we have a functor $\mathcal{M}$ which attaches to any finite separable extension k'/k, an abelian group $\mathcal{M}(k')$ and for any morphism of k-algebras

$$\begin{tikzcd} k' \arrow[rr, "\sigma"] & & k'' \\ & k \arrow[ul] \arrow[ur] & \end{tikzcd}$$

we have a group homomorphism

$$\sigma' : \mathcal{M}(k') \longrightarrow \mathcal{M}(k'').$$

If our separable extension k'/k is not a field, then $k' = \oplus k_i'$ where these k_i are fields and $\mathcal{M}(k') = \oplus \mathcal{M}(k_i')$.

If our separable extension k'/k is not a field, then $k' = \oplus k_i'$ where these k_i are fields and $\mathcal{M}(k' = \oplus \mathcal{M}(k_i')$.

We have to formulate what the sheaf conditions mean. A covering of $\mathrm{Spec}(k') \to \mathrm{Spec}(k)$ is a k-algebra homomorphism

$$\begin{tikzcd} k' \arrow[rr] & & L \\ & k \arrow[ul] \arrow[ur] & \end{tikzcd}$$

i.e. $\mathrm{Spec}(L) \to \mathrm{Spec}(k'$ which sends the identity of k' to the identity of L (Note that k' is not necessarily a field.).

Now it is clear that it suffices to know the value of $\mathcal{M}$ on finite separable field extensions L/k.

If L_1/L is a finite separable and normal extension, then it is clear that the Galois group $\mathrm{Gal}(L_1/L)$ acts upon $\mathcal{M}(L_1)$. The sheaf condition then means: For any $L_1 > L > k$ as above we have

$$\mathcal{M}(L) \hookrightarrow \mathcal{M}(L_1),$$

and $\mathcal{M}(L)$ is exactly the module of fixed points under the action of $\mathrm{Gal}(L_1/L)$ on $\mathcal{M}(L_1)$.

We can reformulate this slightly. If we choose a separable closure k_s/k, then we can restrict the functor $\mathcal{M}$ to finite extensions

$$k \subset L \subset k_s.$$

If now $k \subset L_1 \subset L_2 \subset k_s$, then we have $\mathcal{M}(k) \subset \mathcal{M}(L_1) \subset \mathcal{M}(L_2)$ and we can form the limit

$$\mathcal{M} = \varinjlim_{K/k} \mathcal{M}(L) = \cup\, \mathcal{M}(L).$$

This is now a continous module for the Galois group. The group $\mathrm{Gal}(k_s/k) = \varprojlim \mathrm{Gal}(L/k)$ acts on $\mathcal{M}$ and for any $m \in \mathcal{M}$ the stabilizer of m is an open subgroup in the Galois group. If in turn M is a module on which we have a continous action of $\mathrm{Gal}(k_s/k)$ (i.e. for any $m \in M$ we have a finite extension $k \subset L \subset k_s$ such that $\mathrm{Gal}(k_s/L)m = m$, then we can define $M(L) = M^{\mathrm{Gal}(k_s/L)}$ and $L \to M(L)$ defines an étale sheaf on $\mathrm{Spec}(k)$.)

We return to the more general situation. We have our $X/\,\mathrm{Spec}(k)$. If we consider sheaves with values in AB (or modules over some ring R), then we have the notion of an exact sequence of sheaves on Et_X: This is a sequence

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0,$$

such that for any $\tilde{U}$ the sequence

$$0 \to \mathcal{F}'(U) \to \mathcal{F}(U) \to \mathcal{F}''(U)$$

is exact and such that for any $s'' \in \mathcal{F}''(U)$ we can find a covering

$$\{\tilde{U}_\alpha\}_\alpha \qquad \tilde{U}_\alpha \to U_\alpha \subset U$$

such that for any α the restriction s''_α of s_α to $\tilde{U}_\alpha$ is in the image of $\mathcal{F}(U_\alpha) \to \mathcal{F}''(U_\alpha)$.

Now I claim that we can define the derived functor to the functor

$$\mathcal{F} \longrightarrow \mathcal{F}(X) = H^0(X, \mathcal{F}),$$

and we call the derived group $H^i_{\text{ét}}(X, \mathcal{F})$. A short exact sequence in cohomology

$$\begin{aligned} 0 \to & H^0/(X, \mathcal{F}') \to H^0(X, \mathcal{F}) \to H^0(X, \mathcal{F}'') \xrightarrow{\delta} \\ & H^1_{\text{ét}}(X, \mathcal{F}') \to H^1_{\text{ét}}(X, \mathcal{F}) \to H^1_{\text{ét}}(X, \mathcal{F}'') \xrightarrow{\delta} \end{aligned}$$

(The only thing we need to know is the existence of enough injective objects (see ???).

If for instance $X = \operatorname{Spec}(k)$, then we have see that an étale sheaf with values in (Ab) is simply a $\operatorname{Gal}(k_s/k)$-module M with continous action.

An exact sequence of such sheaves is nothing else than an exact sequence of such modules

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0.$$

But if we look at sections over $X = \operatorname{Spec}(k)$, then we get

$$0 \longrightarrow M'(k) \longrightarrow M(k) \longrightarrow M''(k),$$

and the last arrow needs not to be surjective. For $m'' \in M(k)$ we can find an extension L/k (normal over k) such that m'' lifts to an element m in $M(L)$ and in the section (???) we have seen that the obstruction to lift m'' to an element in $M(k)$ lies in

$$H^1(\operatorname{Gal}(L/k), M'(L)).$$

Hence it is more or less clear that

$$\begin{aligned} H^1_{\text{ét}}(X, M') &= \lim\nolimits_{\longrightarrow} H^1(\operatorname{Gal}(L/k), M(L)) \\ &= H^1(\operatorname{Gal}(k_s/k), M). \end{aligned}$$

In this case we are back in the situation of ordinary group cohomology. Then is a slight difference to the situation in (HA....), since our group is the projective limit of finite groups. This requires some harmless continuity considerations. The groups $H^i_{\text{ét}}(X,M) = H^i(\text{Gal}(k_s/k),M)$ are called the Galois cohomology groups.

The geometric étale cohomology groups.

Let us consider a scheme of finite type X/k. We choose a separable closure k_s and an algebraic closure $\overline{k} > k_s$.

We form the scheme $\overline{X} = X \times_{\text{Spec}(k)} \text{Spec}(\overline{k})$.

Let us assume that we have a sheaf $\mathcal{F}$ over X with values in (Ab) or modules over a ring R. Then it is clear that

$$H^0(\overline{X},\mathcal{F}) = \mathcal{F}(\overline{X}) = \mathcal{F}(X \times_{\text{Spec}(k)} \text{Spec}(k_s)$$

is a module for the Galois group $\text{Gal}(\overline{k}/k) = \text{Gal}(k_s/k)$, and hence we see that all the cohomology groups

$$H^i(\overline{X},\mathcal{F})$$

are modules for $\text{Gal}(\overline{k}/k)$. These are the geometric cohomology groups.

We want to "compute" these cohomology groups in a couple of simple cases. Here "compute" means that we assume that usual tools are available.

We consider a smooth, projective and absolutely irreducible curve C/k. On this curve we have the étale sheaf $\mathcal{O}^*_{C,\text{ét}}$: For any $\tilde{U} \to U \subset C$ in Et_C we put

$$\mathcal{O}^*_{C,\text{ét}}(\tilde{U}) = \mathcal{O}(\tilde{U})^*$$

i.e. we consider the invertible regular functions on $\tilde{U}$.

If ℓ is a prime which is different from the characteristic of k, then we consider the map

$$\mathcal{O}^*_{C,\text{ét}} \longrightarrow \mathcal{O}^*_{C,\text{ét}},$$

and here you see the whole point of the story. This map is certainly not surjective for the sheaf $\mathcal{O}^*_{C,\text{Zar}}$ – this is our sheaf restricted to the Zariski topology. But if we have an open set $U \subset C$ and an étale cover $\tilde{U} \to U$ and an element $f \in \mathcal{O}(\tilde{U})^*$, then we can think of $\tilde{U}$ as an affine scheme and

$$\tilde{U}' = \text{Spec}(\mathcal{O}(\tilde{U})\left[\sqrt[\ell^\alpha]{f}\right]) \longrightarrow \tilde{U} = \text{Spec}(\mathcal{O}(\tilde{U}))$$

is an étale covering. Hence we see that

$$\mathcal{O}^*_{C,\text{ét}} \xrightarrow{\ell^\alpha} \mathcal{O}^*_{C,\text{ét}}$$

is surjective. The kernel is the sheaf μ_{ℓ^α} where

$$\mu_{\ell^\alpha}(\tilde{U}) = \{f \in \mathcal{O}_C(\tilde{U})^* \mid f^{\ell^\alpha} = 1\}.$$

This is the sheaf of ℓ^α-roots of unity. Hence we get an exact sequence

$$1 \to \mu_{\ell^\alpha} \to \mathcal{O}^*_{C,\text{ét}} \to \mathcal{O}^*_{C,\text{ét}} \to 1$$

of sheaves for the étale topology.

We write down the long exact sequence in cohomology, but before we do that I want to make it plausible that

$$H^1_{\text{ét}}(C, \mathcal{O}^*_{C,\text{ét}}) = H^1_{\text{Zar}}(C, \mathcal{O}^*_C) = \text{Pic}(C).$$

This follows from Hilbert's theorem 90 which says:

For any field K and any normal separable extension L/K we have

$$H^1(\text{Gal}(L/K), L^*) = 0.$$

This can be exploited to prove that any class $\xi \in H^1_{\text{ét}}(C, \mathcal{O}^*_{C,\text{ét}})$ becomes trivial on a suitable small Zariski open subset, and this implies the assertion above.

We write our exact cohomology sequence, but we restrict the cohomology to $\overline{C} = C \times_k \overline{k}$. We get

$$\mathcal{O}_{\overline{C},\text{ét}}(\overline{C})^* = \overline{k}^*$$

and

$$\overline{k}^* \xrightarrow{\ell^\alpha} \overline{k}^*$$

is surjective. Hence the sequence starts in degree one and we find

$$\begin{aligned} 0 \to H^1_{\text{ét}}(\overline{C}, \mu_{\ell^\alpha}) &\to H^1_{\text{Zar}}(\overline{C}, \mathcal{O}^*_{\overline{C}}) \to H^1_{\text{Zar}}(\overline{C}, \mathcal{O}^*_{\overline{C}}) \\ \to H^2_{\text{ét}}(\overline{C}, \mu_{\ell^\alpha}) &\to 0. \end{aligned}$$

(Is it entirely obvious why it stops in degree 2?)

Now we have by definition

$$H^1_{\text{Zar}}(\overline{C}, \mathcal{O}^*_{\overline{C}}) = \text{Pic}(\overline{C}),$$

and this sits in the exact sequence

$$0 \to J_{\overline{C}}(\overline{k}) \to \mathrm{Pic}(\overline{C}) \to \mathbb{Z} \to 0,$$

and this implies

$$\begin{array}{rcl} H^1(\overline{C}, \mu_{\ell^\alpha}) & \simeq & J_{\overline{C}}(\overline{k})[\ell^\alpha] \approx (\mathbb{Z}/\ell^\alpha\mathbb{Z})^{2g} \\ H^2(\overline{C}, \mu_{\ell^\alpha}) & \simeq & \mathbb{Z}/\ell^\alpha\mathbb{Z} \end{array}$$

and finally

$$H^0(\overline{C}, \mu_{\ell^\alpha}) = \mu_{\ell^\alpha}(\overline{k}).$$

Now I recall that these étale cohomology groups are not just abelian groups, but they come with an action of the Galois group $\mathrm{Gal}(\overline{k}/k)$, and of course it is as it must be: The exact sequence is an exact sequence of Galois modules.

Therefore we see that the isomorphisms above are isomorphisms of Galois modules.

It may look a little bit strange that we took the coefficient sheaf. Indeed we could also take the constant sheaf $\mathbb{Z}/\ell^\alpha\mathbb{Z}$ on C, this is smiply

$$\mathbb{Z}/\ell^\alpha\mathbb{Z}(\tilde{U}) = \bigoplus_{\text{components of } \tilde{U}} \mathbb{Z}/\ell^\alpha\mathbb{Z}.$$

This sheaf is certainly not isomorphic to μ_{ℓ^α}, but if we restrict to $\overline{C}$, these two sheaves become isomorphic. Hence we can construct an isomorphism

$$H^i_{\text{ét}}(\overline{C}, \mathbb{Z}/\ell^\alpha\mathbb{Z}) \simeq H^i_{\text{ét}}(\overline{C}, \mu_{\ell^\alpha}).$$

But this is not necessarily an isomorphism of Galois modules, this is already clear in degree zero. $H^0_{\text{ét}}(\overline{C}, \mathbb{Z}/\ell^\alpha\mathbb{Z}) = \mathbb{Z}/\ell^\alpha\mathbb{Z}$ is the trivial Galois module

$$H^0_{\text{ét}}(\overline{C}, \mathbb{Z}/\ell^\alpha\mathbb{Z}) = \mu_{\ell^\alpha}(\overline{k})$$

is non trivial, it is the Galois module of ℓ^α-th roots of unity.

This Galoismodule is also denoted by $\mathbb{Z}/\ell^\alpha\mathbb{Z}(1)$, we can define

$$\mathbb{Z}/\ell^\alpha\mathbb{Z}(-1) = \mathrm{Hom}(\mathbb{Z}/\ell^\alpha\mathbb{Z}(1), \mathbb{Z}/\ell^\alpha\mathbb{Z}),$$

and $\mathbb{Z}/\ell^\alpha\mathbb{Z}(n)$ for any integer n.

It is at least very plausible thte we have an isomorphism of Galois modules

$$H^i(\overline{C}, \mathbb{Z}/\ell^\alpha\mathbb{Z}) \simeq H^i(\overline{C}, \mu_{\ell^\alpha}) \otimes \mathbb{Z}/\ell^\alpha\mathbb{Z}(-1)$$

and especially

$$H^1_{\text{ét}}(\overline{C},\mathbb{Z}/\ell^\alpha\mathbb{Z}) \simeq J_{\overline{C}}(\overline{k})[\ell^\alpha] \otimes \mathbb{Z}/\ell^\alpha\mathbb{Z}(-1).$$

Now we may vary the α, we get a projective system, and we put

$$H^i_{\text{ét}}(\overline{C},\mathbb{Z}_\ell) =: \varprojlim H^i_{\text{ét}}(\overline{C},\mathbb{Z}/\ell^\alpha\mathbb{Z}),$$

and we showed

$$\begin{array}{rcl} H^0_{\text{ét}}(\overline{C},\mathbb{Z}_\ell) & = & \mathbb{Z}_\ell \\ H^1_{\text{ét}}(\overline{C},\mathbb{Z}_\ell) & \simeq & T_\ell(J) \otimes \mathbb{Z}_\ell(-1) \\ H_{\text{ét}}(\overline{C},\mathbb{Z}_\ell) & \simeq & \mathbb{Z}_\ell(-1) \end{array}$$

as modules under the Galois group. (It turns out that the reasonable approach is: We start from torsion sheaves und define cohomology with coefficients in $\mathbb{Z}_\ell$ by taking projective limits.)

The above definition of ℓ-adic cohomology groups works in full generality. We can take an arbitrary scheme X/k of finite type and we define

$$H^i_{\text{ét}}(\overline{X},\mathbb{Z}_\ell) = \varprojlim H^i_{\text{ét}}(\overline{X},\mathbb{Z}\ell^\alpha\mathbb{Z}),$$

and these modules are modules for the Galois group $\text{Gal}(\overline{k}/k)$. (We may of course also consider cohomology groups

$$H^i_{\text{ét}}(X,\mathbb{Z}_\ell) = \varprojlim H^i_{\text{ét}}(\mathcal{X}.\mathbb{Z}/\ell^\alpha\mathbb{Z})$$

but they are still much more complicated.)

Now one has to prove many theorems to develop this theory and in principle I want to stop here.

I want to formulate some of the important theorems:

1) If we have a scheme X/k of finite type, then the cohomology groups

$$H^i(\overline{X},\mathbb{Z}_\ell)$$

are finitely generated, and if X is irreducible, then $H^i(\overline{X},\mathbb{Z}_\ell) = 0$ for $i > 2\dim X$.

2) If X/k is an irreducible affine scheme, then we even have

$$H^i(\overline{X},\mathbb{Z}_\ell) = 0 \quad \text{for} \quad i > \dim X$$

and

$$H^i(\mathbb{A}^n_{\overline{k}},\mathbb{Z}_\ell) = \begin{cases} 0 & i > 0 \\ \mathbb{Z}_\ell & \text{for } i = 0. \end{cases}$$

3) If the field $k = \mathbb{C}$, then we can consider the étale topology on X but we also have the analytic topology on $X(\mathbb{C})$.

A little bit of thinking yields that we can have a continous map

$$X_{\text{ét}} \longleftarrow X_{\text{an}}(\mathbb{C})$$

which induces a map in cohomology

$$H^i(X_{\text{ét}}, \mathbb{Z}/\ell^\alpha\mathbb{Z}) \longrightarrow H^1(X_{\text{an}}(\mathbb{C}), \mathbb{Z}/\ell^\alpha\mathbb{Z}),$$

and the comparison theorem asserts that this map is an isomorphism.

4) If X/k is projective non singular and absolutely irreducible, then we have in addition

$$H^{2d}(\overline{X}, \mathbb{Z}_\ell) = \mathbb{Z}_\ell(-d),$$

and the cup product (which of course has to be defined) induces a non degenerate pairing

$$H^i(\overline{X}, \mathbb{Z}_\ell) \otimes \mathbb{Q}_\ell \times H^{2d-i}(\overline{X}, \mathbb{Z}_\ell) \otimes \mathbb{Q}_\ell \xrightarrow{\sim} H^{2d}(\overline{X}, \mathbb{Z}_\ell) \otimes \mathbb{Q}_\ell \simeq \mathbb{Q}_\ell(-d).$$

5) We have a Künneth formula for products

$$H^m(\overline{X} \times \overline{Y}, \mathbb{Q}_\ell) \simeq \bigoplus H^i(\overline{X}, \mathbb{Q}_\ell) \otimes H^{m-1}(\overline{X}, \mathbb{Q}_\ell).$$

6) If we have a subscheme $Z \subset X$, let us assume that it is absolutely irreducible of codimension r, then we can attach a cycle class

$$c(Y) \in H^{2r}(\overline{X}, \mathbb{Q}_\ell)$$

to it. The subspace $\mathbb{Q}_\ell \subset (Y)$ is isomorphic to $\mathbb{Q}_\ell(-r)$ as a module under the Galois group $\mathrm{Gal}(\overline{k}/k)$.

We can extend this cycle map to arbitrary cycles in codimension r. First of all, if $Y \subset X$ is not irreducible, then $Y \times_k \overline{k} = \sum Y_i$ where the Y_i are permuted by the Galois group. Then

$$c(Y) = c(Y \times_k \overline{k}) = \sum c(Y_i),$$

and $\mathbb{Q}_\ell(Y \subset (Y_i) = \mathbb{Q}_\ell(-r)$ under the Galois group $\mathrm{Gal}(\overline{k}/L_i)$ if $\mathrm{Gal}(\overline{\mathbb{Q}}/L_i)$ is the stabilizer of Y_i.

Finally we extend to arbitrary cycles by linearity.

Of course we want: If we have two cycles Y_1, Y_2 in codimension r_1, r_2 respectively, and if they are intersecting in a reasonable manner, then

$$c(Y_1 \circ Y_2) = c(Y_1) \cup c(Y_2).$$

Especially if D is a divisor, then we want

$$D^d = D \cdot - \cdot D = c_1(D)^d.$$

7) These formal properties imply a Lefschetz fixed point formula. If we have a morphism $f : X \to X$ of a smooth projective, absolutely irreducible variety into itself, then we may consider the two cycles

$$\Gamma_f \subset X \times X \ , \ \Delta \subset X \times X,$$

where Δ is the diagonal. Let us assume (which is not really necessary) that Γ_f and Δ intersect transversally, this means that at any geometric point $p \in \overline{\Gamma}_f \cap \overline{\Delta}$, the tangent spaces span the tangent space of $\overline{X} \times \overline{X}$. Then the intersection number

$$\overline{\Gamma}_f \cdot \overline{\Delta} = \#(\overline{\Gamma}_f \cap \overline{\Delta}) = \#\text{fixed points of } f.$$

Now we look at the cycle classes and apply the Künneth formal, then we get

$$c(\overline{\Gamma}_f) \cup c(\overline{\Delta}) = \sum_{i=0}^{2d} (-1)^i \operatorname{tr}(f^x \mid H^i(\overline{X}, \mathbb{Q}_\ell)),$$

and we get (under our assumtion #) fixed points of $f = \sum_{i=0}^{2d} (-1)^i \operatorname{tr}(f^* \mid H^i(\overline{X}, \mathbb{Q}_\ell))$.

If the intersection is not proper, then we have to be more careful in counting the fixed points.

We apply these theorems to the special case of an absolutely irreducible projective, smooth variety over a finite field $\mathbb{F}_q$, let us write $X/\mathbb{F}_q$

In this case we have the Frobenius morphism $\Phi : x \to x^q$, and the intersection of Γ_Φ and the diagonal is transversal. this gives us

$$\#\text{fixed points of} \Gamma_\Phi = \#X(\mathbb{F}_q) = \sum (-1)^i \operatorname{tr}(\Phi^* \mid H^i(\overline{X}, \mathbb{Q}_\ell)).$$

Of course we can apply this to all powers Φ_q^n, and then

$$t \cdot \frac{Z'_X(t)}{Z_X(t)} = \sum_{n=0}^{\infty} \#X(\mathbb{F}_{q^n}) t^n =$$
$$\sum_{n=0}^{\infty} (\sum_{i=0}^{2g} (-1)^i \operatorname{tr}(\Phi_q^n \mid H^i(\overline{X}, \mathbb{Q}_\ell))) t^n$$

and this yields the formula

$$Z_X(t) = \prod_{i=0}^{2g} \det(\mathrm{Id} - t\Phi_q \mid H^i(\overline{X}, \mathbb{Q}_\ell)^{(-1)^{i+1}}$$

for the Z-function of X. This is Grothendiecks theorem (???).

At this point we have

$$Z_X(t) = \frac{P_1 \cdots P_{2g-1}}{P_0 \cdots P_{2d}}$$

where

$$Z_X(t) \in \mathbb{Z}[[t]]$$

but a priori we have

$$P_i(t) = \det(\mathrm{Id} - t\Phi_q \mid H^i(\overline{X}, \mathbb{Q}_\ell)) \in \mathbb{Z}_\ell[t].$$

But then in 1973 P. Deligne proved the following theorem which was anticipated or conjectured by A. Weil in (???).

Theorem (Weil conjectures): 1) The polynomials

$$P_i(t) = \det(\mathrm{Id} - t\Phi_q \mid H^i(\overline{X}, \mathbb{Q}_\ell)) \in \mathbb{Z}[t]$$

and they are independent of ℓ.

2) The reciprocical roots

$$P_i(t) = \prod (1 - \omega_{i\nu} t)$$

are algebraic integers and

$$|\omega_{i\nu}| = q^{i/2} \quad \text{for all} \nu.$$

In principle we have proved this theorem for the case of curves, except that we replaced $H^i(\overline{C}, \mathbb{Q}_\ell)$ by the ad hoc defined group $\mathbb{Q}_\ell(0)$, $T_\ell(J) \otimes \mathbb{Q}_\ell(-1)$, $\mathbb{Q}_\ell(-1)$.