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SCHOOL ON ALGEBRAIC GEOMETRY

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Editor

Lothar Göttsche
The Abdus Salam ICTP
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SCHOOL ON ALGEBRAIC GEOMETRY – First edition

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PREFACE

One of the main missions of the Abdus Salam International Centre for Theoretical Physics in Trieste, Italy, founded in 1964 by Abdus Salam, is to foster the growth of advanced studies and research in developing countries. To this aim, the Centre organizes a large number of schools and workshops in a great variety of physical and mathematical disciplines.

Since unpublished material presented at the meetings might prove of great interest also to scientists who did not take part in the schools the Centre has decided to make it available through a new publication titled ICTP Lecture Note Series. It is hoped that this formally structured pedagogical material in advanced topics will be helpful to young students and researchers, in particular to those working under less favourable conditions.

The Centre is grateful to all lecturers and editors who kindly authorize the ICTP to publish their notes as a contribution to the series.

Since the initiative is new, comments and suggestions are most welcome and greatly appreciated. Information can be obtained from the Publications Section or by e-mail to "`pub_off@ictp.trieste.it`". The series is published in house and also made available on-line via the ICTP web site: "`http://www.ictp.trieste.it`".

M.A. Virasoro
Director

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Introduction

This is the first volume of a new series of lecture notes of the Abdus Salam International Centre for Theoretical Physics. These new lecture notes are put onto the web pages of the ICTP to allow people from all over the world to access them freely. In addition a limited number of hard copies is printed to be distributed to scientists and institutions which otherwise possibly do not have access to the web pages.

This first volume contains the lecture notes of the School on Algebraic Geometry which took place at the Abdus Salam International Centre for Theoretical Physics from 26 July to 13 August 1999 under the direction of Lothar Göttsche (ICTP), Joseph Le Potier (Université Paris 7), Eduard Looijenga (University of Utrecht), M.S. Narasimhan (ICTP).

The school consisted of 2 weeks of lecture courses and one week of conference. This volume contains the notes of most of the lecture courses in the first two weeks. The topic of the school was moduli spaces. More specifically the lectures were devided into three subtopics: principal bundles on Riemann surfaces, moduli spaces of vector bundles and sheaves on projective varieties, and moduli spaces of curves.

The school was financially supported by ICTP and by a grant of the European commission. I take this opportunity to thank the other organizers. We are grateful to all the lecturers and speakers at the conference for their contribution to the success of the school.

Lothar Göttsche
April, 2000

Lectures on moduli of principal G-bundles over algebraic curves

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Lecture given at the

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1. Introduction

These notes are supposed to be an introduction to the moduli of G-bundles on curves. Therefore I will lay stress on ideas in order to make these notes more readable. My presentation of the subject is strongly influenced by the work of several mathematicians as Beauville, Laszlo, Faltings, Beilinson, Drinfeld, Kumar, Narasimhan and others.

In the last years the moduli spaces of G-bundles over algebraic curves have attracted some attention from various subjects like from conformal field theory or Beilinson and Drinfeld's geometric Langlands program [5]. In both subjects it turned out that the "stacky" point of view is more convenient and as the basic motivation of these notes is to introduce to the latter subject our moduli spaces will be moduli stacks (and not coarse moduli spaces). As people may feel uncomfortable with stacks I have included a small introduction to them. Actually there is a forthcoming book of Laumon and Moret-Bailly based on their preprint [15] and my introduction merely does the step -1, *i.e.* explains why we are forced to use them here and recalls the basic results I need later.

So here is the plan of the lectures: after some generalities on G-bundles, I will classify them topologically. Actually the proof is more interesting than the result as it will give a flavor of the basic theorem on G-bundles which describes the moduli stack as a double quotient of loop-groups. This "uniformization theorem", which goes back to A. Weil as a bijection on sets, will be proved in the section following the topological classification.

Then I will introduce two line bundles on the moduli stack: the determinant and the pfaffian bundle. The first one can be used to describe the canonical bundle on the moduli stack and the second to define a square-root of it. Unless G is simply connected the square root depends on the choice of a theta-characteristic. This square root plays an important role in the geometric Langlands program. Actually, in order to get global differential operators on the moduli stack one has to consider twisted differential operators with values in these square-roots.

The rest of the lectures will be dedicated to describe the various objects involved in the uniformization theorem as loop groups or the infinite Grassmannian in some more detail.

2. Generalities on principal G-bundles

In this section I define principal G-bundles and recall the necessary background I need later. Principal G-bundles were introduced in their generality by Serre in Chevalley's seminar in 1958 [19] based on Weil's "espaces fibrés algébriques" (see remark 2.1.2).

2.1. **Basic definitions.**

Let Z be a scheme over an algebraically closed field k, G be an affine algebraic group over k.

2.1.1. Definition. By a *G-fibration* over Z, we understand a scheme E on which G acts from the right and a G-invariant morphism $\pi : E \to Z$. A morphism between G-fibrations $\pi : E \to Z$ and $\pi' : E' \to Z$ is a morphism of schemes $\varphi : Z \to Z'$ such that $\pi = \pi' \circ \varphi$.

A G-fibration is *trivial* if it is isomorphic to $pr_1 : Z \times G \to Z$, where G acts on $Z \times G$ by $(z, g).\gamma = (z, g\gamma)$.

A *principal G-bundle* in the *Zariski*, resp. *étale*, resp. *fppf*, resp. *fpqc* sense is a G-fibration which is locally trivial in the Zariski, resp. étale, resp. fppf, resp. fpqc topology. This means that for any $z \in Z$ there is a neighborhood U of z such that $E_{|U}$ is trivial, resp. that there is an étale, resp. flat of finite presentation, resp. flat quasi-compact covering $U' \xrightarrow{\varphi} U$ such that $\varphi^*(E_{|U}) = U' \times_U E_{|U}$ is trivial.

2.1.2. *Remark.* In the above definition, local triviality in the Zariski sense is the strongest whereas in the fpqc sense is the weakest condition. If G is smooth, then a principal bundle in the fpqc sense is even a principal bundle in the étale sense ([9], §6). In the following we will always suppose G to be smooth and we will simply call *G-bundle* a principal G-bundle in the étale sense. If $G = \mathbf{GL}_r$ or if Z is a smooth curve (see Springer's result in [22], 1.9), such a bundle is even locally trivial in the Zariski sense, but it was Serre's observation that in general it is not. He defined those groups for which local triviality in the Zariski sense implies always local triviality in the étale sense to be *special*. Then, for semi-simple G, Grothendieck (same seminar, some exposés later) classified the special groups: these are exactly the direct products of SL_r's and Sp_{2r}'s.

Remark that if the G-bundle E admits a section, then E is trivial. Define the following pointed (by the trivial bundle) set :

$$H^1_{ét}(Z, G) = \{\text{G-bundles over } Z\}/\text{isomorphism}.$$

2.2. **Associated bundles.** If F is a quasi-projective k-scheme on which G acts on the left and E is a G-bundle, we can form $E(F) = E \times^G F$ the *associated bundle with fiber* F. It is the quotient of $E \times F$ under the action of G defined by $g.(e, f) = (e.g, g^{-1}f)$. The quasi-projectivity[1] of F is needed in order to assure that this quotient actually exists as a scheme.

There are two important cases of this construction.

2.2.1. *The associated vector bundle.* Let F be a vector space of dimension n. Suppose $G = \mathbf{GL}(F)$. Then G acts on F from the left and we can form for a G-bundle E the associated bundle $V = E(F)$. This is actually a vector bundle of rank n. Conversely, for any vector bundle V of rank n the associated frame bundle E (*i.e.* $\underline{\mathrm{Isom}}_{\mathcal{O}_Z}(\mathcal{O}_Z^n, V)$) is a GL_n-bundle.

2.2.2. *Extension of structure group.* Let $\rho : G \to H$ be a morphism of algebraic groups. Then G acts on H via ρ, we can form the *extension of the structure group* of a G-bundle E, that is the H-bundle $E(H)$. Thus, we have defined a map of pointed sets

$$H^1_{\acute{e}t}(Z, G) \longrightarrow H^1_{\acute{e}t}(Z, H)$$

Conversely, if F is an H-bundle, a *reduction of structure group* is a G-bundle E together with an isomorphism of G-bundles $\tau : E(H) \xrightarrow{\sim} F$.

2.2.3. Lemma. *Suppose $\rho : G \hookrightarrow H$ is a closed immersion. If F is an H-bundle, denote $F(H/G)$ simply by F/G. There is a natural one to one correspondence between sections $\sigma : Z \to F/G$ and reductions of the structure group of F to G.*

Proof. View $F \to F/G$ as a G-bundle and consider for $\sigma : Z \to F/G$ the pullback diagram

$$\begin{tikzcd} \sigma^*F \arrow[r, dash] \arrow[d, dash, "G"'] & F \arrow[d, dash, "G"] \\ Z \arrow[r, dash, "\sigma"] & F/G \end{tikzcd}$$

which defines the requested reduction of the structure group. □

[1] In fact it is enough to suppose that F satisfies the property that any finite subset of F lies in an affine open subset of F.

2.3. **G-bundles on a curve.** Let X be a smooth and connected curve. By the above quoted theorem of Springer ([22], 1.9), all G-bundles over X are locally trivial in the Zariski topology, so the reader might ask why I insisted on the étale topology in the above definition. The reason is that in order to study G-bundles on X, we will study families of G-bundles parameterized by some k-scheme S. By definition, these are G-bundles on $X_S = X \times S$, and here is where we need the étale topology.

A warning: it is not a good idea to define families point-wise. Let's look at the example of $\mathbf{O}_r$. Then we may view (considering $\mathbf{O}_r \subset \mathbf{GL}_r$ and using Lemma 2.2.3) an $\mathbf{O}_r$-bundle as a vector bundle E together with an isomorphism $\sigma : E \to E^*$ such that $\sigma = \sigma^*$ (I denote here and later the transposed map of σ by σ^*). The point is as follows. If E is a vector bundle over X_S together with an isomorphism $\sigma : E \to E^*$ such that for all closed point $s \in S$ the induced pair (E_s, σ_s) is an $\mathbf{O}_r$-bundle, this does not imply in general that (E, σ) itself is an $\mathbf{O}_r$-bundle.

3. Algebraic stacks

3.1. **Motivation.** Given a moduli problem such as classifying vector bundles over a curve, there are essentially two approaches to its solution: coarse moduli spaces and algebraic stacks. The former, introduced by Mumford, are schemes and are constructed, after restricting to a certain class of objects such as semi-stable bundles in the above example, as quotients of some parameter scheme by a reductive group using geometric invariant theory. However they do not - in general - carry a universal family and may have artificial singularities coming from the quotient process in their construction. So in order to construct objects on the coarse moduli space, one considers generally first the parameter space (which carries a universal family) and then tries to descend the constructed object to the moduli space which might be tricky or impossible.

In our case of the geometric Langlands program a special line bundle on the moduli space (*i.e.* a certain square root of the dualising sheaf) will play a particular role. However, it can be shown, that even if there is a functorial construction of this line bundle (hence a line bundle on the parameter scheme), it does *not* - for general G - descend to the coarse moduli space of semi-stable G-bundles.

It turns out, for this and other reasons, that in order to study the questions related to the geometric Langlands program, one has to consider the latter, *i.e.* the "stacky" solution to the moduli problem. So in my lectures I will

concentrate on the *moduli stack* of principal G-bundles and as there are not many references for stacks for the moment, I will recall in this section the ideas and properties of stacks I need in order to properly state and prove the basic results for the program.

3.1.1. *The moduli problem.* The basic moduli problem for G-bundles on a projective, connected, and smooth curve X/k is to try to represent the functor which associates to a scheme S/k the set of isomorphism classes of families of G-bundles parameterized by S:

$$\begin{array}{rcl} \underline{\mathcal{M}_{G,X}} : (\mathbf{Sch}/k)^{op} & \longrightarrow & \mathbf{Set} \\ S & \mapsto & \left\{ \begin{array}{c} E \\ \downarrow \mathrm{G} \\ S \times X \end{array} \right\} / \sim \end{array}$$

Now, as G-bundles admit in general non trivial automorphisms (the automorphism group of a G-bundle contains the center of G), we can't expect to be able to solve the above problem, *i.e.* find a scheme M that represents the above functor. Loosely speaking, if it would exist we should be able, given any morphism φ from any scheme S to M, to recover uniquely a family E parameterized by S such that the map defined by $s \mapsto [E_s]$ defines the morphism φ. As this should in particular apply to the closed points $\mathrm{Spec}(k) \in M$, the above translates that not only for every G-bundle one is able to choose an element in its isomorphism class with the property that this choice behaves well under families, but also that there is only one such choice with this property. This clearly is an obstruction which makes the existence of M unlikely and can be turned into a rigorous argument.

However, I will not do this here, but rather discuss the first non trivial case of $G = k^*$, *i.e.* the case of rank 1 vector bundles. Then a possible candidate for M is the jacobian $J(X)$. We know that $J(X)$ parameterizes isomorphism classes of line bundles on X and that there is a Poincaré bundle P on $J(X) \times X$. Hence we get, for every $j \in J(X)$, a canonical element in the isomorphism class it represents, namely P_j, and this choice is compatible with families (pullback P to the family). The point in this example is that this choice is not unique as $P \otimes pr_1^*(A)$ is also a Poincaré bundle for $A \in \mathrm{Pic}(J(X))$.

Actually what we can do here is to consider a slightly different functor, by fixing a point $x \in X$ and looking at pairs (L, α) of line bundles together with an isomorphism $\alpha : L_x \xrightarrow{\sim} k$. Such a choice determines uniquely a Poincaré bundle P and $J(X)$ (together with P) actually represents the functor defined

by such pairs. The above process of adding structure to the functor in order to force the automorphism group of the considered objects to be trivial is sometimes called to "rigidify" the functor.

Let us return to our original moduli problem. As I explained above, the main problem is the existence of non trivial automorphisms and there is nothing much we can do about this, without adding additional structures which may be complicated in the general case and definitively changes the moduli problem. Grothendieck's idea to avoid the difficulties posed by the existence of these non trivial automorphisms is simple : keep them. However, as we will see, carrying out this idea is technically quite involved.

So how to keep the automorphisms? If we do not want to mod out the automorphisms what we can do is to replace the set of automorphisms classes of G-bundles over $S \times X$ by the category which has as objects such bundles and as morphisms the isomorphisms between them.

By definition, the categories we obtain have the property that all arrows are invertible; categories with this property are called groupoids. In the following the category of groupoids will be denoted by **Gpd**. It will be convenient to write groupoids and categories in the form $\{\text{objects}\} + \{\text{arrows}\}$.

Applying the above idea to our moduli problem gives a "functor"

$$\begin{aligned} \mathcal{M}_{G,X} : \mathbf{Sch}/k^{op} &\longrightarrow \mathbf{Gpd} \\ S &\longmapsto \{G\text{-bundles on } X \times S\} + \\ &\qquad \{\text{isomorphisms of } G\text{-bundles on } X \times S\} \end{aligned}$$

Actually this is not really a functor as before, but a broader object, called a "lax functor": if $f : S' \to S$ is a morphism of k-schemes the pullback defines a functor $f^* : \mathcal{M}_{G,X}(S) \to \mathcal{M}_{G,X}(S')$. If $g :$ $S) \to$ $\in \not\leftarrow \leftarrow \{\otimes \not\leftarrow \in \forall \leftarrow \leftarrow \mathcal{D} \otimes \langle | \otimes \not\leftarrow$ $\cdot \forall \leftarrow \leftarrow \in$

maps between sets), and natural transformations between morphisms (this is new).

3.1.2. *The quotient problem.* Suppose that the linear group H acts on the scheme Z. Suppose moreover that the action is free. Then Z/H exists as a scheme and the quotient morphism $\pi : Z \to Z/H$ is actually an H-bundle. What are the points of Z/H? If $S \xrightarrow{f} Z/H$, we get a cartesian diagram

$$\begin{tikzcd} Z' \arrow[r, dash, "\alpha"] \arrow[d, dash, "H"'] & Z \arrow[d, dash, "H"] \\ S \arrow[r, dash, "f"] & Z/H \end{tikzcd}$$

So f defines an H-bundle $Z' \xrightarrow{H} S$ and an H-equivariant morphism α. If the action of H is not free, the quotient Z/H does not, in general, exist as a scheme, however what we can do is to consider the following lax functor

$$\begin{aligned} [Z/H] : \mathbf{Sch}/k^{op} &\longrightarrow \mathbf{Gpd} \\ S &\longmapsto \{(Z', \alpha) \ / Z' \xrightarrow{H} S \text{ is a } H\text{-bundle and } \alpha : Z' \xrightarrow{\alpha} Z \\ &\qquad \text{is a } H\text{-equivariant morphism}\} + \\ &\quad \{\text{isomorphisms of pairs}\} \end{aligned}$$

This definition makes sense for any action of H on Z and the "quotient map" $Z \longrightarrow [Z/H]$ (we will see in a moment what this means) behaves like an H-bundle.

3.1.3. The idea then is to *define* a stack exactly as such lax functors, after imposing some natural topological conditions on them. Of course this may seem to be somehow cheating but Grothendieck showed us that one can actually do geometry with a certain class of such stacks which he called algebraic.

After the above motivation, the plan for the rest of the section is:

- Grothendieck Topologies
- k-spaces and k-stacks
- Descent
- Algebraic stacks

3.2. **Grothendieck Topologies.** Sometimes in algebraic geometry we need to use topologies which are finer than the Zariski topology, especially when interested in an analogue of the inverse function theorem. Over $\mathbb{C}$, there is the classical topology, although using it leads to worries about the algebraicity of analytically defined constructions. Otherwise one has to use a Grothendieck topology such as the étale topology.

A Grothendieck topology is a topology on a category. The category might be similar to the category $\mathbf{Zar}(Z)$ of Zariski open sets of a k-scheme Z, or it might be an ambient category like $\mathbf{Sch}/k$ or $\mathbf{Aff}/k$. Grothendieck topologies are most intuitively described using covering families, which describe a basis or a pretopology for the topology.

3.2.1. *Covering families.* In this approach a *Grothendieck topology* (or pretopology) on a category C with fiber products is a function T which assigns to each object U of C a collection $T(U)$ consisting of families $\{U_i \xrightarrow{\varphi_i} U\}_{i\in I}$ of morphisms with target U such that

- if $U' \to U$ is an isomorphism, then $\{U' \to U\}$ is in $T(U)$;
- if $\{U_i \xrightarrow{\varphi_i} U\}_{i\in I}$ is in $T(U)$, and if $U' \to U$ is any morphism, then the family $\{U_i \times_U U' \to U'\}_{i\in I}$ is in $T(U')$;
- if $\{U_i \xrightarrow{\varphi_i} U\}_{i\in I}$ is in $T(U)$, and if for each $i \in I$ one has a family $\{V_{ij} \to U_i\}_{j\in I_i}$ in $T(U_i)$, then $\{V_{ij} \to U_i \to U\}_{i\in I, j\in I_j}$ is in $T(U)$.

The families in $T(U)$ are called *covering families* for U in the T-topology.

A *site* is a category with a Grothendieck topology.

3.2.2. *Small sites.* Let's look at some examples:

(i) If Z is a k-scheme consider the category of Zariski open subsets of Z. A family $\{U_i \subset U\}_{i\in I}$ is a covering family for U if $\bigcup_{i\in I} U_i = U$. The resulting site is the *small Zariski site* or Zariski topology on Z written Z_{Zar}.

(ii) If Z is a k-scheme, let $\mathbf{Et}/Z$ be the category whose objects are étale maps $U \to Z$ and whose morphisms are étale maps $U' \to U$ compatible with the projections to Z. A family $\{U_i \to U\}_{i\in I}$ is a covering family if the union of the images of the U_i is U (such a family is called a surjective family). This is the *small étale site* or étale topology on Z written $Z_{\text{ét}}$.

(iii) Replacing "étale" by "smooth" gives a topology on $\mathbf{Smooth}/Z$ called the smooth topology. The small smooth site on a scheme is Z_{sm}. Using "flat of finite presentation" gives the *fppf topology* and a small

site Z_{fppf}. The letters "fppf" stand for "fidèlement plat de présentation finie." There are also letters "fpqc" standing for "fidèlement plat et quasi-compact." Intuitively, each of these successive topologies is finer than the previous one because there are more open sets.

3.2.3. *Big sites.* One can also define a topology on all schemes at once. The category $\mathbf{Sch}/k$ of all k-schemes may be given the Zariski, étale, smooth, fppf, and fpqc topologies. In these topologies the covering families of a scheme U are surjective families $\{U_i \xrightarrow{\varphi_i} U\}_{i\in I}$ of, respectively, inclusions of open subschemes, étale maps, smooth maps, flat maps of finite presentation, and flat quasi-compact maps. Each successive topology has more covering families than the previous one and so is finer.

One can do the same thing to the category $\mathbf{Aff}/k$ of affine k-schemes.

3.2.4. *Sheaves.* A presheaf of sets on a category C with a Grothendieck topology (of covering families) is a functor $F : C^{\mathrm{op}} \to \mathbf{Set}$. A presheaf is separated if for all objects U in C, all $f, g \in F(U)$, and all covering families $\{U_i \xrightarrow{\varphi_i} U\}_{i\in I}$ of U in the topology, the condition $f|_{U_i} = g|_{U_i}$ for all i implies $f = g$. A presheaf is a sheaf if it is separated and in addition, whenever one has a covering family $\{U_i \xrightarrow{\varphi_i} U\}_{i\in I}$ in the topology and a system $\{f_i \in F(U_i)\}_{i\in I}$ such that for all i, j, one has $F(p^1_{i,j})(f_i) = F(p^2_{i,j})(f_j)$ in $F(U_i \times_U U_j)$, then there exists an $f \in F(U)$ such that $f|_{U_i} = f_i$ for all i. A compact way to say the above is to say that $F(U)$ is the kernel of the following double arrow

$$\prod_{i\in I} F(U_i) \underset{F(p^2_{i,j})}{\overset{F(p^1_{i,j})}{\rightrightarrows}} \prod_{i,j} F(U_i \times_U U_j)$$

3.3. **k-spaces and k-stacks.** By a *k-space* (resp. *k-group*) we understand a sheaf of sets (rep. groups) over the big site $(\mathbf{Aff}/k)_{fppf}$. A *lax functor* $\mathfrak{X}$ from $\mathbf{Aff}/k^{op}$ to $\mathbf{Gpd}$ associates to any $U \in \mathrm{ob}(\mathbf{Aff}/k)$ a groupoid $\mathfrak{X}(U)$ and to every arrow $f : U' \to U$ in $\mathbf{Aff}/k$ a functor $f^* : \mathfrak{X}(U) \to \mathfrak{X}(U')$ together with isomorphisms of functors $g^* \circ f^* \simeq (f \circ g)^*$ for every arrow $g : U'' \to U'$ in $\mathbf{Aff}/k$. These isomorphisms should satisfy the following compatibility relation: for $h : U''' \to U''$ the following diagram commutes

$$\begin{tikzcd}
h^* \circ g^* \circ f^* \arrow[r, dash, "\sim"] \arrow[d, dash, "\wr"'] & h^*(f \circ g)^* \arrow[d, dash, "\wr"] \\
(g \circ h)^* f^* \arrow[r, dash, "\sim"'] & (f \circ g \circ h)^*
\end{tikzcd}$$

If $x \in \mathrm{ob}(\mathfrak{X}(U))$ and $f : U' \to U$ it is convenient to denote $f^*x \in \mathrm{ob}(\mathfrak{X}(U')$ by $x_{|U'}$. A lax functor will be called a *k-stack* if it satisfies the following two topological properties:

(i) for every $U \in \mathrm{ob}(\mathbf{Aff}/k)$ and all $x, y \in \mathrm{ob}(\mathfrak{X}(U))$ the presheaf

$$\begin{aligned} \underline{\mathrm{Isom}}(x,y) : \mathbf{Aff}/U &\longrightarrow \mathbf{Set} \\ (U' \to U) &\longmapsto \mathrm{Hom}_{\mathfrak{X}(U')}(x_{|U'}, y_{|U'}) \end{aligned}$$

is a sheaf (with respect to the *fppf* topology on $\mathbf{Aff}/U$).

(ii) Every descent datum is effective.

Recall that a *descent datum* for $\mathfrak{X}$ for a covering family $\{U_i \xrightarrow{\varphi_i} U\}_{i \in I}$ is a system of the form $(x_i, \phi_{ji})_{i,j \in I}$ with the following properties: each x_i is an object of $\mathfrak{X}(U_i)$, and each $\phi_{ji} : x_{i|U_{ji}} \to x_{j|U_{ji}}$ is an arrow in $\mathfrak{X}(U_{ji})$. Moreover, we have the co-cycle condition

$$\phi_{ki|U_{kji}} = \phi_{kj|U_{kji}} \circ \phi_{ji|U_{kji}}$$

where $U_{ji} = U_j \times_U U_i$ and $U_{kji} = U_k \times_U U_j \times_U U_i$, for all i, j, k.

A descent datum is *effective* if there exists an object $x \in \mathfrak{X}(U)$ and invertible arrows $\phi_i : x|\, U_i \xrightarrow{\sim} x_i$ in $\mathfrak{X}(U_i)$ for each i such that

$$\phi_{j|U_{ji}} = \phi_{ji} \circ \phi_{i|U_{ji}}$$

for all $i, j \in I$.

Any k-space X may be seen as a k-stack, by considering a set as a groupoid (with the identity as the only morphism). Conversely, any k-stack $\mathfrak{X}$ such that $\mathfrak{X}(R)$ is a discrete groupoid (*i.e.* has only the identity as automorphisms) for all affine k-schemes U, is a k-space.

3.3.1. Example. (The quotient stack) Let us consider again the quotient problem of (3.1.2), in the more general setup of a k-group Γ acting on a k-space Z, which we will actually need in the sequel. The quotient stack $[Z/\Gamma]$ is defined as follows. Let $U \in \mathrm{ob}(\mathbf{Aff}/k)$. The objects of $[Z/\Gamma](U)$ are pairs (Z', α) where Z' is a Γ-bundle over U and $\alpha : Z' \to Z$ is Γ-equivariant, the arrows are defined in the obvious way and so are the functors $[Z/\Gamma](U) \to [Z/\Gamma](U')$.

3.4. **Morphisms.** A *1-morphism* $F : \mathfrak{X} \to \mathfrak{Y}$ will associate, for every $U \in \mathrm{ob}(\mathbf{Aff}/k)$, a functor $F(U) : \mathfrak{X}(U) \to \mathfrak{Y}(U)$ and for every arrow $U' \xrightarrow{f} U$ an

isomorphism of functors $\alpha(f) : f^*_{\mathfrak{X}} \circ F(U') \xrightarrow{\sim} F(U) \circ f^*_{\mathfrak{Y}}$

$$\begin{tikzcd}
\mathfrak{X}(U) \arrow[r, "F(U)"] \arrow[d, "f^*_{\mathfrak{X}}"'] \arrow[dr, bend left, "\alpha(f)"] & \mathfrak{Y}(U) \arrow[d, "f^*_{\mathfrak{Y}}"] \\
\mathfrak{X}(U') \arrow[r, "F(U')"'] & \mathfrak{Y}(U')
\end{tikzcd}$$

satisfying the obvious compatibility conditions: (i) if $f = 1_U$ is an identity, then $\alpha(1_U) = 1_{F(U)}$ is an identity and (ii) if f and g are composable, then $F(gf)$ is the composite of the squares $\alpha(f)$ and $\alpha(g)$ further composed with the composition of pullback isomorphisms $g^* \circ f^* \simeq (f \circ g)^*$ for $\mathfrak{X}$ and $\mathfrak{Y}$ (I will not draw the diagram here).

A *2-morphism* between 1-morphisms $\phi : F \to G$ associates for every $U \in \mathrm{ob}(\mathbf{Aff}/k)$, an isomorphism of functors $\phi(U) : F(U) \to G(U)$:

$$\begin{tikzcd}
\mathfrak{X}(U) \arrow[r, bend left, "F(U)"] \arrow[r, bend right, "G(U)"'] \arrow[r, phantom, "\phi(U)\Downarrow"] & \mathfrak{Y}(U)
\end{tikzcd}$$

There is an obvious compatibility condition which I leave to the reader.

3.4.1. *Remark.* The above definitions of 1- and 2-morphisms make sense for any lax functor. The compatibility conditions, which will be automatically satisfied in our examples, may seem complicated, however can't be avoided with this approach. The point is that typically in nature the pullback objects f^*x for every $x \in \mathrm{ob}(\mathfrak{X})(U)$ and $U' \xrightarrow{f} U$ are well defined up to isomorphism, but that the actual object f^*x is arbitrary in its isomorphism class. Let's have a closer look at our example $\mathcal{M}_{G,X}$. In this case taking the pullback $(f \times \mathrm{id})^*E$ of a G-bundle E on $X \times U$ to $X \times U'$ corresponds to take a tensor product. This is well defined up to canonical isomorphism (it is the solution of a universal problem) and we are so used to choose an element in its isomorphism class that we generally (and safely) forget about this choice. However, when comparing the functors $g^* \circ f^*$ and $(f \circ g)^*$ this choice comes up inherently and we get only something very near to "equality" namely a canonical isomorphism of functors. So once we see that our functors are only lax (as opposed to strict) in general we see that we have to choose these isomorphisms of functors in the definitions and then all sorts of compatibility conditions pop up naturally.

There is another, less intuitive but more intrinsic approach to lax functors using *k-groupoids*. This is an essentially equivalent formalism which avoids the choice of a pullback object, hence reduces the compatibility conditions. As this is the point of view of [15], I will describe briefly the relation between the two (which may also help to facilitate the reading of the first chapter of [15]). I start with a lax functor $\mathfrak{X} : (\mathbf{Aff}/k)^{op} \to \mathbf{Gpd}$ to which I will associate a category $\underline{\mathfrak{X}}$ together with a functor $\phi : \underline{\mathfrak{X}} \to (\mathbf{Aff}/k)$ (actually I should denote $\underline{\mathfrak{X}}$ by $\mathfrak{X}$ as well, but here I want to distinguish the two). The objects of $\underline{\mathfrak{X}}$ are

$$\coprod_{U \in \mathrm{ob}(\mathbf{Aff}/k)} \mathrm{ob}\, \mathfrak{X}(U)$$

the morphisms going from $x \in \mathrm{ob}\, \mathfrak{X}(U)$ to $y \in \mathrm{ob}\, \mathfrak{X}(V)$ are pairs (α, f) with $f : U \to V$ an arrow in $(\mathbf{Aff}/k)$ and α an arrow in $\mathfrak{X}(U)$ from x to f^*y. A convenient way to encode these pairs is as follows[2]

$$x \overset{\alpha}{\text{———}} f^*y \overset{f}{\cdots\cdots} y$$

With these notations, the composite of two arrows

$$x \overset{\alpha}{\text{———}} f^*y \overset{f}{\cdots\cdots} y \overset{\beta}{\text{———}} g^*z \overset{g}{\cdots\cdots} z$$

is defined to be

$$x \overset{\alpha}{\text{———}} f^*y \overset{f^*\beta}{\text{———}} f^*g^*z \overset{\sim}{\text{———}} (gf)^*z \overset{gf}{\cdots\cdots} z$$

The functor ϕ is defined to send an object of $\mathfrak{X}(U)$ to U and an arrow (α, f) to f. Looking at $\phi : \underline{\mathfrak{X}} \to (\mathbf{Aff}/k)$ we see that the categories $\mathfrak{X}(U)$ are the *fiber categories* $\underline{\mathfrak{X}}_U$ with objects the objects x of $\underline{\mathfrak{X}}$ such that $\phi(x) = U$ and arrows the arrows f of $\underline{\mathfrak{X}}$ such that $\phi(f) = 1_{|U}$.

The functor $\phi : \underline{\mathfrak{X}} \to (\mathbf{Aff}/k)$ satisfies the following two properties (exercise: prove this)

(i) for every arrow $U' \xrightarrow{f} U$ in $(\mathbf{Aff}/k)$ and every object x in $\underline{\mathfrak{X}}_U$, there is an arrow $y \xrightarrow{u} x$ in $\underline{\mathfrak{X}}$ such that $\phi(u) = f$

(ii) for every diagram $z \xrightarrow{v} x \xleftarrow{u} y$ in $\underline{\mathfrak{X}}$ with image $U'' \xrightarrow{f} U \xleftarrow{g} U'$ in $(\mathbf{Aff}/k)$ there is for every arrow $U'' \xrightarrow{h} U'$ such that $f = gh$ a unique arrow $z \xrightarrow{w} y$ such that $u = vw$ and $\phi(w) = h$.

[2] I learned this from Charles Walter's lectures on stacks in Trento some years ago.

A functor $\phi : \underline{\mathfrak{X}} \to (\mathbf{Aff}/k)$ satisfying (i) and (ii) is called a k-groupoid in [15]. So a lax functor defines a k-groupoid. On the other hand, given a k-groupoid we can define a lax functor as follows. To every $U \in \mathrm{ob}(\mathbf{Aff}/k)$ we associate the fiber category $\underline{\mathfrak{X}}_U$. If $U' \xrightarrow{f} U$ is an arrow in $(\mathbf{Aff}/k)$ and $x \in \mathrm{ob}\,\underline{\mathfrak{X}}_U$ then by (i) we know that there is $y \xrightarrow{u} x$ in $\underline{\mathfrak{X}}$. From (ii) it follows that $y \xrightarrow{u} x$ is unique up to isomorphism. Now we *choose* – once and for all – for every f and x such an arrow $y \xrightarrow{u} x$ which we denote by $f^*x \xrightarrow{u} x$. Moreover, for every arrow $x' \xrightarrow{u} x$ in $\underline{\mathfrak{X}}$, we denote by $f^*(u)$ the unique arrow which make the following diagram commutative

$$\begin{tikzcd} f^*x' \arrow[r, dash] \arrow[d, dash, "f^*(u)"'] & x' \arrow[d, dash, "u"] \\ f^*x \arrow[r, dash] & x \end{tikzcd}$$

We get a functor $f^* : \underline{\mathfrak{X}}_U \to \underline{\mathfrak{X}}_{U'}$, and also for $U'' \xrightarrow{g} U' \xrightarrow{f} U$ an isomorphism of functors $g^* \circ f^* \xrightarrow{\sim} (f \circ g)^*$ satisfying the conditions of a lax functor.

For k-groupoids most of the basic definitions such as 1- and 2-morphisms are more elegant: a 1-morphism is a functor $F : \mathfrak{X} \to \mathfrak{Y}$ strictly compatible with the projection to $(\mathbf{Aff}/k)$; the 2-morphisms are the isomorphisms of 1-morphisms.

3.5. **Descent.** The word "descent" is just another name for gluing appropriate for situations in which the "open sets" are morphisms (as in the étale topology) rather inclusions of subsets (as in the Zariski topology). The basic descent theorem says that morphisms of schemes can be "glued" together in the flat topology if they agree on the "intersections". The same applies to flat families of quasi-coherent sheaves. Having the notion of a sheaf and a stack to our disposition, faithfully flat descent can be stated as follows:

Theorem. Faithfully flat descent ([SGA 1], VIII 5.1, 1.1 and 1.2):

(i) (Faithfully flat descent for morphisms) For any k-scheme Z the functor of points
$\mathrm{Hom}_{(\mathbf{Aff}/k)}(-, Z) : (\mathbf{Aff}/k)^{\mathrm{op}} \to \mathbf{Set}$ is a k-space.

(ii) (Faithfully flat descent for flat families of quasi-coherent sheaves) For any scheme Z, the lax functor $(\mathbf{Aff}/k)^{\mathrm{op}} \to \mathbf{Gpd}$ defined by

$$S \mapsto \{\text{quasi-coherent } \mathcal{O}_{Z \times_k S}\text{-modules flat over} S\} + \{\text{isomorphisms}\}$$

is a k-stack.

Other descent results can be derived from these two. For instance, faithfully flat descent for principal G-bundles follows from (ii), *i.e.* the lax functor $\mathcal{M}_{G,X}$ of (3.1.1) is a k-stack.

3.6. **Algebraic stacks.** I now come to the definition of an algebraic stack, then I will show in the next section that our k-stack $\mathcal{M}_{G,X}$ is actually algebraic.

3.6.1. *The fiber of a morphism of stacks.* Fiber products exist in the category of k-stacks. I will not define them here, but rather explain what is the fiber of a morphisms of stacks, as this is all I need here. Let $F : \mathfrak{X} \to \mathfrak{Y}$ be a morphisms of stacks, let $U \in \mathrm{ob}(\mathbf{Aff}/k)$ and consider a morphism $\eta : U \to \mathfrak{Y}$, that is an object η of $\mathfrak{Y}(U)$. The *fiber* $\mathfrak{X}_\eta$ is the following stack over U:

$$\begin{aligned}
\mathfrak{X}_\eta : \mathbf{Aff}/U &\longrightarrow \mathbf{Gpd} \\
(U' \to U) &\longmapsto \{(\xi, \alpha) \ / \ \xi \in \mathrm{ob}(\mathfrak{X})(U'), \alpha : F(\xi) \xrightarrow{\sim} \eta_{|U'}\} + \\
&\qquad \{(\xi, \alpha) \xrightarrow{f} (\xi', \alpha') \ / \ \xi \xrightarrow{f} \xi' \text{ s.t. } \alpha \circ F(f) = \alpha'\}
\end{aligned}$$

3.6.2. *Representable morphisms.* The morphism F is *representable* if $\mathfrak{X}_\eta$ is representable as a scheme for all $U \in \mathrm{ob}(\mathbf{Aff}/k)$ and all $\eta \in \mathrm{ob}\,\mathfrak{Y}(U)$, *i.e.* "the fibers are schemes". All properties P of morphisms of schemes which are stable under base change and of local nature for the *fppf* topology make sense for representable morphisms of stacks. Indeed, one *defines* F to have P if for every $U \in \mathrm{ob}(\mathbf{Aff}/k)$ and every $\eta \in \mathrm{ob}(Y(\eta))$ the canonical morphism of schemes $\mathfrak{X}_\eta \to U$ has P. Examples of such properties are flat, smooth, surjective, étale, etc. ; the reader may find a quite complete list in [15].

3.6.3. *Definition.* A k-stack $\mathfrak{X}$ is *algebraic* if

(i) the diagonal morphism $\mathfrak{X} \to \mathfrak{X} \times \mathfrak{X}$ is representable, separated and quasi-compact
(ii) there is a k-scheme P and a smooth, surjective morphism $P \xrightarrow{p} \mathfrak{X}$.

Actually the representability of the diagonal is equivalent to the following: for all $U \in \mathrm{ob}(\mathbf{Aff}/k)$ and all $\eta \in \mathrm{ob}\,\mathfrak{Y}(U)$ the morphism of stacks $U \xrightarrow{\eta} \mathfrak{X}$ is representable. Hence (i) implies that p is representable (and so smoothness and surjectivity of p make sense)

Suppose $F : \mathfrak{X} \to \mathfrak{Y}$ is a *representable* morphism of algebraic k-stacks and that $\mathfrak{Y}$ is algebraic. Then $\mathfrak{X}$ is algebraic also.

3.6.4. Proposition. *Suppose Z is a k-scheme and H is a linear algebraic group over k acting on Z. Then the quotient k-stack $[Z/H]$ is algebraic:*

Proof. This follows from the definitions : a presentation is given by the morphism $p : Z \to [Z/H]$ defined by the trivial H-bundle on Z. □

3.6.5. Proposition. *The k-stack $\mathcal{M}_{GL_r,X}$ of 3.1.1 is algebraic.*

Proof. ([15],4.14.2.1) □

3.6.6. Corollary. *The k-stack $\mathcal{M}_{G,X}$ of 3.1.1 is algebraic.*

Proof. Choose an embedding $G \subset \mathbf{GL}_r$. Using Lemma 2.2.3 we may (and will) view a G-bundle E over a k-scheme Z as a $\mathbf{GL}_r$-bundle V together with a section $\sigma \in H^0(Z, V/G)$. Consider the morphism of k-stacks

$$\varphi : \mathcal{M}_{G,X} \longrightarrow \mathcal{M}_{GL_r,X}$$

defined by extension of the structure group. The corollary follows from the above proposition and the following remark:

3.6.7. *The above morphism is representable.* Let U be a k-scheme and $\eta : U \to \mathcal{M}_{GL_r,X}$ be a morphism, that is a $\mathbf{GL}_r$-bundle F over $X_U = X \times_k U$. For any arrow $U' \to U$ in $\mathbf{Aff}/k$ the $\mathbf{GL}_r$-bundle F defines a $\mathbf{GL}_r$-bundle over $X_{U'}$ which we denote by $F_{U'}$.

We have to show that the fiber $\mathcal{M}_{G,X}(\eta)$, as defined in (3.6.2), is representable as a scheme over U. As a U-stack, $\mathcal{M}_{G,X}(\eta)$ associates to every arrow $U' \to U$ the groupoid defined on the level of objects by pairs (E, α) where E is a G-bundle over $X_{U'}$ and $\alpha : E(\mathbf{GL}_r) \xrightarrow{\sim} F_{U'}$ is an isomorphism of $\mathbf{GL}_r$-bundles. On the level of morphisms we have the isomorphisms of such pairs, defined as follows: the pair (E_1, α_1) is isomorphic to the pair (E_2, α_2) if there is an isomorphism $\phi : E_1 \to E_2$ such that $\alpha_2 \circ \phi(\mathbf{GL}_r) = \alpha_1$. Such an isomorphism is, if it exists, unique for, since G acts faithfully on $\mathbf{GL}_r$, $\phi(\mathbf{GL}_r) = \alpha_2^{-1} \circ \alpha_1$ uniquely determines ϕ. Therefore, the fiber is a U-space. Moreover, the set of pairs (E, α) is canonically bijective to the set $\mathrm{Hom}_{X_{U'}}(X_{U'}, (F/G)_{U'})$. An easy verification shows that this bijection is functorial, *i.e.* defines an isomorphism between the U-space of the above pairs and the functor which associates to $U' \to U$ the above set of sections. So we are reduced to show that the latter functor is representable. But this follows from Grothendiecks theory of Hilbert schemes ([10], pp. 19–20), once we know that $F/G \to X_U$ is quasi-projective. In order to see this last statement we use Chevalley's theorem on semi-invariants: there is a representation V of $\mathbf{GL}_r$ with a line ℓ such that G is the stabilizer (in $\mathbf{GL}_r$) of ℓ. We get an embedding $\mathbf{GL}_r/G \subset \mathbb{P}(V^*)$, hence the required embedding

$$F/G \subset \mathbb{P}(F(V^*)).$$

(Actually the line bundle on F/G which corresponds to the above embedding is nothing else than the line defined by extension of the structure group of the G-bundle $F \to F/G$ via χ^{-1} where χ is the character defined by the action of G on ℓ.) $\square$

3.6.8. Proposition. *Suppose G is reductive. The algebraic stack $\mathcal{M}_{G,X}$ is smooth of dimension* $\dim(G)(g-1)$.

This follows from deformation theory. I will be rather sketchy here as rendering precise the arguments below is quite long. Let E be a G-bundle. Consider the action of G on $\mathfrak{g}$ given by the adjoint representation and then the vector bundle $E(\mathfrak{g})$. The obstruction to smoothness of $\mathcal{M}_{G,X}$ lives in $H^2(X, E(\mathfrak{g}))$ which vanishes since X is of dimension 1. The infinitesimal deformations of E are parameterized by $H^1(X, E(\mathfrak{g}))$ with global automorphisms parameterized by $H^0(X, E(\mathfrak{g}))$. Over schemes in order to calculate the dimension we would calculate the rank of its tangent bundle. We can do this here also but on stacks one has to be careful about how one understands the "tangent bundle". We see this readily here: for example for $G = GL_r$ the tangent space $H^1(X, \underline{\mathrm{End}}(E,E))$ is not of constant dimension over the connected components but only over the open substack of simple vector bundles. Of course $\dim H^1(X, \underline{\mathrm{End}}(E,E))$ jumps exactly when $\dim H^0(X, \underline{\mathrm{End}}(E,E))$ jumps, so again one has to take care of global automorphisms. However, we may consider the tangent complex on $\mathcal{M}_{G,X}$. In our case this complex is $Rpr_{1*}(\mathcal{E}(\mathfrak{g}))$ where $\mathcal{E}$ is the universal G-bundle over $\mathcal{M}_{G,X} \times X$, which may be represented by a perfect complex of length one (see section 6.1.1 for this). By definition, the dimension of the stack $\mathcal{M}_{G,X}$ at the point E is the rank of the cotangent complex at E, which is $-\chi(E(\mathfrak{g}))$. If G is reductive there is an isomorphism $\mathfrak{g} \to \mathfrak{g}^*$ of G-modules. Therefore we know that $\deg(E(\mathfrak{g})) = 0$ and then Riemann-Roch gives $\dim \mathcal{M}_{G,X} = \dim(G)(g-1)$. If $g(X) = 0$, then its dimension is $-\dim(G)$, which may be surprising, but which is, in view of the above, the only reasonable result we may ask for (the standard example of a stack with negative dimension is $BG = [\cdot/H]$ which is of dimension $-\dim H$).

4. Topological classification

Here X is a compact connected oriented smooth real surface of genus g and G a connected topological group. A topological G bundle E over X is a topological space E on which G acts from the right together with a G-invariant continuous map $E \xrightarrow{\pi} X$ such that for every $x \in X$ there is an

open neighborhood U of x such that $E_{|U}$ is trivial, *i.e.* isomorphic to $U \times G$ as a G-homogeneous space where G acts on $U \times G$ by right multiplication.

4.1. **Topological loop groups.** Let $x_0 \in X$ and let D be a neighborhood of x homeomorphic to a disc. Define $D^* = D - x_0$ and $X^* = X - x_0$. Associated are the following three groups

$$L^{top}G = \{f : D^* \longrightarrow G/f \text{ is continuous}\}$$
$$L^{top}_+G = \{f : D \longrightarrow G/f \text{ is continuous}\}$$
$$L^{top}_XG = \{f : X^* \longrightarrow G/f \text{ is continuous}\}$$

By definition, we have the following inclusions:

$$L^{top}_XG \subset L^{top}G \supset L^{top}_+G$$

Let $\mathcal{M}^{top}_{G,X}$ be the set of isomorphism classes of topological G-bundles on X.

4.1.1. Proposition. *There is a canonical bijection*

$$L^{top}_XG \backslash L^{top}G / L^{top}_+G \xrightarrow{\sim} \mathcal{M}^{top}_{G,X}$$

Proof. The basic observation is that if E is a topological G-bundle on X then the restrictions of E to D and X^* are trivial. For the restriction to D this is clear, since D is contractible; for the restriction to X^* we view X as a CW-complex of dimension 2 and remark that, since G is connected, there is no obstruction to the existence of a section of a G-bundle on X^*. It follows that if we choose trivialization $\sigma : E_{|D} \xrightarrow{\sim} D \times G$ and $\tau : E_{|X^*} \xrightarrow{\sim} X^* \times G$ then the transition function $\gamma = \tau \circ \sigma^{-1}_{|D^*}$ is an element of $L^{top}G$. On the other hand, we may take trivial bundles on D and X^* and patch them together by γ in order to get a G-bundle E on X. Therefore there is a canonical bijection

$$L^{top}G = \{(E, \sigma, \tau) \ /E \xrightarrow{G} X,\ \sigma : E_{|D} \xrightarrow{\sim} D \times G,\ \tau : E_{|X^*} \xrightarrow{\sim} X^* \times G\}$$

Now, by construction, multiplying $\gamma \in L^{top}G$ from the right by $\alpha \in L^{top}_+G$ corresponds under this bijection to changing the trivialization σ by $\alpha^{\#} \circ \sigma$, where $\alpha^{\#}$ is the map $D \times G \to D \times G$ defined by $(z, g) \mapsto (z, g\alpha(z))$ and analogously multiplying from the left by $\beta^{-1} \in L^{top}_XG$ corresponds to change the trivialization τ. It follows that dividing by L^{top}_+G forgets about the trivialization σ and dividing by L^{top}_XG forgets about the trivialization τ, hence the proposition. □

4.1.2. Corollary. *The set $\mathcal{M}^{top}_{G,X}$ is in bijective correspondence with $\pi_1(G)$.*

Proof. If $\gamma \in L^{top}G$, we denote by $\gamma_* : \pi_1(D^*) \to \pi_1(G)$ the induced map. Let δ be the positive generator of $\pi_1(D^*)$ and consider the map

$$\begin{aligned} f : L^{top}G &\longrightarrow \pi_1(G) \\ \gamma &\longmapsto \gamma_*(\delta) \end{aligned}$$

Now f depends only on the double classes. In order to see this consider for $\alpha \in L_+^{top}G$ and $\beta \in L_X^{top}G$ the element $\beta^{-1}\gamma\alpha$ which we view as an element of $L^{top}G$ as follows: $z \mapsto \beta^{-1}(z)\gamma(z)\alpha(z)$. Then remark that the composite $D^* \to D \xrightarrow{\alpha} G$ is homotopically trivial since it extends to D. For the composite $D^* \to X^* \xrightarrow{\beta} G$ consider the induced map $\pi_1(D^*) \to \pi_1(X^*) \to \pi_1(G)$ and remark (exercise) that the image of $\pi_1(D^*)$ in $\pi_1(X^*)$ has to sit inside the commutator subgroup. It follows that its image in $\pi_1(G)$ is trivial, since $\pi_1(G)$ is abelian. Thus $D^* \to X^* \xrightarrow{\beta} G$ is also homotopically trivial. Therefore $\beta^{-1}\gamma\alpha$ is homotopic to γ, hence f depends only on the double classes. Then it is an easy exercise to see that the induced map on the double quotient is indeed a bijection. □

5. Uniformization

The uniformization theorem is the analogue of proposition 4.1.1 in the algebraic setup. Let k be an algebraically closed field, X be a smooth, connected and complete algebraic curve over k and G be an affine algebraic group over k. We choose a closed point $x_0 \in X$ and consider $X^* = X - \{x_0\}$. Remark that X^* is affine (map X to $\mathbb{P}^1$ using a rational function f with pole of some order at x_0 and regular elsewhere and remark that $f^{-1}(\mathbb{A}^1) = X^*$).

What is the algebraic analogue of the "neighborhood of x_0 homeomorphic to a disc" of section 4? What we can do is to look at the local ring $\mathcal{O}_{X,x_0}$ and then consider its completion $\widehat{\mathcal{O}}_{X,x_0}$. Then $D_{x_0} = \mathrm{Spec}(\widehat{\mathcal{O}}_{X,x_0})$ will be convenient for if we choose a local coordinate z at $x_0 \in X$ then we may identify $\widehat{\mathcal{O}}_{X,x_0}$ with $k[[z]]$, hence D_{x_0} with the "formal disc" $D = \mathrm{Spec}\left(k[[z]]\right)$.

Moreover, $D_{x_0}^* = D - \{x_0\}$ is $\mathrm{Spec}(K_{x_0})$, where K_{x_0} is the field of fractions of $\widehat{\mathcal{O}}_{X,x_0}$. Using our local coordinate z we see that K_{x_0} identifies to $k((z))$, hence $D_{x_0}^*$ to $D^* = \mathrm{Spec}\left(k((z))\right)$.

It will be convenient in the following to introduce the following notations: if $U = \mathrm{Spec}(R)$ then we will denote $D_U^* = \mathrm{Spec}\left(R((z))\right)$, $D_U = \mathrm{Spec}\left(R[[z]]\right)$ and $X_U^* = X^* \times U$.

5.1. **Algebraic loop groups.** The algebraic analogue of the topological loop group $L^{top}G$ is $\mathrm{Hom}_{alg}(D^*, G)$, that is, the points of G with values in

D^*, *i.e.* $G\big(k((z))\big)$. This has to be made functorial so we will consider the functor

$$\begin{aligned} LG : (\mathbf{Aff}/k) &\longrightarrow \mathbf{Grp} \\ U = \mathrm{Spec}(R) &\longmapsto G\big(R((z))\big) \end{aligned}$$

Actually that is a k-group (in the sense of 3.3). We define the k-groups $L_X G$ and L^+G as well by $U \mapsto G\big(\mathcal{O}(X_U^*)\big)$ and $U \mapsto G\big(R[[z]]\big)$ respectively.

We denote $\mathcal{Q}_G$ the quotient k-space LG/L^+G: this is the sheafification of the presheaf

$$U = \mathrm{Spec}(R) \longmapsto G\big(R((z))\big)/G\big(R[[z]]\big).$$

The k-group $L_X G$ acts on the k-space $\mathcal{Q}_G$; let $[L_X G \backslash \mathcal{Q}_G]$ be the quotient k-stack of 3.3.1.

5.1.1. Theorem. *(Uniformization) Suppose G is semi-simple. Then there is a canonical isomorphism of stacks*

$$[{}_{L_X G}\backslash LG/_{L^+G}] \xrightarrow{\sim} \mathcal{M}_{G,X}$$

Moreover, the $L_X G$-bundle $\mathcal{Q}_G \xrightarrow{L_X G} \mathcal{M}_{G,X}$ is even locally trivial for the étale topology if the characteristic of k does not divide the order of $\pi_1(G(\mathbb{C}))$.

5.2. **Key inputs.** The theorem has two main inputs in its proof:

- Trivializing G-bundles over X_U^* (for this we need G semi-simple)
- Gluing trivial G-bundles over X_U^* and D_U to a G-bundle over X_U.

Both properties are highly non trivial in our functorial setup where U may be any affine k-scheme, not necessarily noetherian. So I discuss them first.

5.2.1. *Trivializing G-bundles over the open curve.* For general G it is not correct that the restriction of a G-bundle to X^* is trivial. The basic examples are of course line bundles. However, if we consider vector bundles with trivial determinant (*i.e.* $\mathbf{SL}_r$-bundles) then this becomes true. The reason is that a vector bundle E over X^* may be written as the direct sum $\mathcal{O}_{X^*}^{\oplus r} \oplus \det(E_{|X^*})$ (translate to the analogue statement of finite module over a ring and use that $\mathcal{O}(X^*)$ is Dedekind as X^* is a smooth curve). Now if E is a vector bundle with trivial determinant on X_U we may ask whether, locally (for an appropriate topology) on U, the restriction of E to X_U^* is trivial. This is indeed true (for the Zariski topology on U) and the argument proceeds by induction on the rank r of E ([2], 3.5), the rank 1 case being trivial: consider the divisor $d = \{x_0\} \times U$ of X_U and choose an integer n such that $E(nd)$ has

no higher cohomology and is generated by its global sections. Then consider a point $u \in U$ and a nowhere vanishing section s of $E(nd)_{|X\times\{u\}}$ (count dimensions in order to see its existence). Shrinking U, one may suppose that this section is the restriction to $E(nd)$ of a section which does not vanish on X_U. When restricting to X_U^* we get an exact sequence

$$0 \to \mathcal{O}_{X_U^*} \longrightarrow E_{|X_U^*} \longrightarrow F \to 0$$

where F is a vector bundle. But after shrinking U again we may assume that F is trivial by induction and that the sequence splits, hence $E_{|X_U^*}$ is trivial.

The natural guess then is that the above trivialization property is true for semi-simple G at least for the appropriate topology on U. This has been proved by Drinfeld and Simpson.

Theorem (Drinfeld-Simpson). [7] *Suppose G is semi-simple. Let E be a G-bundle over X_U. Then the restriction of E to X_U^* is trivial, locally for the fppf topology over U. If $char(k)$ does not divide the order of $\pi_1(G(\mathbb{C}))$, then this is even true locally for the étale topology over U.*

I will not enter into the proof, however I will invite the reader to have a closer look at their note, as it uses some techniques which are quite useful also in other contexts.

5.2.2. *Gluing.* Consider the following cartesian diagram

$$\begin{array}{ccc} D_U^* & \text{——} & D_U \\ | & & | \\ X_U^* & \text{——} & X_U \end{array}$$

Given trivial G-bundles on X_U^* and D_U and an element $\gamma \in G\big(R((z))\big)$ we want to glue them to a G-bundle E on X_U. The reader might say that this is easy: just apply what we have learned about descent in section 3. However some care has to be taken here: if U is not noetherian, then the morphism $D_U \to X_U$ is *not* flat! Nevertheless the gluing statement we need is true:

Theorem (Beauville-Laszlo). [3] *Let $\gamma \in G\big(R((z))\big)$. Then there exists a G-bundle E on X_U and trivializations $\sigma : E_{|D_U} \to D_U \times G$, $\tau : E_{|X_U^*} \to X_U^* \times G$. Moreover the triple (E, σ, τ) is uniquely determined up to unique isomorphism.*

Actually the above theorem is proved for vector bundles in [3] but the generalization to G-bundles is immediate. Again, I will not enter into the proof, but invite the reader to have a look at their note.

5.3. **Proof of the uniformization theorem.** Once the above two key inputs are known, the proof of the uniformization theorem is essentially formal.

We start considering the functor T_G of triples:

$$T_G : (\mathbf{Aff}/k) \longrightarrow \mathbf{Set}$$

$$U \longmapsto \{(E,\tau,\sigma) \ / \ E \xrightarrow{G} X_U \text{ is a } G\text{-bundle with trivializations } \sigma : E_{|D_U} \xrightarrow{\sim} D_U \times G, \tau : E_{|X_U^*} \xrightarrow{\sim} X_U^* \times G.\}/\sim$$

5.3.1. Proposition. *The k-group LG represents the functor T_G.*

Proof. Let (E,τ,σ) be an element of $T_G(U)$. Pulling back the trivializations τ and σ to D_U^* provides two trivializations τ^* and σ^* of the pullback of E over D_U^*: these trivializations differ by an element $\gamma = \tau^{*-1} \circ \sigma^*$ of $G\big(R((z))\big)$ (as usual $U = \mathrm{Spec}(R)$). Conversely, if $\gamma \in G\big(R((z))\big)$, we get an element of $T_G(U)$ by the Beauville-Laszlo theorem. These constructions are inverse to each other by construction. □

Now consider the functor of pairs P_G:

$$P_G : (\mathbf{Aff}/k) \longrightarrow \mathbf{Set}$$

$$U \longmapsto \{(E,\tau) \ / \ E \xrightarrow{G} X_U \text{ is a } G\text{-bundle with trivialization } \tau : E_{|X_U^*} \xrightarrow{\sim} X_U^* \times G.\}/\sim$$

5.3.2. Proposition. *The k-space $\mathcal{Q}_G$ represents the functor P_G.*

Proof. Let $U = \mathrm{Spec}(R)$ be an affine k-scheme and q be an element of $\mathcal{Q}_G(U)$. By definition of $\mathcal{Q}_G$ as a quotient k-space, there exists a faithfully flat homomorphism $U' \to U$ and an element γ of $G\big(R'((z))\big)$ ($U' = \mathrm{Spec}(R')$) such that the image of q in $\mathcal{Q}_G(U')$ is the class of γ. To γ corresponds by 5.3.1 a triple (E',τ',σ') over $X_{U'}$. Let $U'' = U' \times_U U'$, and let (E_1'',τ_1''), (E_2'',τ_2'') denote the pullbacks of (E',τ') by the two projections of $X_{U''}$ onto $X_{U'}$. Since the two images of γ in $G\big(R''((z))\big)$ differ by an element of $G\big(R''[[z]]\big)$, these pairs are isomorphic. So the isomorphism $\tau_2''\tau_1''^{-1}$ over $X_{U''}^*$ extends to an isomorphism $u : E_1'' \to E_2''$ over $X_{U''}$, satisfying the usual co-cycle condition (it is enough to check this over X^*, where it is obvious). Therefore

(E', τ') descends to a pair (E, τ) on X_R as in the above statement. Conversely, given a pair (E, τ) as above over X_U, we can find a faithfully flat homomorphism $U' \to U$ and a trivialization σ' of the pullback of E over $D_{U'}$ (after base change, we may assume that the central fiber of the restriction of E to D_U has a section then use smoothness to extend this section to D_U). By 5.3.1 we get an element γ' of $G\big(R'((z))\big)$ such that the two images of γ' in $G\big(R''((z))\big)$ (with $R'' = R' \otimes_R R'$) differ by an element of $G\big(R''[[z]]\big)$; this gives an element of $\mathcal{Q}_G(U)$. These constructions are inverse to each other by construction. $\square$

5.3.3. *End of the proof.* The universal G-bundle over $X \times \mathcal{Q}_G$ (see 5.3.2), gives rise to a map $\pi : \mathcal{Q}_G \to \mathcal{M}_{G,X}$. This map is $L_X G$-invariant, hence induces a morphism of stacks $\overline{\pi} : L_X G \backslash \mathcal{Q}_G \to \mathcal{M}_{G,X}$. On the other hand we can define a map $\mathcal{M}_{G,X} \to L_X G \backslash \mathcal{Q}_G$ as follows. Let U be an affine k-scheme, E a G-bundle over X_U. For any arrow $U' \to U$, let $T(U')$ be the set of trivializations τ of $E_{U'}$ over $X^*_{U'}$. This defines a U-space T on which the group $L_X G$ acts. By Drinfeld-Simpson's theorem, it is an $L_X G$-bundle. To any element of $T(U')$ corresponds a pair $(E_{U'}, \tau)$, hence by 5.3.2 an element of $\mathcal{Q}_G(U')$. In this way we associate functorially to an object E of $\mathcal{M}_{G,X}(U)$ an $L_X G$-equivariant map $\alpha : T \to \mathcal{Q}_G$. This defines a morphism of stacks $\mathcal{M}_{G,X} \to L_X G \backslash \mathcal{Q}_G$ which is the inverse of $\overline{\pi}$. The second assertion means that for any scheme U over k (resp. over k such that $char(k)$ does not divide the order of $\pi_1(G(\mathbb{C}))$) and any morphism $f : U \to \mathcal{M}_{G,X}$, the pullback to U of the fibration π is *fppf* (resp. *étale*) locally trivial, i.e. admits local sections for the *fppf* (resp. *étale*) topology. Now f corresponds to a G-bundle E over X_U. Let $u \in U$. Again by the Drinfeld-Simpson theorem, we can find an *fppf* (resp. *étale*) neighborhood U' of u in U and a trivialization τ of $E_{|X^*_{U'}}$. The pair (E, τ) defines a morphism $g : U' \to \mathcal{Q}_G$ (by 5.3.2) such that $\pi \circ g = f$, that is a section over U' of the pullback of the fibration π.

6. The determinant and the pfaffian line bundles

Let X be a projective curve, smooth and connected over the algebraically closed field k.

6.1. **The determinant bundle.** Let F be a vector bundle over $X_S = X \times_k S$, where S is a locally noetherian k-scheme. As usual we think of F as a family of vector bundles parameterized by S.

6.1.1. *Representatives of the cohomology.* In the following I will call a complex $K^\bullet$ of coherent locally free $\mathcal{O}_S$-modules

$$0 \to K^0 \xrightarrow{\gamma} K^1 \to 0$$

a *representative* of the cohomology of F if for every base change $T \xrightarrow{f} S$

$$\begin{array}{ccc} X_T & \overset{g}{\text{———}} & X_S \\ {\scriptstyle u}\Big| & & \Big|{\scriptstyle p} \\ T & \overset{f}{\text{———}} & S \end{array}$$

we have $H^i(f^*K^\bullet) = R^i u_* g^* F$. In particular, if $s \in S$ is a closed point:

$$H^i(K^\bullet_s) = H^i(X, F_s)$$

Representatives of the cohomology of F are easy to construct in our setup. Indeed, we may choose a resolution

$$0 \to P_1 \longrightarrow P_0 \longrightarrow F \to 0$$

of F by S-flat coherent $\mathcal{O}_{X_S}$-modules such that $p_* P_0 = 0$ (use Serre's theorem A in its relative version to see its existence). Then we have $p_* P_1 = 0$ and, by base change for coherent cohomology, the complex

$$0 \to R^1 p_* P_1 \longrightarrow R^1 p_* P_0 \to 0$$

is convenient. This result is generally quoted as choosing a perfect complex of length one representing Rp_*F in the derived category[3] $\mathcal{D}_c(S)$

6.1.2. *The determinant bundle.* The determinant of a complex $K^\bullet$ of locally free coherent $\mathcal{O}_S$-modules $0 \to K^0 \longrightarrow K^1 \to 0$ if defined by

$$\det(K^\bullet) = \bigwedge^{max} K^0 \otimes (\bigwedge^{max} K^1)^{-1}$$

The determinant of our family F of vector bundles parameterized by S is defined by[4]

$$\mathcal{D}_F = \det(Rp_*F)^{-1}$$

[3] All the derived category theory I need here and in the proof of 6.2.2 is in ([6],§1). The category of complexes of $\mathcal{O}_S$-modules will be denoted by $\mathcal{C}(S)$; the category with the same objects $\mathcal{C}(S)$ but morphisms homotopy classes of morphisms of $\mathcal{C}(S)$ will be denoted by $\mathcal{K}(S)$. Finally $\mathcal{D}(S)$ is obtained by inverting the quasi-isomorphisms in $\mathcal{K}(S)$. A superscript b (resp. subscript c) means that we consider the full sub-categories of bounded complexes (resp. complexes with coherent cohomology).

[4] The minus sign is chosen in order to get the "positive" determinant bundle.

In general, in order to calculate $\mathcal{D}_F$, we choose a representative $K^\bullet$ of the cohomology of F and then calculate $\det(K^\bullet)^{-1}$. This does not depend, up to canonical isomorphism, on the choice of $K^\bullet$ (and this is the reason why the above definition makes sense) [11].

By construction, the fiber of $\mathcal{D}_F$ at $s \in S$ is given as follows:

$$\mathcal{D}_F(s) = (\bigwedge^{max} H^0(X, F_s))^{-1} \otimes \bigwedge^{max} H^1(X, F_s)$$

We may also twist our family F by bundles coming from X, *i.e.* consider $F \otimes q^*E$ where E is a vector bundle on X. We obtain the line bundle $\mathcal{D}_{F\otimes q^*E}$, and this line bundle actually depends only on the class of E in the Grothendieck group $K(X)$ of X (check this!). It follows that we get a group morphism, Le Potier's determinant morphism [16]

$$\begin{aligned} \lambda_F : K(X) &\longrightarrow \mathrm{Pic}(S) \\ u &\longmapsto \mathcal{D}_{F\otimes q^*u} \end{aligned}$$

If our bundle F comes from a $\mathbf{SL}_r$-bundle, *i.e.* has trivial determinant, twisting F by an element $u \in K(X)$ then taking determinants just means taking the $r(u)$-th power of $\mathcal{D}_F$:

6.1.3. Lemma. *Suppose F is a vector bundle on X_S such that $\bigwedge^{max} F$ is the pullback of some line bundle on X. Then*

$$\mathcal{D}_{F\otimes q^*u} = \mathcal{D}_F^{\otimes r(u)} \text{ in } \mathrm{Pic}(S)$$

where $r(u)$ is the rank of u.

Proof. We may suppose that u is represented by a vector bundle L and even – after writing L as an extension – that L is a line bundle. But then it is enough to check it for $L = \mathcal{O}_X(-p)$, for $p \in X$, where it follows, after considering $0 \to \mathcal{O}_X(-p) \to \mathcal{O}_X \to \mathcal{O}_p \to 0$, from the fact $\mathcal{D}_{F\otimes q^*\mathcal{O}_p}$ is trivial under our hypothesis on F. □

6.1.4. *Theta-functions.* Twisting is particularly useful in order to produce sections of (powers of) the determinant bundle. Suppose S is integral and that F is a vector bundle on X_S with trivial determinant. Choose a vector bundle E such that $F_s \otimes q^*E$ has trivial Euler characteristic for some s. If

$$0 \to K^0 \xrightarrow{\gamma} K^1 \to 0$$

is a representative of the cohomology of $F \otimes q^*E$, then we know that the rank n of K^0 is equal to the rank of K^1, hence γ may be locally represented as a $n \times n$-matrix. We get a section $\theta_E = \det(\gamma)$ of $D_F^{r(E)}$, well defined

up to an invertible function on S: the *theta-function associated to* E. In particular, its divisor Θ_E is well defined with support the points $s \in S$ such that $H^0(F_s \otimes E) \neq 0$.

If we suppose moreover that $F_t \otimes q^*E$ has trivial cohomology for some $t \in S$ then $\Theta_E \neq S$, *i.e.* the section θ_E is non trivial; if there is $t' \in S$ such that $H^0(X, E_{t'} \otimes E) \neq 0$ then $\Theta_E \neq \emptyset$.

6.2. **The pfaffian line bundle.** Suppose $\mathrm{char}(k) \neq 2$ in this subsection. Let F be a vector bundle over $X_S = X \times S$, together with a quadratic non degenerate form σ with values in the canonical bundle ω_X. We will view σ as an isomorphism $F \xrightarrow{\sim} F^\vee$ such that $\sigma = \sigma^\vee$, where $F^\vee = \underline{\mathrm{Hom}}_{\mathcal{O}_{X_S}}(F, q^*\omega_X)$.

6.2.1. Lemma. *If $K^\bullet$ is a representative of the cohomology of F, then $K^{\bullet *}[-1]$ is a representative of the cohomology of $F^\vee$.*

Here[5] $K^{\bullet *}[-1]$ denotes the complex supported in degrees 0 and 1

$$0 \to K^{1*} \xrightarrow{-\gamma^*} K^{0*} \to 0.$$

Proof. In the derived category $\mathcal{D}_c(S)$, we have

$$\begin{aligned} Rp_*(F^\vee) &\xrightarrow{\sim} Rp_*(R\underline{\mathrm{Hom}}_{\mathcal{O}_{X_S}}(F, q^*\omega_X)) && (F \text{ is locally free}) \\ &\xrightarrow{\sim} R\underline{\mathrm{Hom}}(Rp_*(F), \mathcal{O}_S)[-1] && (\text{Grothendieck-Serre duality}) \end{aligned}$$

Now if $K^\bullet$ represents the cohomology of F we see that $R\underline{\mathrm{Hom}}(K^\bullet, \mathcal{O}_S)[-1]$ represents the cohomology of $F^\vee$. But this is nothing else than $K^{\bullet *}[-1]$ as the K^i are locally free. □

6.2.2. Proposition. *There exists, locally for the Zariski topology on S, a representative of the cohomology $K^\bullet$ of F and a symmetric isomorphism:*

$$\tau : K^\bullet \xrightarrow{\sim} K^{\bullet *}[-1]$$

such that τ and σ induce the same map in cohomology.

Proof. Choose a representative $\widetilde{K}^\bullet$ of the cohomology of F and remark that σ induces an isomorphism $\widetilde{\tau}$ in the derived category $\mathcal{D}^b_c(S)$

$$\widetilde{K}^\bullet \xrightarrow{\sim} Rp_*F \xrightarrow{\sigma} Rp_*(F^\vee) \xrightarrow{\sim} \widetilde{K}^{\bullet *}[-1]$$

which is still symmetric (this follows from the symmetry of σ and standard properties of Grothendieck-Serre duality).

[5] This is compatible with the usual signs: the dual of $K^\bullet$ is supported in degrees -1 and 0; when translated to the right by 1, the differential acquires a -1 sign.

The problem here is that this isomorphism is only defined in the derived category: the proposition actually claims that we can get a symmetric isomorphism of complexes and this we only get Zariski locally.

First we may suppose that S is affine. Then the category of coherent sheaves on S has enough projectives and as the $\widetilde{K}^i$ are locally free we see that $\widetilde{\tau}$ is an isomorphism in $\mathcal{K}_c^b(S)$. Let φ be a lift of $\widetilde{\tau}$ to $\mathcal{C}_c^b(S)$. We get a morphism of complexes

$$\begin{tikzcd} \widetilde{K}^0 \arrow[r, dash, "\gamma"] \arrow[d, dash, "\varphi_0"'] & \widetilde{K}^1 \arrow[d, dash, "\varphi_1"] \\ \widetilde{K}^{1*} \arrow[r, dash, "-\gamma^*"] & \widetilde{K}^{0*} \end{tikzcd}$$

which needs neither to be symmetric nor an isomorphism (it is only a quasi-isomorphism). First we symmetrize: $\phi_i = (\varphi_i + \varphi_{1-i}^*)/2$ for $i = 0, 1$. Remark that ϕ is still a quasi-isomorphism, inducing σ in cohomology. Then we fix $s \in S$. A standard argument shows that there is, in a neighborhood of s, another length one complex $K^\bullet$ of free coherent $\mathcal{O}_S$-modules together with a quasi-isomorphism $u : K^\bullet \to \widetilde{K}^\bullet$, such that for the differential d we have $d_{|s} = 0$. Now

$$\tau = u^*[-1]\phi u : K^\bullet \longrightarrow K^{\bullet *}[-1]$$

is a symmetric quasi-isomorphism, inducing σ in cohomology, and $\tau_{|s}$ is an isomorphism. Then, in a neighborhood of s, $\tau_{|s}$ will remain an isomorphism which proves the proposition. $\square$

Let $(K^\bullet, \tau)$ be as in the proposition and consider the following diagram

$$\begin{tikzcd} K^0 \arrow[r, dash, "\gamma"] \arrow[d, dash, "\tau_0"', "\wr"] \arrow[dr, dotted, dash, "\alpha"] & K^1 \arrow[d, dash, "\tau_0^*", "\wr"'] \\ K^{1*} \arrow[r, dash, "-\gamma^*"'] & K^{0*} \end{tikzcd}$$

It follows that α is *skew-symmetric*. Therefore the cohomology of F may be represented, locally for the Zariski topology on S, by complexes of free coherent $\mathcal{O}_S$-modules

$$0 \to K \xrightarrow{\alpha} K^* \to 0$$

with α skew. Such complexes will be called *special* in the following.

An immediate corollary is Riemann's invariance mod 2 theorem:

6.2.3. Corollary. [6] *Let F be a vector bundle on X_S equipped with a non degenerate quadratic form with values in ω_X. Then the function*

$$s \mapsto \dim H^0(X, F_s) \bmod 2$$

is locally constant.

Proof. Locally there is a special representative $K^\bullet$ of the cohomology of F.

$$\dim H^0(X, F_s) = \operatorname{rank} K - \operatorname{rank} \alpha$$

Now use that the rank of α is even as α is skew. □

6.3. **The pfaffian bundle.** Let F be a vector bundle on X_S equipped with a non degenerate quadratic form with values in ω_X and cover S by Zariski open subsets U_i such that F admits a special representative $K_i^\bullet$ of the cohomology of F on U_i. Over U_i

$$\mathcal{D}_{F_{|U_i}} = \bigwedge^{max} K_i^* \otimes \bigwedge^{max} K_i^*$$

which is a square. It turns out, because the $K^\bullet$ are *special* complexes, that the $\bigwedge^{max} K_i^*$ glue together over S and define a canonical square root of $\mathcal{D}_F$, called the *pfaffian bundle*.

This gluing requires quite some work and is the content of ([14], §7). I will not enter into the proof here: *(loc.cit.)* is self contained.

6.3.1. Theorem. *Let F be a vector bundle over X_S equipped with a non degenerate quadratic form σ with values in ω_X. Then the determinant bundle $\mathcal{D}_F$ admits a canonical square root $\mathcal{P}_{(F,\sigma)}$. Moreover, if $f : S' \to S$ is a morphism of locally noetherian k-schemes then we have $\mathcal{P}_{(f^*F, f^*\sigma)} = f^*\mathcal{P}_{(F,\sigma)}$.*

6.4. **The pfaffian bundle on the moduli stack.** Let $r \geq 3$ and (F, σ) be the universal $\mathbf{SO}_r$-bundle over $\mathcal{M}_{SO_r,X} \times X$. If we twist by a theta-characteristic κ (*i.e.* a line bundle such that $\kappa \otimes \kappa = \omega_X$), then $F_\kappa = F \otimes q^*\kappa$ comes with a non-degenerate form with values in ω_X. Then we may apply 6.3.1 in order to get the pfaffian bundle $\mathcal{P}_{(F_\kappa,\sigma)}$ which we denote simply by $\mathcal{P}_\kappa$.

[6] In fact the above arguments are valid for any smooth proper morphism $Y \to S$ of relative dimension 1. I only consider the situation of a product $Y = X \times S$ here as this is the one I need in order to define the determinant resp. pfaffian bundles.

6.5. **The square-root of the dualizing sheaf.** Suppose G is semi-simple and consider its action on $\mathfrak{g}$ given by the adjoint representation. It follows from the proof of Proposition 3.6.8, that the dualizing sheaf $\omega_{\mathcal{M}_{G,X}}$ is $\mathcal{D}_{\mathcal{E}(\mathfrak{g})}$ where $\mathcal{E}$ is the universal G-bundle on $\mathcal{M}_{G,X}$. Remark that the bundle $\mathcal{E}(\mathfrak{g})$ comes with a natural quadratic form given by the Cartan-Killing form. Hence the choice of a theta-characteristic κ defines, by the above, a square root $\omega^{1/2}_{\mathcal{M}_{G,X}}(\kappa)$ of $\omega_{\mathcal{M}_{G,X}}$.

6.6. **The pfaffian divisor.** It may seem easier to construct the pfaffian bundle looking at it from a divisorial point of view using smoothness of $\mathcal{M}_{SO_r,X}$. Suppose F is a vector bundle on X_S equipped with a non degenerate quadratic form σ with values in ω_X. Suppose moreover that S is smooth and that there are points $s, t \in S$ such that $H^0(X, F_s) = 0$ and $H^0(X, F_t) \neq 0$. We know from section 6.1.4 that if $K^0 \xrightarrow{\gamma} K^1$ represents the cohomology of F, then $\mathcal{D}_F$ is the line bundle associated to the divisor defined by $\det(\gamma) = 0$. Now, locally γ may be represented by a skew-symmetric matrix α, so we may take its pfaffian. This defines a local equation for a divisor, which will be called the pfaffian divisor, hence, by smoothness of S, our pfaffian line bundle. Of course the preceding sentence has to be made rigorous, but this may seem easier than the (rather formal) considerations of ([14], §7) which lead to Theorem 6.3.1.

However[7], even if the hypothesis of smoothness is satisfied for $\mathcal{M}_{SO_r,X}$, this approach fails to define line bundles $\mathcal{P}_\kappa$ for all theta-characteristics κ.

The point is that the hypotheses of the existence of $s \in S$ such that $H^0(X, F_s) = 0$ is not always satisfied: the equation $\det(\gamma) = 0$ may not define a divisor (but the whole space).

In order to see this, consider the component $\mathcal{M}_{SO^0_r,X}$ of $\mathcal{M}_{SO_r,X}$, containing the trivial $\mathbf{SO}_r$-bundle. Actually we haven't seen yet that the connected components of $\mathcal{M}_{G,X}$ are parameterized by $\pi_1(G)$ (*i.e.* by their topological type), but for the moment let's just use that $\mathcal{M}_{SO_r,X}$ has two components: $\mathcal{M}_{SO^0_r,X}$ and $\mathcal{M}_{SO^1_r,X}$. They are distinguished by the second Stiefel-Whitney class

$$w_2 : H^1_{\acute{e}t}(X, SO_r) \to H^2_{\acute{e}t}(X, \mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}. \tag{6.6 a}$$

Let $r \geq 3$ and (F, σ) be the universal quadratic bundle over $\mathcal{M}_{SO^0_r,X} \times X$. For κ a theta-characteristic, consider the substack Θ_κ of section 6.1.4.

[7] There are of course many other reasons to prefer to construct a line bundle directly and not as a line bundle associated to a divisor.

6.6.1. Proposition. *The substack Θ_κ of $\mathcal{M}_{SO^0_r,X}$ is a divisor if and only if r or κ are even.*

Proof. We start with a useful lemma.

6.6.2. Lemma. *Let $A = (E, q)$ be an SO_r-bundle, $r \geq 3$ and κ be a theta-characteristic. Then*

$$w_2(A) = h^0(E \otimes \kappa) + rh^0(\kappa) \bmod 2. \tag{6.6 b}$$

Proof. Indeed, by Riemann's invariance mod 2 theorem, the right-hand side of (6.6 b), denoted $w_2'(A)$ in the following, is constant over the 2 connected components of $\mathcal{M}_{SO_r,X}$. Because (6.6 b) is true at the trivial SO_r-bundle T, it is enough to prove that w_2' is not constant. As $w_2'(T) = 0$, we have to construct an $\mathbf{SO}_r$-bundle A such that $w_2'(A) \neq 0$. In order to do this, let L, M be points of order 2 of the jacobian, such that for the Weil pairing we have $< L, M >= 1$. The choice of a trivialization of their squares defines a non degenerated quadratic form on

$$E = (L \otimes M) \oplus L \oplus M \oplus (r-3)\mathcal{O}_X$$

hence an SO_r-bundle A. By [17], we know that we have $w_2'(A) =< L, M >= 1$ which proves (6.6 b). □

Now choose an ineffective theta-characteristic κ_0 and set $L = \kappa_0 \otimes \kappa^{-1}$. If r is even, there exists a $\mathbf{SO}_r$-bundle $A = (E, q)$ such that $H^0(E \otimes \kappa) = 0$ and $w_2(A) = 0$ (choose $E = rL$ with the obvious quadratic form and use (6.6 b)). If r is odd and κ is even, there exists a $\mathbf{SO}_r$-bundle $A = (E, q)$ such that $H^0(E \otimes \kappa) = 0$ and $w_2(A) = 0$ (by Lemma 1.5 of [1], there is an SL_2-bundle F on X such that $H^0(X, \mathrm{ad}(F) \otimes \kappa) = 0$, then choose $E = \mathrm{ad}(F) \oplus (r-3)L$ with the obvious quadratic form). If r and κ are odd, then $H^0(E \otimes \kappa)$ is odd for all $A \in \mathcal{M}_{SO^0_r,X}$. □

7. Affine Lie algebras and groups

In the following sections I suppose $k = \mathbb{C}$. In order to study the infinite Grassmannian I need some basic material on (affine) Lie algebras which I will recall briefly. I start fixing the notations I will use in the rest of these notes. The reader who is not very familiar with Lie algebras may have a closer look at [8].

7.1. **Basic notations.**

7.1.1. *Lie groups.* From here to the end of these notes, G will be a simple (not necessarily simply connected) complex algebraic group. Let $\widetilde{G} \to G$ be the universal cover of G; its kernel is a subgroup of the center $Z(\widetilde{G}) \subset \widetilde{G}$, canonically isomorphic to $\pi_1(G)$. We will denote the adjoint group $\widetilde{G}/Z(\widetilde{G})$ by $\overline{G}$. We will fix a Cartan subgroup $\overline{H} \subset \overline{G}$ (and denote by H and $\widetilde{H}$ its inverse image in G and $\widetilde{G}$ respectively) as well as a Borel subgroup $\overline{B} \subset \overline{G}$ (and denote by B and $\widetilde{B}$ its inverse image in G and $\widetilde{G}$ respectively).

7.1.2. *Lie algebras.* Let $\mathfrak{g} = \mathrm{Lie}(G)$, $\mathfrak{b} = \mathrm{Lie}(B)$ and $\mathfrak{h} = \mathrm{Lie}(H)$. By the roots of $\mathfrak{g}$ we understand the set R of linear forms α on $\mathfrak{h}$ such that $\mathfrak{g}^\alpha = \{X \in \mathfrak{g}/[X,H] = \alpha(H)\ \forall H \in \mathfrak{h}\}$ is non trivial. We have the root decomposition

$$\mathfrak{g} = \mathfrak{h} \oplus \big(\bigoplus_{\alpha \in R} \mathfrak{g}^\alpha\big).$$

Let $\Delta = \{\alpha_1, ..., \alpha_r\}$ be the basis of R defined by B and θ be the corresponding highest root; we denote $\alpha_0 = -\theta$. Let $(\ ,\ )$ be the Cartan-Killing form, normalized such that $(\theta, \theta) = 2$. Using $(\ ,\ )$ we will identify $\mathfrak{h}$ and $\mathfrak{h}^*$ in the sequel. The coroots of $\mathfrak{g}$ are the elements of $\mathfrak{h}$ defined by $\alpha^\vee = \frac{2\alpha}{(\alpha,\alpha)}$; they form the dual root system $R^\vee$.

Let $Q(R)$ and $Q(R^\vee)$ be the root and coroot lattices with basis given by $\{\alpha_1, ..., \alpha_r\}$ and $\{\alpha_1^\vee, ..., \alpha_r^\vee\}$ respectively . We denote $P(R)$ and $P(R^\vee)$ the weight and coweight lattices, *i.e.* the lattices dual to $Q(R^\vee)$ and $Q(R)$ respectively. They have basis given by the fundamental weights ϖ_i and coweights $\varpi_i^\vee$ defined by

$$< \varpi_i, \alpha_j^\vee > = < \varpi_i^\vee, \alpha_j > = \delta_{ij}.$$

Note that $Q(R^\vee) \subset Q(R)$ and $P(R^\vee) \subset P(R)$ and that we have equality if all roots are of equal length, *i.e.* if we are in the A-D-E case.

7.1.3. *Representations.* We denote by $P_+ \subset P(R)$ the set of dominant weights and by $\{\varpi_1, \ldots, \varpi_r\}$ the fundamental weights. The set P_+ is in bijection with the set of simple $\mathfrak{g}$-modules; denote by $L(\lambda)$ the $\mathfrak{g}$-module associated to the dominant weight λ.

7.1.4. *The center.* We will identify the quotient $P(R^\vee)/Q(R^\vee)$ with $Z(\widetilde{G})$ through the exponential map; its Pontrjagin dual $\mathrm{Hom}(Z(\widetilde{G}), \mathbf{C}^*)$ identifies to $P(R)/Q(R)$. Recall from ([Bourbaki], VIII, SS3, prop. 8) that a system of representatives of $P(R^\vee)/Q(R^\vee)$ is given by the miniscule coweights of

R : these are exactly the fundamental coweights $\varpi_j^\vee$, corresponding to the roots $\alpha_j \in \Delta$ having coefficient 1 when writing

$$\theta = \sum_{\alpha_i \in \Delta} n_i \alpha_i.$$

We will denote $J(\widetilde{G}) = \{i \in \{1, \ldots, r\}/n_i = 1\}$ and $J_0(\widetilde{G}) = J(\widetilde{G}) \cup \{0\}$. Then the set $J_0(\widetilde{G})$ has a natural group structure provided by the group structure on $P(R^\vee)/Q(R^\vee)$ which we will denote additively. Recall, for further reference, that the miniscule coweights are given by

Type of $\mathfrak{g}$	A_r	B_r $(r \geq 2)$	C_r $(r \geq 2)$	D_r $(r \geq 3)$	E_6	E_7	E_8	F_4	G_2
J	$\{1, \ldots, r\}$	$\{1\}$	$\{r\}$	$\{1, r-1, r\}$	$\{1, 6\}$	$\{7\}$	$\emptyset$	$\emptyset$	$\emptyset$

For $j \in J_0(\widetilde{G})$ we will denote the corresponding element of $Z(\widetilde{G})$, $\pi_1(\overline{G})$, $P(R)/Q(R)$ or $P(R^\vee)/Q(R^\vee)$ under the above identifications by z_j, τ_j, ϖ_j, $\varpi_j^\vee$ or w_j respectively.

The subgroup of $Z(\widetilde{G})$ corresponding to $\pi_1(G)$ will be denoted by Z, the corresponding subgroup of $J_0(\widetilde{G})$ by J_0 and the lattice generated by $Q(R^\vee)$ and $\varpi_j^\vee$ for $j \in J_0$ by $\Lambda_J(R^\vee)$.

7.1.5. *Dynkin diagrams.* Associated to the Lie algebra $\mathfrak{g}$ is its Cartan matrix A with coefficients $a_{ij} = \langle \alpha_i, \alpha_j^\vee \rangle$, $i, j = 1, \ldots, r$. This matrix is invertible; its determinant is the *connection index* I_c, *i.e.* the index of the root lattice $Q(R)$ in $P(R)$. The associated Dynkin diagram is constructed as follows. The nodes are the simple roots $\alpha_i \in \Pi$; the nodes α_i and α_j are connected by $\max\{|a_{ij}|, |a_{ji}|\}$ lines. Moreover these lines are labeled by "$>$" if $a_{ij} \neq 0$ and $|a_{ij}| > |a_{ji}|$. These diagrams have various interpretations: the i-th node may be seen representing α_i or ϖ_i or $\alpha_i^\vee$ or been labeled, for example by the dual Coxeter numbers $c_i^\vee$ defined by $\theta^\vee = \sum_{i=1}^r c_i^\vee \alpha_i^\vee$. Note that if ρ is the half sum of the roots $\rho = \sum_{i=1}^r \varpi_i$, then $\langle \rho, \theta^\vee \rangle = \sum_{i=1}^r c_i^\vee$. The number $h^\vee = 1 + \langle \rho, \theta^\vee \rangle$ is called the *dual Coxeter number.* The possible Dynkin diagrams, as well as their connection indexes and dual Coxeter numbers are resumed in table A.

7.2. **Affine Lie algebras and groups.**

7.2.1. *Loop algebras and central extensions.* Let $L\mathfrak{g} = \mathfrak{g} \otimes_{\mathbb{C}} \mathbb{C}((z))$ be the *loop algebra* of $\mathfrak{g}$. It has a canonical 2-cocycle defined by

$$\psi_{\mathfrak{g}} : (X \otimes f, Y \otimes g) \mapsto (X, Y) \operatorname*{Res}_{z=0}(gdf), \tag{7.2 a}$$

hence a central extension

$$0 \to \mathbb{C} \longrightarrow \widehat{L\mathfrak{g}} \longrightarrow L\mathfrak{g} \to 0 \tag{7.2 b}$$

In other words, this means that on the level of vector spaces $\widehat{L\mathfrak{g}} = \mathbb{C}c \oplus L\mathfrak{g}$ with Lie bracket given by (c central):

$$[X \otimes f, Y \otimes g] = [X, Y] \otimes fg + (X, Y) \operatorname*{Res}_{z=0}(gdf). \tag{7.2 c}$$

In the following, we denote $X[f]$ the element $X \otimes f$ of $L\mathfrak{g}$; if $f = z^n$ it is also denoted by $X(n)$. The Lie algebra $L\mathfrak{g}$ has several subalgebras which will be important for us. Define

$$L^+\mathfrak{g} = \mathfrak{g} \otimes_{\mathbb{C}} \mathbb{C}[[z]],\ L^{>0}\mathfrak{g} = \mathfrak{g} \otimes_{\mathbb{C}} z\mathbb{C}[[z]],\ L^{<0}\mathfrak{g} = \mathfrak{g} \otimes_{\mathbb{C}} z^{-1}\mathbb{C}[z^{-1}]$$

These are in fact subalgebras of $\widehat{L\mathfrak{g}}$.

7.2.2. *Irreducible and integrable representations.* In nature, representations of $L\mathfrak{g}$ are projective; this is why we look at (true) representations of $\widehat{L\mathfrak{g}}$. Fix an integer ℓ. Call a representation of $\widehat{L\mathfrak{g}}$ *of level* ℓ, if the center acts by multiplication by ℓ. In order to construct such representations we start with a finite dimensional representation $L(\lambda)$, which we may view as an $L^+\mathfrak{g}$-module by evaluation. As the cocycle (7.2 a) is trivial over $L^+\mathfrak{g}$ the central extension $\widehat{L^+\mathfrak{g}}$ obtained by restriction from (7.2 b) splits. Hence we may consider $L(\lambda)$ as an $\widehat{L^+\mathfrak{g}}$-module of level ℓ by letting the center act by multiplication by ℓ; denote this module $L_\ell(\lambda)$. Now consider the generalized Verma module:

$$M_\ell(\lambda) = \operatorname{Ind}_{\widehat{L^+\mathfrak{g}}}^{\widehat{L\mathfrak{g}}} L_\ell(\lambda) = U(\widehat{L\mathfrak{g}}) \otimes_{U(\widehat{L^+\mathfrak{g}})} L_\ell(\lambda)$$

In the case when ℓ is not the critical level $-h^\vee$ (the dual Coxeter number), $M_\ell(\lambda)$ has a unique irreducible quotient $H_\ell(\lambda)$. Moreover, if $\ell \geq (\lambda, \theta)$ then $H_\ell(\lambda)$ has an important finiteness condition: for all $X \in \mathfrak{g}^\alpha$ and all $f \in \mathbb{C}((z))$ the element $X[f]$ acts locally nilpotent on $H_\ell(\lambda)$, *i.e.* for all $u \in H_\ell(\lambda)$ there is N such that $X[f]^N.u = 0$. Such $\widehat{L\mathfrak{g}}$-modules are called

integrable and it can be shown that all irreducible integrable $\widehat{L\mathfrak{g}}$-modules arise in this way. In view of this it is convenient to define the following set.

$$P_\ell = \{\lambda \in P(R)/(\lambda, \alpha_i) \geq 0 \text{ for } i \in I \text{ and } (\lambda, \theta) \leq \ell\}.$$

In the rest of this subsection we restrict to positive[8] ℓ. Actually if $\lambda \in P_\ell$ then $H_\ell(\lambda)$ is the quotient of $H_\ell(\lambda)$ by the sub-module $Z_\lambda(\ell)$ generated by $X_\theta(-1)^{\ell+1-(\lambda,\theta)} \otimes v_\lambda$, where v_λ is a highest weight vector of $L(\lambda)$. By Poincaré-Birkhoff-Witt, $M_\lambda(\ell) = U(L^{<0}\mathfrak{g}) \otimes_{\mathbb{C}} L_\lambda$. It follows that we have the exact sequence

$$0 \to Z_\lambda(\ell) \longrightarrow U(L^{<0}\mathfrak{g}) \otimes_{\mathbb{C}} L_\ell(\lambda) \longrightarrow H_\ell(\lambda) \to 0. \tag{7.2 d}$$

In other words:

$$[H_\ell(\lambda)]^{L^{>0}\mathfrak{g}} = L_\ell(\lambda) = \{v \in H_\ell(\lambda)/L^{>0}\mathfrak{g}.v = 0\} \tag{7.2 e}$$

(7.2 f) $H_\ell(\lambda)$ is generated by $L(\lambda)$ over $L^{<0}\mathfrak{g}$ with only one relation:

$$X_\theta(-1)^{\ell+1-(\lambda,\theta)} \otimes v_\lambda = 0.$$

7.3. **Loop groups and central extensions.** We already have defined the loop groups LG and L^+G in 5.1. The Lie algebra of LG is $L\mathfrak{g}$, as the kernel of the homomorphism $LG(R[\varepsilon]) \to LG(R)$ is $L\mathfrak{g}(R) = \mathfrak{g} \otimes_{\mathbb{C}} R((z))$. For the same reason we have $\mathrm{Lie}(L^+G) = L^+\mathfrak{g}$.

7.3.1. *The adjoint action.* Let H be an infinite dimensional vector space over $\mathbb{C}$. We define the $\mathbb{C}$-space $\mathrm{End}(H)$ by $R \mapsto \mathrm{End}(H \otimes_{\mathbb{C}} R)$, the $\mathbb{C}$-group $\mathbf{GL}(H)$ as the group of its units and $\mathbf{PGL}(H)$ by $\mathbf{GL}(H)/G_m$. The $\mathbb{C}$-group $L\widetilde{G}$ acts on $L\mathfrak{g}$ by the adjoint action. We define the adjoint action of $L\widetilde{G}$ on $\widehat{L\mathfrak{g}}$ as follows:

$$\mathrm{Ad}(\gamma).(\alpha', s) = \big(\mathrm{Ad}(\gamma).\alpha', s + \operatorname*{Res}_{z=0}(\gamma^{-1}\frac{d}{dz}\gamma, \alpha')\big)$$

where $\gamma \in L\widetilde{G}(R)$, $\alpha = (\alpha', s) \in \widehat{L\mathfrak{g}}(R)$ and $(\ ,\ )$ is the $R((z))$-bilinear extension of the Cartan-Killing form. Consider an integral highest weight representation $\bar{\pi} : \widehat{L\mathfrak{g}} \to \mathrm{End}(H)$. The basic result we will use in the sequel is the following:

[8] The interesting case for us is actually $\ell = -h^\vee$; I will come back to this later. See also Frenkel's lectures.

7.3.2. Proposition. *(Faltings) Let R be a $\mathbb{C}$-algebra, $\gamma \in L\widetilde{G}(R)$. Locally over* $\mathrm{Spec}(R)$*, there is an automorphism u_γ of $H_R = H \otimes_{\mathbb{C}} R$, unique up to R^*, such that*

$$\begin{tikzcd} H_R \arrow[r, dash, "\bar{\pi}(\alpha)"] \arrow[d, dash, "u_\gamma"'] & H_R \arrow[d, dash, "u_\gamma"] \\ H_R \arrow[r, dash, "\bar{\pi}(\mathrm{Ad}(\gamma).\alpha)"] & H_R \end{tikzcd} \tag{7.3 a}$$

for any $\alpha \in \widehat{L\mathfrak{g}}(R)$.

Again, the important fact here is that we work over any $\mathbb{C}$-algebra (and not only over $\mathbb{C}$.) The above proposition is proved in ([2], App. A) in the case SL_r; its generalization to G is straightforward.

7.3.3. *Integration.* An immediate corollary of the above proposition is that the representation $\bar{\pi}$ may be "integrated" to a (unique) *algebraic* projective representation of $L\widetilde{G}$, *i.e.* that there is a morphism of $\mathbb{C}$-groups $\pi : L\widetilde{G} \to \mathbf{PGL}(H)$ whose derivate coincides with $\bar{\pi}$ up to homothety. Indeed, thanks to the unicity property the automorphisms u associated locally to γ glue together to define an element $\pi(\gamma) \in \mathbf{PGL}(H)(R)$ and still because of the unicity property, π defines a morphism of $\mathbb{C}$-groups. The assertion on the derivative is a consequence of (7.3 a).

7.3.4. *Central extensions.* We are now looking for a central extension of $L\widetilde{G}$ such that its derivative is the canonical central extension (7.2 b). In order to do this, we apply the above to the basic representation $H_1(0)$ of $\widehat{L\mathfrak{g}}$. Consider the central extension

$$1 \to G_m \longrightarrow \mathbf{GL}(H_1(0)) \longrightarrow \mathbf{PGL}(H_1(0)) \to 1. \tag{7.3 b}$$

Then the pullback of (7.3 b) to $L\widetilde{G}$ is convenient: it defines a central extension to which we refer to as the *canonical* central extension of $L\widetilde{G}$:

$$1 \to G_m \longrightarrow \widehat{L\widetilde{G}} \longrightarrow L\widetilde{G} \to 1 \tag{7.3 c}$$

What happens if we restrict to $L^+\widetilde{G}$?

7.3.5. Lemma. *The extension (7.3 c) splits canonically over $L^+\widetilde{G}$.*

Proof. ([14], 4.9) By construction of (7.3 c), it is enough to prove that the representation $\bar{\pi} : L^+\widetilde{G} \to \mathrm{End}(H_1(0))$ integrates to a representation $\pi : L^+\mathfrak{g} \to \mathbf{GL}(H_1(0))$. This will follow from the fact that in the case $\gamma \in$

$L^+\widetilde{G}(R)$ we can *normalize* the automorphism u of Proposition 7.3.2. Indeed, as $L(0) = [H_1(0)]^{L^+\mathfrak{g}}$ by (7.2 e), it follows from (7.3 a) that u maps $L(0)_R$ to $L(0)_R$. Now $L(0)_R$ is a free R-module of rank one, hence we may choose u (in a unique way) such that it induces the identity on $L(0)_R$. □

8. The infinite Grassmannian

We will now study in some more detail the infinite Grassmannian $\mathcal{Q}_G$ for connected reductive groups over the complex numbers.

8.1. **Ind-schemes.** The category of $\mathbb{C}$-spaces is closed under direct limits. A $\mathbb{C}$-space (resp. $\mathbb{C}$-group) will be called a (strict) *ind-scheme* (resp. *ind-group*) if it is the direct limit of a directed system of quasi-compact $\mathbb{C}$-schemes $(Z_\alpha)_{\alpha\in I}$ such that all the maps $i_{\alpha,\beta} : Z_\alpha \to Z_\beta$ are closed embeddings. Remark that an ind-group is in general not a direct limit of a directed system of *groups*. Any property P of schemes which is stable under passage to closed subschemes make sense for ind-schemes: We say that Z satisfies the ind-property P if each Z_α does. In particular we may define Z to be of ind-finite type or ind-proper.

An ind-scheme is integral (resp. reduced, irreducible) if it is the direct limit of an increasing sequence of integral (resp. reduced, irreducible) $\mathbb{C}$-schemes.

A $\mathbb{C}$-space Z is *formally smooth* if for every $\mathbb{C}$-algebra R and for every nilpotent ideal $I \subset R$ the map $Z(\mathrm{Spec}(R)) \to Z(\mathrm{Spec}(R/I))$ is surjective. If R is an ind-scheme of ind-finite type, then formal smoothness is a local property[9].

8.1.1. Lemma. *Let Z be an ind-scheme, direct limit of an increasing sequence of $\mathbb{C}$-schemes. Then the following is true (see* [2]*, 6.3 for a proof).*

(i) *If Z is reduced and is the direct limit of an increasing sequence of $\mathbb{C}$-schemes (Z_n) then $Z = \varinjlim(Z_n)$.*
(ii) *If Z is covered by reduced open sub-ind-schemes, Z is reduced.*
(iii) *The ind-scheme Z is integral if and only if Z is reduced and irreducible.*
(iv) *If U is a $\mathbb{C}$-scheme and $U \times Z$ is integral, Z is integral.*

[9] Formal smoothness is a weak property in our infinite dimensional setup. For instance, we will see that LGL_r is formally smooth *but not reduced.*

8.2. **The ind-structure of loop groups.** Ind-schemes and ind-groups will be important for us as the $\mathbb{C}$-groups LG, $L_X G$ will be ind-groups, as well as $\mathcal{Q}_G$ which will be an ind-scheme. Actually L^+G is an infinite product of affine schemes.

8.2.1. Lemma. *The $\mathbb{C}$-group L^+GL_r is represented by*

$$\mathbf{GL}_r(\mathbb{C}) \times \prod_1^\infty M_r(\mathbb{C})$$

where $M_r(\mathbb{C})$ is the set of $r \times r$-matrices with entries in $\mathbb{C}$.

Proof. This follows from the fact that for any $\mathbb{C}$-algebra R the set $GL_r(R[[z]])$ consists of the matrices of the form $A(z) = \sum_{n=0}^\infty A_n z^n$ with $A_0 \in \mathbf{GL}_r(R)$ and $A_n \in M_r(R)$ for $n \geq 1$. □

Consider more generally the sub-$\mathbb{C}$-space $LGL_r^{(N)}$ of LGL_r defined for any $\mathbb{C}$-algebra R by the set $\mathbf{GL}_r^{(N)}(R)$ of matrices $A(z)$ such that both $A(z)$ and $A(z)^{-1}$ have poles of order $\leq N$. Of course, by definition $LGL_r^{(0)} = L^+GL_r$.

8.2.2. Lemma. *The $\mathbb{C}$-space $LGL_r^{(N)}$ is representable as an affine scheme.*

Proof. If $M^{(N)}(R)$ is the set of $r \times r$-matrices with coefficients in R, then the corresponding $\mathbb{C}$-space $\mathbf{M}^{(N)}$ is represented by the affine scheme

$$\prod_{n=-N}^\infty M_r(\mathbb{C})$$

Now remark that $LGL_r^{(N)}$ is represented by the closed affine subscheme of $\mathbf{M}^{(N)} \times \mathbf{M}^{(N)}$ of pairs of matrices $(A(z), B(z))$ such that $A(z)B(z) = I$. □

8.2.3. Corollary. *The $\mathbb{C}$-group LGL_r is an ind-group of ind-finite type, direct limit of the increasing sequence of schemes $(LGL_r^{(N)})_{N \geq 0}$*

For a general reductive group choose an embedding $G \subset \mathbf{GL}_r$. Then the ind-structure of LGL_r induces an ind-structure on LG.

8.3. **The ind-structure of the infinite Grassmannian.** The following theorem describes the ind-structure of $\mathcal{Q}_G$

8.3.1. Theorem. *Let G be a connected reductive complex group. Then*

(i) *The $\mathbb{C}$-space $\mathcal{Q}_G$ is an ind-scheme, ind-proper of ind-finite type.*
(ii) *The projection $\pi : LG \to \mathcal{Q}_G$ admits, locally for the Zariski topology, a section.*

(iii) *The ind-scheme $\mathcal{Q}_G$ is formally smooth.*
(iv) *The ind-scheme $\mathcal{Q}_G$ is reduced if and only if* $\mathrm{Hom}(G, G_m) = 0$.

Proof. It follows from corollary 8.2.3 that $\mathcal{Q}_{GL_r}$ is an ind-scheme of ind-finite type: if $\mathcal{Q}_{GL_r}^{(N)} = LGL_r^{(N)}/L^+GL_r$,

$$\mathcal{Q}_{GL_r} = \varinjlim \mathcal{Q}_{GL_r}^{(N)}$$

In order to see that $\mathcal{Q}_{GL_r}$ is ind-proper we use the following lattice approach to $\mathcal{Q}_{GL_r}$ (see [2],§2). For any $\mathbb{C}$-algebra R we define a *lattice* in $R((z)))^r$ to be a sub-$R[[z]]$-module W of $R((z)))^r$ such that

$$z^N R[[z]]^r \subset W \subset z^{-N} R[[z]]^r$$

for some integer N and such that $W/z^N R[[z]]^r$ is projective.

8.3.2. Proposition. *The $\mathbb{C}$-space $\mathcal{Q}_{GL_r}$ represents the functor which associates to any affine $\mathbb{C}$-scheme $U = \mathrm{Spec}(R)$ the set of lattices $W \subset R((z)))^r$.*

Proof. This is a consequence of Proposition (5.3.2). Indeed if E is a vector bundle over X_U together with a trivialization $\tau : E_{|X_U^*} \xrightarrow{\sim} U \times \mathbb{C}^r$, we get by restriction an isomorphism $\tau^* : R((z))^r \to H^0(D_U^*, E_{|D_U^*})$. The inverse image W of $H^0(D_U, E_{|D_U})$ is a lattice in $R((z))^r$. On the other hand, given a lattice $W \subset R((z))^r$ we get a vector bundle E_W on X by gluing the trivial bundle over X_U^* with the bundle on D_R associated to the $R[[z]]$-module W; the gluing isomorphism is given by $W \times_{R[[z]]} R((z)) \to R((z))^r$ coming from the inclusion $W \subset R((z))^r$. By definition, E_W comes with a trivialization τ_W and it is easy to see that both constructions are inverse to each other. $\square$

It follows from the above that $\mathcal{Q}_{GL_r}$ is ind-proper as the latter functor is.

Let us look at the special case of $G = \mathbf{SL}_r$. Then we obtain *special* lattices, *i.e.* the lattices such that the projective module $W/z^N R[[z]]^r$ is of rank Nr. Let F_N be the complex vector space $z^{-N}\mathbb{C}[[z]]^r/z^N\mathbb{C}[[z]]^r$. Then $\dim(F_N) = 2Nr$. Multiplication by z induces a (nilpotent) endomorphism ζ_N of F_N. If follows that $1 + \zeta_N$ is an automorphism of F_N, hence we get an automorphism of $\mathrm{Grass}(Nr, 2Nr)$; denote by V_N its fixed points. Then it is easy to see from the above proposition that the $\mathbb{C}$-space $LSL_r^{(N)}$ is isomorphic to the projective variety V_N.

Once we know (i) for $\mathbf{GL}_r$ it follows for a general reductive group after choosing an embedding $G \subset \mathbf{GL}_r$ from the following lemma.

8.3.3. Lemma. *Suppose $G \subset H$ is an inclusion of affine algebraic groups such that H/G is affine and such that $\mathcal{Q}_H$ is an ind-scheme of ind-finite*

type. Then $\mathcal{Q}_G$ is also an ind-scheme of ind-finite type and the morphism $\mathcal{Q}_G \to \mathcal{Q}_H$ is a closed embedding. In particular, if $\mathcal{Q}_H$ is ind-proper, $\mathcal{Q}_G$ is.

Proof. Left to the reader. □

In order to see (ii), one reduces (use that π is LG-invariant) to show that π admits a section over a Zariski open neighborhood of $[e] \in \mathcal{Q}_G$, which follows from the fact that if $L^{<0}G$ is the $\mathbb{C}$-group defined by $R \mapsto G\big(z^{-1}R[z^{-1}]\big)$, the multiplication map

$$\mu : L^{<0}G \times L^{+}G \longrightarrow LG \tag{8.3 a}$$

is an open immersion. The last statement is proved in ([2], 1.11) using that $\mathcal{Q}_G$ may be seen as parameterizing G-bundles over $\mathbb{P}^1$ and that if E is a G-bundle over $\mathbb{P}^1_U$ for some U then the set of points $u \in U$ such that E_u is trivial is an open subset of U as $H^1(\mathbb{P}^1, \mathcal{O} \otimes \mathfrak{g}) = 0$.

Formal smoothness of $\mathcal{Q}_G$ follows from formal smoothness of LG and the above, so it remains to prove (iv). Actually to see that $\mathcal{Q}_G$ is reduced whenever $\mathrm{Hom}(G, \mathbb{G}_m) = 0$ is quite delicate. It is proved in ([2],6.4) for SL_r where it is deduced from the corresponding statement for $L_{\mathbb{P}^1}G$ and a direct calculation. For general G it is proved in ([14], 4.6) where it is deduced from a theorem of Šafarevič [24]. I will not enter into the proof here, let me just say why $\mathcal{Q}_{\mathbb{G}_m}$ is *not* reduced. Actually, as $L^+\mathbb{G}_m$ is reduced it is equivalent to show that $L\mathbb{G}_m$ is not reduced. Consider $\mu : \mathbb{G}_m \to \mathbb{G}_m$ defined by $\mu(x) = x^n$ and the induced morphism $L\mathbb{G}_m \to L\mathbb{G}_m$. Then the image is not contained in $(L\mathbb{G}_m)_{red}$, hence $L\mathbb{G}_m$ is not reduced. □

8.4. **The connected components of the infinite Grassmannian.** Suppose in this subsection that G is semi-simple and recall the notations of section 7.1.

8.4.1. Lemma. *([4], 1.2)*

(i) *The group $\pi_0(LG)$ is canonically isomorphic to $\pi_1(G)$.*
(ii) *The projection $\pi : LG \to \mathcal{Q}_G$ induces a bijection $\pi_0(LG) \to \pi_0(\mathcal{Q}_G)$. Each connected component of $\mathcal{Q}_G$ is isomorphic to $\mathcal{Q}_{\widetilde{G}}$.*

Proof. By [21], there exists a finite family of homomorphisms $x_\alpha : \mathbb{G}_a \to \widetilde{G}$ such that for any extension K of $\mathbb{C}$, the subgroups $x_\alpha(K)$ generate $\widetilde{G}(K)$. Since the ind-group $\mathbb{G}_a(\mathbb{C}((z)))$ is connected, it follows that $L\widetilde{G}$ is connected.

In the general case, consider the exact sequence $1 \to \pi_1(G) \to \widetilde{G} \to G \to 1$ as an exact sequence of étale sheaves on $D^* := \mathrm{Spec}\,\mathbf{C}((z))$. Since $H^1(D^*, \widetilde{G})$

is trivial [22], it gives rise to an exact sequence of $\mathbb{C}$-groups

$$1 \to L\widetilde{G}/\pi_1(G) \longrightarrow LG \longrightarrow H^1(D^*, \pi_1(G)) \to 1 \tag{8.4 a}$$

Then (i) follows from the connectedness of $L\widetilde{G}$ and the canonical isomorphism $H^1(D^*, \pi_1(G)) \xrightarrow{\sim} \pi_1(G)$ (Puiseux theorem).

To prove (ii), we first observe that the group L^+G is connected: for any $\gamma \in L^+G(\mathbb{C})$, the map $F_\gamma : G \times \mathbb{A}^1 \to L^+G$ defined by $F_\gamma(g,t) = g^{-1}\gamma(tz)$ satisfies $F_\gamma(\gamma(0),0) = 1$ and $F_\gamma(1,1) = \gamma$, hence connects γ to the origin. Therefore the canonical map $\pi_0(LG) \to \pi_0(\mathcal{Q}_G)$ is bijective. Moreover it follows from (8.4 a) that $(LG)^{\mathrm{o}}$ is isomorphic to $L\widetilde{G}/\pi_1(G)$. □

9. The ind-group of loops coming from the open curve

Let G be a connected simple complex group, X be a connected smooth projective complex curve. Recall the notations of 7.1.

9.1. **The simply connected case.**

9.1.1. Proposition. *([14],5.1) The ind-group $L_X\widetilde{G}$ is integral.*

Proof. To see that $L_X\widetilde{G}$ is reduced, consider the morphism $\bar{\pi} : \mathcal{Q}_{\widetilde{G}} \to \mathcal{M}_{\widetilde{G},X}$, which we know to be locally trivial for the étale topology by the uniformization theorem 5.1.1. Hence, locally for the étale topology, $\bar{\pi}$ is $U \times L_X\widetilde{G} \to U$. Now use that $\mathcal{Q}_{\widetilde{G}}$ is reduced (Theorem 8.3.1) and Lemma 8.1.1 (iv).

To prove that $L_X\widetilde{G}$ is irreducible it is enough, as connected ind-groups are irreducible by Proposition 3 of [24], to show that $L_X\widetilde{G}$ is connected. The idea of its proof is due to V. Drinfeld: consider distinct points $p_1, \ldots, p_i$ of X which are all distinct from p. Define $X_i^* = X$ -$\{p, p_1, \ldots, p_i\}$ and, for every affine k-scheme $U = \mathrm{Spec}(R)$, define $X_{i,U}^* = X_i^* \times_k U$. Denote by $A_{X_{i,U}}$ the $\mathbb{C}$-algebra $\Gamma(X_{i,U}^*, \mathcal{O}_{X_{i,U}^*})$ and by $L_X^i\widetilde{G}$ the $\mathbb{C}$-group $R \mapsto \widetilde{G}(A_{X_{i,U}})$. As $L_X\widetilde{G}$, the $\mathbb{C}$-group $L_X^i\widetilde{G}$ is an ind-group. The natural inclusion $A_{X_{i,R}} \subset A_{X_{i+1,R}}$ defines a closed immersion $f : L_X^i\widetilde{G} \to L_X^{i+1}\widetilde{G}$.

9.1.2. Lemma. *([14], 5.3) The map $f : L_X^i\widetilde{G} \to L_X^{i+1}\widetilde{G}$ defines a bijection*

$$\pi_0(L_X^i\widetilde{G}) \xrightarrow{\sim} \pi_0(L_X^{i+1}\widetilde{G}). \tag{9.1 a}$$

Once we know the lemma, we do the following. Let $g \in L_X\widetilde{G}(\mathbb{C})$ and let K be the field of rational functions on X. Using the fact (*cf.* [21]) that $\widetilde{G}(K)$ is generated by the standard unipotent subgroups $U_\alpha(K)$, $\alpha \in \Delta$, we may suppose that g is of the form $\prod_{j\in J} \exp(f_j n_j)$ where the n_j are nilpotent

elements of $\mathfrak{g}$ and $f_j \in K$. Let $\{p_1, \ldots, p_i\}$ be the poles of the functions $f_j, j \in J$. Then the morphism

$$\begin{aligned} \mathbb{A}^1 &\longrightarrow L_X^i \widetilde{G} \\ t &\mapsto \prod_{j \in J} \exp(t f_j n_j) \end{aligned}$$

is a path from g to 1 in $L_X^i \widetilde{G}$. By the above, the morphism $\pi_0(L_X \widetilde{G}) \to \pi_0(L_X^i \widetilde{G})$ is bijective which shows that g and 1 are in the same connected component of $L_X \widetilde{G}$, hence $L_X \widetilde{G}$ is connected. □

9.1.3. Corollary. *([14],5.2) Every character $\chi : L_X \widetilde{G} \to G_m$ is trivial.*

Proof. The differential of χ, considered as a function on $L_X \widetilde{G}$, is everywhere vanishing. Indeed, since χ is a group morphism, this means that the deduced Lie algebra morphism $\mathfrak{g} \otimes A_X \to \mathbb{C}$ is zero (with $A_X = \mathcal{O}(X^*)$). The derived algebra $[\mathfrak{g} \otimes A_X, \mathfrak{g} \otimes A_X]$ is $[\mathfrak{g}, \mathfrak{g}] \otimes A_X$ and therefore equal to $\mathfrak{g} \otimes A_X$ (as $\mathfrak{g}$ is simple). Therefore any Lie algebra morphism $\mathfrak{g} \otimes A_X \to k$ is trivial. As $L_X \widetilde{G}$ is integral we can write $L_X \widetilde{G}$ as the direct limit of an increasing sequence of integral varieties V_n. The restriction of χ to V_n has again zero derivative and is therefore constant. For large n, the varieties V_n contain 1. This implies $\chi_{|V_n} = 1$ and we are done. □

9.2. **The general case.**

9.2.1. Lemma. *([4], 1.2)*

(i) *The group $\pi_0(L_X G)$ is canonically isomorphic to $H^1(X, \pi_1(G))$.*
(ii) *The group $L_X G$ is contained in the neutral component $(LG)^{\circ}$ of LG.*

Proof. Consider the cohomology exact sequence on X^* associated to the exact sequence $1 \to \pi_1(G) \to \widetilde{G} \to G \to 1$. As $H^1(X^*, \widetilde{G})$ is trivial, we get the following exact sequence of $\mathbb{C}$-groups

$$1 \to L_X \widetilde{G}/\pi_1(G) \to L_X G \to H^1(X^*, \pi_1(G)) \to 1 \tag{9.2 a}$$

Now using that the restriction $H^1(X, \pi_1(G)) \to H^1(X^*, \pi_1(G))$ is bijective and that $L_X \widetilde{G}$ is connected by 9.1.1 we get (i).

It follows from (8.4 a) and (9.2 a) that (ii) is equivalent to claim that the restriction map $H^1(X^*, \pi_1(G)) \to H^1(D^*, \pi_1(G))$ is zero. But this is a

consequence of the commutative diagram of restriction maps

$$\begin{array}{ccc} H^1(X,\pi_1(G)) & \overset{\sim}{\text{———}} & H^1(X^*,\pi_1(G)) \\ \Big| & & \Big| \\ H^1(D,\pi_1(G)) & \text{———} & H^1(D^*,\pi_1(G)) \end{array}$$

and the vanishing of $H^1(D,\pi_1(G))$. □

9.2.2. Corollary. *There is a canonical bijection* $\pi_0(\mathcal{M}_{G,X}) \xrightarrow{\sim} \pi_1(G)$.

Proof. This follows from the uniformization theorem and Lemma 8.4.1, (i), (ii) and Lemma 9.2.1 (iv). □

10. The line bundles on the moduli stack of G-bundles

Let G be a connected simple complex group, X be a connected smooth projective complex curve. Recall the notations of 7.1.

10.1. **The line bundles on the infinite Grassmannian.**

10.2. **A natural line bundle.** Consider the canonical central extension (7.3 c) of $L\widetilde{G}$ and its restriction to $L^+\widetilde{G}$. Then we may write

$$\mathcal{Q}_{\widetilde{G}} = \widehat{L\widetilde{G}}/\widehat{L^+\widetilde{G}} \tag{10.2 a}$$

By Lemma 7.3.5 we have a canonical character

$$\chi : \widehat{L^+\widetilde{G}} \xrightarrow{\sim} \mathbb{G}_m \times L^+\widetilde{G} \xrightarrow{p_1} \mathbb{G}_m, \tag{10.2 b}$$

hence a line bundle $\mathcal{L}_{\chi^{-1}}$ on the homogeneous space $\mathcal{Q}_{\widetilde{G}}$.

10.2.1. *A line in the infinite Grassmannian.* Consider the morphism of $\mathbb{C}$-groups $\varphi : SL_2 \to LSL_2$ defined by (for R a $\mathbb{C}$-algebra)

$$\begin{aligned} \phi : SL_2(R) &\longrightarrow SL_2\big(R((z))\big) \\ \begin{pmatrix} a & b \\ c & d \end{pmatrix} &\longmapsto \begin{pmatrix} d & cz^{-1} \\ bz & a \end{pmatrix} \end{aligned}$$

and moreover the morphism of $\mathbb{C}$-groups $\psi : LSL_2 \to L\widetilde{G}$ deduced from the map $SL_2 \to \widetilde{G}$ associated to the highest root θ. Let

$$\varphi = \psi \circ \phi : SL_2 \to L\widetilde{G}. \tag{10.2 c}$$

The Borel subgroup $B_2 \subset SL_2$ of upper triangular matrices maps to $L^+\widetilde{G}$ by construction, hence we get a morphism $\overline{\varphi} : \mathbb{P}^1_{\mathbb{C}} \to \mathcal{Q}_{\widetilde{G}}$. An easy calculation

shows that the derivative $\mathrm{Lie}(\varphi)$ maps the standard $\mathfrak{sl}_2$-triplet $\{e, f, h\} = \{X_\theta, X_{-\theta}, H_\theta\}$ to the $\mathfrak{sl}_2$-triplet $\{X_{-\theta} \otimes z, X_\theta \otimes z^{-1}, -H_\theta\}$ of $L\mathfrak{g}$.

10.2.2. Proposition.

(i) *The pullback defines an isomorphism* $\overline{\varphi}^* : \mathrm{Pic}(\mathcal{Q}_G) \xrightarrow{\sim} \mathrm{Pic}(\mathbb{P}^1_{\mathbb{C}})$

(ii) *We have* $\overline{\varphi}^*(\mathcal{L}_\chi) = \mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}(1)$, *i.e.* $\mathrm{Pic}(\mathcal{Q}_G) = \mathbb{Z}\mathcal{L}_\chi$

Proof. (i) follows from [12]. In order to prove (ii), we use that the restriction of (7.3 c) to SL_2 splits, hence φ lifts to a morphism $\widetilde{\varphi} : SL_2 \to \widehat{L\widetilde{G}}$ and all we have to do is to calculate the character of $B_2 \to \widehat{L^+\widetilde{G}} \xrightarrow{\chi} G_m$. For this it is enough to calculate the character of B_2 on the SL_2-module generated by v_0. By (7.2 f) this is the standard representation, so we are done. □

In the following we denote, in view of the above, $\mathcal{L}_\chi$ by $\mathcal{O}_{\mathcal{Q}_G}(1)$.

10.3. **Linearized line bundles on the infinite Grassmannian.** Consider the group $\mathrm{Pic}_{L_XG}(\mathcal{Q}_G)$ of L_XG-linearized line bundles on $\mathcal{Q}_G$. Recall that a L_XG-linearization of $\mathcal{L}$ is an isomorphism $m^*\mathcal{L} \xrightarrow{\sim} pr_2^*\mathcal{L}$, where $m : L_XG \times \mathcal{Q}_G \to \mathcal{Q}_G$ is the action of L_XG on $\mathcal{Q}_G$, satisfying the usual cocycle condition. It follows from the section on stacks that

10.3.1. Proposition. *The map* $\pi : \mathcal{Q}_G \to \mathcal{M}_{G,X}$ *induces an isomorphism*

$$\pi^* : \mathrm{Pic}(\mathcal{M}_{G,X}) \xrightarrow{\sim} \mathrm{Pic}_{L_XG}(\mathcal{Q}_G).$$

Hence, once we know $\mathrm{Pic}_{L_XG}(\mathcal{Q}_G)$, we know $\mathrm{Pic}(\mathcal{M}_{G,X})$.

10.4. **The case of simply connected groups.** In order to determine the group $\mathrm{Pic}_{L_X\widetilde{G}}(\mathcal{Q}_{\widetilde{G}})$, consider the forgetful morphism $f : \mathrm{Pic}_{L_X\widetilde{G}}(\mathcal{Q}_{\widetilde{G}}) \to \mathrm{Pic}(\mathcal{Q}_{\widetilde{G}})$.

10.4.1. Proposition. *The map* $f : \mathrm{Pic}_{L_X\widetilde{G}}(\mathcal{Q}_{\widetilde{G}}) \to \mathrm{Pic}(\mathcal{Q}_{\widetilde{G}})$ *is injective.*

Proof. The kernel of this morphism consists of the $L_X\widetilde{G}$-linearizations of the trivial bundle. Any two such trivializations differ by an automorphism of $pr_2^*\mathcal{O}_{\mathcal{Q}_G}$ that is by an invertible function on $L_X\widetilde{G} \times \mathcal{Q}_{\widetilde{G}}$. Since $\mathcal{Q}_{\widetilde{G}}$ is integral, it is the direct limit of the integral projective varieties and this function is the pullback of an invertible function f on $L_X\widetilde{G}$. The cocycle conditions on the linearizations imply that f is a character, hence $f = 1$ by Corollary 9.1.3. □

Once we know that f is injective, we may ask whether f is surjective, *i.e.* whether $\mathcal{O}_{\mathcal{Q}_{\widetilde{G}}}(1)$ admits an $L_X\widetilde{G}$-linearization.

10.4.2. Lemma. *The line bundle $\mathcal{O}_{\mathcal{Q}_{\widetilde{G}}}(1)$ admits an $L_X\widetilde{G}$-linearization if and only if the restriction of the central extension* (7.3 c) *to $L_X\widetilde{G}$ splits.*

Proof. Let $\mathrm{Mum}_{L_X\widetilde{G}}(\mathcal{O}_{\mathcal{Q}_{\widetilde{G}}}(1))$ be the Mumford group of $\mathcal{O}_{\mathcal{Q}_{\widetilde{G}}}(1)$ under the action of $L_X\widetilde{G}$ on $\mathcal{Q}_{\widetilde{G}}$. This is the group of pairs (f,g) with $g \in L_X\widetilde{G}$ and $f : g^*\mathcal{O}_{\mathcal{Q}_{\widetilde{G}}}(1) \xrightarrow{\sim} \mathcal{O}_{\mathcal{Q}_{\widetilde{G}}}(1)$. As $\mathcal{Q}_{\widetilde{G}}$ is direct limit of integral projective schemes, we get a central extension

$$1 \longrightarrow G_m \longrightarrow \mathrm{Mum}_{L_X\widetilde{G}}(\mathcal{O}_{\mathcal{Q}_{\widetilde{G}}}(1)) \longrightarrow L_XG \longrightarrow 1. \tag{10.4 a}$$

In this setup, an L_XG-linearization of $\mathcal{O}_{\mathcal{Q}_{\widetilde{G}}}(1)$ corresponds to a splitting of (10.4 a). Such a construction works in general[10] and is functorial. Now observe that $\widehat{L\widetilde{G}}$ is $\mathrm{Mum}_{LG}(\mathcal{O}_{\mathcal{Q}_{\widetilde{G}}}(1))$. It follows that the extension (10.4 a) is the pullback to $L_X\widetilde{G}$ of (7.3 c), which proves the lemma. □

Now our question of the surjectivity of f has a positive answer in view of the following.

10.4.3. Theorem. *The restriction of* (7.3 c) *to L_XG splits.*

Proof. Consider the inclusion $i : L_X\widetilde{G} \hookrightarrow L\widetilde{G}$. The map $\mathrm{Lie}(i) : L_X\mathfrak{g} \hookrightarrow L\mathfrak{g}$ sends $X \otimes f$ to $X \otimes \widehat{f}_{/0}$ where $\widehat{f}_{/0}$ is the Laurent development of f at x_0. By the residue theorem the cocycle (7.2 a) is trivial over $L_X\mathfrak{g}$, hence $L_X\mathfrak{g}$ may be seen as a subalgebra of $\widehat{L\mathfrak{g}}$. Consider the basic highest weight representation $H_1(0)$ of level one of $\widehat{L\mathfrak{g}}$ and take coinvariants:

$$B = [H_1(0)]_{L_X\mathfrak{g}} = H_1(0)/_{L_X\mathfrak{g}.H_1(0)}.$$

The crucial fact[11] I will use is that $B \neq 0$.

Remark that the commutativity of (7.3 a) implies that for $\gamma \in L_XG(R)$ the associated automorphism u_γ of H maps coinvariants to coinvariants. We get a morphism of $\mathbb{C}$-groups $\pi : L_XG \to \mathbf{PGL}(B)$ hence we may consider the diagram

$$\begin{array}{ccccccccc}
1 & — & G_m & — & \widehat{L_X\widetilde{G}} & — & L_X\widetilde{G} & — & 1 \\
 & & | & & | & & | & & \\
1 & — & G_m & — & \mathbf{GL}(B) & — & \mathbf{PGL}(B) & — & 1
\end{array}$$

[10] The reader may consider to define $\mathbf{GL}_2$ as the Mumford group of $\mathcal{O}_{\mathbb{P}^1}(1)$ under the action of $\mathbf{PGL}_2$ on $\mathbb{P}^1$.

[11] This follows from the decomposition formulas of conformal field theory where B is seen as a space of conformal blocks (see [20]).

By construction, the central extension of $L_X\widetilde{G}$ above coincides with the central extension obtained by restriction of (7.3 c) to $L_X\widetilde{G}$. By definition of B, the derivative of π is trivial. As $L_X\widetilde{G}$ is an *integral* ind-group by proposition 9.1.1 it follows that π has to be the constant map identity. Indeed, write L_XG as the direct limit of integral schemes V_n and remark that π has to be constant on V_n; for large n, as V_n contains 1, this constant is $\pi(1) = 1$. So π being the identity, $\hat{\pi}$ factors through G_m which gives the desired splitting. □

10.4.4. Corollary. *The line bundle $\mathcal{O}_{\mathcal{Q}_{\widetilde{G}}}(1)$ descends to a line bundle denoted $\mathcal{O}_{\mathcal{M}_{\widetilde{G},X}}(1)$ on $\mathcal{M}_{\widetilde{G},X}$. Moreover*

$$\mathrm{Pic}(\mathcal{M}_{\widetilde{G},X}) = \mathcal{O}_{\mathcal{M}_{\widetilde{G},X}}(1)\mathbb{Z} \tag{10.4 b}$$

10.5. **The case of the special linear group.** Now that we know that $\mathrm{Pic}(\mathcal{M}_{\widetilde{G},X}) = \mathcal{O}_{\mathcal{M}_{\widetilde{G},X}}(1)\mathbb{Z}$ we may ask what happens to our determinant bundle $\mathcal{D}$.

10.5.1. Lemma. *Let $\mathcal{D}$ be the determinant line bundle on $\mathcal{M}_{SL_r,X}$. Then*

$$\mathcal{D} = \mathcal{O}_{\mathcal{M}_{SL_r,X}}(1)$$

Proof. Consider the morphism $\overline{\varphi}$ of 10.2.1:

$$\begin{tikzcd} \mathbb{P}^1_{\mathbb{C}} \arrow[r, dash, "\overline{\varphi}"] & \mathcal{Q}_{SL_r} \arrow[d, dash, "\pi"] \\ & \mathcal{M}_{SL_r,X} \arrow[u, dash, bend left, "\gamma"] \end{tikzcd}$$

Using γ, we get a family E of SL_r-bundles parameterized by $\mathbb{P}^1_{\mathbb{C}}$ and, by the above, we have to show that the determinant line bundle of this family is $\mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}(1)$. By definition of φ it is enough to treat the rank 2 case in which this family is easily identified: if we think of $\mathcal{Q}_{SL_2}$ as parameterizing special lattices as in Proposition 8.3.2 and the remarks following it. Then $E_{[a:c]}$ is defined by the inclusion

$$W = \begin{pmatrix} d & cz^{-1} \\ bz & a \end{pmatrix} (\mathbb{C}[[z]] \oplus \mathbb{C}[[z]]) \hookrightarrow \mathbb{C}((z)) \oplus \mathbb{C}((z)).$$

As the lattice

$$V = z^{-1}\mathbb{C}[[z]] \oplus \mathbb{C}[[z]] \hookrightarrow \mathbb{C}((z)) \oplus \mathbb{C}((z))$$

defines the rank 2-bundle $F = \mathcal{O}_X(p) \oplus \mathcal{O}_X$, we may view, via the inclusion $W \subset V$, the family $E_{[a:c]}$ as the kernel of the morphism $F \to \mathbb{C}_p$ which maps

the local sections $(z^{-1}f, g)$ to $af(p) - cg(p)$. But then it is easy to see that $\mathcal{D}_E = \mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}(1)$ ([1], 3.4). $\square$

10.6. **The Dynkin index.** Let $\rho : \widetilde{G} \to \mathbf{SL}_r$ be a representation. By extension of structure group we get a morphism of stacks $f_\rho : \mathcal{M}_{\widetilde{G},X} \to \mathcal{M}_{SL_r,X}$, hence by pullback

$$f_\rho^* : \mathrm{Pic}(\mathcal{M}_{SL_r,X}) \longrightarrow \mathrm{Pic}(\mathcal{M}_{\widetilde{G},X}).$$

As we have seen, both groups are canonically isomorphic to $\mathbb{Z}$, so f_ρ^* is an injection. The index d_ρ of f_ρ^* is called the *Dynkin index* of ρ. It has been introduced to the theory of G-bundles over curves by Kumar, Narasimhan and Ramanathan [12].

This index may be calculated as follows. Looking at the commutative diagram

$$\begin{array}{ccc} \mathcal{Q}_{\widetilde{G}} & \overset{\widetilde{f}_\rho}{\text{———}} & \mathcal{Q}_{SL_r} \\ \Big| & & \Big| \\ \mathcal{M}_{\widetilde{G},X} & \overset{f_\rho}{\text{———}} & \mathcal{M}_{SL_r,X} \end{array}$$

we see that $\widetilde{f}_\rho^*(\mathcal{O}_{\mathcal{Q}_{SL_r}}(1)) = \mathcal{O}_{\mathcal{Q}_{\widetilde{G}}}(d_\rho)$. As the canonical central extension (7.3 c) is $\mathrm{Mum}_{LSL_r}(\mathcal{O}_{\mathcal{Q}_{SL_r}}(1))$, by functoriality of the Mumford group the restriction of (7.3 c) to $L\widetilde{G}$ under $L\rho : L\widetilde{G} \to LSL_r$ defines the Mumford group $\widetilde{L\widetilde{G}} = \mathrm{Mum}_{L\widetilde{G}}(\mathcal{O}_{\mathcal{Q}_{\widetilde{G}}}(d_\rho))$. Looking at the differentials we see that if we restrict the canonical central extension (7.2 b) to $L\mathfrak{g}$

$$\begin{array}{ccccccccc} 0 & \text{———} & \mathbb{C} & \text{———} & \widetilde{L\mathfrak{g}} & \text{———} & L\mathfrak{g} & \text{———} & 0 \\ & & \Big\| & & \Big| & & \Big| \; L\rho & & \\ 0 & \text{———} & \mathbb{C} & \text{———} & \widehat{L\mathfrak{sl}_r} & \text{———} & L\mathfrak{sl}_r & \text{———} & 0 \end{array}$$

all we have to do is to determine the extension $\widetilde{L\mathfrak{g}}$, *i.e.* calculate its cocycle.

10.6.1. Lemma. *Let $\rho : \mathfrak{g} \to \mathfrak{sl}(V)$ be a representation of $\mathfrak{g}$ and consider the central extension obtained by restriction of* (7.2 b) *to $L\mathfrak{g}$. Then, if $V = \sum_\lambda n_\lambda e^\lambda$ is the formal character of V, its cocycle is given by*

$$\Big(\frac{1}{2}\sum_\lambda n_\lambda \lambda(H_\theta)^2\Big)\psi_{\mathfrak{g}} \tag{10.6 a}$$

where $\psi_{\mathfrak{g}}$ is the cocycle of (7.2 a).

Proof. By definition the cocycle is given by $\mathrm{Tr}(\rho(X_\theta)\rho(X_{-\theta}))\psi_{\mathfrak{g}}$, so all we have to do is to calculate this number. For this, decompose the $\mathfrak{sl}_2(\theta)$-module V as $\oplus_i V^{(d_i)}$, where $V^{(d_i)}$ is the standard irreducible $\mathfrak{sl}_2$-module with highest weight d_i. As usual, we may realize $V^{(d_i)}$ as the vector space of homogeneous polynomials in 2 variables x and y of degree d_i. Then X_θ acts as $x\partial/\partial y$, and $X_{-\theta}$ as $y\partial/\partial x$. Using the basis $x^l y^{d_i-l}, l = 0, \dots, d_i$ of $V(d_i)$, we see

$$\mathrm{Tr}(\rho(X_\theta)\rho(X_{-\theta})) = \sum_i \sum_{k=0}^{d_i} k(d_i + 1 - k).$$

The formal character of the $\mathfrak{sl}_2(\theta)$-module $V^{(d)}$ is $\sum_{k=0}^{d} e^{d\rho_\theta - k\alpha_\theta}$ where α_θ is the positive root of $\mathfrak{sl}_2(\theta)$ and $\rho_\theta = \frac{1}{2}\alpha_\theta$. Therefore we are reduced to prove the equality

$$\sum_{k=0}^{d} k(d+1-k) = \frac{1}{2}\sum_{k=0}^{d}\big((d\rho_\theta - k\alpha_\theta)(H_\theta)\big)^2 = \frac{1}{2}\sum_{k=0}^{d}(d-2k)^2$$

which is easy. □

Define the Dynkin index $d_{\mathfrak{g}}$ of $\mathfrak{g}$ itself by $\gcd(d_\rho)$ where ρ runs over all representations of $\mathfrak{g}$. The Dynkin indices of the fundamental and the adjoint representations, as well as of $\mathfrak{g}$ itself are listed in Table B. If ρ is a representation of $\mathfrak{g}$, we denote by $\mathcal{D}_\rho$ the pullback of the determinant bundle under the morphism $\mathcal{M}_{\widetilde{G},X} \to \mathcal{M}_{SL_r,X}$. Let $\mathrm{Pic}_{\det}(\mathcal{M}_{\widetilde{G},X})$ be the subgroup generated by the $\mathcal{D}_\rho$, where ρ runs over all representations of $\mathfrak{g}$.

10.6.2. Corollary. *The index of* $\mathrm{Pic}_{\det}(\mathcal{M}_{\widetilde{G},X})$ *in* $\mathrm{Pic}(\mathcal{M}_{\widetilde{G},X})$ *is* $d_{\mathfrak{g}}$.

If $\widetilde{G}$ is of type B,D or G_2, choosing a theta-characteristic κ defines a square-root $\mathcal{P}_\kappa$ of the determinant bundle $\mathcal{D} = \mathcal{D}_{\varpi_1}$ (see section 6). As the Picard group is $\mathbb{Z}$ for simply connected groups we see that $\mathcal{P}_\kappa$ does not depend on κ in this case, hence we may denote it simply by $\mathcal{P}$. Looking at Table B, we see that $d_{\mathfrak{g}}$ is 2 in the B,D or G_2, hence

10.6.3. Corollary. *Suppose* $\widetilde{G}$ *is of type* B,D *or* G_2. *Then*

$$\mathrm{Pic}(\mathcal{M}_{\widetilde{G},X}) = \mathbb{Z}\mathcal{P}$$

In particular, in the B,D or G_2 case there are no other line bundles than (powers of) the determinant and the pfaffian line bundles.

We saw in 6.5 that $\omega_{\mathcal{M}_{\widetilde{G},X}} = \mathcal{D}_{\mathrm{Ad}}^{-1}$ admits a square-root $\omega^{\frac{1}{2}}_{\mathcal{M}_{\widetilde{G},X}}(\kappa)$. Again, in the simply connected case, this square root does not depend on κ. Looking at the Dynkin index of the adjoint representation in Table B, we see

10.6.4. Corollary. *Let $\omega_{\mathcal{M}_{\widetilde{G},X}}$ be the dualizing sheaf, $\omega^{\frac{1}{2}}_{\mathcal{M}_{\widetilde{G},X}}$ its canonical (G is simply connected) square root. Then*

$$\omega^{\frac{1}{2}}_{\mathcal{M}_{\widetilde{G},X}} = \mathcal{O}_{\mathcal{M}_{\widetilde{G},X}}(-h^\vee) \tag{10.6 b}$$

where $h^\vee$ is the dual Coxeter number of $\mathfrak{g}$.

10.7. **The non simply connected case.** In the non simply connected case, $\mathcal{M}_{G,X}$ acquires $\pi_1(G)$ connected components. I will restrict[12] myself here, for simplicity of the notations, to the component containing the trivial bundle $\mathcal{M}^0_{G,X}$.

10.7.1. *The basic index.* We start by defining a number which will be useful in the sequel. Define the *basic index* $\ell_b(G)$ of G to be the smallest positive integer such that $\ell_b(\varpi_j^\vee, \varpi_{j'}^\vee)$ is an integer for all $j, j' \in J_0$ (recall the notations of 7.1). An easy calculation (see [23], Proposition 2.6.3), shows that this number is given by Table C. In order to state the next theorem correctly, I have to modify one of these numbers[13]: define $\ell_b(\mathbf{SO}^{\pm}_{4m}) = 2$ if m is even.

If A is a finite abelian group, denote $A^\wedge = \mathrm{Hom}(A, \mathbb{G}_m)$ its Pontrjagin dual.

10.7.2. Theorem. *([4]) Suppose $g(X) \geq 1$.*

(i) *Let $\mathrm{Pic}_t(\mathcal{M}^0_{G,X})$ be the torsion subgroup of $\mathrm{Pic}(\mathcal{M}^0_{G,X})$. Then we have the canonical isomorphism*

$$\mathrm{Pic}_t(\mathcal{M}^0_{G,X}) \xrightarrow{\sim} H^1(X, \pi_1(G))^\wedge$$

(ii) *The quotient $\mathrm{Pic}(\mathcal{M}^0_{G,X})/\mathrm{Pic}_t(\mathcal{M}^0_{G,X})$ is infinite cyclic. For its positive generator $\mathcal{L}$ we have*

$$f_\pi^*\mathcal{L} = \mathcal{O}_{\mathcal{M}_{\widetilde{G},X}}(\ell_b)$$

where $\pi : \widetilde{G} \to G$ and $f_\pi : \mathcal{M}_{\widetilde{G},X} \to \mathcal{M}^0_{G,X}$ is the morphism defined by extension of the structure group.

Proof. It follows from the proof of Proposition 10.4.1 that the kernel of the forgetful map $f : \mathrm{Pic}_{L_X\widetilde{G}}(\mathcal{Q}_{\widetilde{G}}) \to \mathrm{Pic}(\mathcal{Q}_{\widetilde{G}})$ identifies to the character group

[12] This is not really a restriction: actually the result is the same for the other components.

[13] This is related to the fact that the center of $\mathbf{Spin}_{4m}$ is $\mathbb{Z}_2 \times \mathbb{Z}_2$ which has a non trivial central extension.

$\mathcal{X}(L_X G)$ of $L_X G$. Then (i) follows from (9.2 a) and the fact that $L_X \widetilde{G}$ has no non trivial characters by Corollary 9.1.3.

I will not prove (ii) here. Actually one shows, using central extensions of LG, that an obstruction to the existence of $\mathcal{L}$ is that the following pairing (recall the notations of section 7.1.4)

$$\begin{aligned} c : Z \times Z &\longrightarrow \mathbb{G}_m \\ (z_j, z'_j) &\longmapsto e^{2\pi i(\varpi_j^\vee, \varpi_{j'}^\vee)} \end{aligned}$$

is trivial (see also [18], 4.6.3).

Once we know that we can't do better than ℓ_b, we have to show that $\mathcal{O}_{\mathcal{M}_{\widetilde{G},X}}(\ell_b)$ actually descends. This may be easy, as for $G = \mathbf{PGL}_r$, where a pfaffian of $\mathcal{D}_{Ad}$ is convenient (just look at the numbers of Tables B and C) or more complicated, as for SL_r/μ_s with $s \mid r$ (see [13]). □

10.8. **The case of the special orthogonal group.** We close the section by looking in more detail at $G = \mathbf{SO}_r$. According to Theorem 10.7.2, there is a canonical exact sequence

$$0 \to J_2 \xrightarrow{\lambda} \mathrm{Pic}(\mathcal{M}_{SO_r^0,X}) \longrightarrow \mathbb{Z} \to 0 \ , \tag{10.8 a}$$

where the torsion free quotient is generated by any of the $\mathcal{P}_\kappa$'s.

Denote by $\theta(X) \subset \mathrm{Pic}(X)$ the subgroup of $\mathrm{Pic}(X)$ generated by the theta-characteristics; it is an extension of $\mathbb{Z}$ by J_2.

10.8.1. Proposition. *The map $\kappa \mapsto \mathcal{P}_\kappa$ defines an isomorphism*

$$\mathcal{P} : \theta(X) \xrightarrow{\sim} \mathrm{Pic}(\mathcal{M}_{SO_r^0,X}) \ , \tag{10.8 b}$$

which coincides with λ on J_2.

This means that we have a canonical isomorphism of extensions

$$\begin{tikzcd} 0 \arrow[r, dash] & J_2 \arrow[r, dash] \arrow[d, equal] & \theta(X) \arrow[r, dash] \arrow[d, dash, "\mathcal{P}"'] & \mathbb{Z} \arrow[r, dash] \arrow[d, equal] & 0 \\ 0 \arrow[r, dash] & J_2 \arrow[r, dash, "\lambda"] & \mathrm{Pic}(\mathcal{M}_{SO_r^0,X}) \arrow[r, dash] & \mathbb{Z} \arrow[r, dash] & 0 \end{tikzcd}$$

Proof. ([4], 5.2) □

TABLE A

Type	Dual Coxeter (*Coxeter*) numbers	I_c	$h^\vee$
A_r	1 — 1 — 1 — ····· — 1 — 1	$r+1$	$r+1$
B_r	1 — 2 — 2 — ····· — 2 ⟹ 1(2)	2	$2r-1$
C_r	1(2) — 1(2) — 1(2) — ····· — 1(2) ⟸ 1	2	$r+1$
D_r	1 — 2 — 2 — ····· — 2 (1, 1)	4	$2r-2$
E_6	1 — 2 — 3 — 2 — 1; 2	3	12
E_7	2 — 3 — 4 — 3 — 2 — 1; 2	2	18
E_8	2 — 4 — 6 — 5 — 4 — 3 — 2; 3	1	30
F_4	2 — 3 ⟹ 2(4) — 1(2)	1	4
G_2	1(3) ⟸ 2	1	9

TABLE B

Type	Dynkin index	d_{Ad}	$d_{\mathfrak{g}}$
A_r	$\binom{r-1}{0}$ $\binom{r-1}{1}$ $\binom{r-1}{2}$ ····· $\binom{r-1}{r-2}$ $\binom{r-1}{r-1}$	$2r+2$	1
B_r	$2\binom{2r-1}{0}$ $2\binom{2r-1}{1}$ $2\binom{2r-1}{2}$ ····· $2\binom{2r-1}{r-2}$ $\Rightarrow$ $2^{(r-2)}$	$4r-2$	2
C_r	$\binom{2r-2}{0}$ $\binom{2r-2}{1}$ $\binom{2r-2}{2}-\binom{2r-2}{0}$ ····· $\binom{2r-2}{r-2}-\binom{2r-2}{r-4}$ $\Leftarrow$ $\binom{2r-2}{r-1}-\binom{2r-2}{r-3}$	$2r+2$	1
D_r	$2\binom{2r-2}{0}$ $2\binom{2r-2}{1}$ $2\binom{2r-2}{2}$ ····· $2\binom{2r-2}{r-3}$ $2^{(r-3)}$ $2^{(r-3)}$	$4r-4$	2
E_6	6 150 1800 150 6 (24)	24	6
E_7	36 4680 297000 17160 648 12 (360)	36	12
E_8	1500 5292000 8345660400 141605100 1778400 14700 60 (85500)	60	60
F_4	18 882 $\Rightarrow$ 126 6	8	6
G_2	2 $\Leftarrow$ 8	18	2

TABLE C

Type	$Z(\widetilde{G})$	Z	ℓ_b	$G = \widetilde{G}/Z$
A_r	$\mathbf{Z}_{r+1}$	$\mathbf{Z}_{r+1}$	$r+1$	$\mathbf{PGL}_{r+1}$
		$\mathbf{Z}_s,\ s\|(r+1)$	smallest k s.t. $\frac{k(r+1)r}{s^2} \in \mathbf{Z}$	
B_r	$\mathbf{Z}_2$	$\mathbf{Z}_2$	1	$\mathbf{SO}_{2r+1}$
C_r	$\mathbf{Z}_2$	$\mathbf{Z}_2$	1 for r even, 2 for r odd	$\mathbf{PSp}_{2r}$
D_{2m}	$\mathbf{Z}_2 \times \mathbf{Z}_2$	$\mathbf{Z}_2^0$	1	$\mathbf{SO}_{4m}$
		$\mathbf{Z}_2^+$	1 for m even, 2 for m odd	$\mathbf{SO}_{4m}^+$
		$\mathbf{Z}_2^-$	1 for m even, 2 for m odd	$\mathbf{SO}_{4m}^-$
		$\mathbf{Z}_2 \times \mathbf{Z}_2$	2	$\mathbf{PSO}_{4m}$
D_{2m+1}	$\mathbf{Z}_4$	$\mathbf{Z}_2$	1	$\mathbf{SO}_{4m+2}$
		$\mathbf{Z}_4$	4	$\mathbf{PSO}_{4m+2}$
E_6	$\mathbf{Z}_3$	$\mathbf{Z}_3$	3	$\mathbf{PE}_6$
E_7	$\mathbf{Z}_2$	$\mathbf{Z}_2$	2	$\mathbf{PE}_7$

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Hilbert schemes of points and Heisenberg algebras

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Abstract

Let $X^{[n]}$ be the Hilbert scheme of n points on a smooth projective surface X over the complex numbers. In these lectures we describe the action of the Heisenberg algebra on the direct sum of the cohomologies of all the $X^{[n]}$, which has been constructed by Nakajima. In the second half of the lectures we study the relation of the Heisenberg algebra action and the ring structures of the cohomologies of the $X^{[n]}$, following recent work of Lehn. In particular we study the Chern and Segre classes of tautological vector bundles on the Hilbert schemes $X^{[n]}$.

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1. Introduction

In these notes X will be a smooth and projective surface over the complex numbers. The object of our interest will be the Hilbert scheme of points on X. For any nonnegative integer n there is such a Hilbert scheme $X^{[n]}$ which parameterizes finite subschemes of X of length n.

If $W \subset X$ is a finite subscheme of length n, we shall also denote the corresponding point in $X^{[n]}$ by W.

There is a universal subscheme $Z_n \subset X^{[n]} \times X$ whose underlying set is given as $Z_n = \{(W, P) \mid P \in W\}$. The first projection from $Z_n \subset X^{[n]} \times X$ onto $X^{[n]}$ induces a finite and flat map $\pi : Z_n \to X^{[n]}$. Let $\mathcal{O}^{[n]} := \pi_*(\mathcal{O}_{Z_n})$. It is a locally free sheaf on $X^{[n]}$ of rank n.

The Hilbert scheme $X^{[n]}$ enjoys several nice geometric properties, the most basic one being:

Theorem 1.1. *The Hilbert scheme $X^{[n]}$ is smooth, connected and of dimension $2n$.*

The first proof of this result was given in [6]. Once connectedness is established, that the dimension of $X^{[n]}$ is $2n$, is clear: Each of the n points has two degrees of freedom.

Any subscheme $W \in X^{[n]}$ can be written as $W = \bigcup_i W_i$ where the W_i are mutually disjoint subschemes each having support in just one point. If $\operatorname{Supp} W_i = \{P_i\}$, we may define the 0-cycle

$$\rho(W) := \sum_i (\operatorname{length} W_i) P_i = \sum_{W \in X} (\operatorname{length} \mathcal{O}_{W,P}) P.$$

This 0-cycle is an element of the symmetric power $X^{(n)} := X^n/\mathfrak{S}_n$; the quotient of X^n by the symmetric group $\mathfrak{S}_n$ acting on X^n by permutation. In this way we get a map $\rho : X^{[n]} \to X^{(n)}$, which turns out to be a morphism (see [6]). It is called the Hilbert-Chow morphism.

Contrary to $X^{[n]}$ the symmetric power $X^{(n)}$ is singular. Along the diagonals, where two or more points come together, the action of the symmetric group has nontrivial isotropy, and because this happens in codimension two or more, the quotient will be singular.

It is easy to see that the Hilbert-Chow morphism is birational; indeed it is an isomorphism between the set of reduced subschemes in $X^{[n]}$ and the subset of $X^{(n)}$ consisting of 0-cycles all whose points are different.

Theorem 1.2. *The Hilbert-Chow morphism is a resolution of the singularities. In fact it is even a semismall resolution; which means that*

$$\operatorname{codim}\{z \mid \dim \rho^{-1}(z) \geq r\} \geq 2r$$

for any natural number r.

For any point $P \in X$ we let the closed subscheme $M_n(P) \subset X^{[n]}$ be the set of subschemes whose support is the single point P. In other words

$$M_n(P) = \big\{W \in X^{[n]} \;\big|\; \operatorname{Supp}(W) = \{P\}\big\}.$$

This is set-theoretically the same as $\rho^{-1}(nP)$, and $M_n(P)$ is indeed closed. We also give a name to the closed subset of $X^{[n]}$ whose elements are the subschemes with support in one (unspecified) point, and define

$$M_n := \big\{W \in X^{[n]} \;\big|\; \operatorname{Supp}(W) \text{ contains just one point}\big\}.$$

There is an obvious map $M_n \to X$ which sends a one-point-supported subscheme to the point where it is supported. The following is a basic result which now has several proofs. The first one was given by Briançon in [2], For other proofs see [4] or [5].

Theorem 1.3. *$M_n(P)$ is irreducible of dimension $n-1$, and M_n is irreducible of dimension $n+1$.*

When studying the Hilbert schemes $X^{[n]}$ of points, it is often a good idea to look at all the $X^{[n]}$ at the same time, because they are all related and therefore there is hope that a new structure emerges. One instance of this is the fact that there is a nice generating function for all the Betti numbers of all the $X^{[n]}$. We shall see that this is a reflection of the fact that the direct sum of all the cohomologies of all the $X^{[n]}$ has an additional structure. It is an irreducible module for a Heisenberg algebra action. This has been shown by Nakajima [13]. This Heisenberg action is constructed by means of correspondences between the Hilbert schemes, and the varieties M_n and $M_n(P)$ play a big role. In fact the idea is that one can go from the cohomology of $X^{[k]}$ to that of $X^{[k+n]}$, by adding subschemes of length n supported in one point of X.

In the second part of these lecture notes we will investigate how this Heisenberg action is related to the ring structure of the cohomology rings of the Hilbert schemes. Here we follow the work [11] of Lehn. We are particularly interested in the Chern classes of so-called tautological vector bundles on the Hilbert schemes. For every vector bundle V on X one has an associated tautological vector bundle $V^{[n]}$ on $X^{[n]}$ whose fibers over the

points $W \in X^{[n]}$ are naturally identified with $H^0(W, V|_W)$. In particular, if V has rank r, then $V^{[n]}$ is a vector bundle of rank nr. The Chern classes and Chern numbers of these tautological bundles have interesting geometrical and enumerative interpretations.

We study the operators of multiplication with the Chern classes of the tautological sheaves, and express them in terms of the operators of the Heisenberg algebra action. It is easy to see that the Heisenberg algebra action induces an action of a Virasoro algebra and an important step in the argument is a geometric interpretation of the Virasoro operators. Finally, we restrict to the case of tautological vector bundles associated to a line bundle L on X. We find a generating function for all the Chern classes in terms of the Heisenberg operators and, at least conjecturally, a generating function for the top Segre classes of the $L^{[n]}$.

2. The Betti numbers of $X^{[n]}$

If one is interested in the cohomology of $X^{[n]}$, the first question to ask is what are the Betti numbers of $X^{[n]}$; i.e., what are the dimensions $b_i(X^{[n]}) := \dim H^i(X^{[n]})$? (In these notes we will only be interested in homology and cohomology with coefficients in $\mathbb{Q}$, so for any space Y we write $H^i(Y)$ for $H^i(Y, \mathbb{Q})$ and $H_i(Y)$ for $H_i(Y, \mathbb{Q})$.)

The Betti numbers of the Hilbert schemes $X^{[n]}$ were determined in [8]. There the following generating series for the Betti numbers was obtained:

Theorem 2.1.

$$\sum_{n\geq 0, i\geq 0} b_i(X^{[n]})\, t^i q^n = \prod_{m>0, i\geq 0} (1 - (-1)^i t^{2m-2+i} q^m)^{(-1)^{i+1} b_i(X^{[n]})}.$$

There are several proofs of this formula. The original proof is by using the Weil-conjectures and counting subschemes over finite fields. A second proof, based on intersection cohomology, was given by Göttsche and Soergel in [9], and finally in [3] Cheah gave a third proof using the so-called virtual Hodge polynomials. In addition to the Betti numbers, the last two proofs also give the Hodge numbers of the Hilbert schemes.

If one puts $t = -1$ in Theorem 2.1, one gets an expression for the topological Euler characteristic $e(X^{[n]})$ of the spaces $X^{[n]}$:

$$\sum_{n\geq 0} e(X^{[n]})\, q^n = \prod_{m>0} (1 - q^m)^{-e(X)}$$

and by putting $t = 1$ one gets the generating series for the total dimensions of the cohomology of $X^{[n]}$:

$$\sum_{n\geq 0} \dim_{\mathbb{Q}} H^\bullet(X^{[n]})\, q^n = \prod_{m>0} \frac{(1+q^m)^{d_-}}{(1-q^m)^{d_+}}. \tag{1}$$

Here d_- and d_+ are respectively the dimensions of the even and odd part of $H^\bullet(X)$, i.e.,

$$d_+ = \sum_i \dim H^{2i}(X), \quad d_- = \sum_i \dim H^{2i+1}(X).$$

Later in these notes we shall come back to these formulas and give indications on how one can prove them.

One should note that we got nice generating functions for the Betti numbers and Euler numbers by looking at all the Hilbert schemes $X^{[n]}$ at once. This is a first indication that one should also look at all the cohomologies of the Hilbert schemes at the same time.

3. The Fock space and the current algebra

Let

$$\mathbb{H}(X) = \bigoplus_{n\geq 0} H^\bullet(X^{[n]})$$

be the direct sum of all the cohomologies of all the Hilbert schemes $X^{[n]}$. This is a bigraded vector space over $\mathbb{Q}$ whose homogeneous parts are the cohomology groups $H^i(X^{[n]})$ for $n \geq 0$ and $i \geq 0$. For any class $\alpha \in H^i(X^{[n]})$ we will call n the *weight* of α and i the *cohomological degree* or for short the *degree* of α. Sometimes we will write $\deg \alpha = (n, i)$.

The Hilbert scheme $X^{[0]}$ is just one point — the empty set is the only subscheme of length zero. Hence $H^\bullet(X^{[0]}) \cong \mathbb{Q}$ in a canonical way. We let $\mathbf{1}$ denote the fundamental class $[X^{[0]}]$. It corresponds to $1 \in \mathbb{Q}$, and we call it the *vacuum vector*.

The space $\mathbb{H}(X)$ has a parity structure, or a super structure as many call it: There is a decomposition

$$\mathbb{H}(X) = \mathbb{H}^+(X) \oplus \mathbb{H}^-(X)$$

where $\mathbb{H}^+(X)$ and $\mathbb{H}^-(X)$ are respectively the sums of the even and odd part of the cohomology $H^\bullet(X^{[n]})$; that is

$$\mathbb{H}^+(X) = \bigoplus_{n\geq 0, i\geq 0} H^{2i}(X^{[n]}),$$
$$\mathbb{H}^-(X) = \bigoplus_{n\geq 0, i\geq 0} H^{2i+1}(X^{[n]}).$$

The intersection form

$$\int_{X^{[n]}} \alpha\beta =: \langle \alpha, \beta \rangle$$

induces an intersection form on $\mathbb{H}(X)$ which respects the parity structure, which means that it is symmetric on $\mathbb{H}^+(X)$ and antisymmetric on $\mathbb{H}^-(X)$, and that the two spaces $\mathbb{H}^+(X)$ and $\mathbb{H}^-(X)$ are orthogonal.

The Poincaré series of $\mathbb{H}(X)$ with respect to the weight-grading is given by Göttsche's formula with $t = 1$ as in (1):

$$\sum_{n\geq 0} \dim_{\mathbb{Q}} H^\bullet(X^{[n]})\, q^n = \prod_{m>0} \frac{(1+q^m)^{d_-}}{(1-q^m)^{d_+}}.$$

This series also appears naturally in a construction in the theory of Lie algebras: Let V be a $\mathbb{Q}$-vector space with a parity structure, or a super space if you want; that is a decomposition $V = V^+ \oplus V^-$ of V into an odd and an even part. Assume that V comes equipped with a bilinear form $\langle\ ,\ \rangle$ respecting the parity structure. The cohomology $H^\bullet(X)$ with the pairing $\int_X \alpha \cdot \beta$ is our prototype of such a V.

Associated to V one constructs *the Fock space* $\mathbb{F}(V)$ in the following way: First we take a look at $V \otimes_{\mathbb{Q}} t\,\mathbb{Q}[t]$. A typical element of this space looks like $\sum_{i=1}^m v_i \otimes t^i$. Let T be the full tensor algebra on $V \otimes_{\mathbb{Q}} t\,\mathbb{Q}[t]$. To construct $\mathbb{F}(V)$ we impose in T the (super-)commutation relations:

$$[u \otimes t^i, v \otimes t^j] := (u \otimes t^i)(v \otimes t^j) - (-1)^{p(u)p(v)}(v \otimes t^j)(u \otimes t^i) = 0 \tag{2}$$

where u and v are any homogeneous elements in V, i.e., elements either in V^+ or V^-, and where $i \geq 1$ and $j \geq 1$ are any integers. By $p(w)$ we mean the parity of a homogeneous element w, i.e., $p(w) = 0$ when $w \in V^+$ and $p(w) = 1$ when $w \in V^-$. In order not to get confused with having two different $\otimes$-signs around, one from $V \otimes_{\mathbb{Q}} t\,\mathbb{Q}[t]$ and one from T, we have suppressed the $\otimes$-signs from the tensor algebra T in equation (2).

The formal way to impose the relations above, is to divide T by the two-sided ideal generated by the relations in (2). Clearly $\mathbb{F}(V)$ is an algebra. The unit element $\mathbf{1} \in \mathbb{F}^0(V)$ is called the *vacuum vector*.

There is a natural grading on $V \otimes_{\mathbb{Q}} t\,\mathbb{Q}[t]$ for which the degree of $v \otimes t^i$ is i. This grading induces, in an obvious way, a grading on the tensor algebra T. As the relations (2) are homogeneous of degree $i+j$, the Fock space $\mathbb{F}(V)$ is graded.

The elements of $\mathbb{F}(V)$ are linear combinations of monomials of the form

$$(v_1 \otimes t^{j_1})\,(v_2 \otimes t^{j_2}) \dots (v_p \otimes t^{j_p})$$

where each v_m is either an even or an odd element. The degree of such a monomial is $\sum j_m$. The Fock space also has a parity structure. A monomial as the one above is even (resp. odd) if the number of odd v_m's is even (resp. odd).

One may then easily check that there is an isomorphism of graded vector spaces

$$\mathbb{F}(V) \cong \bigotimes_{m=0}^{\infty} S(V^+ \otimes t^m) \otimes \Lambda(V^- \otimes t^m).$$

Here

$$S(V) := \bigoplus_{i \geq 0} S^i(V), \quad \Lambda(V) := \bigoplus_{i \geq 0} \Lambda^i(V),$$

are the symmetric and alternating algebra on V.

From this the Poincaré series of $\mathbb{F}(V)$ is readily found to be

$$\sum_{m \geq 0} \dim_{\mathbb{Q}} \mathbb{F}^m(V) = \prod_{m>0} \frac{(1+q^m)^{\dim V^-}}{(1-q^m)^{\dim V^+}}.$$

There is another algebra one may associate to V called the *current algebra*. To construct this we start by setting $V[t,t^{-1}] = V \otimes \mathbb{Q}[t,t^{-1}]$. The elements of $V[t,t^{-1}]$ are linear combinations of the elements $q_i[v] := v \otimes t^i$ for $v \in V$ and $i \in \mathbb{Z}$.

Let now T be the full tensor algebra on $V[t,t^{-1}]$. Elements of T are linear combinations of monomials $q_{i_1}[v_1]\,q_{i_2}[v_2] \dots q_{i_p}[v_p]$ where we again suppress the $\otimes$-signs.

By declaring the degree (or weight) of $q_i[v]$ to be i, we get a grading on T. There is also a parity structure on T: We declare $q_i[v]$ to be even if v is even and odd if v is odd; and a monomial $q_{i_1}[v_1]\,q_{i_2}[v_2] \dots q_{i_p}[v_p]$ is even (resp. odd) if it contains an even (resp. odd) number of odd $q_i[v]$'s.

We get the current algebra $\mathbb{S}(V)$ by imposing the following relations in T:

$$[q_i[u], q_j[v]] = i\delta_{i+j}\langle u, v\rangle \mathbf{e} \tag{3}$$

where $\mathbf{e}$ is the unit element in $T^0V[t, t^{-1}] = \mathbb{Q}$, and where u and v are any elements either in V^+ or in V^-. The bracket is the supercommutator

$$[A, B] = AB - (-1)^{p(A)p(B)}BA.$$

We also use the convention that $\delta_m = 0$ if $m \neq 0$ and $\delta_0 = 1$.

The current algebra $\mathbb{S}(V)$ acts on the Fock space $\mathbb{F}(V)$ in the following way. Recall that the Fock space is an algebra.

If $i > 0$, we let the element $q_i[u]$ act as multiplication by $u \otimes t^i$ in the algebra $\mathbb{F}(V)$, i.e., $q_i[u]\, w = (u \otimes t^i)\, w$ for any $w \in \mathbb{F}(V)$. In particular $q_i[u]\mathbf{1} = u \otimes t^i$.

For $i = 0$, we simply put $q_0[u]\, w = 0$ for any u and w.

To define the action of the operators $q_{-i}[u]$, with $i > 0$, it is sufficient to state that $q_{-i}[u]\mathbf{1} = 0$ for any $i > 0$ and any u. Indeed by the relations (3) we get

$$\begin{aligned} q_{-i}[u]\,(v \otimes t^j) &= q_{-i}[u]\, q_j[v]\mathbf{1} \\ &= \pm q_j[v]\, q_{-i}[u]\mathbf{1} - i\delta_{j-i}\langle u, v\rangle \mathbf{1} \\ &= -i\delta_{j-i}\langle u, v\rangle \mathbf{1}. \end{aligned}$$

Thus the action is given by the formula

$$q_{-i}[u]\,(v \otimes t^j) = -i\delta_{j-i}\langle u, v\rangle \mathbf{1}. \tag{4}$$

We call the operators $q_i[u]$ *creation* operators if $i > 0$ and *annihilation* operators if $i < 0$. One has the following lemma:

Lemma 3.1. *If the pairing $\langle\ ,\ \rangle$ is non-degenerate, the $\mathbb{S}(V)$-module $\mathbb{F}(V)$ is irreducible, i.e., there is no proper, nonzero subspace invariant under $\mathbb{S}(V)$.*

Proof. It is clear that the vacuum vector $\mathbf{1}$ is a generator for $\mathbb{F}(V)$ as a module over $\mathbb{S}(V)$. On the other hand, by applying an appropriate sequence of annihilation operators $q_{-i}[u]$ to any element w of $\mathbb{F}(V)$, we may bring it back to the vacuum $\mathbf{1}$. Indeed if $\{v_\bullet\}$ and $\{v'_\bullet\}$ are dual bases for V, then by equation (4) above we get

$$\begin{aligned} q_{-i_p}[v'_{i_p}]\, q_{-i_{p-1}}[v'_{i_{p-1}}] \dots q_{-i_1}[v'_{i_1}](v_{i_1} \otimes t^{i_1})\,(v_{i_2} \otimes t^{i_2}) \dots (v_{i_p} \otimes t^{i_p}) = \\ = (-1)^p i_1 \cdot i_2 \cdot \ldots \cdot i_p\, \mathbf{1} \end{aligned}$$

where the v_i's and the v_i''s are elements from the bases $\{v_\bullet\}$ and $\{v_\bullet'\}$. The operator
$q_{-i_p}[u_{i_p}]q_{-i_{p-1}}[u_{i_{p-1}}]\dots q_{-i_1}[u_{i_1}]$ kills any other monomial made from elements in $\{v_\bullet\}$, again by the relation (4). Hence any nonzero and invariant subspace contains the vacuum, and consequently equals $\mathbb{F}(V)$ because the vacuum generates $\mathbb{F}(V)$ as an $\mathbb{S}(V)$-module. $\square$

4. The Nakajima operators

We now come back to our space $\mathbb{H}(X)$. It has the same Poincaré series as the Fock space modelled on the cohomology $H^\bullet(X)$ of X. The aim of this section is to define an action of the current algebra $\mathbb{S}\big(H^\bullet(X)\big)$ on the space $\mathbb{H}(X)$ in a geometric way, making $\mathbb{H}(X)$ and $\mathbb{F}\big(H^\bullet(X)\big)$ isomorphic as $\mathbb{S}\big(H^\bullet(X)\big)$-modules.

We need to define operators $q_i[u]$ for $i \in \mathbb{Z}$ and $u \in H^\bullet(X)$ satisfying the relations (3). The operator $q_i[u]$ changes the weight by i, hence is given by a map $H^\bullet(X^{[n]}) \to H^\bullet(X^{[n+i]})$ for any $n \geq 0$. In order to define these maps, we introduce the *incidence scheme* $X^{[n,n+i]} \subset X^{[n]} \times X^{[n+i]}$, where now $i \geq 0$. It is defined as

$$X^{[n,n+i]} := \big\{(W, W') \;\big|\; W \subset W', W \in X^{[n]} \text{ and } W' \in X^{[n+i]}\big\}$$

Here, as also in future $W \subset W'$ means that W is a subscheme of W'. This is easily seen to be a closed subset of the product, and we give it the reduced scheme structure.

The two projections induce two maps $p_n : X^{[n,n+i]} \to X^{[n]}$ and $q_{n+i} : X^{[n,n+i]} \to X^{[n+i]}$. There also is a morphism $\rho : X^{[n,n+i]} \to X^{(i)}$ which is a variant of the Hilbert-Chow-map. If $W \subset W'$, then for the ideals I_W and $I_{W'}$ of I_W and $I_{W'}$, we do have the inclusion $I_{W'} \subset I_W$, and the quotient $I_W/I_{W'}$ is an $\mathcal{O}_X$-module of finite length which is supported at the points where the two subschemes W and W' differ. We define

$$\rho(W, W') := \sum_{P \in X} \operatorname{length}(I_W/I_{W'})\, P \in X^{(i)}.$$

One may show that ρ is a morphism.

Inside $X^{(i)}$ there is the small diagonal $\Delta = \big\{iP \;\big|\; P \in X\big\}$ which is isomorphic to X.

We have the following diagram:

$$
\begin{array}{ccccccc}
X = & \Delta & \subset & X^{(i)} & & \\
 & {\scriptstyle f}\uparrow & & {\scriptstyle \rho}\uparrow & & \\
 & Z_{n,i} & \subset & X^{[n,n+i]} & \xrightarrow{q_{n+i}} & X^{[n+i]} \\
 & & & {\scriptstyle p_n}\downarrow & & \\
 & & & X^{[n]} & &
\end{array}
\tag{5}
$$

where $Z_{n,i}$ is the component[1] of

$$\rho^{-1}(\Delta) = \big\{(W, W') \mid W \subset W', I_W/I_{W'} \text{ is supported in one point}\big\}$$

which is the closure of the subset where $\operatorname{Supp}(I_W/I_{W'})$ is disjoint from W. We give it the reduced scheme structure.[2] One easily checks that

$$\dim Z_{n,i} = 2n + i + 1, \tag{6}$$

indeed W is arbitrary in $X^{[n]}$, but $W' - W$ is confined to M_i.

We may pull back any class $u \in H^\bullet(X)$ along f to get a cohomology class f^*u on $Z_{n,i}$. Applying this to the fundamental class $[Z_{n,i}]$, we get the homology class $f^*u \cap [Z_{n,i}]$. This in turn we may push forward to $X^{[n,n+i]}$ via the inclusion $j : Z_{n,i} \to X^{[n,n+i]}$, and in this way we get the homology class

$$Q_{n,i}(u) := j_*(f^*u \cap [Z_{n,i}])$$

on $X^{[n,n+i]}$.

Now we are ready to define the Nakajima *creation operators*; i.e., the operators $q_i[u]$ with $i \geq 0$. We define their action on an element $\alpha \in H^\bullet(X^{[n]})$ by

$$q_i[u]\,\alpha := q_{n+i\,*}(p_n^*\,\alpha \cap Q_{n,i}(u)),$$

which we regard as an element in $H^\bullet(X^{[n+i]})$ by Poincaré duality.

This definition is similar to the classical way of defining the *correspondence* between $X^{[n]}$ and $X^{[n+i]}$ associated to a class on their product — if one insists on $q_i[u]$ being a correspondence, one has

$$q_i[u]\,\alpha = pr_{2\,*}(pr_1^*\,\alpha \cap \nu_* Q_{n,i}(u))$$

where $\nu : Z_{n,i} \to X^{[n]} \times X^{[n+i]}$ is the inclusion map, and where pr_1 and pr_2 are the two projections.

[1] To our knowledge it is unknown whether $\rho^{-1}(\Delta)$ is irreducible or not.

[2] The scheme-theoretical inverse image $\rho^{-1}(\Delta)$ is not reduced.

In order to get some geometric feeling for what these operators do, we assume that u and α are represented by submanifolds $U \subset X$ and $A \subset X^{[n]}$. Then $q_i[u]\,\alpha$ is represented by the subspace

$$(7)\quad \big\{W' \in X^{[n+i]} \;\big|\; \text{there is a } W \in A \text{ with } W \subset W', \\ W \text{ and } W' \text{ such that they differ in one point in } U\big\}.$$

To put it loosely, the creation operator $q_i[u]$ sends A to the set of subschemes which we obtain by adding a subscheme of length i supported in just one point from U to a subscheme in A. As an illustration we prove the following lemma

Lemma 4.1.

$$q_i[\mathrm{pt}]\mathbf{1} = [M_i(P)].$$
$$q_i[X]\mathbf{1} = [M_i].$$

Proof. To explain the first equality, we observe that $\mathbf{1}$ is represented by the empty set. Hence by (7) the class $q_i[\mathrm{pt}]\mathbf{1}$ is represented by

$$\big\{W' \in X^{[i]} \;\big|\; \emptyset \subset W', \emptyset \text{ and } W' \text{ differ only in } P\big\},$$

where P is any point in X, and this is clearly $M_i(P)$; we are just adding subschemes supported at P to the empty set.

The second equality is similar. We add subschemes of length i supported in one point to the empty set, but this time without any constraint on the point. □

We now come to the definition of the Nakajima *annihilation operators* $q_{-i}[u]$, where $i > 0$. We shall, except for a sign factor, literally go the other way around in the diagram (5). For any class $\beta \in H^\bullet(X^{[n+i]})$ we define

$$q_{-i}[u]\beta := (-1)^i p_{n*}\big(q_{n+i}^*\,\beta \cap Q_{n,i}(u)\big).$$

The geometrical interpretation of these annihilation operators is analogous to that of the creation operators. If the class β is represented by a submanifold $B \subset X^{[n+i]}$, then $q_{-i}[u]\beta$ will be represented by the subspace

$$(8)\quad \big\{W \in X^{[n]} \;\big|\; \text{there is a } W' \in B \text{ with } W \subset W' \text{ such that they} \\ \text{differ in just one point in } U\big\}.$$

In other words, the annihilation operator $q_{-i}[u]$ sends B to the set of the subschemes we get by throwing away subschemes supported in one point in U from subschemes in B. Of course this is possible only for some of the subschemes in B.

We will give one example. Let $C \subset X$ be a smooth curve, and let $\sigma = [C]$ be its fundamental class in $H^2(X)$. For every $n \geq 0$ the symmetric product $C^{(n)}$ is naturally embedded in the Hilbert scheme $X^{[n]}$. Put $\sigma^n = [C^{(n)}]$. Let C' be another smooth curve, and assume that $\langle C, C' \rangle = a$. Let $\sigma' = [C']$.

Lemma 4.2.

$$q_{-i}[\sigma']\sigma^n = (-1)^i a \sigma^{n-i}$$

Proof. We assume for simplicity that C and C' intersect transversally in just one point. Because C is smooth, a subscheme $W \subset C$ is uniquely determined by the associated 0-cycle $\sum_{P \in C} \text{length}\,(W_P)P$. Hence there is just one subscheme W' of length i in $C^{(i)}$, whose support is $C \cap C'$. Splitting off W' from the subschemes in $C^{(n)}$ containing it, obviously gives an isomorphism from $\{W \cup W' \in C^{(n)} \mid W \in C^{(n-i)}\}$ to C^{n-i}. This concludes the proof. □

The operators $q_i[u]$ and $q_{-i}[u]$ behave very well with respect to the intersection pairings on $X^{[n]}$ and $X^{[n+i]}$:

Lemma 4.3. *For classes $\alpha \in H^\bullet(X^{[n]})$ and $\beta \in H^\bullet(X^{[n+i]})$ we have the equality*

$$(-1)^i \int_{X^{[n]}} \alpha \cdot q_{-i}[u]\beta = \int_{X^{[n+i]}} (q_i[u]\alpha) \cdot \beta.$$

Proof. By the definition of the operators and the projection formula, both are equal to

$$\int_{X^{[n,n+i]}} p_n^* \alpha \cdot q_{n+i}^* \beta \cap Q_{n,i}(u).$$

□

The following lemma is easily deduced from the definition of the Nakajima operators

Lemma 4.4. *The operator $q_i[u]$ is of bidegree $(i, \deg u + 2(i-1))$.*

5. The relations

The basic result of Nakajima in [13] is that his creation and annihilation operators satisfy the relations of the current algebra. Below we shall sketch a proof of that, closely following the proof that Nakajima gave in [14].

Theorem 5.1. *(Nakajima, Grojnowski) For all integers i and j and all classes u and v in $H^\bullet(X)$ the following relation holds*

$$\big[q_i[u], q_j[v]\big] = i\delta_{i+j}\langle u, v\rangle \mathrm{id}.$$

The proof is in two steps. The first is to establish

Proposition 5.2. *There are universal non-zero constants c_i such that*

$$\big[q_i[u], q_j[v]\big] = c_i \delta_{i+j} \langle u, v \rangle \mathrm{id}.$$

Here by universal we mean that the c_i's neither depend on u or v nor on the surface X. A sketch of the proof of this proposition, will occupy section 6. The next step is — naturally enough — to establish

Proposition 5.3. $c_i = i$.

The last proposition can be proved in two different ways. The constants c_i have a natural interpretation as intersection numbers on the Hilbert scheme. Recall that $\dim M_i = i+1$ and $\dim M_i(P) = i-1$. Therefore M_i and $M_i(P)$ are of complementary dimension, and their intersection gives a number. However $M_i(P) \subset M_i$ so they do not intersect properly and $\int_{X^{[i]}} [M_i(P)][M_i]$ is not entirely obvious to compute. By induction one may prove (see [5]):

Proposition 5.4. *(Ellingsrud–Strømme)*

$$\int_{X^{[i]}} [M_i(P)][M_i] = (-1)^{i-1} i.$$

The following lemma then proves Proposition 5.3.

Lemma 5.5. *If $i > 0$ then $c_i = (-1)^{i-1} \int_{X^{[i]}} [M_i(P)][M_i]$.*

Proof. Recall that by Lemma 4.1 we have $[M_i(P)] = q_i[\mathrm{pt}]\mathbf{1}$ and $[M_i] = q_i[X]\mathbf{1}$. The Nakajima relation for the operators $q_{-i}[X]$ and $q_i[X]$ reads

$$q_i[X]\, q_{-i}[\mathrm{pt}] - q_{-i}[\mathrm{pt}]\, q_i[X] = c_i \cdot \mathrm{id}.$$

When we apply this to the vacuum vector, we obtain

$$q_{-i}[\mathrm{pt}]\, q_i[X]\mathbf{1} = -c_i$$

because any annihilation-operator kills the vacuum. Now, by Lemma 4.3, we get

$$\begin{aligned}\int_{X^{[i]}} [M_i(P)][M_i] &= \int_{X^{[i]}} q_i[\mathrm{pt}]\mathbf{1} \cdot q_i[X]\mathbf{1} = \\ &= (-1)^i \int_{X^{[0]}} \mathbf{1} \cdot q_{-i}[\mathrm{pt}] q_i[X]\mathbf{1} = \\ &= (-1)^i \int_{X^{[0]}} (-c_i)\mathbf{1} = (-1)^{i-1} c_i.\end{aligned}$$

□

There is also another and very elegant approach to Proposition 5.3 due to Nakajima where he uses vertex operators. We shall give this later on.

The main consequence of the Nakajima-Grojnowski theorem is the following:

Theorem 5.6. *The space $\mathbb{H}(X)$ and the Fock-space $\mathbb{F}(H^\bullet(X))$ are isomorphic as $\mathbb{S}(H^\bullet(X))$-modules.*

Proof. There is a map as $\mathbb{S}(H^\bullet(X))$-modules from $\mathbb{F}(H^\bullet(X))$ to $\mathbb{H}(X)$ defined by sending $u \otimes t^i$ to $q_i[u]\mathbf{1}$. The two spaces have the same Poincaré series, and $\mathbb{F}(H^\bullet(X))$ is an irreducible $\mathbb{S}(H^\bullet(X))$-module. □

6. Indication of how to get the relations

In this section we explain in a sketchy way why the commutation relations in Theorem 5.1 hold.

We will first treat the case when i and j have the same sign, for example both are positive. This is the case of the composition of two creation-operators.

Then $\delta_{i+j} = 0$, and we have to prove that $q_i[u]$ and $q_j[v]$ commute up to the correct sign. For simplicity we also assume that $u = [U]$ and $v = [V]$ where U and V are submanifolds of X intersecting transversally.

In the definition of the Nakajima operators we made use of the subvariety $Z_{n,i} \subset X^{[n]} \times X^{[n+i]}$. Recall that it was given as

$$Z_{n,i} = \big\{(W, W') \;\big|\; W \subset W' \text{ differ in one point}\big\}.$$

We are going to compare the two operators $q_j[v]q_i[u]$ and $q_i[u]q_j[v]$, which both map the cohomology of $X^{[n]}$ to the cohomology of $X^{[n+i+j]}$. The natural place to describe the operator $q_j[v]q_i[u]$, which is the composition of two correspondences, is on the product $X^{[n]} \times X^{[n+i]} \times X^{[n+i+j]}$. In the description the following subvariety of this product will play a role:

$$Z_1 = p_{12}^{-1}(Z_{n,i}) \cap p_{23}^{-1}(Z_{n+i,j}). \tag{9}$$

It consists of triples (W, W', W'') of nested subschemes — i.e., $W \subset W' \subset W''$ — such that W and W' just differ in one point which we call P, and at the same time W' and W'' are different only in one point that we call Q. The quotient $I_W/I_{W'}$ has support $\{P\}$ and satisfies length $I_W/I_{W'} = i$. Similarly, the quotient $I_{W'}/I_{W''}$ has support $\{Q\}$ and is of length j.

There is a map $f_1 : Z_1 \to X \times X$ sending the triple (W, W', W'') to the pair (P, Q).

In a similar manner we let $Z_2 \subset X^{[n]} \times X^{[n+j]} \times X^{[n+i+j]}$ be the subvariety given by

$$Z_2 = p_{12}^{-1} Z_{n,j} \cap p_{23}^{-1} Z_{n+j,i}. \tag{10}$$

Its elements are the triples (W, W', W'') of nested subschemes with $I_W/I_{W'}$ and $I_{W'}/I_{W''}$ both having one-point-support in, say, Q and P respectively; the first one of length j and the other one of length i. As above there is a morphism $f_2 : Z_2 \to X \times X$, sending the triple (W, W', W'') to the pair (Q, P).

Lemma 6.1. *Let α be a class on $X^{[n]}$.*

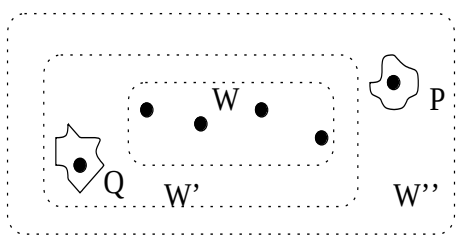
$$q_i[u]\, q_j[v]\alpha = p_{3*}\big(p_1^*\alpha \cdot f_2^*(v \times u) \cap [Z_2]\big), \tag{11}$$

$$q_j[v]\, q_i[u]\alpha = p_{3*}\big(p_1^*\alpha \cdot f_1^*(u \times v) \cap [Z_1]\big), \tag{12}$$

where p_i denotes the restriction of the i-th projection to Z_1 in the first line, and of Z_2 in the second.

Proof. This is just the formula for composing correspondences; the only point to check is that the intersections in (9) and (10) are both proper. □

Let $Z_1' \subset Z_1$ and $Z_2' \subset Z_2$ be the two open subsets where the two points P and Q are different. A typical element of Z_1', for example, may be drawn as



It has a 'central' part W and two 'fuzzy' ends, one in P and one in Q. The 'fuzzy' end at P is a subscheme of length i supported there, and the other 'fuzzy' end is a subscheme supported at Q of length j. The subscheme W' is the union of the 'central' part and the 'fuzzy' end at P. Of course P or Q may belong to the central part, but still the above statement makes sense if interpreted in the right way.

The drawing above might as well represent a typical element in Z_2'. The only difference being that in that case the 'fuzzy' part of length j at Q would belong to W' instead of the one of length i at P. Hence to any nested triple (W, V, W'') in Z_1' we may associate the triple (W, V', W'') where we get V'

from V by swapping the 'fuzzy' parts at P and Q. With a little thought one may convince oneself that this swapping is well defined even if the 'central' part touches P or Q. In this way we get an isomorphism $g : Z_1' \cong Z_2'$.

Clearly this isomorphism respects both p_1 and p_3 — it doesn't change the extreme subschemes W and W'' — and up to permutation of the two factors of $X \times X$, it respects f_1 and f_2. By the projection formula we therefore get the following equality

$$g_*\big(p_1^*\alpha \cdot f_1^*(u \times v) \cap [Z_1']\big) = (-1)^{\deg u \deg v} p_2^*\alpha \cdot f_2^*(v \times u) \cap [Z_2'].$$

The sign comes from the following: $u \times v = pr_1^* u \cdot pr_2^* v$ and via g^* this is mapped to $pr_2^* u \cdot pr_1^* v = (-1)^{\deg u \deg v} v \times u$.

It only remains to see that there is no contribution from the boundaries, i.e., when $P = Q$. The easy case is when $U \cap V = \emptyset$, then the boundary is empty — indeed $P \in U$ and $Q \in V$.

In general, a dimension estimate will show that all components of the boundary are — with good margin — of too small dimension to contribute. We shall need

$$\dim Z_1' = \dim Z_2' = 2n + i + j + 2.$$

Indeed, the n points in the 'central' part each have 2 degrees of freedom, and we are free to choose the 'fuzzy' ends from M_i and M_j, and these two varieties are of dimension $i + 1$ and $j + 1$ respectively.

By the transversality of U and V we know that

$$\dim_{\mathbb{R}} U \cap V = \dim_{\mathbb{R}} U + \dim_{\mathbb{R}} V - 4$$

We now give the dimension count for $f^{-1}(U \times V) \cap (Z - Z')$, where we have suppressed the indices and only write f, Z, Z'; the suppressed index can be either 1 or 2. The 'central' part is of length n and gives a contribution of $4n$ to the (real) dimension. Now $P = Q$, so the two 'fuzzy' parts live at the same point. If they could be chosen independently, their contribution to the dimension would be

$$\dim_{\mathbb{R}}(M_i(P) \times M_j(P) = 2(i-1) + 2(j-1)$$

as long as P is fixed, and P can only move in $U \cap V$. As this gives an upper bound of their contribution, we get

$$\begin{aligned}\dim_{\mathbb{R}}(f^{-1}(U \times V) \cap (Z - Z')) &\leq \dim_{\mathbb{R}} M_i(P) \times M_j(P) + \dim_{\mathbb{R}} U \cap V \\ &\leq 4n + 2i + 2j + \dim_{\mathbb{R}} U + \dim_{\mathbb{R}} V - 8 \\ &< 4n + 2i + 2j + \dim_{\mathbb{R}} U + \dim_{\mathbb{R}} V - 4.\end{aligned}$$

The class $f^*(u \times v) \cap [Z]$ lives in $H_r(Z)$ where

$$\begin{aligned} r &= \dim_{\mathbb{R}} Z - (4 - \dim_{\mathbb{R}} U) - (4 - \dim_{\mathbb{R}} V) \\ &= 4n + 2i + 2j + \dim_{\mathbb{R}} U + \dim_{\mathbb{R}} V - 4. \end{aligned}$$

After the dimension count, we know that the map $H_r(Z - Z') \to H_r(Z)$ induced by the inclusion is an isomorphism. Hence

$$g_* f_1^*(u \times v) \cap [Z_1] = (-1)^{\deg u \deg v} f_2^*(u \times v) \cap [Z_2],$$

and we are done.

Now we shall treat the perhaps more interesting — at least more subtle — case of the composition of one creation and one annihilation operator. That is, the composition of one operator of the form $q_{-i}[u]$ and one of the form $q_j[v]$ where $i \geq 0$ and $j \geq 0$.

We have to explain why

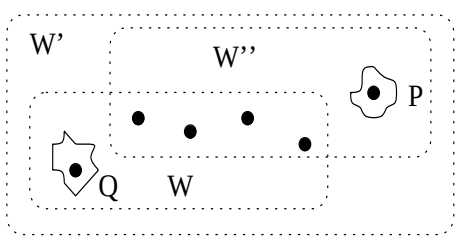
$$q_{-i}[u]\, q_j[v] + (-1)^{\deg u \deg v} q_j[v]\, q_{-i}[u] = -i\langle u, v\rangle \delta_{j-i} \mathrm{id},$$

and we start by examining the composition $q_{-i}[u]\, q_j[v]$. For any $n \geq 0$ it induces a map from $H^\bullet(X^{[n]})$ to $H^\bullet(X^{[n+j-i]})$. As in the preceding case, it is natural to look at the subvariety

$$Z_1 = p_{12}^{-1} Z_{n,j} \cap p_{23}^{-1} Z_{n+j-i,i} \subset X^{[n]} \times X^{[n+j]} \times X^{[n+j-i]}.$$

It may be described as the variety of triples $(W, W', W'') \in X^{[n]} \times X^{[n+j]} \times X^{[n+j-i]}$ with $W \subset W'$ and $W'' \subset W'$ — this time the one in the middle is bigger than the two on the sides — such that W' and W'' differ in just one point, and at the same time W' and W'' also differ only in one point. Call those points P and Q respectively.

The picture now looks like



This time the big one in the middle — W' — is the whole subscheme. The one to the left — W — is the whole except the 'fuzzy' part at P, and the one to the right — W'' — is the whole except the 'fuzzy' part supported at Q. As before there is a map $f_1 : Z_1 \to X \times X$ sending a triple to the two points (P, Q) and there is the lemma

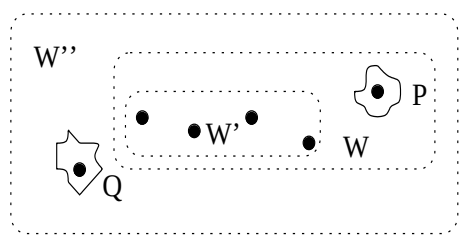
Lemma 6.2.

$$q_{-i}[u]\, q_j[v]\alpha = p_{3*}\big(p_1^*\alpha \cdot f_2^*(v \times u) \cap [Z_1]\big).$$

To understand the composition $q_{-i}[u]\, q_j[v]$, we introduce the subvariety

$$Z_2 = p_{12}^{-1} Z_{n-i,i} \cap p_{23}^{-1} Z_{n-i,j} \subset X^{[n]} \times X^{[n-i]} \times X^{[n+j-i]}.$$

This time the points in Z_2 are triples (W, W', W'') of subschemes with $W' \subset W$ and $W' \subset W''$ — the one in the middle is smaller than the other two — and as usual W' and W are different only at a point P and W' and W'' differ only at a point Q. The picture looks like



The little one in the middle — W' — is the 'central' part, and the two extremes — W and W'' — are subschemes we get by adding the 'fuzzy' part located at P respectively Q.

Just as before one checks that

$$\dim Z_1 = \dim Z_2 = 2n + i + j + 2,$$

for the complex dimensions, and there is the usual map $f_2 : Z_2 \to X \times X$.

We follow the same track as in the creation-creation process, and define $Z' \subset Z$ — where the missing index is either 1 or 2 — as the open subsets where $P \neq Q$. Then there is an isomorphism $g : Z_1' \cong Z_2'$. Indeed we keep the two extremes and exchange the smallest 'central' part with the whole. Writing W_P for the part of W supported at P and similarly for Q and W', W'', this amounts to sending the biggest one, W', to $(W' \setminus W_P' \setminus W_Q') \cup W_P \cup W_Q''$ which has a meaning as long as $P \neq Q$. In the same way, it is easy to write down the inverse of g.

Lemma 6.3.

$$g_*\big(p_1^*\alpha \cdot f_1^*(u \times v) \cap [Z_1']\big) = (-1)^{\deg u \deg v} p_2^*\alpha \cdot f_2^*(v \times u) \cap [Z_2'].$$

Now we come to the more subtle point of analyzing the boundaries where $P = Q$. Because when we compute the composition, we apply p_{13*}, what really matters is the dimension of $p_{13}(Z \setminus Z')$ — for missing index equal 1 and

2. In the case of $p_{13}(Z_2 \setminus Z_2')$ everything works as in the creation-creation case, and there will be no contribution from the boundary, so let us turn our attention to the subtle case $p_{13}(Z_1 \setminus Z_1')$. The case $U \cap V = \emptyset$ gives no boundary at all, but if $U \cap V = \{P\}$ something happens. If in addition $i = j$ we may take $W = W''$. There always exists a subscheme of length $n + j$ containing any subscheme of length n which is supported at p. Hence in this case $p_{13}(Z_1 \setminus Z_1')$ will be supported along the diagonal in $X^{[n]} \times X^{[n]}$. One may check by dimension count as before that this is the only possible contribution from the boundary. It follows that

$$\big[q_{-i}[u], q_i[v]\big] = \mu \,\mathrm{id}$$

for some number μ.

7. Vertex operators and Nakajimas computation of the constants

For any class $u \in H^\bullet(X)$ and any sequence $\mathbf{d} = \{d_m\}_{m \geq 0}$ of numbers we introduce the following operator, often called a vertex operator,

$$E_{\mathbf{d},u}(z) = \exp\Big(\sum_{m>0} d_m q_m[u] z^m\Big) = \exp(P(z)).$$

where $P(z) = \sum_{m>0} d_m q_m[u] z^m$. When we apply $E_{\mathbf{d},u}(z)$ to the vacuum vector, we obtain a sequence $\{\alpha_m\}_{m \geq 0}$ of classes in $\mathbb{H}(X)$, with α_m of weight m and $\alpha_0 = \mathbf{1}$, which are defined by the expression

$$\sum_{m \geq 0} \alpha_m z^m := \exp\Big(\sum_{m>0} d_m z^m q_m[u]\Big)\mathbf{1} = \exp(P(z)) \cdot \mathbf{1}.$$

We have

Proposition 7.1. *For any two classes u, v in $H^\bullet(X)$, and any natural number i, the element $\exp(P(z)) \cdot \mathbf{1}$ is an eigenvector for $q_i[v]$ with eigenvalue $-c_i d_i (\int_X u \cdot v) z^i$. That is, for $m \geq 0$, we have the equality*

$$q_i[v]\alpha_m = -c_i d_i \Big(\int_X u \cdot v\Big)\alpha_{m-i}.$$

In the proof of the proposition we shall need the following easy lemma:

Lemma 7.2. *If A and B are two operators commuting with their commutator, then for any $p \geq 1$*

$$[A, B^p] = p[A, B]B^{p-1}.$$

Furthermore

$$[A, \exp B] = [A, B] \exp B.$$

Proof. Exercise. □

To prove Proposition 7.1 we do the following computation:

$$\begin{aligned}
q_{-i}[v] \exp(P(z)) \cdot \mathbf{1} &= \big[q_{-i}[v], \exp(P(z))\big]\mathbf{1} && \text{ann. oper. kill vacuum}\\
&= [q_{-i}[v], P(z)] \exp(P(z)) \cdot \mathbf{1} && \text{Lemma 7.2}\\
&= \Big(\sum_{m>0} d_m \big[q_{-i}[v], q_m[u]\big] z^m\Big) \exp(P(z)) \cdot \mathbf{1} && \text{definition of } P(z)\\
&= -\, d_i c_i (\int_X uv) z^i \exp(P(z)) \cdot \mathbf{1} && \text{Nakajima relations.}
\end{aligned}$$

By the definition of $\{\alpha_m\}$, this completes the proof.

The property in Proposition 7.1 is very strong. In fact, it determines the sequence α_m completely.

Lemma 7.3. *Let the two sequences $\{\alpha_m\}$ and $\{\beta_m\}$ from $\mathbb{H}(X)$ be given, with α_m and β_m both of weight m and $\alpha_0 = \beta_0 = \mathbf{1}$. Assume that for any $i > 0$ and any class v in $H^\bullet(X)$, there is a number $e_{i,v}$ such that both α_m and β_m satisfy the equation*

$$q_i[v] x_m = e_{i,v} x_{m-i}$$

for all $n \geq 0$. Then $\alpha_m = \beta_m$ for all $m \geq 1$.

Proof. The proof goes by induction on m. We assume that $\alpha_j = \beta_j$ for $j < m$. Then for any $i \geq 0$ and any class v on X we have

$$q_{-i}[v](\alpha_m - \beta_m) = e_{i,v}(\alpha_{m-i} - \beta_{m-i}) = 0$$

by induction. Hence $\mathbb{S}(H^\bullet(X))(\alpha_m - \beta_m)$ will be a sub $\mathbb{S}(H^\bullet(X))$-module all of whose elements are of weight greater than or equal to m. Now if $m \geq 1$, the vacuum, being of weight 0, cannot be in this module which consequently must be trivial, since $\mathbb{H}(X)$ is an irreducible $\mathbb{S}(H^\bullet(X))$-module. Hence $\alpha_m = \beta_m$, and we are done. □

We shall need the following variant of the above lemma:

Lemma 7.4. *Let $\{\alpha_m\}$ and $\{\beta_m\}$ be two sequences in $\mathbb{H}(X)$ with α_m and β_m both of weight m and $\alpha_0 = \beta_0$. Assume that for all $i \geq 0$ and all classes v in $H^\bullet(X)$ there are numbers $e_{i,v}$ with $e_{i,v} = 0$ if $\deg v < 2$, such that the following two conditions are satisfied.*

1. $q_{-i}[v]\beta_m = e_{i,v}\beta_{m-i}$ *for all* $i \geq 0$ *and all classes* v *in* $h^\bullet(X)$,
2. $\deg \alpha_m = 2m$ *and*

$$q_{-i}[v]\alpha_m = e_{i,v}\alpha_{m-i}$$

whenever $\deg v \geq 2$ *and* $i > 0$.

Then $\alpha_m = \beta_m$ *for all* $m \geq 0$.

Proof. Again we use induction on m and assume that $\alpha_{m-i} = \beta_{m-i}$ for all $i > 0$. Just as in the proof above, it is sufficient to see that the vacuum vector is not contained in the $\mathbb{S}(H^\bullet(X))$-module spanned by $\alpha_m - \beta_m$. In other words we must check that any sequence of 'backwards' moves kills $\alpha_m - \beta_m$; to that end let

$$z = q_{-i_1}[v_1]q_{-i_2}[v_2]\dots q_{-i_p}[v_p](\alpha_m - \beta_m)$$

be the result of p 'backwards' moves applied to $\alpha_m - \beta_m$. If one of the v_i's is of degree greater than or equal to 2, we know that $z = 0$. Indeed, this follows by induction from two conditions in the lemma since the annihilation operators involved all commute — we can move the annihilation $q_{i_j}[v_j]$ with $\deg v_j \geq 2$ to the right in the 'backwards' sequence. Hence we may assume that all the v_i's are of degree less than 2. Then by condition 1. in the lemma, we have $q_{-i_1}[v_1]q_{-i_2}[v_2]\dots q_{-i_p}[v_p]\beta_m = 0$ and hence

$$z = q_{-i_1}[v_1]q_{-i_2}[v_2]\dots q_{-i_p}[v_p]\alpha_m.$$

We want to see that the case $z = \mathbf{1}$ cannot happen. Indeed, if $z = \mathbf{1}$, then $\sum_{j=1}^p i_j = m$. By computing the degree of z from the expression above, we obtain

$$\begin{aligned}\deg z =& \deg \alpha_m + \sum \deg\big(v_j - 2(i_j+1)\big) = \\ =& 2m + \sum(\deg v_j - 2) - 2\sum i_j = \\ =& \sum(\deg v_i - 2),\end{aligned}$$

from which it follows that $\deg z < 0$, and thus $z = 0$. □

Let now $C \subset X$ be a smooth curve whose class in $H^\bullet(X)$ is σ. Let σ_n denote the class of the n-th symmetric power $C^{(n)}$ of C in $X^{[n]}$. The classes σ_n may be computed in terms of the Nakajima creation operators as in the following theorem which appeared in [13] and [10].

Theorem 7.5. *(Nakajima, Grojnowski)*

$$\sum_{n\geq 0}\sigma_n z^n = \exp\Big(\sum_{m>0}\frac{(-1)^{m-1}}{c_m}q_m[\sigma]z^m\Big)\cdot\mathbf{1}.$$

Proof. By Proposition 7.1 we know that the sequence $\{\alpha_m\}$ defined by the identity

$$\sum_{n\geq 0}\alpha_n z^n = \exp\Big(\sum_{m>0}\frac{(-1)^{m-1}}{c_m}q_m[\sigma]z^m\Big)\cdot\mathbf{1}$$

satisfies

$$q_i[v]\alpha_m = (-1)^i\Big(\int_X \sigma v\Big)\alpha_{m-i}$$

for all $i>0$ and all $v\in H^\bullet(X)$. From Lemma 4.2 we know that $q_{-i}[v]\sigma^n = (-1)^i a\sigma^{n-i}$ for any curve class v satisfying $\int_X \sigma v = a$. It is also clear that if $v=[V]$ for V a submanifold of X with $C\cap V=\emptyset$, then $q_{-i}[v]\sigma^n=0$; hence we know that

$$q_i[v]\sigma^n = (-1)^i(\int_X v\sigma)\sigma^{n-i}$$

holds for all $i>0$ and all classes v on X of degree 2 or more. The theorem then follows from Lemma 7.4. □

Finally we will give the second computation — due to Nakajima — of the constants c_i as we promised. We start by computing derivatives in Theorem 7.5 to obtain

$$\sum_{n\geq 1} n\sigma^n z^{n-1} = \Big(\frac{d}{dz}\exp P(z)\Big)\mathbf{1} = \Big(\frac{d}{dz}P(z)\Big)\exp(P(z))\cdot\mathbf{1} =$$
$$= \Big(\Big(\sum_{m>0}\frac{(-1)^{m-1}m}{c_m}q_m[\sigma]z^{m-1}\Big)\cdot\sum_{n=0}^{\infty}\sigma^n z^n\Big)\cdot\mathbf{1}.$$

From this we obtain

$$n\sigma_n = \sum_{m=1}^{n}\frac{(-1)^{m-1}m}{c_m}q_m[\sigma]\sigma_{n-m}. \tag{13}$$

As the constants c_m are universal, we may very well assume that $X=\mathbb{P}^2$ and that C is a line.

Lemma 7.6. *Let C and C' be two curves in X intersecting transversally in one point; e.g., two different lines in $\mathbb{P}^2$. Then*

$$\int_{X^{[n]}} \sigma_n \cdot \sigma'_n = \begin{cases} 1 & \text{if } n \leq 1 \\ 0 & \text{else} \end{cases}$$

Proof. If $t = 0$ and $t' = 0$ are local equations for C and C' at the common point, a subscheme in $C^{(n)}$ supported at this point is necessarily of the form $\mathbb{C}[t, t']/(t, t'^n)$ and one in $C'^{(n)}$ must be of the form $\mathbb{C}[t, t']/(t^n, t')$. If a subscheme W simultaneously is of these two forms, necessarily $n \leq 1$. □

Finally we prove

Theorem 7.7.

$$c_i = i.$$

Proof. The idea is to intersect (13) with σ_n. For $n = 1$ we get

$$\begin{aligned} 1 = \int_X \sigma\sigma &= \frac{1}{c_1} \int_X \sigma_1 \cdot q_1[\sigma]\sigma_0 \\ &= \frac{1}{c_1} \int_X (-q_{-1}[\sigma]\sigma_1) \cdot \sigma_0 \\ &= \frac{1}{c_1} \int_X \sigma_0 \cdot \sigma_0 = \frac{1}{c_1}. \end{aligned}$$

This gives $c_1 = 1$. Assume now that $n \geq 2$. Then we obtain

$$\begin{aligned} 0 = \int_{X^{[n]}} \sigma_n \cdot \sigma_n &= \sum_{m=1}^{n} \frac{(-1)^{m-1} m}{c_m} \int_{X^{[n]}} \sigma_n \cdot q_m[\sigma]\sigma_{n-m} \\ &= \sum_{m=1}^{n} \frac{(-1)^{m-1} m}{c_m} (-1)^m \int_{X^{[n-m]}} q_{-m}[\sigma]\sigma_n \cdot \sigma_{n-m} \\ &= \sum_{m=1}^{n} \frac{(-1)^{m-1} m}{c_m} \int_{X^{[n-m]}} \sigma^{n-m} \cdot \sigma^{n-m} \\ &= \frac{(-1)^{n-1} n}{c_n} + \frac{(-1)^{n-2}(n-1)}{c_{n-1}}. \end{aligned}$$

Hence

$$\frac{c_n}{n} = \frac{c_{n-1}}{n-1}$$

from which we get $c_n = n$. □

8. Computation of the Betti numbers of $X^{[n]}$

As before, let X be a smooth projective surface over $\mathbb{C}$. We will now show formula (1) for the Betti numbers of the Hilbert scheme $X^{[n]}$ of points. We needed it in the first part to show that

$$\mathbb{H}(X) := \bigoplus_{n\geq 0} H^\bullet(X^{[n]})$$

is an irreducible representation of the Heisenberg algebra. There are at least three possible different approaches which have been used to prove this result; using the Weil conjectures [8], using perverse sheaves and intersection cohomology [9], or finally one can use the so-called virtual Hodge polynomials [3]. The last two approaches will in addition give the Hodge numbers of the Hilbert schemes. In these notes we will use the second approach. It has the advantage of leading to the shortest and most elegant proof, and to almost completely avoid any computations. The disadvantage is that it requires very deep results about intersection cohomology and perverse sheaves. We will first briefly describe these results and then show how one can use them as a black box, which with rather little effort gives the desired result.

Let Y be an algebraic variety over $\mathbb{C}$. In this section we only use the complex (strong) topology on Y. We want to stress again that all the cohomology that we consider is with $\mathbb{Q}$-coefficients. In particular $H^i(Y)$ stands for $H^i(Y,\mathbb{Q})$. There exists a complex IC_Y of sheaves on Y (for the strong topology), such that

$$\mathrm{IH}^\bullet(Y) := H^\bullet(Y, IC_Y)$$

is the *intersection homology* of Y (strictly speaking IC_Y is an element in the derived category of Y). Recall that the intersection cohomology groups $\mathrm{IH}^i(Y)$ are defined for any algebraic variety and fulfill Poincaré duality (between $\mathrm{IH}^{-i}(Y)$ and $\mathrm{IH}^i(Y)$). IC_Y is called the *intersection cohomology complex* of Y. If Y is smooth and projective of dimension n, then

$$IC_Y = \mathbb{Q}_Y[n],$$

is just the constant sheaf $\mathbb{Q}$ on Y put in degree n. Therefore $\mathrm{IH}^{i-n}(Y) = H^i(X,\mathbb{Q})$. More generally, if $Y = X/G$ is a quotient of a smooth variety of dimension n by a finite group, then $IC_Y = \mathbb{Q}_Y[n]$, and thus again $\mathrm{IH}^{i-n}(Y) = H^i(X,\mathbb{Q})$.

Let now $f : X \to Y$ be a projective morphism of varieties over $\mathbb{C}$. Suppose that Y has a stratification

$$Y = \coprod_\alpha Y_\alpha$$

into locally closed strata. Let $X_\alpha := f^{-1}(Y_\alpha)$. Assume that $f : X_\alpha \to Y_\alpha$ is a locally trivial bundle with fiber F_α (in the strong topology).

Definition 8.1. f is called *strictly semismall* (with respect to the stratification), if, for all α,

$$2dim(F_\alpha) = codim(Y_\alpha).$$

We will use the following facts:

: Fact 1. Assume that $f : X \to Y$ is strictly semismall, and that the F_α are irreducible, then

$$Rf_*(IC_X) = \sum_\alpha IC_{\overline{Y}_\alpha}.$$

(see [9]). Here Rf_* is the push-forward in the derived category, and $\overline{Y}_\alpha$ is the closure of Y_α. This is a consequence of the *Decomposition Theorem* of Beilinson-Bernstein-Deligne [1].

: Fact 2. Let $\kappa : X \to Y$ be a finite birational map of irreducible algebraic varieties, then

$$R\kappa_*(IC_X) = IC_Y$$

(see [9]).

Now we want to see how these facts about the intersection cohomology complex can be applied to compute the Betti numbers of the Hilbert schemes of points.

Let $\rho : X^{[n]} \to X^{(n)}$ be the Hilbert-Chow morphism. The symmetric power $X^{(n)}$ is stratified as follows: Let $\nu = (n_1, \dots, n_r)$ be a partition of n. We also write $\nu = (1^{\alpha_1}, 2^{\alpha_2}, \dots n^{\alpha_n})$, where α_i is the number of l such that $n_l = i$. We put

$$X_\nu^{(n)} := \left\{ \sum n_i x_i \in X^{(n)} \;\middle|\; \text{the } x_i \text{ are distinct} \right\},$$

and $X_\nu^{[n]} := \rho^{-1}(X_\nu^{(n)})$. The $X_\nu^{(n)}$ form a stratification of $X^{(n)}$ and similarly the $X_\nu^{[n]}$ form a stratification of $X^{[n]}$. The smallest stratum

$$X_{(n)}^{[n]} := \left\{ W \in X^{[n]} \;\middle|\; \mathrm{Supp}(W) \text{ is a point} \right\}$$

is just the variety M_n. It is a locally trivial fiber bundle (in the strong topology) over $X^{(n)} \simeq X$, with fiber

$$F_{(n)} := M_n(P).$$

In particular the fiber is independent of X. This is because finite length subschemes concentrated in a point depend only on an analytic neighborhood

of the point. It follows that each stratum $X^{[n]}_{(n_1,\dots,n_r)}$ is a locally trivial fiber bundle over the corresponding stratum $X^{(n)}_{(n_1,\dots,n_r)}$, with fiber $F_{(n_1)} \times \dots \times F_{(n_r)}$.

By Theorem 1.3 $M_n(P)$ is irreducible of dimension $(n-1)$, which is half the codimension of $X^{(n)}_{(n)}$ in $X^{(n)}$. It follows that $\rho : X^{[n]} \to X^{(n)}$ is strictly semismall with respect to the stratification by partitions. Therefore we obtain by Fact 1. above

$$R\rho_*(\mathbb{Q}_{X^{[n]}}[2n]) = R\rho_*(IC_{X^{[n]}}) = \bigoplus_{\nu} IC_{\overline{X}^{(n)}_{\nu}}.$$

We write

$$\nu = (n_1, \dots, n_r) = (1^{\alpha_1}, 2^{\alpha_2}, \dots, n^{\alpha_n}),$$

and denote $(\alpha) := (\alpha_1, \alpha_2, \dots, \alpha_n)$. Then there is a morphism

$$\begin{aligned} \kappa_\alpha : X^{(\alpha)} := X^{(\alpha_1)} \times \dots \times X^{(\alpha_n)} &\to \overline{X}^{(n)}_{\nu} \\ (\xi_1, \dots, \xi_n) &\mapsto \sum_{i=1}^{n} i \cdot \xi_i. \end{aligned}$$

It is easy to see that κ_α is the normalization of $\overline{X}^{(n)}_{\nu}$. Therefore Fact 2. above implies

$$IC_{\overline{X}^{(n)}_{\nu}} = R(\kappa_\alpha)_*(IC_{X^{(\alpha)}}) = R(\kappa_\alpha)_*\mathbb{Q}_{X^{(\alpha)}}\left[2|\alpha|\right],$$

where $|\alpha| = \sum_i \alpha_i$. Putting this together, we get that

$$R\rho_*(\mathbb{Q}_{X^{[n]}}[2n]) = \bigoplus_{\alpha} R(\kappa_\alpha)_*\left(\mathbb{Q}_{X^{(\alpha)}}\left[2|\alpha|\right]\right). \tag{14}$$

Here the sum runs through all $(\alpha) = (\alpha_1, \dots, \alpha_n)$ with $\sum_i i\alpha_i = n$. Finally we take the cohomology of relation (14). We recall that taking the cohomology of a complex of sheaves commutes with push-forward. Therefore we obtain

$$H^{i+2n}(X^{[n]}) = \bigoplus_{\alpha} H^{i+2|\alpha|}(X^{(\alpha)}).$$

So with this we have completely determined the additive structure of the cohomology of the Hilbert schemes $X^{[n]}$ in terms of that of the symmetric powers $X^{(k)}$. The cohomology of the symmetric powers is well known. As $X^{(n)}$ is the quotient of X^n by the action of the symmetric group $\mathfrak{S}_n$ by permuting the factors, we see that $H^i(X^{(n)}) = H^i(X^n)^{\mathfrak{S}_n}$ is the invariant part of the cohomology of X^n under the action of $\mathfrak{S}_n$.

Now we want to turn this into a generating function for the Betti numbers of the Hilbert schemes $X^{[n]}$.

Let $p(Y) := \sum_i dim(H^{i+dim(Y)}(Y))z^i$ be the (shifted) Poincaré polynomial of a variety Y. The description above of the cohomology of the symmetric powers leads, by Macdonald's formula [12], to a generating function for their Poincaré polynomials.

$$\sum_{n=0}^{\infty} p(X^{(n)})t^n = \frac{(1+z^{-1}t)^{b_1(X)}(1+zt)^{b_3(X)}}{(1-z^{-2}t)^{b_0(X)}(1-t)^{b_2(X)}(1-z^2t)^{b_4(X)}}.$$

Here the $b_i(X) = dim(H^i(X))$ are the Betti numbers of X. We are now able to put all the ingredients together to get our desired generating function for the Betti numbers of the Hilbert schemes.

$$\begin{aligned}\sum_{n=0}^{\infty} p(X^{[n]})t^n &= \sum_{n=0}^{\infty} \sum_{\alpha_1+2\alpha_2+\ldots n\alpha_n=n} p(X^{(\alpha_1)})p(X^{(\alpha_2)})\ldots p(X^{(\alpha_n)})t^{\alpha_1+2\alpha_2+\ldots n\alpha_n} \\ &= \prod_{k=1}^{\infty}\left(\sum_l p(X^{(l)})t^{kl}\right) \\ &= \prod_{k=1}^{\infty} \frac{(1+z^{-1}t^k)^{b_1(X)}(1+zt^k)^{b_3(X)}}{(1-z^{-2}t^k)^{b_0(X)}(1-t^k)^{b_2(X)}(1-z^2t^k)^{b_4(X)}}.\end{aligned}$$

This (keeping track of the shift in the Poincaré polynomial) is the formula of Theorem 2.1.

9. The Virasoro algebra

The rest of these lectures is mostly based on the paper [11] of Lehn. Before we got a nice description of the additive structure (+ the intersection pairing) of the Hilbert schemes, which put all the Hilbert schemes together into one structure. Our aim now is to get some insight into the ring structure of the cohomology rings of the Hilbert schemes of points $X^{[n]}$. We want to see how the ring structure is related to the action of the Heisenberg algebra. That is; for any cohomology class $\alpha \in H^\bullet(X^{[n]})$ we can look at the operator of multiplying by α. We want to try to express these operators in terms of the Heisenberg operators. In particular we will be interested in the Chern classes of tautological sheaves on the Hilbert schemes, which are useful in many applications of Hilbert schemes.

As a first step we will construct an action of a Virasoro algebra on the cohomologies of the Hilbert schemes. This is not such a surprising result: There is a standard construction, which associates to a Heisenberg algebra a

Virasoro algebra. This construction is essentially translated into geometric terms. One of the main technical results will be a geometric interpretation of the Virasoro generators.

We will, in the future, ignore all signs coming from odd-degree cohomology classes.

Definition 9.1. Let $\delta : H^\bullet(X) \to H^\bullet(X \times X) = H^\bullet(X) \otimes H^\bullet(X)$ be the push-forward via the diagonal embedding $\delta : X \to X \times X$. If $\delta(\alpha) = \sum_i \beta_i \otimes \gamma_i$, we write

$$q_n q_m \delta(\alpha) := \sum_i q_n[\beta_i] q_m[\gamma_i].$$

We define operators $L_n : H^\bullet(X) \to End(\mathbb{H}(X))$ by

$$L_n := \frac{1}{2} \sum_{\nu \in \mathbb{Z}} q_\nu q_{n-\nu} \delta \quad , \text{if } n \neq 0$$

$$L_0 := \sum_{\nu > 0} q_\nu q_{-\nu} \delta.$$

The sums appear to be infinite, but, for fixed $y \in \mathbb{H}(X)$ and $\alpha \in H^\bullet(X)$, only finitely many terms contribute to $L_n[\alpha]y$.

Theorem 9.2. 1. $\big[L_n[u], q_m[w]\big] = -m q_{m+n}[uw]$.

2.

$$\big[L_n[u], L_m[w]\big] = (n-m) L_{n+m}[uw] - \frac{n^3 - n}{12} \left(\int_X c_2(X) uw \right) \mathbf{1}.$$

Part 2. can be viewed as saying that the Virasoro algebra given by the $L_n[X]$ acts on $\mathbb{H}(X)$ with central charge $c_2(X)$.

The proof of the theorem is mostly formal. We will show part 1. in case $n \neq 0$. Writing

$$\delta(u) = \sum_i s_i \otimes t_i,$$

we get

$$\begin{aligned} \big[q_\nu[s_i] q_{n-\nu}[t_i], q_m[w]\big] &= q_\nu[s_i] \big[q_{n-\nu}[t_i], q_m[w]\big] + \big[q_\nu[s_i], q_m[w]\big] q_{n-\nu}[t_i] \\ &= (-m) \delta_{n+m-\nu} q_{n+m}[s_i] \left(\int_X t_i w \right) + (-m) \delta_{m+\nu} \left(\int_X w s_i \right) q_{n+m}[t_i]. \end{aligned}$$

We sum this up over all ν and i, to obtain

$$2\big[L_n[u], q_m[w]\big] = (-m) q_{n+m}[Z],$$

with

$$Z = \sum_i s_i \int_X t_i w + \sum_i t_i \int_X w u_i.$$

Each of the sums on the right-hand side equals uw. This shows part 1. Part 2. can easily be reduced to part 1.

10. Tautological sheaves

We can, in a natural way, associate a tautological sheaf $F^{[n]}$ on $X^{[n]}$ to a vector bundle F on X. These sheaves are very important in geometric applications of the Hilbert scheme $X^{[n]}$. Let again

$$Z_n := \left\{ (W, x) \in X^{[n]} \times X \;\middle|\; x \in W \right\}$$

be the universal family with the projections $p : Z_n \to X^{[n]}$, $q : Z_n \to X$. Then the *tautological sheaf*

$$F^{[n]} := p_* q^*(F)$$

is a locally free sheaf of rank rn on $X^{[n]}$, where r is the rank of F. (This is because $p : Z_n \to X^{[n]}$ is flat of degree n.) In particular $F^{[1]} = F$. By definition the fiber $F^{[n]}(W)$ of $F^{[n]}$ over a point $W \in X^{[n]}$ is naturally identified with $H^0(W, F|_W)$.

If $0 \to F \to E \to G \to 0$ is an exact sequence of locally free sheaves, then so is

$$0 \to F^{[n]} \to E^{[n]} \to G^{[n]} \to 0.$$

Therefore $(\ )^{[n]} : F \mapsto F^{[n]}$ defines a homomorphism from the Grothendieck group $K(X)$ of locally free sheaves on X to $K(X^{[n]})$.

The Chern classes of the tautological sheaves have interesting geometric interpretations.

1. Let L be a line bundle on X. Then $c_n(L^{[n]}) \in H^n(X^{[n]})$ is the Poincaré dual of the class of $C^{[n]} = C^{(n)}$, where $C \in |L|$ is a smooth curve.
2. More generally $c_{n-l}(L^{[n]})$ is the Poincaré dual of the class of all $W \in X^{[n]}$ with $W \subset C_t$ for C_t a curve in a general l-dimensional linear subsystem of $|L|$.
3. The top Segre class $s_{2n}(L^{[n]})$ is by definition just the top Chern class $c_{2n}(-L^{[n]})$ (here $(-L^{[n]})$ is the negative of $L^{[n]}$ in the $K(X^{[n]})$). In other words that means that $s_{2n}(L^{[n]})$ is the part of degree $2n$ of

$$1/(1 + c_1(L^{[n]}) + c_2(L^{[n]}) + \dots).$$

The degree of $s_{2n}(L^{[n]})$ is the number of all $W \in X^{[n]}$, which do not impose independent conditions on curves in a general $(3n-2)$-dimensional sub-linear system of $|L|$.

All these identifications are under the assumption that L is sufficiently ample. It is e.g. sufficient, but not necessary that L is the n-th tensor power of a very ample line bundle on X. The identifications are proven by using the Thom-Porteous formula ([7] Theorem 4.4), which gives the class of the degeneracy locus of a map of vector bundles in terms of their Chern classes. There is a natural evaluation map

$$ev_n : H^0(X, L) \otimes \mathcal{O}_{X^{[n]}} \to L^{[n]}, (s, W) \mapsto s|_W \in H^0(Z, L|_W) = L^{[n]}(W)$$

from the trivial bundle with fiber $H^0(X, L)$ to $L^{[n]}$. The assumption that L is sufficiently ample ensures that ev_n is surjective. In this situation the Thom-Porteous formula says that $c_{n-l}(L^{[n]})$ is the class of the locus where the restriction of ev_n to a trivial vector subbundle of rank $l+1$ is not injective. Such a vector subbundle corresponds to an l-dimensional linear subsystem M of $|L|$, and the locus where the map is not injective is easily seen to be the locus of $W \in X^{[n]}$ with $W \in C$ for a curve $C \in M$. This shows parts 1. and 2.

Part 3 is similar. In this case we look at the dual map

$$(ev_n)^\vee : (L^{[n]})^\vee \to H^0(X, L) \otimes \mathcal{O}_{X^{[n]}},$$

and the locus we are looking for is the locus where $(ev_n)^\vee$ is not injective.

So we get in particular

$$\int_X s_2(L) = \#\{\text{base points in a pencil of } |L|\} = c_1(L)^2.$$

$\int_{X^{[2]}} s_4(L^{[2]})$ is the number of double points of the map $X \to \mathbb{P}_4$ given by a general 4-dimensional linear subsystem of $|L|$. The numbers $s_{2n}(L^{[n]})$ are, for instance, interesting from the point of view of Donaldson invariants.

11. Geometric interpretation of the Virasoro operators

Our aim is to give a more geometric interpretation of the action of the Virasoro algebra, which was defined in section 9. We shall see that they are related to the "boundary" of $X^{[n]}$, i.e. the locus of subschemes of X with support less then n points. If we write ∂ for the operation of multiplying by the cohomology class of the boundary, then L_n will turn out to be essentially the commutator $q_n\partial - \partial q_n$. In order to be able to prove this result we have

to give another description of the class ∂, which relates it to tautological sheaves. This is done by looking at the incidence scheme

$$X^{[n,n+1]} := \Big\{ (Z,W) \in X^{[n]} \times X^{[n+1]} \;\Big|\; Z \subset W \Big\}.$$

As a special case of diagram (5) we have the diagram

$$\begin{array}{ccccc} X & \xleftarrow{\rho} & X^{[n,n+1]} & \xrightarrow{p_{n+1}} & X^{[n+1]} \\ & & {\scriptstyle p_n}\downarrow & & \\ & & X^{[n]}. & & \end{array} \tag{15}$$

In particular there is a morphism $\pi := (p_n, \rho) : X^{[n,n+1]} \to X^{[n]} \times X$, which sends a pair (Z,W) of subschemes of X to Z and the residual point. It is evident that π is an isomorphism over the open subset of all $(Z,x) \in X^{[n]} \times X$ with $x \neq Z$, i.e. over the complement of the universal family

$$Z_n := \Big\{ (Z,x) \in X^{[n]} \times X \;\Big|\; x \in Z \Big\}.$$

More precisely we have the following theorem:

Theorem 11.1. [5] *$X^{[n,n+1]}$ is the blowup of $X^{[n]} \times X$ along the universal family Z_n.*

Proof. Let $\pi : Y \to X^{[n]} \times X$ be the blowup along Z_n with exceptional divisor E. On $X^{[n]} \times X \times X$, let W_n be the pull-back of Z_n from the first and third factor. On $Y \times X$, let

$$\widetilde{\Delta} := (\pi \times 1_X)^{-1}\Delta, \quad \widetilde{W}_n := (\pi \times 1_X)^{-1} W_n.$$

Then the projection $p_Y|_{\widetilde{\Delta}} : \widetilde{\Delta} \to Y$ is an isomorphism, which maps $\widetilde{\Delta} \cap \widetilde{W}_n$ isomorphically onto the exceptional divisor E. Therefore $\widetilde{Z}_{n+1} := \widetilde{\Delta} \cup \widetilde{W}_n$ is a flat family of degree $(n+1)$ over Y, and on $Y \times X$ we have a sequence

$$0 \to \mathcal{O}_{\widetilde{\Delta}}(-E) \to \mathcal{O}_{\widetilde{Z}_{n+1}} \to \mathcal{O}_{\widetilde{W}_n} \to 0. \tag{16}$$

The flat family $\widetilde{Z}_{n+1}$ induces a morphism $Y \to X^{[n+1]}$, which together with the projection $Y \to X^{[n]}$ gives a morphism $Y \to X^{[n]} \times X^{[n+1]}$ with image $X^{[n,n+1]}$. One checks that the induced morphism $Y \to X^{[n,n+1]}$ is an isomorphism. $\square$

Let E be the exceptional divisor of the blowup $X^{[n,n+1]} \to X^{[n]} \times X$. Then E can be described as

$$E = \Big\{ (Z,W) \in X^{[n,n+1]} \;\Big|\; supp(Z) = supp(W) \Big\}.$$

Let F be a vector bundle on X. Then tensoring the sequence (16) with p_X^*F and pushing down to $Y = X^{[n,n+1]}$ gives the exact sequence

$$0 \to \rho^*F(-E) \to p_{n+1}^*F^{[n+1]} \to p_n^*F^{[n]} \to 0 \tag{17}$$

which relates the tautological bundles $F^{[n]}$ and $F^{[n+1]}$. This makes it possible to try to treat the tautological bundles via an inductive argument. In particular we get

$$\mathcal{O}_{X^{[n,n+1]}}(-E) = p_{n+1}^*\mathcal{O}^{[n+1]} - p_n^*\mathcal{O}^{[n]}$$

in the Grothendieck group $K(X^{[n,n+1]})$.

Let $\partial X^{[n]}$ be the closure of the stratum $X^{[n]}_{(2,1,\dots,1)}$, i.e. the locus in $X^{[n]}$, where the subscheme does not consist of n distinct points. The class of $[\partial X^{[n]}]$ (i.e. the class Poincaré dual to it) is related to the first Chern class of the tautological sheaves.

Lemma 11.2. $[\partial X^{[n]}] = -2c_1(\mathcal{O}_X^{[n]})$.

Proof. $\partial X^{[n]}$ is the branch divisor of the projection $p : Z_n \to X^{[n]}$, therefore $-2c_1(p_*\mathcal{O}_{Z_n}) = [\partial X^{[n]}]$. □

Definition 11.3. Let $d : \mathbb{H}(X) \to \mathbb{H}(X)$ be the operator of multiplying by $c_1(\mathcal{O}_{X^{[n]}})$, i.e. for $y \in H^\bullet(X^{[n]})$ we have $dy = c_1(\mathcal{O}_{X^{[n]}}) \cdot y$. For $f \in End(\mathbb{H}(X))$ the *derivative* f' of f is defined to be

$$f' := [d, f].$$

It is easy to check that

$$(fg)' = f'g + fg', \quad [f,g]' = [f',g] + [f,g'],$$

which gives some justification for calling it derivative.

We have the following geometric interpretation of the derivative in terms of tautological sheaves. Let $X^{[n,m]} \subset X^{[n]} \times X^{[m]}$ be the incidence variety of pairs of subschemes (Z, W) with $Z \subset W$ (in particular $n < m$). Let p_n and be p_m be the projections of $X^{[n,m]}$ to $X^{[n]}$ and $X^{[m]}$.

Then taking the derivative of $f \in End(\mathbb{H}(X))$ amounts to multiplying with $c_1(p_m^*\mathcal{O}_X^{[m]}) - c_1(p_n^*\mathcal{O}_X^{[n]})$.

Proposition 11.4. *Let $f : H^\bullet(X^{[n]}) \to H^\bullet(X^{[m]})$ be a homomorphism which is given by $f(\alpha) := p_{m*}(p_n^*\alpha \cap u)$, for a suitable $u \in H_\bullet(X^{[n,m]})$. Then*

$$f'(\alpha) = p_{m*}\Big(p_n^*(\alpha) \cdot \big(c_1(p_m^*\mathcal{O}_X^{[m]}) - c_1(p_n^*\mathcal{O}_X^{[n]})\big) \cap u\Big).$$

In particular, in case $m = n+1$, we get

$$f'(\alpha) = p_{n+1\,*}\big(p_n^*(\alpha)\cdot(-E)\cap u\big).$$

Proof.

$$\begin{aligned} f'(\alpha) &= df(\alpha) - f\,d\alpha \\ &= c_1(\mathcal{O}_X^{[m]})\cdot p_{m\,*}(p_n^*(\alpha)\cap u) - p_{m\,*}\Big(p_n^*\big(\alpha\cdot c_1(\mathcal{O}_X^{[n]})\big)\cap u\Big) \end{aligned}$$

Now we apply the projection formula. □

$X^{[n,n+m]}\times X$ carries two universal families $Z_n\subset Z_{m+n}$. The above result can also be reinterpreted as saying that we multiply by the first Chern class of the push-forward to $X^{[n,n+m]}$ of the ideal sheaf $\mathcal{I}_{Z_n/Z_{n+m}}$.

Now we come to the most important technical result of Lehns paper. It gives a geometric interpretation of the Virasoro operators L_n.

Theorem 11.5. 1.

$$\big[q_n'[u], q_m[w]\big] = -nm\left(q_{n+m}[uw] + \frac{|n|-1}{2}\delta_{n+m}\left(\int_X K_X uw\right)\mathbf{1}\right).$$

2.

$$q_n'[u] = nL_n[u] + \frac{n(|n|-1)}{2}q_n[K_X u].$$

Part 2. Says that the Virasoro generators $L_n[u]$ are essentially the derivatives of the $q_n[u]$.

Proof. We show that 1. implies 2. By the Heisenberg relations for the q_n and from the formula $\big[L_n[u], q_m[w]\big] = -mq_{n+m}[uw]$ from Theorem 9.2, we get

$$\begin{aligned} &\Big[nL_n[u] + \frac{n(|n|-1)}{2}q_n[K_X u], q_m[w]\Big] \\ &\qquad = -nmq_{n+m}[uw] + \frac{n^2(|n|-1)}{2}\delta_{n+m}\left(\int_X K_X uw\right)\mathbf{1}. \end{aligned}$$

Therefore the difference between the right-hand side and the left-hand side in 2. commutes with all the $q_m[u]$. Since $\mathbb{H}(X)$ is an irreducible Heisenberg module, it follows by Schurs lemma that the difference is the multiplication by a scalar. This scalar must be zero, because the difference has weight n (i.e. sends $H^\bullet(X^{[l]})$ to $H^\bullet(X^{[l+n]})$).

The proof of part 1. requires a complicated geometric argument, and it is also difficult to keep track of the indices. The most difficult part is the

case $n = -m$ (when the Theorem also has an extra term). We will sketch the proof of

$$\big[q_1'[X], q_n[u]\big]\mathbf{1} = -nq_{n+1}[u]\mathbf{1},$$

which illustrates some of the geometric ideas, without running into any of the technicalities. In the application to Chern classes of tautological sheaves, we mostly use $q_1'[X]$.

Let $U \subset X$ be the submanifold represented by u. Let

$$M_n(U) := \Big\{ Z \in X^{[n]} \;\Big|\; supp(Z) \text{ is one point of } U \Big\},$$

$$M_{n,n+1}(U) := \Big\{ (Z,W) \in X^{[n,n+1]} \;\Big|\; supp(Z) = supp(W) \text{ is one point of } U \Big\}.$$

By definition and by Proposition 11.4 we obtain

$$q_1'[X]q_n[u]\mathbf{1} = q_1'[X][M_n(U)] = p_{n+1*}((-E) \cap p_n^*[M_n(U)]).$$

We recall that

$$E = \Big\{ (Z,W) \in X^{[n,n+1]} \;\Big|\; supp(Z) = supp(W) \Big\}.$$

Therefore, set-theoretically $M_{n,n+1} = M_n \times_{X^{[n]}} E$, but the map $E \to X^{[n]}$ has degree n, and the map $M_{n,n+1} \to M_n$ has degree 1. Therefore

$$\begin{aligned} p_{n+1*}((-E) \cap p_n^*[M_n(U)]) &= -p_{n+1*}(n[M_{n,n+1}(U)]) \\ &= -np_{n+1*}[M_{n+1}(U)] \\ &= -nq_{n+1}[U]\mathbf{1}. \end{aligned}$$

On the other hand $q_1'[X]\mathbf{1} = 0$. □

Corollary 11.6. *d and the $q_1[u]$ for $u \in H^\bullet(X)$ suffice to generate $\mathbb{H}(X)$ from $\mathbf{1}$.*

12. Chern classes of tautological sheaves

We define operators on $\mathbb{H}(X)$ of multiplying by the Chern classes of the tautological sheaves $F^{[n]}$ on $X^{[n]}$. If we can understand how these commute with the q_n, this allows us to compute the Chern numbers of all tautological sheaves, and to partially understand the ring structure of the $H^\bullet(X^{[n]})$.

Definition 12.1. Let $u \in K(X)$. We define operators $\bar{c}[u] \in End(\mathbb{H}(X))$ by

$$\bar{c}[u]y = c(u^{[n]}) \cdot y \quad \text{for } y \in H^\bullet(X^{[n]}).$$

So if u is the class of a vector bundle on X, then $\bar{c}[u]$ just multiplies for each n a class on $X^{[n]}$ with the total Chern class of the corresponding tautological

sheaf $F^{[n]}$. We also write $\bar{c}_k[u]y$ for $c_k(u^{[n]}) \cdot y$. Note that by definition $d = \bar{c}_1[\mathcal{O}_X]$. Obviously the $\bar{c}[u]$ commute among each other (and therefore they also commute with d). We put

$$\bar{C}[u] := \bar{c}[u] q_1[X] \bar{c}[u]^{-1}.$$

We can use the operator $\bar{C}[u]$ to write down the total Chern classes of the tautological sheaves in a compact way.

Proposition 12.2.

$$\sum_{n \geq 0} c(u^{[n]}) = \exp(\bar{C}[u])\mathbf{1}.$$

Proof. We note that

$$\frac{q_1[X]^n}{n!}\mathbf{1} = 1_{X^{[n]}}.$$

Therefore

$$\begin{aligned}\sum_{n \geq 0} c(u^{[n]}) &= \bar{c}[u] \exp(q_1[X])\mathbf{1} \\ &= \bar{c}[u] \exp(q_1[X]) \bar{c}[u]^{-1}\mathbf{1} \\ &= \exp(\bar{c}[u] q_1[X] \bar{c}[u]^{-1})\mathbf{1}.\end{aligned}$$

□

Now we express $\bar{C}[u]$ in terms of the derivatives of the Heisenberg operator q_1 applied to the Chern classes of u. This establishes a relation between the Chern classes of the tautological sheaves and the Heisenberg generators.

Theorem 12.3.

$$\bar{C}[u] = \sum_{\nu, k \geq 0} \binom{r-k}{\nu} q_1^{(\nu)}[c_k(u)],$$

(here $q_1^{(\nu)}[c_k(u)]$ is the ν-th derivative of $q_1[c_k(u)]$).

Proof. Let F be a locally free sheaf on X. Recall the incidence variety

$$\begin{tikzcd} X & X^{[n,n+1]} \arrow[l, "\rho"'] \arrow[r, "p_{n+1}"] \arrow[d, "p_n"'] & X^{[n+1]} \\ & X^{[n]} & \end{tikzcd}$$

and the exact sequence

$$0 \to \rho^* F(-E) \to p_{n+1}^* F^{[n+1]} \to p_n^* F^{[n]} \to 0.$$

This gives

$$p_{n+1}^* c(F^{[n+1]}) = p_n^* F^{[n]} \cdot \sum_{\nu,k\geq 0} \binom{r-k}{\nu} (-E)^\nu \rho^* c_k(F). \tag{18}$$

So, for $y \in H^\bullet(X^{[n]})$, we get

$$\begin{aligned}\bar{C}[F]y &= c(F^{[n+1]}) \cdot p_{n+1\,*}\big(p_n^*(y \cdot c(F^{[n]})^{-1})\big)\\ &= p_{n+1\,*}\big(p_{n+1}^* c(F^{[n+1]}) \cdot p_n^* c(F^{[n]})^{-1} \cdot p_n^* y\big).\end{aligned}$$

We insert (18) into this formula and apply Proposition 11.4, which says that multiplying by $(-E)$ corresponds to taking derivatives. □

At least in the case of a line bundle L on X, the results obtained so far are enough for finding an elegant formula for the Chern classes of $L^{[n]}$.

Theorem 12.4.

$$\sum_{n\geq 0} c(L^{[n]}) = \exp\left(\sum_{m\geq 1} \frac{(-1)^{m-1}}{m} q_m[c(L)]\right)\mathbf{1}.$$

Remark 12.5. Note that for the top Chern classes this gives the following. Let $D \in |L|$ be a smooth curve, then $c_n(L^{[n]}) = [D^{[n]}] = [D^{(n)}]$. Then the theorem gives

$$\sum_{n\geq 0} [D^{(n)}] = \exp\left(\sum_{m\geq 1} \frac{(-1)^{m-1}}{m} q_m[c_1(L)]\right)\mathbf{1}.$$

This is Theorem 7.5, which was used to determine the constant in the Heisenberg relations.

Proof. Let

$$U(t) := \sum_{n\geq 0} c(L^{[n]}) t^n = \exp(\bar{C}[L]t)\mathbf{1}.$$

The second equality is by Proposition 12.2. Therefore U satisfies the differential equation

$$\frac{d}{dt} U(t) = \bar{C}[L]U(t), \qquad U(0) = \mathbf{1}.$$

Now let

$$S(t) := \exp\left(\sum_{m\geq 1} \frac{(-1)^{m-1}}{m} q_m[c(L)] t^m\right);$$

we want to show that $S(t)\mathbf{1}$ satisfies the same differential equation. By definition

$$\frac{d}{dt}S(t) = S(t) \cdot \sum_{m\geq 0}(-1)^m q_{m+1}[c(L)]t^m.$$

By the Lehns Main Theorem 11.5, we have

$$\big[q_1'[X], q_m[c(L)]\big] = -mq_{m+1}[c(L)].$$

As this commutes with $q_m[c(L)]$, we get

$$\Big[q_1'[X], \frac{q_m[c(L)]^n}{n!}\Big] = \frac{q_m[c(L)]^{n-1}}{(n-1)!}(-m)q_{m+1}[c(L)].$$

Therefore we obtain

$$\big[q_1'[X], S(t)\big] = S(t) \cdot \sum_{m\geq 1}(-1)^m q_{m+1}[c(L)]t^m.$$

We recall from Theorem 12.3 that

$$\bar{C}(L) = q_1[c(L)] + q_1'[X].$$

So we finally get by putting everything together

$$\begin{aligned}\bar{C}(L)S(t)\mathbf{1} &= \big[q_1'[X], S(t)\big]\mathbf{1} + q_1[c(L)]S(t)\\ &= S(t) \cdot \sum_{m\geq 0}(-1)^m q_{m+1}[c(L)]t^m.\end{aligned}$$

□

Let L again be a line bundle on X. We want to compute the top Segre classes

$$N_n := \int_{X^{[n]}} s_{2n}(L^{[n]})$$

as polynomials in the intersection numbers L^2, LK_X, K_X^2, $c_2(X)$ on X. A priory it is not clear that this should be possible. We rewrite

$$N_n = \int_{X^{[n]}} c_{2n}((-L)^{[n]}) = \int_{X^{[n]}} \frac{\bar{C}[-L]^n}{n!} \cdot 1.$$

By Theorem 12.3 we get

$$\bar{C}[-L] = \sum_{\nu\geq 0}(-1)^\nu q_1^{(\nu)}[c(-L)^{\nu+1}].$$

By the main theorem 11.5 we can express the derivatives of q_1 in terms of the Virasoro generators L_n and the Heisenberg generators q_n. Applying the definitions 9.1 of the Virasoro generators, we can express this in terms of the Heisenberg generators. We can do all these computations explicitly to

compute the N_n for sufficiently small n. The calculation shows that the following conjecture is true until $n = 7$.

Conjecture 12.6. *(Lehn) Let k be the inverse power series to*

$$t = \frac{k(1-k)(1-2k)^4}{(1-6k+6k^2)}.$$

Then

$$\sum_{n\geq 0} N_n t^n = \frac{(1-k)^{LK_X - 2K_X^2}(1-2k)^{(L-K_X)^2+3\chi(\mathcal{O}_X)}}{(1-6k+6k^2)^{\chi(L)}}.$$

(Here $\chi(L) = L(L-K_X)/2 + (K_X^2 + c_2(X))/12$ is the holomorphic Euler characteristic of L.)

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Donaldson invariants in Algebraic Geometry

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Abstract

In these lectures I want to give an introduction to the relation of Donaldson invariants with algebraic geometry: Donaldson invariants are differentiable invariants of smooth compact 4-manifolds X, defined via moduli spaces of anti-self-dual connections. If X is an algebraic surface, then these moduli spaces can for a suitable choice of the metric be identified with moduli spaces of stable vector bundles on X. This can be used to compute Donaldson invariants via methods of algebraic geometry and has lead to a lot of activity on moduli spaces of vector bundles and coherent sheaves on algebraic surfaces.

We will first recall the definition of the Donaldson invariants via gauge theory. Then we will show the relation between moduli spaces of anti-self-dual connections and moduli spaces of vector bundles on algebraic surfaces, and how this makes it possible to compute Donaldson invariants via algebraic geometry methods. Finally we concentrate on the case that the number b_+ of positive eigenvalues of the intersection form on the second homology of the 4-manifold is 1. In this case the Donaldson invariants depend on the metric (or in the algebraic geometric case on the polarization) via a system of walls and chambers. We will study the change of the invariants under wall-crossing, and use this in particular to compute the Donaldson invariants of rational algebraic surfaces.

Keywords: Donaldson invariants, moduli spaces of sheaves.

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1 Introduction

Donaldson invariants [D1] have played an important role in the study and classification of compact differentiable 4-manifolds X. The most comprehensive introduction to Donaldson invariants is [D-Kr]. Discrete invariants of 4-manifolds are the fundamental group $\pi_1(X)$ and the intersection form on $H_2(X, \mathbb{Z})$. If X is simply-connected, then the homotopy type of X is essentially determined by the intersection form. Friedman showed that in this case X is determined up to homeomorphism by its homotopy type.

In order to attempt to make a differentiable classification, ones needs additional invariants. The Donaldson invariants are defined via gauge theory in terms of moduli spaces of anti-self-dual connections on differentiable bundles on X. If X is an algebraic surface, then these moduli spaces can be identified with moduli spaces of stable vector bundles on X. This makes it possible to apply methods of algebraic geometry to compute the Donaldson invariants. In fact, because of this, for a long time most of the calculations of Donaldson invariants have been carried out in the case of algebraic surfaces. On the other hand the Donaldson invariants have provided a lot of interest for the study of moduli spaces of vector bundles and coherent sheaves on algebraic surfaces.

Some results obtained with Donaldson invariants are:

1. Algebraic surfaces are essentially indecomposable: If a simply-connected algebraic surface X is the connected sum $X = Y \# Z$ of two 4-manifolds, then either Y or Z must have a negative definite intersection form. An example where this happens is when X is the blow up of Y in a point.

2. The differentiable classification of elliptic surfaces.

3. The Kodaira dimension of an algebraic surface is a differentiable invariant.

Recently the Seiberg-Witten invariants have appeared, which are also defined via gauge theory, but are often easier to compute [W],[D2]. A number of conjectures from Donaldson theory were immediately proved, e.g.

1. The plurigenera of algebraic surfaces are differentiable invariants.

2. The generalized Thom conjecture: Let X be an algebraic surface, then each smooth algebraic curve in X minimizes the genus of embedded 2-manifolds in its homology class.

Conjecturally the Donaldson- and Seiberg-Witten invariants are very closely related and in particular the Donaldson invariants can be computed in terms of the Seiberg-Witten invariants.

Since the appearance of the Seiberg-Witten invariants the interest in the Donaldson invariants has become a bit less, but there is still a large number of interesting open questions.

2 Definition and properties of the Donaldson invariants

In this lecture we define the Donaldson invariants via gauge theory and state some of their most important properties. We prefer here to formulate everything in terms of vector bundles, which should be more familiar to the audience, instead of principal bundles, which would be more natural.

2.1 Moduli spaces of connections

Let X be a smooth simply-connected compact oriented 4-manifold. Let P be a principal $SU(2)$- or $SO(3)$-bundle on X. The Donaldson invariants are defined via intersection theory on a moduli space of anti-self-dual connections on P.

$SU(2)$-bundles on X are classified by their second Chern class $c_2(P)$, and $SO(3)$-bundles on X are classified by their second Stiefel-Whitney class $w_2(P) \in H^2(X, \mathbb{Z}/2)$ and their first Pontrjagin class $p_1(P) \in H^4(X, \mathbb{Z})$.

In the $SU(2)$-case the moduli space of anti-self-dual connections on P can be identified with the moduli space of anti-self-dual connections on the associated complex vector bundle on E with first Chern class $c_1 = 0$. In the $SO(3)$-case (after choosing a lift $c_1 \in H^2(X, \mathbb{Z})$ of $w_2(P)$) it corresponds to a moduli space of Hermitian Yang-Mills connections on the associated complex vector bundle with Chern classes c_1, c_2 (with $c_1^2 - 4c_2 = p_1(P)$). For simplicity, in the following we will concentrate on the $SU(2)$-case.

Let E be a rank 2 complex differentiable vector bundle on X. We fix a hermitian metric h on E. (That is for each $x \in X$ we have a hermitian inner product on the fiber $E(x)$, varying smoothly with x.) We denote by $\Omega^i(E)$ the space of $\mathbb{C}^\infty$-sections of $E \otimes \Lambda^i T_X^*$.

A hermitian connection on E is a connection $D : \Omega^0(E) \to \Omega^1(E)$, which is compatible with h. (That D is a connection means that it is a linear map

satisfying the Leibniz rule

$$D(f \cdot s) = d(f \otimes s) + f \cdot D(S)$$

and that D is compatible with the metric means furthermore that

$$d(h(s_1, s_2)) = h(D(s_1), s_2) + h(s_1, D(s_2)) \ .)$$

D is called *reducible* if E is the direct sum $L_1 \oplus L_2$ of two line bundles, and $D = D_1 \oplus D_2$ with D_i a connection on L_i.

We write $\mathcal{A}(E)$ for the space of hermitian connections on E, (which are trivial on $det(E)$ in the case $c_1(E) = 0$ and equal to a fixed connection on $det(E)$ otherwise). $\mathcal{A}^*(E) \subset \mathcal{A}(E)$ is the subspace of irreducible connections. The *gauge group* $\mathcal{G}$ is the set of $\mathbb{C}^\infty$-automorphisms of E which are compatible with h and act as the identity on $det(E)$. $\mathcal{G}$ acts on $\mathcal{A}(E)$ and $\mathcal{A}^*(E)$ via

$$\begin{array}{ccc} \Omega^0(E) & \xrightarrow{D} & \Omega^1(E) \\ \downarrow \alpha & & \downarrow \alpha \\ \Omega^0(E) & \xrightarrow{\alpha(D)} & \Omega^1(E). \end{array}$$

Let $\mathcal{B}(E) := \mathcal{A}(E)/\mathcal{G}$, $\mathcal{B}^*(E) := \mathcal{A}^*(E)/\mathcal{G}$.

2.2 ASD-connections

We assume in this part that $c_1(E) = 0$. Now fix a Riemannian metric g on X. It gives rise to a Hodge star operator

$$*_g : \Lambda^2 T_X^* \to \Lambda^2 T_X^*, \quad *_g^2 = 1.$$

We write Λ_+ for the $(+1)$-eigenbundle and Λ_- for the (-1)-eigenbundle of $*_g$.

Definition 2.1 For $D \in \mathcal{A}^*(E)$, let $F(D) = D \circ D \in \Omega^2(End(E))$ be it's curvature. F is called anti-self-dual (ASD), if

$$*F(D) = -F(D).$$

In other words, writing $F(D) := F_-(D) + F_+(D)$, with $F_-(D)$ a section of $\Lambda_-(End(E))$ (and similarly for $F_+(D)$), the condition is $F_+(D) = 0$. We write $N_g(E)$ for the moduli space

$$N_g(E) := \{\text{ASD-connections on } E\}/\mathcal{G} \subset \mathcal{B}^*(E).$$

In the case $c_1(E) \neq 0$ we have instead to take the moduli space of Hermitian Yang-Mills connections on E, because only these correspond to the moduli space of ASD-connections on the corresponding principal bundle. The differentiable type of E is determined by its Chern classes $c_1(E)$, and $c_2(E)$. Therefore we also write $N_g(c_1, c_2)$ for $N_g(E)$.

If D is an ASD-connection (or Hermitian Yang-Mills in case $c_1(E) \neq 0$) on E, then by Chern-Weil theory

$$4c_2(E) - c_1^2(E) = -p_1(E) = \frac{1}{4\pi^2}\int_X tr(F(D) \wedge F(D)) = \frac{1}{4\pi^2}\int_X \|F_-(D)\|^2 > 0.$$

Let $b_+(X)$ be the number of positive eigenvalues of the intersection form on $H_2(X, \mathbb{R})$. We write

$$k := (c_2 - c_1^2/4)(E).$$

Then we have the following generic smoothness result:

Theorem 2.2 *If g is generic, then $N_g(E)$ is a smooth manifold of dimension*

$$2d = 8k - 3(1 + b_+(X)).$$

For a generic path g_t of metrics, the corresponding parameterized moduli space is smooth.

Furthermore $N_E(g)$ is orientable. The orientation depends on the choice of an orientation of a maximal-dimensional subspace $H^2(X, \mathbb{R})^+$ of $H^2(X, \mathbb{R})$ on which the intersection form is positive definite.

2.3 Relations to holomorphic vector bundles

Assume here, that $c_1(E) = 0$. Let X be a projective algebraic surface. Let H be an ample divisor on X. Let $g(H)$ be the corresponding Hodge metric and ω the Kähler form. We write $\Lambda^{p,q}$ for the bundle of (p, q) forms. Then we get

$$\begin{aligned} \Lambda_+ &= \Re(\Lambda^{2,0} \oplus \Lambda^{0,2}) \oplus \mathbb{R}\omega \\ \Lambda_- &= \omega^\perp \qquad \text{in } \Lambda^{1,1}. \end{aligned}$$

We can write $D := \partial_D + \overline{\partial}_D$, where $\partial_D : \Omega^0(E) \to \Omega^{1,0}(E)$ and $\overline{\partial}_D : \Omega^0(E) \to \Omega^{0,1}(E)$. So we get

$$F(D) = \partial_D^2 + (\partial_D \overline{\partial}_D + \overline{\partial}_D \partial_D) + \overline{\partial}_D^2.$$

D is ASD if

1. $\overline{\partial}_D^2 = 0$.

2. $\partial_D^2 = 0$ and $F(D) \wedge \omega = 0$.

1. means that $\overline{\partial}_D$ defines a holomorphic structure on E. 2. implies after some work that $(E, \overline{\partial}_D)$ is μ-stable with respect to H.

Recall that a vector bundle E of rank 2 on an algebraic surface X is called μ-stable (slope stable) with respect to an ample divisor H, if

$$Hc_1(L) < \frac{Hc_1(E)}{2}$$

for all locally free subsheaves L of rank 1 of E. We denote by $M_H^X(c_1, c_2)^\mu$ the moduli space of μ-stable rank 2 bundles on X with Chern classes c_1 and c_2. We have motivated (at least in the case $c_1 = 0$) that there is a map

$$\Psi : N_{g(H)}(c_1, c_2) \to M_H^X(c_1, c_2)^\mu.$$

In fact this map exists for any c_1, and furthermore we get:

Theorem 2.3 *(Donaldson) Ψ is a homeomorphism.*

This will give a relation between the Donaldson invariants (which we will define via moduli spaces of ASD-connections) and moduli of vector bundles.

2.4 Uhlenbeck compactification

We want to define the Donaldson invariants as intersection numbers on $N_g(E)$ which is usually not compact. We have therefore to compactify.

Theorem 2.4 *Let $(A_i)_i$ be a sequence in $N_g(E)$. After passing to a subsequence we obtain: There is a finite collection of points $p_1, \dots p_l \in X$ with multiplicities $n_1, \dots, n_l > 0$, such that up to gauge transformation $A_i|_{X \setminus \{p_1, \dots, p_l\}}$ converges to an ASD-connection A_∞. A_∞ can be extended to an ASD-connection on a vector bundle E' with*

$$det(E') = det(E), \quad c_2(E) = c_2(E') + \sum_{i=1}^{l} n_i.$$

This leads to the Uhlenbeck compactification:

Theorem 2.5 *There exists a topology on*

$$\coprod_{n\geq 0} N_g(c_1, c_2 - n) \times X^{(n)}$$

such that the closure $\overline{N}_g(c_1, c_2)$ *is compact.*

Here $X^{(n)} = X^n/\mathfrak{S}(n)$ denotes the n-th symmetric power of X, the quotient of the n-th power of X by the action of the symmetric group $\mathfrak{S}(n)$ via permuting the factors. It parameterizes effective 0-cycles on X of degree n.

2.5 Definition of the invariants

We write $H^*(X) := H^*(X, \mathbb{Q})$ and $H_*(X) = H_*(X, \mathbb{Q})$. If on $X \times N_g(E)$ there exists a universal bundle $\mathcal{E}$ with a universal connection $\mathcal{D}$ with $\mathcal{D}|_{X\times\{D\}} = D$, then we can define the μ-map as follows.

$$\mu : H_*(X) \to H^*(N_g(E)); \quad \mu(\alpha) = -\frac{1}{4}p_1(\mathcal{E})/\alpha.$$

Here $-\frac{1}{4}p_1(\mathcal{E}) = (c_2(\mathcal{E}) - c_1(\mathcal{E})^2)/4$, and the slant product $-\frac{1}{4}p_1(\mathcal{E})/\alpha$ means: write

$$-\frac{1}{4}p_1(\mathcal{E}) = \sum_i \beta_i \otimes \gamma_i, \quad \beta_i \in H^*(X), \quad \gamma_i \in H^*(N_g(E)).$$

Then

$$-\frac{1}{4}p_1(\mathcal{E})/\alpha = \sum_i \langle \beta_i, \alpha\rangle \gamma_i.$$

If the universal bundle does not exist, its endomorphism bundle $End(\mathcal{E})$ will still exist, and we can define μ by replacing $(c_2(\mathcal{E}) - c_1^2(\mathcal{E}))/4$ with $-c_2(End(\mathcal{E}))/4$.

It can be shown that $\mu(\alpha)$ extends over the Uhlenbeck compactification $\overline{N}_g(E)$. For generic metric g, $\overline{N}_g(E)$ is a stratified space with smooth strata, and the submaximal stratum has codimension at least 4. Therefore $\overline{N}_g(E)$ has a fundamental class.

Now let

$$d := 4c_2 - c_1^2 - \frac{3}{2}(1 + b_+(X))$$

and write $d = l + 2m$. Let $\alpha_1, \ldots \alpha_l \in H_2(X)$ and let $p \in H_0(X)$ be the class of a point. Then we define the *Donaldson invariant*

$$\Phi^{X,g}_{c_1,d}(\alpha_1 \cdot \ldots \cdot \alpha_l \cdot p^m) := \int_{[\overline{N}_g(E)]} \mu(\alpha_1) \cup \ldots \cup \mu(\alpha_l) \cup \mu(p)^m.$$

More generally let

$$A_*(X) := Sym^*(H_2(X) \oplus H_0(X)).$$

This is graded by giving degree $(2 - i/2)$ to elements in $H_i(X)$. We denote by $A_d(X)$ the part of degree d. By linear extension we get a map $\Phi^{X,g}_{c_1,d}$: $A_d(X) \to \mathbb{Q}$ and

$$\Phi^{X,g}_{c_1} := \sum_{d \geq 0} \Phi^{X,g}_{c_1,d} : A_*(X) \to \mathbb{Q}.$$

By definition the Donaldson invariants depend on the choice of the metric g, because the ASD-equation uses the Hodge $*$ operator, which depends on g. We have however

Theorem 2.6 *1. If $b_+(X) > 1$, then $\Phi^{X,g}_{C,d}$ is independent of the generic metric g.*

2. If $b_+(X) = 1$, then $\Phi^{X,g}_{C,d}$ depends only on the chamber of the period point of g.

We will discuss walls and chambers later. The result means that the Donaldson invariants are really invariants of the differentiable structure of X. In the case $b_+(X) > 1$, we can therefore drop the g from our notation.

The argument for showing the theorem is that one connects two generic metrics by a generic path in order to make a cobordism. Reducible connections occur in codimension $b_+(X)$, so they make no problem for $b_+(X) > 1$, but can disconnect the path for $b_+(X) = 1$.

2.6 Structure theorems

It is often useful to look at generating functions for the Donaldson invariants. For $a \in H_2(X)$ and $\alpha \in A_*(X)$ and a variable z we write

$$\Phi^X_C(\alpha e^{az}) := \sum_{n \geq 0} \Phi^X_C(\alpha a^n/n!)z^n.$$

Definition 2.7 A 4-manifold X is *of simple type* if

$$\Phi^X_C(\alpha(p^2 - 4)) = 0$$

for all $\alpha \in A_*(X)$ and all $C \in H^2(X, \mathbb{Z})$.

Many 4-manifolds like $K3$ surfaces and complete intersections are known to be of simple type, and it is possible that all simply-connected 4-manifolds are of simple type. The famous structure theorem of Kronheimer and Mrowka [Kr-Mr] says that all the Donaldson invariants of a manifold of simple type organize themselves in a nice generating function, which depends only on a finite amount of data: a finite number of cohomology classes in $H^2(X,\mathbb{Z})$ (the *basic classes*) and rational multiplicities associated to these numbers.

Theorem 2.8 *Let X be a simply-connected 4-manifold of simple type. Then there exist so-called basic classes $K_1,\dots,K_l \in H^2(X,\mathbb{Z})$ and rational numbers $\alpha_1(C),\dots\alpha_l(C)$, such that for all $a \in H_2(X)$*

$$\Phi_C^X(e^{at}(1+p/2)) = e^{(a\cdot a)t^2/2}\sum_{i=1}^{l}\alpha_i(C)e^{\langle K_i,a\rangle t}.$$

(Here $(a\cdot a)$ denotes the intersection form on $H_2(X)$, and $\langle K_i,a\rangle$ the dual pairing between cohomology and homology.)

3 Algebro-geometric definition of Donaldson invariants

Let X be a simply-connected algebraic surface, and let H be an ample divisor on X. For a sheaf F and a line bundle L on X we denote $F(nL) := F\otimes L^{\otimes n}$. Let $\chi\mathcal{F}) = \sum_i(-1)^i dim H^i(X,F)$ be the holomorphic Euler characteristic of $\mathcal{F}$. Recall that a torsion-free coherent sheaf F on X is μ-stable with respect to H if $(c_1(G)\cdot H)/rk(G) < (c_1(F)\cdot H)/rk(F)$ for all non-zero strict subsheaves of F. F is called (Gieseker) H-semistable if $\chi(G(nH)) \leq \chi(F(nH)$ for all nonzero strict subsheaves G of F.

We denote by $M := M_H^X(C,c_2)$ the moduli space of (Gieseker) H-semistable rank 2 torsion-free coherent sheaves F on X with $c_1(F) = C$ and $c_2(F) = c_2$. We want to relate M to the Uhlenbeck compactification $N := N_{g(H)}(C,c_2)$. Here $g(H)$ is the Fubini-Study metric associated to H. As the Donaldson invariants are defined in terms of the Uhlenbeck compactification, this allows us to compute them on the moduli space M of sheaves.

The steps of the argument are as follows:

1. Introduce the determinant bundles $L_1(nH)$ on M for $n \gg 0$.

2. Construct sections of $L_1(nH)^{\otimes m}$ for $n, m \gg 0$ and show that the corresponding linear system is base-point free, thus giving a morphism

$$\Psi : M \to \mathbb{P}(H^0(M, L_1(nH)^{\otimes m})^\vee).$$

3. Show that $Im(\Psi)$ is homeomorphic to N.

4. Apply this to the computation of the Donaldson invariants.

3.1 Determinant line bundles

We will assume for simplicity that there is a universal sheaf $\mathcal{E}$ over $X \times M$. For instance, this is the case if H is general and either C is not divisible by 2 or $c_2 - C^2/4$ is odd.

For a coherent sheaf $\mathcal{F}$ on $X \times M$, let

$$0 \to G_l \to \ldots \to G_s \to 0$$

be a finite complex of locally free sheaves which is quasi-isomorphic to $Rp_{2*}(\mathcal{F})$. Then we put

$$det(p_{2!}(\mathcal{F})) := \bigotimes det(G_j)^{(-1)^j} \in Pic(M).$$

Definition 3.1 Let $D \in |nH|$ be a smooth curve. For a general $E \in M$ let $\chi_1 := \chi(E|_D)$. Let $a \in X$ be a point. Then we put

$$L_1(nH) := det(p_{2!}(\mathcal{E}|_{D\times M}))^{\otimes -2} \otimes det(\mathcal{E}|_{\{a\}\times M})^{\otimes \chi_1}.$$

Let M_D be the moduli space of semistable rank 2 vector bundles on D of degree $D \cdot C$. Assume for simplicity that also on $D \times M_D$ there is a universal sheaf $\mathcal{G}$. Let G be any element in M_D. Then we define

$$L_0 := det(p_{2!}\mathcal{G})^{\otimes -2} \otimes det(\mathcal{G}|_{\{a\}\times M_D})^{\otimes \chi(G)}.$$

Remark 3.2 $L_1(nH)$ is independent of the choice of $\mathcal{E}$ (and also of D and a). Any other choice of a universal sheaf $\mathcal{F}$ can be written as $\mathcal{F} = \mathcal{E} \otimes p_2^*\lambda$ for λ a line bundle on M. Then the projection formula implies that $Rp_{2*}(\mathcal{E} \otimes p_2^*\lambda) = Rp_{2*}(\mathcal{E}) \otimes \lambda$, and therefore

$$det(p_{2!}(\mathcal{E} \otimes p_2^*\lambda)) = \lambda^{\otimes \chi(E)} \otimes det(p_{2!}(\mathcal{E})).$$

So $L_1(nH)$ stays unchanged if we replace $\mathcal{E}$ by $\mathcal{E} \otimes \lambda$. In fact we do not need the existence of $\mathcal{E}$ in order to define $L_1(nH)$. The definition is part of a more general formalism of determinant sheaves as was explained in the lectures of Huybrechts and Lehn (see [LP], [H-L] where these line bundles are defined via descent from the corresponding Quot scheme).

In the same way we see that L_0 is independent of the choice of $\mathcal{F}$ and indeed we do not need the existence of $\mathcal{G}$ to define L_0.

3.2 Construction of sections of $L_1(nH)$

We have the following theorem

Theorem 3.3 *[D-N] L_0 is ample on M_D.*

Let $U(D) \subset M$ be the open subset of all sheaves E such that $E|_D$ is semistable. Thus for $E \in U(D)$, we get that $E|_D \in M_D$. We obtain therefore a rational map

$$j : M \to M_D,$$

which is defined on $U(D)$. By definition we see that

$$j^*(L_0) = L_1(nH) \quad \text{on } U(D).$$

Fix an integer $m \gg 0$. As L_0 is ample, $L_0^{\otimes m}$ will have many sections. So we want to extend the pullbacks $j^*(s)$ of sections $s \in H^0(M_D, L_0^{\otimes m})$ to sections $\widetilde{s} \in H^0(M, L_1(nH))^{\otimes m}$. By Bogomolovs theorem ([H-L] p. 174) we have the following: For $n \gg 0$ and all $E \in M$ the restriction $E|_D$ is semistable, unless E is not locally free over D. For $c_2 \gg 0$ the general element in M is locally free. If $E \in M$ is not locally free, then its singularities occur in codimension 2. Therefore the condition that $E|_D$ is not locally free has codimension 1 in the locus of not locally free sheaves. So, putting things together, we see that the complement $M \setminus U(D)$ has codimension ≥ 2 in M. Furthermore M is normal. Therefore every $j^*(s)$ for $s \in H^0(M_D, L_0^{\otimes m})$ extends to $\widetilde{s} \in H^0(M, L_1(nH))^{\otimes m}$.

More precisely one can show the following ([Li], Prop. 2.5).

Lemma 3.4 *For every $s \in H^0(M_D, L_0^{\otimes m})$ the pullback $j^*(s)$ extends to $\widetilde{s} \in H^0(M, L(nH)^{\otimes m})$. Furthermore the vanishing locus of $\widetilde{s}$ is*

$$Z(\widetilde{s}) = \big\{E \in M \,\big|\, E|_D \text{ is semistable and } s(E|_D) = 0 \\ \text{or } E|_D \text{ is not semistable}\big\}.$$

Now choose $m, n \gg 0$.

Proposition 3.5 *$H^0(M, L_1(nH)^{\otimes m})$ is base-point free.*

Proof. Let $E \in M$. By the theorem of Mehta and Ramanathan (see [H-L] Theorem. 7.2.1), we can find a smooth curve $D \in |nH|$ such that $E|_D$ is semistable. Choose $s \in H^0(M_D, L_0^{\otimes m})$, such that $s(E|_D) \neq 0$. Then $\widetilde{s}(E) \neq 0$. □

3.3 Uhlenbeck compactification

$L_1(nH)^{\otimes m}$ defines a morphism

$$\Psi : M \to \mathbb{P}(H^0(M, L_1(nH)^{\otimes m})^\vee).$$

Theorem 3.6 *$\Psi(M)$ is homeomorphic to the Uhlenbeck compactification N.*

We want to give a brief sketch of the proof of this theorem.

For $E \in M$, we introduce the pair $(A(E), Z(E))$, where

1. If E is μ-stable, then

$$A(E) = E^{\vee\vee}, \qquad Z(E) = \sum_{p \in X} l(E^{\vee\vee}/E)_p \cdot p.$$

   $l(E^{\vee\vee}/E)_p$ is the length of the sheaf $E^{\vee\vee}/E$ at p. $Z(E)$ is an effective 0-cycle of length $k := c_2(E) - c_2(E^{\vee\vee})$ on X, i.e. a point in the symmetric power $X^{(k)}$.

2. If E is not μ-stable, we have the Harder-Narasimhan filtration

$$0 \to F \to E \to G \to 0,$$

   where F and G are rank 1 sheaves with $deg_H(F) = deg_H(G)$. We put

$$A(E) = F^{\vee\vee} \oplus G^{\vee\vee}, \qquad Z(E) = \sum_{p \in X} l\big((F^{\vee\vee} \oplus G^{\vee\vee})/(F \oplus G)\big)_p \cdot p \in X^{(k)}.$$

Claim: For $E_1, E_2 \in M$ we have $\Psi(E_1) = \Psi(E_2)$ if and only if $(A(E_1), Z(E_1)) = (A(E_2), Z(E_2))$. In other words: the sets $\Psi(M)$ and N are equal.

We want to check the claim in a special case. Assume $\Psi(E_1) = \Psi(E_2)$, where E_1 and E_2 are μ-stable. Take $D \in |nH|$ general, then $E_1|_D = E_2|_D$ (otherwise, as $L_0^{\otimes m}$ is very ample on M_D, we can find a section $s \in H^0(M_D, L_0^{\otimes m})$, such that $0 = s(E_1|_D) \neq s(E_2|_D)$. Then $\widetilde{s}(E_1) = 0, \widetilde{s}(E_2) \neq 0$). The exact sequence

$$Hom(E_1^{\vee\vee}, E_2^{\vee\vee}) \to Hom(E_1|_D, E_2|_D) \to H^1(Hom(E_1^{\vee\vee}, E_2^{\vee\vee}(-nH))) = 0$$

implies that $Hom(E_1^{\vee\vee}, E_2^{\vee\vee}) \neq 0$. The μ-stability of E_1, E_2 and therefore also of $E_1^{\vee\vee}, E_2^{\vee\vee}$ then implies $E_1^{\vee\vee} = E_2^{\vee\vee}$. Now assume $p \in Z(E_1)$ but $p \notin Z(E_2)$. Then we choose $D \in |nH|$ such that $p \in D$ and $E_2|_D$ is semistable. Then $E_1|_D$ is not semistable and therefore we can find $s \in H^0(M_D, L_0^{\otimes m})$, such that $\widetilde{s}(E_1) = 0$ and $\widetilde{s}(E_2) \neq 0$.

3.4 Donaldson invariants via algebraic geometry

Let again $M := M_H^X(C, c_2)$ be the moduli space of Gieseker H-semistable sheaves with Chern classes C and c_2. Assume that there is a universal sheaf $\mathcal{E}$ over $X \times M$. Write

$$d := 4c_2 - C^2 - 3(1 + p_g(X)).$$

Let $\nu : H_*(X) \to H^{4-*}(M)$ be defined by

$$\nu(a) := c_2(\mathcal{E}) - \frac{1}{4}c_1(\mathcal{E})^2/a,$$

(i.e. we write

$$c_2(\mathcal{E}) - \frac{1}{4}c_1(\mathcal{E})^2 := \sum_i \beta_i \otimes \gamma_i, \quad \beta_i \in H^*(X, \mathbb{Q}),\ \gamma_i \in H^*(M, \mathbb{Q}),$$

then

$$\nu(\alpha) = \sum_i \langle \beta_i, a \rangle \gamma_i\ ).$$

Again ν is independent of the choice of a universal sheaf, and, if no universal sheaf exists, ν can be defined without using it. We denote again by $A_d(X)$ the set of elements of degree d in $Sym^*(H_0(X) \oplus H_2(X))$, where the class p of a point in $H_0(X)$ is given weight 2 and a class in $H_2(X)$ is given weight 1. For $\alpha := a_1 \cdot \ldots \cdot a_k \in A_d(X)$, we define

$$\nu(\alpha) := \nu(a_1) \cup \ldots \cup \nu(a_k) \in H^{2d}(M)$$

and

$$\gamma_{C,d}^{X,H}(\alpha) := \int_M \nu(\alpha).$$

Theorem 3.7 *[M],[Li] Under the conditions specified below we have*

$$\Phi_{C,d}^{X,g(H)} = (-1)^{(C^2+K_XC)/2}\gamma_{C,d}^{X,H}.$$

Conditions:

1. Locally-free μ-stable sheaves are dense in M (otherwise replace M by the closure of the locus of locally free sheaves).

2. Every L in $Pic(S) \setminus \{0\}$ with $L \equiv C$ mod 2 and $LH = 0$ satisfies $L^2 < -(4c_2 - C^2)$ (this means that H does not lie on a wall, see below).

3. $M_H^X(C, c_2)$ has dimension $4c_2 - c_1^2 - 3\chi(\mathcal{O}_X)$ and

$$dim(M_H^X(C,n)) + 2(c_2 - n) < dim(M_H^X(C,c_2)), \quad \text{for all } n < c_2.$$

4. If $C \in 2H^2(X,\mathbb{Z})$ there is an extra condition; e.g. for $\alpha \in Sym^d(H_2(X))$, the condition is $d > 2c_2 - C^2/2$.

The point is that the classes $\mu(\alpha)$ and $\nu(\alpha)$ are related by Ψ^*: $\Psi^*(\mu(\alpha)) = \nu(\alpha)$. Furthermore the fundamental classes of M and N are related by Ψ_*: up to different sign convention $\Psi_*([M]) = [N]$. Then the theorem follows from the projection formula.

4 Flips of moduli spaces and wall-crossing for Donaldson invariants

In this and the next lecture we want to determine the dependence of the Donaldson invariants on the metric in the case $b_+ = 1$ when they indeed depend on the metric. In this lecture we will restrict to the case of algebraic surfaces. In this case the change of metric corresponds to a change of ample divisor H. So we study how the moduli spaces $M_H^X(C, c_2)$ vary with H. We will find out that under suitable assumptions, the variation is described by an explicit series of blow ups and blow downs, with centers projective bundles over Hilbert schemes of points. We use this to determine the change

of the Donaldson invariants as an explicit intersection number on a suitable Hilbert scheme of points on X. Finally one can compute the leading terms of this intersection number. We will follow mostly [E-G1]. A similar approach can be found in [Fr-Q].

4.1 Walls and chambers

We start by reviewing general results about the dependence on the metric. Let X be a compact simply-connected differentiable 4-manifold.

In the case $b_+(X) > 1$ the Donaldson invariants $\Phi^{X,g}_{C,d}$ are independent of the metric g (as long as it is generic). Now assume $b_+(X) = 1$. In this case the Donaldson invariants will indeed depend on the metric g. Let $H^2(X,\mathbb{R})^+$ be the set of all $\alpha \in H^2(X,\mathbb{R})$ with $\alpha^2 > 0$. In fact the Donaldson invariants depend on g via a system of walls and chambers in $H^2(X,\mathbb{R})^+$.

We fix $C \in H^2(X,\mathbb{Z})$ and $d \in \mathbb{Z}_{\geq 0}$. The positive cone $H^2(X,\mathbb{R})^+/\mathbb{R}^+$ has two connected components Ω^+ and Ω^-. A homology orientation (i.e. the choice of an orientation on a maximal-dimensional linear subspace of $H^2(X,\mathbb{R})$ on which the intersection form is positive definite), which is needed to define an orientation on the moduli space of ASD-connections, is equivalent to the choice of one of them, say Ω^+.

Definition 4.1 Let g be a Riemannian metric on X. The *period point* $\omega(g)$ is the point in Ω^+ defined by the one-dimensional subspace of g-self-dual g-harmonic 2-forms. I.e these are the harmonic two forms $\eta \in \Omega^2(X)$ with $*_g\eta = \eta$. By the Hodge theorem this is a 1-dimensional subspace of $H^2(X,\mathbb{R})$. An element $\xi \in H^2(X,\mathbb{Z}) + C/2$ is called of *type* (C,d) if

$$(d+3)/4 + \xi^2 \in \mathbb{Z}_{\geq 0}.$$

In this case

$$W^\xi := \{L \in \Omega^+ \mid \xi \cdot L = 0\}$$

is called the corresponding wall of type (C,d). The *chambers of type* (C,d) are the connected components of complement of the walls of type (C,d) in Ω^+.

It turns out that the Donaldson invariants with respect to the Fubini-Study metric corresponding to H depend only on the chamber of the period point of H.

Theorem 4.2 *[K-M]*

1. $\Phi^{X,g}_{C,d}$ *depends only on the chamber (of type* (C,d)*) of* $\omega(g)$.
2. *For all* ξ *of type* (C,d) *there exists a linear map* $\delta^X_{\xi,d} : A_d(X) \to \mathbb{C}$. *such that*

$$\Phi^{X,g_1}_{C,d} - \Phi^{X,g_2}_{C,d} = \sum_{\xi\omega(g_2)<0<\xi\omega(g_2)} (-1)^{(\xi-C/4)C}\delta^X_{\xi,d}.$$

4.2 Interpretation of the walls in algebraic geometry

Now let X be a simply-connected algebraic surface with geometric genus $p_g = 0$ (this is equivalent to $b_+(X) = (1+2p_g(X)) = 1$). Let H be an ample divisor on X. Let $\mathcal{C}$ be the ample cone of X. We choose Ω^+ as the connected component of $H^2(X,\mathbb{R})^+/\mathbb{R}^+$, which contains $\mathcal{C}/\mathbb{R}^+$. Then the period point of the Fubini-Study metric $g(H)$ is $\omega(g(H)) = \mathbb{R}^+H \in \mathcal{C}/\mathbb{R}^+$. Fix $C \in H^2(X,\mathbb{Z})$ and $c_2 \in \mathbb{Z}$, such that $d := 4c_2 - C^2 - 3$ is a nonnegative integer. By Section 3 we can compute $\Phi^{X,g(H)}_{C,d}$ on $M^X_H(C,c_2)$. So we now need to know how $M^X_H(C,c_2)$ depends on H.

Let E be a torsion-free rank 2 sheaf on X with Chern classes C and c_2. Let H_+ and H_- be two ample line bundles on X, and assume that E is Gieseker stable with respect to H_-, but Gieseker unstable with respect to H_+. Then the Harder-Narasimhan filtration of E with respect to H_+ gives an exact sequence

$$0 \to \mathcal{I}_{Z_1}(F) \to E \to \mathcal{I}_{Z_2}(G) \to 0,$$

where

1. The class $\xi := (F-G)/2$ satisfies

$$\xi H_- < 0 < \xi H_+.$$

2. $\mathcal{I}_{Z_1}$ and $\mathcal{I}_{Z_2}$ are the ideal sheaves of 0-dimensional subschemes $Z_1 \in X^{[n]}$ and $Z_2 \in X^{[m]}$ and $c_2(E) = FG + n + m$ or equivalently

$$c_2 - C^2/4 + \xi^2 = n + m \geq 0.$$

This means that ξ is a class of type (C,d) and there exists an ample line bundle H between H_+ and H_- with $\xi H = 0$. In other words ξ defines a wall of type (C,d), such that W^ξ intersects $\mathcal{C}$.

Definition 4.3 Let $E_\xi^{n,m}$ be the set of all sheaves E lying in extensions

$$0 \to \mathcal{I}_{Z_1}(F) \to E \to \mathcal{I}_{Z_2}(G) \to 0,$$

with $\xi := (F-G)/2$, $Z_1 \in X^{[n]}$, $Z_2 \in X^{[m]}$.

Then we conclude

1. $M_H^X(C, c_2)$ depends only on the chamber of type (C, d) of H. In particular the Donaldson invariants are constant on each chamber.

2. $$M_{H_-}^X(C, c_2) \setminus M_{H_+}^X(C, c_2) \subset \coprod_\xi \coprod_{n,m} E_\xi^{n,m}.$$

   Here the sums run over all ξ of type (C, d) such that $H_- < 0 < \xi H_+$, and over all $n, m \in \mathbb{Z}_{\geq 0}$ with $n + m = c_2 - C^2/4 + \xi^2$.

We would like to say that $M_{H_+}^X(C, c_2)$ is obtained from $M_{H_-}^X(C, c_2)$, by removing the $E_\xi^{n,m}$ and replacing them by the $E_{-\xi}^{m,n}$ (for ξ classes of type (C, d) with $\xi H_- < 0 < \xi H_+$). This however is not quite true. The problem is that $E_\xi^{n,m}$ and $E_{-\xi}^{l,r}$ can intersect, i.e. we can have a diagram

$$\begin{tikzcd}
 & & 0 \arrow[d] & & \\
 & & \mathcal{I}_{W_1}(G) \arrow[d] \arrow[dr] & & \\
0 \arrow[r] & \mathcal{I}_{Z_1}(F) \arrow[r] \arrow[dr] & E \arrow[r] \arrow[d] & \mathcal{I}_{Z_2}(G) \arrow[r] & 0 \\
 & & \mathcal{I}_{W_2}(F) \arrow[d] & & \\
 & & 0 & &
\end{tikzcd}$$

To deal with this, we need a finer notion of stability. We use: Gieseker stability is not invariant under tensorizing by a line bundle.

Assume H_- and H_+ are separated by a unique wall W^ξ with $\xi H_- < 0 < \xi H_+$. Let H lie between H_- and H_+ with $H\xi = 0$. If E is a torsion-free H-semistable sheaf of rank 2 with Chern classes C and c_2, then

1. E is H_--semistable, if and only if $E(l(H_- - H_+))$ is H-semistable for $l \gg 0$.

2. E is H_+-semistable, if and only if $E(l(H_+ - H_-))$ is H-semistable for $l \gg 0$.

This gives us a finer notion of stability. By using a parabolic structure of length 1 (which essentially amounts to tensorizing with a fractional power of $H_- - H_+$), we get moduli spaces

$$M_a, \quad a \in [-1,1], \quad M_{-1} = M^X_{H_-}(C, c_2), \quad M_1 = M^X_{H_+}(C, c_2).$$

There are *miniwalls* $a_i \in [-1,1]$ such that for all i and $0 < \epsilon \ll 1$

$$M_{a_i+\epsilon} = (M_{a_i-\epsilon} \setminus E^{n,m}_\xi) \amalg E^{m,n}_{-\xi}$$

for suitable n, m (for more details look at [E-G1]).

4.3 Flip construction

Now we want to see more precisely what happens to $M^X_H(C, c_2)$ when H crosses a wall. We want to see that its change can be described by a sequence of blow ups and blow downs. For this we need an additional assumption which essentially guarantees that all the involved moduli spaces will be smooth near the locus where they change.

Definition 4.4 A class ξ of type (C, d) defines a *good* wall if W^ξ contains ample divisors and $2\xi + K_X$ and $-2\xi + K_X$ are not effective. In particular if $-K_X$ is effective, then all walls in the ample cone are good.

We want to describe the wall-crossing through a good wall defined by ξ. Let b be a miniwall, as above, and let

$$M_- := M_{b-\epsilon}, \quad M_+ := M_{b+\epsilon} = (M_- \setminus E^{n,m}_\xi) \amalg E^{m,n}_{-\xi}.$$

We write

$$E_- := E^{n,m}_\xi, \quad E_+ := E^{m,n}_{-\xi}.$$

Let $T := X^{[n]} \times X^{[m]}$, let $\mathbb{Z}_1, \mathbb{Z}_2 \subset S \times T$ be the two universal families i.e.

$$\mathbb{Z}_1 := \{(x, Z, W) \in X \times T \mid x \in Z\}$$
$$\mathbb{Z}_2 := \{(x, Z, W) \in X \times T \mid x \in W\}.$$

Let q be the projection $S \times T \to T$. We write $C := F + G$, $\xi := F - G/2$ and write $\mathcal{F} := \mathcal{I}_{\mathbb{Z}_1}(F)$, $\mathcal{G} := \mathcal{I}_{\mathbb{Z}_2}(G)$, (these are sheaves on $X \times T$ and we suppress the various pullbacks in the notation). Finally we write

$$\mathcal{A}_- := \mathrm{Ext}^1_q(\mathcal{F}, \mathcal{G}), \quad \mathcal{A}_+ := \mathrm{Ext}^1_q(\mathcal{G}, \mathcal{F}).$$

These are the relative Ext sheaves. Under our assumptions these sheaves are locally free and the fibers over a point $(Z, W) \in T$ are

$$\mathcal{A}_-(Z, W) := \mathrm{Ext}^1(\mathcal{I}_Z(F), \mathcal{I}_W(G)), \quad \mathcal{A}_+(Z, W) := \mathrm{Ext}^1(\mathcal{I}_W(G), \mathcal{I}_Z(F)).$$

We denote the tautological sub-line bundles on $\mathbb{P}(\mathcal{A}_-)$ and $\mathbb{P}(\mathcal{A}_+)$ by τ_- and τ_+.

Lemma 4.5 *1. $\mathcal{A}_-$ is locally free of rank $-\xi(2\xi - K_S) + n + m - 1$.*

2. The tautological extension

$$0 \to \mathcal{F} \to \mathcal{E} \to \mathcal{G}(\tau_-) \to 0$$

gives an isomorphism $\mathbb{P}(\mathcal{A}_-) \simeq E_-$.

3. $N_{E_-/M_-} = \mathcal{A}_+(\tau_-)$.

Proof. 1. follows from Riemann-Roch, because the condition of a good wall implies $\mathrm{Hom}_q(\mathcal{F}, \mathcal{G}) = 0$ and $\mathrm{Ext}^2_q(\mathcal{F}, \mathcal{G}) = 0$.

2. is then easy.

For 3. we use that $T_{M_-} = \mathrm{Ext}^1_q(\mathcal{E}, \mathcal{E})$, where $\mathcal{E}$ is the universal sheaf on $X \times M_-$. Then one has to do some diagram-chasing. □

Part 3. of this lemma lets us hope that the blow up of M_- along E_- and the blow up of M_+ along E_+ might be the same. So let $\widetilde{M}_-$ be the blow up of M_- along E_- and $\widetilde{M}_+$ the blow up of M_+ along E_+. Let $D \simeq \mathbb{P}(\mathcal{A}_-) \times_T \mathbb{P}(\mathcal{A}_+)$ be the exceptional divisor. We see that $\mathcal{O}(D)|_D = \mathcal{O}_D(\tau_- + \tau_+)$.

Theorem 4.6 $\widetilde{M}_- = \widetilde{M}_+$.

Proof. Let $\mathcal{E}_-$ be the universal family on $X \times M_-$ (and denote by the same symbol its pullback to $\widetilde{M}_-$). Let $\mathcal{E}_+$ be the kernel of the composition $\mathcal{E}_- \to \mathcal{E}_-|_D \to \mathcal{G}(\tau_-)_D$, where $\mathcal{G}(\tau_-)_D$ is the pullback of $\mathcal{G}(\tau)$ from $S \times T$ to $S \times D$. So we define $\mathcal{E}_+$ via elementary transform along the exceptional divisor D. Then we check that $\mathcal{E}_+$ is a $b + \epsilon$ stable family for $1 \gg \epsilon > 0$.

Thus $\mathcal{E}_+$ defines a morphism $\widetilde{M}_- \to M_+$. It is not difficult to check that it is the blow up along E_+. $\square$

So we see that $M^X_{H_+}(C, c_2) = M_1$ is obtained from $M^X_{H_-}(C, c_2) = M_{-1}$ via a sequence of blow ups along smooth subvarieties of the form $E^{n,m}_\xi$ followed by a blow up of the exceptional divisor in another direction to $E^{m,n}_{-\xi}$.

4.4 Computation of the wall-crossing

Now we want to compute the wall-crossing terms $\delta^X_{\xi,d}$. For simplicity we restrict to $Sym^d(H_2(X))$. Let $a \in H_2(X)$. Let b run through the miniwalls corresponding to ξ and write $\widetilde{M}_b$ for the blow up $\widetilde{M}_{b-\epsilon} = \widetilde{M}_{b+\epsilon}$. From the definitions we see that

$$\delta^X_{\xi,d}(a^d) = \pm\left(\int_{M^X_{H_+}(C,c_2)} \nu(a)^d - \int_{M^X_{H_-}(C,c_2)} \nu(a)^d\right) = \sum_b \int_{\widetilde{M}_b} (\nu_+(a)^d - \nu_-(a)^d).$$

Here $\nu_-(a) := (c_2(\mathcal{E}_-) - c_1(\mathcal{E}_-)^2/4)/a$, and similarly for ν_+. (The $\mathcal{E}_-$ and $\mathcal{E}_+$ and therefore the ν_- and ν_+ also depend on b.)

Let us again put ourselves in the situation of the previous section: $\widetilde{M}_-$ is the blow up of M_- along E_- and $\widetilde{M}_+$ the blow up of M_+ along E_+, and D is the exceptional divisor.

Lemma 4.7 *1. $\nu_+(a) - \nu_-(a) = -\langle \xi, a\rangle D$.*

2. $\int_{\widetilde{M}_-} (\nu_+(a)^d - \nu_-(a)^d)$ is the evaluation of a suitable (explicitly computable) cohomology class on $X^{[n]} \times X^{[m]}$.

Proof. 1. Is an easy application of Riemann-Roch without denominators see [Fu] (which tells how to compute the Chern classes of sheaves supported on subvarieties).

2.

$$\nu_+(a)^d - \nu_-(a)^d = (\nu_+(a) - \nu_-(a))(\nu_+(a)^{d-1} + \ldots + \nu_-(a)^{d-1})$$

is by 1. divisible by D, so we can view it as a class on D. We can then push the class from D down to T. $\square$

Putting all this together and summing over all the miniwalls corresponding to a given wall ξ we obtain the following: Note that the various $X^{[n]} \times X^{[m]}$ with $m + n = l$ can be collected to $(X \sqcup X)^{[l]} = \coprod_{n+m=l} X^{[n]} \times X^{[m]}$).

Theorem 4.8

$$\delta^X_{\xi,d}(a^d) = \pm \sum_{b=0}^{d} 2^b \binom{d}{b} \langle \xi, a\rangle^{d-b} \cdot$$
$$\int_{(X\sqcup X)^{[l]}} \alpha^b s_{2l-b}(\mathrm{Ext}^1_q(\mathcal{I}_{\mathbb{Z}_1}, \mathcal{I}_{\mathbb{Z}_2} \otimes (\mathcal{O}(-2\xi) \oplus \mathcal{O}(-2\xi + K_X)))).$$

Here p and q are the projections of $X \times (X \sqcup X)^{[l]}$ to X and $(X \sqcup X)^{[n]}$ respectively and $l := c_2 - C^2/4 + \xi^2$. $\mathbb{Z}_1$ and $\mathbb{Z}_2 \subset X \times (X \sqcup X)^{[n]}$ are the universal families

$$\mathbb{Z}_1 := \{(x, Z, W) \in X \times (X \sqcup X)^{[l]} \mid x \in Z\},$$
$$\mathbb{Z}_2 := \{(x, Z, W) \in X \times (X \sqcup X)^{[l]} \mid x \in W\}.$$

$s_i(E)$ denotes the i-th Segre class, defined by

$$1 + s_1(E) + s_2(E) + \ldots = 1/(1 + c_1(E) + c_2(E) + \ldots)$$

and $\alpha := q_*(p^*\alpha \cdot ([\mathbb{Z}_1] + [\mathbb{Z}_2]))$. So we are reduced to a (very complicated) intersection computation on the Hilbert scheme of points on X. The intersection theory of $X^{[n]}$ is in general not understood. It gets harder very fast as n grows. So in our case the difficulty of the computation depends on the number $l := c_2 - C^2/4 + \xi^2$. The intersection number above can be computed for l not too large, say $l \leq 3$. For $l = 0$ we get for instance

$$\delta^X_{\xi,d}(a^d) = \pm\langle \xi, a\rangle^d.$$

There is an alternative way of carrying out the final step of the computation, i.e. the computation in the cohomology ring of the Hilbert scheme of points. Assume X is a blow up of $\mathbb{P}_2$. On $\mathbb{P}_2$ we have actions of $\mathbb{C}^*$ with finitely many fixpoints. We can do the blow up in such a way that X still carries an action of $\mathbb{C}^*$ with finitely many fixpoints (at each step it is enough to only blow up fixpoints). This action lifts to an action on $X^{[n]}$, which still has only finitely many fixpoints. All the intersection numbers we have to compute for the wall-crossing are indeed intersection numbers of Chern classes of equivariant bundles for this action.

We can therefore apply the Bott residue formula. This allows us to compute the intersection numbers by looking at the weights of the action on the fibers of the equivariant bundles over the fixpoints. This gives an algorithm

for computing the wall-crossing for rational surfaces. We used this in [E-G2] to compute the Donaldson invariants of $\mathbb{P}_2$ of degree ≤ 50.

The fact that we can compute the Donaldson invariants of $\mathbb{P}_2$, where there are no walls might seem surprising. We use the blow up formulas (see the next lecture) to relate the Donaldson invariants of $\mathbb{P}_2$ to those of the blow up of $\mathbb{P}_2$ in a point. On this blow up we can then apply the wall-crossing in order to do the computation.

5 Wall-crossing and modular forms

Let X be a simply-connected 4-manifold with $b_+(X) = 1$. In this case the Donaldson invariants were first studied in [K]. In this lecture I want to give a generating function for the wall-crossing terms $\delta^X_{\xi,d}$. We will see that such a generating function can be found in terms of modular forms. This is the contents of the paper [G]. The strategy will be to compare the wall-crossing an X and on the connected sum of X with $\mathbb{P}_2$ with the opposite orientation. This will give us recursion formulas for the $\delta^X_{\xi,d}$.

5.1 Ingredients

There are several ingredients which have to be put together in order to compute the generating function.

(1) Kotschik-Morgan conjecture. In their paper [K-M], where they show that the Donaldson invariants $\Phi^{X,g}_{C,d}$ depend only on the chamber of the period point of the metric g, Kotschik and Morgan also make a conjecture about the structure of the wall-crossing terms $\delta^X_{\xi,d}$. For a class $\xi \in H^2(X)$ and $a \in H_2(X)$, we denote by $\langle \xi, a\rangle$ the pairing of $H^2(X)$ with $H_2(X)$ and by $(a \cdot a)$ the intersection form on the middle homology $H_2(X)$.

Conjecture 5.1 *[K-M] $\delta^X_{\xi,d}(a^d)$ is for $a \in H_2(X)$ a polynomial in $\langle \xi, a\rangle$ and $(a \cdot a)$, whose coefficients depend only on ξ^2, d and the homotopy type of X.*

In a series of papers Fehan and Leness are working on a proof of this conjecture [Fe-Le1],[Fe-Le2-4].

(2) Blow up formulas. The blow up formulas relate the Donaldson invariants of a 4-manifold X with those of the connected sum $\widehat{X} := X\#\overline{\mathbb{P}}_2$ of X with $\mathbb{P}_2$ with the opposite orientation. In the case that X is an algebraic

surface, we can take $\widehat{X}$ to be the blow up of X in a point. In the case $b_+(X) = 1$, when the Donaldson invariants depend on the choice of a metric, we need to choose the metric on $\widehat{X}$ to be very close to the pullback of a metric on X, in order to make the blow up formulas applicable. Let E be the class of the exceptional divisor, then $H^2(\widehat{X}, \mathbb{R}) = H^2(X, \mathbb{R}) \oplus \mathbb{R}E$. We will identify $H^2(X, \mathbb{R})$ with the classes in $H^2(\widehat{X}, \mathbb{R})$ orthogonal to E.

If $L \in H^2(X, \mathbb{R})^+$ is (a representative of) the period point of the metric g, we write

$$\Phi_{C,d}^{X,L} := \Phi_{C,d}^{X,g}.$$

For $C \in H^2(\widehat{X}, \mathbb{Z})$, $H \in H^2(X, \mathbb{R})^+$, we write

$$\Phi_{C,d}^{\widehat{X},H} := \Phi_{C,d}^{\widehat{X},H-\epsilon E}$$

for $0 < \epsilon \ll 1$. (This will be independent of ϵ for sufficiently small $\epsilon > 0$.)

Theorem 5.2 *Let $C \in H^2(X, \mathbb{Z})$, $a \in H_2(X)$, $H \in H^2(X, \mathbb{R})^+$. We write $e \in H_2(\widehat{X}, \mathbb{Z})$ for the Poincaré dual of E. Then*

1. $\Phi_{C,d}^{\widehat{X},H}(a^d) = \Phi_{C,d}^{X,H}(a^d)$.
2. $\Phi_{C+E,d+1}^{\widehat{X},H}(e\, a^d) = \Phi_{C,d}^{X,H}(a^d)$.
3. $\Phi_{C,d}^{\widehat{X},H}(e^2 a^{d-2}) = 0$.

This result holds also if $b_+ > 1$. In fact this is the case in which it was originally proved. In the case $b_+ > 1$ the Donaldson invariants are independent of the metric, so one does not have to worry about which metric to choose on $\widehat{X}$ for a given metric on X.

More generally Fintushel and Stern [F-S] found generating functions for the blow up formulas: Let $p \in H_0(X)$ be the class of a point. Then there are power series

$$B(x,t) = \sum_k B_k(x) t^k,$$
$$S(x,t) = \sum_k S_k(x) t^k,$$

such that

$$\Phi_C^{\widehat{X},H}(a^d e^k) = \Phi_C^{X,H}(a^d B_k(p)),$$
$$\Phi_{C+E}^{\widehat{X},H}(a^d e^k) = \Phi_C^{X,H}(a^d S_k(p)).$$

$B(x,t)$ and $S(x,t)$ can be expressed in terms of elliptic functions, e.g. $S(x,t) = e^{-t^2x/6}\sigma(t)$, where σ is the Weierstrass σ function.

(3) Vanishing results. The last ingredient is that in certain cases (for rational ruled surfaces) the Donaldson invariants vanish. This will give a starting point for the calculations.

Lemma 5.3 *Let X be a rational ruled surface. Let F be the class of a fiber and assume $CF = 1$. Let H be an ample divisor on X. Then $M^X_{F+\epsilon H}(C, c_2) = \emptyset$ for $0 < \epsilon \ll 1$. In particular $\Phi^{X,F+\epsilon H}_{C,d} = 0$ for $0 < \epsilon \ll 1$.*

More generally the following holds: Let $f : X \to C$ be a surjective morphism of an algebraic surface to a curve. Let F be the class of a fiber and let H be ample on X. Then a vector bundle E over X is semistable with respect to $F+\epsilon H$ if and only if the restriction of E to the generic fiber of f is semistable. This fact is also e.g. used by Friedman to study the Donaldson invariants of elliptic surfaces.

5.2 The result

Our aim is to show:

Theorem 5.4 *Let $a \in H_2(X)$ and let t be a variable. Then*

$$\delta^X_\xi(\exp(at)) = \mathrm{Coeff}_{q^0}\left[f(\tau)R(\tau)\theta(\tau)^{\sigma(X)}q^{-\xi^2/2}\exp\Big(-\frac{\langle\xi,a\rangle t}{f(\tau)} - \frac{(a\cdot a)G(\tau)t^2}{f(\tau)^2}\Big)\right].$$

Here $\sigma(X)$ is the signature of X. For the rest of the notations I briefly review modular forms.

Review of modular forms: Let $\mathbb{H} := \{\tau \in \mathbb{C} \mid \Im(\tau) > 0\}$ be the complex upper half plane. For τ in $\mathbb{H}$ we denote $q := e^{2\pi i\tau}$. The group $SL_2(\mathbb{Z})$ acts on $\mathbb{H}$ by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\tau = \frac{a\tau+b}{c\tau+d}.$$

A function $g : \mathbb{H} \to \mathbb{C}$ is called a modular form of weight k for $SL_2(\mathbb{Z})$, if

$$g\Big(\frac{a\tau+b}{c\tau+d}\Big) = (c\tau+d)^k f(\tau), \quad \text{for all } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}),$$

and furthermore g has a q-development

$$f(\tau) = \sum_{n=0}^{\infty} a_n q^n.$$

One can associate an elliptic curve $E_\tau := \mathbb{C}/(\mathbb{Z} + \mathbb{Z}\tau)$ to $\tau \in \mathbb{H}$, and E_τ and $E_{\tau'}$ are isomorphic if and only if τ and τ' are related by an element of $SL_2(\mathbb{Z})$. Therefore modular forms are related to moduli of elliptic curves.

One can also talk about modular forms for subgroups Γ of finite index of $SL_2(\mathbb{Z})$. In this case one requires the transformation behavior only for the elements in Γ and the requirement on the q-development has to be modified.

All the functions appearing in the theorem are (related to) modular forms.

$$\Delta(\tau) := q \prod_{k=1}^{\infty} (1 - q^k)^{24}$$

is the discriminant, a modular form for $SL_2(\mathbb{Z})$.

$$\eta(\tau) = \Delta(\tau)^{1/24}$$

is the Dirichlet eta-function.

$$\theta(\tau) := \sum_{n \in \mathbb{Z}} q^{n^2/2}$$

is the theta function for $\mathbb{Z}$.

$$G_2(\tau) := -\frac{1}{24} + \sum_{n=1}^{\infty} \Big(\sum_{d|n} d \Big) q^n$$

is an Eisenstein series and

$$e_3(\tau) := \frac{1}{12} + \sum_{n=1}^{\infty} \Big(\sum_{d|n, d \text{ odd}} d \Big) q^n,$$

the value of the Weierstrass $\wp$-function at one of the two-division points. We put

$$R(\tau) := \frac{\Delta(\tau)^2}{\Delta(\tau/2)\Delta(2\tau)},$$

$$f(\tau) := e^{-\pi i/4} \frac{\eta(\tau)^3}{\theta(\tau)},$$

$$G(\tau) := G_2(\tau) + e_3(\tau)/2.$$

As a corollary to this result we can compute all the Donaldson invariants of the projective plane $\mathbb{P}_2$. The projective plane is in some respects the simplest algebraic surface. Therefore, if one wants to understand the Donaldson invariants of algebraic surfaces, one should at least be able to compute them for $\mathbb{P}_2$.

Let $H \in H^2(\mathbb{P}_2, \mathbb{Z})$ be the hyperplane class, and let h be its Poincaré dual.

Corollary 5.5

$$\Phi_H^{\mathbb{P}_2,H}(\exp(ht)) = \mathrm{Coeff}_{q^0}\Bigg[f(\tau)R(\tau) \sum_{a \geq n > 0} (-1)^{n+\frac{1}{4}} q^{\frac{1}{2}((a\cdot a)-(n-\frac{1}{2})^2)} \exp\Big(-\frac{(n+\frac{1}{2})t}{f(\tau)} - \frac{G(\tau)t^2}{f(\tau)^2}\Big)\Bigg].$$

There is a similar formula for $\Phi_0^{\mathbb{P}_2,H}$.

Proof. The blow up X of $\mathbb{P}_2$ at a point is a ruled surface, the class of the fiber is $F = H - E$. So we get $\Phi_H^{X,F+\epsilon H} = 0$ by the vanishing result above. On the other hand the blow up formulas give that $\Phi_H^{\mathbb{P}_2,H}(h^d) = \Phi_H^{X,H}(h^d)$, and the last can be computed by adding all the wall-crossing terms $\delta_{\xi,d}^X$ for all classes ξ of type (H,d) with $\xi H > 0 > \xi F$. □

5.3 Proof of the theorem

Now I want to sketch the proof of the theorem. The idea is as follows: We want to relate the wall-crossing on X and its blow up $\widehat{X}$. So fix $C \in H^2(X,\mathbb{Z})$ and let ξ define the only wall of type (C,d) on X between H_- and H_+. Instead of directly applying the wall-crossing formula for the wall W^ξ, we can also first apply the blow up formulas, then cross all the walls between H_- and H_+ on $\widehat{X}$ and then apply the blow up formula again to get back to X. This gives us two different ways to compute the wall-crossing term δ_ξ^X, which will give us recursion formulas.

By definition we see that the classes η of type (C,d) on $\widehat{X}$ with $H_-\eta < 0 < H_+\eta$ are precisely the

$$\eta = \xi + nE, \quad n \in \mathbb{Z},\ n^2 \leq (d+3)/4 + \xi^2,$$

and the classes of type $(C+E, d+1)$ are precisely the

$$\eta = \xi + (n+1/2)E, \quad n \in \mathbb{Z},\ (n+1/2)^2 \leq (d+4)/4 + \xi^2.$$

We write

$$\delta_\xi^X := \sum_{d \geq 0} \delta_{\xi,d}^X.$$

Then together with the above discussion the blow up formulas give:

$$\delta_\xi^X(a^d) = \sum_{n\in\mathbb{Z}} \delta_{\xi+nE}^{\widehat{X}}(a^d), \tag{5.0.1}$$

$$\delta_\xi^X(a^d) = \sum_{n\in\mathbb{Z}} (-1)^{n-1} \delta_{\xi+(n+1/2)E}^{\widehat{X}}(e\, a^d), \tag{5.0.2}$$

$$0 = \sum_{n\in\mathbb{Z}} \delta_{\xi+nE}^{\widehat{X}}(e^2\, a^{d-2}). \tag{5.0.3}$$

Now we use the Kotschick-Morgan conjecture. Let $X(b)$ be the blow up of X in b points. The Kotschick-Morgan conjecture allows us to write

$$\delta_\xi^{X(b)}(a^d/d!) = \sum_{l+2k=d} \frac{\langle \xi, a\rangle^l}{l!} \frac{(a\cdot a)^k}{k!} P(l,k,b,\xi^2),$$

for universal constants $P(l,k,b,w)$ for $l,k,b \in \mathbb{Z}$, $w \in \mathbb{Z}/4$. Then the relations (5.0.1), (5.0.2), (5.0.3) imply in turn

$$P(l,k,b,w) = \sum_{n\in\mathbb{Z}} P(l,k,b+1,w-n^2), \tag{5.0.4}$$

$$P(l,k,b,w) = \sum_{n\in\mathbb{Z}} (-1)^n (n+1/2) P(l,k,b+1,w-(n+1/2)^2), \tag{5.0.5}$$

$$\sum_{n\in\mathbb{Z}} n^2 P(l,k,b,w-n^2) = 2\sum_{n\in\mathbb{Z}} P(l,k,b,w-n^2). \tag{5.0.6}$$

Now we put

$$\Lambda_X := \sum_{l,k,b,w} P(l,k,b,w) q^{w/2} \frac{L^l Q^k t^b}{l!k!b!},$$

for variables q, L, Q, t. We see that we have encoded all the information about the wall-crossing formulas into the generating function Λ_X. So our task is to determine Λ_X explicitly.

The formulas (5.0.4), (5.0.5), (5.0.6) for the $P(l,k,b,w)$ translate into the following differential equations for Λ_X.

$$\begin{aligned}\theta(\tau)\frac{\partial}{\partial t}\Lambda_X &= \Lambda_X,\\ \eta(\tau)^3\frac{\partial}{\partial L}\frac{\partial}{\partial t}\Lambda_X &= \Lambda_X,\\ 2\theta(\tau)\frac{\partial}{\partial Q}\Lambda_X &= \Big(q\frac{\partial}{\partial q}\theta(\tau)\Big)\frac{\partial^2}{\partial L^2}\Lambda_X.\end{aligned}$$

These differential equations are trivial to solve: Writing

$$\lambda_X(q) := \Lambda_X(0,0,0,q)$$

we get

$$\Lambda_X = \exp\Big(-\frac{L}{f(\tau)} - \frac{Q\,G(\tau)}{f(\tau)^2} + \frac{t}{\theta}\Big)\lambda_X(q).$$

Finally we need to determine $\lambda_X(q)$. It is enough to do this in the case $X = \mathbb{P}_1 \times \mathbb{P}_1$: For every simply-connected 4-manifold with $b_+ = 1$, the blow up Y of X in two points is homotopy-equivalent to the blow up of $\mathbb{P}_1 \times \mathbb{P}_1$ in a number of points. The Kotschick-Morgan conjecture says that the wall-crossing terms only depend on the homotopy type of X.

Let F and G be the fibers of the two projections of $\mathbb{P}_1 \times \mathbb{P}_1$ onto its factors. By the vanishing result we get

$$\Phi^{\mathbb{P}_1\times\mathbb{P}_1, F+\epsilon G}_{F+G,d} = \Phi^{\mathbb{P}_1\times\mathbb{P}_1, G+\epsilon F}_{F+G,d} = 0.$$

Therefore the sum of all the wall-crossing terms for all the walls between F and G has to vanish. This is enough to determine all the coefficients of $\Lambda_{\mathbb{P}_1\times\mathbb{P}_1}(q)$. This part of the calculation is slightly more difficult and involves some tricks with modular forms.

5.4 Further results

This result has later been used in [G-Z] to prove structure theorems (like those of Kronheimer and Mrowka in the $b_+ > 1$ case) also for manifolds with $b_+ = 1$. These results work when one takes the limit of the Donaldson invariants $\Phi^{X,H}_C$ as H tends to a class F with $F^2 = 0$. We write $\Phi^{X,F}_C$ for this limit.

We get for instance the following: Let X be a rational elliptic surface (i.e. the blow up of $\mathbb{P}_2$ in the 9 points of intersection of two smooth cubics). Let F be the class of a fiber. Then for all $a \in H_2(X)$ we get

$$\Phi_H^{X,F}(e^{at}(1+p/2)) = -\frac{e^{(a\cdot a)t^2/2}}{\cosh(\langle F, a\rangle t)}.$$

To prove such results, one has to sum over all walls between two classes F, G with $F^2 = G^2 = 0$. These sums organize themselves into theta functions, and somewhat complicated arguments with modular forms will give the result.

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Holomorphic bundles over elliptic manifolds

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Introduction:

In this series of lectures we shall examine holomorphic bundles over compact elliptically fibered manifolds. We shall examine constructions of such bundles as well as (duality) relations between such bundles and other geometric objects, namely $K3$-surfaces and del Pezzo surfaces.

We shall be dealing throughout with holomorphic principal bundles with structure group $G_{\mathbf{C}}$ where G is a compact, simple (usually simply connected) Lie group and $G_{\mathbf{C}}$ is the associated complex simple algebraic group. Of course, in the special case $G = SU(n)$ and hence $G_{\mathbf{C}} = SL_n(\mathbf{C})$, we are considering holomorphic vector bundles with trivial determinant. In the other cases of classical groups, $G = SO(n)$ or $G = \mathrm{Sympl}(2n)$ we are considering holomorphic vector bundles with trivial determinant equipped with a non-degenerate symmetric, or skew symmetric pairing. In addition to these classical cases there are the finite number of exceptional groups. Amazingly enough, motivated by questions in physics, much interest centers around the group E_8 and its subgroups. For these applications it does not suffice to consider only the classical groups. Thus, while often first doing the case of $SU(n)$ or more generally of the classical groups, we shall extend our discussions to the general semi-simple group. Also, we shall spend a good deal of time considering elliptically fibered manifolds of the simplest type – namely, elliptic curves.

The basic references for the material covered in these lectures are:

1. M. Atiyah, *Vectors bundles over an elliptic curve*, Proc. London Math. Soc. **7** (1967) 414-452.

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1 Lie Groups and Holomorphic Principal $G_{\mathbf{C}}$-bundles

In this lecture we review the classification of compact simple groups or equivalently of complex linear simple groups. Then we turn to a review of the

basics of holomorphic principal bundles over complex manifolds. We finish the section with a discussion of isomorphism classes of G-bundles (G compact) over the circle.

1.1 Generalities on roots, the Weyl group, etc.

A good general reference for Root Systems, Weyl groups, etc. is [3]. Let G be a compact group and T a maximal torus for G. Of course, T is unique up to conjugation in G. The rank of G is by definition the dimension of T. We denote by W the Weyl group of T in G, i.e., the quotient of the subgroup of G conjugating T to itself modulo the normal subgroup of elements commuting with T (which is T itself). This is a finite group. We denote by $\mathfrak{g}$ the Lie algebra of G and by $\mathfrak{t} \subset \mathfrak{g}$ the Lie algebra of T. The group G acts on its Lie algebra $\mathfrak{g}$ by the adjoint representation. The exponential mapping $\exp\colon \mathfrak{t} \to T$ is a covering projection with kernel $\Lambda \subset \mathfrak{t}$, where Λ is the fundamental group of T. The adjoint action of W on $\mathfrak{t}$ covers the conjugation action of W on T.

The complexification $\mathfrak{g}_{\mathbf{C}}$ decomposes into the direct summand of subspaces invariant under the conjugation action of the maximal torus T. The subspace on which the action is trivial is the complexification of the Lie algebra $\mathfrak{t}$ of the maximal torus. All other subspaces are one dimensional and are called the *root spaces*. The non-trivial character by which the torus acts on a root space is called a *root* of G (with respect to T) for this subspace. By definition the roots of G are non-zero elements of the character group (dual group) of T. This character group is a free abelian group of dimension equal to the rank of G. In the case when G is semi-simple, the roots of G span a subgroup of finite index inside entire character group. When G is not semi-simple the center of G is positive dimensional, and the roots span a sublattice of the group of characters of codimension equal to the dimension of the center of G.

Equivalently, the roots can be viewed as elements of the dual space $\mathfrak{t}^*$ of the Lie algebra of T, taking integral values on Λ. Since all the root spaces are one-dimensional, we see that the dimension of G as a group is equal to the rank of G plus the number of roots of G.

The collection of all roots forms an algebraic object inside $\mathfrak{t}^*$ called a *root system*. By definition a root system on a vector space V is a finite subset $\Phi \subset V^*$ of roots such that for each root $a \in /\Phi$ there is a dual coroot $/a^\vee \in V$ such that the 'reflection' $r_a\colon V^* \to V^*$ defined by $r_a(b) = b - \langle b, a^\vee \rangle a$

normalizes the set Φ. It is easy to see that if such $a^\vee$ exists then it is unique and furthermore that the $\{r_a\}_{a\in\Phi}$ generate a finite group. The element r_a is called the *reflection in the wall perpendicular to* a. The wall, denoted W_a^*, fixed by r_a is simply the kernel of the linear map $a^\vee$ on V^*. The group W generated by these reflections is the Weyl group. Dually we can consider the wall $W_a \subset V$ determined by $a = 0$. There is the dual reflection $r_a\colon V \to V$ given by the formula $r_a(v) = v - \langle a, v\rangle a^\vee$. These reflections generate the adjoint action of W on V. The set $\Phi \subset V^*$ generates a lattice in V^* called the *root lattice* and denoted Λ^{root}. Dually the coroots span a lattice in V called the coroot lattice and denoted Λ. Back to the case of a root system of a Lie group G with maximal torus T, since any root of G must take integral values on $\pi_1(T)$ we see that the lattice $\pi_1(T) \subset \mathfrak{t}$ is contained in the integral dual of the root lattice. Furthermore, one can show that the coroot lattice Λ is contained in $\pi_1(T)$.

The walls $\{W_a\}$ divide V into regions called Weyl chambers. Each chamber has a set of walls. We say that a set of roots is a set of *simple roots* if (i) the walls associated with the roots are exactly the walls of some chamber C, and (ii) the roots are non-negative on this chamber. It turns out that a set of simple roots is in fact an integral basis for the root lattice. Also, every root is either a non-negative or a non-positive linear combination of the simple roots, and roots are called positive or negative roots depending on the sign of the coefficients when they are expressed as a linear combination of the simple roots. (Of course, these notions are relative to the choice of simple roots.)

Since the Weyl group action on V^* is finite, there is a Weyl-invariant inner product on V^*. This allows us to identify V and V^* in a Weyl-invariant fashion and consider the roots and coroots as lying in the same space. When we do this the relative lengths of the simple roots and the angles between their walls are recorded in a Dynkin diagram which completely classifies the root system and also the Lie algebra up to isomorphism. If the group is simple, then this inner product is unique up to a positive scalar factor. In general, we can use this inner product to identify $\mathfrak{t}$ and $\mathfrak{t}^*$. When we do we have $\alpha^\vee = \frac{2\alpha}{\langle\alpha,\alpha\rangle}$.

There is one set of simple roots for each Weyl chamber. It is a simple geometric exercise to show that the group generated by the reflections in the walls acts simply transitively on the set of Weyl chambers, so that all sets of simple roots are conjugate under the Weyl group, and the stabilizer in the Weyl group of a set of simple roots is trivial. Thus, the quotient $\mathfrak{t}/W$ is

identified with any Weyl chamber.

A Lie algebra is said to be *simple* if it is not one-dimensional and has no non-trivial normal subalgebras. A Lie algebra that is a direct sum of simple algebras is said to be *semi-simple*. A Lie algebra is semi-simple if and only if it has no non-trivial normal abelian subalgebras. A compact Lie group is said to be simple, resp., semi-simple, if and only if its Lie algebra is simple, resp., semi-simple. Notice that a simple Lie group is not necessarily simple as a group. It can have a non-trivial (finite) center, when then produces finite normal subgroups of G. These are the only normal subgroups of G if G is simple as a Lie group.

A simply connected compact semi-simple group is a product of simple groups. In general a compact semi-simple group is finitely covered by a product of simple groups.

There are a finite number of simple Lie groups with a given Lie algebra $\mathfrak{g}$. All are obtained in the following fashion. There is exactly one simply connected group G with $\mathfrak{g}$ as Lie algebra. For this group the fundamental group of the maximal torus is identified with the coroot lattice Λ of the group. This group has a finite center $\mathcal{C}G$ which is in fact identified with the dual to the root lattice modulo the coroot lattice. Any other group with the same Lie algebra is of the form G/C where $C \subset \mathcal{C}G$ is a subgroup. Thus, the fundamental group of the maximal torus of G/C is a lattice in $\mathfrak{t}$ containing the coroot lattice and contained in the dual to the root lattice. Its quotient by the coroot lattice is C.

Each compact simple group G embeds as the maximal compact subgroup of a simple complex linear group $G_{\mathbf{C}}$. The Lie algebra of $G_{\mathbf{C}}$ is the complexification of the Lie algebra of G and there are maximal complex tori of $G_{\mathbf{C}}$ containing maximal tori of G as maximal compact subgroups. For example, the complexification of $SU(n)$ is $SL_n(C)$ whereas the complexification of $SO(n)$ is the complex special orthogonal group $SO(n, \mathbf{C})$. We can recover the compact group from the semi-simple complex group by taking a maximal compact subgroup (all such are conjugate in the complex group and hence are isomorphic). Any such maximal compact subgroup is called the *compact form* of the group. The fundamental group of a complex semi-simple group is the same as the fundamental group of its compact form.

1.2 Classification of simple groups

Let us look at the classification of such objects.

1.2.1 Groups of A_n-type

The first series of groups is the series $SU(n+1)$, $n \geq 1$. These are the groups of A_n-type. The maximal torus of $SU(n+1)$ is usually taken to be the group of all diagonal matrices with entries in S^1 and the product of the entries being one. We identify this in the obvious way with $\{(\lambda_1, \ldots, \lambda_{n+1}) | \prod_{i=1}^{n+1} \lambda_i = 1\}$. The rank of $SU(n+1)$ is n. The Lie algebra $\mathfrak{su}(n)$ is the space of matrices of trace zero, and the root space $\mathfrak{g}^{ij}; 1 \leq i, j \leq n, i \neq j$ consists of matrices with non-trivial entry only in the ij position. The root associated to this root space is denoted α_{ij} and is given by $\alpha_{ij}(\lambda_1, \ldots, \lambda_{n+1}) = \lambda_i \lambda_j^{-1}$. Writing things additively, we identify $\mathfrak{t}$ with $\{(z_1, \ldots, z_{n+1}) | \sum_i z_i = 0\}$ and then $\alpha_{ij} = e_i - e_j$ where e_i is the linear map which is projection onto the ith coordinate. The usual choice of simple roots are $\alpha_{12}, \alpha_{23}, \ldots, \alpha_{nn+1}$. With this choice the positive roots are the α_{ij} where $i < j$. When $i < j$ we have $\alpha_{ij} = \alpha_{i(i+1)} + \cdots + \alpha_{(j-1)j}$. The Weyl group is the symmetric group on $n+1$ letters. This group acts in the obvious way on $\mathbf{R}^{n+1}$ and leaves invariant the subspace we have identified with $\mathfrak{t}$. This is the Weyl group action on $\mathfrak{t}$. In particular, the restriction of the standard inner product on $\mathbf{R}^{n+1}$ to $\mathfrak{t}$ is Weyl invariant. Also, it is easy to see that all the roots are conjugate under the Weyl action. Notice that each simple root has length 2 and meets the previous simple root and the succeeding simple root (in the obvious ordering) in -1, and is perpendicular to all other simple roots. All this information is recorded in the Dynkin diagram for A_n. Since each root has length two, under the induced identification of $\mathfrak{t}$ with $\mathfrak{t}^*$ every root α is identified with its coroot $\alpha^\vee$. Thus, in this case, and as we shall see, in all other simply laced cases, one can identify the roots and coroots and hence their lattices.

$$\begin{array}{ccccccccc} 1 & & 1 & & & & 1 & & 1 & & 1 \\ \bullet & \text{———} & \bullet & \text{———} & \cdots & \text{———} & \bullet & \text{———} & \bullet & \text{———} & \bullet \end{array}$$

The center $\mathcal{C}(SU(n+1))$ is the cyclic group of order $n+1$ consisting of diagonal matrices with diagonal entry ζ an $n+1$ root of unity. Thus for each cyclic subgroup of $\mathbf{Z}/(n+1)\mathbf{Z}$ there is a form of $SU(n)$ with fundamental group this cyclic subgroup. The full quotient $SU(n+1)/\mathcal{C}SU(n+1)$ is often called $PU(n)$. The intermediate quotients are not given names.

1.2.2 Groups of B_n-type

For any $n \geq 3$, the group $Spin(2n+1)$ is a simple group of type B_n. The standard maximal torus of this group is the subgroup of matrices that project into $SO(2n+1)$ to block diagonal matrices $SO(2) \times SO(2) \times \cdots \times SO(2) \times \{1\}$. The Lie algebra of this torus is naturally a product of n copies of $\mathbf{R}$, one for each $SO(2)$, and we let e_i be the projection of $\mathfrak{t}$ onto the ith-factor. The Lie algebra is all skew symmetric matrices. For any $1 \leq i, j \leq n$ let $\mathfrak{g}^{ij}, 1 \leq i < j \leq n, i \neq j$ be the subspace of skew symmetric matrices with non-zero entries only in places $(2i-1, 2j-1), (2i-1, 2j), (2i, 2j-1), (2i, 2j)$ and the symmetric lower diagonal positions. This is a four-dimensional subspace of the Lie algebra of $SPin(2n+1)$. Thus, there are four roots associated with this space, they are $\pm e_i \pm e_j$. There are also subspaces $\mathfrak{g}^i; 1 \leq i \leq n$, where $\mathfrak{g}^i$ has non-zero entries only in positions $(2i-1, 2n+1)$ and $(2i, 2n+1)$ as well as the symmetric lower diagonal positions. The two roots associated to $\mathfrak{g}^i$ are $\pm e_i$. Thus, $Spin(2n+1)$ is of rank n. We can identify the Lie algebra of its maximal torus with $\mathbf{R}^n$ in such a way that the roots are $\pm e_i \pm e_j$ for $i \neq j$ and $\pm e_i$, where e_i is the projection onto the ith coordinate. The dual coroots to these roots are $\pm e_i \pm e_j$ and $\pm 2e_i$, so that the coroot lattice Λ for $Spin(2n+1)$ is the even integral lattice in $\mathbf{R}^n$, whereas the fundamental group of the maximal torus of $SO(2n+1)$ is the full integral lattice $\mathbf{Z}^n$. Of course, the quotient $\Lambda/\mathbf{Z}^n = \mathbf{Z}/2\mathbf{Z}$ is the fundamental group of $SO(2n+1)$. The Weyl group of $Spin(2n+1)$ is the group generated by reflections in the simple roots. It is easy to see that these reflections generate the group of all permutations of the coordinates and also all sign changes of the coordinates. Thus, abstractly the group is $(\{\pm 1\}^n \times \Sigma_n$. Clearly, the standard metric on $\mathbf{R}^n$ is a Weyl invariant metric. Notice that something new happens here – not all the roots have the same length. In our normalization $\pm e_i \pm e_j$ has length squared 2, whereas $\pm e_i$ has length squared 1. In particular, not all the roots are conjugate under the action of the Weyl group. In this case there are two orbits – one orbit for each length. This fact is expressed by saying that the group is *not simply laced.*

Since the center of $Spin(2n+1)$ is the cyclic group of order 2, there are only two groups with this Lie algebra $spin(2n+1)$ and $SO(2n+1)$.

The Dynkin diagram of type B_n is

```
1       2       2                2       2       2
•———————•———————•——————···——————•———————•=>=•
```

1.3 Groups of C_n-type

Let J be the $2n \times 2n$ matrix of block 2×2 diagonal entries

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

The symplectic group consists of all matrices A such that $A^{tr} JA = J$. These are the linear transformations that leave invariant the standard skew symmetric pairing on $\mathbf{R}^{2n}$, the one given by J. The Lie algebra of this group consists of all matrices A such that

$$A^{tr} = -JAJ^{-1} = JAJ.$$

For any $n \geq 2$ the symplectic group $\mathbf{Sympl}(2n)$ is a group of type C_n so that the complex symplectic group is a complex semi-simple group. There is a complication in that the real symplectic group is non-compact; it is rather what is called the $\mathbf{R}$-split form of the group. Its maximal algebraic torus is a product of n copies of $\mathbf{R}^*$ and is given by the group of diagonal matrices with diagonal entries $(\lambda_1, \lambda^{-1}, \ldots, \lambda_n, \lambda_n^{-1})$. For example, $Sympl(2)$ is identified with $SL_2(\mathbf{R})$. By general theory there is a compact form for the complex symplectic group $Sympl_{\mathbf{C}}(2n)$. It is given as the group of quaternion linear transformations of $\mathbf{H}^n$, so that as one would expect, the compact form of $Sympl_{\mathbf{C}}(2)$ is $SU(2)$. The maximal torus of this group is again a product of circles so that $\mathfrak{t}$ is again identified with $\mathbf{R}^n$. The roots are $\pm e_i \pm e_j$ for $1 \leq i < j \leq n$ and $\pm 2e_i$. Thus, once again the group is non-simply laced. Its Weyl group is the same as the Weyl group of B_n. The coroots dual to the roots are $\pm e_i \pm e_j$ and $\pm e_i$ so that the coroot lattice Λ is the integral lattice $\mathbf{Z}^n$. The dual to the root lattice consists of all $\{x_1, \ldots, x_n\}$ such that $x_i \in (1/2)\mathbf{Z}$ for all i and $x_i \cong x_j \pmod{\mathbf{Z}}$ for all i, j. This lattice contains the coroot lattice with index two, so that the center of the simply connected form of this group is $\mathbf{Z}/2\mathbf{Z}$ and there is one non-simply connected form of these groups.

The Dynkin diagram of type C_n is

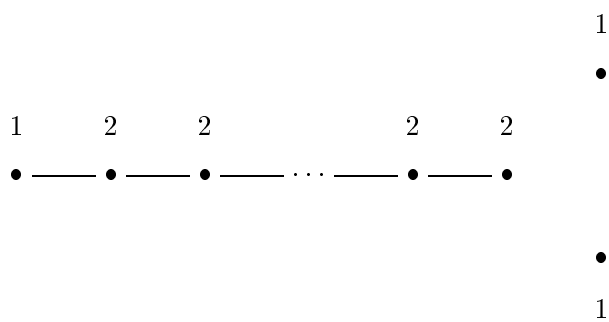
$$\begin{array}{ccccccccc} 1 & & 1 & & & & 1 & & 1 & & 1 \\ \bullet & \text{——} & \bullet & \text{——} & \cdots & \text{——} & \bullet & \text{——} & \bullet & =<= & \bullet \end{array}$$

It turns out that $Spin(5)$ is isomorphic to $\mathbf{Sympl}(4)$ which is why we start the B-series at $n = 3$. The group $\mathbf{Symp}(2)$ is isomorphic to $SU(2)$ which is why we begin the C-series at $n = 2$.

1.3.1 Groups of D_n-type

For any $n \geq 4$ the group $Spin(2n)$ is a group of type D_n. The usual maximal torus for $Spin(2n)$ is the subgroup that projects onto $SO(2) \times \cdots \times SO(2) \subset SO(2n)$. Thus, $\mathfrak{t}$ is identified with $\mathbf{R}^{2n}$ with the factors being tangent to the factors in this decomposition. The roots of $Spin(2n)$ are $\pm e_i \pm e_j$ for $1 \leq i < j \leq n$. This group is simply laced and in the given Weyl invariant inner product all roots have length $\sqrt{2}$. Thus, we can identify the roots with their dual coroots in this case. The coroot lattice is then the even integral lattice. The fundamental group of the maximal torus for $SO(2n)$ is the integral lattice which contains the coroot lattice with index two reflecting the fact that the fundamental group of $SO(2n)$ is $\mathbf{z}/2\mathbf{Z}$. The Weyl group consists of all permutations of the coordinates and all <u>even</u> sign changes of the coordinates. This is a simply laced group.

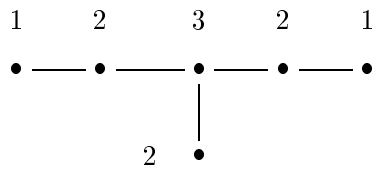
The Dynkin diagram of type D_n is



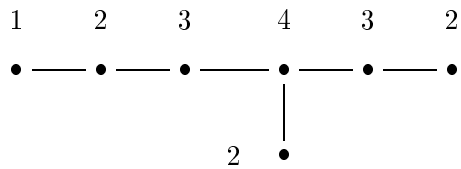
1.3.2 The exceptional groups

A good reference for the exceptional Lie groups is [1]. In addition to the classical groups there are five exceptional simply connected simple groups. Their names are E_6, E_7, E_8, G_2 and F_4. The subscript is the rank of the group. There are natural inclusions $D_5 \subset E_6 \subset E_7 \subset E_8$. The fundamental group of E_r is a cyclic group of order $9 - r$. Both G_2 and F_4 are simply connected.

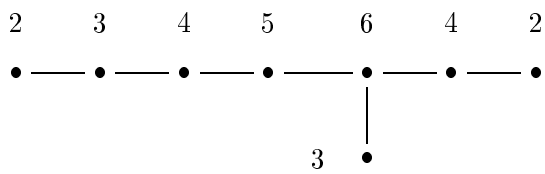
Here are their Dynkin diagrams:



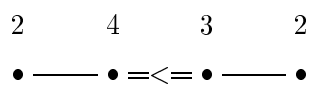
E_6



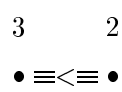
E_7



E_8



F_4



G_2

We shall not say too much about these groups now, but let me give the lattices E_6, E_7, E_8. These are viewed as the fundamental group of the maximal torus of the simply connected form of the group. We give these lattices with an inner product. This is the Weyl invariant inner product. In all cases these groups are simply laced and the coroots are the elements in the lattice of square two. We describe all these lattices at once – for any $r \leq 8$ consider the indefinite integral quadratic form $q(x, a_1, \ldots, a_r) = \sum_{i=1}^{r} a_i^2 - x^2$

on $\mathbf{Z}^{r+1}$. We let $k = (3, 1, 1, \ldots, 1)$. Then $q(k) = r - 9 < 0$. For $6 \leq r \leq 8$, the lattice E_r is the orthogonal subspace in $\mathbf{Z}^{r+1}$ of k. It is of rank r and has the induced quadratic form which is easily seen to be even, positive definite, and of discriminant $9 - r$. For lower values of r it turns out that the lattice defined this way is the lattice of a classical group: $E_5 = D_5$, $E_4 = A_4$, $E_3 = A_2 \times A_1$.

1.4 Principal holomorphic $G_{\mathbf{C}}$-bundles

Let $G_{\mathbf{C}}$ be a complex linear algebraic group and let X be a complex manifold. A holomorphic principal $G_{\mathbf{C}}$-bundle over X is determined by an open covering $\{U_i\}$ of X and transition functions $g_{ij}\colon U_i \cap U_j \to G_{\mathbf{C}}$. The transition functions are required to be holomorphic and to satisfy the cocycle conditions: $g_{ji} = g_{ij}^{-1}$ and $g_{jk}(z) \cdot g_{ij}(z) = g_{ik}(z)$ for all $z \in U_i \cap U_j \cap U_k$. As usual, we can use the g_{ij} as gluing data to glue $U_i \times G_{\mathbf{C}}$ to $U_j \times G_{\mathbf{C}}$ along $u_{ij} \times G_{\mathbf{C}}$, by the rule $(z, g) \in U_i \times G_{\mathbf{C}}$ maps to $(z, g_{ij}(z) \cdot g)$ in $U_j \times G_{\mathbf{C}}$ provided that $z \in U_i \cap U_j$. The cocycle condition tells us that the triple gluings are compatible so that we have defined an equivalence relation and the result of gluing E is a Hausdorff space. The projection mappings $U_i \times G_{\mathbf{C}} \to U_i$ then fit together to define a continuous map $p\colon E \to X$. The fact that the g_{ij} are holomorphic implies that the natural complex structures on the $U_i \times G_{\mathbf{C}}$ are compatible and hence define a complex structure on E for which p is a holomorphic submersion with each fiber isomorphic to $G_{\mathbf{C}}$. The complex manifold E is called the total space of the principal bundle and p is called the projection. There is a natural (right) free, holomorphic $G_{\mathbf{C}}$-action on E such that p is the quotient projection of this action. Two open coverings and gluing functions define the isomorphic principal $G_{\mathbf{C}}$-bundles if the resulting total spaces are biholomorphic by a $G_{\mathbf{C}}$-equivariant mapping commuting with the projections to X.

If ξ is a holomorphic principal $G_{\mathbf{C}}$-bundle over X and $\rho\colon G_{\mathbf{C}} \to \mathrm{Aut}(V)$ is a complex linear representation, then there is an associated holomorphic vector bundle $\xi(\rho)$. If E is the total space of ξ then the total space of $\xi(\rho)$ is $\xi \times_{G_{\mathbf{C}}} V$ where $G_{\mathbf{C}}$ acts on V via the representation ρ.

Example: Let $G_{\mathbf{C}}$ be $\mathbf{C}^*$. (Notice that this is not a simple group.) Then a holomorphic principal $\mathbf{C}^*$ bundle determines a holomorphic line bundle under the natural representation given by complex multiplication $\mathbf{C}^* \times \mathbf{C} \to \mathbf{C}$. The holomorphic line bundles associated to ξ and ξ' are isomorphic as holomorphic line bundles if and only if ξ and ξ' are isomorphic as holomorphic

principal $\mathbf{C}^*$-bundles. Notice that the total space of the $\mathbf{C}^*$-bundle can be identified with the complement of the zero section in the corresponding holomorphic line bundle.

Along the same lines, let $G_{\mathbf{C}}$ be $SL_n(\mathbf{C})$. Then using the defining complex n-dimensional representation, a principal $SL_n(\mathbf{C})$-bundle ξ over X determines a holomorphic n-dimensional vector bundle V over X. But this bundle has the property that its determinant line bundle $\wedge^n V$ is trivialized as a holomorphic line bundle. Of course, given a holomorphic n-dimensional vector bundle V over X with a trivialization of its determinant line bundle we can define the associated bundle of special linear frames in V. The fiber over $x \in X$ consists of all bases $\{f_1, \ldots, f_n\}$ for V_x such that $f_1 \wedge \cdots \wedge f_n$ is identified with $1 \in \mathbf{C}$ under the given trivialization of $\wedge^n(V_x)$. The local trivialization of the vector bundle, produces a local trivialization of this bundle of frames. The holomorphic structure on the total space of V determines a holomorphic structure on the bundle of frames. The obvious $SL_n(\mathbf{C})$-action on the bundle of frames then makes it a holomorphic principal $SL_n(\mathbf{C})$-bundle. This sets up a bijection between isomorphism classes of holomorphic principal $SL_n(\mathbf{C})$-bundles over X and holomorphic vector bundles over X with trivialized determinant line bundle.

In the same way we can identify a holomorphic principal $SO(n, \mathbf{C})$-bundle over X with a holomorphic rank n vector bundle V over X with holomorphically trivialized determinant and with a holomorphic symmetric form $V \otimes V \to \mathbf{C}$ which is non-degenerate on each fiber. A holomorphic principal $\mathrm{Symp}(2n)$-bundle over X is identified with a holomorphic rank $2n$ vector bundle V over X with holomorphically trivialized determinant and with a holomorphically varying skew-symmetric bilinear form on the fibers which is non-degenerate on each fiber.

Up to questions of finite covering groups, this exhausts the list of classical simple groups: $SL_n(\mathbf{C})$, $SO(n, \mathbf{C})$, and $Sympl(2n, \mathbf{C})$.

Associated to any complex group $G_{\mathbf{C}}$ there is the adjoint representation of $G_{\mathbf{C}}$ on its Lie algebra $\mathfrak{g}_{\mathbf{C}}$. Thus, associated to any holomorphic principal $G_{\mathbf{C}}$-bundle ξ is a vector bundle denoted $\mathrm{ad}\xi$. Its rank is the dimension of $G_{\mathbf{C}}$ as a group. In the case of E_8 this is the smallest dimensional representation; it is of course of the same dimension as the group 248. All other simple groups have smaller representations: G_2 has a seven dimensional representation even though it has dimension 14; F_4 has a 28-dimensional representation; E_6 has a 27 dimensional representation (which we discuss later), and E_7 has a 54-dimensional representation. Still, it is not clear that the best way to

study principal bundles over these groups is to look at the vector bundles associated to these representations. For example, it is not obvious what extra structure a 248-dimensional vector bundle carries if it comes from a principal E_8-bundle, nor what extra information it takes to determine the structure of that bundle.

1.5 Principal G-bundles over S^1

Let us begin with a simple problem. Fix a compact, simply connected, simple group G. Let ξ be a principal G-bundle over S^1 and let A be a flat G-connection on ξ. (The reason for the passage from G_C to G and the introduction of a flat connection will be explained in the next lecture.) The holonomy of A around the base circle is an element of G, determined up to conjugacy, which completely determines the isomorphism class of (ξ, A) and sets up an isomorphism between the space of conjugacy classes of elements in G and the space of isomorphism classes of principal G-bundles with flat connections over S^1. Thus, we have reduced our problem to that of understanding the space of conjugacy classes of elements in a compact group. To some extent this is a classical and well-understood problem, as we show in the next section.

1.5.1 The affine Weyl group and the alcove structure

This leads us to the question of what the space of conjugacy classes of elements in G looks like. To answer this question we introduce the affine Weyl group and the alcove structure on the Lie algebra $\mathfrak{t}$ of the maximal torus T of G. For a root α and $k \in \mathbf{Z}$ we denote by $W_{\alpha,k}$ the codimension-one affine-linear subspace of $\mathfrak{t}$ determined by the equation $\{\alpha = k\}$. By definition, the affine Weyl group, W_{aff}, is the group of affine isometries of $\mathfrak{t}$ generated by reflections in all walls of the form $W_{\alpha,k}$. It is easy to see that there is an exact sequence of groups

$$0 \to \Lambda \to W_{\text{aff}} \to W \to 0.$$

(Recall that $\Lambda \subset \mathfrak{t}$ is the coroot lattice.) Here, the map $W_{\text{aff}} \to W$ is the differential or linearization of the affine map. (Recall that Λ is the coroot lattice, i.e., $\pi_1(T) \subset \mathfrak{t}$.) This sequence is split by including W as the group generated by the reflections in the walls $W_{\alpha,0}$, but the action of W on Λ is non-trivial: it is the obvious action. Thus, W_{aff} is isomorphic to the semi-direct product $\Lambda \times W$ with the natural action of W on Λ.

The set of walls $W_{\alpha,k}$ is a locally finite set and divides $\mathfrak{t}$ into (an infinite number of) regions called *alcoves*. If G is simple (or even semi-simple), then the alcoves are compact. In the case when G is simple, the alcoves are simplices. Clearly, each alcove is contained in some Weyl chamber. An alcove containing the origin in fact contains a neighborhood of the origin in the Weyl chamber that contains it. Its walls are the walls of the Weyl chamber containing it together with one more. This extra wall is given by an equation of the form $\tilde{\alpha} = 1$ for some root α determined by the Weyl chamber (or equivalently by the set of simple roots $\{\alpha_1, \ldots, \alpha_r\}$ determined by the Weyl chamber). It turns out that $\tilde{\alpha}$ is a positive linear combination of the α_i and it has the largest coefficients. In particular, it is the unique nonnegative linear combination of the simple roots with the property that its sum with any simple root is not a root. This root $\tilde{\alpha}$ is called the *highest root* of G. The numbers displayed on the Dynkin diagrams above are the coefficients of the simple roots in their unique linear combination which is the highest root. If Δ is a set of simple roots for G, then the associated set $\Delta \cup \tilde{\alpha}$ is denoted by $\widetilde{\Delta}$ and is called the extended set of simple roots.

As in the case of the Weyl group, it is a nice geometric argument to show that W_{aff} acts simply transitively on the set of alcoves and that the quotient $\mathfrak{t}/W_{\text{aff}}$ is identified with any alcove.

Lemma 1.5.1 *Let G be a compact, simply connected semi-simple group. Then the space of conjugacy classes of elements in G is identified with an alcove A in $\mathfrak{t}$. The identification associates to $t \in A$ the conjugacy class of* $\exp(t) \in T$.

Proof. Every $g \in G$ is conjugate to a point $t \in T$. Two points of T are conjugate in G if and only if they are in the same orbit of the Weyl group action on T. Thus, the space of conjugacy classes of elements in G is identified with T/W. Since G is simply connected, $T = \mathfrak{t}/\Lambda$ and hence $T/W = \mathfrak{t}/W_{\text{aff}}$ which we have just seen is identified with an alcove. $\square$

Example: If $G = SU(n+1)$, then the alcove is the subset $(t_1, \ldots, t_{n+1}) \in \mathbf{R}^{n+1}$ satisfying $\sum_j t_j = 0$ and $t_j \geq 0$ and $t_j \leq t_{j+1}$ for all $1 \leq j \leq n+1$. Every element in $SU(n+1)$ is conjugate to a diagonal matrix with diagonal entries $(\lambda_1, \ldots, \lambda_{n+1})$, the λ_j being elements of S^1. We can do a further conjugation under $\lambda_j = \exp(2\pi i t_j$ for $0 \leq t_j \leq 1$. The determinant condition is $\prod_j \lambda_j = 1$, which translates into $\sum_j t_j = 0$. By a further Weyl conjugation

we can arrange that the t_j are in increasing order. This makes explicit the isomorphism between the simplex in $\mathbf{R}^{n+1}$ and the space of conjugacy classes in $SU(n+_1)$.

1.6 Flat G-bundles over T^2

Let G be a compact, simply connected group.

Lemma 1.6.1 *Let $x, y \in G$ be commuting elements. Show that there is a torus $T \subset G$ containing both x and y. If (x, y) and (x', y') are pairs of elements in a torus $T \subset G$ and if there is $g \in G$ such that $g(x, y)g^{-1} = (x', y')$, then there is an element n normalizing T and conjugating (x, y) to (x', y').*

The proof is an exercise.

Corollary 1.6.2 *Let G be a compact simply connected group. Then the space of isomorphism classes of principal G-bundles with flat connections over T^2 is identified with the space $(T \times T)/W$, where T is a maximal torus of G and W is the Weyl group acting on $T \times T$ by simultaneous conjugation.*

Notice that this implies that the space of such bundles is connected. Of course, this is not too surprising since for G simply connected, all G-bundles over T^2 are topologically trivial.

1.7 Exercises

1. Let G be a compact connected Lie group. Show that the exponential mapping from the Lie algebra $\mathfrak{g}$ to G is onto. Use this to show that every element of G is contained in a maximal torus.
2. Show all maximal tori of G are conjugate.
3. Suppose that G is compact. Show that the centralizer of a maximal torus in G is the torus itself.
4. Let G be a compact simply connected group. Show that the center of G is the intersection of the kernels of all the roots. Show that if G is simply connected and semi-simple then the center of G is identified with the quotient $(\Lambda_{\text{root}})^*/\Lambda$ where $\Lambda \subset \mathfrak{t}$ is the coroot lattice, where $\Lambda_{\text{root}} \subset \mathfrak{t}^*$ is the root lattice, and $(\Lambda_{\text{root}})^*$ is the algebraically dual lattice in $\mathfrak{t}$. Use this to show that the center of a semi-simple group is finite.

5. Show that if G is a compact Lie group whose Lie algebra is semi-simple, then any normal subgroup of G is a finite central subgroup.
6. Count the number of roots for groups of A_n, B_n, C_n, and D_n type and determine the dimension of each of these groups.
6. For any $r, 3 \leq r \leq 8$, let $\mathbf{Z}^{r+1}$ be the free abelian group of rank $r+1$ with basis $h, e_1, \ldots, e_r$ and with non-degenerate quadratic form Q with $Q(h) = -1$, $Q(e_i) = 1, 1 \leq i \leq r$, and the basis being mutually orthogonal. Let $k = 3h - \sum_{i=1}^{r} e_i$. Then $k^{\perp}$ is a lattice of rank r with a positive definite pairing of determinant $9 - r$. Show that a basis for $k^{\perp}$ is $e_1 - e_2, \ldots e_{r-1} - e_r, H - e_1 - e_2 - e_3$ and that these vectors are all of square 2. Count the number of vectors in this lattice of square two. Show that for each vector of square 2, reflection in the vector is an integral isomorphism of the lattice $k^{\perp}$ and its form. Show that for $r = 3$ the lattice is the coroot lattice of $A_2 \times A_1$, for $r = 4$, the lattice is the coroot lattice of A_4, for $r = 5$ the lattice is the coroot lattice for D_5. In all cases the coroots are exactly the vectors of square 2. It follows of course that the group generated by reflections in the vectors of square two is the Weyl group. It turns out that for $r = 6, 7, 8$ the lattice is the coroot lattice of E_r. The statements about the roots, coroots and the Weyl group remain true for these cases as well. Assuming this, compute the dimensions of the Lie groups E_6, E_7, E_8.
8. Let E be an elliptic curve. Give a 1-cocycle which represents the generator of $H^1(E; \mathcal{O}_E))$. Give a 1-cocycle that represents the principal $\mathbf{C}^*$-bundle $\mathcal{O}(q - p_0)$.
9. Show that if V is a semi-stable vector bundle of degree zero over an elliptic curve E with a non-degenerate skew-form, then $\det(V)$ is trivial. Show that this is not necessarily true it V supports a non-degenerated quadratic form instead.
10. Define the compact form of $\mathrm{Sympl}_{\mathbf{C}}(2n)$ in terms of quaternion linear mappings of a quaternionic vector space.
11. Show that the conjugacy class of the holonomy representation determines an isomorphism between the space of isomorphism classes of principal G-bundles over S^1 with flat connections and the space of conjugacy classes of elements in G.
12. Show that the space of conjugacy classes of elements in G is identified with the alcove of the affine Weyl group action on $\mathfrak{t}$.
13. Show that if x, y are commuting elements in a compact simply connected group G then there is a torus in G containing both x and y. [Hint: Show that the centralizer of x, $Z_G(x)$ is connected.] Show that this fails to be true

if G is compact but not simply connected (e.g. $G = SO(3)$). Show that if X and X' are ordered subsets of T which are conjugate in G, then they are conjugate by an element of the normalizer of T.

14. Show that if G is a compact semi-simple group, then G-bundles over T^2 are classified up to topological equivalence by a single characteristic class $w_2 \in H^2(T^2; \pi_1(G)) = \pi_1(G)$. Show that if a G-bundle admits a flat connection with holonomy $(\overline{x}, \overline{y})$ around the generating circles, then its characteristic class in $\pi_1(G) \subset G$ is computed as follows. One chooses lifts x, y in the universal covering group $\tilde{G}$ of G for $\overline{x}, \overline{y}$. Then $[x, y] \in \tilde{G}$ lies in the kernel of the projection to G, i.e., lies in $\pi_1(G) \subset \tilde{G}$. This is the characteristic class of the bundle.

15. Show that the space of isomorphism classes of $SO(3)$-bundles with flat connection over T^2 has two components. Show that one of these components has dimension 2 and the other is a single point.

2 Semi-Stable $G_{\mathbf{C}}$-Bundles over Elliptic Curves

In this lecture we introduce the classical notions of stability, semi-stability, and S-equivalence for vector bundles. We then consider in some detail semi-stable vector bundles over an elliptic curve and the moduli space of their S-equivalence classes. We extend these results to the one situation where it has an easy and direct analogue – namely complex symplectic bundles. Then we switch and consider flat $SU(n)$-bundles and state the Narasimhan-Seshadri result relating these bundles to semi-stable holomorphic bundles. Then we generalize this result to compare flat G-bundles and semi-stable holomorphic $G_{\mathbf{C}}$-bundles. Lastly, we discuss Looijenga's theorem which, in the case that G is simple and simply connected, describes quite explicitly these moduli spaces in terms of the coroot integers for the group G.

2.1 Stability

Let C be a compact complex curve. The slope of a holomorphic vector bundle $V \to C$ is $\mu(V) = \deg(V)/\mathrm{rank}(V)$ where $\deg(V)$ is the degree of the determinant line bundle of V. A vector bundle $V \to C$ is said to be *stable* if for every proper subbundle $W \subset V$ we have $\mu(W) < \mu(V)$. The bundle V is said to be *semi-stable* if $\mu(W) \leq \mu(V)$ for every proper subbundle $W \subset V$. A subbundle W which violates these inequalities is called destabilizing or de-semistabilizing.

Note: One usually requires $\mu(W) < \mu(V)$, resp., $\leq \mu(V)$ for every vector bundle over C which admits a vector bundle mapping $W \to V$ which is injective on the generic fiber. The image of such a mapping will not necessarily be a subbundle of V, but rather is a subsheaf of its sheaf of local holomorphic sections. Nevertheless, there is a subbundle $W' \subset V$ containing the image of W under the given mapping with the property that W' modulo the image of W is supported at a finite set of points (a sky-scraper sheaf). In this case $\deg(W') = \deg(W) + \ell(W'/W)$ where $\ell(W'/W)$ is the total length of the sky-scraper sheaf. It follows that $\mu(W') \geq \mu(W)$ so that if W destabilizes or de-semistabilizes V then so does W'. Thus, in the case of curves it suffices to work exclusively with subbundles. Let me re-iterate that this is special to the case of curves.

The main reason for introducing stability is that the space of all bundles can be studied in terms on stable bundles (the so-called Harder-Narasimhan filtration) and that the space of isomorphism classes of stable bundles forms a reasonable space.

Over high dimensional manifolds X^n we need a couple of modifications. First of all we need a Kähler class ω so that we can define the degree of a bundle V, $\deg(V)$, to be $\int_X \omega^{n-1} \wedge c_1(V)$. (Of course, the degree, and hence the slope depends on the choice of Kähler class.) Then as indicated above we must also consider all torsion-free subsheaves W of V. Each of these has a first Chern class and hence a degree, again depending on ω and computed by the same formula as above. With these modifications one defines *slope stability* and *slope semi-stability* exactly as before. There are more refined notions, for example Gieseker stability, which are often needed over higher dimensional bases, especially if one hopes to obtain compact moduli spaces.

2.2 Line bundles of degree zero over an elliptic curve

We fix a smooth elliptic curve E. By this we mean that E is a compact complex (smooth) curve of genus 1, and we have fixed a point $p_0 \in E$. This determines an abelian group law on E for which p_0 is the identity element.

Consider a line bundle $L \to E$ of degree zero. According to Riemann-Roch,

$$\operatorname{rank} H^1(E; L) = \operatorname{rank} H^0(E; L)$$

which tells us nothing about whether L has a holomorphic section. However, if we consider $L \otimes \mathcal{O}(p_0)$, then RR tells us that this bundle has at least one holomorphic section. Let $\sigma\colon \mathcal{O} \to L \otimes \mathcal{O}(p_0)$ be such a section. Of course, since the degree of $L \otimes \mathcal{O}(p_0)$ is 1, σ vanishes once, say at a point $q \in E$. This means that σ factors to give a holomorphic mapping $\sigma'\colon \mathcal{O}(q) \to L \otimes \mathcal{O}(p_0)$ which is generically an isomorphism. In general a map between line bundles which is generically one-to-one has torsion cokernel and the total length of the cokernel is the difference of the degrees of the two bundles. In our case, the domain and range both have degree one, so that the cokernel is trivial, i.e., σ' is an isomorphism. This proves that L is isomorphic to $\mathcal{O}(q) \otimes \mathcal{O}(-p_0)$, which is also written as $\mathcal{O}(q - p_0)$. It is easy to see that associating to L the point q sets up an isomorphism between the space of line bundles of degree zero on E, $\operatorname{Pic}^0(E)$, and E itself.

2.3 Semi-stable $SL_n(\mathbf{C})$-bundles over E

Let V be a semi-stable vector bundle of rank n and degree zero. Semi-stability implies that any subbundle of V has non-positive degree. Let us first show that there is a line bundle of degree zero mapping into V. We

proceed in the same manner as with line bundles. The bundle $V \otimes \mathcal{O}(p_0)$ is also easily seen to be semi-stable and of slope 1. By RR $V \otimes \mathcal{O}(p_0)$ has n holomorphic sections. Let H^0 be the space of these sections. Then evaluation determines a vector bundle mapping from the trivial bundle $H^0 \otimes \mathcal{O}_E$ to $V \otimes \mathcal{O}_E(p_0)$. If no non-trivial section of $V \otimes \mathcal{O}_E(p_0)$ vanished at any point, then this map would be an isomorphism of vector bundles, which is absurd since the bundles have different degree. We conclude that there is a non-zero section of $V \otimes \mathcal{O}_E(p_0)$ vanishing at some point. Consider the cokernel of this section. Generically it is a vector bundle of rank $n-1$, but it has non-trivial torsion, say T, corresponding to the zeros of the section. Then there is a line bundle L_T fitting into an exact sequence

$$0 \to \mathcal{O} \to L_T \to T \to 0$$

and an extension of σ to a map $\sigma' \colon L_T \to V \otimes \mathcal{O}(p_0)$ whose quotient is a vector bundle W of rank one less than that of V. Of course, the degree of L_T is the total length of T. By the fact that the slope of $V \otimes \mathcal{O}_E(p_0)$ is one and is semi-stability, we know the total length of T is at most one. But on the other hand we know that T is nontrivial, so that it has length exactly one. It follows that it is of the form $\mathcal{O}/\mathcal{O}(q)$ for some point $q \in E$. Thus, we have a map $\mathcal{O}(q) \to V$ with torsion-free cokernel and hence a map $\mathcal{O}(q - p_0) \to V$ whose quotient is a semi-stable bundle of degree $n-1$.

Continuing inductively we see that V is written as a successive extension of n line bundles of degree zero. We can associate to V, the n points that are identified with these line bundles. Let us now think about the extensions. For any line bundle of degree zero, RR tells us that since $H^1(E; L) = 0$ unless L is trivial. It follows immediately that if L and L' are non-isomorphic line bundles of degree zero then any extension

$$0 \to L \to V \to L' \to 0$$

is trivial. An easy inductive argument shows that if W_1 written as a successive extension of line bundles L_i of degree zero and W_2 as a successive extension of line bundles M_j of degree zero, and no L_i is isomorphic to any M_j, then any extension

$$0 \to W_1 \to V \to W_2 \to 0$$

is trivial. Consequently, we can split V into pieces $\oplus_{q \in E} V_q$ where V_q is a successive extension of line bundles isomorphic to $\mathcal{O}(q - p_0)$. Lastly, there

is, up to equivalence, exactly one non-trivial extension

$$0 \to \mathcal{O}(q - p_0) \to V \to \mathcal{O}(q - p_0) \to 0.$$

An easy argument shows that if V is such a nontrivial extension, then it has a unique nontrivial extension by $\mathcal{O}(q - p_0)$ and so forth.

Notice that since every vector bundle of degree zero over an elliptic curve E has a subline bundle of degree zero, the only stable degree zero vector bundles over E are line bundles. Thus, the best we can hope for in higher rank and degree zero is that the bundle be semi-stable. As we shall eventually see, there are stable bundles of positive degree.

This allows us to establish the following theorem first proved by Atiyah:

Theorem 2.3.1 *Any semi-stable vector bundle of degree zero over E is isomorphic to a direct sum of bundles of the form $\mathcal{O}(q - p_0) \otimes I_r$ where the I_r are defined inductively as follows; $I_1 = \mathcal{O}$ and I_r is the unique non-trivial extension of I_{r-1} by $\mathcal{O}$.*

Clearly, the determinant of $\oplus_{q \in E} \mathcal{O}(q - p_0) \otimes I_{r(q)}$ is $\otimes_{q \in E} \mathcal{O}(q - p_0)^{\otimes r(q)}$, or under our identification of line bundles of degree zero with points of E, the determinant of V is identified with $\sum_{q \in E} r(q)q$, where the sum is taken in the group law of the elliptic curve. Thus, V has a trivial determinant if and only if $\sum_q r(q)q = 0$ in E.

2.4 S-equivalence

We just classified semi-stable vector bundles of degree zero on an elliptic curve in the sense that we enumerated the isomorphism classes. But we have not produced a moduli space (even a coarse one) of all such bundles. The trouble, as always in these problems, is that the natural space of isomorphism classes is not separated, i.e., not a Hausdorff space. The reason is exemplified by the fact that there is a bundle over $E \times H^1(E; \mathcal{O})$ whose restriction to $E \times \{a\}$ is the bundle which is the extension of $\mathcal{O}$ by $\mathcal{O}$ given by the extension class a. This bundle is isomorphic to I_2 for all $\{a\} \neq 0$ and is isomorphic to $\mathcal{O} \oplus \mathcal{O}$ if $\{a\} = 0$. Thus, we see that any Hausdorff quotient of the space of isomorphism classes of bundles I_2 and $\mathcal{O} \oplus \mathcal{O}$ must be identified.

This phenomenon is an example of S-equivalence. We say that a semi-stable bundle V is S-equivalent to a semi-stable bundle V' if there is a family of $\mathcal{V} \to E \times C$, where C is a connected smooth curve, so that for generic $c \in C$ the bundle $\mathcal{V}|C \times \{c\}$ is isomorphic to V and so that there is $c_0 \in C$

for which $\mathcal{V}|E \times \{c_0\}$ is isomorphic to V'. This relation is not an equivalence relation (it is not symmetric), so we take S-equivalence to be the equivalence relation generated by this relation.

Then we take as the moduli space of semi-stable vector bundles of degree zero the space of S-equivalence classes. Since I_k is S-equivalent to $\oplus_k \mathcal{O}$, it follows immediately from Atiyah's theorem that:

Theorem 2.4.1 *The set of S-equivalence classes of semi-stable bundles of rank n is identified with the set of unordered n-tuples of points $(e_1, \ldots, e_n) \subset E$. The subset of those with trivial determinant is the subset of unordered n-tuples $(e_1, \ldots, e_n)$ for which $\sum_{i=1}^{b} e_i = 0$ in the group law of E.*

2.5 The moduli space of semi-stable bundles

Everything we have done so far is at the level of points – that is to say we are describing all isomorphism classes or S-equivalence classes of bundles. Now we wish to see that the symmetric product of E with itself n-times is actually a coarse moduli space for the S-equivalence classes of semi-stable bundles of rank n and degree zero. Since we have already established a one-to-one correspondence between the points of this symmetric product and the set of S-equivalence classes of bundles, the question is whether this identification varies holomorphically with parameters. By this we mean that any time we have a holomorphic family $\mathcal{V} \to E \times X$ of semi-stable bundles rank n bundles on E (that is to say $\mathcal{V}$ is a rank n vector bundle and its restriction to each slice $E \times \{x\}$ is a semi-stable bundle), there should be a unique holomorphic mapping $X \to (E \times \cdots \times E)/S_n$ which associates to each $x \in X$ the point of the symmetric product which characterizes the S-equivalence class of $\mathcal{V}|E \times \{x\}$. Of course, this does define a function from X to the symmetric product; the only issue is whether it is always holomorphic.

To establish this we need a direct algebraic construction which goes from a vector bundle to an unordered n-tuple of points in E. Let V be a semi-stable vector bundle of degree 0 and rank n. Then $H^0(E; V \otimes \mathcal{O}(p_0))$ is n-dimensional. We have the evaluation map from sections to the bundle which we can view as a map from the trivial bundle $H^0(E; V \otimes \mathcal{O}(p_0)) \otimes \mathcal{O}_E$ to $V \otimes \mathcal{O}(p_0)$. Taking the determinants we get a map of line bundles

$$\wedge^n H^0(E; V \otimes \mathcal{O}) \otimes \mathcal{O}_E \to \det V \otimes \mathcal{O}(np_0).$$

This map is non-trivial. The domain is a trivial line bundle and the range has degree n. Thus, the cokernel of the map is a torsion sheaf of total degree n.

Taking the support of this torsion module, counted with multiplicity gives the unordered n-tuple in E. As one can see directly, for any sum of line bundles of degree zero, this map exactly picks out the unordered n points in E associated with the n-line bundles. More generally, one can easily check that the sections of $I_r \otimes \mathcal{O}(q)$ all vanish to first order at p_0 so that such a factor produces a zero of order r at q in the above determinant map.

Theorem 2.5.1 *The coarse moduli space of S-equivalence classes of semi-stable vector bundles of rank n and degree zero over an elliptic curve E is identified with $(E \times \cdots \times E)/S_n$. The coarse moduli space of those with trivial determinant is identified with the subspace of unordered n-tuples which sum to zero in the group law of E.*

Proof. To each semi-stable vector bundle of degree zero we associate the unordered n-tuple of points in E which corresponds to the set of line bundles of degree zero on E which are the successive quotients of V. By Atiyah's classification we see that this is a well-defined function. Clearly, S-equivalent semi-stable bundles are mapped to the same point and in light of Atiyah's classification, two bundles which map to the same point are S-equivalent.

It remains to see that if $\mathcal{V} \to E \times X$ is an algebraic family of semi-stable rank n bundles of degree zero over E, parametrized by X, then the resulting map $X \to (E \times \cdots \times E)/S_n$ is holomorphic. To see this let $\pi\colon E \times X \to X$ be the projection and consider the cohomology of along the fibers of $V \otimes \mathcal{O}(p_0)$. Let $p\colon E \times X \to E$ be the projection to the other component. Since we have already seen that for each $x \in X$, the cohomology $H^0(E; V|(E \times \{x\} \otimes \mathcal{O}(p_0))$ has rank n, it follows that the cohomology along the fibers $R^0\pi_*(V \otimes p^*\mathcal{O}(p_0))$ is a vector bundle of rank n over X. We take its nth exterior power and pull back to a line bundle $\mathcal{L}$ on $E \times X$, trivial on each $E \times \{x\}$. As before, the evaluation mapping induces a map from $\mathrm{ev}\colon \mathcal{L} \to \wedge^n V \otimes p^*\mathcal{O}(np_0)$, which fiber-by-fiber in X is the map we considered above. In particular, the zero locus of ev is a subvariety of $E \times X$ whose projection to X is an n-sheeted ramified covering. Its intersection (counted with multiplicity) with each $E \times \{x\}$ gives the unordered n-points in E associated with the bundle $\mathcal{V}|E \times \{x\}$. This proves that the map $X \to (E \times \cdots \times E)/S_n$ is algebraic. $\square$

(This argument works in either the classical analytic topology or in the Zariski topology.)

The space we just obtained of S-equivalence classes of semi-stable vector bundles of rank n with trivial determinant has another, extremely useful description.

Theorem 2.5.2 *The coarse moduli space of S-equivalence classes of semi-stable rank n bundles with trivial determinant on an elliptic curve E is identified with the projective space associated with the vector space $H^0(E; \mathcal{O}(np_0))$.*

Proof. Given n points $e_1, \ldots, e_n$ in E whose sum is zero there is a meromorphic function on E vanishing at these n points (with the correct multiplicities) with a pole only at p_0. Of course, the order of this pole is at most n and is exactly equal to the number of $i, 1 \leq i \leq n$, for which e_i is distinct from p_0. (Our evaluation mapping constructed such a function.) But a non-zero meromorphic function on E is determined up multiple by its zeros and poles. □

Corollary 2.5.3 *The coarse moduli space of semi-stable vector bundles on E of rank n and trivial determinant is isomorphic to a projective space $\mathbf{P}^{n-1}$. In particular, as we vary the complex structure on E, the complex structure on this moduli space is unchanged.*

2.6 The spectral cover construction

Let us construct a family of semi-stable bundles on E parametrized by $\mathbf{P}(H^1(E; \mathcal{O}_E(p_0))$, which to simplify notation we denote by $\mathbf{P}_n$. There is a covering $T \to \mathbf{P}_n$ defined as follows: a point of T consists of a pair $([f] \in |\mathcal{O}_E(np_0)|, e)$ where $e \in E$ is a point of the support of the zero locus of f. In other words, letting $S_{n-1} \subset S_n$ be the stabilizer of the first point, $T = \underbrace{(E \times \cdots \times E)}_{n-\text{times}}/S_{n-1}$. Clearly, $T \to \mathbf{P}_n$ is an n-sheeted ramified covering. There is of course a natural mapping $g: T \to E$ which associates to (f, e) to point e. We let $\mathcal{L}$ be pullback of the Poincaré bundles $\mathcal{O}_{E \times E}(\Delta - E \times \{p_0\})$ to $T \times E$ under $g \times \mathrm{Id}$. Then the pushforward $\mathcal{V}_n = (g \times \mathrm{Id})_*(\mathcal{L})$ of $\mathcal{L}$ over $T \times E$ to $\mathbf{P}_n \times E$ is a rank n vector bundle. A generic point of $\mathbf{P}_n$ has n distinct preimages in T and at such points $\mathcal{V}_n$ restricts to be a sum of n distinct line bundles of degree zero – the sum of bundles associated with the n-points in the preimage. If one asks what happens at points where g ramifies, it turns out that the pushforward bundle develops factors of the form $\mathcal{O}(q - p_0) \otimes I_r$

when r points come together. Thus, in fact, this construction produces a family of regular semi-stable bundles on E. Let $q: T \times E \to T$ be the projection to the first factor. Notice that for any line bundle $\mathcal{M}$ on T, we have that $(g \times \mathrm{Id})_*(\mathcal{L} \otimes q^*\mathcal{M})$ is also a rank n vector bundle on $\mathbf{P}_n \times E$ which is isomorphic fiber-by-fiber to the bundle $(g \times \mathrm{Id})_*\mathcal{L}$. Of course, globally these bundles can be quite different, and for example have different characteristic classes over $\mathbf{P}_n \times E$.

This construction is universal in the following sense, which we shall not prove (see Theorem 2.8 in Vector Bundles over elliptic fibrations).

Lemma 2.6.1 *Suppose that S is an analytic space and $\mathcal{U} \to S \times E$ is a rank n vector bundle which is regular and semi-stable with trivial determinant on each fiber $\{s\} \times E$. Let $\varphi_{\mathcal{U}}: S \to \mathbf{P}_n$ be the map that associates to each $s \in S$ the S-equivalence class of $\mathcal{U}|_{\{s\} \times E}$. Let $\tilde{S} = S \times_{\mathbf{P}_n} T$ and let $\tilde{\varphi}: \tilde{S} \to S$ and Φ be the natural maps. Then there is a line bundle $\mathcal{M}$ over $\tilde{S}$ such that $\mathcal{U}$ is isomorphic to*

$$(\varphi \times \mathrm{Id})_*((\Phi \times \mathrm{Id})^*\mathcal{L} \otimes p_1^*\mathcal{M}),$$

where $p_1: \tilde{S} \times E \to \tilde{S}$ is the projection.

2.7 Symplectic bundles

Let us make an analysis of principal $\mathrm{Sympl}_{\mathbf{C}}(2n)$-bundles which follows the same lines as the analysis for SL_n. We can view a holomorphic principal $\mathrm{Sympl}(2n)$-bundle over E as a holomorphic vector bundle V with a nondegenerate skew symmetric (holomorphic) pairing $V \otimes V \to \mathbf{C}$. Equivalently, we can view the pairing as an isomorphism $\varphi: V \to V^*$ which is skew-adjoint in the sense that $\varphi^* = -\varphi$. We say that such a bundle is semi-stable if the underlying vector bundle is. If we consider first a sum of line bundles of degree zero, $\oplus_{i=1}^{2n} L_i$ this bundle will support a skew symmetric pairing if and only if we can number the line bundles so that L_{2i-1} is isomorphic to L_{2i}^*. In this case we take the pairing to be an orthogonal sum of rank two pairings, the individual rank two pairings pairing L_{2i-1} and L_{2i} via the duality isomorphism. This means that if a semi-stable rank $2n$ vector bundle with trivial determinant supports a symplectic form then the associated points $(e_1, \ldots, e_{2n})$ in E are invariant (up to permutation) under the map $e \mapsto -e$. It turns out that we can identify the coarse moduli space of rank $2n$ semi-stable symplectic bundles over E with the subset of n unordered points in $E/\{e \cong -e\}$. Of course, $E/\{e \cong -e\}$ is the projective line P^1.

Then the space of n unordered points in P^1 is the n-fold symmetric product of the projective line, which is well-known to be projective n-space P^n. In terms of linear systems, n unordered points on P^1 is the projective space of $H^0(P^1; \mathcal{O}(n))$.

The double covering map from E to P^1 is given by the Weierstrass p-function, and the set of $2n$ points in E invariant under $e \mapsto -e$ is the zeros of a polynomial of degree at most n in p. This polynomial has a pole only at p_0 and that pole has order at most $2n$.

Once again one can make a spectral covering construction. We define T_{Symp} as the space of pairs $(f \in |\mathcal{O}_E(2np_0) f(-x) = -f(x), \pm e)$ where $\pm e$ is in the support of the zero locus of f. Then over $T \times E$ there is a two plane bundle with a non-degenerate skew symmetric pairing. The pushforward of this bundle to the projective space of even functions on E with pole less than $2np_0$ times E is then a family of symplectic bundles over E.

It turns out that these are the only two families of simply connected groups for which a direct construction like this, relating the moduli space to the projective space of a linear series can be made. For the other groups the coarse moduli space is not a projective space, but rather a space of a type which is a slight generalization called a weighted projective space. For the groups $SO(n)$, which of course are not simply connected, it is possible to make a similar construction producing a projective space, but it is somewhat delicate and I shall not give it here.

2.8 Flat $SU(n)$-connections

Let us switch gears now to state a general result linking principal holomorphic vector bundles to $SU(n)$-bundles equipped with a flat connection. This is a variant of the famous Narasimhan-Seshadri theorem.

Suppose that $W \to E$ is an $SU(n)$-bundle equipped with a connection A. Consider the complex vector bundle associated to the defining n-dimensional representation: $W_{\mathbf{C}} = W \times_{SU(n)} \mathbf{C}^n$. Let $d_A \colon \Omega^0(E; W_{\mathbf{C}}) \to \Omega^1(E; W_{\mathbf{C}})$ be the covariant derivative determined by the connection A. We can take the $(0,1)$-part of the covariant derivative $\overline{\partial}_A \colon \Omega^0(E; W_{\mathbf{C}}) \to \Omega^{0,1}(E; W_{\mathbf{C}})$. Since the base is a curve, for dimension reasons the square of this operator vanishes, and hence it defines a holomorphic structure on $W_{\mathbf{C}}$. It is a standard argument to see that this bundle is semi-stable, and in fact is a sum of stable bundles, i.e. of line bundles of degree zero. The Narasimhan-Seshadri result is a converse to this computation.

Theorem 2.8.1 *Let V be a semi-stable holomorphic vector bundle of rank n with trivial determinant over an elliptic curve E. Then there is an $SU(n)$-bundle $W \to E$ and a flat $SU(n)$-connection on W such that the induced holomorphic structure on the bundle $W_{\mathbf{C}}$ is S-equivalent to V. This bundle and flat connection are unique up to isomorphism. Thus, the space of isomorphism classes of flat $SU(n)$-bundles is identified with the moduli space of S-equivalence classes of semi-stable holomorphic vector bundles of rank n with trivial determinant.*

Remarks: (1) As we have already pointed out the holomorphic bundles produced by this construction are sums of line bundles of degree zero. Generically, of course, these are the unique representatives in their S-equivalence class, but when two or more of the line bundles coincide there are other isomorphism classes in the same S-equivalence class.
(2) This theorem is usually stated for curves of genus at least 2. Over such curves there are stable (as opposed to properly semi-stable) vector bundles. In this context, the notion of S-equivalence is not necessary (or more precisely it coincides with isomorphism). The result is that associating to a flat $SU(n)$-bundle its associated holomorphic vector bundle determines an isomorphism between the space of the space of conjugacy classes of irreducible representations of $\pi_1(C) \to SU(n)$ and the space of stable holomorphic vector bundles of rank n and trivial determinant. In our case, when the base is a curve of genus one, there are now irreducible representations (for $n > 1$) reflecting the fact that there are no stable vector bundles with trivial determinant. In our case we have an equivalence between the S-equivalence classes of properly semi-stable bundles.
(3) A flat connection on an $SU(n)$-bundle over E is given by homomorphisms $\pi_1(E) \to SU(n)$ up to conjugation. Of course, choosing a basis for $H_1(E)$, or equivalently choosing a pair of one-cycles on E which intersect transversely in a single point, with $+1$ interesection at that point, identifies $\pi_1(E)$ with a free abelian group on two generators. A homomorphism of this group into $SU(n)$ is then simply a pair of commuting elements in $SU(n)$. We consider two pairs as equivalent if they are conjugate by a single element of $SU(n)$. As is well-known, two commuting elements in $SU(n)$ are simultaneously conjugate into the maximal torus T (the diagonal matrices) of $SU(n)$. Thus, we can assume that our elements lie in this maximal torus. The only conjugation remaining is simultaneous Weyl conjugation. Thus, the moduli space of homomorphisms of $\pi_1(E) \to SU(n)$ up to conjugation is identified with

$(T \times T)/W$. (Notice that in this description we have ignored the complex structure.) This generalizes the picture we established in the last lecture for flat $SU(n)$-bundles over S^1. Still we have one more step to complete this picture. Before we identified T/W with the alcove for the affine Weyl group action on $\mathfrak{t}$. We have not yet identified $(T \times T)/W$.

Let us give another description of $(T \times T)/W$ in a way that will keep track of the complex structure. We write $E = \mathbf{C}/\pi_1(E)$ and $T = \mathfrak{t}/\Lambda$ where $\mathfrak{t} \subset \mathbf{R}^n$ is the subspace of points whose coordinates sum to zero, and where the coroot lattice Λ is the intersection of this subspace with the integral lattice. A homomorphism $\pi_1(E) \to T$ dualizes under Pontrjagin duality to a homomorphism $\mathrm{Hom}(T, S^1) \to \mathrm{Hom}(\pi_1(E), S^1)$. The first group is identified with the dual Λ^* to the lattice Λ and the second is identified with the dual curve E^* to E. Since we have chosen a point p_0 on E we have identified E and E^* holomorphically. Thus, the Pontrjagin dual of the map $\pi_1(E) \to T$ is a map $\Lambda^* \to E$, or equivalently an element of $\Lambda \otimes E$. We have the exact sequence

$$0 \to \Lambda \to \mathbf{Z}^n \to \mathbf{Z} \to 0$$

where the last map is the sum of the coordinates. Tensoring with E yields

$$0 \to \Lambda \otimes E \to \times_n E \to E$$

where again the last map is the sum of the coordinates. Thus, we see that $\Lambda \otimes E$ is identified with the subset $(e_1, \ldots, e_n) \in \times_n E$ of points which sum to zero. Following through the action of the Weyl group, which is the symmetric group on n letters acting in the obvious way on $\Lambda \subset \mathbf{Z}^n$, we see that the Weyl action on $\mathrm{Hom}(\pi_1(E), T)$ becomes the permutation action on the space on n points summing to zero. Thus, we see a direct isomorphism

$$\mathrm{Hom}(\pi_1(E), T)/W \to (\Lambda \otimes E)/W$$

which realizes the Narasimhan-Seshadri theorem. Notice that the complex structure on E induces a complex structure on $\Lambda \otimes E$ which is clearly invariant under the Weyl action. Thus, this structure descends to a complex structure on $(\Lambda \otimes E)/W$, and hence determines a holomorphic structure on the moduli space. This structure agrees with the usual functorial one of the coarse moduli space. Notice that since, as we have already seen by different methods, the quotient is a projective space, in the end, the complex structure is independent of the complex structure on E.

2.9 Flat G-bundles and holomorphic $G_{\mathbf{C}}$-bundles

The above result for vector bundles generalizes to an arbitrary complex semi-simple group. Let $G_{\mathbf{C}}$ be a complex semi-simple group with compact form G. Suppose that $W \to E$ is a principal G-bundle equipped with a flat G-connection A. We take an open covering of E by contractible open sets $\{U_i\}$. Then for each i, there is a trivialization of $W|U_i$ in which A is the product connection. This induces a trivialization of $W \times_G G_{\mathbf{C}}|U_i$. The overlap functions on $g_{ij}\colon U_i \cap U_j \to G$ in the given trivializations for W are locally constant and hence holomorphic functions when viewed as maps of $U_i \cap U_j$ into $G \times_G G_{\mathbf{C}}$. Thus, we have produced a holomorphic bundle structure. (A better argument shows that any G-connection produces a holomorphic bundle structure on the associated $G_{\mathbf{C}}$-bundle.)

Definition 2.9.1 A holomorphic $G_{\mathbf{C}}$-bundle over E is *semi-stable* if its adjoint bundle is semi-stable. We say that two semi-stable $G_{\mathbf{C}}$-bundles V_1, V_2 on E are S-equivalent if there is a connected holomorphic family of semi-stable $G_{\mathbf{C}}$-bundles on E containing bundles isomorphic to each of V_1 and V_2.

Here is the general version of the Narasimhan-Seshadri result [8] for holomorphic principal $G_{\mathbf{C}}$-bundles over an elliptic curve.

Theorem 2.9.2 *Let E be an elliptic curve and let $G_{\mathbf{C}}$ be a complex semi-simple group (not necessarily simply connected). Let $V \to E$ be a semi-stable holomorphic $G_{\mathbf{C}}$-bundle. Then there is a flat G-bundle $W \to E$ such that the induced holomorphic $G_{\mathbf{C}}$-bundle structure on $W \times_G G_{\mathbf{C}}$ is S-equivalent to V. This flat G-bundle is uniquely determined up to isomorphism.*

Once again this result is usually stated for smooth curves of genus at least two and establishes an isomorphism between the space of conjugacy classes of irreducible representations of $\pi_1(C)$ into G and the space of isomorphism classes of stable $G_{\mathbf{C}}$-bundles. But over an elliptic curve there are no stable $G_{\mathbf{C}}$-bundles (and no irreducible representations of $\pi_1(E)$ into G) for any semi-simple group G, and we must consider semi-stable bundles. As in the case of vector bundles, we are then forced to work with the weaker equivalence relation of S-equivalence instead of isomorphism.

We can examine flat G-bundles analogously to the way we did when G is $SU(n)$. First, it is a classical result [2] that in a simple connected Lie group

G any pair of commuting elements can be conjugated into the maximal torus T, and any two pairs of elements in T are simultaneously conjugate if and only if they are conjugate by a Weyl element. Thus:

Theorem 2.9.3 *Let G be a compact simply connected semi-simple group with maximal torus T, Weyl group W and coroot lattice $\Lambda \subset \mathfrak{t}$. Then, the space of isomorphism classes of G-bundles with flat connections is identified with $(T \times T)/W$. By Pontrjagen duality this space is identified with $(\Lambda \otimes E)/W$ where W acts trivially on E and in the natural fashion on Λ.*

We have already unraveled all this for $SU(n)$. Let us see what it says in the case of Sympl$(2n)$. In this case the coroot lattice Λ is the integral lattice in $\mathbf{R}^n$ and the Weyl group is the group $(\pm 1)^n \rtimes S_n$ with the ± 1's acting as sign changes of the various coordinates and S_n permuting the coordinates. Thus, $\Lambda \otimes E$ is identified with $\times_n E$ and the action of the Weyl group is by ± 1 in each factor and permutations of the factors. The quotient $\times_n E/(\pm 1)^n$ is $\times_n \mathbf{P}^1$ and the symmetric group acts on this to produce a quotient naturally identified with $\mathbf{P}^n$. This recaptures what we saw directly in terms of holomorphic bundles.

Theorem 2.9.3 does not hold for non-simply connected groups. The reason is that we cannot simultaneously conjugate a pair of commuting elements in a non-simply connected group into a maximal torus. For example, if $G = SO(3)$ then rotations by π radians in two perpendicular planes commute but cannot be simultaneously conjugated into a maximal torus (a circle) of $SO(3)$. The reason is that if we lift these elements in any manner to the double covering $SU(2)$, then they generate a quaternion group of order 8, i.e., the commutation of the lifts in $SU(2)$ is the non-trivial central element of $SU(2)$. If we could put both elements in a maximal torus of $SO(3)$, then they would lift to elements in a maximal torus of $SU(2)$, and hence there would be lifts which commuted. A similar phenomenon occurs in any non-simply connected group G/C. It is an interesting problem to determine the dimension of the space of representations of $\pi_1(E) \to G/C$ which produce bundles of a given nontrivial topological type. There is a purely lattice theoretic description of conjugacy classes of commuting elements in a non-simply connected compact group, and hence using the Narasimhan-Seshdari result, a lattice-theoretic description of the coarse moduli space of bundles over a non-simply connected semi-simple complex group. This description is quite interesting but much more complicated.

2.9.1 Looijenga's theorem

We have seen that in the cases of $G = SU(n)$ and $G = \mathrm{Sympl}(2n)$ the space of S-equivalence classes of semi-stable principal $G_{\mathbf{C}}$-bundles or equivalently the space of conjugacy classes of representations of $\pi_1(E) \to G$ is a projective space. In fact, in each case we explicitly identified the moduli space with the projective space of a linear system either on the elliptic curve or on the P^1 quotient of the elliptic curve by ± 1. We now give a theorem which determines the nature of these moduli spaces for an arbitrary simply connected and simple group G.

First, let us recall the notion of a weighted projective space. Suppose that V is a k-dimensional complex vector space with a linear action of $\mathbf{C}^*$. Of course, this action can be diagonalized in an appropriate basis of V and hence the action is completely determined up to isomorphism by k characters on $\mathbf{C}^*$. Any character of $\mathbf{C}^*$ is automatically of the form $\lambda \mapsto \lambda^r$ for some integer r, and in this fashion the characters of $\mathbf{C}^*$ are identified with the integers. The characters arising in the action on V are called the weights of the action. If all the weights are non-zero and have the same sign, let us say positive, then we say that the action is an action with positive weights. In this case, there is a nice compact quotient space: $V - \{0\}/\mathbf{C}^*$ which is called a *weighted projective space*. For any set $(g_0, g_1, \ldots, g_k)$ of positive integers, the symbol $\mathbf{P}(g_0, g_1, \ldots, g_k)$ denotes the quotient of the action of $\mathbf{C}^*$ with weights $(g_0, g_1, \ldots, g_k)$. This quotient space is compact and of dimension k. Indeed, is finitely covered in a ramified fashion by an ordinary projective space $\mathbf{P}^k$. A weighted projective space is not in general a smooth complex variety, since the action of $\mathbf{C}^*$ has finite cyclic isotropy groups along certain subspaces. Rather, the quotient space has cyclic orbifold-type singularities. (The quotient space is locally isomorphic to the quotient of $\mathbf{C}^k$ by a finite cyclic group.)

Theorem 2.9.4 [6,7] *Let G be a compact, simple, simply connected group and let E be an elliptic curve. Then the space of conjugacy classes of homomorphisms $\pi_1(E) \to G$ has a natural complex structure and with this structure is isomorphic to a weighted projective space $\mathbf{P}(1, g_1, \ldots, g_r)$ where $g_1, \ldots, g_r$ are the coefficients that occur when the coroot dual to the highest root of G is expressed as a linear combination of the coroots dual to the simple roots.*

As we have seen, the space of conjugacy classes of homomorphisms $\pi_1(E) \to G$ is identified with $(\Lambda \otimes E)/W$. Since Λ is abstractly a free abelian

group of rank r, $\Lambda \otimes E$ is $\times_r E$ and hence has a natural complex structure inherited from that of E. Clearly, the Weyl action is holomorphic, and thus there is a possibly singular complex structure on the quotient space. This is the one referred to in Looijenga's theorem.

2.10 The coarse moduli space for semi-stable holomorphic $G_{\mathbf{C}}$-bundles

As we explained in the case of vector bundles, it is not enough simply to find the set of S-equivalence classes, one would like to identify a coarse moduli space (one has to worry whether or not such a coarse moduli space even exists). As a first step in constructing the coarse moduli space for semi-stable holomorphic $G_{\mathbf{C}}$-bundles, we claim that there is a holomorphic family of $G_{\mathbf{C}}$-bundles over E parametrized by $\Lambda \otimes E$. Actually, this family is a holomorphic family of $T_{\mathbf{C}}$-bundles over E. To construct this family we choose an integral basis for Λ, hence identifying it with $\mathbf{Z}^r$ and hence identifying $\Lambda \otimes E$ with $\times_r E$. This also identifies the complexification $T_{\mathbf{C}}$ of the maximal torus T of G with a product $\times_r \mathbf{C}^*$. Then we let $\mathcal{P} \to E \times E$ be the Poincaré line bundle $\mathcal{O}(\Delta - E \times \{p_0\})$ (where Δ is the divisor given by the diagonal embedding of E). This is a family of line bundles of degree zero over the second factor parametrized by the first factor. Over $(\times_r E) \times E$ we form $\oplus_{i=1}^r p_i^* \mathcal{P}$ where $p_i \colon (\times_r E) \times E \to E \times E$ is given by the product of projection onto the ith-component in the first factor and the identity in the second factor. This sum of line bundles is then equivalent to a family of principal $\times_r \mathbf{C}^*$-bundles, and though our identification, with a family of $T_{\mathbf{C}}$-bundles. It is easy to trace through the identifications and see that the resulting family of $T_{\mathbf{C}}$-bundles is independent of the choice of basis for Λ.

The Weyl group W acts, as we have already used several times, in the obvious way on the parameter space $\Lambda \otimes E$. It also acts as outer automorphism of the $T_{\mathbf{C}}$ and hence changes one $T_{\mathbf{C}}$-principal bundle into a different one. The family is equivariant under these actions: the $T_{\mathbf{C}}$-bundle parametrized by $\lambda \otimes e$ is transformed by $w \in W$ acting on the set of $T_{\mathbf{C}}$-bundles to the $T_{\mathbf{C}}$-bundle parametrized by $w(\lambda) \otimes e$. Thus, when we extend the structure group from $T_{\mathbf{C}}$ to $G_{\mathbf{C}}$ the bundles parametrized by points in the same Weyl orbit are isomorphic. Thus, our family of $G_{\mathbf{C}}$-bundles is equivariant under the Weyl action. Consequently, if there is a coarse moduli space $\mathcal{M}_{G_{\mathbf{C}}}$ for S-equivalence classes of semi-stable holomorphic $G_{\mathbf{C}}$-bundles over E, then we obtain a Weyl invariant holomorphic mapping $\Lambda \otimes E \to \mathcal{M}_{G_{\mathbf{C}}}$, and

hence holomorphic mapping $(\Lambda \otimes E)/W \to \mathcal{M}_{G_{\mathbf{C}}}$. Since the set of points of $(\Lambda \otimes E)/W$ are identified with the set of S-equivalence classes of such bundles, this map would then be a bijection. In fact there is such a moduli space (as can be established by rather general algebro-geometric arguments) and it is $(\Lambda \otimes E)/W$. I will not establish this here, but I will assume it in what follows. The precise statement is:

Theorem 2.10.1 *The set of points of $(\Lambda \otimes E)/W$ is naturally identified with the set of S-equivalence classes of semi-stable $G_{\mathbf{C}}$-bundles over E. If $W \to E \times X$ is a holomorphic family of semi-stable $G_{\mathbf{C}}$-bundles over E parametrized by X, then the function $X \to (\Lambda \otimes E)/W$ induced by associating to each $x \in X$ the point of $(\Lambda \otimes E)/W$ identified with the S-equivalence class of $W|E \times \{x\}$ is a holomorphic mapping. This makes $(\Lambda \otimes E)/W$ the coarse moduli space for S-equivalence classes of semi-stable $G_{\mathbf{C}}$-bundles over E.*

This completes the problem of understanding semi-stable holomorphic $G_{\mathbf{C}}$-bundles over E. There is a coarse moduli space which is $(\Lambda \otimes E)/W$ with the identification of its points with bundles as given above. Furthermore, by Looijenga's theorem, this complex space is a weighted projective space with positive weights given by the coefficients of the simple coroots in the linear combination which is the coroot dual to the highest root. The only cases when this weighted projective space is in fact an honest projective space is when all the weights are 1 and this occurs only for the groups $SU(n)$ of A-type and the groups $\mathrm{Sympl}(2n)$ of C-type. In these two cases we directly identified with projective space as being associated with an appropriate linear series.

In the next lecture, we will give a construction which will describe the other moduli spaces in terms of a $\mathbf{C}^*$-action on an affine space. In the two special cases given above this affine space will be a linear space and the action will be the usual $\mathbf{C}^*$-action hence producing a quotient projective space. In the other cases, we will show that the affine action can be linearized to produce a quotient which is a weighted projective space. This will provide a proof of Looijenga's theorem, different from his original proof.

2.11 Exercises:

1. Show that if V, W are vector bundles over a smooth curve and that if $W \to V$ is a holomorphic map which is one-to-one on the generic fiber, then there is a subbundle $\hat{W} \subset V$ which contains the image of W and so that

$\hat{W}/W$ is a sky-scraper sheaf. Show that the degree of $\hat{W}$ is the degree of W plus the total length of $\hat{W}/W$.

2. Let E be a smooth projective curve over the complex numbers of genus one. Show that the universal covering of E is analytically isomorphic to $\mathbf{C}$ and that the fundamental group of E is identified with a lattice $\Lambda \subset \mathbf{C}$. Show that fixing a point $p_0 \in E$ there is a holomorphic group law on E which is abelian and for which p_0 is the origin. Show this group law is unique given p_0. Using the Weierstrass p-function associated with this lattice show that E can be embedded as a cubic curve in $\mathbf{P}^2$ with equation of the form

$$y^2 = 4x^3 + g_2 x + g_3$$

for appropriate constants g_2, g_3.

3. Using the RR theorem show that if V is a semi-stable vector bundle of positive degree over E, then $H^1(E;V) = 0$ and $H^0(E;V)$ has rank equal to the degree of V. Formulate and prove the corresponding result for semi-stable vector bundles of negative degree.

4. Show that there is a unique non-trivial extension I_2 of $\mathcal{O}_E$ by $\mathcal{O}_E$. Show more generally that there is a unique bundle I_r of rank r which is an iterated extension of $\mathcal{O}_E$ where each extension is non-trivial. Show that $H^1(E;I_r)$ is rank one. The iterated extensions give an increasing filtration $|calF_*$ of I_r by subbundles so that for each $s \leq r$, we have $\mathcal{F}_s/\mathcal{F}_{s-1} = \mathcal{O}_E$. Show that this filtration is stable under any automorphism of I_r and is preserved under any endomorphism of I_r.

5. Show that if V is a vector bundle over any base and if every nonzero section of V vanishes nowhere then the union of the images of the sections of V produce a trivial subbundle of V with torsion-free cokernel. In particular, in this case the number of linearly independent sections is at most the rank of V and if it is equal to the rank of V, then V is a trivial bundle.

6. Show that there is a rank two vector bundle over $E \times H^1(E;\mathcal{O}_E)$ whose restriction to $E \times \{0\}$ is $\mathcal{O}_E \oplus \mathcal{O}_E$ and whose restriction to any other fiber $E \times \{x\}$, $x \neq 0$ is isomorphic to I_2.

7. By a coarse moduli space of equivalence classes of bundles of a certain type we mean the following: we have a reduced analytic space X and a bijection between the points of X and the equivalence classes of the bundles in question. Furthermore, if $V \to E \times Y$ is a holomorphic bundle such that the restriction V_y of V to each slice $E \times \{y\}$ is of the type under consideration, then the map $Y \to X$ defined by sending y to the point of X corresponding to the equivalence class of V_y is a holomorphic mapping. Show that if a coarse

moduli space exists, then it is unique up to unique isomorphism. Show that there cannot be a coarse moduli space for isomorphism classes of semi-stable bundles over E.

8. Show that $H^0(E; \mathcal{O}(q-p_0) \otimes \mathcal{O}(p_0)$ is one-dimensional and that any non-zero section of this bundle vanishes to order one at q. Show that $H^0(E; I_r \otimes \mathcal{O}(q-p_0) \otimes \mathcal{O}(p_0))$ has rank r and that every section of this bundle vanishes to order one at q. Thus, the determinant map for this bundle has a zero of order r at q.

9. Show that points $(e_1, \ldots, e_r)$ in E are the zeros of a meromorphic function $f: E \to \mathbf{P}^1$ with a pole only at p_0 if and only if $\sum_i e_i = 0$ in the group law of E.

10. Let E be embedded in $\mathbf{P}^2$ so that it is given by an equation in Weierstrass form:

$$y^2 = 4x^3 + g_2 x + g_3.$$

Show that any meromorphic function on E with pole only at infinity in this affine model is a polynomial expression in x and y. Show that a meromorphic function with pole only at infinity which is invariant under $e \mapsto -e$ is a polynomial expression in x.

11. Let $g: T \to |np_0|$ be the n-sheeted ramified covering constructed in Section 2.6. Let $\mathcal{L}$ be the line bundle over $T \times E$ obtained by pulling back the Poincaré line bundle over $E \times E$. Show that if $(e_1, \ldots, e_n)$ are distinct points of E then the restriction of $(g \times \mathrm{Id})_* \mathcal{L}$ to $\{e_1, \ldots, e_n\} \times E$ is isomorphic to $\oplus_i \mathcal{O}_E(e_i - p_0)$. Show that if $e_1 = e_2$ but otherwise the e_i are distinct then $(g \times \mathrm{Id})_*(\mathcal{L}$ restricted to $\{e_1, \ldots, e_n\} \times E$ is isomorphic to $\mathcal{O}_E(e_1 - p_0) \otimes I_2 \oplus_{i=3}^n \mathcal{O}(e_i - p_0)$

12. Show that the Pontrjagen dual of a homomorphism $\pi_1(E) \to T$ is a homomorphism $\Lambda \to \mathrm{Hom}(\pi_1(E), S^1)$. Show that the choice of an origin p_0 allows us to identify E with $\mathrm{Hom}(\pi_1(E), S^1)$.

13. Show that the quotient of E by $e \cong -e$ is $\mathbf{P}^1$. Show that the quotient of $(\mathbf{P}^1)^n$ under the action of the symmetric group on n letters is $\mathbf{P}^n$.

14. Show that a weighted projective space with positive weights is a compact complex variety. Show that it is finitely covered by an ordinary projective space. Show that in general these varieties are singular, but that their singularities are modeled by quotients of finite linear group actions on a vector space.

3 The Parabolic Construction

In this section we are going to give a completely different construction of semi-stable bundles over an elliptic curve. We begin first with the case of vector bundles of degree zero.

3.1 The parabolic construction for vector bundles

Lemma 3.1.1 *Let E be an elliptic curve with p_0 as origin. Then for each integer $d \geq 1$ there is, up to isomorphism, a unique vector W_d over E with the following properties:*

1. $\mathrm{rank}(W_d) = d$.
2. $\det(W_d) = \mathcal{O}(p_0)$.
3. *W_d is stable.*

Furthermore, $H^0(E; W_d)$ is of dimension one.

Proof. The proof is by induction on d. If $d = 1$, then it is clear that there is exactly one bundle, up to isomorphism, which satisfies the first and second item, namely $\mathcal{O}(p_0)$. Since this bundle is stable, we have established the existence and uniqueness when $d = 1$. Since $\mathcal{O}(-p_0)$ is of negative degree, it has no holomorphic sections and hence by RR, $H^1(E; \mathcal{O}(-p_0))$ is one-dimensional. By Serre duality, it follows that $H^0(E : \mathcal{O}(p_0))$ is also one-dimensional. This completes the proof of the result for $d = 1$.

Suppose inductively for $d \geq 2$, there is a unique W_{d-1} as required. By the inductive hypothesis and Serre duality we have $H^1(E; W_{d-1}^*)$ is one-dimensional. Thus, there is a unique non-trivial extension

$$0 \to \mathcal{O} \to W_d \to W_{d-1} \to 0.$$

Clearly, using the inductive hypothesis we see that W_d is of rank d and its determinant line bundle is isomorphic to $\mathcal{O}(p_0)$. In particular, the degree of W_d is one. Suppose that W_d has a destabilizing subbundle U. Then $\deg(U) > 0$. The intersection of U with $\mathcal{O}$ is a subsheaf of $\mathcal{O}$ and hence has non-positive degree. Thus, the image $p(U)$ of U in W_{d-1} has positive degree, and hence degree at least one. In particular, it is non-zero. Since the rank of U is at most the rank of W_{d-1}, it follows that $\mu(p(U)) \geq \mu(W_{d-1})$. Since

$p(U)$ is non-trivial, it follows that $p(U) = W_{d-1}$. Thus, the rank of U is either $d-1$ or d. If it is of rank d, the $U = W_d$ and does not destabilize. Thus, it must be of rank $d-1$. This means that $p: U \to W_{d-1}$ is an isomorphism and hence that U splits the exact sequence for W_d. This is a contradiction since the sequence was a nontrivial extension. This contradiction proves that W_d is stable.

A direct cohomology computation using the given exact sequence shows that $H^0(E; W_d)$ has rank one, completing the proof. □

Note (1): We have seen that W_d is given as successive extensions with all quotients except the last one being $\mathcal{O}$. The last one is $\mathcal{O}(p_0)$.

(2): There is a one-parameter family of stable bundles of rank d and degree 1. The determinant of such a bundle is of the form $\mathcal{O}(q)$ for some point $q \in E$ and the isomorphism class of the bundle is completely determined by q.

(3) The bundle W_d^* is then a stable bundle of degree minus one and determinant $\mathcal{O}(-p_0)$.

Corollary 3.1.2 *The automorphism group of W_d is $\mathbf{C}^*$.*

Proof. Suppose that $\varphi: W_d \to W_d$ is an endomorphism of W_d. Then we see that if φ is not an isomorphism, then $\deg(\mathrm{Ker}(\varphi)) = \deg(\mathrm{Coker}(\varphi))$. But by stability, $\deg(\mathrm{Ker}(\varphi)) \leq 0$ or $\varphi = 0$. Similarly, stability implies that $\deg(\mathrm{Coker}(\varphi)) \geq 1$ or φ is trivial. We conclude that either φ is an isomorphism or $\varphi = 0$. If φ is an endomorphism and λ is an eigenvalue of φ, then applying the previous to $\varphi - \lambda \cdot \mathrm{Id}$ we conclude that $\varphi = \lambda \cdot \mathrm{Id}$. This shows that all endomorphisms of W_d are multiplication by scalars. The result follows. □

Now we are ready to construct semi-stable vector bundles of rank n and degree zero.

Proposition 3.1.3 *Let E be an elliptic curve and $p_0 \in E$ an origin for the group law on E. Fix integers $d, n; 1 \leq d \leq n-1$. Let W_d and W_{n-d} be the bundles of the last lemma. Then any vector bundle V over E which fits in a non-trivial extension*

$$0 \to W_d^* \to V \to W_{n-d} \to 0$$

is semi-stable.

Proof. Clearly, any bundle V as above has rank n and degree zero. Suppose that $U \subset V$ is a destabilizing subbundle. Then $\deg(U) \geq 1$. Since W_d^* is stable, the intersection $U \cap W_d^*$ has negative degree (or is trivial). In either case, it is not all of U. This means that the image of U in W_{n-d} is nontrivial and has degree at least one. Since W_{n-d} is stable, this means the image of U in W_{n-d} is all of W_{n-d}. Hence, the degree of U is one more than the degree of $U \cap W_d^*$. Since $U \cap W_d^*$ has negative degree or is trivial, the only way that U can have positive degree is for $U \cap W_d^* = 0$, which would mean that U splits the sequence. This proves that all nontrivial extensions of the form given are semi-stable bundles. □

Lemma 3.1.4 *Suppose that V is a non-trivial extension of W_{n-d} by W_d^*. Then for any line bundle λ over E of degree zero, we have* $\mathrm{Hom}(V, \lambda)$ *has rank either zero or one.*

Proof. The bundle $Hom(W_{n-d}, \lambda)$ is of degree -1 and is semi-stable. Thus, it has no sections. Similarly, $Hom(W_d^*, \lambda)$ has a one-dimensional space of sections. From the long exact sequence

$$0 \to \mathrm{Hom}(W_{n-d}, \lambda) \to \mathrm{Hom}(W, \lambda) \to \mathrm{Hom}(W_d^*, \lambda) \to \cdots$$

we see that $\mathrm{Hom}(V, \lambda)$ has rank at most one. □

Corollary 3.1.5 *If V is a non-trivial extension of W_{n-d} by W_d^*, then V is isomorphic to a direct sum of bundles of the form $\mathcal{O}(q_i - p_0) \otimes I_{r_i}$ for distinct points $q_i \in E$.*

Proof. If V has two irreducible factors of the form $\mathcal{O}(q-p_0) \otimes I_{r_1}$ and $\mathcal{O}(q-p_0) \otimes I_{r_2}$ then $\mathrm{Hom}(V, \mathcal{O}(q - p_0)$ would be rank at least two, contradicting the previous result. □

3.2 Automorphism group of a vector bundle over an elliptic curve

The following is an easy direct exercise:

Lemma 3.2.1 *Let E be an elliptic curve and let L and M be non-isomorphic line bundles of degree zero over E. Then* $\mathrm{Hom}(L, M) = 0$. *Also,* $\mathrm{Hom}(L, L) = \mathbf{C}$.

Let V be a semi-stable bundle of degree zero over E. The *support* of V is the subset of points $e \in E$ at which some non-zero section of V vanishes. As we have seen in Atiyah's theorem, every semi-stable vector bundle of degree zero over E decomposes as a direct sum of bundles with support a single point.

Corollary 3.2.2 *Let q, q' be distinct points of E and let V_q and $V'_{q'}$ be vector bundles of degree zero over E supported at q and q' respectively. Then* $\mathrm{Hom}(V_q, V'_{q'}) = 0$.

Corollary 3.2.3 *Let V be a semi-stable bundle of degree zero over and elliptic curve E and let $V = \oplus_{q \in E} V_q$ be its decomposition into bundles with support at single points. Then* $\mathrm{Hom}(V, V) = \oplus_{q \in E} \mathrm{Hom}(V_q, v_q)$.

Now let us analyze the individual terms in this decomposition.

Lemma 3.2.4 *With I_r as in Lecture 1,* $\mathrm{Hom}(I_r, I_r)$ *is an abelian algebra* $\mathbf{C}[t]/(t^{r+1})$ *of dimension r.*

Proof. Recall that I_r comes equipped with a filtration $\mathcal{F}_0 \subset \mathcal{F}_1 \subset \cdots \subset \mathcal{F}_r = I_r$ with associated quotients $\mathcal{O}$. The quotient of $I_r/\mathcal{F}_s$ is identified with I_{r-s}. The first thing to prove is that this filtration is preserved under any endomorphism. It suffices by an straightforward inductive argument to show that $\mathcal{F}_1$ is preserved by an endomorphisms. But $\mathcal{F}_1$ is the image of the unique (up to scalar multiples) non-zero section of I_r.

Let $t\colon I_r \to I_r$ be the map of the form $I_r \to I_r/\mathcal{F}_1 = I_{r-1} = \mathcal{F}_{r-1} \subset I_r$. Clearly, the image of t^k is contained in $\mathcal{F}_{r-k}$ so that $t^{r+1} = 0$. We claim that every endomorphism of I_r is a linear combination of $\{1, t, t^2, \ldots, t^r\}$. Suppose that $f\colon I_r \to \mathcal{F}_{r-s} \subset I_r$. Then there is an induced mapping $I_r/I_{r-1} \to I_{r-s}/I_{r-s-1}$. Since both of these quotients are isomorphic to $\mathcal{O}$, this map and some multiple of the map between these quotients induced by t^s are equal. Subtracting this multiple of t^s from f we produce a map $I_r \to \mathcal{F}_{r-s-1}$. Continuing inductively proves the result. $\square$

Notice that an endomorphism is an automorphism if and only if its image is not contained in I_{r-1} if and only if it is not an element of the ideal generated by t.

From this description the following is easy to establish.

Corollary 3.2.5 *Let V be a semi-stable bundle of degree zero and rank n over an elliptic curve E. Then the dimension of the automorphism group is at least n. It is exactly n if and only if for each $q \in E$, the subbundle $V_q \subset V$ supported by q is of the form $\mathcal{O}(q - p_0) \otimes I_{r(q)}$ for some $r(q) \geq 0$.*

Proof. We decompose V into a direct sum of bundles V_i which themselves are indecomposable under direct sum. Thus, each V_i is of the form $\mathcal{O}(q - p_0) \otimes I_r$ for some $q \in E$ and some $r \geq 1$. Thus, by the previous result the automorphism group of V_i has dimension equal to the rank of V_i. Clearly, then the automorphism group of V preserving this decomposition has dimension equal to the rank of V. This will be the entire automorphism group if and only if $\mathrm{Hom}(V_i, V_j) = 0$ for all $i \neq j$. This will be the case if and only if the V_i have disjoint support. □

Note that if, in the above notation, two or more of the V_i have the same support then the automorphism group has dimension at least two more than the rank of V.

Definition 3.2.6 A semi-stable vector bundle over an elliptic curve whose automorphism group has dimension equal to the rank of the bundle is called a *regular* bundle.

We are now in a position to prove the main result along these lines.

Theorem 3.2.7 *Fix n, d with $1 \leq d < n$. A vector bundle V of rank n can be written as a non-trivial extension*

$$0 \to W_d^* \to V \to W_{n-d} \to 0$$

if and only if

1. *The determinant of V is trivial.*
2. *V is semi-stable.*
3. *V is regular.*

Proof. The first condition is obviously necessary and in Proposition 3.1.3 and Corollary 3.1.5 we established that the second and third are also necessary.

Suppose that V satisfies all the conditions. Condition (iii) says that $V = \oplus_{q\in E} V_q$ where each V_q is of the form $\mathcal{O}(q-p_0)\otimes I_{r(q)}$. We claim that there is a map $W_d \to \mathcal{O}(q-p_0)\otimes I_{r(q)}$ whose image is not contained in $\mathcal{O}(q-p_0)\otimes I_{r(q)-1}$. The reason for this is that $W_d \otimes \mathcal{O}(q-p_0)\otimes I_r$ is stable and has degree r. Thus, its space of sections has dimension exactly r. Applying this with $r = r(q)$ and $r = r(q)-1$ we see that there is a homomorphism $W_d \to \mathcal{O}(q-p_0)\otimes I_{r(q)}$ which does not factor through $\mathcal{O}(q-p_0)\otimes I_{r(q)-1}$.

Claim 3.2.8 *If $\mathcal{F}$ is a subsheaf of V of degree zero and if the projection of $\mathcal{F}$ onto each V_q is not contained in a proper subsheaf, then $\mathcal{F} = V$.*

Proof. Let W be the smallest subbundle of V containing $\mathcal{F}$. It has degree at least zero and is equal to $\mathcal{F}$ if and only if its degree is zero. By stability it has degree zero and hence is equal to $\mathcal{F}$. This shows that $\mathcal{F}$ is in fact a subbundle. Since there are no non-trivial maps between subbundles of V_q and $V_{q'}$ for $q \neq q'$, it follows that any subbundle of $\oplus V_q$ is in fact a direct sum of its intersections with the various V_q. If the image of the subbundle under projection to V_q is all of V_q then its intersection with V_q is all of V_q. The claim now follows. □

Let $W_d^* \to V$ be a map whose projection onto each V_q is not contained in any proper subbundle of V_q. Let us consider the image of this map. It is a subsheaf of V which is proper since $d < n$. This means it has degree at most zero. By the above it cannot be of degree zero. Thus, it is of degree at most -1. Hence the kernel of the map is either trivial or has degree at least 0. This latter possibility contradicts the stability of W_d^*. This shows that the map is an isomorphism onto its image; that is to say it is an embedding of $W_d^* \subset V$.

Next, let us consider the cokernel X. We have already seen that the cokernel is a bundle. Clearly, its determinant is $\mathcal{O}(p_0)$ and its rank is $n-d$. To show that it is W_{n-d} we need only see that it is stable. Suppose that $U \subset X$ is destabilizing. Then the degree of U is positive. Let $\tilde{U} \subset V$ be the preimage of U. It has degree one less than U and hence has degree at least zero. But since it contains the image of W_d, the previous claim implies that it is all of V, which implies that U is all of W_{n-d}, contradicting the assumption that U was destabilizing. □

Now we shall show that, given V, there is only one such extension with V in the middle, up to automorphisms of V.

Proposition 3.2.9 *Let V be a semi-stable rank n-vector bundle with trivial determinant. Suppose that the group of automorphisms of V has dimension n. Then V can be written as an extension*

$$0 \to W_d^* \to V \to W_{n-d} \to 0.$$

This extension is unique up to the action of the automorphism group of V.

Proof. Suppose we have an extension as above for V. First notice that if the image of W_d^* were contained in a proper subsheaf of degree zero of V, then this subsheaf would project into W_{n-d} destabilizing it. Thus, the only subsheaf of degree zero of V that contains the image of W_d^* is all of V. That is to say the image of W_d^* in each V_q is not contained in a lower filtration level, i.e. a proper subsheaf of degree zero. Now we need to show that all maps $W_d^* \to V_q$ which are not contained in a proper subsheaf of degree zero are equivalent under the action of the automorphisms of V_q. This is easily established from the structure of the automorphism sheaf of V_q given above. □

Corollary 3.2.10 *For any $1 \leq d < n$, the projective space of $H^1(E; Hom(W_{n-d}, W_d^*)) = H^1(E; W_{n-d}^* \otimes W_d^*)$ is identified with the space of isomorphism classes of regular semi-stable vector bundles of rank n and trivial determinant.*

Notice that it is not apparent, *a priori*, that for different $d < n$ that the above projective spaces can be identified in some natural manner.

The association to each regular semi-stable vector bundle of rank n and trivial determinant of its S-equivalence class then induces a holomorphic map from $\mathbf{P}(H^1(E; W_{n-d}^* \otimes W_d^*))$ to the coarse moduli space $\mathbf{P}(\mathcal{O}(np_0))$ of S-equivalence classes of such bundles. It follows immediately from Atiyah's theorem that each S-equivalence class contains a unique regular representative up to isomorphism, so that this map is bijective. Since it is a map between projective spaces it is in fact a holomorphic isomorphism. Thus, for any d, $1 \leq d < n$, we can view the projective space of $H^1(E; W_{n-d}^* \otimes W_d^*)$ as yet another description of the coarse moduli space of S-equivalence classes

of semi-stable rank n vector bundles of trivial determinant. Notice that the actual bundles produced by this construction are the same as those produced by the spectral covering construction since both are families of regular semi-stable bundles. On the other hand, using the Narasimhan-Seshadri result gives a different set of bundles – namely the direct sum of line bundles. These families agree generically, but differ along the codimension-one subvariety of the parameter space where two or more points come together. We have seen two constructions of holomorphic families of semi-stable vector bundles – the spectral covering construction and the parabolic construction and both create regular semi-stable bundles. It is not clear that the flat connection point of view can be carried out holomorphically in families (indeed it cannot). A hint to this fact is that it is producing different bundles and these do not in general fit together to make holomorphic families.

The reason for the name 'parabolic' will become clear after we extend to the general semi-simple group. Before we can give this generalization we need to discuss parabolic subgroups of a semi-simple group.

3.3 Parabolics in $G_{\mathbf{C}}$

Let $G_{\mathbf{C}}$ be a complex semi-simple group. A *Borel subgroup* of $G_{\mathbf{C}}$ is a connected complex subgroup whose Lie algebra contains a Cartan subalgebra (the Lie algebra of a maximal complex torus) together with the root spaces of all positive roots with respect to some basis of simple roots. All Borel subgroups in $G_{\mathbf{C}}$ are conjugate. By definition *a parabolic subgroup* is a connected complex proper subgroup of $G_{\mathbf{C}}$ that contains a Borel subgroup. Up to conjugation parabolic subgroups of $G_{\mathbf{C}}$ are classified by proper (and possibly empty) subdiagrams of the Dynkin diagram of G. Fix a maximal torus of $G_{\mathbf{C}}$ and a set of simple roots $\{\alpha_1, \ldots, \alpha_n\}$. A subdiagram is given simply by a subset $\{\alpha_1, \ldots, \alpha_r\}$ of the set of simple roots. The Lie algebra of the parabolic is the Cartan subalgebra tangent to the maximal torus, together with all the positive root spaces and all the root spaces associated with negative linear combinations of the $\{\alpha_1, \ldots, \alpha_r\}$. Thus, a Borel subgroup corresponds to the empty subdiagram. The full diagram gives $G_{\mathbf{C}}$ and hence is not a parabolic subgroup. Up to conjugation a parabolic P is contained in a parabolic P' if and only if the diagram corresponding to P is a subdiagram of that corresponding to P'. It follows that the maximal parabolic subgroups of $G_{\mathbf{C}}$ up to conjugation are in one-to-one correspondence with the subdiagrams of the Dynkin diagram of G obtained by deleting

a single vertex. This sets up a bijective correspondence between conjugacy classes of maximal parabolic subgroups of $G_{\mathbf{C}}$ and vertices of the Dynkin diagram, or equivalently with the set of simple roots for $G_{\mathbf{C}}$.

A parabolic subgroup P has a maximal unipotent subgroup U whose Lie algebra is the sum of the roots spaces of positive roots whose negatives are not roots of P. This subgroup is normal and its quotient is a reductive group called the Levi factor L of P. There is always a splitting so that P can be written as a semi-direct product $U \cdot L$. The derived subgroup of L is a semi-simple group whose Dynkin diagram is the subdiagram that determined P in the first place. A maximal torus of P is the original maximal torus of G. If P is a maximal parabolic then the character group of P is isomorphic to the integers, and the component of the identity of the center of P is $\mathbf{C}^*$, and any nontrivial character of P is non-trivial on the center. On the level of the Lie algebra the generating character of P is given by the weight dual to the coroot associated with the simple root α_i that is omitted from the Dynkin diagram in order to create the subdiagram that determines P. The value of this weight on any root α is simply the coefficient of α_i in the linear combination of the simple roots which is α. The root spaces of the Lie algebra of P are those ones which the character is non-negative, and the Lie algebra of the unipotent radical is the sum of the root spaces of roots on which this character is positive.

Example: The maximal parabolics of $SL_n(\mathbf{C})$ correspond to nodes of its diagram. Counting from one end we index these by integers $1 \leq d < n$. The parabolic subgroup corresponding to the integer d is the subgroup of block diagonal matrices with the lower left $d \times (n-d)$ block being zero. The Levi factor is the block diagonal matrices or equivalently pairs $(A, B) \in GL_d(\mathbf{C}) \times GL_{n-d}(\mathbf{C})$ with $\det(A) = \det(B)$. A vector bundle with structure reduced to this parabolic is simply a bundle with a rank d subbundle, or equivalently a bundle written as an extension of a rank d bundle by a rank $(n-d)$ bundle. In this case, the unipotent subgroup is a vector group $Hom(\mathbf{C}^{n-d}, \mathbf{C}^d)$.

3.4 The distinguished maximal parabolic

For all simple groups except those of A_n type we shall work with a distinguished maximal parabolic. It is described as follows: If the group is simply laced, then the node of the Dynkin diagram that is omitted is the trivalent one. If the group is non-simply laced, then either vertex which is omitted

is the long one connected to the multiple bond. It is easy to see that in all cases the Levi factor of this parabolic is written as the subgroup of a product of GL_{k_i} of matrices with a common determinant.

Examples: (i) For a group of type C_n, there is a unique long root. The Levi factor of the corresponding subgroup is $GL_n(\mathbf{C})$ and the unipotent radical is the self-adjoint maps $\mathbf{C}^n$ to its dual. In terms of complex symplectic $2n$-dimensional bundles, a reduction of structure group of V^{2n} to this parabolic means the choice of a self-annihilating n-dimensional subbundle W^n. This bundle has structure group the Levi factor $GL_n(\mathbf{C})$ of the parabolic. The quotient of the bundle by this subbundle is simply the dual bundle W_n^*. The extension class that determines the bundle and its symplectic form is an element in $H^1(E; \mathrm{SymHom}(W, W^*))$, where SymHom means the self-adjoint homomorphisms.

(ii) For a group of type B_n the distinguished maximal parabolic has Levi factor the subgroup of $GL_{n-1}(\mathbf{C}) \times GL_2(\mathbf{C})$ consisting of matrices of the same determinant. Let us consider the orthogonal group instead of the spin group. Then a reduction in the structure group of an orthogonal bundle V^{2n+1} to this parabolic is a self-annihilating subspace $W_1 \subset V^{2n+1}$ of dimension n. This produces a three term filtration $W_1 \subset W_2 \subset W_3$ where $W_2 = W_1^{\perp}$. Under the orthogonal pairing W_1 and W_3/W_2 are dually paired and W_2/W_1, which is three-dimensional, has a self-dual pairing and is identified with the adjoint of the bundle over the $GL_2(\mathbf{C})$-factor. The subbundle W_1 is the bundle over the $GL_n(\mathbf{C})$-factor of the Levi. There are two levels of extension data one giving the extension comparing $W_1 \subset W_2$ which is an element of $H^1(E; (W_2/W_1)^* \otimes W_1))$ and the other an extension class in $H^1(E; \mathrm{SkewHom}(W_3/W_2, W_1))$, where SkewHom refers to the anti-self adjoint mappings under the given pairing.

(iii) There is a similar description for D_{2n}. Here the Levi factor of the distinguished parabolic is the subgroup of matrices in $GL_{n-2}(\mathbf{C}) \times GL_2(\mathbf{C}) \times GL_2(\mathbf{C})$ consisting of matrices with a common determinant. This time a reduction of the structure group to P corresponds to a self-annihilating subspace W_1 of dimension $n-2$, it is the bundle over the $GL_{n-2}(\mathbf{C})$-factor of the Levi. The quotient W_2/W_1 is four-dimensional and self-dually paired. It is identified with the tensor product of the bundle over one of the $GL_2(\mathbf{C})$-factor with the inverse of the bundle over the other. Once again the cohomology describing the extension data is two step – one giving the extension which is $W_1 \subset W_2$ and the other a self-dual extension class for W_3/W_2 by W_1.

(iv) In the case of E_r, $r = 6, 7, 8$ the Levi factor is the subgroup of matrices in $GL_2(\mathbf{C}) \times GL_3(\mathbf{C}) \times GL_{r-3}(\mathbf{C})$ with the same determinants. It is more difficult to describe what a reduction of the structure group to this parabolic means since we have no standard linear representation to use. For E_6 there is the 27-dimensional representation, which would then have a three-step filtration with various properties.

In the case of $SL_n(\mathbf{C})$ we began with a particular, minimally unstable vector bundle $W_d \times W_{n-d}$ whose structure group has been reduced to the Levi factor of the parabolic subgroup. We then considered extensions

$$0 \to W_d^* \to V \to W_{n-d} \to 0.$$

These extensions have structure group the entire parabolic. They also have the property that modulo the unipotent subgroup they become the unstable bundle $W_d^* \oplus W_{n-d}$ with structure group reduced to the Levi factor L of this maximal parabolic.

3.5 The unipotent subgroup

Let us consider the unipotent subgroup of a maximal parabolic group. Fix a maximal torus T of $G_{\mathbf{C}}$ and a set of simple roots $\{\alpha_1, \ldots, \alpha_n\}$. Suppose that this parabolic is the one determined by deleting the simple root α_i. We begin with its Lie algebra. Consider the direct sum of all the root spaces $\mathfrak{g}^u$ associated with positive roots whose negatives are not roots of P. These are exactly the positive roots which, when expressed as a linear combination of the simple roots have a positive coefficient times α_i. Clearly, these roots form a subset which is closed under addition, in the sense that if the sum of two roots of this type is a root, then that root is also of this type. This means that the sum $\tilde{U}$ of the root spaces for these roots makes a Lie subalgebra of $\mathfrak{g}_{\mathbf{C}}$. Furthermore, there is an integer $k > 0$ such that any sum of at least k roots of this type is not a root. (The integer k can be taken to be the largest coefficient of α_i in any root of $\mathfrak{g}_{\mathbf{C}}$.) This means that the Lie algebra $\tilde{U}$ is in fact nilpotent of index of nilpotency at most k. It follows that the restriction of the exponential map to $\tilde{U}$ is a holomorphic isomorphism from $\tilde{U}$ to a unipotent subgroup $U \subset G_{\mathbf{C}}$. The dimension of this group is equal to the number of roots with positive coefficient on α_i. Furthermore, U is filtered by a chain of normal subgroups $\{1\} \subset U_k \subset U_{k-1} \subset \cdot \subset U_1$ where U_i is the unipotent subgroup whose Lie algebra is the root spaces of roots whose α_i-coefficient is at least i. Clearly, U_k is contained in the center of the

group and U_i/U_{i-1} is contained in the center of U/U_i. The entire structure of the unipotent group can be directly read off from the set of roots with positive coefficient on α_i together with the information about which sums of roots are roots.

Examples: (i) For $SL_n(\mathbf{C})$ and any maximal parabolic the filtration is trivial $U_1 = U; U_2 = \{1\}$. The reason is of course that all positive roots are linear combinations of the simple roots with coefficients $0, 1$ only. Thus, U is a vector group. It is $\mathrm{Hom}(\mathbf{C}^d, \mathbf{C}^{n-d})$.

(ii) For groups of type C_n and maximal parabolics obtained by deleting the vertex corresponding to the unique long root, again $U = U_1; U_2 = \{1\}$ (all roots have coefficient ± 1 or 0 on this simple root). The unipotent radical U is the vector group $\wedge^2 \mathbf{C}^n$. For all other maximal parabolics of groups of type C_n, the unipotent radical has a two-step filtration and is not abelian.

(iii) For groups of type B_n and for maximal parabolics obtained by deleting the simple root α_i which is long and which corresponds to a vertex of the double bond in the Dynkin diagram, the filtration is $\{1\} \subset U_2 \subset U_1 = U$. The dimension of U_2 is $(n-1)(n-2)/2$ and the dimension of U_1/U_2 is $2(n-1)$. The Lie bracket mapping $U_1/U_2 \otimes U_1/U_2 \to U_2$ is onto, so that the unipotent group is not a vector group, i.e., it is unipotent but not abelian.

(iv) For groups of type D_n and maximal parabolics obtained by deleting the simple root corresponding to the trivalent vertex, once again the filtration is of length 2: we have $\{1\} \subset U_2 \subset U_1 = U$. The dimension of U_2 is $(n-2)(n-3)/2$ and the dimension of U_1/U_2 is $4(n-2)$. Once again the bracket mapping $U_1/U_2 \otimes U_1/U_2 \to U_2$ is onto, and hence the group is non-abelian.

(iv) Once we leave the classical groups, the filtrations become more complicated. For E_6 the filtration of the unipotent subgroup of the distinguished maximal parabolic is $\{1\} \subset U_3 \subset U_2 \subset U_1$ where the dimension of U_3 is two, the dimension of U_2/U_3 is 9 and the dimension of U_1/U_2 is 18. For E_7 and the distinguished maximal parabolic, the filtration of the unipotent radical begins at U_4 which is three-dimensional and descends to U_1 with U_1/U_2 being 24 dimensional. For E_8 and the distinguished maximal parabolic, the filtration begins at U_6 which is five-dimensional and descends all the way to U_1 with U_1/U_2 being of dimension 30.

3.6 Unipotent cohomology

Fix a simple group $G_{\mathbf{C}}$ and a maximal parabolic subgroup P, and fix a splitting $P = U \cdot L$. Also, fix a holomorphic principal bundle $\eta_L \to E$ with structure the Levi factor L of P. We wish to study holomorphic bundles $\xi \to E$ with structure group P with a given isomorphism $\xi/U \to \eta_L$. Let us choose a covering of the elliptic curve by small analytic open subsets $\{U_i\}$. The bundle η_L is described by a cocycle $n_{ij}: U_i \cap U_j \to L$. A bundle ξ and isomorphism $\xi/U \to \eta_L$ is given by maps $u_{ij}: U_i \cap U_j \to U$ satisfying the cocycle condition:

$$u_{ij} n_{ij} u_{jk} n_{jk} = u_{ik} n_{ik}.$$

Since the $\{n_{ij}\}$ are already a cocycle, we can rewrite this condition as

$$u_{ij} u_{jk}^{n_{ij}} = u_{ik},$$

where $u^n = nun^{-1}$ for $u \in U$ and $n \in L$. This is the twisted cocycle condition associated with the bundle η_L and the action of L (by conjugation) on the unipotent subgroup U. A zero cochain is simply a collection of holomorphic maps $v_i: U_i \to U$. Varying a twisted cocycle $\{u_{ij}\}$ by replacing the coboundary of this zero cochain means replacing it by $v_i u_{ij} (v_j^{-1})^{n_{ij}}$.

The set of cocycles modulo the equivalence relation of coboundary makes a set, denoted $H^1(E; U(\eta_L))$. In fact, it is a pointed set since we have the trivial cocycle: $u_{ij} = 1$ for all i, j. In the case when U is abelian, associated to η_L and the action of L on U (which is linear), there is a vector bundle $U(\eta_L)$. The twisted cocycles modulo coboundaries are exactly the usual Cech cohomology of this vector bundle, $H^1(E; U(\eta_L))$, and hence this cohomology space is in fact a vector group. The general situation is not quite this nice. But since U is filtered by normal subgroups with the associated gradeds being vector groups, we can filter the twisted cohomology and the associated gradeds are naturally the usual cohomology of the vector bundles $H^1(E; (U_i/U_{i-1}(\eta_L))$. In this situation, the entire cohomology $H^1(E; U(\eta_L))$ can be given the structure on an affine space which has an origin, and which is filtered with associated gradeds being vector bundles.

The center of P is $\mathbf{C}^*$ (more precisely, the component of the identity of the center of P) and hence acts on U and on $H^1(E; U(\eta_L))$. This action preserves the origin, and the filtration and on each associated graded is a linear action of homogeneous weight. That weight is given by the index of that filtration level (weight i on U_i/U_{i-1}). It is a general theorem that since all these weights of the $\mathbf{C}^*$-action are positive, there is in fact an isomorphism

of this affine space with a vector space in such a way that the $\mathbf{C}^*$-action becomes linearized. In particular, the quotient of $\left(H^1(E;U(\eta_L)) - \{0\}\right)/\mathbf{C}^*$ is isomorphic as a projective variety to a weighted projective space. The dimension of the subprojective space of weight i is equal to one less that the dimension of $H^1(E;U_i/U_{i-1})$).

This is a fairly formal construction and it is not clear that it has anything to do with stable $G_{\mathbf{C}}$-bundles. Here is a theorem that tells us that using the distinguished maximal parabolic identified above and a special unstable bundle with structure group the Levi subgroup of this parabolic in fact leads to semi-stable $G_{\mathbf{C}}$-bundles. It is a generalization of what we have established directly for vector bundles.

Theorem 3.6.1 *Let $G_{\mathbf{C}}$ be a simply connected simple group and let $P \subset G_{\mathbf{C}}$ be the distinguished parabolic subgroup as above. Then the Levi factor of P is isomorphic to the $L \subset \prod_i GL_{n-i}$ consisting of all $\{A_i \in GL_{n_i}\}_i$ such that $\det(A_i) = \det(A_j)$ for all i,j. Let W_{n_i} be the unique stable bundle of rank n_i and determinant $\mathcal{O}(p_0)$. Then $\eta_L = \times_i W^*_{n_i}$ is naturally a holomorphic principal L-bundle over E. Every principal P-bundle which is obtained from a non-trivial cohomology class in $H^1(E;U(\eta_L))$ becomes semi-stable when extended to a $G_{\mathbf{C}}$-bundle. Cohomology classes in the same $\mathbf{C}^*$-orbit determine isomorphic $G_{\mathbf{C}}$-bundles. This sets up an isomorphism between $\left(H^1(E;U(\eta_L)) - \{0\}\right)/\mathbf{C}^*$ and the coarse moduli space of S-equivalence classes of semi-stable $G_{\mathbf{C}}$-bundles over E. Every $G_{\mathbf{C}}$-bundle constructed this way is regular in the sense that its $G_{\mathbf{C}}$-automorphism group has dimension equal to the rank of G, and any regular semi-stable $G_{\mathbf{C}}$-bundle arises from this construction. Any non-regular semi-stable $G_{\mathbf{C}}$-bundle has automorphism group of dimension at least two more than the rank of G.*

This result gives a different proof of Looijenga's theorem. It identifies the coarse moduli space as weighted projective space associated with a non-abelian cohomology space. It is easy to check given the information about the roots and their coefficients over the distinguished simple root that the weights of this weighted projective space are as given in Looijenga's theorem.

3.7 Exercises:

1. Suppose that we have a non-trivial extension

$$0 \to \mathcal{O} \to X \to W_{d-1} \to 0,$$

where W_{d-1} is as in the first lemma of this lecture. Show that $H^0(E;X)$ is one-dimensional and hence that $H^1(E;X^*)$ is also of dimension one.
2. Let V be a holomorphic vector bundle. Show that $\mathrm{Aut}(V)$ is a complex Lie group and that its Lie algebra is identified with $\mathrm{End}(V) = H^0(V \otimes V^*)$.
3. Show that if V_1 and V_2 are semi-stable bundles over E, then so is $V_1 \otimes V_2$. Compute the degree of $V_1 \otimes V_2$ in terms of the degrees and ranks of V_1 and V_2.
4. Prove Lemma 3.2.1 and Corollary 3.2.2.
5. Show that if V is a semi-stable vector bundle of rank n over E which is not regular, then the dimension of the automorphism group of V is at least $r+2$.
6. Let V_{q_i} be semi-stable vector bundles over E of degree zero and disjoint support. Show that any subbundle of degree zero in $\oplus V_{q_i}$ is in fact a direct sum of subbundles of the V_{q_i}.
7. Show that if any two homomorphisms $W_d^* \to I_r$ which have image not contained in I_{r-1} differ by an automorphism of I_r.
8. Show that a Borel subgroup of $G_{\mathbf{C}}$ is determined by a choice of a maximal torus for $G_{\mathbf{C}}$ and a choice of simple roots for that torus. Show all Borel subgroups of $G_{\mathbf{C}}$ are conjugate.
9. Up to conjugation, describe explicitly all parabolic subgroups of $SL_n(\mathbf{C})$.
10. Let $G_{\mathbf{C}}$ be a semi-simple group. Show that the character group of a maximal parabolic subgroup of $G_{\mathbf{C}}$ is isomorphic to $\mathbf{Z}$. Show that the center of a maximal parabolic subgroup of $G_{\mathbf{C}}$ is one-dimensional.
11. For E_6, E_7, E_8, G_2, F_4 work out the dimensions of the various filtration levels in the unipotent subgroups associated with the distinguished maximal parabolic subgroups.
12. Check that in the formula given for the action by a coboundary on a twisted cocylce that the resulting one-cochain is still a twisted cocycle.
13. For groups of type B_n and D_n and the distinguished parabolic and the given bundle η_L over the Levi factor, compute the cohomology vector spaces $H^1(E;U_2(\eta_L))$ and $H^1(E;U_1/U_2(\eta_L))$.

4 Bundles over Families of Elliptic Curves

In this lecture we will generalize the constructions for the case of vector bundles over an elliptic curve to vector bundles over families of elliptic curves.

4.1 Families of elliptic curves

The first thing that we need to do is to decide what we shall mean by a family of elliptic curves. The best choice for our context is a family of Weierstrass cubic curves. Recall that a single Weierstrass cubic is an equation of the form

$$y^2 = 4x^3 + g_2 x + g_3,$$

or written in homogeneous coordinates is given by:

$$zy^2 = 4x^2 + g_2 xz^2 + g_3 z^3.$$

This equation defines a cubic curve in the projective plane with homogeneous coordinates (x, y, z). The point at infinity, i.e., the point with homogeneous coordinates $(0, 1, 0)$ is always a smooth point of the curve. In the case when the curve is itself smooth, this point is taken to be the identity element of the group law on the curve. More generally, there are only two types of singular curves which can occur as Weierstrass cubics – a rational curve with a single node – which occurs when $\Delta(g_2, g_3) = 0$ where $\Delta(g_2, g_3) = g_2^3 + 27g_3^2$ is the discriminant, and the cubic cusp when $g_2 = g_3 = 0$. In each of these cases the subvariety of smooth points of the curve forms a group ($\mathbf{C}^*$ in the nodal case and $\mathbf{C}$ in the cuspidal case), and again we use the point at infinity as the origin of the group law on the subvariety of smooth points.

Now suppose that we wish to study a family of such cubic curves parametrized by a base B which we take to be a smooth variety. Then we fix a line bundle L over B. We interpret the variables x, y, z as follows: let $\mathcal{E}$ be the three-plane bundle $\mathcal{O}_B \oplus L^2 \oplus L^3$ over B; $z\colon \mathcal{E} \to \mathcal{O}_B$, $x\colon \mathcal{E} \to L^2$, and $y\colon \mathcal{E} \to L^3$ are the natural projections. Furthermore, g_2 is a global section of L^4 and g_3 is a global section of L^6. With these definitions

$$zy^2 - (4x^3 + g_2 xz^2 + g_3 z^3)$$

is a section of $\mathrm{Sym}^3(\mathcal{E}^*) \otimes L^6$. Its vanishing locus projectivizes to give a subvariety $Z \subset \mathbf{P}(\mathcal{E})$ over B, which fiber-by-fiber is the elliptic curve (possible singular) given by trivializing the bundle L over the point $b \in B$ in question

and viewing $g_2(b)$ and $g_3(b)$ as complex numbers so that the above cubic equation with values in L^6 becomes an ordinary cubic equation depending on b. While the actual equation associated to b will of course depend on the trivialization of $L|_{\{b\}}$, the homogeneous cubic curve it defines will be independent of this choice.

Thus, as long as the sections g_2 and g_3 are generic enough so as not to always lie in the discriminant locus, $Z \to B$ is an elliptic fibration (which by definition is a flat family of curves over B whose generic member is an elliptic curve). This family of elliptic curves comes equipped with a choice of base point, i.e., there is a given section of $Z \to B$. It is the section given by $\{z = x = 0\}$ or equivalently, by the section $[L^3] \in \mathbf{P}(\mathcal{O}_B \oplus L^2 \oplus L^3)$. (This is the globalization of the point $(0, 1, 0)$ in a single Weierstrass curve.) This does indeed define a section σ of $Z \to B$. The image of this section is always a smooth point of the fiber. If we use local fiberwise coordinates $(u = x/y, v = z/y)$ near this section, then the local equation is $v = 4u^3 + g_2uv^2 + g_3v^3$, and its gradient at the point $(0, 0)$ points in the direction of the v-axis. This means that along σ the surface Z is tangent to the u-axis. Since what we are calling the u-axis actually has coordinate x/y, these lines fit together to form the line bundle $L^2 \otimes (L^3)^{-1} = L^{-1}$, which then is the normal bundle of σ in Z. This bundle is also of course the relative tangent bundle of the fibers along the section σ. Since the tangent bundle of each fiber is trivialized, it follows that the pushforward, $\pi_* T_{\text{fibers}}$, is isomorphic to L^{-1}. Also important for us will be the relative dualizing sheaf. It is $\pi_* T^*_{\text{fibers}}$. (Of course, as I have presented it, we are working only at smooth fibers. But because the singular curves have sufficiently mild singularities the relative dualizing sheaf is still a line bundle, and in fact is the bundle L.)

We have proved:

Lemma 4.1.1 *Let ν_σ be the normal bundle of σ in Z. Let $\pi\colon Z \to B$ be the natural projection. Then $\nu_\sigma = \mathcal{O}_Z(\sigma)|_\sigma$ and $\pi_*(\mathcal{O}_Z(\sigma)|_\sigma) = \pi_*(\nu_\sigma) = L^{-1}$. The bundle L is the relative dualizing line bundle.*

N.B. The subvariety of B consisting of $b \in B$ for which the Weierstrass curve parametrized by b is singular, resp., a cuspidal curve, is a subvariety. For generic g_2 and g_3 the codimension of these subvarieties are one and two, respectively.

4.2 Globalization of the spectral covering construction

Having said how we shall replace our single elliptic curve by a family of elliptic curves with a section, we now turn to globalizing the vector bundle constructions. Our first attempt at globalizing the previous constructions would be to try to find the analogue for $\underbrace{(E \times \cdots \times E)}_{n-\text{times}}/S_n$. The obvious candidate is $\underbrace{(Z \times_B \cdots \times_B Z)}_{n-\text{times}}/S_n$. This works fine as long as Z is smooth over B but does not give a good result at the singular fibers. There is in fact a way to globalize this construction, at least across the nodes. It involves considering $Z_{\text{reg}} \times_B \cdots \times_B Z_{\text{reg}}$, where Z_{reg} is the open subvariety of points regular in their fibers, and then given an appropriate toroidal compactification at the nodal fibers. I shall not discuss this construction here.

There is however another way to view n points on E which sum to zero, up to permutation. Namely, as we have already seen, these points are naturally the points of the projective space $H^0(E; \mathcal{O}_E(np_0))$. Thus, a better way to globalize is to replace $\mathcal{O}(p_0)$ by $\mathcal{O}_Z(\sigma)$ and thus consider $R^0\pi_*(\mathcal{O}_Z(n\sigma))$. This is a vector bundle of rank n on B. Its associated projective space bundle is then a locally trivial $\mathbf{P}^{n-1}$ bundle over B. The fiber of this projective bundle over a point $b \in B$ is canonically identified with the projective bundle of the linear system $|np_0|$ on E.

As the next result shows, this pushed-forward bundle splits naturally as a sum of line bundles.

Claim 4.2.1 *The bundle $R^0\pi_*(\mathcal{O}_Z(n\sigma))$ is naturally split as a sum of line bundles: $\mathcal{O}_B \oplus L^{-2} \oplus L^{-3} \oplus \cdots \oplus L^{-n}$.*

Proof. By definition we are considering the bundle whose sections over an open subset $U \subset B$ are the analytic functions on $Z|_U$ with poles only along $\sigma \cap (Z|_U)$ and those being of order at most n. We have already at our disposal functions with this property: $1, x, x^2, \ldots, x^{[n/2]}, y, xy, \ldots, x^{[(n-3)/2]}y$. Given any function with this property over U, we can subtract (uniquely) a multiple of one of these basic functions, x^a or $x^a y$, so that the order of the pole is reduced by at least one. The multiple will have a coefficient which is a section of the line bundle L^{-2a} in the first case and L^{-2a+3} in the second. In this way we identify the sections of our vector bundle over U with expressions of the form

$$a_0 + a_1 x + \cdots + a_{[n/2]}x^{[n/2]} + b_0 y + \cdots + b_{[(n-3)/2]}x^{[(n-3)/2]}y.$$

The coefficient of x^a lies in L^{-a} and the coefficient of $x^a y$ lies in $L^{-(2a+3)}$. This identifies the space of sections with the sum $\mathcal{O}_B \oplus L^{-2} \oplus L^{-3} \oplus \cdots \oplus L^{-n}$. $\square$

Notice that a section of this n-plane bundle is then a family of S-equivalence classes of semi-stable bundles on the fibers of $Z//B$, but that it is not yet a vector bundle on Z. Nevertheless, the spectral covering construction generalizes to produce a vector bundle. Let $\mathbf{P}_n$ be the bundle of projective spaces associated to the vector bundle $R^0\pi_*(\mathcal{O}_Z(n\sigma))$. This is the bundle whose fiber over $b \in B$ is the projective space of the linear system $\mathcal{O}_{E_b}(np_0)$. Consider the natural map $\pi^*\pi_*\mathcal{O}_Z(n\sigma) \to \mathcal{O}_Z(n\sigma)$. It is surjective and we denote by $\mathcal{E}$ its kernel which is a vector bundle of rank $n-1$. Define $\mathcal{T} = \mathbf{P}(\mathcal{E})$. A point of $\pi^*\pi_*\mathcal{O}_Z(n\sigma)$ consists of an element $f \in |\mathcal{O}_{E_b}(n\sigma(b))|$ together with a point $z \in E_b$. The fiber $\mathcal{E}$ consists of all pairs for which $f(z) = 0$. The bundle $\mathcal{T}$ is a $\mathbf{P}^{n-2}$-bundle over Z whose fiber over any $z \in E_b$ is the projective space of the linear system $\mathcal{O}_{E_b}(n\sigma(b) - z)$ on E_b. The composition of the inclusion $\mathcal{T} \to \mathbf{P}_n \times_B Z$ followed by the projection onto $\mathbf{P}_n$ is a ramified n-sheeted covering denoted g, which fiber-by-fiber is the map we constructed before for a single elliptic curve.

Using this map we can construct a family of vector bundles over Z semi-stable on each fiber. Namely, we consider the pullback Δ to $\mathcal{T} \times_B Z$ of the diagonal $\Delta_0 \subset Z \times_B Z$. Then we have a line bundle

$$\mathcal{L} = \mathcal{O}_{\mathcal{T} \times_B Z}(\Delta - \mathcal{T} \times_B \sigma).$$

The pushforward $(g \times_B \mathrm{Id})_*(\mathcal{L})$ is a rank n vector bundle on Z which is regular semi-stable and of trivial determinant on each fiber. Analogous to our result for a single curve we have the following universal property for this construction.

Theorem 4.2.2 *Let $\mathcal{U} \to Z$ be a vector bundle which is regular, semi-stable with trivial determinant on each fiber of $Z//B$. Then associating to each $b \in B$ the class of $\mathcal{U}|E_b$ determines a section $s_A\colon B \to \mathbf{P}_n$. Let $\mathcal{T}_A$ be the pullback of $\mathcal{T} \to \mathbf{P}_n$ via this section. Then the natural projection $T_A \to B$ is an n-sheeted ramified covering. Let $\mathcal{L}_A$ be the pullback to $\mathcal{T}_A \times_B Z$ of the line bundle $\mathcal{L}$ over $\mathcal{T} \times_B Z$ by $s_A \times_B \mathrm{Id}$. Then there is a line bundle M over $\mathcal{T}_A$ such that $\mathcal{U}$ is isomorphic to $(g \times \mathrm{Id})_*(\mathcal{L}_A \otimes p_1^* M)$, where $p_1\colon \mathcal{T}_A \times_B Z \to \mathcal{T}_A$.*

Notice that there are in essence two ingredients in this construction: the first is a section A of $\mathbf{P}_n \to B$ and the second is a line bundle over the induced

ramified covering T_A of B. The section A is equivalent to the information of the isomorphism class of the bundle on each fiber of $Z//B$. The line bundle over T_A gives us the allowable twists of the bundle on Z which do not change the isomorphism class on each fiber.

This completes the spectral covering construction. It has the advantage that it produces all vector bundles over Z which are regular and semi-stable with trivial determinant on each fiber. Its main drawback is that it does not easily generalize to other simple groups. The construction that does generalize easily is the parabolic construction to which we turn now.

4.3 Globalization of the parabolic construction

It turns out that (except in the case of E_8-bundles and cuspidal fibers) that the parabolic construction of vector bundles globalizes in a natural way. The first step in establishing this is to globalize the bundles W_d which are an essential part of the construction, both for vector bundles and for more general principal G-bundles.

4.3.1 Globalization of the bundles W_d

We define inductively the global versions of the bundles W_d. The globalization of $W_1 = \mathcal{O}_E(p_0)$ is of course $\mathcal{W}_1 = \mathcal{O}_Z(\sigma)$, so that the way we have chosen to globalize curves has already given us a natural globalization of W_1. Clearly, the restriction of this line bundle to any fiber E of $Z//B$ is the bundle $\mathcal{O}_E(p_0)$. (Notice that even if the fiber is singular, p_0 is a smooth point of it, so that $\mathcal{O}_E(p_0)$ still makes sense as a line bundle.)

Claim 4.3.1 *There is, up to non-zero scalar multiples, a unique non-trivial extension*

$$0 \to \pi^* L \to \mathcal{X} \to \mathcal{W}_1 \to 0.$$

The restriction of $\mathcal{X}$ to any fiber is isomorphic to W_2 of that fiber.

Proof. Let us compute the global extension group $\mathrm{Ext}^1(\mathcal{O}_Z(\sigma), \pi^* L)$. Since both the terms are vector bundles, the extension group is identified with the cohomology group $H^1(Z; \mathcal{O}_Z(\sigma)^* \otimes \pi^* L)$. The local-to-global spectral sequence produces an exact sequence

$$0 \to H^1(B; \pi_*(\mathcal{O}_Z(\sigma)^* \otimes \pi^* L) \to H^1(Z; \mathcal{O}_Z(\sigma)^* \otimes \pi^* L)$$
$$\to H^0(B; R^1\pi_* \mathcal{O}_Z(\sigma)^* \otimes \pi^* L) \to H^2(B; \pi_* \mathcal{O}_Z(\sigma)^* \otimes \pi^* L) \to .$$

Since the restriction of $\mathcal{O}_Z(\sigma)^*$ to each fiber is semi-stable of negative degree, it follows that the first term and the fourth term are both zero, and hence we have an isomorphism

$$H^1(\mathcal{O}_Z(\sigma)^* \otimes \pi^* L) \to H^0(B; R^1(\pi_* \mathcal{O}_Z(\sigma) \otimes \pi^* L)) = H^0(B; R^1\pi_*(\mathcal{O}_Z(\sigma)) \otimes L).$$

But we have already seen that $R^1\pi_*(\mathcal{O}_Z(-\sigma)) = L^{-1}$, so that we are considering $H^0(B; (L^{-1} \otimes L) = H^0(B; \mathcal{O}_B) = \mathbf{C}$. Since any non-trivial section of this bundle is nonzero at each point, any non-trivial extension class has non-trivial restriction to each fiber and hence any non-trivial extension of the form $0 \to L \to \mathcal{W}_1 \to 0$ restricts to each fiber E_b to give a nontrivial restriction of W_1 by $\mathcal{O}_{E_b}$ and hence restricts to each fiber to give a bundle isomorphic to W_2 on that fiber. □

Now let us continue this construction. The following is easily established by induction.

Proposition 4.3.2 *For each integer $n \geq 1$ there is a bundle $\mathcal{W}_n$ over Z with the following properties:*

1. $\mathcal{W}_1 = \mathcal{O}_Z(\sigma)$
2. *For any $n \geq 2$ we have a non-split exact sequence*
$$0 \to L^{n-1} \to \mathcal{W}_n \to \mathcal{W}_{n-1} \to 0.$$
3. $R^1\pi_* \mathcal{W}_n^* = L^{-n}$.
4. $R^0\pi_* \mathcal{W}_n^* = 0$.

For these bundles the restriction of $\mathcal{W}_n$ to any fiber of $Z//B$ is isomorphic to the bundle W_n of that Weierstrass cubic curve.

Proof. The proof is by induction on d, with the case $d = 1$ being the last claim. Suppose inductively we have constructed W_{d-1} as required. Since W_{d-1}^* is semi-stable of negative degree on each fiber, and since $R^1\pi_* W_{d-1} = L^{1-d}$, it follows by exactly the same local-to-global spectral sequence argument as in the claim that $H^1(W_{d-1}^* \otimes \pi^* L^{d-1}) = H^0(B; L^{1-d} \otimes L^{d-1}) = H^0(B; \mathcal{O}_B) = \mathbf{C}$. Thus, there is a unique (up to scalar multiples) nontrivial extension of the form

$$0 \to L^{d-1} \to \mathcal{X} \to W_{d-1} \to 0$$

and the restriction of this extension to each fiber of $Z//B$ is nontrivial. We let $\mathcal{W}_d$ be the bundle which is such a nontrivial extension. The computations of $R^i\pi_*\mathcal{W}_d^*$ are straightforward from the extension sequence. □

Notice that $\mathcal{W}_n$ is not the only bundle that restricts to each fiber to give W_n. Any bundle of the form $\mathcal{W}_n \otimes \pi^* M$ for any line bundle M on B will also have that property. Since the endomorphism group of W_n is $\mathbf{C}^*$, one shows easily that these are the only bundles with that property.

N.B If we assume that B is simply connected then there are no torsion line bundles on B. In this case requiring that the determinant of $\mathcal{W}_d$ be $\pi^* L^{d(d-1)/2} \otimes \mathcal{O})Z(\sigma)$ will determine $\mathcal{W}_d$ up to isomorphism.

4.3.2 Globalizing the construction of vector bundles

Lemma 4.3.3 $\mathrm{Ext}^1(\mathcal{W}_{n-d}, \mathcal{W}_d^*)$ *is identified with the space of global sections of the sheaf*

$$R^1\pi_*(\mathcal{W}_{n-d}^*, \mathcal{W}_d^*)$$

on B.

Proof. First of all since $\mathcal{W}_{n-d}$ and $\mathcal{W}_d^*$ are vector bundles, we can identify $\mathrm{Ext}^1(\mathcal{W}_{n-d}, \mathcal{W}_d^*)$ with $H^1(Z; \mathcal{W}_{n-d}^* \otimes \mathcal{W}_d^*)$. The local-to-global spectral sequence produces an exact sequence

$$0 \to H^1(B; R^0\pi_*(\mathcal{W}_{n-d}^* \otimes \mathcal{W}_d^*)) \to H^1(Z; \mathcal{W}_{n-d}^* \otimes \mathcal{W}_d^*)$$
$$\to H^0(B; R^1\pi_*(\mathcal{W}_{n-d} \otimes \mathcal{W}_d^*)) \to H^2(B; R^0\pi_*\mathcal{W}_{n-d}^* \otimes \mathcal{W}_d^*).$$

Since $\mathcal{W}_d^*$ and $\mathcal{W}_{n-d}^*$ are both semi-stable and of negative degree on each fiber, the restriction of their tensor product to each fiber has no sections. It follows that $R^0\pi_*(\mathcal{W}_{n-d}^* \otimes \mathcal{W}_d^*)$ is trivial. Thus, we have an isomorphism

$$H^1(Z; \mathcal{W}_{n-d}^* \otimes \mathcal{W}_d^*) \to H^0(B; R^1\pi_*(\mathcal{W}_{n-d} \otimes \mathcal{W}_d^*)),$$

as claimed in the statement. □

Next we need to compute the sheaf $R^1\pi_*(B; \mathcal{W}_d^* \otimes \mathcal{W}_{n-d}^*)$ on B.

Proposition 4.3.4 $R^1\pi_*(B; \mathcal{W}_d^* \otimes \mathcal{W}_{n-d}^*)$ *is a vector bundle and is isomorphic to the direct sum of line bundles $L \oplus L^{-1} \oplus L^{-2} \oplus \cdots \oplus L^{1-n}$.*

First we consider a special case:

Lemma 4.3.5 *$R^1\pi_*(B;\mathcal{O}_Z(-\sigma)\otimes\mathcal{W}^*_{n-1})$ is isomorphic to $L\oplus L^{-1}\oplus L^{-2}\cdots\oplus L^{1-n}$.*

Proof. Let $R^0_{n-1}=R^0\pi_*(\mathcal{O}_Z(\sigma)\otimes\mathcal{W}_{n-1})$. Since the restriction of $\mathcal{O}_Z(\sigma)\otimes\mathcal{W}_{n-1}$ to each fiber is a semi-stable bundle of degree $-n$, R^0_{n-1} is a vector bundle of rank n over B.

The relative dualizing sheaf for $Z//B$ is π^*L and $R^1\pi_*L=\mathcal{O}_B$. Thus, relative Serre duality is a map

$$S\colon R^1\pi_*(\mathcal{O}_Z(-\sigma)\otimes\mathcal{W}^*_{n-1})\to R^0\pi_*(\mathcal{O}_{\mathcal{Z}}(\sigma)\otimes\mathcal{W}_{\backslash-\infty}\otimes\pi^*\mathcal{L}^{-\infty})$$
$$\otimes R^1\pi_*L=R^0_{n-1}\otimes L^{-1}.$$

Consider the composition of S with the map

$$(R^0_{n-1})^*\otimes L^{-1}\xrightarrow{A}\bigwedge^{n-1}R^0_{n-1}\otimes\det(R^0_{n-1})^{-1}\otimes L^{-1}$$
$$\xrightarrow{\mathrm{ev}\otimes\mathrm{Id}\otimes\mathrm{Id}}R^0\pi_*(\det(\mathcal{O}_Z(\sigma)\otimes\mathcal{W}_{n-1}))\otimes\det(R^0_{n-1})^{-1}\otimes L^{-1}$$
$$=R^0\pi_*(\mathcal{O}_Z(n\sigma))\otimes L^{(n-1)(n-2)/2}\otimes\det(R^0_{n-1})^{-1}\otimes L^{-1},$$

where the map A is induced by taking adjoints from the natural pairing

$$R^0_{n-1}\otimes\bigwedge^{n-1}R^0_{n-1}\to\det(R^0_{n-1}),$$

and ev is the map

$$\mathrm{ev}\colon\bigwedge^{n-1}R^0\pi_*(\mathcal{O}_Z(\sigma\otimes\mathcal{W}_{n-1})\to R^0\pi_*(\bigwedge^{n-1}\mathcal{O}_Z(\sigma)\otimes\mathcal{W}_{n-1})$$

obtained by evaluating sections. Clearly, both S and A are isomorphisms. It is not so clear, but it is still true that ev is also an isomorphism. I shall not prove this result – it is somewhat involved but fairly straightforward. A reference is Proposition 3.13 in Vector Bundles over Elliptic Fibrations.

Assuming this result, we see that the vector bundle we are interested in computing differs from $R^0\pi_*(\mathcal{O}_Z(n\sigma)$ by twisting by the line bundle $L^{-1}\otimes\det R^0_{n-1}$.

According to Claim 4.2.1 $R^0(\mathcal{O}_Z(n\sigma))$ splits as a sum of line bundles $\mathcal{O}\oplus L^{-2}\oplus L^{-3}\oplus\cdots\oplus L^{1-n}$. Now to complete the evaluation of $R^1\pi_*(\mathcal{W}_d\otimes\mathcal{W}^*_{n-d})$ we need only to compute the line bundle $\det R^0\pi_*(\mathcal{O}_Z(n\sigma)\otimes\mathcal{W}_{n-1})$.

Claim 4.3.6 $\det R^0\pi_*(\mathcal{O}_Z(n\sigma)\otimes\mathcal{W}_{n-1})$ *is equal to* $L^{(n-2)(n-1)/2-2}$.

Proof. In computing the determinants we can assume that all sequences split. This allows us to replace $\mathcal{W}_{n-1}$ by $\mathcal{O}_Z(\sigma)\oplus L\oplus L^2\cdots\oplus L^{n-2}$. Since $\mathcal{O}_Z(2\sigma)$ sits in an exact sequence

$$0\to\mathcal{O}_Z(\sigma)\to\mathcal{O}_Z(\sigma)\to\mathcal{O}_Z(\sigma)|_\sigma\to 0$$

and since $R^0\pi_*\mathcal{O}_Z(\sigma))=L$ and $R^0\pi_*\mathcal{O}_Z(\sigma)|_\sigma=L^{-1}$, and $R^0\pi_*(\mathcal{O}_Z(\sigma)\otimes\pi^*L^a)=L^{a-1}$, the result follows easily. □

Putting all this together we see that

$$R^1\pi_*(B;\mathcal{O}_Z(-\sigma)\otimes\mathcal{W}_{n-1})=R^0\pi_*(B;\mathcal{O}_Z(n\sigma))\otimes L.$$

This completes the proof of Lemma 4.3.5 □

Now we are ready to complete the proof of Proposition 4.3.4. This is done by induction on d. The case $d=1$ is exactly the case covered by Lemma 4.3.5. Suppose inductively that we have established the result for $\mathcal{W}_d^*\otimes W_{n-d}^*$ for some $d\geq 1$. We consider the commutative diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{W}_d^*\otimes\mathcal{W}_{n-d-1}^* & \longrightarrow & \mathcal{W}_{d+1}^*\otimes\mathcal{W}_{n-d-1}^* & \longrightarrow & L^{-d}\otimes\mathcal{W}_{n-d-1}^* & \longrightarrow & 0\\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{W}_d^*\otimes\mathcal{W}_{n-d}^* & \longrightarrow & \mathcal{W}_{d+1}^*\otimes\mathcal{W}_{n-d}^* & \longrightarrow & L^{-d}\otimes\mathcal{W}_{n-d}^* & \longrightarrow & 0\\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{W}_d^*\otimes L^{1+d-n} & \longrightarrow & \mathcal{W}_{d+1}^*\otimes L^{1+d-n} & \longrightarrow & L^{-d}\otimes L^{1+d-n} & \longrightarrow & 0\\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

The natural maps $R^1\pi_*(\mathcal{W}_{d+1}^*\otimes L^{1+d-n})\to R^1\pi_*(L^{1-d}\otimes L^{1+d-n})$ and $R^1\pi_*(L^{-d}\otimes\mathcal{W}_{n-d}^*)\to R^1\pi_*(L^{-d}\otimes L^{1+d-n})$ are both isomorphisms. It follows that the images of $R^1\pi_*(\mathcal{W}_{d+1}\otimes\mathcal{W}_{n-d-1}^*)$ and of $R^1\pi_*(\mathcal{W}_d^*\otimes\mathcal{W}_{n-d}^*)$ in $R^1\pi_*(\mathcal{W}_{d+1}^*\otimes\mathcal{W}_{n-d})$ are equal to the kernel of the natural map $R^1\pi_*(\mathcal{W}_{d+1}^*\otimes$

$\mathcal{W}^*_{n-d} \to R^1\pi_*(L^{-d} \otimes L^{1+d-n})$. Since all the bundles in question are semi-stable and of negative degree on each fiber, they all have trivial $R^0\pi_*$. Thus the maps $R^1\pi_*(\mathcal{W}_{d+1} \otimes \mathcal{W}^*_{n-d-1})$ and $R^1\pi_*(\mathcal{W}^*_d \otimes \mathcal{W}^*_{n-d})$ to $R^1\pi_*(\mathcal{W}^*_{d+1} \otimes \mathcal{W}_{n-d})$ are injections. It follows that $R^1\pi_*(\mathcal{W}_{d+1} \otimes \mathcal{W}^*_{n-d-1})$ and $R^1\pi_*(\mathcal{W}^*_d \otimes \mathcal{W}^*_{n-d})$ are identified This completes the inductive step and hence the proof of the theorem.

4.4 The parabolic construction of vector bundles regular and semi-stable with trivial determinant on each fiber

Let $Z \to B$ be a family of Weierstrass cubic curves with σ the section at infinity. Fix a line bundle M on B and sections t_i of $L^{-i} \otimes M$ for $i = 0, 2, 3, 4, \ldots, n$. Supposing that there is no point of B where all these sections vanish we can construct a vector bundle as follows.

The identification of $\mathrm{Ext}^1(\mathcal{W}_{n-d}, \mathcal{W}^*_d)$ with $L \oplus L^{-1} \oplus L^{-2} \cdots \oplus L^{1-n}$ can be twisted by tensoring with M so as to produce an identification of $\mathrm{Ext}^1(\mathcal{W}_{n-d}, W^*_d \otimes \pi^* M)$ with $M \otimes L \oplus M \otimes L^{-1} \oplus \cdots \oplus M \otimes L^{1-n}$. Thus, the sections t_i determine an element of $\mathrm{Ext}^1(\mathcal{W}_{n-d}, \mathcal{W}^*_d \otimes \pi^* M)$ and hence determine an extension

$$0 \to \mathcal{W}^*_d \otimes \pi^* M \to \mathcal{V} \to \mathcal{W}_{n-d} \to 0.$$

Since we are assuming that not all the sections t_i vanish at the same point of B, the restriction of $\mathcal{V}$ to each fiber is a non-trivial extension of W_{n-d} by W^*_d. Thus, the restriction of $\mathcal{V}$ to each fiber is in fact semi-stable, regular and with trivial determinant.

This parabolic construction thus produces one particular vector bundle associated with each line bundle M on B and each non-zero section of $R^0\pi_*(\mathcal{O}_{\mathcal{Z}}(n\sigma)) \otimes M$. This bundle is automatically regular and semi-stable on each fiber and has trivial determinant on each fiber. Conversely, given the bundle regular and semi-stable and with trivial determinant of each fiber, it determines a section of the projective bundle $\mathbf{P}_n \to B$, to which we can apply the parabolic construction. The result of the parabolic construction may not agree with the original bundle – but they will have isomorphic restrictions to each fiber. Thus, they will differ by twisting by a line bundle on the spectral covering corresponding to the section. That is to say to construct all bundles corresponding to a given section we begin with the one produced by the parabolic construction. The section also gives us a spectral covering $T \to B$. We are then free to twist the bundle constructed by the parabolic

construction by any line bundle on T, just as in the spectral covering construction. Thus, the moduli space of bundles that we are constructing fibers over the projective space of $H^1(Z; \mathcal{W}_d^* \otimes \mathcal{W}_{n-d}^*)$ with fibers being Jacobians of the spectral coverings $T \to B$ produced by the section. This twisting corresponds to finding all bundles which agree with the given one fiber-by-fiber. By general theory all such bundles are obtained by twisting with the sheaf of groups $H^1(B; \pi_*(Aut(\mathcal{V}, \mathcal{V}))$.

4.5 Exercises:

1. Show that a Weierstrass cubic has at most one singularity, and that is either a node or a cusp. Show that the cusp appears only if $g_2 = g_3 = 0$. Show that the node appears when $\Delta(g_2, g_3)$, as defined in the lecture, vanishes. Show that the point at infinity is always a smooth point.
2. Show that for any Weierstrass cubic the usual geometric law defines a group structure on the subset of smooth points with the point at infinity being the origin for the group law. Show that this algebraic group is isomorphic to $\mathbf{C}^*$ if the curve is nodal and isomorphic to $\mathbf{C}$ if the curve is cuspidal.
3. Show that any family of Weierstrass cubics is a flat family of curves over the base.
4. Prove Lemma 4.1.1.
5. Describe the singularities of $Z \times_B \times \cdots \times_B Z$ at the nodes and cusps of $Z//B$.
6. Show that if $V \to Z$ is a vector bundle and for each fiber E_b of $Z//B$ we have $H^i(E_b; V|E_b)$ is of dimension k, show that $R^i\pi_*(V)$ is a vector bundle of rank k on B.
7. Let M be a line bundle over B and let $\mathcal{V}$ fit in an exact sequence

$$0 \to \mathcal{W}_d^* \otimes \pi^* M \to \mathcal{V} \to \mathcal{W}_{n-d} \to 0.$$

Compute the Chern classes of $\mathcal{V}$.
8. Show that if V and U are vector bundles over a smooth variety, then $\mathrm{Ext}^1(U, V) = H^1(U^* \otimes V)$.
9. Show that if V is a rank n vector bundle then $\bigwedge^{n-1} V$ is isomorphic to $V^* \otimes \det(V)$.
10. State relative Serre duality and show that it is correctly applied to produce the map S given in the proof of Lemma 4.3.5.
11. Suppose that V is a vector bundle. Show that to first order the deformations of V are given by $H^1(Hom(V, V))$.

5 The Global Parabolic Construction for Holomorphic Principal Bundles

In this section we wish to generalize the parabolic construction to families of Weierstrass cubics. In the last lecture we did this for vector bundles, here we consider principal bundles over an arbitrary semi-simple group $G_{\mathbf{C}}$. This construction will produce holomorphic principal bundles on the total space Z of the family of Weierstrass cubics which have the property that they are regular semi-stable $G_{\mathbf{C}}$-bundles on each fiber of $Z//B$. Of course, this construction can also be viewed as a generalization of the construction given in the third lecture for a single elliptic curve. It is important to note that we do not give an analogue of the spectral covering construction for $G_{\mathbf{C}}$-bundles. We do not know whether such a construction exists for groups other than $SL_n(\mathbf{C})$ and $\mathrm{Sympl}(2n)$.

5.1 The parabolic construction in families

We let $Z \to B$ be a family of Weierstrass cubics with section $\sigma\colon B \to Z$ as before. Let $G_{\mathbf{C}}$ be a simply connected simple group. Fix a maximal torus and a set of simple roots for G, and let $P \subset G$ be the distinguished maximal parabolic subgroup with respect to these choices. Then the Levi factor L of P is isomorphic to the subgroup of a product of general linear groups $\prod_{i=1}^{s} GL_{n_i}$ consisting of matrices with a common determinant. The character group of P and of L is $\mathbf{Z}$ and the generator is the character that takes the common determinant. We consider the bundle $\mathcal{W}^*_{n_1} \times \cdots \times \mathcal{W}^*_{n_s}$. This naturally determines a holomorphic principal L-bundle η_L over Z. Viewed as a bundle over $G_{\mathbf{C}}$ it is unstable since the $G_{\mathbf{C}}$-adjoint bundle associated with this L bundle splits into three pieces: the adjoint $\mathrm{ad}(\eta_L)$ of the L-bundle, the vector bundle associated with the tangent space to the unipotent radical $U_+(\eta_L)$ and the vector bundle associated to the root spaces negative to those in U_+, $U_-(\eta_L)$. The first bundle has degree zero, the second has negative degree and the third has positive degree. The degree of the entire bundle is zero. This makes it clear that $\mathrm{ad}(\eta_L \times_L G_{\mathbf{C}})$ is unstable, and hence according to our definition that $\eta_{\mathbf{C}} \times_L G_{\mathbf{C}}$ is an unstable principal $G_{\mathbf{C}}$-bundle. (Notice that the L-bundle is stable as an L-bundle.)

Once again we are interested in deformations of η_L to P-bundles ξ with identifications $\xi/U = \eta_L$. Just as in the case of a single elliptic curve, these deformations are classified by equivalence classes of twisted cocycles, which

we denote by $H^1(Z; U(\eta_L))$. Recall that U is filtered $0 \subset U_n \subset U_{n-1} \subset \cdots \subset U_1 = U$ where the center $\mathbf{C}^*$ acts on U and has homogeneous weight i on U_i/U_{i-1}. Furthermore, U_i/U_{i-1} is abelian and hence is a vector space which lies in the center of U/U_{i-1}. Thus, once again we can filter the cohomology by $H^1(Z; U_i(\eta_L))$ with the associated gradeds being ordinary cohomology of vector bundles $H^1(Z; U_i/U_{i-1}(\eta_L))$. Since $\det(\eta_L)$ as measured with respect to the generating dominant character of P is negative, it follows that $R^0\pi_*(U_i/U_{i-1}(\eta_L))$ is trivial for all i. A simply inductive argument then shows that $R^0\pi_*(U(\eta_L))$ is a bundle of zero dimensional affine spaces over B, and hence $H^0(Z; U(\eta_L))$ has only the trivial element.

A similar inductive discussion shows that $R^1\pi_*(U(\eta_L))$ is filtered with the associated gradeds being the vector bundles $R^1\pi_*(U_i/U_{i-1}(\eta_L))$. This implies that $R^1\pi_*(U(\eta_L))$ is in fact a bundle of affine spaces over B, with a distinguished element – the trivial cohomology class on each fiber. The local-to-global spectral sequence, the vanishing of the $R^0\pi_*(U(\eta_L))$ and an inductive argument shows that in fact the cohomology set $H^1(Z; U(\eta_L))$ is identified with the global sections of $R^1\pi_*(U(\eta_L))$ over B.

5.2 Evaluation of the cohomology group

In all cases except $G = E_8$ and over the cuspidal fibers we can in fact split the bundle $R^1\pi_*(U(\eta_L))$ of affine spaces so that it becomes a direct sum of vector bundles. Under this splitting the $\mathbf{C}^*$ action becomes linear.

Theorem 5.2.1 *Let G be a compact simply connected, simple group and let $Z \to B$ be a family of Weierstrass cubic curves. Assume either that G is not isomorphic to E_8 or that no fiber of $Z//B$ is a cuspidal curve. Then there is an isomorphism $R^1\pi_*U(\eta_L)$ with a direct sum of line bundles $\oplus_i L^{1-d_i}$ where $d_1 = 0$ and $d_2, \ldots, d_r$ are the Casimir weights associated to the group G. Furthermore, the $\mathbf{C}^*$ action that produces the weighted projective space is diagonal with respect to this decomposition and is a linear action on each line bundle.*

Corollary 5.2.2 *The cohomology $H^1(Z; U(\eta_L))$ is identified with the space of sections of a sum of line bundles over B, and hence the space of extensions is identified with a bundle of weighted projective spaces over B. The fibers are weighted projective spaces of type $\mathbf{P}(g_0, g_1, \ldots, g_r)$ where $g_0 = 1$ and for $i = 1, \ldots, r$ the g_i are the coroot integers.*

Here is a table of the Casimir weights grouped by $\mathbf{C}^*$-weights

Group	1	2	3	4
A_n	$0, 2, 3, \ldots, n$			
B_n	$0, 2, 4$	$6, 8, \ldots, 2n$		
C_n	$0, 2, 4, \ldots, 2n$			
D_n	$0, 2, 4, n$	$6, 8, \ldots, 2n-2$		
E_6	$0, 2, 5$	$6, 8, 9$	12	
E_7	$0, 2$	$6, 8, 10$	$12, 14$	18
G_2	$0, 2$	6		
F_4	$0, 2$	$6, 8$	12	

Thus, with this choice of splitting for the unipotent cohomology, a choice of a line bundle M over B and sections t_i of $M \otimes L^{1-d_i}$ will determine a section of $R^1\pi_*(U(\eta_L))$ and a P-bundle over Z deforming the original L-bundle η_L. By construction there will be a given isomorphism from the quotient of the deformed bundle modulo the unipotent subgroup back to η_L. Furthermore, if the sections t_i never all vanish at the same point of B, then the resulting P-bundle will extend to a $G_{\mathbf{C}}$-bundle which is regular and semi-stable on each fiber of $Z//B$. The resulting section of the weighted projective space bundle is equivalent to the data of the S-equivalence class of the restriction of the principal $G_{\mathbf{C}}$-bundle to each fiber of $Z//B$. Of course, since these bundles are regular, it is equivalent to the isomorphism class of the restriction of the $G_{\mathbf{C}}$-bundle to each fiber.

5.3 Concluding remarks

Thus, for each collection of sections we are able to construct a G_{bfC}-bundle which is regular semi-stable on each fiber. The study of all bundles which agree with one of this type fiber-by-fiber is more delicate. From the parabolic point of view, it requires a study of the sheaf $R^1\pi_*(Aut(\xi))$ which can be quite complicated, and is only partially understood at best.

Even assuming this, we are far from knowing the entire story – one would like to have control over the automorphism sheaf so as to find all bundles which are the same fiber-by-fiber. Then one would like to complete the space of bundles by adding those which become unstable on some fibers (but remain semi-stable on the generic fiber). Finally, to complete the space it is surely necessary to add in torsion-free sheaves of some sort. All these issues are ripe for investigation – little if anything is currently known.

The study of these bundles is an interesting problem in its own right. After varieties themselves bundles are probably the next most studied objects in algebraic geometry. Constructions, invariants, classification, moduli spaces are the main sources of interest. The study we have been describing here fits perfectly in that pattern. Nevertheless, from my point of view, there is another completely different motivation for this study. That motivation is the connection with other differential geometric, algebro-geometric, and theoretical physical questions.

The study of stable G-bundles over surfaces is closely related to the study of anti-self-dual connections on G-bundles (this is a variant of the Narasimhan-Seshadri theorem in for surfaces rather than curves and was first established by Donaldson [4]) and whence to the Donaldson polynomial invariants of these algebraic surfaces. Thus, the study described here can be used to compute the Donaldson invariants of elliptic surfaces. These were the first such computations of those invariants, see [5].

More recently, there has been a connection proposed, see [9], between algebraic n-manifolds elliptically fibered over a base B with $E_8 \times E_8$-bundle and algebraic $(n+1)$-dimensional manifolds fibered over the same base with fiber an elliptically fibered $K3$ with a section. The physics of this later set-up is called F-theory. The precise mathematical statements underlying this physically suggested correspondence are not well understood yet, and this work is an attempt to clarify the relationship between these two seemingly disparate mathematical objects. All the evidence to date is extremely positive – the two theories $E_8 \times E_8$-bundles over families elliptic curves line up perfectly as far as we can tell with families of elliptically fibered $K3$-surfaces with sections over the same base. Yet, there is still much that is not understood in this correspondence. Sorting it out will lead to much interesting mathematics around these natural algebro-geometric objects.

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Degenerations of the moduli spaces of vector bundles on curves

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1 Introduction

Let Y be a smooth projective curve of genus g and $U_Y = U_Y(n,d)$ the moduli space of (semi-stable) vector bundles on Y of rank n and degree d. One strategy for studying the variety U_Y in depth is by the method of degeneration (or specialization), namely one specializes Y to a curve X_0, say with only one singularity which is an ordinary double point. One would have a moduli object U_{X_0} on X_0 such that U_Y specializes to U_{X_0} and one expects a close relationship between U_{X_0} and the moduli space U_X on the normalisation X of X_0. Since the genus of X is $(g-1)$, one would then obtain a machinery for studying U_Y, especially its properties which are amenable to specialization, by induction on g.

This strategy was employed by Gieseker to prove a conjecture of Newstead and Ramanan for moduli spaces in rank 2, namely that the Chern classes c_i of the smooth projective variety $U_Y(2,1)$ vanish for $i > \frac{1}{2}\dim U_Y(2,1)$, i.e. $i \geq (2g-1)$, since $\dim U_Y(2,1) = (4g-3)$ (see [G]). A similar one was employed by M.S. Narasimhan and T.R. Ramadas to prove what is called the factorisation rule in the rank two case and recently it has been generalized to arbitrary rank by Xiaotao Sun (see [NR] and [Su]).

Gieseker constructed a moduli object G_{X_0} on X_0 (we do not denote this by U_{X_0} since this will stand for a moduli space of torsion free sheaves on X_0) such that it has nice singularities and $U_Y(2,1)$ specializes to G_{X_0}. Further, he gave a concrete realization of G_{X_0} via the moduli space $U_X(2,1)$, which helps in solving the conjecture of Newstead and Ramanan by induction on the genus.

Recently, in collaboration with D.S. Nagaraj, we have been able to generalize Gieseker's construction of G_{X_0} for arbitrary rank (see [NS]). A good part of these lectures is devoted to outlining this construction. Our method for the global construction is quite different from that of Gieseker; it consists in relating the Gieseker moduli space to that of torsion free sheaves on X_0, an aspect which does not figure in Gieseker's work.

We give also a brief sketch of the proof of the conjecture of Newstead and Ramanan in the rank two case. This is essentially on the lines as done by Gieseker ([G]). However, our proof for the concrete realization of the Gieseker moduli space (via the moduli space $U_X(2,1)$) connects it with the moduli space of GPB's (generalized parabolic bundles) and shows a close

relationship with good compactifications of the full linear group. In this sense it appears more conceptual and suggests a natural candidate for the concrete realization in arbitrary rank.

For related work see [Te] and especially the very recent one [K].

2 Review of basic facts of the moduli space $U(n,d)$ on X_0

We work over an algebraically closed base field K, which we can take to be $\mathbb{C}$ - the field of complex numbers, as the emphasis is not on the characteristic of K.

Let X_0 be a projective, irreducible curve with only one singularity at $p \in X_0$, which is an ordinary double point. We fix an ample line bundle $\mathcal{O}_{X_0}(1)$ on X_0. Unless otherwise stated we assume $g =$ arithmetic genus of $X_0 \geq 2$.

Let F be a torsion free (coherent) $\mathcal{O}_{X_0}$-module on X_0. We have the notion of the *degree* of F (say defined by $\deg F = \chi(F) - \operatorname{rk} F \cdot \chi(\mathcal{O}_\chi)$). Following Mumford, we say F is *semi-stable* (resp. *stable*) if for every (resp. proper) subsheaf G of F, we have

$$\frac{\deg G}{\operatorname{rk} G} \leq \frac{\deg F}{\operatorname{rk} F} \text{ (resp. } \quad <).$$

We have a natural structure of a projective variety (irreducible and reduced) on certain equivalence classes of semi-stable torsion free sheaves on X_0 of rank n and deg. d. We this by $U(n,d) = U_{X_0}(n,d)$. If $(n,d) = 1$, $U(n,d)$ is, in fact, the set of isomorphism classes of stable torsion free sheaves of rank n and degree d.

The variety $U(n,d)$ has good spe0ialization properties. This means, for example, the following: Let $S = \operatorname{Spe0} A$, where A is a d.v.r. (discrete valuation ring, in fact wean even take A as a K-algebra) with residue field the ground field K. Let $\mathcal{X} \longrightarrow S$ be a flat family of projectiveurves such that the closed fibre $\mathcal{X}_{s_0} \simeq X_0$ and the generic fibre $\mathcal{X}_\xi$ is smooth (of genus g), s_0 (resp. ξ) being the closed point (resp. generic point) of S. (One may have to fix a section passing through the smooth points of the fibres). Then whave well-defined moduli s0heme $U(n,d)_S$ which is flat and projective

over S such that its generic fibre $U(n,d)_\xi$ can be identified with the moduli space of vector bundles on $\mathcal{X}_\xi$ and we have a bijective morphism

$$\text{(i)} \qquad U_{X_0}(n,d) \longrightarrow U(n,d)_{s_0}$$

$U(n,d)_{s_0}$ being the closed fibre of $U(n,d)_S \longrightarrow S$. All these follow by appealing to GIT over arbitrary base. We do not know whether, in general, the map (i) is an isomorphism, even if we suppose that $\mathcal{X} \longrightarrow S$ is smooth i.e. the closed fibre is smooth. However, if char $K = 0$, it is easily seen that (i) is an isomorphism. Also if $(n,d) = 1$ the map (i) is an isomorphism.

The facts that $U_{X_0}(n,d)$ is a variety (in particular reduced) and that it has good specialization properties are not very general results. For example, if Y_0 is a projective curve whose singularities are not ordinary double points, the moduli space on Y_0 may have irreducible components whose dimensions are bigger than the expected dimension (this happens even in the rank one case). To show that $U_{X_0}(n,d)$ is reduced (scheme theoretically) one has to have a closer study of deformations of torsion free sheaves. It is known that, locally at the singular point 'p' of X_0, a torsion free sheaf F is of the form

$$F \simeq \left(\bigoplus_{i=1}^{a} m\right) \oplus \left(\bigoplus_{j=1}^{b} \mathcal{O}\right), \quad m \text{ maximal ideal of } \mathcal{O} = \mathcal{O}_{X_0,p}.$$

We refer to 'a' as the *type* of F (at p). It is known (see [S] and [F]) that the miniversal deformation space of F (as an $\mathcal{O}$-module) is formally smooth to the singularity defined by

$$\text{(ii)} \qquad XY = YX = 0 \ (X, Y \text{ space of '}a \times a\text{'-matrices}).$$

i.e. the analytic local ring at the origin of the closed subscheme of the $2a^2$ dimensional affine scheme, defined by the equations in (ii). One knows that this analytic local ring is reduced and Cohen -Macaulay (see [St.]). The reduced nature of $U_{X_0}(n,d)$ can be deduced from this fact.

Let $U(n,d)^0$ denote the subset of points of $U(n,d)$ which represent vector bundles on X_0. Then $U(n,d)^0$ is open in $U(n,d)$ and if $(n,d) = 1$, $U(n,d) \backslash U(n,d)^0 = \operatorname{Sing} U(n,d)$ (singular locus of $U(n,d)$). It can be seen that the singularities of $U(n,d)$ are not, in general, normal crossings. This is a consequence of (ii) above.

One of the main goals of these lectures is to give a construction (see §4 and §5) of a variety $G(n,d)$ (for $(n,d) = 1$) such that

(i) $G(n,d)$ is projective and $U(n,d)^0$ is open in $G(n,d)$

(ii) $G(n,d)$ has only (analytic) normal crossing singularities and $\operatorname{Sing} G(n,d) = G(n,d) \backslash U(n,d)^0$

(iii) there is a canonical morphism $\pi_* : G(n,d) \longrightarrow U(n,d)$ which is an isomorphism over $U(n,d)^0$.

(iv) $G(n,d)$ has good specialization properties i.e. we have an S-scheme $G(n,d)_S$ associated to a moduli problem on $\mathcal{X} \longrightarrow S$, which is flat and projective over S. Besides, the generic fibre $G(n,d)_\xi$ of $G(n,d)_S$ over S, identifies with $U(n,d)_\xi$ and the closed fibre with $G(n,d)$.

This generalizes Gieseker's construction of G_{X_0}, referred to above.

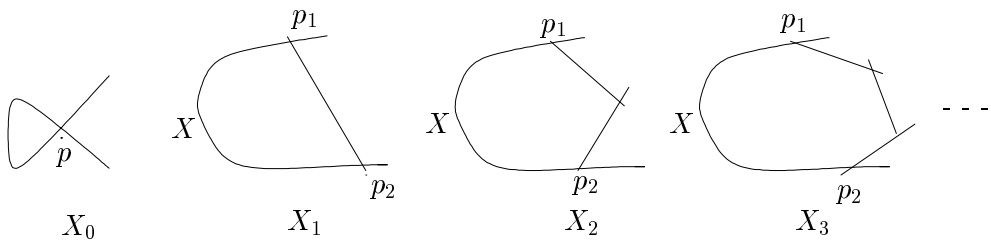
We study also the fibres of π_*. They turn out to be the wonderful compactifications of the projective linear group (Rem. 5.2).

In §6 we give a concrete realization of $G(2,1)$ (Th. 6.1 and Th.6.2) and sketch very briefly how this helps in solving the conjecture of Newstead and Ramanan in the rank two case.

3 Vector bundles on the curves X_k

Definition-Notation 3.1. We call a scheme R, a *chain of projective lines* if $R = \bigcup_{i=1}^n R_i$, $R_i \simeq \mathbb{P}^1$, $R_i \cap R_j$ (for distinct i,j) is a single point if $|i-j| = 1$ and otherwise empty. We call m the *length* of R. Let E be a vector bundle of rank n on R. One knows that $E|_{R_i} = \bigoplus_{j=1}^n \mathcal{O}(a_{ij})$, $a_{ij} \in \mathbb{Z}$. We say that E is *positive* if $a_{ij} \geq 0$ for all i,j. We say that E is *strictly positive* if it is positive and for every i, there is a j such that $a_{ij} > 0$. We say E is *standard* if it is positive and $a_{ij} \leq 1$ for all i,j and *strictly standard* if, moreover, it is strictly positive.

Definition-Notation 3.2. Let X_0 be the curve as in §2 with an ordinary double point singularity at 'p'. Let $\pi : X \longrightarrow X_0$ be the normalisation of X_0 and $\pi^{-1}(p) = \{p_1, p_2\}$. Let X_k be the curves which are "semi-stably equivalent to X_0" i.e. X is a component of X_k ($k \geq 1$) and if $\pi : X_k \longrightarrow X_0$ denotes the canonical morphism, then $\pi^{-1}(p)$ is a chain R of projective lines of length k, passing through p_1, p_2 i.e. they are the curves as follows:



Let Z be a scheme and E a vector bundle on Z of rank n such that $H^0(E)$ generates E i.e. the canonical map $H^0(E) \longrightarrow E_z$ (fibre of E at $z \in Z$) is surjective. Then we get a canonical morphism.

$$\phi_E = \phi : Z \longrightarrow Gr(H^0(E), n) \quad \text{(Grassmannian of } n \text{ dimensional quotients of } H^0(E))$$

such that E is the inverse image by ϕ of the tautological quotient bundle on $Gr(H^0(E), n)$.

Proposition 3.1. Let R be a chain of projective lines. Then we have the following:

(1) if E is a positive vector bundle on R, then $H^0(E)$ generates E and $H^1(E) = 0$. Moreover, if R_1 is a *subchain* of R (in the obvious sense), the canonical map

$$H^0(R, E) \longrightarrow H^0(R_1, E)$$

is surjective.

(2) if E is strictly positive, the canonical morphism

$$\phi : R \longrightarrow Gr(H^0(E), n)$$

is a closed immersion. Conversely, the pull-back of the tautological quotient bundle by a closed immersion of R in a Grassmannian, is strictly positive.

(3) $\chi(E) = \deg E + n$ ($\deg E$ = total degree i.e. sum of $\deg E|_{R_i}$, for this claim E could be any vector bundle on R) so that if E is positive $h^0(E) = \deg E + n$.

(ii) $R^i\pi_*(E) = 0,\ i > 0$

(iii) $H^i(X_k, E) \simeq H^i(X_0, \pi_*(E))$

(iv) If E is trivial on R, then $\pi_*(E)$ is a vector bundle on X_0 and $E \simeq \pi^*(\pi_*(E))$.

Proof. The proof of (i) is immediate. For (ii) take an affine neighbourhood V of p and set $U = \pi^{-1}(V)$, $V' = U \cap X$. Since V' is affine, the canonical map $E|_{V'} \longrightarrow E_{p_1} \oplus E_{p_2}$ is surjective so that we get

$$H^i(E|_U) \simeq H^i(E|_{V'}) \oplus H^i(E|_R), \quad i > 0.$$

The RHS is zero and then (ii) and (iii) follow. When $E|_R$ is trivial, we get canonical isomorphisms of $H^0(E|_R)$ with E_{p_i} and hence a canonical identification $\theta : E_{p_1} \longrightarrow E_{p_2}$. We see that $H^0(E|_U)$ identifies with the subspace of $H^0(E|_{V'})$ consisting of elements 's' such that $\theta \cdot s(p_1) = s(p_2)$. This shows that $\pi_*(E)$ identifies with the vector bundle on X_0 defined by $E|_X$ on the normalisation X of X_0 and the patching condition $\theta : E_{p_1} \longrightarrow E_{p_2}$ and then (iv) follows.

Proposition 3.3. Let E be a vector bundle of rank n on X_k such that $E|_R$ is strictly positive. If $F = E \otimes \pi^*(\mathcal{O}_{X_0}(l))$, then for $l \gg 0$, $H^0(F)$ generates F and the canonical morphism $\phi : X_k \longrightarrow Gr(H^0(F), n)$ is a closed immersion. Further (for all $l \gg 0$) $H^1(X_k, F) = 0$ so that by Prop. 3.2, we have

$$H^0(X_k, F) \simeq H^0(X_0, \pi_*(F)).$$

$$H^i(X_k, F) = H^i(X_0, \pi_*(F)) = 0, \quad i > 0.$$

Note that $E|_R = F|_R$.

Proof. If $\pi^*(\mathcal{O}_{X_0}(1))$ were ample, this proposition would be immediate. However, this is not the case ($k \geq 1$) and the proof requires a little work though it is not difficult. Observe that $\pi^*(\mathcal{O}_{X_0}(1))|_X$ is ample. From this it follows that (for $l \gg 0$) the canonical map $H^0(F|_X) \longrightarrow F_{p_1} \oplus F_{p_2}$ is surjective and $H^1(F|_X) = 0$. Then from the patching exact sequence

$$0 \longrightarrow F \longrightarrow F|_X \oplus F|_R \longrightarrow T \longrightarrow 0$$

we deduce that $H^1(F) \simeq H^1(F|_X) \oplus H^1(F|_R)$, which implies that $H^1(F) = 0$. Then the last assertions follow from the previous Prop. 3.2.

Since $H^0(F|_X) \longrightarrow F_{p_1} \oplus F_{p_2}$ is surjective, it follows that $H^0(X_k, F) \longrightarrow H^0(F|_R)$ is surjective. We cannot say that the canonical map $H^0(X_k, F) \longrightarrow H^0(F|_X)$ is surjective, which causes the little complication. However sections of $F|_X$ which vanish at p_1, p_2 can be extended to the whole of X_k (by putting zero on R) and one uses the sheaf $I_{p_1,p_2}F|_X$ (where I_{p_1,p_2} denotes the ideal sheaf of the closed subscheme $\{p_1, p_2\}$ of X). We can suppose that for $l \gg 0$, $H^0(I_{p_1,p_2}F|_X)$ generates $I_{p_1,p_2}F|_X$ and the canonical morphism $X \longrightarrow Gr(H^0(I_{p_1,p_2}F|_X), n)$ is a closed immersion. We identify $H^0(X, I_{p_1,p_2}F|_X)$ with the subspace of $H^0(X_k, F)$ vanishing on R. Then with these observations and the surjectivity of $H^0(X_k, F) \longrightarrow H^0(X_k, F|_R)$, we see that $H^0(X_k, F)$ generates F, that the canonical morphism $\phi : X_k \longrightarrow Gr(H^0(x_k, F), n)$ is injective (we see easily that given x, y, $x \neq y$, there exist sections on X_k such that $(s_1 \wedge \cdots \wedge s_n)(x) \neq 0$ and $(s_1 \wedge \cdots \wedge s_n)(y) = 0$) and that $\mathrm{d}\phi$ is injective at all the points except $\{p_1, p_2\}$. It is not difficult to show that $\mathrm{d}\phi$ is injective at $\{p_1, p_2\}$ (see [NS]) and the proposition follows.

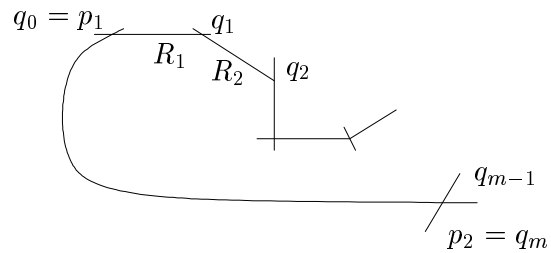
Consider the vector bundles E on X_k such that $E|_R$ are strictly positive. Our next aim is to characterize those E such that $\pi_*(E)$ are torsion free. This characterization involves only properties of $E|_R$.

Remark 3.1.

(a) Let E be a strictly standard vector bundle on $\mathbb{P}^1$. We have then a well-defined sub-bundle K of E, which we call the *canonical subbundle* of E such that K is a direct sum $\mathcal{O}(1)$'s and $E/K = Q$ (called the *canonical quotient bundle*) is free. Let $x, y \in \mathbb{P}^1$ such that $x \neq y$ and L_x a linear subspace of E_x. Then we have a well-defined subbundle F of E such that $K \subset F$ and $F_x|_{K_x} =$ Image of L_x in $Q_x = E_x|_{K_x}$. Then if V is the linear subspace of $H^0(E)$ consisting of sections 's' such that $s(x) \in L_x$, then $V \subset H^0(F)$ and the image of V in E_y in F_y. *We say that F_y is the subspace of E_y determined by L_x.* We see that if $L_x = 0$, then $F_y = K_y$. Note also that if $s \in H^0(E)$ and $s(x) = s(y) = 0$, then s is identically zero.

(b) Let E be a strictly standard vector bundle on a chain R of projective lines of length m. Let K_i (resp. Q_i) be the canonical sub-bundle (resp. quotient bundle) of $E|_{R_i}$ (R_i the $\mathbb{P}^1$-components of R). We denote by

q_i the points $R_i \cap R_{i+1}$ $1 \leq i \leq (m-1)$ and $q_0 = p_1$, $q_m = p_2$. Let us take $L_{q_0} = (0)$. We set



L_{q_1} as the linear subspace of E_{q_1} determined by L_{q_0} for $E|_{R_1}$. We write L_{q_2} for the linear subspace of E_{q_2} defined by L_{q_1} for $E|_{R_2}$. Inductively, we thus define a linear subspace L_{q_i} of E_{q_i}. We write M for the subspace $L_{q_m} \subset E_{p_2} = E_{q_m}$. Consider the condition:

$$(*) \qquad \dim M = \operatorname{rk} K_1 + \cdots + \operatorname{rk} K_m.$$

We see that

$$(*) \Longleftrightarrow \dim L_{q_i} = \operatorname{rk} K_1 + \cdots + rkK_i, i \leq j \leq m.$$

Take for example $m = 2$. Then $(*)$ means that

$$(K_1)_{q_1} \cap (K_2)_{q_1} = (0).$$

Lemma 3.1. Let E be a strictly positive vector bundle on a chain R of projective lines. Consider the property:

$$(**) \qquad s \in H^0(E), \quad s(p_1) = s(p_2) = 0 \Longrightarrow s \equiv 0.$$

Then $(**)$ holds if and only if E is strictly standard and the property $(*)$ of Remark 3.1 holds.

Proof. Suppose that $E|_{R_i} = \oplus \mathcal{O}(a_{ij})$ and some $a_{ij} \geq 2$. Then it is immediate that there exists $s \in H^0(E|_{R_1})$ such that $s(q_0) = s(q_1) = 0$ and s is not identically zero. Then we can extend s to a section of E vanishing identically on all R_i, $i \geq 2$. Thus $(**) \Longrightarrow E$ is strictly standard.

Let us suppose, for simplicity, that the length m of R is 2. The proof in the general case is quite similar (see [NS]). Suppose that $(**)$ holds and

that $(K_1)_{q_1} \cap (K_2)_{q_1} = L \neq (0)$. Then it is clear that given $l \in L$, $l \neq (0)$, there exist $s_1 \in H^0(E|_{R_1})$ and $s_2 \in H^0(E|_{R_2})$ such that $s_1(q_1) = l = s_2(q_1)$ and $s_1(q_0) = 0$ and $s_2(q_2) = 0$. Hence s_1, s_2 define a section s of E such that $s(q_0) = s(q_2) = 0$ and s is not identically zero. Hence we shall have $(K_1)_q = (K_2)_q = (0)$, which shows that $(**)$ implies that $(*)$ holds and E is strictly standard.

Suppose on the other hand that E is strictly standard and $(*)$ holds. Let $s \in H^0(E)$ such that $s(p_1) = s(p_2) = 0$. Then $s(q_1)$ is in $(K_1)_{q_1}$ as well as $(K_2)_{q_1}$. Since $(*)$ holds, this implies that $s(q_1) = 0$. Then we see that the restriction of s to R_1 as well as R_2 is identically zero. Hence s is identically zero and $(**)$ holds.

Proposition 3.4. Let E be a vector bundle on X_k such that $E|_R$ is strictly positive. Then we have the following:

(A) $\pi_*(E)$ is torsion free on X_0 if and only if the property $(**)$ (of Lemma 3.1) holds. Then by Lemma 3.1, we see that $\pi_*(E)$ is torsion free if and only if E is strictly standard and the property $(*)$ of Remark 3.1 holds. Note that $(*)$ implies that

$$\begin{cases} \text{length of } R = m \leq n = \operatorname{rk} E; \text{ in fact} \\ m \leq \deg E|_R = \sum_i \deg E|_{R_i} \leq n. \end{cases}$$

(B) if $\pi_*(E)$ is torsion free, then its type (at p) is $\deg E|_R$.

Proof. We have the following exact sequence of $\mathcal{O}_{X_k}$-modules

$$0 \longrightarrow I_X E \longrightarrow E \longmapsto E|_X \longrightarrow 0. \tag{a}$$

I_X being the ideal sheaf of X. Note that $I_X E$ can be identified with $I_{p_1,p_2} E|_R$ - the sheaf of sections of $E|_R$ vanishing at p_1, p_2. Then we have the exact sequence

$$0 \longrightarrow \pi_*(I_{p_1,p_2} E|_R) \longrightarrow \pi_*(E) \longrightarrow \pi_*(E|_X). \tag{b}$$

Now $\pi_*(E|_X)$ is torsion free on X_0 and it is clear that $\pi_*(I_{p_1,p_2} E|_R)$ is a torsion sheaf, in fact its support is at p. Hence it follows that the torsion subsheaf of $\pi_*(E)$ is precisely $\pi_*(I_{p_1,p_2} E|_R)$. It is clear that $\pi_*(I_{p_1,p_2} E|_R)$ is

the sheaf determined by the vector space $H^0(R, I_{p_1,p_2}E|_R)$ considered as an $\mathcal{O}_{X_0,p}$ module (through its residue field). From these remarks the assertion (A) follows.

Now if $\pi_*(E)$ is torsion free, continuing the exact sequence above we get the exact sequence

$$0 \longrightarrow \pi_*(E) \longrightarrow \pi_*(E|_X) \longrightarrow R^1\pi_*(I_{p_1,p_2}E|_X) \longrightarrow 0$$

since $R^1\pi_*(E) = 0$ by Prop. 3.2. Further, we see that $R^1\pi_*(I_{p_1,p_2}E|_R)$ is the "sky-scraper" sheaf with support at p and defined by the vector space $H^1(I_{p_1,p_2}E|_R)$.

Let us first suppose that $\deg E|_R = n$, then $\deg I_{p_1,p_2}E|_R = -n$ and $\chi(I_{p_2,p_2}E|_R) = 0$. Since $H^0(I_{p_1,p_2}E|_R) = 0$, we see that $H^1(I_{p_1,p_2}E|_R) = 0$ (see Prop 3.1.). Hence we get

$$\text{(c)} \qquad \pi_*(E) \simeq \pi_*(E|_X)$$

and it is an easy exercise that the RHS is of type n at p. Thus in this case the assertion (B) above follows. We have then to consider the case $\deg E|_R < n$. Then by the considerations in Remark 3.1, it is not difficult to see that $E|_R$ has a direct summand which is a trivial vector bundle of rank $t = n - \deg E|_R$. Then we see that we can choose a suitable neighbourhood V of p such that in $\pi^{-1}(V)$, E has a trivial direct summand of rank t. Thus we see that to prove (B) we are reduced to the case $\deg E|_R = n$.

Remark 3.2.

(a) Let E be a vector bundle on X_k such that $E|_R$ is strictly standard and $\pi_*(E)$ is torsion free. Then we have

$m_{X_0,p}(\pi_*(E)_p) = \pi_*(I_{p_1,p_2}E|_X)_p$, or equivalently ($m_{X_0,p}$ denotes the maximal ideal of $\mathcal{O}_{X_0,p}$, $\pi_*(E)_p$ the stalk of $\pi_*(E)$ at p)

$$I_{X_0,p}(\pi_*(E)) = \pi_*(I_{p_1,p_2}E|_X)$$

(b) if $\pi : X \longrightarrow X_0$ is the normalisation map, the functor π_*: (Vector bundles on X) $\longrightarrow$ (Torsion free sheaves on X_0) is faithful i.e.

$$\mathrm{Hom}\,(V_1, V_2) \simeq \mathrm{Hom}\,(\pi_*(V_1), \pi_*(V_2))$$

in particular $V_1 \simeq V_2 \Longleftrightarrow \pi_*(V_1) \simeq \pi_*(V_2)$.

(c) $\pi_*(E)$ determines $E|_X$ i.e. if E_1, E_2 on X_k (possibly for different k with $E|_R$ strictly positive) are such that $\pi_*(E_i)$ are torsion free and $\pi_*(E_1) \simeq \pi_*(E_2)$, then $E_1|_X \simeq E_2|_X$

(d) if we have a family of vector bundles $\{E\}$ on X_k (possibly for different k with $E|_R$ strictly positive) such that $\{\pi_*(E)\}$ is a bounded family of torsion free sheaves on X_0, then for $l \gg 0$ (independent of E) $\{E|_X\}$ is a bounded family and for $F = E \otimes \pi^*(\mathcal{O}_{X_0}(l))$, $H^0(F)$ generates F and the canonical morphism $\phi : X_k \longrightarrow Gr(H^0(F), n)$ is a closed immersion. Besides we have

(i) $H^0(X_k, F) \simeq H^0(X_0, \pi_*(F))$

(ii) $H^i(X_k, F) \simeq H^i(X, \pi_*(F)) = 0, \quad i > 1.$

(i.e. the properties of Prop. 3.2 hold. In a sense, we may say that $\{E\}$ is a bounded family if $\pi_*(E)$ is a bounded family).

Proof.

(a) Consider the exact sequence

$$0 \longrightarrow I_R E \longrightarrow E \longrightarrow E|_R \longrightarrow 0.$$

We see that $I_R E \simeq I_{p_1,p_2} E|_X$. Then we get the following exact sequence of $\mathcal{O}_{X_0,p}$-modules

$$0 \longrightarrow \pi_*(I_{p_1,p_2} E|_X)_p \longrightarrow \pi_*(E)_p \longrightarrow \pi_*(E|_R)_p \longrightarrow 0$$

since $R^1\pi_*(I_{p_1,p_2} E|_X) = 0$ ($X \longrightarrow X_0$ being an affine morphism). We see that $\pi_*(E|_R)_p$ is the sky-scraper sheaf at p associated to the vector space $H^0(E|_R)$. We have $\dim H^0(E|_R) = \deg E|_R + n$ and we saw in Prop. 3.4 above that $\deg E|_R$ is the type 'a' of $\pi_*(E)$. On the other hand we see that

$$\dim(\pi_*(E)_p / m_{X_0,p} \pi_*(E)_p) = a + n.$$

Then the assertion (a) follows.

(b) Let $A = \mathcal{O}_{X_0,p}$ and let B the semi-local ring of X at p_1, p_2 (integral closure of A). Then V_i are represented by free B-modules and to prove (b) is to show that an A-module homomorphism $B^m \longrightarrow B^n$ is in fact a B-module homomorphism. This follows from the fact that $\mathrm{Hom}_A(B, B) \simeq B$ (multiplication by elements of B).

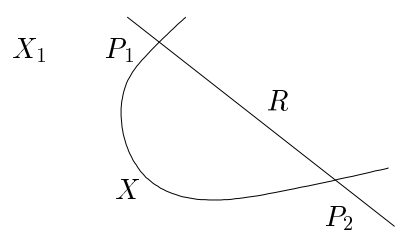
(c) By (a) and (b), it follows that if $\pi_*(E_1) \simeq \pi_*(E_2)$, then $I_{p_1,p_2}E_1|_X \simeq I_{p_1,p_2}E_2|_X$ and then by multiplying by I_{p_1,p_2}^{-1} the assertion (c) follows.

(d) Now if $\{\pi_*(E)\}$ is bounded, by (a) and (b) it follows easily that $\{E|_X\}$ is a bounded family and all the other assertions also follow (see [NS] for more details).

4 The moduli space

We shall now illustrate our construction of the moduli space for the case of line bundles. This is quite simple but provides a motivation for the general considerations.

The curves X_k and X_0 are as in Def. 3.2. For a line bundle L of X_1, we write $L_1 = L|_X$ and $L_2 = L|_R$ ($R \simeq \mathbb{P}^1$).



The line bundle L is defined by the following isomorphisms of 1-dimensional spaces:

$$\theta_1 : (L_1)_{p_2} \longrightarrow (L_2)_{p_1}, \quad \theta_2(L_1)_{p_2} \longrightarrow (L_2)_{p_2}.$$

We represent L by the 4-tuple $(L_1, L_2, \theta_1, \theta_2)$. If we modify the 4-tuple by an automorphism of L_1, as well as an automorphism of L_2, the resulting 4-tuple represents a line bundle isomorphic to L. We see that

$$\text{(i)} \qquad \begin{cases} (L_1, L_2, \theta_1, \theta_2) \sim (L_1, L_2, \mu_1, \mu_2) \\ \iff \mu_1 = a\theta_1, \quad \mu_2 = a\theta_2, \quad \text{a non-zero scalar.} \end{cases}$$

Let $P^{a,b}$ denote the set of isomorphism classes of line bundle L on X_1 such that $\deg L_1 = a$ and $\deg L_2 = b$. Then we see that the canonical morphism $P^{a,b} \longrightarrow \operatorname{Pic}^a(X)$ $(L \longmapsto L_1)$ is a principal $\mathbb{G}_m$ fibration. Also note that

we have an identification

$$\text{(ii)} \qquad \begin{cases} \pi_* : P^{a,0} \xrightarrow{\sim} \operatorname{Pic}^a(X_0) \\ L \longmapsto \pi_*(L), \quad \pi : X_1 \longrightarrow X_0 \end{cases}$$

since in this case L_2 is trivial. In particular, if we write $G(1,0)_0 = \operatorname{Pic}^0(X_0)$, we have $G(1,0)_0 \simeq P^{0,0}$.

Let g be an automorphism of X_1 such that it is identity on the component X. If $L \in P^{a,0}$ we see that $g^*(L) \simeq L$. Suppose now that $L \in P^{a,1}$. Then $g^*(L)|_R \simeq L_2 = \mathcal{O}(1)$. We have $R \simeq \mathbb{P}^1 \simeq \mathbb{P}(V)$, where $V^* = H^0(L_2)$. Then identifying p_1, p_2 with $\{0, \infty\}$, the automorphism g on R is represented by an automorphism of V as a diagonal matrix

$$\begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}.$$

From these considerations, it follows that if L is represented by $(L_1, L_2, \theta_1, \theta_2)$, then $g^*(L)$ is represented by a 4-tuple

$$(L_1, L_2, t_1\theta_1, t_2\theta_2), \quad t_1 \neq 0, t_2 \neq 0.$$

Let us introduce an equivalence relation in $P^{a,1}$, namely $L \sim M$ if $L \simeq g^*(M)$ for some $g \in \operatorname{Aut} X_1$ such that g is identity on X. Let us take the case $a = -1$ so that we deal with the case $\deg L$ (total degree of L) $= 0$. Let $G(1,0)_1$ denote the set of equivalence classes in $P^{-1,1}$. Then we see that $G(1,0)_1$ identifies with $\operatorname{Pic}^{-1} X$ (by $L \longmapsto L_1$). We see also that if $L \sim M$ (equivalence relation) then $\pi_*(L) \simeq \pi_*(M)$. In fact, we have a bijection

$$\text{(iii)} \qquad \pi_* : G(1,0)_1 \simeq P_1$$

where P_1 is the set of isomorphism classes of torsion free sheaves of rank one, degree zero and type one on X_0.

If $U(1,0)$ is the moduli space of isomorphism classes of torsion free sheaves of rank one and degree zero on X_0, then one knows that

$$U(1,0) \backslash \operatorname{Pic}^0(X_0) \simeq P_1.$$

Let $G(1,0)$ denote the disjoint union

$$G(1,0) = G(1,0)_0 \coprod G(1,0)_1, \quad G(1,0)_0 = \operatorname{Pic}^0(X_0).$$

Thus we get a bijection

(iv) $$\pi_* : G(1,0) \longrightarrow U(1,0).$$

As we shall see, we have a similar but more complicated phenomenon in higher rank. The generalisation of π_* is no longer a bijection.

Definition 4.1.

(i) Let E be a vector bundle on X_k such that $E|_R$ is strictly positive. It is said to be *stable* if $\pi_*(E)$ is a stable (torsion free) sheaf on X_0. Note that it has all the nice properties stated in Prop. 3.4, in particular $E|_R$ is strictly standard.

(ii) We call two vector bundles E_1, E_2 on X_k *equivalent* if $E_1 \simeq g^*(E_2)$, where g is an automorphism of X_k, which is identity on the component X (g could move points on R).

(iii) We set

$$\begin{aligned} G(n,d)_k &= \left\{ \begin{array}{l} \text{equivalence classes of stable vector bundles on } X_k \\ \text{of rank } n \text{ and degree (total degree) } d \end{array} \right. \\ G(n,d) &= \coprod_{0 \le k \le n} G(n,d)_k \quad \text{(disjoint sum).} \end{aligned}$$

Note that $G(n,d)_0$ is the set of isomorphism classes of stable vector bundles of rank n and degree d on X_0. We shall see that if $(n,d) = 1$, $G(n,d)$ has a natural structure of a projective variety with a birational morphism onto the projective variety $U(n,d) = U_X(n,d)$ (the moduli space of stable torsion free sheaves of rank n and degree d on X_0) and that it has all the good properties like specialization (stated in §2).

Let L be a line bundle on X_0. If E is a vector bundle on X_k, then since $\pi_*(E \otimes \pi^*(L)) \simeq \pi_*(E) \otimes L$, we note that $\pi_*(E)$ is torsion free $\Longleftrightarrow$ $\pi_*(E \otimes \pi^*(L))$ is torsion free. Since stable torsion free sheaves of rank n and degree d form a bounded family, we see that for $l \gg 0$ and $E \in G(n,d)$, $E \otimes \pi^*(L)$ has all the good properties of (d) of Remark 3.2. Thus without loss of generality we may assume that if $E \in G(n,d)$, $H^0(E)$ generates E, the canonical morphism

$$\phi_E : X_k \longrightarrow Gr(H^0(E), n)$$

is a closed immersion and the properties (d) of Remark 3.2 are satisfied.

We can identify $Gr(H^0(E), n)$ with the standard Grassmannian $Gr(m, n)$ (this identification is upto an automorphism of $Gr(m, n)$ i.e. upto an element of $PGL(m)$) and ϕ_E with a morphism (denoted again by ϕ_E).

$$\phi_E : X_k \longrightarrow Gr(m, n) \quad (m = \dim H^0(E)).$$

Now $PGL(m)$ operates canonically on $Gr(m, n)$ and also on $X_0 \times Gr(m, n)$ by taking the identity action on X_0. Now ϕ_E gives rise to a closed immersion:

$$\psi_E : X_k \hookrightarrow X_0 \times Gr(m, n), \quad \psi_E = (\pi, \phi_E).$$

Let $E_1, E_2 \in G(n, d)_k$ and ψ_{E_1}, ψ_{E_2} the imbeddings into $X_0 \times Gr(m, n)$. Then the important remark is the observation:

$$\begin{cases} E_1 \sim E_2 \text{ (equivalence relation, see (ii) of Def. 4.1)} \\ \Longleftrightarrow g\ (\mathrm{Im}\ \psi_{E_1}) = \mathrm{Im}\ \psi_{E_2},\ g \in PGL(m). \end{cases}$$

We observe also that the Hilbert polynomial P_1 of $\mathrm{Im}\,\psi_E$ is the same for all $E \in G(n, d)$. Thus $\mathrm{Im}\,\psi_E \in \mathrm{Hilb}^{P_1}(X_0 \times Gr(m, n))$ (we choose some polarisation on $X_0 \times Gr(m, n)$). Note that the action of $PGL(m)$ on $X_0 \times Gr(m, n)$ induces a canonical action of $PGL(m)$ on $\mathrm{Hilb}^{P_1}(X_0 \times Gr(m, n))$. The foregoing discussion shows that $G(n, d)$ can be identified (set theoretically) as the set of $PGL(m)$ orbits of a certain $PGL(m)$ stable subset of $\mathrm{Hilb}^{P_1}(X_0 \times G(m, n))$. We observe that given ψ_E, E is expressed canonically as a quotient of the trivial rank m vector bundle

$$\mathcal{O}^m_{X_k} \longrightarrow E,\ H^0(\mathcal{O}^m_{X_k}) \xrightarrow{\sim} H^0(E), \quad H^1(E) = 0.$$

Then by (d) of Remark 3.2, $\pi_*(E)$ is a quotient of the trivial vector bundle of rank m on X_0

$$(*) \qquad \mathcal{E} = \mathcal{O}^m_{X_0} \longrightarrow \pi_*(E) \ \text{ and } \ H^0(\mathcal{O}^m_{X_0}) \xrightarrow{\sim} H^0(\pi_*(E))$$

Let P_2 be the Hilbert polynomial of the stable torsion free sheaves on X_0 of rank n and degree d of X_0. Let $Q = Q(\mathcal{E}/P_2)$ be the Quot scheme of quotients of the trivial vector bundle $\mathcal{E}$ of rank m on X_0 and R, R^s the $PGL(m)$ stable open subsets of Q, which are now standard (R is defined by the condition that the corresponding point of the Quot scheme defines a quotient which is torsion free, as well as the second condition in $(*)$. The

subset R^s of R corresponds to these quotients which are moreover stable). Recall that R^s mod $PGL(m) \simeq U(n,d)^s$ and that R^s is a principal bundle over $U(n,d)^s$.

The following are the main steps in giving a canonical structure of a quasi-projective variety on $G(n,d)$:

(I) The subset $Y^s = Y(n,d)^s \subset \mathrm{Hilb}^{P_1}(X_0 \times Gr(m,n))$, $Y^s = \{\mathrm{Im}\,\psi_E\}$ ($E \in G(n,d)$) is $PGL(m)$ stable and has a natural structure of an (irreducible) variety whose singularities are normal crossings.

(II) The map $\theta : Y^s \longrightarrow R^s$ defined by y (represented by ψ_E or $\mathrm{Im}\,\psi_E$) $\longmapsto$ the point of R^s represented by $(*)$ above, is a $PGL(m)$ equivariant morphism.

(III) The morphism θ is proper.

We shall now indicate how admitting I, II and III, we get a nice structure of a variety on $G(n,d)$.

Let $R^{s,0}$ denote the $PGL(m)$ stable open subset of R^s represented by vector bundles on X_0. Then a point of $\theta^{-1}(R^{s,0})$ is represented by Ψ_E such that the equivalence class is in $G(n,d)_0$ i.e. a closed immersion $\psi_E : X_0 \hookrightarrow X_0 \times Gr(m,n)$. Then it is easy to see that the morphism

$$\theta : \theta^{-1}(R^{s,0}) \longrightarrow R^{s,0}$$

is an isomorphism. Hence it follows that θ is a birational morphism. Since $\theta : Y^s \longrightarrow R^s$ in $PGL(m)$ equivariant and $R^s \longrightarrow U(n,d)^s$ is a principal bundle, it is easily seen that the quotient Y^s mod $PGL(n)$ exists and in fact that $Y^s \longrightarrow Y^s$ and $PGL(m)$ is a principal $PGL(m)$ bundle. Thus we get a canonical structure of a variety on $G(n,d)$. Further, since Y^s has normal crossing singularities and $Y^s \longrightarrow Y^s$ and $PGL(m)$ is a principal fibration, we see that $G(n,d)$ has normal crossing singularities.

To prove that $G(n,d)$ is quasi-projective, we use GIT, which can also be used to give simultaneously a variety structure on $G(n,d)$. We can suppose that $Q^s = R^s$, $Q^{ss} = R^{ss}$ (Q^s-stable points of a polarisation on $Q, \cdots$, for example, as has been done recently by Simpson). Then since θ is a projective morphism, it is easily seen that we can find a $PGL(m)$ equivariant factorisation (instead of taking the Quot scheme we can take the closure of

R^{ss} in Q, but we use the same notation):

$$\begin{array}{ccccc} & Y^s & \hookrightarrow & Z & \\ \theta & \downarrow & & \downarrow & \lambda \\ & R^s & \hookrightarrow & Q & \end{array}$$

where Z is a projective variety with an action of $PGL(m)$ lifting to an ample line bundle $\mathcal{O}_Z(1)$. Consider now the polarisation $L = \lambda^*(\mathcal{O}_Q(a)) \otimes \mathcal{O}_Z(1)$ on Z. Then with the usual notations, one knows that for 'a' sufficiently large, we have:

(i) $\lambda^{-1}(R^s)$ $(= \lambda^{-1}(Q^s)) \subset Z(L)^s$.

(ii) λ maps $Z(L)^{ss}$ onto R^{ss}.

It follows then that Y^s mod $PGL(m)$ exists as a quasi-projective variety and thus $G(n,d)$ acquires a canonical structure of a quasi-projective variety. Further θ induces a canonical (birational) projective morphism $G(n,d) \longrightarrow U(n,d)^s$. If $(n,d) = 1$, $U(n,d)^s = U(n,d)$ and it follows that $G(n,d)$ is projective.

For proving I, II, III we require more formal considerations.

Definition 4.2. Let $\mathcal{Y}$ be the functor defined as follows:

$$\mathcal{Y} : (K - \text{schemes}) \longrightarrow \text{Sets}.$$

$\mathcal{Y}(T) =$ set of closed subschemes $\Delta \hookrightarrow X_0 \times T \times Gr(m,n)$ such that:

(i) the induced projection map $p_{23} : \Delta \longrightarrow T \times Gr(m,n)$ is a closed immersion. We denote by E the rank n vector bundle on Δ, obtained as the pull-back of the rank n tautological quotient bundle on $Gr(m,n)$

(ii) the projection $p_2 : \Delta \longrightarrow T$ is a proper flat family of curves $\{\Delta_t\}$, $t \in T$, such that Δ_t is a curve of the form X_k. Besides, the map $(X_k \simeq)\Delta_t \longrightarrow X_0$ (induced by $\Delta \longrightarrow X_i \times T$) is the canonical $\pi : X_k \longrightarrow X_0$ that we have been considering

(iii) the vector bundle E_t on Δ_t $(E_t = E|_{\Delta_t})$ is of degree d (and rank n) with $d = m + n(g-1)$

(iv) by the definition of E, we get a quotient representation $\mathcal{O}^m_{\Delta_t} \longrightarrow E_t$ and we assume that this induces an isomorphism $H^0(\mathcal{O}^m_{\Delta_t}) \xrightarrow{\sim} H^0(E_t)$. In particular, $\dim H^0(E_t) = m$ and it follows that $H^1(E_t) = 0$.

Proposition 4.1. The functor $\mathcal{Y}$ is represented by a $PGL(m)$ stable subscheme Y of $\mathrm{Hilb}^{P_1}(X_0 \times Gr(m,n))$ (P_1 being the Hilbert polynomial of the closed subscheme Δ_t of $X_0 \times Gr(m,n)$, choosing of course a polarisation). Further Y is an (irreducible) variety with (analytic) normal crossing singularities.

This is essentially due to Gieseker and some indication of proof will be given later.

Proposition 4.2. Let Δ be the universal object representing the functor $\mathcal{Y}$ above. Consider the "universal" closed immersion

$$\Delta \hookrightarrow X_0 \times Y \times Gr(m,n)$$

defined by $\mathcal{Y}$. This defines a flat family of curves $\Delta \longrightarrow Y$. We have also a vector bundle E on Δ obtained as the pull-back of the tautological quotient bundle of rank n of $Gr(m,n)$. Then E defines a family $\{E_y\}$ of vector bundles on $\{\Delta_y\}$, $y \in Y$. We denote by $\pi_y : \Delta_y \longrightarrow X_0$ the morphism induced by the first projection p_1, to be consistent with our earlier notation. We observe that $(\pi_y)_*(E_y)$ comes with a quotient representation

$$(*) \qquad \mathcal{O}^n_{X_0} \longrightarrow (\pi_y)_*(E_y) \quad \text{with} \quad H^0(\mathcal{O}^m_{X_0}) \xrightarrow{\sim} H^0((\pi_y)_*(E_y)).$$

Hence $(*)$ defines a point of the open subscheme R of $Q(\mathcal{E}/P_2)$ (Quot scheme mentioned above). Then we claim that the map $\theta : Y \longrightarrow R$, defined by $y \longmapsto$ the point of R defined by $(*)$, is a morphism.

Proof. We give a sketch. We have a commutative diagram

$$\begin{tikzcd} \Delta \arrow[rr, "\pi"] \arrow[dr, "p"'] & & X_0 \times Y \arrow[dl, "q"] \\ & Y & \end{tikzcd}$$

where π is the projection p_{12}, $p =$ projection p_2 and $q =$ canonical projection onto Y. Using the fact that the fibres of π are connected and the nice nature of the singularities of Y, it follows that

$$(a) \qquad \pi_*(\mathcal{O}_\Delta) = \mathcal{O}_{X_0 \times Y}.$$

We have a quotient representation

$$\mathcal{O}_\Delta^m \longrightarrow E \quad \text{on} \quad \Delta.$$

Using (a) and applying π_*, we get a homomorphism

$$(b) \qquad \mathcal{O}_{X_0 \times Y}^m \longrightarrow \pi_*(E).$$

The crucial point is to show that

$$(*) \qquad \begin{cases} \pi_*(E) \text{ behaves well for restriction to fibres over } Y, i.e. \\ \pi_*(E)|_{q^{-1}(y)} \simeq (\pi_y)_*(E_y) \\ q^{-1}(y) \simeq X_0 \times y \simeq X_0 \end{cases}$$

for one sees easily from $(*)$ that (b) is surjective and that the Hilbert polynomial of $\pi_*(E)|_{q^{-1}(y)}$ is P_2. Since Y is reduced, $\pi_*(E)$ is flat over Y. Thus (b) defines a morphism of Y into $Q(\mathcal{E}|P_2)$ which factors enough R. Thus $(*)$ and hence the above proposition is a consequence of the following:

Lemma 4.1. Suppose that we have a commutative diagram

$$\begin{tikzcd} Z \arrow[rr, "\pi"] \arrow[dr, "p"'] & & W \arrow[dl, "q"] \\ & T & \end{tikzcd}$$

such that p, q and π are projective morphisms, $\pi_*(\mathcal{O}_Z) = \mathcal{O}_W$ and p is flat. Let E be a vector bundle on Z such that

$$R^1(\pi_t)_*(E|_{Z_t}) = 0 \quad i \geq 1, \ t \in T.$$

Then $\pi_*(E)$ behaves well for restriction to fibres over T i.e.

$$\pi_*(E)|_{W_t} \simeq (\pi_t)_*(E|_{Z_t}), \quad t \in T.$$

Further

$$H^0(Z_t, E|_{Z_t}) \simeq H^0(W_t, (\pi_t)_*(E|_{Z_t})).$$

Proof. We refer to [NS].

Since $\theta : Y \longrightarrow R$ is a morphism and R^s is open in R, it follows that Y^s is open in Y. If we denote by R^{tf} the open subset of R corresponding to quotients which are torsion free, we see that R^{tf} is open in R and hence $Y^{tf} = \theta^{-1}(R^{tf})$ is also open in Y. We see that we have a canonical morphism

$$\pi_* : (Y^s \mathrm{mod}\, PGL(m)) = G(n,d) \longrightarrow U(n,d)^s = (R^s \mathrm{mod}\, PGL(m).)$$

For properness and good specialization properties, we require to work with a more general base scheme.

Thus admitting I, II and III we have the following:

Theorem 1. We have a natural structure of a quasi-projective variety on $G(n,d)$ and the canonical map $\pi_* : G(n,d) \longrightarrow U(n,d)$ is a proper birational morphism. If $(n,d) = 1$, $G(n,d)$ is projective. Further the singularities of $G(n,d)$ are normal crossings.

5 Properness and specialization

Definition 5.1. Let $S = \mathrm{Spec}\, A$, where A is a d.v.r. (in fact K-algebra) with residue field K. Let $\mathcal{X} \longrightarrow S$ be a proper, flat family of curves such that the closed fibre $\mathcal{X}_{s_0} \simeq X_0$ and the generic fibre $\mathcal{X}_\xi$ is smooth (s_0-closed point of S, ξ generic point of S). We suppose also that $\mathcal{X}_S$ is regular over K (we may also have to fix a section of $\mathcal{X}$ over S passing through the smooth points of the fibres).

Definition 5.2. Same as in Def. 4.2 over the base S. The functor is denoted by $\mathcal{Y}_S$: (S-schemes) $\longrightarrow$ Sets. The one point to remember is that,

for example, the morphism $\Delta_t \longrightarrow (\mathcal{X} \times_S T)_t$ is an isomorphism if t maps to the generic point of S [$\mathcal{Y}_S(T)$ = set of closed subschemes

$$\Delta \hookrightarrow \mathcal{X} \times_S T \times_S Gr(m,n)(Gr(m,n) = Gr(m,n)_S)$$

$\Delta \longrightarrow \mathcal{X} \times_S T$ flat family of curves etc.].

Now Prop. 4.1 generalizes as:

Proposition 5.1. The factor $\mathcal{Y}_S$ is represented by a $PGL(m)$ stable subscheme Y_S of the S-scheme $Hilb^{P_1}(\mathcal{X} \times_S Gr(m,n)$ and $Y_S \longrightarrow S$ has the following properties:

(i) the closed fibre $(Y_S)_{s_0}$ is a variety with (analytic) normal crossing singularities

(ii) the generic fibre $(Y_S)_\xi$ is smooth

(iii) Y_S is regular over K (of course $Y_S \longrightarrow S$ is flat).

This proposition is essentially to be found in Gieseker [G]. Some indication of proof will be given later.

Proposition 5.2. Proposition 4.2 generalizes to give a canonical morphism $\theta : Y_S \longrightarrow R_S$ (R_S - the obvious open subset of the Quot scheme $Q(\mathcal{E}|P_2)$ associated to $\mathcal{X} \longrightarrow S$). Then θ induces morphisms (denoted by the same letter)

$$\begin{cases} Y_S^s \longrightarrow R_S^s, \quad Y_S^{tf} \longrightarrow R_S^{tf}, \quad (R_S^s \subset R_S^{tf}) \\ Y_S^s = \theta^{-1}(R_S^s), \quad Y_S^{tf} = \theta^{-1}(R_S^{tf}). \end{cases}$$

Note that the S-morphisms θ are isomorphisms over $S \backslash \{s_0\}$ i.e. θ induces an isomorphism of their generic fibres over S. (e.g. $\theta_\xi : (Y_S^s)_\xi \longrightarrow (R_S^s)_\xi$). In fact θ is an isomorphism over the bigger open subset R_S^v corresponding to vector bundles i.e. defined by $q \in Q_S(\mathcal{E}|P_2)$ such that the corresponding quotient sheaf is locally free.

Proposition 5.3. The morphism

$$\theta : Y_S^s \longrightarrow R_S^s \quad (\text{resp. } Y_S^{tf} \longrightarrow R_S^{tf})$$

is proper.

Proof. We shall now outline the proof giving the main points.

We see that it suffices to prove that $\theta : Y_S^{tf} \longrightarrow R_S^{tf}$ is proper. We apply the valuation criterion. Let $T = \operatorname{Spec} B$, B d.v.r with residue field K. Given a rational map $\mu : T \longrightarrow Y_S^{tf}$ such that $(\theta \cdot \mu) : T \longrightarrow R_S^{tf}$ is a morphism, then to show that μ is a morphism. We see that $(\theta \cdot \mu)$ defines a family of torsion free sheaves F of rank n on the base change $\mathcal{X}_T = \mathcal{X} \times_S T$ parametrized by T. We can suppose that F is a quotient of the trivial bundle of rank m:

$$\mathcal{O}_{\mathcal{X}_T}^m \longrightarrow F \longrightarrow 0.$$

We can suppose that the morphism $T \longrightarrow S$ does not map to the closed point of S so that for the family of curves $\mathcal{X}_T \longrightarrow T$, the fibre is smooth over the generic point of T (i.e. the generic fibre of $\mathcal{X}_T \longrightarrow T$ is the base change of the generic fibre of $\mathcal{X} \longrightarrow S$). The question is whether we can lift this to a T-valued point of Y_S^{tf} (satisfying the required properties). We will see that there is a canonical candidate for this.

We see also that the closed fibre of $\mathcal{X}_T \longrightarrow T$ identifies with the closed fibre of $\mathcal{X} \longrightarrow S$ and denote by the same 'p' the singular point of the closed fibre of $\mathcal{X}_T \longrightarrow T$, which is $\simeq X_0$. Now F is locally free outside 'p'. The quotient representation gives a T-morphism

$$g : \mathcal{X}_T \backslash \{p\} \longrightarrow Gr(m,n) \quad (Gr(m,n)_T).$$

We can also suppose that g is an immersion. Let Γ_g be the graph of g so that we have a closed immersion of T-schemes

$$(*) \qquad \Gamma_g \hookrightarrow \mathcal{X}_T \times_T Gr(m,n).$$

Let $\pi_g : \Gamma_g \longrightarrow \mathcal{X}_T$ be the canonical projection which is a T-morphism. Obviously π_g induces an isomorphism over the generic fibres, in fact an isomorphism over $\mathcal{X}_T \backslash \{p\}$. Let E be the vector bundle of rank n on Γ_g induced by the quotient bundle on $G_r(m,n)$ (through $(*)$). Then for the required lifting and properness, it suffices to prove the assertion:

(A)

$$\begin{cases} \text{(i) The closed fibres of } \Gamma_g \longrightarrow T \text{ is a curve of the form } X_k \text{ (this} \\ \quad \text{implies that the morphism induced by } \pi_g \text{ on the closed fibres} \\ \quad \text{is the canonical morphism } X_k \longrightarrow X_0). \\ \text{(ii) } (\pi_g)_*(E) = F. \end{cases}$$

Note that π_g is an isomorphism over $\mathcal{X}_T \backslash \{p\}$ and E coincides with F outside $\{p\}$ i.e. E and F coincide on $\mathcal{X}_T - \{p\}$.

We see that the assertions (A) are really local with respect to $\mathcal{X}_T$ i.e. it suffices to check them over a neighbourhood of p in $\mathcal{X}_T$. More precisely let C be the local ring of $\mathcal{X}_T$ at p. Then F is represented by a torsion free C-module which is B-flat ($T = \operatorname{Spec} B$) and F is the quotient of a free module of rank m over C (F is of rank n over B) and Γ_g should be viewed as the graph of a morphism

$$g : \operatorname{Spec} C \backslash \{p\} \longrightarrow Gr(m, n)$$

(restriction of g to $\operatorname{Spec} C \backslash \{p\}$) and denoted by the same letter. Thus we have a commmutative diagram

$$\begin{tikzcd}
\Gamma g \arrow[rr, "\pi_g"] \arrow[dr] & & \operatorname{Spec}\ C \arrow[dl] \\
& T = \operatorname{Spec}\ B &
\end{tikzcd}$$

Let m' denote the minimal number of generators of the C-module F, we have a factorisation

$$\begin{tikzcd}
\operatorname{Spec}\ C - \{p\} \arrow[r] \arrow[dr] & Gr(m, n) \\
& Gr(m', n) \arrow[u, hook]
\end{tikzcd}$$

We can assume without loss of generality that $m =$ the number of minimal generators of F as a C-module.

The point is that we have a concrete description of the C-module F which facilitates the checking of the local version of (A).

We see that we can suppose that C (resp. B and A) are *complete* local rings. It is also not difficult to see that F is equal to its bidual (see [NS]) i.e.

$$F^{**} = F \quad (F^{**} \text{ bidual of } F \text{ as a } C\text{-module}).$$

We suppose that char K is zero, say $K = \mathbb{C}$. Now $A = K[[t]]$ and we see that because of our hypothesis, the completion $\widehat{\mathcal{O}}_{\mathcal{X},p}$ of the local ring at 'p' is given by $K[[x,y]]$ and the canonical morphism $\operatorname{Spec}\widehat{\mathcal{O}}_p \longrightarrow \operatorname{Spec} A(= S)$, is given by

$$A = K[[t]] \longrightarrow K[[x,y]], \quad t \longmapsto xy.$$

The canonical homomorphism $A \longrightarrow B$ $(B \simeq K[[t]])$ is given by $t \longmapsto t^r$. From these considerations it follows easily that

$$C \simeq K[[x,y,y]]/(xy - t^r)$$

and the canonical homomorphism $B \longrightarrow C$ is defined by $t \longmapsto$ image of t in C.

Let $D = K[[u,v]]$. Consider the action of the cyclic group Γ_r of order r operating on D by:

$$\begin{cases} (u,v) \longmapsto (\zeta u, \overline{\zeta} v), \ \zeta \ r\text{th root of unit representing our element of } \Gamma_r \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \overline{\zeta} \text{ complex conjugate of } \zeta. \end{cases}$$

We see that

$$\begin{cases} C = D^{\Gamma_r} \quad (\Gamma_r\text{-invariants in } D), \\ \quad \text{taking } x = u^r,\ y = v^r,\ t = uv. \end{cases}$$

Let f be the canonical morphism $f : \operatorname{Spec} D \longrightarrow \operatorname{Spec} C$ and $f_0 : \operatorname{Spec} D\backslash\{0\} \longrightarrow \operatorname{Spec} C\backslash\{p\}$, the restriction of f to $\operatorname{Spec} D\backslash\{0\}$. Now C is normal with an isolated (rational) singularity at 'p' and of type A. Also f_0 is an unramified covering. Then $f_0^*(F)$ can be extended to a locally free sheaf on $\operatorname{Spec} D$ represented by a free D-modules $\mathcal{F}$. We see that $\mathcal{F}$ is canonically a $(D - \Gamma_r)$ module (i.e. D-module with an action of Γ_r). By the reflexivity of F, we deduce easily that

$$F \simeq (\mathcal{F})^{\Gamma_r}.$$

One knows that a $(D - \Gamma_r)$ free modules is given by a representation of Γ_r, which is a direct sum of 1-dimensional characters. Thus F is a direct sum of Γ_r-line bundle L of the form

$$\begin{cases} L = \operatorname{Spec} D \times \mathbb{C}, \text{ action of } \Gamma_r \text{ is given by} \\ \zeta = \{(u,v), \mathbb{C}\} = \{(\zeta u, \overline{\zeta} v, \zeta^s \theta\}, \quad \theta \in \mathbb{C} \end{cases}$$

(we could view $\operatorname{Spec} D$ as a 2-dimensional disc). A Γ-invariant section of this bundle L is easily identified with a function δ on the line satisfying:

$$\delta(\zeta u, \xi v) = \zeta^s \delta(u,v).$$

Then we check that the Γ_r-invariant sections of L are generated by u^s and v^{r-s} as a C-module. We have

$$u^{r-s}(u^s, v^{r-s}) = (u^r, (uv)^{r-s}) = (x, t^{r-s}).$$

Thus we see easily that the C-module F is of the following form:

$$F \simeq \bigoplus_{i=1}^{n} (t^{a_i}, x), \quad 0 \leq a_1 \leq a_2 \leq \cdots \leq a_n.$$

We see that (t^{a_i}, x) is a principal ideal if and only if $a_i = 0$ so that

$$F \simeq \mathcal{O}_C^b \bigoplus_{i=1}^{n-b} (t^{a_i}, x), 0 < a_1 \leq a_2 \leq \cdots \leq a_{n-b}.$$

As we saw above, we have taken $m =$ minimal number of generators of F which is equal to $b + 2(n-b)$. We see that the morphism Γ_g factorises as:

$$\begin{tikzcd}
\text{Spec } C - \{p\} \arrow[rr, "\Gamma g"] \arrow[dr] & & Gr(m,n) \\
& Gr(2(n-b),(n-b)) \arrow[ur] &
\end{tikzcd}$$

Thus we can suppose without loss of generality that $b = 0$ i.e. $m = 2n$. Then we see that Γ_g factorises as:

$$\begin{tikzcd}
\text{Spec } C - \{p\} \arrow[r] \arrow[dr, "\Gamma g"'] & \mathbb{P}^1 \times \cdots \times \mathbb{P}^1 (n \text{ times}) \arrow[d] \\
& Gr(2n,n)
\end{tikzcd}$$

Let $\alpha_1, \cdots, \alpha_l$ be the distinct ones occurring among the $\{a_i\}$ with multiplicity β_i, $1 \leq i \leq l$. Then we see that we have a factorisation

$$\begin{tikzcd}
Spec\ C\backslash\{p\} \arrow[r] \arrow[dr, "\Gamma g"'] & \mathbb{P}^1 \times \cdots \times \mathbb{P}^1 \quad (l \text{ times}) \arrow[d, hook, "\Delta_1"'] \arrow[d, hook, "\Delta_l"] \\
& (\mathbb{P}^1)^{\beta_1} \times \cdots \times (\mathbb{P}^1)^{\beta_l} \arrow[d, hook] \\
& Gr(2n, n)
\end{tikzcd}$$

where Δ_l are the diagonal morphisms $\mathbb{P}^1 \longrightarrow (\mathbb{P}^1)^{\beta_i}$, $1 \leq i \leq l$. We observe that the pull-back of the tautological quotient bundle on $Gr(2n, n)$ on $(\mathbb{P}^1)^l$ identifies with:

(a) $$\begin{cases} \mathcal{O}(1)^{\beta_1} \boxplus \mathcal{O}(1)^{\beta_2} \boxplus \cdots \boxplus \mathcal{O}(1)^{\beta_l} \quad \text{(external direct sum)} \\ \text{i.e. } \oplus_i \mathcal{O}(1)^{\beta_i},\ \mathcal{O}(1)^{\beta_i} \text{ coming from the } i\text{-th factor.} \end{cases}$$

We see also easily that the fibre of $\Gamma_g \longrightarrow \operatorname{Spec} C$ over the closed point of $\operatorname{Spec} C$ identifies with the chain R of $\mathbb{P}^1$'s in $(\mathbb{P}^1)^l$ (of length l) of the form

(b)
$$\begin{cases} \{\mathbb{P}^1 \times (0,1) \times \cdots \times (0,1)\} \cup \{(1,0) \times \mathbb{P}^1 \times (0,1) \times \cdots \times (0,1)\} \\ \cup \{(1,0) \times (1,0) \times \mathbb{P}^1 \times (0,1) \times \cdots \times (0,1)\} \cup \cdots \cup \{(1,0) \times (1,0) \times \cdots \times \mathbb{P}^1\}. \end{cases}$$

From these considerations one sees that $E|_R$ is strictly standard ($E|_{R_i} = \mathcal{O}(1)^{\beta_i} \oplus$ Trivial) that the good conditions of Remark 3.2 are satisfied and in fact that $(\pi_g)_*(E) = F$ so that (A) follows (see [NS] for more details).

Remark 5.1. *Local Theory (outline of proof of Prop. 5.1)*

The main steps could be formulated as follows:

I Let $\mathcal{Y}'_S$ be the functor obtained from $\mathcal{Y}_S$ by forgetting the imbeddings into Grassmannians (see Def. 5.2) i.e. $\mathcal{Y}'_S(T)$ is a (flat) family of curves $\Delta \longrightarrow T$ together with a T-morphism $\Delta \longrightarrow \mathcal{X} \times_S T$ which induces the canonical morphisms of fibres over $t \in T$.

We claim that the functor $\mathcal{Y}_S \longrightarrow \mathcal{Y}'_S$ is formally smooth. The proof of this is quite standard and comes to extending vector bundles and sections over infinitesimal neighbourhoods, which are possible because of the vanishing of H^1 in Def. 4.2 (and Def. 5.2) and that we are in the case of curves.

II We take $S = \operatorname{Spec} A$, $A = K[[t]]$ and $W = \operatorname{Spec} K[[t_0, \ldots, t_r]]$ endowed with an S-scheme structure by $t \longmapsto t_0 \cdots t_r$. A crucial point is the construction by Gieseker of an element $\theta \in \mathcal{Y}'_S(W)$ defined by a family of curves $S \longrightarrow W$ and $\psi : Z \longrightarrow \mathcal{X} \times_S W$, having the following properties:

(a) the closed fibre of $Z \longrightarrow W$ is a curve X_r

(b) the closed subscheme of W corresponding to the singular fibres of $Z \longrightarrow W$ is the union of $t_i = 0$ so that it has, in particular, normal crossing singularities and is the inverse image of the closed point of S (by the morphism $W \longrightarrow S$)

(c) $Z \longrightarrow W$ provides a miniversal (effective) deformation of the singularities of the closed fibre of $Z \longrightarrow W$. More precisely, let V be an affine open subset of the closed fibre of $Z \longrightarrow W$, containing its singular points (or we can take the semi-local ring at the singular points). Then $Z \longrightarrow W$ defines a deformation $Z_V \longrightarrow W$ of V. The property is that $Z_V \longrightarrow W$ is an (effective) miniversal deformation of V,

(d) $\Psi^*(M) = L$, where L (resp. M) is the dualizing sheaf of Z (resp. $\mathcal{X} \times_S W$) relative to W.

Note that for $r = 0$, $Z \simeq \mathcal{X}$ and the property (C) holds, which is a consequence of the fact that $\mathcal{X}$ is regular.

Roughly speaking Z is obtained by taking the base change of $\mathcal{X}$ by $W \longrightarrow S$ and performing certain blow ups.

III We restrict the functor $\mathcal{Y}'_S$ to Spec of Artin local rings such that $\mathcal{Y}'_S$ (closed point) $\simeq X_r$ i.e. we take only $\mathcal{Y}'_S(T)$ where $T = \operatorname{Spec} B$, B is an Artin local algebra with residue field K and $\mathcal{Y}'_S(\operatorname{Spec} K) \simeq X_r$. In this

way we obtain a functor $\mathcal{Y}'_r \longrightarrow$ Spec (Artin local S-schemes). We view $\mathcal{Y}'_r$ as studying $\mathcal{Y}'_S$ in an infinitesimal neighbourhood of a point which represents X_r. Through the element $\theta \in \mathcal{Y}'_S(W)$ in II above, we get a canonical morphism

$$\text{(i)} \qquad \lambda : W \longrightarrow \mathcal{Y}'_r.$$

Also by (c) of II above, we get a functor

$$\text{(ii)} \qquad \mathcal{Y}'_r \xrightarrow{\alpha} \operatorname{Def} V \quad (\text{Deformationsof } V).$$

The important claim is that α and λ are isomorphisms

$$\text{(iii)} \qquad W \xrightarrow[\lambda]{\sim} \mathcal{Y}'_r \xrightarrow[\alpha]{\sim} \operatorname{Def} V.$$

By (c) of II above $(\alpha \circ \lambda)$ is an isomorphism and to prove (iii) it suffices to show that λ is formally smooth.

To prove that λ is formally smooth we have to do the following. Given an element $\delta \in \mathcal{Y}'_r(T)$ defined by $Z' \longrightarrow T$ and $\psi' : Z' \longrightarrow \mathcal{X} \times_S T$, we suppose further that there is a morphism $\phi_0 : T_0 \longrightarrow W$ such that the pull backs by ϕ_0 of $Z \longrightarrow W$ and $\psi : Z \longrightarrow \mathcal{X} \times_S W$ coincide with the restriction $\delta_0 \in \mathcal{Y}'_r(T_0)$ of δ to T_0 (T_0 closed subscheme of T with length one less than that of T). Then it is required to show that ϕ_0 can be extended to a morphism $\phi : T \longrightarrow W$ such that the pull-back of $Z \longrightarrow W$ and $\psi : Z \longrightarrow \mathcal{X} \times_S W$ are isomorphic to $Z' \longrightarrow T$ and ψ'.

The proof can be sketched as follows: Given Z', ψ' and ϕ_0, we can find a morphism $\phi : T \longrightarrow W$ such that the pull-back (Z_1, ψ_1) of (Z, Ψ) by ϕ is isomorphic to (Z', ψ') over T_0 (i.e. the restrictions to T_0 are isomorphic); besides (Z_1, ψ_1) and (Z', ψ') are locally isomorphic over T. This is a consequence of (c) of II above. Given these local isomorphisms (whose restrictions to T_0 define the given isomorphism of (Z, ψ_1) with (Z', ψ') over T_0), we find that the obstruction to extending these local isomorphisms to a global one over T is an element μ,

$$\mu \in H^1(X_r, \mathcal{H}om\,(\Omega'_{X_r}, \mathcal{O}_{X_r}))$$

where Ω'_{X_r} denotes the sheaf of differentials and $\mathcal{H}om$ denotes the "sheaf Hom". Similarly, the obstruction to extending $\psi'_0 : Z'_0 \longrightarrow \mathcal{X} \times_S T_0$ to a morphism $Z' \longrightarrow \mathcal{X} \times_S T$ is an element μ'

$$\mu' \in H^1(X_r, \mathcal{H}om\,(\pi^* \Omega'_{X_0}, \mathcal{O}_{X_r}))$$

and we see that μ maps to μ' under the canonical homomorphism

$$\text{(iv)} \qquad H^1(X_r, \mathcal{H}om\,(\Omega'_{X_r}, \mathcal{O}_{X_r})) \longrightarrow H^1(X_r, \mathcal{H}om\,(\pi^*\Omega'_{X_0}, \mathcal{O}_{X_r}))$$

where π is the canonical morphism $X_r \longrightarrow X_0$. But since ψ' extends ψ'_0, we see that $\mu' = 0$. One shows that (iv) is injective (see [G]) so that $\mu = 0$.

IV From the preceding discussions, we see that if we show that the functor $\mathcal{Y}_S$ is represented by an open subscheme of $H = \mathrm{Hilb}^{p_1}(\mathcal{X} \times_S Gr(m,n))$, Prop. 5.1 would be proved (of course the closed fibre of $\mathcal{Y}_S \longrightarrow S$ should also be shown to be irreducible. But this follows easily). Let $h \in H$ be a closed point represented by an element $\mathcal{Y}_S$ (Spec K), given by a curve X_r. We denote by

$$\text{(i)} \qquad \begin{cases} p_H : \Delta_H \longrightarrow H \text{ - the universal object over } H \text{ and } \psi_H \text{ the} \\ \text{canonical morphism } \psi_H : \Delta_H \longrightarrow \mathcal{X} \times_S H = \mathcal{X}_H. \end{cases}$$

Let $\mathcal{O}$ be the local ring of H at h. We set $C = \mathrm{Spec}\,\mathcal{O}$ and write

$$\text{(ii)} \qquad p_C : \Delta_C \longrightarrow C, \quad \psi_C : \Delta_C \longrightarrow \mathcal{X}_C$$

for the base changes of (i) by $C \longrightarrow H$. For proving the openness, the crucial point to show is that (ii) represents an element of $\mathcal{Y}_S(C)$. For this one has to use the property of dualizing sheaves given by II(d). One sees that the fibres of p_C have only ordinary double point singularities. Let L (resp. M) denote the dualizing sheaf of Δ (resp. $\mathcal{X}_C$) relative to C. The point is the claim:

$$\text{(iii)} \qquad \Psi_C^*(M) = L.$$

By II(d) and the arguments in III, it follows that

$$\psi_{C_n}^*(M_n) = L_n$$

where the subscript 'n' denote base changes by $C_n \longrightarrow C$ where $C_n = \mathrm{Spec}\,\mathcal{O}/m^n$ (m-maximal ideal of $\mathcal{O}$). Then by an application of Grothendieck's comparison theorem, the assertion (iii) follows (see [NS] for more details). Once one has (iii), that (ii) defines an element of $\mathcal{Y}_S(C)$ is a consequence of the following lemma which is easily seen.

Lemma 5.1. Let Y be a connected projective curve with arithmetic genus g and only ordinary double point singularities. Let $f : Y \longrightarrow D$ be a

morphism, where D is a smooth projective curve of genus g or $D \simeq X_0$. Suppose that the pull-back of the dualizing sheaf D is isomorphic to the dualizing sheaf of Y. Then f is an isomorphism if D is smooth; otherwise $Y \simeq X_r$ and f identifies with the canonical morphism $X_r \longrightarrow X_0$.

Thus we have the following:

Theorem 2. Let $\mathcal{X} \longrightarrow S$ be a proper flat family of curves as in Def. 5.1. Then we have a scheme $G(n,d)_S$ which is quasi-projective and flat over S (and projective if $(n,d) = 1$). Its generic fibre is the moduli space of stable vector bundles of rank n and degree d of the generic fibre of $\mathcal{X} \longrightarrow S$. Its closed fibre is the variety $G(n,d)$ (see Def. 4.1 and Theorem 1) whose singularities are normal crossings. Besides, $G(n,d)_S$ is regular as a scheme over k (recall that we have assumed that $\mathcal{X}$ is regular as a scheme over k).

Remark 5.2. Let $(n,d) = 1$. Consider the canonical morphism $\pi_* : G(n,d) \longrightarrow U(n,d)$. Let $F \in U(n,d)$ such that its type is r. Then we claim the following:

$(*)$

$$\left\{\begin{array}{l}\text{The fibre of } \pi_* \text{ over } F \text{ can be canonically identified with the "won-}\\ \text{derful compactification" of } PGL(r) \text{ (in the sense of De Concini and}\\ \text{Processi [D-P]).}\end{array}\right\}$$

We shall now outline a proof of $(*)$ when F is of type n (for type $\leq n$, the proof is similar). Let $E' \in G(n,d)$, such that $\pi_*(E') = F$. Then we see that $\pi_*(E'|_X) = F$ (see (c) in the proof of Prop. 3.4). It is not difficult to see that $E'|_X$ is a stable vector bundle (intuitively, because of(b) of Remark 3.2). In particular, $\mathrm{Aut}\,(E'|_X) \simeq \mathbb{G}_m$. Fix such an E'. If $E \in G(n,d)$ is an extension to X_k of $E'|_X$ $(1 \leq k \leq n)$, we observe that $\deg E|_R = n$, so that $\pi_*(E)$ is of type n (by (B) of Prop. 3.4). Then by (c) of Remark 3.2, the fibre of π_* over F identifies with the set of all $E \in G(n,d)$ which are extensions to X_k $(1 \leq k \leq n)$ of $E'|_X$. We shall now describe this more concretely. Let $I_1 = E'_{p_1}$ and $I_2 = E'_{p_2}$, so that they are considered as fixed

vector spaces of dimension n. Consider triples (V, α_1, α_2) such that:

$$\text{(i)} \begin{cases} (a) & V \text{ is a strictly standard vector bundle on } R \text{ of rank } n \text{ and degree } n \text{ (it follows that length of } R \leq n) \\ (b) & \alpha_i : V_{P_i} \xrightarrow{\sim} I_i \text{ are isomorphisms } (1 \leq i \leq 2). \\ (c) & \text{if } s \text{ is a section of } V \text{ such that } s(p_1) = s(p_2) = 0, \text{ then } s \text{ vanishes identically.} \end{cases}$$

We have the obvious notion of *isomorphisms* between triples. We see that if $\phi : V \longrightarrow V'$ is an isomorphism and (V, α, α_2) is a triple, then ϕ extends to an isomorphism

$$\theta : (V, \alpha_1, \alpha_2) \longrightarrow (V', \alpha_1', \alpha_2')$$

of triples; besides, θ and α_1', α_2' are uniquely determined. If $(\beta_1, \beta_2) \in \operatorname{Aut} I_1 \times \operatorname{Aut} I_2$, then we get a map of triples:

$$(V, \alpha_1, \alpha_2) \longrightarrow (V, \beta_1 \circ \alpha_1, \beta_2 \circ \alpha_2)$$

and this "action" of $\operatorname{Aut} I_1 \times \operatorname{Aut} I_2$ preserves isomorphism classes of triples. Let Z_1 denote the set of isomorphism classes of triples. Then the above map defines an action of $\operatorname{Aut} I_1 \times \operatorname{Aut} I_2$ on Z_1. Let Z_2 denote the set of *equivalence* classes of triples (as in Def. 4.1). Let $C = \mathbb{G}_m \times \mathbb{G}_m$ be the centre of $\operatorname{Aut} I_1 \times \operatorname{Aut} I_2$. The crucial point is that

$$Z_2 \simeq \text{fibre of } \pi_* \text{ over } F.$$

We have of course a canonical action of $\operatorname{Aut} I_1 \times \operatorname{Aut} I_2$ on Z_2, which is effectively an action of $(\operatorname{Aut} I_1 \times \operatorname{Aut} I_2)/C \simeq PGL(n) \times PGL(n)$.

We know that for V in (i), $\dim H^0(V) = 2n$ and $H^0(V)$ generates V. Let W be a fixed vector space of dimension $2n$ and $\underline{W}$ the trivial vector bundle on R of rank $2n$. Consider the set of triples $T = (\underline{W} \longrightarrow V, \alpha_1, \alpha_2)$ such that

$$\text{(ii)} \begin{cases} \text{(a) } (V, \alpha_1, \alpha_2) \text{ is a triple as in (i) and} \\ \text{(b) } \underline{W} \longrightarrow V \text{ induces an isomorphism } H^0(\underline{W}) \simeq W \xrightarrow{\sim} H^0(V). \end{cases}$$

We have canonical commuting actions of $\operatorname{Aut} W$ and $\operatorname{Aut} I_1 \times \operatorname{Aut} I_2$ on T. We see that:

the orbit space $T/\mathrm{Aut}\, W$ identifies with Z_1.

We fix two linear subspaces K_1 and K_2 of W such that $\dim K_i = n$ and $W = K_1 \oplus K_2$. We fix also identifications $W/K_i \simeq I_i$, $i = 1, 2$. Let S_1 be the set of all $\{\underline{W} \longrightarrow V\}$ such that

(iii)
$$\begin{cases} \text{(a) } V \text{ is as in (i)} \\ \text{(b) the canonical map } H^0(\underline{W}) = W \longrightarrow H^0(V) \text{ is an isomorphism} \\ \text{(c) } \mathrm{Ker}\,(W \longrightarrow V_{p_i}) = K_i, \text{ so that } V_{p_i} \simeq I_i \quad (1 \leq i \leq 2). \end{cases}$$

Set

(iv) $\quad S =$ set of triples $(\underline{W} \longrightarrow V, \alpha_1, \alpha_2)$ such that $\underline{W} \longrightarrow V$ is as in S_1.

We see that $\alpha_i \in \mathrm{Aut}\, I_i$. We see that given $t \in T$, there exists $g \in \mathrm{Aut}\, W$ such that $g \cdot t$ is in S. The subgroup H' of $\mathrm{Aut}\, W$ which leaves S (also S_1) stable, is precisely the subgroup which leaves K_1 as well as K_2 stable and $H' \simeq \mathrm{Aut}\, K_1 \times \mathrm{Aut}\, K_2 \simeq \mathrm{Aut}\, I_1 \times \mathrm{Aut}\, I_2$. Let H be the group (H' mod centre) $\simeq PGL(n) \times PGL(n)$. It is easily seen that for s in S there exists $h \in H'$ such that $h \circ s$ is the triple

$$(\underline{W} \longrightarrow V, id, id), \quad \underline{W} \longrightarrow V \text{ in } S_1$$

and, in fact, that the orbit space T mod $\mathrm{Aut}\, W$ can be identified with the above set of triples. Thus we see that we have a canonical identification $S_1 \simeq Z_1$.

We see that S_1 is the set of all embeddings $\psi : R \longrightarrow Gr(W, n)$ such that

(v)
$$\begin{cases} \text{(a) the pull-back of the tautological bundle on } Gr(W,n) \text{ by } \psi \text{ is a} \\ \quad \text{bundle } V \text{ as in (1), and} \\ \text{(b) the canonical map } H^0(W) = W \longrightarrow H^0(V) \text{ is an isomorphism} \\ \text{(c) } \psi(p_i) = k_i,\ k_i \text{ the points of } Gr(W,n) \text{ defined by } K_i \ (i = 1, 2). \end{cases}$$

Through the canonical identification $S_1 \longrightarrow Z_1$ we get an action of $\mathrm{Aut}\, I_1 \times \mathrm{Aut}\, I_2$ and we see that this identifies with the canonical action of H' of S_1 (H' is the subgroup of $\mathrm{Aut}\, W$ which leaves invariant the points k_i, $i = 1, 2$).

It is now clear that Z_2 can be identified with the set of closed subschemes $\psi(R)$ of $Gr(W,n)$, ψ as in (v). Thus we see that

(vi)
$$\begin{cases} \text{the fibre of } \pi_* \text{ over } F \text{ can be identified with the set of closed sub-} \\ \text{schemes } \psi(R) \text{ of } Gr(W,n),\ \psi \text{ as in (v)} \end{cases}$$

Roughly speaking the above set is the set of all nicely imbedded curves of type R (with length $\leq n$) in $Gr(W,n)$, passing through the two fixed points k_1, k_2 in $Gr(W,n)$. We see that all these subschemes of $Gr(W,n)$ have the same degree. Thus Z_2 can be identified with a closed subscheme of a Hilbert scheme Φ of subschemes of $Gr(W,n)$. We see that $H = (\operatorname{Aut} K_1 \times \operatorname{Aut} K_2)/C \simeq PGL(n) \times PGL(n)$ acts on Z_2 and $(\operatorname{Aut} K_1 \times \operatorname{Aut} K_2)$ identifies with the subgroup of $\operatorname{Aut} W$ fixing k_i, $i = 1, 2$.

Let R be of length one so that $R \simeq \mathbb{P}^1$. Then we see that for all the imbeddings ψ of $\mathbb{P}^1$ as in (v), the inverse image by ψ of the tautological bundle on $Gr(W,n)$ is $V = \bigoplus^n \mathcal{O}(1)$ on $\mathbb{P}^1$. Then the points of Z_2 which correspond to these imbeddings contribute one orbit under $H = PGL(n) \times PGL(n)$ and it is not difficult to show that Z_2 is the closure of this orbit. We can take an imbedding ψ of $\mathbb{P}^1$ as follows. Let D be a 2-dimensional vector space so that $\mathbb{P}^1 = \mathbb{P}(D)$. We fix a basis f_1, f_2 of D and we denote by p_1, p_2 the points of $\mathbb{P}^1$ represented by f_1, f_2. Set $W = D \otimes K$, where K is an n-dimensional vector space and $K_i = f_i \otimes K$ so that $W = K_1 \oplus K_2$. Let $\alpha_1, \ldots, \alpha_n$ be a basis of K. Then we see that

$$e_1 = f_1 \otimes \alpha_1, \quad e_3 = f_1 \otimes \alpha_2, \ldots, e_{2n-1} = f_1 \otimes \alpha_n$$

is a basis of K_1 and that

$$e_2 = f_2 \otimes \alpha_1, \quad e_4 = f_2 \otimes \alpha_2, \ldots, e_{2n} = f_2 \otimes \alpha_n$$

is a basis of K_2. Then the embedding

$$\psi : \mathbb{P}^1 \longrightarrow Gr(n, W)(\text{Grassmannian of } n\text{-dim subspaces of } W) \hookrightarrow \mathbb{P}(\bigwedge^n W)$$

is defined by

$$\begin{aligned} (x,y) = xf_1 + yf_2 &\longmapsto n\text{-dimensional linear subspace of } W \\ &\text{spanned by } (xe_1 + ye_2), (xe_3 + ye_4), \ldots, (xe_{2n-1} + ye_{2n}) \end{aligned}$$

i.e.

$$(x,y) \longmapsto (xe_1 + ye_2) \wedge (xe_3 + ye_4) \wedge \cdots \wedge (xe_{2n-1} + ye_{2n}) \in \bigwedge^n W.$$

We denote by z_0 the point of Z_2 defined by this imbedding. We see that $p_i \longmapsto k_i$ point of $Gr(n,W)$ represented by K_i. If we identify $\operatorname{Aut} K$ as the subgroup (id on D) $\times \operatorname{Aut} K$ of $\operatorname{Aut} W$, we see that $(\operatorname{Aut} K)$ fixes $\psi(\mathbb{P}^1)$ (pointwise). Then the isotropy subgroup of H at z_0 is $(\operatorname{Aut} K)/$ centre, which can be considered as the diagonal subgroup of $H \simeq PGL(n) \times PGL(n)$. Thus Z_2 is a $PGL(n) \times PGL(n)$ equivariant closure of $PGL(n)$ and it is natural to expect that Z_2 is the wonderful compactification of $PGL(n)$.

We now see that if L is the line bundle $\mathcal{O}_{Gr(n,W)}(1)$ on $Gr(n,W)$, then L restricts to the line bundle $\mathcal{O}_{\mathbb{P}^1}(n)$ on $\mathbb{P}^1$ and that

$$H^0(Gr(n,W),L) \longrightarrow H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(n)) \longrightarrow 0$$

is exact. Let us suppose that this phenomenon is more generally true for all the imbeddings ψ of R (which seems to be the case). From this we see that if M_ψ is the linear subspace of $\mathbb{P}(\bigwedge^n W)$ generated by $\psi(R)$ ($\psi(R) \hookrightarrow Gr(n,W) \hookrightarrow \mathbb{P}(\bigwedge^n W)$), then $\dim M_\psi = n+1$. If ψ_z denotes the imbedding of R for $z \in Z_2$, let us suppose that $z \longmapsto M_{\psi_z}$ defines a closed immersion

$$\text{(vii)} \qquad \Delta : Z_2 \longrightarrow Gr(n+1,Q) \hookrightarrow \mathbb{P}(\bigwedge^{n+1} Q) \quad (Q = \bigwedge^n W).$$

It is H equivariant.

We shall now determine $\Delta(z_0)$, where z_0 corresponds to the embedding of $\mathbb{P}^1$ given above. As we saw above, this is given by

$$\begin{aligned} (x,y) \quad &\longmapsto \quad (xe_1 + ye_2) \wedge \cdots \wedge (xe_{2n-1} + ye_{2n}) \\ &= \quad x^n\theta_0 + x^{n-1}y\theta_1 + \cdots + y^n\theta_n \end{aligned}$$

Say for $n = 3$, the above expression is given by

$$\begin{aligned} (x,y) \longmapsto\ & x^3 e_1 \wedge e_3 \wedge e_5 + x^2 y(e_1 \wedge e_3 \wedge e_6 + e_1 \wedge e_4 \wedge e_5 + e_2 \wedge e_3 \wedge e_5) \\ & + xy^2(e_1 \wedge e_4 \wedge e_6 + e_2 \wedge e_3 \wedge e_6 + e_2 \wedge e_4 \wedge e_5) + y^3(e_2 \wedge e_4 \wedge e_6). \end{aligned}$$

If we now identify $K_1 \simeq K_2 \simeq K$ ($f_1 \longmapsto f_2$), we have

$$\begin{aligned} Q = \bigwedge^n W = \bigwedge^n (K \oplus K) &= \bigoplus_{i=0}^n \left(\bigwedge^i K \otimes \bigwedge^{n-i} K\right) \\ &\simeq \bigoplus_{i=0}^n Q_i, \quad Q_i = \operatorname{Hom}\left(\bigwedge^i K, \bigwedge^i K\right) \end{aligned}$$

and we see that θ_i identifies with the identity homomorphism of $\mathrm{Hom}\,(\bigwedge^i K, \bigwedge^i K) = Q_i$ which is $\mathrm{Aut}\,K$ invariant. We see that Q_0 and Q_n are 1-dimensional modules and that the H-irreducible module $\mathrm{Hom}\,(W_\rho, W_\rho)$ ($\rho =$ half sum of positive roots for $SL(n) \simeq SL(K)$) is a direct summand of $\bigotimes_{i=0}^n Q_i$. We have $\bigwedge^{n+1} Q = (\bigotimes_{i=0}^n Q_i) \oplus$ other irreducible components for $\mathrm{Aut}\,K \times \mathrm{Aut}\,K$. Now $\Delta(z_0)$ is defined by

$$(\theta_0 \otimes \cdots \otimes \theta_n) \in \left(\bigotimes_{i=0}^{n} Q_i\right) \hookrightarrow \bigwedge^{n+1} Q.$$

We see that the projection of $(\theta_0 \otimes \cdots \otimes \theta_n)$ in $\mathrm{Hom}\,(W_\rho, W_\rho)$ is non-zero and $\mathrm{Aut}\,K$ (in fact $\mathrm{Aut}\,K/$ centre) invariant. Now $\mathrm{Hom}\,(W_\rho, W_\rho)$ is of *regular special* weight in the sense of De Concini-Processi and then by their work [D-P] it follows that the closure of the H orbit $H \cdot \Delta(z_0)$ in $\mathbb{P}(\bigwedge^{n+1} Q)$ is the wonderful compactification of $PGL(n)$. Hence $Z_2 \simeq \Delta(Z_2)$ is the wonderful compactification of $PGL(n)$ and the principal claim follows.

We have supposed that Δ is a closed immersion. However, we need not use this. By the general theory of Hilbert schemes, Z_2 gets imbedded in the Grassmannian of $(kn+1)$ dimensional quotients ($kn+1 = \dim H^0(\mathbb{P}^1, \mathcal{O}(kn))$ of $H^0(\mathbb{P}(Q), \mathcal{O}_{\mathbb{P}(Q)}(k))$, for $k \gg 0$. This latter vector space is $S^k(Q^*)$. Hence we see that we get an imbedding:

$$\Delta_k : Z_2 \longrightarrow Gr(kn+1, \otimes^k Q), k \gg 0.$$

Now $\Delta_k(z_0)$ is associated to the imbedding

$$\mathbb{P}^1 \overset{j}{\hookrightarrow} \mathbb{P}\left(\bigotimes^{k} Q\right)$$

obtained by composing $\mathbb{P}^1 \longrightarrow \mathbb{P}(Q)$ with the canonical imbeddings $\mathbb{P}(Q) \hookrightarrow \mathbb{P}(\bigotimes^k Q)$. This means that j is defined by:

$$\begin{aligned} (x, y) \quad &\longmapsto \quad \bigotimes^{k}(x^n\theta_0 + x^{n-1}y\theta_1 + \cdots + y^n\theta_n) \\ &= \quad x^{kn}\phi_0 + x^{kn-1}y\phi_1 + \cdots + y^{kn}\phi_{kn} \end{aligned}$$

where $\phi_{kn} \in \bigotimes^k Q$. Then by a similar argument, using the criterion of De Concini and Processi, it follows that Z_2 is the wonderful compactification of $PGL(n)$

6 Concrete descriptions of the moduli spaces and applications

A vector bundle V on X_0 can be described by the following datum on the normalisation X of X_0, namely by $W = \pi^*(V)$ ($\pi : X \longrightarrow X_0$) and an isomorphism $j : V_{p_1} \longrightarrow V_{p_2}$ of the fibres of V at $\{p_i\}$, $i = 1, 2$. One would therefore expect to describe a torsion free sheaf on X_0 as some limits of the isomorphisms j. This leads to the notion of a *generalized parabolic bundle* (GPB) due to U. Bhosle.

Unless otherwise stated we will hereafter restrict to the case of rank 2 vector bundles.

Definition 6.1. A GPB on X is a vector bundle E on X together with an element of $G_r(E_{p_1} \oplus E_{p_2}, 2)$ i.e. a quotient

$$\begin{cases} E_{p_1} \oplus E_{p_2} \longrightarrow Q \longrightarrow 0, \quad \dim Q = 2 \\ \text{or equivalently } 0 \longrightarrow N \longrightarrow E_{p_1} \oplus E_{p_2}, \dim N = 2 \\ (0 \longrightarrow N \longrightarrow E_{p_1} \oplus E_{p_2} \longrightarrow Q \longrightarrow 0). \end{cases}$$

We denote this GPB by (E, Q) (or (E, N)).

To (E, Q) we can canonically associate a torsion free sheaf F on X_0 as follows. Now Q need not have an $\mathcal{O}_X$-module structure. However it has an $\mathcal{O}_{X_0}$-module structure (as a sky-scraper sheaf with support at p) and F is defined by:

$$0 \longrightarrow F \longrightarrow \pi_*(E) \longrightarrow Q \longrightarrow 0.$$

The important point is that $\deg F = \deg E$, since we have

$$\chi(F) = \chi(\pi_*(E)) - 2 = \chi(E) - 2 \text{ i.e.}$$

$$\deg F - 2(g-1) = \deg E - 2[(g-1) - 1] - 2 = \deg E - 2(g-1).$$

Now any torsion free sheaf F on X_0 can be represented by a GPB (E, Q) in this manner. However, this representation is not unique, as we shall see below.

Let E be a vector bundle on X and M a linear subspace of E_x, $x \in X$. Then we have two well-defined vector bundles E', E'' on X called *Hecke*

modifications defined by homomorphisms

$$E' \xrightarrow{j} E, \quad \operatorname{Im} E'_x = M; \quad E \xrightarrow{j} E'', \quad \operatorname{Ker} j_x = M.$$

We can of course define a Hecke modification defined at several points of X.

Proposition 6.1. Let F be a torsion free sheaf on X_0. Then we have the following:

(i) if (E, Q) is a GPB which represents F, then we have a homomorphism $(\pi^*(F)$ mod. torsion$) \longrightarrow E$ which is a Hecke modification at p_1, p_2

(ii) if F is a vector bundle, then the representation (E, Q) is unique, $E = \pi^*(F)$ and N is the graph of an isomorphism $E_{p_1} \longrightarrow E_{p_2}$

(iii) let F be of type 1 (i.e. $F = m \oplus \mathcal{O}$ at p). Then there are precisely two GPB's (E, Q) and (E', Q') which represent F and N (resp. N') is the graph of a rank 1 homomorphism

$$E_{p_1} \xrightarrow{i} E_{p_2} \quad (\text{resp. } \; E'_{p_2} \xrightarrow{i'} E'_{p_1}).$$

We have Hecke modifications

$$\begin{cases} (\pi^*(F) \;\; \text{mod torsion}) \longrightarrow E \text{ (only modification at } p_2 \text{ associated to Im } i) \\ (\pi^*(F) \;\; \text{mod torsion}) \longrightarrow E' \text{ (only modification at } p_1 \text{ associated to ker } i). \end{cases}$$

(iv) Let F be type 2 i.e. $F = m \oplus m$ at p. In this case, there are an infinite number of GPB's which represent F and they are given as follows:

(a) (E_1, Q_1) and N_1 is the graph of the 0-map $(E_1)_{p_1} \longrightarrow (E_1)_{p_2}$.

(b) (E_2, Q_2) and N_2 is the graph of the 0-map $(E_2)_{p_2} \longrightarrow (E_2)_{p_1}$.

(c) Consider (E, Q) with $N = K_1 \oplus K_2$, $\dim K_i = 1$ and $K_1 \hookrightarrow E_{p_1}$, $K_2 \hookrightarrow E_{p_2}$ so that $Q = Q_1 \oplus Q_2$, $Q_i = E_{p_i}/K_i$, $i = 1, 2$. We denote by E' the Hecke modification of E, $E' \longrightarrow E$ such that Im E'_{p_i} in E_{p_i} is K_i, $i = 1, 2$. Then all (E, Q) such that $E' = \pi^*(F)$ mod torsion, represent F. We see that all these (E, Q) are parametrized by $\mathbb{P}^1 \times \mathbb{P}^1$.

Remark 6.1. For the objects in (a) and (b) of (iv), we have the relation

$$\begin{cases} E_1(-p_2) = E' = E_2(-p_1) \quad \text{or} \quad E_2 = E_1(p_1 - p_2) \\ E' = \pi^*(F) \bmod \text{ torsion.} \end{cases}$$

Note that if F is of rank 1 and type 1, then if (E_1, Q_1) and (E_2, Q_2) represent F, we have

$$E_2 = E_1(p_1 - p_2).$$

Definition 6.2. A GPB (E, Q) on X is *semi-stable* (resp. *stable*) if for every (resp. proper $\neq 0$) subsheaf E' of E, we have

$$(*) \qquad \begin{cases} \dfrac{\deg E' - \dim Q^{E'}}{\operatorname{rk} E'} \leq \dfrac{\deg E - \dim Q}{\operatorname{rk} E} \quad (\text{resp. } <), \\ Q^{E'} = \text{the image of } E' \text{ in } Q. \end{cases}$$

Note that E' is a vector bundle and the canonical map $E' \longrightarrow E$ need not be injective everywhere but only generically injective; it could even be a generic isomorphism.

Remark 6.2.

(a) Let (E, Q) represent the torsion free sheaf F on X_0. Then one has (see [NR], [Su]):

$$F \text{semi-stable} \Longleftrightarrow (E, Q) \text{ semi-stable.}$$

However, if F is stable, (E, Q) need not be stable. If (E, Q) is as in (iv)(c) of Prop. 6.1 (which implies F is type 2), consider the Hecke modification $E' \longrightarrow E$ which maps E'_{p_i} onto K_{p_i} and an isomorphism outside $\{p_1, p_2\}$. Then we see that the inequality in $(*)$ is, in fact, an equality. A similar argument works for (iv) (a), (b) of Prop. 6.1. Thus for example, if F is semi-stable and torsion free of degree 1, it is stable; however, say it is of type 2, then any (E, Q) which represents F is only semi-stable and not stable.

(b) We can define families of GPB's and then we obtain a functor

$$(GPB) \longrightarrow (\text{Torsion free sheaves on } X_0).$$

(c) There is a canonical moduli space associated to the semi-stable GPB's of rank n and degree d of X. We denote this by $GPB_X(n,d)$; in particular $GPB_X(2,1)$. It has a structure of a normal projective variety. Further if (E,Q) is a semi-stable GPB and F is the semi-stable torsion free sheaf on X_0 associated to (E,Q) (see (a) of Remark 6.2 above), the map $(E,Q) \longmapsto F$ defines a morphism $GPB_X(n,d) \longrightarrow U_X(n,d)$; in fact $GPB_X(n,d)$ identifies with the normalisation of $U_X(n,d)$ (see [NR] and [Su]). Note that $GPB_X(2,1)$ is not the set of isomorphism classes of semi-stable GPB's. We have to have an equivalence relation.

(d) We have also a forget functor:

$$(GPB) \longrightarrow (\text{vector bundles on } X).$$

This is not well-behaved with respect to semistability. If (E,Q) is semi-stable (as a GPB), the underlying vector bundle need not be semi-stable; further if E is semi-stable, (E,Q) need not be semi-stable.

Remark 6.3. Take two copies V_1, V_2 of a 2-dimensional vector space. Then we have identifications $\operatorname{Aut} V_i \simeq GL(2)$ and $\operatorname{Isom}(V_1,V_2) \simeq GL(2)$. Imbedding $\operatorname{Isom}(V_1,V_2)$ by their graphs in $Gr(V_1 \oplus V_2, 2)$, we get a compactification of $GL(2)$. However, this is not a "good compactification" as the complement of $GL(2)$ is not a variety with normal crossings. Let v_1(resp. v_2) represent the points of $Gr(V_1 \oplus V_2, 2)$ corresponding to the zero homomorphism $V_1 \longrightarrow V_2$ (resp. $V_2 \longrightarrow V_1$) represented by its graph. We blow up $(Gr(V_1 \otimes V_2, 2)$ at these points v_1, v_2 and obtain a variety $\mathcal{P}$ with exceptional divisors H_1 and H_2 mapping to v_1, v_2 respectively. We see that $\mathcal{P} \longrightarrow Gr(V_1 \otimes V_2, 2)$ is $\operatorname{Aut} V_1 \times \operatorname{Aut} V_2$ equivariant. Further, this compactification $\mathcal{P}$ of $GL(2) \simeq Isom(V_1,V_2)$ is "good" as we check easily that its complement has normal crossing singularities, i.e. $\mathcal{P}$ is a "good" compactification of $GL(2)$ unlike $Gr(V_1 \otimes V_2, 2)$.

Definition 6.3.

(a) An $\mathcal{P}$-structure on a vector bundle V on X is just giving a point of $\mathcal{P}$ where $\mathcal{P}$ is the blowing up of $Gr(V_{p_1} \otimes V_{p_2}, 2)$ as defined above, or we can say that an $\mathcal{P}$-*bundle on* X is a pair (V,h), where V is a vector bundle, $h \in \mathcal{P}$, $\mathcal{P}$ being the blowing up of $Gr(V_{p_1} \otimes V_{p_2}, 2)$ as indicated above. We have functors

$$(\mathcal{P} - \text{ bundles }) \longrightarrow (GPB) \longrightarrow (\text{ Torsion free sheaves on } X_0).$$

(b) We say that an $\mathcal{G}$-bundle is *stable* if the corresponding torsion free sheaf on X_0 is stable (recall that we are in the rank 2 case).

Remark 6.4. We have a canonical morphism $\pi_* : G(2,1) \longrightarrow U_{X_0}(2,1)$. As mentioned in Remark 6.2(c), $GPB_X(2,1)$ is the normalisation of $U_{X_0}(2,1)$. Hence π_* induces a canonical morphism

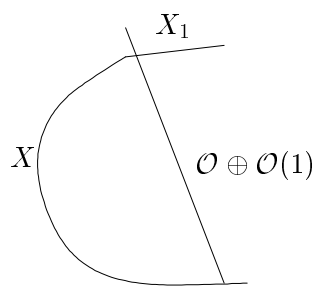
$$(*) : \widetilde{G(2,1)} \longrightarrow GPB_X(2,1)$$

where $\widetilde{G(2,1)}$ denotes the normalisation of $G(2,1)$. This map can be more concretely seen as follows.

Let E be a vector bundle on X_k (of rank 2 and degree 1) such that $E|_R$ is strictly positive and $\pi_*(E)$ is torsion free (which implies $E|_R$ is strictly standard and restricts the possibilities as below). Then to E we shall now associate, upto a finite number of choices, a GPB such that the underlying bundle is of rank 2 and degree 1. This leads to the canonical morphism $(*)$ above.

(i) <u>k=0</u> i.e. E is a bundle on X_0. Then E is given by an element Isom (E'_{p_1}, E'_{p_2}), $E' = \pi^*(E)$ and hence we associate this GPB structure on E'

(ii) $\underline{k = 1, E|_R = \mathcal{O} \oplus \mathcal{O}(1)}$



We set $E' = E|_X$. Let L_i be the 1-dimensional subspace of $E_{p_i} (= E'_{p_i})$ defined by $\mathcal{O}(1)_{p_i}$. Let E_1 (resp. E_2) be the Hecke modification:

$$\begin{cases} E' \longrightarrow E_1 (\text{resp. } E_2), \\ \text{Ker}\,(E'_{p_2} \longrightarrow (E_1)_{p_2}) (\text{resp. Ker } (E'_{p_1} \longrightarrow (E_2)_{p_1}) \text{ is } L_2 \text{ (resp. } L_1) \end{cases}$$

Then we have well-defined rank 1 homomorphisms:

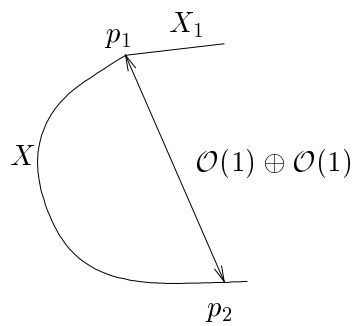
$$(E_1)_{p_1} \xrightarrow{f_1} (E_1)_{p_2} (\text{resp. } (E_2)_{p_2} \xrightarrow{f_2} (E_2)_{p_1}).$$

Let N_1 (resp. N_2) be the graph of f_1(resp. f_2). Then we have exact sequences

$$0 \longrightarrow N_1 \longrightarrow E_{p_1} \oplus E_{p_2} \longrightarrow Q_1 \longrightarrow 0$$
$$0 \longrightarrow N_2 \longrightarrow E_{p_1} \oplus E_{p_2} \longrightarrow Q_2 \longrightarrow 0$$

and $(E_1, Q_1), (E_2, Q_2)$ are the two GPB's which we associate to E. They are the two GPB's which represent $\pi_*(E)$ (see (iii) of Prop. 6.1). One sees that $E' = \pi^*(F)$mod torsion.

(iii) $\underline{k = 1, E|_R = \mathcal{O}(1) \oplus \mathcal{O}(1)}$ *We set* $E' = E|_X$. We check $E' = \pi^*(F)$mod torsion. Let $E' \longrightarrow E_i (i = 1, 2)$ be the Hecke modifications defined by $E_i = E'(p_i)$, $(i = 1, 2)$ and (E_1, Q_1) (resp. (E_2, Q_2)) be the GPB defined by the zero homomorphism $(E_1)_{p_2} \longrightarrow (E_1)_{p_1}$ (resp. $(E_2)_{p_1} \longrightarrow (E_2)_{p_2}$). We assign to E these two GPB's. These two GPB's represent $\pi_*(E)$.

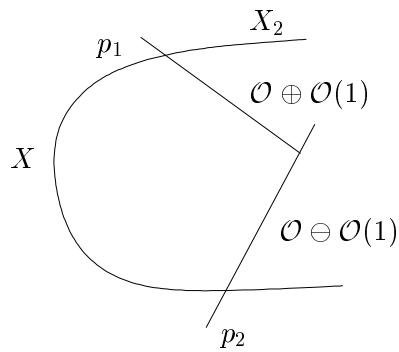


Given E' on X, we can extend E' to E on X_1 with $E|_R \simeq \mathcal{O}(1) \oplus \mathcal{O}(1)$ by giving isomorphisms:

$$g_1 : E'_{p_1} \longrightarrow (\mathcal{O}(1) \oplus \mathcal{O}(1))_{p_1},$$
$$g_2 : E'_{p_2} \longrightarrow (\mathcal{O}(1) \oplus \mathcal{O}(1))_{p_2}$$

Now we can modify g_i by an automorphism of $\mathcal{O}(1) \oplus \mathcal{O}(1)$. Since g_i can be identified with elements of $GL(2)$, we can suppose that $g_2 =$ identity. Since there are two components, we can modify g_1 by an arbitrary scalar. Thus the number of ways of extending E' to E on X_1 (with $E|_R = \mathcal{O}(1) \oplus \mathcal{O}(1)$) depends upto an element of $PGL(2)$.

(iv) $\underline{k = 2, E|_{R_i} = \mathcal{O} \oplus \mathcal{O}(1)}$ We set $E' = E|_X$ and check that $E' \simeq \pi^*(F)$mod torsion.



Let $K_i \subset E'_{p_i}$ be the 1-dimensional subspace by the fibre at $p_i (\simeq \mathcal{O}(1)_{p_1})$ of the canonical sub-bundle of $E|_{R_i}$. Let E'' be the Hecke modification $E' \longrightarrow E''$ such that $\mathrm{Ker}\,(E'_{p_i} \longrightarrow E''_{p_i}) = K_i, i = 1, 2$. Let $N_i = Im\ E'_{p_i}$, $Q_i = E''_{p_i}/N_i$, $Q = Q_1 \oplus Q_2$. Then (E'', Q) is the GPB assigned to E.

The above discussion, especially of cases (iii) and (iv) indicates that a fibre of the morphism $(*)$ is either a point or $\mathbb{P}^3$ (compare also the general result in Remark 5.2). For the morphism $\mathcal{C} \longrightarrow Gr(V_1 \oplus V_2, 2)$, the fibre over v_1 (resp. v_2) is $\mathbb{P}^3$. Thus we may expect the following result which is an important step in the concrete determination of the Gieseker moduli space.

Theorem 6.1. On the set of isomorphism classes of stable $\mathcal{C}$-bundles on X of rank 2 and degree 1 (i.e. the underlying vector bundles have these properties), there is a canonical structure of a smooth projective variety, which can be identified with the normalisation $\widetilde{G(2,1)}$ of the Gieseker variety $G(2,1)$. We denote $\widetilde{G(2,1)}$ by S_{II}.

Proof. We do not give a formal proof but carry it out in a test case. In the following considerations, compare Remark 6.4. Let us take a 1-parameter Gieseker family of the following type : We have a T-morphism

$$\mathcal{X} \xrightarrow{\pi} X_0 \times T$$

where T is a smooth curve and we fix a point $t_0 \in T$ (e.g. $T = \text{Spec}.$ of a d.v.r. and t_0 is the closed point of T. We suppose that for $t \neq t_0, \pi_t : \mathcal{X} \longrightarrow X_0 \times t \approx X_0$ is an isomorphism. The curve $\mathcal{X}_t$ is the form $X_k (0 \leq k \leq 2)$. We have a vector bundle V on $\mathcal{X}$ which defines a Gieseker family of rank 2 and degree 1. A test case for the theorem is that one should be able to associate to this Gieseker family a canonical 1-parameter family of stable $\mathcal{C}^{\prime}$-bundles on X. We can suppose that $1 \leq k \leq 2$.

Let Z denote the normalisation of $\mathcal{X}$. Then we have a canonical morphism.

$$\phi : Z \longrightarrow X \times T \quad (\text{normalisation of } X_0 \times T).$$

Then we have the following possibilities:

(i) Z is obtained by blowing up either (p_1, t_0) or (p_2, t_0). Then the exceptional fibre is $D \simeq \mathbb{P}^1 (k = 1)$

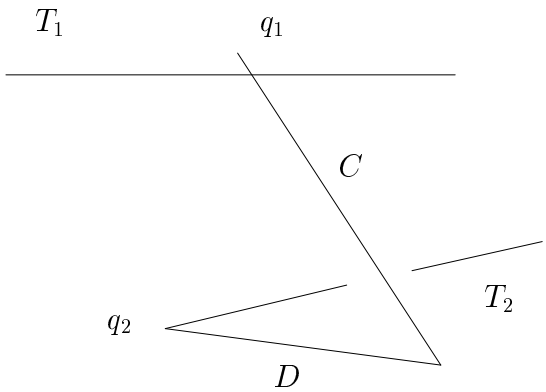
(ii) Z is obtained by blowing up both the points (p_1, t_0) and (p_2, t_0). Then the exceptional fibres are $D_1, D_2 \simeq \mathbb{P}^1$ and $D_1 \cap D_2 = \emptyset$ $(k = 2)$

(iii) Z is obtained by first blowing up say (p_2, t_0) and then blowing up a point on the exceptional fibre. Hence the exceptional fibre is $D_1 \cup D_2$, such that $D_i \simeq \mathbb{P}^1$, the intersection $D_1 \cap D_2$ reduces to a point and the exceptional fibre maps to (p_1, t_0) or (p_2, t_0).

If $T_0 = T \backslash t_0$, we can identify $p \times T_0$ (p singular point of X_0) as a subscheme of $\mathcal{X} \backslash \mathcal{X}_{t_0}$. The closure S of $p \times T_0$ is a section of $\mathcal{X}$ over T and S meets one of the singular points $\mathcal{X}_{t_0} \simeq X_k$. The above possibilities (i), (ii) and (iii) depend upon the way S meets the singular point of X_k.

Let C, T_1, T_2 denote respectively the proper transforms of $X \times t_0,\ p_1 \times T$ and $p_2 \times T$ in Z. Let W denote the vector bundle on Z obtained as the inverse image of V by the canonical morphism $Z \longrightarrow \mathcal{X}$. In the case (i) above the restriction of W to D is either $\mathcal{O} \oplus \mathcal{O}(1)$ or $\mathcal{O}(1) \oplus \mathcal{O}(1)$. In all the other cases the restriction of W to D_i is $\mathcal{O} \oplus \mathcal{O}(1)$.

<u>Case $k = 1$ and $W|_D \simeq \mathcal{O} \oplus \mathcal{O}(1)$</u> Now D maps to (p_1, t_0) or (p_2, t_0), say (p_2, t_0). Then we see that T_2 does not meet C. We see that T_2 meets D, T_1 meets C and C meets D.



$q_1 = C \cap T_1$

$q_2 = D \cap T_2$

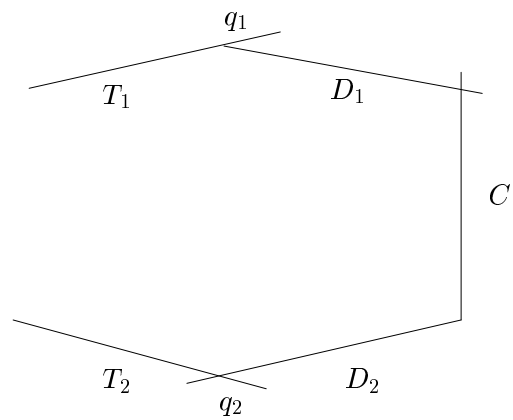
We have a canonical isomorphism $\delta : W|_{T_1} \longrightarrow W|_{T_2}$, which defines the vector bundle V (the non-normal variety χ is obtained by identifying T_1 and T_2). Let F be the direct image $\phi_*(W)$ which is a torsion free sheaf on $X \times T$ and $E = F^{**}$ - the double dual of F. Then E is a vector bundle on $X \times T$. It is not difficult to see that we have a canonical homomorphism $j : W \longrightarrow \phi^*(E)$ such that when restricted to D, the trivial quotient line bundle of $W|_D$ maps onto a line sub-bundle of $\phi^*(E)|_D$ i.e. $\phi^*(E)$ is the "push-forward Hecke modification" of W along D by the trivial quotient line bundle of $W|_D$. Then $j|_{T_2} : W|_{T_2} \longrightarrow \phi^*(E)|_{T_2}$ is the Hecke modification (on T_2) of $W|_{T_2}$ by the canonical 1-dimensional quotient of the fibre of $W|_{T_2}$ at q_2 (namely the fibre of the canonical trivial quotient line bundle of $W|_D$ at q_2). We have identifications

$$W|_{T_1} \simeq \phi^*(E)|_{T_1} \quad \text{and} \quad \phi^*(E)|_{T_i} \simeq E_{p_i \times T} \quad (i = 1, 2).$$

Hence if we set $\theta = (j|_{T_2} \circ \delta)$, θ defines a homomorphism $\theta : E_{p_1 \times T} \longrightarrow E_{p_2 \times T}$ such that θ_t $(t \neq t_0)$ is an isomorphism and $\theta_0 = \theta_{t_0}$ is of rank one. Now θ defines a family of $\mathcal{C}$-bundles, in fact a family of GPB's. This is the required 1-parameter family of $\mathcal{C}$-bundles.

<u>Case $k = 2$ and Z is obtained by blowing up (p_1, t_0) and (p_2, t_0)</u>

In this case we have:



We see that C meets D_1 and D_2, T_1 meets D_1 at q_1, T_2 meets D_2 at q_2 and T_1, T_2 do not meet C. We define E as above and then we get a canonical homomorphism $W \longrightarrow \phi^*(E)$. We have also a canonical isomorphism $\delta : W|_{T_1} \longrightarrow W|_{T_2}$ inducing an isomorphism $\delta_0 : W_{q_1} \longrightarrow W_{q_2}$. We see that the kernel K_i of $W_{q_i} \longrightarrow \phi^*(E)_{q_i}$ identifies with the fibre of the line subbundle of $W|_{D_i} (= \mathcal{O}(1) \oplus \mathcal{O})$ isomorphic to $\mathcal{O}(1)$, so that by Prop. 3.4 (and Remark 3.1) we have $\delta_0(K_1) \cap K_2 = (0)$. From this we deduce that under the canonical homomorphism

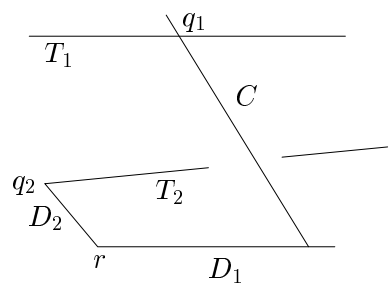
$$W_{q_1} \oplus W_{q_2} \longrightarrow \phi^*(E)_{q_1} \oplus \phi^*(E)_{q_2}$$

the graph of δ_0 maps isomorphically onto the image of the above homomorphism, which is $N_1 \oplus N_2$, $N_i = W_{q_i}/K_i$. This means that the family of GPB's represented by the isomorphisms $\theta_t : E_{p_1,t} \longrightarrow E_{p_2,t} (t \neq t_0)$ specializes to the GPB θ_{t_0} represented by $(E_{t_0}, N_1 \oplus N_2)$. Thus θ_t defines a family of $\mathcal{C}^\infty$-bundles, in fact a family of GPB's. This is the required 1-parameter family of $\mathcal{C}^\infty$-bundles.

<u>Case $k = 1$ and $W_D = \mathcal{O}(1) \oplus \mathcal{O}(1)$</u>. In this case the figure is as in the first case ($k = 1$) considered above. We define E as before. We have canonical isomorphisms $\delta : W|_{T_1} \longrightarrow W|_{T_2}$, $\delta_0 : W_{q_1} \longrightarrow W_{q_2}$ and a canonical homomorphism $j : W \longrightarrow \phi^*(E)$. We observe that $\phi^*(E) = W(D)$ and j is a "push-forward Hecke modification" of W along D inducing the zero homomorphism when restricted to D. We see that $j|_{T_2} : W|_{T_2} \longrightarrow \phi^*(E)|_{T_2}$ is the Hecke modification (on T_2) inducing the zero map on W_{q_2} - the fibre at q_2. Then $\theta = (j|_{T_2} \circ \delta)$ defines a homomorphism $\theta : E_{p_1 \times T} \longrightarrow E_{p_2 \times T}$ such that θ_t $(t \neq t_0)$ is an isomorphism and the local expression for θ_t at

t_0 takes the form $t\Delta$ (t - local coordinate at t_0) such that Δ (essentially δ): $E_{p_1,t_0} \longrightarrow E_{p_2,t_0}$ is an isomorphism. We get a GPB on E_{t_0} by taking the zero homomorphism $E_{p_1,t_0} \longrightarrow E_{p_2,t_0}$ and we endow it with an $\mathcal{C}$-structure by taking the image of Δ in $\mathbb{P}(\mathrm{Hom}\,(E_{p_1,t_0}, E_{p_2,t_0})$ (which is canonically the fibre of $\mathcal{C} \longrightarrow Gr(E_{p_1,t_0} \oplus E_{p_2,t_0}, 2)$ over the point represented by the zero homomorphism $E_{p_1,t_0} \longrightarrow E_{p_2,t_0}$). We check that θ_t ($t \neq t_0$) together with this $\mathcal{C}$-structure on E_{t_0} defines a family of $\mathcal{C}$-bundles. This is the required 1-parameter family of $\mathcal{C}$-bundles.

Case $k = 2$ and as in (iii) above. In this case we have the figure



$r = D_1 \cap D_2$

The first blow up gives D_1 and then the next blow up gives D_2. Let E_1 be the push-forward Hecke modification of W along D_2 for the canonical quotient line bundle of $W|_{D_2}$. We have a canonical homomorphism $W \longrightarrow E_1$ and canonical isomorphisms $\delta : W|_{T_1} \longrightarrow W|_{T_2}$ and $\delta_0 : W_{q_1} \longrightarrow W_{q_2}$. We claim that $E_1|_{D_1} \simeq \mathcal{O}(1) \oplus \mathcal{O}(1)$. To prove this observe first that $W|_{D_1} \longrightarrow E_1|_{D_1}$ is a Hecke modification. Further, for this homomorphism, the kernel of the fibre of $W|_{D_1}$ ($\simeq \mathcal{O} \oplus \mathcal{O}(1)$) at $r = D_1 \cap D_2$, is not the fibre of the canonical line sub-bundle ($\simeq \mathcal{O}(1)$) of $W|_{D_1}$. This is an easy consequence of the fact that the fibre at r of the canonical sub-bundles of $W|_{D_1}$ and $W|_{D_2}$ generate the fibre of W at r. Then the claim follows easily.

The homomorphism $W \longrightarrow E_1$ induces a rank one homomorphism $W_{q_2} \longrightarrow (E_1)_{q_2} \simeq (E_1)_r$ (the restriction of E_1 to D_2 is trivial). Then composing with the isomorphism $\delta_0 : W_{q_1}(\simeq (E_1)_{q_1}) \longrightarrow W_{q_2}$, we get a canonical rank one homomorphism $\eta : (E_1)_{q_1} \longrightarrow (E_1)_{q_2} \simeq (E_1)_r$.

Let Z_1 denote the variety which gives the first blow up D_1 so that we have the factorisation of the morphism ϕ:

$$Z \longrightarrow Z_1 \longrightarrow X \times T.$$

Then E_1 goes down to a vector bundle on Z_1 and we denote this by the same letter E_1. We have $E_1|_{D_1} \simeq \mathcal{O}(1) \oplus \mathcal{O}(1)$ and we are in a situation similar to the case considered above ($k = 1$ and $W|_D = \mathcal{O}(1) \oplus \mathcal{O}(1)$). Set $E_2 = E_1(D_1)$. Then the restriction of E_2 to D_1 is trivial and E goes down to a vector bundle E on $X \times T$. Then we see easily that we have a homomorphism $\theta : E_{p_1 \times T} \longrightarrow E_{p_2 \times T}$ such that θ_t $(t \neq t_0)$ is an isomorphism and the local expression for θ_t at t_0 takes the form $t\Delta \pmod{t^2}$ such that Δ (essentially the above $\eta : E_{p_1,t_0} \longrightarrow E_{p_2,t_0}$ is a rank one homomorphism. By taking the image of Δ in $\mathbb{P}$ $(\mathrm{Hom}\,(E_{p_1,t_0}, E_{p_2,t_0}))$, we get a family of $\mathcal{P}$-bundles as we did in the case above. This is the required family of $\mathcal{P}$-bundles.

One can give a structure of a smooth projective variety on the isomorphism classes of $\mathcal{P}$ stable bundles of rank 2 and degree one on X. We shall outline a proof later (see Remark 6.6). Call this moduli space $\mathcal{P}(2,1)$. The preceding considerations show that we have a morphism $S_{II} = G(2,1) \longrightarrow \mathcal{P}(2,1)$. This is certainly birational. It is not hard to show that the fibres of this morphism are finite. This would show that it is an isomorphism.

Remark 6.5.

(a) Consider the moduli space $U_X = U_X(2,1)$ of stable vector bundles of rank 2 and degree 1 on X. Let P be a Poincaré bundle on $X \times U_X(2,1)$. Then Grass $(P_{p_1} \oplus P_{p_2}, 2)$ is a Grassmann bundle on U_X. Then if $\mathcal{P}_1, \mathcal{P}_2$ are the principal $GL(2)$ bundles associated to the vector bundles P_{p_i} on U_X. Then we can construct a bundle over U_X with fibre $\mathcal{P}$ as a bundle associated to $\mathcal{P}_1 \times \mathcal{P}_2$. We call this variety S_I. This variety is "inductively good" with respect to the genus g i.e. it is a fibre space with fibre $\mathcal{P}$ over the moduli space U_X with X of genus $(g-1)$. We see we have a diagram with morphisms j_1, j_2:

$$S_I \xrightarrow{j_1} Gr(P_{p_i} \oplus P_{p_2}, 2) \xrightarrow{j_2} V_X,$$

We have of course a family of GPB's on X parametrized by $Gr(P_{p_1} \oplus P_{p_2}, 2)$ which is the set of all GPB's over the moduli space U_X. We get then a family of GPB's parametrized by S_I.

(b) If V is a vector bundle on X, the set of all GPB's on V is parametrized by $Gr(V_{p_1} \oplus V_{p_2}, 2)$ and those which yield vector bundles on X_0 correspond to the open subset Isom (V_{p_1}, V_{p_2}). One sees that the motivation

for defining $\mathcal{C}^{\mathcal{A}}$ bundles is to have a better compactification for the moduli space of vector bundles on the singular curve X_0.

(c) We see that S_I and S_{II} are birational. In fact, we have the following. Let $\Delta_1 \hookrightarrow S_I$ be the closed subscheme such that the underlying GPB's are *not* semi-stable or equivalently the corresponding tension free sheaves on X_0 are *not* stable. Similarly let Δ_2 be the closed subscheme of S_{II} such that the underlying vector bundles on X are *not* semi-stable (or equivalently not stable since we are in the degree 1 case). We see that

$$S_I \backslash \Delta_1 \simeq S_{II} \backslash \Delta_2.$$

We have the following result essentially due to Gieseker (see [G]).

Theorem 6.2. The varieties Δ_1, Δ_2 are smooth. Let $\mathcal{S}$ be the blow up S_I along Δ_1 and Δ the inverse image of Δ_1 in S. Then we have a canonical morphism $\phi : \mathcal{S} \longrightarrow S_{II}$ so that we have a diagram: ($\delta : \mathcal{S} \longrightarrow S_I$ canonical map).

$$\begin{tikzcd}
 & \mathcal{S} \arrow[dl, "\delta"'] \arrow[dr, "\phi"] & \\
S_I \arrow[d, "j_1"'] & & S_{II} = \widetilde{G(2,1)} \arrow[d] \\
Gr(P_{p_1} \oplus P_{p_2}, 2) \arrow[d, "j_2"'] & & G(2,1) \arrow[d, "\pi_*"] \\
U_X & & U_{X_0}(2,1)
\end{tikzcd}$$

Further $\mathcal{S}$ is the blow up of S_{II} along Δ_2 and ϕ is the canonical morphism $\mathcal{S} \longrightarrow S_{II}$. Besides, we have the following properties of Δ's.

The variety of $\Delta_1 \simeq \mathbb{P}_{J_0 \times J_1}(F_1)$, where J_i = Jacobian of degree i on X and F_1 is the vector bundle on $J_0 \times J_1$ representing the space of all extensions

$$0 \longrightarrow L_0 \longrightarrow E \longrightarrow L_i \longrightarrow 0, \quad L_i \epsilon J_i.$$

In fact Δ_1 maps onto its image in U_X (we see that these extensions which are non-trivial define *stable* bundles of degree 1 on X. These acquire canonical GPB's defined by the quotients $E_{p_1} \oplus E_{p_2} \longrightarrow (L_1)_{p_1} \oplus (L_1)_{p_2}$. Similarly, one has a vector bundle F_2 on $J_0 \times J_1$ such that $\Delta_2 = \mathbb{P}_{J_0 \times J_1}(F_2)$. We have

$$\Delta = \mathbb{P}_{J_0 \times J_1}(F_1) \times_{J_0 \times J_1} \mathbb{P}_{J_0 \times J_1}(F_2).$$

$$\begin{tikzcd}
 & \Delta \arrow[dl] \arrow[dr] & \\
\Delta_1 \arrow[dr] & & \Delta_2 \arrow[dl] \\
 & J_0 \times J_1 &
\end{tikzcd}$$

Remark 6.6. We shall now indicate how S_I and S_{II} can be viewed as GIT quotients for two different polarizations on the same variety. This gives also the required construction of the variety structure on $\mathcal{M}(2,1)$ $(\simeq S_{II})$ stated in the proof of Theorem 6.1. One knows that we have a process of blowing up and blowing down when comparing GIT quotients for different polarisations (as exploited by Thaddeus [T]). This should clarify the blowing up and blowing down process to obtain S_{II} from S_I.

We call a GPB (E, Q) on X α-semi-stable (resp, α-stable) if for every (resp. proper $\neq 0$) subsheaf E' of E, we have

$$\frac{\deg E' - \alpha \dim Q^{E'}}{\operatorname{rk} E'} \leq \frac{\deg E - \alpha \dim Q^{E}}{\operatorname{rk} E} \quad (\text{resp. } <).$$

In Def. 6.2 we had taken $\alpha = 1$.

We shall now relate this to GIT stability (see [U]) and ([NR]). We restrict to the case of rank 2 and degree 1.

Let Q be the Quot scheme of quotients of the trivial bundle of rank $n (n >> 0)$ with Hilbert polynomial of a rank two vector bundle of sufficiently

large odd degree. We denote by $\mathcal{E}$ the universal quotient on $X \times Q$. For a suitable polarization on Q, we know that the GIT quotient $Q^s \bmod PGL(n)$ is the moduli space $U_X(2,1)$. Let $\widetilde{Q}$ denote the Q-scheme $Gr(\mathcal{E}_{p_1} \otimes \mathcal{E}_{p_2}, 2)$- the Grassmannian of 2 dimensional quotients of $\mathcal{E}_{p_1} = \mathcal{E}|_{p_i \times Q}$. Then we have a family of polarisations L_α parametrized by α, $0 < \alpha < 1$, on $\widetilde{Q}$ such that

$$\begin{cases} GIT \text{ stability (resp. semi-stability) for } L_\alpha \\ \Longleftrightarrow \alpha \text{ stability (resp. semi-stability) as above for } GPB\text{'s.} \end{cases}$$

This implies, in particular, that if $\mathcal{E}_q$ is the quotient sheaf of the trivial rank n vector bundle corresponding to a point $q \in \widetilde{Q}$ which is GIT semi-stable for L_α, then $\mathcal{E}_q$ is locally free. The above assertion is in [U], it follows also from the arguments in [NR]. The notion of α-stability does not figure in [NR] but only in [U]. It is also shown in [NR] that

$$\begin{cases} 1 \text{ stability (resp. semi-stability) i.e. as in Def. 6.2} \\ \Rightarrow GIT \text{ stability (resp. semi-stability) for } L_1. \end{cases}$$

The converse is not true as there are GIT semi-stable objects for L_1 which are not locally free. However, in each GIT semi-stable equivalence class, there are vector bundles and the converse is true for vector bundles (see [NR]). The GIT quotient for the polarisation L_1 is the variety $GPB_X(2,1)$ (see Remark 6.2 (c)).

We see easily that α stable (resp. semi-stable) GPB's remain the same, respectively in the intervals $0 < \alpha < \frac{1}{2}$ and $\frac{1}{2} < \alpha < 1$ and consequently the GIT quotients behave the same way. We denote the GIT quotients respectively by $GPB(\delta_0)$ and $GPB(\delta_1)$. We see also that in these cases stable $\Longleftrightarrow$ semi-stable, so that $GPB(\delta_1)$ (resp. $GPB(\delta_0)$) is a smooth projective variety and its underlying set is the set of *isomorphism classes* of α-stable GPB's with $\frac{1}{2} < \alpha < 1$ (resp. $0 < \alpha < \frac{1}{2}$). We denote the GIT quotients for $L_{\frac{1}{2}}$ and L_1 respectively by $GPB(\frac{1}{2})$ and $GPB(1)(= GPB_X(2,1)$ as mentioned above). By the general process of blowing up and blowing down for GIT quotients for varying polarisations, we get canonical birational morphisms.

$$\begin{tikzcd}
 & & GPB(\delta_1) \arrow[dl] \arrow[dr] & \\
GPB(\delta_0) \arrow[dr] & & & GPB(1) \\
 & GPB(\tfrac{1}{2}) & &
\end{tikzcd}$$

We observe that for $\frac{1}{2} < \alpha < 1$, we have:

$$\begin{cases} GPB\ \alpha\text{-stable} \Rightarrow GPB\ \text{1-semi-stable} \\ GPB\ \text{1-semi-stable} \Rightarrow GPB\alpha\ \text{stable.} \end{cases}$$

From this it follows that $GPB(\delta_1)$ (rather its underlying set) is, in fact, the set of *isomorphism* classes of GPB's which define stable torsion free shears rank 2 and deg 1 on X_0, whereas $GPB(1) = GPB_X(2,1)$ is a further quotient of $GPB(\delta_1)$. It is also easily seen that $GPB(\delta_0)$ is a Grassmann bundle $Gr(P_{p_1} \oplus P_{p_2}, 2)$ over U_X (see Remark 6.5).

The maps (blow ups and blow downs) in (1) above result from the following observation in GIT. Let us fix a polarisation L (on a projective variety with an action of a reductive group etc.). Then if L' is a polarisation which is sufficiently close to L, we have

(a) L' semi-stability $\Longrightarrow$ L semi-stability.

(b) L stability $\Longrightarrow$ L' stability.

(Note the slightly more general observation prior to Def.4.2 which is used for the construction of the variety structure on $G(n.d)$.) The set of L' semi-stable points is contained in that of L-semi-stable points so that we get a canonical morphism.

L' GIT quotient $\longrightarrow$ L GIT quotient.

Further, if the set of L stable points is non-empty, then the above is a (proper) birational morphism. This gives the maps in (1) above. Further, from general considerations, it follows that we can divide the ample cone (rather the one with the group action) into nice regions in each of which stable and semi-stable points remains the same. This is the general phenomenon behind stable and semi-stable GPB's remaining the same in the intervals $0 < \alpha < \frac{1}{2}$ and $\frac{1}{2} < \alpha < 1$.

Now to get the $\mathcal{C}$-stable moduli, we construct a scheme Q' which is proper over the open subset of $\widetilde{Q}$ corresponding to quotients which are locally free, and corresponds to the blowing up of $\mathcal{C}$ of the Grassmannian. We can take a suitable Q^* which contains Q' as an open subset, Q^* being proper over Q and we can suppose that the actions of the group lift to Q', Q^* etc. Fix a θ in such a way that $0 < \theta < \frac{1}{2}$. Let M be a polarisation on Q^* which is relatively ample with respect to $Q^* \xrightarrow{\lambda} Q$. Then (as we saw prior to Def.4.2) if a is sufficiently small ($a > 0$) and $\theta \leq \alpha < 1$, we have

(a) $q \in Q^*$ is $(L_\alpha + aM)$ semi-stable then $\lambda(q)$ is L_α semi-stable.

(b) $q \in Q$ is L_α stable, then $\lambda^{-1}(q)$ is $(L_\alpha + aM)$ stable.

Now it is not difficult to see that the variety S_I (see Remark 6.5) is the GIT quotient for the polarisation for $L_\alpha + aM$ with $\theta \leq \alpha < \frac{1}{2}$ and that S_{II} is the GIT quotient for $L_\alpha + aM$ with $\frac{1}{2} < \alpha < 1$. For all these polarisations we have stability $\Longleftrightarrow$ semi-stability, so that we get, in particular, a canonical structure of a projective variety, namely S_{II}, on the set of *isomorphism* classes of $\mathcal{C}$-stable vector bundles on X of rank 2 and deg. 1 (mentioned in the proof of theorem 6.1).

Let S' be the GIT quotient for $L_{\frac{1}{2}} + aM$. Then we get as in (1) above (by general considerations explained above) canonical birational morphisms.

$$\begin{tikzcd} S_I \arrow[dr] & & S_{II} \arrow[dl] \\ & S' & \end{tikzcd}$$

The variety $\mathcal{S}$ in Theorem 6.2 seems to be the fibre product $S_I \times_{S'} S_{II}$.

Remark 6.7. *Vanishing of Chern classes for the moduli space on a smooth projective curve of genus g.*
We shall now very briefly outline Gieseker's proof of the conjecture of Newstead and Ramanan, namely that

$$c_i(\Omega_{U(2,1)_Y}) = 0, i > 2g - 2,$$

where Y is a smooth curve of genus g, which can be taken as the generic fibre of $\mathcal{X} \longrightarrow S$.

One shows that there is a vector bundle Ω on $G(2,1)_S$ such that the restriction Ω_ξ of this bundle to the generic fibre of $G(2,1)_S$ over S is the cotangent bundle and the restriction Ω_0 to the closed fibre is $\Omega_{G(2,1)}(\log D')$, where D' is the singular locus of $G(n,d)$. This uses the fact that $G(n,d)_S$ is regular and its closed fibre is a divisor with normal crossings. Then one shows that it suffices to prove that $c_i(\Omega_0) = 0$ for $i > 2g-2$ and in fact that it suffices to prove this vanishing for $\widetilde{\Omega}_0$ the pull-back of Ω_0 on the normalisation $\widetilde{G(2,1)}$ of $G(2,1)$. It is seen that $\widetilde{\Omega}_0 = \Omega_{\widetilde{G(2,1)}}(\log \widetilde{D}')$, $\widetilde{D}$ being the inverse image of D' in $\widetilde{G(2,1)}$. Then the problem reduces to proving the vanishing of Chern classes of the pull-back of this bundle on the variety $\mathcal{S}$ (see Theorem 6.2). One has an explicit hold on this bundle. For the vanishing of Chern classes one uses the factorisation: $\mathcal{S} \longrightarrow S_I \longrightarrow Gr(P_{p_1} \oplus P_{p_2}, 2) \longrightarrow U_X$. The proof is by induction on the genus and one can start the induction process, since for $g = 1$, $G(2,1)$ is the curve X_0 itself. Hence we can suppose that the vanishing result holds for the moduli space $U_X(2,1) = U_X$ on X. Then the required vanishing result follows by this inductive argument.

7 Comments

(I) It should be possible to work out generalisations of Gieseker moduli spaces for $(n,d) \neq 1$ say for X_0. For obvious reasons one cannot expect normal crossing singularities since the quotients are not by free actions. Semi-stability has to be more carefully defined. We have to add more conditions besides the condition that the direct image by π is torsion free and semi-stable.

If there is more than one ordinary double point (say the curve is irreducible), even if $(n,d) = 1$ the singularities for the generalized Gieseker moduli spaces are not normal crossings since they are only products of normal crossings.

(II) It should be possible to work out generalisations of Gieseker moduli spaces for any family, say for stable curves over general base schemes,

for any rank and degree.

(III) The method of Gieseker for constructing the moduli space $G(2,1)$ is by giving criteria for semi-stability of the points in the Hilbert scheme represented by imbeddings of curves into a Grassmannian (of two planes). This is a very natural method but seems complicated and therefore difficult to generalize for arbitrary rank. It would be interesting to see if our method would imply some results on the "Hilbert stability" of imbeddings of curves into Grassmannians of n-planes.

(IV) The moduli space $G(n,d)$ can be thought of as a moduli problem of vector bundles on the curve X_n (we need not take all k, $k<n$) modulo the action of the automorphism group of X_n which is identity on X. One knows how to construct the moduli of semi-stable torsion free sheaves on X_n (now thanks to Simpson for general projective schemes). It would be interesting to construct the moduli spaces $G(n,d)$ more directly as quotients of the moduli spaces of vector bundles or torsion free sheaves on X_n.

(V) In the proof of properness, one saw that a torsion free sheaf F on the isolated normal singularity represented by C is the invariant direct image of a vector bundle on a finite covering represented by a disc. In fact, it is easily checked that a torsion free sheaf on X_0 is, locally at p, an invariant direct image of Γ-vector bundle on a ramified Galois covering with Galois group Γ (Γ-cyclic group). This may suggest a good definition for G-objects on a nodal singularity for a semi-simple algebraic group G.

(VI) These moduli spaces should be considered as solutions of the moduli problem associated to the following objects over X_0:

$$\{(\pi,E),\ \pi \text{ proper map } X' \longrightarrow X_0,\ E \text{ a vector bundle on } X_0$$
$$\text{such that } \pi_*(E) \text{ is torsion free and } \pi \text{ is an isomorphism over } X_0\backslash\{p\}.$$

We have of course to fix invariants. It is tempting to ask for generalisations when X_0 is replaced by a higher dimensional variety, say even a smooth surface. We may get compact moduli spaces only with the use of vector bundles but over varying varieties dominating the quasi variety.

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A minicourse on moduli of curves

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Abstract

These are notes that accompany a short course given at the School on Algebraic Geometry 1999 at the ICTP, Trieste. A major goal is to outline various approaches to moduli spaces of curves. In the last part I discuss the algebraic classes that naturally live on these spaces; these can be thought of as the characteristic classes for bundles of curves.

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1. Structures on a surface

We start with two notions from linear algebra. Let T be a real vector space. A *conformal structure* on T is a positive definite inner product $(\,\cdot\,)$ on T given up to multiplication by a positive scalar. The notion of length is lost, but we retain the notion of angle, for if $v_1, v_2 \in T$ are independent, then

$$\angle(v_1, v_2) := \frac{(v_1 \cdot v_2)}{\|v_1\|\|v_2\|}$$

does not change if we multiply $(\,\cdot)$ with a positive scalar. A *complex structure* on T is a linear automorphism J of T such that $J^2 = -1_V$; this makes T a complex vector space by stipulating that multiplication by $\sqrt{-1}$ is given by J. If $\dim T = 2$, then these notions almost coincide: if we are given an orientation plus a conformal structure, then 'rotation over $\frac{\pi}{2}$' is a complex structure on T. Conversely, if we are given a complex structure J, then we have an orientation prescribed by the condition that (v, Jv) is oriented whenever $v \neq 0$ and a conformal structure by taking any nonzero inner product preserved by J.

Let S be an oriented C^∞ surface. A *conformal structure* on S is given by a smooth Riemann metric on S given up to multiplication by a positive C^∞ function on S. By the preceding remark this is equivalent to giving an *almost-complex structure* on S, i.e., an automorphism J of the tangent bundle TS with $J^2 = -1_{TS}$, that is compatible with the given orientation. Given such a structure, then we have a notion of holomorphic function: a C^∞ function $f : U \to \mathbb{C}$ on an open subset U of S open is said to be *holomorphic* if for all $p \in U$, $df_p \circ J_p = \sqrt{-1} df_p : T_pS \to \mathbb{C}$. This generalizes the familiar notion for if S happens to be $\mathbb{C}$ with its standard almost-complex structure, then we are just saying that f satisfies the Cauchy-Riemann equations. It is a special property of dimension two that S admits an atlas consisting of holomorphic charts. (This amounts to the property that for every Riemann metric on a neighborhood of $p \in S$ we can find local coordinates x, y at p such that the metric takes the form $c(x, y)(dx^2 + dy^2)$.) Coordinate changes will be holomorphic also (but now in the conventional sense), and we thus find that S has actually a complex-analytic structure.

A surface equipped with such a structure is called a *Riemann surface*. We shall usually denote such a surface by C. I will assume you are familiar with some of the basic facts regarding compact Riemann surfaces such as the

Riemann-Roch theorem and Serre-duality and the notions that enter here. These give us for instance

Theorem 1.1. *A compact connected Riemann surface C of genus g can be complex-analytically embedded in $\mathbb{P}^{g+1}$ such that the image is a nonsingular complex projective curve of degree $2g+1$. The algebraic structure that C thus receives is canonical.*

The proof may be sketched as follows. Choose $p \in C$. Then the complete linear system generated by $(2g+1)(p)$ is of dimension $g+2$ and defines a complex-analytic embedding of C in $\mathbb{P}^{g+1}$ of degree $2g+1$. A theorem of Chow asserts that a closed analytic subvariety of a complex projective space is algebraic. So the image of this embedding is algebraic. It also shows that the algebraic structure is unique: if C is complex analytically embedded into two projective spaces as $C_1 \subset \mathbb{P}^k$ and $C_2 \subset \mathbb{P}^l$, then consider the diagonal embedding of C in $\mathbb{P}^k \times \mathbb{P}^l$, composed with the Segre embedding of $\mathbb{P}^k \times \mathbb{P}^l$ in $\mathbb{P}^{kl+k+l}$; by Chow's theorem the image is a complex projective curve C_3. The curves C_1 and C_2 are now obtained as images of C_3 under linear projections. These are therefore (algebraic) morphisms that are complex-analytic isomorphisms. Such morphisms are always algebraic isomorphisms.

The above theorem shows in particular that a compact Riemann surface of genus zero resp. one is isomorphic to $\mathbb{P}^1$ resp. to a nonsingular curve in $\mathbb{P}^2$ of degree 3.

What can we say about the automorphism group of a compact Riemann surface C of genus g? If $g=0$, then we can assume $C=\mathbb{P}^1$ and $\mathrm{Aut}(\mathbb{P}^1)$ is then just the group of fractional linear transformations $z \mapsto (az+b)(cz+d)^{-1}$ with $ad-bc \neq 0$. If $g=1$, then the classical theory tells us that C is isomorphic to a complex torus, and so $\mathrm{Aut}(C)$ contains that torus as a 'translation' group. This subgroup is normal and the factor group is finite. In all other cases ($g \geq 2$), $\mathrm{Aut}(C)$ is finite. There are several ways to see this, one could be based on the uniformization theorem, another on the fact that $\mathrm{Aut}(C)$ acts faithfully on $H_1(C,\mathbb{Z})$ (any automorphism acting trivially has Lefschetz number $2-2g<0$, so cannot have a finite fixed point set, hence must be the identity) and preserves a positive definite Hermitian form on $H_1(C,\mathbb{C})$.

We denote by $\mathcal{M}_g$ the *set* of isomorphism classes of nonsingular genus g curves. For the moment it is just that: a set and nothing more, but our aim is to put more structure on $\mathcal{M}_g$ when $g \geq 2$. We will discuss four approaches to this:

- Riemann's original (heuristic) approach, that we will discuss very briefly.
- The approach through Teichmüller theory. Actually there are several of this type, but we mention just one. More is said about this in Hain's lectures.
- The introduction of an orbifold structure on $\mathcal{M}_g$ in the spirit of Grothendieck's formalization and generalization of the Kodaira-Spencer theory.
- The introduction of a quasi-projective structure on $\mathcal{M}_g$ by means of geometric invariant theory.

The last two approaches lead us to consider a compactification of $\mathcal{M}_g$ as well.

2. Riemann's moduli count

Fix integers $g \geq 2$ and $d \geq 2g-1$. Let C be a smooth genus g curve. Choose a point $p \in C$. By Riemann-Roch the linear system $|d(p)|$ has dimension $g-d$. Choose a generic line L in this linear system that passes through $d(p)$, in other words, L is a pencil through $d(p)$. The genericity assumption ensures that this pencil has no fixed points. Choose an affine coordinate w on L such that $w = \infty$ defines $d(p)$. We now have a finite morphism $C \to \mathbb{P}^1$ of degree d that restricts to a finite morphism $f : C - \{p\} \to \mathbb{C}$. We invoke the Riemann-Hurwitz formula (which is basically an euler characteristic computation):

$$\sum_{x \in C - \{p\}} \nu_x(f) = 2g - 1 + d,$$

where $\nu_x(f)$ is the *ramification index* of f at x (= the order of vanishing of df at x). The *discriminant divisor* D_f is $\sum_{x \in C-\{p\}} \nu_x(f)(f(x))$ (so the coefficient of $w \in \mathbb{C}$ is the sum of the ramification indices of the points of $f^{-1}(w)$). Its degree is clearly $2g-1+d$. The passage to the discriminant divisor loses only a finite amount of information: from that divisor we can reconstruct C and the covering $C \to \mathbb{P}^1$ (up to isomorphism) with finite ambiguity. Furthermore, it is easy to convince yourself that in the $(2g-1+d)$-dimensional projective space of effective degree $2g-1+d$ divisors on $\mathbb{P}^1$ the discriminant divisors make up a Zariski open subset. We now count moduli as follows: in order to arrive at f we needed for a given C, the choice of $p \in C$ (one parameter), the choice of a line L in $|d(p)|$ through $d(p)$ ($d-g-1$ parameters) and the affine coordinate w (2 parameters). Hence the number of parameters remaining for C is

$$(2g-1+d) - \big(1 + (d-g-1) + 2\big) = 3g - 3.$$

This suggests that $\mathcal{M}_g$ is like a variety of dimension $3g-3$.

3. Orbifolds and the Teichmüller approach

We begin with a modest discussion of orbifolds. Let G be a Lie group acting smoothly and properly on a manifold M. *Proper* means that the map $(g,p) \in G \times M \to (g(p),p) \in M \times M$ is proper; this guarantees that the orbit space $G\backslash M$ is Hausdorff.

If G acts freely on M, then the orbit space $G\backslash M$ is in a natural way a smooth manifold: for every $p \in M$, choose a submanifold S of M through p such that T_pS supplements the tangent space of the G-orbit of p at p. After shrinking S if necessary, S will meet every orbit transversally and at most once. Hence the map $S \to G\backslash M$ is injective. It is not hard to see that the collection of these maps defines a smooth atlas for $G\backslash M$, making it a manifold.

If G acts only with finite stabilizers, then we can choose S in such a way that it is invariant under the finite group G_p. After shrinking S in a suitable way we can ensure that every G-orbit that meets S, meets it in a G_p-orbit and that the intersection is transversal. So we then have an injection $G_p\backslash S \to G\backslash M$. This is in fact an open embedding and hence $G\backslash M$ is locally like a manifold modulo a finite group. It is often very useful to remember the local genesis of such a space, because this information cannot be recovered from the space itself (example: the obvious action of the nth roots of unity on a one dimensional complex vector space has orbit space isomorphic to $\mathbb{R}^2$, so that we cannot read off n from just the orbit space). This leads to Thurston's notion of *orbifold*: this is a Hausdorff space X for which we are given an 'atlas of charts' of the form $(U_\alpha, G_\alpha, h_\alpha)_\alpha$, where U_α is a smooth manifold on which a finite group G_α acts, and h_α is an open embedding of the orbits space $U_\alpha\backslash G_\alpha$ in X. The images of these open embeddings must cover X and there should be compatibility relations on overlaps. It is understood that two such atlasses whose union is also atlas define the same orbifold structure. So if F is a discrete space on which a finite group H acts simply transitively, then we may add the chart given by $U_\alpha \times F$ with its obvious action of $G_\alpha \times H$ (its orbit space is $U_\alpha\backslash G_\alpha$) and h_α. This allows us to express the compatibility relation simply by saying that the atlas is closed under the formation of fibered products ('intersections'): $U_\alpha \times_X U_\beta$ with its $G_\alpha \times G_\beta$-action and the identification of the orbit space with a subset of X should also be in it. It also implies that we have an atlas of charts for which the group actions are *effective*. I leave it to you to

verify that $G\backslash M$ has that structure. There exist parallel notions in various settings, e.g., complex-analytic and algebraic. In the last two cases, it is often useful to work with a more refined notion of orbifold (a 'stack'), but we will not go into this now.

The case that concerns us is in infinite dimensional analogue of the above situation: we fix a closed oriented surface S_g of genus g and we let the space of conformal structures on S take the role of M and the group of orientation preserving diffeomorphisms take the role of G. It is understood here that these carry certain structures that allow us to think of an action of an infinite dimensional Lie group on an infinite dimensional space. This action turns out to be proper with finite stabilizers. It turns out that all orbits have codimension $6g-6$ and so it is at least plausible that $\mathcal{M}_g$ has the structure of an orbifold of real dimension $6g-6$. This heuristic reasoning has been justified by Earle and Eells.

4. Grothendieck's view point

It is worthwhile to discuss things in a more general setting than is strictly necessary for the present purpose, for the methods and notions that we need come up in virtually all deformation problems.

4.1. **Kodaira-Spencer maps.** Suppose $\pi : \mathcal{C} \to B$ is a proper (holomorphic) submersion between complex manifolds. According to Ehresmann's fibration theorem, π is then locally trivial in the C^∞-category, that is, for every $b \in B$ we can find an open $U \ni b$ and a smooth retraction $h : \mathcal{C}_U = \pi^{-1}U \to C_b = \pi^{-1}(b)$ such that $\tilde{h} = (h, \pi) : \mathcal{C}_U \to C_b \times U$ is a diffeomorphism. In particular, when B is connected, then all fibers of π are mutually diffeomorphic. If both B and the fibers are connected, we will call π a *family of compex manifolds with smooth base*. We assume that this is the case and we wish to address the question whether the fibers C_b are mutually isomorphic as complex manifolds. Suppose that the family is trivial over U, in other words, that the retraction h can be chosen holomorphically so that $\tilde{h}$ is an analytic isomorphism. Then each holomorphic vector field on U lifts to $C_b \times U$ in an obvious way and hence also lifts to $\mathcal{C}_U$ (via $\tilde{h}$). Suppose now the converse, namely, that every holomorphic vector field at b lifts holomorphically. Then π is locally trivial at b. To see this, assume for simplicity that $\dim B = 1$. Choose a nowhere zero vector field on an open $U \ni b$ (always obtainable by means of a coordiante chart) that lifts holomorphically to a vector field on $\mathcal{C}_U$. The properness of π ensures that

this lift is integrable to a holomorphic flow on $\mathcal{C}_U$. The flow surfaces (= complex flow lines) produces the desired retraction $h : \mathcal{C}_U \to C_b$. The case of a higher dimensional base goes by induction and the induction step is a parametrized version of the one dimensional case just discussed.

Liftability issues lead inevitably to cohomology. Let us begin with the noting that the fact that π is a submersion implies that for every $x \in \mathcal{C}$ we have an exact sequence

$$0 \to T_x C_b \to T_x \mathcal{C} \to T_b B \to 0, \quad b := \pi(x).$$

This sheafifies as an exact sequence of $\mathcal{O}_{\mathcal{C}}$-modules

$$0 \to \theta_{\mathcal{C}/B} \to \theta_{\mathcal{C}} \to \pi^* \theta_B \to 0$$

($\theta_{\mathcal{C}}$ stands for the sheaf of holomorphic vector fields on $\mathcal{C}$, $\theta_{\mathcal{C}/B}$ for the subsheaf of $\theta_{\mathcal{C}}$ of vector fields that are tangent to the fibers of π). Now take the direct image under π; since the fibers are connected, we get:

$$0 \to \pi_* \theta_{\mathcal{C}/B} \to \pi_* \theta_{\mathcal{C}} \to \theta_B \to 0.$$

This sequence diplays our lifting problem: an element of θ_B is holomorphically liftable iff it is in the image of $\pi_* \theta_{\mathcal{C}/B}$. But the sequence may fail to be exact at θ_B since π_* is only left exact. We need the right derived functors of π_* in order to continue the sequence in an exact manner:

$$0 \to \pi_* \theta_{\mathcal{C}/B} \to \pi_* \theta_{\mathcal{C}} \to \theta_B \xrightarrow{\delta} R^1 \pi_* \theta_{\mathcal{C}/B} \to \cdots .$$

The sheaf $R^1 \pi_* \theta_{\mathcal{C}/B}$ is a coherent $\mathcal{O}_B$-module, whose value at b is equal to the cohomology group $H^1(C_b, \theta_{C_b})$. So an element of θ_B is holomorphically liftable iff its image under δ vanishes. Hence δ gives us a good idea of how nontrivial the family at a point b is: the family is locally trivial at b iff δ is zero in b. Both δ and its value at a point b, $\delta(b) : T_b B \to H^1(C_b, \theta_{C_b})$, are called the *Kodaira-Spencer map*.

4.2. **The deformation category.** Let us fix a connected compact complex manifold C. A *deformation of C with smooth base* (B, b_0) is given by a proper holomorphic submersion $\pi : \mathcal{C} \to B$ of complex manifolds, a distinguished point $b_0 \in B$ and an isomorphism $\iota : C \cong C_{b_0}$, with the understanding that replacing π by its restriction to a neighborhood of b_0 in B defines the same deformation (in particular, B may be assumed to connected). From the preceding discussion it is clear that we may think of this as a variation of complex structure on C parametrized by the manifold germ (B, b_0).

The deformations of C are objects of a category: a morphism from $(\pi,' \iota')$ to (π, ι) is given by a pair of holomorphic map germs $(\tilde{\phi}, \phi)$ in the diagram

$$\begin{array}{ccc} \mathcal{C}' & \xrightarrow{\tilde{\phi}} & \mathcal{C} \\ {\scriptstyle \pi'}\downarrow & & \downarrow{\scriptstyle \pi} \\ (B', b_0') & \xrightarrow{\phi} & (B, b_0) \end{array}$$

such that the square is cartesian (this basically says that $\tilde{\phi}$ sends the fiber $C'_{b'}$ isomorphically to the fibre $C_{\phi(b')}$) and $\tilde{\phi}\iota' = \iota$. So (π, ι) is more interesting than $(\pi,' \iota')$ as every fiber of the latter is present in the former. In this sense, the most interesting object would be a final object, if it exists (which is often not the case). A deformation (π, ι) is said to be *universal* if it is a final object for this category. So this means that for every deformation (π', ι') of C there is a unique morphism $(\pi', \iota') \to (\pi, \iota)$. A universal deformation is unique up to unique isomorphism (a general property of final objects). We also observe that the automorphism group $\mathrm{Aut}(C)$ acts on (π, ι): if $g \in \mathrm{Aut}(C)$, then $(\pi, \iota g^{-1})$ is another deformation of C and so there is a unique morphism $(\tilde{\phi}_g, \phi_g)) : (\pi, \iota g^{-1}) \to (\pi, \iota)$. The uniqueness implies that $\tilde{\phi}_{gh} = \tilde{\phi}_g \tilde{\phi}_h$ and that $\tilde{\phi}_1$ is the identity. So the action of $\mathrm{Aut}(C)$ extends to $(\mathcal{C}, C_{b_0})$. Similarly, $\mathrm{Aut}(C)$ acts on (B, b_0) such that π is equivariant.

Remark 4.1. The restriction to deformations over a smooth base turns out to be inconvenient. The custom is to allow B to be singular. The submersivity requirement for π is then replaced by the condition that π be locally trivial on $\mathcal{C}$: for every $x \in C_{b_o}$, there is a local holomorphic retraction $h : (\mathcal{C}, x) \to (C_{b_o}, x)$ such that $\tilde{h} = (h, \pi)$ is an isomorphism of analytic germs. This enlarges the deformation category and consequently the notion of universal deformation changes. Our restriction to deformations with smooth base was only for didactical purposes: a universal deformation is always understood to be the final object of this bigger category (and therefore need not have a smooth base).

We can go a step further and allow C to be singular as well (in fact, we shall have to deal with that case). Then the right condition to impose on π is that it be *flat*, which is an algebraic way of saying that the map must be open. In contrast to the situation considered above, the topological type of C can now change (simple example: take the family of conics in $\mathbb{P}^2$ with affine equation $y^2 = x^2 + t$). The Kodaira-Spencer theory has to be modified as well. For example, $H^1(C, \theta_C)$ must be replaced by $\mathrm{Ext}^1(\Omega_C, \mathcal{O}_C)$.

Remark 4.2. As said earlier, a universal deformation need not exist. But what always exists is a deformation (π, ι) with the property that for any deformation (π, ι') there exists a morphism $(\tilde{\phi}, \phi) : (\pi', \iota') \to (\pi, \iota)$ with ϕ *unique up to first order only.* This is called a *semi-universal deformation* of C (others call it a *Kuranishi family* for C). Such a deformation is still unique, but may have automorphisms (inducing the identity on C and the Zariski tangent space of the base).

4.3. **Orbifold structure on $\mathcal{M}_g$.** We can now state the basic

Theorem 4.3. *For a smooth curve C of genus $g \geq 2$ we have*

(i) *C has a universal deformation with smooth base.*
(ii) *A deformation (π, ι) of C is universal iff its Kodaira-Spencer map $T_{b_0}B \to H^1(C_{b_0}, \theta_{C_{b_0}}) \cong H^1(C, \theta_C)$ is an isomorphism.*
(iii) *A universal deformation of C can be represented by a family $\mathcal{C} \to B$ such that $\mathrm{Aut}(C)$ acts on this family (and not just on the germ) so that every isomorphism between fibers $C_{b_1} \to C_{b_2}$ is the restriction of the action of an automorphism of C.*

So the universal deformation of C is smooth of dimension $h^1(\theta_C)$. A simple application of Riemann-Roch shows that this number is $3g - 3$.

A universal deformation as in (iii) defines a map $B \to \mathcal{M}_g$ that factorizes over an injection $\mathrm{Aut}(C)\backslash B \to \mathcal{M}_g$. We give $\mathcal{M}_g$ the finest topology that makes all those maps continuous. It is not difficult to derive from the above theorem that with this topology the maps $\mathrm{Aut}(C)\backslash B \to \mathcal{M}_g$ become open embeddings. It is harder to prove that the topology is Hausdorff. Thus $\mathcal{M}_g$ acquires the structure of a (complex-analytic) orbifold of complex dimension $3g - 3$.

Remark 4.4. The orbifold structure on $\mathcal{M}_g$ has the property that every orbifold chart $(U, G, h : G\backslash U \to \mathcal{M}_g$ is induced by a family of genus g curves over U, provided that $g \geq 3$. For $g = 2$ we run into trouble since every genus 2 curve has a nontrivial involution (it is hyperelliptic). For this reason, the orbifold structure as defined here is not quite adequate and we have to resort to a more sophisticated version: ultimately we want only charts that support honest families of curves, with a change of charts covered by an isomorphism of families.

The space $\mathcal{M}_g$ is not compact. The reason is that one easily defines families of smooth genus g curves over the punctured unit disk $\mathcal{C} \to \Delta - \{0\}$ with monodromy of infinite order. A classic example is that of degerating

family of genus one curves given by the affine equation $y^2 = x^3 + x^2 + t$, with t the parameter of the unit disk. I admit that we dismissed genus one, but it is not difficult to generalize to higher genus: for example, replace x^3 by x^{2g+1}. Such a family defines an analytic map $\Delta - \{0\} \to \mathcal{M}_g$ such that the image of its intersection with a closed disk of radius < 1 is closed. To see this, suppose the opposite. One then shows that the map $\Delta - \{0\} \to \mathcal{M}_g$ must extend holomorphically over Δ. If the image of the origin is represented by the curve C, then a finite ramified cover of $(\Delta, 0)$ will map to the universal deformation of C so that the family on this finite cover will have trivial monodromy. This contradicts our assumption. The remedy is simple in principle: compactify $\mathcal{M}_g$ by allowing the curves to degenerate (as mildly as possible). This leads us to the next topic.

4.4. **Stable curves.** A *stable curve* is by definition a *nodal* curve C (that is, a connected complex projective whose singularities are normal crossings (nodes), analytically locally isomorphic to the union of the two coordinate axes in $\mathbb{C}^2$ at the origin) such that

- the euler characteristic of every connected component of the smooth part of C is negative.

The *genus* of such a curve can be defined algebro-geometrically as $h^1(\mathcal{O}_C)$ or topologically by the formula $2 - 2g = e(C_{\text{reg}})$. So it has to be ≥ 2. The itemized condition is equivalent to each of the following ones:

- $g \geq 2$ and C_{reg} has no connected component isomorphic to $\mathbb{P}^1 - \{\infty\}$ or $\mathbb{P}^1 - \{0, \infty\}$,
- $\mathrm{Aut}(C)$ is finite,
- C has no infinitesimal automorphisms.

Topologically a stable genus g curve is obtained as follows. Let S_g be a closed oriented genus g surface. Choose on S a finite collection of embedded circles in distinct isotopy classes and such that none of these is trivial in the sense that it bounds a disk. Then the space obtained by contracting each of the circles underlies a stable curve and all these topological types are thus obtained. (Note that removal of an embedded circle from a surface does not alter its euler characteristic.) There is a deformation theory for stable curves which is almost as good as if the curve were smooth:

Theorem 4.5. *A stable curve of genus $g \geq 2$ has a universal deformation with smooth base of dimension $3g - 3$. This deformation can be represented by a family $\mathcal{C} \to B$ such that $\mathrm{Aut}(C)$ acts on this family in such a way that*

(i) *every isomorphism between fibers* $C_{b_1} \to C_{b_2}$ *is the restriction of the action of an automorphism of* C,
(ii) *all fibers are stable genus* g *curves,*
(iii) *for any singular point* x *of* C, *the locus* $\Delta_x \subset B$ *that parametrizes the curves for which* x *persists as a singular point is a smooth hypersurface and the* $(\Delta_x)_{x \in C_{\mathrm{sing}}}$ *cross normally.*

So the complement of the normal crossing hypersurface $\Delta := \cup_{x \in C_{\mathrm{sing}}} \Delta_x$ in B parametrizes smooth genus g curves. If there is just one singular point, then you may picture the degeneration from a smooth curve C_b to the singular curve $C_{b_0} \cong C$ in metric terms as by letting the circumference of the embedded circle on S_g that defines the topological type of C go to zero. The monodromy around B_x is in this picture the Dehn twist along that circle (see the lectures by Hain).

Let $\overline{\mathcal{M}}_g$ be the set of isomorphism classes of stable genus g curves. The above theorem leads to a compact orbifold structure on this set:

Theorem 4.6 (Deligne-Mumford). *The universal deformations of stable genus* g *curves put a complex-analytic orbifold structure on* $\overline{\mathcal{M}}_g$ *of dimension* $3g-3$. *The space* $\overline{\mathcal{M}}_g$ *is compact and the locus* $\partial\mathcal{M}_g = \overline{\mathcal{M}}_g - \mathcal{M}_g$ *parametrizing singular curves is a normal crossing divisor in the orbifold sense.*

This is why $\overline{\mathcal{M}}_g$ is called the *Deligne-Mumford* compactification of $\mathcal{M}_g$. A generic point of the boundary divisor corresponds to a stable curve C with just one singular point. The underlying topological type of such a curve is determined by a nontrivial isotopy class of an embedded circle δ on S_g. The following cases occur:

Δ_0: C is irreducible ($S_g - \delta$ is connected) or
$\Delta_{\{g',g''\}}$: C is the one point union of two smooth curves of positive genera g', g'' with sum $g' + g'' = g$ ($S_g - \delta$ disconnected with components punctured surfaces of genera g' and g'').

These cases correspond to irreducible components of $\partial\mathcal{M}_g$. We denote them by Δ_0 and $\Delta_{\{g',g''\}}$.

5. The approach through geometric invariant theory

Perhaps the most appealing way to arrive at $\mathcal{M}_g$ and its Deligne-Mumford compactification is by means of geometric invariant theory. Conceptually this approach is more direct than the one discussed in the previous section. Best of all, we stay in the projective category. The disadvantage is that we do not know a priori what objects we are parametrizing.

Let us begin with a minimalist discussion of the general theory. Let a semisimple algebraic subgroup G of $\mathrm{SL}(r+1,\mathbb{C})$ be given. That group acts on $\mathbb{P}^r$. Let $X \subset \mathbb{P}^r$ be a closed G-invariant subvariety. A G-orbit in $\mathbb{P}^r$ is called *semistable* if it is the projection of a G-orbit in $\mathbb{C}^{r+1} - \{0\}$ that does not have the origin in its closure. The union X^{ss} of the semistable orbits contained in X is a subvariety of X. This subvariety need not be closed and may be empty (in which case there is little reason to proceed). The basic results of Hilbert and Mumford are as follows:

1. Every semistable orbit in X^{ss} has in its closure a unique semistable orbit that is closed in X^{ss}.
2. There exists a positive integer N such that the semistable orbits that are closed in X^{ss} can be separated by the G-invariant homogeneous polynomials of degree N.
3. If R_X stands for the homogeneous coordinate ring of X, then the subring of its G-invariants R_X^G is noetherian.

Let $G\backslash\backslash X$ denote the set of semistable G-orbits in X that are closed in X^{ss}. Property 1 implies that there is a natural quotient map $X^{ss} \to G\backslash\backslash X$. If G happens to act properly on X^{ss}, then every orbit in X^{ss} is closed in X^{ss} and so $G\backslash\backslash X$ will be just the orbit set $G\backslash X^{ss}$. From property 2 it follows that if $f_0, \ldots, f_m$ is a basis of the degree N part of R_X^G, then the map $[f_0 : \cdots : f_m] : X^{ss} \to \mathbb{P}^m$ is well defined and factorizes over an injection $G\backslash\backslash X \to \mathbb{P}^m$. The image of this injection is a closed subvariety of $\mathbb{P}^m$ and thus $G\backslash\backslash X$ acquires the structure of a projective variety. (A more intrinsic way to give it that structure is to identify it with $\mathrm{Proj}(R_X^G)$.)

So much for the general theory. For the case that interests us, you need to know what the *dualizing sheaf* ω_C of a nodal curve C is: it is the coherent subsheaf of the sheaf of meromorphic differentials characterized by the property that on C_{reg} it is the sheaf of regular differentials, whereas at a node p we allow a local section to have on each of the two branches a pole of order one, provided that the residues sum up to zero. It is easy to see that ω_C is always a line bundle (as opposed to Ω_C) and that its degree is $2g(C) - 2$. It is ample precisely when C is stable and in that case $\omega_C^{\otimes k}$ is very ample for $k \geq 3$. (The name dualizing sheaf has to do with the fact that it governs Serre duality. But that property is of no concern to us; what matters here is that every stable curve comes naturally with an ample line bundle.)

If C is stable of genus g, then a small computation shows that for $k \geq 2$, $h^0(\omega_C^{\otimes k}) = (2k-1)(g-1)$. Let us fix for each $k \geq 2$ a complex vector space

V_k of dimension $d_k := (2k-1)(g-1)$ $(k \geq 2)$. Choose $k \geq 3$. Since $\omega_C^{\otimes k}$ is very ample, we have an embedding of C in $\mathbb{P}(H^0(C, \omega_C^{\otimes k})^*)$. The choice of an isomorphism $\sigma : H^0(C, \omega_C^{\otimes k})^* \cong V_k$ allows us to identify C with a curve $C_\sigma \subset \mathbb{P}(V_k)$. It is a standard result of projective geometry that for m large enough,

1. the degree m hypersurfaces in $\mathbb{P}(V_k)$ induce on C_σ a complete linear system,
2. C_σ is an intersection of degree m hypersurfaces.

In other words, the natural map

$$\mathrm{Sym}^m(V_k^*) \cong \mathrm{Sym}^m H^0(C, \omega_C^{\otimes k}) \to H^0(C, \omega_C^{\otimes mk})$$

is surjective (property 1) and its kernel defines C_σ (property 2). So the image $W_\sigma \subset \mathrm{Sym}^m V_k$ of the dual of this map is of dimension d_{mk} and determines C_σ. Nothing is lost if we take the d_{mk}th exterior power of W_σ and regard it as a point of $\mathbb{P}(\wedge^{d_{mk}} \mathrm{Sym}^m V_k)$. Now let $X_{k,m}$ be the set of points in $\mathbb{P}(\wedge^{d_{mk}} \mathrm{Sym}^m V_k)$ that we obtain by letting C run over all the stable genus g curves and σ over all choices of isomorphism. This is a (not necessarily closed) subvariety that is invariant under the obvious $SL(V_k)$-action on $\mathbb{P}(\wedge^{d_{mk}} \mathrm{Sym}^m V_k)$. One can show that $SL(V_k)$ acts properly on $X_{k,m}$. It is clear from the construction that as a set, $SL(V_k)\backslash X_{k,m}$ may be identified with $\overline{\mathcal{M}}_g$. The fundamental result is

Theorem 5.1 (Gieseker). *For k and m sufficiently large, the semistable locus of the closure of $X_{k,m}$ in $\mathbb{P}(\wedge^{d_{mk}} \mathrm{Sym}^m V_k)$ is $X_{k,m}$ itself.*

Corollary 5.2. *The set $\overline{\mathcal{M}}_g$ is in a natural way a projective variety containing $\mathcal{M}_g$ as an open dense subvariety*

In particular, $\mathcal{M}_g$ acquires a quasi-projective structure. As one may expect, the structure of projective variety $\overline{\mathcal{M}}_g$ is compatible with the analytic structure defined before. Incidentally, geometric invariant theory also allows us to put the orbifold structure on $\overline{\mathcal{M}}_g$, but we shall not discuss that here.

6. Pointed stable curves

It is quite natural (and very worthwhile) to extend the preceding to the case of pointed curves. If n is a nonnegative integer, then an *n-pointed curve* is a curve C together with n numbered points $x_1, \ldots, x_n$ on its smooth part C_{reg}. If $(C; x_1, \ldots, x_n)$ is an n-pointed smooth projective genus g curve, then its automorphism group is finite unless $2g - 2 + n \leq 0$ (so the exceptions

are $(g,n) = (0,0),(0,1),(0,2),(1,0))$. Therefore we always assume that $2g-2+n>0$. In much the same way as for $\mathcal{M}_g$ one shows that the set of isomorphism classes $\mathcal{M}_{g,n}$ of n-pointed smooth projective genus g curves has the structure of a smooth orbifold of dimension $3g-3+n$. Just as we did for $\mathcal{M}_g$, we compactify $\mathcal{M}_{g,n}$ by allowing mild degenerations. The relevant definition is as follows:

An n-pointed curve $(C; x_1, \ldots, x_n)$ is said to be *stable* if C is a nodal curve C such that

- the euler characteristic of every connected component of $C_{\mathrm{reg}} - \{x_1, \ldots, x_n\}$ is negative,

a condition that is equivalent to each of the following ones:

- $2g-2+n>0$ (where g is the genus defined as before) and C_{reg} has no connected component isomorphic to $\mathbb{P}^1 - \{\infty\}$ or $\mathbb{P}^1 - \{0,\infty\}$,
- $\mathrm{Aut}(C; x_1, \ldots, x_n)$ is finite,
- $(C; x_1, \ldots, x_n)$ has no infinitesimal automorphisms.

We will also refer to a stable n-pointed genus g curve as a *stable curve of type* (g,n). The underlying topology is obtained as follows: fix $p_1, \ldots, p_n$ distinct points of our surface S_g and choose on $S_g - \{p_1, \ldots, p_n\}$ a finite collection of embedded circles $(\delta_e)_{e\in E}$ in distinct isotopy classes relative to $S_g - \{p_1, \ldots, p_n\}$ such that none of these bounds a disk on S_g containing at most one p_i, and contract each of these circles. There is a more combinatorial way of describing the topological type that we will use later. It is given by a finite connected graph Γ that may have multiple bonds and for which we allow *loose ends* (that is, edges of which only one end is attached to a vertex, some call them legs). We need the additional data consisting of

- for each vertex a nonnegative integer g_v and
- a numbering of the loose ends by $\{1, 2, \ldots, n\}$.

We say that these data define a *stable graph* if

- for every vertex v we have $2g_v - 2 + \deg(v) > 0$.

We define the *genus* by $g(\Gamma) := \sum_v g_v + b_1(\Gamma)$, and we call the pair $(g(\Gamma), n)$ the *type* of the stable graph. The recipe for assigning a stable graph of type (g,n) to $(S_g; p_1, \ldots, p_n; (\delta_e)_{e\in E})$ is as follows: the vertex set is the set of connected components of $S_g - \{p_1, \ldots, p_n\} - \cup_e \delta_e$, the set of bonds is indexed by E: the two sides of δ_e define one or two connected components and we insert a bond between the corresponding vertices (so this might be a loop), and we attach the ith loose end to the vertex v if the corresponding connected component contains p_i. You should check that the topological

type is faithfully represented in this way. Note that the stable graph of a smooth curve of type (g, n) is like a star: n loose ends attached to a single vertex of weight g.

Let $\overline{\mathcal{M}}_{g,n}$ denote the set of isomorphism classes of stable curves of type (g, n). We have the expected theorem:

Theorem 6.1 (Knudsen-Mumford). *The universal deformations of stable curves of type (g, n) put a complex-analytic orbifold structure on $\overline{\mathcal{M}}_{g,n}$ of dimension $3g - 3 + n$. The space $\overline{\mathcal{M}}_{g,n}$ is compact and the locus $\partial\mathcal{M}_{g,n}$ parametrizing singular curves is a normal crossing divisor in the orbifold sense.*

The construction of $\overline{\mathcal{M}}_{g,n}$ can also be obtained by means of geometric invariant theory, which implies that it is a projective orbifold. (The role of the dualizing sheaf is taken by $\omega_C(x_1 + \cdots + x_n)$; details can be found in a forthcoming sequel to [1].) The irreducible components of the boundary $\partial\mathcal{M}_{g,n}$ are in bijective correspondence with the embedded circles in $S - \{p_1, \ldots, p_n\}$ given up to orientation preserving diffeomorphism. These are:

Δ_0**:** C is irreducible or

$\Delta_{\{(g',I'),(g'',I'')\}}$**:** C is a one point union of smooth curves of genera g' and g'' with the former containing the points x_i with $i \in I'$ and the latter the points indexed by I'' (so $\{I', I''\}$ is a partition of $\{1, \ldots, n\}$). We allow g' to be zero, provided that $|I'| \geq 2$ and similarly for g''.

6.1. **The universal stable curve.** Let $(C; x_1, \ldots, x_n)$ be a stable curve of type (g, n). Let us show that any $x \in C$ determines a stable curve $(\tilde{C}; \tilde{x}_1, \ldots, \tilde{x}_{n+1})$ of type $(g, n + 1)$.

- If $x \in C_{\text{reg}} - \{x_1, \ldots, x_n\}$, then take $(\tilde{C}; \tilde{x}_1, \ldots, \tilde{x}_{n+1}) = (C; x_1, \ldots, x_n, x)$.
- If $x = x_i$ for some i, we let $\tilde{C}$ be the disjoint union of C and $\mathbb{P}^1$ with the points x_i and ∞ indentified. We let $\tilde{x}_i = 1 \in \mathbb{P}^1$ and $\tilde{x}_{n+1} = 0 \in \mathbb{P}^1$; whereas for $j \neq i, n + 1$, $\tilde{x}_j = x_j$, viewed as a point of $\tilde{C}$. We denote this $(n + 1)$-pointed curve by $\sigma_i(C; x_1, \ldots, x_n)$.
- If $x \in C_{\text{sing}}$, then $\tilde{C}$ is obtained by separating the branches of C in x (i.e., we normalize C in this point only) and by putting back a copy of $\mathbb{P}^1$ with $\{0, \infty\}$ identified with the preimage of x. Then $\tilde{x}_{n+1} = 1 \in \mathbb{P}^1$ and for $i \leq n$, $\tilde{x}_i = x_i$, viewed as a point of $\tilde{C}$.

We thus have defined a map $C \to \overline{\mathcal{M}}_{g,n+1}$ that maps x_i to $\sigma_i((C; x_1, \ldots, x_n)$.

There is also a converse construction: given stable curve $(\tilde{C}; \tilde{x}_1, \ldots, \tilde{x}_{n+1})$ of type $(g, n + 1)$, then we can associate to it a stable curve $(C; x_1, \ldots, x_n)$

of type (g, n) basically by forgetting $\tilde{x}_{n+1}$; this yields a stable pointed curve unless $\tilde{x}_{n+1}$ lies on a smooth rational component which has only two other special points. Let $\tilde{C}$ be obtained by contracting this component and let x_i be the image of $\tilde{x}_i$ $(i \leq n)$. This defines a map $\pi : \overline{\mathcal{M}}_{g,n+1} \to \overline{\mathcal{M}}_{g,n}$. Notice that the map $C \to \overline{\mathcal{M}}_{g,n+1}$ defined above parametrizes the fiber of π over the point defined by $(C; x_1, \ldots, x_n)$.

Proposition 6.2. *The map $\pi : \overline{\mathcal{M}}_{g,n+1} \to \overline{\mathcal{M}}_{g,n}$ is morphism and so are its sections $\sigma_1, \ldots, \sigma_n$. The fiber of π over the point defined by $(C; x_1, \ldots, x_n)$ can be identified with the quotient of C by* $\mathrm{Aut}(C; x_1, \ldots, x_n)$.

This Proposition says that in a sense the projection π with its n sections defines the universal stable curve of type (g, n). For this reason we often refer to $\mathcal{M}_{g,1}$ as the *universal smooth genus g curve* $(g \geq 2)$ and denote it by $\mathcal{C}_g$. Likewise $\overline{\mathcal{C}}_g := \overline{\mathcal{M}}_{g,1}$ is the *universal stable genus g curve.*

6.2. **Stratification of $\overline{\mathcal{M}}_{g,n}$.** The normal crossing boundary $\partial\mathcal{M}_{g,n}$ determines a stratification of $\overline{\mathcal{M}}_{g,n}$ in an obvious way: a *stratum* is by definition a connected component of the locus of points of $\overline{\mathcal{M}}_{g,n}$ where the number of local branches of $\partial\mathcal{M}_{g,n}$ at that point is equal to fixed number. That number may be zero, so that $\mathcal{M}_{g,n}$ is a stratum. It is clear that the strata decompose $\overline{\mathcal{M}}_{g,n}$ into subvarieties. A stratum of codimension k parametrizes stable curves of type (g, n) with fixed topological type (and k singular points). You may check that distinct strata correspond to distinct topological types.

We next show that the closure of every stratum is naturally covered by a product of moduli spaces of stable curves. Let Y be a stratum. If $(C; x_1, \ldots, x_n)$ represents a point of Y, then consider the normalization $n : \hat{C} \to C$ and the preimage of the set of special points $\hat{X} := n^{-1}(C_{\mathrm{sing}} \cup \{x_1, \ldots, x_n\})$. The connected components of the pair $(\hat{C}, \hat{X})$ are stable curves, at least after suitably (re)numbering the points of $\hat{X}$, for every component of $\hat{C} - \hat{X}$ maps homeomorphically to a component of $C_{\mathrm{reg}} - \{x_1, \ldots, x_n\}$, hence has negative euler characteristic. So if the types of the stable curves are $(g_i, n_i)_{i \in I}$, then we find an element of $\prod_{i \in I} \mathcal{M}_{g_i, n_i}$. Conversely if we are given a finite collection of smooth curves of type $(g_i, n_i)_{i \in I}$, then by identifying some pairs of points we find a stable curve of type (g, n). In terms of stable graphs: the first procedure amounts to cutting all the bonds in the middle of the stable graph associated to Y so that we end up with a finite set of stars, whereas the second builds out of stars a stable graph by identifying certain pairs of edges. This recipe defines a map

$\prod_i \mathcal{M}_{g_i,n_i} \to Y$ of which it is not difficult to see that it is a finite surjective morphism. The same recipe can be used to glue stable (not necessarily smooth) curves of type (g_i, n_i), so that the map extends to

$$f : \prod_i \overline{\mathcal{M}}_{g_i,n_i} \to \overline{\mathcal{M}}_{g,n}.$$

It is easy to verify that this is a finite morphism with image the closure of Y. (It is in fact an orbifold cover of that closure.) The section σ_i of the universal curve is a special case of this construction.

7. Tautological classes

Throughout the discussion that follows we take rational coefficients for our cohomology groups. One reason is that the space underlying an orbifold is a rational homology manifold, so that it satisfies Poincaré duality when oriented (which is always the case in the complex-analytic setting). We shall be dealing with complex quasi-projective orbifolds and the coholomology classes that we consider happen to have Poincaré duals that are $\mathbb{Q}$-linear combinations of closed subvarieties. Such classes are called *algebraic classes*.

7.1. **The Witten classes.** Given a stable curve $(C; x_1, \ldots, x_n)$, then for a fixed $i \in \{1, \ldots, n\}$, we may associate to it the one dimensional complex vector space $T^*_{x_i}C$. This generalizes to families: if $(\mathcal{C} \to B : (\xi_i : B \to \mathcal{C})_{i=1}^n)$ is a family of stable curves, then for a fixed i, the conormal bundle of ξ_i defines a line bundle over B. In the universal example $(\overline{\mathcal{M}}_{g,n+1} \to \mathcal{M}_{g,n}; \sigma_1, \ldots, \sigma_n)$ this produces an orbifold line bundle over $\mathcal{M}_{g,n}$. Its first Chern class, denoted $\psi_i \in H^2(\overline{\mathcal{M}}_{g,n})$, is called the *ith Witten class*. Since the notation wants to travel lightly, it is a little ambiguous. For example, it is not true that the image of ψ_i in $H^2(\overline{\mathcal{M}}_{g,n+1}$ is the ith Witten class of $\overline{\mathcal{M}}_{g,n+1}$. The euler class of the normal bundle of a parametrization $f : \prod_i \overline{\mathcal{M}}_{g_i,n_i} \to \overline{\mathcal{M}}_{g,n}$ of a closed stratum is a product of Witten classes of the factors. This illustrates a point we are going to make, namely, that all classes of interest appear to be obtainable from the Witten classes.

7.2. **The Mumford classes.** Mumford defined these classes for $\overline{\mathcal{M}}_g$ before the Witten classes were considered; Arbarello-Cornalba used the Witten classes to extend Mumford's definition to the pointed case. It goes as follows: if $\tilde{\psi}_{n+1}$ denotes the last Witten class on $H^2(\overline{\mathcal{M}}_{g,n+1})$, and r is a nonnegative integer, then the rth Mumford class is

$$\kappa_r := \pi_*(\tilde{\psi}_{n+1}^{r+1}) \in H^{2r}(\overline{\mathcal{M}}_{g,n}).$$

(It is an interesting exercise to check that $\kappa_0 = 2g - 2 + n$.) Arbarello and Cornalba showed that κ_1 is the first Chern class of an ample line bundle. They also proved that $H^2(\overline{\mathcal{M}}_{g,n})$ is generated by $\psi_1, \ldots, \psi_n, \kappa_1$ and the Poincaré duals of the irreducible components of the boundary $\partial\overline{\mathcal{M}}_{g,n}$ and that these classes form a basis when $g \geq 3$.

7.3. **The tautological algebra.** It is convenient to make an auxiliary definition first: define the *basic algebra* $\mathcal{B}(\overline{\mathcal{M}}_{g,n})$ of $\overline{\mathcal{M}}_{g,n}$ as the subalgebra of $H^\bullet(\overline{\mathcal{M}}_{g,n})$ generated by the Witten classes $(\psi_j)_j$ and the Mumford classes $(\kappa_r)_{r\geq 0}$. Then the *tautological algebra* of $\overline{\mathcal{M}}_{g,n}$, $\mathcal{R}(\overline{\mathcal{M}}_{g,n})$, is by definition the subalgebra of $H^\bullet(\overline{\mathcal{M}}_{g,n})$ generated by the direct images $f_*(\otimes_i \mathcal{B}(\overline{\mathcal{M}}_{g_i,n_i}) \subset H^{\text{even}}(\overline{\mathcal{M}}_{g,n})$, where the maps $f : \prod_i \overline{\mathcal{M}}_{g_i,n_i} \to \overline{\mathcal{M}}_{g,n}$ run over the parametrizations of the strata. Since the tautological algebra is made of algebraic classes we grade it by half the cohomological degree: $\mathcal{R}^k(\overline{\mathcal{M}}_{g,n}) \subset H^{2k}(\overline{\mathcal{M}}_{g,n})$. It is clear that the tautological algebra and the basic algebra have the same restriction to $\mathcal{M}_{g,n}$; we denote that restriction by $\mathcal{R}(\mathcal{M}_{g,n})$ and call it the *tautological algebra* of $\mathcal{M}_{g,n}$.

It is remarkable that all the known algebraic classes on $\overline{\mathcal{M}}_{g,n}$ are in the tautological subalgebra. We illustrate this with two examples.

Example 7.1 (The Hodge bundle)**.** If C is a stable curve of genus $g \geq 2$, then $H^0(C, \omega_C)$ is a g-dimensional vector space. On the universal example this gives a rank g vector bundle $\mathbb{E}$ over $\overline{\mathcal{M}}_g$ called the *Hodge bundle*. (Since $H^0(C, \omega_C)$ only depends on the (generalized) Jacobian of C, $\mathbb{E}$ is the pullback of a bundle that is naturally defined on a certain compactification of the moduli space of principally polarized abelian varieties of dimension g.) Mumford [3] expressed the Chern class $\lambda_i := c_i(\mathbb{E}) \in H^{2i}(\overline{\mathcal{M}}_g)$ as an element of $\mathcal{R}^i(\overline{\mathcal{M}}_g)$.

Example 7.2 (The Weierstraß loci)**.** Suppose C is a smooth, connected projective curve C of genus $g \geq 2$, $p \in C$, and an l a positive integer. It is easy to see that the following conditions are equivalent:

- The linear system $|l(p)|$ is of dimension ≥ 1.
- There exists a nonconstant regular function on $C - \{p\}$ that has in p with a pole of order $\leq l$.
- There exists a finite morphism $C \to \mathbb{P}^1$ of degree $\leq l$ that is totally ramified in p.

By Riemann-Roch these conditions are always fulfilled if $l \geq g + 1$. If $l = 2$, then the morphism f appearing in the last item must have degree 2 so that C is hyperelliptic and p is a Weierstraß point.

These equivalent conditions define a closed subvariety W_l^* of the universal smooth curve of genus g, $\mathcal{C}_g$. Arbarello, who introduced these varieties, noted that W_l^* is irreducible of codimension $g+1-l$ in $\mathcal{C}_g$. Mumford expressed the corresponding class in $H^{2(g+1-l)}(\mathcal{C}_g)$ as an element of $\mathcal{R}^{g+1-l}(\mathcal{C}_g)$.

The validity of Grothendieck's standard conjectures implies that the algebraic classes on $\overline{\mathcal{M}}_{g,n}$ make up a nondegenerate subspace of $H^{\text{even}}(\overline{\mathcal{M}}_{g,n})$ with respect to the intersection product. Since we do not know any algebraic class that is not tautological, we ask:

Question 7.3. Does $\mathcal{R}(\overline{\mathcal{M}}_{g,n})$ satisfy Poincaré duality?

Since all tautological classes originate from Witten classes, this question could, in principle, be answered for a given pair (g_0, n_0) if we would know all the intersection numbers

$$\int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{k_1} \cdots \psi_n^{k_n} \in \mathbb{Q},$$

(where it is of course understood that this number is zero if the degree $k_1 + \cdots + k_n$ of the integrand fails to equal $3g-3+n$) for all (g,n) with $2g+n \leq 2g_0+n_0$. A marvelous conjecture of Witten (which is a conjecture no longer) predicts the values of these numbers. We shall state it in a form that exhibits the algebro-geometric content best (this formulation is due to Dijkgraaf-E. Verlinde-H. Verlinde). For this purpose it is convenient to renormalize the intersection numbers as follows:

$$[\tau_{k_1}\tau_{k_2}\cdots\tau_{k_n}]_g := (2k_1+1)!!(2k_2+1)!!\cdots(2k_n+1)!! \int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{k_1}\cdots\psi_n^{k_n},$$

where $(2k+1)!! = 1.3.5.\cdots.(2k+1)$. We use these to form the series in the variables $t_0, t_1, t_2, \ldots$:

$$F_g := \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{k_1\geq 0,,k_2\geq 0,\ldots,k_n\geq 0} [\tau_{k_1}\tau_{k_2}\cdots\tau_{k_n}]_g t_{k_1}t_{k_2}\cdots t_{k_n}.$$

It is invariant under permutation of variables. The Witten conjectures assert that these polynomials satisfy a series of differential equations indexed by the integers ≥ -1. For index -1 this is the *string equation*:

$$\frac{\partial F_g}{\partial t_0} = \sum_{m\geq 1}(2m+1)t_m\frac{\partial F_g}{\partial t_{m-1}} + \tfrac{1}{2}\delta_{0,g}t_0^2,$$

for index 0 the *dilaton equation*:

$$\frac{\partial F_g}{\partial t_1} = \sum_{m\geq 0}(2m+1)t_m\frac{\partial F_g}{\partial t_m} + \tfrac{1}{8}\delta_{1,g}t_0^2$$

and for $k \geq 1$ we get:

$$\begin{aligned}\frac{\partial F_g}{\partial t_{k+1}} &= \sum_{m\geq 1}(2m+1)t_m\frac{\partial F_g}{\partial t_{m+k}}\\ &> +\tfrac{1}{2}\sum_{m'+m''=k-1}\frac{\partial F_{g-1}}{\partial t_{m'}\partial t_{m''}}\\ &\quad +\tfrac{1}{2}\sum_{m'+m''=k-1}\sum_{g'+g''=g}\frac{\partial F_{g'}}{\partial t_{m'}}\frac{\partial F_{g''}}{\partial t_{m''}}.\end{aligned}$$

You may verify that these equations determine the functions F_g completely (note that F_g has no constant term). The string equation and the dilaton equation involve a single genus only and were verified by Witten using standard arguments from algebriac geometry. But the equations for $k \geq 1$ were proved in an entirely different manner: Kontsevich gave an amazing proof based on a triangulation of Teichmüller space on which the intersection numbers appear as integrals of explicitly given differential forms. Yet there are reasons to wish for a proof within the realm of algebraic geometry. The form of the equations is suggestive in this respect: the first line involves the $\overline{\mathcal{M}}_{g,n}$, but the second and third seem to be about intersection numbers formed on irreducible components of $\partial\mathcal{M}g, n$ (Δ_0 and the $\Delta_{\{(g',I'),(g'',I'')\}}$ respectively). Consider this a challenge.

7.4. **Faber's conjectures.** These concern the structure of $\mathcal{R}(\mathcal{M}_g)$. From the definition it is clear that these are generated by the restrictions of the Mumford classes. Let us for convenience denote these classes κ_r instead of $\kappa_r|\mathcal{M}_g$. Faber made the essential part of his conjectures around 1993. We shall not state them in their most precise form (we refer to [4] for that).

1. $\mathcal{R}_g$ is zero in degree $\geq g-1$ and is of dimension one in degree $g-2$ and the cup product
$$\mathcal{R}^i(\mathcal{M})g) \times \mathcal{R}^{g-2-i}(\mathcal{M}_g) \to \mathcal{R}^{g-2}(\mathcal{M}_g) \cong \mathbb{Q}$$
is nondegenerate.
2. The classes $\kappa_1, \ldots, \kappa_{[g/3]}$ generate $\mathcal{R}(\mathcal{M}_g)$ and satisfy no polynomial relation in degree $\leq [g/3]$.

Let us review the status of these conjectures. First, there is the evidence provided Faber himself: he checked his conjectures up to genus 15.

As to conjecture 1: the vanishing assertion was proved in a paper of mine where it was also shown that $\dim \mathcal{R}^{g-2}(\mathcal{M}_g) \leq 1$. Subsequently Faber proved that the dimension is in fact equal to one. The rest of conjecture 1 remains open. Conjecture 2 now seems settled: Harer had shown around the time that Faber made his conjectures that the kappa classes have no polynomial relations in degree $\leq [g/3]$ and Morita has recently announced that $\kappa_1, \ldots, \kappa_{[g/3]}$ generate $\mathcal{R}(\mathcal{M}_g)$.

Let me close with saying a bit more about Faber's nonvanishing proof. He observes that the class $\lambda_g \lambda_{g-1} \in \mathcal{R}^{2g-1}(\overline{\mathcal{M}}_g)$ restricts to zero on the boundary $\partial \mathcal{M}_g$. This implies that for every $u \in \mathcal{R}^{g-2}(\overline{\mathcal{M}}_g)$, the intersection product $\int_{\overline{\mathcal{M}}_g} \lambda_g \lambda_{g-1} u$ only depends on $u|\mathcal{M}_g$. So this defines a 'trace' $t : \mathcal{R}(\mathcal{M}_g) \to \mathbb{Q}$. Faber proves that this trace is nonzero on κ_{g-2}. The unproven part of the conjecture can be phrased as saying that the associated form $(u, v) \in \mathcal{R}(\mathcal{M}_g) \times \mathcal{R}(\mathcal{M}_g) \mapsto t(uv)$ is nondegenerate. Faber has also an explicit proposal for the value of the trace on any monomial in the Mumford classes.

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Moduli of Riemann surfaces, transcendental aspects

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These notes are an informal introduction to moduli spaces of compact Riemann surfaces via complex analysis, topology and Hodge Theory. The prerequisites for the first lecture are just basic complex variables, basic Riemann surface theory up to at least the Riemann-Roch formula, and some algebraic topology, especially covering space theory. Some good references for this material include [1] for complex analysis, [8] and [9] for the basic theory of Riemann surfaces, and [11] for algebraic topology. For later lectures I will assume more. The book by Clemens [5] and Chapter 2 of Griffiths and Harris [12] are excellent and are highly recommended. Other useful references include the surveys [1] and [14] and the book [17].

The first lecture covers moduli in genus 0 and genus 1 as these can be understood using relatively elementary methods, but illustrate many of the points which arise in higher genus. The notes cover more material than was covered in the lectures, and sometimes the order of topics in the notes differs from that in the lectures. I hope to add the material from the last lecture on the Torelli group and Morita's approach to the tautological classes in a future version.

Lecture 1: Low Genus Examples

Suppose that g and n are non-negative integers. An *n-pointed Riemann surface $(C; x_1, \ldots, x_n)$ of genus g* is a compact Riemann surface C of genus g together with an ordered n-tuple of distinct points $(x_1, \ldots, x_n)$ of C. Two n-pointed Riemann surfaces $(C; x_1, \ldots, x_n)$ and $(C'; x'_1, \ldots, x'_n)$ are *isomorphic* if there is a biholomorphism $f : C \to C'$ such that $f(x_j) = x'_j$ when $1 \leq j \leq n$. The principal objects of study in these lectures are the spaces

$$\mathcal{M}_{g,n} = \left\{ \begin{array}{l} \text{isomorphism classes of } n\text{-pointed compact} \\ \text{Riemann surfaces } C \text{ of genus } g \end{array} \right\}$$

At the moment all we can say is that these are sets. One of the main objectives of these lectures is to show that each $\mathcal{M}_{g,n}$ is a complex analytic variety with very mild singularities.

Later we will only consider $\mathcal{M}_{g,n}$ when the stability condition

$$2g - 2 + n > 0 \tag{1}$$

is satisfied. But for the time being we will consider all possible values of g and n. When $n = 0$, we will simply write $\mathcal{M}_g$ instead of $\mathcal{M}_{g,0}$.

The space $\mathcal{M}_{g,n}$ is called the *moduli space of n-pointed curves (or Riemann surfaces) of genus g*. The isomorphism class of $(C; x_1, \ldots, x_n)$ is called the *moduli point* of $(C; x_1, \ldots, x_n)$ and will be denoted by $[C; x_1, \ldots, x_n]$.

There are (at least) two notions of the genus of a compact Riemann surface C. First there is the (analytic) genus

$$g(C) := \dim H^0(C, \Omega^1_C),$$

the dimension of the space of global holomorphic 1-forms on C. Second there is the *topological genus*

$$g_{\mathrm{top}}(C) := \frac{1}{2} \operatorname{rank} H_1(C, \mathbb{Z}).$$

Intuitively, this is the 'number of holes' in C. A basic fact is that these are equal. There are various ways to prove this, but perhaps the most standard is to use the Hodge Theorem (reference) which implies that

$$H^1(C, \mathbb{C}) \cong \{\text{holomorphic 1-forms}\} \oplus \{\text{anti-holomorphic 1-forms}\}.$$

The equality of $g_{\mathrm{top}}(C)$ and $g(C)$ follows immediately as complex conjugation interchanges the holomorphic and antiholomorphic differentials.

Finally, we shall use the terms "complex curve" and "Riemann surface" interchangeably.

1. Genus 0

It follows from Riemann-Roch formula that if X is a compact Riemann surface of genus 0, then X is biholomorphic to the Riemann sphere $\mathbb{P}^1$. So $\mathcal{M}_0$ consists of a single point.

An *automorphism* of a Riemann surface X is simply a biholomorphism $f : X \to X$. The set of all automorphisms of X forms a group $\operatorname{Aut} X$. The group $GL_2(\mathbb{C})$ acts in $\mathbb{P}^1$ via fractional linear transformations:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} : z \mapsto \frac{az+b}{cz+d}$$

The scalar matrices S act trivially, and so we have a homomorphism

$$PGL_2(\mathbb{C}) \to \operatorname{Aut} \mathbb{P}^1$$

where for any field F

$$PGL_n(F) = GL_n(F)/\{\text{scalar matrices}\}$$

and

$$PSL_n(F) = SL_n(F)/\{\text{scalar matrices of determinant } 1\}.$$

Exercise 1.1. Prove that $PGL_2(\mathbb{C}) \cong PSL_2(\mathbb{C})$ and that these are isomorphic to $\operatorname{Aut} \mathbb{P}^1$.

Exercise 1.2. Prove that $\operatorname{Aut}\mathbb{P}^1$ acts 3-transitively on $\mathbb{P}^1$. That is, given any two ordered 3-tuples (a_1, a_2, a_3) and (b_1, b_2, b_3) of distinct points of $\mathbb{P}^1$, there is an element f of $\operatorname{Aut}\mathbb{P}^1$ such that $f(a_j) = b_j$ for $j = 1, 2, 3$. Show that f is unique.

Exercise 1.3. Prove that if X is a compact Riemann surface of genus 0, then X is biholomorphic to the Riemann sphere.

Exercise 1.4. Show that the automorphism group of an n-pointed curve of genus g is finite if and only if the stability condition (1) is satisfied. (Depending on what you know, you may find this a little difficult at present. More techniques will become available soon.)

Since $\operatorname{Aut}\mathbb{P}^1$ acts 3-transitively on $\mathbb{P}^1$, we have:

Proposition 1.5. *Every n-pointed Riemann surface of genus 0 is isomorphic to*

$$\begin{array}{ll} (\mathbb{P}^1; \infty) & \textit{if } n = 1; \\ (\mathbb{P}^1; 0, \infty) & \textit{if } n = 2; \\ (\mathbb{P}^1; 0, 1, \infty) & \textit{if } n = 3. \end{array}$$

□

Corollary 1.6. *If $0 \le n \le 3$, then $\mathcal{M}_{0,n}$ consists of a single point.* □

The first interesting case is when $n = 4$. If $(X; x_1, x_2, x_3, x_4)$ is a 4-pointed Riemann surface of genus 0, then there is a unique biholomorphism $f : X \to \mathbb{P}^1$ with $f(x_2) = 1$, $f(x_3) = 0$ and $f(x_4) = \infty$. The value of $f(x_1)$ is forced by these conditions. Since the x_j are distinct and f is a biholomorphism, $f(x_1) \in \mathbb{C} - \{0, 1\}$. It is therefore an invariant of $(X; x_1, x_2, x_3, x_4)$.

Exercise 1.7. Show that if $g : X \to \mathbb{P}^1$ is any biholomorphism, then $f(x_1)$ is the cross ratio

$$(g(x_1) : g(x_2) : g(x_3) : g(x_4))$$

of $g(x_1)$, $g(x_2)$, $g(x_3)$, $g(x_4)$. Recall that the cross ratio of four distinct points x_1, x_2, x_3, x_4 in $\mathbb{P}^1$ is defined by

$$(x_1 : x_2 : x_3 : x_4) = \frac{(x_1 - x_3)/(x_2 - x_3)}{(x_1 - x_4)/(x_2 - x_4)}$$

The result of the previous exercise can be rephrased as a statement about moduli spaces:

Proposition 1.8. *The moduli space $\mathcal{M}_{0,4}$ can be identified naturally with $\mathbb{C} - \{0, 1\}$. The moduli point $[\mathbb{P}^1; x_1, x_2, x_3, x_4]$ is identified with the cross ratio $(x_1 : x_2 : x_3 : x_4) \in \mathbb{C} - \{0, 1\}$.* □

It is now easy to generalize this to general $n \geq 4$. Since every genus 0 Riemann surface is biholomorphic to $\mathbb{P}^1$, we need only consider n-pointed curves of the form $(\mathbb{P}^1; x_1, \ldots, x_n)$. There is a unique automorphism f of $\mathbb{P}^1$ such that $f(x_1) = 0, f(x_2) = 1$ and $f(x_3) = \infty$. So every n-pointed Riemann surface of genus 0 is isomorphic to exactly one of the form

$$(\mathbb{P}^1; 0, 1, \infty, y_1, \ldots, y_{n-3}).$$

To say that this is an n-pointed curve is to say that the points $0, 1, \infty$, $y_1, \ldots, y_{n-3}$ are distinct. That is,

$$(y_1, \ldots, y_{n-3}) \in (\mathbb{C} - \{0, 1\})^{n-3} - \Delta$$

where $\Delta = \cup_{j<k} \Delta_{jk}$ is the union of the diagonals

$$\Delta_{jk} = \{(y_1, \ldots, y_{n-3}) : y_j = y_k\}.$$

This is an affine algebraic variety as it is the complement of a divisor in an affine space. This shows that:

Theorem 1.9. *If $n \geq 3$, then $\mathcal{M}_{0,n}$ is a smooth affine algebraic variety of dimension $n-3$ isomorphic to*

$$(\mathbb{C} - \{0, 1\})^{n-3} - \Delta.$$

The symmetric group Σ_n acts on $\mathcal{M}_{0,n}$ by

$$\sigma : (\mathbb{P}^1; x_1, \ldots, x_n) \mapsto (\mathbb{P}^1; x_{\sigma(1)}, \ldots, x_{\sigma(n)}).$$

Exercise 1.10. Show that each $\sigma \in \Sigma_n$ acts on $\mathcal{M}_{0,n}$ as a regular mapping. (Hint: it suffices to consider the case of a transposition.)

Exercise 1.11. Suppose that $n \geq 3$. Construct a universal n-pointed genus 0 curve $\mathcal{M}_{0,n} \times \mathbb{P}^1 \to \mathcal{M}_{0,n}$ that is equipped with n disjoint sections $\sigma_1, \ldots, \sigma_n$ such that

$$\sigma_j([\mathbb{P}^1; x_1, \ldots, x_n]) = ([\mathbb{P}^1; x_1, \ldots, x_n], x_j).$$

Show that it is universal in the sense that if $f : X \to T$ is a family of smooth genus 0 curves over a smooth variety T and if the family has n sections $s_1, \ldots, s_n$ that are disjoint, then there is a holomorphic mapping $\phi_f : T \to \mathcal{M}_{0,n}$ such that the pullback of the universal family is f and the pullback of σ_j is s_j.

2. Genus 1

The study of the moduli space of genus 1 compact Riemann surfaces is very rich and has a long history because of its fundamental connections to number theory and the theory of plane cubic curves. We will take a transcendental approach to understanding $\mathcal{M}_1$ which will reveal the connection with modular forms. Our first task is show that genus 1 Riemann surfaces can always be represented as the quotient of $\mathbb{C}$ by a lattice.

One way to construct a Riemann surface of genus 1 is to take the quotient of $\mathbb{C}$ by a lattice. Recall that a *lattice* in a finite dimensional real vector space V is a finitely generated (and therefore free abelian) subgroup Λ of V with the property that a basis of Λ as an abelian group is also a basis of V as a real vector space. A lattice in $\mathbb{C}$ is thus a subgroup Λ of $\mathbb{C}$ that is isomorphic to $\mathbb{Z}^2$ and is generated by two complex numbers that are not real multiples of each other.

Exercise 2.1. Show that if Λ is a lattice in V and if $\dim_{\mathbb{R}} V = d$, then V/Λ is a compact manifold of real dimension d, which is diffeomorphic to the d-torus $(\mathbb{R}/\mathbb{Z})^d$.

If Λ is a lattice in $\mathbb{C}$, the quotient group $\mathbb{C}/\Lambda$ is a compact Riemann surface which is diffeomorphic to the product of two circles, and so of genus 1.

Theorem 2.2. *If C is a compact Riemann surface of genus 1, then there is a lattice Λ in $\mathbb{C}$ and an isomorphism $\mu : C \to \mathbb{C}/\Lambda$. If $x_o \in C$, then we may choose μ such that $\mu(x_o) = 0$.*

The proof follows from the sequence of exercises below. Let C be a compact Riemann surface of genus 1.

Exercise 2.3. Show that a non-zero holomorphic differential on C has no zeros. Hint: Use Riemann-Roch.

Since $g_{\text{top}}(C) = g(C) = 1$, we know that $H_1(C, \mathbb{Z})$ is free of rank 2. Fix a non-zero holomorphic differential w on C. Every other holomorphic differential is a multiple of w. The *period lattice* of C is defined to be

$$\Lambda = \{ \int_c w : c \in H_1(C, \mathbb{Z}) \}.$$

This is easily seen to be a subgroup of $\mathbb{C}$.

Exercise 2.4. Show that Λ is a lattice in $\mathbb{C}$. Hint: Choose a basis a, b of $H_1(C, \mathbb{Z})$. Show that if $\int_a w$ and $\int_b w$ are linearly independent over $\mathbb{R}$, then this would contradict the Hodge decomposition of $H^1(C, \mathbb{C})$.

Let $E = \mathbb{C}/\Lambda$. Our next task is to construct a holomorphic mapping from C to E.

Exercise 2.5. Fix a base point x_o of C. Define a mapping $\nu : C \to E$ by

$$\nu(x) = \int_\gamma w$$

where γ is any smooth path in C that goes from x_o to x. Show that

(i) ν is well defined;
(ii) ν is holomorphic;
(iii) ν has nowhere vanishing differential, and is therefore a covering map;
(iv) the homomorphism $\nu_* : \pi_1(C, x_o) \to \pi_1(E, 0) \cong \Lambda$ is surjective, and therefore an isomorphism.

Deduce that ν is a biholomorphism.

This completes the proof of Theorem 2.2. It has the following important consequence:

Corollary 2.6. *If C is a compact Riemann surface of genus 1, then the automorphism group of C acts transitively on C. Consequently, the natural mapping $\mathcal{M}_{1,1} \to \mathcal{M}_1$ that takes $[C; x]$ to $[C]$ is a bijection.*

Proof. This follows as every genus 1 Riemann surface is isomorphic to one of the form $\mathbb{C}/\Lambda$. For such Riemann surfaces, we have the homomorphism

$$\mathbb{C}/\Lambda \to \mathrm{Aut}(\mathbb{C}/\Lambda)$$

that takes the coset $a + \Lambda$ to the translation $z + \Lambda \mapsto z + a + \Lambda$. □

Definition 2.7. An *elliptic curve* is a a genus 1 curve C together with a point $x_o \in C$.

The previous result says that if C is a genus 1 curve and x_o and y_o are points of C, then the elliptic curves $(C; x_o)$ and $(C; y_o)$ are isomorphic.

The moduli space of elliptic curves is $\mathcal{M}_{1,1}$.

Exercise 2.8. Suppose that $f : C \to \mathbb{C}/\Lambda$ is a holomorphic mapping from an arbitrary Riemann surface to $\mathbb{C}/\Lambda$. Let x_o be a base point of C. The 1-form dz on $\mathbb{C}$ descends to a holomorphic differential w on $\mathbb{C}/\Lambda$. Its pullback

f^*w is a holomorphic differential on C. Show that for all $x \in C$,

$$f(x) = f(x_o) + \int_\gamma f^*w + \Lambda$$

where γ is a path in C from x_o to x.

Exercise 2.9. Use the results of the previous exercise to prove the following result.

Corollary 2.10. *If Λ_1 and Λ_2 are lattices in $\mathbb{C}$, then $\mathbb{C}/\Lambda_1$ is isomorphic to $\mathbb{C}/\Lambda_2$ if and only if there exists $\lambda \in \mathbb{C}^*$ such that $\Lambda_1 = \lambda\Lambda_2$.* □

Exercise 2.11. Show that if $(C; x_o)$ is an elliptic curve, then C has a natural group structure with identity x_o.

We are finally ready to give a construction of $\mathcal{M}_1$. Recall that the complex structure on a Riemann surface C gives it a canonical orientation. This can be thought of as giving a direction of "positive rotation" about each point in the surface — the positive direction being that given by turning counter-clockwise about the point in any local holomorphic coordinate system. If C is compact, this orientation allows us to define the intersection number of two transversally intersecting closed curves on C. It depends only on the homology classes of the two curves and therefore defines the *intersection pairing*

$$\langle\ ,\ \rangle : H_1(C, \mathbb{Z}) \otimes H_1(C, \mathbb{Z}) \to \mathbb{Z}.$$

If α, β is a basis of $H_1(C, \mathbb{Z})$, then $\langle \alpha, \beta \rangle = \pm 1$. We shall call the basis *positive* if $\langle \alpha, \beta \rangle = 1$.

A *framing* of Riemann surface C of genus 1 is a positive basis α, β of its first homology group. We will refer to $(C : \alpha, \beta)$ as a *framed* genus 1 Riemann surface. Two framed genus 1 Riemann surfaces $(C : \alpha, \beta)$ and $(C' : \alpha', \beta')$ are isomorphic if there is a biholomorphism $f : C \to C'$ such that $\alpha' = f_*\alpha$ and $\beta' = f_*\beta$.

Let

$$\mathcal{X}_1 = \left\{ \begin{matrix} \text{isomorphism classes of framed} \\ \text{Riemann surfaces of genus 1} \end{matrix} \right\}.$$

At the moment, this is just a set. But soon we will see that it is itself a Riemann surface. Note that forgetting the framing defines a a function

$$\phi : \mathcal{X}_1 \to \mathcal{M}_1.$$

Denote the isomorphism class of $(C : \alpha, \beta)$ by $[C : \alpha, \beta]$. If C is a genus 1 Riemann surface, then

$$\phi^{-1}([C]) = \{[C : \alpha, \beta] : (\alpha, \beta) \text{ is a positive basis of } H_1(C, \mathbb{Z})\}.$$

Exercise 2.12. Show that if (α, β) and (α', β') are two positive bases of $H_1(C, \mathbb{Z})$, then there is a unique element

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

of $SL_2(\mathbb{Z})$ such that

$$\begin{pmatrix} \beta' \\ \alpha' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \beta \\ \alpha \end{pmatrix}$$

(The reason for writing the basis vectors in the reverse order will become apparent shortly.)

Define an action of $SL_2(\mathbb{Z})$ on $\mathcal{X}_1$ by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} [C : \alpha, \beta] = [C : \alpha', b']$$

where

$$\begin{pmatrix} \beta' \\ \alpha' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \beta \\ \alpha \end{pmatrix}$$

Exercise 2.13. Show that there is a natural bijection

$$\mathcal{M}_1 \cong SL_2(\mathbb{Z}) \backslash \mathcal{X}_1.$$

At present, $\mathcal{X}_1$ is just a set, but we now show that it is naturally a Riemann surface. We know from Theorem 2.2 that every element of $\mathcal{X}_1$ is of the form $[\mathbb{C}/\Lambda : \alpha, \beta]$. But, by standard algebraic topology, there is a natural isomorphism

$$\Lambda \cong H_1(C, \mathbb{Z}).$$

Thus a basis of $H_1(C, \mathbb{Z})$ corresponds to a basis of Λ.

Exercise 2.14. Suppose that α, β is a basis of $H_1(\mathbb{C}/\Lambda, \mathbb{Z})$ and that ω_1, ω_2 is the corresponding basis of Λ. Show that α, β is positive if and only if ω_2/ω_1 has positive imaginary part.

It follows from this and Corollary 2.10 that

$$\mathcal{X}_1 = \{[\mathbb{C}/\Lambda : \omega_1, \omega_2] : \mathrm{Im}(\omega_2/\omega_1) > 0\}/\mathbb{C}^*$$

where the $\mathbb{C}^*$-action is defined by

$$\lambda \cdot [\mathbb{C}/\Lambda : \omega_1, \omega_2] = [\mathbb{C}/\lambda\Lambda : \lambda\omega_1, \lambda\omega_2].$$

We can go even further: since the basis ω_1, ω_2 determines the lattice,

$$\Lambda = \mathbb{Z}\omega_1 \oplus \mathbb{Z}\omega_2,$$

we can dispense with the lattice altogether. We have:

$$\mathcal{X}_1 = \{(\omega_1, \omega_2) : \omega_1, \omega_2 \in \mathbb{C} \text{ and } \operatorname{Im}(\omega_2/\omega_1) > 0\}/\mathbb{C}^* \tag{2}$$

where $\mathbb{C}^*$ acts on (ω_1, ω_2) by scalar multiplication.

Denote the upper half plane $\{z \in \mathbb{C} : \operatorname{Im} z > 0\}$ by $\mathbb{H}$. Each $\tau \in \mathbb{H}$ determines the element

$$[\mathbb{C}/(\mathbb{Z} \oplus \mathbb{Z}\tau) : 1, \tau]$$

of $\mathcal{X}_1$. This defines a function $\psi : \mathbb{H} \to \mathcal{X}_1$. Under the identification (2),

$$\psi(\tau) = \text{ the } \mathbb{C}^*\text{-orbit of } (1, \tau)$$

Since (ω_1, ω_2) and $(1, \omega_2/\omega_1)$ are in the same orbit, we have proved:

Theorem 2.15. *The function $\psi : \mathbb{H} \to \mathcal{X}_1$ is a bijection.* □

The group $PSL_2(\mathbb{C})$ acts on $\mathbb{P}^1$ by fractional linear transformations.

Exercise 2.16. Show that $T \in PSL_2(\mathbb{C})$ satisfies $T(\mathbb{H}) \subseteq \mathbb{H}$ if and only if $T \in PSL_2(\mathbb{R})$.

Thus the group $SL_2(\mathbb{R})$ acts on $\mathbb{H}$ by fractional linear transformations:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \tau = \frac{a\tau + b}{c\tau + d}$$

Exercise 2.17. Show that $\psi : \mathbb{H} \to \mathcal{X}_1$ is $SL_2(\mathbb{Z})$-equivariant. That is, if $T \in SL_2(\mathbb{Z})$, then $T\psi(\tau) = \psi(T\tau)$.

Theorem 2.18. *There are natural bijections*

$$\mathcal{M}_1 \cong \mathcal{M}_{1,1} \cong SL_2(\mathbb{Z})\backslash\mathbb{H}.$$

Proof. The first bijection was established in Corollary 2. . The second follows from Exercise 2.13, Theorem 2.15 and Exercise 2.17. □

Exercise 2.19. Suppose that C is a genus 1 Riemann surface. Show that the point of $SL_2(\mathbb{Z})\backslash\mathbb{H}$ that corresponds to $[C] \in \mathcal{M}_1$ is the $SL_2(\mathbb{Z})$ orbit of

$$\left(\int_\beta w \Big/ \int_\alpha w \right) \in \mathbb{H}$$

where w is any non-zero element of $H^0(C, \Omega^1_C)$ and α, β is a positive basis of $H_1(C, \mathbb{Z})$.

2.1. **Understanding** $SL_2(\mathbb{Z})\backslash\mathbb{H}$. The Riemann surface structure on $\mathbb{H}$ descends to a Riemann surface structure on $SL_2(\mathbb{Z})\backslash\mathbb{H}$. Good references for this are Chapter VII of Serre's book [24], and Chapter 3 of Clemens' book [5]. We'll sketch part of the proof of the following fundamental theorem.

Theorem 2.20. *The quotient of $\mathbb{H}$ by $SL_2(\mathbb{Z})$ has a unique structure of a Riemann surface such that the projection $\mathbb{H} \to SL_2(\mathbb{Z})\backslash\mathbb{H}$ is holomorphic. Moreover, there is a biholomorphism between $SL_2(\mathbb{Z})\backslash\mathbb{H}$ and $\mathbb{C}$ which can be given by the modular function $j : \mathbb{H} \to \mathbb{C}$, where*

$$j(\tau) = \frac{1}{q} + 744 + 19\ \ 884\, q + 21\,493\,7\ \ 0\, q^2 + \cdots$$

and $q = e^{2\pi i\tau}$.

The following exercises will allow you to construct most of the proof. The rest can be found in [24] and [5].

Let $\mathbb{P}^1(\mathbb{R}) = \mathbb{R} \cup \{\infty\}$. This is a circle on the Riemann sphere which forms the boundary of $\mathbb{H}$. Let $\overline{\mathbb{H}}$ be the closure of $\mathbb{H}$ in the Riemann sphere $\mathbb{P}^1$; it is the union of $\mathbb{H}$ and $\mathbb{P}^1(\mathbb{R})$. Recall that every non-trivial element of $PSL_2(\mathbb{C})$ has at most two fixed points in $\mathbb{P}^1$. Note that the fixed points of elements of $PSL_2(\mathbb{R})$ are real or occur in complex conjugate pairs.

Exercise 2.21. Suppose that $T \in SL_2(\mathbb{Z})$ is not a scalar matrix. Show that T has exactly

(i) one fixed point in $\mathbb{H}$ if and only if $|\operatorname{tr} T| < 2$;
(ii) one fixed point in $\mathbb{P}^1(\mathbb{R})$ if and only if $|\operatorname{tr} T| = 2$;
(iii) two fixed points in $\mathbb{P}^1(\mathbb{R})$ if and only if $|\operatorname{tr} T| > 2$.

Show that $T \in SL_2(\mathbb{R})$ has finite order if and only if T has a fixed point in $\mathbb{H}$.

Fix an integer $l \geq 0$. The *level l subgroup* of $SL_2(\mathbb{Z})$ is the subgroup of $SL_2(\mathbb{Z})$ consisting of those matrices congruent to the identity mod l. We shall denote it by $SL_2(\mathbb{Z})[l]$. Since it is the kernel of the homomorphism $SL_2(\mathbb{Z}) \to SL_2(\mathbb{Z}/l)$, it is normal and of finite index in $SL_2(\mathbb{Z})$.

Exercise 2.22. Show that $SL_2(\mathbb{Z})[l]$ is torsion free for all $l \geq 3$. Hint: use the previous exercise. Deduce that $SL_2(\mathbb{Z})[l]\backslash\mathbb{H}$ is a Riemann surface with fundamental group $SL_2(\mathbb{Z})[l]$ and universal covering $\mathbb{H}$ whenever $l \geq 3$.

The quotient of $SL_2(\mathbb{Z})$ by its level l subgroup is $SL_2(\mathbb{Z}/l)$, which is a finite group. It follows that the projection

$$SL_2(\mathbb{Z})[l]\backslash\mathbb{H} \to SL_2(\mathbb{Z})\backslash\mathbb{H}$$

is finite-to-one of degree equal to the half order of $SL_2(\mathbb{Z}/l)$ when $l > 2$.

Exercise 2.23. Show that the quotient $G\backslash C$ of a Riemann surface C by a finite subgroup G of $\operatorname{Aut} C$ has the structure of a Riemann surface such that the projection $C \to G\backslash C$ is holomorphic. Hint: first show that for each point x of C, the isotropy group

$$G_x := \{g \in G : gx = x\}$$

is cyclic. This can be done by considering the actions of G_x on T_xX and $\mathcal{O}_{X,x}$.

This result, combined with the previous exercises, establishes that $\mathcal{M}_1 = SL_2(\mathbb{Z})\backslash\mathbb{H}$ has a natural structure of a Riemann surface such that the projection $\mathbb{H} \to \mathcal{M}_1$ is holomorphic.

Serre [24, p. 78] proves that a fundamental domain for the action of $SL_2(\mathbb{Z})$ on $\mathbb{H}$ is the region

$$F = \{\tau \in \mathbb{H} : |\operatorname{Re}\tau| \leq 1/2, |\tau| \geq 1\}.$$

Points of F can be thought of as giving a canonical framing of a lattice Λ

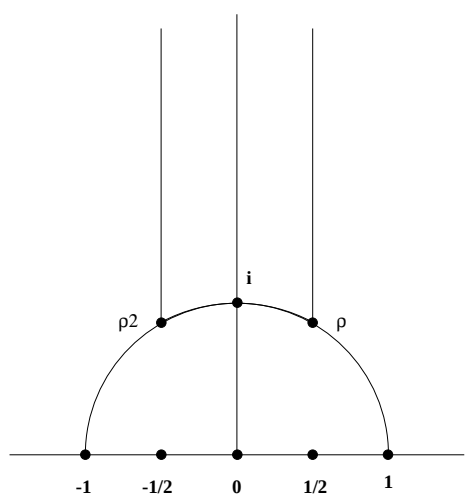

FIGURE 1. A a fundamental domain of $SL_2(\mathbb{Z})$ in $\mathbb{H}$

in $\mathbb{C}$. Such a framing is given as follows: the first basis element is a non-zero vector of shortest length in Λ, the second basis element is a shortest vector in Λ that is not a multiple of the first.

Exercise 2.24. Show that $\tau \in F$ if and only if $(1, \tau)$ is such a canonical basis of the lattice $\mathbb{Z} \oplus \mathbb{Z}\tau$.

Serre [24, p. 78] also proves that $PSL_2(\mathbb{Z})$ is generated by $\tau \mapsto -1/\tau$ and $\tau \mapsto \tau + 1$.

Exercise 2.25. Use this to prove that $SL_2(\mathbb{Z})\backslash\mathbb{H}$ is the quotient of the fundamental domain F obtained by identifying the opposite vertical sides, and by identifying the arc of the circle $|\tau| = 1$ from ρ to i with the arc from ρ^2 to i. Deduce that $\mathcal{M}_1 = SL_2(\mathbb{Z})\backslash\mathbb{H}$ is a Riemann surface homeomorphic to a disk.

It is conceivable that $\mathcal{M}_1$ is biholomorphic to a disk, for example. But this is not the case as $\mathcal{M}_1$ can be compactified by adding one point.

Exercise 2.26. Show that $\mathcal{M}_1$ can be compactified by adding a single point ∞. A coordinate neighbourhood of ∞ is the unit disk Δ. Denote the holomorphic coordinate in it by q. The point τ of $\mathcal{M}_1 = SL_2(\mathbb{Z})\backslash\mathbb{H}$ is identified with the point $e^{2\pi i\tau}$ of Δ. Show that

$$\mathcal{M}_1 \cup \{\infty\}$$

is a compact Riemann surface of genus 0 where q is a local parameter about ∞ and where $\mathcal{M}_1$ holomorphically embedded. Deduce that $\mathcal{M}_1$ is biholomorphic to $\mathbb{C}$.

Remark 2.27. This compactification is the moduli space $\overline{\mathcal{M}}_{1,1}$.

Since every compact Riemann surface is canonically a complex algebraic curve, this shows that $\mathcal{M}_1$ is an algebraic variety.

2.2. **Automorphisms.** The automorphisms of an elliptic curve are intimately related with the set of elements of $SL_2(\mathbb{Z})$ that stabilize the points corresponding to it in $\mathbb{H}$.

Exercise 2.28. Suppose that C is a genus 1 Riemann surface curve. Suppose that $\phi : C \to C$ is an automorphism of C. Show that

(i) ϕ is a translation if and only if $\phi_* : H_1(C, \mathbb{Z}) \to H_1(C, \mathbb{Z})$ is the identity;
(ii) if α, β is a positive basis of $H_1(C, \mathbb{Z})$, then

$$[C : \alpha, \beta] = [C : \phi_*\alpha, \phi_*\beta] \in \mathcal{X}_1;$$

Deduce that there is a natural isomorphism

$$\mathrm{Aut}(C; x_o) \cong \text{ stabilizer in } SL_2(\mathbb{Z}) \text{ of } [C : \alpha, \beta] \in \mathcal{X}_1.$$

Note that any two points of $\mathbb{H}$ that lie in the same orbit of $SL_2(\mathbb{Z})$ have isomorphic stabilizers. Consequently, to find all elliptic curves with automorphism groups larger than $\mathbb{Z}/2\mathbb{Z}$, one only has to look for points in the fundamental domain with stabilizers larger than $\mathbb{Z}/2\mathbb{Z}$.

Exercise 2.29. Show that the stabilizer in $SL_2(\mathbb{Z})$ of $\tau \in \mathbb{H}$ is

$$\begin{cases} \mathbb{Z}/2\mathbb{Z} & \tau \notin \text{ orbit of } i \text{ and } \rho; \\ \mathbb{Z}/4\mathbb{Z} & \tau \in \text{ orbit of } i; \\ \mathbb{Z}/\ \mathbb{Z} & \tau \in \text{ orbit of } \rho. \end{cases}$$

An immediate consequence of this computation is the following:

Theorem 2.30. *If $(C; x_o)$ is an elliptic curve, then*

$$\mathrm{Aut}(C, x_o) \cong \begin{cases} \mathbb{Z}/4\mathbb{Z} & \textit{if } (C; x_o) \cong (\mathbb{C}/\mathbb{Z}[i]; 0) \\ \mathbb{Z}/\ \mathbb{Z} & \textit{if } (C; x_o) \cong (\mathbb{C}/\mathbb{Z}[\rho]; 0) \\ \mathbb{Z}/2\mathbb{Z} & \textit{otherwise.} \end{cases}$$

It is easy to see from this, for example, that the Fermat cubic

$$x^3 + y^3 + z^3 = 0$$

in $\mathbb{P}^2$ has automorphism group $\mathbb{Z}/\ \mathbb{Z}$ and therefore is isomorphic to $\mathbb{Z}/\mathbb{Z}[\rho]$.

2.3. **Families of Genus 1 and Elliptic Curves.** Suppose that X is a complex analytic manifold and that $f : X \to T$ is a holomorphic mapping to another complex manifold each of whose fibers is a genus 1 curve. We say that f is a family of genus 1 curves. If there is a section $s : T \to X$ of f, then the fiber of f over $t \in T$ is an elliptic curve with identity $s(t)$. We say that f is a family of elliptic curves.

Each such family gives rise to a function

$$\phi_f : T \to \mathcal{M}_1$$

that is defined by taking $t \in T$ to the moduli point $[X_t]$ of the fiber X_t of f over t. We shall call it the *period mapping* of the family.

Theorem 2.31. *The period map ϕ_f is holomorphic.*

Sketch of Proof. For simplicity, we suppose that T has complex dimension 1. The relative holomorphic tangent bundle of f is the holomorphic line bundle T'_f on X consisting of holomorphic tangent vectors to X that are tangent to the fibers of f. That is,

$$T'_f = \ker\{T'X \to T'T\}.$$

The sheaf of holomorphic sections of its dual is called the relative dualizing sheaf and is often denoted by $\omega_{X/T}$. The push forward $f_*\omega_{X/T}$ of this sheaf to T is a holomorphic line bundle over T and has fiber

$$H^0(X_t, \Omega^1_{X_t})$$

over $t \in T$. Fix a reference point $t_o \in T$. Let $w(t)$ be a local holomorphic section of $\omega_{X/T}$ defined in a contractible neighbourhood U of t_o. Since bundle is locally topologically trivial, $f^{-1}(U)$ is homeomorphic to $U \times X_{t_o}$. We can thus identify $H_1(X_t, \mathbb{Z})$ with $H_1(X_{t_o}, \mathbb{Z})$ for each $t \in U$. Fix a positive basis α, β of $H_1(X_{t_o}, \mathbb{Z})$. This can be viewed as a positive basis of $H_1(X_t, \mathbb{Z})$ for all $t \in U$.

It is not difficult to show that

$$\int_\alpha w(t) \text{ and } \int_\beta w(t)$$

vary holomorphically with $t \in U$. It follows that

$$\tau(t) := \left\{ \int_\beta w(t) \Big/ \int_\alpha w(t) \right\}$$

varies holomorphically with $t \in U$. This shows that map ϕ_f is holomorphic in the neighbourhood U of t_o. It follows that ϕ_f is holomorphic. $\square$

If you examine the proof, you will see that we really proved two extra facts. First, the period mapping $\phi_f : T \to \mathcal{M}_1$ associated to every family $f : X \to T$ of genus 1 curves is *locally liftable* to a holomorphic mapping to $\mathbb{H}$. If, in addition, f is a family of elliptic curves, then the period mapping ϕ_f determines the family f and the section s:

Proposition 2.32. *The period mapping ϕ_f associated to a family $f : X \to T$ of genus 1 curves is locally liftable to a holomorphic mapping to $\mathbb{H}$. If f is a family of elliptic curves, then ϕ_f can be globally lifted to a holomorphic mapping $\tilde{\phi}_f : \tilde{T} \to \mathbb{H}$ and there is a homomorphism $\phi_{f*} : \pi_1(T, t_o) \to SL_2(\mathbb{Z})$ (unique up to conjugacy) such that the diagram*

$$\begin{array}{ccc} \tilde{T} & \xrightarrow{\tilde{\phi}_f} & \mathbb{H} \\ \downarrow & & \downarrow \\ T & \xrightarrow[\phi_f]{} & \mathcal{M}_1 \end{array}$$

commutes and such that

$$\tilde{\phi}_f(\gamma \cdot x) = \phi_{f*}(\gamma) \cdot \tilde{\phi}_f(x)$$

for all $x \in \tilde{T}$ *and all* $\gamma \in \pi_1(T, t_o)$. $\square$

The proof is left as an exercise. We shall see in a moment that the converse of this result is also true.

This result has an important consequence — and that is that not every holomorphic mapping $T \to \mathcal{M}_1$ is the period mapping of a holomorphic family of elliptic (or even genus 1) curves. The reason for this is that not every mapping $T \to \mathcal{M}_1$ is locally liftable.

Exercise 2.33. Show that the identity mapping $\mathcal{M}_1 \to \mathcal{M}_1$ is not locally liftable. Deduce that there is no family of genus 1 curves over $\mathcal{M}_1$ whose period mapping is the identity mapping. In particular, show that there is no universal elliptic curve over $\mathcal{M}_1$.

To each genus 1 curve C, we can canonically associate the elliptic curve $\operatorname{Jac} C := \operatorname{Pic}^0 C$ which we shall call the *jacobian* of C. Abel's Theorem tells us that C and $\operatorname{Jac} C$ are isomorphic as genus 1 curves, but the isomorphism depends on the choice of a base point of C.

Exercise 2.34. Show that for each family $f : X \to T$ of genus 1 curves the corresponding family of jacobians is a family of elliptic curves. Show that these two families have the same period mapping. Show that if f is a family of elliptic curves, then the family of jacobians is canonically isomorphic to the original family f.

3. Orbifolds

The discussion in the previous section suggests that $\mathcal{M}_1$ should be viewed as $SL_2(\mathbb{Z})\backslash\mathbb{H}$ rather than as $\mathbb{C}$ — or that $\mathcal{M}_1$ is not an algebraic variety or a manifold, but rather something whose local structure includes the information of how it is locally the quotient of a disk by a finite group. In topology such objects are called *orbifolds* and in algebraic geometry *stacks*. Very roughly speaking, orbifolds are to manifolds as stacks are to varieties. The moduli spaces $\mathcal{M}_{g,n}$ are often conveniently viewed as orbifolds or as stacks.

For us, an orbifold is a topological space that is the quotient of a simply connected topological space X by a group Γ. The group is required to act properly discontinuously on X and all isotropy groups are required to be finite.[1] For example, $\mathcal{M}_1$ can be viewed as an orbifold as the quotient of $\mathbb{H}$ by $SL_2(\mathbb{Z})$.

[1]We could have added some other natural conditions, such as requiring that there be a finite index subgroup of Γ that acts on X fixed point freely. We could also require that

This definition is not as general as it could be, but since all of our moduli spaces $\mathcal{M}_{g,n}$ are of this form, it is good enough for our purposes. The general definition is obtained by "sheafifying" this one — i.e., general orbifolds are locally the quotient of a simply connected space by a finite group. A general definition along these lines can be found in Chapter 13 of [25]. Mumford's definition of stacks can be found in [23].

A morphism $f : \Gamma_1 \backslash X_1 \to \Gamma_2 \backslash X_2$ of orbifolds is a continuous map that arises as follows: there is a homomorphism $f_* : \Gamma_1 \to \Gamma_2$ and a continuous mapping $\tilde{f} : X_1 \to X_2$ that is equivariant with respect to f_*; that is, the diagram

$$\begin{array}{ccc} X_1 & \xrightarrow{\tilde{f}} & X_2 \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f_*(g)} \\ X_1 & \xrightarrow{\tilde{f}} & X_2 \end{array}$$

commutes for all $g \in \Gamma_1$.

At this stage, I should point out that every reasonable topological space X can be viewed as the quotient $\pi_1(X, x)\backslash\tilde{X}$ of its universal covering by its fundamental group. In this way, every topological space can be regarded as an orbifold. It thus it makes sense to talk about orbifold mappings between orbifolds and ordinary topological spaces.

An orbifold $\Gamma\backslash X$ can have an enriched structure — such as a smooth, Riemannian, Kähler, or algebraic structure. One just insists that its "orbifold universal covering" X has such a structure and that Γ act on X as automorphisms of this structure. Maps between two orbifolds with the same kind of enriched structure are defined in the obvious way. For example, a map between two orbifolds with complex structures is given by an equivariant holomorphic map between their orbifold universal coverings.

Many orbifolds are given as quotients of non-simply connected spaces by a group that acts discontinuously, but not fixed point freely. Such quotients have canonical orbifold structures: If M is a topological space and G a group that acts discontinuously on M, then

$$G\backslash M \cong \Gamma\backslash\widetilde{M}$$

---

the set of elements of G that act trivially on X is central in Γ. Both of these conditions are natural and are satisfied by all of our primary examples of orbifolds, the $\mathcal{M}_{g,n}$.

where $p : \widetilde{M} \to M$ is the universal covering[2] of M and

$$\Gamma = \{(\phi, g) : \text{ where } \phi : \widetilde{M} \to \widetilde{M} \text{ covers } g \in G\}.$$

Here ϕ covers $g \in G$ means that the diagram

$$\begin{tikzcd} \widetilde{M} \arrow[r, "\phi"] \arrow[d, "p"'] & \widetilde{M} \arrow[d, "p"] \\ M \arrow[r, "g"'] & M \end{tikzcd}$$

commutes.

There is a notion of the *orbifold fundamental* group $\pi_1^{\mathrm{orb}}(\Gamma\backslash X, x_o)$ of a connected pointed orbifold. It is a variant of the definition of the fundamental group of a pointed topological space.

First, we'll fix the convention that the composition $\alpha\beta$ of two paths α and β in a space is defined when $\alpha(1) = \beta(0)$. The product path is the one obtained by traversing α first, then β. This is the convention used in [11], for example, and is the opposite of the convention used by many algebraic geometers such as Deligne.

We need to impose several conditions on X and on x_o in order for the definition to make sense. Let $p : X \to \Gamma\backslash X$ be the projection. Note that if $x, y \in p^{-1}(x_o)$, then the isotropy groups Γ_y and Γ_z are conjugate in Γ. Our first assumption is that Γ_y is contained in the center of Γ for one (and hence all) $y \in p^{-1}(x_o)$. For such x_o, we shall denote the common isotropy group of the points lying over x_o by Γ_{x_o}. Our second assumption is that X is connected, locally path connected, and locally simply connected. These conditions are satisfied by all connected complex algebraic varieties. Finally, we shall assume that if $g \in \Gamma$ acts trivially in the neighbourhood of some point of X, it acts trivially on all of X. All three of these conditions are natural and will be satisfied in cases of interest to us.

Let

$$P(x_o) = \{\text{homotopy classes of paths } ([0,1], \{0,1\}) \to (X, p^{-1}(x_o))\}.$$

Since Γ acts on the left of X, this is a left Γ-set. Let

$$Q(x_o) = \{(g, \gamma) \in \Gamma \times P(x_o) : g^{-1} \cdot \gamma(0) = \gamma(1)\}.$$

[2]Of course, I am assuming that M is nice enough as a topological space to have a universal covering. We assume, for example, that M is locally simply connected, a condition satisfied by all complex algebraic varieties and all manifolds.

This has a natural left Γ-action given by

$$g : (h, \gamma) \mapsto (ghg^{-1}, g \cdot \gamma).$$

Define $\pi_1^{\mathrm{orb}}(\Gamma \backslash X, x_o)$ to be the quotient $\Gamma \backslash Q(x_o)$. This has a natural group structure which can be understood by noting that $Q(x_o)$ is a groupoid. The composition of two elements (g, γ) and (h, μ) is defined when $\gamma(1) = \mu(0)$. It is then given by

$$(g, \gamma) \cdot (h, \mu) = (gh, \gamma\mu).$$

To multiply two elements of $\pi_1^{\mathrm{orb}}(G \backslash X, x_o)$, translate one of them until they are composable. For example, if we fix a point $\tilde{x}_o$ of $p^{-1}(x_o)$, then each element of $\pi_1^{\mathrm{orb}}(G \backslash X, x_o)$ has a representative of the form (g, γ) where $\gamma(0) = \tilde{x}_o$. This representation is unique up to translation by an element of Γ_{x_o}. To multiply two elements (g, γ) and (h, μ) starting at $\tilde{x}_o$, multiply (h, μ) by g so that it can be composed with (g, γ). Then the product of these two elements in $\pi_1^{\mathrm{orb}}(G \backslash X, x_o)$ is represented by the path

$$(g, \gamma) \cdot (g^{-1}hg, g^{-1} \cdot \mu) = (hg, \gamma\,(g^{-1} \cdot \mu)).$$

Exercise 3.1. Show that if $k \in \Gamma_{x_o}$, then k acts trivially on $P(x_o)$. Deduce that the multiplication above is well defined. (This is where we need Γ_{x_o} to be central in Γ.)

The following exercises should help give some understanding of orbifold fundamental groups.

Exercise 3.2. Show that if Γ acts fixed point freely on X, then there is a natural isomorphism between $\pi_1^{\mathrm{orb}}(\Gamma \backslash X, x_o)$ and the usual fundamental group $\pi_1(\Gamma \backslash X, x_o)$. In particular, if we view a topological space X as an orbifold, then the two notions of fundamental group agree.

Exercise 3.3. Show that if Γ is an abelian group that acts trivially on a one point space $X = \{*\}$, then $\pi_1^{\mathrm{orb}}(\Gamma \backslash X, x_o)$ is defined and there is a natural isomorphism

$$\pi_1^{\mathrm{orb}}(\Gamma \backslash X, x_o) \cong \Gamma.$$

Exercise 3.4. Let $X = \mathbb{C}$ and $\Gamma = \mathbb{Z}/n\mathbb{Z}$. Define an action Γ on X by letting the generator 1 of $\mathbb{Z}/n\mathbb{Z}$ act by multiplication by $e^{2\pi i/n}$. Show that $\pi_1^{\mathrm{orb}}(\Gamma \backslash X, x_o)$ is defined for all $x_o \in \Gamma \backslash \mathbb{C}$, and for each x_o there is a natural isomorphism

$$\pi_1^{\mathrm{orb}}(\Gamma \backslash X, x_o) \cong \mathbb{Z}/n\mathbb{Z}.$$

Exercise 3.5. Show that for each choice of a point $x \in p^{-1}(x_o)$, there is a natural isomorphism

$$\phi_x : \pi_1^{\mathrm{orb}}(\Gamma\backslash X, x_o) \to \Gamma$$

defined by $\phi_x(g,\gamma) = g^{-1}$ when $\gamma(0) = x$. Show that if $y = hx$, then $\phi_y = h\phi_x h^{-1}$.

Remark 3. . One can also define the orbifold fundamental group of $\Gamma\backslash X$ using the Borel construction. First, find a contractible space $E\Gamma$ on which Γ acts discontinuously and fixed point freely (any one will do). There is a canonical construction of such spaces. (See for example [3, p. 19].) Fix a base point $e_o \in E\Gamma$. The diagonal action of Γ on $E\Gamma \times X$, a simply connected space, is fixed point free and the map

$$q : E\Gamma \times X \to \Gamma\backslash(E\Gamma \times X)$$

is a covering mapping with Galois group Γ. For $x \in X$, define

$$\pi_1^{\mathrm{orb}}(\Gamma\backslash X, q(e_o, x)) = \pi_1(\Gamma(E\Gamma \times X), q(e_o, x)).$$

If Γ_x is central in Γ, then this depends only on $x_o = p(x)$. It is not difficult to show that, in this case, this definition agrees with the more elementary one given above.

The moduli space $\mathcal{M}_1$ is naturally an orbifold, being the quotient of $\mathbb{H}$ by $SL_2(\mathbb{Z})$. The condition on x_o is satisfied for all points not in the orbit of i or ρ. Consequently, $\pi_1^{\mathrm{orb}}(\mathcal{M}_1, x_o)$ is defined for all x_o other than those corresponding to the orbits of i and ρ, and for each such x_o, there is an isomorphism

$$\pi_1^{\mathrm{orb}}(\mathcal{M}_1, x_o) \cong SL_2(\mathbb{Z})$$

which is well defined up to conjugacy.

This gives the following restatement of Proposition 2.32.

Proposition 3.7. *If $f : X \to T$ is a holomorphic family of genus 1 curves, then the period mapping $\phi_f : T \to \mathcal{M}_1$ is a morphism of orbifolds.*

Remark 3.8. Since $\mathcal{M}_1$ can also be written as $PSL_2(\mathbb{Z})\backslash\mathbb{H}$, it is natural to ask why we are not giving $\mathcal{M}_1$ this orbifold structure. This question will be fully answered in the section on the universal elliptic curve and in subsequent sections on curves of higher genus. For the time being, just note that if we give $\mathcal{M}_1$ the orbifold structure $PSL_2(\mathbb{Z})\backslash\mathbb{H}$, then $\pi^{\mathrm{orb}}(\mathcal{M}_1)$ would be $PSL_2(\mathbb{Z})$ instead of $SL_2(\mathbb{Z})$.

3.1. **The Universal Elliptic Curve.** Let's attempt to construct a "universal elliptic curve" over $\mathcal{M}_1$. We begin by constructing one over $\mathbb{H}$. The group $\mathbb{Z}^2$ acts on $\mathbb{C} \times \mathbb{H}$ by

$$(n, m) : (z, \tau) \mapsto (z + n\tau + m, \tau).$$

This action is fixed point free, so the quotient $\mathbb{Z}^2 \backslash (\mathbb{C} \times \mathbb{H})$ is a complex manifold. The fiber of the projection

$$\mathbb{Z}^2 \backslash (\mathbb{C} \times \mathbb{H}) \to \mathbb{H}$$

over τ is simply the elliptic curve $\mathbb{C}/\mathbb{Z} \oplus \mathbb{Z}\tau$. The family has the section that takes $\tau \in \mathbb{H}$ to the identity element of the fiber lying over it. So this really is a family of elliptic curves.

Let's see what happens if we try to quotient out by $\mathbb{Z}^2$ and $SL_2(\mathbb{Z})$ at the same time. First note that $SL_2(\mathbb{Z})$ acts on $\mathbb{Z}^2$ on the right by matrix multiplication. We can thus form the semi-direct product $SL_2(\mathbb{Z}) \ltimes \mathbb{Z}^2$. This is the group whose underlying set is $SL_2(\mathbb{Z}) \times \mathbb{Z}^2$ and whose multiplication is given by

$$(A_1, (n_1, m_1))(A_1, (n_2, m_2)) = (A_1 A_2, (n_1, m_1)A_2 + (n_2, m_2)).$$

Exercise 3.9. Show that the action of $SL_2(\mathbb{Z}) \ltimes \mathbb{Z}^2$ on $\mathbb{C} \times \mathbb{H}$ given by

$$\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, (n, m) \right) : (z, \tau) \mapsto \left(\frac{z + n\tau + m}{c\tau + d}, \frac{a\tau + b}{c\tau + b} \right)$$

is well defined. Show that there is a well defined projection

$$(SL_2(\mathbb{Z}) \ltimes \mathbb{Z}^2) \backslash (\mathbb{C} \times \mathbb{H}) \to SL_2(\mathbb{Z}) \backslash \mathbb{H}.$$

This is a reasonable candidate for the universal curve. But we should be careful.

Exercise 3.10. Show that the fiber of the natural projection

$$(SL_2(\mathbb{Z}) \ltimes \mathbb{Z}^2) \backslash (\mathbb{C} \times \mathbb{H}) \to SL_2(\mathbb{Z}) \backslash \mathbb{H} \tag{3}$$

over the point corresponding to the elliptic curve $(C; 0)$ is the quotient of C by the finite group $\mathrm{Aut}(C; 0)$. In particular, the fiber of the projection over $[(C; 0)]$ is always a quotient of $C/\pm \cong \mathbb{P}^1$, and is never C.

However, it is more natural to regard $\mathcal{E} := (SL_2(\mathbb{Z}) \ltimes \mathbb{Z}^2) \ltimes \mathbb{H}$ as an orbifold. We shall do this. First note that the projection (3) has an orbifold section that is induced by the mapping

$$\mathbb{H} \to \mathbb{C} \times \mathbb{H}$$

that takes τ to $(0,\tau)$. Note that if we consider $\mathcal{M}_1$ as the orbifold $PSL_2(\mathbb{Z})\backslash\mathbb{H}$, then the section would not exist. It is for this reason that we give $\mathcal{M}_1$ the orbifold structure with orbifold fundamental group $SL_2(\mathbb{Z})$ instead of $PSL_2(\mathbb{Z})$.

The following should illustrate why it is more natural to view $\mathcal{M}_1$ as an orbifold than as a variety.

Theorem 3.11. *There is a natural one-to-one correspondence between holomorphic orbifold mappings from a smooth complex curve (or variety) T to $\mathcal{M}_1$ and families of elliptic curves over T. The correspondence is given by pullback.* □

We could define a 'geometric points' of an orbifold X to be an orbifold map from a point with trivial fundamental group to X. For example, suppose τ is any point of $\mathbb{H}$. Denote its isotropy group in $SL_2(\mathbb{Z})$ by Γ_τ. Then $\Gamma_\tau\backslash\{\tau\}$ is a point of $SL_2(\mathbb{Z})\backslash\mathbb{H}$, but is not a geometric point as it has fundamental group Γ_τ, which is always non-trivial as it contains $\mathbb{Z}/2\mathbb{Z}$. The corresponding geometric point corresponds to the 'universal covering' $\{\tau\}\to\Gamma_\tau\backslash\{\tau\}$ of this point. One can define pullbacks of orbifolds in such a way that the fiber of the pullback of the universal curve to the geometric point τ of $\mathcal{M}_1$ is the corresponding elliptic curve $\mathbb{C}/(\mathbb{Z}\oplus\mathbb{Z}\tau)$.

Denote the n-fold fibered product of $\mathcal{E}\to\mathcal{M}_1$ with itself by $\mathcal{E}^{(n)}\to\mathcal{M}_1$. It is naturally an orbifold (exercise). This has divisors Δ_{jk} that consist of the points of $\mathcal{E}^{(n)}$ where the jth and kth points agree. It also has divisors Δ_j where the jth point is zero.

Exercise 3.12. Show that the moduli space $\mathcal{M}_{1,n}$ can be identified with

$$\mathcal{E}^{(n-1)} - \Big(\bigcup_{j<k}\Delta_{jk}\cup\bigcup_j\Delta_j\Big).$$

3.2. **Modular Forms.** There is a natural orbifold line bundle $\mathcal{L}$ over $\mathcal{M}_1$. It is the quotient of $\mathbb{H}\times\mathbb{C}$ by $SL_2(\mathbb{Z})$ where $SL_2(\mathbb{Z})$ acts via the formula

$$\begin{pmatrix} a & b\\ c & d\end{pmatrix} : (\tau,z)\to\left(\frac{a\tau+b}{c\tau+d},(c\tau+d)z\right).$$

Exercise 3.13. Show that this is indeed an action. Show that the kth power $\mathcal{L}^{\otimes k}$ of this line bundle is the quotient of $\mathbb{H}\times\mathbb{C}$ by the action

$$\begin{pmatrix} a & b\\ c & d\end{pmatrix} : (\tau,z)\to\left(\frac{a\tau+b}{c\tau+d},(c\tau+d)^k z\right).$$

Orbifold sections of $\mathcal{L}^{\otimes k}$ correspond to holomorphic functions $f : \mathbb{H} \to \mathbb{C}$ for which the mapping

$$\mathbb{H} \to \mathbb{H} \times \mathbb{C} \qquad \tau \mapsto (\tau, f(\tau))$$

is $SL_2(\mathbb{Z})$-equivariant.

Exercise 3.14. Show that $f : \mathbb{H} \to \mathbb{C}$ corresponds to a section of $\mathcal{L}^{\otimes k}$ if and only if

$$f\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \tau\right) = (c\tau + d)^k f(\tau).$$

Such a function is called a *modular form of weight* k for $SL_2(\mathbb{Z})$.

Exercise 3.15. Show that $f : \mathbb{H} \to \mathbb{C}$ is a modular form of weight $2k$ if and only if the k-differential (i.e., section of the kth power of the canonical bundle)

$$f(\tau) d\tau^{\otimes k}$$

is invariant under $SL_2(\mathbb{Z})$. Deduce that the (orbifold) canonical bundle of $\mathcal{M}_1$ is isomorphic to $\mathcal{L}^{\otimes 2}$.

Some basic properties and applications of modular forms are given in Chapter VII of [24].

Just as one can define the Picard group of a complex analytic variety to be the group of isomorphism classes of holomorphic line bundles over it, one can define the Picard group $\mathrm{Pic}_{\mathrm{orb}} X$ of a holomorphic orbifold X as the group of isomorphism classes of orbifold line bundles over X. The details are left as a straightforward exercise. The following result will be proved later in the lectures.

Theorem 3.16. *The Picard group of the orbifold $\mathcal{M}_1$ is cyclic of order* 12 *and generated by the class of $\mathcal{L}$.*

It is easy to see that the class of $\mathcal{L}$ in $\mathrm{Pic}_{\mathrm{orb}} \mathcal{M}_1$ as $\mathcal{L}^{\otimes 12}$ is trivialized by the cusp form

$$\Delta(q) = (2\pi)^{12} q \prod_{n=1}^{\infty} (1 - q^n)^{24}$$

of weight 12 which has no zeros or poles in $\mathbb{H}$. Also, the non-existence of meromorphic modular forms of weight k with $0 < k < 12$ and no zeros or poles in $\mathbb{H}$ implies that $\mathcal{L}$ is an element of order 12 in $\mathrm{Pic}_{\mathrm{orb}} \mathcal{M}_1$. We will show later that the order of $\mathrm{Pic}_{\mathrm{orb}} \mathcal{M}_1$ is of order 12, from which it will follow that $\mathrm{Pic}_{\mathrm{orb}} \mathcal{M}_1$ is isomorphic to $\mathbb{Z}/12$ and generated by the class of $\mathcal{L}$.

Lecture 2: Teichmüller Theory

We have seen in genus 1 case that $\mathcal{M}_1$ is the quotient

$$\Gamma_1 \backslash \mathcal{X}_1$$

of a contractible complex manifold $\mathcal{X}_1 = \mathbb{H}$ by a discrete group $\Gamma_1 = SL_2(\mathbb{Z})$. The action of Γ_1 on $\mathcal{X}_1$ is said to be *virtually free* — that is, Γ_1 has a finite index subgroup which acts (fixed point) freely on $\mathcal{X}_1$.[3] In this section we will generalize this to all $g \geq 1$ — we will sketch a proof that there is a contractible complex manifold $\mathcal{X}_g$, called *Teichmüller space*, and a group Γ_g, called the *mapping class group*, which acts virtually freely on $\mathcal{X}_g$. The moduli space of genus g compact Riemann surfaces is the quotient:

$$\mathcal{M}_g = \Gamma_g \backslash \mathcal{X}_g.$$

This will imply that $\mathcal{M}_g$ has the structure of a complex analytic variety with finite quotient singularities.

Teichmüller theory is a difficult and technical subject. Because of this, it is only possible to give an overview.

4. The Uniformization Theorem

Our basic tool is the the generalization of the Riemann Mapping Theorem known as the Uniformization Theorem. You can find a proof, for example, in [9] and [8].

Theorem 4.1. *Every simply connected Riemann surface*[4] *is biholomorphic to either* $\mathbb{P}^1$, $\mathbb{C}$ *or* $\mathbb{H}$.

Exercise 4.2. Show that no two of $\mathbb{P}^1$, $\mathbb{H}$ and $\mathbb{C}$ are isomorphic as Riemann surfaces.

We have already seen that $\operatorname{Aut}\mathbb{P}^1 \cong PSL_2(\mathbb{C})$ and that $PSL_2(\mathbb{R}) \subseteq \operatorname{Aut}\mathbb{H}$.

Exercise 4.3. (i) Show that

$$\operatorname{Aut}\mathbb{C} = \{z \mapsto az + b : a \in \mathbb{C}^* \text{ and } b \in \mathbb{C}\}.$$

(ii) Show that there is an element T of $\operatorname{Aut}\mathbb{P}^1$ that restricts to an isomorphism $T : \Delta \to \mathbb{H}$ between the unit disk and the upper half plane.

[3] More generally, we say that a group G has a property P *virtually* if P holds for some finite index subgroup of G.

[4] We follow the convention that every Riemann surface is, by definition, connected.

(iii) Show that every element of $\operatorname{Aut}\Delta$ is a fractional linear transformation. Hint: Use the Schwartz Lemma.
(iv) Deduce that every element of $\operatorname{Aut}\mathbb{H}$ is a fractional linear transformation and that $\operatorname{Aut}\mathbb{H} \cong PSL_2(\mathbb{R})$.

If X is a Riemann surface and $x \in X$, then $\pi_1(X, x)$ acts on the universal covering $\widetilde{X}$ as a group of biholomorphisms. This action is fixed point free.

Exercise 4.4. Show that if X is a Riemann surface whose universal covering is isomorphic to $\mathbb{H}$ and $x \in X$, then there is a natural injective homomorphism $\rho : \pi_1(X, x) \to PSL_2(\mathbb{R})$ which is injective and has discrete image. Show that this homomorphism is unique up to conjugation by an element of $PSL_2(\mathbb{R})$, and that the conjugacy class of ρ is independent of the choice of x. Show that X is isomorphic to $\operatorname{im}\rho\backslash\mathbb{H}$ and that the conjugacy class of ρ determines X up to isomorphism.

This will give a direct method of putting a topology on $\mathcal{M}_g$ when $g \geq 2$. But first some preliminaries.

Exercise 4.5. Show that if X is a Riemann surface whose universal covering is

(i) $\mathbb{P}^1$, then X is isomorphic to $\mathbb{P}^1$;
(ii) $\mathbb{C}$, then X is isomorphic to $\mathbb{C}$, $\mathbb{C}^*$ or is a genus 1 curve.

Hint: Classify the subgroups of $\operatorname{Aut}\mathbb{P}^1$ and $\operatorname{Aut}\mathbb{C}$ that act properly discontinuously and freely.

So we come to the striking conclusion that the universal covering of a Riemann surface not isomorphic to $\mathbb{P}^1$, $\mathbb{C}$, $\mathbb{C}^*$ or a genus 1 curves must be isomorphic to $\mathbb{H}$. In particular, the universal covering of $\mathbb{C} - \{0, 1\}$ is $\mathbb{H}$. Picard's Little Theorem is a consequence.

Exercise 4.6. Show that if $f : \mathbb{C} \to \mathbb{C}$ is a holomorphic mapping that omits 2 distinct points of $\mathbb{C}$, then f is constant.

For our immediate purposes, the most important fact is that the universal covering of every Riemann surface of genus $g \geq 2$ is isomorphic to $\mathbb{H}$. More generally, we have the following interpretation of the stability condition $2g - 2 + n > 0$.

Exercise 4.7. Show that if U is a Riemann surface of the form $X - F$ where X is a compact Riemann surface genus g and F a finite subset of X of cardinality n, then the universal covering of U is isomorphic to $\mathbb{H}$ if and

only if $2g-2+n>0$. Observe that $2-2g-n$ is the topological Euler characteristic of U.

5. Teichmüller Space

Suppose that $g \geq 2$ and that S is a compact oriented surface of genus $g \geq 2$. Fix a base point $x_o \in S$ and set $\pi = \pi_1(S, x_o)$. Let

$$\mathcal{X}_g = \left\{ \begin{array}{l} \text{conjugacy classes of injective representations } \rho : \\ \pi \hookrightarrow PSL_2(\mathbb{R}) \text{ such that } \operatorname{im}\rho \text{ acts freely on } \mathbb{H} \text{ and} \\ \operatorname{im}\rho\backslash\mathbb{H} \text{ is a compact Riemann surface of genus } g \end{array} \right\}$$

It is standard that we can choose generators $a_1, \ldots, a_g$, $b_1, \ldots, b_g$ of π that are subject to the relation

$$\prod_{j=1}^{g} [a_j, b_j] = 1.$$

where $[x, y]$ denotes the commutator $xyx^{-1}y^{-1}$.

Exercise 5.1. Show that $\mathcal{X}_g$ can also be identified naturally with the set

$$\left\{ \begin{array}{l} \text{compact Riemann surfaces } C \text{ plus a conjugacy class of isomor-} \\ \text{phisms } \pi \cong \pi_1(C) \text{ modulo the obvious isomorphisms} \end{array} \right\}$$

The group $PSL_2(\mathbb{R})$ is a real affine algebraic group. It can be realized as a closed subgroup of $GL_3(\mathbb{R})$ defined by real polynomial equations.

Exercise 5.2. Denote the standard 2-dimensional representation of $SL_2(\mathbb{R})$ by V. Denote the second symmetric power of V by S^2V. Show that this is a 3-dimensional representation of $SL_2(\mathbb{R})$. Show that $-I \in SL_2(\mathbb{R})$ acts trivially on S^2V, so that S^2V is a 3-dimensional representation of $PSL_2(\mathbb{R})$. Show that this representation is faithful and that its image is defined by polynomial equations. Deduce that $PSL_2(\mathbb{R})$ is a real affine algebraic group.

Because of this, we will think of elements of $PSL_2(\mathbb{R})$ as 3×3 matrices whose entries satisfy certain polynomial equations. A representation $\rho : \pi \to PSL_2(\mathbb{R})$ thus corresponds to a collection of matrices

$$A_1, \ldots, A_g, B_1, \ldots, B_g$$

in $PSL_2(\mathbb{R}) \subset GL_3(\mathbb{R})$ that satisfy the polynomial equation

$$I - \prod_{j=1}^{g} [A_j, B_j] = 0; \tag{4}$$

the correspondence is given by setting $A_j = \rho(a_j)$ and $B_j = \rho(b_j)$.

The set of all representations $\rho : \pi \to PSL_2(\mathbb{R})$ is the real algebraic subvariety $\mathcal{R}$ of $PSL_2(\mathbb{R})^{2g}$ consisting of all $2g$-tuples

$$(A_1, \ldots, A_g, B_1, \ldots, B_g)$$

that satisfy (4). This is a closed subvariety of $PSL_2(\mathbb{R})^{2g}$ and is therefore an affine variety. Note that[5]

$$\dim \mathcal{R} \geq 2g \cdot \dim PSL_2(\mathbb{R}) - \dim PSL_2(\mathbb{R}) = \ g - 3.$$

Observe that $PSL_2(\mathbb{R})$ acts on $\mathcal{R}$ on the right by conjugation:

$$A : \rho \mapsto \{g \mapsto A^{-1}\rho(g)A\}.$$

Exercise 5.3. Show that

$$U := \left\{ \begin{array}{l} \rho : \pi \hookrightarrow PSL_2(\mathbb{R}) \text{ such that } \operatorname{im}\rho \text{ acts} \\ \text{freely on } \mathbb{H} \text{ and } \operatorname{im}\rho\backslash\mathbb{H} \text{ is a compact Rie-} \\ \text{mann surface of genus } g \end{array} \right\}$$

is an open subset of $\mathcal{R}$ and that it is closed under the $PSL_2(\mathbb{R})$-action. Hint: One can construct a fundamental domain F for the action of π on $\mathbb{H}$ given by ρ as follows. First choose a point $x \in \mathbb{H}$. Then take F to be all points in $\mathbb{H}$ that are closer to x than to any of the points $\rho(g)x$ (with respect to hyperbolic distance) where $g \in \pi$ is non-trivial. Then F is a compact subset of $\mathbb{H}$ whose boundary is a union of geodesic segments. As ρ varies continuously, the orbit of x varies continuously. Now choose x to be generic enough and study the change in F as ρ varies.

Note that $\mathcal{X}_g = U/PSL_2(\mathbb{R})$.

Exercise 5.4. Show that the center of $PSL_2(\mathbb{R})$ is trivial.

Proposition 5.5. *If $\rho \in U$, then the stabilizer in $PSL_2(\mathbb{R})$ of ρ is trivial.*

Proof. The first step is to show that if $\rho \in U$, then $\operatorname{im}\rho$ is Zariski dense in $PSL_2(\mathbb{R})$. To prove this, it suffices to show that $\operatorname{im}\rho$ is Zariski dense in the set of complex points $PSL_2(\mathbb{C})$. From Lie theory (or algebraic group theory), we know that all proper subgroups of $PSL_2(\mathbb{C})$ are extensions of a finite group by a solvable group. Since $\operatorname{im}\rho$ is isomorphic to π, and since π is not solvable (as it contains a free group of rank g), $\operatorname{im}\rho$ cannot be contained in any proper algebraic subgroup of $PSL_2(\mathbb{R})$, and is therefore Zariski dense.

[5] One has to be more careful here as, in real algebraic geometry, affine varieties of higher codimension can all be cut out by just one equation. Probably the best way to proceed is to compute the derivative of the product of commutators map $PSL_2(\mathbb{R})^{2g} \to PSL_2(\mathbb{R})$ at points in the fiber over the identity.

An element A of $PSL_2(\mathbb{R})$ stabilizes ρ if and only if $A^{-1}\rho(g)A = \rho(g)$ for all $g \in \pi$ — that is, if and only if A centralizes $\operatorname{im}\rho$. But A centralizes $\operatorname{im}\rho$ if and only if it centralizes the Zariski closure of $\operatorname{im}\rho$. Since $\operatorname{im}\rho$ is Zariski dense in $PSL_2(\mathbb{R})$, and since the center of $PSL_2(\mathbb{R})$ is trivial, it follows that A is trivial. □

Theorem 5.6. *The set U is a smooth manifold of real dimension $g-3$. The group $PGL_2(\mathbb{R})$ acts principally on U, so that quotient $\mathcal{X}_g = U/PSL_2(\mathbb{R})$ is a manifold of real dimension $g-$.*

Sketch of Proof. It can be shown, using deformation theory, that the Zariski tangent space of $\mathcal{R}$ at the representation ρ of π is given by the relative cohomology group:[6]

$$T_\rho\mathcal{R} = H^1(S, x_o; \mathbb{A})$$

where $\mathbb{A}$ is the local system (i.e., locally constant sheaf) over S whose fiber over x_o is $\mathfrak{sl}_1(\mathbb{R})$ and whose monodromy representation is the homomorphism

$$\pi \xrightarrow{\rho} PSL_2(\mathbb{R}) \xrightarrow{\mathrm{Ad}} \operatorname{Aut}\mathfrak{sl}_2(\mathbb{R}).$$

Here $\mathfrak{sl}_2(\mathbb{R})$ denotes the Lie algebra of $PSL_2(\mathbb{R})$, which is the set of 2×2 matrices of trace 0, and Ad is the adjoint representation

$$A \mapsto \{X \mapsto AXA^{-1}\}.$$

A result equivalent to this was first proved by André Weil [2]. The long exact sequence of the pair (S, x_o) with coefficients in $\mathbb{A}$ gives a short exact sequence

$$0 \to \mathfrak{sl}_2(\mathbb{R}) \to H^1(S, x_o; \mathbb{A}) \to H^1(S; \mathbb{A}) \to 0$$

and an isomorphism $H^2(S, x_o; \mathbb{A}) \cong H^2(S; \mathbb{A})$.

[6] This is a non-standard way of giving this group — Weil used cocycles. Here $A^\bullet(S;\mathbb{A})$ is the complex of differential forms on S with coefficients in the local system $\mathbb{A}$. It is a differential graded Lie algebra. Restricting to the base point x_o one obtains an augmentation $A^\bullet(S;\mathbb{A}) \to \mathfrak{sl}_2(\mathbb{R})$. The complex $A^\bullet(S, x_o;\mathbb{A})$ is the kernel of this, and $H^\bullet(S, x_o;\mathbb{A})$ is defined to be the cohomology of this complex. Briefly, this is related to deformations of ρ as follows: when the representation ρ is deformed, the vector bundle underlying the local system corresponding to ρ does not change, only the connection on it. But a new connection on the bundle corresponding to ρ will differ from the original connection by an element of $A^1(S;\mathbb{A})$. So a deformation of ρ corresponds to a family of sections $w(t)$ of $A^1(S;\mathbb{A})$ where $w(0) = 0$. For each t the new connection is flat. This will imply that the t derivative of $w(t)$ at $t = 0$ is closed in $A^\bullet(S,\mathbb{A})$. Changes of gauge that do not alter the marking of the fiber over x_o are, to first order, elements of $A^0(S, x_o;\mathbb{A})$. The t derivative of such first order changes of gauge gives the trivial first order deformations of ρ.

Deformation theory also tells us that if $H^2(S, x_o; \mathbb{A})$ vanishes, then $\mathcal{R}$ is smooth at $[\rho] \in \mathcal{R}$.[7] The Killing form is the symmetric bilinear form on $\mathfrak{sl}_2(\mathbb{R})$ given by

$$b(X, Y) = \operatorname{tr}(XY).$$

It is non-degenerate and invariant under $\operatorname{Ad} \circ \rho$. It follows that $\mathbb{A}$ is isomorphic to its dual $\mathbb{A}^*$ as a local system. Thus it follows from Poinaré duality (with twisted coefficients) that

$$H^2(S, \mathbb{A}) \cong H^0(S, \mathbb{A})^*.$$

But if ρ is an element of U, it follows from Proposition 5.5 that $H^0(S, \mathbb{A})$ vanishes. Consequently, if $\rho \in U$, then

$$\dim H^1(S, \mathbb{A}) = -\chi(S, \mathbb{A}) = -(\operatorname{rank} \mathbb{A}) \cdot \chi(S) = 3(2g-2) = \ g -$$

and

$$\dim H^1(S, x_o; \mathbb{A}) = \ g - 3.$$

We saw above that U has dimension $\geq \ g - 3$, but $\dim U$ is not bigger than the dimension of any of its Zariski tangent spaces, which we have just shown is $\ g - 3$. It follows that $\dim U = \ g - 3$ and that it is smooth as its dimension equals the rank of its Zariski tangent space. Finally, since the $PSL_2(\mathbb{R})$-action on U is principal, $\mathcal{X}_g = U/PSL_2(\mathbb{R})$ is smooth of dimension $(\ g - 3) - \dim PSL_2(\mathbb{R}) = \ g - \ .$ □

It is not very difficult to show that the tangent space of $\mathcal{X}_g$ at $[\rho]$ is naturally isomorphic to $H^1(S, \mathbb{A})$.

. Mapping Class Groups

The mapping class group Γ_g plays a role for higher genus ($g \geq 2$) Riemann surfaces analogous to that played by $SL_2(\mathbb{Z})$ in the theory of genus 1 curves. As in the previous section, S is a fixed compact oriented surface of genus g, $x_o \in S$, and $\pi = \pi_1(S, x_o)$. But unlike the last section, we allow any $g \geq 1$.

The mapping class group Γ_S of S is defined to be the group of connected components of the group of orientation preserving diffeomorphisms of S:

$$\Gamma_S = \pi_0 \operatorname{Diff}^+ S.$$

[7] A proof of this and also a proof of Weil's result can be found in the beautiful paper of Goldman and Millson [10].

Here Diff S is given the compact open topology. It is the finest topology on Diff S such that a function $f : K \to \operatorname{Diff} S$ from a compact space into Diff S is continuous if and only if the map $\phi_f : K \times S \to S$ defined by

$$\phi_f(k, x) = f(k)(x)$$

is continuous.

Exercise 6.1. Suppose that M is a smooth manifold. Show that a path in Diff M from ϕ to ψ is a homotopy $\Psi : [0,1] \times M \to M$ from ϕ to ψ such that for each $t \in [0,1]$, the function $\Psi(t, \ ) : M \to M$ that takes x to $\Psi(t, x)$ is a diffeomorphism. Such a homotopy is called an *isotopy* between ϕ and ψ.

Recall that the outer automorphism group of a group G is defined to be the quotient

$$\operatorname{Out} G = \operatorname{Aut} G / \operatorname{Inn} G$$

of the automorphism group $\operatorname{Aut} G$ of G by the subgroup $\operatorname{Inn} G$ of inner automorphisms.

Exercise 6.2. Suppose that M is a smooth manifold. Show that each diffeomorphism of M induces an outer automorphism of its fundamental group and that the corresponding function

$$\operatorname{Diff} M \to \operatorname{Out} \pi(M)$$

is a homomorphism. (Hint: Fix a base point of M and show that every diffeomorphism of M is isotopic to one that fixes the base point.) Show that the kernel of this homomorphism contains the subgroup of all diffeomorphisms isotopic to the identity. Deduce that there is a homomorphism

$$\pi_0(\operatorname{Diff} M) \to \operatorname{Out} \pi_1(M).$$

Taking M to be S, our reference surface, we see that there is a natural homomorphism

$$\Phi : \Gamma_S \to \operatorname{Out} \pi.$$

It is a remarkable fact, due to Dehn [ ] and Nielsen [22], that this map is almost an isomorphism. (I believe there is a proof of this in the translation of some papers of Dehn on topology by Stillwell.)

Theorem 6.3 (Dehn-Nielsen). *For all $g \geq 1$, the homomorphism Φ is injective and the image of Φ is a subgroup of index 2.* □

One can describe the index 2 subgroup of $\operatorname{Out}\pi$ as follows. Since S has genus ≥ 1, the universal covering of S is contractible, and there is a natural isomorphism $H_i(S,\mathbb{Z}) \cong H_i(\pi,\mathbb{Z})$. Each element of $\operatorname{Out}\pi$ induces an automorphism of $H_i(\pi,\mathbb{Z})$ as inner automorphisms of a group act trivially on its cohomology. The image of Φ is the kernel of the natural homomorphism

$$\operatorname{Out}\pi \to \operatorname{Aut} H_2(\pi,\mathbb{Z}) \cong \{\pm 1\}.$$

When $g = 1$, $\pi \cong \mathbb{Z}^2$, and

$$\operatorname{Out}\pi = \operatorname{Aut}\pi \cong GL_2(\mathbb{Z}).$$

Exercise 6.4. Show that when $g = 1$, the homomorphism $\operatorname{Out}\pi \to \{\pm 1\}$ corresponds to the determinant $GL_2(\mathbb{Z}) \to \{\pm 1\}$.

The genus 1 case of the Dehn-Nielsen Theorem can be proved by elementary means:

Exercise 6.5. Show that if $g = 1$, then the homomorphism $\Gamma_S \to SL_2(\mathbb{Z})$ is an isomorphism. Hint: Construct a homomorphism from $SL_2(\mathbb{Z})$ to $\operatorname{Diff} S$.

Since two compact orientable surfaces are diffeomorphic if and only if they have the same genus, it follows that the group Γ_S depends only on the genus of S. For this reason, we define Γ_g to be Γ_S where S is any compact genus g surface.

7. The Moduli Space

In this section, S is once again a compact oriented surface of genus $g \geq 2$ and π denotes its fundamental group. The mapping class group $\Gamma_g = \Gamma_S$ acts smoothly on Teichmüller space $\mathcal{X}_g$ on the left via the homomorphism $\Gamma_g \to \operatorname{Out}\pi$.

Exercise 7.1. Describe this action explicitly. Show that the isotropy group of $[\rho] \in \mathcal{X}_g$ is naturally isomorphic to the automorphism group of the Riemann surface $\operatorname{im}\rho\backslash\mathbb{H}$. (Compare this with the genus 1 case.) Deduce that all isotropy groups are finite.

Exercise 7.2. Show that two points in $\mathcal{X}_g$ are in the same Γ_g orbit if and only if they determine the same Riemann surface. Hint: Use the Uniformization Theorem.

Rephrasing this, we get:

Theorem 7.3. *If $g \geq 2$, then $\mathcal{M}_g$ is naturally isomorphic to the quotient of $\mathcal{X}_g$ by Γ_g.* $\square$

This result allows us to put a topology on $\mathcal{M}_g$. We give it the unique topology so that $\mathcal{X}_g \to \mathcal{M}_g$ is a quotient mapping. To try to understand the topology of $\mathcal{X}_g$ we shall need the following fundamental result.

Theorem 7.4 (Teichmüller). *For all $g \geq 2$, the Teichmüller space $\mathcal{X}_g$ is contractible and the action of Γ_g on it is properly discontinuous.*

A natural way to approach the proof of this theorem is via hyperbolic geometry. We do this in the next two sections.

8. Hyperbolic Geometry

Perhaps the most direct way to approach the study of Teichmüller space is via hyperbolic geometry. The link between complex analysis and hyperbolic geometry comes from the fact that the upper half plane has a complete metric

$$ds^2 = \frac{1}{y^2}(dx^2 + dy^2)$$

of constant curvature -1 whose group of orientation preserving isometries is $PSL_2(\mathbb{R})$. The Riemannian manifold $(\mathbb{H}, ds^2)$ is called the *Poincaré upper half plane*. A good reference for hyperbolic geometry is the book by Beardon [2].

Exercise 8.1. Show that

(i) every element of $PSL_2(\mathbb{R})$ is an orientation preserving isometry of $(\mathbb{H}, ds^2)$;

(ii) $PSL_2(\mathbb{R})$ acts transitively on $\mathbb{H}$ and that the stabilizer of one point (say i), and therefore all points, of $\mathbb{H}$ is isomorphic to $SO(2)$ and acts transitively on the unit tangent space of $\mathbb{H}$ at this point.

Deduce that the Poincaré metric ds^2 has constant curvature and that $PSL_2(\mathbb{R})$ is the group of all orientation preserving isometries of the Poincaré upper half plane.

The fact that the group of biholomorphisms of $\mathbb{H}$ coincides with the group of orientation preserving isometries of $\mathbb{H}$ is fundamental and is used as follows. Suppose that X is a Riemann surface whose universal covering is $\mathbb{H}$. Then, by Exercise 4.4, X is biholomorphic to the quotient of $\mathbb{H}$ by a discrete subgroup Γ of $PSL_2(\mathbb{R})$ isomorphic to $\pi_1(X)$. Now, since $PSL_2(\mathbb{R})$ is also the group of orientation preserving isometries of $\mathbb{H}$, the Poincaré metric ds^2 is invariant under Γ and therefore descends to $X = \Gamma\backslash\mathbb{H}$. This metric is complete as the Poincaré metric is.

Exercise 8.2. Prove the converse of this: if X is an oriented surface with a complete hyperbolic metric, then X has a natural complex structure whose canonical orientation agrees with the original orientation.

These two constructions are mutually inverse. Thus we conclude that when $g \geq 2$, there is a natural one-to-one correspondence

$$\left\{\begin{matrix}\text{isomorphism classes of compact}\\ \text{Riemann surfaces of genus } g\end{matrix}\right\} \leftrightarrow \left\{\begin{matrix}\text{isometry classes of compact oriented surfaces}\\ \text{of genus } g \text{ with a hyperbolic metric}\end{matrix}\right\} \tag{5}$$

It follows that if $g \geq 2$, then

$$\mathcal{M}_g = \left\{\begin{matrix}\text{isometry classes of compact oriented surfaces}\\ \text{of genus } g \text{ with a hyperbolic metric}\end{matrix}\right\}.$$

Remark 8.3. Likewise, a Riemann surface of genus 1 has a flat metric (unique up to rescaling) which determines the complex structure. So $\mathcal{M}_1$ can be regarded as the moduli space of (conformal classes of) flat tori.

Exercise 8.4. Show that if X is a Riemann surface whose universal covering is $\mathbb{H}$, then the group of orientation preserving isometries of X equals the group of biholomorphisms of X.

9. Fenchel-Nielsen Coordinates

Once again, we assume that $g \geq 2$ and that S is a reference surface of genus g. The interpretation of Teichmüller space in terms of hyperbolic geometry allows us to define coordinates on $\mathcal{X}_g$. To do this we decompose the surface into "pants."

A *pair of pants* is a compact oriented surface of genus 0 with 3 boundary components. Alternatively, a pair of pants is a disk with 2 holes. A *simple*

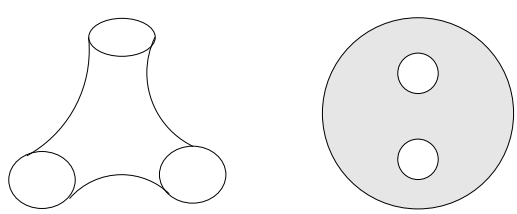

Figure 2. pairs of pants

closed curve (SCC) on S is an imbedded circle. A *pants decomposition* of

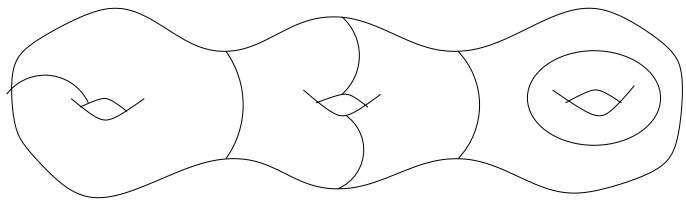

FIGURE 3. a pants decomposition

S is a set of disjoint simple closed curves in S that divides S into pairs of pants.

Proposition 9.1. *If the simple closed curves $C_1, \ldots, C_n$ divide S into m pairs of pants, then $n = 3g - 3$ and $m = 2g - 2$.*

Proof. Since a pair of pants has the homotopy type of a bouquet of 2 circles, it has Euler characteristic -1. Since the Euler characteristic of a disjoint union of circles is 0, we have

$$2 - 2g = \chi(S) = m \cdot \chi(\text{a pair of pants}) = -m.$$

Thus $m = 2g - 2$. Since each pair of pants has 3 boundary components, and since each circle lies on the boundary of two pairs of pants, we see that

$$n = 3m/2 = 3g - 3.$$

□

If T is a compact oriented surface with boundary and $\chi(T) < 0$, then T has a hyperbolic metric such that each boundary component is totally geodesic. As in the case where the boundary of T is empty, the set of all hyperbolic structures on T (modulo diffeomorphisms isotopic to the identity) can be described as a representation variety. I will omit the details. This space is called the Teichmüller space of T and will be denoted by $\mathcal{X}_T$.

Proposition 9.2. *If P is a pair of pants, then $\mathcal{X}_P$ is a manifold diffeomorphic to $\mathbb{R}^3$. The diffeomorphism is given by taking a hyperbolic structure on P to the lengths of its 3 boundary components.*

Discussion of the Proof. A proof can be found in the Exposé 3, partie II by Poénaru in the book [7]. The basic idea is that a pair of pants can be cut into two isomorphic hyperbolic right hexagons, and that two right hyperbolic hexagons are equivalent if the lengths of every other side of one equal the lengths of the corresponding sides of the other. Also, one can construct

hyperbolic right hexagons where these lengths are arbitrary positive real numbers. □

A basic fact in hyperbolic geometry is that if one has a compact hyperbolic surface T (not assumed to be connected) with totally geodesic boundary, then one can identify pairs of boundary components of the same lengths to obtain a hyperbolic surface whose new hyperbolic structure agrees with that on $T - \partial T$. Another elementary fact we shall need is that if C is a SCC in a compact hyperbolic surface T with totally geodesic boundary, then there is a unique closed geodesic $\gamma : S^1 \to T$ that is freely homotopic to C.

Now suppose that $\mathcal{P} = \{P_1, \ldots, P_{2g-2}\}$ is a pants decomposition of S determined by the SCCs $C_1, \ldots, C_{3g-3}$. For each hyperbolic structure on S, we can find closed geodesics $\gamma_1, \ldots, \gamma_{3g-3}$ that are isotopic to the C_j. Taking the lengths of these, we obtain a function

$$\ell : \mathcal{X}_S \to \mathbb{R}_+^{3g-3}.$$

It is not difficult to show that this mapping is continuous. (See [7], for example.) On the other hand, we can define an action of $\mathbb{R}^{3g-3}$ on $\mathcal{X}_S$ and whose orbits lie in the fibers of ℓ. To define the action of $(\theta_1, \ldots, \theta_{3g-3})$ on a hyperbolic surface S, write S as the union of $2g-2$ pairs of hyperbolic pants whose boundaries are represented by geodesics $\gamma_1, \ldots, \gamma_{3g-3}$ isotopic to the C_j. Then twist the gluing map at γ_j by an angle θ_j. This is also continuous.

Theorem 9.3 (Douady,[7, Exposé 7]). *If $g \geq 2$, then the map $\ell : \mathcal{X}_S \to \mathbb{R}_+^{3g-3}$ is a principal $\mathbb{R}^{3g-3}$ bundle with the action described above.* □

Corollary 9.4. *For all $g \geq 2$, Teichmüller space $\mathcal{X}_g$ is diffeomorphic to $\mathbb{R}^{6g-6}$.* □

Exercise 9.5. At first it may appear that $\ell : \mathcal{X}_g \to \mathbb{R}_+^{3g-3}$ should be a principal $(S^1)^{3g-3}$ bundle. Show that rotation by 2π about one of the SCCs C_j alters the representation ρ by an automorphism of π that is not inner. Hint: it may help to first read the part of Section 15 on Dehn twists.

10. The Complex Structure

Suppose that C is a compact Riemann surface. Recall that a deformation of C is the germ about $t_o \in T$ of a proper analytic mapping $f : \mathcal{C} \to T$, where T is an analytic variety, and an isomorphism $j : C \to f^{-1}(t_o)$. One defines maps between deformations to be cartesian squares. A deformation of C is called universal if every other deformation is pulled back from it.

A standard result in deformation theory is that every Riemann surface of genus $g \geq 2$ has a universal deformation. This has the property that T is smooth at t_o and that T is of complex dimension $3g-3$ with tangent space at t_o canonically isomorphic to $H^1(C, \Theta_C)$, where Θ_C denotes the sheaf of holomorphic sections of the tangent bundle of C.

As explained in Looijenga's lectures, $\mathcal{M}_g$ can be obtained by patching such local deformation spaces together. If C is a smooth projective curve of genus g and $\rho : \pi \to PSL_2(\mathbb{R})$ is a representation such that $C \cong \operatorname{im}\rho\backslash\mathbb{H}$, then $[\rho] \in \mathcal{X}_g$ goes to $[C] \in \mathcal{M}_g$ under the projection $\mathcal{X}_g \to \mathcal{M}_g$. The following fact can be proved using partial differential equations.

Theorem 10.1. *If (T, t_o) is a universal deformation space for C, then there is a smooth mapping $\psi : T \to \mathcal{X}_g$ such that $\psi(t_o) = [\rho]$ which is a diffeomorphism in a neighbourhood of t_o. Moreover, the composition of ψ with the projection $\mathcal{X}_g \to \mathcal{M}_g$ is the canonical mapping that classifies the universal deformation of C.* □

Corollary 10.2. *There is a canonical Γ_g invariant complex structure on $\mathcal{X}_g$ such that the projection $\mathcal{X}_g \to \mathcal{M}_g$ is a complex analytic mapping.* □

It should be noted that, although $\mathcal{X}_g$ is a complex manifold diffeomorphic to $\mathbb{R}^{6g-6}$, it is not biholomorphic to either a complex $3g-3$ ball or $\mathbb{C}^{3g-3}$ when $g > 1$.

Combining these results, we obtain the following basic result:

Theorem 10.3. *The moduli space $\mathcal{M}_g$ is a complex analytic variety whose singularities are all finite quotient singularities. Furthermore, $\mathcal{M}_g$ can be regarded as an orbifold whose universal covering is the complex manifold $\mathcal{X}_g$ and whose orbifold fundamental groups can be identified, up to conjugacy, with Γ_g. More precisely, if C is a compact genus g curve with no non-trivial automorphisms, then there is a canonical isomorphism*

$$\pi_1^{\mathrm{orb}}(\mathcal{M}_g, [C]) \cong \pi_0 \operatorname{Diff}^+ C. \quad \square$$

11. The Teichmüller Space $\mathcal{X}_{g,n}$

It is natural to expect that there is a complex manifold $\mathcal{X}_{g,n}$ and a discrete group $\Gamma_{g,n}$ that acts holomorphically on $\mathcal{X}_{g,n}$ in such a way that $\mathcal{M}_{g,n}$ is isomorphic to the quotient $\Gamma_{g,n}\backslash\mathcal{X}_{g,n}$. We give a brief sketch of how this may be deduced from the results when $n = 0$ and $g \geq 2$.

First, we fix an n-pointed reference surface of genus g where $g \geq 2$. That is, we fix a compact oriented surface S and a subset $P = \{x_1, \ldots, x_n\}$ of n

distinct points of S. Define

$$\Gamma_{g,n} := \pi_0 \operatorname{Diff}^+(S, P)$$

By definition, elements of $\operatorname{Diff}^+(S, P)$ are orientation preserving and act trivially on P.

Here is a sketch of a construction of $\mathcal{X}_{g,1}$. One can construct the $\mathcal{X}_{g,n}$ when $n > 1$ in a similar fashion.

There is a universal curve $\mathcal{C} \to \mathcal{X}_g$. This can be constructed using deformation theory. This is a fiber bundle in the topological sense and it is not difficult to see that the action of the mapping class group Γ_g on $\mathcal{X}_g$ can be lifted to an action on $\mathcal{C}$ such that the projection is equivariant. Since the fiber of $\mathcal{C} \to \mathcal{X}_g$ is a compact surface of genus g, and since $\mathcal{X}_g$ is contractible, $\mathcal{C}$ has the homotopy type of a surface of genus g and therefore has fundamental group isomorphic to $\pi_1(S)$. Define $\mathcal{X}_{g,1}$ to be the universal covering of $\mathcal{C}$. This is a complex manifold as $\mathcal{C}$ is. The fibers of the projection $\mathcal{X}_{g,1}$ are all isomorphic to $\mathbb{H}$. Since $\mathcal{X}_g$ is contractible, so is $\mathcal{X}_{g,1}$.

Exercise 11.1. Show that there is a natural bijection

$$\mathcal{X}_{g,1} = \left\{ \begin{array}{l} \text{conjugacy classes of representations } \rho : \pi \to \\ PSL_2(\mathbb{R}) \text{ such that } C_\rho := \operatorname{im} \rho \backslash \mathbb{H} \text{ is of genus } g, \\ \text{plus a point } x \in C_\rho \end{array} \right\}.$$

Use this (or otherwise) to show that $\Gamma_{g,1}$ acts on $\mathcal{X}_{g,1}$ and that the quotient is $\mathcal{M}_{g,1}$. Show that the isotropy group of any point of $\mathcal{X}_{g,n}$ lying above $[C; x] \in \mathcal{M}_{g,n}$ is naturally isomorphic to $\operatorname{Aut}(C, x)$.

We shall regard $\mathcal{M}_{g,n}$ as the orbifold $\Gamma_{g,n} \backslash \mathcal{X}_{g,n}$. There is a natural isomorphism

$$\pi_1^{\mathrm{orb}}(\mathcal{M}_{g,n}, [C; x_1, \ldots, x_n]) \cong \pi_0 \operatorname{Diff}^+(C, \{x_1, \ldots, x_n\})$$

provided $\operatorname{Aut}(C; x_1, \ldots, x_n)$ is trivial.

12. Level Structures

Level structures are useful technical devices for rigidifying curves. Suppose that ℓ is a positive integer. A *level ℓ structure* on a compact Riemann surface is the choice of a symplectic basis of $H_1(C, \mathbb{Z}/\ell\mathbb{Z})$.

Exercise 12.1. Show that if C is a compact Riemann surface of genus g, then there is a canonical isomorphism between $H_1(C, \mathbb{Z}/\ell\mathbb{Z})$ and the ℓ-torsion points in $\operatorname{Pic}^0 C$. (Remark: the intersection pairing on $H_1(C, \mathbb{Z}/\ell\mathbb{Z})$ corresponds to the Weil pairing on the ℓ-torsion points.)

Denote the moduli space of n-pointed, smooth, genus g curves with a level ℓ structure by $\mathcal{M}_{g,n}[l]$. This can be described as a quotient of Teichmüller space by a subgroup of $\Gamma_{g,n}$ that we now describe.

Fix an n-pointed compact oriented reference surface S of genus g. The mapping class group $\Gamma_{g,n}$ acts naturally on $H_1(S,\mathbb{Z})$. Since it preserves the intersection pairing, this leads to a homomorphism

$$\rho : \Gamma_{g,n} \to \mathrm{Aut}(H_1(S,\mathbb{Z}), \text{intersection form}).$$

Define the level ℓ subgroup of $\Gamma_{g,n}$ to be the kernel of the homomorphism

$$\rho_\ell : \Gamma_{g,n} \to \mathrm{Aut}(H_1(S,\mathbb{Z}/\ell\mathbb{Z}), \text{intersection form}).$$

This homomorphism is surjective.

Exercise 12.2. Show that $\Gamma_{g,n}[\ell]$ is a normal subgroup of finite index in $\Gamma_{g,n}$ and that the quotient is isomorphic to $Sp_g(\mathbb{Z}/\ell\mathbb{Z})$.

Exercise 12.3. Show that there is a natural bijection between $\mathcal{M}_{g,n}[\ell]$ and the quotient of $\mathcal{X}_{g,n}$ by $\Gamma_{g,n}[\ell]$. Show that the quotient mapping $\mathcal{M}_{g,n}[\ell] \to \mathcal{M}_{g,n}$ that forgets the level structure has finite degree and is Galois with Galois group $Sp_g(\mathbb{Z}/\ell\mathbb{Z})$.

Choosing a symplectic basis of $H_1(S,\mathbb{Z})$ gives an isomorphism

$$\mathrm{Aut}(H_1(S,\mathbb{Z}), \text{intersection form}) \cong Sp_g(\mathbb{Z}).$$

We therefore have a homomorphism

$$\rho : \Gamma_{g,n} \to Sp_g(\mathbb{Z}).$$

Define the level ℓ subgroup $Sp_g(\mathbb{Z})[\ell]$ of $Sp_g(\mathbb{Z})$ to be the kernel of the reduction mod ℓ homomorphism

$$Sp_g(\mathbb{Z}) \to Sp_g(\mathbb{Z}/\ell\mathbb{Z}).$$

This homomorphism is surjective.

Exercise 12.4. (i) Show that $\Gamma_{g,n}[\ell]$ is the inverse image of $Sp_g(\mathbb{Z})[\ell]$ under ρ.

(ii) Show that $Sp_1(\mathbb{Z})$ is isomorphic to $SL_2(\mathbb{Z})$ and that ρ is the standard representation when $g = 1$.

Theorem 12.5 (Minkowski). *The group $Sp_g(\mathbb{Z})[\ell]$ is torsion free when $\ell \geq 3$.* □

Proposition 12.6. *If $2g - 2 + n > 0$, $g \geq 1$ and $\ell \geq 3$, then the mapping class group $\Gamma_{g,n}[\ell]$ is torsion free and acts fixed point freely on $\mathcal{X}_{g,n}$.*

Sketch of Proof. The case $g = 1$ is left as an exercise. Suppose that $g \geq 2$. We first show that $\Gamma_{g,n}[\ell]$ acts fixed point freely on $\mathcal{X}_{g,n}$. The isotropy group of a point in $\mathcal{X}_{g,n}$ lying over $[C; x_1, \ldots, x_n]$ is isomorphic to $\mathrm{Aut}(C; x_1, \ldots, x_n)$. This is a finite group, and is a subgroup of $\mathrm{Aut}\, C$. It is standard that the natural representation $\mathrm{Aut}\, C \to \mathrm{Aut}\, H^0(C, \Omega^1_C)$ is injective (Exercise: prove this. Hint: use Riemann-Roch). It follows that the natural representation

$$\mathrm{Aut}\, C \to \mathrm{Aut}(H_1(C, \mathbb{Z}), \text{intersection pairing}) \cong Sp_g(\mathbb{Z}).$$

is injective and that $\mathrm{Aut}(C; x_1, \ldots, x_n) \cap \Gamma_{g,n}[\ell]$ is trivial. (Here we are realizing $\mathrm{Aut}(C; x_1, \ldots, x_n)$ as a subgroup of $\Gamma_{g,n}$ as an isotropy group.) It follows from Minkowski's Theorem that if $\ell \geq 3$, then $\Gamma_{g,n}[\ell]$ acts fixed point freely on $\mathcal{X}_{g,n}$.

The rest of the proof is standard topology. If $\Gamma_{g,n}[\ell]$ has a torsion element, then it contains a subgroup G of prime order, p say. Since this acts fixed point freely on the contractible space $\mathcal{X}_{g,n}$, it follows that the quotient $G \backslash \mathcal{X}_{g,n}$ is a model of the classifying space $B(\mathbb{Z}/p\mathbb{Z})$ of the cyclic group of order p. Since the model is a manifold of real dimension $g - \quad + 2n$, this implies that $H^k(\mathbb{Z}/p\mathbb{Z}, \mathbb{Z}/p\mathbb{Z})$ vanishes when $k > \quad g - \quad + 2n$. But this contradicts the known computation that $H^k(\mathbb{Z}/p\mathbb{Z}, \mathbb{Z}/p\mathbb{Z})$ is non-trivial for all $k \geq 0$. The result follows. □

Putting together the results of this section, we have:

Corollary 12.7. *If $g \geq 1$, $n \geq 0$ and $\ell \geq 3$, then $\mathcal{M}_{g,n}[\ell]$ is smooth and the mapping $\mathcal{X}_{g,n} \to \mathcal{M}_{g,n}[\ell]$ is unramified with Galois group $\Gamma_{g,n}$; the covering $\mathcal{M}_{g,n}[\ell] \to \mathcal{M}_{g,n}$ is a finite (ramified) Galois covering with Galois group $Sp_g(\mathbb{Z}/\ell\mathbb{Z})$.* □

With this information, we are able to prove a non-trivial result about the mapping class group.

Corollary 12.8 (McCool, Hatcher-Thurston)**.** *For all g and n, the mapping class group is finitely presented.*

Proof. We shall use the fact that each $\mathcal{M}_{g,n}[\ell]$ is a quasi-projective variety. As we have seen, this is smooth when $\ell \geq 3$ and has fundamental group isomorphic to $\Gamma_{g,n}[\ell]$. But a well known result Lojasiewicz [19] (see also [18]) implies that every smooth quasi-projective variety has the homotopy type of a finite complex. It follows that $\Gamma_{g,n}[\ell]$ is finitely presented when $\ell \geq 3$. But since $\Gamma_{g,n}[\ell]$ has finite index in $\Gamma_{g,n}$, this implies that $\Gamma_{g,n}$ is also finitely presented. □

13. Cohomology

One way to define the homology and cohomology of a group G is to find a topological space X such that

$$\pi_j(X,*) = \begin{cases} G & j = 1; \\ 0 & j \neq 1. \end{cases}$$

This occurs, for example, when the universal covering of X is contractible. Such a space is called a *classifying space of* G. Under some mild hypotheses, it is unique up to homotopy. One then defines the cohomology of G with coefficients in the G-module V by

$$H^j(G,V) = H^j(X,\mathbb{V})$$

where $\mathbb{V}$ is the locally constant sheaf over X whose fiber is V and whose monodromy is given by the action of G on V. It is well defined. Homology is defined similarly in terms of the homology of X with coefficients in $\mathbb{V}$.

Exercise 13.1. Suppose that V is a $\Gamma_{g,n}$-module and $\mathbb{V}$ is the corresponding locally constant sheaf over $\mathcal{M}_{g,n}[\ell]$, where $\ell \geq 3$. Show that there is a natural isomorphism

$$H^\bullet(\mathcal{M}_{g,n}[\ell],\mathbb{V}) \cong H^\bullet(\Gamma_{g,n}[\ell],V).$$

Show that the group $Sp_g(\mathbb{Z}/\ell\mathbb{Z})$ acts on both sides and that the isomorphism is $Sp_g(\mathbb{Z}/\ell\mathbb{Z})$-equivariant.

Since $\Gamma_{g,n}$ does not act fixed point freely on $\mathcal{X}_{g,n}$, $\mathcal{M}_{g,n}$ is not a classifying space for $\Gamma_{g,n}$. Nonetheless, standard topological arguments give

$$H^\bullet(\mathcal{M}_{g,n},\mathbb{Q}) \cong H^\bullet(\mathcal{M}_{g,n},\mathbb{Q})^{Sp_g(\mathbb{Z}/\ell\mathbb{Z})}$$

and

$$H^\bullet(\Gamma_{g,n},\mathbb{Q}) \cong H^\bullet(\Gamma_{g,n},\mathbb{Q})^{Sp_g(\mathbb{Z}/\ell\mathbb{Z})}.$$

It follows that:

Theorem 13.2. *There is a natural isomorphism*

$$H^\bullet(\Gamma_{g,n},\mathbb{Q}) \cong H^\bullet(\mathcal{M}_{g,n},\mathbb{Q}). \quad \square$$

If V is a $\Gamma_{g,n}$-module and $\Gamma_{g,n}$ has fixed points in $\mathcal{X}_{g,n}$, we cannot always define a local system $\mathbb{V}$ over $\mathcal{M}_{g,n}$ corresponding to V. However, we can formally define

$$H^\bullet(\mathcal{M}_g,\mathbb{V}\otimes\mathbb{Q}) := H^\bullet(\mathcal{M}_{g,n}[\ell],\mathbb{V}\otimes\mathbb{Q})^{Sp_g(\mathbb{Z}/\ell\mathbb{Z})}.$$

where the superscript $Sp_g(\mathbb{Z}/\ell\mathbb{Z})$ means that we take the $Sp_g(\mathbb{Z}/\ell\mathbb{Z})$ invariant part. This should be regarded as the cohomology of the orbifold $\mathcal{M}_{g,n}$ with coefficients in the orbifold local system corresponding to V.

Exercise 13.3. Show that this definition is independent of the choice of the level $\ell \geq 3$.

Let A be the locus of curves in $\mathcal{M}_g$ with non-trivial automorphisms. The goal of the following exercise is to show that A is an analytic subvariety of $\mathcal{M}_g$, each of whose components has codimension $\geq g-2$. Set

$$\mathcal{M}_g' = \mathcal{M}_g - A.$$

Exercise 13.4. Suppose that X is a Riemann surface of genus $g \geq 2$ and that G is a finite subgroup of $\operatorname{Aut} X$.

(i) Set $Y = G\backslash X$. Show that if X is compact, then

$$g(X) - 1 = d(g(Y) - 1) + \sum_{\mathcal{O}} (d - |\mathcal{O}|)/2$$

where $\mathcal{O}$ ranges over the orbits of G acting on X and where d is the order of G.

(ii) Show that if $g \geq 3$, then each component of the locus of curves in $\mathcal{M}_g$ that have automorphisms is a proper subvariety of $\mathcal{M}_g$.

(iii) Show that the hyperelliptic locus in $\mathcal{M}_g$ has codimension $g-2$.

(iv) Show that codimension of each component of the locus in $\mathcal{M}_g$ with curves with a non-trivial automorphism is $\geq g-2$, with equality if and only if the component is the hyperelliptic locus. Hint: reduce to the case where d is prime.

(v) Give an argument that there are only finitely many components of the locus in $\mathcal{M}_g$ of curves with a non-trivial automorphism.

Deduce that when $g \geq 3$, the set of points of $\mathcal{M}_g$ corresponding to curves without automorphisms is Zariski dense in $\mathcal{M}_g$.

Since each component of A has real codimension $\geq 2g-4$, it follows from standard topological arguments (transversality) that the inverse image $\mathcal{X}_g'$ of $\mathcal{M}_g$ in $\mathcal{X}_g$ has the property

$$\pi_j(\mathcal{X}_g') = 0 \text{ if } j < 2g-5.$$

From this, one can use standard topology to show that if $g \geq 3$, and if V is any Γ_g-module, then there is a natural mapping

$$H^k(\Gamma_g, V) \to H^k(\mathcal{M}_g', \mathbb{V})$$

which is an isomorphism when $k < 2g - 5$ and injective when $k = 2g - 5$.[8] In particular, there is a natural isomorphism

$$H^k(\Gamma_g, \mathbb{Z}) \cong H^k(\mathcal{M}'_g, \mathbb{Z})$$

when $k \leq 2g - 5$.

There are similar results for $\mathcal{M}_{g,n}$, but the codimension of the locus of curves with automorphisms rises with n. For example, if $n > 2g + 2$, then $\mathrm{Aut}(C; x_1, \ldots, x_n)$ is always trivial by the Lefschetz fixed point formula. (Exercise: prove this.)

Lecture 3: The Picard Group

In this lecture, we compute the orbifold Picard group of $\mathcal{M}_g$ for all $g \geq 1$. Recall that an orbifold line bundle over $\mathcal{M}_g$ is a holomorphic line bundle $\mathcal{L}$ over Teichmüller space $\mathcal{X}_g$ together with an action of the mapping class group Γ_g on it such that the projection $\mathcal{L} \to \mathcal{X}_g$ is Γ_g-equivariant. An orbifold section of this line bundle is a holomorphic Γ_g-equivariant section $\mathcal{X}_g \to \mathcal{L}$ of $\mathcal{L}$. This is easily seen to be equivalent to fixing a level $\ell \geq 3$ and considering holomorphic line bundles over $\mathcal{M}_g[\ell]$ with an $Sp_g(\mathbb{Z}/\ell\mathbb{Z})$-action such that the projection is $Sp_g(\mathbb{Z}/\ell\mathbb{Z})$-equivariant. Working on $\mathcal{M}_g[\ell]$ has the advantage that we can talk about algebraic line bundles more easily.

An *algebraic orbifold line bundle* over $\mathcal{M}_g$ is an algebraic line bundle over $\mathcal{M}_g[\ell]$ for some ℓ equipped with an action of $Sp_g(\mathbb{Z}/\ell\mathbb{Z})$ such that the projection to $\mathcal{M}_g[\ell]$ is $Sp_g(\mathbb{Z}/\ell\mathbb{Z})$equivariant. A section of such a line bundle is simply an $Sp_g(\mathbb{Z}/\ell\mathbb{Z})$-equivariant section defined over $\mathcal{M}_g[\ell]$. Isomorphism of such orbifold line bundles is defined in the obvious way. Let

$$\mathrm{Pic}_{\mathrm{orb}}\, \mathcal{M}_g$$

denote the group of isomorphisms classes of algebraic orbifold line bundles over $\mathcal{M}_g$. Our goal in this lecture is to compute this group. It is first useful to review some facts about the Picard group of a smooth projective variety.

14. General Facts

Recall that if X is a compact Kähler manifold (such as a smooth projective variety), then the exponential sequence gives an exact sequence

$$0 \to H^1(X, \mathbb{Z}) \to H^1(X, \mathcal{O}_X) \to \mathrm{Pic}\, X \to H^2(X, \mathbb{Z})$$

[8] There is a similar result for homology with the arrows reversed and injectivity replaced by surjectivity.

where the last map is the first Chern class c_1. The quotient

$$\operatorname{Pic}^0 X := H^1(X, \mathcal{O}_X)/H^1(X, \mathbb{Z})$$

is the group of topologically trivial complex line bundles and is a compact complex torus (in fact, an abelian variety). Note that if $H^1(X, \mathbb{Z})$ vanishes, then $\operatorname{Pic}^0 X$ vanishes and the first Chern class $c_1 : \operatorname{Pic} X \to H^2(X, \mathbb{Z})$ is injective.

If $H^1(X, \mathbb{Z})$ vanishes, it follows from the Universal Coefficient Theorem that the torsion subgroup of $H^2(X, \mathbb{Z})$ is $\operatorname{Hom}(H_1(X, \mathbb{Z}), \mathbb{C}^*)$. Every torsion element of $H^2(X, \mathbb{Z})$ is the Chern class of a holomorphic line bundle as a homomorphism $\chi : H_1(X, \mathbb{Z}) \to \mathbb{C}^*$ gives rise to a flat (and therefore holomorphic) line bundle over X.

Exercise 14.1. Suppose that X is a compact Kähler manifold with $H^1(X, \mathbb{Z}) = 0$. Show that if $\mathcal{L}$ is the flat line bundle over whose monodromy is given by the homomorphism $\chi : H_1(X, \mathbb{Z}) \to \mathbb{C}^*$, then

$$c_1(\mathcal{L}) = \chi \in \operatorname{Hom}(H_1(X, \mathbb{Z}), \mathbb{C}^*) \subseteq H^2(X, \mathbb{Z}).$$

These basic facts generalize to non-compact varieties. Suppose that X is a smooth quasi-projective variety. Define $\operatorname{Pic} X$ to be the group of isomorphism classes of algebraic line bundles over X, and $\operatorname{Pic}^0 X$ to be the kernel of the Chern class mapping

$$c_1 : \operatorname{Pic} X \to H^2(X, \mathbb{Z}).$$

Theorem 14.2. *Suppose that X is a smooth quasi-projective variety. If $H^1(X, \mathbb{Q})$ vanishes, then $\operatorname{Pic}^0 X = 0$ and the torsion subgroup of $\operatorname{Pic} X$ is naturally isomorphic to $\operatorname{Hom}(H_1(X, \mathbb{Z}), \mathbb{C}^*)$.*

Sketch of Proof. There are several ways to prove this. One is to use Deligne cohomology which gives a Hodge theoretic computation of $\operatorname{Pic} X$. Details can be found in [13], for example. A more elementary approach goes as follows. First pick a smooth compactification $\overline{X}$ of X. Each line bundle $\mathcal{L}$ over X can be extended to a line bundle over $\overline{X}$. Any two extensions differ by twists by the divisors in X that lie in $\overline{X} - X$. After twisting by suitable boundary components, we may assume that the extended line bundle also has trivial c_1 in $H^2(\overline{X}, \mathbb{Z})$. (Prove this using the Gysin sequence.) It therefore gives an element of $\operatorname{Pic}^0 \overline{X}$, which, by the discussion at the beginning of the section, is flat. This implies that the original line bundle $\mathcal{L}$ over X is also flat. But if $H^1(X, \mathbb{Q})$ vanishes, then $H_1(X, \mathbb{Z})$ is torsion and $c_1(\mathcal{L}) \in H^2(X, \mathbb{Z})$ is the

corresponding character $\chi : H_1(X, \mathbb{Z}) \to \mathbb{C}^*$. But since $c_1(\mathcal{L})$ is trivial, this implies that χ is the trivial character and that $\mathcal{L}$ is trivial. $\square$

This result extends to the orbifold case. By a smooth quasi-projective orbifold, we mean an orbifold $\Gamma\backslash X$ where Γ has a subgroup Γ' of finite index that acts fixed point freely on X and where $\Gamma'\backslash X$ is a smooth quasi-projective variety. There is a Chern class mapping

$$c_1 : \mathrm{Pic}_{\mathrm{orb}}(\Gamma\backslash X) \to H^2(\Gamma, \mathbb{Z}).$$

(The Picard group is constructed using equivariant algebraic line bundles on finite covers of $\Gamma\backslash X$.) The Chern class c_1 can be constructed using the Borel construction, for example. Define $\mathrm{Pic}^0_{\mathrm{orb}}(\Gamma\backslash X)$ to be the kernel of c_1.

Theorem 14.3. *Suppose that $\Gamma\backslash X$ is a smooth quasi-projective orbifold. If $H^1(X, \mathbb{Q})$ vanishes, then $\mathrm{Pic}^0_{\mathrm{orb}} X = 0$ and the torsion subgroup of $\mathrm{Pic}\, X$ is naturally isomorphic to $\mathrm{Hom}(H_1(\Gamma, \mathbb{Z}), \mathbb{C}^*)$.*

Sketch of Proof. Fix a finite orbifold covering of $\Gamma'\backslash X$ of $\Gamma\backslash X$ where Γ' is normal in Γ and acts fixed point freely on X. The Galois group of the covering is $G = \Gamma/\Gamma'$. Suppose that $\mathcal{L}$ is an orbifold line bundle over $\Gamma\backslash X$ with $c_1(\mathcal{L}) = 0$ in $H^2(\Gamma, \mathbb{Z})$. This implies that the first Chern class of the pullback of $\mathcal{L}$ to $\Gamma'\backslash X$ is also trivial. Since this pullback has a natural G-action, this means that the corresponding point in $\mathrm{Pic}^0(\Gamma'\backslash X)$ is G-invariant. Since

$$H^1(\Gamma'\backslash X, \mathbb{Q})^G \cong H^1(\Gamma\backslash X, \mathbb{Q}) = 0$$

it follows that the G-invariant part of $\mathrm{Pic}(\Gamma'\backslash X)$ is finite. It follows that some power $\mathcal{L}^{\otimes N}$ of the pullback of $\mathcal{L}$ to $\Gamma'\backslash X$ is trivial. This also has a G-action. If s is a trivializing section of $\mathcal{L}^{\otimes N}$, then the product

$$\bigotimes_{g \in G} g \cdot s$$

is a G-invariant section of $\mathcal{L}^{\otimes N|G|}$. It follows that $\mathcal{L}$ has a flat structure invariant under the G-action. But since $\mathcal{L}$ has trivial c_1, it must have trivial monodromy. It is therefore trivial. It follows that $\mathrm{Pic}_{\mathrm{orb}}(\Gamma\backslash X)$ is trivial. $\square$

Assembling the pieces, we have:

Corollary 14.4. *If $H^1(\Gamma_g, \mathbb{Q})$ vanishes, then the Chern class mapping*

$$\mathrm{Pic}_{\mathrm{orb}}\, \mathcal{M}_g \to H^2(\Gamma_g, \mathbb{Z})$$

is injective. $\square$

15. Relations in Γ_g

In this section we write down some well known relations that hold in various mapping class groups. These will be enough to compute $H_1(\Gamma_g)$, which we shall do in the next section.

First some notation. We shall let S denote any compact oriented surface with (possibly empty) boundary. The corresponding mapping class group is defined to be

$$\Gamma_S := \pi_0 \operatorname{Diff}^+(S, \partial S).$$

That is, Γ_S consists of isotopy classes of orientation preserving diffeomorphisms that act trivially on the boundary ∂S of S.

Exercise 15.1. Show that elements of Γ_S can be represented by diffeomorphisms that equal the identity in a neighbourhood of ∂S.

Exercise 15.2. Show that if S is a compact oriented surface and T is a compact subsurface, then there is a natural homomorphism $\Gamma_T \to \Gamma_S$ obtained by extending elements of Γ_T to be the identity outside T.

An important special case is where we take T to be the cylinder $[0,1] \times S^1$ (with the product orientation). One element of Γ_T is the diffeomorphism

$$\tau : (t, \theta) \mapsto (t, \theta + 2\pi t).$$

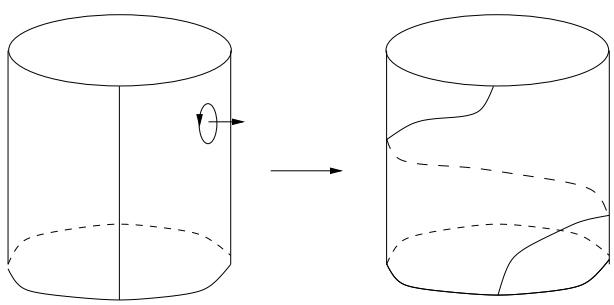

Figure 4. a positive Dehn twist

Theorem 15.3. *If T is the cylinder, then Γ_T is infinite cyclic and is generated by τ.* $\square$

Now, if S is any surface, and A is a (smoothly imbedded) simple closed curve (SCC), then A has a neighbourhood that is diffeomorphic to the cylinder $T = [0,1] \times S^1$ as in Figure 5. Denote the image of τ under the homomorphism $\Gamma_T \to \Gamma_S$ by t_A. It is called the *Dehn twist about* A.

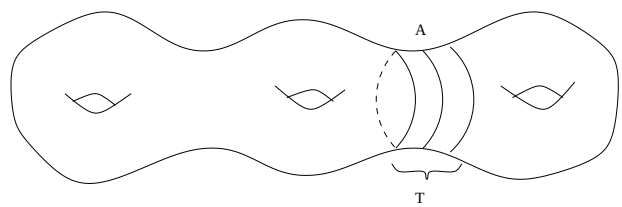

FIGURE 5

Theorem 15.4. *The class of t_A in Γ_S is independent of the choice of the tubular neighbourhood T of A and depends only on the isotopy class of A. Every mapping class group is generated by Dehn twists.* □

Exercise 15.5. Show that if $\phi \in \Gamma_S$ and A is a SCC in S, then

$$t_{\phi(A)} = \phi t_A \phi^{-1}.$$

Observe that if two elements of Γ_S can be represented by diffeomorphisms with disjoint support, then they commute. In particular:

Proposition 15.6. *If A and B are disjoint SCCs on S, then t_A and t_B commute in Γ_S.* □

Exercise 15.7. Show that if S is a surface with boundary and A and B are each non-separating SCCs in S, then there is an orientation preserving diffeomorphism ϕ of S that takes one onto the other. Deduce that t_A and t_B are conjugate in Γ_S. Give an example to show that not all Dehn twists about bounding SCCs are conjugate in Γ_S when $g \geq 3$.

Next we consider what happens when A and B are two simple closed curves in S that meet transversally in one point as in Figure .

Exercise 15.8. Show that if A and B are two SCCs in S that meet transversally in one point, then there is a neighbourhood of their union that is a compact genus 1 surface with one boundary component. Hint: Compute the homology of a small regular neighbourhood of the union and then apply the classification of compact oriented surfaces.

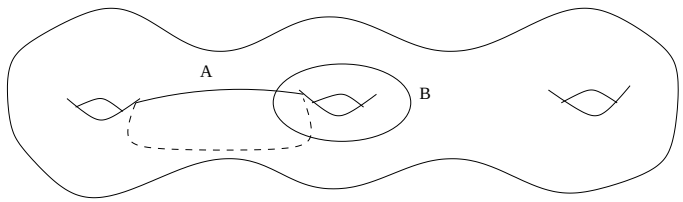

FIGURE 6

So any relation that holds between Dehn twists such curves in a genus one surface with one boundary component will hold in all surfaces. Let S be a compact genus 1 surface with one boundary component and let A and B be the two SCCs in the diagram:

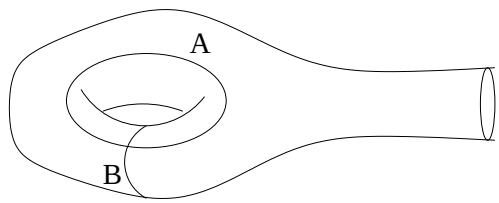

FIGURE 7

Theorem 15.9. *With notation as above, we have*

$$t_A t_B t_A = t_B t_A t_B$$

in Γ_S *and therefore in all mapping class groups.*

This relation is called the *braid relation* as it comes from the braid group on 3 strings[9] using the following technique.

Denote the unit disk by D. We view this as a manifold with boundary. Suppose that P is a set of n distinct points of D, none of which lies on ∂D. The braid group B_n is defined to be

$$B_n := \pi_0 \operatorname{Diff}^+(D, (P)),$$

where $\operatorname{Diff}^+(D, (P))$ denotes the group of orientation preserving diffeomorphisms of D that fix P as a set, but may permute its elements. There is a surjective homomorphism

$$B_n \to \operatorname{Aut} P \cong \Sigma_n$$

[9] A good reference for braid groups is Joan Birman's book [4].

onto the symmetric group on n letters.

Suppose that U is a disk imbedded in D such that

$$\partial U \cap P = \emptyset$$

and $U \cap P$ consists of two distinct points x and y of P. Then there is an element σ_U of B_n whose square is the Dehn twist $t_{\partial U}$ about the boundary of U and which swaps x and y.

It can be represented schematically as in Figure 8: The braid group B_n

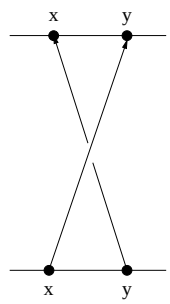

FIGURE 8. a basic braid

is generated by the braids $\sigma_1, \ldots, \sigma_{n-1}$ illustrated in Figure 9. Note that σ_i

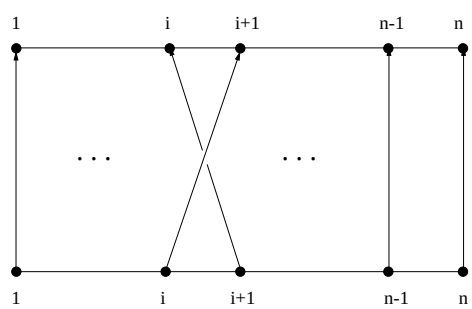

FIGURE 9. the generator σ_i

and σ_j commute when $|i-j| > 1$, and that

$$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}. \tag{ }$$

Now suppose that $(S, \partial S) \to (D, \partial D)$ is a branched covering, unramified over the boundary. Suppose that the image of the branch points is P. Then there is a natural homomorphism

$$B_n \to \Gamma_S.$$

So relations in B_n will give relations in Γ_S. The relations we are interested in come from certain double branched coverings of the disk.

Suppose that $(S, \partial S) \to (D, \partial D)$ is a 2-fold branched covering. The inverse image of a smooth arc α joining two critical values $p_1, p_2 \in P$ and avoiding P otherwise, is an SCC, say A, in S. There is a small neighbourhood U of α that is diffeomorphic to a disk and whose intersection with P is $\{p_1, p_2\}$ as in Figure 10. There is a braid σ supported in U that swaps p_1 and p_2 by rotating in the positive direction about α.

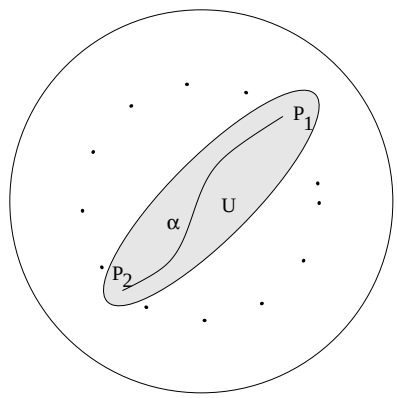

FIGURE 10

Proposition 15.10. *The image of σ under the homomorphism $B_n \to \Gamma_S$ is the Dehn twist t_A about A.* □

Exercise 15.11. Show that a genus 1 surface S with one boundary component can be realized as a 2:1 covering of D, branched over 3 points.

We therefore have a homomorphism $B_3 \to \Gamma_S$. Note that the inverse image of the two arcs α and β in Figure 11 under the covering of the disk branched over $\{p_1, p_2, p_3\}$ is a pair of SCCs in S that intersect transversally in one point. The braid relation in Γ_S now follows as we have the braid relation ( )

$$\sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2$$

in B_3.

More relations can be obtained this way. Suppose that S is a compact genus 1 surface with 2 boundary components.

Let A, B, C, T_1 and T_2 be the SCCs in Figure 12. Denote the Dehn twists about A, B and C by a, b and c, and those about T_1 and T_2 by t_1 and t_2.

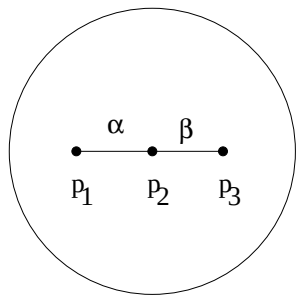

FIGURE 11

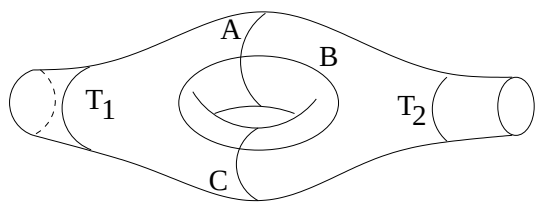

FIGURE 12

Theorem 15.12. *The relation*

$$(abc)^4 = t_1 t_2$$

holds in the mapping class group Γ_S.

Corollary 15.13. *If S is a compact genus 1 surface with one boundary component, then the relation*

$$(ab)^6 = t$$

holds in Γ_S, where a and b denote Dehn twists about a pair of SCCs that intersect transversally in one point and t denotes a Dehn twist about the boundary.

Exercise 15.14. Deduce Corollary 15.13 from Theorem 15.12 by capping off one boundary component and using the braid relation, Theorem 15.9.

Theorem 15.12 is proved in the following exercise.

Exercise 15.15. Suppose that S is a genus 1 surface with two boundary components. Show that S is a double covering of the disk, branched over 4 points. Use this to construct a homomorphism from the braid group B_4 on 4 strings into Γ_S. Use this to prove the relation

$$(abc)^4 = t_1 t_2$$

where a, b, c, t_1 and t_2 denote Dehn twists on the SCCs A, B, C, T_1 and T_2 in the diagram above. Note that in the braid group, we have the relation

$$t = \sigma_1\sigma_2\sigma_3$$

where t is Dehn twist about the boundary of the disk and σ_i is the ith standard generator of B_4.

There is one final relation. It is called the *lantern relation* and is due to Johnson and Harer independently. Let S be a disk with 3 holes. (That is, a genus 0 surface with 4 boundary components.) Consider the SCCs in Figure 13.

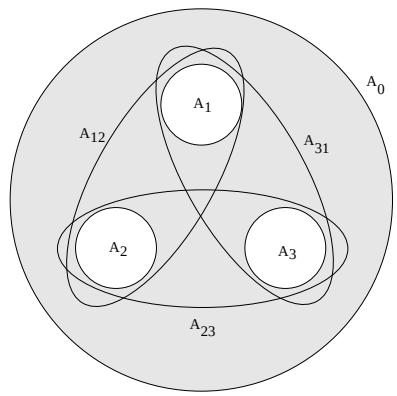

FIGURE 13. the lantern configuration

Theorem 15.16. *The relation*

$$a_0a_1a_2a_3 = a_{12}a_{23}a_{31}$$

holds in Γ_S *where* a_i *denotes the Dehn twist about* A_i *and* a_{ij} *denotes the Dehn twist about* A_{ij}. □

1 . Computation of $H_1(\Gamma_g, \mathbb{Z})$

The fact Γ_g is generated by Dehn twists and the relations given in the previous section allow the computation of $H_1(\Gamma_g)$.

Theorem 16.1 (Harer). *If* $g \geq 1$, *then*

$$H_1(\Gamma_g, \mathbb{Z}) \cong \begin{cases} \mathbb{Z}/12\mathbb{Z} & g = 1; \\ \mathbb{Z}/10\mathbb{Z} & g = 2; \\ 0 & g \geq 3. \end{cases}$$

Proof. We begin with the observation that if S is a compact oriented genus g surface, then all Dehn twists on non-separating SCCs lie in the same homology class as they are conjugate by Exercise 15.5. Denote their common homology class by L. Next, using the relations coming from an imbedded genus 1 surface with one boundary component, we see that the homology class of any separating SCC that divides S into a genus 1 and genus $g-1$ surfaces has homology class $12L$. Using the relation that comes from an imbedded genus 1 surface with 2 boundary components, we see that the homology class of every separating SCC is an integer multiple of L. It follows that $H_1(\Gamma_g)$ is cyclic and generated by L.

Now suppose that $g \geq 3$. Then we can find an imbedded lantern as in Figure 14 Since each of the curves in this lantern is non-separating, the

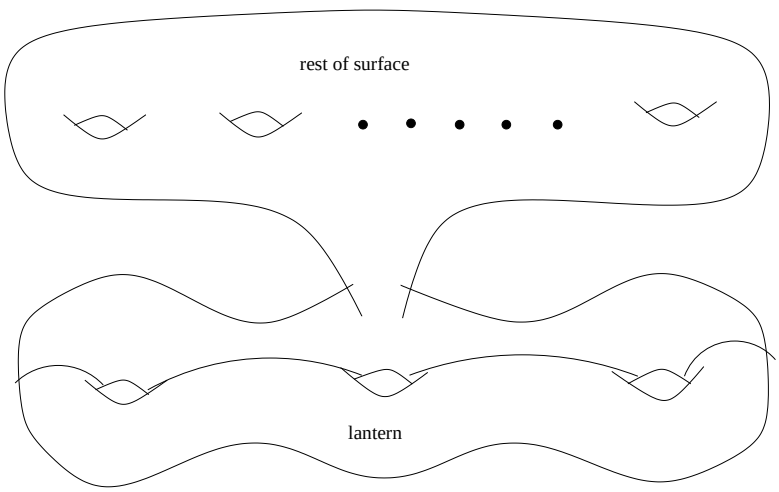

FIGURE 14

lantern relation tells us that $3L = 4L$, which implies that $L = 0$. This proves the vanishing of $H_1(\Gamma_g)$ when $g \geq 3$.

When $g = 1$, the relation for a genus 1 surface with one boundary component implies that $12L = 0$ as the twist about the boundary is trivial in Γ_1. Thus $H_1(\Gamma_1)$ is a quotient of $\mathbb{Z}/12\mathbb{Z}$. But, as we shall explain a little later, the fact that $\mathrm{Pic}_{\mathrm{orb}}\,\mathcal{M}_1$ is at least as big as $\mathbb{Z}/12$ implies that $H_1(\Gamma_1) = \mathbb{Z}/12\mathbb{Z}$.[10]

[10]The abelianization of $SL_2(\mathbb{Z})$ can also be computed using, for example, the presentation of $PSL_2(\mathbb{Z})$ given in [24].

A genus 2 surface can be obtained from a genus 1 surface with two boundary components by identifying the boundary components. The relation obtained for a genus 1 surface with 2 boundary components gives $12L = 2L$, so that $10L = 0$. This shows that $H_1(\Gamma_2)$ is a quotient of $\mathbb{Z}/10\mathbb{Z}$. But, as in the genus 1 case, the theory of Siegel modular forms shows that it cannot be any smaller. So we have $H_1(\Gamma_2) = \mathbb{Z}/10\mathbb{Z}$. □

Corollary 16.2. *For all $g \geq 1$, $H^1(\Gamma_g, \mathbb{Z})$ vanishes.* □

17. Computation of $\operatorname{Pic}_{\mathrm{orb}} \mathcal{M}_g$

Since $H^1(\Gamma_g, \mathbb{Q})$ is torsion, it follows from Corollary 14.4 that

$$c_1 : \operatorname{Pic}_{\mathrm{orb}} \mathcal{M}_g \to H^2(\Gamma_g, \mathbb{Z})$$

is injective. Since $H_1(\Gamma_g)$ is torsion, the Universal Coefficient Theorem implies that the sequence

$$0 \to \operatorname{Hom}(H_1(\Gamma_g, \mathbb{Z}), \mathbb{C}^*) \to H^2(\Gamma_g, \mathbb{Z}) \to \operatorname{Hom}(H_2(\Gamma_g), \mathbb{Z}) \to 1$$

is exact. To determine the rank of $H^2(\Gamma_g)$, we need the following fundamental and difficult result of Harer [15].

Theorem 17.1. *The rank of $H_2(\Gamma_g, \mathbb{Q})$ is 0 if $g \leq 2$ and 1 if $g \geq 3$.* □

Combining this with our previous discussion, we have:

Theorem 17.2. *If $g \geq 1$, then*

$$H^2(\Gamma_g, \mathbb{Z}) \cong \begin{cases} \mathbb{Z}/12\mathbb{Z} & g = 1; \\ \mathbb{Z}/10\mathbb{Z} & g = 2; \\ \mathbb{Z} & g \geq 3. \end{cases}$$

There is one obvious orbifold line bundle over $\mathcal{M}_g$. Namely, if we take the universal curve $\pi : \mathcal{C} \to \mathcal{M}_g$ over the orbifold $\mathcal{M}_g$, then we can form the line bundle

$$\mathcal{L} := \det \pi_* \omega_{\mathcal{C}/\mathcal{M}_g}$$

over $\mathcal{M}_g$, where $\omega_{\mathcal{C}/\mathcal{M}_g}$ denotes the relative dualizing sheaf. The fiber over $[C] \in \mathcal{M}_g$ is

$$\Lambda^g H^0(C, \Omega^1_C).$$

Theorem 17.3. *If $g \geq 1$, the orbifold Picard group of $\mathcal{M}_g$ is cyclic, generated by $\mathcal{L}$ and given by*

$$\operatorname{Pic}_{\mathrm{orb}} \mathcal{M}_g \cong \begin{cases} \mathbb{Z}/12\mathbb{Z} & g = 1; \\ \mathbb{Z}/10\mathbb{Z} & g = 2; \\ \mathbb{Z} & g \geq 3. \end{cases}$$

Proof. All but the generation by $\mathcal{L}$ follows from preceding results. In genus 1 and 2, the generation by $\mathcal{L}$ follows from the theory of modular forms.

Suppose that $g \geq 3$. Denote the first Chern class of $\mathcal{L}$ by λ. To prove that $\mathcal{L}$ generates $\operatorname{Pic}_{\mathrm{orb}} \mathcal{M}_g$, it suffices to show that λ generates $H^2(\Gamma_g, \mathbb{Z})$. The following proof of this I learned from Shigeyuki Morita. It assumes some knowledge of characteristic classes. A good reference for this topic is the book [21] by Milnor and Stasheff.

We begin by recalling the definition of the signature of a compact oriented 4-manifold. Every symmetric bilinear form on a real vector space can be represented by a symmetric matrix. The signature of a non-degenerate symmetric bilinear form is the number of positive eigenvalues of a representing matrix minus the number of negative eigenvalues. To each compact oriented 4-manifold X, we associate the symmetric bilinear form

$$H^2(X, \mathbb{R}) \otimes H^2(X, \mathbb{R}) \to \mathbb{R}$$

defined by

$$\xi_1 \otimes \xi_2 \mapsto \int_X \xi_1 \wedge \xi_2.$$

Poincaré duality implies that it is non-degenerate. The signature $\tau(X)$ of X is defined to be the signature of this bilinear form. It is a cobordism invariant.

The Hirzebruch Signature Theorem (see [21, Theorem 19.4], for example) asserts that

$$\tau(X) = \frac{1}{3} \int_X p_1(X)$$

where $p_1(X) \in H^4(X, \mathbb{Z})$ is the first Pontrjagin class of X. When X is a complex manifold,

$$p_1(X) = c_1(X)^2 - 2c_2(X). \tag{7}$$

Now suppose that X is a smooth algebraic surface and that T is a smooth algebraic curve. Suppose that $\pi : X \to T$ is a family whose fibers are smooth curves of genus $g \geq 3$.[11] Denote the relative cotangent bundle $\omega_{X/T}$ of π by

[11] Such families exist — see [17, p. 45, p. 55].

ω. Then it follows from the exact sequence

$$0 \to \pi^*\Omega^1_S \to \Omega^2_X \to \omega_{X/T} \to 0$$

that

$$c_1(X) = \pi^* c_1(T) - c_1(\omega) \text{ and } c_2(X) = -c_1(\omega) \wedge \pi^* c_1(T).$$

Plugging these into (7) we see that $p_1(X) = c_1(\omega)^2$. Using integration over the fiber, we have

$$\tau(X) = \frac{1}{3}\int_X c_1(\omega)^2 = \frac{1}{3}\int_T \pi_*(c_1(\omega)^2).$$

It is standard to denote $\pi_*(c_1(\omega)^2)$ by κ_1. Thus we have

$$\tau(X) = \frac{1}{3}\int_T \kappa_1.$$

An easy consequence of the Grothendieck-Riemann-Roch Theorem is that $\kappa_1 = 12\lambda$. This is proved in detail in the book of Harris and Morrison [17, pp. 155–15]. It follows that for a family $\pi : X \to T$ of smooth curves

$$\tau(X) = 4\int_T \lambda.$$

The last step is topological. Suppose that F is a compact oriented surface and that $p : W \to F$ is an oriented surface bundle over F where the fibers of p are compact oriented surfaces of genus $g \geq 3$. Denote the local system of the first integral homology groups of the fibers by $\mathbb{H}$. There is a symmetric bilinear form

$$H^1(F, \mathbb{H}_{\mathbb{R}}) \otimes H^1(F, \mathbb{H}_{\mathbb{R}}) \to \mathbb{R} \tag{8}$$

obtained from the cup product and the intersection form. Poincaré duality implies that it is non-degenerate. It follows from the Leray-Serre spectral sequence of p that

$$\tau(W) = - \text{ the signature of the pairing (8).}$$

The local system $\mathbb{H}$ over F corresponds to a mapping $\phi : F \to BSp_g(\mathbb{Z})$ into the classifying space of the symplectic group. Meyer [20] shows that there is a cohomology class $m \in H^2(Sp_g(\mathbb{Z}), \mathbb{Z})$ whose value on ϕ is the signature of the pairing (8). It follows from this and the discussion above, that under the mapping

$$\rho^* : H^2(Sp_g(\mathbb{Z}), \mathbb{Z}) \to H^2(\Gamma_g, \mathbb{Z})$$

induced by the canonical homomorphism $\rho : \Gamma_g \to Sp_g(\mathbb{Z})$, m goes to -4λ. Mayer also shows that the image of

$$m : H^2(\Gamma_g, \mathbb{Z}) \to \mathbb{Z}$$

is exactly $4\mathbb{Z}$, which implies that λ generates $H^2(\Gamma_g, \mathbb{Z})$. □

As a corollary of the proof, we have:

Corollary 17.4. *For all $g \geq 3$, both $H^2(Sp_g(\mathbb{Z}), \mathbb{Z})$ and $H^2(\Gamma_g, \mathbb{Z})$ are generated by λ and the natural mapping*

$$\rho^* : H^2(Sp_g(\mathbb{Z}), \mathbb{Z}) \to H^2(\Gamma_g, \mathbb{Z})$$

is an isomorphism. □

Note that $H^2(Sp_2(\mathbb{Z}), \mathbb{Z})$ is infinite cyclic, while $H^2(\Gamma_2, \mathbb{Z})$ is cyclic of order 10. So ρ^* is not an isomorphism in genus 2.

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Arithmetic fundamental groups and moduli of curves

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Abstract

This is a short note on the algebraic (or sometimes called arithmetic) fundamental groups of an algebraic variety, which connects classical fundamental groups with Galois groups of fields. A large part of this note describes the algebraic fundamental groups in a concrete manner.

This note gives only a sketch of the fundamental groups of the algebraic stack of moduli of curves. Some application to a purely topological statement, i.e., an obstruction to the surjectivity of Johnson homomorphisms in the mapping class groups, which comes from Galois group of $\mathbb{Q}$, is explained.

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1. Galois groups of field extension

We recall some basic notions of classical Galois theory of fields. Let K be a field, and fix its algebraic closure $\bar{K}$. Let L be a finite extension of K. An extension L/K is said to be *separable* if the cardinality of $\mathrm{Hom}_K(L, \bar{K})$ is the same as the extension degree $[L : K]$. Let $\mathrm{Aut}(L/K)$ be the group of automorphisms of L as a field, fixing each element in K. Then, $\mathrm{Aut}(L/K)$ acts on $\mathrm{Hom}_K(L, \bar{K})$. If the action is transitive, then the extension L/K is said to be a *normal* extension. A finite separable normal extension L/K is called a finite *Galois extension*, and $\mathrm{Aut}(L/K)$ is called the *Galois group* of L/K and denoted by $G(L/K)$.

Suppose that L/K is a finite Galois extension. Galois theory asserts that there exists a one-to-one correspondence between

$$\{M : \text{intermediate field between } L \text{ and } K\}$$

and

$$\{N : \text{subgroup of } G(L/K)\}$$

given by $M \mapsto G(L/M)$ and $N \mapsto L^N$, where L^N denotes the subfield of elements fixed by the action of all elements of N. The correspondence reverses the inclusion relation, i.e., if $M \subset M'$ then $G(L/M) \supset G(L/M')$. If we use the terminology of categories, then the correspondence is a contravariant equivalence.

There is an infinite version. Let L/K be an algebraic extension, which is allowed to be infinite. It is called an (infinite) Galois extension, if any finite extension of K in L is finite Galois. Then, $\mathrm{Aut}(L/K)$ is called the Galois group of L/K, and denoted by $G(L/K)$ again.

This infinite Galois group is equipped with a natural topology, called profinite topology, as will be stated later. For $L \supset M \supset K$ with M being Galois over K, we have a short exact sequence

$$1 \to G(L/M) \to G(L/K) \to G(M/K) \to 1. \tag{1.1}$$

Exercise 1.1. *Let $K = \mathbb{Q}$, $L = \mathbb{Q}(\xi_n)$, with $\xi_n = e^{2\pi\sqrt{-1}/n}$ being an n-th primitive root of unity. Show that $\chi : G(L/K) \cong (\mathbb{Z}/n)^\times$, where $\sigma \in G(L/K)$ is mapped to the unique $\chi(\sigma) \in (\mathbb{Z}/n)^\times$ such that $\sigma(\xi_n) = \xi_n^{\chi(n)}$, by using the irreducibility of the minimal polynomial of ξ_n.*

If we put $L' := \mathbb{Q}(\xi_n | n \in \mathbb{N})$, what is $G(L'/K)$?

2. A short way to arithmetic fundamental groups

Let X be an arcwise connected topological space, and let a be a point of X. The *fundamental group* of X with base point a, denoted by $\pi_1(X, a)$, is defined to be the set of homotopy equivalence classes of closed paths from a to itself. By the usual composition of paths, taking the homotopy classes, $\pi_1(X, a)$ becomes a group. For the composition rule, we denote by $\gamma' \circ \gamma$ the path first going along γ and then along γ'.

For another choice of base point b, we have $\pi_1(X, a) \cong \pi_1(X, b)$, where the isomorphism is well-defined up to an inner automorphism.

More generally, let $\pi_1(X; a, b)$ denote the homotopy classes of paths from a to b. Two paths $\gamma \in \pi_1(X; a, b)$ and $\gamma' \in \pi_1(X; b, c)$ can be composed to get an element $\gamma' \circ \gamma \in \pi_1(X; a, c)$, and the system $\{\pi_1(X; a, b) | (a, b \in X)\}$ constitutes a *groupoid* (i.e. a category where every homomorphism is invertible).

Assume that X is a smooth complex algebraic variety. Let a be a point of X.

Definition 2.1. *Let M_a be the field of the germs of meromorphic algebraic functions at a on X, such that the germ has analytic continuation to a* finitely *multivalued meromorphic function along any path on X.*

Algebraic means that every $h \in M_a$ is algebraic over the rational function field $\mathbb{C}(X)$ of algebraic variety X. Functions like $\exp(z)$ are not in $\mathbb{C}(\mathbb{A}^1)$.

Clearly M_a is a field, which contains the rational function field $\mathbb{C}(X)$ as the single-valued meromophic functions. By analytic continuation, we have an action

$$\pi_1(X; a, b) \times M_a \to M_b,$$

with $h \in M_a \mapsto {}^{\gamma}h \in M_b$ being obtained by analytic continuation of the germ h along the path γ. In particular, $\pi_1(X, a)$ acts on M_a from the left, so we have a group homomorphism

$$\pi_1(X, a) \to \mathrm{Aut}(M_a/\mathbb{C}(X)),$$

since $\mathbb{C}(X)$ is single-valued and is fixed elementwise by analytic continuation along a closed path. Because $h \in M_a$ is assumed to be finitely multivalued, the set of all the branches $S := \{{}^{\gamma}h | \gamma \in \pi_1(X, a)\}$ is finite. Hence, the fundamental symmetric polynomials of $S =: \{f_1, f_2, \ldots, f_n\}$ are single-valued, i.e. in $\mathbb{C}(X)$, since $\pi_1(X, a)$ acts on S by permutations. Thus,

$F(T) := \prod_{i=1}^{n}(T - f_i) \in \mathbb{C}(X)[T]$, which is divisible by the minimal polynomial of h over $\mathbb{C}(X)$. Thus, any algebraic conjugate of h over $\mathbb{C}(X)$ is one of the f_i, and hence $M_a/\mathbb{C}(X)$ is a Galois extension.

Now we have a homomorphism

$$\pi_1(X, a) \to G(M_a/\mathbb{C}(X)).$$

Definition 2.2. *We define the algebraic fundamental group of a connected smooth algebraic variety X over $\mathbb{C}$ by*

$$\pi_1^{alg}(X, a) := G(M_a/\mathbb{C}(X)).$$

We have a group homomorphism

$$\pi_1(X, a) \to \pi_1^{alg}(X, a),$$

by analytic continuation.

It can be proved that this morphism is "completion," i.e., the right-hand side is obtained by a purely group-theoretical operation called "profinite completion" of the left-hand side, and thus depends only on the homotopy type of X, as we shall see in the next section. The arithmetic part will come into sight when we consider X over a non algebraically closed field K in §2.2.

2.1. **Profinite groups and algebraic fundamental groups.** We want to describe the (possibly infinite) Galois group $G(L/K)$ for $L := M_a$, $K := \mathbb{C}(X)$. The answer is that it is the profinite completion of $\pi_1(X, a)$.

To see this, for a while, let L/K be a general infinite Galois extension. We look at all finite Galois subextensions M/K in L. The surjectivity of $G(L/K) \to G(M/K)$ in (1.1) says that $\sigma \in G(L/K)$ determines a family $\sigma_M \in G(M/K)$ for each finite Galois extension M, so that they are compatible in the sense that if $M \supset M'$, then $\sigma_M \mapsto \sigma_{M'}$ via natural morphism $G(M/K) \to G(M'/K)$.

In general, let Λ be a directed set (i.e. a partially ordered set such that for any $\lambda, \lambda' \in \Lambda$ there is a common upper element $\lambda'' \geq \lambda, \lambda'$ in Λ), and assume that we are given a family of finite groups G_λ ($\lambda \in \Lambda$) together with a group homomorphism $G_\lambda \to G_\mu$ for $\lambda \geq \mu$, with the following compatibility condition: the composition of $G_\lambda \to G_\mu$ with $G_\mu \to G_\nu$ coincides with $G_\lambda \to G_\nu$. We call this family *a projective system* of finite groups. (In the terminology of category theory, this is merely a functor from Λ to the category of finite groups.) We define its *projective limit* $\operatorname{proj\,lim}_{\lambda\in\Lambda} G_\lambda$ as follows. An element of $\operatorname{proj\,lim}_\lambda G_\lambda$ is a family $(\sigma_\lambda)_{\lambda\in\Lambda}$, $\sigma_\lambda \in G_\lambda$, with the

property $\sigma_\lambda \mapsto \sigma_\mu$ for $\lambda \geq \mu$. This is a subset of the direct product:

$$\operatorname*{proj\,lim}_{\lambda\in\Lambda} G_\lambda \subset \prod_{\lambda\in\Lambda} G_\lambda.$$

We equip G_λ with the discrete (but compact, being finite) topology. Then, the product is compact by Tikhonov's Theorem, and so is the projective limit, being a closed subgroup of the compact group.

If we apply this notion to the case: Λ is the set of finite Galois subextension M/K equipped with the inclusion ordering $M \geq M' \Leftrightarrow M \supset M'$, we obtain a projective system of finite groups $G(M/K)$. Our previous observation says that there is a morphism

$$G(L/K) \to \operatorname*{proj\,lim}_{M} G(M/K),$$

$\sigma \in G(L/K) \mapsto (\sigma|_M \in G(M/K))$. This is injective, since L being algebraic, L is the union of all finite Galois extensions M. The above morphism is surjective, since if we have an element of $(\sigma_M) \in \operatorname{proj\,lim}_M G(M/K)$, then the compatibility assures that the action of σ_M restricts to that of $\sigma_{M'}$ if $M \supset M'$. Thus, the actions σ_M patch together to give an element of $G(L/K)$. Thus we have

$$G(L/K) = \operatorname*{proj\,lim}_{M} G(M/K).$$

A topological group which can be written as the projective limit of finite groups is called a *profinite group*. $G(L/K)$ is an example. It is known that G is a profinite group if and only if it is a compact totally disconnected Hausdorff topological group.

Let G be an abstract group. The set of finite-index normal subgroups $\Lambda := \{N \triangleleft G \mid G/N : \text{finite group}\}$ is a directed set by $N \geq N'$ if and only if $N \subset N'$. Then, we have a projective system of finite groups G/N, $N \in \Lambda$.

Definition 2.3. *For an abstract group G, we define its profinite completion $\widehat{G}$ to be*

$$\widehat{G} := \operatorname*{proj\,lim}_{N}(G/N),$$

where N runs over the finite index normal subgroups of G. We have a natural group homomorphism

$$G \to \widehat{G}$$

by patching $G \to G/N$ together, and its image is dense.

Fix a prime l. Let N run over the finite index normal subgroups of G with the quotient G/N being an l-group, and take the similar projective limit. Then we obtain the pro-l completion of G, denoted by G^l.

Theorem 2.1. (SGA1[7, Corollary 5.2, p.337]) *Let X be a connected algebraic variety over $\mathbb{C}$. Then*

$$\pi_1(X,a) \to \pi_1^{alg}(X,a)$$

gives an isomorphism of the profinite completion of the topological fundamental group and the algebraic fundamental group,

$$\pi_1(X,a) \to \pi_1(\widehat{X,a}) \cong \pi_1^{alg}(X,a).$$

A rough sketch of the proof is as follows. It is enough to prove that there is a one-to-one correspondence between

$$\{N \triangleleft \pi_1^{alg}(X,a) \mid \pi_1^{alg}(X,a)/N : \text{finite group}\}$$

and

$$\{M \subset M_a \mid M/K \text{ is finite Galois }\}$$

so that

$$\pi_1^{alg}(X,a)/N \cong G(M/K) \tag{2.1}$$

holds through

$$\pi_1^{alg}(X,a) \to G(M_a/K) \to G(M/K),$$

since then by taking the projective limit of (2.1) we have the desired isomorphism.

For N, we construct a finite topological unramified covering $p_N : Y_N \to X$ with $a_N \in p_N^{-1}(a)$ fixed. This can be done as follows. Let $\tilde{X}$ be the universal covering of X, and $\tilde{a} \in \tilde{X}$ be a point above a. (These topological notions are recalled in §3.1 below.) It is well known that $\pi_1(X,a)$ acts on the covering $\tilde{X}/X$ from the right. Then, by taking the quotient of $\tilde{X}$ by N, we have a finite connected unramified covering $p_N : Y_N \to X$, with a point $a_N \in Y_N$ being fixed as the image of $\tilde{a}$. A "Riemann existence theorem" in SGA1[7, Theorem 5.1, p.332] asserts that Y_N is actually an algebraic variety over $\mathbb{C}$. We have a homomorphism $\mathbb{C}(Y) \to M_a$ induced by a_N, namely, a rational function $h \in \mathbb{C}(Y)$ is a multivalued meromorphic function on X, and it can be regarded as a germ at a by restricting h to a neighbourhood of $a_N \in Y$. This gives a correspondence $N \mapsto \mathbb{C}(Y) \subset M_a$, such that $\pi_1(X,a)/N \cong G(\mathbb{C}(Y)/\mathbb{C}(X))$, as desired. The converse correspondence is $M/K \mapsto \mathrm{Ker}(\pi_1(X,a) \to G(M/K))$.

We shall again discuss this construction in a later section §3.

2.2. **Arithmetic fundamental groups.** So far, the algebraic fundamental group is obtained from the topological one by profinite completion, and it ignores the algebraic structure altogether. An interesting part comes from the absolute Galois groups of the field over which the algebraic variety is defined.

Let $K \subset \mathbb{C}$ be a subfield, $\bar{K}$ the algebraic closure of K in $\mathbb{C}$. Assume that X is a smooth algebraic variety over K, which is geometrically connected. This can be paraphrased by saying that the defining polynomials of X have coefficients in K, and $X(\mathbb{C})$ is connected as a complex variety.

If X is defined over K, then $K(X)$ denotes the rational function field of X over K. If X is defined by polynomials, $K(X)$ is the set of the functions which can be described as rational expressions of the coordinate variables with coefficients in K.

Theorem 2.2. (c.f. SGA1[7, Chapter XIII p.393]) *If $Y \to X$ is a finite unramified covering of a complex algebraic variety, and X is defined over $\bar{K}$, then there is a model $Y_{\bar{K}} \to X_{\bar{K}}$ of varieties over $\bar{K}$, giving $Y \to X$ by base extension $\otimes_{\bar{K}}\mathbb{C}$. If $\mathbb{C}(Y)/\mathbb{C}(X)$ is Galois, then so is $\bar{K}(Y)/\bar{K}(X)$ and their Galois groups are isomorphic.*

Corollary 2.1. *Let M_a^{alg} be the subfield of M_a of the algebraic elements over $\bar{K}(X)$. Then*

$$G(M_a^{alg}/\bar{K}(X)) = G(M_a/\mathbb{C}(X)).$$

One can show that if X is defined over K, then $M_a^{alg}/K(X)$ is a Galois extension.

Definition 2.4. *If X is a smooth algebraic variety over K, then we define its arithmetic fundamental group*

$$\pi_1^{alg}(X, a) := G(M_a^{alg}/K(X)).$$

Corollary 2.2. *We have a short exact sequence*

$$1 \to \pi_1^{alg}(X \otimes_K \bar{K}, a) \to \pi_1^{alg}(X, a) \to G(\bar{K}/K) \to 1. \tag{2.2}$$

This is the exact sequence coming from Galois extensions $M_a^{alg} \supset \bar{K}(X) \supset K(X)$. Note that $G(\bar{K}(X)/K(X)) = G(\bar{K}/K)$ since $K(X) \otimes_K \bar{K} = \bar{K}(X)$, i.e., since X is geometrically connected.

Now the left term of the short exact sequence (2.2) is the profinite completion of the topological fundamental group $\pi_1(X, a)$, which depends only on the homotopy type of X, while the right term $G(\bar{K}/K)$ is the absolute

Galois group of K, which controls the arithmetic of K and depends only on the field K.

But the middle term, or the extension, highly depends on the structure of X as an algebraic variety, for example if K is a number field. Grothendieck conjectured [8] that X is recoverable from the exact sequence of profinite group, if X is "anabelian." Recently much progress has been done in this direction, by H. Nakamura, F. Pop, A. Tamagawa, and S. Mochizuki, and others. For example, the conjecture is true for curves over p-adic fields with nonabelian fundamental groups. An exposition on these researches in Japanese is available, and its English translation is to appear [25].

2.3. **Arithmetic monodromy.** The exact sequence (2.2) can be considered to be an algebraic version of the "fiber exact sequence" for $X \to \mathrm{Spec}K$.

Let us consider a topological fibration $F \to B$ which is locally trivial. Take a point $b \in B$, and let F_b be the fiber at b. Fix $x \in F_b$. Assume $\pi_2(B) = \{1\}$. Then, we have the so-called homotopy exact sequence

$$1 \to \pi_1(F_b, x) \to \pi_1(F, x) \to \pi_1(B, b) \to 1.$$

In the arithmetic case, $F \to B$ is $X \to \mathrm{Spec}K$, b is $a : \mathrm{Spec}\bar{K} \to \mathrm{Spec}K$, and F_b is $X \times_K \bar{K}$, therefore (2.2) is an analogue of the above. Note that $\pi_1^{alg}(\mathrm{Spec}K, a) = G(\bar{K}/K)$ holds, if we argue in the etale fundamental group, see §3 below.

In topology theory, it is often more convenient to consider the *monodromy* of the fibration $p : F \to B$, as follows. The rough idea is as follows: take a closed path $\gamma \in \pi_1(B, b)$. Then, because of the local triviality of the fibration, we can consider the family $F_c := p^{-1}(c)$ where c moves along γ as a deformation of F_b. Then, γ induces a homeomorphism of F_b to itself. Such a homeomorphism of F_b is not unique, but its isotopy class is well defined. Thus, $\pi_1(B, b)$ acts on various homotopy invariants of F_b, like cohomologies and homotopy groups. Such a representation of $\pi_1(B, b)$ is called monodromy representation associated to $F \to B$. Let us concentrate on the monodromy on the fundamental group of F_b. In this case, we take $x \in F_b$, and take any lift $\gamma \in \pi_1(B, b)$ to F with starting point x, and call it $\tilde{\gamma}$. Since F_b is connected (i.e. $\pi_0(F_b) = \{1\}$), we may adjust $\tilde{\gamma}$ so that it lies in $\pi_1(F, x)$. The ambiguity of $\tilde{\gamma}$ is up to a composition with an element of $\pi_1(F_b, x)$. Take an element $\beta \in \pi_1(F_b, x)$. Then, its deformation along γ can be considered to be $\tilde{\gamma}\beta\tilde{\gamma}^{-1}$. In this way $\pi_1(B, b)$ acts on $\pi_1(F_b, x)$, but the action is well defined up to the choice of $\tilde{\gamma}$, i.e., up to the inner automorphism of $\pi_1(F_b, x)$.

Then we have an outer monodromy representation associated to $F \to B$,

$$\rho : \pi_1(B, b) \to \mathrm{Aut}(\pi_1(F_b, x))/\mathrm{Inn}(\pi_1(F_b, x)) =: \mathrm{Out}(\pi_1(F_b, x)).$$

This is easier to treat than the exact sequence itself, since often the fundamental groups of a fiber and the base are well understood.

An analogue can be considered for the arithmetic version. Take $\sigma \in G(\bar{K}/K)$, then take any lift $\tilde{\sigma} \in \pi_1^{alg}(X, b)$, and let it act by conjugation $\beta \mapsto \tilde{\sigma}(\beta)\tilde{\sigma}^{-1}$ on $\beta \in \pi_1^{alg}(X \otimes \bar{K}, b)$. This gives an arithmetic version of the monodromy representation

$$\rho_X : G(\bar{K}/K) \to \mathrm{Out}(\pi_1^{alg}(X \otimes \bar{K}, b)). \tag{2.3}$$

The left-hand side depends only on K, and the right-hand side depends only on the homotopy type of $X \otimes \mathbb{C}$, and thus ρ_X connects arithmetic and topology. Sometimes ρ_X is called the *outer Galois representation* associated with X.

2.4. **The projective line minus two points.** As a simplest example, we consider the case where $X/\mathbb{Q}$ is an affine line minus one point $\{0\}$, with coordinate function z, and take $a = 1$ as the base point. The topological fundamental group of this space is $\mathbb{Z}$. An element h of M_a^{alg} is finitely multivalued, say, N-valued. Then, h is a single-valued function of w, if we take an N-th root w of z. Since we assumed that h is algebraic over $\mathbb{Q}(X)$, h is a rational function of w with coefficients in $\overline{\mathbb{Q}}$. Thus, M_a^{alg} is generated by the functions $z^{1/N} := \exp(\log z/N)$, with branch fixed so as to have positive real values around 1. Put $\xi_N := \exp(2\pi\sqrt{-1}/N)$. The field extensions

$$M_a^{alg} = \overline{\mathbb{Q}}(z^{1/N} | N \in \mathbb{N}) \supset \overline{\mathbb{Q}}(X) = \overline{\mathbb{Q}}(z) \supset \mathbb{Q}(X) = \mathbb{Q}(z)$$

give

$$\pi_1^{alg}(X \otimes \overline{\mathbb{Q}}, a) = \operatorname*{proj\,lim}_N G(\overline{\mathbb{Q}}(z^{1/N})/\overline{\mathbb{Q}}(z)) = \operatorname*{proj\,lim}_N (\mathbb{Z}/N) =: \widehat{\mathbb{Z}}.$$

Here, the identification $G(\overline{\mathbb{Q}}(z^{1/N})/\overline{\mathbb{Q}}(z)) = \mathbb{Z}/N$ is given by $\beta \in G(\overline{\mathbb{Q}}(z^{1/N})/\overline{\mathbb{Q}}(z)) \mapsto b \in \mathbb{Z}/N$, where b is the unique element such that $\beta(z^{1/N}) \mapsto \xi_N^b z^{1/N}$. Then, the generator γ of $\pi_1(X, a)$ which goes around 0 counterclockwise gives an element of the Galois group which maps

$$\gamma : z^{1/N} \mapsto \exp(2\pi\sqrt{-1}/N) z^{1/N},$$

so γ is, through $G(\overline{\mathbb{Q}}(z^{1/N})/\overline{\mathbb{Q}}(z)) = \mathbb{Z}/N$, identified with $1 \in \widehat{\mathbb{Z}}$. To compute

$$\rho_X : G(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{Out}(\widehat{\mathbb{Z}}) = \mathrm{Aut}(\widehat{\mathbb{Z}}),$$

it suffices to see the action on $1 \in \widehat{\mathbb{Z}}$, or equivalently on $\gamma \in G(\overline{\mathbb{Q}}(z^{1/N})/\overline{\mathbb{Q}}(z))$, since it generates $\widehat{\mathbb{Z}}$ topologically. We take a lift of $\sigma \in G(\overline{\mathbb{Q}}/\mathbb{Q}) = G(\overline{\mathbb{Q}}(z)/\mathbb{Q}(z))$ to $\tilde{\sigma} \in G(\overline{\mathbb{Q}}(z^{1/N}|N \in \mathbb{N})/\mathbb{Q}(z))$. For this, for example, we may take $\tilde{\sigma}$ acting trivially on $z^{1/N}$. Then, $\tilde{\sigma}\gamma\tilde{\sigma}^{-1}$ maps

$$z^{1/N} \overset{\tilde{\sigma}}{\mapsto} z^{1/N} \overset{\gamma}{\mapsto} \exp(2\pi\sqrt{-1}/N)z^{1/N} \overset{\tilde{\sigma}}{\mapsto} \exp(2\pi\sqrt{-1}/N)^{\chi(\sigma)}z^{1/N} = \gamma^{\chi(\sigma)}(z^{1/N}).$$

Here,

$$\chi(\sigma) = (\chi(\sigma)_N) \in \operatorname*{proj\,lim}_N (\mathbb{Z}/N)^\times = \widehat{\mathbb{Z}}^\times$$

is the cyclotomic character $G(\overline{\mathbb{Q}}/\mathbb{Q}) \to \widehat{\mathbb{Z}}^\times$, that is, $\chi(\sigma)_N \in \mathbb{Z}/N$ is uniquely determined by $\sigma : \xi_N \mapsto \xi_N^{\chi(\sigma)_N}$. This shows $\sigma(\gamma) = \gamma^{\chi(\sigma)}$, so ρ_X is nothing but the cyclotomic character

$$\chi : G(\overline{\mathbb{Q}}/\mathbb{Q}) \to \widehat{\mathbb{Z}}^\times = \mathrm{Aut}(\widehat{\mathbb{Z}}).$$

2.5. **The projective line minus three points.** A basic result by Belyi [3] says

Theorem 2.3. *Let X be the projective line minus three points $0, 1, \infty$ over $\mathbb{Q}$, and take an $a \in X$. Then,*

$$\rho_X : G(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{Out}(\pi_1(\widehat{X, a}))$$

is injective.

The right-hand side $\pi_1(\widehat{X, a})$ is the profinite completion of the free group F_2 with two generators, say, x, y. This implies that $G(\overline{\mathbb{Q}}/\mathbb{Q})$ is contained in a purely group-theoretic object. One can fix a lift

$$\rho_X : G(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{Aut}(\widehat{F}_2)$$

from $\mathrm{Out}(\widehat{F}_2)$, then two elements $\rho_X(\sigma)(x)$, $\rho_X(\sigma)(y)$ in $\widehat{F}_2$ characterise σ. It is convenient to fix the lift so that $\rho_X(\sigma) : x \mapsto x^{\chi(\sigma)}, y \mapsto f_\sigma(x, y)y^{\chi(\sigma)}f_\sigma(x, y)^{-1}$ with $f_\sigma(x, y) \in [\widehat{F}_2, \widehat{F}_2]$. This is possible in a unique way [3] [12].

There are roughly two directions of research:

(i) Characterize the image of ρ_X in a group-theoretic way.
(ii) Use $\chi(\sigma)$, $f_\sigma(x, y)$ to describe $\rho_Y(\sigma)$ for other varieties Y.

For (i), the Grothendieck-Teichmüller group was introduced by Drinfeld [6], and its profinite version $\widehat{\mathrm{GT}}$ was given by Ihara [12] [13]. $\widehat{\mathrm{GT}}$ is the subgroup of $\mathrm{Aut}(\widehat{F}_2)$ given by three conditions [12], with $G(\overline{\mathbb{Q}}/\mathbb{Q}) \hookrightarrow \widehat{\mathrm{GT}} \subset \mathrm{Aut}(\widehat{F}_2)$ being Belyi's injection.

It is known that $\widehat{\mathrm{GT}}$ acts on the profinite completion of the braid groups $\widehat{B_n}$ ([6], for the profinite case, [15, Appendix]). $G(\overline{\mathbb{Q}}/\mathbb{Q}) \to \widehat{\mathrm{GT}} \to \mathrm{Aut}(\widehat{B_n})$ is

known to come from the representation on the algebraic fundamental group of the configuration space of n points on the affine line [15].

Recently, Hatcher, Lochak, Nakamura, Schneps [10] [24] gave a subgroup Γ of $\widehat{\mathrm{GT}}$, $G(\overline{\mathbb{Q}}/\mathbb{Q}) \subset \Gamma \subset \widehat{\mathrm{GT}}$ (actually there are several variations of Γ, but whether $\Gamma = \widehat{\mathrm{GT}}$ or not is still open), so that Γ acts on the profinite completion of mapping class groups of type (g, n) systematically.

These researches are closely related to the program by Grothendieck [8], which tries to study $G(\overline{\mathbb{Q}}/\mathbb{Q})$ by towers of moduli spaces, but here we don't pursue this direction.

These studies go with (ii): first the Galois action is described in terms of $\chi(\sigma)$ and $f_\sigma(x, y)$, then $\widehat{\mathrm{GT}}$-action is described so that it generalizes the Galois action.

It is still open whether $G(\overline{\mathbb{Q}}/\mathbb{Q}) = \widehat{\mathrm{GT}}$ or not.

3. Arithmetic fundamental groups by etale topology

The definition of arithmetic fundamental groups in the previous section is concrete, but not intrinsic. For example, it cannot be applied for the positive characteristic case. A more sophisticated general definition uses the notion of Galois category [7, Chapter V].

3.1. Unramified coverings of a topological space and fiber functors.

In this section, we forget about algebraic varieties and consider only topological spaces. Though our aim is the theory of fundamental groups *without paths*, this section illustrates how the categorical machinary works.

Let X be an arcwise connected topological space. Let $p : Y \to X$ be an *unramified covering* of X. That is, $p : Y \to X$ is a surjective continuous map such that for any $x \in X$ there exists an open neighbourhood U of x with each connected component of $p^{-1}(U)$ being homeomorphic to U via p. It is a *finite* unramified covering if the number of the connected components is finite. The map p is called a covering map.

It is well known that the *connected* unramified coverings of X and the subgroups of $\pi_1(X, a)$ are in one-to-one correspondence, but we shall start by recalling this fact.

Fix a point a on X, and consider the category of connected unramified coverings with one point specified. Its objects are the pairs of an unramified connected covering $p_Y : Y \to X$ and a point $b \in p_Y^{-1}(a)$, and its morphisms are the continuous maps $f : Y' \to Y$ compatible with the covering maps: $p_Y \circ f = p_{Y'}$ and $f(b') = b$. We denote this category by $Con(X, a)$.

Assume that X is arcwise connected, locally arcwise connected, and locally simply connected. Then there is a category equivalence between $Con(X,a)$ and $\{N : \text{subgroup of } \pi_1(X,a)\}$. The correspondence is: for $(Y,b) \in Con(X,a)$, we have $\pi_1(Y,b) \to \pi_1(X,a)$, which is injective, so $\pi_1(Y,b)$ considered as a subgroup corresponds to (Y,b). For the converse, for $N \subset \pi_1(X,a)$, we consider the set of classes of paths

$$Y_N := \{\text{a path starting from } a \text{ with arbitrary end point in } X\}/\sim_N,$$

where the equivalence is given by $\gamma \sim_N \gamma'$ if and only if they have the same end point and the closed path $\gamma^{-1} \circ \gamma'$ lies in the homotopy class in $N \subset \pi_1(X,a)$. Y_N has a specified point a_N, which corresponds to the trivial path at a. Taking the end point of the path gives a map $p_N : Y_N \to X$, which is locally a homeomorphism by the assumption on locally arcwise simply connectedness. Thus, $(Y_N, a_N) \to (X,a)$ is an object of $Con(X,a)$.

If N is a normal subgroup of $\pi_1(X,a)$, then $p_N : Y_N \to X$ is called a Galois covering. In this case, $\pi_1(X,a)$ acts on Y_N/X from the right. Take $\alpha \in \pi_1(X,a)$, and let α act by $[\gamma] \in Y_N \mapsto [\gamma \circ \alpha] \in Y_N$. We need to check the well-definedness; if $\gamma \sim_N \gamma'$ then $\gamma\alpha \sim_N \gamma'\alpha$ should hold, which requires that, if $\gamma^{-1}\gamma' \in N$, then $\alpha^{-1}\gamma^{-1}\gamma'\alpha \in N$, that is, N must be normal. In this case, we have

$$\pi_1(X,a)/N \cong \mathrm{Aut}(Y_N/X)^o,$$

where ^{o} denotes the opposite, i.e., the group obtained by reversing the order of multiplication.

If N is $\{1\}$, then $Y_{\{1\}}$ is called the universal covering of X, and denoted by $\tilde{X}$. Then, we have

$$\pi_1(X,a) \cong \mathrm{Aut}(\tilde{X}/X)^o.$$

Thus, if we can define a universal covering *without using paths*, then we can define the fundamental group. Essentially this is the case for schemes, a projective system (or a pro-object) plays the role of the universal covering. However, if we adopt $\mathrm{Aut}(\tilde{X}/X)$ as the definition of the fundamental group, it is not clear where the dependence on the choice of a disappeared. The functoriality of π_1 is also not clear, and the definition of fundamental groupoid is difficult.

A smart idea is to use all (possibly non connected) coverings of X.

We consider the category $\mathcal{C}_X$ of unramified coverings of X, i.e., an object is a covering $f : Y \to X$, and a morphism is $Y' \to Y$ compatible with f', f. Thus, differently from $Con(X,a)$, a point is not specified, and the covering

may not be connected. Let $a, b \in X$ be points. The discrete set $f^{-1}(a) \subset Y$ is called the fiber of f at a. Take an element $a' \in f^{-1}(a)$. By lifting of the path $\gamma \in \pi_1(X; a, b)$ to a path starting from a' in Y we have $\tilde{\gamma} \in \pi_1(Y; a', b')$, and the end point b' of $\tilde{\gamma}$ is uniquely determined. Thus, we obtain a bijection

$$c_f(\gamma) : f^{-1}(a) \to f^{-1}(b), a' \mapsto b',$$

in other words, an action of groupoid

$$c_f : \pi_1(a, b) \times f^{-1}(a) \to f^{-1}(b)$$

with $(\gamma, a') \mapsto c_f(\gamma)(a') = b'$. $c_f(\gamma)$ is a bijection from $f^{-1}(a)$ to $f^{-1}(b)$, which is compatible with morphisms $Y' \to Y$. The assignment $(f : Y \to X) \mapsto f^{-1}(a)$ gives a covariant functor

$$\mathcal{C}_X \to Sets$$

from the category of unramified coverings to the category of sets, which we denote by F_a. Each element of $\pi_1(X; a, b)$ gives a natural transformation $F_a \mapsto F_b$, which is a natural equivalence.

On the other hand, a natural transformation $F_a \mapsto F_b$ always comes from some element of $\pi_1(X; a, b)$. To see this, note that $\mathcal{C}_X$ has a special object, namely, the universal covering $p : \tilde{X} \to X$. Actually, the object $\tilde{X}$ represents the fiber functor F_a, i.e., we have a natural isomorphism $F_a(Y) \cong \mathrm{Hom}_{\mathcal{C}_X}(\tilde{X}, Y)$. To fix the isomorphism, it is enough to fix a point $\tilde{a} \in \tilde{X}$ above a. Then, for every point $b \in f^{-1}(a)$, there exists a unique homomorphism $\tilde{X} \to Y$ with $\tilde{a} \mapsto b$, by the property of the universal covering.

We shall see that a natural transformation $F_a \to F_b$ comes from an element of $\pi_1(X; a, b)$. A natural transformation $\gamma : F_a \to F_b$ gives a map $\gamma(\tilde{X}) : F_a(\tilde{X}) = p^{-1}(a) \to F_b(\tilde{X}) = p^{-1}(b)$. Then, once we choose a $\tilde{a} \in p^{-1}(a)$, we have $\gamma(\tilde{X})(\tilde{a}) \in p^{-1}(b)$, which we denote $\tilde{b}$. Now, by the naturality of γ, it commutes with any automorphism of $\tilde{X}/X$. Since $\tilde{X}/X$ is a Galois cover, for any $\tilde{a}' \in p^{-1}(a)$, there is an automorphism $\tau : \tilde{X} \to \tilde{X}$ such that $\tau(\tilde{a}) = \tilde{a}'$. This shows that the image of $\tilde{a}'$ by $\gamma(\tilde{X})$ must be $\tau(\tilde{b})$. Thus, $\gamma(\tilde{X}) : p^{-1}(a) \to p^{-1}(b)$ is bijective, and it is uniquely determined, once $\tilde{b}$ is chosen, by compatibility with τ. Let $\tilde{\gamma}$ be a path from $\tilde{a}$ to $\tilde{b}$ in $\tilde{X}$. $\tau(\tilde{\gamma})$ for $\tau \in \mathrm{Aut}(\tilde{X}/X)$ gives the bijection $F_a(\tilde{X}) \to F_b(\tilde{X})$.

Since connected components of other coverings are quotients of the universal covering, it is easy to see that any choice of $\tilde{b}$ uniquely gives a natural transformation $F_a \to F_b$, and it comes from the path from a to b which is the projection of $\tilde{\gamma}$ to X. Thus, we have $\mathrm{Hom}(F_a, F_b) \cong \pi_1(X; a, b)$ canonically.

This identifies the groupoid $\{\pi_1(X;a,b)|a,b \in X\}$ with the groupoid whose objects are $F_a : \mathcal{C}_X \to Sets$, and the morphisms are the natural transformations $F_a \to F_b$, which are automatically invertible.

Since F_a is representable by $\tilde{X}$, we can identify $\pi_1(X,a) = Aut(F_a)$ with the opposite group of $\mathrm{Aut}(\tilde{X}/X)$, because of the Yoneda lemma

$$\mathrm{Aut}(F_a) \cong \mathrm{Aut}(\mathrm{Hom}_{\mathcal{C}}(\tilde{X},-)) = \mathrm{Aut}(\tilde{X})^o,$$

where $\cong$ is determined once $\tilde{a}$ is specified.

Now, $F_a : \mathcal{C}_X \to Sets$ induces a functor from $\mathcal{C}_X$ to the category $\pi_1(X,a)$-set, whose objects are the sets with an action of $\pi_1(X,a) = \mathrm{Aut}(F_a)$, and whose morphisms are the maps compatible with this action. This functor gives a categorical equivalence. To establish the equivalence, let S be a $\pi_1(X,a)$-set. We decompose S to orbits, and construct a covering corresponding to each orbit, then take the direct sum. For an orbit, take one point and let H be the stabilizer of the point. The covering corresponding to H gives the desired covering for that orbit.

Proposition 3.1. *The category $\mathcal{C}_X$ of unramified coverings of an arcwise and locally arcwise simply connected topological space X is categorically equivalent to $\pi_1(X,a)$-set. The equivalence is given by the fiber functor*

$$F_a : (f : Y \to X) \mapsto f^{-1}(a).$$

The $\pi_1(X,a)$ action on $f^{-1}(a)$ comes from $\pi_1(X,a) = \mathrm{Aut}(F_a)$.

3.2. **Finite coverings.** Roughly speaking, the etale fundamental groupoid of a connected scheme is defined by using the category of unramified covers in an algebraic sense. We don't have an analogue of real-one dimensional "path", say, in the positive characteristic world, but we have a good category and functors which allow us a categorical formulation of the algebraic (or etale) fundamental groupoid.

Before proceeding to etale fundamental groups, we note what will occur if we restrict $\mathcal{C}_X$ to the category $\mathcal{C}_{X,fin}$ of finite unramified coverings and the fiber functor

$$F_a : \mathcal{C}_{X,fin} \to Finsets,$$

where $Finsets$ is the category of finite sets. This modification is essential when we work in the category of algebraic varieties.

A problem is that this functor F_a is not representable by an object in $\mathcal{C}_{X,fin}$. So, instead of the universal covering, we use a projective system, called a *pro-object*, which represents the functor F_a. Let $(P_\lambda)_{(\lambda\in\Lambda)}$ be the system of all *connected* finite Galois coverings of X with one point above a

specified. That is, an object of P_λ is a pair (a_Y, Y) with a connected finite unramified covering $p_Y : Y \to X$ and $a_Y \mapsto a$, and a morphism is $Y \mapsto Y'$ which maps a_Y to a'_Y. We consider (P_λ) as a projective system in $\mathcal{C}_{X,fin}$ (thus, a_Y for each Y gives no restriction on morphisms in $\mathcal{C}_{X,fin}$).

It can be shown that

$$\mathrm{Hom}_{pro-\mathcal{C}_{X,fin}}((P_\lambda), Z) := \lim_{\to\lambda} \mathrm{Hom}_{\mathcal{C}_{X,fin}}(P_\lambda, Z) \cong p_Z^{-1}(a).$$

Then, one can show

$$\mathrm{Aut}(F_a) \overset{opposite}{=} \mathrm{Aut}((P_\lambda)) = \mathrm{proj}\lim_{\lambda} \mathrm{Aut}(P_\lambda) \overset{opposite}{=} \pi_1(\widehat{X, a}).$$

The first identity comes from the Yoneda lemma, and the last equality is because $\mathrm{Aut}(P_\lambda)^o$ is the finite quotient of $\pi_1(X, a)$ correspondinng to P_λ. Both equalities are fixed because a system (a_λ) is fixed. In this case, the category $\mathcal{C}_{X,fin}$ is equivalent to the category of finite sets with continuous action by $\pi_1(\widehat{X, a})$, which we call the category of finite $\pi_1(\widehat{X, a})$-sets.

Proposition 3.2. *$\mathcal{C}_{X,fin}$ is categorically equivalent to the category of finite $\pi_1(\widehat{X, a})$-sets. The equivalence is given by the fiber functor F_a.*

3.3. **Etale fundamental groups.** In the following, we work in the category of schemes. We shall only sketch the story of the etale fundamental groups here. For the precise notions, see SGA1 [7].

Definition 3.1. *Let $f : X \to Y$ be a morphism of finite type, $x \in X$, $y := f(x) \in Y$. We say f is unramified at x, if $\mathcal{O}_{X,x}/f(m_y)\mathcal{O}_{X,x}$ is a finite direct sum of finite separable field extensions of $k(y)$. If moreover f is flat at x, then f is said to be etale at x. If f is etale at every point $x \in X$, then f is called etale. If moreover f is finite and Y is connected, then $f : X \to Y$ is called an etale covering.*

If X and Y are algebraic varieties over an algebraically closed field $K \subset \mathbb{C}$, and if x is a closed point, then it is known that f is etale at $x \in X$ if and only if f is finite and unramified as an analytic morphism [7, Chapter XII]. Thus, the etale morphisms correctly generalize the notion of unramified coverings.

Definition 3.2. *Let X be a locally noetherian connected scheme. Let C_X be the category of etale coverings of X, i.e., objects are finite etale $f : Y \to X$, and morphisms are $Y' \to Y$ compatible with f', f.*

This category is an analogue of that of finite unramified coverings of a connected topological space X.

There is a notion of Galois category. It consists of a category C and a functor called the fiber functor, $F : C \to Finsets$, and satisfies six axioms stated in [7, Chapter V-§4], which we shall omit here. Once we have a Galois category, we can define its *fundamental group* with base point F as $\mathrm{Aut}(F)$, i.e., the group of natural transformation from F to itself. This becomes a profinite group. Two fiber functors $F, G : C \to Finsets$ are non-canonically isomorphic, and the set of natural transformations from F to G is a groupoid, with objects fiber functors and morphisms natural transformations.

Similarly to the topological case, one can show the category equivalence between C_X and the category of the finite sets with $\mathrm{Aut}(F)$-continuous action.

Theorem 3.1. [7, Chapter V] *Let X be a locally noetherian connected scheme. Take a geometric point $a : \mathrm{Spec}\Omega \to X$, where Ω is an algebraically closed field. Then, the category C_X of finite etale covers of X, with fiber functor $F_a : C_X \to Finset$, $(f : Y \to X) \mapsto f^{-1}(a) = Y \times_X \mathrm{Spec}\Omega$, is a Galois category.*

Definition 3.3.

$$\pi_1^{alg}(X, a) := \mathrm{Aut}(F_a).$$

Thus, the category of finite etale covers of X is equivalent to the cagegory of finite sets with $\pi_1^{alg}(X, a)$-action. Similarly to the topological case, we may use (P_λ), the projective system of *connected* finite Galois cover of X, with a geometric point a_λ above a compatibly specified. Then, by forgetting a_λ, we may regard (P_λ) as a projective system in C_X, which pro-represents F_a. Then, we have

$$\pi_1^{alg}(X, a) := \mathrm{Aut}(F_a) = \operatorname*{proj\,lim}_{\lambda}(P_\lambda).$$

In this setting, the functoriality of π_1^{alg} is easy. For $f : X \to Y$, we have $\pi_1^{alg}(X, a) \to \pi_1^{alg}(Y, f(a))$, since $(-) \times_Y X$ is a functor $f^* : C_Y \to C_X$, and $F_a \circ f^* = F_{f(a)} : C_Y \to Finsets$ holds, so an element of $\mathrm{Aut}(F_a)$ gives an element of $\mathrm{Aut}(F_{f(a)})$, inducing

$$\mathrm{Aut}(F_a) = \pi_1^{alg}(X, a) \to \mathrm{Aut}(F_{f(a)}) = \pi_1^{alg}(Y, f(a)).$$

Let X be a geometrically connected scheme over a field K. Let $a : \mathrm{Spec}\bar{K} \to X$ be a geometric point. The sequence

$$X \otimes \bar{K} \to X \to \mathrm{Spec}K$$

gives a short exact sequence

$$1 \to \pi_1^{alg}(X \otimes_K \bar{K}, a) \to \pi_1^{alg}(X, a) \to \pi_1^{alg}(\mathrm{Spec}K, \mathrm{Spec}\bar{K}) \to 1,$$

which is nothing but (2.2) (for a proof, see [7, Chapter X, XIII]).

It is easy to show that an object of $C_{\mathrm{Spec}K}$ is a direct sum of a finite number of finite separable extensions of K, and morphisms are usual homomorphisms of algebras over K. A connected object is the spectrum of a field. Once we fix a geometric point $a : \mathrm{Spec}\Omega \to \mathrm{Spec}K$, the pro-object which represents the fiber functor F_a is the system of finite Galois extensions of K inside Ω (this inclusion into Ω corresponds to choosing a point a_N in the fiber $Y_N \to X$ above a). Thus, we have

$$\pi_1^{alg}(\mathrm{Spec}K, \mathrm{Spec}\Omega) \overset{contra}{=} \operatorname*{proj\,lim}_L \mathrm{Aut}(\mathrm{Spec}L/\mathrm{Spec}K) \overset{contra}{=} \operatorname*{proj\,lim}_L G(L/K),$$

where L runs through the finite Galois extensions of K in Ω, and it is nothing but $G(K^{sep}/K)$ where $K^{sep} \subset \Omega$ is the separable closure of K in Ω.

Proposition 3.3. *Let Ω be an algebraically closed field, and let $K \subset \Omega$ be a subfield. Then*

$$\pi_1(\mathrm{Spec}K, \mathrm{Spec}\Omega) = G(K^{sep}/K)$$

holds, where K^{sep} is the separable closure of K in Ω.

The geometric part of the fundamental group can be obtained as follows. A theorem called "Riemann's existence theorem" in SGA1[7, Theorem 5.1 P.332] assures that a finite unramified covering of an algebraic variety X over $\mathbb{C}$ is algebraic and etale over X, i.e., C_X and $\mathcal{C}_{X,fin}$ are categorically equivalent. An argument in SGA1[7, Chapter XIII] says that if X is an algebraic variety over an algebraically closed field $\bar{K} \subset \mathbb{C}$, then the base change $(-)\otimes_{\bar{K}}\mathbb{C}$ gives a category equivalence between $C_{X\otimes\mathbb{C}}$ and C_X. These category equivalences are compatible with fiber functors, so we have

$$\pi_1^{alg}(X, a) = \pi_1^{alg}(X \otimes \mathbb{C}, a) = \mathrm{Aut}_{\mathcal{C}_{X,fin}}(F_a) = \pi_1(\widehat{X, a}).$$

Exercise 3.1. *Show that the universal covering of the complex plane $\mathbb{C}$ minus 0 is still an algebraic variety, but that of $\mathbb{C}$ minus 0 and 1 is not. Even in the former case, the covering map is not algebraic.*

Exercise 3.2. *Describe the category C_X of finite etale coverings where X is the affine line minus one point 0 over $\mathbb{Q}$.*

4. Arithmetic mapping class groups

4.1. **The algebraic stack $M_{g,n}$ over Spec$\mathbb{Z}$.** I do not give the definition of an algebraic stack, the definition of the fundamental group of an algebraic stack, and so on, in this note, simply because of my lack of ability to state it concisely. I would just like to refer to [5] for the definition of an algebraic stack, the moduli stack of genus g curves and the moduli stack of stable curves and to [19] for the case of n pointed genus g curves. For the arithmetic fundamental group of the moduli stack, see [26] (but this article requires prerequisites on etale homotopy [1]).

We just sketch the picture. Let g, n be integers with $2g-2+n>0$. We want to introduce the universal property of the moduli stack of n pointed genus g curves.

Definition 4.1. *A family of n pointed genus g curves over a scheme S (a family of (g,n)-curves in short), $C \to S$, is a proper smooth morphism $C^* \to S$, whose fibers are a proper smooth curves of genus g, with n sections $s_1, s_2, \dots, s_n : S \to C^*$ given, where the images of the s_i do not intersect each other, and $C \to S$ is the complement of the image of these sections in C^*.*

What we want is the universal family $\mathcal{C}_{g,n} \to \mathcal{M}_{g,n}$, which itself is a family of (g,n)-curves, with the universal property that for any family of (g,n)-curves $C \to S$, we have a unique morphism $S \to \mathcal{M}_{g,n}$ such that C is isomorphic to the base change $\mathcal{C}_{g,n} \times_{\mathcal{M}_{g,n}} S$. Unfortunately, we don't have such a universal family in the category of schemes. So, we need to enlarge the category to that of algebraic stacks.

I just describe some properties of algebraic stacks here. The category of algebraic stacks contains the category of schemes as a full subcategory, and algebraic stacks behave similarly to schemes. In the category of algebraic stacks, we have the correct universal family $\mathcal{C}_{g,n} \to \mathcal{M}_{g,n}$.

The notion of finite morphisms, etale morphisms, connectedness, etc. can be defined for algebraic stacks. In particular, for a connected algebraic stack, we have the category of its finite etale covers. It becomes a Galois category, and we have its etale fundamental group.

The algebraic stack $\mathcal{M}_{g,n}$ is defined over Spec$\mathbb{Z}$. But from now on, we consider $\mathcal{M}_{g,n}$ over Spec$\mathbb{Q}$.

4.2. **The arithmetic fundamental group of the moduli stack.** Takayuki Oda [26] showed that the etale homotopy type of the algebraic stack $\mathcal{M}_{g,n} \otimes \overline{\mathbb{Q}}$

is the same as that of the analytic stack $\mathcal{M}_{g,n}^{an}$, and using Teichmüller space, showed that the latter object has the etale homotopy type $K(\widehat{\Gamma_{g,n}, 1})$ in the sense of Artin-Mazur [1], where $\Gamma_{g,n}$ is the Teichmüller-modular group or the mapping class group of n-punctured genus g Riemann surfaces.

This shows as a corollary

$$\pi_1^{alg}(\mathcal{M}_{g,n} \times \overline{\mathbb{Q}}, a) \cong \widehat{\Gamma}_{g,n},$$

and gives a short exact sequence

$$1 \to \pi_1^{alg}(\mathcal{M}_{g,n} \times \overline{\mathbb{Q}}, a) \to \pi_1^{alg}(\mathcal{M}_{g,n}, a) \to G(\overline{\mathbb{Q}}/\mathbb{Q}) \to 1. \tag{4.1}$$

Also, the vanishing of π_2 of $\mathcal{M}_{g,n}$ gives a short exact sequence

$$1 \to \pi_1^{alg}(C_{g,n}, b) \to \pi_1^{alg}(\mathcal{C}_{g,n}, b) \to \pi_1^{alg}(\mathcal{M}_{g,n}, a) \to 1, \tag{4.2}$$

where a is a geometric point of $\mathcal{M}_{g,n}$, $C_{g,n}$ is the fiber on a, b a geometric point of $C_{g,n}$. Hence, $C_{g,n}$ is a (g,n)-curve over an algebraically closed field, and $\pi_1^{alg}(C_{g,n}, b)$ is isomorphic to the profinite completion of the orientable surface of (g,n)-type, i.e., the profinite completion of

$$\Pi_{g,n} :=< \alpha_1, \beta_1, \dots, \alpha_g, \beta_g, \gamma_1, \gamma_2, \dots, \gamma_n;$$

$$[\alpha_1, \beta_1][\alpha_2, \beta_2] \cdots [\alpha_g, \beta_g]\gamma_1 \cdots \gamma_n = 1 >, \tag{4.3}$$

where γ_i are paths around the punctures, α_i, β_i are usual generators of π_1 of an orientable surface.

Once we are given a short exact sequence (4.2), in the same way as §2.3, we have the monodromy representation

$$\rho_{g,n} : \pi_1^{alg}(\mathcal{M}_{g,n}, a) \to \mathrm{Out}(\pi_1^{alg}(C_{g,n}, b)) \cong \mathrm{Out}(\widehat{\Pi}_{g,n}),$$

which is called *arithmetic universal monodromy representation.* This contains the usual representation of the mapping class group $\Gamma_{g,n}$ in the fundamental group of the orientable surface $\Pi_{g,n}$, since the restriction of $\rho_{g,n}$ to

$$\pi_1^{alg}(\mathcal{M}_{g,n} \otimes \overline{\mathbb{Q}}, b) \subset \pi_1^{alg}(\mathcal{M}_{g,n}, b)$$

coincides with

$$\widehat{\Gamma}_{g,n} \to \mathrm{Out}(\widehat{\Pi_{g,n}}),$$

which comes from the natural homomorphism

$$\Gamma_{g,n} \to \mathrm{Out}(\Pi_{g,n}).$$

This latter may be called the *topological universal monodromy.* What do we get if we consider the arithmetic universal monodromy instead of the

topological one? There is an interesting phenomenon: "arithmetic action gives an obstruction to topological action."

5. A conjecture of Takayuki Oda

5.1. **Weight filtration on the fundamental group.** Let C be a (g,n)-curve, and $\Pi_{g,n}$ be its (classical) fundamental group. We define its weight filtration as follows.

Definition 5.1. (*Weight filtration on* $\Pi_{g,n}$.)
We define a filtration on $\Pi_{g,n}$

$$\Pi_{g,n} = W_{-1}\Pi_{g,n} \supset W_{-2}\Pi_{g,n} \supset W_{-3}\Pi_{g,n} \supset \cdots$$

By $W_{-1} := \Pi_{g,n}$,

$$W_{-2} :=< [\Pi_{g,n}, \Pi_{g,n}], \gamma_1, \gamma_2, \ldots, \gamma_n >_{norm},$$

where $<>_{norm}$ *denotes the normal subgroup generated by elements inside* $<>$ *and* $[,]$ *denotes the commutator product,* γ_i *are elements in the presentation (4.3), and then*

$$W_{-N} :=< [W_{-i}, W_{-j}] | i+j = N >_{norm}$$

inductively for $N \geq 3$.

Fix a prime l. *We define a similar filtration on the pro-*l *completion of* $\Pi^l_{g,n}$. *There,* $<>_{norm}$ *and* $[,]$ *are the topological closure of the normal subgroup generated by the elements inside* $<>$, *the commutators, respectively.*

It is easy to check that $gr_j(\Pi_{g,n}) := W_{-j}/W_{-j-1}$ is abelian, and is central in $\Pi_{g,n}/W_{-j-1}$. In other words, $W_\bullet$ is the fastest decreasing central filtration with W_{-2} containing $\gamma_1, \ldots, \gamma_n$. It is known that each gr_j is a free $\mathbb{Z}$-module (free $\mathbb{Z}_l$-module, respectively for pro-l case) of finite rank [2] [18].

This notion of weight filtration came from the study of the mixed Hodge structure on the fundamental groups, by Morgan and Hain [9], but for the particular case of $\mathbb{P}^1 - \{0, 1, \infty\}$, Ihara had worked on this [11] independently, from an arithmetic motivation.

For $x \in W_{-i}, y \in W_{-j}$, $[x,y] \in W_{-i-j}$ holds, and this defines a $\mathbb{Z}$-bilinear product $\mathrm{gr}_{-i} \otimes \mathrm{gr}_{-j} \to gr_{-i-j}$. We define

$$Gr\Pi_{g,n} := \oplus_{i=1}^{\infty} gr_{-i}(\Pi_{g,n}).$$

With the product $[x,y]$, $Gr\Pi_{g,n}$ becomes a Lie algebra over $\mathbb{Z}$. For $\Pi^l_{g,n}$, we have a Lie algebra over $\mathbb{Z}_l$.

Definition 5.2. (Induced filtration) *We equip* $\Gamma := \mathrm{Aut}(\Pi_{g,n})$*, with the following filtration* $\Gamma = I_0 \supset I_{-1} \supset I_{-2} \supset \cdots$*, called induced filtration:*

$\gamma \in I_{-j} \Leftrightarrow$ *for any* $k \in \mathbb{N}$ *and* $x \in W_{-k}(\Pi_{g,n})$*,* $\tilde{\gamma}(x)x^{-1} \in W_{-k-j}(\Pi_{g,n})$ *holds.*

We pushout this filtration to $\mathrm{Out}(\Pi_{g,n})$*. For an outer representation*

$$\rho : G \to \mathrm{Out}(\Pi_{g,n})$$

of any group G*, we pullback the filtration to* G*, and call it induced filtration:*

$$G = I_{-0}(G) \supset I_{-1}(G) \supset I_{-2}(G) \supset \cdots .$$

The same kind of filtration is defined for $G \to \mathrm{Out}(\Pi^l_{g,n})$*.*

In this case, we define

$$Gr(G) := \oplus_{i=1}^{\infty} gr_{-i}(G) = \oplus_{i=1}^{\infty} I_{-i}(G)/I_{-i-1}(G)$$

(note that i starts from 1, not 0), then $Gr(G)$ becomes a Lie algebra. By definition, if we induce filtrations by $G \to G' \to \mathrm{Out}(\Pi_{g,n})$, then $Gr(G) \hookrightarrow Gr(G')$. By [18] [2], $Gr\mathrm{Out}(\Pi_{g,n})$ injects to $Gr\mathrm{Out}(\Pi^l_{g,n})$, and hence if

$$G \to \mathrm{Out}(\Pi_{g,n}) \to \mathrm{Out}(\Pi^l_{g,n})$$

factors through $G' \to \mathrm{Out}(\Pi^l_{g,n})$, then $GrG \hookrightarrow GrG'$ holds.

The natural homomorphism

$$\Gamma_{g,n} \to \mathrm{Out}(\Pi_{g,n})$$

gives a natural filtration to $\Gamma_{g,n}$, which seems to go back to D. Johnson [17].

By composing with the natural morphism

$$\mathrm{Out}(\widehat{\Pi_{g,n}}) \to \mathrm{Out}(\Pi^l_{g,n}),$$

we have

$$\pi_1^{alg}(\mathcal{M}_{g,n}, a) \to \mathrm{Out}(\Pi^l_{g,n}),$$

hence $\pi_1^{alg}(\mathcal{M}_{g,n}, a)$, $\pi_1^{alg}(\mathcal{M}_{g,n} \otimes \overline{\mathbb{Q}}, a)$, is equipped with an induced filtration, and we have a natural injection

$$Gr\pi_1^{alg}(\mathcal{M}_{g,n}, a) \hookrightarrow Gr\pi_1^{alg}(\mathcal{M}_{g,n} \otimes \overline{\mathbb{Q}}, a),$$

and the image is a Lie algebra ideal.

Conjecture 5.1. *(Conjectured by Takayuki Oda) The quotient of*

$$Gr(\pi_1^{alg}(\mathcal{M}_{g,n}, a))$$

by the ideal

$$Gr(\pi_1^{alg}(\mathcal{M}_{g,n} \otimes \overline{\mathbb{Q}}, a))$$

is independent of g, n *for* $2g - 2 + n \geq 0$*.*

This conjecture is almost proved by a collection of works by Nakamura, Ihara, Takao, myself, et. al. [21] [23] [16],

Theorem 5.1. *The quotient of*

$$Gr(\pi_1^{alg}(\mathcal{M}_{g,n}, a)) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l$$

by

$$Gr(\pi_1^{alg}(\mathcal{M}_{g,n} \otimes \overline{\mathbb{Q}}, a)) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l$$

is independent of g, n *for* $2g - 2 + n \geq 0$.

The significance of this result is that for $\mathcal{M}_{0,3} = \mathrm{Spec}\mathbb{Q}$, the Lie algebra is understood to some extent by deep results such as Anderson-Coleman-Ihara's power series and Soulé's non-vanishing of Galois cohomology, and it implies a purely topological consequence: an obstruction to the surjectivity of the Johnson homomorphisms.

5.2. **Obstruction to the surjectivity of Johnson morphisms.** For simplicity, assume $n = 0$, and hence Γ_g denotes the mapping class group of genus g Riemann surfaces. Take $\sigma \in I_{-m}\Gamma_g$, and take a suitable lift $\tilde{\sigma} \in I_{-m}\mathrm{Aut}(\Pi_g)$ as in Definition 5.2. Then, $\tilde{\sigma}(\alpha)\alpha^{-1} \in W_{-m-1}\Pi_g$ for any $\alpha \in \Pi_g$. The map

$$\Pi_g \to W_{-m-1}\Pi_g; \quad \alpha \mapsto \tilde{\sigma}(\alpha)\alpha^{-1}$$

gives a linear map $\Pi_g/W_{-1}\Pi_g \to gr_{-m-1}\Pi_g$. We denote $H := \Pi_g/W_{-1}\Pi_g$ for homology, then we have Poincare duality $H^* \cong H$, and define

$$h_{g,*}(m) := \mathrm{Ker}(\mathrm{Hom}(H, gr_{-m-1}\Pi_g) \to \mathrm{gr}_{-m-2}\pi_g),$$

where

$$\mathrm{Hom}(H, gr_{-m-1}\Pi_g) \to \mathrm{gr}_{-m-2}\pi_g$$

comes from

$$\mathrm{Hom}(H, gr_{-m-1}\Pi_g) \cong H \otimes gr_{-m-1}\Pi_g \xrightarrow{[,]} gr_{-m-2}\Pi_g.$$

The lift $\tilde{\sigma}$ in $\mathrm{Aut}(\Pi_g)$ is mapped into $h_{g,*}(m)$. The ambiguity of taking the lift in Aut is absorbed by taking the quotient by the action of $gr_{-m}\Pi_g$ by $\alpha \mapsto [\alpha, x]$ for $x \in gr_{-m}\Pi_g$, and we have an injective morphism

$$gr_{-m}(\Gamma_g) \hookrightarrow h_{g,*}/gr_{-m}\Pi_g \quad (\subset \mathrm{Hom}(H, gr_{-m-1}\Pi_g)/gr_{-m}\Pi_g).$$

This is called the Johnson homomorphism [17] (see Morita [22]).

D. Johnson proved that this is an isomorphism for $m = 1$, but for general m it is not necessarily surjective; actually S. Morita gave an obstruction called Morita-trace [22] for m odd, $m \geq 3$.

We can define the same filtration for

$$\pi_1^{alg}(\mathcal{M}_g) \to \mathrm{Out}(\Pi_g^l),$$

and then we have an injection

$$gr_{-m}(\pi_1^{alg}(\mathcal{M}_g)) \hookrightarrow (h_{g,*}/gr_{-m}\Pi_g) \otimes \mathbb{Z}_l \subset \mathrm{Hom}(H, gr_{-m-1}\Pi_g^l)/gr_{-m}\Pi_g^l.$$

Theorem 5.1 asserts that

$$gr_{-m}(\pi_1^{alg}(\mathcal{M}_{g,n} \otimes \overline{\mathbb{Q}})) \hookrightarrow gr_{-m}(\pi_1^{alg}(\mathcal{M}_{g,n}))$$

is not surjective for some m, it has cokernel of rank independent of g, n. As I am going to explain in the next section, for $(g, n) = (3, 0)$, it is known that this cokernel is nontrivial at least for $m = 4k + 2$, $k \geq 1$ (and the rank has a lower bound which is a linear function of m). Thus,

$$gr_{-m}(\pi_1^{alg}(\mathcal{M}_{g,n} \otimes \overline{\mathbb{Q}})) \hookrightarrow h_{g,*}/gr_{-m}\Pi_g \otimes \mathbb{Z}_l$$

has also cokernel of at least that rank. This homomorphism is given by $\otimes\mathbb{Z}_l$ from the Johnson homomorphism, hence this gives an obstruction to the surjectivity of Johnson homomorphisms, which is different from Morita's trace. The existence of such an obstruction was conjectured by Takayuki Oda, and proved by myself [21] and H. Nakamura [23] independently.

5.3. **The projective line minus three points again.** Let $\mathbb{P}^1_{01\infty}$ denote the projective line minus three points over $\mathbb{Q}$. This curve does not deform over $\overline{\mathbb{Q}}$, and hence the universal family is trivial,

$$\mathcal{C}_{0,3} = \mathbb{P}^1_{01\infty} \text{ and } \mathcal{M}_{0,3} = \mathrm{Spec}\mathbb{Q}.$$

Geometrically, thus, there is no monodromy, but arithmetically this has huge monodromy as proved by Belyi (see §2.3).

Fix a prime l. We shall consider pro-l completion F_2^l of the free group F_2 in two generators, so we have

$$\pi_1^{alg}(\mathbb{P}^1_{01\infty} \otimes \overline{\mathbb{Q}}, a) = \widehat{F}_2 \to F_2^l.$$

Then, we have a group homomorphism

$$\rho^l_{\mathbb{P}^1_{01\infty}} : G(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{Out}(F_2^l).$$

The weight filtration for the $(g, n) = (0, 3)$ curve essentially coincides with the lower central series

$$F_2^l = F_2^l(1) \supset F_2^l(2) \supset F_2^l(3) \supset \cdots$$

defined inductively by $F_2^l(1) = F_2^l$, $F_2^l(m) = [F_2^l(m-1), F_2^l]$ (here $[,]$ denotes the closure of the commutator); the correspondence is

$$W_{-2m+1}(F_2^l) = W_{-2m}(F_2^l) = F_2^l(m) \quad (m \geq 1).$$

Ihara [11] started to study the filtration of $G(\overline{\mathbb{Q}}/\mathbb{Q})$ induced by this filtration, independently of the notion of weight etc. Note that, the case of $(0,3)$ in Theorem 5.1, the geometric part vanishes, so the quotient in the theorem is nothing but just $GrG(\overline{\mathbb{Q}}/\mathbb{Q})$ in this case.

The following is a corollary of the theory of power-series by Anderson, Coleman, Ihara, together with Soulé's nonvanishing of Galois cohomology (there is a list of references, see the references in [11]).

Theorem 5.2. *In the Lie algebra $Gr(G(\overline{\mathbb{Q}}/\mathbb{Q})$, each $gr_{-m}(G(\overline{\mathbb{Q}}/\mathbb{Q}))$ does not vanish for odd $m \geq 3$.*

Roughly speaking, by using Anderson-Coleman-Ihara's power-series, one can construct a homomorphism

$$gr_{-2m}(G(\overline{\mathbb{Q}}/\mathbb{Q})) \to \mathrm{Hom}_{G(\overline{\mathbb{Q}}/\mathbb{Q})}(\pi_1(\mathrm{Spec}\mathbb{Z}[1/l]), \mathbb{Z}_l(m)).$$

It can be described as a particular Kummer cocycle, and the morphism does not vanish for odd $m \geq 3$ by Soulé's result. The right-hand side is rank 1 up to torsion. An element $\sigma_{2m} \in gr_{-2m}(G(\overline{\mathbb{Q}}/\mathbb{Q}))$ which does not vanish in the right-hand side is called a Soulé element.

The following conjecture is often contributed to Deligne [4].

Conjecture 5.2. (i) $\mathrm{Gr}(G(\overline{\mathbb{Q}}/\mathbb{Q})) \otimes \mathbb{Q}_l$ *is generated by σ_{2m} ($m \geq 3$, odd).*
(ii) $\mathrm{Gr}(G(\overline{\mathbb{Q}}/\mathbb{Q})) \otimes \mathbb{Q}_l$ *is a free graded Lie algebra.*

The rank of $\mathrm{Gr}_m(G(\overline{\mathbb{Q}}/\mathbb{Q}))$ as $\mathbb{Z}_l$-module has a lower bound which is a linear function of m, and these conjectures are verified for $m \leq 11$ [20] [27], but both conjectures seem to be still open. Ihara [14] recently showed that (ii) implies (i).

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